Finite Difference Method Applicable to Non-Linear Ordinary Differential Equations

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MS THESIS WORK

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Dedicated to my beloved parents.

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Abstract

The key motive behind this work was to observe the precision of findings of finite difference method (FDM) of the system of non-linear second order ordinary differential equations (ODEs).

This dissertation has two parts. In the first part, by applying the adequate similarity transformation the governing partial differential equations reduces to ordinary differential eqautions (ODEs). Then the solution of resulting ordinary differential equations acquires through finite difference method (FDM) and comparisn is made with the outcomes received from MATLAB's built-in function *bvp4c* to assess the accuracy. Presented the impact of different parameters through graphs. Comparison of findings with the literature is done, and our findings are in favorable agreement.

In second part, we considered nonlinear ordinary differential equations (ODEs) from fluid mechanics. Conversion of continuous nonlinear coupled ordinary differential equations into discrete differences equations aligned boundary conditions is made and solved in MATLAB. To validate our results, the coupled ordinary differential equations are also solved with MATLAB builtin function *bvp4c*. And we are in agreement with the already published research articles.

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Chapter 1

Introduction

In the following chapter, we will be briefly discussing some fundamental concepts of differential equation with its types and introduction of some numerical methods to solve differential equations.

1.1 Differential Equations

Mathematical expressions explain a large number of physical and theoretical phenomena that exist in the universe. Differential equations are among the most important tools in science and engineering as they explain several relationships and physical laws in terms of differential equations. Differential equations are categorized into different types. Such forms of differential equations, besides defining the attributes of the equation itself, helps to choose the method for finding a solution. Generally categorized differential equations are: Ordinary/Partial, Linear/Non-linear, and Homogeneous/Nonhomogeneous. This list is far from comprehensive, many other properties and categories of differential equations exist. Depending on the type of conditions, differential equations are subcategorized into further two types: initial value problem and boundary value problem.

1.1.1 Ordinary Differential Equation (ODE)

Is an equation that incorporates derivative w.r.t only one independent variable of the function.

Example:

A simple example of ordinary differential equation is Newton's second law of motion, that is

$$m\frac{d^2\tilde{x}}{dt^2} = f(\tilde{x}), \tag{1.1}$$

where m is the mass of the particle in motion.

1.1.2 Partial Differential Equation (PDE)

Is an equation that incorporates derivative w.r.t two independent variables or more of the function.

Examples:

Some examples of PDEs are as follows;

$$m\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2}, \tag{1.2}$$

is heat equation, where D is the material property.

$$v^2\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}, \tag{1.3}$$

is wave equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.4}$$

is laplace's equation.

1.1.3 Intial Value Problem (IVP)

Is a problem consisting of ODEs or PDEs that has conditions specified at same point in the domain of the independent variable.

Example

$$f'' - 10f' + 9f = 5t, \tag{1.5}$$

$$f(0) = 0, \qquad f'(0) = 2. \tag{1.6}$$

1.1.4 Boundary Value Problem (BVP)

A problem of ODE or PDE that has conditions specified at the end points in the domain of the independent variable.

Example

$$f'' + 4f = 0, \tag{1.7}$$

$$f(0) = -2, \quad f(2\pi) = -2. \tag{1.8}$$

1.2 Numerical Methods

In order of solve differential equations, researchers develop a number of different methods. Nevertheless, there are limited available methods to get the exact solution of differential equations. It is not easy to find exact solutions to differential equations, particularly, for the system of differential equations. So whenever there is no analytical method available to solve a differential equation we use numerical technique.

There are different methods for finding numerical solutions, such as

- Finite difference method (FDM)
- Finite volume method (FVM)
- Finite element method (FEM)
- shooting method
- *bvp4c*
  etc

In this thesis, we will be discussing FDM and *bvp4c*.

1.2.1 Finite Difference Method (FDM)

Among the first methods that were adopted to numerically solve of differential equations was the FDM. Euler used it for the very first time, possibly in 1768. A wider range of problems have been solved through using finite difference method, which includes linear/non linear, time-independent/dependent, homogeneous/nonhomogeneous etc kinds problem. The fundamental principle of the FDM is to transform derivatives of a function into difference approximations of the unknown function.

The Taylor series expansions was the source to derived the difference approximations. Suppose that a function ξ have continuous derivatives of every order on an interval $[u, v]$. Let $\tilde{x} \in [u, v]$ and let $k > 0$ be step size, then by Taylor series

$$\xi(\tilde{x} + k) = \xi(\tilde{x}) + k\xi'(\tilde{x}) + \frac{k^2}{2!}\xi''(\tilde{x}) + ... + \frac{k^{n-1}}{(n-1)!}\xi^{(n-1)}(\tilde{x}) + R_n, \tag{1.9}$$

where R_n is the remainder after n terms in Lagrange's or Cauchy's form. If $R_n \to 0$ as $n \to \infty$, we get

$$\xi(\tilde{x} + k) = \xi(\tilde{x}) + k\xi'(\tilde{x}) + \frac{k^2}{2!}\xi''(\tilde{x}) + ... + \frac{k^{n-1}}{(n-1)!}\xi^{(n-1)}(\tilde{x}) + ... \tag{1.10}$$

and the infinite series called Taylor series.

If we truncate Taylor series after the first derivative, we have

$$\xi(\tilde{x} + k) \approx \xi(\tilde{x}) + k\xi'(\tilde{x}), \tag{1.11}$$

Rearranging above

$$\xi'(\tilde{x}) \approx \frac{\xi(\tilde{x} + k) - \xi(\tilde{x})}{k}. \tag{1.12}$$

is known as first-order forward difference approximation of $\xi'(\tilde{x})$.

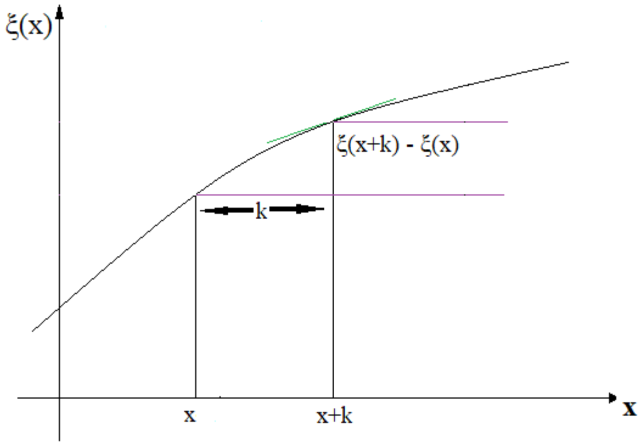

Figure 1.1: Forward difference approximation.

The backward expansion of Taylor series can be also be made, in order to calculate value on the previous point on the basis of a current point value.

$$\xi(\tilde{x}-k) = \xi(\tilde{x}) - k\xi'(\tilde{x}) + \frac{k^2}{2!}\xi''(\tilde{x}) - \frac{k^3}{3!}\xi'''(\tilde{x}) + ... \tag{1.13}$$

Truncating the expansion in the above equation after the first derivative gives:

$$\xi(\tilde{x}-k) \approx \xi(\tilde{x}) - k\xi'(\tilde{x}), \tag{1.14}$$

Rearranging above

$$\xi'(\tilde{x}) \approx \frac{\xi(\tilde{x}) - \xi(\tilde{x}-k)}{k}. \tag{1.15}$$

is known as first-order backward differnece approxomation of $\xi'(\tilde{x})$.

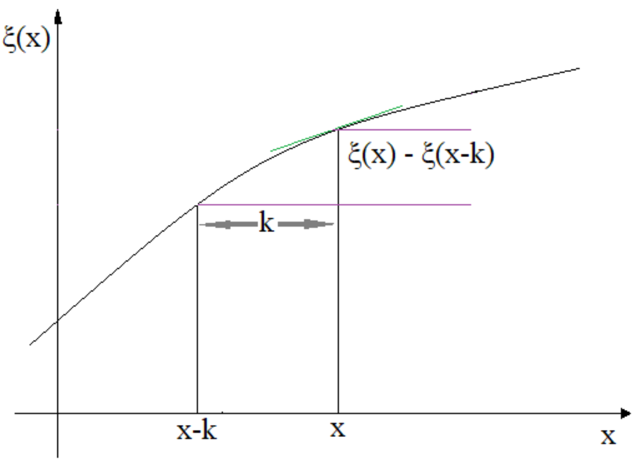

Figure 1.2: Backward difference approximation.

Subtracting Eq. (1.13) from Eq. (1.10), we get

$$\xi(\tilde{x}+k) - \xi(\tilde{x}-k) = 2k\xi'(\tilde{x}) - 2\frac{k^3}{3!}\xi'''(\tilde{x}) + ... \tag{1.16}$$

Truncating the above equation after first derivative and rearranging , we get

$$\xi'(\tilde{x}) \approx \frac{\xi(\tilde{x}+k) - \xi(\tilde{x}-k)}{2k} \tag{1.17}$$

which is known as second-order central difference approximation of $\xi'(\tilde{x})$.

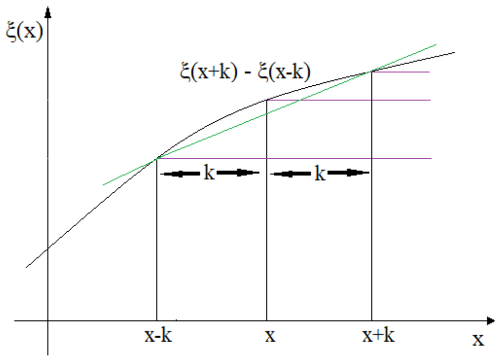

Figure 1.3: Central difference approximation.

Approximation of second derivative $\xi''(\tilde{x})$ in a similar way. For example,

$$\xi(\tilde{x}+k) = \xi(\tilde{x}) + k\xi'(\tilde{x}) + \frac{k^2}{2!}\xi''(\tilde{x}) + \frac{k^3}{(3)!}\xi^{(3)}(\tilde{x}) + ... \tag{1.18}$$

$$\xi(\tilde{x}-k) = \xi(\tilde{x}) - k\xi'(\tilde{x}) + \frac{k^2}{2!}\xi''(\tilde{x}) - \frac{k^3}{(3)!}\xi^{(3)}(\tilde{x}) + ... \tag{1.19}$$

Truncating eqs (1.18) and (1.19) after the second derivative, we get

$$\xi(\tilde{x}+k) + \xi(\tilde{x}-k) \approx 2\xi(\tilde{x}) + k^2\xi''(\tilde{x}) \tag{1.20}$$

Rearranging

$$\xi''(\tilde{x}) = \frac{\xi(\tilde{x}-k) - 2\xi(\tilde{x}) + \xi(\tilde{x}+k)}{k^2} \tag{1.21}$$

is known as second-order central difference approximation of $\xi''(\tilde{x})$.

Procedural Steps of FDM:

The four steps to solve DEs using finite difference method are as follows:

1. discretization of domain,
2. satisfying the equations at discrete points,
3. using finite differences to replace derivatives,
4. recursive algorithm is being developed.

1.2.2 Finite Volume Method (FVM)

The name of finite volume method is derived from the feature of its solution, that the governing PDE that is continuous are divide into finite-sized control volumes, instead of at points. Splitting the computational domain into a set of control volumes, is the basic idea. Then the derivaton of finite volume equation initiated by integration of orignal PDE over the set of control volumes.

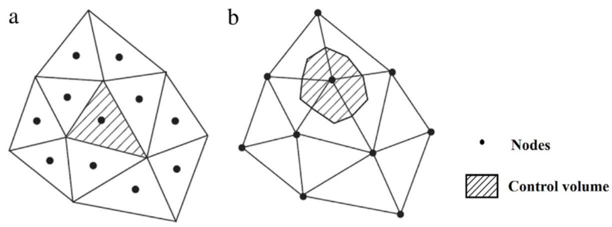

Figure 1.4: Control volume variants used in a finite volume method (Internet).

1.2.3 *bvp4c*

bvp4c is basically a built-in function of MATLAB, for solving the boundary value problems. The *bvp4c* code consists of three-stage Lobatto IIIa formula. It is basically a collocation formula and the collocation polynomial gives a continuous solution which is fourth order precise uniformly in the domain.

Procedural Steps of *bvp4c*:

The algorithm of *bvp4c* depends on the three phases of imformation:

1. the equations to be solve (of first-order),
2. their associated boundary conditions,
3. initial guess of the solutions.

Chapter 2

Literaure Review

Finite difference approximation for derivatives is one of the simplest and oldest approaches for solving differential equations. In 1768, L. Euler recognised finite difference scheme in one dimension of space and it was presume that, In 1908 C. Runge expanded the scheme to two dimension. In the early 1950s, the emergence of finite difference approximation in computational applications tried and their development was accelerated with the advent of softwares generated in computers that provided a convenient tool to deal with complicated technical and scientific issues. During the last five decades, experimental findings about the precision, consistency and convergence of the finite difference method for DEs were obtained.

In 1999, Gamet et al.[26] researched to establish a compact FDM of high order capable of handling variable grid sizes. They provide a numerical truncation error review. It takes into account the convection equation for the first derivative and the diffusion equation for the second derivative. The potential of compact schemes to replicate results is shown by non-uniform mesh generalization.

In 2000, Abarbanel et al.[27] the rate of convergence of error bounds and temporal behaviour of approximations of the finite difference to partial differential equations are studied in their research paper. They calculated the dependency on mesh size and time of the error boundaries. Hyperbolic and parabolic partial differential equations are used for this purpose.

Further Mickens, R. in (2001), provides an introduction to non-standard methods of

finite difference that are helpful in constructing differential equations. He defined the precise finite difference scheme in his article, even the rules for building non-standard scheme with its implementation.
Farjadpur, A. and Roundy (2006) took modelling of discontinuous dielectric materials, the finite difference time domain approach suffers from decreased precision due to discretization. They demonstrate that by using sub pixel smoothing, if it is correctly designed, accuracy can be increased. This scheme completes quadratic convergence as well.
Thankane et al.[30] in 2009,they present successful algorithms for linear and non linear beam equations based on the finite difference approach. They also have an overview of the algorithm's convergence. Designing the Mathmatica Module gives a solution to the number of beam equations.
In 2010, Dolicanin et al.[31] studied in their research paper, the method of finite difference is used in the theory of thin plates to analyse phenomena. FDM is based on substituting the equation of difference into the equation of difference. The problem of bending of thin plates can be solved effectively by this approach.
Sungu et al. [32] in 2012, introduced hybrid methodology for non linear partial differential equation. The goal of hybrid method is to merge differential transformation stability and finite differential efficiency. They observed that hybrid technique is quicker than the accompanying method of finite difference. The hybrid approach provides an iterative method for the estimation of precise numerical solutions.
Later in 2012, El-Azab et al. [33] introduce the new FDM, which is known as logarithmic FDM. This method is expanded for solving partial differential equations in linear or non-linear higher order. The Kortweg de Vries Burger equation was solved (KdVB) in their research paper. They made a comparison of explicit FDM and exponential FDM. The results showed that log FDM solution gives high accuracy and no need for more conditions, also closed to analytical solution.
In 2013, approximately for the first time Lakshmi et al. [34] worked for the coding of FDM in MATLAB software for the ODEs. The results were then compared with the analytic solution and had an agreement with exact solution.

In this dissertation, I tried to construct FDM code in MATLAB, that works for nonlinear coupled ODEs associated with boundary conditions, and it has an agreement with the already published work.

Chapter 3

Some Computational Results for Steady Flow Problem

In the following chapter, we considered a mathematical problem of megnetohydrodynamic steady flow from fluid mechancis. In Section 3.1, we will describe the governing partial differtial equation. In Section 3.2, we investigated the problem by converting the governing PDEs into ODEs via some similarity transformations. In Section 3.3, there will be numerical solutions of the resultant ODEs, first we use finite difference method (FDM) and then *bvp4c* will be used to solve the resultant ODEs. In Section 3.4, comparison of the evaluated findings with previously published research have been done, and some graphical representations of the findings are also shown.

3.1 Governing Partial Differential Equations

The considered governing equations of the current fluid problem from [2], by letting $\frac{\partial \tilde{u}_1}{\partial \tilde{t}} = 0$, $\frac{\partial \tilde{\tilde{T}}}{\partial \tilde{t}} = 0$ and $\frac{\partial \tilde{\tilde{C}}}{\partial \tilde{t}} = 0$, we have

$$\frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{\partial \tilde{u}_2}{\partial \tilde{y}} = 0, \tag{3.1}$$

$$\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \tilde{u}_2 \frac{\partial \tilde{u}_1}{\partial \tilde{y}} = \nu \frac{\partial^2 \tilde{u}_1}{\partial \tilde{y}^2} - \frac{\sigma B_0^2 \tilde{u}_1}{\rho} - \frac{\nu \tilde{u}_1}{K} + g_0 \beta (\tilde{\tilde{T}} - \tilde{\tilde{T}}_\infty), \tag{3.2}$$

$$\tilde{u_1}\frac{\partial\tilde{\tilde{T}}}{\partial\tilde{x}}+\tilde{u_2}\frac{\partial\tilde{\tilde{T}}}{\partial\tilde{y}}=\frac{k}{\rho C_p}\frac{\partial^2\tilde{\tilde{T}}}{\partial\tilde{y}^2}+\frac{\nu}{C_p}(\frac{\partial\tilde{u_1}}{\partial\tilde{y}})^2+\frac{\nu\tilde{u_1}^2}{C_pK}$$
$$+\hat{\tau}\left[\frac{D_T}{\tilde{T_\infty}}\left(\frac{\partial\tilde{\tilde{T}}}{\partial\tilde{y}}\right)^2+D_B\frac{\partial\tilde{\tilde{C}}}{\partial\tilde{y}}\frac{\partial\tilde{\tilde{T}}}{\partial\tilde{y}}\right]+\frac{\sigma B_0^2\tilde{u_1}^2}{\rho C_p}-\frac{1}{\rho C_p}\frac{\partial q_r}{\partial\tilde{y}}+\frac{Q(\tilde{\tilde{T}}-\tilde{\tilde{T}}_\infty)}{\rho C_p}, \quad (3.3)$$

$$\tilde{u_1}\frac{\partial\tilde{\tilde{C}}}{\partial\tilde{x}}+\tilde{u_2}\frac{\partial\tilde{\tilde{C}}}{\partial\tilde{y}}=\frac{D_T}{\tilde{\tilde{T}}_\infty}\frac{\partial^2\tilde{\tilde{T}}}{\partial\tilde{y}^2}+D_B\frac{\partial^2\tilde{\tilde{C}}}{\partial\tilde{y}^2} \quad (3.4)$$

where,

$\tilde{u_1}$: Component of velocity in x direction,

$\tilde{u_2}$: Component of velocity in y direction,

$\tilde{\tilde{T}}$: fluid's temperature,

$\tilde{\tilde{C}}$: fluid's concentration,

K : porous medium permeability,

μ : Viscosity's coefficient ,

ν : Kinematic viscosity,

k : Thermal conductivity,

β : Thermal expansion coefficient,

ρ : fluid's density,

q_r : Radiative heat flux,

C_p : Specific heat at constant pressure,

σ : fluid's electrical conductivity,

Q : Heat source coefficient,

D_B : coefficient of brownian diffusion,

D_T : coefficient of thermophoretic diffusion,

$\tilde{\tilde{T}}_\infty$: Ambient fluid temperature,

$\tilde{\tilde{C}}_\infty$: Ambient fluid concentration.

Following are the given conditions on boundary:

$$\tilde{u_1} = U_w + \frac{\mu}{L}\frac{\partial \tilde{u_1}}{\partial \tilde{y}}, \qquad \tilde{u_2} = 0, \qquad \tilde{\tilde{T}} = \tilde{\tilde{T}}_w, \qquad \tilde{\tilde{C}} = \tilde{\tilde{C}}_w, \qquad as \qquad \tilde{y} \to 0. \tag{3.5}$$

$$\tilde{u_1} \to 0, \qquad \tilde{\tilde{T}} \to \tilde{\tilde{T}}_\infty, \qquad \tilde{\tilde{C}} \to \tilde{\tilde{C}}_\infty \qquad as \qquad \tilde{y} \to \infty. \tag{3.6}$$

where $U_w = a\tilde{x}$, $\tilde{\tilde{T}}_w = \tilde{\tilde{T}}_\infty + b\tilde{x}$, $\tilde{\tilde{T}}_\infty$ is a constant, $\tilde{\tilde{C}}_w = \tilde{\tilde{C}}_\infty + b\tilde{x}$, $\tilde{\tilde{C}}_\infty$ is a constant.

Skin Friction Coefficient ($C_{\tilde{\tilde{g}}}$)

The $C_{\tilde{\tilde{g}}}$ is defined in [2] as

$$C_{\tilde{\tilde{g}}} = \frac{\hat{\tau}_w}{\rho \tilde{u}_w^2}, \qquad \text{where} \qquad \tilde{u}_w = \frac{\partial \tilde{u_1}}{\partial \tilde{y}} \tag{3.7}$$

Local Nusselt Number ($N\tilde{u}_{\tilde{x}}$)

The $N\tilde{u}_{\tilde{x}}$ is defined in [2] as

$$N\tilde{u}_{\tilde{x}} = \frac{\tilde{x} q_w}{k(\tilde{\tilde{T}}_w - \tilde{\tilde{T}}_\infty)}, \qquad \text{where} \qquad q_w = -k\left(1 + \frac{16\sigma^* \tilde{\tilde{T}}_\infty^3}{3k^* k}\right)\frac{\partial \tilde{\tilde{f}}}{\partial \tilde{y}} \tag{3.8}$$

Local Sherwood Number ($Sh_{\tilde{x}}$)

The $Sh_{\tilde{x}}$ is given by [2] as

$$Sh_{\tilde{x}} = \frac{\tilde{x} j_w}{k(\tilde{\tilde{C}}_w - \tilde{\tilde{C}}_\infty)}, \qquad where \qquad j_w = -D\frac{\partial \tilde{\tilde{C}}}{\partial \tilde{y}}|_{\tilde{y}=0} \tag{3.9}$$

3.2 Transformed Ordinary Differential Equations

In this Section, we will be considering some similarity transformations to convert the governing PDEs (3.1)−(3.6) into ODEs of a steady flow problem. Considered similarity transformations are as follows

$$\tilde{\eta} = \sqrt{\frac{a}{\nu}}\tilde{y}, \qquad \tilde{\psi}(\tilde{\eta}) = \sqrt{a\nu}\tilde{x}\tilde{\tilde{g}}(\tilde{\eta}), \qquad \tilde{u}_1 = a\tilde{x}\frac{\partial\tilde{\tilde{g}}}{\partial\tilde{\eta}}, \qquad \tilde{u}_2 = -\sqrt{a\nu}\tilde{\tilde{g}}(\tilde{\eta}),$$
$$\tilde{\tilde{\theta}}(\tilde{\eta}) = \frac{\tilde{\tilde{T}} - \tilde{\tilde{T}}_\infty}{\tilde{\tilde{T}}_w - \tilde{\tilde{T}}_\infty}, \qquad \tilde{\tilde{\phi}}(\tilde{\eta}) = \frac{\tilde{\tilde{C}} - \tilde{\tilde{C}}_\infty}{\tilde{\tilde{C}}_w - \tilde{\tilde{C}}_\infty}.$$

After using the similarity tansformation in governing system of equations (3.1)-(3.6), we obtain

$$\frac{d^3\tilde{\tilde{g}}}{d\tilde{\eta}^3} + \tilde{\tilde{g}}\frac{d^2\tilde{\tilde{g}}}{d\tilde{\eta}^2} - \left(\frac{d\tilde{\tilde{g}}}{d\tilde{\eta}}\right)^2 - \left(M + \frac{1}{Da}\right)\frac{d\tilde{\tilde{g}}}{d\tilde{\eta}} + G_r\tilde{\tilde{\theta}} = 0 \tag{3.10}$$

$$Pr\left(\tilde{\tilde{\theta}}\frac{d\tilde{\tilde{g}}}{d\tilde{\eta}} - \tilde{\tilde{g}}\frac{d\tilde{\tilde{\theta}}}{d\tilde{\eta}}\right) - (1+Nr)\frac{d^2\tilde{\tilde{\theta}}}{d\tilde{\eta}^2} - EcPr\left(\frac{d^2\tilde{\tilde{g}}}{d\tilde{\eta}^2}\right)^2 - EcPr\left(M + \frac{1}{Da}\right)\left(\frac{d\tilde{\tilde{g}}}{d\tilde{\eta}}\right)^2$$
$$- PrS\tilde{\tilde{\theta}} - PrNb\frac{d\tilde{\tilde{\theta}}}{d\tilde{\eta}}\frac{d\tilde{\tilde{\phi}}}{d\tilde{\eta}} - NtPr\left(\frac{d\tilde{\tilde{\theta}}}{d\tilde{\eta}}\right)^2 = 0, \tag{3.11}$$

$$Le\left(\tilde{\tilde{\phi}}\frac{d\tilde{\tilde{g}}}{d\tilde{\eta}} - \tilde{\tilde{g}}\frac{d\tilde{\tilde{\phi}}}{d\tilde{\eta}}\right) - \frac{d^2\tilde{\tilde{\phi}}}{d\tilde{\eta}^2} - \frac{Nt}{Nb}\frac{d^2\tilde{\tilde{\theta}}}{d\tilde{\eta}^2} = 0. \tag{3.12}$$

And

$$\tilde{\tilde{g}}(0) = 0, \qquad \frac{d\tilde{\tilde{g}}}{d\tilde{\eta}}(0) = 1 + \lambda\frac{d^2\tilde{\tilde{g}}}{d\tilde{\eta}^2}(0), \qquad \tilde{\tilde{\theta}}(0) = 1, \qquad \tilde{\tilde{\phi}}(0) = 1, \tag{3.13}$$

$$\frac{d\tilde{\tilde{g}}}{d\tilde{\eta}} \to 0, \qquad \tilde{\tilde{\theta}} \to 0, \qquad \tilde{\tilde{\phi}} \to 0, \qquad as \qquad \tilde{\eta} \to \infty. \tag{3.14}$$

Here $M = \frac{\sigma B_0^2}{\rho a}$ is Magnetic number, $G_r = \frac{\beta g_0 b}{a^2}$ is Grashof number, $Pr = \frac{\rho \nu C_p}{k}$ is Prandtl number, $Da = \frac{Ka}{\nu}$ is Darcy number, $Ec = \frac{a u_w}{b C_p}$ is Eckert number, $S = \frac{Q}{a \rho C_p}$ is Heat source, $Le = \frac{\nu}{D_B}$ is Lewis number, $Nt = \dfrac{\tau D_T(\tilde{T_w} - \tilde{T_\infty})}{\nu \tilde{T_\infty}}$ is Thermophoresis parameter, $Nb = \dfrac{\hat{\tau} D_B(\tilde{C_w} - \tilde{C_\infty})}{\nu}$ is Brownian parameter.

After using the similarity transformation the equations (3.7) - (3.9) becomes

$$\sqrt{Re_{\tilde{x}}} C_{\tilde{\tilde{g}}} = -\tilde{\tilde{g}}''(0), \tag{3.15}$$

$$\frac{Nu_{\tilde{x}}}{\sqrt{Re_{\tilde{x}}}} = -(1 + Nr)\tilde{\tilde{\theta}}'(0), \tag{3.16}$$

$$\frac{Sh_{\tilde{x}}}{\sqrt{Re_{\tilde{x}}}} = -\tilde{\tilde{\phi}}'(0). \tag{3.17}$$

3.3 Numerical Solutions

In this Section, we will be solving the resultant ordinary differential equations ODEs (3.10) – (3.12) with the subjected boundary conditions (3.13) – (3.14) through finite difference method (FDM). Furthermore, MATLAB built-in function *bvp4c* is used in order to verify our findings.

3.3.1 Finite Difference Method

One of the easiest and oldest methods for solving differential equations is finite difference method (FDM). In finite difference approximation, we convert the differetial equation into difference equation by interchanging the derivatives with difference quotients. For implementation of FDM, the order of equation (3.10) have to reduce to second order. Letting $\dfrac{d\tilde{\tilde{g}}}{d\tilde{\eta}} = \tilde{\tilde{G}}$, the Eq. (3.10)-(3.14) takes the form as follows

$$\frac{d^2\tilde{\tilde{G}}}{d\tilde{\eta}^2} + \tilde{\tilde{g}}\frac{d\tilde{\tilde{G}}}{d\tilde{\eta}} - (\tilde{\tilde{G}})^2 - \left(M + \frac{1}{Da}\right)\tilde{\tilde{G}} + G_r\tilde{\tilde{\theta}} = 0, \tag{3.18}$$

$$Pr\left(\tilde{\tilde{\theta}}\tilde{\tilde{G}} - \tilde{\tilde{g}}\frac{d\tilde{\tilde{\theta}}}{d\tilde{\eta}}\right) - (1+Nr)\frac{d^2\tilde{\tilde{\theta}}}{d\tilde{\eta}^2} - EcPr\left(\frac{d\tilde{\tilde{G}}}{d\tilde{\eta}}\right)^2$$
$$- EcPr\left(M + \frac{1}{Da}\right)(\tilde{\tilde{G}})^2 - PrS\tilde{\tilde{\theta}} - PrNb\frac{d\tilde{\tilde{\theta}}}{d\tilde{\eta}}\frac{d\tilde{\tilde{\phi}}}{d\tilde{\eta}} - NtPr\left(\frac{d\tilde{\tilde{\theta}}}{d\tilde{\eta}}\right)^2 = 0, \tag{3.19}$$

$$Le\left(\tilde{\tilde{\phi}}\tilde{\tilde{G}} - \tilde{\tilde{g}}\frac{d\tilde{\tilde{\phi}}}{d\tilde{\eta}}\right) - \frac{d^2\tilde{\tilde{\phi}}}{d\tilde{\eta}^2} - \frac{Nt}{Nb}\frac{d^2\tilde{\tilde{\theta}}}{d\tilde{\eta}^2} = 0. \tag{3.20}$$

The related bounday condition are

$$\tilde{\tilde{G}}(0) = 1 + \lambda\tilde{\tilde{G}}'(0), \qquad \tilde{\tilde{g}}(0) = 0, \qquad \tilde{\tilde{\theta}}(0) = 1, \qquad \tilde{\tilde{\phi}}(0) = 1, \tag{3.21}$$

$$\tilde{\tilde{G}} \to 0, \qquad \tilde{\tilde{\theta}} \to 0, \qquad \tilde{\tilde{\phi}} \to 0, \qquad as \qquad \tilde{\eta} \to \infty. \tag{3.22}$$

The domain of the flow have to split into grid of finite lines in order to have the coupled ODE solution.

$$\begin{aligned}
\left(\frac{d\tilde{\tilde{G}}}{d\tilde{\eta}}\right) &= \frac{\tilde{\tilde{G}}_{i+1} - \tilde{\tilde{G}}_i}{\Delta\tilde{\eta}}, & \left(\frac{d^2\tilde{\tilde{G}}}{d\tilde{\eta}^2}\right) &= \frac{\tilde{\tilde{G}}_{i+1} - 2\tilde{\tilde{G}}_i + \tilde{\tilde{G}}_{i-1}}{(\Delta\tilde{\eta})^2}, \\
\left(\frac{d\tilde{\tilde{\theta}}}{d\tilde{\eta}}\right) &= \frac{\tilde{\tilde{\theta}}_{i+1} - \tilde{\tilde{\theta}}_i}{\Delta\tilde{\eta}}, & \left(\frac{d^2\tilde{\tilde{\theta}}}{d\tilde{\eta}^2}\right) &= \frac{\tilde{\tilde{\theta}}_{i+1} - 2\tilde{\tilde{\theta}}_i + \tilde{\tilde{\theta}}_{i-1}}{(\Delta\tilde{\eta})^2}, \\
\left(\frac{d\tilde{\tilde{\phi}}}{d\tilde{\eta}}\right) &= \frac{\tilde{\tilde{\phi}}_{i+1} - \tilde{\tilde{\phi}}_i}{\Delta\tilde{\eta}}, & \left(\frac{d^2\tilde{\tilde{\phi}}}{d\tilde{\eta}^2}\right) &= \frac{\tilde{\tilde{\phi}}_{i+1} - 2\tilde{\tilde{\phi}}_i + \tilde{\tilde{\phi}}_{i-1}}{(\Delta\tilde{\eta})^2}.
\end{aligned}$$

Using the discretizations as mentioned above, Eqs. (3.18) – (3.22) has becomes

$$\frac{\tilde{\tilde{G}}_{i+1} - 2\tilde{\tilde{G}}_i + \tilde{\tilde{G}}_{i-1}}{(\Delta\tilde{\eta})^2} + \tilde{\tilde{g}}_i \left(\frac{\tilde{\tilde{G}}_{i+1} - \tilde{\tilde{G}}_i}{\Delta\tilde{\eta}} \right) - (\tilde{\tilde{G}}_i)^2 - \left(M + \frac{1}{Da} \right) \tilde{\tilde{G}}_i + G_r \tilde{\tilde{\theta}}_i = 0, \quad (3.23)$$

$$Pr\tilde{\tilde{\theta}}_i \tilde{\tilde{G}}_i - Pr\tilde{\tilde{g}}_i \left(\frac{\tilde{\tilde{\theta}}_{i+1} - \tilde{\tilde{\theta}}_i}{\Delta\tilde{\eta}} \right) - (1+Nr) \frac{\tilde{\tilde{\theta}}_{i+1} - 2\tilde{\tilde{\theta}}_i + \tilde{\tilde{\theta}}_{i-1}}{(\Delta\tilde{\eta})^2} - EcPr \left(\frac{\tilde{\tilde{G}}_{i+1} - \tilde{\tilde{G}}_i}{\Delta\tilde{\eta}} \right)^2 - PrS\tilde{\tilde{\theta}}_i$$
$$-EcPr \left(M + \frac{1}{Da} \right) (\tilde{\tilde{G}}_i)^2 - PrNb \left(\frac{\tilde{\tilde{\theta}}_{i+1} - \tilde{\tilde{\theta}}_i}{\Delta\tilde{\eta}} \right) \left(\frac{\tilde{\tilde{\phi}}_{i+1} - \tilde{\tilde{\phi}}_i}{\Delta\tilde{\eta}} \right) - NtPr \left(\frac{\tilde{\tilde{\theta}}_{i+1} - \tilde{\tilde{\theta}}_i}{\Delta\tilde{\eta}} \right)^2 = 0, \quad (3.24)$$

$$Le \left(\tilde{\tilde{\phi}}_i \tilde{\tilde{G}}_i - \tilde{\tilde{g}}_i \frac{\tilde{\tilde{\phi}}_{i+1} - \tilde{\tilde{\phi}}_i}{\Delta\tilde{\eta}} \right) - \frac{\tilde{\tilde{\phi}}_{i+1} - 2\tilde{\tilde{\phi}}_i + \tilde{\tilde{\phi}}_{i-1}}{(\Delta\tilde{\eta})^2} - \frac{Nt}{Nb} \left(\frac{\tilde{\tilde{\theta}}_{i+1} - 2\tilde{\tilde{\theta}}_i + \tilde{\tilde{\theta}}_{i-1}}{(\Delta\tilde{\eta})^2} \right) = 0. \quad (3.25)$$

Also

$$\tilde{\eta} = 0: \qquad \tilde{\tilde{G}}_0 = 1 + \lambda \frac{\tilde{\tilde{G}}_1 - \tilde{\tilde{G}}_0}{\Delta\tilde{\eta}}, \qquad \tilde{\tilde{g}}_0 = 0, \qquad \tilde{\tilde{\theta}}_0 = 1, \qquad \tilde{\tilde{\phi}}_0 = 1, \quad (3.26)$$

$$\tilde{\eta} \to \infty: \qquad \tilde{\tilde{G}}_\infty = 0, \qquad \tilde{\tilde{\theta}}_\infty = 0, \qquad \tilde{\tilde{\phi}}_\infty = 0. \quad (3.27)$$

3.3.2 *bvp4c*

It is difficult to solve the system of non-linear ODEs by analytical methods. So we've developed the numerical solution of the ODEs by using *bvp4c*, which is a built-in solver in MATLAB. *bvp4c* have the ability to solve quite complex problems. The algorithm of *bvp4c* depends on the three phases of information: the equations to be solved; their associated boundary conditions; and the initial guess of the solutions and the guess for unknown parameters. In the initial lines of the code we will assign the initial guess for the unknown parameter whilst the following lines of the code execute the remaining three functions.

Firstly, the equations converted to form a system of first-order ordinary differential equations. For that, we introduce the variables

$$
\begin{aligned}
\tilde{\kappa}_1 &= \tilde{\tilde{g}}, \\
\tilde{\kappa}_2 &= \tilde{\tilde{g}}', \\
\tilde{\kappa}_3 &= \tilde{\tilde{g}}'', \\
\tilde{F}1 &= \tilde{\tilde{g}}''', \\
\tilde{\kappa}_4 &= \tilde{\tilde{\theta}}, \\
\tilde{\kappa}_5 &= \tilde{\tilde{\theta}}', \\
\tilde{F}2 &= \tilde{\tilde{\theta}}'', \\
\tilde{\kappa}_6 &= \tilde{\tilde{\phi}}, \\
\tilde{\kappa}_7 &= \tilde{\tilde{\phi}}', \\
\tilde{F}3 &= \tilde{\tilde{\phi}}''.
\end{aligned}
$$

Then, we can rewrite the ordinary differential equations (3.10) to (3.12)

$$\tilde{\kappa}_1' = \tilde{\kappa}_2 \tag{3.28}$$

$$\tilde{\kappa}_2' = \tilde{\kappa}_3 \tag{3.29}$$

$$\tilde{F}1 = \tilde{\kappa}_2^2 - \tilde{\kappa}_1\tilde{\kappa}_3 + \left(M + \frac{1}{Da}\right)\tilde{\kappa}_2 - Gr\tilde{\kappa}_4 \tag{3.30}$$

$$\tilde{\kappa}_4' = \tilde{\kappa}_5 \tag{3.31}$$

$$\tilde{F}2 = \frac{Pr}{1+Nr}\left[\tilde{\kappa}_4\tilde{\kappa}_2 - \tilde{\kappa}_1\tilde{\kappa}_5 - Ec\tilde{\kappa}_3^2 - Ec\left(M + \frac{1}{Da}\right)\tilde{\kappa}_2^2 - S\tilde{\kappa}_4 - Nb\tilde{\kappa}_5\tilde{\kappa}_7 - Nt\tilde{\kappa}_5^2\right] \tag{3.32}$$

$$\tilde{\kappa}_6' = \tilde{\kappa}_7 \tag{3.33}$$

$$\tilde{F}3 = Le(\tilde{\kappa}_6\tilde{\kappa}_2 - \tilde{\kappa}_1\tilde{\kappa}_7) - \frac{Nt}{Nb}\tilde{F}2 \tag{3.34}$$

along with the transformed initial and boundary conditions (3.13) to (3.14)

$$\tilde{\kappa}_2(0) = 1 + \lambda\tilde{\kappa}_3; \qquad \tilde{\kappa}_1(0) = 0; \qquad \tilde{\kappa}_4(0) = 1; \qquad \tilde{\kappa}_6(0) = 1, \tag{3.35}$$

$$\tilde{\kappa}_2(\infty) = 0; \qquad \tilde{\kappa}_4(\infty) = 0; \qquad \tilde{\kappa}_6(\infty) = 0. \tag{3.36}$$

Here we do not know the values of the parameters M, Da, Gr, Pr, Nr, Ec, S, Nb, Nt, Le and λ. Therefore it has to be determined along with the solution. As with the initial guesses for solution, we will also guess an initial value of parameters, and let *bvp4c* converge to corresponding parameter values. The efficacy of *bvp4c* depends almost entirely on a decent initial guess.

3.4 Results and Discussion

The discretized governing equations $(3.23) - (3.25)$ along with boundary conditions $(3.26) - (3.27)$ were solved numerically by creating FDM code in MATLAB, and the transformed ordinary differential equations $(3.28) - (3.34)$ subjected to the boundary conditions equations $(3.35) - (3.36)$ were solved numerically by MATLAB built-in function *bvp4c*. Find the values of the derived physical quantities, such as, $\tilde{\tilde{g}}''(0)$, $-\tilde{\tilde{\theta}}'(0)$ and $-\tilde{\tilde{\phi}}'(0)$ as in $(3.15), (3.16) and (3.17)$ respectively. And the comparison of $-\tilde{\tilde{\theta}}'(0)$ has been made with the previously published research works.

Table 3.1 Comparison for $-\tilde{\tilde{\theta}}'(0)$ for some values of Pr when $G_r = Nr = Ec = S = Nt = Nb = \lambda = 0$.

Pr	Ishak et al.[9]	Elbashbeshy[5]	Exact Solution[35]	Ibrahim et al.[7]	*bvp4c*	FDM
0.72	0.8086	0.808	0.8086	0.8095	0.8086	0.8104
1	1.0	1.0	1.0	1.001	1.0000	1.0001
10	3.7202	3.7207	3.7206	3.7208	3.7206	3.6691

Table 3.1: Comparison for $-\tilde{\tilde{\theta}}'(0)$.

To assess the precision of the present analysis through FDM and *bvp4c*, comparisons are developed with the available results of the local Nusselt number $-\tilde{\tilde{\theta}}'(0)$ for the steady flow problem. In table 3.1, we compared results of $-\tilde{\tilde{\theta}}'(0)$ provided by Ibrahim et al [7]. In this comparison, we have observed that our findings of *bvp4c* are in good agreement. While the difference in results of FDM indicates to discretization error.

The results are also graphically illustrated for further interpretation of the findings through *bvp4c* when Nt = 0.2, Nb = 0.1, Nr = 0.2, Gr = 0.5, Ec = 0.2, lambda = 0.1, S = 0, Pr = 1, Da = 0.2, M = 0.5 and Le = 1.5.

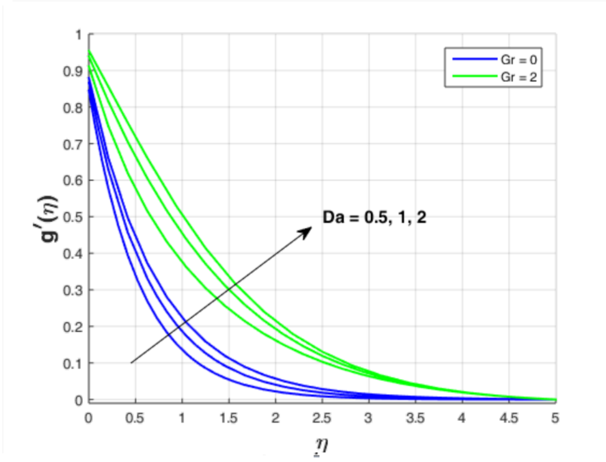

Figure 3.1: Computational velocity profile $\tilde{\tilde{g}}'$.

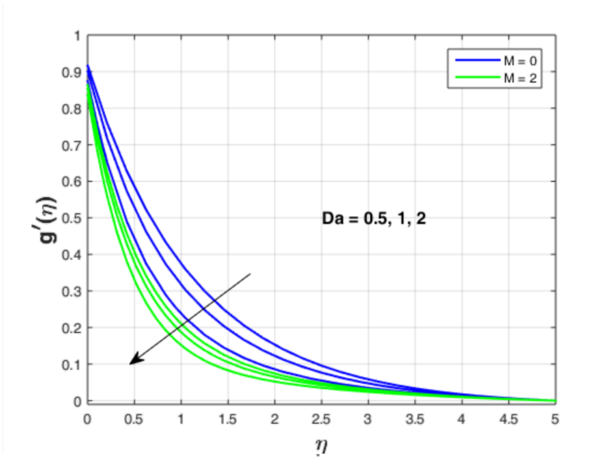

Figure 3.2: Computational velocity profile $\tilde{\tilde{g}}'$.

In figure 3.1, the graph of $\tilde{\tilde{g}}'$ is observed against variation in parameter Da under the non-appearance and appearance of parameter G_r. With the increasing value of G_r, the $\tilde{\tilde{g}}'$ increases. In figure 3.2, the graph of $\tilde{\tilde{g}}'$ is also observed against variations in parameter Da under the non-appearance and appearance of parameter M. The $\tilde{\tilde{g}}'$ decreases with the increasing value of M.

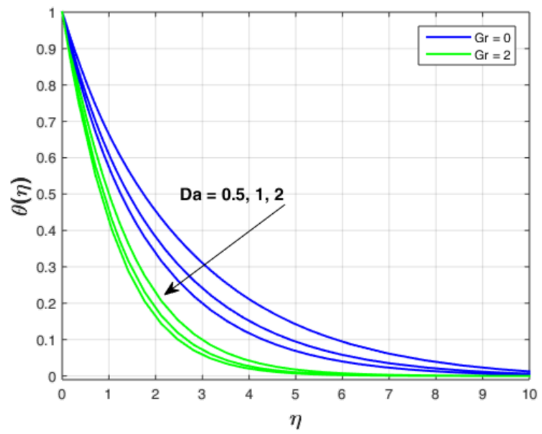

Figure 3.3: Computational temperature profile $\tilde{\tilde{\theta}}$.

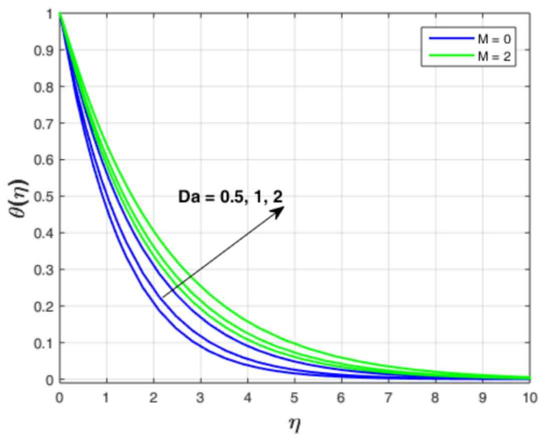

Figure 3.4: Computational temperature profile $\tilde{\tilde{\theta}}$.

In figure 3.3, the graph of $\tilde{\tilde{\theta}}$ is observed against variation of parameter Da under the absence and presence of parameter G_r. With the increasing value of G_r, the $\tilde{\tilde{\theta}}$ decreases. In figure 3.4, the graph of $\tilde{\tilde{\theta}}$ is also observed against variable value of Da under the appearance and nonappearance of parameter M. The $\tilde{\tilde{\theta}}$ increases with the increasing value of M.

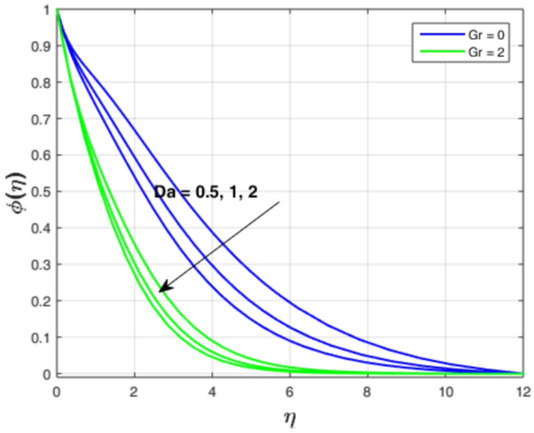

Figure 3.5: Computational concentration profile $\tilde{\tilde{\phi}}$.

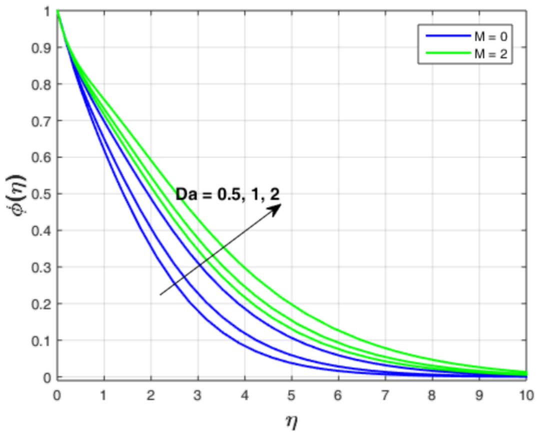

Figure 3.6: Computational concentration profile $\tilde{\tilde{\phi}}$.

In figure 3.5, the graph of $\tilde{\tilde{\phi}}$ is observed against variable value of Da under the absence and presence of G_r. With the increasing value of G_r, the $\tilde{\tilde{\phi}}$ decreases. In figure 3.6, the graph $\tilde{\tilde{\phi}}$ is also observed against variable value of Da under the appearance and nonappearance of parameter M. The graph of $\tilde{\tilde{\phi}}$ moves upward as there is increament in the value of parameter M.

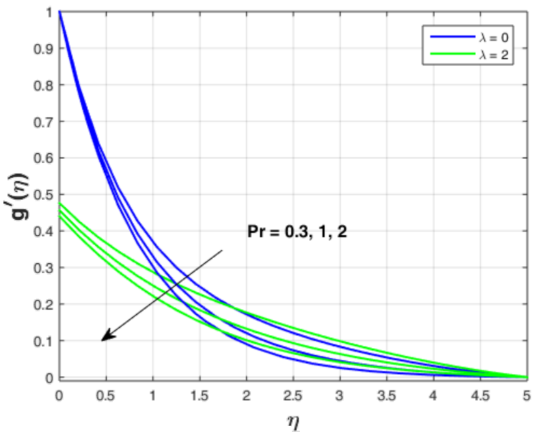

Figure 3.7: Computational velocity profile $\tilde{\tilde{g}}'$.

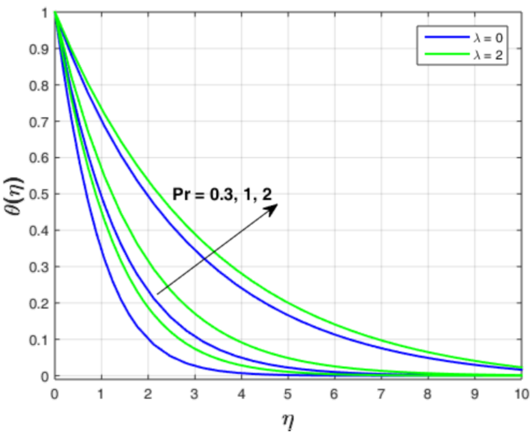

Figure 3.8: Computational temperature profile $\tilde{\tilde{\theta}}$.

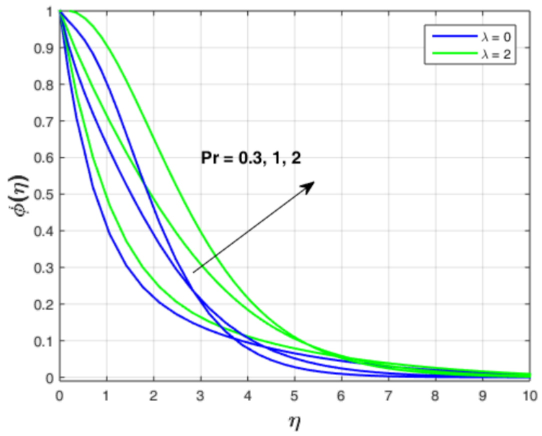

Figure 3.9: Computational concentration $\tilde{\tilde{\phi}}$.

In figures 3.7, 3.8 and 3.9 the graphs of $\tilde{\tilde{g}}'(\tilde{\eta})$, $\tilde{\tilde{\theta}}(\tilde{\eta})$ and $\tilde{\tilde{\phi}}(\tilde{\eta})$ respectively are observed against variation in values of parameter Pr under the appearance and nonappearance of the parameter λ. With the increasing value of the parameter λ, the $\tilde{\tilde{g}}'$ profile moves downward while the temperature and concentration profiles moves upward.

Chapter 4

Finite Difference Method for Ordinary Differential Equations Appearing in Bionanofluid Flow Problem

In this chapter, we considered coupled ordinary differential equations from fluid mechanics regarding bionanofluid. The considered ODEs were converted into second order ODEs. The resultant second order ODEs are further discritized and acquired the values of physical values.

4.1 Governing Equations

Let us consider coupled ODEs taken from the fluid mechanics problem.

$$(1+h_2-h_2\theta)f^{'''}-h_2\theta^{'}f^{''}+ff^{''}-f^{'2}-A(f^{'}+\frac{\eta}{2}f^{''})-(Kp(1+h_2-h_2\theta)+M)f^{'}=0 \tag{4.1}$$

$$\begin{aligned}&(1+h_4\theta+\frac{4}{3}Rd)\theta^{''}+h_4\theta^{'2}+Nb(1+h_6\phi)\theta^{'}\phi^{'}+N_t\theta^{'2}+Pr_{\infty}(f\theta^{'}-\frac{A\eta}{2}\theta^{'}\\&+Ec(1+h_2-\theta h_2)f^{''2}+MEcf^{'2}+KpEc(1+h_2-h_2\theta)f^{'2}+s\theta)=0,\end{aligned} \tag{4.2}$$

$$(1+h_6\phi)\phi^{''}+h_6\phi^{'2}+Sc(f\phi^{'}-\frac{A\eta}{2}\phi^{'}-Kr\phi)+\frac{Nt}{Nb}\theta^{''}=0, \tag{4.3}$$

$$(1+h_8\phi)\chi^{''}+h_8\phi^{'}\chi^{'}-Pe(\chi\phi^{''}+\phi^{'}\chi^{'})+Sb(f\chi^{'}-\frac{A\eta}{2}\chi^{'})=0. \tag{4.4}$$

and the conditions on boundary are

$$f_0=0, \qquad f_0^{'}=1+\delta f_0^{''}, \qquad \theta_0=1+\gamma_0^{'}, \qquad \phi_0=1, \qquad \chi_0=1$$

$$f_\infty^{'}=0, \qquad \theta_\infty=0, \qquad \phi_\infty=0, \qquad \chi_\infty=0 \tag{4.5}$$

4.2 Numerical Solution

In this Section, finite difference method (FDM) will be used to solve the coupled ordinary differential equations $(4.1)-(4.4)$ subjected to the boundary conditions (4.5). Since eq(4.2), $(4.3)and(4.4)$ are already in second order, so we have to reduce the order of eq (4.1) by one with the aim that we have to implement finite differences. Therefore, let us consider

$$f'=F,$$

then eqs$(4.1)-(4.5)$ takes the form

$$(1+h_2-h_2\theta)F''-h_2\theta'F'-F^2+fF'-A(F+\frac{\eta}{2}F')-(M+Kp(1+h_2-h_2\theta))F=0, \tag{4.6}$$

$$\begin{aligned}(1+h_4\theta+\frac{4}{3}Rd)\theta''+h_4\theta'^2+Nb(1+h_6\phi)\theta'\phi'+Nt\theta'^2+Pr_\infty(f\theta'-\frac{A\eta}{2}\theta'\\ +Ec(1+h_2-\theta h_2)F'^2+MEcF^2+KpEc(1+h_2-h_2\theta)F^2+s\theta)=0,\end{aligned} \tag{4.7}$$

$$(1+h_6\phi)\phi''+h_6\phi'^2+Sc(f\phi'-Kr\phi-\frac{A\eta}{2}\phi')+\frac{Nt}{Nb}\theta''=0, \tag{4.8}$$

$$(1+h_8\phi)\chi''+h_8\phi'\chi'-Pe(\chi\phi''+\phi'\chi')+Sb(f\chi'-\frac{A\eta}{2}\chi')=0. \tag{4.9}$$

and the boundary conditions are

$$f_0 = 0, \qquad F_0 = 1 + \delta F_0', \qquad \theta_0 = 1 + \gamma\theta_0', \qquad \phi_0 = 1, \qquad \chi_0 = 1,$$

$$F_\infty = 0, \qquad \theta_\infty = 0, \qquad \phi_\infty = 0, \qquad \chi_\infty = 0. \tag{4.10}$$

4.3 Discretization

To obtain a suitable solution to a mathematical problem, discretization is necessary. Discretization is the method of converting continuous functions, models, variables, and equations to discrete counterparts. This method is generally carried out as a first step towards making them suitable for implementation of numerical methods. In order to discretize the eqs (4.6) – (4.10), we will going to use forward finite difference approximation for first-order derivatives and central finite difference approximation for second- order derivatives.

$$\begin{aligned}
\left(\frac{dF}{d\eta}\right) &= \frac{F_{i+1} - F_i}{\Delta\eta}, & \left(\frac{d^2F}{d\eta^2}\right) &= \frac{F_{i+2} - 2F_{i+1} + F_i}{(\Delta\eta)^2},\\
\left(\frac{d\theta}{d\eta}\right) &= \frac{\theta_{i+1} - \theta_i}{\Delta\eta}, & \left(\frac{d^2\theta}{d\eta^2}\right) &= \frac{\theta_{i+2} - 2\theta_{i+1} + \theta_i}{(\Delta\eta)^2},\\
\left(\frac{d\phi}{d\eta}\right) &= \frac{\phi_{i+1} - \phi_i}{\Delta\eta}, & \left(\frac{d^2\phi}{d\eta^2}\right) &= \frac{\phi_{i+2} - 2\phi_{i+1} + \phi_i}{(\Delta\eta)^2},\\
\left(\frac{d\chi}{d\eta}\right) &= \frac{\chi_{i+1} - \chi_i}{\Delta\eta}, & \left(\frac{d^2\chi}{d\eta^2}\right) &= \frac{\chi_{i+2} - 2\chi_{i+1} + \chi_i}{(\Delta\eta)^2}.
\end{aligned}$$

Implementing the above mentioned approximations to eqs (4.6) – (4.10), we get

$$\begin{aligned}
(1 + h_2 - h_2\theta_i)\left(\frac{F_{i+2} - 2F_{i+1} + F_i}{(\Delta\eta)^2}\right) - h_2\theta_i'\left(\frac{F_{i+1} - F_i}{\Delta\eta}\right) - F_i^2 + f_i\left(\frac{F_{i+1} - F_i}{\Delta\eta}\right)\\
- A\left(F_i + \frac{\eta}{2}\left(\frac{F_{i+1} - F_i}{\Delta\eta}\right)\right) - (M - Kp(1 + h_2 - h_2\theta_i))F_i = 0,
\end{aligned} \tag{4.11}$$

$$(1+h_4\theta_i+\frac{4}{3}Rd)\left(\frac{\theta_{i+2}-2\theta_{i+1}+\theta_i}{(\Delta\eta)^2}\right)+(h_4+Nt)\left(\frac{\theta_{i+1}-\theta_i}{\Delta\eta}\right)^2$$
$$+Nb(1+h_6\phi_i)\left(\frac{\theta_{i+1}-\theta_i}{\Delta\eta}\right)\left(\frac{\phi_{i+1}-\phi_i}{\Delta\eta}\right)+Pr_\infty(f_i\left(\frac{\theta_{i+1}-\theta_i}{\Delta\eta}\right)-A\frac{\eta}{2}\left(\frac{\theta_{i+1}-\theta_i}{\Delta\eta}\right)$$
$$+KpEc(1+h_2-h_2\theta_i)F_i^2+Ec(1+h_2-h_2\theta_i)\left(\frac{F_{i+1}-F_i}{\Delta\eta}\right)^2+s\theta_i+MEcF_i^2)=0, \tag{4.12}$$

$$(1+h_6\phi_i)\left(\frac{\phi_{i+2}-2\phi_{i+1}+\phi_i}{(\Delta\eta)^2}\right)+h_6\left(\frac{\phi_{i+1}-\phi_i}{\Delta\eta}\right)^2+Sc(f_i\left(\frac{\phi_{i+1}-\phi_i}{\Delta\eta}\right)-A\frac{\eta}{2}\left(\frac{\phi_{i+1}-\phi_i}{\Delta\eta}\right)$$
$$-Kr\phi_i)+\frac{Nt}{Nb}\left(\frac{\theta_{i+2}-2\theta_{i+1}+\theta_i}{(\Delta\eta)^2}\right)=0, \tag{4.13}$$

$$(1+h_8\phi_i)\left(\frac{\chi_{i+2}-2\chi_{i+1}+\chi_i}{(\Delta\eta)^2}\right)+h_8\left(\frac{\phi_{i+1}-\phi_i}{\Delta\eta}\right)\left(\frac{\chi_{i+1}-\chi_i}{\Delta\eta}\right)+Sb(f_i-A\frac{\eta}{2})\left(\frac{\chi_{i+1}-\chi_i}{\Delta\eta}\right)$$
$$-Pe\left(\left(\frac{\phi_{i+1}-\phi_i}{\Delta\eta}\right)\left(\frac{\chi_{i+1}-\chi_i}{\Delta\eta}\right)+\chi_i\left(\frac{\phi_{i+2}-2\phi_{i+1}+\phi_i}{(\Delta\eta)^2}\right)\right)=0. \tag{4.14}$$

and (4.10) becames

$$f_0=0, \qquad F_0=1+\delta\left(\frac{F_1-F_0}{\Delta\eta}\right), \qquad \theta_0=1+\gamma\left(\frac{\theta_1-\theta_0}{\Delta\eta}\right), \qquad \phi_0=1, \qquad \chi_0=1.$$
$$F_\infty=0, \qquad \theta_\infty=0, \qquad \phi_\infty=0, \qquad \chi_\infty=0. \tag{4.15}$$

In this research, the physical quantities of interest are $C_{f_{\breve{x}}}$, $Nu_{\breve{x}}$, $Sh_{\breve{x}}$ and $Nn_{\breve{x}}$ are local skin friction coefficient, the local Nusselt number, the local Sherwood number and the local density number of motile microorganisms respectively. We will be using the drived form of the physical quantities, are defined as follows

$$Re_{\breve{x}}^{1/2}C_{f\breve{x}}=-(1+h_2\phi)f''(0), \qquad Re_{\breve{x}}^{-1/2}Nu_{\breve{x}}=-\left(1+\frac{4}{3}Rd\right)\theta'(0),$$
$$Re_{\breve{x}}^{-1/2}Sh_{\breve{x}}=-\phi'(0), \qquad Re_{\breve{x}}^{-1/2}Nn_{\breve{x}}=-\chi'(0). \tag{4.16}$$

4.4 Result and Dicussion

The dicretized form of the governing equations (4.11)−(4.14) affiliated with the boundary conditions (4.15) are numerically solved by finite difference method using MATLAB coding. In order to validate and determine the reliability our present results, comparison is made with the already available results in literature of the the skin friction coefficient $-f''(0)$. In Table 4.1 and 4.2, demonstrated the findings for the skin friction coefficient $-f''(0)$ with variation of magnetic parameter M and the first order slip parameter δ when $Pr_\infty = 1$. In Table 4.1, it is observed that $-f''(0)$ continuously increases as there in increament in values of M. In Table 4.2, the observation is that $-f''(0)$ continuously decreases as the values δ increases.

For further validation, the coupled ordinary differential equations (4.1) − (4.4) with boundary conditions (5) are also numerically solved by using MATLAB builtin function *bvp4c*.

Table 4.1 Comparison of $-f''(0)$ while assigning some variable values to M when Kp $= \delta = \gamma =$ h2 $=$ h4 $=$ h6 $=$ h8 $= 0$ and $Pr_\infty = 1$.

M	Hayat et al. [19]	Amirsom et al. [20]	Mabood and Mastroberardino[21]	*bvp4c*	FDM
0	1.0000	1.0000002	1.000008	1.0001	1.0001
1	1.41421	1.4142221	1.4142135	1.4142	1.4142
5	2.44948	2.4494901	2.4494897	2.4495	2.4495
10	3.31662	3.3166229	3.3166247	3.3166	3.3166
50	7.14142	7.1414279	7.1414284	7.1414	7.1414
100	10.04987	10.049868	10.049875	10.0499	10.0499
500	22.38302	22.383031	22.383029	22.3830	22.3830
1000	31.63858	31.638578	31.638584	31.6386	31.6386

Table 4.1: Comparison for $-f''(0)$ for different values of M.

In table 4.1, our findings through *bvp4c* and FDM are in an examplary agreement. To get the desired outcomes through FDM, we had to made changes in number of mesh points.

Table 4.2 Comparison of $-f''(0)$ for different value of δ when Kp = M = γ = h2 = h4 = h6 = h8 = 0 and $Pr_\infty = 1$.

δ	Amirsom et al. [20]	Andersson [23]	Hamad et al. [22]	*bvp4c*
0.0	1.00000000	1.0000	1.00000000	1.0001
0.1	0.87204247	0.8721	0.87208247	0.8722
0.2	0.77593307	0.7764	0.77637707	0.7765
0.5	0.59119589	0.5912	0.59119548	0.5913
1.0	0.43016000	0.4302	0.43015970	0.4303
2.0	0.28398932	0.2840	0.28397959	0.2841
5.0	0.14464015	0.1448	0.14484019	0.1449
10.0	0.08124091	0.0812	0.08124198	0.0813
20.0	0.04378790	0.0438	0.04378834	0.0439
50.0	0.01857868	0.0186	0.01859623	0.0186
100.0	0.00954677	0.0095	0.00954997	0.0096

Table 4.2: Comparison for $-f''(0)$ for different values of δ.

In table 4.2, findings of *bvp4c* are in good agreement with literature. While FDM failed to respond to variational values of δ.

Table 4.3 Comparison of $Nu_{\tilde{x}}$ and $Sh_{\tilde{x}}$ between FDM and *bvp4c* findings, when Kp = γ = δ = h2 = h4 = h6 = h8 = 0 and s = Ec = Nb = 0.1, Nt = 0.2, Sc = 4, Kr = 0.2, Sb = 2, Pe = 1, M = 0.5.

Pr_∞	Rd	*bvp4c*		FDM	
		$-\left(1+\frac{4}{3}Rd\right)\theta'(0)$	$-\phi'(0)$	$-\left(1+\frac{4}{3}Rd\right)\theta'(0)$	$-\phi'(0)$
1.7	0.5	0.5158	1.5441	0.5148	1.5059
2	0.5	0.6026	1.5103	0.6668	1.5016
4	0.5	0.9977	1.3061	0.9752	1.3082
3	1	0.9022	1.4801	0.9030	1.4172
3	1.5	0.9729	1.5231	0.9737	1.5218
3	2	1.1153	1.5413	1.1189	1.5244

Table 4.3: Comparison for $Nu_{\tilde{x}}$ and $Sh_{\tilde{x}}$.

The following figures 4.1, 4.2, 4.3 and 4.4 represents the profiles of $f'(\eta)$, $\theta(\eta)$, $\phi(\eta)$ and $\chi(\eta)$ respectively, while having Kp $= \gamma = \delta =$ h2 $=$ h4 $=$ h6 $=$ h8 $= 0$, s $=$ Ec $=$ Nb $= 0.1$, Nt $= 0.2$, Sc $= 4$, Kr $= 0.2$, Sb $= 2$, Pe $= 1$, M $= 0.5$, Pr $= 3$ and Rd $= 0.5$. It is observed that the graphs are steadily converges to zero from the left boundary. Generally, we considered the length of the domain 5 and grid points were 30.

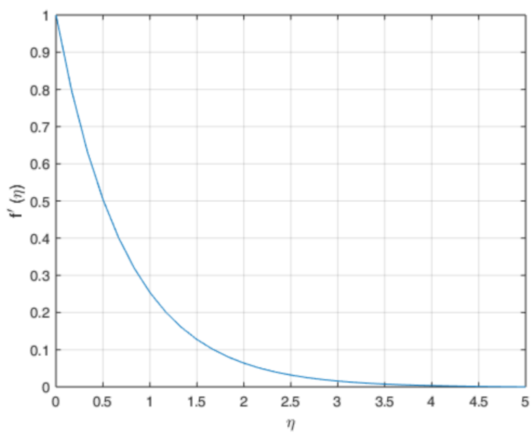

Figure 4.1: Graph of $f'(\eta)$.

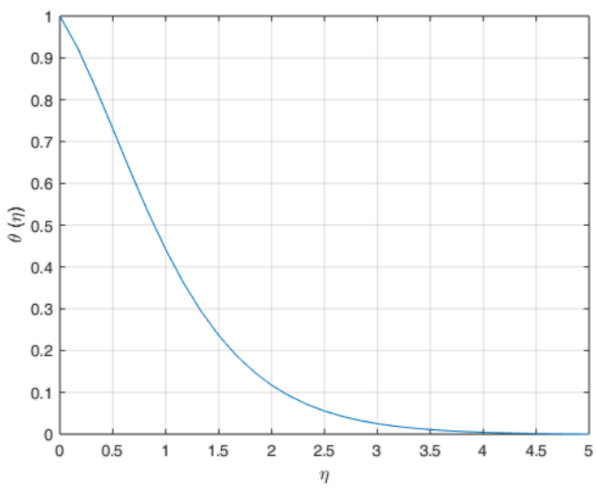

Figure 4.2: Graph of $\theta(\eta)$.

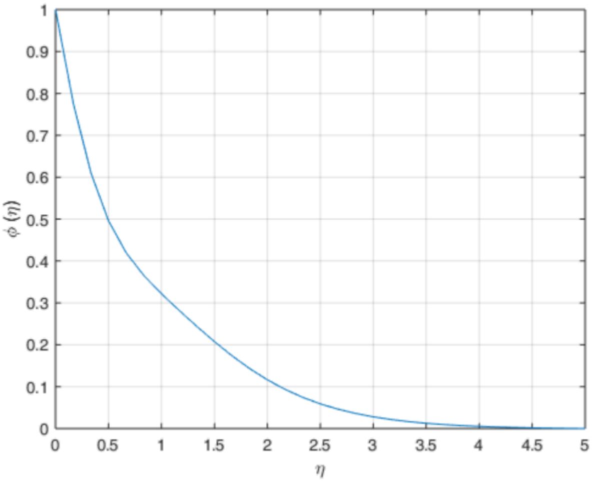

Figure 4.3: Graph of $\phi(\eta)$.

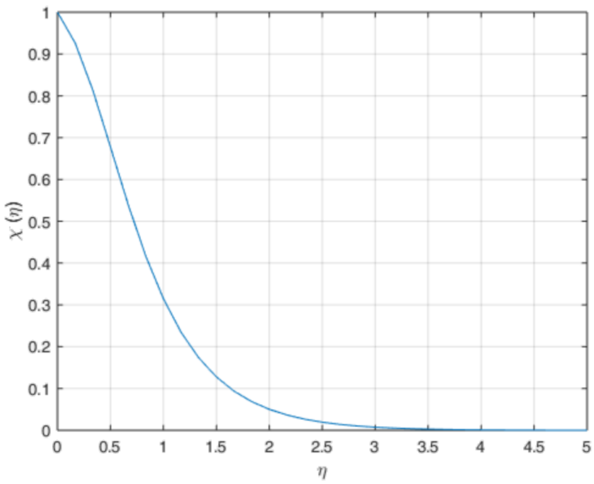

Figure 4.4: Graph of $\chi(\eta)$.

Chapter 5

Conclusion

In this thesis, we considered two different mathematical problems from fluid mechancis. Both were numerically solved by finite difference method and MATLAB builtin function *bvp4c*.

A mathematical model of two-dimensional magnetohydrodynamic unsteady fluid flow were considered. Suitable similarity transformation were used to convert the system of governing PDEs into system of ODEs to numerically solve using finite difference method after discretizing the ODEs. We also investigate the same system of ODEs by using MATLAB builtin function *bvp4c*. Find the values of the derived physical quantities, which are skin friction coefficient $C_{\tilde{\tilde{g}}}$, local Nusselt number Nu_x and local Sherwood number Sh_x. And the comparison of $-\tilde{\tilde{\theta}}'(0)$ has been made with the previously published research works. Furthermore, we presented some graphical visualization of the findings.

Another coupled ordinary differential equations were considered from fluid mechanics of bionanofluid. Since the considered coupled ordinary differential eqautions are nonlinear, therefore they have to be solve numerically. To build a relaible argument we choose two numerical methods to solve the problem. Initially the coupled ordinary differential equations were solved by finite difference method, after the coupled ordinary differential equations were converted into discretized form. To validate our results for the considered coupled ordinary differential equations, MATLAB builtin function *bvp4c* has been used. Derive the values for physical quantities C_{fx}, Nu_x, Sh_x and

Nn_x. Further, comparison of $-f''(0)$ has been made for different values of M and δ and achieved an agreement with literature.

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