Exact Solutions of the Einstein Maxwell Field Equations in Paraboloidal Geometry

By

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Dean/Principal

Dedicated

to my

Grandmother,

my beloved

Parents & Brothers

for their endless love and support

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All praise and gratitude belong to **Allah**, the Omniscient, the Omnipotent, the Omnipresent and the Most Compassionate. Indeed, the Almighty has sanctified me with His innumerable blessings, the faith that He is inexorably by my side as my guardian, overseer and the architect of my fortunate destiny, has always been my greatest strength. I, express my profound thanks to **Prof. Tooba Feroze**, my supervisor, for her unwavering and treasured upkeep supervision and a constant offering of her time and energy. I consider myself privileged to have been blessed by the opportunity to work under her dynamic supervision, which had been a longing since my under graduation days. She has always made herself available and helped me in every possible manner. Her persistent and compassionate supervision has always inspired me to keep going, may **Allah's** great blessings, incredible pleasure, and excellent health be bestowed upon her.

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Abstract

In this thesis, assuming the generalized polytropic equation of state and suitable form of electric field intensity, new classes of charged anisotropic perfect fluid solutions to the Einstein-Maxwell equations in paraboloidal geometry are obtained, representing relativistic charged compact stellar objects. These exact polytropic solutions for different variations of the adjustable parameter, known as the polytropic index η ($\eta = 1/2, 1, 2$), satisfy all physically admissible conditions. The matter composition obeys all stability conditions; including the hydrostatic equilibrium by means of the Tolman–Oppenheimer–Volkoff equation, viable trends of stability through the relativistic adiabatic index and Abreu's criterion are fulfilled. The profiles are displayed by assuming the estimated radius, the geometric parameter L, and other constants for which compact stars obeying the mass-radius ratio are obtained.

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Chapter 1

Introduction

1.1 Theory of Relativity and Beyond

The concept of space and time are enlightened by several scientists and philosophers since ancient times. In 4th century BC, Aristotle's perspectives regarding this were that space is comparative with its matter substance and time to the sequence of occasions, he concluded "*time as the potentiality of the motion of matter*" [1]. In 17th/18th century, the word "physics" significantly changed its importance, by the revelations and significant work of Galileo and Newton. It was Galileo initially who in 1632 gave a logical connection between previous fuzzy thoughts and resulting developments. Relativity expressed in his point of view was free from any reference to the physics of natural motion. He characterized equivalence of physical laws concerning all inertial reference frames. Newtonian mechanics added several other concepts, including laws of motion, gravitation, and asserted in new notion of an absoluteness of space and time.

The principle of relativity (Newtonian or Galilean) articulates that "*the laws of mechanics are invariant under change of inertial frame*" [2]. The physical experiment should necessarily be identical when performed in two inertial frames of references to all sorts of physics (kinematics or non-gravitational). This implies that there is no preferred initial frame or absolute motion, space is well-defined by three coordinates

(x, y, z) and is regarded as a phase where events occur, with time being considered as a universal entity. In the pre-relativistic era, Newtonian mechanics used the Galilean transformations for the change in inertial frames due to the significant fact that "*Newton's laws are invariant under Galilee transformations*"[2]. In short, before the 20th century the classical world related with the names like Aristotle, Newton etc. view was that the rules governing space and time were absolute always, energy was considered moving through the medium called ether, matter was deliberated to be made up of inseparable and immutable atoms. These concepts lurked many unanswered queries especially after the development of Maxwell's theory of electrodynamics

1.1.1 Relativity and Role of Electrodynamics

In 1864, Maxwell came across the equations manifesting phenomena of electricity, magnetism and light traveling as a wave into a single frame called electromagnetism. Before in Newtonian physics, light waves were considered to appear motionless meaning that oscillation of electric and magnetic field will go nowhere, where as Maxwell equations gave no such indication. Electromagnetic field arises from two sources mainly that is electric charge (Q) and current (I), the stationary charge creates electric field and the moving charge creates magnetic field. Typically charge and current densities are utilized in Maxwell's equations to quantify the effects of fields. The equations split up in two pairs the "source equations"

$$\nabla \cdot \mathbf{E} = \rho, \tag{1.1}$$

$$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{j}, \tag{1.2}$$

and the "internal equations"

$$\nabla \cdot \mathbf{B} = 0, \tag{1.3}$$

$$\nabla \times \mathbf{E} - \partial_t \mathbf{B} = 0, \tag{1.4}$$

here, ρ is the electric charge density, $\mathbf{j}$ is the current density, $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field [3]. These equations were not Galilee-invariant which lead to the incompatibility of Newtons theory with electromagnetism in a sense that it was only effectual in the limit $|v| \ll c$. One of the major question of incompatibility was raised when an experiment to measure the speed of earth relative to the ether was performed by Michelson and Morley, it resulted as an unsuccessful testing in 1887. This leads to certain possibilities, firstly that the ether is attached firmly to the earth, which created some further problems and the other possibility was that there is no ether, this leads speed of light to be the fundamental quantity of the medium.

Another crisis occurred for the immutable atoms in the early 1900s by the understanding of the concept of radioactive decay. In that period Maxwell's equations were being investigated and the heat emitted by dark objects when they absorb light was being observed. Furthermore, energy was considered as a continuous wave, however, the wave-based theory contradicted this point as according to it there might exist infinite transmitted energy. All these ongoing events disregarded the recently settled laws of thermodynamics and in order to describe the strange results about light and heat. Planck speculated that light may be a series of particles or quantum units rather than a wave.

But still no proper explanation paved into the minds until a proposal led by Einstein worked out, it severely discredited the ether theories and got rid of all reference frames for space and time. In 1905, with the four publications including the revolutionary work about the Brownian motion, the photoelectric effect, the equivalence of mass and energy, and the special relativity theory published in the paper "*On the Electrodynamics of Moving Bodies*" [4] Einstein changed physics overnight. He substantiated that nothing can move faster than the speed of light 'c' and that there does not exist any universal space. In accordance with his results all the measurements became relative to the position and speed of the observer, space and time became one entity space-time. Einstein's thinking basically revolved around electrodynamics He figured out that Maxwell's condition of electromagnetism required special theory of relativity. The two postulates of the theory of relativity are [2]:

1. In all inertial reference frames, the laws of physics assume the same form and all observers are equivalent.

2. In all inertial reference frames regardless of the state of motion or source, all observers will measure same speed of light.

Various consequences like contraction of length, time dilation and mass expansion were derived from this theory. Later on, in 1907, events happening in the universe were expressed in a four-dimensional worldview by Minkowski. In Minkowski spacetime event was considered as a point simply. As in the classical era Galilean transformations were used, here, Lorentz transformations became the center of this new theory of relativity. These are basically linear coordinate transformations relating two frames relative to each other moving with the constant speed. In flat spacetime (no acceleration) the Minkowski line element is invariant under Lorentzian transformations, it is the square of infinitesimal interval between two events $(ct, \mathbf{x})$ and $(ct + cdt, \mathbf{x} + d\mathbf{x})$ that are separated infinitesimally given as [2]

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1.5}$$

One of the importance of these transformations in special theory of relativity is that Maxwell equations are invariant under them. But this was not the end as soon after this revolutionary work Einstein began to wonder about gravity, whether the Newton's theory of gravitation i.e. (between the interactions of two massive objects gravity acts as an attractive force) is fully consistent with his theory or not, since in the frames of special relativity idea of gravitation propagation appeared slightly absurd.

1.1.2 Principles of General Relativity

Laboriously working for almost ten years Einstein ended up again with the series of four papers in 1915, forming the general theory of relativity [5]. Few founding principles that guided to construct the theory directly or indirectly are

Mach's Principle

Mach's principle may be regarded as founding principles that pretty much influenced Einstein when he was in the development phase of general theory of relativity. We began our explanation with the famous bucket experiment of Newton: in an absolute space imagining a bucket of water, if the bucket is rotating around its own vertical axis the rotation of water in it will form a concave surface due to the centrifugal force and when one imagines sitting on the edge of bucket no rotation would be observed but actually it still has the concave shape. This was objectionable by Mach's, since it hypothesized absolute space and reference, he was of the view that it was acceptable in hypothetically empty universe and the matter content in the real universe did not support such existence. He insisted on the point that absolute space and absolute motion are not to be used in scientific context as they are meaningless metaphysical concepts. Mach was of the view that in rotating water the concavity can be caused because of the presence of nearby large masses, according to him physics of the small scale actually depends on the physics of large scale. Summary of the results of Mach's principle [3] is:

1. Matter distribution determines the geometry of spacetime.
2. Simply no matter then there exists no geometry.
3. Universe containing a single body should not possess any inertial properties.

Equivalence Principle

The particles inertial and gravitational equivalence of masses led to a thought that crossed Einstein's mind in 1907 later which he called "*the happiest thought of my life*". He envisioned a person falling off from the roof top would feel weightless when plummeting downwards, this made him realize that by eliminating gravity various complications can be reduced as well and concluded that accelerated motion and gravitational fields are one and the same. The equivalence principle was stated following this

"*The laws of physics are the same in uniform static gravitational field and in the accelerated frame of reference*".

Principal of General Covariance

Einstein proposed that "all observers are equivalent" is the complete logical form of the principle of special relativity and he incorporated it into his general theory. This means that physics must have the tensorial form and any observer independent of its property are acceptable to determine laws of physics. It is stated as

"*The laws of physics are invariant under any set of coordinate transformation and must be specified in tensorial form*".

1.2 Tensors

The last mentioned principle derives the attention towards the importance of tensor theory in relativity. What are tensors? and why they are considered essential part of theory? These are the intriguing questions that need to be answered before the development of main field equations proposed by Einstein, that changed the meaning of physics and the eye to look at various aspects of physics.

Tensors are mathematical objects, entirely described in respective coordinate systems by transformation properties. Tensor formalism is studied in relativity to get deeper geometrical awareness as it is the language required to explain how the quantities transform under coordinate transformation as they are valid ia all systems. Tensors are fundamentally the generalization of vectors and dual vectors. If T is (r, s) tensor then in component notation it is written as

$$T = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_r} \otimes \hat{v}^{j_1} \otimes \cdots \otimes \hat{v}^{j_s}, \tag{1.6}$$

where

$$\hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_r} \otimes \hat{v}^{j_1} \otimes \cdots \otimes \hat{v}^{j_s}, \tag{1.7}$$

are basis of (r, s) tensor, given by the tensor product of vectors and dual vectors. Similarly one can define components of contravariant and covariant tensors having all upper and lower indices respectively. The transformation law for the components of mixed tensor is defined as

$$T^{i'_1 \cdots i'_r}{}_{j'_1 \cdots j'_s} = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_s}}{\partial x^{j'_s}}. \tag{1.8}$$

Properties

1. The fundamental operations like addition, subtraction are valid for the tensors of same rank and type, while the product of two tensors given by its components whose upper and lower indices consist of all upper and lower indices of the original tensor components is valid. Expressed as follows

$$T^{ab}{}_{c} = U^{ab}{}_{c} \pm V^{ab}{}_{c}, \tag{1.9}$$

$$T^{abc}{}_{ef} = U^{ab}{}_{e}\, V^{c}_{f}. \tag{1.10}$$

2. Contraction is summation of one upper and one lower index (of the same type) otherwise tensor would not hold the transformation laws.
Rank n tensor $\xrightarrow{1\ Contrac}$ Rank $(n-2)$ tensor $\xrightarrow{1\ Contrac}$Rank $(n-4)$ tensor $\cdots$
As an example consider

$$S^{ab}{}_{c} = T^{adb}{}_{cd}. \tag{1.11}$$

3. Given a tensor, its symmetric and skew-symmetric parts are defined as follows

$$T_{(i_1 \cdots i_n)} = \frac{1}{n!}\left(T_{i_1 \cdots i_n} + sum\ over\ permutation\ of\ indices\ i_1 \cdots i_n\right), \tag{1.12}$$

$$T_{[i_1 \cdots i_n]} = \frac{1}{n!}\left(T_{i_1 \cdots i_n} + alternating\ sum\ over\ permutation\ of\ indices\ i_1 \cdots i_n\right). \tag{1.13}$$

1.2.1 Metric Tensor

It is the fundamental object of study in relativity, specially when studying the geometric properties of manifolds (in any finite no of dimension) such as distance or curvature of manifold we define the quantity called metric tensor. It is defined in term of basis vector

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b. \tag{1.14}$$

As the dot product of $\mathbf{e}_a \cdot \mathbf{e}_b = \mathbf{e}_b \cdot \mathbf{e}_a$ this implies symmetry of second rank tensor i.e.

$$g_{ab} = g_{ba}. \tag{1.15}$$

The metric tensor is also defined as the square of infinitesimal distance between two points $R(x^a)$ and $S(x^a + dx^a)$ on a manifold given as

$$ds^2 = g_{ab}\, dx^a dx^b. \tag{1.16}$$

Properties

1. g^{ab} is the inverse metric of g_{ab} since $g^{ab} g_{bc} = \delta^a{}_c$.

2. Metric determinant 'g' is the determinant of $n \times n$ symmetric non-degenerate matrix i.e. $g = Det(g_{ab})$.

3. It plays a key role in raising and lowering of index.

4. Law of transformation for second rank tensor (g_{ab}) can be easily figured out from equation (1.8).

1.2.2 Curvature Tensor

General theory gave a profound connection between accelerated observer and gravity. It was proposed that deformation in spacetime is caused due to the existence of immense bodies. It is not a flat structure rather massive objects produce curvature which is described by the Riemann tensor. The Riemann curvature tensor is expressed in terms of covariant derivative (';') that is the generalization of ∂_a for a curved spacetime. For example the components of covariant derivative of a tensor of rank $(1,1)$ is defined as

$$A^a{}_{b;c} = A^a{}_{b,c} + \Gamma^a{}_{bd}A^d{}_c - \Gamma^d{}_{cb}A^a{}_d. \tag{1.17}$$

Here $\Gamma^a{}_{bc}$ is the Christoffel symbol, defined as an array of numbers depicting the metric association to surfaces or manifolds permitting distances to be estimated on that particular manifold or surface. In the general theory of relativity vital role of this association is the correspondence between gravitational field and the gravitational potential being metric tensor. Christoffel symbol of second kind is expressed as.

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{da}\left(g_{cd,b} + g_{bd,c} - g_{bc,d}\right). \tag{1.18}$$

The Riemann curvature tensor is expressed by the covariant derivative as

$$A^a{}_{;d;c} - A^a{}_{;c;d} = A^a{}_{;d,c} + \Gamma^a{}_{ce}A^e{}_{;d} - \Gamma^e{}_{dc}A^a{}_{;e} - A^a{}_{;c,d} - \Gamma^a{}_{de}A^e{}_{;c} + \Gamma^e{}_{cd}A^a{}_{;e}, \tag{1.19}$$

$$A^a{}_{;d;c} - A^a{}_{;c;d} = \left((\Gamma^a{}_{bd})_{,c} - (\Gamma^a{}_{bc})_{,d} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}\right)A^b + (\Gamma^e{}_{dc} - \Gamma^e{}_{cd})A^a{}_e, \tag{1.20}$$

Torsion is defined as $\Gamma^a{}_{bc} - \Gamma^a{}_{cb} = T^a_{bc}$ and in the framework of general relativity the spacetime is torsion free. So the required results are attained by utilizing the symmetry connection $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$. The rank-4 Riemann curvature tensor $\mathbf{R}$ in components form is defined as

$$R^a{}_{bcd} = (\Gamma^a{}_{bd})_{,c} - (\Gamma^a{}_{bc})_{,d} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}, \tag{1.21}$$

Thus equation (1.20) reduces to

$$A^a{}_{;d;c} - A^a{}_{;c;d} = R^a{}_{bcd}A^b. \tag{1.22}$$

The covariant derivatives do not commute and satisfies the condition $g_{ab;c} = 0$ unlike the partial derivatives. This condition is used to determine the deviation from flat spacetime by the components of Riemann tensor $R^{a}{}_{bcd}$. In simple words, for $R^{a}{}_{bcd} = 0$ there is flat geometry and for $R^{a}{}_{bcd} \neq 0$ there is curved space. $R^{a}{}_{bcd}$ can be transformed into covariant tensor form by using the transformation

$$R_{abcd} = g_{ae} R^{e}{}_{bcd}, \tag{1.23}$$

where $R^{a}{}_{bcd} \neq R_{abcd}$. The components in explicit form by performing some algebra can be written as

$$R_{abcd} = \frac{1}{2}\left(g_{da,bc} + g_{bc,da} - g_{bd,ac} - g_{ca,bd}\right) + g_{pe}\left(\Gamma^{e}{}_{da}\Gamma^{p}{}_{cb} - \Gamma^{e}{}_{ca}\Gamma^{p}{}_{db}\right). \tag{1.24}$$

Properties of curvature tensor are as follows:

1. It is skew-symmetric either the order of first two indices is swapped or of the last two.

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc}. \tag{1.25}$$

2. It satisfies Bianchi identity of first and second kind given as

$$R_{a[bcd]} = R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{1.26}$$

$$R^{a}_{p[bc;d]} = R^{a}_{pbc;d} + R^{a}_{pdb;c} + R^{a}_{pcd;b} = 0. \tag{1.27}$$

3. It is symmetric if first two pair of indices are swapped with the last two

$$R_{abcd} = R_{cdab}. \tag{1.28}$$

By contracting the components of Riemann tensor one can construct Ricci tensor

$$R_{ab} = R^{c}{}_{acb}, \tag{1.29}$$

and the trace of Ricci tensor is the Ricci scalar (or curvature scalar) as given below

$$R = g^{ab}R_{ab}. \tag{1.30}$$

Ricci tensor is symmetric and an interesting feature of Ricci scalar is that it determines the nature of singularity as it is invariant under coordinate transformation. There are two types of singularities coordinate and essential, one arises with the bad choice of coordinates and is removable, other occurs due to problem in geometry which can not be removed. By examining the invariant quantities given below, one can conclude that if there are finite curvature invariants then there is coordinate singularity otherwise essential (or spacetime).

$$R_1 = R, \tag{1.31}$$

$$R_2 = R^{ab}_{cd}R^{cd}_{ab}, \tag{1.32}$$

$$R_3 = R^{ab}_{cd}R^{cd}_{ef}R^{ef}_{ab}, \tag{1.33}$$

$$R_4 = R^{ab}_{cd}R^{cd}_{ef}R^{ef}_{gh}R^{gh}_{ab}. \tag{1.34}$$

Now as curvature tensor is discussed in detail then the principle of minimal gravitational coupling implicitly used by Einstein is quite easier to understand, which is stated as "*No term explicitly containing the curvature tensor should be added in making the transition from special to general theory of relativity*" [3].

1.2.3 The Einstein Tensor

The Einstein tensor gives symmetric, linear and divergence free function of the curvature. It can be derived by the second kind of Bianchi identity stated in equation (1.27), firstly contracting a and b in it

$$R^{a}{}_{pac;d} + R^{a}{}_{pda;c} + R^{a}{}_{pcd;a} = 0. \tag{1.35}$$

Substituting $R^a{}_{pac} = R_{pc}$ and $R^a{}_{pda} = -R^a{}_{pad} = -R_{pd}$ in equation (1.20)

$$R_{pc;d} - R_{pd;c} + R^a{}_{pcd;a} = 0, \tag{1.36}$$

multiplying the above by g^{pe}

$$R^e{}_{c;d} - R^e{}_{d;c} + R^{ae}{}_{cd;a} = 0. \tag{1.37}$$

Carrying out some contractions, equation (1.37) reduces to

$$\left(R^a{}_c - \frac{1}{2}\delta^a{}_c R\right)_{;a} = 0, \tag{1.38}$$

where

$$G^a{}_c = R^a{}_c - \frac{1}{2}\delta^a{}_c R, \tag{1.39}$$

Here the Einstein tensor $\mathbf{G}$ in components form is given in equation (1.39) where equation (1.38) implies $G^a{}_{c;a} = 0$. This tensor is symmetric, divergence free and in covariant form is given as

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}, \tag{1.40}$$

1.2.4 The Maxwell Tensor

The Maxwell tensor is also known as electromagnetic field tensor, for its construction initially define the four vector potential $\mathbf{A}$ and skew symmetric tensor $\mathbf{F} = F_{ab}$ as

$$A_a = (\phi, \mathbf{A}), \tag{1.41}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a. \tag{1.42}$$

Here, ϕ is the scalar potential and $\mathbf{A}$ is 3-vector potential. Electric and magnetic fields in terms of ϕ and $\mathbf{A}$ become

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial_t \mathbf{A}, \tag{1.43}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1.44}$$

By the equations (1.42)-(1.44), we get $F_{0i} = E_i$ and $F_{ij} = \epsilon_{ijk} B^k$ where $i, j, k = 1, 2, 3$ and ϵ_{ijk} is the Levi-Civita tensor (0,3) defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutaion of 123} \\ \text{-1} & \text{if } ijk \text{ is an odd permutaion of 123} \\ 0 & \text{otherwise} \end{cases} \tag{1.45}$$

Further simplification yields the covariant form of electromagnetic field tensor

$$F_{ab} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B^3 & -B^2 \\ -E_2 & -B^3 & 0 & B^1 \\ -E_3 & B^2 & -B^1 & 0 \end{bmatrix}. \tag{1.46}$$

The contravariant form can be obtained by $F^{ab} = g^{ac} g^{bd} F_{cd}$.

1.3 The Maxwell Equation in Relativity

Now the main goal is to observe electromagnetism in context of relativity. In order to show that the equations (1.1)-(1.4) need to be transformed into tensor notation, by the electromagnetic field tensor given by equation (1.46) and four vector J^a defined as $J^a = (\rho, \mathbf{J})$. Thus the set of traditional Maxwell equations in tensor form are reduced to two equations as follows

$$\partial_b F^{ab} = J^a, \tag{1.47}$$

$$\partial_{[b} F_{ac]} = 0. \tag{1.48}$$

Source equation (1.47) is valid in Minkowski space (inertial coordinates), one can express it in coordinate invariant (arbitrary coordinates) way as

$$\nabla_b F^{ab} = J^a, \tag{1.49}$$

where as the internal equation (1.48) subject to continuity equation is satisfied automatically as there is no change in $\nabla_{[b} F_{ac]} = \partial_{[b} F_{ac]} = 0$. If these covariant tensor equations are valid in one coordinate system then they are valid for all.

1.4 The Energy Momentum Tensor

The energy momentum tensor (**T**) plays a key role in constructing gravitational field equations. It is related to Einsteins tensor (**G**) and delineates the matter distribution in spacetime. The most commonly known energy momentum tensor is "dust", where the distribution can be characterized by matter density ρ and four velocity (u^a) of fluid in some coordinate x^a as

$$T^{ab} = \rho u^a u^b. \tag{1.50}$$

The detail explanation of the components and their physical meaning are stated as [15]

1. T^{00} represents energy density ρ.
2. T^{i0} is flow of i^{th} component of momentum called the momentum density (momentum per unit volume).
3. T^{0i} is flow of energy across the surface x^i called the energy flux.
4. T^{ij} is flow of i^{th} component of momentum crossing the interface in j^{th} direction, which gives force per unit area called stress.

Due to this reason energy momentum tensor is known as stress-energy tensor as well. Other energy momentum tensor related to matter distribution is "perfect fluid", defined as "*The fluid in which there are no forces between the particles, and no heat conduction or viscosity in the inertial reference frame*" [6]. It can be characterized by adding scalar pressure p along with energy density ρ and flow vector u^a as

$$T^{ab} = (\rho_0 + p)u^a u^b - p g^{ab}. \tag{1.51}$$

T^{ab} is symmetric rank-2 tensor and note that perfect fluid becomes dust if $p \to 0$. The Maxwell energy momentum tensor is given as

$$T^{ab} = -F^{ad}F^{b}_{\ d} + \frac{1}{4}g^{ab}F_{cd}F^{cd}. \tag{1.52}$$

The conservation equation of energy and momentum is given as

$$T^{ab}{}_{;b} = 0, \tag{1.53}$$

and for flat space time it becomes

$$T^{ab}{}_{,b} = 0. \tag{1.54}$$

1.5 The Einstein Field Equation

Beginning the derivation of field equation by defining the general form of Einstein-Hilbert action as

$$S = \int L\sqrt{-g}\, d^4x. \tag{1.55}$$

The Lagrangian L for gravitational source and matter is taken as $L = L_G + L_m$ where $L_G = \frac{1}{2k}R$ and $K = \frac{8\pi G}{c^4}$. Originally L_G was defined without the cosmological constant, later on it was added by Einstein to develop a static model of universe. Throughout this dissertation the speed of light c, the gravitational constant G are taken to be 1 and cosmological constant is zero ($\Lambda = 0$). Thus action becomes

$$S = \int \frac{1}{2k}R\sqrt{-g}\, d^4x + \int L_M\sqrt{-g}\, d^4x. \tag{1.56}$$

Since $\delta S = 0$, using this and equation (1.30) in (1.56) gives

$$\begin{aligned} \delta S = \frac{1}{2k}\int &\left(R_{ab}g^{ab}\delta\sqrt{-g} + R_{ab}\sqrt{-g}\delta g^{ab} + \sqrt{-g}g^{ab}(\delta R_{ab})\right) d^4x \\ + \int &\left(L_M\delta\sqrt{-g} + \sqrt{-g}\delta(L_M)\right) d^4x = 0. \end{aligned} \tag{1.57}$$

Now considering geodesic coordinate where $\Gamma^a{}_{bc} = 0$ at an arbitrary point P. This then reduces Riemann tensor as

$$R^c{}_{adb} = \Gamma^c{}_{ab,d} - \Gamma^c{}_{ad,b}. \tag{1.58}$$

δ on both sides of equation (1.58) yields

$$\delta R^c{}_{adb} = \delta\Gamma^c{}_{ab,d} - \delta\Gamma^c{}_{ad,b}. \tag{1.59}$$

Since in geodesic coordinates the partial derivative is equivalent to covariant derivative and commutes with variation. So famous Palatini equation is obtained given as

$$\delta R^c{}_{adb} = \delta\Gamma^c{}_{ab;d} - \delta\Gamma^c{}_{ad;b}. \tag{1.60}$$

Contraction of c and d gives

$$\delta R_{ab} = \delta\Gamma^c{}_{ab;c} - \delta\Gamma^c{}_{ac;b}, \tag{1.61}$$

multiplying above equation by g^{ab}

$$g^{ab}\delta R_{ab} = g^{ab}\delta\Gamma^c{}_{ab;c} - g^{ab}\delta\Gamma^c{}_{ac;b}, \tag{1.62}$$

$$= g^{ab}\delta\Gamma^c{}_{ab;c} - g^{ac}\delta\Gamma^b{}_{ab;c}, \tag{1.63}$$

$$= \left(g^{ab}\delta\Gamma^c{}_{ab} - g^{ac}\delta\Gamma^b{}_{ab}\right)_{;c}. \tag{1.64}$$

Consider

$$A^c = g^{ab}\delta\Gamma^c{}_{ab} - g^{ac}\delta\Gamma^b{}_{ab}, \tag{1.65}$$

further simplification and integration on both sides yields

$$\int_v g^{ab}\delta R_{ab}\sqrt{-g}\, d^4x = \int_v A^c{}_{;c}\sqrt{-g}\, d^4x, \tag{1.66}$$

using divergence theorem above equation becomes

$$\int_v g^{ab}\delta R_{ab}\sqrt{-g}\, d^4x = 0. \tag{1.67}$$

By equation (1.67) and substitution of the identity $\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g}g_{ab}\delta g^{ab}$, equation (1.57) becomes

$$\begin{aligned} &\frac{1}{2k}\int_v \left(R_{ab}g^{ab}\left(\frac{-1}{2}\sqrt{-g}\, g_{ab}\delta g^{ab}\right) + R_{ab}\sqrt{-g}\,\delta g^{ab}\right) d^4x \\ &+ \int_v \left(L_m\left(\frac{-1}{2}\sqrt{-g}\, g_{ab}\delta g^{ab}\right) + \sqrt{-g}\,\delta(L_m)\right) d^4x = 0. \end{aligned} \tag{1.68}$$

As $L_m = L_m(g^{ab})$ this implies $\delta L_m = \frac{\partial L_m}{\partial g^{ab}}\delta g^{ab}$, thus equation (1.68) becomes

$$\begin{aligned}&\frac{1}{2k}\int_v \left(\frac{-1}{2}Rg_{ab}\delta g^{ab} + R_{ab}\delta g^{ab}\right)\sqrt{-g}\; d^4x \\ &-\frac{1}{2}\int_v \left(-2\frac{\partial L_m}{\partial g^{ab}} + L_m g_{ab}\right)\delta g^{ab}\sqrt{-g}\; d^4x = 0.\end{aligned} \tag{1.69}$$

Energy momentum tensor is terms of Lagrangian is defined as

$$T_{ab} = -2\frac{\partial L_m}{\partial g^{ab}} + L_m g^{ab}. \tag{1.70}$$

Thus equation (1.69) takes the form

$$\frac{1}{2k}\int_v \left(R_{ab} - \frac{1}{2}g_{ab}R - kT_{ab}\right)\delta g^{ab}\sqrt{-g}\; d^4x = 0, \tag{1.71}$$

this implies

$$R_{ab} - \frac{1}{2}g_{ab}R - kT_{ab} = 0, \tag{1.72}$$

$$R_{ab} - \frac{1}{2}g_{ab}R = kT_{ab}. \tag{1.73}$$

Equation (1.73) are famous Einstein equations on left we have curvature that determines the presence of gravitational source and on right energy momentum tensor leads to the representation of matter in given space.

Chapter 2

Exact Solutions for Anisotropic Compact Objects in Relativity

2.1 Exact Solutions

From the time general theory was developed, due to its essential feature of non-linearity, few approaches are taken to understand it deeply. Most important being worked out still is the exact solutions to the Einstein field equations as said by Mason and Woodhouse "*they combine tractability with non-linearity, so they make it possible to explore nonlinear phenomena while working with explicit solutions*" [7]. Although exact solutions are special cases that are very useful, there are still no adequate conditions or a fixed method for obtaining viable exact solutions of the theory after a century of analysis. Since there are two rank-two tensors T_{ab} and g_{ab}, each with up to ten individual components, the derivation of solutions is a very broad term. As a result, to proceed certain logical assumptions must be made, the known exact solutions are all obtained by imposing certain constraints. In general, to minimize complexity some symmetry conditions are imposed on metrics. Some other conditions such as static, stationary or non-charged spacetimes may also be considered [8]. Moreover, for proceeding towards the solution many suitable assumptions can be made about T_{ab} like fixing its form dust, perfect fluid, etc. As mathematical and physical assumptions are applied to get the solution then it is essential for solution to be physically admissible. The most physical way of finding the solutions is by the equation of state, where it relates pressure in terms of energy density $p = p(\rho)$. The polytropic equation of state i.e. $\big(P = \kappa\rho^{\Gamma}$ where $\Gamma = 1 + \frac{1}{\eta}$ and η is the polytropic index$\big)$ along with the equation of hydrostatic equilibrium is often considered playing important part in obeying the astrophysical systems [9] which is discussed in detail later on.

2.2 Compact Objects

Earlier in 1798, Laplace introduced the theoretical concept of the existence of a massive object from whose gravitational field no particle could escape even light, but it did not receive much attention due to some strange properties. Astronomers believe that stars are formed by the gravitational collapse of gaseous mass. In 18th century, it was known as Laplace's nebular hypothesis which was overruled lately in 1970's by solar nebular disk model, mainly due to its lack of information about the dissemination of angular momentum in between planets and sun. Formation of star by nebula means that there is the dense interstellar disc of cosmic dust and gasses present in form of molecular clouds in the universe composed of 75% hydrogen and 23% helium. This cloud starts spinning due to the conservation of momentum from movement of particles, this drastic spin flattens the cloud into proplanetary disk. The formation of a protostar occurs as regions of increased gravity allow gas and dust to condense, when these regions grow more massive, they crumble under the enhanced gravitational field, creating an increase in temperature and thereby becoming a protostar. Finally the star remains in this state for several thousands of years, until the nuclear fusion process ignites crushing hydrogen atoms into helium then carbon until the formation of iron. Fusion process that creates iron does not generate any energy. Due to nuclear fusion process if the gravitational force pressing inwards is grater than the outwards push of internal pressure, the core collapses under the dominant gravitational field [10], from this point star are formed often called the "*compact objects*".

2.2.1 White Dwarfs, Neutron Stars and Black Holes

The study of compact objects can be categorized as

1. **White Dwarfs**: If the star is small its core will turn into a white dwarf. It is one of the densest forms of matter present with mass comparable to that of the Sun and a volume comparable to that of the Earth. Just three white dwarfs were discovered until 1926, they are held up against gravity not by heat but by electrons repelling each other. Chandrasekhar later developed the degenerate electron equation of state in 1930, taking into account special relativistic results. The amount of maximum mass they can hold was to be $1.4M_{\odot}$, where $M_{\odot}$ is the solar mass [11].

2. **Neutron Stars**:

   When Chandrasekhar limit is reached massive stars which have finished burning there fuel undergo a supernova explosion. This explosion blows off the outer layers of a star into a supernova remnant. The central region collapses so much that protons and electrons combine to form neutrons resulting in neutron stars, with neutron degeneracy pressure partially supporting against further collapse. Neutron stars are partially supported against further collapse by neutron degeneracy pressure, a phenomenon described by the Pauli exclusion principle, just as white dwarfs are supported against collapse by electron degeneracy pressure. They were discovered as radio pulsars at the end of the 1960s and as X-ray stars at the start of the 1970s, also planets have been discovered in one neutron star [11]. Typically they have radius about 10km and the maximum mass estimated to be is $3M_\odot$, massive stars could not resist the gravitational pull and continue to collapse.

3. **Black Holes**: As fluid in these are the densest material so in the massive stars ($M > 3M_\odot$) entire mass of core collapses into a black hole. These are so dense objects with extremely strong gravity that even light cannot escape through. The most important feature in a black hole is event horizon, defined as "*a hypersurface separating those spacetime points that are connected to infinity by a timelike path from those that are not*" [9]. If something crosses this it falls into the black hole singularity (it is infinitely small and dense where space time and laws of physics do not apply). Escaping through it one needs to move faster than speed of light. The outside observer dose not get effected by events happening inside an event horizon. Depending on the mass distribution black holes can be defined as

   • Stellar black holes: These black holes are smaller in size and to grow in size they consume gas and dust present around them. These are the most common black holes, according to scientists millions of them can be only present in Milky Way galaxy, and the mass range lies between $10^{\frac{1}{2}}$ to $10^2 M_\odot$.

   • Intermediate black holes: Presence of this medium size black hole is still debatable astronomers believe that these are formed by collision of cluster of stars in a chain reaction. Usually there mass ranges between 10^3 to $10^5 M_\odot$.

   • Supermassive black holes: These black holes are located at the heart of each galaxy and are formed by the merger of hundred thousands of stellar and intermediate black holes. The known supermassive black hole is $S50014 + 81$ which is 40 billions time mass of sun and its diameter is 236.7 billion km. The range of such black holes is 10^6 to $10^9 M_\odot$.

2.3 Some Known Black Hole Solutions

The structure of spacetime followed in general relativity is of a four-dimensional pseudo-Riemannian manifold M, the related metric is not positive definite thus the signature representation will be $(-,+,+,+)$ or $(+,-,-,-)$, given the general line element in metric component form by equation (1.16). In this thesis work is carried out under $(-,+,+,+)$ signatures and on static spherically symmetric paraboloidal spacetime metric unless mentioned otherwise. When contemplating solutions especially exact solutions, symmetry is a crucial assumption to make because of the mathematical simplifications. Many good theoretical predictions for spherically symmetric solutions have been made, proving Einstein's theory [10]. So assuming spherical symmetry the line element (1.16) in spherical coordinates will thus become

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2, \tag{2.1}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. Now as for the metric to reach the requirement of being static consider time independence, static spacetime is defined as "*In a static spacetime all metric components g_{ab} are independent for some time like coordinate say x^0 and the line element is invariant under transformation $x^0 \rightarrow -x^0$*" [6]. Next if there is no preferable angular direction in space then the metric is spherically symmetric i.e $dx^a \rightarrow -dx^a$ where x^a are spatial coordinates. Thus, when $r \rightarrow \infty$ and presence of spherical symmetry the metric in equation (2.1) becomes

$$ds^2 = -e^{2\nu(r)}c^2dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.2}$$

2.3.1 Schwarzschild Black Hole

In 1916, Schwarzschild presented a simplest exact solution by considering vacuum, thus the Einstein field equation becomes

$$R_{ab} = 0. \tag{2.3}$$

For the simplest form of metric given in equation (2.2) which is time independent and spherically symmetric, the metric tensor and its inverse are

$$g_{ab} = \left(-e^{2\nu(r)}, e^{2\lambda(r)}, r^2, r^2 sin^2\theta\right),$$
$$g^{ab} = \left(-e^{-2\nu(r)}, e^{-2\lambda(r)}, \frac{1}{r^2}, \frac{1}{r^2 sin^2\theta}\right). \tag{2.4}$$

The total independent Christoffel symbols are 40 but non zero components are

$$\Gamma^1_{00} = \nu' e^{2(\nu-\lambda)}, \quad \Gamma^1_{22} = -re^{2\lambda}, \quad \Gamma^1_{11} = \lambda', \quad \Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r},$$
$$\Gamma^1_{33} = -re^{-2\lambda} sin^2\theta, \quad \Gamma^2_{33} = -sin\theta cos\theta, \quad \Gamma^3_{32} = cot\theta, \quad \Gamma^0_{01} = \nu'. \tag{2.5}$$

Thus in equation (2.3) the non-vanishing components are

$$R_{00} = \nu'' + \nu'(\nu' - \lambda') + \frac{2\nu'}{r} = 0, \tag{2.6}$$

$$R_{11} = -\nu'' + \nu'(\lambda' - \nu') + \frac{2\lambda'}{r} = 0, \tag{2.7}$$

$$R_{22} = 1 - e^{-2\lambda} + re^{-2\lambda}(\lambda' - \nu') = 0, \tag{2.8}$$

$$R_{33} = R_{22} sin^2\theta = 0. \tag{2.9}$$

Simplifying equations (2.6) and (2.7) and further substituting it in equation (2.8) yields

$$\nu = -\lambda, \tag{2.10}$$

$$(re^{-2\lambda})' = 1. \tag{2.11}$$

This implies

$$e^{2\nu} = e^{-2\lambda} = \left(1 + \frac{\alpha}{r}\right), \tag{2.12}$$

where $\alpha = \frac{-2Gm}{c^2}$. Thus metric in equation (2.2) takes the form

$$ds^2 = -\left(1 - \frac{2Gm}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.13}$$

The above metric for $G = c = 1$ becomes

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.14}$$

which is the Schwarzschild line element [12]. Event horizon of Schwarzschild black hole is given by $r_s = \frac{2Gm}{c^2}$. There exist two singularities in the metric (2.14). At $r = 0$ there is essential singularity which can not be removed and at $r = 2m$ there is coordinate singularity which can be removed by appropriate choice of coordinates.

2.3.2 Reissner–Nordstrom Black Hole

The analogue of above solution having charged point mass was discovered in 1916 by Reissner [13] and in 1918 by Nordstrom [14] independently, hence the solution is known as Reissner-Nordstrom. Therefore, adding charge into the previous assumptions the Einstein-Maxwell field equation becomes

$$R_{ab} = 8\pi T_{ab}. \tag{2.15}$$

Here due to spherical symmetry and point charge placed at origin the components of electrostatic field are in the radial direction i.e. $E = E(r)$ with magnetic field being zero. The Maxwell tensor F_{ab} in this case takes the form

$$F_{ab} = \begin{bmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.16}$$

It should satisfy the Maxwell equations, by plugging the assumptions and using the metric and Christoffel symbols given in equations (2.4) and (2.5), $\nabla_b F^{ab} = F^{ab}_{;b} = 0$ reduces to

$$(e^{-2(\nu+\lambda)} r^2 E)' = 0. \tag{2.17}$$

For $r \to \infty$ with $\nu, \lambda \to 0$ we get

$$E(r) = \frac{Qe^{2(\nu+\lambda)}}{r^2}. \tag{2.18}$$

Components of Maxwell energy momentum tensor T_{ab} become

$$T_{ab} = (-E^2, -E^2, E^2, E^2). \tag{2.19}$$

Similarly using the set of equations (2.6) to (2.8) along with equation (2.19) in (2.15) and following the detailed procedure again these equations then yield

$$\nu = -\lambda, \tag{2.20}$$

$$(re^{-2\lambda})' = 1 - \frac{Q^2}{r^2}. \tag{2.21}$$

Simplifying further we get

$$e^{2\nu} = e^{-2\lambda} = \left(1 + \frac{constant}{r} + \frac{Q^2}{r^2}\right), \tag{2.22}$$

for $Q = 0$ the solution reduces to Schwarzschild which implies $constant = -2m$, thus, equation (2.22) becomes

$$e^{2\nu} = e^{-2\lambda} = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right). \tag{2.23}$$

Therefore, the Reissner-Nordstrom metric is

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.24}$$

Here again at $r = 0$ essential singularity occurs and for finding other singularities take

$$1 - \frac{2m}{r} + \frac{Q^2}{r^2} = 0, \tag{2.25}$$

this implies

$$r_\pm = m \pm \sqrt{m^2 - Q^2}. \tag{2.26}$$

Here on surface $r = r_\pm$ coordinate singularity occurs, r_+ is called outer horizon and r_- is called inner horizon. There exist three cases depending on the values of m and Q, which are as follows [6]:

1. $m^2 < Q^2$: No coordinate singularity exists this is called naked singularity and is considered not to be physically realistic.

2. $m^2 > Q^2$: On surface $r = r_\pm$ two coordinate singularities exist. This case is called as usual Ressiner-Nordstrom black hole

3. $m^2 = Q^2$: This case is called as extreme Reissner-Nordstrom black hole and is similar to second case but with the region $r_- < r < r_+$ removed.

2.4 Permissible Conditions for Relativistic Stellar Models

As discussed earlier that exact solution require to be physically consistent with the assumptions applied to get the solution. For a compact stellar model to be acceptable the following physical and stability criteria needs to be fulfilled:

1. Metric potential in the interior part of star should be free from geometrical singularities.

2. The nature of energy density and radial pressure needs to be decreasing monotonically and should definitely be positive throughout whereas tangential pressure decreases as we move towards the boundary of the star but it may have increasing values in the central region of stellar configuration.

3. The tangential and radial pressure at origin surely needs to be equal and at the boundary of object radial pressure should necessarily vanish which is not a requisite condition to be obeyed for tangential pressure.

4. The electric field intensity and anisotropic factor at center are zero and should definitely portray increasing nature moving towards the surface of the star.

5. The profiles of mass radius, compactification factor, and surface redshift increase monotonically whereas gravitational redshift decreases.

6. The density and pressure gradient ($\frac{d\rho}{dr}$, $\frac{dp_r}{dr}$, $\frac{dp_t}{dr}$) must be negative through out the stellar interior.

7. The trace of energy tensor must be positive inside the stellar configuration.

8. For stellar interior to be stable the energy conditions, adiabatic index, causality, and hydrostatic equilibrium condition must be satisfied.

2.5 Review of Some Known Solutions by Polytropic Equation of State

Previously polytropic models were studied under Newtonian gravity, which describes the internal pressure and distribution of fluid etc. in various conditions. It is considered important because it can help model stars made of real materials, such as natural gas, photon gas, decaying Fermi gas and quark matter. Special polytropic indicators are used earlier to show correspondence to low mass white dwarfs, neutron stars, isothermal sphere and different main sequence stars. Chandrasekhar gave a comprehensive examination of polytropes, polytropic index $\eta = 0, 1, 3$ and 5 are considered of extraordinary importance in astronomy. The main sequence stars are pretty well modeled by using $\eta = 3$ whereas $\eta = 0, 1$ can be tackled in complete generality [15]. Earlier studies have revealed that polytropic models have finite radius for polytropic index $\eta < 5$. The key features of the neutron stars are described by Oppenheimer and Volkoff using $\eta = 3/2$. The polytropic fluid sphere structure along with numerical solutions for η = 1, 3/2, 5/2, and 3 were discussed in detail by Tooper [16]. Detailed outcomes of polytropes in the span of $1/2 \leq \eta \leq 3$ were found by Pandey *et al* [17]. It was discussed by Kippenhahn [18] that the polytrope has an infinite radius for models where $\eta \geq$

5 and the isothermal sphere is generated when the index $\eta \to \infty$ i.e. an isothermal self-gravitating gas sphere.

Now will be discussing some known solutions presented by Takisa and Maharaj [19], they presented an exact solutions for Einstein-Maxwell equation in the presence of static, spherically symmetric, charged anisotropic fluid distribution using polytropic equation of state ($p_r = \kappa\rho^{1+\frac{1}{\eta}}$). Moreover, Thirukkanesh *et al* [20] analyzed solutions to Einstein equations using polytropic equation of state ($p_r = k\rho^{1+\frac{1}{\eta}} - \beta$) in a spherically symmetric paraboloidal spacetime with uncharged anisotropic fluid distribution.

2.5.1 Some Charged Polytropic Models

Mafa Takisa and S. D. Maharaj [19]:

The line element in standard coordinate is given by equation (2.2) and the components of energy momentum tensor for this distribution in the presence of anisotropy and electromagnetic field takes the form

$$T_{ab} = diag = \left(-\rho - E^2, p_r - E^2, p_t + E^2, p_t + E^2\right). \tag{2.27}$$

These quantities are studied in connection with the co-moving fluid velocity $u^a = e^{\lambda}$ δ^a_0 and radial four vector $v^a = e^{-\nu}$ δ^a_1, with ρ being the energy density, E the electric field intensity and (p_r, p_t) are radial and tangential pressure respectively. Thus the Einstein-Maxwell equations become

$$\frac{1}{r^2}[r(1 - e^{-2\lambda})]' = \rho + E^2, \tag{2.28}$$

$$-\frac{1}{r^2}(1 - e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p_r - E^2, \tag{2.29}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r'}\right) = p_t + E^2, \tag{2.30}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}\left(r^2 E\right)'. \tag{2.31}$$

Assuming the polytropic equation of state along with the transformations

$$p_r = \kappa\rho^{1+\frac{1}{\eta}}, \qquad x = Cr^2, \qquad Z(x) = e^{-2\lambda(r)}, \qquad A^2y^2(x) = e^{2\nu(r)}. \tag{2.32}$$

The gravitational behavior observed by the set of transformed equations is given as

$$\frac{\rho}{C} = \frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{C}, \tag{2.33}$$
$$p_r = \kappa\rho^{1+\frac{1}{\eta}}, \tag{2.34}$$
$$p_t = p_r + \Delta, \tag{2.35}$$
$$\frac{\Delta}{C} = 4xZ\frac{\ddot{y}}{y} + \dot{Z}\left(1 + 2x\frac{\dot{y}}{y}\right) + \frac{1-Z}{x} - \frac{2E^2}{C}, \tag{2.36}$$

$$\frac{\dot{y}}{y} = \frac{1-Z}{4xZ} - \frac{E^2}{2CZ} + \frac{\kappa C^{1+1/\eta}}{4Z}\left(\frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{C}\right)^{1+\frac{1}{\eta}}, \tag{2.37}$$
$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2, \tag{2.38}$$

where Δ is the ansiotropic factor. A physically reasonable form of gravitational potential Z and electric field intensity E are chosen i.e.

$$Z = \frac{1+bx}{1+ax}, \quad a \neq b, \ b \neq 0 \tag{2.39}$$
$$\frac{E^2}{C} = \frac{\varepsilon x}{(1+ax)^2}. \tag{2.40}$$

Therefore, equations (2.34), (2.38) and (2.39) by substituting Z and E become

$$\frac{\rho}{C} = \frac{(a-b)(3+ax) - \varepsilon x}{(1+ax)^2}, \tag{2.41}$$
$$\frac{\dot{y}}{y} = \frac{a-b}{4(1+bx)} - \frac{\varepsilon x}{(1+ax)(1+bx)} + \frac{kC^{1+\frac{1}{\eta}(1+ax)}}{4(1+bx)}\left(\frac{(a-b)(3+ax) - \varepsilon x}{(1+ax)^2}\right)^{1+\frac{1}{\eta}}, \tag{2.42}$$
$$\frac{\sigma^2}{C} = \frac{2\varepsilon(1+bx)(3+ax)^2}{(1+ax)^5}. \tag{2.43}$$

Polytropic Models

Following are the models obtained by varying values of $\eta = 1, 2, 1/2, 2/3$ in this particular article by the authors.

Case I:

When $\eta = 1$, radial and tangential pressure along are given as

$$p_r = \kappa\rho^2, \tag{2.44}$$

$$p_t = \frac{4xC(1+bx)}{1+ax}\left(\frac{\kappa(\kappa-1)a^2}{(1+ax)^2} + \frac{2\kappa lab}{(1+ax)(1+bx)} + \frac{2\kappa a\dot{F}(x)}{1+ax} + \frac{b^2 l(l-1)}{(1+bx)^2} + \frac{2lb\dot{F}(x)}{1+bx} + \ddot{F}(x) + \dot{F}(x)^2\right) + 2xC\left(\frac{a\kappa}{1+ax} + \frac{b}{1+bx} + \dot{F}(x)\right) + \frac{C(a-b)ax - 4\varepsilon x}{(1+ax)^2} + \kappa C^2\left(\frac{(a-b)(3+ax) - 2\varepsilon x}{(1+ax)^2}\right)^2. \tag{2.45}$$

Difference of these pressures reveal the expression for anisotropic factor and integrating equation (2.42) yields the gravitational potential y in this case as

$$y = B(1+ax)^{\kappa}(1+bx)^{l} exp(F(x)). \tag{2.46}$$

The variable $F(x)$ and the constants k and l are given as

$$F(x) = \frac{C^2 k(2(2b-a)(1+ax) + (b-a))}{2(b-a)^2(1+ax)^2} - \frac{C^2 2\kappa\varepsilon(4a(a-b) + 2\varepsilon)}{8a^2(a-b)(1+ax)} - \frac{C^2\kappa\varepsilon(2a(a^2-4\varepsilon) + b(2ab-2\varepsilon))}{4a^2(a-b)^2(1+ax)},$$

$$k = C^2\kappa(2(a-b))^2\left(\frac{b^2}{(b-a)^3} + \frac{b}{(b-a)^2} + \frac{1}{4}\right) - \frac{4\varepsilon((a-b)^2 + C^2 2\kappa a\varepsilon)}{a} - 8C^2\kappa a\varepsilon(1 + b(4-3b)),$$

$$l = \frac{(a-b)}{4b} + C^2\kappa(2(a-b))^2\left(\frac{b^2}{(b-a)^3} + \frac{b}{(b-a)^2} + \frac{1}{4}\right) - \frac{2\varepsilon((a-b)^2 + C^2\kappa b\varepsilon)}{b} - 4C^2\kappa\varepsilon((a-b)(a-3b)).$$

For $A^2B^2 = D$ and $C = 1$ the line element becomes

$$ds^2 = -D(1+ar^2)^{2k}(1+br^2)^{2l} exp(2F(r^2))dt^2 + \frac{1+ar^2}{1+br^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.47}$$

Now the uncharged model in this case was generated by taking $\varepsilon = 0$ then E becomes zero

$$ds^2 = -D(1+ar^2)^{2\kappa(2(a-b))^2\left(\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right)} \times (1+br^2)^{2\frac{a-b}{4b}+\kappa(2(a-b))^2\left(\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right)}$$
$$\times\, exp\left(\frac{\kappa(2(b-a))(1+ax)+(b-a)}{(b-a)^2(1+ax)^2}\right)dt^2 + \frac{1+ar^2}{1+br^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.48}$$

Case II:

When $\eta = 2$, radial and tangential pressure are given as

$$p_r = \kappa\rho^{\frac{3}{2}}, \tag{2.49}$$

$$p_t = \frac{4xC(1+bx)}{1+ax}\left[\frac{d}{dx}\left(\frac{b((a-b)^2+2\varepsilon)}{4b(a-b)(1+bx)} - \frac{2a\varepsilon}{4a(a-b)(1+ax)}\right.\right.$$
$$\left.\left.-\frac{(m+w)\sqrt{b(a(a-b)\varepsilon)}}{2U(\sqrt{2a(a-b)+2\varepsilon})}\right) + \frac{\dot{y}^2}{y^2}\right] + \frac{C(b-a)}{(1+ax)^2}\times[1+2x$$
$$\left.\left(\frac{b((a-b)^2+2\varepsilon)}{4b(a-b)(1+bx)} - \frac{2a\varepsilon}{4a(a-b)(1+ax)} - \frac{(m+w)\sqrt{2b(a(a-b)\varepsilon)}}{2U(\sqrt{2a(a-b)+2\varepsilon})}\right)\right]$$
$$+\frac{(a-b)(1+ax)-2\varepsilon x}{(1+ax)^2} + \kappa C^{\frac{3}{2}}\left(\frac{(a-b)(3+ax)-\varepsilon x}{(1+ax)^2}\right)^{\frac{3}{2}}. \tag{2.50}$$

Here $U = \sqrt{(3+ax)(a-b)-\varepsilon x}$, difference of these pressures reveal the expression for anisotropic factor and integrating equation (2.42) yields the gravitational potential y in this case as

$$y = B\frac{(1+bx)^{\frac{(a-b)^2+2\varepsilon}{4b(a-b)}}}{(1+ax)^{\frac{-2\varepsilon}{4a(a-b)}}}\left(\frac{\sqrt{(2a(a-b)+2\varepsilon)}-\sqrt{b((3+ax)(a-b)-2\varepsilon x)}}{\sqrt{(2a(a-b)+2\varepsilon)}+\sqrt{b((3+ax)(a-b)-2\varepsilon x)}}\right)^{m+w} exp(G(x)). \tag{2.51}$$

The variable $G(x)$ and the constants m and w are given as

$$G(x) = \frac{C^{\frac{3}{2}}\kappa}{2(1+ax)} - \frac{C^3 2\kappa\varepsilon\sqrt{(3+ax)(a-b)-2\varepsilon x}}{4a(a-b)(1+ax)},$$
$$m = \frac{C^{\frac{3}{2}}\kappa((a-b)(3b-a)-2\varepsilon)^{\frac{3}{2}}}{2\sqrt{b}\,(a-b)},$$
$$w = \frac{C^{\frac{3}{2}}\kappa(2a^2(a-b)(3a+7b)-2a\varepsilon(3a+5b))-2\varepsilon^2(b-3a)}{4a^{\frac{3}{2}}(a-b)\sqrt{2a(a-b)+2\varepsilon}}.$$

For $A^2B^2 = D$ and $C = 1$ the line element becomes

$$
\begin{aligned}
ds^2 = &-D(1+br^2)^{\frac{(a-b)^2+4\varepsilon}{2b(a-b)}}(1+ar^2)^{\frac{2\varepsilon}{2a(a-b)}}exp(2G(r^2))dt^2 \\
&\left(\frac{\sqrt{(2a(a-b)+2\varepsilon)}-\sqrt{b((3+ar^2)(a-b)-2\varepsilon r^2)}}{\sqrt{(2a(a-b)+2\varepsilon)}+\sqrt{b((3+ar^2)(a-b)-2\varepsilon r^2)}}\right)^{2(m+w)} \\
&+\frac{1+ar^2}{1+br^2}dr^2+r^2(d\theta^2+sin^2\theta d\phi^2).
\end{aligned}
\tag{2.52}
$$

The uncharged model in this case was generated by taking $\varepsilon = 0 \implies E = 0$

$$
\begin{aligned}
ds^2 = &-D(1+br^2)^{\frac{(a-b)^2}{2b(a-b)}}exp\left(\frac{-\kappa}{(1+ar^2)}\right)dt^2 \\
&\times\left(\frac{\sqrt{(2a(a-b))}-\sqrt{b(3+ar^2)(a-b)}}{\sqrt{(2a(a-b))}+\sqrt{b(3+ar^2)(a-b)}}\right)^{\frac{\kappa(3b-a)\sqrt{(a-b)(3b-a)}}{\sqrt{b}}+\frac{\kappa\sqrt{a}\ (3a+7b)}{\sqrt{2a(a-b)}}} \\
&+\frac{1+ar^2}{1+br^2}dr^2+r^2(d\theta^2+sin^2\theta d\phi^2).
\end{aligned}
\tag{2.53}
$$

Case III:

When $\eta = 2/3$, radial and tangential pressure are given as

$$
p_r = \kappa\rho^{\frac{5}{2}}, \tag{2.54}
$$

$$
\begin{aligned}
p_t = &\frac{4xC(1+bx)}{1+ax}\left[\frac{d}{dx}\left(\frac{b((a-b)^2+2\varepsilon)}{4b(a-b)(1+bx)}-\frac{2a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\
&\left.\left.-\frac{(p+q)\sqrt{2b(a(a-b)\varepsilon)}}{2T(\sqrt{2a(a-b)+2\varepsilon}+\sqrt{b}T(x))}\right)+\frac{\dot{y}^2}{y^2}\right]+\frac{C(b-a)}{(1+ax)^2}\times\left[1+2x\right. \\
&\left.\left(\frac{b((a-b)^2+2\varepsilon)}{4b(a-b)(1+bx)}-\frac{2a\varepsilon}{4a(a-b)(1+ax)}-\frac{(p+q)\sqrt{2b(a(a-b)\varepsilon)}}{2T(\sqrt{2a(a-b)+2\varepsilon}+\sqrt{b}T(x))}\right)\right] \\
&+\frac{(a-b)(1+ax)-4\varepsilon x}{(1+ax)^2}+\kappa C^{\frac{5}{2}}\left(\frac{(a-b)(3+ax)-2\varepsilon x}{(1+ax)^2}\right)^{\frac{5}{2}}.
\end{aligned}
\tag{2.55}
$$

Difference of these pressures reveal the expression for anisotropic factor and integrating equation (2.42) yields the gravitational potential y in this case as

$$
y = B\frac{(1+bx)^{\frac{(a-b)^2+2\varepsilon}{4b(a-b)}}}{(1+ax)^{\frac{-2\varepsilon}{4a(a-b)}}}\left(\frac{\sqrt{(2a(a-b)+2\varepsilon)}-\sqrt{b((3+ax)(a-b)-2\varepsilon x)}}{\sqrt{(2a(a-b)+2\varepsilon)}+\sqrt{b((3+ax)(a-b)-2\varepsilon x)}}\right)^{p+q}exp(H(x)).
\tag{2.56}
$$

The variable $H(x)$ and the constants p and q are given as

$$H(x) = -\frac{C^{\frac{5}{2}}\kappa(2a(a-b)+2\varepsilon)((a-b)(13a^2-25ab)+2\varepsilon(13a+7b))\mathcal{A}}{48a^2(a-b)^2(1+ax)^3}$$
$$-\frac{C^{\frac{5}{2}}\kappa(206a^3b^2(a-b^3)+70a^2b^2(b^2-2\varepsilon)+2a\varepsilon(8a^2-14\varepsilon)-a^3(3a^6-b^3))\mathcal{A}}{32a^2(a-b)^3(1+ax)}$$
$$-\frac{C^{\frac{5}{2}}\kappa(2a(a-b)+2\varepsilon)^2\mathcal{A}}{12a^2(a-b)(1+ax)^3}-\frac{C^{\frac{5}{2}}\kappa((a-b)(8a^3(a^2-2\varepsilon)+8a\varepsilon(2\varepsilon-1))-9a^2b^3)\mathcal{A}}{32a^2(a-b)^3(1+ax)},$$

$$p = \frac{C^{\frac{5}{2}}\kappa\sqrt{b}((a-b)(3b-a)+2\varepsilon)^{\frac{5}{2}}}{32a^2(a-b)^3(1+ax)},$$
$$q = -\frac{C^{\frac{5}{2}}\kappa(51a^2b^4 2\varepsilon+60a^3b\varepsilon^2+1468a^5b^3)}{32a^{\frac{5}{2}}(a-b)^3\sqrt{2a(a-b)+2\varepsilon}}-\frac{C^{\frac{5}{2}}\kappa(a^4b^5(353ab-1354))}{32a^{\frac{5}{2}}(a-b)^4\sqrt{2a(a-b)+2\varepsilon}}$$
$$+\frac{C^{\frac{5}{2}}\kappa[(a+b)(498a^4 2b\varepsilon+15a^6s+10ab\varepsilon^3-5a^8-30a^4\varepsilon^2)+a^5b(16b^3-85a^3)]}{32a^{\frac{5}{2}}(a-b)^4\sqrt{2a(a-b)+2\varepsilon}}$$
$$+\frac{C^{\frac{5}{2}}\kappa[-b(42b+75a^4)-2\varepsilon^2(2a+5b)+6b^2(a^3-3b^3)]}{16a^{\frac{1}{2}}(a-b)^4\sqrt{2a(a-b)+2\varepsilon}}$$
$$+\frac{C^{\frac{5}{2}}\kappa[2\varepsilon^3(a^3+b^3)+2ab\varepsilon^2(9b^3+15a^3)+535b^4(a^5+b^5)]}{32a^{\frac{5}{2}}(a-b)^4\sqrt{2a(a-b)+2\varepsilon}}.$$

Here $\mathcal{A}=\sqrt{(3+ax)(a-b)-\varepsilon x}$, for $A^2B^2=D$ and $C=1$ the line element becomes

$$\begin{aligned} ds^2 &= -D(1+br^2)^{\frac{(a-b)^2+4\varepsilon}{2b(a-b)}}(1+ar^2)^{\frac{2\varepsilon}{2a(a-b)}}exp(2H(r^2))dt^2 \\ &\times\left(\frac{\sqrt{(2a(a-b)+2\varepsilon)}-\sqrt{b((3+ar^2)(a-b)-2\varepsilon r^2)}}{\sqrt{(2a(a-b)+2\varepsilon)}+\sqrt{b((3+ar^2)(a-b)-2\varepsilon r^2)}}\right)^{2(p+q)} \\ &+\frac{1+ar^2}{1+br^2}dr^2+r^2(d\theta^2+sin^2\theta d\phi^2). \end{aligned} \tag{2.57}$$

The uncharged model in this case was generated by taking $\varepsilon=0$ then E becomes zero

$$\begin{aligned} ds^2 &= -D(1+br^2)^{\frac{(a-b)^2}{2b(a-b)}}\left(\frac{\sqrt{(2a(a-b))}-\sqrt{b(3+ar^2)(a-b)}}{\sqrt{(2a(a-b))}+\sqrt{b(3+ar^2)(a-b)}}\right)^{2(p+q)} \\ &\times exp(2H(r^2))dt^2+\frac{1+ar^2}{1+br^2}dr^2+r^2(d\theta^2+sin^2\theta d\phi^2). \end{aligned} \tag{2.58}$$

Case IV:

When $\eta = 1/2$, radial and tangential pressure are given as

$$p_r = \kappa\rho^3, \tag{2.59}$$

$$p_t = \frac{4xC(1+bx)}{1+ax}\left(\frac{s(s-1)a^2}{(1+ax)^2} + \frac{2suab}{(1+ax)(1+bx)} + \frac{2sa\dot{I}(x)}{1+ax} + \frac{b^2u(u-1)}{(1+bx)^2} + \frac{2ub\dot{I}(x)}{1+bx} + \ddot{I}(x) + \dot{I}(x)^2\right) + 2xC\left(\frac{as}{1+ax} + \frac{b}{1+bx} + \dot{I}(x)\right) + \frac{C(a-b)ax - 4\varepsilon x}{(1+ax)^2} + \kappa C^2\left(\frac{(a-b)(3+ax) - 2\varepsilon x}{(1+ax)^2}\right)^3, \tag{2.60}$$

Difference of these pressures reveal the expression for anisotropic factor and integrating equation (2.42) yields the gravitational potential y in this case as

$$y = B(1+ax)^s(1+bx)^u exp(I(x)). \tag{2.61}$$

The variable $I(x)$ and the constants s and u are given as

$$
\begin{aligned}
I(x) =& -\frac{C^2\kappa(2a(a-b)+2\varepsilon)^3}{16a^3(a-b)(1+ax)^4} - \frac{C^3\kappa(2a(a-b)+2\varepsilon)^2[(a-b)(a(3a-5b)-4\varepsilon)-2a\varepsilon]}{12a^3(a-b)^2(1+ax)^3} \\
&- \frac{C^3\kappa((a-b)(a-3b)-2\varepsilon)^3}{4(a-b)^4(1+ax)} - \frac{C^3\kappa[6a^4(a^3+12b\varepsilon)4a^4b^2(29a+10b)+6a\varepsilon^2]}{8a^3(a-b)^3(1+ax)^2} \\
&- \frac{C^3\kappa a^2(6\varepsilon(3a-b^2)+14b)}{8a^3(a-b)^3(1+ax)^2} - \frac{C^3 2\kappa b\varepsilon[b\eta^2+3a(2\varepsilon(b^2+3a^2)+ab(b^2-a^2))]}{8a^3(a-b)^3(1+ax)^2} \\
&- \frac{C^3\kappa[36a^4b^3(a^3-b^3)+12a^2b^2(2\varepsilon(ab-1)-a^2b^2)]}{8a^3(a-b)^3(1+ax)^2}, \\
s =& -\frac{2\varepsilon[(a^2-b^2)^2-4b(a^2(a-b)+b^2)]}{4a(a-b)^5} + \frac{C^3\kappa[a^2b^2(a^2+b^2)(136b^2+11a^2)]}{4a(a-b)^5} \\
&- \frac{C^3\kappa a^2 b[a^2+17ab^2-44b\varepsilon]}{4(a-b)^4} - \frac{C^3\kappa[9ab^5(3a^2-19b^2)+6b^2\varepsilon^2(4a^2+3b)]}{4(a-b)^5} \\
&+ \frac{C^3\kappa[6ab^4\varepsilon(4a+9b)+2ab\varepsilon(a^3b+2\varepsilon^2)]}{4(a-b)^5} - \frac{C^3\kappa[6a^3b\varepsilon(a^2+1)+40a^3b^3(a^2-2\varepsilon)]}{4(a-b)^5}, \\
u =& \frac{(a^2+b^2)(8ab\varepsilon+15a^2b^2)-2\varepsilon(a^4+b^4)}{4b(a-b)^5} + \frac{(a^3-b^3)-6ab(a^4+b^4+ab(3+2\varepsilon))}{4b(a-b)^5} \\
&+ \frac{C^3\kappa[2a^3b^3(a^2-b^2)+2b^2\varepsilon(6ab-2\varepsilon^2)]}{4b(a-b)^4} + \frac{C^3\kappa[27b^3(1+b^3)+a^3b^2(a^3-4b^3)]}{4b(a-b)^4} \\
&+ \frac{C^3\kappa[ab^4(57a^2+108b^2)-6b^2\varepsilon^2(1+3b)-3ab^3(21ab^2+44b\varepsilon)]}{4b(a-b)^4}.
\end{aligned}
$$

For $A^2B^2 = D$ and $C = 1$ the line element becomes

$$ds^2 = -D(1+ar^2)^{2s}(1+br^2)^{2u}exp(2I(r^2))dt^2 + \frac{1+ar^2}{1+br^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.62}$$

The uncharged model in this case was generated by taking $\varepsilon = 0 \implies E = 0$

$$ds^2 = -D(1+ar^2)^{2s}(1+br^2)^{2u}exp(2I(r^2))dt^2 + \frac{1+ar^2}{1+br^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.63}$$

Discussion

In [19] authors, have found out some new classes of solution for different polytropic indices. From Case I, for $\varepsilon = 0$, the models of Feroze and Siddiqui [21] and Maharaj and Mafa Takisa [22] are regained. Rest of all are acceptable new solutions that can be utilized in modeling charged anisotropic compact objects. The gravitational potential are regular and continuous in stellar interior with choice of electric field E being physically acceptable. Density and radial pressure are monotonically decreasing and also p_r vanishes at boundary. The tangential pressure at center has increasing values due to conservation of momentum in quasi equilibrium contraction as pointed by Karmarkar *et al* [23], pressure anisotropy is finite at center and increases while moving towards the boundary. Lastly for stability of models the causality condition i.e. the speed of sound $\frac{dp_r}{d\rho}$ is less than one. So in all the above models different solutions have been found for charged anisotropic fluid distribution satisfying the acceptability conditions of compact objects which can also be further reduced to the uncharged models.

2.5.2 Review of Model of a Static Spherically Symmetric Anisotropic Fluid Distribution in Paraboloidal Spacetime Admitting a Polytropic Equation of State

***S. Thirukkanesh, Ranjan Sharma and Shyam Das* [20]**

If parabolidal spacetime is embedded in a spherically symmetric static metric given by equation (2.2). Initiating with the Cartesian equation of a Euclidean space characterizing as being four dimensional with an immersed three paraboloid is given as

$$x^2 + y^2 + z^2 = 2wL, \tag{2.64}$$

here L is a constant, where as a three-paraboloid is specified by the constants x, y, z and the constant w represents the sections of sphere. The 3-paraboloid immersed in four-dimensional space is parameterized by utilizing the following parameterizations earlier stated by Thomas and Pandya [24]

$$x = rsin\theta cos\phi, \quad y = rsin\theta sin\phi, \quad z = rcos\theta, \quad w = \frac{r^2}{2L}. \tag{2.65}$$

Thus by equation (2.65) the Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2, \tag{2.66}$$

takes the form

$$ds^2 = \left(1 + \frac{r^2}{L^2}\right) dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.67}$$

Here comparing equation (2.67) with equation (2.2) yields

$$e^{2\lambda} = 1 + \frac{r^2}{L^2}, \tag{2.68}$$

Thus, the interior stellar structure in a spherically symmetric, parabolidal spacetime is given by metric

$$ds^2 = -e^{2\nu}dt^2 + \left(1 + \frac{r^2}{L^2}\right) dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.69}$$

For anisotropic fluid distribution the components of energy momentum tensor T_{ab} are

$$T_{ab} = diag(-\rho, p_r, p_t, p_t). \tag{2.70}$$

So the field equations along with mass are given as

$$\frac{1}{r^2}\left(r\left(1 - e^{-2\lambda}\right)\right)' = \rho, \tag{2.71}$$

$$e^{-2\lambda}\frac{2}{r}\nu' - \frac{1}{r^2}((1 - e^{-2\lambda}) = p_r \tag{2.72}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{1}{r}(\nu' - \lambda')\right) = p_t, \tag{2.73}$$

$$\frac{1}{2}\int_0^r w^2(\rho(w))\, dw = m(r). \tag{2.74}$$

here '$\prime$' shows derivative with respect to r. Assuming the polytropic equation of state along with the transformations

$$p_r = k\rho^{1+1/\eta} - \beta, \tag{2.75}$$

$$x = \frac{r^2}{L^2},$$

$$y^2(x) = e^{2\nu(r)}, \tag{2.76}$$

$$z(x) = e^{-2\lambda(r)},$$

here k and β are constants. Thus, by these substitutions the system of equations are given as

$$\rho = \frac{1}{L^2}\left(\frac{1-z}{x} - 2\dot{z}\right), \tag{2.77}$$

$$p_r = k\rho^{1+\frac{1}{\eta}} - \beta, \tag{2.78}$$

$$\Delta = \frac{1}{L^2}\left(4xz\frac{\ddot{y}}{y} + \dot{z}\left(1 + 2x\frac{\dot{y}}{y}\right) + \frac{1-z}{x}\right), \tag{2.79}$$

$$\frac{\dot{y}}{y} = \frac{k}{4L^{2/\eta}z}\left(\frac{1-z}{x} - 2\dot{z}\right)^{1+\frac{1}{\eta}} + \frac{1-z}{4xz} - \frac{-\beta L^2}{4z}, \tag{2.80}$$

$$m(x) = \frac{L^3}{4}\int_0^x \sqrt{w}\,(\rho(w))\,dw, \tag{2.81}$$

where '$\cdot$' represents derivative with respect to x. Rewriting the gravitational potential z by utilizing the transformation given by (2.76) in equation (2.68)

$$z = \frac{1}{1+x}. \tag{2.82}$$

Polytropic Models

This paper proposes following models for different variations of η i.e. $\eta = 1, 2$.

Case: I

When $\eta = 1$, then equation (2.80) implies

$$\frac{\dot{y}}{y} = \frac{k}{4L^2}\frac{(3+x)^2}{(1+x)^3} - \frac{\beta L^2(1+x)}{4} + \frac{1}{4}, \tag{2.83}$$

integrating above equation yields the gravitational potential y as

$$y = d_1(1+x)^{\frac{k}{4L^2}} \times exp\left[\frac{1}{8}\left(x(2-\beta L^2(2+x)) - \frac{4k(3+2x)}{L^2(1+x)^2}\right)\right]. \tag{2.84}$$

The system becomes

$$e^{2\lambda} = 1 + x, \tag{2.85}$$

$$e^{2\nu} = d_1^2(1+x)^{\frac{k}{2L^2}} \times exp\left[\frac{1}{4}\left(x(2-\beta L^2(2+x)) - \frac{4k(3+2x)}{L^2(1+x)^2}\right)\right], \tag{2.86}$$

$$\rho = \frac{3+x}{L^2(1+x)^2}, \tag{2.87}$$

$$p_r = k\rho^2 - \beta, \tag{2.88}$$

$$p_t = p_r + \Delta, \tag{2.89}$$

$$\Delta = \frac{f_1(x)}{4L^6(1+x)^7}, \tag{2.90}$$

$$f_1(x) = x[(3+x)(L^4(1+x)^5 + k^2(3+x)^3 + 2kL^2(1+x)^2(-14+x+x^2))]$$
$$- 2L^2(1+x)^4(L^2(1+x)^2(2+x) + k(3+x)^2 + \beta + L^8(1+x)^8\beta^2). \tag{2.91}$$

Case: II

When $\eta = 2$, then equation (2.80) implies

$$\frac{\dot{y}}{y} = \frac{k}{4L}\frac{(3+x)^{\frac{3}{2}}}{(1+x)^2} - \frac{\beta L^2(1+x)}{4} + \frac{1}{4}, \tag{2.92}$$

integrating above equation yields the gravitational potential y as

$$y = d_2\left(\frac{\sqrt{3+x}-\sqrt{2}}{\sqrt{3+x}+\sqrt{2}}\right)^{\frac{3k}{4\sqrt{2L}}} \times exp\left[\frac{x}{8}\left(x(2-\beta L^2(2+x)) - \frac{4k(3+2x)}{L^2(1+x)^2}\right)\right]. \tag{2.93}$$

The system becomes

$$e^{2\lambda} = 1+x, \tag{2.94}$$

$$e^{2\nu} = d_2^2\left(\frac{\sqrt{3+x}-\sqrt{2}}{\sqrt{3+x}+\sqrt{2}}\right)^{\frac{3k}{2\sqrt{2L}}} \times exp\left[\frac{x}{4}\left(x(2-\beta L^2(2+x)) - \frac{4k(3+2x)}{L^2(1+x)^2}\right)\right], \tag{2.95}$$

$$\rho = \frac{3+x}{L^2(1+x)^2}, \tag{2.96}$$

$$p_r = k\rho^{\frac{3}{2}} - \beta, \tag{2.97}$$

$$p_t = p_r + \Delta, \tag{2.98}$$

$$\Delta = \frac{f_2(x)}{4L^4(1+x)^5\sqrt{3+x}}, \tag{2.99}$$

$$f_2(x) = x\left[k^2(3+x)^{\frac{7}{2}} + 2kL(-27-30x+2x^2+6x^3+x^4 - L^2(1+x)^3(3+x)^2\beta)\right.$$
$$\left.+L^2(1+x)^3\sqrt{3+x} \times (3+x-2L^2(1+x)(2+x)\beta + L^4(1+x)^3\beta^2)\right]. \tag{2.100}$$

Discussion

At boundary $r = R$ authors have matched their solution to the Schwarzschild exterior spacetime i.e.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.101}$$

from which the total mass M and model parameter L are found as follows

$$\begin{aligned} M = m(R) = \frac{R^3}{2(L^2 + R^2)}, \\ L = \sqrt{\frac{R^3}{2M}\left(1 - \frac{2M}{R}\right)}. \end{aligned} \tag{2.102}$$

Further the boundary conditions imply the expression for constant of integration and β in both cases as follow

$$d_1^2 = \left(1 + \frac{R^2}{L^2}\right)^{-\left(1+\frac{k}{2L^2}\right)} \times exp\left[\frac{4k(3L^2 + 2R^2)}{(L^2 + R^2)^2} - \frac{R^2}{4L^2}(2 - \beta(2L^2 + R^2))\right], \tag{2.103}$$

$$\begin{aligned} d_2^2 = &\left(\frac{L^2}{L^2 + R^2}\right)\left(\frac{\sqrt{3L^2 + R^2} - \sqrt{2}L}{\sqrt{3L^2 + R^2} + \sqrt{2}L}\right)^{-\frac{3k}{2\sqrt{2}L}} \\ &\times exp\left[-\frac{R^2}{4L^2}\left(\frac{4k\sqrt{3L^2 + 2R^2}}{(L^2 + R^2)^2} + 2 - \beta(2L^2 + R^2)\right)\right], \end{aligned} \tag{2.104}$$

$$\beta_I = \frac{4kM^2(4M - 3R^2)}{R^8}, \tag{2.105}$$

$$\beta_{II} = \sqrt{8}k\left(\frac{M(3R - 4M)}{R^4}\right)^{\frac{3}{2}}. \tag{2.106}$$

Utilizing the mass $M = 1.58M_{\odot}$ and radius $R = 9.1$km of the pulsar $4U1820 - 30$ in the above expressions, the conditions of acceptability are checked. The gravitational potentials are regular whereas density and pressures at center $(r = 0)$ yields $\rho(0) = \frac{3}{L^2}$ and $p_r(0) = p_t(0) = \frac{9k}{4} - \beta$. This implies that (ρ, p_r, p_t) are positive and monotonically decreasing throughout the stellar interior, and pressure anisotropy being 0 at origin.

In these models from Δ one figures out that for a certain region in stellar interior the radial pressure is greater than the tangential pressure implying that core $p_r > p_t$ that is more unstable as compared to the crust region where $p_t > p_r$. Moving further, the gradients of density and pressure are negative given as follows

$$\frac{d\rho}{dr} = -\frac{2r(5L^2 + r^2)}{(L^2 + r^2)^3}, \tag{2.107}$$

$$\frac{dp_r}{dr} = -\frac{4kr(3L^2 + r^2)(5L^2 + r^2)}{(L^2 + r^2)^5}, \tag{2.108}$$

$$\begin{aligned}\frac{dp_t}{dr} &= \frac{1}{2L^2(L^2 + r^2)^8}\left[r(k^2(3L^2 + r^2)^3)(3L^4 - 13L^2 + r^2 - 2r^4)\right.\\ &-2k(L^2 + r^2)^2(102L^8 - 54L^6r^2 - 9L^4r^4 + 8L^2r^6 + r^8\\ &+3L^2(L - r)(L + r)(L^2 + r^2)^2(3L^2 + r^2)\beta)\\ &+(L^2 + r^2)(3L^4 - L^2 + r^2 - 2(L^2 + r^2)(2L^4 + 2L^2r^2 + r^4)\beta\\ &\left.+(L^2 + r^2)^3(L^2 + 2r^2)\beta^2)\right].\end{aligned} \tag{2.109}$$

From these the causality condition i.e radial and tangential speed of sound is less than speed of light in these models ($0 < v_r^2 = \frac{dp_r}{d\rho}, v_t^2 = \frac{dp_t}{d\rho} < 1$), also the two models are satisfying the null, weak and strong energy conditions. Thus two acceptable models for the uncharged anisotropic fluid distribution in parabolidal spacetime are found.

Chapter 3

Charged Ansisotropic Models with Generalized Polytropic Equation of State

3.1 The Einstein-Maxwell Field Equations

In this chapter, the extension of work earlier presented by Thirukkanesh *et al* [20] is contemplated by generalizing the polytropic equation of state in the presence of electromagnetic field and anisotropy, on background of paraboloidal geometry. The modified form of field equations (2.28)-(2.31) is attained by considering the transformations given by equation (2.76) in which x is the new independent coordinate and y, z are new metric functions.

$$\frac{1-z}{x} - 2\dot{z} = L^2(\rho + E^2), \tag{3.1}$$

$$4z\frac{\dot{y}}{y} - \frac{1-z}{x} = L^2(p_r - E^2), \tag{3.2}$$

$$4xz\frac{\ddot{y}}{y} + (4z + 2x\dot{z})\frac{\dot{y}}{y} + \dot{z} = L^2(p_t + E^2), \tag{3.3}$$

$$\sigma^2 = \frac{4z}{xL^2}(x\dot{E} + E)^2, \tag{3.4}$$

where '$\cdot$' represents derivative with respect to x. The mass function on the interior region of sphere having radius r is given as

$$M(r) = \frac{1}{2}\int_0^r \tilde{r}^2 \left(\rho(\tilde{r}) + E^2\right) d\tilde{r}. \tag{3.5}$$

utilizing the transformation (2.76) mass becomes

$$M(x) = \frac{L^3}{4}\int_0^x \sqrt{\tilde{x}} \left(\rho(\tilde{x}) + E^2\right) d\tilde{x}. \tag{3.6}$$

3.2 Exact Solutions with Generalized Polytropic Equation of State

Considering a common class of barotropic model (i.e. polytropic model) where density is a function of pressure, the generalized polytropic equation of state is written as

$$p_r = \alpha\rho^\Gamma + \beta\rho - \gamma, \tag{3.7}$$

for the arbitrary constants α, β, γ. Using the generalized polytropic equation of state and energy density given by equations (3.7) and (3.1) in equation (3.2), we get the expression

$$\alpha\left(\frac{1-z}{xL^2} - \frac{2\dot{z}}{L^2} - E^2\right)^\Gamma + (1+\beta)\left(\frac{1-z}{xL^2} - E^2\right) - \frac{2\beta\dot{z}}{L^2} - \frac{4z}{L^2}\frac{\dot{y}}{y} - \gamma = 0. \tag{3.8}$$

Equation (3.8) implies

$$\frac{\dot{y}}{y} = \frac{\alpha L^2}{4z}\left(\frac{1-z}{xL^2} - \frac{2\dot{z}}{L^2} - E^2\right)^\Gamma + \frac{(1+\beta)L^2}{4z}\left(\frac{1-z}{xL^2} - E^2\right) - \frac{\beta\dot{z}}{2z} - \frac{\gamma L^2}{4z}. \tag{3.9}$$

For the sake of exact solution, taking the ansatz E

$$E^2 = \frac{2\xi a x}{(1+x)}, \tag{3.10}$$

where ξ and a are arbitrary real constants. Substituting the values of z given by equation (2.82) and E in (3.9), we get

$$\begin{aligned}\frac{\dot{y}}{y} &= \frac{(1+\beta)(x+1)(1-2\xi a x L^2)+2\beta-\gamma L^2(x+1)^2}{4(x+1)} \\ &+ \frac{\alpha L^2(1+x)}{4}\left(\frac{3+x-2\xi a x L^2(x+1)}{L^2(x+1)^2}\right)^{\Gamma}.\end{aligned} \tag{3.11}$$

The expressions of energy density, charge density given by equations (3.1) and (3.4) take the form

$$\rho = \frac{3+x-2\xi a x L^2(1+x)}{L^2(1+x)^2}, \tag{3.12}$$

$$\sigma^2 = \frac{2a\xi(3x+2)^2}{L^2(1+x)^4}. \tag{3.13}$$

3.2.1 Polytropic Models

In this section exact solutions to field equations using polytropic indices $\eta = 1/2,\ 1,\ 2$ in the presence of anisotropy and electromagnetic field are presented

Model I: $\eta = 1/2$

For $\eta = 1/2$, equation (3.7) becomes

$$p_r = \alpha\rho^3 + \beta\rho - \gamma. \tag{3.14}$$

For the present model using z and E, the tangential pressure takes the form

$$\begin{aligned}p_t = &\left(\frac{3\alpha x+2\alpha}{2(x+1)}\right)\times\left(\frac{3+x-2\xi a x L^2(x+1)}{L^2(x+1)^2}\right)^3 - \frac{(3\alpha x)(5+x+2\xi a L^2(x+1))}{L^2(x+1)^3}\times \\ &\left(\frac{3+x-2\xi a x L^2(x+1)}{L^2(x+1)^2}\right)^2 + \frac{xL^2(x+1)^2(-2a\xi(1+\beta)-\gamma)-2\beta x}{L^2(x+1)^3} \\ &+ \frac{4x}{L^2(x+1)}\times\left(\frac{\alpha L^2(x+1)}{4}\left(\frac{3+x-2\xi a x L^2(x+1)}{L^2(x+1)^2}\right)^3 + \frac{(1+\beta)(1-2\xi a x L^2)}{4}\right.\end{aligned}$$

$$+\frac{2\beta-\gamma L^2(x+1)^2}{4(x+1)}\Bigg)^2+\frac{(x^2+3x+2)[(1+\beta)(1-2\xi axL^2)-\gamma L^2(x+1)]}{2L^2(x+1)^3}$$
$$+\frac{2\beta(x+2)-2(x+1)[1+2\xi axL^2(x+1)]}{2L^2(x+1)^3}. \tag{3.15}$$

On integrating equation (3.11) by substituting $\eta = 1/2$, we get the gravitational potential y as

$$y = C_1(1+x)^m \times exp\left[U(x)\right], \tag{3.16}$$

here constant of integration is C_1 whereas m and $U(x)$ are defined as

$$m = \frac{-\beta\alpha(24a^3L^6\xi^3+6aL^2\xi)}{8L^4},$$
$$U(x) = \frac{\alpha x(24aL^6\xi^3+12a^2L^2\xi^2)}{4L^4}+\frac{\alpha(-24a^2L^4\xi^2-6)}{8L^4(x+1)^2}-\alpha a^3L^2\xi^3(x+1)^2$$
$$+\frac{\alpha(-8a^3L^6\xi^3+36a^2L^4\xi^2+18aL^2\xi-1)}{4L^4(x+1)}+\frac{\alpha(-24aL^2\xi-12)}{12L^4(x+1)^3}$$
$$-\frac{\alpha}{2L^4(x+1)^4}+\frac{(1+\beta)x}{4}-\frac{\xi aL^2(1+\beta)x^2}{4}-\frac{\gamma L^2(2x+x^2)}{8}.$$

By means of gravitational potential y and $e^{2\lambda}$ given by equations (3.16) and (2.68) the line element becomes

$$ds^2 = -C_1^2\left(1+\frac{r^2}{L^2}\right)^{2m} \times exp\left[2U\left(\frac{r^2}{L^2}\right)\right]dt^2+\left(1+\frac{r^2}{L^2}\right)dr^2+r^2(d\theta^2+sin^2\theta d\phi^2). \tag{3.17}$$

Model II: $\eta = 1$

For $\eta = 1$, equation (3.7) becomes

$$p_r = \alpha\rho^2+\beta\rho-\gamma. \tag{3.18}$$

For the present model using z and E, the tangential pressure takes the form

$$p_t = \left(\frac{3\alpha x+2\alpha}{2(x+1)}\right)\times\left(\frac{3+x-2\xi axL^2(x+1)}{L^2(x+1)^2}\right)^2-\frac{(2\alpha x)(5+x+2\xi aL^2(x+1))}{L^2(x+1)^3}$$

$$
\begin{aligned}
&\times \left(\frac{3+x-2\xi axL^2(x+1)}{L^2(x+1)^2}\right) + \frac{xL^2(x+1)^2(-2a\xi(1+\beta)-\gamma)-2\beta x}{L^2(x+1)^3} \\
&+ \frac{4x}{L^2(x+1)} \times \left(\frac{\alpha L^2(x+1)}{4}\left(\frac{3+x-2\xi axL^2(x+1)}{L^2(x+1)^2}\right)^2 + \frac{(1+\beta)(1-2\xi axL^2)}{4}\right. \\
&\left.+ \frac{2\beta-\gamma L^2(x+1)^2}{4(x+1)}\right)^2 + \frac{(x^2+3x+2)[(1+\beta)(1-2\xi axL^2)-\gamma L^2(x+1)]}{2L^2(x+1)^3} \\
&+ \frac{2\beta(2+x)-2(x+1)[1+2\xi axL^2(x+1)]}{2L^2(x+1)^3}.
\end{aligned} \tag{3.19}
$$

On integrating equation (3.11) by substituting $\eta = 1$, we get the gravitational potential y as

$$
y = C_2(1+x)^n \times exp\left[V(x)\right], \tag{3.20}
$$

here constant of integration is C_2 whereas n and $V(x)$ are defined as

$$
\begin{aligned}
n &= \frac{\beta\alpha(2aL^2\xi-1)^2}{8L^2}, \\
V(x) &= \frac{2x\alpha(aL^2\xi(x(x(aL^2\xi x-2)-2aL^2\xi-4)-6)-2)}{4L^2(1+x)^2} + \frac{(1+\beta)x}{4} \\
&+ \frac{2a\alpha L^2\xi(aL^2\xi-4)-6\alpha}{4L^2(1+x)^2} - \frac{\xi aL^2(1+\beta)x^2}{4} - \frac{\gamma L^2(2x+x^2)}{8}.
\end{aligned}
$$

By means of gravitational potential y and $e^{2\lambda}$ given by equations (3.20) and (2.68) the line element becomes

$$
ds^2 = -C_2^2\left(1+\frac{r^2}{L^2}\right)^{2n} \times exp\left[2V\left(\frac{r^2}{L^2}\right)\right]dt^2 + \left(1+\frac{r^2}{L^2}\right)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.21}
$$

Model III: $\eta = 2$

For $\eta = 2$, equation (3.7) becomes

$$
p_r = \alpha\rho^{\left(\frac{3}{2}\right)} + \beta\rho - \gamma. \tag{3.22}
$$

For the present model using z and E, the tangential pressure takes the form

$$
\begin{aligned}
p_t = & \left(\frac{3\alpha x+2\alpha}{2(x+1)}\right)\times\left(\frac{3+x-2\xi a x L^2(x+1)}{L^2(x+1)^2}\right)^{\frac{3}{2}}-\frac{(3\alpha x)(5+x+2\xi a L^2(x+1))}{2L^2(x+1)^3}\\
& \times\left(\frac{3+x-2\xi a x L^2(x+1)}{L^2(x+1)^2}\right)^{\frac{1}{2}}+\frac{xL^2(x+1)^2(-2a\xi(1+\beta)-\gamma)-2\beta x}{L^2(x+1)^3}\\
& +\frac{4x}{L^2(x+1)}\times\left(\frac{\alpha L^2(x+1)}{4}\left(\frac{3+x-2\xi a x L^2(x+1)}{L^2(x+1)^2}\right)^{\frac{3}{2}}+\frac{(1+\beta)(1-2\xi a x L^2)}{4}\right.\\
& \left.+\frac{2\beta-\gamma L^2(x+1)^2}{4(x+1)}\right)^2+\frac{(x^2+3x+2)[(1+\beta)(1-2\xi a x L^2)-\gamma L^2(x+1)]}{2L^2(x+1)^3}\\
& +\frac{2\beta(x+2)-2(x+1)[1+2\xi a x L^2(x+1)]}{2L^2(x+1)^3}.
\end{aligned}
\tag{3.23}
$$

On integrating equation (3.11) by substituting $\eta = 2$, we get the gravitational potential y as

$$
y = C_3(1+x)^{\frac{\beta}{2}}\times exp\left[W(x)\right], \tag{3.24}
$$

here constant of integration is C_3 and $W(x)$ is defined as

$$
\begin{aligned}
W(x) = & \frac{-3\alpha I arcsin\left(\frac{J}{\sqrt{I}}\right)-3\alpha J\sqrt{I-J^2 I}}{2^{\frac{11}{2}}\sqrt{a}L^2\xi}-\frac{-3\alpha K^2 arcsin\left(\frac{J}{\sqrt{I}}\right)}{2^{\frac{9}{2}}\sqrt{a\xi}L^2}-\frac{\xi a L^2(1+\beta)x^2}{4}\\
& -\frac{3\alpha(-K)\sqrt{I}}{2^{\frac{7}{2}}\sqrt{a\xi}L^2\left(\frac{I\left(1-\sqrt{1-\frac{J^2}{I}}\right)^2}{J^2}+1\right)}-\frac{3(-K)\alpha\sqrt{2a\xi}arctan\left(\frac{\frac{K\sqrt{I}\left(1-\sqrt{1-\frac{J^2}{I}}\right)}{J}+I}{\sqrt{K^2-I}}\right)}{\sqrt{K^2-I}}\\
& -\frac{\alpha(-2aL^2\xi x(1+x)+x+3)^{\frac{3}{2}}}{4L(1+x)}+\frac{(1+\beta)x}{4}-\frac{\gamma L^2(2x+x^2)}{8},
\end{aligned}
$$

where I, J and K are given as

$$I = 4aL^4\xi + 20aL^2\xi + 1,$$
$$J = 4aL^2\xi x + 2aL^2\xi - 1,$$
$$K = 2aL^2\xi + 1.$$

By means of gravitational potential y and $e^{2\lambda}$ given by equations (3.24) and (2.68) the line element becomes

$$ds^2 = -C_3^2\left(1+\frac{r^2}{L^2}\right)^{\beta} \times exp\left[2W\left(\frac{r^2}{L^2}\right)\right]dt^2 + \left(1+\frac{r^2}{L^2}\right)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.25}$$

3.3 Analysis

Now examining certain conditions in order to check whether the new models are acceptable or not.

3.3.1 Boundary Conditions

The exterior stellar structure of spherically symmetric, charged anisotropic spacetime is described by the Reissner-Nordstrom metric

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.26}$$

On matching the interior spacetime metric (2.2) at boundary $r = R$ with exterior spacetime metric given by (3.26), we get

$$e^{2\nu} = e^{-2\lambda} = \left(1-\frac{2M}{R}+\frac{Q^2}{R^2}\right), \tag{3.27}$$

where the total mass is given by M and charge by Q. The total mass at boundary in our case by using equation (3.27) becomes

$$M = \frac{R^3 + 2\xi aR^5}{2(L^2+R^2)}. \tag{3.28}$$

Thus by (3.28), the constant L takes the form

$$L = \left[\frac{R^3}{2M}\left((1 - 2a\xi R^2) - \frac{2M}{R}\right)\right]^{1/2}. \tag{3.29}$$

By the boundary condition (3.27), we get the value of constant of integration for all models as

$$C_1^2 = \left(\frac{L^2}{L^2 + R^2}\right)^{(1-2m)} \times \exp\left(-2U\left(\frac{R^2}{L^2}\right)\right), \tag{3.30}$$

$$C_2^2 = \left(\frac{L^2}{L^2 + R^2}\right)^{(1+2n)} \times \exp\left(-2V\left(\frac{R^2}{L^2}\right)\right), \tag{3.31}$$

$$C_3^2 = \left(\frac{L^2}{L^2 + R^2}\right)^{(1+\beta)} \times \exp\left(-2W\left(\frac{R^2}{L^2}\right)\right). \tag{3.32}$$

At boundary of star the radial pressure must vanish i.e. $p_r(r = R) = 0$, this fixes another constant to find the solutions

$$\beta = \left(\frac{\gamma(L^2 + R^2)^2}{L^2(3 - 2\xi aR^2) + R^2(1 - 2\xi aR^2)}\right) - \alpha\left(\frac{L^2(3 - 2\xi aR^2) + R^2(1 - 2\xi aR^2)}{(L^2 + R^2)^2}\right)^{\Gamma-1}. \tag{3.33}$$

Constants Assumed for Analyzing the Physical and Stability Conditions

In Section 2.4, the acceptability criteria is mentioned in order to check that the profiles of all conditions are plotted for the constants $\xi = 1$, $a = 0.0029$, $L = 4.95$, $\alpha = 0.232$ and $\gamma = 0.001422$ for radius r from 0 to 3.5. The value of β and constant of integration for all three models are computed from equations (3.30)-(3.33) respectively.

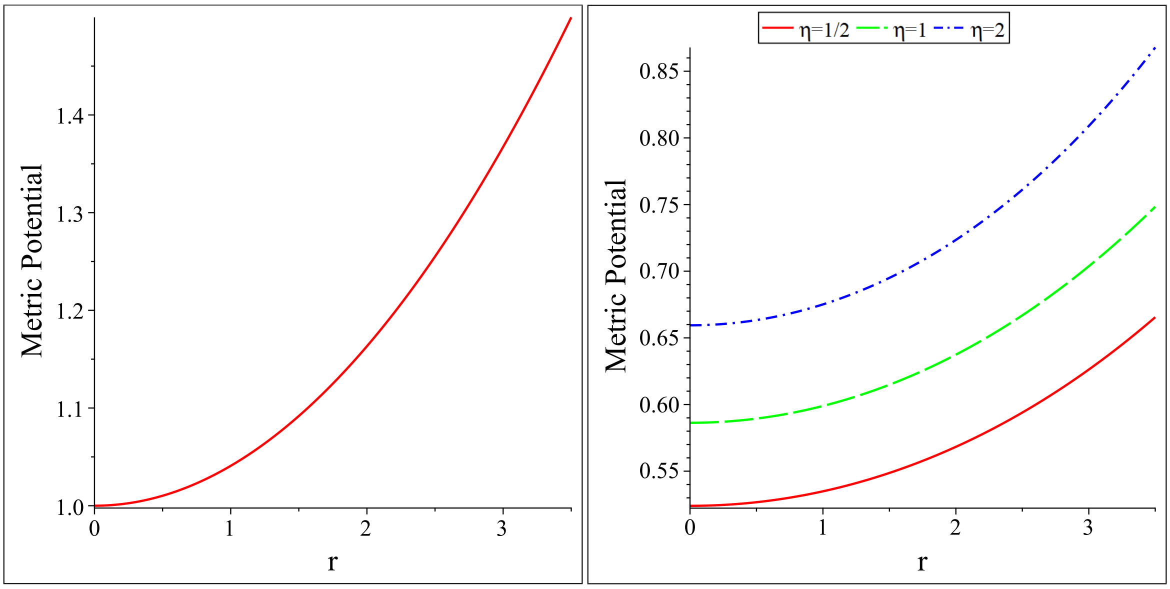
Models	$I : \eta = 1/2$	$II : \eta = 1$	$III : \eta = 2$
C	0.725982118494	0.777671082366	0.621610450317
β	0.0222219256901	0.00882005986932	−0.0344592129661

Table 3.1: Constants β and C

3.3.2 Physical Conditions

Metric potential

The most basic and important condition is to check for the regularity of metric potentials i.e ($e^{2\lambda}$ & $e^{2\nu} > 0$). At the center ($r = 0$) of star the metric potential $e^{2\lambda}$ is 1 and $e^{2\nu}$ for all three models is equivalent to some positive constant which implies $(e^{2\lambda})' = (e^{2\nu})' = 0$. Figure 3.1 exhibits the singularity free nature of model parameters as they satisfy the above mentioned criteria and are increasing monotonically throughout the stellar interior.



(a) $e^{2\lambda}$ (b) $e^{2\nu}$

Figure 3.1: Graph of metric potentials with respect to r

Electric Field Intensity

The behavior of electric field intensity is represented in Figure 3.2, where at origin it is zero and increases while moving towards the boundary of star.

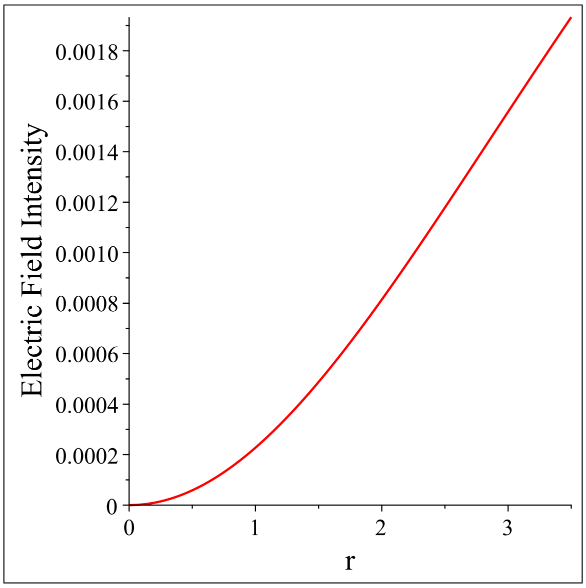


Figure 3.2: Graph of electric field intensity (E^2) with respect to r

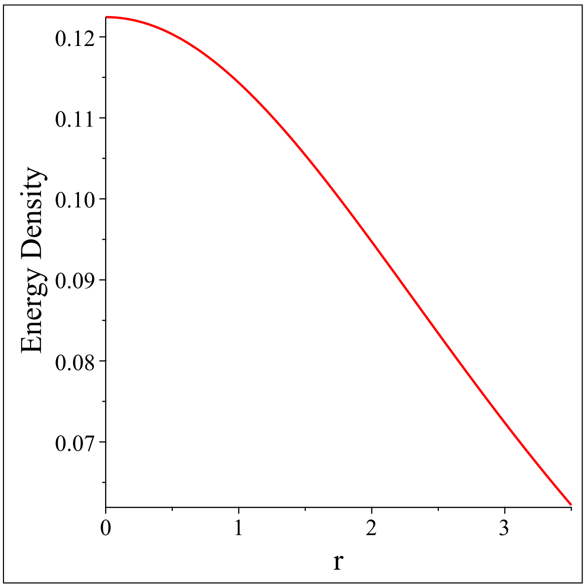


Figure 3.3: Graph of energy density (ρ) with respect to r

Energy Density

The expression of density at the center is obtained as

$$\rho_c = \frac{3}{L^2} > 0. \tag{3.34}$$

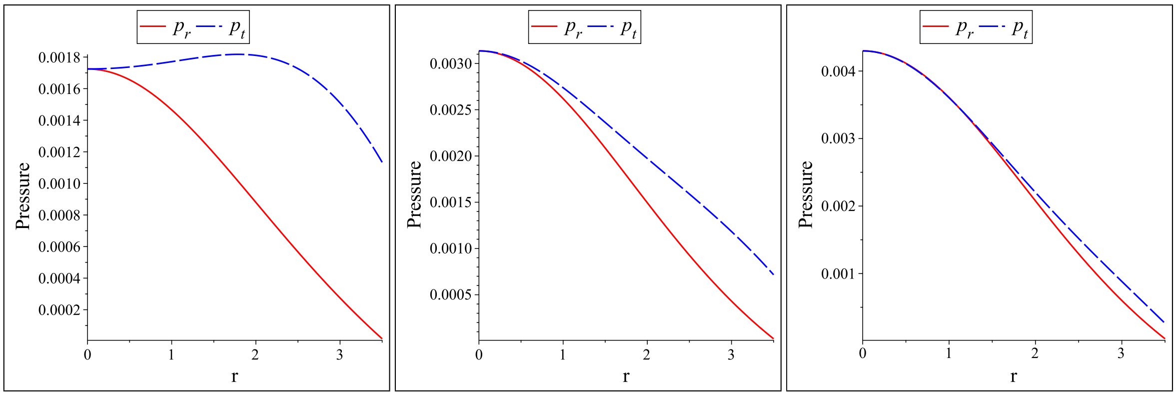
Thus this implies that energy density is free from central singularity and the profile is shown in Figure 3.3, which depicts its decreasing nature.

Radial and Tangential Pressure

The expression of radial and tangential pressure at the center are obtained as

$$p_r(r=0) = p_t(r=0) = \alpha\left(\frac{3}{L^2}\right)^{\Gamma} + \beta\left(\frac{3}{L^2}\right) - \gamma > 0. \tag{3.35}$$

Thus equation (3.35) implies that the pressures are equal at center and free from the central singularity. The profile in Figure 3.4 show the decreasing behavior of radial pressure. As for tangential pressure, it is decreasing eventually as we move towards the boundary but in Model I for $\eta = 1/2$ we have increasing nature in central regions which is acceptable as identified by Karmakar *et al* [24] because the quasi-equilibrium contraction conserves angular momentum of a compact body. Such increasing nature of tangential pressure earlier showed up in Mafa Takisa and Maharaj [20] charged models presented using the polytropic equation of state ($p_r = k\rho^{\Gamma}$) on static spherically symmetric spacetime.



(a) $\eta = 1/2$ (b) $\eta = 1$ (c) $\eta = 2$

Figure 3.4: Graph of pressure with respect to r

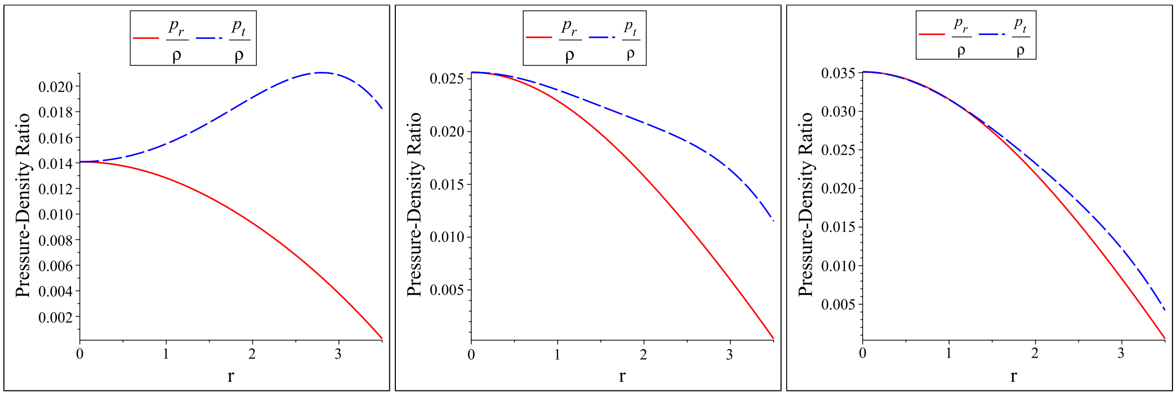
To substantiate the regularity of any physical solution the Zeldovich's [25] criteria must be satisfied that is the pressure density ratios at the center must be less than 1 continuous and positive throughout the stellar interior $\left(\frac{p_r}{\rho}, \frac{p_t}{\rho}\right)|_{r=0} \leq 1$. Therefore,

$$\alpha \left(\frac{3}{L^2}\right)^{\Gamma-1} - \frac{\gamma L^2}{3} + \beta \leq 1. \tag{3.36}$$

The constants assumed to find out the solutions satisfy the condition i.e. values at the center are obtained for all models to be 0.0141, 0.0256, and 0.0351 respectively, as shown in Figure 3.5. Equations (3.35) and (3.36) imply

$$\frac{L^{(2/n)}(L^2\gamma - 3\beta)}{3^{\Gamma}} < \alpha \leq \frac{L^{(2/n)}(3 - 3\beta + L^2\gamma)}{3^{\Gamma}}. \tag{3.37}$$

Now by equation (3.37) physical constraint emerges for α, in all three cases our choice of α lies within the range.



(a) $\eta = 1/2$ (b) $\eta = 1$ (c) $\eta = 2$

Figure 3.5: Graph of pressure-density ratios with respect to r

Trace of Energy Tensor

The trace of energy tensor for our proposed models of compact stars is plotted against r in Figure 3.7, which is positive in all cases and satisfies Bondi's [26] condition for the anisotropic fluid sphere that is $\rho - p_r - 2p_t > 0$.

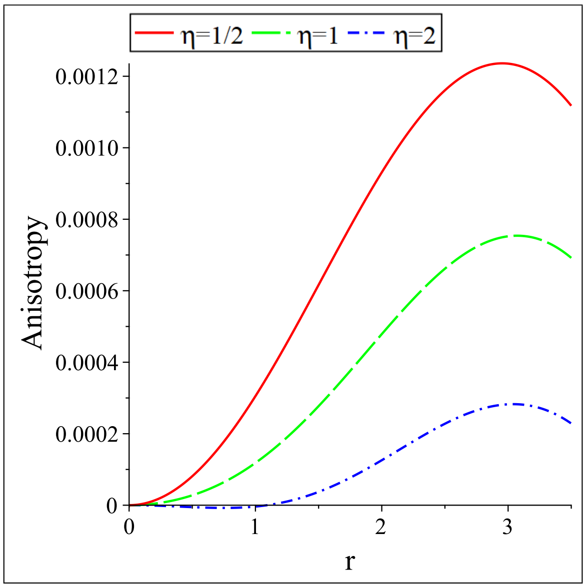


Figure 3.6: Graph of anisotropy (Δ) with respect to r

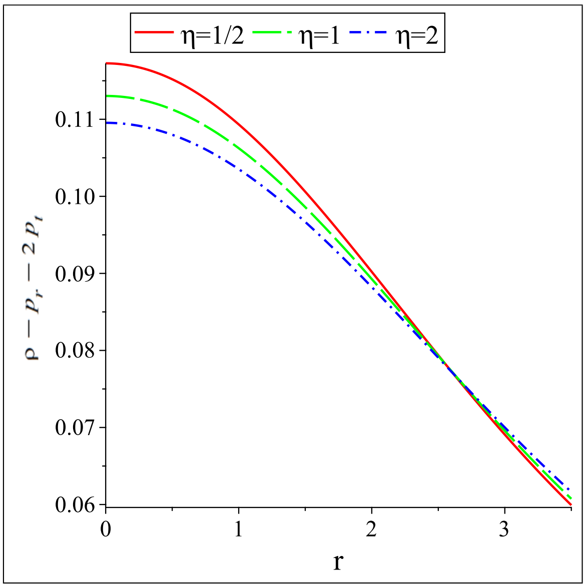


Figure 3.7: Graph of the trace of energy tensor with respect to r

Anisotropic Factor

Anisotropy is defined as the difference between tangential and radial pressure i.e. ($\Delta= p_t - p_r$), the spherical symmetry $p_r(0) = p_t(0)$ implies $\Delta_{r=0}$ is zero. For the compact object it is necessary that $\Delta > 0$ in the interior structure meaning that the ansiotropic force is repulsive in nature, it is possible in the case when $(p_t > p_r)$. Otherwise, if radial pressure dominates the tangential one i.e. $(p_r > p_t)$ then this implies presence of a new force which is attractive in nature. The profile of Δ for all three cases satisfying the required condition is expressed in Figure 3.6.

Gradients

The generalized expressions for density gradient and pressure gradients in radial and transverse direction for our models are stated below

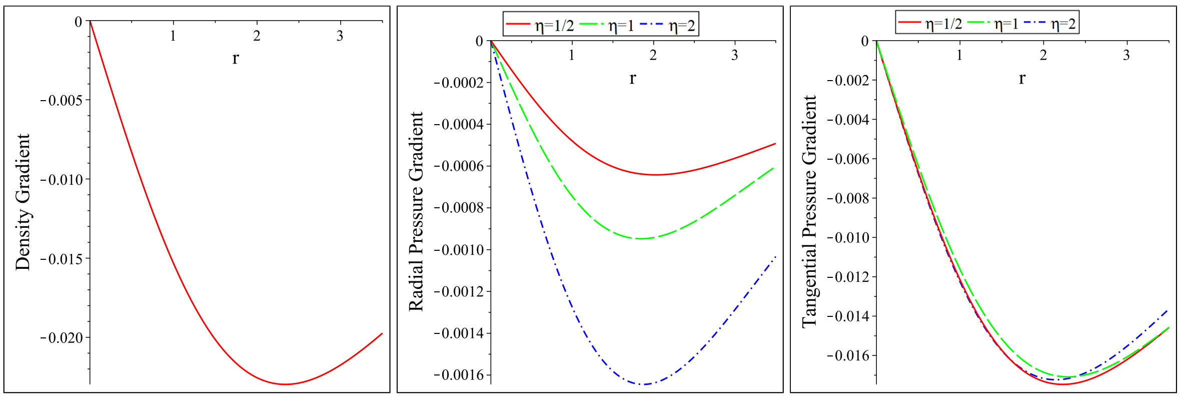
$$\frac{d\rho}{dr} = \frac{-4\xi arL^2(L^2+r^2) - 2r(5L^2+r^2)}{(L^2+r^2)^3}, \tag{3.38}$$

$$\frac{dp_r}{dr} = \left(\alpha\Gamma\left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{\Gamma-1} + \beta\right) \times \frac{-4\xi arL^2(L^2+r^2) - 2r(5L^2+r^2)}{(L^2+r^2)^3}. \tag{3.39}$$

$$\begin{aligned}\frac{dp_t}{dr} =& \left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{\Gamma}\left(\frac{\alpha r(1+\beta)(L^2+r^2)^2(1-4\xi ar^2)}{L^2(L^2+r^2)^2} - \frac{\gamma(2L^2+r^2)}{2L^2}\right.\\ &+\frac{\beta r^2+5\alpha L^2}{(L^2+r^2)^2} - \left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{-1} \times \frac{2r\alpha\Gamma((2\xi aL^2(L^4-r^4)-r^4))}{(L^2+r^2)^4}\\ &\left.+\frac{2r\alpha\Gamma(5L^4-8r^2L^2)}{(L^2+r^2)^4}\right) + \left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{\Gamma-1}\left(\frac{-4\xi ar(L^4+r^2L^2)}{(L^2+r^2)^3}\right.\\ &\left.+\frac{2r(-5L^2-r^2)-4\xi ar^5}{(L^2+r^2)^3}\right) \times \left(\frac{\alpha\Gamma(4rL^4(L^2+\beta L^2+r^2)-2r^2\gamma(L^2+r^2)^3)}{4L^2(L^2+r^2)^2}\right.\\ &\left.+\frac{\alpha\Gamma(2L^4+2r^2L^2+r^4)(1+\beta)(1-2\xi ar^2)}{2L^2(L^2+r^2)}\right) - \left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{-1} \times\\ &\left(\frac{(\Gamma-1)ar^2\Gamma(5L^2+r^2+2\xi aL^2(r^2+L^2))}{(L^2+r^2)^3}\right) - \frac{4a\xi r(2+\beta)+3\gamma}{L^2+r^2} - \frac{6r^3\beta L^2}{(L^2+r^2)^4}\end{aligned}$$

$$
\begin{aligned}
&+\frac{r(1+\beta)\left((L^2+r^2)(1-4\xi a(L^2+r^2))-(2L^2+r^2)(1-2\xi a r^2)\right)}{(L^2+r^2)}\\
&+\frac{r(4\xi a r^2(2+\beta)+\gamma(2r^2-L^2))}{(L^2+r^2)^2}+\frac{r(6\beta L^2-\gamma(2L^4+3r^2L^2+r^4)-2L^2)}{(L^2+r^2)^3}\\
&+\left(\frac{\alpha^2 r(L^2+2r^2)}{2L^2}+\frac{\Gamma\alpha^2 r(L^2+r^2)}{L^4}\times\left(\frac{3L^2+r^2-2\xi a r^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{-1}\right)\times\\
&\left(\frac{3L^2+r^2-2\xi a r^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{2\Gamma}+\left(\frac{2r}{L^2}+\frac{6\beta r}{L^2}+\frac{2r^3(1+\beta)(1-2\xi a L^2)}{L^4}\right.\\
&\left.-\frac{4\xi a r(r^2L^2+r^4)}{L^4}-\frac{2r\gamma(L^4+2r^2L^2+r^4)}{L^6}\right)\times\left(\frac{L^4(L^2-2r^2)}{4(L^2+r^2)^4}\times\left[1+3\beta\right.\right.\\
&\left.+\frac{r^2(1+\beta)(1-2\xi a L^2)}{L^2}-\frac{2\xi a(r^2L^2+r^4)}{L^2}-\frac{\gamma(L^4+2r^2L^2+r^4)}{L^2}\right]+\frac{L^2r^2}{2(L^2+r^2)^3}\\
&\left.\times\left[(1+\beta)(1-2\xi a L^2)-2\xi a(L^2+2r^2)-2\gamma(L^2+r^2)\right]\right).
\end{aligned}
\tag{3.40}
$$

Clearly from the equations decreasing nature of gradients is observed, which is displayed for all models in Figure 3.8.



(a) $\frac{d\rho}{dr}$ (b) $\frac{dp_r}{dr}$ (c) $\frac{dp_t}{dr}$

Figure 3.8: Graph of density and pressure gradients with respect to r

Mass-Radius Relation

We have earlier obtained the expression of mass function given in equation (3.28). Buchdal [27] introduced the concept that mass radius ratio inside a compact star must be less than 4/9 i.e. ($M/R < 4/9$). This is true for any solution with an energy density greater than zero, for present models the mass radius ratio is $\frac{M}{R} = 0.1785 < 4/9$. The profile of mass function is shown in Figure 3.9 which is positive and regular inside the stellar interior and has an increasing nature with m. The compactness factor for stellar configuration is given as

$$u(r) = \frac{M(r)}{r}. \tag{3.41}$$

It classifies the compact object as follows [28]

- normal stars: $M/r \sim 10^{-5}$,
- white dwarfs: $M/r \sim 10^{-3}$,
- neutron star: $10^{-1} < M/r < 0.25$,
- ultra-compact star: $0.25 < M/r < 0.5$,
- blackhole: $M/r = 0.5$,

The profile for our case is shown in Figure 3.10 which is increasing monotonically and is less then 4/9. The redshifts are defined as

$$z_s = e^{\lambda} - 1 = \left(1 + \frac{R^2}{L^2}\right)^{1/2} - 1, \tag{3.42}$$

$$z = e^{-\nu} - 1. \tag{3.43}$$

Here, surface redshift (z_s) is less than 1 as required for physical acceptance of the models and is plotted in Figure 3.11a where moving towards the boundary its increasing nature is depicted. Moreover, gravitational redshift (z) as shown in Figure 3.11b has decreasing nature in all three models as per requirement.

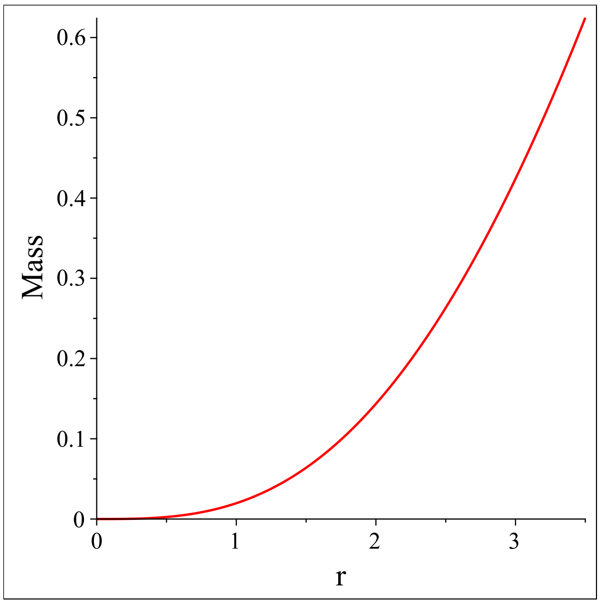


Figure 3.9: Graph of mass function with respect to r

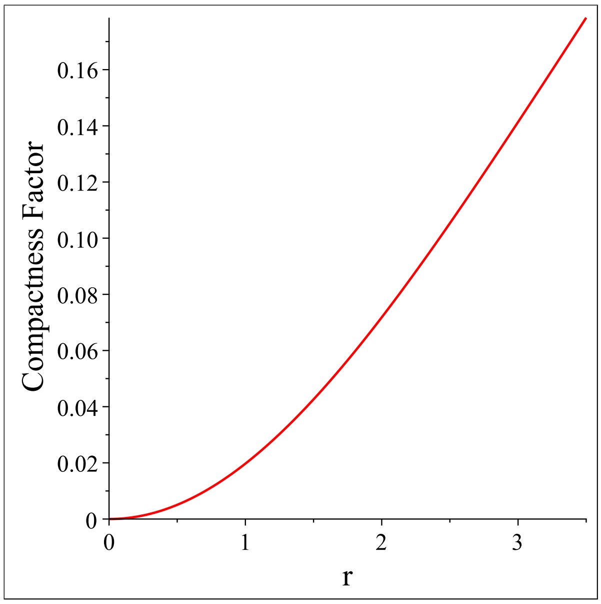
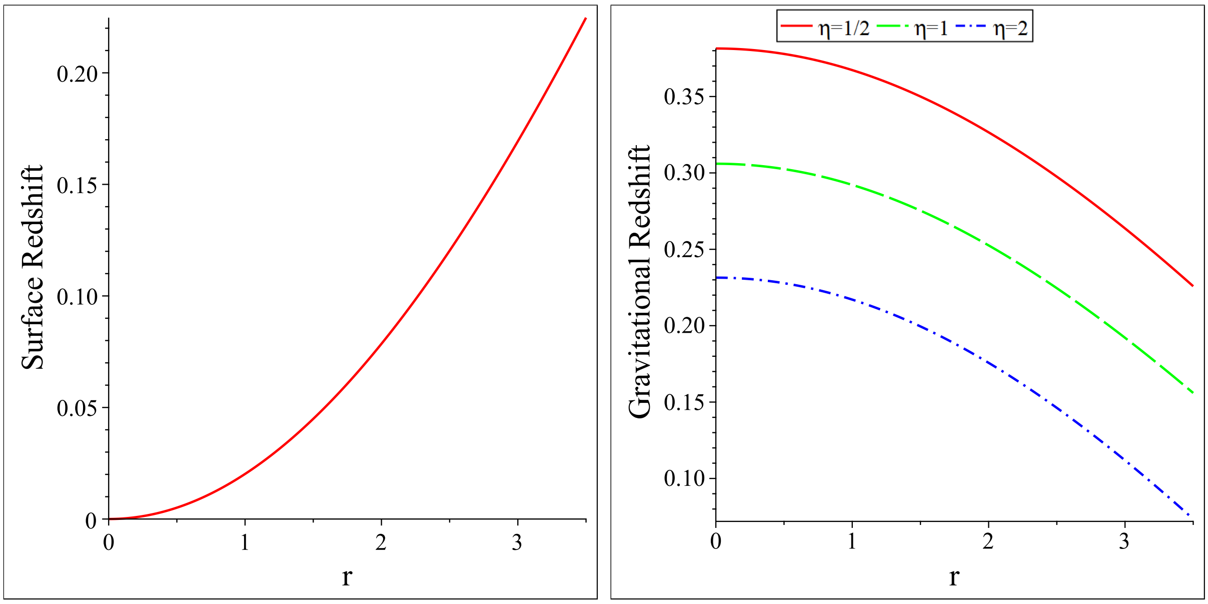


Figure 3.10: Graph of compactification factor with respect to r



(a) z_s

(b) z

Figure 3.11: Graph of surface and gravitational redshifts with respect to r

3.3.3 Stability Conditions

Casuality Condition

The speed of sound in radial and tangential direction for anisotropic fluid distribution are obtained as

$$v_r^2 = \frac{dp_r}{d\rho} = \left(\frac{dp_r/dr}{d\rho/dr}\right), \qquad (3.44)$$

$$v_t^2 = \frac{dp_t}{d\rho} = \left(\frac{dp_t/dr}{d\rho/dr}\right), \qquad (3.45)$$

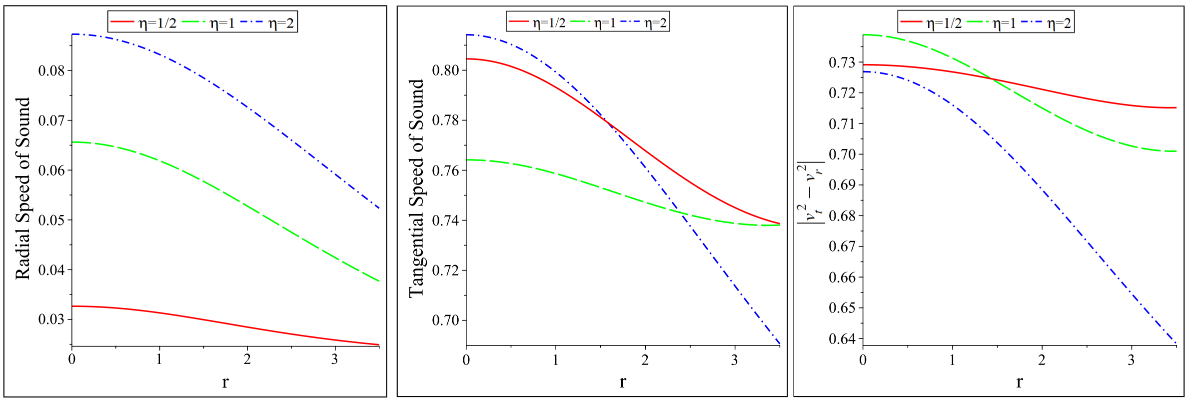
where in our cases the value of radial and tangential speeds of sound takes the form

$$\frac{dp_r}{d\rho} = \alpha\Gamma\left(\frac{L^2(3-2\xi aR^2)+R^2(1-2\xi aR^2)}{(L^2+R^2)^2}\right)^{\Gamma-1}+\beta, \qquad (3.46)$$

$$\frac{dp_t}{d\rho} = \Bigg[\left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{\Gamma}\left(\frac{\alpha r(1+\beta)(1-4\xi ar^2)}{L^2}+\frac{\beta r^2+5\alpha L^2}{(L^2+r^2)^2}-\right.$$
$$\frac{\gamma(2L^2+r^2)}{2L^2}-\left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{-1}\times\frac{2r\alpha\Gamma((2\xi aL^2(L^4-r^4)-r^4))}{(L^2+r^2)^4}$$
$$\left.+\frac{2r\alpha\Gamma(5L^4-8r^2L^2)}{(L^2+r^2)^4}\right)+\left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{\Gamma-1}\left(\frac{-4\xi ar(L^4+r^2L^2)}{(L^2+r^2)^3}\right.$$
$$\left.+\frac{2r(-5L^2-r^2)-4\xi ar^5}{(L^2+r^2)^3}\right)\times\left(\frac{\alpha\Gamma(4rL^4(L^2+\beta L^2+r^2)-2r^2\gamma(L^2+r^2)^3)}{4L^2(L^2+r^2)^2}\right.$$
$$\left.\frac{\alpha\Gamma(2L^4+2r^2L^2+r^4)(1+\beta)(1-2\xi ar^2)}{2L^2(L^2+r^2)}\right)-\left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{-1}\times$$
$$\left(\frac{(\Gamma-1)ar^2\Gamma(5L^2+r^2+2\xi aL^2(r^2+L^2))}{(L^2+r^2)^3}\right)-\frac{4a\xi r(2+\beta)+3\gamma}{L^2+r^2}-\frac{6r^3\beta L^2}{(L^2+r^2)^4}$$
$$+\frac{r(1+\beta)\left((L^2+r^2)(1-4\xi a(L^2+r^2))-(2L^2+r^2)(1-2\xi ar^2)\right)}{(L^2+r^2)}$$
$$+\frac{r(4\xi ar^2(2+\beta)+\gamma(2r^2-L^2))}{(L^2+r^2)^2}+\frac{r(6\beta L^2-\gamma(2L^4+3r^2L^2+r^4)-2L^2)}{(L^2+r^2)^3}$$
$$+\left(\frac{\alpha^2r(L^2+2r^2)}{2L^2}+\frac{\Gamma\alpha^2r(L^2+r^2)}{L^4}\times\left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{-1}\right)\times$$
$$\left(\frac{3L^2+r^2-2\xi ar^2(L^2+r^2)}{(L^2+r^2)^2}\right)^{2\Gamma}+\left(\frac{2r}{L^2}+\frac{6\beta r}{L^2}+\frac{2r^3(1+\beta)(1-2\xi aL^2)}{L^4}\right.$$

$$-\frac{4\xi ar(r^2L^2+r^4)}{L^4}-\frac{2r\gamma(L^4+2r^2L^2+r^4)}{L^6}\Bigg)\times\left(\frac{L^4(L^2-2r^2)}{4(L^2+r^2)^4}\times\Big[1+3\beta\right.$$
$$+\frac{r^2(1+\beta)(1-2\xi aL^2)}{L^2}-\frac{2\xi a(r^2L^2+r^4)}{L^2}-\frac{\gamma(L^4+2r^2L^2+r^4)}{L^2}\Bigg]+\frac{L^2r^2}{2(L^2+r^2)^3}$$
$$\times\left[(1+\beta)(1-2\xi aL^2)-2\xi a(L^2+2r^2)-2\gamma(L^2+r^2)\right]\Big)\Big]$$
$$\times\left(\frac{(L^2+r^2)^3}{-4\xi arL^2(L^2+r^2)-2r(5L^2+r^2)}\right). \tag{3.47}$$

For physical acceptance the speed of sound inside the interior of relativistic stellar model must necessarily be less than the speed of light "$c = 1$" both in radial and transverse direction i.e. $0 < v_r^2, v_t^2 < 1$. The concept of "cracking" was proposed by Herrara [29] for anisotropic distribution of matter. Later on Abrue *et al* [30] using the "cracking" concept proved that the region is potentially stable where $-1 < v_t^2 - v_r^2 < 0$ and potentially unstable where $0 < v_t^2 - v_r^2 < 1$ inside the anisotropic fluid sphere, this implies $0 < |v_t^2 - v_r^2| < 1$. The graphical behaviors for all models are shown in Figure 3.12 from which we clearly observe that our models satisfy causality conditions and the region inside the star is potentially stable.



(a) v_r^2 (b) v_t^2 (c) $|v_t^2 - v_r^2|$

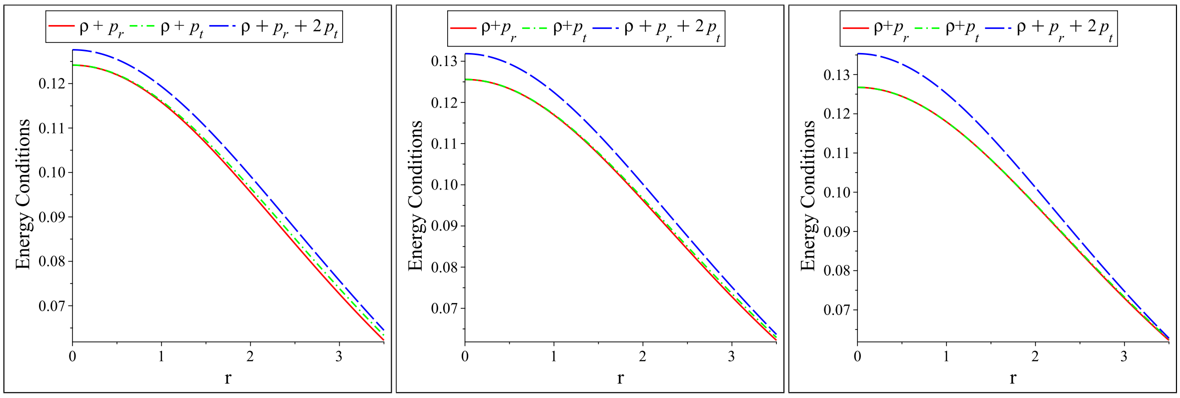
Figure 3.12: Graph of radial, tangential speed of sound and stability factor with respect to r

Energy Conditions

For a system to be physically allowable, it should certainly satisfy the following energy requirements throughout the stellar interior

1. Null Energy Condition: $\rho(r) \geq 0$,
2. Weak Energy Condition: $\rho(r) + p_r \geq 0$, $\rho(r) + p_t \geq 0$,
3. Strong Energy Condition: $\rho(r) + p_r + 2p_t \geq 0$.

For stable configuration Figures 3.3 & 3.13 clearly exhibits the well behaved nature of energy conditions for all cases.



(a) $\eta = 1/2$ (b) $\eta = 1$ (c) $\eta = 2$

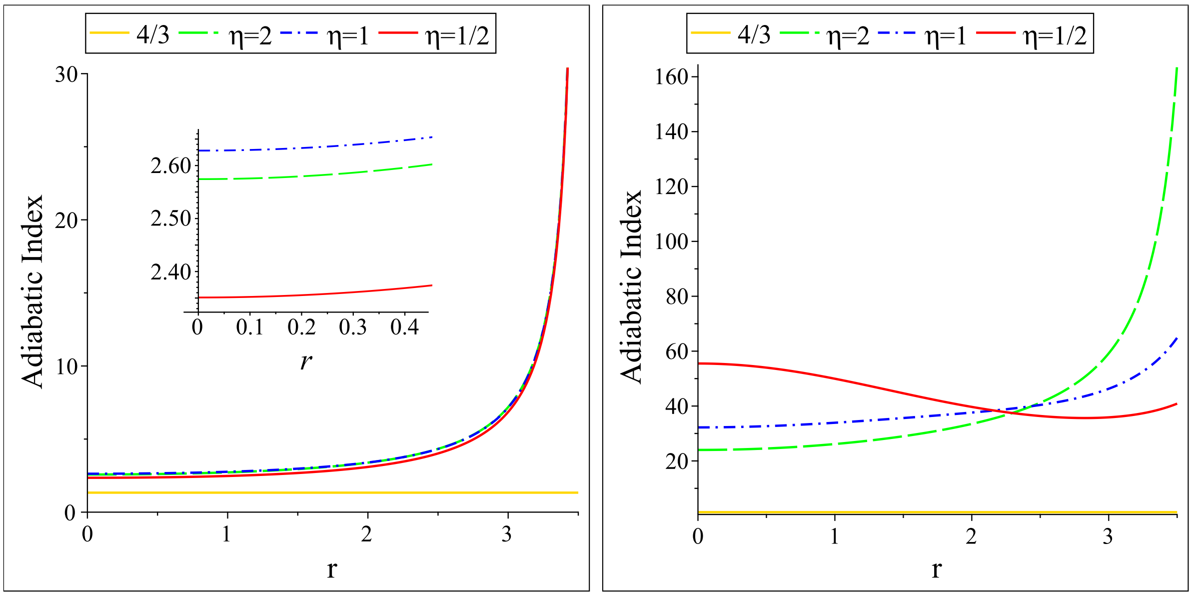
Figure 3.13: Graph of energy condition with respect to r

Adiabatic Index

Two specific heats ratio as explicated by Heintzmann and Hillebrandt [31] and Chan *et al.* [32] for the stable system is

$$\Gamma_r = \frac{\rho + p_r}{p_r}\frac{dp_r}{d\rho}, \tag{3.48}$$
$$\Gamma_t = \frac{\rho + p_t}{p_t}\frac{dp_t}{d\rho}.$$

The model is physically relevant if the value of Γ_i is greater than $4/3$. In Figure 3.14 the condition for all models are exhibited in which we can clearly observe that condition for stability within the relativistic fluid sphere ($\Gamma_i > 4/3$) is satisfied.



(a) Γ_r (b) Γ_t

Figure 3.14: Graph of adiabatic index with respect to r

Equilibrium State Under Various Forces

If the equation given by Tolman-Oppenheimer-Volkoff [33] & [34] is satisfied then the stellar system is said to be in the state of equilibrium. It is also known as TOV equation, given as

$$\frac{2}{r}(p_t - p_r) - \frac{dp_r}{dr} - (\rho + p_r)\nu' + \sigma E e^{\lambda} = 0. \tag{3.49}$$

Writing

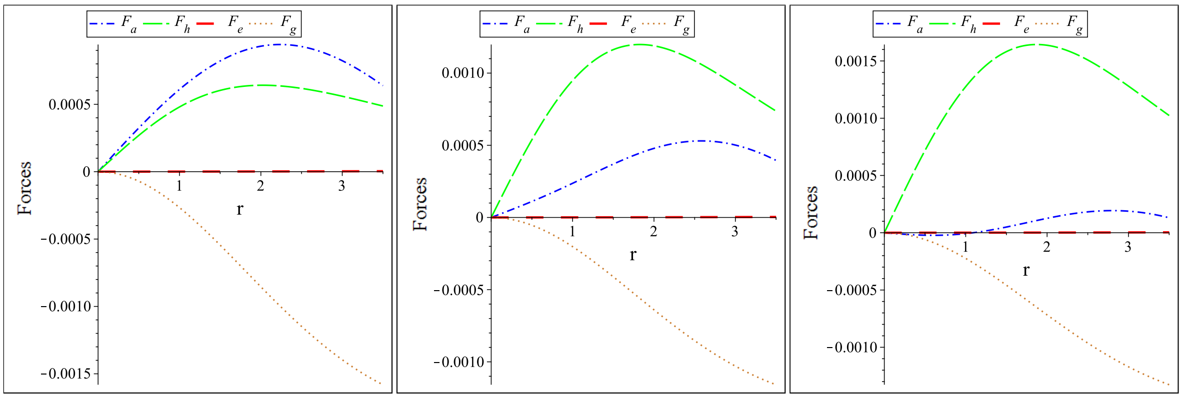
$$F_h = -\frac{dp_r}{dr},$$
$$F_g = -(\rho + p_r)\nu',$$

$$F_a = \frac{2}{r}(p_t - p_r),$$
$$F_e = \sigma E e^{\lambda},$$

thus equation (3.49) takes the form

$$F_h + F_g + F_a + F_e = 0, \tag{3.50}$$

where F_h, F_g, F_a and F_e are hydrostaic, gravitational, anisotropic and electric forces respectively. The profile of four different forces is shown in Figure 3.15 in which we can clearly see that by our presented solutions, the systems are stable and in static equilibrium since electric force is very small to create any impact. The hydrostatic and anisotropic forces are dominated by gravitational force which is negative.



(a) $\eta = 1/2$ (b) $\eta = 1$ (c) $\eta = 2$

Figure 3.15: Graph of different forces with respect to r

Chapter 4

Conclusion

In this thesis, at first the foundation of relativity is talked about from where the theory started and how it transformed over the years until it took its final shape by the genius mind of Einstein. Since the formulation of the famous Einsteins field equations, the work is being carried out to find the exact solutions. Some of the earlier presented solutions are discussed in detail in this thesis, the most important one being the Schwarzschild solution for point mass and the Reissner-Nordstrom for the charged point mass presented in 1916 and 1918 respectively. In a research carried out by Delgaty and Lake [35] only 16 solutions out of 127 passed the acceptability test and from these 16 only 9 solutions were seen satisfying the decreasing nature of the speed of sound from center to the boundary of the star. Moreover, in this thesis analysis of some exact solutions determined earlier using the polytropic equation of state ($p_r = k\rho^\Gamma$ & $p_r = k\rho^\Gamma - \beta$) are discussed precisely. These solutions fulfilled all acceptability criteria along with the decreasing nature of the speed of sound.

In Chapter 3, some new classes of solutions to the charged anisotropic relativistic stars on paraboloidal spacetime geometry using the generalized polytropic equation of state are deliberated. A suitable form of electric field intensity E^2 was introduced for this purpose which is finite at origin, if the constant ξ in it is 0 then the uncharged models can be obtained. The inner geometry of the star described by metric potentials has a singularity-free nature which gives a well-behaved form of energy density. Moreover, the tangential and radial pressure are free from central singularity, are positive inside the stellar interior, and there equal nature at the center leaves pressure anisotropy to be finite at the origin. The notable point here is that for $\eta = 1/2$ and 1, more significant compact models are developed since $p_t > p_r$, this is a piece of important factual evidence as it induces positive anisotropic factor as shown in Figure 3.4. According to Gokhroo and Mehra [36] a positive Δ allows to form more compact objects, in these models there exists the repulsive force which prevents the system from gravitational re-collapse. For $\eta = 2$ we have radial pressure dominating the tangential pressure over a small range in the interior of a star which depicts there exists an attractive force. Nevertheless, for each model all the physical and stability conditions are verified including compactification factor which is less than 4/9, redshift being less than 1 and adiabatic index greater than 4/3. Furthermore, the speed of sound in radial and transverse direction is less than one and the stability factor i.e. $0 < |v_t^2 - v_r^2| < 1$ is satisfied implying potential stability of models. Also, energy conditions and stability conditions using the Tolman-Oppenheimer-Volkoff equation are acceptable and have smooth variations in the stellar configuration.

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