On Stability Analysis of Fractional Difference Equations

By

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To my beloved Mother

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'In the name of Allah, the Most Gracious and the Most Merciful'

"Surely, Allah is with those who are As-Saabiroon (the patient)".

[Al Quran 6:46]

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Rabia Ilyas Butt

Abstract

In this dissertation, we discussed existence, uniqueness and stability of different types of fractional difference equations in delta and nabla sense.

Existence and stability results for a class of non-linear Caputo nabla fractional difference equations are obtained using fixed point theorems including Schauder's fixed point theorem, the Banach contraction principle and Krasnoselskii's fixed point theorem. Furthermore, results depends on the structure of nabla discrete Mittag-Leffler functions. Existence and uniqueness of solution for impulsive fractional difference equation is investigated through fixed point theorems including the Banach contraction principle, Schaefer's fixed point theorem and nonlinear alternative Leray Schauder theorem. Moreover, Ulam's type stability of problem in delta sense is discussed using newly developed Gronwall inequality. Using existing q-fractional Gronwall inequality, Ulam-Hyers stability and the Ulam-Hyers-Rassias stability is discussed for a delay Caputo q-fractional difference system.

Using newly developed Gronwall-Bellman inequality, we discussed Ulam-Hyers stability of Caputo nabla fractional difference system. Existence of solution of p-Laplacian fractional difference equations in nabla sense is discussed using Schaefer's fixed point theorem and then Ulam-Hyers stability is examined. Furthermore, we discussed Ulam-Hyers-Mittag-Leffler and Ulam-Hyers-Rassias-Mittag-Leffler stability for a class of Caputo nabla fractional order delay difference equation using Banach fixed point theorem in generalized complete metric space and using Chebyshev norm. Moreover, we obtained existence and stability results for a fractional difference Langevin equation within nabla Caputo fractional difference and subject to non-local boundary conditions using fixed point theorems.

Contents

List of Publications

1. Rabia Ilyas Butt, Thabet Abdeljawad, Manar A. Alqudah and Mujeeb ur Rehman, Ulam Stability of Caputo q-Fractional Delay Difference Equation: q-Fractional Gronwall Inequality Approach, Journal of Inequalities and Applications, 2019 (2019) 1–13.

2. Rabia Ilyas Butt, Thabet Abdeljawad and Mujeeb ur Rehman, Stability Analysis by Fixed Point Theorems for a Class of Non-Linear Caputo Nabla Fractional Difference Equation, Advances in Difference Equations, 2020 (2020) 1–11.

3. Rabia Ilyas Butt, Jehad Alzabut, Mujeeb ur Rehman and Jagan Mohan Jonnalagadda, On Fractional Difference Langevin Equations Involving Non-Local Boundary Conditions, Dynamic Systems and Applications, 29 (2020) 305–326.

4. Rabia Ilyas Butt, Mujeeb ur Rehman, Thabet Abdeljawad and Gülsen Kılınç, Stability Analysis of p-Laplacian Fractional Difference Equation, Dynamic Systems and Applications, 30 (2021) No.1, 17–31.

5. Rabia Ilyas Butt, Thabet Abdeljawad and Mujeeb ur Rehman, Discrete Version of Gronwall-Bellman Type Inequality and its Application in Stability Analysis, Dynamic Systems and Applications, 30 (2021) No.3, 449–459.

6. Rabia Ilyas Butt, Mujeeb ur Rehman, Ulam-Hyers-Mittag-Leffler Stability of Fractional Difference Equation with Delay, Rocky Mountain Journal of Mathematics, 51 (2021), No. 3, 891–901.

7. Rabia Ilyas Butt, Mujeeb ur Rehman, Ulam's Type Stability Analysis of Fractional Difference Equation with Impulse: Gronwall Inequality Approach (Submitted).

Chapter 1

Introduction

Fractional Calculus (FC) is a branch of mathematics that studies non-integer powers of derivatives and integrals. The historical backdrop of the hypothesis returns to seventeenth century, when in 1695 the derivative of order $n = 1/2$ was depicted by Leibniz in his letter to L'Hospital. From that point forward, the new hypothesis ended up being alluring to mathematicians just as physicists, researcher, engineers and financial analysts. By the late nineteenth century, the joined endeavors of various mathematicians, most strikingly Liouville, Grünwald, Letnikov furthermore, Riemann delivered a genuinely strong hypothesis of FC. Fractional Calculus has applications in biophysics, quantum mechanics, wave hypothesis, polymers, continuum mechanics, Lie hypothesis, field hypothesis, spectroscopy and in group theory, among different applications [77, 78, 100, 118].

Discrete fractional calculus manages sums and differences of arbitrary orders and this hypothesis intially started by Diaz and Osler [61]. They introduce a discrete fractional difference operator defined as an infinite series, a generalization of the binomial formula for the N^{th}-order difference operator Δ^N. Two fundamental approaches to manage discrete fractional calculus are using delta and nabla fractional difference approaches. The progression of the theory of nabla fractional difference calculus is ascribed to Gray and Zhang [74]. Miller et al. [116] defined a fractional sum of positive order via the solution of a linear difference equation and proved some basic properties of this operator. The math rising up out of this definition has gotten speaking to numerous creators and now it involves solid examination in different ways. Later, Atici and Eloe [32, 34] have turned out to be a portion of the fundamental operational properties of the discrete fractional calculus. From that point onward, few creators began to manage discrete fractional calculus [33, 36, 28, 27, 2, 3].

Numerical methods for solving fractional differential equations spurred the hypothetical advancement of Discrete Fractional Calculus. Real-time calculations based on available current and previous information force the application of the backward difference and sum [119]. There are several recent areas of specialized research in mathematical biology: Enzyme kinetics, biological tissue analysis, cancer modeling, heart and arterial disease modeling being among the popular ones. Gompertz fractional difference equation is used to model tumor growth [36]. Recently the discrete fractional calculus (DFC) began to acquire a lot of significance due to its applications to the mathematical modeling of real world phenomena with memory effect. The logistic equation is discretized by utilizing the DFC approach and the related discrete chaos is reported in [140, 141, 80].

This rising part of discrete fractional calculus (DFC) is still in its early stages and analyst are adding to its turn of events. At present, the research in this field is prevalently centered around the advancement of operational properties, existence theory and stability analysis.

Application of Discrete Fractional Calculus: An Example

Before we go to an itemized investigation of the mathematical properties of fractional difference operators and fractional difference equations, let us investigate a straightforward yet not ridiculous illustration of a model emerging in science where fractional differences can be utilized effectively.

The allure of discrete fractional calculus is that the fractional difference of a function relies upon its entire time history, and not on its immediate conduct. Such a characteristic of the fractional difference operator in biosciences is impeccably appropriate for the depiction of materials with memory. Improvement of further developed numerical models which depict and anticipate tumor development energy is a persistent investigation area for researchers, particularly for applied mathematicians.

The Gompertz differential equation [72] has the following form:

$$\varsigma^{'}(\eta) = \big(d - e\ ln\varsigma(\eta)\big)\varsigma(\eta),$$

where d and e are parameters. The solution of this equation acts as a sigmoidal curve which increases exponentially first then followed by a direct sluggish increment. We divide each side of the above equation by $\varsigma(\eta)$ and then using the substitution $\psi(\eta) := ln\varsigma(\eta)$, we get the following differential equation

$$\psi^{'}(\eta) = d - e\ \psi(\eta).$$

To obtain the first order Gompertz difference equation, replace first order derivative by Δ

$$\Delta\psi(\eta) = d - e\ \psi(\eta).$$

Later, to obtain fractional difference equation the first order difference operator Δ was replaced by discrete fractional operator of ζ-th order $(0 < \zeta < 1)$. This is the general idea for fractionalizing the given first order differential or difference equation. In this fractionalizing process, one must be exceptionally cautious with not losing the subjective and actual importance of the condition. This has been discussed in the paper by Magin et al. [62].

The solution, which is known as sigmoidal curve, of the above differential equation has been broadly utilized in clinical exploration to depict and to anticipate tumor development. In this sense, $\varsigma(\eta)$ represents the tumor volume at time η, $e > 0$ is the growth deceleration factor and the parameter $d > 0$ is the intrinsic growth rate of the tumor.

1.1 Special Functions

In this Section, we will discuss about two basic special functions that will be helpful throughout our dissertation. One of them is Gamma function and other one is Mittag Leffler function.

1.1.1 Gamma Function

The gamma work was first presented by the Swiss mathematician Leonhard Euler $(1707-1783)$ in his objective to generalize the factorial to non integer values. Afterward, due to its incredible significance, it was concentrated by other prominent mathematicians like Legendre, Gauss, Gudermann, Liouville, Weierstrass and Hermite, just as numerous others.

The gamma function belongs to the category of the special transcendental functions and some famous mathematical constants are occurring in its study. It additionally shows up in different regions as asymptotic arrangement, definite integration, hypergeometric arrangement, Riemann zeta work, number hypothesis, and so on.

The gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-\eta}\eta^{z-1}d\eta,$$

for those complex numbers z for which the real part of z is positive.

Integration by parts implies

$$\Gamma(z+1) = z\Gamma(z), \quad z \neq 0, -1, -2, ...$$

It follows that

$$\lim_{z\to -n} |\Gamma(z)| = \infty \quad n = 0, 1, 2, ...$$

Another important property of gamma function due to which it is called generalization of factorial function is

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, ...$$

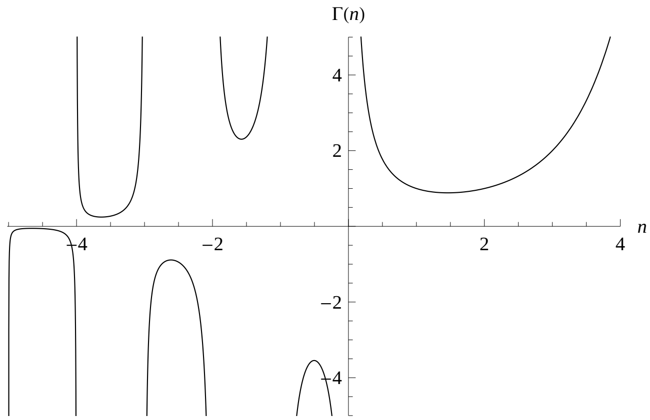

Figure 1.1: Gamma Function

1.1.2 Mittag-Leffler Function

Gosta Mittag Leffler (1846 – 1927) introduced a function as a generalization of exponential function, known as Mittag-Leffler function. The one parameter Mittag-Leffler function $E_{\alpha,1}$ is defined by the power series

$$E_{\alpha,1} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad \alpha \in \mathbb{C}, Re(\alpha) > 0.$$

In 1905, Wiman introduced the generalization of one parameter Mittag-Leffler function. The two parameter Mittag-Leffler function $E_{\alpha,\beta}$ is defined as

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0.$$

Discrete Mittag Leffler function

In 2003, Nagai [117] discussed about fractional difference operator and fisrt of all he gave the concept of discrete Mittag-Leffler function as an eigen function of defined difference operator. After long time, in 2011, Abdeljawad first of all discussed about Delta discrete Mittag-Leffler function as a solution of Delta Caputo fractional difference equation. In the same year, Atici worked on Nabla discrete Mittag-Leffler function as a solution of Nabla Reimann-Liouville fractional difference equation. Then, in 2013, Nabla discrete Mittag-Leffler function as a solution of Caputo Nabla fractional difference equation was discussed by Abdeljawad.

Abdeljawad [2] introduced two parameter Delta discrete Mittag-Leffler function which is defined as follows, for $\lambda \in \mathbb{R},\ \alpha, \beta, z \in \mathbb{C},\ Re(\alpha) > 0$

$$E_{\underline{\alpha,\beta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + (k-1)(\alpha-1))^{\underline{k\alpha}}(z + k(\alpha-1))^{\underline{\beta-1}}}{\Gamma(\alpha k + \beta)}, \tag{1.1}$$

which is genearalization of one parameter Mittag-Leffler function,

$$E_{\underline{\alpha}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + (k-1)(\alpha-1))^{\underline{k\alpha}}}{\Gamma(\alpha k + 1)}. \tag{1.2}$$

Abdeljawad [3] solved nabla Caputo type non-homogeneous fractional difference equation to formulate Nabla type discrete Mittag-Leffler functions.

For $\lambda \in \mathbb{R},\ \alpha, \beta, z \in \mathbb{C},\ Re(\alpha) > 0$, two parameter Nabla discrete Mittag-Leffler function is defined as,

$$E_{\overline{\alpha,\beta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{\alpha k + \beta - 1}}}{\Gamma(\alpha k + \beta)}, \tag{1.3}$$

which is generalization of one parameter Nabla discrete Mittag-Leffler function

$$E_{\overline{\alpha}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{z^{\overline{\alpha k}}}{\Gamma(\alpha k + 1)}. \tag{1.4}$$

In 2011, Abdeljawad et al. [10] considered a Caputo-type q-fractional initial value problem and expressed its solution by means of a new introduced q-Mittag–Leffler function.

Let z, z_0 are complex numbers with $Re(\alpha) > 0$, then q-Mittag–Leffler function is defined by

$${}_qE_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z - z_0)^{\alpha k}}{\Gamma_q(\alpha k + \beta)}. \tag{1.5}$$

In 2011, Čermák et al. [52] introduced (q, h)-Mittag-Leffler function, when they were solving discrete fractional difference equation in suitable domain.

Let $\alpha, \beta, \lambda \in \mathbb{R}$. (q,h)-Mittag-Leffler function is defined as

$$E^{s,\lambda}_{\alpha,\beta}(\eta) = \sum_{k=0}^{\infty} \lambda^k \frac{(\eta - s)^{(\alpha k+\beta-1)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(\alpha k + \beta)}, \quad s, \eta \in \tilde{\mathbb{T}}^a_{(q,h)}, \eta \geq s. \tag{1.6}$$

1.2 Basic Definitions

In this section, we present some fundamental documentations, definitions and lemmas that are useful in demonstrating our primary outcomes.

1.2.1 Basic Difference Calculus

In this dissertation, we will frequently consider functions defined on sets $\mathbb{N}_a$ or $\mathbb{N}^b_a$, where

$$\mathbb{N}_a := \{a, a+1, a+2, ...\}, \quad a \in \mathbb{R},$$

$$\mathbb{N}^b_a := \{a, a+1, ..., b\}, \quad a, b \in \mathbb{R}, b - a \in \mathbb{Z}^+.$$

Definition 1.2.1. *For a function $\mathscr{H}$ defined on $\mathbb{N}^b_a$ with $b > a$, we define* ***delta operator (forward difference operator)*** Δ *as*

$$\Delta\mathscr{H}(\eta) = \mathscr{H}(\eta + 1) - \mathscr{H}(\eta), \quad \eta \in \mathbb{N}^{b-1}_a.$$

$(\Delta\mathscr{H})(\eta)$ means $(\Delta\mathscr{H})$ is a function being evaluated at the point η. We will use $\Delta\mathscr{H}(\eta)$ instead of $(\Delta\mathscr{H})(\eta)$ throughout this dissertation to simplify the notation.

Definition 1.2.2. ***Forward jump operator*** σ *is represented as*

$$\sigma(\eta) = \eta + 1, \quad \eta \in \mathbb{N}^{b-1}_a.$$

Definition 1.2.3. *For n being positive integer, **falling function** is represented as*

$$\eta^{\underline{n}} := \eta(\eta-1)(\eta-2)(\eta-3)...(\eta-(n-1)), \quad \eta^{\underline{0}} = 1.$$

We can write above expression as

$$\begin{aligned}\eta^{\underline{n}} &= \eta(\eta-1)(\eta-2)(\eta-3)...(\eta-(n-1)) \\ &= \frac{\eta(\eta-1)(\eta-2)(\eta-3)...(\eta-(n-1))\Gamma(\eta-(n-1))}{\Gamma(\eta-(n-1))} \\ &= \frac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)}.\end{aligned}$$

*Motivated by above definition, **generalized falling function** is defined as*

$$r^{\underline{\xi}} := \frac{\Gamma(r+1)}{\Gamma(r-\xi+1)},$$

for those estimations of r and ξ with the end goal that right-hand side of this equation bodes well.

Definition 1.2.4. *For a function $\mathscr{H}$ defined on $\mathbb{N}_a$, **nabla operator (backward difference operator)** ∇ is represented by*

$$\nabla\mathscr{H}(\eta) = \mathscr{H}(\eta) - \mathscr{H}(\eta-1), \quad \eta \in \mathbb{N}_{a+1}.$$

Definition 1.2.5. ***Backward jump operator** ϱ defined on $\mathbb{N}_{a+1}$ is given by*

$$\varrho(\eta) = \eta - 1.$$

Definition 1.2.6. *For n being positive integer and $\eta \in \mathbb{R}$, **rising function** $\eta^{\overline{n}}$ is represented as*

$$\eta^{\overline{n}} := \eta(\eta+1)(\eta+2)...(\eta+n-1).$$

For $n \in \mathbb{N}_1$, we can write above expression as follows

$$\begin{aligned}
\eta^{\overline{n}} &= \eta(\eta+1)(\eta+2)...(\eta+n-1) \\
&= (\eta+n-1)...(\eta+2)(\eta+1)\eta \\
&= \frac{(\eta+n-1)...(\eta+2)(\eta+1)\eta\Gamma(\eta)}{\Gamma(\eta)} \\
&= \frac{\Gamma(\eta+n)}{\Gamma(\eta)}, \quad \eta \notin \{0,-1,-2,...\}.
\end{aligned}$$

Motivated by above definition, we can define ***generalized rising function*** *as follows:*

$$r^{\overline{\xi}} := \frac{\Gamma(r+\xi)}{\Gamma(r)},$$

for those estimations of r and ξ with the end goal that right-hand side of this equation bodes well.

1.2.2 Discrete Delta Fractional Calculus

In this subsection, we present some fundamental definitions and lemmas for the motivation behind acquainting with essential discrete fractional calculus hypothesis in the delta sense [79, 2, 3, 73, 142].

Definition 1.2.7. *[79] Let $\mathscr{H} : \mathbb{N}_a \to \mathbb{R}$ and $0 < \zeta,\ a \in \mathbb{R}$. The ζ-th order sum is given by*

$$\Delta_a^{-\zeta}\mathscr{H}(\eta) := \frac{1}{\Gamma(\zeta)}\sum_{s=a}^{\eta-\zeta}(\eta-\sigma(s))^{\underline{\zeta-1}}\mathscr{H}(s), \quad \eta \in \mathbb{N}_{a+\zeta}.$$

Definition 1.2.8. *[79] Let $\mathscr{H}$ be defined on $\mathbb{N}_a$ and ζ being positive real number. Choose positive integer N such that $N-1 < \zeta \leq N$, then the Riemann-Liouville difference is provided by*

$$\Delta_a^{\zeta}\mathscr{H}(\eta) := \Delta^N\Delta_a^{-(N-\zeta)}\mathscr{H}(\eta),\ \eta \in \mathbb{N}_{a+N-\zeta},\ N = [\zeta]+1.$$

For $0 < \zeta \leq 1$, we have following

$$\Delta_a^{\zeta}\mathscr{H}(\eta) := \frac{1}{\Gamma(-\zeta)}\sum_{s=a}^{\eta+\zeta}(\eta - \sigma(s))^{\underline{-\zeta-1}}\mathscr{H}(s),\ \eta \in \mathbb{N}_{a+1-\zeta}.$$

Definition 1.2.9. *Let $\mathscr{H}$ be defined on $\mathbb{N}_a$ and $0 < \zeta,\ \zeta \notin \mathbb{N}$, the Caputo fractional difference is given by*

$$\begin{aligned}{}^{c}\Delta_a^{\zeta}\mathscr{H}(\eta) &= \Delta_a^{-(m-\zeta)}\ \Delta^m\mathscr{H}(\eta)\\ &= \frac{1}{\Gamma(m-\zeta)}\sum_{s=a}^{\eta-m+\zeta}(\eta - \sigma(s))^{\underline{m-\zeta-1}}\Delta^m\mathscr{H}(s), \quad \eta \in \mathbb{N}_{a+m-\zeta},\ m = [\zeta]+1.\end{aligned}$$

For $\zeta = m$, ${}^{c}\Delta_a^{\zeta}\mathscr{H}(\eta) := \Delta^m\mathscr{H}(\eta)$.

Definition 1.2.10. *For $0 < \zeta \leq 1$, Discrete Leibniz integral law is defined as:*

$$\Delta_{a+1-\zeta}^{-\zeta}\ {}^{c}\Delta_a^{\zeta}\mathscr{H}(\eta) = \mathscr{H}(\eta) - \mathscr{H}(a),\ \eta \in \mathbb{N}_{a+1}.$$

Lemma 1.2.11. *[2] Let $\zeta > 0$ and $\mathscr{H}$ is defined on $\mathbb{N}_a$. Then*

$$\Delta_{a+n-\zeta}^{-\zeta}\ {}^{c}\Delta_a^{\zeta}\mathscr{H}(\eta) = \mathscr{H}(\eta) - \sum_{k=0}^{n-1}\frac{(\eta-a)^{\underline{k}}}{k!}\Delta^k\mathscr{H}(a), \quad n = [\,\zeta\,]+1,$$

n being smallest integer greater than or equal to ζ.

Lemma 1.2.12. *[112] Let k be a positive integer. Then*

$$(\eta - a)^{\underline{k}} = \sum_{q=0}^{k}\binom{k}{q}(-1)^q a^{\overline{q}}\eta^{\underline{k-q}}, \quad \eta \in \mathbb{N}_a. \tag{1.7}$$

Proof. To prove our statement, we will use mathematical induction.

For $k = 1$, Eq. (1.7) holds.

Let us assume that Eq. (1.7) holds for $k = p$, that is

$$(\eta - a)^{\underline{p}} = \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p-q}}, \quad \eta \in \mathbb{N}_a.$$

Now we will prove for $k = p + 1$,

$$\begin{aligned}
(\eta - a)^{\underline{p+1}} &= (\eta - a)^{\underline{p}}(\eta - a - p) \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p-q}}(\eta - a - p) \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p-q}}(\eta - p + q) - \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p-q}}(a + q) \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p+1-q}} - \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\overline{q+1}} \eta^{\underline{p-q}} \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p+1-q}} + \sum_{q=0}^{p} \binom{p}{q} (-1)^{q+1} a^{\overline{q+1}} \eta^{\underline{p-q}} \\
&= \eta^{\underline{p+1}} + \sum_{q=1}^{p} \binom{p}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p+1-q}} + \sum_{q=1}^{p} \binom{p}{q-1} (-1)^q a^{\overline{q}} \eta^{\underline{p+1-q}} + (-1)^{p+1} a^{\overline{p+1}} \\
&= \sum_{q=0}^{p+1} \binom{p+1}{q} (-1)^q a^{\overline{q}} \eta^{\underline{p+1-q}}, \quad \eta \in \mathbb{N}_a.
\end{aligned}$$

Therefore, Eq. (1.7) holds for $k = p + 1$. Hence Eq. (1.7) holds for any positive integer. □

Lemma 1.2.13. *[112] Let for $\zeta > 0$ and $\mathscr{H}$ defined on $\mathbb{N}_a$. Then*

$$\Delta_{a+n-\zeta}^{-\zeta}\ {}^{c}\Delta_a^{\zeta} \mathscr{H}(\eta) = \mathscr{H}(\eta) + c_0 + c_1\eta + ... + c_{n-1}\eta^{\underline{n-1}},$$

where all coefficients are real numbers and n is the smallest integer greater than or equal to ζ.

Lemma 1.2.14. ***(Power Rules)***

(i) Let us suppose that $\mu \geq 0$ and $\zeta > 0$. Then fractional sum power rule is defined as

$$\Delta_{a+\mu}^{-\zeta}(\eta - a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\zeta+1)}(\eta - a)^{\underline{\mu+\zeta}}, \quad \eta \in \mathbb{N}_{a+\mu+\zeta}.$$

(ii) Let us suppose that $\mu > 0$ and $\zeta \geq 0$, $N-1 < \zeta < N$. Then fractional difference power rule is defined as

$$\Delta_{a+\mu}^{\zeta}(\eta - a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\zeta+1)}(\eta - a)^{\underline{\mu-\zeta}}, \quad \eta \in \mathbb{N}_{a+\mu+N-\zeta}.$$

Theorem 1.2.1. ***(Composition Rules)***

(i) Assume that $\mathscr{H}$ is defined on $\mathbb{N}_a$ and μ, ζ are positive numbers. Then composition of fractional sums is defined as

$$\left[\Delta_{a+\zeta}^{-\mu}(\Delta_a^{-\zeta}\mathscr{H})\right](\eta) = (\Delta_a^{-(\mu+\zeta)}\mathscr{H})(\eta) = \left[\Delta_{a+\mu}^{-\zeta}(\Delta_a^{-\mu}\mathscr{H})\right](\eta), \quad \eta \in \mathbb{N}_{a+\mu+\zeta}.$$

(ii) Assume that $\mathscr{H}$ is defined on $\mathbb{N}_a$ and μ, ζ are positive numbers and $N-1 < \zeta \leq N$, $N \in \mathbb{N}_1$. Then composition of fractional difference with fractional sum is defined as

$$\Delta_{a+\mu}^{\zeta}\Delta_a^{-\mu}\mathscr{H}(\eta) = \Delta_a^{\zeta-\mu}\mathscr{H}(\eta), \quad \eta \in \mathbb{N}_{a+\mu+N-\zeta}.$$

(iii) Assume that $\mathscr{H}$ is defined on $\mathbb{N}_a$ and μ, ζ are positive numbers with $\zeta = N$ and $M-1 < \mu \leq M$, then composition of two fractional differences is defined as

$$\Delta_{a+M-\mu}^{\zeta}\Delta_a^{\mu}\mathscr{H}(\eta) = \Delta_a^{\zeta+\mu}\mathscr{H}(\eta), \quad \eta \in \mathbb{N}_{a+M-\mu}.$$

1.2.3 Discrete Nabla Fractional Calculus

In this subsection, we present some fundamental definitions and lemmas for the motivation behind acquainting with essential discrete fractional calculus hypothesis in the nabla sense [2, 3, 73].

Definition 1.2.15. *Let $\mathscr{H}$ be defined on $\mathbb{N}_{a+1}$ and $\zeta > 0$, the nabla fractional sum is defined as*

$$\nabla_a^{-\zeta}\mathscr{H}(\eta) = \frac{1}{\Gamma(\zeta)} \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \mathscr{H}(s), \quad \eta \in \mathbb{N}_a,$$

whereas ∇_a^{-0} is taken as the identity operator.

Definition 1.2.16. *Let $\mathscr{H}$ be defined on $\mathbb{N}_a$, ζ being positive real number. Choose a natural number n such that $n-1 < \zeta < n$, then the ζ-th order nabla fractional difference is defined as*

$$\nabla_a^{\zeta}\mathscr{H}(\eta) = \frac{1}{\Gamma(-\zeta)} \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{-\zeta-1}} \mathscr{H}(s), \quad \eta \in \mathbb{N}_{a+1},$$

whereas ∇_a^{0} is taken as the identity operator.

Definition 1.2.17. *Let $\mathscr{H}$ be defined on $\mathbb{N}_{a-n+1}$. The Caputo nabla fractional difference of $\mathscr{H}$ of order $\zeta > 0$ is given by*

$${}^{c}\nabla_a^{\zeta}\mathscr{H}(\eta) = \nabla_a^{-(n-\zeta)} \left[\nabla^n \mathscr{H}(\eta)\right], \quad \eta \in \mathbb{N}_{a+1}, \quad n = \lceil \zeta \rceil.$$

Whereas ${}^{c}\nabla_a^{0}$ is taken as the identity operator.

Lemma 1.2.18. *Let $\zeta > 0$ and $\mathscr{H}$ is defined on $\mathbb{N}_a$. Then*

$$\nabla_a^{-\zeta}\, {}^{c}\nabla_a^{\zeta}\mathscr{H}(\eta) = \mathscr{H}(\eta) - \sum_{k=0}^{n-1} \frac{(\eta - a)^{\overline{k}}}{k!} \nabla^k \mathscr{H}(a),$$

n being smallest integer greater than or equal to ζ.

Lemma 1.2.19. *Let k be a positive integer. Then*

$$(\eta - a)^{\overline{k}} = \sum_{q=0}^{k} \binom{k}{q} (-1)^q a^{\underline{q}} \eta^{\overline{k-q}}, \quad \eta \in \mathbb{N}_a. \tag{1.8}$$

Proof. We will use mathematical induction to prove the statement.

For $k = 1$, Eq. (1.8) is the same as $(\eta - a)^{\overline{1}} = \eta - a$.

Let us assume that Eq. (1.8) is true for $k = p$, that is

$$(\eta - a)^{\overline{p}} = \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p-q}}, \quad \eta \in \mathbb{N}_a.$$

Now we will prove for $k = p + 1$,

$$\begin{aligned}
(\eta - a)^{\overline{p+1}} &= (\eta - a)^{\overline{p}}(\eta - a + p) \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p-q}} (\eta - a + p) \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p-q}} (\eta + p - q) - \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p-q}} (a - q) \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p+1-q}} - \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\underline{q+1}} \eta^{\overline{p-q}} \\
&= \sum_{q=0}^{p} \binom{p}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p+1-q}} + \sum_{q=0}^{p} \binom{p}{q} (-1)^{q+1} a^{\underline{q+1}} \eta^{\overline{p-q}} \\
&= \eta^{\overline{p+1}} + \sum_{q=1}^{p} \binom{p}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p+1-q}} + \sum_{q=1}^{p} \binom{p}{q-1} (-1)^q a^{\underline{q}} \eta^{\overline{p+1-q}} + (-1)^{p+1} a^{\underline{p+1}} \\
&= \sum_{q=0}^{p+1} \binom{p+1}{q} (-1)^q a^{\underline{q}} \eta^{\overline{p+1-q}}, \quad \eta \in \mathbb{N}_a.
\end{aligned}$$

Therefore Eq. (1.8) holds for $k = p + 1$. Hence Eq. (1.8) holds for any positive integer. □

Lemma 1.2.20. *Let for $\zeta > 0$ and $\mathscr{H}$ defined on $\mathbb{N}_a$. Then*

$$\nabla_a^{-\zeta}\, {}^c\nabla_a^{\zeta} \mathscr{H}(\eta) = \mathscr{H}(\eta) + c_0 + c_1\eta + ... + c_{n-1}\eta^{\overline{n-1}},$$

where all coefficients are real numbers and n is the smallest integer greater than or equal to ζ.

Lemma 1.2.21. *(Power Rules) Let ζ being positive real number and $\nu \in \mathbb{R}$ such that $\nu, \nu + \zeta, \nu - \zeta$ are nonnegative integers. Then for $\eta \in \mathbb{N}_a$, generalized power rules are given as*
(i) $\nabla_a^{-\zeta}(\eta - a)^{\overline{\nu}} = \dfrac{\Gamma(\nu + 1)}{\Gamma(\nu + \zeta + 1)}(\eta - a)^{\overline{\nu+\zeta}}$,
(ii) $\nabla_a^{\zeta}(\eta - a)^{\overline{\nu}} = \dfrac{\Gamma(\nu + 1)}{\Gamma(\nu - \zeta + 1)}(\eta - a)^{\overline{\nu-\zeta}}$.

Theorem 1.2.2. *(Composition Rules)*

(i) Let us suppose that $\mathscr{H}$ *is defined on* $\mathbb{N}_{a+1}$ *and* $\zeta, \mu > 0$*. Then composition of nabla fractional sums is defined as*

$$\nabla_a^{-\zeta} \nabla_a^{-\mu} \mathscr{H}(\eta) = \nabla_a^{-\zeta-\mu} \mathscr{H}(\eta), \quad \eta \in \mathbb{N}_a.$$

(ii) Let us suppose that $\mathscr{H}$ *is defined on* $\mathbb{N}_a$ *and* $\zeta, \mu > 0$ *with* $n-1 < \zeta \leq n$*, where* n *is a natural number. Then composition of nabla fractional difference with sum is defined as*

$$\nabla_a^{\zeta} \nabla_a^{-\mu} \mathscr{H}(\eta) = \nabla_a^{\zeta-\mu} \mathscr{H}(\eta), \quad \eta \in \mathbb{N}_a.$$

(iii) Replacing μ *by* $-\mu$ *in (ii), we get*

$$\nabla_a^{\zeta} \nabla_a^{\mu} \mathscr{H}(\eta) = \nabla_a^{\zeta+\mu} \mathscr{H}(\eta), \quad \eta \in \mathbb{N}_a.$$

1.2.4 q-Calculus

In this subsection, we provide some basic concepts of q-fractional calculus that are essential to proving our main results. For more details on the theory of q-calculus and q-fractional calculus, we refer to [39, 31, 10, 7] (and references therein). For more remarkable basic articles in q-fractional calculus, we refer to [17, 16, 18, 13, 123]. The book [29] is also recommended for readers.

$\mathbb{T}_q$ is defined as $\mathbb{T}_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $q \in (0,1)$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$. If $n_0 \in \mathbb{Z}$ and $a = q^{n_0}$, $\mathbb{T}_a$ can be written as $\mathbb{T}_a = [a, \infty)_q = \{q^{-i}a : i = 0, 1, 2, ...\}$. Further define $\mathbb{I}_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, ..., a\}$ and $\mathbb{T}_{\tau a} = [\tau a, \infty)_q = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, ...\}$ where $\tau = q^d \in \mathbb{T}_q$, $d \in \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ and $\mathbb{I}_\tau = \{a\}$ with $d = 0$ is the non-delay case.

Definition 1.2.22. *For* $\mathscr{H}$ *defined on* $\mathbb{T}_q$*, the nabla* q*-derivative of* $\mathscr{H}$ *is defined as*

$$\nabla_q \mathscr{H}(\eta) = \frac{\mathscr{H}(\eta) - \mathscr{H}(q\eta)}{(1-q)\eta}, \quad \eta \in \mathbb{T}_q - \{0\}.$$

Definition 1.2.23. *For $\mathscr{H} : \mathbb{T}_q \to \mathbb{R}$, the nabla q-integral of $\mathscr{H}$ is defined as*

$$\int_0^\eta \mathscr{H}(s)\nabla_q s = (1-q)\eta \sum_{i=0}^{\infty} q^i \mathscr{H}(\eta q^i).$$

For $0 \leq a \in \mathbb{T}_q$,

$$\int_a^\eta \mathscr{H}(s)\nabla_q s = \int_0^\eta \mathscr{H}(s)\nabla_q s - \int_0^a \mathscr{H}(s)\nabla_q s.$$

Definition 1.2.24. *For $\zeta \in \mathbb{R}$, the nabla q-derivative of the q-factorial function*

(i) with respect to η is given by

$$\nabla_q (\eta - s)_q^\zeta = \frac{1-q^\zeta}{1-q}(\eta - s)_q^{\zeta-1},$$

(ii) with respect to s is given by

$$\nabla_q (\eta - s)_q^\zeta = -\frac{1-q^\zeta}{1-q}(\eta - qs)_q^{\zeta-1}.$$

Definition 1.2.25. *Left q-fractional integral of order ζ for function $\mathscr{H}$ defined on $\mathbb{T}_q$ is defined as*

$${}_q\nabla_a^{-\zeta}\mathscr{H}(\eta) = \frac{1}{\Gamma_q(\zeta)}\int_a^\eta (\eta - qs)_q^{\zeta-1}\mathscr{H}(s)\nabla_q s,$$

where $\zeta \neq 0, -1, -2, -3, ...,$ and

$$\Gamma_q(\zeta+1) = \frac{1-q^\zeta}{1-q}\Gamma_q(\zeta), \quad \Gamma_q(1) = 1,\ \zeta > 0.$$

Definition 1.2.26. *Let $\mathscr{H} : \mathbb{T}_q \to \mathbb{R}$, where $0 < \zeta \notin \mathbb{N}$. Then the Caputo left q-fractional derivative of order ζ is given by*

$${}_q^c\nabla_a^\zeta \mathscr{H}(\eta) = {}_q\nabla_a^{-(n-\zeta)}\nabla_q^n \mathscr{H}(\eta) = \frac{1}{\Gamma_q(n-\zeta)}\int_a^\eta (\eta - qs)_q^{n-\zeta-1}\nabla_q^n\mathscr{H}(s)\nabla_q s,$$

where $n = [\zeta] + 1$.

Lemma 1.2.27. *[10] For $\zeta > 0$ and $\mathscr{H}$ defined on suitable domain, we have*

$$ {}_q\nabla_a^{-\zeta} \; {}_q^c\nabla_a^{\zeta}\mathscr{H}(\eta) = \mathscr{H}(\eta) - \sum_{k=0}^{n-1} \frac{(\eta - a)_q^k}{\Gamma_q(k+1)} \nabla_q^k \mathscr{H}(a). $$

In particular for $\zeta \in (0, 1]$, we have

$$ {}_q\nabla_a^{-\zeta} \; {}_q^c\nabla_a^{\zeta}\mathscr{H}(\eta) = \mathscr{H}(\eta) - \mathscr{H}(a). $$

Lemma 1.2.28. *[10] For $\zeta \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$, we have*

$$ {}_q\nabla_a^{-\zeta}(\psi - a)_q^{\mu} = \frac{\Gamma_q(\mu + 1)}{\Gamma_q(\zeta + \mu + 1)}(\psi - a)_q^{\mu+\zeta}, \quad 0 < a < \psi < b. $$

1.2.5 Discrete Fractional Operators with Discrete Mittag-Leffler Kernel

A class of fractional difference operators with discrete Mittag-Leffler kernel is a new topic to be explored.

First of all we will provide ABC fractional derivative and integral.

Definition 1.2.29. *The Atangana-Baleanu fractional derivative with variable-order ζ in Liouville-Caputo sense (ABC) is defined as*

$$ ({}_0^{ABC}D_\eta^{\zeta}\mathscr{H})(\eta) = \frac{B(\zeta)}{1-\zeta} \int_0^{\eta} E_{\zeta}\left(\frac{-\zeta(\eta - \tau)^{\zeta}}{1-\zeta} \right) \mathscr{H}(\tau) d\tau \qquad 0 < \zeta \leq 1. $$

Besides, the fractional integral of Atangana-Baleanu of order ζ is defined as

$$ ({}_0^{ABC}J_t^{\zeta}\mathscr{H})(\eta) = \frac{1-\zeta}{B(\zeta)}\mathscr{H}(\eta) + \frac{\zeta}{\Gamma(\zeta)B(\zeta)} \int_0^{\eta} \mathscr{H}(\tau)(\eta - \tau)^{\zeta - 1} d\tau \qquad 0 < \zeta \leq 1. $$

where $B(\zeta) = 1 - \zeta + \dfrac{\zeta}{\Gamma(\zeta)}$ is a normalization function with $B(0) = B(1) = 1$.

Definition 1.2.30. *Let $\mathscr{H}$ be defined on $\mathbb{N}_a \cap {}_b\mathbb{N}$ with $a < b$. $b \equiv a(\mod 1)$, and $\zeta \in (0, 1/2)$,* *then the ABC nabla discrete new (left Caputo) fractional difference is*

$$\begin{aligned}({}_{a}^{ABC}\nabla^{\zeta}\mathscr{H})(\eta) &= \frac{B(\zeta)}{1-\zeta}\sum_{s=a+1}^{\eta}\nabla_s\mathscr{H}(s)E_{\overline{\zeta}}\left(\frac{-\zeta}{1-\zeta},\eta-\varrho(s)\right)\\ &= \frac{B(\zeta)}{1-\zeta}\left[\nabla\mathscr{H}(\eta)*E_{\overline{\zeta}}\left(\frac{-\zeta}{1-\zeta},\eta\right)\right].\end{aligned}$$

We have the following expression in the left Riemann sense:

$$\begin{aligned}({}_{a}^{ABR}\nabla^{\zeta}\mathscr{H})(\eta) &= \frac{B(\zeta)}{1-\zeta}\nabla_{\eta}\sum_{s=a+1}^{\eta}\mathscr{H}(s)E_{\overline{\zeta}}\left(\frac{-\zeta}{1-\zeta},\eta-\varrho(s)\right)\\ &= \frac{B(\zeta)}{1-\zeta}\nabla_{\eta}\left[\mathscr{H}(\eta)*E_{\overline{\zeta}}\left(\frac{-\zeta}{1-\zeta},\eta\right)\right],\end{aligned}$$

where $B(\zeta)$ is a normalization constant with $B(0) = B(1) = 1$.

Definition 1.2.31. *The fractional sum associate to $({}_{a}^{ABR}\nabla^{\zeta}\mathscr{H})(\eta)$ with order $0 < \zeta < 1$ is defined by*

$$({}_{a}^{AB}\nabla^{-\zeta}\mathscr{H})(\eta) = \frac{1-\zeta}{B(\zeta)}\mathscr{H}(\eta) + \frac{\zeta}{B(\zeta)}(\nabla_a^{-\zeta}\mathscr{H})(\eta).$$

Definition 1.2.32. *The relation between the Caputo and Riemann fractional differences with Mittag-Leffler kernel is*

$$({}_{a}^{ABC}\nabla^{\zeta}\mathscr{H})(\eta) = ({}_{a}^{ABR}\nabla^{\zeta}\mathscr{H})(\eta) - \mathscr{H}(a)\frac{B(\zeta)}{1-\zeta}E_{\overline{\zeta}}(\lambda,\eta-a).$$

Proposition 1. *Let $0 < \zeta < 1$, we have*

$$\begin{aligned}\left({}_{a}^{AB}\nabla^{-\zeta}\,{}_{a}^{ABC}\nabla^{\zeta}\mathscr{H}\right)(\eta) =& \mathscr{H}(\eta) - \mathscr{H}(a)E_{\overline{\zeta}}(\lambda,\eta-a) - \frac{\zeta}{1-\zeta}\mathscr{H}(a)E_{\overline{\zeta,\zeta+1}}(\lambda,\eta-a)\\ =& \mathscr{H}(\eta) - \mathscr{H}(a).\end{aligned}$$

1.3 Results from Analysis

This section contains some significant outcomes from literature to encourage the peruser for better comprehension of this work.

Definition 1.3.1. *[124] Any subset of sequences in l_0^∞ is called uniformly Cauchy (or equi Cauchy) if for every $\epsilon > 0$, we have an integer N such that for any sequence $\psi = \{\psi(n)\}$ and $i, j > N$, we must have $|\psi(i) - \psi(j)| < \epsilon$.*

Theorem 1.3.1. *([124], Discrete Arzela-Ascoli's Theorem) Any subset of l_0^∞ which is bounded and uniformly Cauchy is called relatively compact.*

Theorem 1.3.2. *([150], Schauder's fixed point theorem) Let $\mathscr{N}$ be a non-empty, closed and convex subset of a Banach space S. Further assume a continuous mapping $\mathscr{T} : \mathscr{N} \to \mathscr{N}$ such that $\mathscr{T}\mathscr{N}$ is a relatively compact set and contained in S. Then $\mathscr{T}$ admits a unique fixed point in $\mathscr{N}$. That is, $\mathscr{T}\psi = \psi$ for $\psi \in \mathscr{N}$.*

Theorem 1.3.3. *(Banach contraction principle) Let $\mathscr{N}$ be a non-empty complete metric space with a contraction mapping $\mathscr{T} : \mathscr{N} \to \mathscr{N}$. Then $\mathscr{T}$ has at least one fixed point in $\mathscr{N}$. That is, there exists $\psi \in \mathscr{N}$ such that $\mathscr{T}\psi = \psi$.*

Now we state Krasnoselskii's fixed point theorem.

Theorem 1.3.4. *[9] Let $\mathscr{N}$ be a non-empty, closed and convex subset of a Banach space $(S, \|.\|)$. Suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ maps $\mathscr{N}$ into S such that*
(i) $\mathscr{T}_1\psi + \mathscr{T}_2\varsigma \in \mathscr{N}$ for all $\psi, \varsigma \in \mathscr{N}$,
(ii) $\mathscr{T}_1$ is continuous and $\mathscr{T}_1\mathscr{N}$ is contained in a compact set of S,
(iii) $\mathscr{T}_2$ is a contraction.
Then $\mathscr{T}$ admits a fixed point in $\mathscr{N}$ such that $\mathscr{T}_1 z + \mathscr{T}_2 z = z$.

Lemma 1.3.2. *([110], Schaefer's fixed point theorem) Let $\mathcal{E}$ be a linear normed space and $\phi : \mathcal{E} \to \mathcal{E}$ be a compact operator. Suppose that the set*

$$\mathcal{S} = \{z \in \mathcal{E} \mid z = \lambda\phi z, \text{ for some } \lambda \in (0, 1)\},$$

is bounded. Then ϕ has a fixed point in $\mathcal{E}$.

Lemma 1.3.3. *[76] Let φ_p be a nonlinear operator. Then*

(1) *for $1 < p \leq 2$, $\delta_1^* \delta_2^* > 0$ and $|\delta_1^*|, |\delta_2^*| \geq \rho > 0$, then*

$$|\varphi_p(\delta_1^*) - \varphi_p(\delta_2^*)| \leq (p-1)\rho^{p-2}|\delta_1^* - \delta_2^*|,$$

(2) *for $p > 2$, and $|\delta_1^*|, |\delta_2^*| \leq \rho^*$, then*

$$|\varphi_p(\delta_1^*) - \varphi_p(\delta_2^*)| \leq (p-1)\rho^{*p-2}|\delta_1^* - \delta_2^*|.$$

Chapter 2

Stability Analysis by Fixed Point Theorems

Discrete fractional calculus has generated interest in recent years (see [73] and the references therein). Specially, fractional difference equations have become a popular topic recently. Using different approaches existence, uniqueness and then analysis of attractivity, stability and asymptotic stability of the solutions of fractional difference equations are most important topics in this direction [82, 54, 55, 142, 143]. The work done by Čermák [52, 53, 49, 50, 51] toward this path is extraordinary. In this chapter, we investigate the existence and stability results for a class of non-linear Caputo nabla fractional difference equation.

Fixed point theorems are widely used for existence and uniqueness purposes. Furthermore, they are used to investigate the attractivity of solutions as well as stability and the asymptotic stability. The original idea related to stability analysis of functional as well as fractional differential equations using fixed point theorem approach was proposed by Burton [40, 41, 42, 43]. In 2017, Abdeljawad et al. [9] used Krasnoselskii fixed point theorem for Caputo q-difference equation to investigate existence of solution. In 2018, Zhang et al. [150] investigated existence and attractivity of solutions of Riemann-Liouville-like fractional difference equations using Picard iteration method and then they used Schauder's fixed point theorem for required results. Furthermore, they also proved results using weighted space. In 2018, Ardjouni et al. [30] used

Krasnoselskii fixed point theorem for stability analysis of nabla fractional difference equation. In 2018, Arjumand worked on existence and stability analysis of fractional differential equation using Krasnoselskii fixed point theorem by considering solution of fractional differential equation [125]. Besides, for help on stability one can follow the latest articles and references in that [20, 103, 19, 132, 128, 96]. Motivated by all above mentioned papers, here we consider the following fractional difference equation with initial condition

$$^{c}\nabla_0^{\zeta}\psi(\eta) = \lambda\psi(\eta) + \mathscr{Q}(\eta, \psi(\eta)), \quad \psi(0) = \psi_0, \quad \eta \in \mathbb{N}_0. \tag{2.1}$$

Where $^{c}\nabla_0^{\zeta}$ represents Caputo nabla fractional difference operator with $0 < \zeta \leq 1$ and a continuous function $\mathscr{Q}$ is defined as $\mathscr{Q} : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ with $\mathscr{Q}(\eta, 0) = 0$. Here we use the approach as mentioned in [125] by considering the solution of Eq. (2.1) in terms of nabla discrete Mittag-Leffler function and then using its properties, we discuss existence of solutions and then attractivity, stability and finally asymptotic stability of solutions of above mentioned problem. This approach to study the stability using fixed point theorem has never been studied by any researcher. Results of this chapter are published in [45]

We arranged our current chapter as follows: Section 2.1 contains some basic definitions, notations, lemmas and graphic analysis for the nabla discrete Mittag-Leffler functions. Section 2.2, 2.3 intend to investigate the existence and stability results. Finally an example is provided.

2.1 Fundamental Results

In this section, we present basic definitions and lemma that are helpful in proving our main results. The graphs of the nabla discrete Mittag-Leffler functions are shown and the behavior at infinity is also studied.

Lemma 2.1.1. *[3] Let $0 < \zeta \leq 1$, $a \in \mathbb{R}$ and consider the nabla Caputo nonhomogeneous* *fractional difference equation*

$$ {}^{c}\nabla_{a}^{\zeta} z(\eta) = \lambda z(\eta) + \mathscr{H}(\eta), \quad z(a) = a_0, \quad \eta \in \mathbb{N}_a, \tag{2.2}$$

at that point its answer is given by

$$ z(\eta) = a_0 E_{\overline{\zeta}}(\lambda, \eta - a) + \sum_{\tau=a+1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{H}(\tau). \tag{2.3}$$

Remark 1. *Following the above lemma, the solution of problem* (2.1) *is given by*

$$ \psi(\eta) = \psi_0 E_{\overline{\zeta}}(\lambda, \eta) + \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau)). \tag{2.4}$$

Definition 2.1.2. *[3] For $\lambda \in \mathbb{R}$, $|\lambda| < 1$ and $\zeta, \beta, \eta \in \mathbb{C}$ with $Re(\zeta) > 0$, the nabla discrete Mittag-Leffler functions are defined as follows:*

$$ E_{\overline{\zeta,\beta}}(\lambda, \eta) = \sum_{m=0}^{\infty} \lambda^m \frac{\eta^{\overline{m\zeta+\beta-1}}}{\Gamma(m\zeta+\beta)}. $$

For $\beta = 1$, one parameter nabla discrete Mittag-Leffler function can be written as

$$ E_{\overline{\zeta}}(\lambda, \eta) = \sum_{m=0}^{\infty} \lambda^m \frac{\eta^{\overline{m\zeta}}}{\Gamma(m\zeta+1)}. $$

Definition 2.1.3. *[125] $\psi = \varphi(\eta)$ being solution of Eq.* (2.1) *is called*

(i) stable, if for every $\epsilon > 0$ and $\eta_0 \geq 0$, there exists a $\delta > 0$ depending on η_0 and ϵ such that for $|\psi_0 - \varphi(\eta_0)| \leq \delta(\eta_0, \epsilon)$, we have for all $\eta \geq \eta_0$, $|\psi(\eta, \psi_0, \eta_0) - \varphi(\eta)| < \epsilon$;

(ii) attractive, if there exists $\xi(\eta_0) > 0$ such that $\|\psi_0\| \leq \xi$ implies $\lim_{\eta\to\infty} \psi(\eta, \psi_0, \eta_0) = 0$;

(iii) asymptotically stable if it is attractive and stable.

Remark 2. *(The graphs and behavior of nabla discrete Mittag-Leffler functions at ∞) Moti-*

vated by Remark 1 in [11] and for the sake of benefiting in verification of our primary outcomes, the numerical evidences as illustrated in Figure 2.1 *and Figure* 2.2*, show that for* $\eta \in \mathbb{R}^+$, $0 < \zeta \leq 1$*, the one and two parameter nabla discrete Mittag-Leffler functions are decreasing functions of* η *and are bounded above by* 1*. That is* $E_{\overline{\zeta}}(\lambda, \eta) \leq 1$ *and* $E_{\overline{\zeta,\zeta}}(\lambda, \eta) \leq 1$*, where* $-1 < \lambda < 0$ *and* $-\zeta < \lambda < 0$*. Moreover, it is to be noted that* $\lim_{\eta\to\infty} E_{\overline{\zeta}}(\lambda, \eta) = 0$ *and* $\lim_{\eta\to\infty} E_{\overline{\zeta,\zeta}}(\lambda, \eta) = 0$.

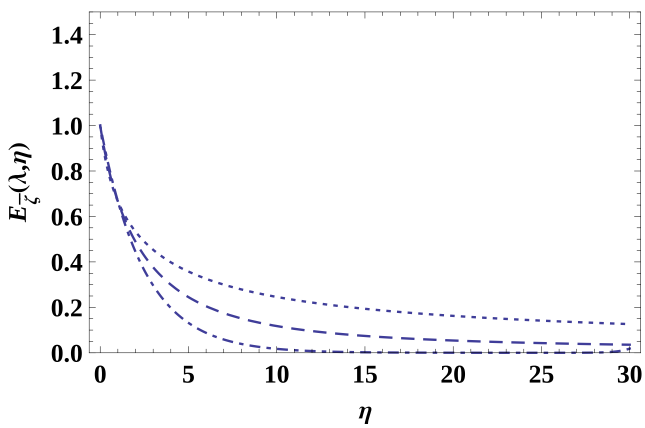

Figure 2.1: $E_{\overline{\zeta}}(\lambda, \eta)$ for $\lambda = -0.5, \zeta = 0.6(Dotted), 0.8(Dashed), 1(Dotted - Dashed)$.

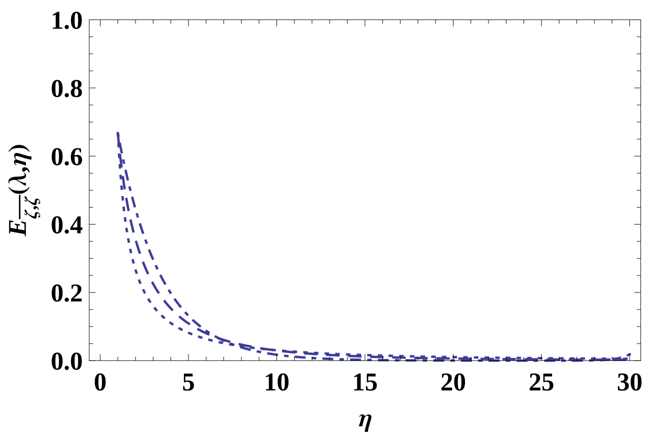

Figure 2.2: $E_{\overline{\zeta,\zeta}}(\lambda, \eta)$ for $\lambda = -0.5, \zeta = 0.6(Dotted), 0.8(Dashed), 1(Dotted - Dashed)$.

2.2 The Existence and Uniqueness Theorems

Let l_0^∞ consists of all real sequences $\psi = \{\psi(\eta)\}_{\eta=0}^{\infty}$ from the starting point $\eta = 0$. The space is endowed with the supremum norm $\|\psi\| = \sup_{\eta\in\mathbb{N}_0} |\psi(\eta)|$. l_0^∞ is a Banach space.

Theorem 2.2.1. *Let us consider the equivalent form of problem* (2.1) *as mentioned in Eq.* (2.4). *Assume a continuous and bounded function* $\mathscr{Q}$,

(A_1) $\mathscr{Q} : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ *satisfying Lipschitz condition with* $\mathscr{L} > 0$ *being Lipschitz constant*

$$\|\mathscr{Q}(\eta, \psi) - \mathscr{Q}(\eta, \varsigma)\| \leq \mathscr{L} \|\psi - \varsigma\|, \quad \forall\ \eta \in \mathbb{N}_0^b,\ b > 0.$$

At that point utilizing Schauder's fixed point hypothesis, there exists at least one solution of Eq. (2.1).

Proof. Let us consider non-empty, closed and convex subset $\mathscr{K} = \{\psi : \psi \in l_0^\infty, \|\psi\| \leq \Lambda\}$, also assume that $\|\mathscr{Q}(\eta, \psi)\| \leq \mathscr{R},\ \forall (\eta, \psi) \in \mathbb{N}_0^b \times \mathbb{R}$. Furthermore, consider a mapping $\mathscr{T}$ on $\mathscr{K}$ as given below:

$$\mathscr{T}\psi(\eta) = \psi_0 E_{\overline{\zeta}}(\lambda, \eta) + \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau)).$$

First of all we show that $\mathscr{T}$ maps $\mathscr{K}$ into $\mathscr{K}$.

$$\begin{aligned}
|\mathscr{T}\psi(\eta)| &= \left| \psi_0 E_{\overline{\zeta}}(\lambda, \eta) + \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau)) \right| \\
&\leq |\psi_0||E_{\overline{\zeta}}(\lambda, \eta)| + \int_0^{\eta} \left|E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\right| |\mathscr{Q}(\tau, \psi(\tau))| \, \nabla\tau \\
&\leq |\psi_0| + \mathscr{R}b \leq \Lambda.
\end{aligned}$$

Now we have to show that $\mathscr{T}$ is relatively compact, for this consider $0 \leq \eta_1 \leq \eta_2 \leq b$, so we get

$$\begin{aligned}
&|\mathscr{T}\psi(\eta_2) - \mathscr{T}\psi(\eta_1)| \\
&\leq \left|\psi_0\big(E_{\overline{\zeta}}(\lambda,\eta_2) - E_{\overline{\zeta}}(\lambda,\eta_1)\big)\right| \\
&\quad + \left|\left(\sum_{\tau=1}^{\eta_2} E_{\overline{\zeta,\zeta}}(\lambda,\eta_2-\varrho(\tau)) - \sum_{\tau=1}^{\eta_1} E_{\overline{\zeta,\zeta}}(\lambda,\eta_1-\varrho(\tau))\right)\mathscr{Q}(\tau,\psi(\tau))\right| \\
&\leq \left|\psi_0\big(E_{\overline{\zeta}}(\lambda,\eta_2) - E_{\overline{\zeta}}(\lambda,\eta_1)\big)\right| \\
&\quad + \left|\left(\sum_{\tau=1}^{\eta_1} E_{\overline{\zeta,\zeta}}(\lambda,\eta_2-\varrho(\tau)) - \sum_{\tau=1}^{\eta_1} E_{\overline{\zeta,\zeta}}(\lambda,\eta_1-\varrho(\tau))\right)\mathscr{Q}(\tau,\psi(\tau))\right| \\
&\quad + \left|\sum_{\tau=\eta_1+1}^{\eta_2} E_{\overline{\zeta,\zeta}}(\lambda,\eta_2-\varrho(\tau))\mathscr{Q}(\tau,\psi(\tau))\right| \\
&\leq \left|\psi_0\big(E_{\overline{\zeta}}(\lambda,\eta_2) - E_{\overline{\zeta}}(\lambda,\eta_1)\big)\right| \\
&\quad + \int_0^{\eta_1}\left|\big(E_{\overline{\zeta,\zeta}}(\lambda,\eta_2-\varrho(\tau)) - E_{\overline{\zeta,\zeta}}(\lambda,\eta_1-\varrho(\tau))\big)\right| |\mathscr{Q}(\tau,\psi(\tau))|\,\nabla\tau \\
&\quad + \int_{\eta_1}^{\eta_2}\left|E_{\overline{\zeta,\zeta}}(\lambda,\eta_2-\varrho(\tau))\right| |\mathscr{Q}(\tau,\psi(\tau))|\,\nabla\tau \\
&\leq \left|\psi_0\big(E_{\overline{\zeta}}(\lambda,\eta_2) - E_{\overline{\zeta}}(\lambda,\eta_1)\big)\right| \\
&\quad + \int_0^{\eta_1}\left|\big(E_{\overline{\zeta,\zeta}}(\lambda,\eta_2-\varrho(\tau)) - E_{\overline{\zeta,\zeta}}(\lambda,\eta_1-\varrho(\tau))\big)\right| \mathscr{R}\nabla\tau \\
&\quad + \mathscr{R}(\eta_2-\eta_1).
\end{aligned}$$

Hence it follows that $|\mathscr{T}\psi(\eta_2) - \mathscr{T}\psi(\eta_1)| \to 0$ as $\eta_1 \to \eta_2$.

Now in order to show the continuity of $\mathscr{T}$, let us consider a sequence ψ_n which converges to ψ, then we get

$$\begin{aligned}
|\mathscr{T}\psi_n(\eta) - \mathscr{T}\psi(\eta)| &\leq \left|\sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))\Big(\mathscr{Q}(\tau,\psi_n(\tau)) - \mathscr{Q}(\tau,\psi(\tau))\Big)\right| \\
&\leq \int_0^{\eta}\left|E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))\right| |\mathscr{Q}(\tau,\psi_n(\tau)) - \mathscr{Q}(\tau,\psi(\tau))|\,\nabla\tau \\
&\leq \mathscr{L}\,\|\psi_n-\psi\|\int_0^{\eta}\nabla\tau \leq \mathscr{L}b\,\|\psi_n-\psi\|,
\end{aligned}$$

we can see easily $\mathscr{T}\psi_n \to \mathscr{T}\psi$ as $\psi_n \to \psi$. Hence using Arzela Ascoli's Theorem, $\mathscr{T}\mathscr{K}$ is relatively compact due to boundedness and being uniformly cauchy subset of l_0^∞.
Hence by Schauder's fixed point theorem there exists at least one fixed point of $\mathscr{T}$ in $\mathscr{K}$. Furthermore, if all functions ψ in $\mathscr{K}$ tends to 0 as $\eta \to \infty$ then solutions of Eq. (2.1) tends to zero as $\eta \to \infty$ hence called attractive solutions. □

2.3 The Stability Results

On the light of the fixed point results presented in the above section, we present some stability results in what follows.

Theorem 2.3.1. *Assume that a bounded, continuous function $\mathscr{Q}$ satisfies the following conditions,*

(A_2) $\|\mathscr{Q}(\eta,\psi) - \mathscr{Q}(\eta,\varsigma)\| \leq \mathscr{L}(\eta)\|\psi - \varsigma\|$,

(A_3) $\int_0^\eta \mathscr{L}(\tau)\nabla\tau \to 0$ *as* $\eta \to \infty$, *where* $\mathscr{M} = \sup_{\eta\in\mathbb{N}_0}\int_0^\eta \mathscr{L}(\tau)\nabla\tau$,

then using Banach contraction principle, there exits a unique solution of Eq. (2.1). *Moreover, the solution is attractive.*

Proof. Define a set $\mathscr{E} = \{\psi \in l_0^\infty, \|\psi\| \leq \epsilon$ and $\psi(\eta) \to 0$ as $\eta \to \infty\}$. Further assume that

$$|\mathscr{Q}(\eta,\psi)| \leq |\mathscr{Q}(\eta,\psi) - \mathscr{Q}(\eta,0)| + |\mathscr{Q}(\eta,0)| \leq \mathscr{L}(\eta)|\psi - 0| + 0 = \mathscr{L}(\eta)\|\psi\|.$$

Now we define a mapping $\mathscr{T}$ on $\mathscr{E}$ as follows:

$$\mathscr{T}\psi(\eta) = \psi_0 E_{\overline{\zeta}}(\lambda,\eta) + \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))\mathscr{Q}(\tau,\psi(\tau)).$$

Continuity of $\mathscr{T}\psi$ is followed by $\psi \in \mathscr{E}$. First of all we prove that $\mathscr{T}$ maps $\mathscr{E}$ into itself.

$$\begin{aligned}
|\mathscr{T}\psi(\eta)| &= \left|\psi_0 E_{\overline{\zeta}}(\lambda,\eta) + \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))\mathscr{Q}(\tau,\psi(\tau))\right| \\
&\leq |\psi_0||E_{\overline{\zeta}}(\lambda,\eta)| + \int_0^{\eta} \left|E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))\right| |\mathscr{Q}(\tau,\psi(\tau))| \nabla\tau \\
&\leq |\psi_0| + \int_0^{\eta} \mathscr{L}(\tau) \|\psi\| \nabla\tau \\
&\leq |\psi_0| + \epsilon \int_0^{\eta} \mathscr{L}(\tau)\nabla\tau \leq |\psi_0| + \epsilon\mathscr{M} \leq \epsilon.
\end{aligned}$$

Hence $\mathscr{T}$ maps $\mathscr{E}$ into itself. Now we show that $\mathscr{T}\psi(\eta) \to 0$ as $\eta \to \infty$. Since we can see easily in Figure 2.1 and 2.2, $\lim_{\eta\to\infty} E_{\overline{\zeta}}(\lambda,\eta) \to 0$ and $\lim_{\eta\to\infty} E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau)) \to 0$. So we have $\lim_{\eta\to\infty} \psi_0 E_{\overline{\zeta}}(\lambda,\eta) \to 0$. Also

$$\begin{aligned}
\left|\int_0^{\eta} E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))\mathscr{Q}(\tau,\psi(\tau))\nabla\tau\right| &\leq \int_0^{\eta} |E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))||\mathscr{Q}(\tau,\psi(\tau))|\nabla\tau \\
&\leq \int_0^{\eta} |E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))|\mathscr{L}(\tau) \|\psi\| \nabla\tau \\
&\leq \epsilon \int_0^{\eta} \mathscr{L}(\tau)\nabla\tau \leq \epsilon\mathscr{M}.
\end{aligned}$$

Thus $\left|\int_0^{\eta} E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))\mathscr{Q}(\tau,\psi(\tau))\nabla\tau\right| \to 0$ as $\eta \to \infty$. Hence $\mathscr{T}\psi(\eta) \to 0$ as $\eta \to \infty$. Now we show that $\mathscr{T}$ is contraction mapping.

$$\begin{aligned}
|\mathscr{T}\psi(\eta) - \mathscr{T}\varsigma(\eta)| &\leq \int_0^{\eta} |E_{\overline{\zeta,\zeta}}(\lambda,\eta-\varrho(\tau))||\mathscr{Q}(\tau,\psi(\tau)) - \mathscr{Q}(\tau,\varsigma(\tau))|\nabla\tau \\
&\leq \|\psi-\varsigma\| \int_0^{\eta} \mathscr{L}(\tau)\nabla\tau \leq \mathscr{M}\|\psi-\varsigma\|.
\end{aligned}$$

Since for $\mathscr{M} < 1$, $\mathscr{T}$ is contraction. Hence by contraction mapping principle Eq. (2.1) has a unique solution and furthermore, since all functions ψ in $\mathscr{E}$ tends to 0 as $\eta \to \infty$, so solution of Eq. (2.1) tends to zero as $\eta \to \infty$, hence called attractive solution. □

Theorem 2.3.2. *Let ψ be solution of Eq. (2.1) and $\hat{\psi}$ be a solution of Eq. (2.1) satisfying the initial condition $\hat{\psi}(0) = \hat{\psi}_0$. Moreover, let for very small $\epsilon > 0$, there exists $\delta = (1-\mathscr{M})\epsilon$.*

Then solutions of Eq. (2.1) *are stable.*

Proof. Since ψ is a solution of Eq. (2.1) and $\hat{\psi}$ is also a solution of Eq. (2.1) satisfying the initial condition $\hat{\psi}(0) = \hat{\psi}_0$. Then

$$\begin{aligned}
|\psi(\eta) - \hat{\psi}(\eta)| &\leq \left|\psi_0 E_{\overline{\zeta}}(\lambda, \eta) - \hat{\psi}_0 E_{\overline{\zeta}}(\lambda, \eta)\right| \\
&\quad + \left|\sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau)) - \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \hat{\psi}(\tau))\right| \\
&\leq |\psi_0 - \hat{\psi}_0||E_{\overline{\zeta}}(\lambda, \eta)| + \int_0^{\eta} |E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))||\mathscr{Q}(\tau, \psi(\tau)) - \mathscr{Q}(\tau, \hat{\psi}(\tau))|\nabla\tau \\
&\leq |\psi_0 - \hat{\psi}_0| + \int_0^{\eta} \mathscr{L}(\tau)|\psi(\tau) - \hat{\psi}(\tau)|\nabla\tau \\
&\leq |\psi_0 - \hat{\psi}_0| + \left\|\psi - \hat{\psi}\right\| \int_0^{\eta} \mathscr{L}(\tau)\nabla\tau \\
&\leq |\psi_0 - \hat{\psi}_0| + \left\|\psi - \hat{\psi}\right\| \mathscr{M}.
\end{aligned}$$

Hence we have

$$\left\|\psi - \hat{\psi}\right\| \leq \frac{1}{1 - \mathscr{M}} \left\|\psi_0 - \hat{\psi}_0\right\|.$$

Then for any $\epsilon > 0$, let $\delta = (1 - \mathscr{M})\epsilon$ so for $\left\|\psi_0 - \hat{\psi}_0\right\| < \delta$ we have $\left\|\psi - \hat{\psi}\right\| < \epsilon$. Therefore, the solutions of Eq. (2.1) are stable. This completes the proof. □

Remark 3. *From Theorem* (2.3.1) *and Theorem* (2.3.2)*, it is clear that solution of Eq.* (2.1) *is asymptotically stable.*

Theorem 2.3.3. *Assume that a bounded, continuous function $\mathscr{Q}$ satisfying assumptions* (A_2) *and* (A_3)*. Then problem* (2.1) *has at least one solution.*

Proof. Let us consider a non-empty, closed, convex subset of a Banach space l_0^{∞} defined as $\mathscr{N} = \{\psi : \psi \in l_0^{\infty},\ |\psi(\eta)| \leq m,\ \forall\, \eta \in \mathbb{N}_0\}$. Further, define operators $\mathscr{T}_1$ and $\mathscr{T}_2$ on $\mathscr{N}$ as follows:

$$\mathscr{T}_1\psi(\eta) = \psi_0 E_{\overline{\zeta}}(\lambda, \eta),$$

$$\mathscr{T}_2\psi(\eta) = \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau)) = \int_0^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau))\nabla\tau.$$

As we know that ψ being fixed point of the operator $\mathscr{T}\psi = \mathscr{T}_1\psi + \mathscr{T}_2\psi$ is a solution of Eq. (2.1). Following three steps as mentioned in Theorem (1.3.4), we present our proof as follows.

In the first step, we prove that $\mathscr{T}$ maps $\mathscr{N}$ into $\mathscr{N}$ i.e. for any $\psi, \varsigma \in \mathscr{N}$, we have to show that $\mathscr{T}_1\psi(\eta) + \mathscr{T}_2\varsigma(\eta) \in \mathscr{N}$.

$$\begin{aligned}
|\mathscr{T}_1\psi(\eta) + \mathscr{T}_2\varsigma(\eta)| &= \left|\psi_0 E_{\overline{\zeta}}(\lambda,\eta) + \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \varsigma(\tau))\right| \\
&\leq |\psi_0 E_{\overline{\zeta}}(\lambda,\eta)| + |\sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \varsigma(\tau))| \\
&\leq |\psi_0| + \int_0^{\eta} |E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))||\mathscr{Q}(\tau, \varsigma(\tau))|\nabla\tau \\
&\leq |\psi_0| + \int_0^{\eta} \mathscr{L}(\tau)|\varsigma(\tau)|\nabla\tau \\
&\leq |\psi_0| + \mathscr{M}m \leq m.
\end{aligned}$$

Hence $\mathscr{T}\mathscr{N} \subset \mathscr{N}$. In the second step, in order to prove that $\mathscr{T}_1$ is continuous, let us consider a sequence ψ_n such that $\psi_n \to \psi$.

$$|\mathscr{T}_1\psi_n(\eta) - \mathscr{T}_1\psi(\eta)| = |\psi_0 E_{\overline{\zeta}}(\lambda,\eta) - \psi_0 E_{\overline{\zeta}}(\lambda,\eta)| = 0,$$

so for $\psi_n \to \psi$, $\mathscr{T}_1\psi_n \to \mathscr{T}_1\psi$. Hence $\mathscr{T}_1$ is continuous. Now we show that $\mathscr{T}_1(\mathscr{N})$ resides in relatively compact set of l_0^∞. Taking $\eta_1 \leq \eta_2 \leq H$, we have

$$\begin{aligned}
|\mathscr{T}_1\psi(\eta_2) - \mathscr{T}_1\psi(\eta_1)| &= |\psi_0 E_{\overline{\zeta}}(\lambda,\eta_2) - \psi_0 E_{\overline{\zeta}}(\lambda,\eta_1)| \\
&= |\psi_0||E_{\overline{\zeta}}(\lambda,\eta_2) - E_{\overline{\zeta}}(\lambda,\eta_1)|
\end{aligned}$$

as $\eta_1 \to \eta_2$, we get $|\mathscr{T}_1\psi(\eta_2) - \mathscr{T}_1\psi(\eta_1)| \to 0$. Hence $\mathscr{T}_1(\mathscr{N})$ resides in relatively compact set of l_0^∞.

In the last step, we show that $\mathscr{T}_2$ is contraction. Letting $\psi, \varsigma \in \mathscr{N}$, we have

$$\begin{aligned}
&|\mathscr{T}_2\psi(\eta) - \mathscr{T}_2\varsigma(\eta)| \\
&= \left| \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau)) - \sum_{\tau=1}^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \varsigma(\tau)) \right| \\
&= \left| \int_0^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \psi(\tau))\nabla\tau - \int_0^{\eta} E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))\mathscr{Q}(\tau, \varsigma(\tau))\nabla\tau \right| \\
&\leq \int_0^{\eta} |E_{\overline{\zeta,\zeta}}(\lambda, \eta - \varrho(\tau))||\mathscr{Q}(\tau, \psi(\tau)) - \mathscr{Q}(\tau, \varsigma(\tau))|\nabla\tau \\
&\leq \int_0^{\eta} \mathscr{L}(\tau)|\psi(\tau) - \varsigma(\tau)|\nabla\tau \\
&\leq \|\psi - \varsigma\| \int_0^{\eta} \mathscr{L}(\tau)\nabla\tau \leq \mathscr{M} \|\psi - \varsigma\|.
\end{aligned}$$

For $\mathscr{M} < 1$, $\mathscr{T}_2$ is contraction.

Hence according to the Theorem (1.3.4), $\mathscr{T}$ has a fixed point in $\mathscr{N}$ which is solution of Eq. (2.1). □

Remark 4. *According to Theorem* (2.3.3)*, the solutions of Eq.* (2.1) *exists and are in* $\mathscr{N}$*. Furthermore, if all functions* ψ *in* $\mathscr{N}$ *tends to* 0 *as* $\eta \to \infty$*. Then, the solutions of Eq.* (2.1) *are attractive.*

Remark 5. *From Theorem* (2.3.2) *and Theorem* (2.3.3)*, it is clear that the solution of Eq.* (2.1) *is asymptotically stable.*

Example 2.3.1. *Consider fractional difference equation as follows:*

$$^{c}\nabla_0^{0.6}\psi(\eta) = -0.5\psi(\eta) + \eta^{\overline{-1.7}} \sin\psi(\eta), \quad \psi(0) = \psi_0, \quad \eta \in \mathbb{N}_1, \tag{2.5}$$

where $\mathscr{Q}(\eta, \psi(\eta)) = \eta^{\overline{-1.7}} \sin\psi(\eta), \quad \eta \in \mathbb{N}_1$.

According to Lemma (2.1.1)*, solution of problem under consideration can be written as*

$$\psi(\eta) = \psi_0 E_{\overline{0.6}}(-0.5, \eta) + \sum_{\tau=1}^{\eta} E_{\overline{0.6,0.6}}(-0.5, \eta - \varrho(\tau))\tau^{\overline{-1.7}} \sin\psi(\tau). \tag{2.6}$$

We can see easily that function $\mathscr{Q}(\eta,\psi(\eta))$ *satisfies condition* (A_1)*, so by Theorem* (2.2.1)*, there exists atleast one solution of problem* (2.5)*.*
Moreover,

$$\begin{aligned}\|\mathscr{Q}(\eta,\psi)-\mathscr{Q}(\eta,\varsigma)\| &= \left\|\eta^{\overline{-1.7}}\sin\psi-\eta^{\overline{-1.7}}\sin\varsigma\right\| \\ &\leq \mathscr{L}(\eta)\left\|\psi-\varsigma\right\|,\end{aligned}$$

where $\mathscr{L}(\eta)=\eta^{\overline{-1.7}}$*. Calculations shows that* $\mathscr{M}=\sup_{\eta\in\mathbb{N}_1}\int_0^\eta \mathscr{L}(\tau)\nabla\tau<1$ *and* $\int_0^\eta \mathscr{L}(\tau)\nabla\tau\to 0$ *as* $\eta\to\infty$*. Hence condition* (A_2) *and* (A_3) *are satisfied. So by Theorem* (2.3.1) *and* (2.3.3)*, there exists atleast one solution of problem* (2.5)*. Moreover, by Theorem* (2.3.2) *the solution is stable. Furthermore, solution* ψ *of problem* (2.5) *tends to* 0 *as* $\eta\to\infty$*. Hence solution is asymptotically stable.*

Chapter 3

Ulam's Type Stability Analysis of Fractional Difference Equation with Impulse

In current chapter, we have introduced a new Gronwall inequality with impulsive effect. Existence and uniqueness of solution is investigated through fixed point theorems. Moreover, stability analysis of impulsive fractional difference equation is investigated. Stability criterion is obtained with the help of newly developed Gronwall inequality. At last a model is given to help the hypothetical outcome.

Nowadays, study of discrete fractional calculus has become part of great interest due to its consolidation in several scientific areas. Discretized real-life models are new concepts introduced by discrete fractional calculus. Discrete fractional calculus is more applicable for study population growth, image processing and tumor growth [79, 36] etc. Having no truncation errors is the main feature of discrete fractional calculus while functions are defined on a discrete set.

Fractional difference equations becomes a most attractive topic nowadays. Stability analysis is one of the most significant and well developed topics in continuous fractional calculus but rather rare in discrete case. In order to discuss stability of fractional differential/difference equations, various research articles have been appeared [85, 56, 60, 104, 142, 144, 82, 45, 23, 81] but still this direction is an open dilemma for researchers. In the most recent articles, Koksal [102] has investigated stability analysis of fractional differential equations with unknown parameters. The fractional input stability of the electrical circuit equations described by the fractional derivative operators has been investigated by Sene [126]. Ameen et al. [26] have investigated Ulam stability for delay fractional differential equations with a generalized Caputo derivative. Khan et al. [95] discussed existence-uniqueness of a solution and the Hyers-Ulam stability.

Researchers are using different approaches to check stability of fractional differnential as well as difference equations. Gronwall inequality is one of the most widely used approach to check stability. Gronwall inequality is discussed by many authors in continuous as well as discrete fractional calculus [25, 70, 6, 146, 44, 5]. Wang et al. [131] used Gronwall inequality to analyze Ulam's type stability of impulsive ordinary differential equations. Ulam-Hyers stability of differential equations of second order was discussed by Alqifiary et al. [22]. Wang et al. [135] investigated nonlinear impulsive problems for fractional differential equations and then Ulam stability via Gronwall inequality. Wang et al. [136] investigated stability analysis of impulsive fractional differential systems with delay.

Motivated by all above mentioned articles, in this chapter, we develop a new fractional difference inequality of Gronwall type with impulsive effect which is further used to obtain stability criteria. This inequality has not been previously produced in the literature. We consider a fractional difference equation with impulsive effects and investigate the Ulam's type stability. The problem under consideration is as follows:

$$\begin{cases} {}^{c}\Delta_a^{\zeta}\psi(\eta) & = \mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),\ \eta \in \mathbb{N}_{a+1-\zeta},\ \eta \neq a+n_l+1-\zeta,\ 0<\zeta\leq 1, \\ \psi_{n_l+1} & = \overline{\psi}_{n_l+1} + I_l(\overline{\psi}_{n_l+1}),\ \eta = a+n_l+1-\zeta,\ l \in \mathbb{N}_1, \\ \psi(a) & = \psi_0 = \xi, \end{cases} \tag{3.1}$$

where ${}^{c}\Delta_a^{\zeta}$ is delta Caputo fractional difference with $0 < \zeta \leq 1$. Furthermore, function $\mathscr{Q}$ is defined on $\mathbb{N}_a \times \mathbb{R}$ and $I_l : \mathbb{R} \to \mathbb{R}$.

This chapter is formulated in following manner: In Section 3.1, we list some basic definitions and lemmas which are necessary for proving main results. In Section 3.2, we provide main results of article including Gronwall inequality and existence-uniqueness of solution of above mentioned problem using Banach contraction principle, Schaefer's fixed point theorem and Leray Schauder theorem. Further in Section 3.3, newly developed Gronwall inequality is used to obtain stability criteria for fractional difference equation with impulse effects. To demonstrarte the theoretical results, an example is also provided.

3.1 Fundamental Results

In the following section, some necessary lemmas are provided that are helpul in proving our main results.

Lemma 3.1.1. *[142] Let ψ defined on $\mathbb{N}_a$ be a solution of fractional sum equation*

$$\psi(\eta) = \psi(\eta^*) - \frac{1}{\Gamma(\zeta)} \sum_{s=a+1-\zeta}^{\eta^*-\zeta} (\eta^* - \sigma(s))^{\underline{\zeta-1}} \mathscr{Q}(s+\zeta, \psi(s+\zeta)) + \frac{1}{\Gamma(\zeta)} \sum_{s=a+1-\zeta}^{\eta-\zeta} (\eta - \sigma(s))^{\underline{\zeta-1}} \mathscr{Q}(s+\zeta, \psi(s+\zeta)), \ \eta \in \mathbb{N}_{a+1},$$

if and only if ψ is a solution of the following Cauchy problem

$${}^c\Delta_a^{\zeta} \psi(\eta) = \mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)), \ \eta \in \mathbb{N}_{a+1-\zeta}, \ 0 < \zeta \leq 1,$$

subject to

$$\psi(a) = \psi(\eta^*) - \frac{1}{\Gamma(\zeta)} \sum_{s=a+1-\zeta}^{\eta^*-\zeta} (\eta^* - \sigma(s))^{\underline{\zeta-1}} \mathscr{Q}(s+\zeta, \psi(s+\zeta)).$$

Lemma 3.1.2. *[144] Let us suppose discrete functions $\mathscr{F}$ and $\mathscr{G}$ which are nonnegtive and nondecreasing. Furthermore, $\mathscr{G}(\eta) \leq M$ with M being a positive constan and $\eta \in \mathbb{N}_a$. If we* *have following sum inequality*

$$\psi(\eta) \leq \mathscr{F}(\eta) + \mathscr{G}(\eta)\Delta_{a+1-\zeta}^{-\zeta}\psi(\eta+\zeta),$$

where $\eta \in \mathbb{N}_{a+1}$*, then*

$$\psi(\eta) \leq \mathscr{F}(\eta)e_{\zeta}(\mathscr{G}(\eta), \eta),$$

where $e_{\zeta}(\mathscr{G}(\eta), \eta)$ *is discrete Mittag-Leffler function in Delta sense.*

Lemma 3.1.3. *[142] Let* ψ *be defined on* $\mathbb{N}_a$ *and suppose that following inequality holds,*

$$\psi(\eta) \leq \mathscr{F}(\eta) + \mathscr{G}(\eta)\Delta_{a+1-\zeta}^{-\zeta}\psi(\eta+\zeta),\ \eta \in \mathbb{N}_a,\ 0 < \zeta \leq 1,$$

$$\psi_{n_l+1} = \overline{\psi}_{n_l+1} + q_l\overline{\psi}_{n_l+1}, -1 < q_l < 0,\ \eta = a + n_l + 1 - \zeta,\ l \in \mathbb{N}_1,\ \psi(a) = \psi_0.$$

then

$$\psi(\eta) \leq \mathscr{F}(\eta)e_{\zeta}^{*}(\mathscr{G}(\eta), \eta - a),\ \eta \in \{a + n_l + 1, ..., a + n_{l+1}\},\ l = 1, 2, ..., N, ...,$$

where discrete functions $\mathscr{F}$ *and* $\mathscr{G}$ *are nonnegtive and nondecreasing. Furthermore,* $\mathscr{G}(\eta) \leq M$ *with* M *being a positive constan and* $\eta \in \mathbb{N}_a$*.*

Now to investigate Ulam's type stability concepts for fractional difference equation having impulsive effects. Consider the following inequalities with a positve real number ϵ and a continuous function ϕ defined on $\mathbb{N}_a$.

$$\begin{cases} |{}^{c}\Delta_a^{\zeta}\varsigma(\eta) - \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta))| & \leq \epsilon,\ \eta \in \mathbb{N}_{a+1-\zeta},\ \eta \neq a + n_l + 1 - \zeta. \\ |\varsigma_{n_l+1} - \overline{\varsigma}_{n_l+1} - I_l(\overline{\varsigma}_{n_l+1})| & \leq \epsilon,\ \eta = a + n_l + 1 - \zeta,\ l \in \mathbb{N}_1. \end{cases} \tag{3.2}$$

$$\begin{cases} |{}^{c}\Delta_a^{\zeta}\varsigma(\eta) - \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta))| & \leq \phi(\eta),\ \eta \in \mathbb{N}_{a+1-\zeta},\ \eta \neq a + n_l + 1 - \zeta. \\ |\varsigma_{n_l+1} - \overline{\varsigma}_{n_l+1} - I_l(\overline{\varsigma}_{n_l+1})| & \leq \phi(\eta),\ \eta = a + n_l + 1 - \zeta,\ l \in \mathbb{N}_1. \end{cases} \tag{3.3}$$

$$\begin{cases} |^{c}\Delta_{a}^{\zeta}\varsigma(\eta) - \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta))| & \leq \epsilon\phi(\eta),\ \eta \in \mathbb{N}_{a+1-\zeta}, \eta \neq a+n_l+1-\zeta. \\ |\varsigma_{n_l+1} - \overline{\varsigma}_{n_l+1} - I_l(\overline{\varsigma}_{n_l+1})| & \leq \epsilon\phi(\eta),\ \eta = a+n_l+1-\zeta,\ l \in \mathbb{N}_1. \end{cases} \tag{3.4}$$

Definition 3.1.4. *Let ψ and ς defined on $\mathbb{N}_a$ being solutions of problem* (3.1) *and inequality* (3.2) *respectively. For each $\epsilon > 0$ and real number $c > 0$,*

$$|\varsigma(\eta) - \psi(\eta)| \leq c\epsilon,$$

then problem (3.1) *is Ulam-Hyers stable.*

Definition 3.1.5. *Let ψ and ς defined on $\mathbb{N}_a$ being solutions of problem* (3.1) *and inequality* (3.2) *respectively. If there exists θ defined on $\mathbb{N}_a$ such that $\theta(0) = 0$, then problem* (3.1) *is called generalized Ulam-Hyers stable, provided*

$$|\varsigma(\eta) - \psi(\eta)| \leq \theta(\epsilon).$$

Definition 3.1.6. *Let ψ and ς defined on $\mathbb{N}_a$ being solutions of problem* (3.1) *and inequality* (3.4) *respectively. If there exists $\chi > 0$ such that for each $\epsilon > 0$, problem* (3.1) *is Ulam-Hyers-Rassias stable with respect to ϕ, if following holds*

$$|\varsigma(\eta) - \psi(\eta)| \leq \chi\epsilon\phi(\eta).$$

Definition 3.1.7. *Let ψ and ς defined on $\mathbb{N}_a$ being solutions of problem* (3.1) *and inequality* (3.3) *respectively. If there exists $\chi > 0$ such that*

$$|\varsigma(\eta) - \psi(\eta)| \leq \chi\phi(\eta),$$

then problem (3.1) *is called generalized Ulam-Hyers-Rassias stable with respect to ϕ.*

3.2 Main Results

In this section, we start by proving the existence and uniqueness of the solution of problem (3.1). Then we developed a new delta discrete fractional inequality of Gronwall type to check Ulam's type stability.

3.2.1 Existence and Uniqueness

Theorem 3.2.1. *A function ψ defined on $\mathbb{N}_a$ is a solution of fractional difference equation with impulse condition*

$$
\begin{gathered}
{}^{c}\Delta_a^{\zeta}\psi(\eta) = \mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),\ \eta \in \mathbb{N}_{a+1-\zeta},\ 0<\zeta\leq 1,\ \eta \neq a+n_l+1-\zeta, \\
\psi_{n_l+1} = \overline{\psi}_{n_l+1} + I_l(\overline{\psi}_{n_l+1}),\ \eta = a+n_l+1-\zeta,\ l \in \mathbb{N}_1,\ \psi(a)=\psi_0.
\end{gathered}
$$

iff ψ is solution of summation equation

$$
\psi(\eta) = \begin{cases}
\psi_0 + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),\ \eta \in \{a+n_0+1, ..., a+n_1\}, \\
\vdots \\
\psi_0 + \sum_{i=1}^{l} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)), \\
\qquad\qquad \eta \in \{a+n_l+1, ..., a+n_{l+1}\}, \\
\vdots \\
\psi_0 + \sum_{i=1}^{N} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),\ \eta \in \{a+n_N+1, ...\},
\end{cases}
$$

where $l = 1, ..., N-1$ and $N \to \infty$ and $I_i : \mathbb{R} \to \mathbb{R}$.

Proof. For $\eta \in \{a+n_0+1, ..., a+n_1\}$, we get

$$
\psi(\eta) = \psi_0 + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),
$$

$$\overline{\psi}_{n_1+1} = \psi(a+n_1+1) = \psi_0 + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))|_{\eta=a+n_1+1}. \tag{3.5}$$

For $\eta \in \{a+n_1+1,...,a+n_2\}$, we use Lemma 3.1.1 and Eq. (3.5) to obtain the following

$$\begin{aligned}\psi(\eta) &= \psi_{n_1+1} - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))|_{\eta=a+n_1+1} + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))\\ &= \overline{\psi}_{n_1+1} + I_1(\overline{\psi}_{n_1+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))|_{\eta=a+n_1+1}\\ &\quad + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))\\ &= \psi_0 + I_1(\overline{\psi}_{n_1+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta)).\end{aligned}$$

Let us suppose, for $\eta \in \{a+n_k+1,...,a+n_{k+1}\}$, following holds

$$\psi(\eta) = \psi_0 + \sum_{i=1}^{k} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta)).$$

For $\eta \in \{a+n_{k+1}+1,...,a+n_{k+2}\}$,we have

$$\begin{aligned}\psi(\eta) &= \psi_{n_{k+1}+1} - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))|_{\eta=a+n_{k+1}+1} + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))\\ &= \overline{\psi}_{n_{k+1}+1} + I_{k+1}(\overline{\psi}_{n_{k+1}+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))|_{\eta=a+n_{k+1}+1}\\ &\quad + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))\\ &= \psi_0 + I_{k+1}(\overline{\psi}_{n_{k+1}+1}) + \sum_{i=1}^{k} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))\\ &= \psi_0 + \sum_{i=1}^{k+1} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta)).\end{aligned}$$

Hence by mathematical induction, the evidence is finished. □

Now we will prove existence and uniqueness of solution using fixed point theorems. Let us consider a Banach space of all continuous functions $\psi : \mathbb{N}_a \to \mathbb{R}$ with norm defined as $\|\psi\| = \sup_{\eta\in\mathbb{N}_a} |\psi(\eta)|$. First of all we will use Banach fixed point theorem.

Theorem 3.2.2. *Let us assume that*

(A_1) *There exists a constant* $\mathcal{L} > 0$ *such that*

$$|\mathscr{Q}(\eta, \psi) - \mathscr{Q}(\eta, \varsigma)| \leq \mathcal{L}|\psi - \varsigma|,$$

(A_2) *There exists a constant* $\mathcal{L}^* > 0$ *such that*

$$|I_k(\psi) - I_k(\varsigma)| \leq \mathcal{L}^*|\psi - \varsigma|, \qquad k = 1, 2, ...N - 1.$$

If

$$\left((N-1)\mathcal{L}^* + \mathcal{L}\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\right) < 1, \tag{3.6}$$

then problem (3.1) *has a unique solution.*

Proof. Consider the operator $\mathcal{T} : C(\mathbb{N}_a, \mathbb{R}) \to C(\mathbb{N}_a, \mathbb{R})$ defined by

$$\mathcal{T}(\psi)(\eta) = \psi_0 + \sum_{i=1}^{k} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)).$$

Clearly, fixed points of the opertor $\mathcal{T}$ are solutions of problem (3.1). Now by showing that $\mathscr{Q}$ is a contraction, we will prove that $\mathcal{T}$ has fixed point using Banach fixed point theorem. For ψ, ς defined on $\mathbb{N}_a$, we have

$$\begin{aligned}
&|\mathcal{T}(\psi)(\eta) - \mathcal{T}(\varsigma)(\eta)| \\
&\quad\leq \sum_{i=1}^{k} |I_i(\overline{\psi}_{n_i+1}) - I_i(\overline{\varsigma}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta}|\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta) - \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta))| \\
&\quad\leq \sum_{i=1}^{k} \mathcal{L}^*|\overline{\psi}_{n_i+1} - \overline{\varsigma}_{n_i+1}| + \mathcal{L}\Delta_{a+1-\zeta}^{-\zeta}|\psi(\eta+\zeta) - \varsigma(\eta+\zeta)| \\
&\quad\leq (N-1)\mathcal{L}^* \|\psi - \varsigma\| + \mathcal{L}\frac{(\eta-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)} \|\psi - \varsigma\| \\
&\quad\leq \left((N-1)\mathcal{L}^* + \mathcal{L}\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\right) \|\psi - \varsigma\|, \qquad \eta < T.
\end{aligned}$$

Therefore,

$$\|\mathcal{T}(\psi)-\mathcal{T}(\varsigma)\| \leq \left((N-1)\mathcal{L}^* + \mathcal{L}\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\right)\|\psi-\varsigma\|.$$

Using inequality (3.6), $\mathcal{T}$ is a contraction. Hence using Banach contraction principle, operator $\mathcal{T}$ has fixed point which is solution of problem (3.1). □

Now, we will use Schaefer's fixed point theorem.

Theorem 3.2.3. *Assume that*

(A_3) *The function* $\mathscr{Q} : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(A_4) *There exists a constant* $\mathcal{M} > 0$ *such that* $|\mathscr{Q}(\eta, \psi)| \leq \mathcal{M}$,

(A_5) *The real-valued functions* I_k *are continuous and there exists a positive number* $\mathcal{M}^*$ *such that* $|I_k(\psi)| \leq \mathcal{M}^*, \quad k = 1, 2, ...N-1.$

Then problem (3.1) *has atleast one solution.*

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathcal{T}$ has a fixed point. The proof is given in several steps.

Step 1: $\mathcal{T}$ is continuous. Let $\{\psi_n\}$ be a sequence such that $\psi_n \to \psi$.

$$\begin{aligned}
&|\mathcal{T}(\psi_n)(\eta) - \mathcal{T}(\psi)(\eta)| \\
&\quad \leq \sum_{i=1}^{k} |I_i(\overline{\psi}_{n_{n_i+1}}) - I_i(\overline{\psi}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta}|\mathscr{Q}(\eta+\zeta, \psi_n(\eta+\zeta)) - \mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta))|.
\end{aligned}$$

Since $\mathscr{Q}$ and I_k are continuous functions, we have

$$\|\mathcal{T}(\psi_n) - \mathcal{T}(\psi)\| \to 0 \text{ as } n \to \infty.$$

Step 2: $\mathcal{T}$ maps bounded sets into bounded sets. Indeed, it is enough to show that for any $\mu > 0$, there exists a positive number l such that for each $\psi \in B_\mu = \{\psi \in C(\mathbb{N}_a, \mathbb{R}) : \|\psi\| \leq \mu\}$, we have $\|\mathcal{T}(\psi)\| \leq l$. Now using assumptions (A_4) and (A_5), we have

$$\begin{aligned}|\mathcal{T}(\psi)(\eta)| &\leq |\psi_0| + \sum_{i=1}^{k} |I_i(\overline{\psi}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta}|\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta))| \\ &\leq |\psi_0| + (N-1)\mathcal{M}^* + \frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\mathcal{M} := l.\end{aligned}$$

Thus

$$\|\mathcal{T}(\psi)\| \leq l.$$

Step 3: $\mathcal{T}$ maps bounded sets into equicontinuous sets. Let $\eta_1, \eta_2 \in \mathbb{N}_a$, $\eta_1 < \eta_2$, B_μ is bounded as mentioned in Step 2, and let $\psi \in B_\mu$. Then

$$\begin{aligned}&|\mathcal{T}(\psi)(\eta_2) - \mathcal{T}(\psi)(\eta_1)| \\ &\leq \sum_{0<\eta_k<\eta_2-\eta_1} |I_i(\overline{\psi}_{n_i+1})| \\ &\quad + \frac{1}{\Gamma(\zeta)} \int_{a+1-\zeta}^{\eta_1-\zeta+1} \left|(\eta_2-\sigma(\tau))^{\underline{\zeta-1}} - (\eta_1-\sigma(\tau))^{\underline{\zeta-1}}\right| \left|\mathscr{Q}(\tau+\zeta, \psi(\tau+\zeta))\right| \Delta\tau \\ &\quad + \frac{1}{\Gamma(\zeta)} \int_{\eta_1+1-\zeta}^{\eta_2-\zeta+1} \left|(\eta_2-\sigma(\tau))^{\underline{\zeta-1}}\right| \left|\mathscr{Q}(\tau+\zeta, \psi(\tau+\zeta))\right| \Delta\tau \\ &\leq \sum_{0<\eta_k<\eta_2-\eta_1} |I_i(\overline{\psi}_{n_i+1})| + \frac{\mathcal{M}}{\Gamma(\zeta)} \int_{a+1-\zeta}^{\eta_1-\zeta+1} \left|(\eta_2-\sigma(\tau))^{\underline{\zeta-1}} - (\eta_1-\sigma(\tau))^{\underline{\zeta-1}}\right| \Delta\tau \\ &\quad + \frac{\mathcal{M}}{\Gamma(\zeta)} \int_{\eta_1+1-\zeta}^{\eta_2-\zeta+1} \left|(\eta_2-\sigma(\tau))^{\underline{\zeta-1}}\right| \Delta\tau.\end{aligned}$$

As $\eta_1 \to \eta_2$, the right-hand side of the above inequality tends to zero. As an outcome of Steps 1 to 3 along with the Arzela-Ascoli hypothesis, we can conclude that $\mathcal{T}$ is completely continuous.

Step 4: Now it remains to show that the set

$$\mathcal{E} = \{\psi \in C(\mathbb{N}_a, \mathbb{R}) : \psi = \lambda\mathcal{T}(\psi) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $\psi \in \mathcal{E}$, then $\psi = \lambda \mathcal{T}(\psi)$ for some $0 < \lambda < 1$. Thus, we have

$$\psi(\eta) = \lambda\psi_0 + \lambda \sum_{i=1}^{k} I_i(\overline{\psi}_{n_i+1}) + \lambda \Delta_{a+1-\zeta}^{-\zeta} \mathscr{Q}(\eta + \zeta, \psi(\eta + \zeta)).$$

Now using the assupmtions (A_4) and (A_5) as in Step 2, we get

$$|\psi(\eta)| \leq |\psi_0| + (N-1)\mathcal{M}^* + \frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\mathcal{M}.$$

Therefore

$$\|\psi\| \leq |\psi_0| + (N-1)\mathcal{M}^* + \frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}\mathcal{M} := \mathcal{R}.$$

Hence, we get boundedness of the set $\mathcal{E}$. As a result of Schaefer's fixed point hypothesis, we conclude that $\mathcal{T}$ has a fixed point which is a solution of the problem (3.1). □

In the accompanying hypothesis we give an existence result for the problem (3.1) by applying the nonlinear alternative of Leray-Schauder type theorem.

Theorem 3.2.4. *Assume that assumption (A_2) with the following assumptions hold:*

(A_6) *There exists continuous and nondecreasing functions $\phi_{\mathscr{Q}} : \mathbb{N}_a \to \mathbb{R}^+$ and $\Psi : \mathbb{R}^+ \to (0, \infty)$ such that*

$$|\mathscr{Q}(\eta, \psi)| \leq \phi_{\mathscr{Q}}(\eta)\Psi(|\psi|).$$

(A_7) *There exists continuous and nondecreasing functions $\phi^* : \mathbb{R}^+ \to (0, \infty)$ such that*

$$|I_k(\psi)| \leq \phi^*(|\psi|).$$

(A_8) *There exists positive number $\overline{M}$ such that*

$$\frac{\overline{M}}{|\psi_0| + (N-1)\phi^*(\overline{M}) + \overline{\phi}_{\mathscr{Q}}\Psi(\overline{M})\dfrac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}} > 1,$$

where $\overline{\phi}_{\mathscr{Q}} = \sup\{\phi_{\mathscr{Q}}(\eta) : \eta \in \mathbb{N}_a\}$.

Then problem (3.1) *has atleast one solution.*

Proof. We can see easily that $\mathcal{T}$ is continuous, whereas, the operator $\mathcal{T}$ is defined as in previous theorems. For $\lambda \in [0,1]$, let ψ be such that for each $\eta \in \mathbb{N}_a$, we have $\psi(\eta) = \lambda(\mathcal{T}\psi)(\eta)$. Then from $(A_6) - (A_7)$ we have

$$
\begin{aligned}
|\psi(\eta)| &\leq |\psi_0| + \sum_{i=1}^{k} |I_i(\overline{\psi}_{n_i+1})| + \Delta_{a+1-\zeta}^{-\zeta} |\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta))| \\
&\leq |\psi_0| + \sum_{i=1}^{k} \phi^*(|\overline{\psi}_{n_i+1}|) + \Delta_{a+1-\zeta}^{-\zeta} \phi_{\mathscr{Q}}(\eta+\zeta)\Psi(|\psi(\eta+\zeta)|) \\
&\leq |\psi_0| + (N-1)\phi^*(\|\psi\|) + \overline{\phi}_{\mathscr{Q}}\Psi(\|\psi\|)\frac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}.
\end{aligned}
$$

Thus

$$
\frac{\|\psi\|}{|\psi_0| + (N-1)\phi^*(\|\psi\|) + \overline{\phi}_{\mathscr{Q}}\Psi(\|\psi\|)\dfrac{(T-(a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)}} \leq 1.
$$

Then by assupmtion (A_8), there exists $\overline{M}$ such that $\|\psi\| \neq \overline{M}$. Let

$$
\mathcal{U} = \left\{\psi \in C(\mathbb{N}_a, \mathbb{R}) : \|\psi\| < \overline{M}\right\}.
$$

The operator $\mathcal{T} : \overline{\mathcal{U}} \to C(\mathbb{N}_a, \mathbb{R})$ is continuous and completely continuous. There is no $\psi \in \partial\mathcal{U}$ such that $\psi = \lambda\mathcal{T}(\psi)$, where $0 < \lambda < 1$. As a result of the nonlinear alternative of Leray-Schauder type, we arrived at point that $\mathcal{T}$ has a fixed point ψ in $\overline{\mathcal{U}}$ which is a solution of the problem (3.1). This completes the proof. □

3.2.2 Gronwall Inequality

Theorem 3.2.5. *Let* ψ *be defined on* $\mathbb{N}_a$ *satisfying the inequality given below*

$$
|\psi(\eta)| \leq \mathfrak{d}_1(\eta) + \mathfrak{d}_2 \Delta_{a+1-\zeta}^{-\zeta} \psi(\eta+\zeta) + \sum_{0<\eta_l<\eta} \theta_l \left|\overline{\psi}_{n_l+1}\right|, \tag{3.7}
$$

where $\mathfrak{d}_1(\eta)$ *is non-negative discrete and non-decreasing on* $\mathbb{N}_a$ *and* $\mathfrak{d}_2,\ \theta_l \geq 0$ *are constants. Then*

$$|\psi(\eta)| \leq \mathfrak{d}_1(\eta)\left(1+\theta e_\zeta(\mathfrak{d}_2, \eta-a)\right)^l e_\zeta(\mathfrak{d}_2, \eta-a), \tag{3.8}$$

for $\eta \in \{a+n_l+1, ..., a+n_{l+1}\}$.

Moreover, $l = 1, 2, ..., N-1$ *and* $\theta = max\left\{\theta_l : l = 1, ...N-1\right\}$.

Proof. We will prove the Gronwall inequality in (3.8), using mathematical induction. For $\eta \in \{a+n_0+1, ..., a+n_1\}$, using Lemma 3.1.2 we derive,

$$|\psi(\eta)| \leq \mathfrak{d}_1(\eta) e_\zeta(\mathfrak{d}_2, \eta-a). \tag{3.9}$$

For $\eta \in \{a+n_l+1, ..., a+n_{l+1}\}$, $l = 1, 2, ..., N-1$,

$$|\psi(\eta)| \leq \left(\mathfrak{d}_1(\eta) + \sum_{i=1}^{l} \theta_i |\overline{\psi}_{n_i+1}|\right) e_\zeta(\mathfrak{d}_2, \eta-a). \tag{3.10}$$

For $l = 0$, inequality (3.8) holds using inequality (3.9).

Let us suppose that inequality (3.8) holds for l, where $l = 1, 2, ..., N-1$. Then by inequality (3.10) and since $\mathfrak{d}_1, e_\zeta(\mathfrak{d}_2, \eta-a)$ are nondecreasing.

For $\eta \in \{a+n_l+1+1, ..., a+n_{l+2}\}$, we have following estimate for ψ,

$$\begin{aligned}
|\psi(\eta)| &\leq \left(\mathfrak{d}_1(\eta) + \sum_{i=1}^{l+1} \theta_i \left|\overline{\psi}_{n_i+1}\right|\right) e_\zeta(\mathfrak{d}_2, \eta-a) \\
&\leq \left(\mathfrak{d}_1(\eta) + \sum_{i=1}^{l+1} \theta_i \mathfrak{d}_1(a+n_i+1)\left(1+\theta e_\zeta(\mathfrak{d}_2, n_i+1)\right)^{i-1} e_\zeta(\mathfrak{d}_2, n_i+1)\right) e_\zeta(\mathfrak{d}_2, \eta-a) \\
&\leq \left(\mathfrak{d}_1(\eta) + \theta \sum_{i=1}^{l+1} \mathfrak{d}_1(\eta)(1+\theta e_\zeta(\mathfrak{d}_2, \eta-a))^{i-1} e_\zeta(\mathfrak{d}_2, \eta-a)\right) e_\zeta(\mathfrak{d}_2, \eta-a) \\
&\leq \mathfrak{d}_1(\eta)\left(1+\theta e_\zeta(\mathfrak{d}_2, \eta-a)\right)^{l+1} e_\zeta(\mathfrak{d}_2, \eta-a).
\end{aligned}$$

Hence completing the proof. □

3.3 Stability Analysis

Remark 6. *Solution of inequality* (3.2) *is a discrete function* ς *defined on* $\mathbb{N}_a$ *iff there exists a discrete function* $\mathscr{G}$ *defined on* $\mathbb{N}_a$ *and a sequence* $\mathscr{G}_l$ *satisfying*

(i) $|\mathscr{G}(\eta)| \leq \epsilon,\ \eta \in \mathbb{N}_a$ *and* $|\mathscr{G}_l| \leq \epsilon,\ l = 1, 2, ..., N-1,$

(ii) ${}^c\Delta_a^\zeta \varsigma(\eta) = \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta)) + \mathscr{G}(\eta),\ 0 < \zeta \leq 1,\ \eta \in \mathbb{N}_{a+1-\zeta},\ \eta \neq a + n_l + 1 - \zeta,$

(iii) $\varsigma_{n_l+1} = \overline{\varsigma}_{n_l+1} + I_l(\overline{\varsigma}_{n_l+1}) + \mathscr{G}_l,\ \eta = a + n_l + 1 - \zeta,\ l = 1, 2, ..., N-1.$

Remark 7. *As* ς *is a solution of inequality* (3.2), *so the following inequality must be satisfied by* ς

$$|\varsigma(\eta) - \varsigma(a) - \sum_{i=1}^{l} I_i(\overline{\varsigma}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta} \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta))| \leq \left(N - 1 + \frac{(\eta - (a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)} \right) \epsilon.$$

By previous Remark, we have

$${}^c\Delta_a^\zeta \varsigma(\eta) = \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta)) + \mathscr{G}(\eta),\ 0 < \zeta \leq 1,\ \eta \in \mathbb{N}_{a+1-\zeta},\ \eta \neq a + n_l + 1 - \zeta,$$

$$\varsigma_{n_l+1} = \overline{\varsigma}_{n_l+1} + I_l(\overline{\varsigma}_{n_l+1}) + \mathscr{G}_l,\ \eta = a + n_l + 1 - \zeta,\ l = 1, 2, ..., N-1.$$

Then for $\eta \in \{a + n_l + 1, ..., a + n_{l+1}\}$, *where* $l = 1, 2, ..., N-1,$

$$\varsigma(\eta) = \varsigma(a) + \sum_{i=1}^{l} I_i(\overline{\varsigma}_{n_i+1}) + \sum_{i=1}^{l} \mathscr{G}_i + \Delta_{a+1-\zeta}^{-\zeta} \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta)) + \Delta_{a+1-\zeta}^{-\zeta} \mathscr{G}(\eta),$$

$$\begin{aligned} |\varsigma(\eta) - \varsigma(a) - \sum_{i=1}^{l} I_i(\overline{\varsigma}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta} \mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta))| &= |\sum_{i=1}^{l} \mathscr{G}_i + \Delta_{a+1-\zeta}^{-\zeta} \mathscr{G}(\eta)| \\ \leq |\sum_{i=1}^{N-1} \mathscr{G}_i| + |\Delta_{a+1-\zeta}^{-\zeta} \mathscr{G}(\eta)| \leq (N-1)\epsilon + \Delta_{a+1-\zeta}^{-\zeta} \epsilon &= \left(N - 1 + \frac{(\eta - (a+1-\zeta))^{\underline{\zeta}}}{\Gamma(\zeta+1)} \right) \epsilon. \end{aligned}$$

Theorem 3.3.1. *Assume* $\mathscr{Q}$ *defined on* $\mathbb{N}_a$ *satisfies Lipschitz condition with respect to second variable i.e* $|\mathscr{Q}(\eta, p) - \mathscr{Q}(\eta, q)| \leq L_{\mathscr{Q}} |p - q|$ *for all* $p, q \in \mathbb{R}$, *where* $L_{\mathscr{Q}} > 0$ *is Lipschitz constant.*

Moreover, there exists a constant $\rho_k > 0$ and a real-valued function I_k such that $|I_k(p) - I_k(q)| \leq \rho_k |p - q|$ for all $p, q \in \mathbb{R}$. Furthermore, if for $\lambda_\varphi > 0$ we have $\Delta_{a+1-\zeta}^{-\zeta}\varphi(\eta) \leq \lambda_\varphi \varphi(\eta)$, where $\varphi : \mathbb{N}_a \to \mathbb{R}^+$ is nondecreaing. Then problem (3.1) is generalized Ulam-Hyers-Rassias stable with respect to φ.

Proof. Let ψ be a unique solution of the following fractional difference equation with impulse condition whereas ς be a solution of inequality (3.3),

$$\begin{cases} {}^c\Delta_a^\zeta \psi(\eta) & = \mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),\ \eta \in \mathbb{N}_{a+1-\zeta},\ \eta \neq a+n_l+1-\zeta,\ 0<\zeta\leq 1, \\ \psi_{n_l+1} & = \overline{\psi}_{n_l+1} + I_l(\overline{\psi}_{n_l+1}),\ \eta = a+n_l+1-\zeta,\ l = 1,2,...,N-1, \\ \psi(a) & = \varsigma(a). \end{cases} \tag{3.11}$$

Then we have

$$\psi(\eta) = \begin{cases} \varsigma(a) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),\ \eta \in \{a+n_0+1, ..., a+n_1\}, \\ \vdots \\ \varsigma(a) + \sum_{i=1}^{l} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)), \\ \qquad \eta \in \{a+n_l+1, ..., a+n_{l+1}\}, \\ \vdots \\ \varsigma(a) + \sum_{i=1}^{N} I_i(\overline{\psi}_{n_i+1}) + \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \psi(\eta+\zeta)),\ \eta \in \{a+n_N+1, ...\}, \end{cases} \tag{3.12}$$

where $l = 1, ..., N-1$ *and* $N \to \infty$.

Like in Remark 7, for $\eta \in \{a+n_l+1, ..., a+n_{l+1}\}$ where $l = 1, 2, ..., N-1$, we have

$$\begin{aligned} \left|\varsigma(\eta) - \varsigma(a) - \sum_{i=1}^{l} I_i(\overline{\varsigma}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta, \varsigma(\eta+\zeta))\right| &\leq \sum_{i=1}^{N-1} |\mathscr{G}_i| + \Delta_{a+1-\zeta}^{-\zeta}\varphi(\eta) \\ &\leq (N-1+\lambda_\varphi)\varphi(\eta). \end{aligned}$$

Hence, for $\eta \in \{a+n_l+1, ..., a+n_{l+1}\}$, $l = 1, 2, ..., N-1$, we have

$$\begin{aligned}
|\varsigma(\eta)-\psi(\eta)| &= |\varsigma(\eta)-\varsigma(a)-\sum_{i=1}^{l} I_i(\overline{\psi}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))| \\
&\leq |\varsigma(\eta)-\varsigma(a)-\sum_{i=1}^{l} I_i(\overline{\varsigma}_{n_i+1}) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\varsigma(\eta+\zeta))| \\
&\quad + |\sum_{i=1}^{l} I_i(\overline{\varsigma}_{n_i+1}) - \sum_{i=1}^{l} I_i(\overline{\psi}_{n_i+1})| \\
&\quad + |\Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\varsigma(\eta+\zeta)) - \Delta_{a+1-\zeta}^{-\zeta}\mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))| \\
&\leq (N-1+\lambda_\varphi)\varphi(\eta) + \sum_{i=1}^{l} |I_i(\overline{\varsigma}_{n_i+1}) - I_i(\overline{\psi}_{n_i+1})| \\
&\quad + \Delta_{a+1-\zeta}^{-\zeta}|\mathscr{Q}(\eta+\zeta,\varsigma(\eta+\zeta)) - \mathscr{Q}(\eta+\zeta,\psi(\eta+\zeta))| \\
|\varsigma(\eta)-\psi(\eta)| &\leq (N-1+\lambda_\varphi)\varphi(\eta) + \sum_{i=1}^{l} \rho_i|\overline{\varsigma}_{n_i+1} - \overline{\psi}_{n_i+1}| \\
&\quad + \Delta_{a+1-\zeta}^{-\zeta} L_{\mathscr{Q}}|\varsigma(\eta+\zeta)-\psi(\eta+\zeta)|.
\end{aligned}$$

Using Theorem 3.2.5 (Gronwall inequality), there exists a positive constant $\mathscr{M}^*$, independant of $\lambda_\varphi\varphi(\eta)$, so we have

$|\varsigma(\eta)-\psi(\eta)| \leq \mathscr{M}^*(N-1+\lambda_\varphi)\varphi(\eta) = \chi\varphi(\eta)$.

So problem (3.1) is generalized Ulam-Hyers-Rassias stable. Hence completing the proof. □

Example 3.3.1. *Let us consider the fractional difference equation with impulsive effect*

$$^c\Delta_0^\zeta\psi(\eta) = 0,\ \eta \in (0,1] \cap \mathbb{N}_0 \setminus \left\{\frac{1}{2}\right\},\ \psi\left(\frac{1}{2}^+\right) - \psi\left(\frac{1}{2}^-\right) = \frac{|\psi(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|\psi(\frac{1}{2}^-)|^{\frac{1}{2}}}, \tag{3.13}$$

also consider the following inequalities

$$|^c\Delta_0^\zeta\varsigma(\eta)| \leq \epsilon,\ \eta \in (0,1] \cap \mathbb{N}_0 \setminus \left\{\frac{1}{2}\right\},\ \left|\varsigma\left(\frac{1}{2}^+\right) - \varsigma\left(\frac{1}{2}^-\right) - \frac{|\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}}{1+|\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}}\right| \leq \epsilon,\ \epsilon > 0. \tag{3.14}$$

Let discrete function ς be a solution of inequality (3.14)*. Then there exists a discrete function $\mathscr{G}$ and $\mathscr{G}_1 \in \mathbb{R}$, so we have following* *(i)* $|\mathscr{G}(\eta)| \leq \epsilon,\ \eta \in [0,1] \cap \mathbb{N}_0,\ |\mathscr{G}_1| \leq \epsilon,$

(ii) ${}^c\Delta_0^{\zeta}\varsigma(\eta) = \mathscr{G}(\eta),\ \eta \in [0,1] \cap \mathbb{N}_0 \setminus \left\{\frac{1}{2}\right\},$

(iii) $\varsigma\left(\frac{1}{2}^+\right) - \varsigma\left(\frac{1}{2}^-\right) = \dfrac{|\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}}{1 + |\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}} + \mathscr{G}_1.$

For $(i), (ii)$ *and* (iii), *using Remark 7, we have*

$$\varsigma(\eta) = \varsigma(0) + \varphi_{(1/2,1]\cap\mathbb{N}_0}(\eta)\left(\frac{|\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}}{1 + |\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}} + \mathscr{G}_1\right) + \Delta_{1-\zeta}^{-\zeta}\mathscr{G}(\eta),$$

where $\varphi_{(1/2,1]\cap\mathbb{N}_0}(\eta)$ *is characteristic function of* $(1/2, 1] \cap \mathbb{N}_0$.

Let ψ *be the unique solution of problem* (3.13),
$\psi(\eta) = \varsigma(0) + \varphi_{(1/2,1]\cap\mathbb{N}_0}(\eta)\dfrac{|\psi(\frac{1}{2}^-)|^{\frac{1}{2}}}{1 + |\psi(\frac{1}{2}^-)|^{\frac{1}{2}}}.$

$$\begin{aligned}
|\varsigma(\eta) - \psi(\eta)| &= \left|\varphi_{(1/2,1]\cap\mathbb{N}_0}(\eta)\left(\frac{|\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}}{1 + |\varsigma(\frac{1}{2}^-)|^{\frac{1}{2}}} - \frac{|\psi(\frac{1}{2}^-)|^{\frac{1}{2}}}{1 + |\psi(\frac{1}{2}^-)|^{\frac{1}{2}}} + \mathscr{G}_1\right) + \Delta_{1-\zeta}^{-\zeta}\mathscr{G}(\eta)\right| \\
&\leq \varphi_{(1/2,1]\cap\mathbb{N}_0}(\eta)\left|\varsigma\left(\frac{1}{2}^-\right) - \psi\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}} + |\mathscr{G}_1| + \Delta_{1-\zeta}^{-\zeta}|\mathscr{G}(\eta)| \\
&\leq \varphi_{(1/2,1]\cap\mathbb{N}_0}(\eta)\left|\varsigma\left(\frac{1}{2}^-\right) - \psi\left(\frac{1}{2}^-\right)\right|^{\frac{1}{2}} + \epsilon + \epsilon,\ \eta \in [0,1] \cap \mathbb{N}_0 \\
&\leq \sqrt{2\epsilon} + 2\epsilon,\ \eta \in [0,1] \cap \mathbb{N}_0.
\end{aligned}$$

Hence problem (3.13) *is generalized Ulam-Hyers stable.*

Chapter 4

Stability Analysis of Caputo q-Fractional Delay Difference Equation Using q-Fractional Gronwall Inequality

In this chapter, we talk about the existence and uniqueness of solution of a delay Caputo q-fractional difference system. Ulam-Hyers stability and Ulam-Hyers-Rassias stability are established using existing q-fractional Gronwall inequality. The consequences of this chapter are published in [44].

In the theory of differential equations, Gronwall's inequality is one of the most important tools. In 1919, for the first time, Gronwall worked on this type of inequality [75]. As time passed, many extensions of the Gronwall inequality have started to take part of the literature on mathematical inequalities. In 1935, Mikeladze published about the discrete fractional Gronwall inequality for the first time [115]. Gronwall's inequality is useful in the analysis of qualitative and quantitative properties of the ordinary and fractional dynamical systems. That is why it attracted many researchers to work on it. Haiping Ye et al. [147] presented a generalized Gronwall inequality and studied the dependence of the solution on the order and the initial condition of a fractional differential equation. Very recently, the authors in [25], proved a Gronwall inequality for the generalized proportional fractional operators. A class of stochastic

Gronwall inequalities have been studied by Wang et al. in [138]. Luo et al. in [111] studied the uniqueness and novel finite-time stability of solutions of delay difference equations using Gronwall inequality approach. Recently, Almeida et al. in [21] and Yassine et al. in [12] presented an extension of the fractional Gronwall inequality and used it in the qualitative analysis of the solutions to generalized fractional differential equations. Difference equations have appeared in mathematical modelling to describe many real life problems, e.g., queueing problems, electrical networks, economics etc. For that reason, many researchers have proved discrete versions of Gronwall type inequalities in fractional calculus and applied them to study the qualitative and quantitative properties of fractional difference equations [35, 70, 146, 66, 5, 24]. Moreover, Gronwall's inequality is widely used for the analysis of stability of fractional differential as well as fractional difference equations. In one of the most recent works, Ameen et al. discussed the Ulam stability of delay fractional differential equations with a generalized Caputo derivative using a Gronwall inequality approach [26]. Liu et al. [108] also presented Ulam-Hyers stability of solutions for differential equations with Caputo-Fabrizio fractional derivative with the help of Gronwall's inequality. For further assistance in stability analysis using Gronwall inequality approach, one can follow the articles cited in [144, 131, 135]. As for the stability results without any Gronwall approach for the q-fractional systems, we refer to the first two works [81, 83]. Since stability, and specially Ulam-Hyers stability, is of high priority for researchers and has been studied for applied as well as mathematical problems, one can follow the most recent articles on stability cited in [88, 92, 87]. For recent operator and mathematical models, whose stability analysis is an open dilemma, we refer to [94, 98, 89].

In q-fractional calculus most probably the first article on q-fractional Gronwall inequality was presented by Abdeljawad et al. in [6]. Later on, another new Gronwall inequality in q-fractional calculus was proved in [8], where the authors considered a nonlinear delay Caputo q-fractional difference system and discussed the uniqueness and estimates for the solutions of the system under consideration.

The objective of this chapter is to study the existence, uniqueness and then analyze the Ulam-Hyers and Ulam-Hyers-Rassias stability of Caputo q-fractional difference equation with delay of the form

$$\begin{cases} {}^{c}_{q}\nabla^{\zeta}_{a}\psi(\eta) &= \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))), \quad \eta \in \mathbb{T}_a, \\ \psi(\eta) &= \Phi(\eta), \quad \eta \in \mathbb{I}_\tau, \end{cases} \tag{4.1}$$

where $\psi : \mathbb{T}_{\tau a} \to \mathbb{R}$, $\mathscr{Q} : \mathbb{T}_a \times \mathbb{R}^2 \to \mathbb{R}$, $\mathscr{G} : \mathbb{T}_a \to \mathbb{T}_{\tau a}$, $\Phi : \mathbb{I}_\tau \to \mathbb{R}$ and ${}^{c}_{q}\nabla^{\zeta}_{a}$ denotes the Caputo q-fractional difference operator of order $0 < \zeta < 1$. To demonstrate our fundamental outcomes, we use the q-fractional Gronwall inequality proved in [8].

This chapter is composed as follows: In Section 4.1, we present some basic definitions, notations, lemmas and remarks that are important for proving our main results. In Section 4.2, we discuss existence and uniqueness of solution of problem (4.1). Then, we discuss the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of above mentioned problem in Section 4.3 and 4.4. Example is also provided to illustrate the theoretical results.

4.1 Fundamental results

In this section, we provide basic results that are helpful in proving our main results.

Definition 4.1.1. *If for all $\epsilon > 0$ there exists a real number c such that for each $\varsigma(\eta)$ defined on $\mathbb{T}_{\tau a}$, satisfying the inequality*

$$|{}^{c}_{q}\nabla^{\zeta}_{a}\varsigma(\eta) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))| \leq \epsilon, \quad \eta \in \mathbb{T}_a, \tag{4.2}$$

there exists a solution $\psi(\eta)$ defined on $\mathbb{T}_{\tau a}$ of Eq. (4.1) satisfying

$$|\varsigma(\eta) - \psi(\eta)| \leq c\epsilon, \quad \eta \in \mathbb{T}_{\tau a}.$$

Then Eq. (4.1) is said to be Ulam-Hyers stable.

Definition 4.1.2. *If for all $\epsilon > 0$ there exists a real number c such that for each $\varsigma(\eta)$ defined* *on $\mathbb{T}_{\tau a}$ satisfying the inequality*

$$|{}_q^c\nabla_a^\zeta\varsigma(\eta) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))| \leq \epsilon\Psi(\eta), \quad \eta \in \mathbb{T}_a, \tag{4.3}$$

then there exists a solution $\psi(\eta)$ defined on $\mathbb{T}_{\tau a}$ of Eq. (4.1) *satisfying*

$$|\varsigma(\eta) - \psi(\eta)| \leq c\epsilon\Psi(\eta), \quad \eta \in \mathbb{T}_{\tau a}.$$

Then Eq. (4.1) is said to be Ulam-Hyers-Rassias stable with respect to $\Psi(\eta)$ where $\Psi : \mathbb{T}_a \to \mathbb{R}^+$.

The following lemma is the key to proceed.

Lemma 4.1.3. *[8] Let $\zeta > 0$, $\psi(\eta)$ and $\varsigma(\eta)$ be nonnegative functions, $w(\eta)$ be nonnegative and nondecreaing function for $\eta \in \mathbb{T}_a$ such that $w(\eta) \leq M$, where M is a constant. If*

$$\psi(\eta) \leq \varsigma(\eta) + w(\eta){}_q\nabla_a^{-\zeta}\psi(\eta),$$

then

$$\psi(\eta) \leq \varsigma(\eta) + \sum_{k=1}^{\infty}(w(\eta)\Gamma_q(\zeta))^k \; {}_q\nabla_a^{-k\zeta}\varsigma(\eta).$$

Corollary 4.1.4. *[8] Under the hypothesis of Lemma 4.1.3, assume further that $\varsigma(\eta)$ is a nondecreasing function for $\eta \in \mathbb{T}_a$, then*

$$\psi(\eta) \leq \varsigma(\eta) \; {}_qE_\zeta(w(\eta)\Gamma_q(\zeta), \eta - a), \quad \eta \in \mathbb{T}_a,$$

where ${}_qE_\zeta(\lambda, \eta - a) = \sum_{k=0}^{\infty} \lambda^k \frac{(\eta-a)_q^{k\zeta}}{\Gamma_q(k\zeta+1)}$ is the q-Mittag-Leffler function.

4.2 Existence and Uniqueness Results

Consider the space $X = l_\infty(\mathbb{T}_{\tau a})$ of bounded functions (sequences) on $\mathbb{T}_{\tau a}$, where $\mathbb{T}_{\tau a} = [\tau a, \infty)_q$. The space X is a Banach space with the norm defined by $\|z\|_X = \sup_{\eta \in \mathbb{T}_{\tau a}} |z(\eta)|$. In the following lemma we present the solution representation.

Lemma 4.2.1. *$\psi(\eta)$ satisfies Eq. (4.1) if and only if it satisfies the following q-sum equation*

$$\psi(\eta) = \begin{cases} \Phi(\eta) & \eta \in \mathbb{I}_\tau, \\ \Phi(a) + {}_q\nabla_a^{-\zeta} \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))), & \eta \in \mathbb{T}_a. \end{cases} \tag{4.4}$$

Proof. For $\eta \in \mathbb{I}_\tau$, it is clear that $\psi(\eta) = \Phi(\eta)$ is the solution of Eq. (4.1). Now for $\eta \in \mathbb{T}_a$, applying ${}_q\nabla_a^\zeta$ on both sides of Eq. (4.4), we get

$${}_q\nabla_a^\zeta \psi(\eta) = {}_q\nabla_a^\zeta \Phi(a) + {}_q\nabla_a^\zeta ({}_q\nabla_a^{-\zeta}) \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))).$$

Using the fact ${}_q\nabla_a^\zeta ({}_q\nabla_a^{-\zeta})\psi(\eta) = \psi(\eta)$, we have

$${}_q^c\nabla_a^\zeta \psi(\eta) = \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))).$$

On the other hand, from Eq. (4.4), for $\eta \in \mathbb{I}_\tau$, we have $\psi(\eta) = \Phi(\eta)$. Also, by applying ${}_q\nabla_a^{-\zeta}$ on both sides of Eq. (4.1) and making use of Lemma 1.2.27, we get

$$\psi(\eta) = \psi(a) + {}_q\nabla_a^{-\zeta} \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))),$$

hence we get

$$\psi(\eta) = \Phi(a) + {}_q\nabla_a^{-\zeta} \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))).$$

□

Now we present the following uniqueness theorem.

Theorem 4.2.1. *Assume the following:*

(A_1) $\mathscr{Q}$ and Φ are continuous functions defined as $\mathscr{Q} : \mathbb{T}_a \times \mathbb{R}^2 \to \mathbb{R}$ and $\Phi : \mathbb{I}_\tau \to \mathbb{R}$;

(A_2) *$\mathscr{Q}$ satisfies Lipschitz condition with $\mathscr{L} > 0$ such that for $\eta \in \mathbb{T}_a$,*

$$\|\mathscr{Q}(\eta, \psi_1, \psi_2) - \mathscr{Q}(\eta, \varsigma_1, \varsigma_2)\| \leq \mathscr{L}(\|\psi_1 - \varsigma_1\| + \|\psi_2 - \varsigma_2\|);$$

(A_3) *The map $\mathscr{G}$ preserves the delay interval $\mathbb{I}_\tau$.*

If $\psi(\eta)$ and $\varsigma(\eta)$ satisfy the problem (4.1), then $\psi(\eta) = \varsigma(\eta)$.

Proof. Let $z(\eta) = \psi(\eta) - \varsigma(\eta)$, then we have to show that $z(\eta) = 0$. It is immediate that $z(\eta) = 0$ for $\eta \in \mathbb{I}_\tau$. For $\eta \in \mathbb{T}_a$, we have

$$z(\eta) = {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))).$$

If $\eta \in \mathbb{I}_{\tau^{-1}} = \{a, ..., \tau^{-1}a\}$, then (A_3) forces $z(\mathscr{G}(\tau\eta)) = 0$. Then, together with the other assumptions it will imply that

$$\begin{aligned}
\|z(\eta)\| &= \left\|{}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))\right\| \\
&\leq {}_q\nabla_a^{-\zeta}\|\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))\| \\
&\leq {}_q\nabla_a^{-\zeta}\mathscr{L}\left(\|\psi(\eta) - \varsigma(\eta)\| + \|\psi(\mathscr{G}(\tau\eta)) - \varsigma(\mathscr{G}(\tau\eta))\|\right) \\
&\leq {}_q\nabla_a^{-\zeta}\mathscr{L}\left(\|z(\eta)\| + \|z(\mathscr{G}(\tau\eta))\|\right) \\
&\leq \frac{\mathscr{L}}{\Gamma_q(\zeta)}\int_a^\eta (\eta - qs)_q^{\zeta-1}\|z(s)\|\,\nabla_q s.
\end{aligned}$$

An application of Corollary 4.1.4 will imply

$$\|z(\eta)\| \leq 0.{}_qE_\zeta[\mathscr{L}\Gamma_q(\zeta), \eta - a],$$

and hence $z(\eta) = 0$ for $\eta \in \mathbb{I}_{\tau^{-1}}$. Next for $\eta \in [\tau^{-1}a, \infty)_q$, we have

$$\begin{aligned}
\|z(\eta)\| &= \left\| {}_q\nabla_a^{-\zeta} \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))) - {}_q\nabla_a^{-\zeta} \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) \right\| \\
&\leq \ {}_q\nabla_a^{-\zeta} \left\| \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) \right\| \\
&\leq \ {}_q\nabla_a^{-\zeta} \mathscr{L} \left(\|\psi(\eta) - \varsigma(\eta)\| + \|\psi(\mathscr{G}(\tau\eta)) - \varsigma(\mathscr{G}(\tau\eta))\| \right) \\
&\leq \frac{\mathscr{L}}{\Gamma_q(\zeta)} \int_a^\eta (\eta - qs)_q^{\zeta-1} (\|z(s)\| + \|z(\mathscr{G}(\tau s))\|) \nabla_q s.
\end{aligned}$$

If we let $\hat{z}(\eta) = \sup_{\beta \in \mathbb{I}_\tau} \|z(\mathscr{G}(\beta\eta))\|$, then we have

$$\begin{aligned}
\hat{z}(\eta) &\leq \frac{\mathscr{L}}{\Gamma_q(\zeta)} \int_a^\eta (\eta - qs)_q^{\zeta-1} (\hat{z}(s) + \hat{z}(s)) \nabla_q s \\
&\leq \frac{2\mathscr{L}}{\Gamma_q(\zeta)} \int_a^\eta (\eta - qs)_q^{\zeta-1} \hat{z}(s) \nabla_q s.
\end{aligned}$$

Finally, Corollary 4.1.4 implies that

$$\|z(\eta)\| \leq \hat{z}(\eta) \leq 0. {}_qE_\zeta[2\mathscr{L}\Gamma_q(\zeta), \eta - a].$$

Hence again we have $z(\eta) = 0$, i.e., $\psi(\eta) = \varsigma(\eta)$ for $\eta \in \mathbb{T}_{\tau a}$. □

Now we present the following existence and uniqueness theorem for the problem (4.1).

Theorem 4.2.2. *With the assumptions* (A_1), (A_2) *and* (A_3), *let us assume that*
(A_4) $\dfrac{2\mathscr{L}(T-a)_q^\zeta}{\Gamma_q(\zeta+1)} < 1$, *for some* $T > a$.

Then problem (4.1) *has a unique solution in* $\mathbb{T}_{\tau a}$.

Proof. On the Banach space $X = l_\infty(\mathbb{T}_{\tau a})$ define an operator as follows:

$$\mathscr{T}\psi(\eta) = \begin{cases} \Phi(\eta), & \eta \in \mathbb{I}_\tau, \\ \Phi(a) + {}_q\nabla_a^{-\zeta} \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))), & \eta \in \mathbb{T}_a. \end{cases}$$

For $\eta \in \mathbb{I}_\tau$, we have $|\mathscr{T}\varsigma(\eta) - \mathscr{T}\psi(\eta)| = 0$, $\psi, \varsigma \in X$. Now for $\eta \in \mathbb{T}_a$,

$$\begin{aligned}
|\mathscr{T}\varsigma(\eta) - \mathscr{T}\psi(\eta)| &= |{}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta,\varsigma(\eta),\varsigma(\mathscr{G}(\tau\eta))) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta,\psi(\eta),\psi(\mathscr{G}(\tau\eta)))| \\
&\leq {}_q\nabla_a^{-\zeta}|\mathscr{Q}(\eta,\varsigma(\eta),\varsigma(\mathscr{G}(\tau\eta))) - \mathscr{Q}(\eta,\psi(\eta),\psi(\mathscr{G}(\tau\eta)))| \\
&\leq \mathscr{L}(\sup_{\eta\in\mathbb{T}_{\tau a}}|\varsigma(\eta)-\psi(\eta)| + \sup_{\eta\in\mathbb{T}_{\tau a}}|\varsigma(\mathscr{G}(\tau\eta))-\psi(\mathscr{G}(\tau\eta))|)\ {}_q\nabla_a^{-\zeta}(1) \\
&\leq \frac{2\mathscr{L}(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}\|\varsigma-\psi\|_X \\
&\leq \frac{2\mathscr{L}(T-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}\|\varsigma-\psi\|_X, \qquad \eta < T.
\end{aligned}$$

Since $\dfrac{2\mathscr{L}(T-a)_q^{\zeta}}{\Gamma_q(\zeta+1)} < 1$, then the operator $\mathscr{T}$ is a contraction and by Banach fixed point theorem there exists a unique fixed point. Clearly this unique fixed point is the unique solution of the problem (4.1). □

4.3 Ulam-Hyers Stability

In the remainder of what follows, we demonstrate the Ulam-Hyers stability of problem (4.1).

Lemma 4.3.1. *If a function $\varsigma(\eta)$ defined on $\mathbb{T}_{\tau a}$ is a solution of inequality* (4.2)*, then $\varsigma(\eta)$ satisfies the following inequality*

$$|\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta,\varsigma(\eta),\varsigma(\mathscr{G}(\tau\eta)))| \leq \frac{\epsilon(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}.$$

Proof. As we know that $\varsigma(\eta)$ satisfies inequality (4.2) if and only if there exists a function $h(\eta)$ such that $|h(\eta)| \leq \epsilon$ and

$$ {}_q^c\nabla_a^{\zeta}\varsigma(\eta) - \mathscr{Q}(\eta,\varsigma(\eta),\varsigma(\mathscr{G}(\tau\eta))) = h(\eta), \quad \eta \in \mathbb{T}_a. \tag{4.5}$$

Now applying ${}_q\nabla_a^{-\zeta}$ on both sides of Eq. (4.5), we get

$$({}_q\nabla_a^{-\zeta})({}_q^c\nabla_a^{\zeta})\varsigma(\eta) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) = {}_q\nabla_a^{-\zeta}h(\eta),$$

$$\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) = {}_q\nabla_a^{-\zeta}h(\eta).$$

Hence it follows that

$$\begin{aligned}|({}_q\nabla_a^{-\zeta})({}_q^c\nabla_a^{\zeta})\varsigma(\eta) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))| &\leq {}_q\nabla_a^{-\zeta}|h(\eta)| \leq \epsilon\, {}_q\nabla_a^{-\zeta}(1)\\ &= \frac{\epsilon(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)} = \frac{\epsilon(T-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}, \qquad \eta < T.\end{aligned}$$

□

Theorem 4.3.1. *Under the assumptions* (A_1)*,* (A_2)*,* (A_3) *and* (A_4)*, Eq.* (4.1) *is Ulam-Hyers stable.*

Proof. Assume $\varsigma(\eta)$ is solution of inequality (4.2) and $\psi(\eta)$ be unique solution of Eq. (4.1) satisfying the condition $\psi(\eta) = \varsigma(\eta)$ for $\eta \in \mathbb{I}_\tau$. Then we have

$$\psi(\eta) = \begin{cases} \varsigma(\eta), & \eta \in \mathbb{I}_\tau, \\ \varsigma(a) + {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))), & \eta \in \mathbb{T}_a. \end{cases}$$

For $\eta \in \mathbb{I}_\tau$, we have $|\varsigma(\eta) - \psi(\eta)| = 0$. Now for $\eta \in \mathbb{I}_{\tau^{-1}} = \{a, ..., \tau^{-1}a\}$, we have

$$\begin{aligned}|\varsigma(\eta) - \psi(\eta)| &= |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta)))|\\ &\leq |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))|\\ &\quad + |{}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta)))|\\ &\leq |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))|\\ &\quad + |{}_q\nabla_a^{-\zeta}(\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) - \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))))|\\ &\leq |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))| + \mathscr{L}\, {}_q\nabla_a^{-\zeta}(|\varsigma(\eta) - \psi(\eta)|),\end{aligned}$$

where we have used $\varsigma(\mathscr{G}(\tau\eta))-\psi(\mathscr{G}(\tau\eta))=0$ for $\eta\in\mathbb{I}_{\tau^{-1}}$. Now using Lemma 4.3.1, we conclude that

$$|\varsigma(\eta)-\psi(\eta)|\leq\frac{\epsilon(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}+\mathscr{L}\ {}_q\nabla_a^{-\zeta}(|\varsigma(\eta)-\psi(\eta)|).$$

Since $\dfrac{\epsilon(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}$ is non-negative and non-decreasing function $\forall\ \eta\in\mathbb{I}_{\tau^{-1}}$, then using Gronwall's inequality in Corollary 4.1.4, we see that

$$|\varsigma(\eta)-\psi(\eta)|\leq\left[\frac{(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}\ {}_qE_{\zeta}(\mathscr{L}\Gamma_q(\zeta),\eta-a)\right]\epsilon,\quad\forall\ \eta\in\mathbb{I}_{\tau^{-1}}.$$

Now for $\eta\in[\tau^{-1}a,\infty)_q$, following the same steps as mentioned above, we get

$$\begin{aligned}|\varsigma(\eta)-\psi(\eta)|&\leq|\varsigma(\eta)-\varsigma(a)-\ {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta,\varsigma(\eta),\varsigma(\mathscr{G}(\tau\eta)))|\\&\quad+\mathscr{L}\ {}_q\nabla_a^{-\zeta}(|\varsigma(\eta)-\psi(\eta)|+|\varsigma(\mathscr{G}(\tau\eta))-\psi(\mathscr{G}(\tau\eta))|).\end{aligned}$$

Let $\hat{z}(\eta)=\sup_{\beta\in\mathbb{I}_\tau}|\varsigma(\mathscr{G}(\beta\eta))-\psi(\mathscr{G}(\beta\eta))|$, so we get

$$\begin{aligned}\hat{z}(\eta)&\leq\frac{\epsilon(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}+\mathscr{L}\ {}_q\nabla_a^{-\zeta}(\hat{z}(\eta)+\hat{z}(\eta))\\&\leq\frac{\epsilon(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}+2\mathscr{L}\ {}_q\nabla_a^{-\zeta}\hat{z}(\eta).\end{aligned}$$

Similarly, the use of Gronwall's inequality in Corollary 4.1.4, will lead to

$$\hat{z}(\eta)\leq\left[\frac{(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}\ {}_qE_{\zeta}(2\mathscr{L}\Gamma_q(\zeta),\eta-a)\right]\epsilon,\quad\forall\ \eta\in[\tau^{-1}a,\infty)_q.$$

$$|\varsigma(\eta)-\psi(\eta)|\leq\hat{z}(\eta)\leq\left[\frac{(\eta-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}\ {}_qE_{\zeta}(2\mathscr{L}\Gamma_q(\zeta),\eta-a)\right]\epsilon,\quad\forall\ \eta,$$

or

$$|\varsigma(\eta)-\psi(\eta)|\leq\left[\frac{(T-a)_q^{\zeta}}{\Gamma_q(\zeta+1)}\ {}_qE_{\zeta}(2\mathscr{L}\Gamma_q(\zeta),T-a)\right]\epsilon,\quad\forall\ \eta<T.$$

That is

$$|\varsigma(\eta) - \psi(\eta)| \leq c\epsilon.$$

Hence completing the proof. □

Example 4.3.2. *The following example will illustrate Theorem 4.3.1.*

Let us consider the following problem

$$\begin{aligned} {}_q^c\nabla_a^{\frac{1}{2}}\psi(\eta) &= \frac{\sin(\psi(\eta)) + \sin(\psi(\tau\eta))}{200}, && \eta \in \mathbb{T}_a, \\ \psi(\eta) &= \cos 2\eta, && \eta \in \mathbb{I}_\tau. \end{aligned}$$

(A_1) $\mathscr{Q}(\eta, \psi, \psi^*) = \dfrac{\sin\psi + \sin\psi^*}{200}$ *and* $\Phi(\eta) = \cos 2\eta$ *are continuous functions.*
(A_2) $\mathscr{Q}$ *satisfies Lipschitz condition with Lipschitz constant* $\mathscr{L} = \dfrac{1}{200}$ *as follows:*

$$\begin{aligned} |\mathscr{Q}(\eta, \psi_1, \psi_2) - \mathscr{Q}(\eta, \varsigma_1, \varsigma_2)| &\leq \frac{1}{200}(|\sin\psi_1 - \sin\varsigma_1| + |\sin\psi_2 - \sin\varsigma_2|) \\ &\leq \frac{1}{200}(|\psi_1 - \varsigma_1| + |\psi_2 - \varsigma_2|). \end{aligned}$$

(A_3) $\mathscr{G}(\tau\eta) = \tau\eta$ *preserves the delay interval* $\mathbb{I}_\tau$,
(A_4) $\dfrac{2\mathcal{L}(T-a)_q^{\zeta}}{\Gamma_q(\zeta+1)} = \dfrac{2(\frac{1}{200})(T-a)_q^{\frac{1}{2}}}{\Gamma_q(\frac{1}{2}+1)} < 1$, *for some* $T > a$.
Hence all conditions are satisfied, so the problem under consideration is Ulam-Hyers stable.

The Lotka-Volterra model set up by Lotka and Volterra during the 1920s is huge in present day biology hypothesis. Due to applicability of fractional operators in real world problems numerous researchers have started to contemplate fractional Lotka-Volterra systems [15, 64]. This model in q-calculus was discussed by Abdeljawad et al. [9] for existence of solution using fixed point theorem. Now we employ Theorem 4.3.1 to prove existence and then stability results for Lotka-Volterra model using q-fractional Gronwall inequality approach.

Example 4.3.3. *Consider the model as follows:*

$$
\begin{aligned}
{}^{c}_{q}\nabla^{\zeta}_{a}\psi(\eta) &= \psi(\eta)\big(\gamma(\eta) - \beta(\eta)\psi(\tau\eta)\big), && \eta \in \mathbb{T}_a, \\
\psi(\eta) &= \Phi(\eta), && \eta \in \mathbb{I}_\tau.
\end{aligned} \tag{4.6}
$$

where $\mathscr{Q}(\eta, \psi(\eta), \psi(\tau\eta)) = \psi(\eta)\big(\gamma(\eta) - \beta(\eta)\psi(\tau\eta)\big)$ *and the coefficients* γ *and* β *satisfy the boundedness realtions*

$\inf_{\eta\in\mathbb{T}_a} \gamma(t) = \gamma^- \le \gamma(t) \le \gamma^+ = \sup_{\eta\in\mathbb{T}_a} \gamma(t)$ *and* $\inf_{\eta\in\mathbb{T}_a} \beta(t) = \beta^- \le \beta(t) \le \beta^+ = \sup_{\eta\in\mathbb{T}_a} \beta(t)$ *which are medically and biologically feasible. Model (4.6) represents the inter-specific competition in single species with* τ *denotes the maturity time period.*

(A_1) *Let* $\mathscr{Q}(\eta, \psi, \psi^*) = \psi\big(\gamma(\eta) - \beta(\eta)\psi^*\big)$ *and* $\Phi(\eta)$ *are continuous functions.*

(A_2) $\mathscr{Q}$ *satisfies Lipschitz condition with Lipschitz constant* $\mathscr{L} > 0$ *as follows:*

$$
\begin{aligned}
|\mathscr{Q}(\eta, \psi_1, \psi_2) - \mathscr{Q}(\eta, \varsigma_1, \varsigma_2)| &= |\psi_1\gamma(\eta) - \psi_1\beta(\eta)\psi_2 - \varsigma_1\gamma(\eta) + \varsigma_1\beta(\eta)\varsigma_2| \\
&\le \gamma^+|\psi_1 - \varsigma_1| + \beta^+|\psi_2||\psi_1 - \varsigma_1| + \beta^+|\varsigma_1||\psi_2 - \varsigma_2| \\
&\le \max\left\{\gamma^+ + \beta^+|\psi_2|, \beta^+|\varsigma_1|\right\}\left(|\psi_1 - \varsigma_1| + |\psi_2 - \varsigma_2|\right) \\
&\le \mathscr{L}\big(|\psi_1 - \varsigma_1| + |\psi_2 - \varsigma_2|\big).
\end{aligned}
$$

(A_3) $\mathscr{G}(\tau\eta) = \tau\eta$ *preserves the delay interval* $\mathbb{I}_\tau$,

(A_4) $\dfrac{2\mathcal{L}(T-a)_q^{\zeta}}{\Gamma_q(\zeta+1)} < 1$, *for some* $T > a$.

If all conditions $(A_1 - A_4)$ *are satisfied, then the model under consideration is Ulam-Hyers stable.*

Remark 8. *The above result can be extended to* n *species competitive Caputo q-fractional Lotka-Volterra system of the form*

$$
\begin{aligned}
{}^{c}_{q}\nabla^{\zeta}_{a}\psi_i(\eta) &= \psi_i(\eta)\Big(\gamma_i(\eta) - \sum_{j=1}^{n}\beta_{ij}(\eta)\psi_j(\tau_{ij}\eta)\Big), && \eta \in \mathbb{T}_a, i = 1, 2, ..., n, , \\
\psi_i(\eta) &= \Phi_i(\eta), && \eta \in \mathbb{I}_\tau, 0 < \zeta < 1, \tau_i = \max_{1\le j\le n} \tau_{ij}.
\end{aligned}
$$

4.4 Ulam-Hyers-Rassias Stability

In this section, Ulam-Hyers-Rassias stability of the problem (4.1) is analyzed.

Theorem 4.4.1. *Let us consider assumptions* (A_1), (A_2), (A_3) *and* (A_4) *are satisfied. Furthermore,*

(A_5) *there exists a continuous function* $\Psi : \mathbb{T}_a \to \mathbb{R}^+$ *and* $\lambda_\Psi \in \mathbb{R}^+$ *such that*

$$ {}_q\nabla_a^{-\zeta}\Psi(\eta) \leq \lambda_\Psi \Psi(\eta). $$

Then, Eq. (4.1) *is Ulam-Hyers-Rassias stable with respect to* Ψ*.*

Proof. Let $\varsigma(\eta)$ be solution of inequality (4.3). So we have

$$ -\epsilon\Psi(\eta) \leq {}_q^c\nabla_a^\zeta \varsigma(\eta) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) \leq \epsilon\Psi(\eta). $$

Applying ${}_q\nabla_a^{-\zeta}$, we get

$$ -\epsilon \, {}_q\nabla_a^{-\zeta}\Psi(\eta) \leq ({}_q\nabla_a^{-\zeta})({}_q^c\nabla_a^\zeta)\varsigma(\eta) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) \leq \epsilon \, {}_q\nabla_a^{-\zeta}\Psi(\eta). $$

Now using assumption (A_5), we get

$$ -\epsilon\lambda_\Psi\Psi(\eta) \leq \varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) \leq \epsilon\lambda_\Psi\Psi(\eta). $$

This implies

$$ |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))| \leq \epsilon\lambda_\Psi\Psi(\eta). \tag{4.7} $$

Let us consider $\psi(\eta)$ such that

$$ \begin{aligned} {}_q^c\nabla_a^\zeta \psi(\eta) &= \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))), \quad && \eta \in \mathbb{T}_a, \\ \psi(\eta) &= \varsigma(\eta), && \eta \in \mathbb{I}_\tau. \end{aligned} $$

For $\eta \in \mathbb{I}_\tau$, we have $|\varsigma(\eta) - \psi(\eta)| = 0$. Now for $\eta \in \mathbb{I}_{\tau^{-1}}$, we have

$$\begin{aligned}|\varsigma(\eta) - \psi(\eta)| &= |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta)))| \\ &\leq |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))| \\ &\quad + |{}_q\nabla_a^{-\zeta}(\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta))) - \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\tau\eta))))|.\end{aligned}$$

Using inequality (4.7) and the observation that $\varsigma(\mathscr{G}(\tau\eta)) - \psi(\mathscr{G}(\tau\eta)) = 0$ for $\eta \in \mathbb{I}_{\tau^{-1}}$, we have

$$|\varsigma(\eta) - \psi(\eta)| \leq \epsilon\lambda_\Psi \Psi(\eta) + \mathscr{L}\ {}_q\nabla_a^{-\zeta}(|\varsigma(\eta) - \psi(\eta)|).$$

Now by using Gronwall's inequality in Corollary 4.1.4, we conclude that

$$|\varsigma(\eta) - \psi(\eta)| \leq \epsilon\lambda_\Psi \Psi(\eta)\ {}_qE_\zeta(\mathscr{L}\Gamma_q(\zeta), \eta - a).$$

For $\eta \in [\tau^{-1}a, \infty)_q$, using same steps as mentioned above we have

$$\begin{aligned}|\varsigma(\eta) - \psi(\eta)| &\leq |\varsigma(\eta) - \varsigma(a) - {}_q\nabla_a^{-\zeta}\mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\tau\eta)))| \\ &\quad + \mathscr{L}\ {}_q\nabla_a^{-\zeta}(|\varsigma(\eta) - \psi(\eta)| + |\varsigma(\mathscr{G}(\tau\eta)) - \psi(\mathscr{G}(\tau\eta))|) \\ &\leq \epsilon\lambda_\Psi \Psi(\eta) + \mathscr{L}\ {}_q\nabla_a^{-\zeta}(|\varsigma(\eta) - \psi(\eta)| + |\varsigma(\mathscr{G}(\tau\eta)) - \psi(\mathscr{G}(\tau\eta))|).\end{aligned}$$

Again letting $\hat{z}(\eta) = \sup_{\beta \in \mathbb{I}_\tau} |\varsigma(\mathscr{G}(\beta\eta)) - \psi(\mathscr{G}(\beta\eta))|$, we see that

$$\hat{z}(\eta) \leq \epsilon\lambda_\Psi \Psi(\eta) + 2\mathscr{L}\ {}_q\nabla_a^{-\zeta}\hat{z}(\eta).$$

Finally, the use of the Gronwall's inequality as in Corollary 4.1.4 implies that

$$\begin{aligned}\hat{z}(\eta) &\leq \epsilon\lambda_\Psi \Psi(\eta)\ {}_qE_\zeta(2\mathscr{L}\Gamma_q(\zeta), \eta - a) \\ &\leq \epsilon\Psi(\eta)\left[\lambda_\Psi\ {}_qE_\zeta(2\mathscr{L}\Gamma_q(\zeta), \eta - a)\right].\end{aligned}$$

So we have

$$|\varsigma(\eta) - \psi(\eta)| \leq \hat{z}(\eta) \leq \epsilon\Psi(\eta) \left[\lambda_{\Psi\ q}E_{\zeta}(2\mathscr{L}\Gamma_q(\zeta), \eta - a)\right], \quad \forall\ \eta.$$

or we may write it as

$$|\varsigma(\eta) - \psi(\eta)| \leq \epsilon\Psi(\eta) \left[\lambda_{\Psi\ q}E_{\zeta}(2\mathscr{L}\Gamma_q(\zeta), T - a)\right], \quad \forall\ \eta < T.$$

This implies that

$$|\varsigma(\eta) - \psi(\eta)| \leq c\epsilon\Psi(\eta).$$

Hence Eq. (4.1) is Ulam-Hyers-Rassias stable. □

Chapter 5

Discrete Version of Gronwall-Bellman Type Inequality and its Application in Stability Analysis

In this chapter, we develop a new discrete version of Gronwall-Bellman type inequality. Then, using the newly developed inequality to discuss Ulam-Hyers stability of a Caputo nabla fractional difference system. A model is given to show the hypothetical outcomes. Our analysis is applied on an isolated time scale equivalent to $h\mathbb{Z}$. The consequences of this chapter are distributed in [46].

A notable number of integral and difference inequalities [120, 57, 101, 113, 121, 71, 24, 106, 137, 139, 122, 147, 68, 1, 25, 151] have been disclosed with the advancement in theory of fractional calculus. These type of inequalities are essential tools in investigation of boundedness, existence, stability of solutions of differential equations as well as difference equations. In these inequalities, Gronwall-Bellman inequality is one of the most important inequality. Many authors have been working on these type of inequalities and their generalizations. These type of inequalities play important role in the analysis of qualitative and quantitative properties of solutions of differential equations as well as certain difference equations.

Few recent works on Gronwall-Bellman type inequalities are given below: Using properties of the modified Riemann-Liouville fractional derivative, Bin Zheng [151] researched some new Gronwall-Bellman-type inequalities. Zheng et al. [154] developed a new Gronwall-Bellman inequality to investigate explicit bounds for the unknown function. Wu [145] investigated a new Gronwall-Bellman inequality having its application in fractional stochastic differential equations. In [153], Zheng investigated a new Gronwall-Bellman inequality. In discrete fractional calculus, these type of inequalities are still an open dilemma for researchers, however few researchers have their contribution in this direction. Deekshitulu et al. [59] established some fractional difference inequalities of Gronwall-Bellman type. Feng [67] investigated a Gronwall-Bellman inequality including fractional sum and fractional difference inequalities in order to analyze explicit bounds for unknown functions. Bin Zheng [152] developed a new Gronwall-Bellman inequality in terms of fractional sum and difference for qualitative and quantitative analysis of fractional difference system. Feng [69] developed a Gronwall-Bellman inequality in discrete fractional calculus for qualitative analysis of solutions of fractional difference equations. The current chapter is prepared as follows: Section 5.1 consists of some basic definitions, lemmas, theorem while Section 5.2 contains main results of the article consisting of derivation of Gronwall-Bellman inequality and then in Section 5.3 using newly developed inequality, we present Ulam-Hyers stability analysis of Caputo nabla fractional difference system.

5.1 Fundamental Results

In this section, we present some fundamental nabla notations, definitions [2, 3, 73, 4] and lemmas that are helpful in proving our main results.

Definition 5.1.1. *[4] The backward difference operator of a function ψ on $h\mathbb{Z}$ is given by*

$$\nabla_h \psi(\eta) = \frac{\psi(\eta) - \psi(\eta - h)}{h}.$$

For $h = 1$, we get the backward difference operator $\nabla\psi(\eta) = \psi(\eta) - \psi(\eta - 1)$. The backward jumping operator on time scale $h\mathbb{Z}$ is $\varrho(\eta) = \eta - h$.

Definition 5.1.2. *[4] Let $r, \zeta \in \mathbb{R}$ and $h > 0$, the nabla h-factorial is defined as*

$$r_h^{\overline{\zeta}} = h^{\zeta} \frac{\Gamma(\frac{r}{h} + \zeta)}{\Gamma(\frac{r}{h})}.$$

For $h = 1$, we write $r^{\overline{\zeta}} = \dfrac{\Gamma(r+\zeta)}{\Gamma(r)}$. Furthermore, $(r + h\zeta - h)_h^{(\zeta)} = r_h^{\overline{\zeta}}$.
We accept that division by a pole of the Gamma function will lead to 0 provided that the numerator is not a pole at the same time.
We use the notation $\mathbb{N}_{a,h} = \{a, a+h, a+2h, ...\}$, where $a \in \mathbb{R}$ and $h > 0$.

Definition 5.1.3. *[4](Nabla h-fractional sums) Let function $\psi : \mathbb{N}_{a,h} \to \mathbb{R}$, the nabla left h-fractional sum of order $\zeta > 0$ is given by*

$$\begin{aligned}({}_a\nabla_h^{-\zeta}\psi)(\eta) &= \frac{1}{\Gamma(\zeta)} \int_a^{\eta} (\eta - \varrho(\tau))_h^{\overline{\zeta-1}} \psi(\tau) \nabla_h \tau \\ &= \frac{1}{\Gamma(\zeta)} \sum_{k=\frac{a}{h}+1}^{\frac{\eta}{h}} (\eta - \varrho(kh))_h^{\overline{\zeta-1}} \psi(kh)h, \quad \eta \in \mathbb{N}_{a+h,h}.\end{aligned}$$

Definition 5.1.4. *[4] (Nabla h-fractional difference) The nabla left h-fractional difference of order $\zeta > 0$ has the form*

$$\begin{aligned}({}_a\nabla_h^{\zeta}\psi)(\eta) &= (\nabla_h^n \, {}_a\nabla_h^{-(n-\zeta)}\psi)(\eta) \\ &= \frac{\nabla_h^n}{\Gamma(n-\zeta)} \sum_{k=\frac{a}{h}+1}^{\frac{\eta}{h}} (\eta - \varrho(kh))_h^{\overline{n-\zeta-1}} \psi(kh)h, \quad \eta \in \mathbb{N}_{a+h,h}.\end{aligned}$$

By the help of the Leibniz rule on $h\mathbb{Z}$, and by assuming that the nabla fractional sum is extendable on whole $\mathbb{R}$, it turns that

$$({}_a\nabla_h^{\zeta}\psi)(\eta) = ({}_a\nabla_h^{-(-\zeta)}\psi)(\eta) = \frac{1}{\Gamma(-\zeta)} \int_a^{\eta} (\eta - \varrho(\tau))_h^{\overline{-\zeta-1}} \psi(\tau) \nabla_h \tau.$$

Definition 5.1.5. *[4](h-Caputo fractional difference) Let $\zeta > 0$, $n = [\,\zeta]\, +1$, $h > 0$, $a < b \in \mathbb{R}$, $a_h(\zeta) = a + (n-1)h$. For $0 < \zeta < 1$ then $a_h(\zeta) = a$.*

The left h-Caputo fractional difference of order ζ starting at $a_h(\zeta)$ is defined by

$$({}^{c}_{a_h(\zeta)}\nabla^{\zeta}\psi)(\eta) = ({}_{a_h(\zeta)}\nabla^{-(n-\zeta)}\nabla_h^n\psi)(\eta), \quad \eta \in \mathbb{N}_{a+nh,h}.$$

Theorem 5.1.1. *[4] Let $\zeta > 0$, $\beta > -1$, $h > 0$. Then,*

$${}_a\nabla_h^{-\zeta}(\eta - a)_h^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\zeta)}(\eta - a)_h^{\overline{\zeta+\beta}}.$$

$${}_a\nabla_h^{\zeta}(\eta - a)_h^{\overline{\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\zeta)}(\eta - a)_h^{\overline{\beta-\zeta}}.$$

Theorem 5.1.2. *[4] Let $\zeta > 0$, $\mu > 0$, $h > 0$. If ψ is defined on $\mathbb{N}_{a,h}$, then for all $\eta \in \mathbb{N}_{a+h,h}$, we have*

$$({}_a\nabla_h^{-\mu}\ {}_a\nabla_h^{-\zeta}\psi)(\eta) = ({}_a\nabla_h^{-(\zeta+\mu)}\psi)(\eta) = ({}_a\nabla_h^{-\zeta}\ {}_a\nabla_h^{-\mu}\psi)(\eta)$$

Lemma 5.1.6. *[84] Let $c \geq 0$, $m \geq n \geq 0$ with $m \neq 0$. Then, for any $d > 0$, we have*

$$c^{\frac{n}{m}} \leq \frac{n}{m} d^{\frac{n-m}{m}} c + \frac{m-n}{m} d^{\frac{n}{m}}. \tag{5.1}$$

5.2 Gronwall-Bellman Inequality

In this section, we estabilish a new discrete version of Gronwall-Bellman type inequality.

Theorem 5.2.1. *Let us consider constants $p \geq 1$, $\zeta > 0$. Assume $\mathcal{F}$ is a nonnegative and continuous function defined as $\mathcal{F} : \mathbb{N}_{0,h} \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the Lipschitz condition $|\mathcal{F}(\eta, \psi) - \mathcal{F}(\eta, \varsigma)| \leq \mathcal{L}|\psi - \varsigma|$, where $\psi \geq \varsigma \geq 0$, $\mathcal{L}$ being Lipschitz constant. Further, assume that the function $\mathcal{G}$ is bounded and nondecreasing, and $\mathcal{H}$ is positive real-valued. If the following inequality holds*

$$\psi^p(\eta) \leq \mathcal{H}(\eta) + \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \mathcal{F}(s, \psi(s)) \nabla_h s, \tag{5.2}$$

at that point we have

$$\psi(\eta) \le \left(\hat{\mathcal{H}}(\eta) + \sum_{k=1}^{\infty} (\frac{1}{p}\mathcal{K}^{\frac{1-p}{p}})^k . \frac{\mathcal{L}^k \mathcal{G}^k(\eta)}{\Gamma(k\zeta)} . \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{k\zeta-1}} \hat{\mathcal{H}}(s) \nabla_h s \right)^{\frac{1}{p}}, \tag{5.3}$$

$\mathcal{K} > 0$, $\eta \in \mathbb{N}_{0,h}$ *and*

$$\hat{\mathcal{H}}(\eta) = \mathcal{H}(\eta) + \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \mathcal{F}(s, \frac{p-1}{p}\mathcal{K}^{\frac{1}{p}}) \nabla_h s. \tag{5.4}$$

Proof. Let us denote $\psi^p(\eta) = \varsigma(\eta)$, then we have $\psi(\eta) = \varsigma^{\frac{1}{p}}(\eta)$ and

$$\varsigma(\eta) \le \mathcal{H}(\eta) + \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \mathcal{F}(s, \varsigma^{\frac{1}{p}}(s)) \nabla_h s. \tag{5.5}$$

Now using Lemma 5.1.6, we get

$$\begin{aligned}
\varsigma(\eta) &\le \mathcal{H}(\eta) + \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \mathcal{F}(s, \frac{1}{p}\mathcal{K}^{\frac{1-p}{p}}\varsigma(s) + \frac{p-1}{p}\mathcal{K}^{\frac{1}{p}}) \nabla_h s \\
&= \mathcal{H}(\eta) + \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} [\mathcal{F}(s, \frac{1}{p}\mathcal{K}^{\frac{1-p}{p}}\varsigma(s) + \frac{p-1}{p}\mathcal{K}^{\frac{1}{p}}) \\
&\qquad - \mathcal{F}(s, \frac{p-1}{p}\mathcal{K}^{\frac{1}{p}}) + \mathcal{F}(s, \frac{p-1}{p}\mathcal{K}^{\frac{1}{p}})] \nabla_h s \\
&\le \mathcal{H}(\eta) + \frac{1}{p}\mathcal{K}^{\frac{1-p}{p}} \mathcal{L} \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \varsigma(s) \nabla_h s \\
&\quad + \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \mathcal{F}(s, \frac{p-1}{p}\mathcal{K}^{\frac{1}{p}}) \nabla_h s \\
&= \hat{\mathcal{H}}(\eta) + \frac{1}{p}\mathcal{K}^{\frac{1-p}{p}} \mathcal{L} \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \varsigma(s) \nabla_h s
\end{aligned}$$

Let us consider

$$\mathcal{B}\Psi(\eta) = \frac{1}{p}\mathcal{K}^{\frac{1-p}{p}} \mathcal{L} \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^{\eta} (\eta - \varrho(s))_h^{\overline{\zeta-1}} \Psi(s) \nabla_h s. \tag{5.6}$$

Then

$$\varsigma(\eta) \le \hat{\mathcal{H}}(\eta) + \mathcal{B}\varsigma(\eta).$$

Now by mathematical induction and noting that the operator $\mathcal{B}$ is non-negative and nondecreasing, we have

$$\begin{aligned}\varsigma(\eta) &\leq \hat{\mathcal{H}}(\eta) + \mathcal{B}(\hat{\mathcal{H}}(\eta) + \mathcal{B}\varsigma(\eta)) \\ &= \hat{\mathcal{H}}(\eta) + \mathcal{B}\hat{\mathcal{H}}(\eta) + \mathcal{B}^2\varsigma(\eta) \\ &\leq \dots \\ &\leq \sum_{m=0}^{k-1} \mathcal{B}^m \hat{\mathcal{H}}(\eta) + \mathcal{B}^k\varsigma(\eta).\end{aligned}$$

Let $\mathcal{R} = \frac{1}{p}\mathcal{K}^{\frac{1-p}{p}}$, then by the help of the fact that

$${}_a\nabla_h^{-\mu}\,{}_a\nabla_h^{-\nu}\varsigma(\eta) = {}_a\nabla_h^{-(\mu+\nu)}\varsigma(\eta),$$

we have

$$\mathcal{B}^k\varsigma(\eta) \leq \mathcal{R}^k \frac{\mathcal{L}^k\mathcal{G}^k(\eta)}{\Gamma(k\zeta)} \int_0^\eta (\eta - \varrho(s))_h^{\overline{k\zeta-1}} \varsigma(s)\nabla_h s, \tag{5.7}$$

for any $k \in \mathbb{N}$ and $\mathcal{B}^k\varsigma(\eta) \to 0$ as $k \to \infty$ for each $\eta \in \mathbb{N}_{0,h}$.

(i) for $k = 1$, we have

$$\begin{aligned}\mathcal{B}\varsigma(\eta) &\leq \mathcal{R}\frac{\mathcal{L}\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^\eta (\eta - \varrho(s))_h^{\overline{\zeta-1}} \varsigma(s)\nabla_h s \\ &= \mathcal{B}\varsigma(\eta),\end{aligned}$$

so inequality (5.7) holds for $k = 1$.

(ii) Let us suppose that inequality (5.7) holds for $k = j$,

$$\mathcal{B}^j\varsigma(\eta) \leq \mathcal{R}^j \frac{\mathcal{L}^j\mathcal{G}^j(\eta)}{\Gamma(j\zeta)} \int_0^\eta (\eta - \varrho(s))_h^{\overline{j\zeta-1}} \varsigma(s)\nabla_h s.$$

(iii) For $k = j+1$, we get

$$\begin{aligned}\mathcal{B}^{j+1}\varsigma(\eta) &= \mathcal{B}(\mathcal{B}^{j}\varsigma(\eta))\\ &\leq \mathcal{R}\frac{\mathcal{L}\mathcal{G}(\eta)}{\Gamma(\zeta)}\int_{0}^{\eta}(\eta-\varrho(s))_{h}^{\overline{\zeta-1}}\Big(\mathcal{R}^{j}\frac{\mathcal{L}^{j}\mathcal{G}^{j}(s)}{\Gamma(j\zeta)}\int_{0}^{s}(s-\varrho(\tau))_{h}^{\overline{j\zeta-1}}\varsigma(\tau)\nabla_{h}\tau\Big)\nabla_{h}s.\end{aligned}$$

Since $\mathcal{G}$ is nondecreasing, so we get

$$\mathcal{B}^{j+1}\varsigma(\eta) \leq \mathcal{R}\frac{\mathcal{L}\mathcal{G}(\eta)}{\Gamma(\zeta)}.\mathcal{R}^{j}\frac{\mathcal{L}^{j}\mathcal{G}^{j}(\eta)}{\Gamma(j\zeta)}.\int_{0}^{\eta}(\eta-\varrho(s))_{h}^{\overline{\zeta-1}}.\int_{0}^{s}(s-\varrho(\tau))_{h}^{\overline{j\zeta-1}}\varsigma(\tau)\nabla_{h}\tau\nabla_{h}s.$$

Using composition rule, we get

$$\mathcal{B}^{j+1}\varsigma(\eta) \leq \mathcal{R}^{j+1}\frac{\mathcal{L}^{j+1}\mathcal{G}^{j+1}(\eta)}{\Gamma((j+1)\zeta)}\int_{0}^{\eta}(\eta-\varrho(s))_{h}^{\overline{(j+1)\zeta-1}}\varsigma(s)\nabla_{h}s.$$

So inequality (5.7) holds for $k = j+1$. Hence inequality (5.7) is satisfied by any natural number k. Since $\mathcal{G}$ is bounded functions i.e $\mathcal{G} < \mathcal{M}$ where $\mathcal{M} > 0$, so we get

$$\mathcal{B}^{k}\varsigma(\eta) \leq \mathcal{R}^{k}\frac{\mathcal{L}^{k}\mathcal{M}^{k}}{\Gamma(k\zeta)}\int_{0}^{\eta}(\eta-\varrho(s))_{h}^{\overline{k\zeta-1}}\varsigma(s)\nabla_{h}s. \tag{5.8}$$

Now using properties of Gamma function, $\mathcal{B}^{k}\varsigma(\eta) \to 0$ as $k \to \infty$, so we get

$$\varsigma(\eta) \leq \hat{\mathcal{H}}(\eta) + \sum_{k=1}^{\infty}\big(\frac{1}{p}\mathcal{K}^{\frac{1-p}{p}}\big)^{k}.\frac{\mathcal{L}^{k}\mathcal{G}^{k}(\eta)}{\Gamma(k\zeta)}\int_{0}^{\eta}(\eta-\varrho(s))_{h}^{\overline{k\zeta-1}}\hat{\mathcal{H}}(s)\nabla_{h}s.$$

Consequently,

$$\psi(\eta) \leq \Big(\hat{\mathcal{H}}(\eta) + \sum_{k=1}^{\infty}\big(\frac{1}{p}\mathcal{K}^{\frac{1-p}{p}}\big)^{k}.\frac{\mathcal{L}^{k}\mathcal{G}^{k}(\eta)}{\Gamma(k\zeta)}\int_{0}^{\eta}(\eta-\varrho(s))_{h}^{\overline{k\zeta-1}}\hat{\mathcal{H}}(s)\nabla_{h}s\Big)^{\frac{1}{p}}.$$

Hence completing the proof. □

Corollary 5.2.1. *If for $\eta \in \mathbb{N}_{0,h}$, we have*

$$\psi(\eta) \leq \mathcal{H}(\eta) + \frac{\mathcal{G}(\eta)}{\Gamma(\zeta)} \int_0^\eta (\eta - \varrho(s))_h^{\overline{\zeta-1}} \psi(s) \nabla_h s, \tag{5.9}$$

where all functions are nonnegative and continuous. ζ is positive constant, $\mathcal{G}$ is bounded and monotonically increasing function on $\mathbb{N}_{0,h}$. Then we have

$$\psi(\eta) \leq \mathcal{H}(\eta) + \sum_{k=1}^{\infty} \frac{\mathcal{G}^k(\eta)}{\Gamma(k\zeta)} \int_0^\eta (\eta - \varrho(s))_h^{\overline{k\zeta-1}} \mathcal{H}(s) \nabla_h s. \tag{5.10}$$

Corollary 5.2.2. *Let us suppose all the conditions of Theorem 5.2.1 are satisfied. Furthermore, if $\mathcal{H}(\eta)$ is increasing function on $\mathbb{N}_{0,h}$, then*

$$\psi(\eta) \leq \left(\hat{\mathcal{H}}(T) \sum_{k=0}^{\infty} \left(\left(\frac{1}{p} \mathcal{K}^{\frac{1-p}{p}} \right)^k . \frac{\mathcal{L}^k \mathcal{G}^k(T)}{\Gamma(k\zeta+1)} . T_h^{\overline{k\zeta}} \right) \right)^{\frac{1}{p}}. \tag{5.11}$$

Proof. If $\mathcal{H}(\eta)$ is increasing then $\hat{\mathcal{H}}(\eta)$ is also increasing,

$$\begin{aligned}
\psi(\eta) &\leq \left(\hat{\mathcal{H}}(\eta) + \sum_{k=1}^{\infty} (\frac{1}{p} \mathcal{K}^{\frac{1-p}{p}})^k . \frac{\mathcal{L}^k \mathcal{G}^k(\eta)}{\Gamma(k\zeta)} . \int_0^\eta (\eta - \varrho(s))_h^{\overline{k\zeta-1}} \hat{\mathcal{H}}(s) \nabla_h s \right)^{\frac{1}{p}} \\
&\leq (\hat{\mathcal{H}}(\eta))^{\frac{1}{p}} \left(1 + \sum_{k=1}^{\infty} (\frac{1}{p} \mathcal{K}^{\frac{1-p}{p}})^k . \frac{\mathcal{L}^k \mathcal{G}^k(\eta)}{\Gamma(k\zeta)} . \int_0^\eta (\eta - \varrho(s))_h^{\overline{k\zeta-1}} \nabla_h s \right)^{\frac{1}{p}} \\
&= \left(\hat{\mathcal{H}}(\eta) \sum_{k=0}^{\infty} (\frac{1}{p} \mathcal{K}^{\frac{1-p}{p}})^k . \frac{\mathcal{L}^k \mathcal{G}^k(\eta)}{\Gamma(k\zeta+1)} . \eta_h^{\overline{k\zeta}} \right)^{\frac{1}{p}} \\
&\leq \left(\hat{\mathcal{H}}(T) \sum_{k=0}^{\infty} \left((\frac{1}{p} \mathcal{K}^{\frac{1-p}{p}})^k . \frac{\mathcal{L}^k \mathcal{G}^k(T)}{\Gamma(k\zeta+1)} . T_h^{\overline{k\zeta}} \right) \right)^{\frac{1}{p}}, \qquad \eta < T.
\end{aligned}$$

Hence completing the proof.

Remark 9. *For $h = 1$, we get Gronwall-Bellman inequality for time scale $\mathbb{N}_0$.*

□

5.3 Ulam-Hyers Stability

In this section, we show that our main result is useful in investigating the Ulam-Hyers stability of fractional difference equations.

Let us consider following fractional difference equation,

$$ {}_{0}^{c}\nabla_{h}^{\zeta}\psi(\eta) = \mathcal{F}(\eta, \psi(\eta)), \quad \eta \in \mathbb{N}_{0,h}, \quad \psi(0) = \xi, \tag{5.12} $$

where $0 < \zeta < 1$, $\eta \in \mathbb{N}_{0,h}$, $\psi : \mathbb{N}_{0,h} \to \mathbb{R}^{+}$, ${}_{0}^{c}\nabla_{h}^{\zeta}$ is Caputo nabla fractional difference operator. Further assume that $\mathcal{F}$ satisfies Lipschitz condition

$$ |\mathcal{F}(\eta, \psi(\eta)) - \mathcal{F}(\eta, \varsigma(\eta))| \leq \mathcal{L}|\psi(\eta) - \varsigma(\eta)|, $$

where $\mathcal{L}$ is Lipschitz constant.

5.3.1 Uniqueness

First of all we discuss how the uniqueness of solution of the above mentioned problem (5.12) can be shown by using the proven Gronwall's inequality.

$$ {}_{0}^{c}\nabla_{h}^{\zeta}\psi(\eta) = \mathcal{F}(\eta, \psi(\eta)), $$

applying ${}_{0}\nabla_{h}^{-\zeta}$ on both sides, making use of Proposition 36 in [3], and using $\psi(0) = \xi$, we get

$$ \begin{aligned} \nabla_{h}^{-\zeta}\, {}_{0}^{c}\nabla_{h}^{\zeta}\psi(\eta) &= \nabla_{h}^{-\zeta}\mathcal{F}(\eta, \psi(\eta)) \\ \psi(\eta) - \psi(0) &= \nabla_{h}^{-\zeta}\mathcal{F}(\eta, \psi(\eta)) \\ \psi(\eta) - \xi &= \nabla_{h}^{-\zeta}\mathcal{F}(\eta, \psi(\eta)) \end{aligned} $$

$$ \psi(\eta) = \nabla_{h}^{-\zeta}\mathcal{F}(\eta, \psi(\eta)) + \xi. \tag{5.13} $$

Now to prove uniqueness of solution, let $\psi_1(\eta)$ and $\psi_2(\eta)$ are two solutions of problem (5.12) then they will satisfy equation (5.13).

$$\begin{aligned}|\psi_1(\eta)-\psi_2(\eta)| &\leq |\nabla_h^{-\zeta}\big(\mathcal{F}(\eta,\psi_1(\eta))-\mathcal{F}(\eta,\psi_2(\eta))\big)| \\ &\leq \frac{1}{\Gamma(\zeta)}\int_0^{\eta}(\eta-\varrho(s))_h^{\overline{\zeta-1}}|\mathcal{F}(s,\psi_1(s))-\mathcal{F}(s,\psi_2(s))|\nabla_h s \\ &\leq \frac{\mathcal{L}}{\Gamma(\zeta)}\int_0^{\eta}(\eta-\varrho(s))_h^{\overline{\zeta-1}}|\psi_1(s)-\psi_2(s)|\nabla_h s.\end{aligned}$$

According to the Theorem 5.2.1, we conclude that

$$|\psi_1(\eta)-\psi_2(\eta)| \leq 0$$

This implies

$$\psi_1(\eta)=\psi_2(\eta).$$

Hence solution of problem (5.12) is unique.

5.3.2 Stability Analysis

Now we study the Ulam-Hyers stability of the above mentioned problem (5.12).

Definition 5.3.1. *Let for $\epsilon>0$ there exists positive real number k such that $\varsigma(\eta)$ defined on $\mathbb{N}_{0,h}$ satisfies the following inequality*

$$\big|\ {}_0^c\nabla_h^{\zeta}\varsigma(\eta)-\mathcal{F}(\eta,\varsigma(\eta))\big| \leq \epsilon, \tag{5.14}$$

if $\psi(\eta)$ is solution of problem (5.12) then

$$|\varsigma(\eta)-\psi(\eta)| \leq k\epsilon, \quad \forall\ \eta\in\mathbb{N}_{0,h}.$$

Then problem (5.12) is called Ulam-Hyers stable.

We can write inequality (5.14) as follows

$$-\nabla_h^{-\zeta}\epsilon \leq \varsigma(\eta) - [\nabla_h^{-\zeta}\mathcal{F}(\eta,\varsigma(\eta)) + \xi] \leq \nabla_h^{-\zeta}\epsilon.$$

Now we have

$$\begin{aligned}
|\varsigma(\eta) - \psi(\eta)| &\leq |\varsigma(\eta) - \nabla_h^{-\zeta}\mathcal{F}(\eta,\psi(\eta)) - \xi| \\
&\leq |\varsigma(\eta) - \nabla_h^{-\zeta}\mathcal{F}(\eta,\varsigma(\eta)) - \xi| + |\nabla_h^{-\zeta}\mathcal{F}(\eta,\varsigma(\eta)) - \nabla_h^{-\zeta}\mathcal{F}(\eta,\psi(\eta))| \\
&\leq \nabla_h^{-\zeta}\epsilon + \mathcal{L}\nabla_h^{-\zeta}|\varsigma(\eta) - \psi(\eta)| \\
&= \frac{\epsilon\eta_h^{\overline{\zeta}}}{\Gamma(\zeta+1)} + \frac{\mathcal{L}}{\Gamma(\zeta)}\int_0^\eta (\eta - \varrho(s))_h^{\overline{\zeta-1}}|\varsigma(s) - \psi(s)|\nabla_h s.
\end{aligned}$$

Now using Theorem 5.2.1, we get

$$\begin{aligned}
|\varsigma(\eta) - \psi(\eta)| &\leq \frac{\epsilon\eta_h^{\overline{\zeta}}}{\Gamma(\zeta+1)} + \sum_{k=1}^{\infty}\left(\frac{\mathcal{L}^k}{\Gamma(k\zeta)}\int_0^\eta (\eta - \varrho(s))_h^{\overline{k\zeta-1}}\frac{\epsilon s_h^{\overline{\zeta}}}{\Gamma(\zeta+1)}\nabla_h s\right) \\
&\leq \frac{\epsilon\eta_h^{\overline{\zeta}}}{\Gamma(\zeta+1)} + \sum_{k=1}^{\infty}\frac{\epsilon\mathcal{L}^k}{\Gamma(\zeta+1)}\nabla_h^{-k\zeta}\eta_h^{\overline{\zeta}} \\
&\leq \frac{\epsilon\eta_h^{\overline{\zeta}}}{\Gamma(\zeta+1)} + \sum_{k=1}^{\infty}\frac{\epsilon\mathcal{L}^k}{\Gamma(\zeta+1)}\frac{\Gamma(\zeta+1)\ \eta_h^{\overline{k\zeta+\zeta}}}{\Gamma((k+1)\zeta+1)},
\end{aligned}$$

using property $\eta_h^{\overline{\zeta+\beta}} = (\eta + h\beta)_h^{\overline{\zeta}}\ \eta_h^{\overline{\beta}}$, we get

$$\begin{aligned}
|\varsigma(\eta) - \psi(\eta)| &\leq \frac{\epsilon\eta_h^{\overline{\zeta}}}{\Gamma(\zeta+1)} + \sum_{k=1}^{\infty}\frac{\epsilon\mathcal{L}^k}{\Gamma(\zeta+1)}\frac{\Gamma(\zeta+1)(\eta+h\zeta)_h^{\overline{k\zeta}}\eta_h^{\overline{\zeta}}}{\Gamma((k+1)\zeta+1)} \\
&\leq \frac{\epsilon\eta_h^{\overline{\zeta}}}{\Gamma(\zeta+1)}\left(1 + \sum_{k=1}^{\infty}\frac{\mathcal{L}^k\Gamma(\zeta+1)(\eta+h\zeta)_h^{\overline{k\zeta}}}{\Gamma((k+1)\zeta+1)}\right) \\
&\leq \frac{\epsilon T_h^{\overline{\zeta}}}{\Gamma(\zeta+1)}\left(1 + \sum_{k=1}^{\infty}\frac{\mathcal{L}^k\Gamma(\zeta+1)(T+h\zeta)_h^{\overline{k\zeta}}}{\Gamma((k+1)\zeta+1)}\right), \qquad \eta < T.
\end{aligned}$$

Hence

$$|\varsigma(\eta) - \psi(\eta)| \leq k\epsilon,$$

where

$$k = \frac{T_h^{\bar{\zeta}}}{\Gamma(\zeta+1)}\left(1 + \sum_{k=1}^{\infty} \frac{\mathcal{L}^k \Gamma(\zeta+1)(T+h\zeta)_h^{\overline{k\zeta}}}{\Gamma((k+1)\zeta+1)}\right).$$

Hence Ulam-Hyers stability is verified.

Chapter 6

Stability Analysis of p-Laplacian Fractional Difference Equation

This chapter manages two main points, existence of solution along with the stability analysis of p-Laplacian fractional difference equations in Nabla sense. As a matter of first importance, the existence of a solution is considered utilizing Schaefer's fixed point hypothesis. Besides, Ulam-Hyers stability is examined. The consequences of this chapter are distributed in [48].

There exists an enormous number of nonlinear mathematical models in the logical fields for the investigation of dynamical systems. One of the most significant nonlinear operator utilized is the traditional p-Laplacian operator, which fulfills

$$\frac{1}{p}+\frac{1}{q}=1,\ \phi_p(s)=|s|^{p-2}s,\ p>1,\ \phi_q(\theta)=\phi_p^{-1}(\theta).$$

For the subtleties and applications as respects the nonlinear p-Laplacian operator, the peruser is alluded to [97, 91, 94, 107, 112, 129] and the references in that.

Liu et al. [109] investigated the boundary value problems for nonlinear fractional impulsive differential equations with p-Laplacian operator. Khan et al. [93, 90] considered coupled fractional differential equations with integral boundary conditions involving p-Laplacian operator and discussed existence of solution as well as Hyers-Ulam stability using different approaches.

Shah et al. [127] worked on existence of solution for nonlinear implicit fractional differential equation with integral boundary conditions involving p-Laplacian operator without compactness and also discussed Ulam-Hyers stability. Recently Khan et al. [99] considered delay fractional differential equation of higher order with singularity and p-Laplacian operator for the existence of solution and stability analysis.

Motivated by all previous mentioned work, in this chapter we consider a nabla Caputo fractional difference equation with p-Laplacian operator having boundary conditions as follows:

$$\begin{cases} {}^c\nabla_0^\zeta[\phi_p({}^c\nabla_0^\nu\psi)](\eta) = \mathcal{G}(\eta, \psi(\eta)), \quad \eta \in \mathbb{N}_0^m, \quad m \in \mathbb{N}_1, \\ \phi_p({}^c\nabla_0^\nu\psi)(\eta)|_{\eta=0} + \phi_p({}^c\nabla_0^\nu\psi)(\eta)|_{\eta=m} = 0, \ \psi(0) + \psi(m) = 0, \end{cases} \tag{6.1}$$

where $0 < \nu, \zeta \leq 1, \ 1 < \nu + \zeta \leq 2$. ${}^c\nabla_0^\zeta$ is nabla Caputo fractional difference operator, function $\mathcal{G}$ is defined as $\mathcal{G} : \mathbb{N}_0^m \times \mathbb{R} \to \mathbb{R}$. ϕ_p is p-Laplacian operator whose properties are discussed earlier. Shortly, $\phi_p(s) = s|s|^{p-2}$ is nonlinear operator such that $\phi_p^{-1} = \phi_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

The chapter under consideration is arranged in following manner: Section 6.1 consists of main results about solution while in Section 6.2, we talk about Ulam-Hyers stability of the problem under consideration. In the end an example is given to illustrate theoretical results.

6.1 Existence Results

In this section, we present solution of problem (6.1) and then discuss existence of solution using Schaefer's fixed point theorem.

6.1.1 The Solution Representation Key

Theorem 6.1.1. *Let us consider the following problem*

$$\begin{cases} {}^c\nabla_0^\zeta[\phi_p({}^c\nabla_0^\nu\psi)](\eta) = \mathcal{Q}(\eta), \quad \eta \in \mathbb{N}_0^m, \quad m \in \mathbb{N}_1, \\ \phi_p({}^c\nabla_0^\nu\psi)(\eta)|_{\eta=0} + \phi_p({}^c\nabla_0^\nu\psi)(\eta)|_{\eta=m} = 0, \ \psi(0) + \psi(m) = 0, \end{cases} \tag{6.2}$$

where $0 < \nu, \zeta \leq 1,\ 1 < \zeta + \nu \leq 2$. *At this point, we get its result as given below*

$$\begin{aligned}\psi(\eta) = \frac{1}{\Gamma(\nu)} \int_0^\eta (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \Big[\frac{1}{\Gamma(\zeta)} \int_0^s (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \\ - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \Big] \nabla s \\ - \frac{1}{2\Gamma(\nu)} \int_0^m (m - \varrho(s))^{\overline{\nu-1}} \phi_q \Big[\frac{1}{\Gamma(\zeta)} \int_0^s (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \\ - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \Big] \nabla s.\end{aligned}$$

Proof. If we apply $\nabla_0^{-\zeta}$ on both sides of Eq. (6.2) and make use of Lemma 1.2.20, we get

$$\phi_p({}^c\nabla_0^\nu \psi)(\eta) = \frac{1}{\Gamma(\zeta)} \int_0^\eta (\eta - \varrho(s))^{\overline{\zeta-1}} \mathcal{Q}(s) \nabla s + c_0. \tag{6.3}$$

Now using the condition

$$\phi_p({}^c\nabla_0^\nu \psi)(\eta)|_{\eta=0} + \phi_p({}^c\nabla_0^\nu \psi)(\eta)|_{\eta=m} = 0,$$

we get

$$c_0 = -\frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(s))^{\overline{\zeta-1}} \mathcal{Q}(s) \nabla s.$$

Hence, Eq. (6.3) becomes

$$\phi_p({}^c\nabla_0^\nu \psi)(\eta) = \frac{1}{\Gamma(\zeta)} \int_0^\eta (\eta - \varrho(s))^{\overline{\zeta-1}} \mathcal{Q}(s) \nabla s - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(s))^{\overline{\zeta-1}} \mathcal{Q}(s) \nabla s.$$

Applying $(\phi_p)^{-1} = \phi_q$, we get

$${}^c\nabla_0^\nu \psi(\eta) = \phi_q \left[\frac{1}{\Gamma(\zeta)} \int_0^\eta (\eta - \varrho(s))^{\overline{\zeta-1}} \mathcal{Q}(s) \nabla s - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(s))^{\overline{\zeta-1}} \mathcal{Q}(s) \nabla s \right].$$

Applying $\nabla_0^{-\nu}$ on both sides and making use of Lemma 1.2.20, we get

$$\psi(\eta) = \frac{1}{\Gamma(\nu)} \int_0^{\eta} (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q [\frac{1}{\Gamma(\zeta)} \int_0^s (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \\ - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau] \nabla s + c_1. \tag{6.4}$$

Now using the condition $\psi(0) + \psi(m) = 0$, we get

$$c_1 = -\frac{1}{2\Gamma(\nu)} \int_0^m (m - \varrho(s))^{\overline{\nu-1}} \phi_q [\frac{1}{\Gamma(\zeta)} \int_0^s (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \\ - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau] \nabla s.$$

Substituting value of c_1, Eq. (6.4) becomes

$$\begin{aligned} \psi(\eta) = &\frac{1}{\Gamma(\nu)} \int_0^{\eta} (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \left[\frac{1}{\Gamma(\zeta)} \int_0^s (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \right. \\ &\left. - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \right] \nabla s \\ &- \frac{1}{2\Gamma(\nu)} \int_0^m (m - \varrho(s))^{\overline{\nu-1}} \phi_q \left[\frac{1}{\Gamma(\zeta)} \int_0^s (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \right. \\ &\left. - \frac{1}{2\Gamma(\zeta)} \int_0^m (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{Q}(\tau) \nabla\tau \right] \nabla s. \end{aligned} \tag{6.5}$$

□

If we replace $\mathcal{Q}(\eta)$ by function $\mathcal{G}(\eta, \psi(\eta))$, we can state the the accompanying hypothesis.

Theorem 6.1.2. *Let us consider fractional difference equation followed by boundary conditions*

$$\begin{cases} {}^c\nabla_0^{\zeta} [\phi_p ({}^c\nabla_0^{\nu} \psi)](\eta) = \mathcal{G}(\eta, \psi(\eta)), \quad \eta \in \mathbb{N}_0^m, \quad m \in \mathbb{N}_1, \\ \phi_p ({}^c\nabla_0^{\nu} \psi)(\eta)|_{\eta=0} + \phi_p ({}^c\nabla_0^{\nu} \psi)(\eta)|_{\eta=m} = 0, \ \psi(0) + \psi(m) = 0, \end{cases} \tag{6.6}$$

where $0 < \nu, \zeta \leq 1,\ 1 < \nu+\zeta \leq 2$. *Then, its solution satisfies the following summation equation*

$$\begin{aligned}
\psi(\eta) = &\frac{1}{\Gamma(\nu)} \int_0^{\eta} (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \Bigg[\frac{1}{\Gamma(\zeta)} \int_0^{s} (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)} \int_0^{m} (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \Bigg] \nabla s \\
&- \frac{1}{2\Gamma(\nu)} \int_0^{m} (m - \varrho(s))^{\overline{\nu-1}} \phi_q \Bigg[\frac{1}{\Gamma(\zeta)} \int_0^{s} (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)} \int_0^{m} (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \Bigg] \nabla s.
\end{aligned}$$

6.1.2 Existence of Solution

In this subsection, we prove the existence of solution of problem (6.1) using Schaefer's fixed point theorem.

Let us define an operator $\mathcal{N} : C(\mathbb{N}_0^m) \to C(\mathbb{N}_0^m)$ as follows:

$$\begin{aligned}
\mathcal{N}\psi(\eta) = &\frac{1}{\Gamma(\nu)} \int_0^{\eta} (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \Bigg[\frac{1}{\Gamma(\zeta)} \int_0^{s} (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)} \int_0^{m} (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \Bigg] \nabla s \\
&- \frac{1}{2\Gamma(\nu)} \int_0^{m} (m - \varrho(s))^{\overline{\nu-1}} \phi_q \Bigg[\frac{1}{\Gamma(\zeta)} \int_0^{s} (s - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)} \int_0^{m} (m - \varrho(\tau))^{\overline{\zeta-1}} \mathcal{G}(\tau, \psi(\tau)) \nabla\tau \Bigg] \nabla s.
\end{aligned}$$

Here $\mathbb{N}_0^m$ is a Banach space of all functions $\psi : \mathbb{N}_0^m \to \mathbb{R}$. Furthermore, norm is defined as $\|\psi\| = \max_{\eta \in \mathbb{N}_0^m} |\psi(\eta)|$. Definiteness of the operator $\mathcal{N}$ can be verified easily. Moreover, the fixed points of the operator $\mathcal{N}$ are solutions of the problem (6.1).

Theorem 6.1.3. *Assume that there exists nonnegative functions* $\mathcal{U}, \mathcal{V} \in C(\mathbb{N}_0^m)$ *such that*

$$|\mathcal{G}(\eta, \psi(\eta))| \leq \mathcal{U}(\eta) + \mathcal{V}(\eta)|\psi|^{p-1}, \quad \forall\ \eta \in \mathbb{N}_0^m. \tag{6.7}$$

Then the problem (6.1) *has atleast one solution, provided that*

$$\left(\left(\frac{3}{2}\right)^q \|\mathcal{V}\|^{q-1} \frac{(m^{\overline{\zeta}})^{q-1} m^{\overline{\nu}}}{(\Gamma(\zeta+1))^{q-1}\Gamma(\nu+1)}\right) < 1. \tag{6.8}$$

Proof. Let us consider a nonempty convex subset $\mathcal{K} = \{\psi : \psi \in C(\mathbb{N}_0^m), \|\psi\| \leq \mathcal{R}\}$.

$$\begin{aligned}
|\mathcal{N}\psi(\eta)| \leq & \frac{1}{\Gamma(\nu)} \int_0^\eta (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \bigg[\frac{1}{\Gamma(\zeta)} \int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}} |\mathcal{G}(\tau,\psi(\tau))| \nabla\tau \\
& + \frac{1}{2\Gamma(\zeta)} \int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}} |\mathcal{G}(\tau,\psi(\tau))| \nabla\tau \bigg] \nabla s \\
& + \frac{1}{2\Gamma(\nu)} \int_0^m (m - \varrho(s))^{\overline{\nu-1}} \phi_q \bigg[\frac{1}{\Gamma(\zeta)} \int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}} |\mathcal{G}(\tau,\psi(\tau))| \nabla\tau \\
& + \frac{1}{2\Gamma(\zeta)} \int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}} |\mathcal{G}(\tau,\psi(\tau))| \nabla\tau \bigg] \nabla s,
\end{aligned}$$

now using condition on $\mathcal{G}$ as mentioned in (6.7), we get

$$\begin{aligned}
|\mathcal{N}\psi(\eta)| \leq & \frac{1}{\Gamma(\nu)} \int_0^\eta (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \bigg[\frac{1}{\Gamma(\zeta)} \int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) \nabla\tau \\
& + \frac{1}{2\Gamma(\zeta)} \int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) \nabla\tau \bigg] \nabla s \\
& + \frac{1}{2\Gamma(\nu)} \int_0^m (m - \varrho(s))^{\overline{\nu-1}} \phi_q \bigg[\frac{1}{\Gamma(\zeta)} \int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) \nabla\tau \\
& + \frac{1}{2\Gamma(\zeta)} \int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) \nabla\tau \bigg] \nabla s, \\
\leq & \frac{1}{\Gamma(\nu)} \int_0^\eta (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \bigg(\frac{(\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) s^{\overline{\zeta}}}{\Gamma(\zeta+1)} \\
& + \frac{(\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) m^{\overline{\zeta}}}{2\Gamma(\zeta+1)} \bigg) \nabla s \\
& + \frac{1}{2\Gamma(\nu)} \int_0^m (m - \varrho(s))^{\overline{\nu-1}} \phi_q \bigg(\frac{(\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) s^{\overline{\zeta}}}{\Gamma(\zeta+1)} \\
& + \frac{(\|\mathcal{U}\| + \|\mathcal{V}\|\,\|\psi\|^{p-1}) m^{\overline{\zeta}}}{2\Gamma(\zeta+1)} \bigg) \nabla s,
\end{aligned}$$

using property of p-Laplacian, we get

$$\begin{aligned}|\mathcal{N}\psi(\eta)| &\leq \phi_q\Big(\frac{3(\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1})m^{\overline{\zeta}}}{2\Gamma(\zeta+1)}\Big)\Big[\frac{1}{\Gamma(\nu)}\int_0^{\eta}(\eta-\varrho(s))^{\overline{\nu-1}}\nabla s \\ &\quad + \frac{1}{2\Gamma(\nu)}\int_0^{m}(m-\varrho(s))^{\overline{\nu-1}}\nabla s\Big] \\ &\leq \left(\frac{3(\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1})m^{\overline{\zeta}}}{2\Gamma(\zeta+1)}\right)^{q-1}\left(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)}\right),\end{aligned}$$

using $\|\psi\| \leq \mathcal{R}$, we get

$$|\mathcal{N}\psi(\eta)| \leq \left((\|\mathcal{U}\| + \|\mathcal{V}\| \, \mathcal{R}^{p-1})\frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)}\right)^{q-1}\left(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)}\right) \leq \mathcal{R}.$$

Hence $\mathcal{N}$ maps bounded sets into bounded sets. Now, we will show that $\mathcal{N}$ is a compact operator.

For this, let $0 \leq \eta_1 \leq \eta_2 \leq m$, we have

$$\begin{aligned}&|\mathcal{N}\psi(\eta_2) - \mathcal{N}\psi(\eta_1)| \\ &\leq \frac{1}{\Gamma(\nu)}\int_0^{\eta_1}((\eta_2-\varrho(s))^{\overline{\nu-1}} - (\eta_1-\varrho(s))^{\overline{\nu-1}})\phi_q\Big[\frac{1}{\Gamma(\zeta)}\int_0^{s}(s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau \\ &\quad + \frac{1}{2\Gamma(\zeta)}\int_0^{m}(m-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau\Big]\nabla s \\ &\quad + \frac{1}{\Gamma(\nu)}\int_{\eta_1}^{\eta_2}(\eta_2-\varrho(s))^{\overline{\nu-1}}\phi_q\Big[\frac{1}{\Gamma(\zeta)}\int_0^{s}(s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau \\ &\quad + \frac{1}{2\Gamma(\zeta)}\int_0^{m}(m-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau\Big]\nabla s \\ &\leq \left((\|\mathcal{U}\| + \|\mathcal{V}\| \, \mathcal{R}^{p-1})\frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)}\right)^{q-1}\frac{1}{\Gamma(\nu)}\int_0^{\eta_1}((\eta_2-\varrho(s))^{\overline{\nu-1}} - (\eta_1-\varrho(s))^{\overline{\nu-1}})\nabla s \\ &\quad + \left((\|\mathcal{U}\| + \|\mathcal{V}\| \, \mathcal{R}^{p-1})\frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)}\right)^{q-1}\frac{1}{\Gamma(\nu)}\int_{\eta_1}^{\eta_2}(\eta_2-\varrho(s))^{\overline{\nu-1}}\nabla s.\end{aligned}$$

$|\mathcal{N}\psi(\eta_2) - \mathcal{N}\psi(\eta_1)| \to 0$ as $\eta_1 \to \eta_2$.

Now we will show that $\mathcal{N}$ is continuous. Let a sequence ψ_n converges to ψ, then using Lipschitz condition with $\mathcal{K}$ being Lipschitz constant, we have

$$\begin{aligned}
&|\mathcal{N}\psi_n(\eta) - \mathcal{N}\psi(\eta)| \\
&\leq (p-1)\rho^{p-2}\frac{1}{\Gamma(\nu)}\int_0^\eta (\eta-\varrho(s))^{\overline{\nu-1}}\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi_n(\tau)) - \mathcal{G}(\tau,\psi(\tau))|\nabla\tau \\
&\quad + \frac{1}{2\Gamma(\zeta)}\int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi_n(\tau)) - \mathcal{G}(\tau,\psi(\tau))|\nabla\tau\Bigg]\nabla s \\
&\quad + \frac{1}{2\Gamma(\nu)}\int_0^m (m-\varrho(s))^{\overline{\nu-1}}\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi_n(\tau)) - \mathcal{G}(\tau,\psi(\tau))|\nabla\tau \\
&\quad + \frac{1}{2\Gamma(\zeta)}\int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi_n(\tau)) - \mathcal{G}(\tau,\psi(\tau))|\nabla\tau\Bigg]\nabla s \\
&\leq (p-1)\rho^{p-2}\frac{1}{\Gamma(\nu)}\int_0^\eta (\eta-\varrho(s))^{\overline{\nu-1}}\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{K}\,\|\psi_n-\psi\|\,\nabla\tau \\
&\quad + \frac{1}{2\Gamma(\zeta)}\int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{K}\,\|\psi_n-\psi\|\,\nabla\tau\Bigg]\nabla s \\
&\quad + \frac{1}{2\Gamma(\nu)}\int_0^m (m-\varrho(s))^{\overline{\nu-1}}\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{K}\,\|\psi_n-\psi\|\,\nabla\tau \\
&\quad + \frac{1}{2\Gamma(\zeta)}\int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{K}\,\|\psi_n-\psi\|\,\nabla\tau\Bigg]\nabla s \\
&\leq (p-1)\rho^{p-2}\left(\mathcal{K}\,\|\psi_n-\psi\|\,\frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)}\right)\left(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)}\right).
\end{aligned}$$

Hence $\mathcal{N}\psi_n \to \mathcal{N}\psi$ as $\psi_n \to \psi$. Hence $\mathcal{N}$ is compact operator using Arzelá-Ascoli theorem. Now we find bounds for operator $\mathcal{N}$.

For this, let set $\mathcal{S} = \{\psi \in C(\mathbb{N}_0^m) | \psi = \lambda\mathcal{N}\psi,\ 0<\lambda<1\}$. Now to show the boundedness of the set $\mathcal{S}$, we proceed as follow:

$$\begin{aligned}
|\psi(\eta)| &= \lambda|\mathcal{N}\psi(\eta)| \\
&\leq \frac{1}{\Gamma(\nu)}\int_0^\eta (\eta-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau \\
&\quad + \frac{1}{2\Gamma(\zeta)}\int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau\Bigg]\nabla s \\
&\quad + \frac{1}{2\Gamma(\nu)}\int_0^m (m-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau \\
&\quad + \frac{1}{2\Gamma(\zeta)}\int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\psi(\tau))|\nabla\tau\Bigg]\nabla s,
\end{aligned}$$

using condition on $\mathcal{G}$ as mentioned in (6.7), we get

$$
\begin{aligned}
|\psi(\eta)| \leq & \frac{1}{\Gamma(\nu)} \int_0^{\eta} (\eta - \varrho(s))^{\overline{\nu-1}} \phi_q \Bigg[\frac{1}{\Gamma(\zeta)} \int_0^{s} (s - \varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1}) \nabla\tau \\
& + \frac{1}{2\Gamma(\zeta)} \int_0^{m} (m - \varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1}) \nabla\tau \Bigg] \nabla s \\
& + \frac{1}{2\Gamma(\nu)} \int_0^{m} (m - \varrho(s))^{\overline{\nu-1}} \phi_q \Bigg[\frac{1}{\Gamma(\zeta)} \int_0^{s} (s - \varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1}) \nabla\tau \\
& + \frac{1}{2\Gamma(\zeta)} \int_0^{m} (m - \varrho(\tau))^{\overline{\zeta-1}} (\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1}) \nabla\tau \Bigg] \nabla s \\
\leq & \, \phi_q \left((\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1}) \frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)} \right) \left(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)} \right) \\
= & \left((\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1}) \frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)} \right)^{q-1} \left(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)} \right) \\
= & \left(\frac{3}{2} \right)^{q} \left(\frac{(m^{\overline{\zeta}})^{q-1} m^{\overline{\nu}}}{(\Gamma(\zeta+1))^{q-1} \Gamma(\nu+1)} \right) \left(\|\mathcal{U}\| + \|\mathcal{V}\| \, \|\psi\|^{p-1} \right)^{q-1}.
\end{aligned}
$$

Boundedness of the set $\mathcal{S}$ can be checked easily for $\|\mathcal{V}\| = 0$. If $\|\mathcal{V}\| \neq 0$, then we have

$$
\|\psi\|^{p-1} \leq \left(\left(\frac{3}{2} \right)^{q} \|\mathcal{V}\|^{q-1} \frac{(m^{\overline{\zeta}})^{q-1} m^{\overline{\nu}}}{(\Gamma(\zeta+1))^{q-1} \Gamma(\nu+1)} \right)^{p-1} \left(\frac{\|\mathcal{U}\|}{\|\mathcal{V}\|} + \|\psi\|^{p-1} \right).
$$

Using (6.8), there exists a positive number $\mathcal{M}$ such that $\|\psi\| \leq \mathcal{M}$. Hence, set $\mathcal{S}$ is bounded. $\mathcal{N}$ satisfies conditions of Schaefer's fixed point theorem, so $\mathcal{N}$ has at least one fixed point which is solution of problem (6.1). Hence completing the proof.

□

6.2 Ulam-Hyers Stability

In this section, we intent to show that problem (6.1) is Ulam-Hyers stable. Let us consider for $\epsilon > 0$, we have a positive real number c such that if we have following inequality

$$\begin{aligned}\bigg|\varsigma(\eta)-\bigg(&\frac{1}{\Gamma(\nu)}\int_0^\eta(\eta-\varrho(s))^{\overline{\nu-1}}\phi_q\bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau \\ &-\frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau\bigg]\nabla s \\ &-\frac{1}{2\Gamma(\nu)}\int_0^m(m-\varrho(s))^{\overline{\nu-1}}\phi_q\bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau \\ &-\frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau\bigg]\nabla s\bigg)\bigg| \leq \epsilon, \end{aligned} \tag{6.9}$$

and there exists $\psi(\eta)$ solution of problem (6.1) satisfying

$$\begin{aligned}\psi(\eta) =&\frac{1}{\Gamma(\nu)}\int_0^\eta(\eta-\varrho(s))^{\overline{\nu-1}}\phi_q\bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau \\ &-\frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau\bigg]\nabla s \\ &-\frac{1}{2\Gamma(\nu)}\int_0^m(m-\varrho(s))^{\overline{\nu-1}}\phi_q\bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau \\ &-\frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau\bigg]\nabla s,\end{aligned}$$

such that

$$|\varsigma(\eta)-\psi(\eta)| \leq c\epsilon. \tag{6.10}$$

Then Problem (6.1) is Ulam-Hyers stable.

Theorem 6.2.1. *Problem* (6.1) *is Ulam-Hyers stable provided following assumptions holds:*

(A_1) *Function* $\mathcal{G} : \mathbb{N}_0^m \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(A_2) $\mathcal{G}$ *satisfying Lipschitz condition with* $\mathcal{K}$ *being Lipschitz constant*

$$|\mathcal{G}(\eta,\varsigma(\eta)) - \mathcal{G}(\eta,\psi(\eta))| \leq \mathcal{K}|\varsigma(\eta)-\psi(\eta)|.$$

Proof. Let $\psi(\eta)$ be solution of problem (6.1) and $\varsigma(\eta)$ be an approximate solution and satisfying inequality (6.9). Then, we have

$$\begin{aligned}
|\varsigma(\eta)-\psi(\eta)| =\Bigg|\varsigma(\eta) &- \Bigg(\frac{1}{\Gamma(\nu)}\int_0^\eta(\eta-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau\Bigg]\nabla s \\
&- \frac{1}{2\Gamma(\nu)}\int_0^m(m-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau\Bigg]\nabla s\Bigg) \\
&+ \Bigg(\frac{1}{\Gamma(\nu)}\int_0^\eta(\eta-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau\Bigg]\nabla s \\
&- \frac{1}{2\Gamma(\nu)}\int_0^m(m-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\varsigma(\tau))\nabla\tau\Bigg]\nabla s\Bigg) \\
&- \Bigg(\frac{1}{\Gamma(\nu)}\int_0^\eta(\eta-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau \\
&- \frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau\Bigg]\nabla s \\
&- \frac{1}{2\Gamma(\nu)}\int_0^m(m-\varrho(s))^{\overline{\nu-1}}\phi_q\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau \\
- \frac{1}{2\Gamma(\zeta)}&\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}\mathcal{G}(\tau,\psi(\tau))\nabla\tau\Bigg]\nabla s\Bigg)\Bigg|,
\end{aligned}$$

using p-Laplacian operator as mentioned in Lemma 1.3.3, we get

$$\begin{aligned}
&|\varsigma(\eta)-\psi(\eta)| \\
&\leq \epsilon+(p-1)\rho^{p-2}\Bigg(\frac{1}{\Gamma(\nu)}\int_0^\eta(\eta-\varrho(s))^{\overline{\nu-1}}\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\varsigma(\tau))-\mathcal{G}(\tau,\psi(\tau))|\nabla\tau \\
&\quad+\frac{1}{2\Gamma(\zeta)}\int_0^m(m-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\varsigma(\tau))-\mathcal{G}(\tau,\psi(\tau))|\nabla\tau\Bigg]\nabla s \\
&\quad+\frac{1}{2\Gamma(\nu)}\int_0^m(m-\varrho(s))^{\overline{\nu-1}}\Bigg[\frac{1}{\Gamma(\zeta)}\int_0^s(s-\varrho(\tau))^{\overline{\zeta-1}}|\mathcal{G}(\tau,\varsigma(\tau))-\mathcal{G}(\tau,\psi(\tau))|\nabla\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2\Gamma(\zeta)} \int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}} |\mathcal{G}(\tau,\varsigma(\tau)) - \mathcal{G}(\tau,\psi(\tau))| \nabla\tau \Big] \nabla s \Big) \\
&\leq \epsilon + (p-1)\rho^{p-2}\mathcal{K}\bigg(\frac{1}{\Gamma(\nu)} \int_0^\eta (\eta-\varrho(s))^{\overline{\nu-1}} \bigg[\frac{1}{\Gamma(\zeta)} \int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}} \nabla\tau \\
&+ \frac{1}{2\Gamma(\zeta)} \int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}} \nabla\tau \bigg] \nabla s \\
&+ \frac{1}{2\Gamma(\nu)} \int_0^m (m-\varrho(s))^{\overline{\nu-1}} \bigg[\frac{1}{\Gamma(\zeta)} \int_0^s (s-\varrho(\tau))^{\overline{\zeta-1}} \nabla\tau \\
&+ \frac{1}{2\Gamma(\zeta)} \int_0^m (m-\varrho(\tau))^{\overline{\zeta-1}} \nabla\tau \bigg] \nabla s \bigg) \|\varsigma - \psi\| \\
&\leq \epsilon + (p-1)\rho^{p-2}\mathcal{K} \left(\frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)} \right) \left(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)} \right) \|\varsigma - \psi\|
\end{aligned}
$$

$$
|\varsigma(\eta) - \psi(\eta)| \leq \frac{\epsilon}{1-(p-1)\rho^{p-2}\mathcal{K}(\frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)})(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)})} \leq c\epsilon,
$$

where $c = \dfrac{1}{1-(p-1)\rho^{p-2}\mathcal{K}(\frac{3m^{\overline{\zeta}}}{2\Gamma(\zeta+1)})(\frac{3m^{\overline{\nu}}}{2\Gamma(\nu+1)})}$.

Hence problem (6.1) is Ulam-Hyers stable. □

Example 6.2.1. *Let us consider the following problem*

$$
\begin{cases}
{}^c\nabla_0^{\frac{1}{2}}[\phi_3({}^c\nabla_0^{\frac{2}{3}}\psi)](\eta) = \dfrac{1}{100}\psi^2(\eta) + \sin(\eta), \quad \eta \in \mathbb{N}_0^1, \\
\phi_3({}^c\nabla_0^{\frac{2}{3}}\psi)(\eta)|_{\eta=0} + \phi_3({}^c\nabla_0^{\frac{2}{3}}\psi)(\eta)|_{\eta=1} = 0, \ \psi(0)+\psi(1) = 0.
\end{cases}
\tag{6.11}
$$

Comparison of above problem with problem (6.1) *shows that* $p = 3,\ q = \frac{3}{2},\ \nu = \frac{2}{3},\ \zeta = \frac{1}{2},$ $\mathcal{G}(\eta,\psi(\eta)) = \dfrac{1}{100}\psi^2(\eta) + \sin(\eta),\ \eta \in \mathbb{N}_0^1,\ \psi \in \mathbb{R}.$

Let $\mathcal{U}(\eta) = 1,\ \mathcal{V}(\eta) = \dfrac{1}{100}$. *After calculation, we get*

$$
\left(\left(\frac{3}{2}\right)^{3/2} \left(\frac{1}{100}\right)^{1/2} \frac{(1^{\overline{1/2}})^{1/2} 1^{\overline{2/3}}}{(\Gamma(3/2))^{1/2}\Gamma(5/3)} \right) < 1.
$$

Hence problem (6.11) *satisfying all the conditions of Theorem* (6.1.3). *Therefore, problem under consideration has atleast one solution.*

Clearly, $\mathcal{G}(\eta,\psi(\eta)) = \dfrac{1}{100}\psi^2(\eta) + \sin(\eta)$ *is continuous function. Furthermore, it satisfies Lips-*

chitz condition,

$$\begin{aligned}|\mathcal{G}(\eta,\psi(\eta))-\mathcal{G}(\eta,\varsigma(\eta))| &= |\frac{1}{100}\psi^2(\eta)+\sin(\eta)-\frac{1}{100}\varsigma^2(\eta)-\sin(\eta)| \\ &\leq \frac{1}{100}|\psi^2(\eta)-\varsigma^2(\eta)| \leq \frac{m_1}{100}|\psi(\eta)-\varsigma(\eta)|, \qquad m_1>0.\end{aligned}$$

Moreover, calculation shows that $c = \dfrac{1}{1-(p-1)\rho^{p-2}\mathcal{K}(\frac{3m^{\bar{\zeta}}}{2\Gamma(\zeta+1)})(\frac{3m^{\bar{\nu}}}{2\Gamma(\nu+1)})} > 0$. *Hence, all conditions of Theorem* (6.2.1) *are satisfied. Therefore, problem under consideration is Ulam-Hyers stable.*

Chapter 7

Ulam-Hyers-Mittag-Leffler Stability of Fractional Difference Equation with Delay

In this chapter, we discussed the stability analysis of a class of Caputo nabla fractional-order delay difference equation. We used Chebyshev norm to obtain Ulam-Hyers-Mittag-Leffler stability. Furthermore, Ulam-Hyers-Rassias-Mittag-Leffler stability of the problem is also discussed.

Motivated by the work [63, 134], the objective of the current chapter is to deliberate Ulam-Hyers-Mittag-Leffler stability of the following problem:

$$\begin{cases} {}^{c}\nabla_{a}^{\zeta}\psi(\eta) = \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\eta))), \ 0 < \zeta \leq 1, \ \eta \in \mathbb{N}_{a}^{b}, \\ \quad \psi(\eta) = \phi(\eta), \qquad \eta \in [m_0, a] \cap \mathbb{N}_{m_0}, \end{cases} \tag{7.1}$$

associated with the inequality:

$$|{}^{c}\nabla_{a}^{\zeta}\varsigma(\eta) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\eta)))| \leq \epsilon E_{\overline{\zeta}}(\eta), \ \eta \in \mathbb{N}_{a}^{b}, \tag{7.2}$$

where ${}^{c}\nabla_{a}^{\zeta}$ is Caputo nabla fractional difference of order ζ. $\mathscr{Q}$ is a function defined as $\mathscr{Q}$: $\mathbb{N}_{a}^{b} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ whereas delay function $\mathscr{G}$ is defined on $\mathbb{N}_{a}^{b}$ satisfying $\mathscr{G}(\eta) \leq \eta$, $a, b \in \mathbb{R}$ with $b > a$ and $m_0 = \inf_{\eta\in\mathbb{N}_{a}^{b}} \{\mathscr{G}(\eta)\}$. We will use $a = 0$ throughout this chapter. Moreover $E_{\overline{\zeta,\beta}}(\eta)$

is nabla discrete Mittag-Leffler function [3] defined by

$$E_{\overline{\zeta,\beta}}(\eta) = \sum_{k=0}^{\infty} \frac{\eta^{\overline{k\zeta+\beta-1}}}{\Gamma(k\zeta+\beta)}.$$

For $\beta = 1$, we can write it as

$$E_{\overline{\zeta,1}}(\eta) = E_{\overline{\zeta}}(\eta) = \sum_{k=0}^{\infty} \frac{\eta^{\overline{k\zeta}}}{\Gamma(k\zeta+1)}.$$

Section 7.1 contains some necessary definitions, lemmas, and theorems that help to prove our major results. Section 7.2 deals with the Ulam-Hyers-Mittag-Leffler stability analysis using the Chebyshev norm followed by example. In Section 7.3, we intend to analyze the Ulam-Hyers-Rassias-Mittag-Leffler stability of the problem (7.1).

7.1 Fundamental Results

The motivation behind this section is, to sum up, some essential outcomes which are significant for the development of main results in later section. For further details on the subject, we recommend [2, 73, 79]. Some necessary results are provided.

Definition 7.1.1. *[63] A function $\mathscr{D} : X \times X \to \mathbb{R}^+$ is a generalized metric on X if and only if it satisfies followig axioms:*

(1) $\mathscr{D}(\psi, \varsigma) = 0$ if and only if $\psi = \varsigma$;

(2) $\mathscr{D}(\psi, \varsigma) = \mathscr{D}(\varsigma, \psi)$ for all $\psi, \varsigma \in X$;

(3) $\mathscr{D}(\psi, z) \leq \mathscr{D}(\psi, \varsigma) + \mathscr{D}(\varsigma, z)$ for all $\psi, \varsigma, z \in X$.

Note: Here $\mathbb{R}^+$= Set of all positive real numbers including "0".

The concept of a generalized complete metric space X is different from the familiar perception of complete metric space in a sense that distance between two points in X is not necessarily finite.

In the accompanying, essential consequences of the Banach fixed point hypothesis are expressed in a generalized complete metric space.

Theorem 7.1.1. *[63] Let $(X, \mathscr{D})$ be a generalized complete metric space, $\Psi : X \to X$ is a strictly contractive operator with $\mathscr{L} < 1$ being Lipschitz constant. Moreover for some $\psi \in X$, $\mathscr{D}(\Psi^{m+1}\psi, \Psi^m\psi) < \infty$, where m is a non negative integer then following properties hold:*

(a) The sequence $\Psi^m\psi$ converges to a fixed point ψ^* of Ψ;

(b) In $X^* = \{\varsigma \in X \mid \mathscr{D}(\Psi^m\psi, \varsigma) < \infty\}$, ψ^* is the unique fixed point of Ψ ;

(c) If $\varsigma \in X^*$, then $\mathscr{D}(\varsigma, \psi^*) \leq \dfrac{1}{1-\mathscr{L}}\mathscr{D}(\Psi\varsigma, \varsigma)$.

7.2 Ulam-Hyers-Mittag-Leffler Stability

Definition 7.2.1. *Eq.* (7.1) *is called Ulam-Hyers-Mittag-Leffler stable with respect to nabla discrete Mittag-Leffler function. If for $\epsilon > 0$, we have positive real number C and for each solution ς of the inequality*

$$\left|\varsigma(\eta) - \varsigma(a) - \frac{1}{\Gamma(\zeta)} \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, \varsigma(s), \varsigma(\mathscr{G}(s)))\right| \leq \epsilon E_{\overline{\zeta}}(\eta), \tag{7.3}$$

then there exists solution ψ of Eq. (7.1) *satisfying the following inequality:*

$$|\varsigma(\eta) - \psi(\eta)| \leq C\epsilon E_{\overline{\zeta}}(\eta), \qquad \eta \in \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b.$$

Remark 10. *A function ς defined on $\mathbb{N}_a^b$ is a solution of the inequality* (7.2) *if and only if we have a function $\tilde{h}$ defined on $\mathbb{N}_a^b$ such that:*

(i) $|\tilde{h}(\eta)| \leq \epsilon E_{\overline{\zeta}}(\eta)$ for all $\eta \in \mathbb{N}_a^b$;

(ii) ${}^c\nabla_a^\zeta \varsigma(\eta) = \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\eta))) + \tilde{h}(\eta)$ for all $\eta \in \mathbb{N}_a^b$.

Remark 11. *From above remark we deduce that* ς *being solution of the inequality* (7.2) *is also solution of following inequality:*

$$
\begin{aligned}
&|\varsigma(\eta) - \varsigma(a) - \frac{1}{\Gamma(\zeta)} \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, \varsigma(s), \varsigma(\mathscr{G}(s))| \\
&\leq \frac{\epsilon}{\Gamma(\zeta)} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} E_{\overline{\zeta}}(s) \nabla s \\
&= \frac{\epsilon}{\Gamma(\zeta)} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \sum_{m=0}^{\infty} \frac{s^{\overline{m\zeta}}}{\Gamma(m\zeta+1)} \nabla s \\
&= \frac{\epsilon}{\Gamma(\zeta)} \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\zeta+1)} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} s^{\overline{m\zeta}} \nabla s \\
&= \epsilon \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\zeta+1)} \nabla_a^{-\zeta} \eta^{\overline{m\zeta}} \\
&= \epsilon \sum_{m=0}^{\infty} \frac{\eta^{\overline{(m+1)\zeta}}}{\Gamma((m+1)\zeta+1)} \\
&\leq \epsilon \sum_{m=0}^{\infty} \frac{\eta^{\overline{m\zeta}}}{\Gamma(m\zeta+1)} \\
&= \epsilon E_{\overline{\zeta}}(\eta).
\end{aligned}
$$

Theorem 7.2.1. *Let us consider following assumptions:*

(A_1) $\mathscr{Q} : \mathbb{N}_a^b \times \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function;*

(A_2) *delay function* $\mathscr{G} : \mathbb{N}_a^b \to \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b$ *satisfying* $\mathscr{G}(\eta) \leq \eta$*;*

(A_3) $\mathscr{Q}$ *satisfy Lipschitz condition with* $\mathscr{L} > 0$ *being Lipschitz constant*

$$|\mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\eta)) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\mathscr{G}(\eta))| \leq \mathscr{L}|\psi(\eta) - \varsigma(\eta)|,$$

$\eta \in \mathbb{N}_a^b$ *and* $\psi, \varsigma \in \mathbb{R}$*;*

(A_4) ς *satisfy inequality* (7.3)*;*

Then Eq. (7.1) *is Ulam-Hyers-Mittag-Leffler stable.*

Proof. Let us define space of continuous functions X and generalized metric $\mathscr{D}$ as follows:

$$X = \left\{ p : \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b \to \mathbb{R} \mid p \text{ is continuous} \right\}.$$

$$\mathscr{D}(p,q) = \inf \left\{ \mathscr{K} \in \mathbb{R}^+ \mid |p(\eta) - q(\eta)| \le \mathscr{K} \epsilon E_{\overline{\zeta}}(\eta), \ \forall \, \eta \in \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b \right\}. \tag{7.4}$$

Here $(X, \mathscr{D})$ is a complete generalized metric space.

Further define an operator $\Psi : X \to X$ by

$$(\Psi p)(\eta) = \begin{cases} \phi(\eta), & \eta \in \ [m_0, a] \cap \mathbb{N}_{m_0}, \\ \phi(a) + \dfrac{1}{\Gamma(\zeta)} \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, p(s), p(\mathscr{G}(s))), & \eta \in \mathbb{N}_a^b, \end{cases} \tag{7.5}$$

for all $p \in X$. Continuity of p signify continuity of Ψp and hence we have a well-defined operator Ψ. Let for any $p, q \in X$, we have $\mathscr{K}_{pq} \in \mathbb{R}^+$ with

$$|p(\eta) - q(\eta)| \le \mathscr{K}_{pq} \epsilon E_{\overline{\zeta}}(\eta), \tag{7.6}$$

for any $\eta \in \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b$. Now using Ψ operator as defined in Eq. (7.5). For $\eta \in [m_0, a] \cap \mathbb{N}_{m_0}$, we get

$$|(\Psi p)(\eta) - (\Psi q)(\eta)| = 0.$$

Now for $\eta \in \mathbb{N}_a^b$, using Lipschitz condition and inequality (7.6), we get

$$\begin{aligned} &|(\Psi p)(\eta) - (\Psi q)(\eta)| \\ &\qquad = \frac{1}{\Gamma(\zeta)} \left| \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} (\mathscr{Q}(s, p(s), p(\mathscr{G}(s))) - \mathscr{Q}(s, q(s), q(\mathscr{G}(s)))) \right| \\ &\qquad = \frac{1}{\Gamma(\zeta)} \left| \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} (\mathscr{Q}(s, p(s), p(\mathscr{G}(s))) - \mathscr{Q}(s, q(s), q(\mathscr{G}(s)))) \nabla s \right| \\ &\qquad \le \frac{\mathscr{L}}{\Gamma(\zeta)} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} |p(s) - q(s)| \nabla s \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mathscr{L}\mathscr{K}_{pq}\epsilon}{\Gamma(\zeta)} \int_a^\eta (\eta-\varrho(s))^{\overline{\zeta-1}} E_{\overline{\zeta}}(s) \nabla s \\
&= \frac{\mathscr{L}\mathscr{K}_{pq}\epsilon}{\Gamma(\zeta)} \int_a^\eta (\eta-\varrho(s))^{\overline{\zeta-1}} \sum_{m=0}^{\infty} \frac{s^{\overline{m\zeta}}}{\Gamma(m\zeta+1)} \nabla s \\
&= \frac{\mathscr{L}\mathscr{K}_{pq}\epsilon}{\Gamma(\zeta)} \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\zeta+1)} \int_a^\eta (\eta-\varrho(s))^{\overline{\zeta-1}} s^{\overline{m\zeta}} \nabla s \\
&= \mathscr{L}\mathscr{K}_{pq}\epsilon \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\zeta+1)} \nabla_a^{-\zeta} \eta^{\overline{m\zeta}} \\
&= \mathscr{L}\mathscr{K}_{pq}\epsilon \sum_{m=0}^{\infty} \frac{1}{\Gamma((m+1)\zeta+1)} \eta^{\overline{(m+1)\zeta}} = \mathscr{L}\mathscr{K}_{pq}\epsilon \sum_{m=1}^{\infty} \frac{1}{\Gamma(m\zeta+1)} \eta^{\overline{m\zeta}} \\
&\leq \mathscr{L}\mathscr{K}_{pq}\epsilon \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\zeta+1)} \eta^{\overline{m\zeta}} = \mathscr{L}\mathscr{K}_{pq}\epsilon E_{\overline{\zeta}}(\eta), \qquad \forall\, \eta \in \mathbb{N}_a^b.
\end{aligned}
$$

Consequently

$$\mathscr{D}(\Psi p, \Psi q) \leq \mathscr{L}\mathscr{K}_{pq}\epsilon E_{\overline{\zeta}}(\eta),$$

$$\mathscr{D}(\Psi p, \Psi q) \leq \mathscr{L}\mathscr{D}(p,q),$$

$\forall\, p, q \in X$ and strict contractive property is followed by $0 < \mathscr{L} < 1$.

Let us consider $p_0 \in X$. Since p_0 and Ψp_0 satisfy continuity condition, so we have $\mathscr{K}_1 \in (0, \infty)$ such that

$$
\begin{aligned}
|(\Psi p_0)(\eta) - p_0(\eta)| &= |\phi(a) + \frac{1}{\Gamma(\zeta)} \sum_{s=a+1}^{\eta} (\eta-\varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, p_0(s), p_0(\mathscr{G}(s))) - p_0(\eta)| \\
&\leq \mathscr{K}_1 E_{\overline{\zeta}}(\eta),
\end{aligned}
$$

for all $\eta \in \mathbb{N}_a^b$. Since $\mathscr{Q}$ and p_0 are bounded on $\mathbb{N}_a^b$ and $\min_{\eta\in\mathbb{N}_a^b} E_{\overline{\zeta}}(\eta) > 0$. Thus Eq. (7.4) implies that $\mathscr{D}(\Psi p_0, p_0) < \infty$.

Therefore, using part (a) of Theorem 7.1.1, $\Psi^m p_0 \to \psi$ in $(X, \mathscr{D})$ as $m \to \infty$ where a continuous function ψ is defined as $\psi : \mathbb{N}_a^b \to \mathbb{R}$. Moreover $\Psi\psi = \psi$; so Eq. (7.1) is satisfied by ψ for every $\eta \in \mathbb{N}_a^b$.

Our next goal is to show that $\{p \in X \mid \mathscr{D}(p_0, p) < \infty\} = X$. If $p \in X$ then p and p_0 are continuous and bounded defined on $\mathbb{N}_a^b$. Moreover $\min_{\eta\in\mathbb{N}_a^b} E_{\overline{\zeta}}(\eta) > 0$, so there exists $\mathscr{C}_{\mathscr{P}} \in$

$(0,\infty)$ with

$$|p_0(\eta)-p(\eta)| \leq \mathscr{C}_{\mathscr{P}} E_{\overline{\zeta}}(\eta),$$

for any $\eta \in \mathbb{N}_a^b$. This implies that $\mathscr{D}(p_0,p)<\infty$, for all $p \in X$ or identically

$$\{p \in X \mid \mathscr{D}(p_0,p)<\infty\} = X.$$

Using part (b) of Theorem (7.1.1), we can say that ψ is the unique continuous function satisfying Eq. (7.1). Now using inequality (7.3), we have $\mathscr{D}(\varsigma, \Psi\varsigma) \leq \epsilon E_{\overline{\zeta}}(\eta)$. Using part (c) of Theorem (7.1.1) together with the above inequality, we get

$$\mathscr{D}(\psi,\varsigma) \leq \frac{1}{1-\mathscr{L}} \mathscr{D}(\Psi\varsigma,\varsigma) \leq \frac{1}{1-\mathscr{L}} \epsilon E_{\overline{\zeta}}(\eta).$$

Hence Ulam-Hyers-Mittag-Leffler stability of Eq. (7.1) is verified. □

In the next theorem, we will use Chebyshev norm defined as $\|\psi\|_c = \max_{\eta\in\mathbb{N}_a^b} |\psi(\eta)|$ to analyze the stability for Eq. (7.1).

Theorem 7.2.2. *If* $(A_1),(A_2),(A_3^*)$, (A_4) *and in addition* (A_5) *are satisfied, where* (A_3^*) *and* (A_5) *are given as follows:*

(A_3^*) $\mathscr{F}$ satisfy Lipschitz condition with $\mathscr{L}^*>0$ being Lipschitz constant

$$|\mathscr{Q}(\eta,u(\eta),u(\mathscr{G}(\eta))) - \mathscr{Q}(\eta,v(\eta),v(\mathscr{G}(\eta)))| \leq \mathscr{L}^*\big(|u(\eta)-v(\eta)| + |u(\mathscr{G}(\eta)) - v(\mathscr{G}(\eta))|\big),$$

$\eta \in \mathbb{N}_a^b$ and $u,v \in \mathbb{R}$;

(A_5) $0 < 2\mathscr{L}^* \dfrac{(b-a)^{\overline{\gamma}}}{\Gamma(1+\gamma)} < 1$*;*

then Eq. (7.1) *is called Ulam-Hyers-Mittag-Leffler stable using Chebyshev norm.*

Proof. Similar to the above proof, we only need to prove that Ψ defined in Eq. (7.5) is a contractive map on X with respect to Chebyshev norm:

$$
\begin{aligned}
&|(\Psi p)(\eta) - (\Psi q)(\eta)| \\
&\quad = \frac{1}{\Gamma(\zeta)}\Big| \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} (\mathscr{Q}(s, p(s), p(\mathscr{G}(s))) - \mathscr{Q}(s, q(s), q(\mathscr{G}(s))))\Big| \\
&\quad = \frac{1}{\Gamma(\zeta)}\Big| \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} (\mathscr{Q}(s, p(s), p(\mathscr{G}(s))) - \mathscr{Q}(s, q(s), q(\mathscr{G}(s))))\nabla s\Big| \\
&\quad \leq \frac{1}{\Gamma(\zeta)} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} |(\mathscr{Q}(s, p(s), p(\mathscr{G}(s))) - \mathscr{Q}(s, q(s), q(\mathscr{G}(s))))|\nabla s \\
&\quad \leq \frac{\mathscr{L}^*}{\Gamma(\zeta)} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} (\max_{\eta\in\mathbb{N}_a^b} |p(s) - q(s)| + \max_{\eta\in\mathbb{N}_a^b} |p(\mathscr{G}(s)) - q(\mathscr{G}(s))|)\nabla s \\
&\quad \leq \frac{2\mathscr{L}^*}{\Gamma(\zeta)} \|p - q\|_c \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \nabla s \\
&\quad = 2\mathscr{L}^* \|p - q\|_c \nabla_a^{-\zeta}(1) \\
&\quad = 2\mathscr{L}^* \|p - q\|_c \frac{(\eta - a)^{\overline{\zeta}}}{\Gamma(1+\zeta)} \\
&\quad \leq 2\mathscr{L}^* \|p - q\|_c \frac{(b - a)^{\overline{\zeta}}}{\Gamma(1+\zeta)},
\end{aligned}
$$

for all $\eta \in \mathbb{N}_a^b$. That is, $\mathscr{D}(\Psi p, \Psi q) \leq 2\mathscr{L}^*\|p-q\|_c \dfrac{(b-a)^{\overline{\zeta}}}{\Gamma(1+\zeta)}$. Hence we conclude that $\mathscr{D}(\Psi p, \Psi q) \leq 2\mathscr{L}^* \dfrac{(b-a)^{\overline{\zeta}}}{\Gamma(1+\zeta)} \mathscr{D}(p, q)$ for any $p, q \in X$. Strict contractive property is verified using $0 < 2\mathscr{L}^* \dfrac{(b-a)^{\overline{\zeta}}}{\Gamma(1+\zeta)} < 1$. Presently continuing likewise as in past hypothesis, we have

$$
\mathscr{D}(\psi, \varsigma) \leq \frac{1}{1 - 2\mathscr{L}^* \frac{(b-a)^{\overline{\zeta}}}{\Gamma(1+\zeta)}} \mathscr{D}(\Psi\varsigma, \varsigma) \leq \frac{1}{1 - 2\mathscr{L}^* \frac{(b-a)^{\overline{\zeta}}}{\Gamma(1+\zeta)}} \epsilon E_{\overline{\zeta}}(\eta) \leq C\epsilon E_{\overline{\zeta}}(\eta).
$$

Hence Ulam-Hyers-Mittag-Leffler stability of Eq. (7.1) is verified using Chebyshev norm. □

Example 7.2.2. *Let us consider following fractional order difference system*

$$
{}^c\nabla_0^{\frac{1}{2}} \psi(\eta) = \frac{1}{5} \frac{\psi^2(\eta - 1)}{1 + \psi^2(\eta - 1)} + \frac{1}{5} \sin(2\psi(\eta)), \;\; \eta \in \mathbb{N}_0^1; \quad \psi(\eta) = 0, \;\; \eta \in [-1, 0] \cap \mathbb{N}_{-1}
$$

and the inequality

$$|^{c}\nabla_0^{\frac{1}{2}}\varsigma(\eta) - \mathscr{Q}(\eta, \varsigma(\eta), \varsigma(\eta-1))| \leq \epsilon E_{\overline{\frac{1}{2}}}(\eta).$$

Here $a = 0,\ b = 1,\ \mathscr{G}(\eta) = \eta - 1,\ \mathscr{Q}(\eta, \psi(\eta), \psi(\mathscr{G}(\eta))) = \frac{1}{5}\frac{\psi^2(\eta-1)}{1+\psi^2(\eta-1)} + \frac{1}{5}\sin(2\psi(\eta)),$ $\zeta = \frac{1}{2}$ *and* $\mathscr{L}^* = \frac{2}{5}$. *Thus* $2\mathscr{L}^*\frac{(b-a)^{\overline{\zeta}}}{\Gamma(1+\zeta)} < 1$. *Since* $(A_1), (A_2), (A_3^*), (A_4)$ *and* (A_5) *of above theorem are satisfied therefore the problem under consideration is Ulam-Hyers-Mittag-Leffler stable with*

$$|\varsigma(\eta) - \psi(\eta)| \leq C\epsilon E_{\overline{\frac{1}{2}}}(\eta), \quad \eta \in \{[-1, 0] \cap \mathbb{N}_{-1}\} \cup \mathbb{N}_0^1.$$

7.3 Ulam-Hyers-Rassias-Mittag-Leffler Stability

Definition 7.3.1. *Eq.* (7.1) *is Ulam-Hyers-Rassias-Mittag-Leffler stable with respect to nabla discrete Mittag-Leffler function, if for each* $\epsilon > 0$ *we have a positive real number* C *and* ς *being solution of the following inequality*

$$|\varsigma(\eta) - \varsigma(a) - \frac{1}{\Gamma(\zeta)}\sum_{s=a+1}^{\eta}(\eta - \varrho(s))^{\overline{\zeta-1}}\mathscr{Q}(s, \varsigma(s), \varsigma(\mathscr{G}(s)))| \leq \Phi(\eta)\epsilon E_{\overline{\zeta}}(\eta). \tag{7.7}$$

We have a unique solution ψ *of Eq.* (7.1) *satisfying*

$$|\varsigma(\eta) - \psi(\eta)| \leq C\Phi(\eta)\epsilon E_{\overline{\zeta}}(\eta),$$

whereas a continuous function Φ *is defined as* $\Phi : X \to \mathbb{R}^+$.

Theorem 7.3.1. *Assume that* $(A_1), (A_2)$ *and* (A_3) *are satisfied.* (A_4) *replaced with*

(A_4) ς *satisfy inequality* (7.7).

(A_6) *there exists a function* $\Phi : \mathbb{N}_a^b \to \mathbb{R}^+$ *satisfying*

$$\nabla_a^{-\zeta}\Phi(\eta) \leq \mathscr{K}\Phi(\eta), \tag{7.8}$$

then Eq. (7.1) *is Ulam-Hyers-Rassias-Mittag-Leffler stable. Hence*

$$|\varsigma(\eta) - \psi(\eta)| \leq C\Phi(\eta)\epsilon E_{\overline{\zeta}}(\eta), \tag{7.9}$$

for all $\eta \in \mathbb{N}_a^b$.

Proof. Let us define space of continuous functions X and generalized metric $\mathscr{D}$ as follows:

$$X = \left\{p : \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b \to \mathbb{R} \mid p \text{ is continuous}\right\}. \tag{7.10}$$

$$\mathscr{D}(p, q) = \inf\left\{\mathscr{K} \in \mathbb{R}^+ \mid |p(\eta) - q(\eta)| \leq \mathscr{K}\Phi(\eta) \ \forall \ \eta \in \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b\right\}. \tag{7.11}$$

Hence $(X, \mathscr{D})$ is a complete generalized metric space.

Further define an operator $\Psi : X \to X$ as follows:

$$(\Psi p)(\eta) = \begin{cases} \phi(\eta), & \eta \in \ [m_0, a] \cap \mathbb{N}_{m_0}, \\ \phi(a) + \dfrac{1}{\Gamma(\zeta)} \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta - 1}} \mathscr{Q}(s, p(s), p(\mathscr{G}(s))), & \eta \in \mathbb{N}_a^b, \end{cases} \tag{7.12}$$

for all $p \in X$ and $\eta \in \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b$. Continuity of function p implies continuity of Ψp and hence Ψ is a well-defined operator. For p, q being elements of X, let $\mathscr{K}_{pq} \in \mathbb{R}^+$ with

$$|p(\eta) - q(\eta)| \leq \mathscr{K}_{pq}\Phi(\eta), \tag{7.13}$$

holds for any $\eta \in \{[m_0, a] \cap \mathbb{N}_{m_0}\} \cup \mathbb{N}_a^b$.

For $\eta \in \ [m_0, a] \cap \mathbb{N}_{m_0}$, we get

$$|(\Psi p)(\eta) - (\Psi q)(\eta)| = 0.$$

Now for $\eta \in \mathbb{N}_a^b$, using definition of Ψ, Lipschitz condition and inequalities (7.8), (7.13), we have

$$\begin{aligned}
&|(\Psi p)(\eta) - (\Psi q)(\eta)| \\
&\qquad = \frac{1}{\Gamma(\zeta)}\Big| \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} (\mathscr{Q}(s, p(s), p(\mathscr{G}(s))) - \mathscr{Q}(s, q(s), q(\mathscr{G}(s))))\Big| \\
&\qquad = \frac{1}{\Gamma(\zeta)}\Big| \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} (\mathscr{Q}(s, p(s), p(\mathscr{G}(s))) - \mathscr{Q}(s, q(s), q(\mathscr{G}(s))))\nabla s\Big| \\
&\qquad \le \frac{\mathscr{L}}{\Gamma(\zeta)} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} |p(s) - q(s)|\nabla s \\
&\qquad \le \frac{\mathscr{L}}{\Gamma(\zeta)} \mathscr{K}_{pq} \int_a^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \Phi(s)\nabla s \\
&\qquad = \mathscr{L}\mathscr{K}_{pq} \nabla_a^{-\zeta}\Phi(\eta) \\
&\qquad \le \mathscr{L}\mathscr{K}_{pq}\mathscr{K}\Phi(\eta),
\end{aligned}$$

for all $\eta \in \mathbb{N}_a^b$, that is $\mathscr{D}(\Psi p, \Psi q) \le \mathscr{K}\mathscr{L}\mathscr{K}_{pq}\Phi(\eta)$. Hence for any $p, q \in X$, we conclude that $\mathscr{D}(\Psi p, \Psi q) \le \mathscr{L}\mathscr{K}\mathscr{D}(p, q)$ and strict contractive property is verified using $0 < \mathscr{L}\mathscr{K} < 1$.

Let us consider function $p_0 \in X$ so we have $0 < \mathscr{K}_1 < \infty$ with

$$|(\Psi p_0)(\eta) - p_0(\eta)| = |\phi(a) + \frac{1}{\Gamma(\zeta)} \sum_{s=a+1}^{\eta} (\eta - \varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, p_0(s), p_0(\mathscr{G}(s))) - p_0(\eta)| \le \mathscr{K}_1\Phi(\eta),$$

for all $\eta \in \mathbb{N}_a^b$, since $\mathscr{Q}, p_0$ are bounded on $\mathbb{N}_a^b$ and $\min_{\eta\in\mathbb{N}_a^b} \Phi(\eta) > 0$. Thus, Eq. (7.11) implies that $\mathscr{D}(\Psi p_0, p_0) < \infty$. Therefore, using part (a) of Theorem 7.1.1, $\Psi^m p_0 \to \psi$ in $(X, \mathscr{D})$ as $m \to \infty$ where a continuous function ψ is defined as $\psi : \mathbb{N}_a^b \to \mathbb{R}$. Moreover $\Psi\psi = \psi$; so Eq. (7.1) is satisfied by ψ for every $\eta \in \mathbb{N}_a^b$.

In next step, we have to show that $\{p \in X \mid \mathscr{D}(p_0, p) < \infty\} = X$. Since p and p_0 are bounded on $\mathbb{N}_a^b$ and $\min_{\eta\in\mathbb{N}_a^b} E_{\overline{\zeta}}(\eta) > 0$, so we have a constant $0 < \mathscr{C}_{\mathscr{P}} < \infty$ with

$$|p_0(\eta) - p(\eta)| \le \mathscr{C}_{\mathscr{P}}\Phi(\eta),$$

for any $\eta \in \mathbb{N}_a^b$. This implies that $\mathscr{D}(p_0, p) < \infty$ for all $p \in X$; or identically,

$$\{p \in X \mid \mathscr{D}(p_0, p) < \infty\} = X.$$

So we can say using part (b) of Theorem (7.1.1) that Eq. (7.1) is satisfied by unique continuous function ψ.

Inspite of this, from inequality (7.7) it follows that

$$d(\varsigma, \Psi\varsigma) \leq \epsilon E_{\overline{\zeta}}(\eta)\Phi(\eta).$$

In the end, using part (c) of Theorem (7.1.1) together with above inequality, we have

$$\mathscr{D}(\varsigma, \psi) \leq \frac{1}{1 - \mathscr{K}\mathscr{L}}\mathscr{D}(\Psi\varsigma, \varsigma) \leq \frac{1}{1 - \mathscr{K}\mathscr{L}}\Phi(\eta)\epsilon E_{\overline{\zeta}}(\eta).$$

Hence Eq. (7.1) is Ulam-Hyers-Rassias-Mittag-Leffler stable. □

Chapter 8

Stability Analysis of Fractional Difference Langevin Equations

In this chapter, we examined stability of a fractional difference Langevin equation within nabla sense with non-local boundary conditions. The results of this chapter are given in our article [47].

Many researchers worked on Langevin fractional differential equations [114, 14, 105, 130, 148, 149, 38, 37, 65, 58, 86, 133, 155], however, the study of fractional difference Langevin equation is rare.

In this chapter, we investigated fractional Langevin difference equation as given below:

$$\begin{cases} {}^c\nabla_0^{\zeta}({}^c\nabla_0^{\beta} + \lambda(\eta))\psi(\eta) = \mathscr{Q}(\eta, \psi(\eta)), \quad \eta \in \mathbb{N}_0^d, \quad d \in \mathbb{N}_1, \\ \psi(0) = 0 = \psi(d), \ {}^c\nabla_0^{\beta}\psi(0) +^c \nabla_0^{\beta}\psi(d) = \mu \sum_{s=1}^{\nu} \psi(s), \end{cases} \tag{8.1}$$

where $\beta \in (0,1)$, $\zeta \in (1,2]$, μ being real number, λ defined on $\mathbb{N}_0^d$ and $\nu \in (0,1)$, ${}^c\nabla_0^{\gamma}$ is the nabla fractional difference of order $\gamma \in \{\beta, \zeta\}$ and $\mathscr{Q}$ defined on $\mathbb{N}_0^d \times \mathbb{R}$.

Current chapter is formulated as follows: section 8.1 contains the essential outcomes about existence-uniqueness of solution for problem (8.1) using fixed point theorems. Section 8.2 contains the Ulam-Hyers stability of the above mentioned problem followed by example.

8.1 Existence-Uniqueness Results

In this section, we present solution of above mentioned problem and then discuss about existence and uniquness of solution.

Theorem 8.1.1. *Let us consider a fractional Langevin difference equation with boundary conditions*

$$\begin{cases} {}^c\nabla_0^{\zeta}({}^c\nabla_0^{\beta} + \lambda(\eta))\psi(\eta) = \mathcal{U}(\eta), \quad \eta \in \mathbb{N}_0^d, \quad d \in \mathbb{N}_1 \\ \psi(0) = 0 = \psi(d), \ {}^c\nabla_0^{\beta}\psi(0) + {}^c\nabla_0^{\beta}\psi(d) = \mu \sum_{s=1}^{\nu} \psi(s), \end{cases} \tag{8.2}$$

where $\beta \in (0,1)$, $\zeta \in (1,2]$, $\mu \in \mathbb{R}$ and $\nu \in (0,1)$. Then, its solution is

$$\begin{aligned}
\psi(\eta) = & \frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta+\zeta-1}} \mathcal{U}(s) - \frac{1}{\Gamma(\beta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s) \\
& + \frac{1}{\Gamma(\zeta)} \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} \mathcal{U}(s) \\
& - \mu \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{\nu} \psi(s) \\
& + \frac{1}{\Gamma(\beta+\zeta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} \mathcal{U}(s) \\
& - \frac{1}{\Gamma(\beta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s), \quad \eta \in \mathbb{N}_0^d.
\end{aligned}$$

Proof. Using $\nabla_0^{-\zeta}$ on both sides of Eq. (8.2), we have

$$ {}^c\nabla_0^{\beta}\psi(\eta) = \nabla_0^{-\zeta}\mathcal{U}(\eta) + c_0 + c_1\eta - \lambda(\eta)\psi(\eta). \tag{8.3}$$

Now, using $\nabla_0^{-\beta}$ on both sides of Eq. (8.3), it gives

$$\psi(\eta) - \psi(0) = \nabla_0^{-(\beta+\zeta)}\mathcal{U}(\eta) + c_0\nabla_0^{-\beta}(1) + c_1\nabla_0^{-\beta}\eta - \nabla_0^{-\beta}\lambda(\eta)\psi(\eta).$$

Using condition $\psi(0) = 0$, we get

$$\psi(\eta) = \frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta+\zeta-1}} \mathcal{U}(s) + c_0 \frac{\eta^{\overline{\beta}}}{\Gamma(\beta+1)} + c_1 \frac{\eta^{\overline{\beta+1}}}{\Gamma(\beta+2)} - \frac{1}{\Gamma(\beta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s). \tag{8.4}$$

Using $\psi(d) = 0$, we get

$$c_0 \frac{d^{\overline{\beta}}}{\Gamma(\beta+1)} + c_1 \frac{d^{\overline{\beta+1}}}{\Gamma(\beta+2)} = -\frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} \mathcal{U}(s) + \frac{1}{\Gamma(\beta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s). \tag{8.5}$$

Using third boundary condition, we get

$$2c_0 + c_1 d = -\frac{1}{\Gamma(\zeta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} \mathcal{U}(s) + \mu \sum_{1}^{\nu} \psi(s). \tag{8.6}$$

Solving Eq. (8.5) and Eq.(8.6), we get

$$\begin{aligned} c_1 = & \frac{\beta+1}{\beta d - 2\beta - d} \left(\frac{-1}{\Gamma(\zeta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} \mathcal{U}(s) + \mu \sum_{1}^{\nu} \psi(s) \right) \\ & + \frac{2\Gamma(\beta+2)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \left(\frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} \mathcal{U}(s) \right. \\ & \left. - \frac{1}{\Gamma(\beta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s) \right), \end{aligned}$$

and

$$
\begin{aligned}
c_0 =& \frac{-d(\beta+1)}{2(\beta d-2\beta-d)}\left(\frac{-1}{\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}\mathcal{U}(s)+\mu\sum_{1}^{\nu}\psi(s)\right)\\
&-\frac{d\Gamma(\beta+2)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\left(\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}\mathcal{U}(s)\right.\\
&\left.-\frac{1}{\Gamma(\beta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s)\right)\\
&-\frac{1}{2\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}\mathcal{U}(s)+\frac{\mu}{2}\sum_{1}^{\nu}\psi(s).
\end{aligned}
$$

Using c_0 and c_1 in Eq. (8.4), we obtain

$$
\begin{aligned}
\psi(\eta) =& \frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}\mathcal{U}(s)-\frac{1}{\Gamma(\beta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s)\\
&+\frac{1}{\Gamma(\zeta)}\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}\mathcal{U}(s)\\
&-\mu\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{\nu}\psi(s)\\
&+\frac{1}{\Gamma(\beta+\zeta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}\mathcal{U}(s)\\
&-\frac{1}{\Gamma(\beta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s).
\end{aligned}
$$

Hence completing the proof. □

Theorem 8.1.2. *The solution of problem* (8.1) *is provided by*

$$
\begin{aligned}
\psi(\eta) =& \frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\psi(s))-\frac{1}{\Gamma(\beta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s)\\
&+\frac{1}{\Gamma(\zeta)}\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}\mathscr{Q}(s,\psi(s))\\
&-\mu\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{\nu}\psi(s)
\end{aligned}
$$

$$+\frac{1}{\Gamma(\beta+\zeta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\psi(s))$$
$$-\frac{1}{\Gamma(\beta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s), \quad \eta\in\mathbb{N}_0^d.$$

The accompanying two lemmas demonstrate support in the subsequent parts of the chapter.

Lemma 8.1.1. *Let for* $0<\beta<1$ *and* $d\in\mathbb{N}_1$. *We have,*

$$\max_{\eta\in\mathbb{N}_0^d}\left[\eta^{\overline{\beta}}(d-\eta)\right]=\alpha^{\overline{\beta}}(d-\alpha),$$

where

$$\alpha=\left\lfloor\frac{1+d\beta}{1+\beta}\right\rfloor.$$

Proof. Denote by

$$G(\eta)=\eta^{\overline{\beta}}(d-\eta), \quad \eta\in\mathbb{N}_0^d.$$

Observe that $G(\eta)\geq 0$ for all $\eta\in\mathbb{N}_0^d$. Consider

$$\begin{aligned}\nabla\left[\eta^{\overline{\beta}}(d-\eta)\right]&=(\eta-1)^{\overline{\beta}}(-1)+\beta\eta^{\overline{\beta-1}}(d-\eta)\\&=-\frac{\Gamma(\eta+\beta-1)}{\Gamma(\eta-1)}+\beta\frac{\Gamma(\eta+\beta-1)}{\Gamma(\eta)}(d-\eta)\\&=\frac{\Gamma(\eta+\beta-1)}{\Gamma(\eta)}\left[-(\eta-1)+\beta(d-\eta)\right].\end{aligned}$$

Clearly,

$$\frac{\Gamma(\eta+\beta-1)}{\Gamma(\eta)}\geq 0, \quad \eta\in\mathbb{N}_0^d.$$

For $0\leq\eta\leq\left\lfloor\frac{1+d\beta}{1+\beta}\right\rfloor$, function G is an increasing function and for $\left\lfloor\frac{1+d\beta}{1+\beta}\right\rfloor\leq\eta\leq d$ function G is decreasing function. Thus,

$$\max_{\eta\in\mathbb{N}_0^d}G(\eta)=G(\alpha)=\alpha^{\overline{\beta}}(d-\alpha).$$

□

Lemma 8.1.2. *Let for $0 < \beta < 1$ and $d \in \mathbb{N}_1$. Denote by*

$$H(\eta) = \eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta), \quad \eta \in \mathbb{N}_0^d.$$

Then,

$$\max_{\eta \in \mathbb{N}_0^d} |H(\eta)| = \max\{|H(\Theta)|, H(d)\},$$

where

$$\Theta = \begin{cases} \left\lfloor \frac{(\beta + \frac{2}{d-2})}{2} \right\rfloor, & d \in \mathbb{N}_3, \\ 1, & d = 2, \\ 0, & d = 1. \end{cases}$$

Proof. Here $H(0) = 0$ and $H(d) = d^{\overline{\beta}}(-d\beta + d + 2\beta) > 0$. Consider

$$\begin{aligned} \nabla H(\eta) &= (\eta - 1)^{\overline{\beta}}(2) + \beta\eta^{\overline{\beta-1}}(-d\beta - d + 2\eta + 2\beta) \\ &= 2\frac{\Gamma(\eta + \beta - 1)}{\Gamma(\eta - 1)} + \beta\frac{\Gamma(\eta + \beta - 1)}{\Gamma(\eta)}(-d\beta - d + 2\eta + 2\beta) \\ &= \frac{\Gamma(\eta + \beta - 1)}{\Gamma(\eta)}\left[2(\eta - 1) + \beta(-d\beta - d + 2\eta + 2\beta)\right]. \end{aligned}$$

Here,

$$\frac{\Gamma(\eta + \beta - 1)}{\Gamma(\eta)} \geq 0, \quad \eta \in \mathbb{N}_0^d.$$

Thus, $\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)$ is a decreasing function for $0 \leq \eta \leq \left\lfloor \frac{(\beta + \frac{2}{d-2})}{2} \right\rfloor$ and increasing function for $\left\lfloor \frac{(\beta + \frac{2}{d-2})}{2} \right\rfloor \leq \eta \leq d$. Then,

$$\max_{\eta \in \mathbb{N}_0^d} |H(\eta)| = \max\{|H(\Theta)|, H(d)\}.$$

□

Let $\mathcal{S} = \left\{\psi | \psi : \mathbb{N}_0^d \to \mathbb{R}\right\}$ be a Banach space with norm defined as

$$\|\psi\| = \max_{\eta \in \mathbb{N}_0^d} |\psi(\eta)|.$$

Let us consider the following assumptions,

(H_1) $\mathscr{Q}$ defined on $\mathbb{N}_0^d \times \mathbb{R}$ is a continuous function.

(H_2) $\mathscr{Q}$ satisfies Lipschitz condition with $\mathcal{L}$ being Lipschitz constant

$$|\mathscr{Q}(\eta, \psi) - \mathscr{Q}(\eta, \varsigma)| \leq \mathcal{L}|\psi - \varsigma|, \quad \forall\, \eta \in \mathbb{N}_0^d, \psi, \varsigma \in \mathbb{R}.$$

(H_3) There exists a nonnegative function $\varphi : \mathbb{N}_0^d \to \mathbb{R}^+$ such that

$$|\mathscr{Q}(\eta, \psi)| \leq \varphi(\eta), \quad \forall\, (\eta, \psi) \in (\mathbb{N}_0^d, \mathbb{R}).$$

(H_4) There exists two nonnegative functions w, w^* such that

$$|\mathscr{Q}(\eta, \psi)| \leq w(\eta)|\psi| + w^*(\eta), \quad (\eta, \psi) \in \mathbb{N}_0^d \times \mathbb{R}.$$

Let us denote by

$$\mathcal{N}_1 = \max_{\eta \in \mathbb{N}_0^d} \left[\frac{\eta^{\overline{\beta}}(d - \eta)}{\Gamma(\beta + 1)(\beta d - 2\beta - d)}\right],\ \mathcal{N}_2 = \max_{\eta \in \mathbb{N}_0^d} \left[\frac{\eta^{\overline{\beta}}(-\beta d - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)}\right],$$

$$\lambda^* = \max_{\eta \in \mathbb{N}_0^d} |\lambda(\eta)|.$$

Suppose

$$\mathcal{J} = \mathcal{J}_1 + \mathcal{L}\mathcal{J}_2, \tag{8.7}$$

$$\mathcal{M} = \frac{\mathcal{L}\mathcal{N}_2 d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)} + \frac{\lambda^* \mathcal{N}_2 d^{\overline{\beta}}}{\Gamma(\beta+1)} + \mathcal{N}_1\left(\nu|\mu| + \frac{\mathcal{L}d^{\overline{\zeta}}}{\Gamma(\zeta+1)}\right), \tag{8.8}$$

$$\mathcal{P} = \mathcal{J}_1 + \frac{1+\mathcal{N}_2}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}w(s) \tag{8.9}$$

$$+ \frac{\mathcal{N}_1}{\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}w(s),$$

$$\mathcal{R} = \frac{1+\mathcal{N}_2}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}w^*(s) + \frac{\mathcal{N}_1}{\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}w^*(s), \tag{8.10}$$

where

$$\mathcal{J}_1 = \frac{\lambda^*(1+\mathcal{N}_2)d^{\overline{\beta}}}{\Gamma(\beta+1)} + \nu|\mu|\mathcal{N}_1, \tag{8.11}$$

$$\mathcal{J}_2 = \frac{(1+\mathcal{N}_2)d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)} + \frac{\mathcal{N}_1 d^{\overline{\zeta}}}{\Gamma(\zeta+1)}. \tag{8.12}$$

Considering Theorem (8.1.2), solution of problem (8.1) can be written as follows

$$\psi = \mathscr{T}(\psi) = \mathscr{T}_1(\psi) + \mathscr{T}_2(\psi). \tag{8.13}$$

Furthermore, operators $\mathscr{T}_1$ and $\mathscr{T}_2$ are defined on Banach space $\mathcal{S}$ as follows:

$$\mathscr{T}_1\psi(\eta) = \frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\psi(s)) - \frac{1}{\Gamma(\beta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s), \tag{8.14}$$

and

$$\mathscr{T}_2\psi(\eta) = \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\zeta)\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}\mathscr{Q}(s,\psi(s)) \tag{8.15}$$

$$- \frac{\mu\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{\nu}\psi(s)$$

$$+ \frac{\eta^{\overline{\beta}}(-\beta d - d + 2\eta + 2\beta)}{\Gamma(\beta+\zeta) d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} \mathscr{Q}(s, \psi(s))$$

$$- \frac{\eta^{\overline{\beta}}(-\beta d - d + 2\eta + 2\beta)}{\Gamma(\beta) d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s).$$

Theorem 8.1.3. *Let us consider* (H_1) *and* (H_2) *are satisfied. For* $\mathcal{J} < 1$*, problem (8.1) has a unique solution, where* $\mathcal{J}$ *is provided by* (8.7).

Proof. Let us consider a set $\mathcal{J}_r = \{\psi \in \mathcal{S} : \|\psi\| \leq r\}$, where $r \geq \dfrac{\mathcal{N}\mathcal{J}_2}{1-\mathcal{J}}$. Let $\mathcal{N} = \max_{\eta \in \mathbb{N}_0^d} |\mathscr{Q}(\eta, 0)|$. Then, for $\psi \in \mathcal{J}_r$, we have

$$\begin{aligned}
\|\mathscr{Q}(\eta, \psi(\eta))\| &= \max_{\eta \in \mathbb{N}_0^d} |\mathscr{Q}(\eta, \psi(\eta)) - \mathscr{Q}(\eta, 0) + \mathscr{Q}(\eta, 0)| \\
&\leq \max_{\eta \in \mathbb{N}_0^d} |\mathscr{Q}(\eta, \psi(\eta)) - \mathscr{Q}(\eta, 0)| + \max_{\eta \in \mathbb{N}_0^d} |\mathscr{Q}(\eta, 0)| \\
&\leq \mathcal{L}\|\psi\| + \mathcal{N} \leq \mathcal{L} r + \mathcal{N}.
\end{aligned}$$

Using Lemmas 8.1.1 and 8.1.2 with the above inequality, we get

$$\begin{aligned}
\|(\mathscr{T}\psi)\eta\| = \max_{\eta \in \mathbb{N}_0^d} \Big| &\frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta+\zeta-1}} \mathscr{Q}(s, \psi(s)) \\
&- \frac{1}{\Gamma(\beta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s) \\
&+ \frac{1}{\Gamma(\zeta)} \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, \psi(s)) \\
&- \mu \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{\nu} \psi(s) \\
&+ \frac{1}{\Gamma(\beta+\zeta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} \mathscr{Q}(s, \psi(s)) \\
&- \frac{1}{\Gamma(\beta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s) \Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\mathcal{L}r+\mathcal{N}}{\Gamma(\beta+\zeta)}\max_{\eta\in\mathbb{N}_0^d}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}+\frac{\lambda^* r}{\Gamma(\beta)}\max_{\eta\in\mathbb{N}_0^d}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\\
&\quad+\frac{\mathcal{N}_1(\mathcal{L}r+\mathcal{N})}{\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}+\mu\mathcal{N}_1 r\nu\\
&\quad+\frac{\mathcal{N}_2(\mathcal{L}r+\mathcal{N})}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}+\frac{\lambda^*\mathcal{N}_2 r}{\Gamma(\beta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\\
&\le \frac{(\mathcal{L}r+\mathcal{N})(1+\mathcal{N}_2)d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)}+\frac{\lambda^* r(1+\mathcal{N}_2)d^{\overline{\beta}}}{\Gamma(\beta+1)}+\mathcal{N}_1\left(\nu|\mu|r+\frac{(\mathcal{L}r+\mathcal{N})d^{\overline{\zeta}}}{\Gamma(\zeta+1)}\right)\\
&\le r,
\end{aligned}
$$

implying that $\|\mathscr{T}\psi\|\le r$. Let $\psi,\varsigma\in\mathcal{S}$, and consider

$$
\begin{aligned}
&\|(\mathscr{T}\psi)(\eta)-(\mathscr{T}\varsigma)(\eta)\|\\
&\le \frac{1}{\Gamma(\beta+\zeta)}\max_{\eta\in\mathbb{N}_0^d}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}|\mathscr{Q}(s,\psi(s))-\mathscr{Q}(s,\varsigma(s))|\\
&\quad+\frac{\lambda^*}{\Gamma(\beta)}\max_{\eta\in\mathbb{N}_0^d}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}|\psi(s)-\varsigma(s)|\\
&\quad+\frac{1}{\Gamma(\zeta)}\max_{\eta\in\mathbb{N}_0^d}\frac{(\eta^{\overline{\beta}}(d-\eta))}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}|\mathscr{Q}(s,\psi(s))-\mathscr{Q}(s,\varsigma(s))|\\
&\quad+\mu\max_{\eta\in\mathbb{N}_0^d}\frac{(\eta^{\overline{\beta}}(d-\eta))}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{\nu}|\psi(s)-\varsigma(s)|\\
&\quad+\max_{\eta\in\mathbb{N}_0^d}\frac{(\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta))}{\Gamma(\beta+\zeta)d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}|\mathscr{Q}(s,\psi(s))-\mathscr{Q}(s,\varsigma(s))|\\
&\quad+\frac{\lambda^*}{\Gamma(\beta)}\max_{\eta\in\mathbb{N}_0^d}\frac{(\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta))}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}|\psi(s)-\varsigma(s)|\\
&\le \|\psi-\varsigma\|\left\{\frac{\mathcal{L}(1+\mathcal{N}_2)}{\Gamma(\beta+\zeta+1)}d^{\overline{\beta+\zeta}}+\frac{\lambda^*(1+\mathcal{N}_2)d^{\overline{\beta}}}{\Gamma(\beta+1)}+\mathcal{N}_1\left(\nu|\mu|+\frac{\mathcal{L}d^{\overline{\zeta}}}{\Gamma(\zeta+1)}\right)\right\}\\
&\le \mathcal{J}\,\|\psi-\varsigma\|\,.
\end{aligned}
$$

Since $\mathcal{J}<1$, so problem (8.1) has a unique solution using Banach contraction hypothesis. □

Theorem 8.1.4. *Let us consider assupmtions (H_1), (H_2) and (H_3) are satisfied. For $\mathcal{M}<1$,* *problem* (8.1) *has at least one solution, where* $\mathcal{M}$ *is provided by* (8.8).

Proof. We consider the operators $\mathscr{T}_1$ and $\mathscr{T}_2$ as defined in (8.14) and (8.15), respectively. Let $\max_{\eta\in\mathbb{N}_0^d}|\varphi(\eta)| \leq \|\varphi\|$. Let us consider closed ball $\mathcal{J}_r = \{\psi \in \mathcal{S} : \|\psi\| \leq r\}$ with $r \geq \mathcal{J}_2 \|\varphi\| (|1-\mathcal{J}_1|)^{-1}$. Let $\psi, \varsigma \in \mathcal{J}_r$, and consider

$$\begin{aligned}
\|\mathscr{T}_1\psi(\eta) + \mathscr{T}_2\varsigma(\eta)\| &\leq \frac{1}{\Gamma(\beta+\zeta)} \max_{\eta\in\mathbb{N}_0^d} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta+\zeta-1}} |\mathscr{Q}(s,\psi(s))| \\
&+ \frac{\lambda^*}{\Gamma(\beta)} \max_{\eta\in\mathbb{N}_0^d} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta-1}} |\psi(s)| \\
&+ \frac{\mathcal{N}_1}{\Gamma(\zeta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} |\mathscr{Q}(s,\varsigma(s))| + |\mu|\mathcal{N}_1 \sum_{1}^{\nu} |\varsigma(s)| \\
&+ \frac{\mathcal{N}_2}{\Gamma(\beta+\zeta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} |\mathscr{Q}(s,\varsigma(s))| \\
&+ \frac{\lambda^*\mathcal{N}_2}{\Gamma(\beta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} |\varsigma(s)| \\
&\leq \frac{\|\varphi\|}{\Gamma(\beta+\zeta+1)} d^{\overline{\beta+\zeta}} + \frac{\lambda^* r}{\Gamma(\beta+1)} d^{\overline{\beta}} + \frac{\mathcal{N}_1 \|\varphi\|}{\Gamma(\zeta+1)} d^{\overline{\zeta}} \\
&+ |\mu|\mathcal{N}_1 \nu r + \frac{\mathcal{N}_2 \|\varphi\|}{\Gamma(\beta+\zeta+1)} d^{\overline{\beta+\zeta}} + \frac{\lambda^*\mathcal{N}_2 r}{\Gamma(\beta+1)} d^{\overline{\beta}} \\
&\leq \left(\frac{(1+\mathcal{N}_2) d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)} + \frac{\mathcal{N}_1 d^{\overline{\zeta}}}{\Gamma(\zeta+1)} \right) \|\varphi\| + \left(\frac{\lambda^*(1+\mathcal{N}_2) d^{\overline{\beta}}}{\Gamma(\beta+1)} + \nu|\mu|\mathcal{N}_1 \right) r \\
&\leq \mathcal{J}_2 \|\varphi\| + r\mathcal{J}_1 \leq r.
\end{aligned}$$

This implies $\mathscr{T}_1\psi + \mathscr{T}_2\varsigma \in \mathcal{J}_r$. Now, we show that $\mathscr{T}_2$ is a contraction mapping using assumption (H_2) provided $\mathcal{M} < 1$.

$$\begin{aligned}
\|\mathscr{T}_2\psi(\eta) - \mathscr{T}_2\varsigma(\eta)\| &\leq \frac{\mathcal{N}_1}{\Gamma(\zeta)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} |\mathscr{Q}(s,\psi(s)) - \mathscr{Q}(s,\varsigma(s))| \\
&+ \mathcal{N}_1 |\mu| \sum_{1}^{\nu} |\psi(s) - \varsigma(s)|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathcal{N}_2}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}|\mathscr{Q}(s,\psi(s))-\mathscr{Q}(s,\varsigma(s))| \\
&+\frac{\lambda^*\mathcal{N}_2}{\Gamma(\beta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}|\psi(s)-\varsigma(s)| \\
\leq &\left(\frac{\mathcal{N}_1\mathcal{L}d^{\overline{\zeta}}}{\Gamma(\zeta+1)}+\mathcal{N}_1|\mu|\nu+\frac{\mathcal{N}_2\mathcal{L}d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)}+\frac{\lambda^*\mathcal{N}_2d^{\overline{\beta}}}{\Gamma(\beta+1)}\right)\|\psi-\varsigma\| \\
= &\mathcal{M}\|\psi-\varsigma\|.
\end{aligned}
$$

Hence for $\mathcal{M}<1$, $\mathscr{T}_2$ is contraction. $\mathscr{T}_1$ is continuous due to continuity of function $\mathscr{Q}$. Now for $\psi\in\mathcal{J}_r$, we have

$$
\begin{aligned}
\|\mathscr{T}_1\psi(\eta)\| &= \max_{\eta\in\mathbb{N}_0^d}\Big|\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\psi(s)) \\
&-\frac{1}{\Gamma(\beta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s)\Big| \\
&\leq \frac{\|\varphi\|\,d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)}+\frac{\lambda^*rd^{\overline{\beta}}}{\Gamma(\beta+1)}.
\end{aligned}
$$

Hence $\mathscr{T}_1$ is uniformly bounded. Now let $0\leq\eta_1<\eta_2\leq d$, then for $\psi\in\mathcal{J}_r$, we have

$$
\begin{aligned}
&\|\mathscr{T}_1\psi(\eta_2)-\mathscr{T}_1\psi(\eta_1)\| \\
&=||\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta_2}(\eta_2-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\psi(s))-\frac{1}{\Gamma(\beta)}\sum_{1}^{\eta_2}(\eta_2-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s) \\
&\quad-\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta_1}(\eta_1-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\psi(s))+\frac{1}{\Gamma(\beta)}\sum_{1}^{\eta_1}(\eta_1-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s)|| \\
&\leq||\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta_1}[(\eta_2-\varrho(s))^{\overline{\beta+\zeta-1}}-(\eta_1-\varrho(s))^{\overline{\beta+\zeta-1}}]\mathscr{Q}(s,\psi(s)) \\
&\quad+\frac{1}{\Gamma(\beta+\zeta)}\sum_{\eta_1+1}^{\eta_2}(\eta_2-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\psi(s)) \\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{1}^{\eta_1}[(\eta_1-\varrho(s))^{\overline{\beta-1}}-(\eta_2-\varrho(s))^{\overline{\beta-1}}]\lambda(s)\psi(s) \\
&\quad-\frac{1}{\Gamma(\beta)}\sum_{\eta_1+1}^{\eta_2}(\eta_2-\varrho(s))^{\overline{\beta-1}}\lambda(s)\psi(s)||
\end{aligned}
$$

$$\leq \frac{\|\varphi\|}{\Gamma(\beta+\zeta+1)}(\eta_2^{\overline{\beta+\zeta}} - \eta_1^{\overline{\beta+\zeta}}) + \frac{\lambda^* r}{\Gamma(\beta+1)}(\eta_1^{\overline{\beta}} - \eta_2^{\overline{\beta}} + 2(\eta_2-\eta_1)^{\overline{\beta}}),$$

above expression approaches to zero as we let $\eta_2 \to \eta_1$. Hence $\mathscr{T}_1$ is relatively compact and with the usage of Arzela-Ascoli Theorem, the operator $\mathscr{T}_1$ is completely continuous on $\mathcal{J}_r$. Therefore, problem (8.1) has at least one solution using the Krasnoselskii hypothesis. The proof is complete. □

Theorem 8.1.5. *Let us consider assumptions* (H_1) *and* (H_4) *are satisfied. For* $\mathcal{P} < 1$*, problem* (8.1) *has at least one solution, where* $\mathcal{P}$ *is given by* (8.9).

Proof. Let us Consider $\mathcal{G} = \{\psi \in \mathcal{S} : \|\psi\| < l\}$ which is an open subset of a Banach space S with $l = \mathcal{R}(1-\mathcal{P})^{-1}$, and $\mathcal{R}$ is given by (8.10). It can be shown easily that operator $\mathscr{T} : \overline{\mathcal{G}} \to \mathcal{S}$ given by (8.13) is completely continuous. Further assume that for $\delta > 1$, $\psi \in \partial\mathcal{G}$ such that $\delta\psi = \mathscr{T}(\psi)$. Then, we have

$$\begin{aligned}
\delta l = \delta\,\|\psi(\eta)\| &= \|\mathscr{T}\psi(\eta)\| = \max_{\eta\in\mathbb{N}_0^d} |\mathscr{T}\psi(\eta)| \\
&\leq \frac{1}{\Gamma(\beta+\zeta)} \max_{\eta\in\mathbb{N}_0^d} \sum_1^{\eta} (\eta-\varrho(s))^{\overline{\beta+\zeta-1}} |\mathscr{Q}(s,\psi(s))| \\
&\quad + \frac{\lambda^*}{\Gamma(\beta)} \max_{\eta\in\mathbb{N}_0^d} \sum_1^{\eta} (\eta-\varrho(s))^{\overline{\beta-1}} |\psi(s)| \\
&\quad + \frac{\mathcal{N}_1}{\Gamma(\zeta)} \sum_1^{d} (d-\varrho(s))^{\overline{\zeta-1}} |\mathscr{Q}(s,\psi(s))| + |\mu|\mathcal{N}_1 \sum_1^{\nu} |\psi(s)| \\
&\quad + \frac{\mathcal{N}_2}{\Gamma(\beta+\zeta)} \sum_1^{d} (d-\varrho(s))^{\overline{\beta+\zeta-1}} |\mathscr{Q}(s,\psi(s))| \\
&\quad + \frac{\lambda^*\mathcal{N}_2}{\Gamma(\beta)} \sum_1^{d} (d-\varrho(s))^{\overline{\beta-1}} |\psi(s)| \\
&\leq \frac{1}{\Gamma(\beta+\zeta)} \max_{\eta\in\mathbb{N}_0^d} \sum_1^{\eta} (\eta-\varrho(s))^{\overline{\beta+\zeta-1}} (w(s)l + w^*(s)) + \frac{\lambda^*}{\Gamma(\beta)} \max_{\eta\in\mathbb{N}_0^d} \sum_1^{\eta} (\eta-\varrho(s))^{\overline{\beta-1}} l \\
&\quad + \frac{\mathcal{N}_1}{\Gamma(\zeta)} \sum_1^{d} (d-\varrho(s))^{\overline{\zeta-1}} (w(s)l + w^*(s)) + |\mu|\mathcal{N}_1 \sum_1^{\nu} l
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathcal{N}_2}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}(w(s)l+w^*(s))+\frac{\lambda^*\mathcal{N}_2}{\Gamma(\beta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}l\\
\leq&\ \frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}(w(s)l+w^*(s))+\frac{\lambda^*}{\Gamma(\beta+1)}d^{\overline{\beta}}l\\
&+\frac{\mathcal{N}_1}{\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}(w(s)l+w^*(s))+|\mu|\mathcal{N}_1\nu l\\
&+\frac{\mathcal{N}_2}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}(w(s)l+w^*(s))+\frac{\lambda^*\mathcal{N}_2}{\Gamma(\beta+1)}d^{\overline{\beta}}l\\
\leq&\ \Bigg(\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}w(s)+\frac{\lambda^*}{\Gamma(\beta+1)}d^{\overline{\beta}}\\
&+\frac{\mathcal{N}_1}{\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}w(s)+|\mu|\mathcal{N}_1\nu+\frac{\mathcal{N}_2}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}w(s)\\
&+\frac{\lambda^*\mathcal{N}_2}{\Gamma(\beta+1)}d^{\overline{\beta}}\Bigg)l+\Bigg(\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}w^*(s)\\
&+\frac{\mathcal{N}_1}{\Gamma(\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}w^*(s)+\frac{\mathcal{N}_2}{\Gamma(\beta+\zeta)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}w^*(s)\Bigg)\\
\leq&\ \mathcal{P}l+\mathcal{R}.
\end{aligned}
$$

Here, assumption is $\delta > 1$, but we arrived at conclusion that $\delta \leq 1$, which is a contradiction. Hence, problem (8.1) has at least one solution using Leray-Schauder hypothesis. □

8.2 Ulam-Hyers Stability

In this section, Ulam–Hyers stability for the problem (8.1) is presented. For ϵ being positive real number, consider problem (8.1) with the following inequality

$$
|{}^c\nabla_0^{\zeta}({}^c\nabla_0^{\beta}+\lambda(\eta))\varsigma(\eta)-\mathscr{Q}(\eta,\varsigma(\eta))|\leq\epsilon,\quad \eta\in\mathbb{N}_0^d. \tag{8.16}
$$

Definition 8.2.1. *A function ς defined on $\mathbb{N}_0^d$ is a solution of the inequality* (8.16) *if and only if there exists a function g (which depends on ς) defined on $\mathbb{N}_0^d$ such that*

(i) $|g(\eta)|\leq\epsilon,\quad \eta\in\mathbb{N}_0^d,$

(ii) ${}^{c}\nabla_{0}^{\zeta}({}^{c}\nabla_{0}^{\beta} + \lambda(\eta))\varsigma(\eta) = \mathscr{Q}(\eta, \varsigma(\eta)) + g(\eta), \quad \eta \in \mathbb{N}_{0}^{d}.$

Lemma 8.2.2. *Let ς being solution of inequality* (8.16) *is also solution of following inequality*

$$\begin{aligned}\Bigg|\varsigma(\eta) - \Bigg(&\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\varsigma(s)) - \frac{1}{\Gamma(\beta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\lambda(s)\varsigma(s)\\
&+\frac{1}{\Gamma(\zeta)}\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}\mathscr{Q}(s,\varsigma(s))\\
&-\mu\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{\nu}\varsigma(s)\\
&+\frac{1}{\Gamma(\beta+\zeta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\varsigma(s))\\
&-\frac{1}{\Gamma(\beta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\lambda(s)\varsigma(s)\Bigg)\Bigg| \le C\epsilon.\end{aligned}$$

Proof. As we know

$${}^{c}\nabla_{0}^{\zeta}({}^{c}\nabla_{0}^{\beta} + \lambda(\eta))\varsigma(\eta) = \mathscr{Q}(\eta, \varsigma(\eta)) + g(\eta),$$

its solution is given by

$$\begin{aligned}\varsigma(\eta) =&\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}(\mathscr{Q}(s,\varsigma(s))+g(s)) - \frac{1}{\Gamma(\beta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\lambda(s)\varsigma(s)\\
&+\frac{1}{\Gamma(\zeta)}\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}(\mathscr{Q}(s,\varsigma(s))+g(s))\\
&-\mu\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{\nu}\varsigma(s)\\
&+\frac{1}{\Gamma(\beta+\zeta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}(\mathscr{Q}(s,\varsigma(s))+g(s))\\
&-\frac{1}{\Gamma(\beta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\lambda(s)\varsigma(s).\end{aligned}$$

Now, we have

$$
\begin{aligned}
&\Bigg|\varsigma(\eta) - \Bigg(\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\varsigma(s)) - \frac{1}{\Gamma(\beta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta-1}}\lambda(s)\varsigma(s) \\
&\quad + \frac{1}{\Gamma(\zeta)}\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}\mathscr{Q}(s,\varsigma(s)) \\
&\quad - \mu\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{\nu}\varsigma(s) \\
&\quad + \frac{1}{\Gamma(\beta+\zeta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}\mathscr{Q}(s,\varsigma(s)) \\
&\quad - \frac{1}{\Gamma(\beta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\lambda(s)\varsigma(s)\Bigg)\Bigg| \\
&= \Bigg|\frac{1}{\Gamma(\beta+\zeta)}\sum_{1}^{\eta}(\eta-\varrho(s))^{\overline{\beta+\zeta-1}}g(s) + \frac{1}{\Gamma(\zeta)}\frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\zeta-1}}g(s) \\
&\quad + \frac{1}{\Gamma(\beta+\zeta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}g(s)\Bigg| \\
&\leq \left(\frac{d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)} + \frac{\mathcal{N}_1 d^{\overline{\zeta}}}{\Gamma(\zeta+1)} + \frac{\mathcal{N}_2 d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)}\right)\epsilon \leq C\epsilon.
\end{aligned}
$$

□

Theorem 8.2.1. *Let us consider assumptions* (H_1) *and* (H_2) *are satisfied. For* $\mathcal{L}C + C_1 < 1$, *problem* (8.1) *is Ulam-Hyers stable.*

Proof. Let ς being solution of inequality (8.16) and ψ being solution of following problem

$$
\begin{cases}
{}^{c}\nabla_0^{\zeta}({}^{c}\nabla_0^{\beta} + \lambda(\eta))\psi(\eta) = \mathscr{Q}(\eta,\psi(\eta)), \quad \eta \in \mathbb{N}_0^d, \quad d \in \mathbb{N}_1, \\
\psi(0) = \varsigma(0), \psi(d) = \varsigma(d), \ {}^{c}\nabla_0^{\beta}\psi(0) + {}^{c}\nabla_0^{\beta}\psi(d) = \mu\sum_{s=1}^{\nu}\varsigma(s),
\end{cases}
\tag{8.17}
$$

then its solution is provided by

$$\psi(\eta) = \frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta+\zeta-1}} \mathscr{Q}(s, \psi(s)) - \frac{1}{\Gamma(\beta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s)$$
$$+ \frac{1}{\Gamma(\zeta)} \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, \psi(s))$$
$$- \mu \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{\nu} \varsigma(s)$$
$$+ \frac{1}{\Gamma(\beta+\zeta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} \mathscr{Q}(s, \psi(s))$$
$$- \frac{1}{\Gamma(\beta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} \lambda(s)\psi(s).$$

Now using Lemma 8.2.2, we obtain

$$|\varsigma(\eta) - \psi(\eta)|$$
$$= \Bigg| \varsigma(\eta) - \Bigg(\frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta+\zeta-1}} \mathscr{Q}(s, \varsigma(s)) - \frac{1}{\Gamma(\beta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta-1}} \lambda(s)\varsigma(s)$$
$$+ \frac{1}{\Gamma(\zeta)} \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} \mathscr{Q}(s, \varsigma(s))$$
$$- \mu \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{\nu} \varsigma(s)$$
$$+ \frac{1}{\Gamma(\beta+\zeta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta+\zeta-1}} \mathscr{Q}(s, \varsigma(s))$$
$$- \frac{1}{\Gamma(\beta)} \frac{\eta^{\overline{\beta}}(-d\beta - d + 2\eta + 2\beta)}{d^{\overline{\beta}}(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\beta-1}} \lambda(s)\varsigma(s) \Bigg)$$
$$+ \Bigg(\frac{1}{\Gamma(\beta+\zeta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta+\zeta-1}} (\mathscr{Q}(s, \varsigma(s)) - \mathscr{Q}(s, \psi(s)))$$
$$- \frac{1}{\Gamma(\beta)} \sum_{1}^{\eta} (\eta - \varrho(s))^{\overline{\beta-1}} \lambda(s)(\varsigma(s) - \psi(s))$$
$$+ \frac{1}{\Gamma(\zeta)} \frac{\eta^{\overline{\beta}}(d-\eta)}{\Gamma(\beta+1)(\beta d - 2\beta - d)} \sum_{1}^{d} (d - \varrho(s))^{\overline{\zeta-1}} (\mathscr{Q}(s, \varsigma(s)) - \mathscr{Q}(s, \psi(s)))$$

$$+\frac{1}{\Gamma(\beta+\zeta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta+\zeta-1}}(\mathscr{Q}(s,\varsigma(s))-\mathscr{Q}(s,\psi(s)))$$

$$-\frac{1}{\Gamma(\beta)}\frac{\eta^{\overline{\beta}}(-d\beta-d+2\eta+2\beta)}{d^{\overline{\beta}}(\beta d-2\beta-d)}\sum_{1}^{d}(d-\varrho(s))^{\overline{\beta-1}}\lambda(s)(\varsigma(s)-\psi(s))\Bigg)\Bigg|$$

$$\leq C\epsilon+\left\{\mathcal{L}\left(\frac{d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)}+\frac{\mathcal{N}_1 d^{\overline{\zeta}}}{\Gamma(\zeta+1)}+\frac{\mathcal{N}_2 d^{\overline{\beta+\zeta}}}{\Gamma(\beta+\zeta+1)}\right)+\left(\frac{\lambda^* d^{\overline{\beta}}}{\Gamma(\beta+1)}+\frac{\lambda^*\mathcal{N}_2 d^{\overline{\beta}}}{\Gamma(\beta+1)}\right)\right\}$$

$$\times\|\varsigma-\psi\|$$

$$\leq C\epsilon+(\mathcal{L}C+C_1)\|\varsigma-\psi\|.$$

This implies $\big(1-(\mathcal{L}C+C_1)\big)\|\varsigma-\psi\|\leq C\epsilon$. Hence we acquire

$$\|\varsigma-\psi\|\leq\frac{C\epsilon}{1-(\mathcal{L}C+C_1)}.$$

Hence problem (8.1) is Ulam–Hyers stable. □

Example 8.2.3. *Let us consider the fractional difference Langevin equation with boundary conditions as follows:*

$$\begin{cases} {}^c\nabla_0^{\frac{3}{2}}\left({}^c\nabla_0^{\frac{2}{5}}+\frac{\eta^2}{8}\right)\psi(\eta)=\frac{1}{2}\Big(1+\eta\sin(\eta\psi)\Big), \quad \eta\in\mathbb{N}_0^1, \\ \psi(0)=0=\psi(1), \quad {}^c\nabla_0^{\frac{2}{5}}\psi(0)+{}^c\nabla_0^{\frac{2}{5}}\psi(1)=2\sum_{s=1}^{\frac{1}{10}}\psi(s). \end{cases} \tag{8.18}$$

Clearly assumptions H_1 and H_2 hold, that is, $\mathscr{Q}(\eta,\psi(\eta))=\frac{1}{2}(1+\eta\sin(\eta\psi))$ is continuous function on $\mathbb{N}_0^1$ and satisfies Lipschitz condition as follows:

$$|\mathscr{Q}(\eta,\psi(\eta))-\mathscr{Q}(\eta,\varsigma(\eta))|=\frac{1}{2}|\eta\sin(\eta\psi)-\eta\sin(\eta\varsigma)|\leq\frac{1}{2}\eta^2\int_{\varsigma}^{\psi}|\cos(\eta s)|\leq\frac{1}{2}|\psi-\varsigma|.$$

Here we have $\lambda(\eta)=\frac{\eta^2}{8}$, $\beta=\frac{2}{5}$, $\zeta=\frac{3}{2}$, $d=1$, so $\mathcal{N}_1=0$, $\mathcal{N}_2=0$, $\lambda^*=\frac{1}{8}$. Hence we get $\mathcal{J}<1$, where $\mathcal{J}$ is provided by Eq. (8.7). Hence there exists a unique solutions of problem (8.18) *using Theorem 8.1.3.*

Furthermore, consider $\varphi(\eta) = 1 + \eta$ *such that assupmtions* H_1, H_2 *and* H_3 *are satisfied with* $\mathcal{M} < 1$, *where* $\mathcal{M}$ *is provided by Eq.* (8.8)*. Then there exists at least one solution of problem* (8.18) *using Theorem 8.1.4.*

Furthermore, $\mathcal{L}C + C_1 < 1$ *such that assumptions* H_1 *and* H_2 *are satisfied. Hence, problem* (8.18) *is Ulam-Hyers stable using Theorem* 8.2.1.

Summary

In the primary part, we give a short prologue to Fractional Calculus (FC) and Discrete Fractional Calculus (DFC). We state essential definitions and essential properties of Basic Difference Calculus, Discrete Delta Fractional Calculus, Discrete Nabla Fractional Calculus, q-Calculus, Discrete fractional operators with discrete Mittag-Leffler kernel. Furthermore, we give valuable data about Gamma function and Discrete Mittag-Leffler function. We likewise give limits to some significant Discrete Mittag-Leffler capacities that we utilized in our next parts. In the end of Chapter 1, we provide essential lemmas from fixed point hypothesis.

In Chapter 2, we investigated the existence and stability results for a class of non-linear Caputo nabla fractional difference equations. To acquire the existence and stability results, we utilize Schauder's fixed point theorem, the Banach contraction principle and Krasnoselskii's fixed point theorem. The investigation of the hypothetical outcomes relies upon the construction of nabla discrete Mittag-Leffler capacities.

In Chapter 3, we have presented a new Gronwall inequality with impulsive effect. Moreover, this study investigates the Ulam's type stability of impulsive fractional difference equation. Stability criterion is obtained with the help of newly developed Gronwall inequality.

In Chapter 4, we examined the existence and uniqueness of solution of a delay Caputo q-fractional difference system. In light of the q-fractional Gronwall inequality, we examined the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability.

In Chapter 5, we developed a new discrete version of Gronwall-Bellman type inequality. Using newly developed Gronwall-Bellman inequality, we discussed Ulam-Hyers stability of Caputo nabla fractional difference system.

Chapter 6, manages two main points, existence of solution along with the stability analysis of p-Laplacian fractional difference equations in Nabla sense. As a matter of first importance, the existence of a solution is considered utilizing Schaefer's fixed point hypothesis. Besides, Ulam-Hyers stability is examined.

In Chapter 7, we discussed the stability analysis of a class of Caputo nabla fractional-order delay difference equation. We used Chebyshev norm to obtain Ulam-Hyers-Mittag-Leffler stability. Furthermore, Ulam-Hyers-Rassias-Mittag-Leffler stability of the problem is also discussed.

In Chapter 8, we studied a fractional difference Langevin equation within nabla sense. The existence-uniqueness results are proved using fixed point theorems. Furthermore, Ulam-Hyers stability of solution is also established.

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