Hermite-Hadamard type integral inequalities for s-convex functions

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Dedication

This thesis is dedicated to my parents.
For their endless love, prayer, support and encouragement.

Acknowledgments

My first thanks to the Lord of heavens and earth. To Whom belongs all knowledge and intellect. I thank Him for making this possible. I am thankful to my supervisor Dr. Matloob Anwar for paving this path for me. His utter guidance, kindness and precious time made this whole process look easier and vivid. I owe to thank him for being the sole guide in my research work. I owe a lot to my family and friends for constantly ushering me to concentrate and progress in my research. Their all frustrating and troublesome phases of each and every research work. Moreover, I thank all my teachers who taught me the skills to research, analyze, review, read and write.

All of the mistakes, negligence, delays, ignorance belongs to me solely and I totally take responsibility for that. I believe that attributing mistakes as experiences is a form of cowardice, hence I open heartedly accept my faults and don't feel guilty. These mistakes eventually lead me to a new avenue of success someday as they original, and a profound work in good faith.

Abstract

One of the impressive application of the theory of convex functions is to the study of classical inequalities. Here, we show that how the theory provides an elementary, elegant, and unified treatment of some of the best known inequalities in mathematics. The fundamental purpose of this thesis is to establish some new Hermite-Hadamard type integral inequalities associated to s-convex function for $(2\omega+1)$ times differentiable functions. We assume definite integrable function that can be differentiated up to $(2\omega+1)$ times on closed interval $[0,1]$. The integrable function that we assumed, has an isolated singularities on 0 and 1. So, it is an improper integral. Our first purpose was to remove these isolated singularities. Henceforth, to remove these isolated singularities we solved this improper integral by famous integration technique namely as integration by parts. After, solving and making some substitution we observed that it has no singularities on 0 and 1. The improper integral turns into proper integral. Here, we also used Binomial expansion to write integrable function in a compact form. The result that we obtain, named as a lemma. Then, we associates that lemma with Hermite-Hadamard type integral inequalities for s-convex function. We introduced several new results associated to s-convex function and extended s-convex functions. We used some famous integral inequalities i.e. classical Hermite-Hadamard integral inequality, power mean's integral inequality, Holder's integral inequality and Jensen integral inequality in order to obtain new results. These famous integral inequalities helps us a lot to solve our problem related to s-convex function.

Contents

List of Abbreviations

Unless specified otherwise, the following notations are used in this thesis.

$\mathbb{R}$	:	$\{x : -\infty < x < \infty\}$
ω	:	Set of all positive integers.
I	:	Any interval in $\mathbb{R}$.
I^o	:	Interior of I (Largest open set contained in I)
$[\alpha_1, \alpha_2]$	:	$\{x \in \mathbb{R} : \alpha_1 \leq x \leq \alpha_2\}$
(α_1, α_2)	:	$\{x \in \mathbb{R} : a < x < b\}$
$h^{(v)}$	:	Five times derivative of a function h.
$h^{(2\omega+1)}$	:	$2\omega+1$ times derivative of a function h.
$L[\alpha_1, \alpha_2]$	:	Lebesgue integrable function on $[\alpha_1, \alpha_2]$.

Chapter 1

Preliminaries of convex and s-convex functions

Introduction

Generally, the word "inequality" refers the difference between two quantities, that interrelate one quantity to another quantity by some relation [25]. In Mathematics these relations are less than or greater than are mostly used. In simple words, someone can say that "inequality" means any two quantities that are not equal. In our daily life when someone is comparing scalar quantities like ages, weights, masses, lengths etc. Actually someone is using inequalities. In Mathematics a lot of work has been started on inequalities in the 19th century, and some well known inequalities came into exist like Holder's inequality, Jensen's inequality, Power mean integral inequality and classical Hermite-Hadamard type inequality etc. In these days inequalities play a very significant role in many fields of engineering and physical sciences. It has several applications in the field of engineering, mathematics and interrelated disciplines many more. One of the very interesting problem in computational mathematics, to asses the definite integral of a real valued function f(t) on a closed interval $[a, b]$. Henceforth, to tackle these type of problem many techniques and methods are appear in literature [32].

We observe that in the theory of convex function the Hermite-Hadamard inequality has a fundamental role. It has been used as a tool to obtain many results in integral inequalities, approximation theory, optimization theory and numerical analysis. Since, many results have been obtained in numerical analysis, optimization theory, by using Hermite-Hadamard type inequality. So, we can say that in the theory of convex function the classical Hermite-Hadamard inequality has a primitive role. It is known that, convexity plays a major role in the evolution of numerous branches of mathematics.

1.1 Convex Sets:

The convex set has an ordinary concept. A set in space X is convex if whenever it contains two points, it also contain the line segment joining them. Ellipses, triangles, cubes, balls, half spaces, parallelograms all are convex. Vertex set of a cube, an annulus, a crescent all are non convex [1].

Definition 1.1.1. *A set X is said to be convex set if for any two pair of points x_1,$x_2 \in X$, line segment joining these two points must contain in X that means for all x_1,$x_2 \in X$ and for any ζ such that $0 \leq \zeta \leq 1$, we obtain* [32]

$$\zeta x_1 + (1-\zeta)x_2 \in X.$$

Remark 1. *The empty set is trivially convex. Every one-point set x is convex* [30].

Example 1.1.1. *Half-spaces are convex* [2]*. We show that the closed half space A in $\mathbb{R}^n$ defined by the inequality $w.x \leq \mu_o$ is convex.*
To prove this, let $x, y \in A$ and $\lambda, \mu \geq 0$ such that $\lambda + \mu = 1$, we have

$$\begin{aligned} w.(\lambda x + \mu y) &= \lambda w.x + \mu w.y \\ &\leq \lambda\mu_o + \mu\mu_o = \mu_o. \end{aligned}$$

Hence, $\lambda x + \mu y \in A$ that proves that A is convex. Similarly, in the same fashion we can prove that the open Half spaces are convex.

1.2 Properties of convex sets:

The following results gives the important properties of convex sets [2].

Theorem 1. *The intersection of an arbitrary family of a convex sets in R^n is also a convex set.*

Theorem 2. *Suppose that $a_1, a_2, a_3 \cdots a_m$ be any arbitrary points of a convex set A in R^n. Let $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_m \geq 0$ such that $\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_m = 1$. Then $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + ... + \lambda_m a_m \in A$.*

Remark 2. *A point **x** is said to be **convex combination** of points $a_1, a_2, a_3 \cdots a_m \in R^n$, if there exists $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_m \geq 0$ be the scalars, such that $\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_m = 1$. Then,* [25]

$$\boldsymbol{x}=\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + ... + \lambda_m a_m.$$

Remark 3. *Every convex combination of points of a convex set in R^n belongs to that set.*

1.3 Convex function:

Definition 1.3.1. *A function Ψ is supposed to be convex on an interval $[a, b]$ if for any pair of points $x_1, x_2 \in [a, b]$, and for any ζ such that $0 \leq \zeta \leq 1$, we have* [3].

$$\Psi[\zeta x_1 + (1-\zeta)x_2] \leq \zeta\Psi(x_1) + (1-\zeta)\Psi(x_2). \tag{1.1}$$

holds.

Graphical interpretation of convex function is the line segment for every pair of points must lie on or above the function's graph [28]. It is not always possible to check convexity or concavity by plotting their graphs. So there is another suitable way to check convexity or concavity through second derivative.

Theorem 3. *If Ψ is twice differentiable on interval $[c, d]$, then a necessary and sufficient condition for function to be convex is that the second derivative is greater than or equal to zero for all $x \in [c, d]$* [28].

Remark 4. *By adding any two convex function that has been defined on the same interval; then the resultant function that we have obtained is again a convex function,if any of them is strictly convex then sum will also be strictly convex* [1].

Remark 5. *Multiplying a positive scalar by strictly convex function we also obtained a strictly convex function* [1].

Example 1.3.1. *Let $h : \mathbb{R} \to \mathbb{R}$ defined as; $h(x) = x^2$. Then $h(x)$ is convex function.*

Proof. We will use the definition of convex function, to show $h(x)$ is convex function.

$$h\big((1-\zeta)x + \zeta y\big) \leq (1-\zeta)h(x) + \zeta h(y). \tag{1.2}$$

Let $1 - \zeta = \mu$, so the above inequality directly becomes;

$$h\big(\mu x + \zeta y\big) \leq \mu h(x) + \zeta h(y).$$

such that $\zeta, \mu \geq 0$ with $\zeta + \mu = 1$. This implies that

$$\mu h(x) + \zeta h(y) - h\big(\mu x + \zeta y\big) \geq 0$$

Now,

$$\begin{aligned}
\mu h(x) + \zeta h(y) - h\big(\mu x + \zeta y\big) &= \mu x^2 + \zeta y^2 - (\mu x + \zeta y)^2 \\
&= \mu x^2 - \mu^2 x^2 + \zeta y^2 - \zeta^2 y^2 - 2\mu\zeta xy \\
&= \mu x^2(1-\mu) + \zeta y^2(1-\zeta) - 2\mu\zeta xy \\
&= \zeta\mu x^2 + \zeta\mu y^2 - 2\mu\zeta xy \\
&= \zeta\mu(x^2 + y^2 - 2xy) \\
&= \zeta\mu(x-y)^2 \geq 0.
\end{aligned}$$

This shows that $h(x) = x^2$ is a convex function. □

1.4 s-convex function:

Definition 1.4.1. *A function $\Psi : [0, \infty) \to \mathbb{R}$ is supposed to be s-convex in the second sense if the inequality* [3],

$$\Psi[\zeta x + (1-\zeta)y] \leq \zeta^s \Psi(x) + (1-\zeta)^s \Psi(y). \tag{1.3}$$

holds for all $x, y \in [0, \infty)$, $\zeta \in [0, 1]$ *and* $s \in (0, 1]$.

Example 1.4.1. *The function* [22] $f(x) = x^s$ *is* s *convex on* $[0, 1]$
We have,

$$\begin{aligned} \int_0^1 x^s dx &= \frac{1}{s+1} \\ &= \frac{f(0) + f(1)}{s+1} \end{aligned} \tag{1.4}$$

1.5 Extended s-convex function:

Definition 1.5.1. *A function* $\Psi : I \subseteq [0, \infty] \to \mathbb{R}$ *is supposed to be extended s-convex function in the second sense if the inequality* [5],

$$\Psi[\zeta x + (1 - \zeta)y] \leq \zeta^s \Psi(x) + (1 - \zeta)^s \Psi(y). \tag{1.5}$$

holds for all $x, y \in I$, *and* $\zeta \in [0, 1]$, *for some* $s \in [-1, 1]$.
Here, we have an example of extended s-convex function.

Example 1.5.1. *Suppose that* [5],

$$h(r) = 1 - r^2$$

for $r \in [0, 1]$.
By definition of extended s-convex function (1.5),

$$\Psi(rx + (1 - r)y) \leq r^s \Psi(x) + (1 - r)^s \Psi(y).$$

$$\Psi(rx + (1 - r)y) - r^s \Psi(x) - (1 - r)^s \Psi(y) \leq 0.$$

For extended s-convex function $s = -1$, *then,*

$$\Psi(rx + (1 - r)y) - \frac{\Psi(x)}{r} - \frac{\Psi(y)}{1 - r} \leq 0.$$

$$r(1 - r)[\Psi(rx + (1 - r)y)] - (1 - r)\Psi(x) - r\Psi(y) \leq 0.$$

$$-(1 - r)(1 - x^2) - r(1 - y^2) + r(1 - r)[1 - (rx + (1 - r)y)^2] \leq 0.$$

$$(1-r)(1-x^2)+r(1-y^2)-r(1-r)\big[1-(rx+(1-r)y)^2\big]\geq 0.$$

for every $x, y \in [0,1]$ *and* $r \in (0,1)$. *This means that h(r) is an extended -1 convex function in the second sense on* $[0,1]$.

The defining inequality inequality for a convex function implies a more general one, known as Jensen's inequality.

1.6 Jensen's Inequality:

The Jensen inequality was derived by Danish mathematician, John Jensen in 1906 [25]. The classical literature of mathematics involves a comprehensive study of the inequalities which is used excessively in mathematics. The critical analysis of inequalities demonstrate the novel feature of current mathematics "inequalities" by Hardy at all was published in 1934 [29]. This book describes the inequalities in a very efficient and sophisticated manner. These famous mathematician not only explained and demonstrated this subject with its due but also they made " inequalities" popular among their peers. An "Introduction to inequalities" by Beckenbach and Bellman brings forth a well described, brief and comprehensive introduction to inequalities in 1975 [30].
Jensen's inequality has several different forms.In simple words, Jensen,s inequality demonstrate that the convex transformation of a mean is less than or equal to the mean applied after convex transformation.

Definition 1.6.1. *Suppose* $\alpha_1, \alpha_2 \cdots \alpha_n$ *are positive numbers such that* $\sum_{\iota=1}^{n} \alpha_\iota = 1$ *and g is real valued continuous convex function, then* [25],

$$g\bigg(\sum_{\iota=1}^{n} \alpha_\iota x_i]\bigg) \leq \sum_{\iota=1}^{n} \alpha_\iota g(x_\iota).$$

Chapter 2

Some well known results of Hermite-Hadamard type inequalities for s-convex function

Originally convexity belongs to geometry but it is wide spread in other mathematical fields simultaneously like calculus of variation, functional analysis, graph theory, probability theory, complex analysis and many other fields. Moreover, convexity has significant interdisciplinary features and occupies essential position in the field of chemistry, biology and other sciences. However this aspect of convexity would not be the main focus in this research.

A short background of convexity is being given here. Convexity has a history going back to Greek, Egypt and Babylonian times. It is assumed that it is quite younger than numbers but basic geometric drawings are traced to the initial stage of human civilization. It is difficult to ascertain the first person who first defined convexity. Supposedly "Archimedes" was the first one to define "Convexity". His definitions and postulates remained in the dark for almost two thousand years. Though, the mathematical experts were aware of these. Though 17th century calculus was at a primitive stage and convexity was not take as a priority.

Convex function and theories of inequalities have a very close relation. Convexity is a broad subject which also includes theory of convex functions. Convexity is a very powerful property of function. It is known as a natural property of functions. Furthermore, its minimization property makes it unique, novel and beneficial. Due to its minimization characteristic it possess a significant status in optimization theory, calculus of variation and probability theory. As the idea of convex functions has placed a significant role in modern Mathematics [25]. Since, we observed that a lot of research articles and books has been dedicated to this field in last number of years. The Hermite-Hadamard inequality has placed a significant role in the study of convex function [7]. In this sense it could be said that, the Hermite-Hadamard inequality is one of the fundamental result for convex functions with a natural geometrical interpretation and has a lot of applications, that attract much interest in elementary mathematics. Many researches have done a lot of work to refinement and extend it for many different classes of functions such as quasi-convex function, s-convex function, log-convex functions, p-functions, and r-convex functions.

In this Chapter, we have some well known results about classical Hermite-Hadamard inequality for s-convex functions. These results are obtained by using Holder-Iscan integral inequality and improved power mean integral inequality that provide better approach as compared to the results obtained by classical Holder's and power mean's inequalities. The followings are some well known theorems in the literature, that has been used to prove our succeeding theorems.

2.1 Fundamental inequalities:

Here, are some of the basic integral inequalities that are mostly used to prove our succeeding theorems.

Theorem 4. *(Holder's inequality for integrals): Assume that* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Supposing that* $g_1(\zeta)$, $g_2(\zeta)$ *are two real valued functions defined in the interval of* $[c, d]$, *and if* $| g_1 |^p$ *and* $| g_2 |^q$ *are integrable on* $[c, d]$, *then,*[31]

$$\int_c^d | g_1(\zeta)g_2(\zeta) | \, d\zeta \leq \left(\int_c^d | g_1(\zeta) |^p \, d\zeta \right)^{\frac{1}{p}} \left(\int_c^d | g_2(\zeta) |^q \, d\zeta \right)^{\frac{1}{q}}. \qquad (2.1)$$

with equality holds if and only if $A | g_1(\zeta) |^p = B | g_2(\zeta) |^q$ *almost everywhere, where* A *and* B *are constants.*

The Holder's inequality has essential significance in numerous branches of pure and applied mathematics [30]*. It also has many applications in the theory of convex functions.*

In some sense the Holder's integral inequality also illustrated in the following way.

Theorem 5. *(Power-mean integral inequality) Suppose that $q \geq 1$. If g_1 and g_2 are two real valued functions define in the interval $[c, d]$, and if $\mid g_1 \mid$ and $\mid g_2 \mid$ are integrable on $[c, d]$, then,* [31].

$$\int_c^d \mid g_1(\zeta) g_2(\zeta) \mid d\zeta \leq \left(\int_c^d \mid g_1(\zeta) \mid d\zeta \right)^{1-\frac{1}{q}} \left(\int_c^d \mid g_1(\zeta) \mid\mid g_2(\zeta) \mid^q d\zeta \right)^{\frac{1}{q}}. \tag{2.2}$$

2.2 Some results of s-convex function:

Lemma 2.2.1. *Suppose that $g : J \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is differentiable function up to three times on J^o and $c,d \in J$ with $c < d$. If $g^{'''} \in L_1[c, d]$ and $0 \leq \zeta \leq 1$, then* [5].

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d g(t)dt - \frac{3\psi^2 - 3\psi + 1}{6}(d-c)^2 g^{''}(\psi c + (1-\psi)d) \\
&- \frac{2\psi - 1}{2}(d-c)g^{'}(\psi c + (1-\psi)d) - g(\psi c + (1-\psi)d) \\
&= \frac{(d-c)^3}{6}\left[\psi^4 \int_0^1 t^3 g^{'''}(t(\psi c + (1-\psi)d) + (1-t)d)dt \right. \\
&\left. - (1-\psi)^4 \int_0^1 t^3 g^{'''}(t(\psi c + (1-\psi)d) + (1-t)c)dt \right].
\end{aligned}$$

Theorem 6. *Suppose that $g : J \subseteq [0, \infty) \to \mathbb{R}$ is a three times differentiable function on J, $c, d \in J$ with $c < d$, $g^{'''} \in L_1[c, d]$, and $0 \leq \psi \leq 1$. If $\mid g^{'''}(t) \mid^q$ is an extended s-convex function in the second sense on $[c, d]$, $s \in [-1, 1]$, and $q \geq 1$, then* [5].

$$\left|\frac{1}{d-c}\int_c^d g(t)dt - \frac{3\psi^2-3\psi+1}{6}(d-c)^2 g^{''}(\psi c+(1-\psi)d)\right.$$
$$\left. - \frac{2\psi-1}{2}(d-c)g^{'}(\psi c+(1-\psi)d) - g(\psi c+(1-\psi)d)\right|$$
$$\leq \frac{(d-c)^3}{24}\left[\frac{2}{3\psi^4(1-\psi)^4}\right]^{\frac{1}{q}}$$
$$\left\{(1-\psi)^4\left[2(1-\psi)^3\psi^4 \mid g^{'''}(d) \mid^q - \psi^4(2(1-\psi)^3+3(1-\psi)^2+(1-\psi)+6ln\psi) \mid g^{'''}(c) \mid^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \psi^4\left[2(1-\psi)^4\psi^3 \mid g^{'''}(c) \mid^q - (1-\psi)^4(2\psi^3+3\psi^2+6\psi+6ln(1-\psi)) \mid g^{'''}(d) \mid^q\right]^{\frac{1}{q}}\right\}. \tag{2.3}$$

Theorem 7. *Suppose that $g: J \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to three times on J, where $c, d \in J$ with $c < d$, $g^{'''} \in L_1[c,d]$, and $0 \leq \psi \leq 1$. If $\mid g^{'''}(t) \mid^q$ is an extended s-convex function in the second sense on $[c,d]$, $-1 < s \leq 1$, and $q \geq 1$, subsequently,* [5].

$$\left|\frac{1}{d-c}\int_c^d g(t)dt - \frac{3\psi^2-3\psi+1}{6}(d-c)^2 g^{''}(\psi c+(1-\psi)d)\right.$$
$$\left. - \frac{2\psi-1}{2}(d-c)g^{'}(\psi c+(1-\psi)d) - g(\psi c+(1-\psi)d)\right|$$
$$\leq \frac{(d-c)^3}{6}\left(\frac{1}{3q+s+1}\right)^{\frac{1}{q}} \tag{2.4}$$
$$\left\{(1-\psi)^4\left[\mid g^{'''}(\psi c+(1-\psi)d) \mid^q + (3q+s+1)\beta(3q+1,s+1) \mid g^{'''}(c) \mid^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \psi^4\left[\mid g^{'''}(\psi c+(1-\psi)d) \mid^q + (3q+s+1)\beta(3q+1,s+1) \mid g^{'''}(b) \mid^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 8. *Suppose that $g: J \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to three times on J, $c, d \in J$ with $c < d$, $g^{'''} \in L_1[c,d]$, and $0 \leq \psi \leq 1$. If $\mid g^{'''}(t) \mid^q$ is an extended s-convex function in the second sense on $[c,d]$, $-1 < s \leq 1$, and $q > 1$,* *subsequently,* [5].

$$\left|\frac{1}{d-c}\int_c^d g(t)dt - \frac{3\psi^2-3\psi+1}{6}(d-c)^2 g^{''}(\psi c+(1-\psi)d)\right.$$
$$\left. - \frac{2\psi-1}{2}(d-c)g^{'}(\psi c+(1-\psi)d) - g(\psi c+(1-\psi)d)\right|$$
$$\leq \frac{(d-c)^3}{6}\left(\frac{q-1}{4q-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left\{(1-\psi)^4[\mid g^{'''}(\psi c+(1-\psi)d)\mid^q + \mid g^{'''}(c)\mid^q]^{\frac{1}{q}}\right.$$
$$\left. + \psi^4[\mid g^{'''}(\psi c+(1-\psi)d)\mid^q + \mid g^{'''}(d)\mid^q]^{\frac{1}{q}}\right\}. \tag{2.5}$$

Lemma 2.2.2. *Suppose that $h : J \subset \mathbb{R}^+ \cup [0\} \to \mathbb{R}$ is r-times differentiable function on J^o with $h^{(r)} \in L^1[c,d]$, subsequently,* [8].

$$\begin{aligned}(-1)^r\int_c^d h(u)du &= \sum_{k=1}^{r}(-1)^{r-k+1}\left[\frac{(t-c)^k-(t-d)^k}{k!}\right]h^{(k-1)}(t)\\ &+\frac{1}{r!}\left[\int_c^t (u-c)^r h^{(r)}(u)du + \int_t^d (u-d)^r h^{(r)}(u)du\right].\end{aligned} \tag{2.6}$$

Theorem 9. *Suppose that $g : J \subset \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to r-times on J^o such that $g^{(r)} \in L^1[c,d]$, where $c,d \in J$, $c<d$. If $\mid g^{(r)} \mid$ is s-convex on $[c,d]$ for some fixed $0 < s \leq 1$, then for every $\psi \in [c,d]$, subsequently,* [8].

$$\begin{aligned}&\left|(-1)^r\int_c^d g(u)du + \sum_{k=1}^{r}(-1)^{r-k+2}\left[\frac{(\psi-c)^k-(\psi-d)^k}{k!}\right]g^{k-1}(\psi)\right|\\ &\leq \frac{1}{k!}\Big[\beta(s+1,r+1)\left((\psi-c)^{r+1}\mid g^r(c)\mid + (d-\psi)^{r+1}\mid g^r(d)\mid\right)\\ &+\beta(1,r+s+1)\left((\psi-c)^{r+1}+(d-\psi)^{r+1}\right)\mid g^{(r)}(\psi)\mid\Big].\end{aligned} \tag{2.7}$$

Theorem 10. *Suppose that $g : J \subset \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to r-times on J^o such that $g^{(r)} \in L^1[c,d]$, where $c,d \in J$, $c<d$. If $\mid g^{(r)} \mid^q$ is s-convex on*

$[c,d]$ *for some fixed* $0 < s \leq 1$, *and* $q \geq 1$. *Then for every* $\psi \in [c,d]$, *we have* [8].

$$\left|(-1)^r \int_c^d g(x)dx + \sum_{k=1}^{r}(-1)^{r-k+2}\left[\frac{(\psi-c)^k-(\psi-d)^k}{k!}\right]g^{(k-1)}(\psi)\right|$$
$$\leq \frac{(r+1)^{-\frac{1}{p}}}{r!}\left[(\psi-c)^{r+1}\left\{\beta(s+1,r+1)\mid g^r(c)\mid^q+\beta(1,r+s+1)\mid g^r(\psi)\mid^q\right\}^{\frac{1}{q}}\right. \quad (2.8)$$
$$\left.+(d-\psi)^{r+1}\left\{\beta(s+1,r+1)\mid g^r(d)\mid^q+\beta(1,r+s+1)\mid g^{(r)}(\psi)\mid^q\right\}^{\frac{1}{q}}\right].$$

Lemma 2.2.3. *Lets* $g : J \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *is differentiable function on* J^o *where* $c,d \in J$ *with* $c < d$. *If* $g' \in L[c,d]$, *then the following inequality holds* [9].

$$\frac{g(c)+g(d)}{2} - \frac{1}{d-c}\int_c^d g(x)dx = \frac{d-c}{2}\int_0^1 (1-2t)g'(tc+(1-t)d)dt. \quad (2.9)$$

Theorem 11. *Suppose that* $g : J^o \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *is a differentiable function on* J^o, *also* $c,d \in J^o$ *with* $c < d$ *and* $g' \in L[c,d]$. *If* $\mid g' \mid^q$ *is s-convex on* $[c,d]$. *If* $p > 1$ *such that* $q = \frac{p}{p-1}$, *subsequently,*[13]

$$\left|\frac{g(c)+g(d)}{2} - \frac{1}{d-c}\int_c^d g(x)dx\right| \leq \frac{d-c}{2(p+1)^{\frac{1}{p}}}\left(\frac{\mid g'(c)\mid^q + \mid g'(d)\mid^q}{s+1}\right)^{\frac{1}{q}}. \quad (2.10)$$

Theorem 12. *Suppose that* $g : J \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *is a differentiable function up to three times on J,* $c,d \in J$ *with* $c < d$, $g''' \in L_1[c,d]$, *and* $0 \leq \psi \leq 1$. *If* $\mid g'''(t) \mid^q$ *is an extended s-convex function in the second sense on [c,d],* $-1 < s \leq 1$, *and* $q \geq 1$, *subsequently,* [5].

$$\left|\frac{1}{d-c}\int_c^d g(t)dt - \frac{3\psi^2-3\psi+1}{6}(d-c)^2 g''(\psi c+(1-\psi)d)\right.$$
$$\left.-\frac{2\psi-1}{2}(d-c)g'(\psi c+(1-\psi)d) - g(\psi c+(1-\psi)d)\right|$$
$$\leq \frac{(d-c)^3}{6}\left(\frac{1}{3q+s+1}\right)^{\frac{1}{q}} \quad (2.11)$$
$$\left\{(1-\psi)^4\Big[\mid g'''(\psi c+(1-\psi)b)\mid^q+(3q+s+1)\beta(3q+1,s+1)\mid g'''(c)\mid^q\Big]^{\frac{1}{q}}\right.$$
$$\left.+\psi^4\Big[\mid g'''(\psi c+(1-\psi)d)\mid^q+(3q+s+1)\beta(3q+1,s+1)\mid g'''(d)\mid^q\Big]^{\frac{1}{q}}\right\}.$$

Chapter 3

Some generalized results of Hermite-Hadamard type inequalities for s-convex function

To obtain results, in more general form we prove a lemma for $(2\omega + 1)$ times differentiable function. We need a lemma for the purpose of establishing main theorems of s-convex functions. We mainly use this for showing results of Hermite-Hadamard type inequalities for s-convex function. This will play a significant role by relating it with Hermite-Hadamard types inequalities for s-convex function as well as the the other types of integral inequalities like power means integral inequalities, Holder's integral inequality, to obtain the results associated to s-convex function.

3.1 Some generalized results for s-convex function:

Lemma 3.1.1. *Assume that $h : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a $(2\omega+1)$ times differentiable function on $\mathbb{R}^+$ in such a way that $\alpha_1, \alpha_2 \in \mathbb{R}^+$. When $h^{(2\omega+1)} \in L_1([\alpha_1, \alpha_2])$, moreover*

$0 \leq \zeta \leq 1$, *subsequently,*

$$\left(\sum_{\gamma=0}^{2\omega+1}(-1)^{\gamma+1}\binom{2\omega+1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega+1)}\right)\frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=o}^{2\omega}(-1)^{\gamma}\binom{2\omega}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{(2\omega-1)}(-1)^{(\gamma+1)}\binom{2\omega-1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-1)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega-2}(-1)^{\gamma}\binom{2\omega-2}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-2)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega-3}(-1)^{(\gamma+1)}\binom{2\omega-3}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-3)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!}h^{(2\omega-4)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\cdots+\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}h(t)dt$$
$$=\frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!}\left[\zeta^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt\right.$$
$$\left.-(1-\zeta)^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt\right]. \tag{3.1}$$

Proof. We assume a following improper definite integral. Here, we consider $0 < \zeta < 1$.

$$\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt. \tag{3.2}$$

Now, we solve the above definite integral by famous technique namely as integration by parts. Here, we consider the internal part of above integral equation (3.2) and make it in a simplified form for the convenience of integration.

$$\begin{aligned}&t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\\&=\frac{-1}{\zeta(\alpha_2-\alpha_1)}t^{(2\omega+1)}dh^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2).\end{aligned} \tag{3.3}$$

So, equation (3.2) becomes;

$$\frac{-1}{\zeta(\alpha_2-\alpha_1)}\int_0^1 t^{(2\omega+1)}dh^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt. \tag{3.4}$$

Using integration by parts equation (3.4) implies

$$\frac{-1}{\zeta(\alpha_2-\alpha_1)}t^{(2\omega+1)}h^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\ |_0^1$$
$$+\frac{2\omega+1}{\zeta(\alpha_2-\alpha_1)}\int_0^1 t^{2\omega}h^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{2\omega}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2}\int_0^1 t^{2\omega}dh^{(2\omega-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2}t^{2\omega}h^{(2\omega-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\ |_0^1$$
$$+\frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2}\int_0^1 2\omega t^{2\omega-1}h^{(2\omega-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\frac{(2\omega+1)(2\omega)}{\zeta^2(\alpha_2-\alpha_1)^2}\int_0^1 t^{2\omega-1}h^{(2n-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{(2\omega+1)(2\omega)}{\zeta^3(\alpha_2-\alpha_1)^3}\int_0^1 t^{2\omega-1}dh^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{(2\omega+1)(2\omega)}{\zeta^3(\alpha_2-\alpha_1)^3}t^{2\omega-1}h^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2)\ |_0^1$$
$$+\frac{(2\omega+1)(2\omega)(2\omega-1)}{\zeta^3(\alpha_2-\alpha_1)^3}\int_0^1 t^{2\omega-2}h^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{(2\omega+1)(2\omega)}{\zeta^3(\alpha_2-\alpha_1)^3}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\frac{(2\omega+1)(2\omega)(2\omega-1)}{\zeta^3(\alpha_2-\alpha_1)^3}\int_0^1 t^{2\omega-2}h^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$

$$
\begin{aligned}
&= \frac{-1}{\zeta(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)}{\zeta^3(\alpha_2-\alpha_1)^3} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)}{\zeta^4(\alpha_2-\alpha_1)^4} \int_0^1 t^{2\omega-2} dh^{(2\omega-3)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt \\
&= \frac{-1}{\zeta(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)}{\zeta^3(\alpha_2-\alpha_1)^3} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)}{\zeta^4(\alpha_2-\alpha_1)^4} t^{2\omega-2} h^{(2\omega-3)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2) \Big|_0^1 \\
&\quad + \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)}{\zeta^4(\alpha_2-\alpha_1)^4} \int_0^1 t^{2\omega-3} h^{(2\omega-3)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt \\
&= \frac{-1}{\zeta(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)}{\zeta^3(\alpha_2-\alpha_1)^3} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{(2\omega+1)(2\omega)(2\omega-1)}{\zeta^4(\alpha_2-\alpha_1)^4} h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)}{\zeta^4(\alpha_2-\alpha_1)^4} \int_0^1 t^{2\omega-3} h^{(2\omega-3)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt
\end{aligned}
$$

Here, we observe the behavior of integration of above definite integral, that looks like in certain definite pattern, so we can write $(2\omega+1)$th term of above definite integral. Particularly, the generalization for $(2\omega+1)$ times is given as follows;

$$
\begin{aligned}
&\int_0^1 t^{(2\omega+1)} h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt \\
&= \frac{-1}{\zeta(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{\zeta^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)}{\zeta^3(\alpha_2-\alpha_1)^3} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{(2\omega+1)(2\omega)(2\omega-1)}{\zeta^4(\alpha_2-\alpha_1)^4} h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)}{\zeta^4(\alpha_2-\alpha_1)^4} h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \cdots - \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)\cdots 5.4.3.2}{\zeta^{2\omega+1}(\alpha_2-\alpha_1)^{2\omega+1}} h(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)\cdots 5.4.3.2.1}{\zeta^{2\omega+1}(\alpha_2-\alpha_1)^{2\omega+1}} \int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt.
\end{aligned} \tag{3.5}
$$

Multiplying equation (3.5) by $\frac{\zeta^{2\omega+2}(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}$.

$$\begin{aligned}
&\frac{\zeta^{2\omega+2}(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt\\
&=\frac{-\zeta^{2\omega+1}(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{\zeta^{2\omega}(\alpha_2-\alpha_1)^{2\omega-1}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&-\frac{\zeta^{2\omega-1}(\alpha_2-\alpha_1)^{2\omega-2}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{\zeta^{2\omega-2}(\alpha_2-\alpha_1)^{2\omega-3}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\cdots-\zeta h(\zeta\alpha_1+(1-\zeta)\alpha_2)+\zeta\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt.
\end{aligned}\tag{3.6}$$

Here, we assume another definite integrable function on $[0,1]$ of the following form;

$$\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt.\tag{3.7}$$

Now, we consider the internal part of above integral equation (3.7) and simplify it for the ease of integration.

$$\begin{aligned}
&t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\\
&=\frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}t^{(2\omega+1)}dh^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1).
\end{aligned}\tag{3.8}$$

So, after simplification the above equation (3.7) becomes;

$$\frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}\int_0^1 t^{(2\omega+1)}dh^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt.\tag{3.9}$$

Using integration by parts equation (3.9) implies,

$$\begin{aligned}
&=\frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}t^{(2\omega+1)}h^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\Big|_0^1\\
&\quad-\frac{2\omega+1}{(1-\zeta)(\alpha_2-\alpha_1)}\int_0^1 t^{2\omega}h^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt\\
&=\frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad-\frac{2\omega+1}{(1-\zeta)(\alpha_2-\alpha_1)}\int_0^1 t^{2\omega}h^{(2\omega)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{2\omega}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} \int_0^1 t^{2\omega} dh^{(2\omega-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \\
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&- \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} t^{2\omega} h^{(2\omega-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1) \mid_0^1 \\
&\quad + \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} \int_0^1 2\omega t^{2\omega-1} h^{(2\omega-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \\
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} \int_0^1 t^{2\omega-1} h^{(2\omega-1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \\
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} \int_0^1 t^{2\omega-1} dh^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \\
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} t^{2\omega-1} h^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2) \mid_0^1 \\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} \int_0^1 t^{2\omega-2} h^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \\
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} \int_0^1 t^{2\omega-2} h^{(2\omega-2)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \\
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)}{(1-\zeta)^4(\alpha_2-\alpha_1)^4} \int_0^1 t^{2\omega-2} dh^{(2\omega-3)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&- \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}\int_0^1 t^{2\omega-3}h^{(2\omega-3)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt\\
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&- \frac{(2\omega+1)(2\omega)(2\omega-1)}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad + \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}\int_0^1 t^{2\omega-3}h^{(2\omega-3)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt
\end{aligned}
$$

Here, we observe the behavior of integration of above definite integral that looks like in certain pattern, so we can write $(2\omega+1)$th term of above definite integral i.e More specifically, Generalizing for $(2\omega+1)$ times.

$$
\begin{aligned}
&= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{2\omega+1}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad + \frac{(2\omega+1)(2\omega)}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad + \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)}{(1-\zeta)^5(\alpha_2-\alpha_1)^5}h^{(2\omega-4)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad + \cdots + \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)\cdots 5.4.3.2}{(1-\zeta)^{2\omega+1}(\alpha_2-\alpha_1)^{2\omega+1}}h(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{(2\omega+1)(2\omega)(2\omega-1)(2\omega-2)\cdots 5.4.3.2.1}{(1-\zeta)^{2\omega+1}(\alpha_2-\alpha_1)^{2\omega+1}}\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt.
\end{aligned}
\tag{3.10}
$$

In order to obtain some specific form, we multiply above equation (3.10) on both sides by $\frac{(1-\zeta)^{2\omega+2}(\alpha_2-a)^{2\omega+1}}{(2\omega+1)!}$. So, after some simplification we obtain a definite integral in the following form.

$$
\begin{aligned}
&\frac{(1-\zeta)^{2\omega+2}(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt\\
&=\frac{(1-\zeta)^{2\omega+1}(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&-\frac{(1-\zeta)^{2\omega}(\alpha_2-\alpha_1)^{2\omega-1}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\frac{(1-\zeta)^{2\omega-1}(\alpha_2-\alpha_1)^{2\omega-2}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&-\frac{(1-\zeta)^{2\omega-2}(\alpha_2-\alpha_1)^{2\omega-3}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\cdots+(1-\zeta)h(\zeta\alpha_1+(1-\zeta)\alpha_2)-(1-\zeta)\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt.
\end{aligned}
\tag{3.11}
$$

Now, we write the above Equation in Binomial series form:

$$
\begin{aligned}
&\frac{(1-\zeta)^{2\omega+2}(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt\\
&=\left(\sum_{\gamma=0}^{2\omega+1}(-1)^\gamma\binom{2\omega+1}{\gamma}\zeta^\gamma\frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right)\\
&-\left(\sum_{\gamma=0}^{2\omega}(-1)^\gamma\binom{2\omega}{\gamma}\zeta^\gamma\frac{(\alpha_2-\alpha_1)^{2\omega-1}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right)\\
&+\left(\sum_{\gamma=0}^{2\omega-1)}(-1)^\gamma\binom{2\omega-1}{\gamma}\zeta^\gamma\frac{(\alpha_2-\alpha_1)^{2\omega-2}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right)\\
&-\left(\sum_{\gamma=0}^{2\omega-2}(-1)^\gamma\binom{2\omega-2}{\gamma}\zeta^\gamma\frac{(\alpha_2-\alpha_1)^{2\omega-3}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right)\\
&+\cdots+(1-\zeta)h(\zeta\alpha_1+(1-\zeta)\alpha_2)-(1-\zeta)\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt
\end{aligned}
\tag{3.12}
$$

By subtracting the equation (3.12) from equation (3.6) term by term.

$$\frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\left[\zeta^{2\omega+2}\int_0^1 t^{2\omega+1}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt\right.$$
$$\left.-(1-\zeta)^{2\omega+2}\int_0^1 t^{2\omega+1}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt\right]$$
$$=\frac{-\zeta^{2\omega+1}(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega+1}(-1)^{\gamma+1}\binom{2\omega+1}{\gamma}\zeta^{\gamma}\right).\frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{2\omega}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{\zeta^{2\omega}(\alpha_2-\alpha_1)^{2\omega-1}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega}(-1)^{\gamma}\binom{2\omega}{\gamma}\zeta^{\gamma}\right)\frac{(\alpha_2-\alpha_1)^{2\omega-1}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{\zeta^{2\omega-1}(\alpha_2-\alpha_1)^{2\omega-2}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega-1}(-1)^{\gamma+1}\binom{2\omega-1}{\gamma}\zeta^{\gamma}\right)\frac{(\alpha_2-\alpha_1)^{2\omega-2}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{\zeta^{2\omega-2}(\alpha_2-\alpha_1)^{2\omega-3}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega-2}(-1)^{\gamma}\binom{2\omega-2}{\gamma}\zeta^{\gamma}\right)\frac{(\alpha_2-\alpha_1)^{2\omega-3}}{(2\omega-2)!}h^{2\omega-3}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\cdots-\zeta h(\zeta\alpha_1+(1-\zeta)\alpha_2)-(1-\zeta)h(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\zeta\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt+(1-\zeta)\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \tag{3.13}$$

So, the right side of above equation (3.13) becomes;

$$
\begin{aligned}
&= \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{2\omega+1} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{2\omega} \right) \frac{(\alpha_2-\alpha_1)^{2\omega-1}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-1} (-1)^{\gamma+1} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{2\omega-1} \right) \frac{(\alpha_2-\alpha_1)^{2\omega-2}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{2\omega-2} \right) \frac{(\alpha_2-\alpha_1)^{2\omega-3}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta a + (1-\zeta)\alpha_2) + \cdots \\
&- \zeta(h(\zeta\alpha_1 + (1-\zeta)\alpha_2)) - h(\zeta\alpha_1 + (1-\zeta)\alpha_2) + h\zeta(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \zeta \int_0^1 h(t(\zeta a + (1-\zeta)\alpha_2) + (1-t)\alpha_2)dt + \int_0^1 h(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1)dt \\
&- \zeta \int_0^1 h(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1)dt
\end{aligned}
$$

By making some simplifications and substitutions, also uses the change of variable and finally write up in the simplest form of the definite integral on $[\alpha_1, \alpha_2]$.

$$\begin{aligned}
&\left(\sum_{\gamma=0}^{2\omega+1}(-1)^{\gamma+1}\binom{2\omega+1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega+1)}\right)\frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta a+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega}(-1)^{\gamma}\binom{2\omega}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!}h^{(2\omega-1)}(\zeta a+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{(2\omega-1)}(-1)^{(\gamma+1)}\binom{2\omega-1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-1)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta a+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega-2}(-1)^{\gamma}\binom{2\omega-2}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-2)}\right)\frac{(\alpha_2-a)^{(2\omega-3)}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta a+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega-3}(-1)^{(\gamma+1)}\binom{2\omega-3}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-3)}\right)\frac{(\alpha_2-a)^{(2\omega-4)}}{(2\omega-3)!}h^{(2\omega-4)}(\zeta a+(1-\zeta)\alpha_2)\\
&+\cdots+\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}h(t)dt\\
&=\frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!}\left[\zeta^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt\right.\\
&\quad\left.-(1-\zeta)^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt\right].
\end{aligned} \tag{3.14}$$

□

Theorem 13. *Assume that $h : I \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to $(2\omega+1)$ times in the interior of I, having $\alpha_1, \alpha_2 \in I$ in order that $h^{(2\omega+1)} \in L([\alpha_1, \alpha_2])$, Furthermore $0 \le \zeta \le 1$. When $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function in the* *second sense on closed interval* $[\alpha_1, \alpha_2]$*, whereas* $s \in [-1, 1]$*, and* $q \geq 1$*, then,*

$$\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg|$$

$$\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{1}{2(\omega+1)} \right)^{1-\frac{1}{q}}$$

$$\Bigg[\zeta^{2\omega+2} \left(\frac{\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{2(\omega+1)+s} + \beta\big(2(\omega+1)+s, s+1\big) \mid h^{(2\omega+1)}(\alpha_1) \mid^q \right)^{\frac{1}{q}}$$

$$+ (1-\zeta)^{2\omega+2} \left(\frac{\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{2(\omega+1)+s} + \beta(2(\omega+1)+s, s+1) \mid h^{(2\omega+1)}(\alpha_2) \mid^q \right)^{\frac{1}{q}} \Bigg]. \tag{3.15}$$

Proof. Since, $0 \leq \zeta \leq 1$ and $-1 < s \leq 1$, $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$, then by (3.14) and the Holder's integrable inequality,

$$\left(\sum_{\gamma=0}^{2\omega+1}(-1)^{\gamma+1}\binom{2\omega+1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega+1)}\right)\frac{(\alpha_2-a)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega}(-1)^{\gamma}\binom{2\omega}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{(2\omega-1)}(-1)^{(\gamma+1)}\binom{2\omega-1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-1)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega-2}(-1)^{\gamma}\binom{2\omega-2}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-2)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\left(\sum_{\gamma=0}^{2\omega-3}(-1)^{(\gamma+1)}\binom{2\omega-3}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-3)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!}h^{(2\omega-4)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\cdots+\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}h(t)dt$$
$$\leq\frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!}\left[\zeta^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}\left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\right|dt\right.$$
$$\left.-(1-\zeta)^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}\left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\right|dt\right]. \tag{3.16}$$

By taking the right side of (3.16) of first integral and using Holder's integrable inequality. Also, we apply the Holder's integral inequality on the R.H.S of (3.16). We also know that $\mid h^{(2\omega+1)} \mid^q$ is an s-convex function in the second sense on closed interval $[\alpha_1, \alpha_2]$.

$$\int_0^1 t^{(2\omega+1)}\left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\right|dt$$
$$\leq\left(\int_0^1 t^{2\omega+1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{2\omega+1}(t^s\mid h^{2\omega+1}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q+\left((1-t)^s\mid h^{2\omega+1}(\alpha_2)\mid^q dt\right)\right)^{\frac{1}{q}}. \tag{3.17}$$

By applying the definition of classical Beta function on (3.17) and after some simplification we have,

$$\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt$$

$$\leq \left(\frac{1}{2(\omega+1)}\right)^{1-\frac{1}{q}} \left(\frac{\mid h^{2\omega+1}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{2(\omega+1)+s} + \beta\left(2(\omega+1)+s, s+1 \mid h^{2\omega+1}(\alpha_2) \mid^q\right)\right)^{\frac{1}{q}}. \tag{3.18}$$

Similarly, in the same fashion on the second integral of right side of inequality (3.16), by applying the definition of classical Beta type function and by doing some simplification we obtain.

$$\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt$$

$$\leq \left(\frac{1}{2(\omega+1)}\right)^{1-\frac{1}{q}} \left(\frac{\mid h^{2\omega+1}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{2(\omega+1)+s} + \beta\left(2(\omega+1)+s, s+1 \mid h^{2\omega+1}(\alpha_1) \mid^q\right)\right)^{\frac{1}{q}}. \tag{3.19}$$

By using the above two inequalities (3.18), (3.19) and do some necessarily simplifications, we have.

$$\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{1}{2(\omega+1)}\right)^{1-\frac{1}{q}}$$

$$\left[\zeta^{2\omega+2} \left(\frac{\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{2(\omega+1)+s} + \beta\left(2(\omega+1)+s, s+1\right) \mid h^{(2\omega+1)}(\alpha_2) \mid^q\right)^{\frac{1}{q}}\right.$$

$$\left. + (1-\zeta)^{2\omega+2} \left(\frac{\mid h^{(2\omega+1)}(\zeta a + (1-\zeta)\alpha_2) \mid^q}{2(\omega+1)+s} + \beta(2(\omega+1)+s, s+1) \mid h^{(2\omega+1)}(\alpha_1) \mid^q\right)^{\frac{1}{q}}\right]. \tag{3.20}$$

□

Theorem 14. *Assume that $h : I \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to $(2\omega+1)$ times is in the interior of $[\alpha_1, \alpha_2]$, having $\alpha_1, \alpha_2 \in I$ in order that $h^{(2\omega+1)} \in L([\alpha_1, \alpha_2])$, furthermore $0 \leq \zeta \leq 1$. When $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function* *in the second sense on closed interval* $[\alpha_1, \alpha_2]$*, whereas* $s \in (-1, 1]$*, and* $q \geq 1$*, then,*

$$\begin{aligned}
&\left(\sum_{\gamma=0}^{2\omega+1}(-1)^{\gamma+1}\binom{2\omega+1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega+1)}\right)\frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega}(-1)^{\gamma}\binom{2\omega}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{(2\omega-1)}(-1)^{(\gamma+1)}\binom{2\omega-1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-1)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega-2}(-1)^{\gamma}\binom{2\omega-2}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-2)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega-3}(-1)^{(\gamma+1)}\binom{2\omega-3}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-3)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!}h^{(2\omega-4)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\cdots+\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}h(t)dt\\
&\leq\frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\left(\frac{1}{(2\omega+1)q+s+1}\right)^{\frac{1}{q}}\\
&\quad\Bigg[(1-\zeta)^{2\omega+2}\Big(\mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^{q}\\
&\quad+((2\omega+1)q+s+1)\beta((2\omega+1)q,s+1)\mid h^{(2\omega+1)}(\alpha_1)\mid^{q}\Big)^{\frac{1}{q}}\\
&\quad+\zeta^{2\omega+2}\Big(\mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^{q}\\
&\quad+((2\omega+1)q+s+1)\beta((2\omega+1)q,s+1)\mid h^{(2\omega+1)}(\alpha_2)\mid^{q}\Big)^{\frac{1}{q}}\Bigg].
\end{aligned} \tag{3.21}$$

Proof. By taking Lemma (3.1.1) and applying the Holder's integral inequality on

(3.1.1),

$$
\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{2\omega-1} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=o}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{2\omega} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!} \Bigg[\zeta^{(2\omega+2)} \int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2) \right| dt \\
&\quad - (1-\zeta)^{(2\omega+2)} \int_0^1 t^{(2\omega_1+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1) \right| dt \Bigg].
\end{aligned} \tag{3.22}
$$

Now, we apply Holder's integral inequality on the R.H.S of (3.22). we also know that $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$.

$$
\begin{aligned}
&\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2) \right| dt \\
&\leq \left(\int_0^1 1dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{(2\omega+1)q} \mid h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2) \mid^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.23}
$$

Since, $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$ for

$t \in [0,1]$,

$$\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt$$
$$\leq \left(\int_0^1 t^{(2\omega+q)}(t^s \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2 \mid^q + (1-t)^s) \mid h^{(2\omega+1)}(\alpha_2) \mid^q) dt \right)^{\frac{1}{q}}. \tag{3.24}$$

By applying the definition of classical Beta function on (3.24), and after some simplifications,

$$\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt$$
$$\leq \left(\frac{\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{(2\omega+1)q + s + 1} + \beta((2\omega+1)q + 1, s+1) \mid h^{(2\omega+1)}(\alpha_2) \mid^q \right)^{\frac{1}{q}}. \tag{3.25}$$

Similarly, in the same fashion on the second integral of right side of above (3.22) inequality, By applying the definition of classical Beta type function and by doing some simplifications we obtain.

$$\begin{aligned} &\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) \right| dt \\ &\leq \left(\frac{\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{(2\omega+1)q + s + 1} + \beta((2\omega+1)q + 1, s+1) \mid h^{(2\omega+1)}(a) \mid^q \right)^{\frac{1}{q}}. \end{aligned} \tag{3.26}$$

Here, by using the inequality (3.25) and (3.26), and by doing some necessarily simplifications.

$$\begin{aligned} &\leq \frac{(\alpha_2 - \alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{1}{(2\omega+1)q + s + 1} \right)^{\frac{1}{q}} \\ &\Bigg[(1-\zeta)^{2\omega+2} \Big(\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q \\ &+ ((2\omega+1)q + s + 1)\beta((2\omega+1)q, s+1) \mid h^{(2\omega+1)}(\alpha_1) \mid^q \Big)^{\frac{1}{q}} \\ &+ \zeta^{2\omega+2} \Big(\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q \\ &+ ((2\omega+1)q + s + 1)\beta((2\omega+1)q, s+1) \mid h^{(2\omega+1)}(\alpha_2) \mid^q \Big)^{\frac{1}{q}} \Bigg]. \end{aligned} \tag{3.27}$$

$\square$

Corollary 3.1.1. *Under the assumption of above theorem (14). If $s = 1$, then,*

$$
\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{1}{((2\omega+1)q+1)((2\omega+1)q+2)} \right)^{\frac{1}{q}} \\
&\quad \Big\{ (1-\zeta)^{2\omega+2} \Big[((2\omega+1)q+1) \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + \mid h^{(2\omega+1)}(\alpha_1) \mid^q \Big]^{\frac{1}{q}} \\
&\quad + \zeta^{2\omega+2} \Big[((2\omega+1)q+1) \mid h^{(2\omega+1)}(\zeta a + (1-\zeta)\alpha_2) \mid^q + \mid h^{(2\omega+1)}(\alpha_2) \mid^q \Big]^{\frac{1}{q}} \Big\}.
\end{aligned} \tag{3.28}
$$

Theorem 15. *Assume that $h : J \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to $(2\omega + 1)$ times having $\alpha_1, \alpha_2 \in J$ in order that $h^{(2\omega+1)} \in L([\alpha_1, \alpha_2])$, and $0 \leq \zeta \leq 1$. When $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function in the second sense on the closed* *interval* $[\alpha_1, \alpha_2]$, *whereas* $s \in (-1, 1]$, *and* $q \geq 1$, *then,*

$$\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{q-1}{2q(\omega+1)-1} \right)^{1-\frac{1}{q}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \\
&\quad \Bigg[(1-\zeta)^{2\omega+2} \left[| h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) |^q + | h^{(2\omega+1)}(\alpha_1) |^q \right]^{\frac{1}{q}} \\
&\quad + \zeta^{2\omega+2} \left[| h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) |^q + | h^{(2\omega+1)}(\alpha_2) |^q \right]^{\frac{1}{q}} \Bigg].
\end{aligned} \tag{3.29}$$

Proof. By taking Lemma (3.1.1) and applying the Holder's integral inequality on

(3.1.1),

$$\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1}(-1)^{\gamma+1}\binom{2\omega+1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega+1)}\right)\frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!}h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega}(-1)^{\gamma}\binom{2\omega}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!}h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{(2\omega-1)}(-1)^{(\gamma+1)}\binom{2\omega-1}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-1)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!}h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega-2}(-1)^{\gamma}\binom{2\omega-2}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-2)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!}h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\left(\sum_{\gamma=0}^{2\omega-3}(-1)^{(\gamma+1)}\binom{2\omega-3}{\gamma}\zeta^{\gamma}-\zeta^{(2\omega-3)}\right)\frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!}h^{(2\omega-4)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&+\cdots+\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}h(t)dt\Bigg|\\
&\leq\frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!}\Bigg[\zeta^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}\left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\right|dt\\
&\quad-(1-\zeta)^{(2\omega+2)}\int_0^1 t^{(2\omega+1)}\left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\right|dt\Bigg].
\end{aligned} \tag{3.30}$$

Applying the Holder's integral inequality on the R.H.S of (3.30). We also know that $\mid h^{(2\omega+1)}\mid^q$ is an extended s-convex function on $[\alpha_1,\alpha_2]$ in the second sense.

$$\begin{aligned}
&\int_0^1 t^{(2\omega+1)}\left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\right|dt\\
&\leq\left(\int_0^1 t^{\frac{(2\omega+1)q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \mid h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2)\mid^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.31}$$

Since $\mid h^{(2n+1)}\mid^q$ is an extended s-convex function in the second sense on $[\alpha_1,\alpha_2]$ for

$t \in [0,1]$, so

$$\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt$$
$$\leq \left(\frac{q-1}{2q(\omega+1)-1} \right)^{1-\frac{1}{q}} \left(\int_0^1 t^s \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + (1-t)^s) \mid h^{(2\omega+1)}(\alpha_2) \mid^q dt \right)^{\frac{1}{q}}. \tag{3.32}$$

Here, we apply the simple rule of integration and obtain the following inequality.

$$\begin{aligned} &\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt \\ &\leq \left(\frac{q-1}{2q(\omega+1)-1} \right)^{1-\frac{1}{q}} \left(\frac{\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{s+1} + \frac{\mid h^{(2\omega+1)}(\alpha_2) \mid^q}{s+1} \right)^{\frac{1}{q}}. \end{aligned} \tag{3.33}$$

Similarly, in the same fashion by applying the Holder's integral inequality on the second integral of right side of (3.30). Also, we know that $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$. So, after some simplification we obtain.

$$\begin{aligned} &\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) \right| dt \\ &\leq \left(\frac{q-1}{2q(\omega+1)-1} \right)^{1-\frac{1}{q}} \left(\frac{\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q}{s+1} + \frac{\mid h^{(2\omega+1)}(\alpha_1) \mid^q}{s+1} \right)^{\frac{1}{q}}. \end{aligned} \tag{3.34}$$

Now, by using above two inequalities (3.33) and (3.34) we have,

$$\begin{aligned} &\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{q-1}{2q(\omega+1)-1} \right)^{1-\frac{1}{q}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \\ &\left[(1-\zeta)^{2\omega+2} \left(\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + \mid h^{(2\omega+1)}(\alpha_1) \mid^q \right)^{\frac{1}{q}} \right. \\ &\left. + \zeta^{2\omega+2} \left(\mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + \mid h^{(2\omega+1)}(\alpha_2) \mid^q \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{3.35}$$

□

Corollary 3.1.2. *Under the assumption of above theorem* (15). *If* $s = 1$, *then,*

$$\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{2\omega} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$

$$+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg|$$

$$= \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{1}{2}\right)^{\frac{1}{q}} \Bigg[(1-\zeta)^{2\omega+2} \left[| h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) |^q + | h^{(2\omega+1)}(\alpha_1) |^q \right]^{\frac{1}{q}}$$

$$+ \zeta^{2\omega+2} \left[| h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) |^q + | h^{(2\omega+1)}(\alpha_2) |^q \right]^{\frac{1}{q}} \Bigg]. \tag{3.36}$$

3.2 Inequalities for (2ω+1) times differentiable function:

Here, some of the inequalities of s-convex function for $(2\omega+1)$ times differentiable function.

$$\int_0^1 t^s (1-t)^{\omega} dt = \beta(s+1, \omega+1). \tag{3.37}$$

The above equation (3.37) may be known in the literature as classical Beta function.

Theorem 16. *Assume that* $h : I \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *is a differentiable function up to* $(2\omega+1)$ *times on* $[\alpha_1, \alpha_2]$ *whereas* $\alpha_1 < \alpha_2$ *and* $\alpha_1, \alpha_2 \in I$. *When* $| \ h^{(2\omega+1)} \ |$ *is an*

s-convex function on $[\alpha_1, \alpha_2]$ for some fixed $s \in (0,1]$, then for every $\zeta \in [0,1]$, we have,

$$
\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \Bigg[\left(\zeta^{2\omega+2} + (1-\zeta)^{2\omega+2}\right) \beta(2(\omega+1)+s, 1) \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid \\
&\quad + \beta(2(\omega+1), s+1) \left[\zeta^{2\omega+2} \mid h^{(2\omega+1)}(\alpha_2) \mid + (1-\zeta)^{2\omega+2} \mid h^{(2\omega+1)}(\alpha_1) \mid \right] \Bigg].
\end{aligned} \tag{3.38}
$$

Proof. Here we take (3.14) and apply Holder's integral inequality on (3.14) and obtain the following result.

$$\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!} \Bigg[\zeta^{(2\omega+2)} \int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2) \right| dt \\
&- (1-\zeta)^{(2\omega+2)} \int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)a) \right| dt \Bigg].
\end{aligned} \tag{3.39}$$

Here, we consider the first part of the R.H.S of above inequality (3.39). Also, $\mid h^{(2\omega+1)} \mid$ is an s-convex function on $[\alpha_1, \alpha_2]$, as $0 \leq t \leq 1$.

$$\mid h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2) \mid \leq t^s \mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \mid +(1-t)^s \mid h^{(2\omega+1)}(\alpha_2) \mid .$$

Similarly, Here we consider the second part of R.H.S of above inequality(3.39). Also, $\mid h^{(2\omega+1)} \mid$ is an s-convex function on $[\alpha_1, \alpha_2]$, as $0 \leq t \leq 1$.

$$\mid h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1) \mid \leq t^s \mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \mid +(1-t)^s \mid h^{(2\omega+1)}(\alpha_1) \mid .$$

Here, we use the above two inequalities in the equation (3.39) and obtain a following inequality.

$$\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\Bigg[\zeta^{2\omega+1}\int_0^1 t^{2\omega+1}\Big[t^s \mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid +(1-t)^s \mid h^{(2\omega+1)}(\alpha_2)\mid\Big]\,dt$$

$$+(1-\zeta)^{2\omega+2}\int_0^1 t^{2\omega+1}\Big[t^s \mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid +(1-t)^s \mid h^{(2\omega+1)}(\alpha_1)\mid\Big]\,dt\Bigg]. \tag{3.40}$$

We observed that some part of above inequality (3.40) is in the form of classical Beta type function. So, we separates that part and apply the definition of classical Beta type function on that part.

$$\int_0^1 t^{2\omega+s+1}(1-t)^0 dt = \beta\left(2(\omega+1)+s,1\right). \tag{3.41}$$

$$\int_0^1 t^{2\omega+1}(1-t)^s dt = \beta\left(2(\omega+1),s+1\right). \tag{3.42}$$

By using equation (3.41), (3.42) and above inequality (3.40) we obtain,

$$\begin{aligned}
&\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\Bigg[\zeta^{2\omega+1}\Bigg(\beta(2(\omega+1)+s,1)\mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid \\
&+\beta(2(\omega+1),s+1)\mid h^{(2\omega+1)}(\alpha_2)\mid\Bigg) \\
&+(1-\zeta)^{2\omega+2}\Bigg(\beta(2(\omega+1)+s,1)\mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid \\
&+\beta(2(\omega+1),s+1)\mid h^{(2\omega+1)}(\alpha_1)\mid\Bigg)\Bigg].
\end{aligned} \tag{3.43}$$

After some simplification, we obtain the following inequality.

$$\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\Bigg[\left(\zeta^{2\omega+2}+(1-\zeta)^{2\omega+2}\right)\beta(2(\omega+1)+s,1)\mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid$$

$$+\beta(2(\omega+1),s+1)\Big[\zeta^{2\omega+2}\mid h^{(2\omega+1)}(\alpha_2)\mid +(1-\zeta)^{2\omega+2}\mid h^{(2\omega+1)}(\alpha_1)\mid\Big]\Bigg]. \tag{3.44}$$

Here, we obtain $(2\omega+1)$ times differentiable s-convex function associated with classical Beta type function. □

Theorem 17. *Assume that $h : I \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a differentiable function up to $(2\omega+1)$ times on $[\alpha_1, \alpha_2]$ whereas $\alpha_1 < \alpha_2$ and $\alpha_1, \alpha_2 \in I$. If $\mid h^{(2\omega+1)} \mid$ is an s-convex function on $[\alpha_1, \alpha_2]$, for some fixed $s \in (0, 1]$, then for every $\zeta \in [0, 1]$, we have,*

$$
\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \left(\frac{1}{2(\omega+1)} \right)^{\frac{1}{p}} \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \\
&\Bigg[\Bigg(\beta(2(\omega+1)+s, 1) \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q \\
&+ \beta(2(\omega+1), s+1) \mid h^{(2\omega+1)}(\alpha_2) \mid^q \Bigg)^{\frac{1}{q}} \\
&+ (1-\zeta)^{2\omega+2} \Bigg(\beta(2(\omega+1)+s, 1) \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q \\
&+ \beta(2(\omega+1), s+1) \mid h^{(2\omega+1)}(\alpha_1) \mid^q \Bigg)^{\frac{1}{q}} \Bigg].
\end{aligned}
\tag{3.45}
$$

Proof. Here, we use the equation (3.14) and apply the Holder's integral inequality and obtain following result,

$$
\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{(\alpha_2-\alpha_1)} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!} \Bigg[\zeta^{(2\omega+2)} \int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt \\
&\quad - (1-\zeta)^{(2\omega+2)} \int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) \right| dt \Bigg].
\end{aligned} \tag{3.46}
$$

By applying the Holder's integral inequality on the R.H.S of above inequality(3.46). We also know that $\mid h^{(2\omega+1)} \mid^{q}$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$.

$$
\begin{aligned}
&\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt \\
&\leq \left(\int_0^1 t^{2\omega+1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{2\omega+1} \mid h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2 + (1-t)\alpha_2) \mid^{q} dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.47}
$$

However, it may be convenient to say that,

i.e $p = \frac{q}{q-1}$ the conjugate exponent. Since, it satisfies,

$\frac{1}{p} + \frac{1}{q} = 1.$

Here;

$$\int_0^1 t^{2\omega+1} dt = \frac{1}{2(\omega+1)}. \tag{3.48}$$

So, above inequality (3.47) becomes,

$$\begin{aligned}
&\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt \\
&\leq \left(\frac{1}{2(\omega+1)} \right)^{\frac{1}{p}} \left(\int_0^1 t^{2n+1} \mid h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2 + (1-t)\alpha_2) \mid^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.49}$$

Similarly, now we consider second integral of right side of above inequality(3.46). We also know that $\mid h^{(2\omega+1)} \mid^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$. Here, we apply the definition of classical Beta type function.

$$\begin{aligned}
&\left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt \\
&\leq t^s \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + (1-t)^s \mid h^{(2\omega+1)}(\alpha_2) \mid^q .
\end{aligned} \tag{3.50}$$

By using (3.49) and definition of classical Beta function.

$$\begin{aligned}
&\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right|^q dt \leq \left(\frac{1}{2(\omega+1)} \right)^{\frac{1}{p}} \\
&\left[\beta(2(\omega+1)+s, 1) \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + \beta(2(\omega+1), s+1) \mid h^{(2\omega+1)}(\alpha_2) \mid^q \right].
\end{aligned} \tag{3.51}$$

In a similar way, we obtain.

$$\begin{aligned}
&\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right|^q dt \leq \left(\frac{1}{2(\omega+1)} \right)^{\frac{1}{p}} \\
&\left[\beta(2(\omega+1)+s, 1) \mid h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + \beta(2(\omega+1), s+1) \mid h^{(2\omega+1)}(\alpha_1) \mid^q \right].
\end{aligned} \tag{3.52}$$

Here, we use above two inequalities (3.51), (3.52) and obtain the following result.

$$\leq \left(\frac{1}{2(\omega+1)}\right)^{\frac{1}{p}} \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\Bigg[\Bigg(\beta(2(\omega+1)+s,1) \mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \mid^q$$
$$+ \beta(2(\omega+1),s+1) \mid h^{(2\omega+1)}(\alpha_2) \mid^q \Bigg)^{\frac{1}{q}}$$
$$+ (1-\zeta)^{2\omega+2}\Bigg(\beta(2(\omega+1)+s,1) \mid h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2) \mid^q$$
$$+ \beta(2(\omega+1),s+1) \mid h^{(2\omega+1)}(\alpha_1) \mid^q \Bigg)^{\frac{1}{q}}\Bigg]. \tag{3.53}$$

The result that we obtain associates s-convex function with classical Beta type function.

□

Theorem 18. *Assume that* $h : I \subseteq \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *is a differentiable function up to* $(2\omega+1)$ *times on* $[\alpha_1, \alpha_2]$*, whereas* $\alpha_1 < \alpha_2$ *and* $\alpha_1, \alpha_2 \in I$*. If* $\mid h^{(2\omega+1)} \mid$ *is an s-convex function on* $[\alpha_1, \alpha_2]$*, for some fixed* $s \in (0, 1]$*, then for every* $0 < \zeta < 1$ *where p,q are* *conjugate numbers and $q \geq 1$, we have,*

$$
\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!} \left(\frac{1}{(2\omega+1)p+1} \right)^{\frac{1}{p}} \left(\frac{1}{s+1} \right)^{\frac{1}{q}} \\
&\quad \Bigg[\zeta^{2\omega+2} \left(\left| h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \right|^q + \left| h^{(2\omega+1)}(\alpha_2) \right|^q \right)^{\frac{1}{q}} \\
&\quad + (1-\zeta)^{2\omega+2} \left(\left| h^{(2\omega+1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \right|^q + \left| h^{(2\omega+1)}(\alpha_1) \right|^q \right)^{\frac{1}{q}} \Bigg].
\end{aligned}
\tag{3.54}
$$

Proof. Here, we use the above (3.14) and apply the Holder's integral inequality and obtain the following result.

$$\begin{aligned}
&\Bigg| \left(\sum_{\gamma=0}^{2\omega+1} (-1)^{\gamma+1} \binom{2\omega+1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega+1)} \right) \frac{(\alpha_2-\alpha_1)^{2\omega}}{(2\omega+1)!} h^{(2\omega)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega} (-1)^{\gamma} \binom{2\omega}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-1)}}{2\omega!} h^{(2\omega-1)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{(2\omega-1)} (-1)^{(\gamma+1)} \binom{2\omega-1}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-1)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-2)}}{(2\omega-1)!} h^{(2\omega-2)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-2} (-1)^{\gamma} \binom{2\omega-2}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-2)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-3)}}{(2\omega-2)!} h^{(2\omega-3)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \left(\sum_{\gamma=0}^{2\omega-3} (-1)^{(\gamma+1)} \binom{2\omega-3}{\gamma} \zeta^{\gamma} - \zeta^{(2\omega-3)} \right) \frac{(\alpha_2-\alpha_1)^{(2\omega-4)}}{(2\omega-3)!} h^{(2\omega-4)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&+ \cdots + \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h(t)dt \Bigg| \\
&\leq \frac{(\alpha_2-\alpha_1)^{(2\omega+1)}}{(2\omega+1)!} \Bigg[\zeta^{(2\omega+2)} \int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt \\
&\quad + (1-\zeta)^{(2\omega+2)} \int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) \right| dt \Bigg].
\end{aligned} \tag{3.55}$$

Here, we consider the first part of the R.H.S of (3.55), and by using Holder's integral inequality on (3.55).

$$\begin{aligned}
&\int_0^1 t^{(2\omega+1)} \left| h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \right| dt \\
&\leq \left(\int_0^1 t^{(2\omega+1)p} dt \right)^{\frac{1}{p}} \left(\int_0^1 \mid h^{(2\omega+1)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2 + (1-t)\alpha_2) \mid^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{3.56}$$

Here,

$$\int_0^1 t^{(2\omega+1)p} dt = \frac{1}{(2\omega+1)p+1}. \tag{3.57}$$

Now, using the s-convexity of $\left| h^{(2\omega+1)} \right|^q$ on $[\alpha_1, \alpha_2]$ in the second integral on the right side of (3.56), we have,

$$\int_0^1 \left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\right| dt$$
$$\leq \int_0^1 \left(t^s \left|h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|^q + (1-t)^s \left|h^{(2\omega+1)}(\alpha_2)\right) dt\right|^q.$$

So finally the equation (3.56) yields,

$$\begin{aligned}&\int_0^1 \left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\right| dt \\ &\leq \frac{1}{s+1}\left[\left|h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|^q + \left|h^{(2\omega+1)}(\alpha_2)\right|^q\right].\end{aligned} \tag{3.58}$$

Henceforth, by using (3.56) and (3.58) we obtain the following final inequality,

$$\begin{aligned}&\int_0^1 \left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\right| dt \\ &\leq \left(\frac{1}{(2\omega+1)p+1}\right)^{\frac{1}{p}} \left(\frac{\left|h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|^q + \left|h^{(2\omega+1)}(\alpha_2)\right|^q}{s+1}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.59}$$

Similarly, now we consider the second integral of above inequality (3.55), and performed the same mathematical process on (3.55) and obtain the following integral inequality.

$$\begin{aligned}&\int_0^1 \left|h^{(2\omega+1)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\right| dt \\ &\leq \left(\frac{1}{(2\omega+1)p+1}\right)^{\frac{1}{p}} \left(\frac{\left|h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|^q + \left|h^{(2\omega+1)}(\alpha_1)\right|^q}{s+1}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.60}$$

By using above two inequalities (3.59) and (3.60), we obtain the final result.

$$\begin{aligned}&\leq \frac{(\alpha_2-\alpha_1)^{2\omega+1}}{(2\omega+1)!}\left(\frac{1}{(2\omega+1)p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\ &\left[\zeta^{2\omega+2}\left(\left|h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|^q + \left|h^{(2\omega+1)}(\alpha_2)\right|^q\right)^{\frac{1}{q}}\right. \\ &\left.+(1-\zeta)^{2\omega+2}\left(\left|h^{(2\omega+1)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|^q + \left|h^{(2\omega+1)}(\alpha_1)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned} \tag{3.61}$$

□

Chapter 4

Hermite-Hadamard type inequalities for s-convex and Extended s-convex function

Hudzik et al.in [3] demonstrate some results associating with s-convex in the second sense. Here, we established some more new results about Hadamard inequality for the class of s-convex function whose fifth derivative at certain powers are s-convex function in the second sense. Although it is seen that many important inequalities connecting with 1-convex function.

4.1 Some results for s-convex function and extended s-convex function:

Lemma 4.1.1. *Assume that h:* $J \subseteq \mathbb{R}^{+} \cup \{0\} \to \mathbb{R}$ *is a differentiable function up to five times in the interior of* J *such that* $\alpha_1, \alpha_2 \in J$. *When* $h^{(v)} \in L_1([\alpha_1, \alpha_2])$ *and*

$0 \leq \zeta \leq 1$, *subsequently,*

$$\left| \frac{1}{(\alpha_2 - \alpha_1)} \int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4 - 10\zeta^3 + 10\zeta^2 - 5\zeta + 1}{120} (\alpha_2 - a)^4 h^{(iv)}(\zeta a + (1-\zeta)\alpha_2 \right.$$
$$- \frac{4\zeta^3 - 6\zeta^2 + 4\zeta - 1}{24} (\alpha_2 - \alpha_1)^3 h^{'''}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$
$$- \frac{3\zeta^2 - 3\zeta + 1}{6} (\alpha_2 - \alpha_1)^2 h''(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$
$$\left. - \frac{2\zeta - 1}{2} (\alpha_2 - \alpha_1) h^{'}(\zeta\alpha_1 + (1-\zeta)\alpha_2) - h(\zeta\alpha_1 + (1-\zeta)\alpha_2) \right|$$
$$\leq \frac{(\alpha_2 - a)^5}{120} \left[\zeta^6 \int_0^1 t^5 \mid h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \mid dt \right.$$
$$\left. - (1-\zeta)^6 \int_0^1 t^5 \mid h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) \mid dt \right]. \tag{4.1}$$

Proof. We assume a following improper integral and integrating this integral on $[0, 1]$. Here, we consider $\zeta \in (0, 1)$.

$$\int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2 + (1-t)\alpha_2))dt. \tag{4.2}$$

Consider the internal part of integral equation (4.2) and make it in a simplified form for the convenience of integration.

$$t^5 h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2 + (1-t)\alpha_2)) = \frac{-1}{\zeta(\alpha_2 - a)} t^5 dh^{(iv)}(t(\zeta\alpha_1 + (1-\zeta\alpha_2) + (1-t)\alpha_2)).$$

So finally, the integral equation (4.2) becomes;

$$\frac{-1}{\zeta(\alpha_2 - \alpha_1)} \int_0^1 t^5 dh^{(iv)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2 + (1-t)\alpha_2))dt. \tag{4.3}$$

Hence the above equation (4.3) implies

$$\frac{-1}{\zeta(\alpha_2-\alpha_1)}[t^5h^{(iv)}(t(\zeta)+(1-\zeta\alpha_2))]\mid_0^1+\frac{5}{\zeta(\alpha_2-\alpha_1)}\int_0^1 t^4h^{(iv)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-a)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}\int_0^1 t^4dh^{'''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}t^4h^{'''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2\mid_0^1$$
$$+\frac{5.4}{\zeta^2(\alpha_2-\alpha_1)^2}\int_0^1 t^3h^{'''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5.4}{\zeta^3(\alpha_2-\alpha_1)^3}\int_0^1 t^3dh^{''}(t(\zeta\alpha_1)+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5.4}{\zeta^3(\alpha_2-\alpha_1)^3}t^3h''(t(\zeta\alpha_1+(1-\zeta)\alpha_2))+(1-t)\alpha_2\mid_0^1$$
$$+\frac{5.4.3}{\zeta^3(\alpha_2-\alpha_1)^3}\int_0^1 t^2h^{''}(t(\zeta\alpha_1)+(1-\zeta\alpha_2))+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5.4}{\zeta^3(\alpha_2-\alpha_1)^3}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5.4.3}{\zeta^4(\alpha_2-\alpha_1)^4}\int_0^1 t^2dh^{'}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5.4}{\zeta^3(\alpha_2-\alpha_1)^3}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{5.4.3}{\zeta^4(\alpha_2-\alpha_1)^4}t^2h'(t(\zeta\alpha_1)+(1-\zeta)\alpha_2)+(1-t)\alpha_2\mid_0^1$$
$$+\frac{5.4.3.2}{\zeta^4(\alpha_2-\alpha_1)^4}\int_0^1 th^{'}(t(\zeta\alpha_1+(1-t)\alpha_2))dt$$
$$=\frac{-1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1(1-\zeta)\alpha_2)-\frac{5}{\zeta^2(\alpha_2-a)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5.4}{\zeta^3(\alpha_2-\alpha_1)^3}h^{''}(\zeta\alpha_1(1-\zeta)\alpha_2)$$
$$-\frac{5.4.3}{\zeta^4(\alpha_2-\alpha_1)^4}h^{'}(\zeta\alpha_1+(1-\zeta\alpha_2))-\frac{5.4.3.2}{\zeta^5(\alpha_2-\alpha_1)^5}\int_0^1 tdh(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$

$$= -\frac{1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{5.4}{\zeta^3(\alpha_2-\alpha_1)^3}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5.4.3}{\zeta^4(\alpha_2-\alpha_1)^4}h^{'}(\zeta\alpha_1)+(1-\zeta)\alpha_2$$
$$- \frac{5.4.3.2}{\zeta^5(\alpha_2-\alpha_1)^5}[t(h(\zeta\alpha_1+(1-\zeta\alpha_2)))+(1-t)\alpha_2]\,|_0^1$$
$$+ \frac{5.4.3.2.1}{\zeta^5(\alpha_2-a)^5}\int_0^1 h(t(\zeta\alpha_1(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt.$$

Finally, we obtain a following integration.

$$\int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2))dt = -\frac{1}{\zeta(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{5}{\zeta^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5.4}{\zeta^3(\alpha_2-\alpha_1)^3}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{5.4.3}{\zeta^4(\alpha_2-\alpha_1)^4}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5.4.3.2}{\zeta^4(\alpha_2-\alpha_1)^4}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \frac{5.4.3.2.1}{\zeta^5(\alpha_2-\alpha_1)^5}\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt. \tag{4.4}$$

Now, by multiplying the equation(4.4) by $\frac{\zeta^6(\alpha_2-\alpha_1)^5}{120}$.

$$\frac{(\alpha_2-\alpha_1)^5}{120}\left[\zeta^6\int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1)+(1-\zeta)\alpha_2)dt\right]$$
$$= -\frac{\zeta^5(\alpha_2-\alpha_1)^4}{120}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{\zeta^4(\alpha_2-a)^3}{24}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{\zeta^3(\alpha_2-a)^2}{6}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{\zeta^2(\alpha_2-\alpha_1)}{2}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \zeta h(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \zeta\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)). \tag{4.5}$$

Here, we assume another improper definite integral on $[0,1]$ of the following form;

$$\int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt. \tag{4.6}$$

Now, we consider the internal part of (4.6) and simplify it for the ease of integration.

$$t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt = \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}t^5 dh^{(iv)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1).$$

So, after simplification the definite integral (4.6) becomes;

$$\frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}\int_0^1 t^5 dh^{(iv)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt. \qquad (4.7)$$

Hence the above equation (4.7) implies,

$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}[t^5 h^{(iv)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2))+(1-t)\alpha_1]\,|_0^1$$

$$- \frac{5}{(1-\zeta(\alpha_2-\alpha_1)}\int_0^1 t^4 h^{(iv)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$

$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$

$$- \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}\int_0^1 t^4 dh^{'''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$

$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$

$$- \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}[t^4 h^{'''}(t(\zeta\alpha_1)+(1-\zeta)\alpha_2)+(1-t)\alpha_1]\,|_0^1$$

$$+ \frac{5.4}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}\int_0^1 t^3 h^{'''}(t(\zeta\alpha_1+(1-t)\alpha_2)+(1-t)\alpha_1)dt$$

$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$

$$+ \frac{5.4}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}\int_0^1 t^3 dh^{''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$

$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$

$$+ \frac{5.4}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}\int_0^1 t^3 dh^{''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$

$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \frac{5.4}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}[t^3h^{''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)] \mid_0^1$$
$$- \frac{5.4.3}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}\int_0^1 t^2h^{''}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$
$$= -\frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \frac{5.4}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{5.4.3}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}\int_0^1 t^2dh^{'}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$
$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \frac{5.4}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{5.4.3}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}[t^2h^{'}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)] \mid_0^1$$
$$+ \frac{5.4.3.2}{(1-\zeta^4(\alpha_2-\alpha_1)^4}\int_0^1 th^{'}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$
$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \frac{5.4}{(1-\zeta)3(\alpha_2-\alpha_1)^3} h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5.4.3}{(1-\zeta)^4(\alpha_2-\alpha_1)^4} h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \frac{5.4.3.2}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}\int_0^1 tdh(t(\zeta\alpha_1)+(1-\zeta)\alpha_2+(1-t)\alpha_1)dt$$
$$= \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)} h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2} h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+ \frac{5.4}{(1-\zeta)^3(\alpha_2-\alpha_1)^3} h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2) - \frac{5.4.3}{(1-\zeta)^4(\alpha_2-\alpha_1)^4} h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{5.4.3.2}{(1-\zeta)^5(\alpha_2-\alpha_1)^5}[th(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)] \mid_0^1$$
$$- \frac{5.4.3.2.1}{(1-\zeta)^5(\alpha_2-\alpha_1)^5}\int_0^1 h(t(\zeta\alpha_1)+(1-\zeta)\alpha_2+(1-t)\alpha_1)dt$$

Finally, we obtain a following definite integral .

$$\int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_1))dt = \frac{1}{(1-\zeta)(\alpha_2-\alpha_1)}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5}{(1-\zeta)^2(\alpha_2-\alpha_1)^2}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)+\frac{5.4}{(1-\zeta)^3(\alpha_2-\alpha_1)^3}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5.4.3}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)+\frac{5.4.3.2.1}{(1-\zeta)^4(\alpha_2-\alpha_1)^4}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{5.4.3.2.1}{(1-\zeta)^5(\alpha_2-\alpha_1)^5}\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt. \tag{4.8}$$

By multiplying the above equation (4.8) on both sides by $\frac{(1-\zeta)^6(\alpha_2-\alpha_1)^5}{120}$. So, after some simplification we obtain a definite integral of the following form.

$$\frac{(1-\zeta)^6(\alpha_2-\alpha_1)^5}{120}\int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_1))dt$$
$$=\frac{(1-\zeta)^5(\alpha_2-\alpha_1)4}{120}h^{(iv)}(\zeta\alpha_1+(1-)\alpha_2)-\frac{(1-\zeta)^4(\alpha_2-\alpha_1)^3}{24}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+\frac{(1-\zeta)^3(\alpha_2-\alpha_1)^2}{6}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{(1-\zeta)^2(\alpha_2-\alpha_1)}{2}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$+(1-\zeta)h(\zeta\alpha_1+(1-\zeta)\alpha_2)-(1-\zeta)\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt. \tag{4.9}$$

Since, we consider two definite integrals and solve these two definite integral simultaneously by using the technique of integration by parts. Hence, we obtain two different results corresponding to two different definite integrals.

Here, we subtract equation (4.9) from equations (4.4) and after some simplifications obtain a final result that we name as lemma, and this will help us a lot for the purpose to prove our succeeding theorems, also we frequently use this lemma for establishing the theorems related to Hermite-Hadamard types integral inequalities for s-convex function.

$$\frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\int_0^1 t^5h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$
$$-(1-\zeta)^6\int_0^1 t^5h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2))dt\Bigg]$$
$$=-\frac{\zeta^5(\alpha_2-\alpha_1)^4}{120}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{\zeta^4(\alpha_2-\alpha_1)^3}{24}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{\zeta^3(\alpha_2-\alpha_1)^2}{6}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\frac{\zeta^2(\alpha_2-\alpha_1)}{2}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)-\zeta(h(\zeta\alpha_1+(1-\zeta)\alpha_2))$$
$$+\zeta\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt$$
$$-\frac{(1-\zeta)^5(\alpha_2-\alpha_1)^4}{120}h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)+\frac{(1-\zeta)^4(\alpha_2-\alpha_1)^3}{24}h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{(1-\zeta)^3}{6}h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)+\frac{(1-\zeta)^2(\alpha_2-\alpha_1)}{2}h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-(1-\zeta)(h(\zeta\alpha_1+(1-\zeta)\alpha_2))+(1-\zeta)\int_0^1 h(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt \tag{4.10}$$

Here, we made some simplifications and also use the change of variable and finally write up the simplest form of (4.10).

$$\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}h(t)dt-\frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2h''(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2h''(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$=\frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\int_0^1 t^5h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)dt$$
$$-(1-\zeta)^6\int_0^1 t^5h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2))dt\Bigg]. \tag{4.11}$$

We mainly use this lemma on Holder's inequalities for integrals, power mean integral inequalities, Holder Iscan integral inequality in order to prove succeeding results related to Hermite-Hadamard types inequalities for s-convex function. □

Remark 6. *Since, we assume* $\zeta \in (0,1)$*. The first definite integral* (4.2) *that we assumed diverge at* $\zeta = 0$*. Similarly, the second definite integral* (4.6) *that we assume diverge at* $\zeta = 1$*. So, in order to avoid this obstacle, we take* $\zeta \in (0,1)$ *in the above Lemma* (4.1.1) *.*
However, if we take $\zeta \in [0,1]$ *we obtain definite value. Now the behavior of integral changes.*

Remark 7. *If we take* $\zeta = 0$ *in above equation* (4.11)*, we obtain.*

$$\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{1}{5!}(\alpha_2-\alpha_1)^4 h^{(iv)}(\alpha_1) - \frac{1}{4!}(\alpha_2-\alpha_1)^3 h^{'''}(\alpha_2) - \frac{1}{3!}(\alpha_2-\alpha_1)^2 h^{''}(\alpha_1)$$
$$-\frac{1}{2!}(\alpha_2-\alpha_1)h^{'}(\alpha_1) - h(\alpha_1) = -\frac{(\alpha_2-\alpha_1)^5}{5!}\int_0^1 t^5 h^{(v)}(t\alpha_2+(1-t)\alpha_1)dt. \tag{4.12}$$

Remark 8. *Similarly, if we take* $\zeta = 1$ *in the above inequality* (4.11)*, we obtain.*

$$\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{1}{5!}(\alpha_2-\alpha_1)^4 h^{(iv)}(\alpha_1) - \frac{1}{4!}(\alpha_2-\alpha_1)^3 h^{'''}(\alpha_2) - \frac{1}{3!}(\alpha_2-\alpha_1)^2 h^{''}(\alpha_1)$$
$$-\frac{1}{2!}(\alpha_2-\alpha_1)h^{'}(\alpha_1) - h(\alpha_1) = \frac{(\alpha_2-\alpha_1)^5}{5!}\int_0^1 t^5 h^{(v)}(t\alpha_1+(1-t)\alpha_2)dt. \tag{4.13}$$

4.2 Inequalities for extended s-convex function in the second sense:

Now, we use Hermite-Hadamard type integral inequalities and Lemma (4.1.1) *to establish some new integral inequalities associated to s-convex function.*

Theorem 19. *Assume that* $h : J \subseteq [0,\infty) \to \mathbb{R}$ *is a differentiable function up to five times in the interval of* J*, such that* $\alpha_1, \alpha_2 \in J$ *with* $\alpha_1 < \alpha_2$*, also* $h^{(v)} \in L_1([\alpha_1,\alpha_2])$*, and* $0 \leq \zeta \leq 1$*. If* $\mid h^{(v)}(t) \mid^q$ *is an extended s-convex function in the second sense on*

$[\alpha_1, \alpha_2]$, $-1 \leq s \leq 1$, *and* $q \geqslant 1$, *then,*

$$\left| \frac{1}{(\alpha_2-\alpha_1)} \int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4 - 10\zeta^3 + 10\zeta^2 - 5\zeta + 1}{120} (\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \right.$$
$$- \frac{4\zeta^3 - 6\zeta^2 + 4\zeta - 1}{24} (\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$
$$- \frac{3\zeta^2 - 3\zeta + 1}{6} (\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$
$$\left. - \frac{2\zeta - 1}{2} (\alpha_2-\alpha_1) h^{'}(\zeta\alpha_1 + (1-\zeta)\alpha_2) - h(\zeta\alpha_1 + (1-\zeta)\alpha_2) \right|$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{720} \left[\frac{6}{(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)} \right]^{\frac{1}{q}}$$
$$\Bigg[(1-\zeta)^6 \big[(s+1)(s+2)(s+3)(s+4)(s+5) \mid h^{(v)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + 120 \mid h^{(v)}(\alpha_1) \mid^q \big]^{\frac{1}{q}}$$
$$+ \zeta^6 \big[(s+1)(s+2)(s+3)(s+4)(s+5) \mid h^{(v)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid^q + 120 \mid h^{(v)}(\alpha_2) \mid^q \big]^{\frac{1}{q}} \Bigg]. \tag{4.14}$$

Proof. We prove the above theorem by taking $0 \leq \zeta \leq 1$ and $-1 < s \leq 1$. So, we take (4.11) and apply the power mean integral inequality on (4.11) to prove the theorem. Also, we know that $\mid h^{(v)}(t) \mid^q$ is an s-convex function in the second sense on $[\alpha_1, \alpha_2]$.

$$\left| \frac{1}{(\alpha_2-\alpha_1)} \int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4 - 10\zeta^3 + 10\zeta^2 - 5\zeta + 1}{120} (\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \right.$$
$$- \frac{4\zeta^3 - 6\zeta^2 + 4\zeta - 1}{24} (\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$
$$- \frac{3\zeta^2 - 3\zeta + 1}{6} (\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1 + (1-\zeta)\alpha_2)$$
$$\left. - \frac{2\zeta - 1}{2} (\alpha_2-\alpha_1) h^{'}(\zeta\alpha_1 + (1-\zeta)\alpha_2) - h(\zeta\alpha_1 + (1-\zeta)\alpha_2) \right|$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{120} \left[\zeta^6 \int_0^1 \mid t^5 h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) \mid \right.$$
$$\left. + (1-\zeta)^6 \int_0^1 \mid t^5 h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) \mid dt \right]. \tag{4.15}$$

Here, by taking right side of above inequality (4.15) and applying power mean's integral inequality on (4.15).

$$\frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\left(\int_0^1 t^5 dt\right)^{(1-\frac{1}{q})}\left[\int_0^1 t^5 \mid h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\mid^q dt\right]^{\frac{1}{q}}$$
$$+(1-\zeta)^6\left(\int_0^1 t^5 dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t^5 \mid h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\mid^q dt\right]^{\frac{1}{q}}\Bigg]$$

Because, $\mid h(t)\mid^q$ is an extended s-convex function in the second sense.

$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\left(\int_0^1 t^5 dt\right)^{(1-\frac{1}{q})}\left[\int_0^1 t^5 \mid h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\mid^q dt\right]^{\frac{1}{q}}$$
$$+(1-\zeta)^6\left(\int_0^1 t^5 dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t^5 \mid h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\mid^q dt\right]^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\left(\int_0^1 t^5 dt\right)^{(1-\frac{1}{q})}\left[\int_0^1 t^5(t^s \mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid +(1-t)^s \mid h^{(v)}(\alpha_2))\mid^q dt\right]$$
$$+(1-\zeta)^6\left(\int_0^1 t^5 dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t^5(t^s \mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid +(1-t)^s \mid h^{(v)}(\alpha_1)\mid^q dt\right]^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\left(\frac{1}{6}\right)^{(1-\frac{1}{q})}\Bigg[\frac{\mid h'''(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q}{s+6}$$
$$+\frac{120\mid h^{(v)}(\alpha_2)\mid^q}{(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)}\Bigg]^{\frac{1}{q}}$$
$$+(1-\zeta)^6\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\Bigg[\frac{\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q}{s+6}$$
$$+\frac{120\mid h^{(v)}(\alpha_1)\mid^{\frac{1}{q}}}{(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)}\Bigg]^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{720}\left[\frac{6}{(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)}\right]^{\frac{1}{q}}$$
$$\Bigg[(1-\zeta)^6[(s+1)(s+2)(s+3)(s+4)(s+5)\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q+120\mid h^{(v)}(\alpha_1)\mid^q]^{\frac{1}{q}}$$
$$+\zeta^6[(s+1)(s+2)(s+3)(s+4)(s+5)\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q+120\mid h^{(v)}(\alpha_2)\mid^q]^{\frac{1}{q}}\Bigg].$$

□

Theorem 20. *Assume that $h : J \subseteq [0,\infty) \to \mathbb{R}$ is a differentiable function up to five times in the interval of J, such that $\alpha_1, \alpha_2 \in J$ with $\alpha_1 < \alpha_2$, also $h^{(v)} \in L_1([\alpha_1,\alpha_2])$, and $0 \leq \zeta \leq 1$. If $\mid h^{(v)}(t) \mid^q$ is an extended s-convex function in the second sense on $[\alpha_1,\alpha_2]$, $-1 \leq s \leq 1$, and $q \geqslant 1$, then,*

$$\Bigg|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{720}\left(\frac{6}{60\zeta^6(1-\zeta)^6}\right)^{\frac{1}{q}} \times \Bigg[(1-\zeta)^6\Big[12(1-\zeta)^5\zeta^6 \mid h^{(v)}(\alpha_2)\mid^q$$
$$-\zeta^6(12(1-\zeta)^5+15(1-\zeta)^4+20(1-\zeta)^3+30(1-\zeta)^2+60(1-\zeta)+60ln\zeta) \mid h^{(v)}(\alpha_1)\mid^q\Big]^{\frac{1}{q}}$$
$$+\zeta^6\Big[12(1-\zeta)^6\zeta^5 \mid h^{(v)}(\alpha_1)\mid^q - (1-\zeta)^6(12\zeta^5+15\zeta^4+20\zeta^3$$
$$+30\zeta^2+60\zeta+60ln(1-\zeta)) \mid h^{(v)}(\alpha_2)\mid^q\Big]^{\frac{1}{q}}\Bigg]. \tag{4.16}$$

Proof. When $\zeta \in (0,1)$ and s=-1, since $\mid h^{(v)}(t) \mid^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$, by (4.9) and holder inequality, we have,

$$\Bigg|\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{4\zeta^3-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)-h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}[\zeta^6\int_0^1 \Big|t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_2)\Big|$$
$$+(1-\zeta)^6\int_0^1 \Big|t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\Big| dt]. \tag{4.17}$$

Here, we consider the right side of above inequality (4.17) and apply the power mean integral inequality on (4.17).

$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}\zeta^6\left(\int_0^1 t^5dt\right)^{(1-\frac{1}{q})}\left[\int_0^1 t^5(t\zeta)^s \mid h^{(v)}(\alpha_1)\mid^q + (1-t\zeta)^s \mid h^{(v)}(\alpha_2)\mid^q\right]^{\frac{1}{q}}$$
$$+(1-\zeta)^6\left(\int_0^1 t^5dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t^5(t\zeta+1-t)^s \mid h^{(v)}(\alpha_1)\mid^q + (t-t\zeta)^s \mid h^{(v)}(\alpha_2)\mid^q dt\right]^{\frac{1}{q}}$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}\zeta^6\left(\int_0^1 t^5dt\right)^{(1-\frac{1}{q})}\left[\int_0^1 t^5(t\zeta)^{-1} \mid h^{(v)}(\alpha_1)\mid^q + t^5(1-t\zeta)^{-1} \mid h^{(v)}(\alpha_2)\mid^q\right]^{\frac{1}{q}}$$
$$+(1-\zeta)^6\left(\int_0^1 t^5dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t^5(t\zeta+1-t)^{-1} \mid h^{(v)}(\alpha_1)\mid^q + t^5(t-t\zeta)^{-1} \mid h^{(v)}(\alpha_2)\mid^q dt\right]^{\frac{1}{q}} \tag{4.18}$$

$$= \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg\{\zeta^6\Big(\frac{1}{6}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|\ h^{(v)}(\alpha_1)\ |^q}{5\zeta}$$

$$- \frac{12\zeta^5+15\zeta^4+20\zeta^3+30\zeta^2+60\zeta+60ln(1-\zeta)}{60\zeta^6}\ |\ h^{(v)}(\alpha_2)\ |^q\Bigg]^{\frac{1}{q}}\Bigg\}$$

$$+ (1-\zeta)^6\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\Bigg[\frac{|\ h^{(v)}(\alpha_2)\ |^q}{5(1-\zeta)}$$

$$- \frac{12(1-\zeta)^5+15(1-\zeta)^4+20(1-\zeta)^3+30(1-\zeta)^2+60(1-\zeta)+60ln\zeta}{60(1-\zeta)^6}\ |\ h^{(v)}(\alpha_1)\ |^q\Bigg]^{\frac{1}{q}}$$

$$= \frac{(\alpha_2-\alpha_1)^5}{720}\left(\frac{6}{60\zeta^6(1-\zeta)^6}\right)^{\frac{1}{q}}$$

$$\times\Bigg[(1-\zeta)^6[12(1-\zeta)^5\zeta^6\ |\ h^{(v)}(\alpha_2)\ |^q - \zeta^6(12(1-\zeta)^5+15(1-\zeta)^4+20(1-\zeta)^3+30(1-\zeta)^2$$

$$+ 60(1-\zeta)+60ln\zeta)\ |\ h^{(v)}(\alpha_2)\ |^q]^{\frac{1}{q}} + \zeta^6[12(1-\zeta)^6\zeta^5\ |\ h^{(v)}(\alpha_1)\ |^q$$

$$- (1-\zeta)^6(12\zeta^5+15\zeta^4+20\zeta^3+30\zeta^2+60\zeta+60ln(1-\zeta))\ |\ h^{(v)}(\alpha_2)\ |^q]^{\frac{1}{q}}\Bigg].$$

□

Corollary 4.2.1. *Under the assumption of above Theorem (19), if $s = 1$. then,*

$$\Bigg|\frac{1}{(\alpha_2-\alpha_1)}\int_0^1 h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$- \frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|$$
$$\leq \frac{(\alpha_2-a)^5}{720}\left(\frac{1}{7}\right)^{\frac{1}{q}}\times\Bigg[(1-\zeta)^6\big[6\ |\ h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\ |^q + |\ h^{(v)}(\alpha_1)\ |^q\ \big]^{\frac{1}{q}}$$
$$+ \zeta^6\big[6\ |\ h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\ |^q + |\ h^{(v)}(\alpha_2)\ |^q\ \big]^{\frac{1}{q}}\Bigg]. \tag{4.19}$$

Corollary 4.2.2. *Under the assumption of above Theorem* (19). *If s=q=1 then,*

$$\left|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right.$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$\left.-\frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)-h(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{5040}\left[(1-\zeta)^6 \mid h^{(v)}(\alpha_1)\mid +6(\zeta^6+(1-\zeta)^6)\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid +\zeta^6 \mid h^{(v)}(\alpha_2)\mid\right]. \tag{4.20}$$

Corollary 4.2.3. *Under the assumption of above Theorem* (19). *If q=1, then,*

$$\left|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right.$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$\left.-\frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)-h(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|$$
$$\leq \frac{(\alpha_2-\alpha_1)^5}{7200}\Big[(12\zeta^5-12(1-\zeta)^5-15(1-\zeta)^4-20(1-\zeta)^3-30(1-\zeta)-60(1-\zeta)-60\ln\zeta)$$
$$\left|h^{(v)}(\alpha_1)\right| + (12(1-\zeta)^5-12\zeta^5-15\zeta^4-20\zeta^3-30\zeta^2-60\zeta-60ln(1-\zeta))\mid h^{(v)}(\alpha_2)\mid\Big].$$

Theorem 21. *Assume that $h: J\subseteq[0,\infty)\to\mathbb{R}$ is a differentiable function up to five times in the interval J, such that $\alpha_1,\alpha_2\in J$ and $\alpha_1<\alpha_2$, moreover $h^{(v)}\in L_1([\alpha_1,\alpha_2])$, and $0\leq\zeta\leq 1$. When $\mid h^{(v)}(t)\mid^q$ is an extended s-convex function in the second sense* *on* $[\alpha_1, \alpha_2]$, $-1 < s \leq 1$, *and* $q \geqslant 1$, *then,*

$$\begin{aligned}
&\Bigg|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{2\zeta-1}{2}(\alpha_2-\alpha_1) h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|\\
&\leq \frac{(\alpha_2-\alpha_1)^5}{120}\left(\frac{1}{5q+s+1}\right)^{\frac{1}{q}}\\
&\quad\Bigg[(1-\zeta)^6[|\, h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\,|^q + (5q+s+1)\beta(5q+1,s+1)\,|\, h^{(v)}(\alpha_1)\,|^q]^{\frac{1}{q}}\\
&\quad + \zeta^6[|\, h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\,|^q + (5q+s+1)\beta(5q+1,s+1)\,|\, h^{(v)}(\alpha_2)\,|^q]^{\frac{1}{q}}\Bigg].
\end{aligned} \tag{4.21}$$

Proof. We take (4.11) to prove the above inequality for s-convex function. We use here power mean integral inequality and the Euler integral of first kind that is commonly known as Beta function.

$$\begin{aligned}
&\Bigg|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h''(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{2\zeta-1}{2}(\alpha_2-\alpha_1) h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|\\
&\quad \leq \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\int_0^1 \Big|t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)dt\\
&\quad - (1-\zeta)^6\int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2+(1-t)\alpha_1))dt\Big|\Bigg].
\end{aligned} \tag{4.22}$$

Now, we apply the power mean integral inequality on the right side of above inequality on (4.22). Also, we used the integral inequality of first kind;

$$\leqslant \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{5q}\mid h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_2)\mid^q dt\right)^{\frac{1}{q}}$$

$$+(1-\zeta)^6\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{5q}\mid h^{(v)}(t(\zeta\alpha_1+(1-\zeta)\alpha_2)+(1-t)\alpha_1)\mid^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leqslant \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\left(\int_0^1 t^{5q}(t^s\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2))\mid^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 t^{5q}(1-t)^s\mid h^{(v)}(\alpha_2)\mid^q dt\right)$$

$$+(1-\zeta)^6\left(\int_0^1 t^{5q}(t^s\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2))\mid^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 t^{5q}(1-t)^s\mid h^{(v)}(\alpha_1)\mid^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}\Bigg[\zeta^6\left(\frac{\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q}{5q+s+1}+\beta(5q+1,s+1)\mid h^{(v)}(\alpha_2)\mid^q\right)^{\frac{1}{q}}$$

$$+(1-\zeta)^6\left(\frac{\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q}{5q+s+1}+\beta(5q+1,s+1)\mid h^{(v)}(\alpha_1)\mid^q\right)^{\frac{1}{q}}\Bigg]$$

We done here a simplification and obtain a result of s-convex that associates with the Euler integral inequality (Beta function).

$$\Bigg|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}h(t)dt-\frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$

$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$

$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$

$$-\frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)-h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|$$

$$\leq \frac{(\alpha_2-\alpha_1)^5}{120}\left(\frac{1}{5q+s+1}\right)^{\frac{1}{q}}$$

$$\Bigg[(1-\zeta)^6[\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q+(5q+s+1)\beta(5q+1,s+1)\mid h^{(v)}(\alpha_1)\mid^q]^{\frac{1}{q}}$$

$$+\zeta^6[\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q+(5q+s+1)\beta(5q+1,s+1)\mid h^{(v)}(\alpha_2)\mid^q]^{\frac{1}{q}}\Bigg]. \tag{4.23}$$

□

Corollary 4.2.4. *Under the assumption of Theorem (21) if $s = 1$, we obtain a following inequality that does not involve Beta function.*

$$\begin{aligned}
&\left|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right.\\
&\quad - \frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h^{''}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad \left.- \frac{2\zeta-1}{2}(\alpha_2-\alpha_1) h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - h(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|\\
&\leq \frac{(\alpha_2-\alpha_1)^5}{120}\left(\frac{1}{(5q+1)(5q+2)}\right)^{\frac{1}{q}} \Big[(1-\zeta)^6\big[(5q+1)\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q\\
&\quad + \mid h^{(v)}(\alpha_1)\mid^q\big]^{\frac{1}{q}} + \zeta^6\big[(5q+1)\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q + \mid h^{(v)}(\alpha_2)\mid^q\big]^{\frac{1}{q}}\Big].
\end{aligned} \tag{4.24}$$

Theorem 22. *Assume that $h : J \subseteq [0,\infty) \to \mathbb{R}$ is a differentiable function up to five times on J, $\alpha_1, \alpha_2 \in J$ with $\alpha_1 < \alpha_2$ moreover $h^{(v)} \in L_1([\alpha_1, \alpha_2])$, and $0 \leq \zeta \leq 1$. When $\mid h^{(v)}(t)\mid^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$, $-1 < s \leq 1$, and $q \geqslant 1$, subsequently,*

$$\begin{aligned}
&\left|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4 h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\right.\\
&\quad - \frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3 h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad - \frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2 h''(\zeta\alpha_1+(1-\zeta)\alpha_2)\\
&\quad \left.- \frac{2\zeta-1}{2}(\alpha_2-\alpha_1) h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2) - h(\zeta\alpha_1+(1-\zeta)\alpha_2)\right|\\
&\leq \frac{(\alpha_2-\alpha_1)^5}{120}\left(\frac{q-1}{6q-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\\
&\quad \Big[(1-\zeta)^6\big[\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q + \mid h^{(v)}(\alpha_1)\mid^q\big]^{\frac{1}{q}}\\
&\quad + \zeta^6\big[\mid h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\mid^q + \mid h^{(v)}(\alpha_2)\mid^q\big]^{\frac{1}{q}}\Big].
\end{aligned} \tag{4.25}$$

Proof. Since $| h^{(v)}(t) |^q$ is an extended s-convex function in the second sense on $[\alpha_1, \alpha_2]$, then by (4.11) and Holder's integral inequality, we have,

$$\begin{aligned}
&\Bigg| \frac{1}{(b-\alpha_1)} \int_{\alpha_1}^{\alpha_2} h(t)dt - \frac{5\zeta^4 - 10\zeta^3 + 10\zeta^2 - 5\zeta + 1}{120} (\alpha_2 - \alpha_1)^4 h^{(iv)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&- \frac{4\zeta^3 - 6\zeta^2 + 4\zeta - 1}{24} (\alpha_2 - \alpha_1)^3 h^{'''}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&- \frac{3\zeta^2 - 3\zeta + 1}{6} (\alpha_2 - \alpha_1)^2 h''(\zeta\alpha_1 + (1-\zeta)\alpha_2) \\
&- \frac{2\zeta - 1}{2} (\alpha_2 - \alpha_1) h^{'}(\zeta\alpha_1 + (1-\zeta)\alpha_2) - h(\zeta\alpha_1 + (1-\zeta)\alpha_2) \Bigg| \\
&\le \frac{(\alpha_2 - \alpha_1)^5}{120} \Bigg[\zeta^6 \int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2) dt \\
&- (1-\zeta)^6 \int_0^1 t^5 h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) dt \Bigg].
\end{aligned} \tag{4.26}$$

Here, we consider the right side of integral inequality (4.26) and apply Holder's integral inequality on (4.26). Also, $| h^{(v)}(t) |^q$ is an s-convex function,

$$\begin{aligned}
&\le \frac{(\alpha_2 - \alpha_1)^5}{120} \Bigg[\zeta^6 \left(\int_0^1 t^{\frac{5q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 | h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_2 |^q \, dt \right)^{\frac{1}{q}} \\
&+ (1-\zeta)^6 \left(\int_0^1 t^{\frac{5q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 | h^{(v)}(t(\zeta\alpha_1 + (1-\zeta)\alpha_2) + (1-t)\alpha_1) |^q \, dt \right)^{\frac{1}{q}} \Bigg]. \\
&\le \frac{(\alpha_2 - \alpha_1)^5}{120} \Bigg[\zeta^6 \left(\int_0^1 t^{\frac{5q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (t^s \mid h^{(v)}(\zeta\alpha_1 + (1-\zeta)\alpha_2) \mid + (1-t)^s \mid h^{(v)}(\alpha_2) |^q \, dt \right)^{\frac{1}{q}} \\
&+ (1-\zeta)^6 \left(\int_0^1 t^{\frac{5q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (t^s \mid h^{(v)}(\zeta\alpha_1 + (1-\zeta)\alpha_2)) |^q + (1-t)^s \mid h^{(v)}(\alpha_1) |^q \, dt \right)^{\frac{1}{q}} \Bigg].
\end{aligned}$$

After some simplifications we obtain.

$$\Bigg|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}h(t)dt-\frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2h''(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)-h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|$$
$$\leq\frac{(\alpha_2-\alpha_1)^5}{120}\left(\frac{q-1}{6q-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\Bigg[(1-\zeta)^6\big[\,|\,h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\,|^q+|\,h^{(v)}(\alpha_1)\,|^q\,\big]^{\frac{1}{q}}$$
$$+\zeta^6\big[\,|\,h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\,|^q+|\,h^{(v)}(\alpha_2)\,|^q\,\big]^{\frac{1}{q}}\Bigg]. \tag{4.27}$$

□

Corollary 4.2.5. *Under the assumption of above theorem (22), if s=1, then,*

$$\Bigg|\frac{1}{(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}h(t)dt-\frac{5\zeta^4-10\zeta^3+10\zeta^2-5\zeta+1}{120}(\alpha_2-\alpha_1)^4h^{(iv)}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{4\zeta^3-6\zeta^2+4\zeta-1}{24}(\alpha_2-\alpha_1)^3h^{'''}(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{3\zeta^2-3\zeta+1}{6}(\alpha_2-\alpha_1)^2h''(\zeta\alpha_1+(1-\zeta)\alpha_2)$$
$$-\frac{2\zeta-1}{2}(\alpha_2-\alpha_1)h^{'}(\zeta\alpha_1+(1-\zeta)\alpha_2)-h(\zeta\alpha_1+(1-\zeta)\alpha_2)\Bigg|$$
$$=\frac{(\alpha_2-\alpha_1)^5}{120}\left(\frac{q-1}{6q-1}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\Bigg[(1-\zeta)^6\Big[\,|\,h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\,|^q+|\,h^{(v)}(\alpha_1)\,|^q\,\Big]^{\frac{1}{q}}$$
$$+\zeta^6\Big[\,|\,h^{(v)}(\zeta\alpha_1+(1-\zeta)\alpha_2)\,|^q+|\,h^{(v)}(\alpha_2)\,|^q\,\Big]^{\frac{1}{q}}\Bigg]. \tag{4.28}$$

Chapter 5

Conclusion

In this thesis, we introduced several new results of Hermite-Hadamard type integral inequalities for $(2\omega + 1)$ times differentiable function associated to s-convex functions and extended s-convex functions. These results are obtained by using the famous integral inequalities i.e Holder's integral inequality and power mean's integral inequality. It is observed that the results obtained here are better than the known results.

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Appendix