Depth, Stanley Depth and Dimension of Edge Ideals Corresponding to Some Classes of Graphs

By

Malik Muhammad Suleman Shahid

Supervised by

Dr. Muhammad Ishaq

Department of Mathematics
School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan
2021

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: MALIK MUHAMMAD SULEMAN SHAHID, Regn No. 00000321192 Titled: **"Depth, Stanley Depth and Dimension of Edge Ideals Corresponding to Some Classes of Graphs"** accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. MUJEEB UR REHMAN Signature:

2. Name: DR. MUHAMMAD QASIM Signature:

External Examiner: DR. WAQAS MAHMOOD Signature:

Supervisor's Name: DR. MUHAMMAD ISHAQ Signature:

Head of Department

07/09/2021
Date

COUNTERSINGED

Date: 08.09.2021

Dean/Principal

I dedicate this thesis to my beloved parents, honourable supervisor, reputable teachers and trustworthy fellows for their unlimited assistance and motivation.

Acknowledgement

In the Name of Allah, the Most Merciful, the Most Compassionate All applause be to Allah, the Lord of the planets; and blessings and serenity be upon Mohammad His assistant and messenger. Primary and foremost, I must affirm my limitless thankfulness to Allah, the Ever-Magnificent; the Ever-Thankful, for His assistance and blessing. I am certain that this work would have never become reality, without His direction. I owe a deep debt of acknowledgment to our university for authorizing us to complete this work. I am pleased to some people, who struggled hard with me from the beginning till the completion of the present research especially my supervisor Dr. Muhammad Ishaq, who has been always generous during all phases of the research. Moreover, my entire understanding and appreciation of this field are mainly because of his accomplishments and positive reaction to my queries. And I highly admire the efforts expended by Dr.Muhammad Qasim and Dr. Mujeeb ur Rehman. I would also like to expand my thanks to the Principal, the Head of the Department, and the whole faculty of the department of mathematics for their support during my research work. My stay at NUST was made thoroughly indelible and delightful by my class fellows and friends. I would like to offer sincere gratitude to Adnan Iqbal and Naeem ud din for their indispensable assistance in the completion of this thesis. How can I forget to quote my classmates Talha Aziz, Saddam Sultan Akbar, Muhammad Hamza, and Safi ur Rehman who were always there when I needed them. I would always cherish their guidance and honest friendship. Of course, everyone in my class has helped to provide me encouragement along with a friendly environment

I would also like to convey my passionate thankfulness to my family. Particularly, my parents; Mr. Shahid Zameer and Salma Kanwal, for the substantial assistance they gave me throughout my whole life and mainly through the process of seeking a master's degree. Because of their unbroken devotion and blessings, I have the opportunity to complete this thesis.

Malik Muhammad Suleman Shahid

Abstract

This dissertation deals with the algebraic invariants such as depth and dimension as well as geometric invariant Stanley depth of some particular classes of graphs. Earlier, we have some general bounds for these invariants. The present thesis is primarily concerned with the value of depth and Stanley depth of edge ideals and their quotient rings (cyclic modules) related to some classes of graphs. In some cases we have an exact value, otherwise, we give very sharp bounds. In the end, we obtain a very strong lower bound for the dimension of the quotient rings of the edge ideals associated with these graphs. our results are general in nature, i.e., they hold for any non-negative integer. Also, we examine conjecture of Herzog and question of Rauf.

Contents

List of Figures

Introduction

In this thesis some exact values and improved lower and upper bounds of depth and Stanley depth are computed for the edge ideals and their quotient rings (cyclic modules) associated with some particular classes of graphs. Also, we have adequate results for the dimension of quotient rings (cyclic modules) of the edge ideals of some graphs. Moreover, in many cases we prove a conjecture of Herzog presented in [25] and question of Rauf given in [18]. This thesis comprises of five chapters.

Chapter 1 offers the overview, definitions and results related to abstract algebra and commutative algebra. This chapter covers the basics of ring and module theory. This chapter also illustrates the quick introduction of fundamental graph theory and the prominent operations of graphs.

Chapter 2 reviews the basic theory of depth and Stanley depth and Stanley decomposition of ideals and modules. Furthermore, previously known results are discussed in detail.

In Chapter 3, quotient rings (cyclic modules) of edge ideals associated with some graphs are introduced and their depth and Stanley depth are computed by using induction and some previously known results and Depth Lemma on short exact sequences.

In Chapter 4, depth and Stanley depth of edge ideals related to some graphs are calculated by using induction and some known results.

In Chapter 5, the results are briefly justified by making comparison with previously known results. It can be seen that there are positive answers in many cases for both conjecture of Herzog and question of Rauf.

Chapter 1

Ring theory, module theory and graph theory

1.1 Introduction

The basic idea of a ring established from an early attempt to prove Fermat's last theorem, which could be traced back to Richard Dedekind [1]. Adolf Fraenkel [2] introduced the first axiomatic definition of ring, however, his axioms were more rigorous than those in the most recent definition. After certain attempts from different fields, primarily number theory, the generalized and modern notion of ring (commutative) was established by Emmy Noether and Wolfgang Krull [3].

Nowadays, the idea of associating a graph with a specific algebraic structure and exploring the interactions between the structure of the algebraic objects and the graph theoretic properties of the graphs connected with them is an absorbing and active area of research. The idea of associating a graph to a commutative ring was initiated by I. Beck in [4].

1.2 Ring theory

In the principality of algebra, the algebraic structures are dealt under the banner of ring theory, which have defined operations of multiplication and addition.

Definition 1.2.1. *A ring $\mathcal{P}$ is a set with two binary operations, addition (denoted by $e+f$) and multiplication (denoted by ef), such that the following axioms hold in $\mathcal{P}$:*

1. *For all $e, f \in \mathcal{P}$, $e+f=f+e$ (commutativity w.r.t addition).*
2. *For all $e, f, g \in \mathcal{P}$, $e+(f+g)=(e+f)+g$ (associativity w.r.t addition).*
3. *There is an additive identity 0, that is, there is an element 0 in R such that $e+0=e$, for all $c \in \mathcal{P}$*
4. *For all $e \in \mathcal{P}$, there is an element $-e \in \mathcal{P}$ such that $e+(-e)=0$*
5. *Multiplication is associative, that is, $e \cdot (f \cdot g)=(e \cdot f) \cdot g$, for all $e, f, g \in \mathcal{P}$.*
6. *For all $e, f, g \in \mathcal{P}$, the left distributive law, $e \cdot (f+g)=(e \cdot f)+(e \cdot g)$ and the right distributive law, $(e+f) \cdot g=(e \cdot g)+(f \cdot g)$ hold.*

Definition 1.2.2. *A ring $\mathcal{P}$ is called commutative if multiplication is commutative in $\mathcal{P}$,that is, $pq=qa$, for all $p, q \in \mathcal{P}$.*

Definition 1.2.3. *If there is an element $e \in \mathcal{P}$ such that $pe=p=ep$, for all $p \in \mathcal{P}$, we say $\mathcal{P}$ is a ring with multiplicative identity (or a ring with unity). Multiplicative identity or unity of $\mathcal{P}$ is denoted by symbol 1.*

Example 1.2.4. We have some examples of ring.

1. $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$ and $\mathbb{Z}$ form rings w.r.t usual addition and multiplication. All these rings are commutative with unity.
2. The set of even integers i.e $2\mathbb{Z}=\{0, \pm 2, \pm 4, \pm 6, \ldots\}$ forms a commutative ring without unity.

1.2.1 Polynomial ring

The polynomial ring is a type of ring which is formed by a set of polynomials. These polynomials are in one or more than one variables, where the coefficients belong to a ring. Polynomial rings are used in different fields of mathematics and the examination of their properties is among the primary insight for the evolution of Commutative Algebra and Ring Theory.

Definition 1.2.5. Let $\mathcal{P}$ be a commutative ring containing unity, a polynomial in variable t has the form

$$p_0 + p_1 t + \cdots + p_{n-1}t^{n-1} + p_n t^n,$$

with $n \in \mathbb{Z}^+ \cup \{0\}$ and every $p_i \in \mathcal{P}$. The polynomial is of degree n if $p_n \neq 0$. The set of such polynomials is denoted by

$$\mathcal{P}[t] = \{p_0 + p_1 t + \cdots + p_{n-1}t^{n-1} + p_n t^n : n \in \mathbb{Z}^+ \cup \{0\}, p_i \in \mathcal{P}\}.$$

$\mathcal{P}[t]$ is a commutative ring with unity under polynomial addition and polynomial multiplication and the unity of $\mathcal{P}[t]$ is the unity of coefficient $\mathcal{P}$.

Definition 1.2.6. The polynomial ring in the variables $u_1, u_2, \ldots, u_n$ and coefficients belonging to $\mathcal{P}$ (commutative with identity) is defined inductively

$$\mathcal{P}[u_1, u_2, \ldots, u_n] = \mathcal{P}[u_1, u_2, \ldots, u_{n-1}][u_n].$$

A ring homomorphism is a map from one ring to another which preserves the same additive and multiplicative structures.

Definition 1.2.7. Consider two rings P_1 and P_2. A ring homomorphism is a map $\mathcal{O} : P_1 \longrightarrow P_2$, provided that for all $g_1, g_2 \in P_1$, the following are satisfied:

- $\mathcal{O}(g_1 + g_2) = \mathcal{O}(g_1) + \mathcal{O}(g_2)$;
- $\mathcal{O}(g_1 g_2) = \mathcal{O}(g_1)\mathcal{O}(g_2)$.

A ring homomorphism which is both injective and surjective is known as ring isomorphism.

1.2.2 Ideals

Proposition 1.2.8. *A non-empty subset J of a ring $\mathcal{P}$ is known to be an ideal if and only if $i_1 - i_2 \in J$, $ip \in J$ and $pi \in J$ for all $i_1, i_2, i \in I$ and $p \in \mathcal{P}$.*

Definition 1.2.9. For a proper ideal J, a quotient ring $\mathcal{P}/J$ can be formed, which consists of cosets $p + J$, where $p \in \mathcal{P}$, and the product of cosets is defined as:

$$(p_1 + J)(p_2 + J) = p_1p_2 + J.$$

There are isomorphism theorems for rings.

Theorem 1.2.10. *(Isomorphism Theorems)*

1. *For a ring homomorphism $\pi : P_1 \to P_2$ of two rings P_1 and P_2, $\pi(P_1)$ is isomorphic to $P_1/\ker(\pi)$, i.e.,*
$$P_1/\ker(\pi) \cong \pi(P_1).$$
2. *Consider the ideals I_1 and I_2 of ring P_1, with $I_1 \subseteq I_2$, then I_2/I_1 is an ideal of P_1/I_1. Also*
$$(P_1/I_1)/(I_2/I_1) \cong P_1/I_2.$$

Definition 1.2.11. Assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ are the ideals of ring $\mathcal{P}$. Product of two ideals, say $\mathcal{I}_1$ and $\mathcal{I}_2$, is a set consisting of all possible finite sums of the form i_1i_2, where $i_1 \in \mathcal{I}_1$ and $i_2 \in \mathcal{I}_2$. It is represented by $\mathcal{I}_1\mathcal{I}_2$.

Example 1.2.12. Let $I_1 = 8\mathbb{Z}$ and $I_2 = 12\mathbb{Z}$ in $\mathbb{Z}$. Then $I_1 + I_2$ comprises all integers of the form $8s_1 + 12s_2$ with $s_1, s_2 \in \mathbb{Z}$. For each such type of integer is divisible by 4, so $8\mathbb{Z} + 12\mathbb{Z} \subseteq 4\mathbb{Z}$. On the other hand, $4 = 8(2) + 12(-1)$ shows that $4\mathbb{Z}$ is contained in $8\mathbb{Z} + 12\mathbb{Z}$, hence $8\mathbb{Z} + 12\mathbb{Z} = 4\mathbb{Z}$. In general, $p_1\mathbb{Z} + p_2\mathbb{Z} = d\mathbb{Z}$, whereas $d = (p_1, p_2)$ is greatest common divisor and p_1, p_2 are any integers. The product I_1I_2 comprises all possible finite sums of the components of the form $(8s_1)(12s_2)$, where $s_1, s_2 \in \mathbb{Z}$, which clearly gives the ideal $96\mathbb{Z}$.

For ideals I and J of the ring $\mathcal{P}$, the set of sums $i + j$ with $i \in I$, $j \in J$ is not only a subring of $\mathcal{P}$ but also is an ideal in $\mathcal{P}$.

Definition 1.2.13. A maximal ideal $\mathcal{M}$ in a ring $\mathcal{P}$ is a ideal such that there is no proper ideal in between $\mathcal{M}$ and $\mathcal{P}$.
In other words, if $\mathcal{J}$ is an ideal containing $\mathcal{M}$, then either $\mathcal{M} = \mathcal{J}$ or $\mathcal{J} = \mathcal{R}$.

Definition 1.2.14. Local ring is a ring $\mathcal{P}$ with unique maximal ideal.

Definition 1.2.15. For a ring $\mathcal{P}$, principal ideal be an ideal with a single element in its generating set. Finitely generated ideal is an ideal with a finite elements in its generating set.

Example 1.2.16. Ideal generated by $(2) = \{0, 2, 4, 6\}$ is the maximal ideal in $\mathbb{Z}_8$. (2) is also the unique maximal ideal in $\mathbb{Z}_8$. So $\mathbb{Z}_8$ is a local ring.

Definition 1.2.17. A prime ideal $\mathcal{J}$ is a proper ideal of a ring $\mathcal{P}$ such that if for $p_1, p_2 \in \mathcal{P}$, $p_1 p_2 \in \mathcal{J}$, then either $p_1 \in \mathcal{J}$ or $p_2 \in \mathcal{J}$.

Definition 1.2.18. Let $\mathcal{P}$ be a ring and $\mathcal{I}$ is an ideal of $\mathcal{P}$. Then $(0 : \mathcal{I})$ is an ideal known as the annihilator of $\mathcal{I}$ represented as $Ann(\mathcal{I})$ defined as

$$Ann(\mathcal{I}) = \{p \in \mathcal{P} : p\mathcal{I} = 0\}.$$

Definition 1.2.19. For a ring $\mathcal{P}$, let us suppose two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$. Then their ideal quotient is defined as

$$(\mathcal{I}_1 : \mathcal{I}_2) = \{p \in \mathcal{P} \ : \ p\mathcal{I}_2 \subseteq \mathcal{I}_1\}.$$

Definition 1.2.20. An ideal $\mathcal{K}$ of $\mathcal{P}$ is primary ideal if $p_1 p_2 \in \mathcal{K}$, for $p_1, p_2 \in \mathcal{P}$, then either $p_1 \in \mathcal{K}$ or $p_2^n \in \mathcal{K}$ for some $n \geq 1$.

When $\mathcal{K}$ is a primary ideal, $\mathcal{J}$ is a prime ideal and also $\mathcal{J} = \sqrt{\mathcal{K}}$, then $\mathcal{K}$ is called $\mathcal{J}$-primary.

1.2.3 Monomial ideal

Let $S = T[\xi_1, \ldots, \xi_n]$ be a ring over field T, monomials forms the natural T-basis for S. Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ where every $c_j \geq 0$. A monomial is any product of the form $\xi_1^{c_1} \dots \xi_n^{c_n}$ with $c_j \in \mathbb{Z}_+$. If $v = \xi_1^{c_1} \dots \xi_n^{c_n}$ is a monomial, then we write $v = \xi^{\mathbf{c}}$ with $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{Z}_+^n$, and

$$\xi^{\mathbf{c_1}} \xi^{\mathbf{c_2}} = \xi^{\mathbf{c_1} + \mathbf{c_2}}.$$

An ideal whose generating set only consists of monomials is said to be a monomial ideal. Mon(S) denotes the set of all monomials in S and it forms the basis of S. For any polynomial $g \in S$ and for $c_w \in T$

$$g = \sum_{w \in Mon(S)} c_w w,$$

where support of g is defined as

$$\operatorname{supp}(g) = \{w \in Mon(S) \, : \, c_w \neq 0\}.$$

Proposition 1.2.21. *Consider two monomial ideals K_1 and K_2. Then*

1. *$K_1 \cap K_2$ is a monomial ideal, and $\{\operatorname{lcm}(a, b) \, : \, a \in G(K_1) \, , \, b \in G(K_2)\}$ is the generating set of $K_1 \cap K_2$.*

2. *$(K_1 : K_2)$ is a monomial ideal and $(K_1 : K_2) = \bigcap_{b \in G(K_2)} (K_1 : (b))$.*

A monomial $y^{\mathbf{c}}$ is said to be squarefree if $\mathbf{c}$ has components 0 and 1. An ideal with a generating set containing only squarefree monomials is known as squarefree monomial ideal.

1.3 Module theory

Definition 1.3.1. Consider a commutative ring $\mathcal{P}$, a $\mathcal{P}$-module $\mathcal{N}$ is a commutative group w.r.t addition, together with a scalar multiplication map $\cdot : \mathcal{P} \times \mathcal{N} \to \mathcal{N}$, defined as $\cdot ((\beta, \varrho)) = \beta \varrho$, which holds the succeeding axioms.

1. $\beta(\varrho_1 + \varrho_2) = \beta \varrho_1 + \beta \varrho_2$,

2. $(\beta_1 + \beta_2)\varrho = \beta_1 \varrho + \beta_2 \varrho$,

3. $(\beta_1\beta_2)\varrho = \beta_1(\beta_2\varrho)$,

4. $1\varrho = \varrho, \qquad \forall \beta_1, \beta_2 \in \mathcal{P}$ *and* $\varrho_1, \varrho_2 \in \mathcal{N}$.

Examples 1.3.2. 1. For a commutative group $\mathcal{U}$, let $j \in \mathcal{U}$, $q \in \mathbb{Z}$ and define $\cdot : \mathbb{Z} \times \mathcal{U} \to \mathcal{U}$, such that

$$\cdot(q, j) = qj = \begin{cases} (-j) + \cdots + (-j)\,, & \text{if} \quad \mathrm{q} < 0; \\ j + j + \cdots + j\,, & \text{if} \quad \mathrm{q} > 0; \\ 0\,, & \text{if} \quad \mathrm{q} = 0. \end{cases}$$

Then $\mathcal{U}$ is a $\mathbb{Z}$-module.

2. The ideals of the ring $\mathcal{P}$ are also $\mathcal{P}$-modules.

Definition 1.3.3. For a ring $\mathcal{P}$, let us assume C and D be $\mathcal{P}$-modules. A function $g : C \to D$ is known as $\mathcal{P}$-module homomorphism if

- $g(\varrho_1 + \varrho_2) = g(\varrho_1) + g(\varrho_2)$, for all $\varrho_1, \varrho_2 \in C$,
- $g(p\varrho) = pg(\varrho)$, for all $p \in \mathcal{P}\,,\ \varrho \in C$.

If g is injective and onto then it becomes a $\mathcal{P}$-module isomorphism.

Examples 1.3.4. 1. For a ring $\mathcal{P}$, consider $\mathcal{P}$-module $\mathcal{P}$. Then $\mathcal{P}$-module homomorphism (even from $\mathcal{P}$ into itself) needs not to be a ring homomorphism. Consider $\mathcal{P} = \mathbb{Z}$, then $\mathbb{Z}$-module homomorphism $u \mapsto 2u$ is not a ring homomorphism.

2. When $\mathcal{P} = F[u]$, the ring homomorphism $\phi : j(u) \mapsto j(u^2)$ is not an $F[u]$-module homomorphism.

Definition 1.3.5. Consider a ring $\mathcal{P}$, and a submodule W of $\mathcal{P}$-module $\mathcal{N}$. Then (additive abelian) quotient group $\mathcal{N}/W$ becomes a $\mathcal{P}$-module by using scalar multiplication defined as

$$o(j + W) = oj + W$$

$\forall\ o \in \mathcal{P}\,,\ j + W \in \mathcal{N}/W$.

1.3.1 Generation of modules

For any subset V of $\mathcal{P}$-module $\mathcal{N}$, let

$$\mathcal{P}V = \{p_1v_1 + \cdots + p_nv_n \,:\, p_1, \ldots, p_n \in \mathcal{P}\,,\, v_1, \ldots, v_n \in V \text{ and } n \in \mathbb{Z}^+\}.$$

If V is a finite set $\{v_1, \ldots, v_n\}$, then $\mathcal{P}V = \mathcal{P}v_1 + \mathcal{P}v_2 + \cdots + \mathcal{P}v_n$. Call $\mathcal{P}V$ the submodule of $\mathcal{N}$ generated by V.

Definition 1.3.6. Let $\mathcal{N}$ be an $\mathcal{P}$-module then it is called free on the subset Y of $\mathcal{N}$ if for $0 \neq n \in \mathcal{N}$, there are unique non-zero elements $p_1, \ldots, p_k$ of $\mathcal{P}$ and unique $\xi_1, \ldots, \xi_k$ in Y, such that

$$n = p_1\xi_1 + \cdots + p_k\xi_k.$$

1.3.2 Noetherian ring and Noetherian module

Proposition 1.3.7. *Let γ be a poset with respect to $\leq$. Then the following are equivalent.*

1. *Any increasing sequence $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_r \leq \ldots$ in γ is stationary, that is there exist $r \in \mathbb{N}$ for which $\alpha_s = \alpha_r$, for all $s \geq r$.*
2. *Any $\emptyset \neq W \subset \gamma$ possesses a maximal element.*

Let γ be the set of submodules of $\mathcal{N}$ which is ordered w.r.t the relation $\subseteq$ then statement *1* is known as *ascending chain condition* and statement *2* is known as the *maximal condition.*

Definition 1.3.8. Let $\mathcal{P}$ be a commutative ring, a $\mathcal{P}$-module $\mathcal{N}$ is known to be Noetherian if each ascending chain of $\mathcal{P}$-submodules of $\mathcal{N}$ is stationary. A ring $\mathcal{P}$ is Noetherian if $\mathcal{P}$ is Noetherian as a $\mathcal{P}$-module.

Examples 1.3.9. 1. A finite abelian group (as $\mathbb{Z}$-module) satisfies both ascending chain condition and descending chain condition. That is, every finite abelian group is noetherian.

2. The polynomial ring $K[\delta_1, \delta_2, \dots]$ over the field K satisfies neither chain condition on ideals, the sequence $(\delta_1) \subset (\delta_1, \delta_2) \subset \dots$ is strictly increasing and the sequence $(\delta_1) \supset (\delta_1^2) \supset (\delta_1^3) \dots$ is strictly decreasing.

Definition 1.3.10. Let $\mathcal{N}$ be finitely generated $\mathcal{P}$-module where $\mathcal{P}$ is a Noetherian ring, an associated prime ideal of a module is a prime ideal Q of the ring $\mathcal{P}$ such that $Q = \text{Ann}(n)$, where $\text{Ann}(n) = \{\varepsilon \in \mathcal{P} : \varepsilon n = 0\}$. *The set of associated prime ideals of* $\mathcal{N}$ *is represented by* $\text{Ass}(\mathcal{N})$.

Definition 1.3.11. Let $\mathcal{P}$ be a commutative ring, consider a chain of prime ideals in the ring

$$Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \dots \subsetneq Q_{a_i},$$

then dimension of ring $\mathcal{P}$ is defined as

$$\dim \mathcal{P} = \sup\{a_i\}.$$

Suppose $\mathcal{N}$ be a $\mathcal{P}$-module, then Krull dimension of $\mathcal{N}$ is

$$\dim(\mathcal{N}) = \dim(\mathcal{P}/\text{Ann}(\mathcal{N})).$$

For the modules of the type $\mathcal{P}/\mathcal{I}$

$$\dim(\mathcal{P}/\mathcal{I}) = \max\{\dim(\mathcal{P}/J_i) : J_i \in \text{Ass}(\mathcal{P}/\mathcal{I})\}.$$

1.3.3 Exact sequences

Definition 1.3.12. Let $\mathcal{P}$ be a commutative ring, consider a sequence of $\mathcal{P}$-homomorphisms on $\mathcal{P}$-modules.

$$\dots \longrightarrow \mathcal{V}_{i-1} \xrightarrow{h_i} \mathcal{V}_i \xrightarrow{h_{i+1}} \mathcal{V}_{i+1} \xrightarrow{h_{i+2}} \dots$$

It is exact at $\mathcal{V}_i$ if $\text{Im}(h_i) = \ker(h_{i+1})$. The sequence is known to be exact if it is observed to be exact at every $\mathcal{V}_i$. Particularly, $0 \longrightarrow U' \xrightarrow{g} \mathcal{V}$ is exact at U' if and only if g is one to one, and $\mathcal{V} \xrightarrow{h} U'' \longrightarrow 0$ is exact at U'' if and only if h is onto.

Proposition 1.3.13. *The sequence*

$$0 \longrightarrow U' \xrightarrow{h} \mathcal{V} \xrightarrow{g} U'' \longrightarrow 0$$

is an exact sequence if and only if h is one to one, g is onto and $Im(h) = ker(g)$.

Remark 1.3.14. *The sequence in Proposition 1.3.13 is called a short exact sequence.*

1.3.4 Graded rings

Consider a commutative semigroup (w.r.t addition) $\mathcal{W}$. A $\mathcal{W}$-graded ring is such type of a ring $\mathcal{P}$ having a decomposition

$$\mathcal{P} = \bigoplus_{w \in \mathcal{W}} \mathcal{P}_w \text{ (as a group)},$$

such that $\mathcal{P}_w \mathcal{P}_v \subset \mathcal{P}_{w+v}$, $\forall\ w, v \in \mathcal{W}$.
Then for $p \in \mathcal{P}$, we can write a unique expression

$$p = \sum_{w \in \mathcal{W}} p_w,$$

where $p_w \in \mathcal{P}_w$ and almost all $p_w = 0$. The element p_w is called the wth homogeneous component and if $p = p_w$, then p is homogeneous of degree w. $\mathcal{P}[c]$ and $\mathcal{P}[c, d]$ are $\mathbb{Z}$-graded rings as:

- $\mathcal{P}[c] = \mathcal{P} \oplus \mathcal{P}c \oplus \mathcal{P}c^2 \oplus \mathcal{P}c^3 \oplus \mathcal{P}c^4 \oplus \mathcal{P}c^5 \oplus \cdots$;
- $\mathcal{P}[c, d] = \mathcal{P} \oplus (\mathcal{P}c + \mathcal{P}d) \oplus (\mathcal{P}c^2 + \mathcal{P}cd + \mathcal{P}d^2) \oplus (\mathcal{P}c^3 + \mathcal{P}c^2d + \mathcal{P}cd^2 + \mathcal{P}d^3) \oplus \cdots$

For a $\mathcal{W}$-graded ring $\mathcal{P}$ and $\mathcal{P}$-module $\mathcal{N}$

$$\mathcal{N} = \bigoplus_{w \in \mathcal{W}} \mathcal{N}_w \text{ (as a group)},$$

with $\mathcal{P}_w \mathcal{N}_v \subset \mathcal{N}_{w+v}$ for all $w, v \in \mathcal{W}$, then $\mathcal{N}$ is said to be a $\mathcal{W}$-graded module. A non zero element of $\mathcal{N}_w$ is called a homogeneous element of degree w.

For a polynomial ring $\mathcal{P}$ defined over the field T, suppose $\mathbf{c} \in \mathbb{Z}^n$, then $p \in \mathcal{P}$

is said to be homogeneous of degree $\mathbf{c}$ when p has the form $\beta\xi^{\mathbf{c}}$, where $\beta \in T$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. Also $\mathcal{P}$ is $\mathbb{Z}^n$-graded with graded components:

$$\mathcal{P}_{\mathbf{c}} = \begin{cases} T\xi^{\mathbf{c}}, & \text{if } \mathbf{c} \in \mathbb{Z}^n_+; \\ 0\,, & \text{otherwise.} \end{cases}$$

An $\mathcal{P}$-module $\mathcal{N}$ is $\mathbb{Z}^n$-graded if $\mathcal{N} = \bigoplus_{\mathbf{c}\in\mathbb{Z}^n} \mathcal{N}_{\mathbf{c}}$ and $\mathcal{P}_{\mathbf{c_1}}\mathcal{N}_{\mathbf{c_2}} \subset \mathcal{N}_{\mathbf{c_1}+\mathbf{c_2}}$ for all $\mathbf{c_1}, \mathbf{c_2} \in \mathbb{Z}^n$.

1.4 Graph theory

Finite graphs are very straightforward formation in Mathematics. For this specific aspect, before any systematic study of graph theory itself, many graph-theoretic problems remained unsolved. Leonhard Euler's 1735 Königsberg bridges Problem [5] is the famous instance of such a problem and the Four-Color Problem which Francis Guthrie initially introduced in 1852, as a coloring problem of the map of England's counties. Such worthy experiments include research on polyhedra cycles by Thomas Kirkman and William Hamilton [6], the circuit laws by Gustav Kirchhoff [7], and research by Arthur Cayley and James Sylvester [8] that had ties to theoretical chemistry to the structure of molecules in particular. In 1878, it was Sylvester who suggested the name of "Graph" to the structure he was researching.

In this chapter primary definition and notion of graph theory are given. This chapter provides a detailed overview of individual types of graphs, distinct operations of graph and results which we will use in our last two chapter.

1.4.1 Fundamental graph theory

Graph theory comprises of the study of graphs, while the graphs are the mathematical structure used to establish the relation among the objects. The fundamental ideas of graph theory are introduced in this section.

Definition 1.4.1. A graph G is an ordered paired $G = (V, E)$ where V is a (finite) set of elements called vertices and E is a set of 2-subsets of V called edges.

Definition 1.4.2. An edge with same end points is known as a loop. The edges with exactly the same set of endpoints are known as multiple edges. A simple graph is a graph with no multiple edges and loops. Given below is a graph with vertices $\{c_1, c_2, c_3, c_4, c_5, c_6\}$ and edges $\{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9\}$.

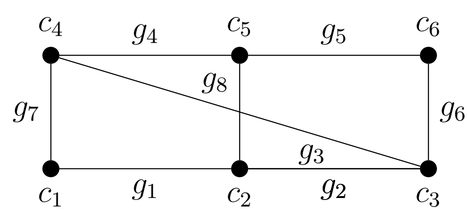

Figure 1.1: A simple Graph

Consider an edge with endpoints c_1, c_2. Then c_1, c_2 are said to be adjacent and they are neighbors of each other. The focus is restricted to only simple graphs in various important applications.

Definition 1.4.3. The total edges incident on vertex u of a graph W is known as degree of u, which is commonly represented by $d_W(u)$ or $d(u)$.

Definition 1.4.4. The total vertices in vertex set $V(W)$ is known as the order of graph W, represented by $n(W)$. Whereas the total edges in edge set $E(W)$ indicates the size of graph, written as $e(W)$.

Definition 1.4.5. *A star graph G is a graph on m vertices, in which one vertex has degree $m - 1$ and all other vertices have degree 1.*

Definition 1.4.6. A path graph is a sequence of vertices $u_1, u_2, \ldots, u_n$ where there is an edge connecting u_z and u_{z+1} for $z = 1, 2, \ldots, n - 1$.

A graph of n vertices ($n \geq 3$) is called cycle graph if we join first and last vertices of path graph by an edge. A cycle and path on n vertices are represented by C_n and P_n, respectively.

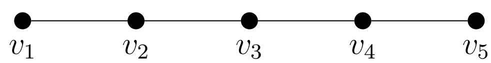

Figure 1.2: P_5

Definition 1.4.7. A subgraph B of a graph C, written as $B \subseteq C$, is a type of graph such that $V(B) \subseteq V(C)$ and $E(B) \subseteq E(C)$ and the endpoints of edges in B are exactly the same as in C.

Lemma 1.4.8. *[1](Handshaking lemma) The sum of the degrees of the vertices of a graph G is twice the number of edges,*

$$\sum_{u \in V(G)} deg(u) = 2E(G).$$

Definition 1.4.9. *(Fusion /Merged/Identified)*
The vertices u_1 and u_2 in a graph W is said to be fused, if these two vertices are replaced by a single new vertex u such that every edge that was adjacent to either u_1 or u_2 or both, is adjacent to u.

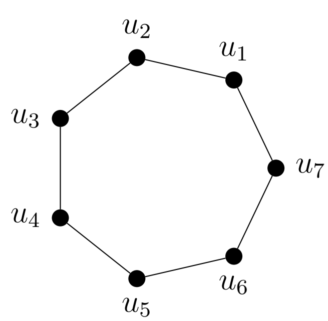

Figure 1.3: C_7

Proposition 1.4.10. Any graph with a vertices and b edges has at least $a - b$ components.

Definition 1.4.11. If the vertex set can be written as a union of two disjoint independent sets then the graph is called bipartite graph.

Definition 1.4.12. Let us have a s, d-path in graph W. The distance from c to d is the minimum length of c, d-path, written as $d(c, d)$. The path with the maximum length in W gives the diameter i.e.,

$$\text{diam}W = \max_{c,d \in V(W)} d(c, d).$$

Definition 1.4.13. *A vertex cover of a graph is a collection of vertices that contains at least one endpoint of each edge of the graph. A minimal vertex cover is an vertex cover of a graph that is not a proper subset of any other vertex cover.*

1.4.2 Graph operations

Definition 1.4.14. Consider two graphs $\mathcal{W}$ and $\mathcal{X}$ with vertex sets $V(\mathcal{W}) = \{w_1, w_2, \ldots, w_n\}$ and $V(\mathcal{X}) = \{\delta_1, \delta_2, \ldots, \delta_n\}$, respectively. The Cartesian product of $\mathcal{W}$ and $\mathcal{X}$ is a graph, with $V(\mathcal{W}\Box\mathcal{X}) = V(\mathcal{W}) \times V(\mathcal{X})$ (the cartesian product of sets), and for $(w_i, \delta_j), (w_k, \delta_l) \in V(\mathcal{W}\Box\mathcal{X})$, $(w_i, \delta_j)(w_k, \delta_l) \in E(\mathcal{W}\Box\mathcal{X})$, whenever

- $\delta_j = \delta_l$ and $w_i w_k \in E(\mathcal{W})$ or
- $\delta_j \delta_l \in E(\mathcal{X})$ and $w_i = w_k$

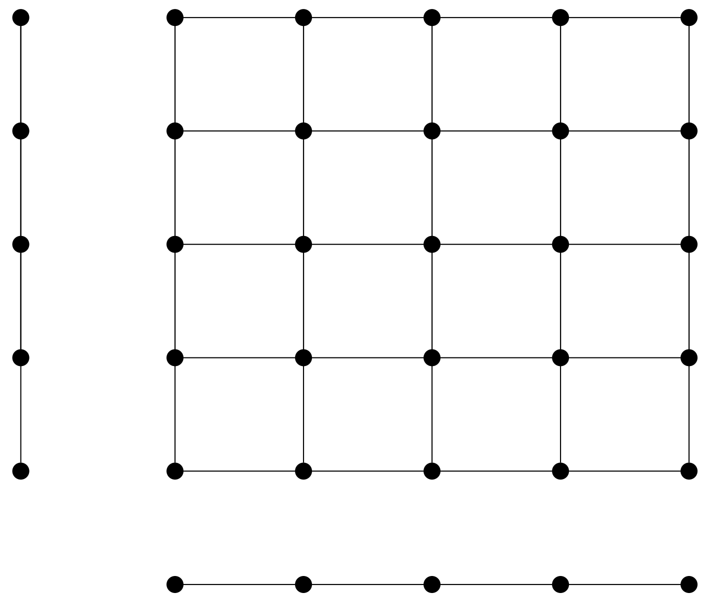

Figure 1.4: Cartesian product of P_5 and P_5 ($P_5 \Box P_5$)

Definition 1.4.15. *The pth power H^p of graph H is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in H is at most p.*

Definition 1.4.16. *The union of two simple graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ is a simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of H_1 and H_2 is denoted by $H_1 \cup H_2$.*

Definition 1.4.17. Consider two graphs $\mathcal{Q}$ and $\mathcal{R}$ with vertex set $V(\mathcal{Q}) = \{a_1, a_2, a_3, \ldots, a_n\}$ and $V(\mathcal{R}) = \{b_1, b_2, b_3, \ldots, b_n\}$ respectively. The standard strong product of $\mathcal{Q}$ and $\mathcal{K}$ is a graph, with $V(\mathcal{Q} \boxtimes \mathcal{R}) = V(\mathcal{Q}) \times V(\mathcal{R})$ (the cartesian product of sets), and for $(a_i, b_j), (a_k, b_l) \in V(\mathcal{Q} \boxtimes \mathcal{R})$, $(a_i, b_j)(a_k, b_l) \in E(\mathcal{Q} \boxtimes \mathcal{R})$, whenever

- $a_j b_l \in E(\mathcal{R})$ and $a_i = a_k$ or
- $b_j = b_l$ and $a_i a_k \in E(\mathcal{Q})$ or
- $a_i \in V(\mathcal{Q}), b_j \in V(\mathcal{R})$, $b_j b_l \in E(\mathcal{R})$ and $a_i a_k \in E(\mathcal{Q})$ or
- $a_k \in V(\mathcal{Q}), b_l \in V(\mathcal{R})$, $b_j b_l \in E(\mathcal{R})$ and $a_i a_k \in E(\mathcal{Q})$.

Definition 1.4.18. *If $n \geq 2$, then the Cartesian product of two paths P_2 and P_n is called ladder graph. Similarly, For $n \geq 3$, the Cartesian product of P_2 and C_n is said to be a circular ladder graph.*

Chapter 2

Depth, Stanley depth and some known results on depth and Stanley depth

This chapter deals with the Stanley depth (named after Richard Stanley [10] in 1982) and depth of $\mathbb{Z}^n$-graded modules over polynomial ring in n variables over a field, including the Stanley's conjecture. From now to onward, ring $\mathcal{P}$ has identity $1 \neq 0$.

2.1 Depth

Definition 2.1.1. Consider a $\mathcal{P}$-module $\mathcal{N}$. A zero divisor of a module $\mathcal{N}$ is an element $0 \neq p \in \mathcal{P}$ such that $pn = 0$, where $0 \neq n \in \mathcal{N}$.

Definition 2.1.2. Let $\mathcal{N}$ be a $\mathcal{P}$-module. An element p of $\mathcal{P}$ which is non-zero is $\mathcal{N}$ regular if for every $n \in \mathcal{N}$, $pn = 0$ implies $n = 0$.

Definition 2.1.3. A sequence $\beta = \beta_1, \ldots, \beta_n$ of elements of $\mathcal{P}$ is said to be $\mathcal{N}$-regular if it satisfies the given axioms:

1. β_j is $\mathcal{N}/(\beta_1, \ldots, \beta_{j-1})\mathcal{N}$ regular for any j;

2. $\mathcal{N} \neq (\beta)\mathcal{N}$.

Example 2.1.4. Consider $\mathcal{R} = K[a_1, a_2, a_3]$ as a module over itself. As a_1 is regular in $\mathcal{R}/(0)\mathcal{R}$, a_2 is regular in $\mathcal{R}/(a_1)\mathcal{R}$, a_3 is regular in $\mathcal{R}/(a_1, a_2)\mathcal{R}$. a_1, a_2, a_3 is the $\mathcal{N}$-regular sequence in $\mathcal{R}$.

Definition 2.1.5. *Let R be a local Noetherian ring with unique maximal ideal m and M be a finitely generated R-module. The common length of all maximal M-sequences in m is called the depth of M and is denoted by* $\operatorname{depth}(M)$.

Lemma 2.1.6. *(Depth Lemma)[11] Given a short exact sequence $0 \to \eta_1 \to \eta_2 \to \eta_3 \to 0$ of $\mathcal{P}$-modules where $\mathcal{P}$ is a local ring, then*

1. $\operatorname{depth}(\eta_2) \geq \min\{\operatorname{depth}(\eta_3), \operatorname{depth}(\eta_1)\}$.
2. $\operatorname{depth}(\eta_3) \geq \min\{\operatorname{depth}(\eta_2), \operatorname{depth}(\eta_1) + 1\}$.
3. $\operatorname{depth}(\eta_1) \geq \min\{\operatorname{depth}(\eta_3) - 1, \operatorname{depth}(\eta_2)\}$.

2.2 Stanley decomposition and Stanley depth

Definition 2.2.1. Let $\mathcal{P} = \mathcal{K}[\beta_1, \ldots, \beta_n]$ be a ring of polynomials over the field $\mathcal{K}$ and consider $\mathbb{Z}^n$-graded $\mathcal{P}$-module $\mathcal{N}$. Suppose $n \in \mathcal{N}$ and also consider $U \subset \{\beta_1, \ldots, \beta_n\}$, then $n\mathcal{K}[U]$ represents the $\mathcal{K}$-subspace of $\mathcal{N}$, whose generating set comprises of elements (homogeneous in degree) of the form nr, where r is a monomial in $\mathcal{K}[U]$. If $n\mathcal{K}[U]$ is a free $\mathcal{K}[U]$-module then it is known as a Stanley space of dimension $|U|$. A Stanley decomposition of $\mathcal{N}$ is defined as:

$$\mathcal{D} : \mathcal{N} = \bigoplus_{i=1}^{j} r_i \mathcal{K}[U_i],$$

and

$$\operatorname{sdepth} \mathcal{D} = \min\{\, |U_i| \,,\, i = 1, \ldots, j\}.$$

Also,

$$\operatorname{sdepth}_s(\mathcal{N}) = \max\{\, \operatorname{sdepth} \mathcal{D} \,:\, \mathcal{D} \text{ is a Stanley decomposition of } \mathcal{N}\}.$$

Lemma 2.2.2. *[18] Let*

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence of $\mathbb{Z}^n$-graded S-modules. Then

$$\operatorname{sdepth}(Y) \geq \min\{\operatorname{sdepth}(X), \operatorname{sdepth}(Z)\}.$$

2.2.1 Stanley's conjecture

In 1982, Stanley [10] presented a conjecture.

$$\operatorname{depth}(\mathcal{N}) \leq \operatorname{sdepth}(\mathcal{N}).$$

It has been extremely noteworthy as it examined a relation between two very different invariants of modules. For a ring of polynomials $\mathcal{P}$ in m number of variables, let $I \subset \mathcal{P}$ be the monomial ideal, then for $m \leq 3$, $m = 4$ and $m = 5$ the conjecture for $\mathcal{P}/I$ is proved by Apel [12], Anwar [13] and Popescu [14], respectively. But in 2016, Duval et al. [15] proved that Stanley's conjecture is generally false, by giving a counter example for the module of type $\mathcal{P}/I$ for which the conjecture does not hold.

Conjecture 2.2.3. *Herzog in [25] presented a conjecture. Let $I \subset S$ be a monomial ideal then*

$$\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I).$$

The above conjecture has been proved in some special cases by Popescu and Qureshi in [27] and Rauf in [18]. Recently, Keller and Young [26] proved the above conjecture for any squarefree monomial ideal in the polynomial ring with at most 7 variables.

Question 2.2.4. *Rauf [18] gave a question that is the strong form of Herzog's conjecture. Let $I \subset S$ be a monomial ideal. Does the following inequality hold*

$$\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I) + 1?$$

2.2.2 Method of computing Stanley depth for squarefree monomial ideals

In 2009, Herzog et al. [16] granted a method of computing the lower bound for Stanley depth of monomial ideals in finite number of steps by using posets. Let A be a squarefree monomial ideal with $G(A) = (a_1, \ldots, a_m)$. The characteristic poset of A w.r.t $g = (1, \ldots, 1)$, written as $\mathcal{H}_A^{(1,\ldots,1)}$ is defined as

$$\mathcal{H}_A^{(1,\ldots,1)} = \{\gamma \subset [m] \mid \gamma \text{ contains } \operatorname{supp}(a_j) \text{ for some } j\},$$

where $\text{supp}(a_j) = \{i : \delta_i | a_j\} \subseteq [m] := \{1, \dots, m\}$. For each $\rho, \sigma \in \mathcal{H}_A^{(1,\dots,1)}$ where $\rho \subseteq \sigma$, and

$$[\rho\,,\,\sigma] = \{\gamma \in \mathcal{H}_A^{(1,\dots,1)} \,:\, \rho \subseteq \gamma \subseteq \sigma\}.$$

Let $\mathcal{H} : \mathcal{H}_A^{(1,\dots,1)} = \cup_{j=1}^{k}[\gamma_j\,,\,\eta_j]$ be a partition of $\mathcal{H}_A^{(1,\dots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\text{supp}(x^{s(j)}) = \gamma_j$, then the Stanley decomposition $\mathcal{D}(\mathcal{H})$ of A is given by

$$\mathcal{D}(\mathcal{H}) \,:\, A = \bigoplus_{j=1}^{r} x^{s(j)}\mathcal{K}[\{x_k \,|\, k \in \eta_j\}].$$

Clearly, $\text{sdepth}\,\mathcal{D}(\mathcal{H}) = \min\{|\eta_1|, \dots, |\eta_r|\}$ and

$$\text{sdepth}(A) = \max\{\text{sdepth}\,\mathcal{D}(\mathcal{H}) \mid \mathcal{H} \text{ is a partition of } \mathcal{H}_A^{(1,\dots,1)}\}.$$

In the following example we are going to find stanley depth of edge ideal as well as quotient of edge ideal.

Example 2.2.5. Consider $\mathcal{P} = K[\psi_1, \psi_2, \psi_3, \psi_4, \psi_5]$, $U = (\psi_1\psi_3, \psi_2\psi_4, \psi_1\psi_4\psi_5) \subset K[\psi_1, \psi_2, \psi_3, \psi_4, \psi_5]$ be a square-free monomial ideal and $J = 0$. Set $\gamma_1 = (1,0,1,0,0)$, $\gamma_2 = (0,1,0,1,0)$ and $\gamma_3 = (1,0,0,1,1)$. Thus U is generated by $\psi^{\gamma_1}, \psi^{\gamma_2}, \psi^{\gamma_3}$ and choose $g = (1,1,1,1,1)$. The poset $\mathcal{H} = \mathcal{H}_{U/J}^{g}$ is given by

$$\begin{aligned}\mathcal{H} = \{&(1,0,1,0,0),(0,1,0,1,0),(1,1,1,0,0),(1,1,0,1,0),(1,0,1,1,0),\\ &(1,0,1,0,1),(1,0,0,1,1),(0,1,1,1,0),(0,1,0,1,1),(1,1,1,1,0),\\ &(1,1,1,0,1),(1,1,0,1,1),(1,0,1,1,1),(0,1,1,1,1),(1,1,1,1,1)\}.\end{aligned}$$

Partitions of $\mathcal{H}$ are given by

$$
\begin{aligned}
\mathcal{H}_1 :\ & [(1,0,1,0,0),(1,0,1,0,0)] \bigcup [(0,1,0,1,0),(0,1,0,1,0)] \bigcup \\
& [(1,1,1,0,0),(1,1,1,0,0)] \bigcup [(1,1,0,1,0),(1,1,0,1,0)] \bigcup \\
& [(1,0,1,1,0),(1,0,1,1,0)] \bigcup [(1,0,1,0,1),(1,0,1,0,1)] \bigcup \\
& [(1,0,0,1,1),(1,0,0,1,1)] \bigcup [(0,1,1,1,0),(0,1,1,1,0)] \bigcup \\
& [(0,1,0,1,1),(0,1,0,1,1)] \bigcup [(1,1,1,1,0),(1,1,1,1,0)] \bigcup \\
& [(1,1,1,0,1),(1,1,1,0,1)] \bigcup [(1,1,0,1,1),(1,1,0,1,1)] \bigcup \\
& [(1,0,1,1,1),(1,0,1,1,1)] \bigcup [(0,1,1,1,1),(0,1,1,1,1)] \bigcup \\
& [(1,1,1,1,1),(1,1,1,1,1)].
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_2 :\ & [(1,0,1,0,0),(1,0,1,1,1)] \bigcup [(0,1,0,1,0),(0,1,1,1,1)] \bigcup \\
& [(1,1,1,0,0),(1,1,0,1,1)] \bigcup [(1,1,0,1,0),(1,1,1,0,1)] \bigcup \\
& [(1,0,0,1,1),(1,1,1,1,0)] \bigcup [(1,1,1,1,1),(1,1,1,1,1)].
\end{aligned}
$$

And the corresponding Stanley decomposition is

$$
\begin{aligned}
\mathcal{D}(\mathcal{H}_1) &= \psi_1\psi_3 K[\psi_1,\psi_3] \oplus \psi_2\psi_4 K[\psi_2,\psi_4] \oplus \psi_1\psi_2\psi_3 K[\psi_1,\psi_2,\psi_3] \oplus \\
& \psi_1\psi_2\psi_4 K[\psi_1,\psi_2,\psi_4] \oplus \psi_1\psi_3\psi_4 K[\psi_1,\psi_3,\psi_4] \oplus \psi_1\psi_3\psi_5 K[\psi_1,\psi_3,\psi_5] \oplus \\
& \psi_1\psi_4\psi_5 K[\psi_1,\psi_4,\psi_5] \oplus \psi_2\psi_3\psi_4 K[\psi_2,\psi_3,\psi_4] \oplus \psi_2\psi_4\psi_5 K[\psi_2,\psi_4,\psi_5] \\
& \psi_1\psi_2\psi_3\psi_4 K[\psi_1,\psi_2,\psi_3,\psi_4] \oplus \psi_1\psi_2\psi_3\psi_5 K[\psi_1,\psi_2,\psi_3,\psi_5] \oplus \\
& \psi_1\psi_2\psi_4\psi_5 K[\psi_1,\psi_2,\psi_4,\psi_5] \oplus \psi_1\psi_3\psi_4\psi_5 K[\psi_1,\psi_3,\psi_4,\psi_5] \oplus \\
& \psi_2\psi_3\psi_4\psi_5 K[\psi_2,\psi_3,\psi_4,\psi_5] \oplus \psi_1\psi_2\psi_3\psi_4\psi_5 K[\psi_1,\psi_2,\psi_3,\psi_4,\psi_5].
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}(\mathcal{H}_2) &= \psi_1\psi_3 K[\psi_1,\psi_3,\psi_4,\psi_5] \oplus \psi_2\psi_4 K[\psi_2,\psi_3,\psi_4,\psi_5] \oplus \\
& \psi_1\psi_2\psi_3 K[\psi_1,\psi_2,\psi_4,\psi_5] \oplus \psi_1\psi_2\psi_4 K[\psi_1,\psi_2,\psi_3,\psi_5] \oplus \\
& \psi_1\psi_4\psi_5 K[\psi_1,\psi_2,\psi_3,\psi_4] \oplus \psi_1\psi_2\psi_3\psi_4\psi_5 K[\psi_1,\psi_2,\psi_3,\psi_4,\psi_5].
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{sdepth}(U) &\geq \max\{\text{sdepth}(\mathcal{D}(\mathcal{H}_1))\,,\, \text{sdepth}(\mathcal{D}(\mathcal{H}_2))\} \\
&= \max\{2,4\} \\
&= 4.
\end{aligned}
$$

Now for $\mathcal{P}/U$, the poset $\mathcal{F} = \mathcal{F}^g_{\mathcal{P}/U}$ is given by

$$\begin{aligned}\mathcal{F} = \{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),\\&(0,0,0,0,1),(1,1,0,0,0),(1,0,0,1,0),(1,0,0,0,1),(0,1,1,0,0),\\&(0,1,0,0,1),(0,0,1,1,0),(0,0,1,0,1),(0,0,0,1,1),(1,1,0,0,1),\\&(0,1,1,0,1),(0,0,1,1,1)\}.\end{aligned}$$

Partitions of $\mathcal{F}$ are given by

$$\begin{aligned}\mathcal{F}_1:\ &[(0,0,0,0,0),(0,0,0,0,0)]\bigcup[(1,0,0,0,0),(1,0,0,0,0)]\bigcup\\&[(0,1,0,0,0),(0,1,0,0,0)]\bigcup[(0,0,1,0,0),(0,0,1,0,0)]\bigcup\\&[(0,0,0,1,0),(0,0,0,1,0)]\bigcup[(0,0,0,0,1),(0,0,0,0,1)]\bigcup\\&[(1,1,0,0,0),(1,1,0,0,0)]\bigcup[(1,0,0,1,0),(1,0,0,1,0)]\bigcup\\&[(1,0,0,0,1),(1,0,0,0,1)]\bigcup[(0,1,1,0,0),(0,1,1,0,0)]\bigcup\\&[(1,1,0,0,1),(1,1,0,0,1)]\bigcup[(0,1,0,0,1),(0,1,0,0,1)]\bigcup\\&[(0,0,1,1,0),(0,0,1,1,0)]\bigcup[(0,0,1,0,1),(0,0,1,0,1)]\bigcup\\&[(0,0,0,1,1),(0,0,0,1,1)]\bigcup[(0,1,1,0,0),(0,1,1,0,0)]\bigcup\\&[(0,0,1,1,1),(0,0,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{F}_2:\ &[(0,0,0,0,0),(0,0,1,1,1)]\bigcup[(1,0,0,0,0),(1,1,0,0,1)]\bigcup\\&[(0,1,0,0,0),(0,1,1,0,1)]\bigcup[(1,0,0,1,0),(1,0,0,1,0)].\end{aligned}$$

And the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{F}_1) \ = \ &\psi_1K[\psi_1]\oplus\psi_2K[\psi_2]\oplus\psi_3K[\psi_3]\oplus\psi_4K[\psi_4]\oplus\psi_1\psi_2K[\psi_1,\psi_2]\oplus\\&\psi_1\psi_4K[\psi_1,\psi_4]\oplus\psi_1\psi_5K[\psi_1,\psi_5]\oplus\psi_2\psi_3K[\psi_2,\psi_3]\oplus\\&\psi_1\psi_2\psi_5K[\psi_1,\psi_2,\psi_5]\oplus\psi_2\psi_5K[\psi_2,\psi_5]\oplus\psi_3\psi_4K[\psi_3,\psi_4]\oplus\\&\psi_3\psi_5K[\psi_3,\psi_5]\oplus\psi_4\psi_5K[\psi_4,\psi_5]\oplus\psi_2\psi_3K[\psi_2,\psi_3,]\oplus\\&\psi_3\psi_4\psi_5K[\psi_3,\psi_4,\psi_5].\end{aligned}$$

$$\mathcal{D}(\mathcal{F}_2) \ = \ K[\psi_3,\psi_4,\psi_5]\oplus\psi_1K[\psi_1,\psi_2,\psi_5]\oplus\psi_2K[\psi_2,\psi_3,\psi_5]\oplus\psi_1\psi_4K[\psi_1,\psi_4].$$

Then

$$\begin{aligned} \text{sdepth}(\mathcal{P}/U) &\geq \max\{\text{sdepth}(\mathcal{D}(\mathcal{F}_1))\,,\,\text{sdepth}(\mathcal{D}(\mathcal{F}_2))\} \\ &= \max\{1,2\} \\ &= 2. \end{aligned}$$

.

Example 2.2.6. Consider $\mathcal{P} = K[\psi_1, \psi_2, \psi_3, \psi_4, \psi_5]$, $V = (\psi_1\psi_4, \psi_2\psi_3, \psi_1\psi_3\psi_5) \subset K[\psi_1, \psi_2, \psi_3, \psi_4, \psi_5]$ be a square-free monomial ideal and $J = 0$. Set $\gamma_1 = (1,0,0,1,0)$, $\gamma_2 = (0,1,1,0,0)$ and $\gamma_3 = (1,0,1,0,1)$. Thus V is generated by $\psi^{\gamma_1}, \psi^{\gamma_2}, \psi^{\gamma_3}$ and choose $g = (1,1,1,1,1)$. The poset $\mathcal{H} = \mathcal{H}^g_{V/J}$ is given by

$$\begin{aligned} \mathcal{H} = \{&(1,0,0,1,0),(0,1,1,0,0),(1,0,1,0,1),(1,1,0,1,0),(1,0,0,1,1),(1,0,1,1,0), \\ &(1,1,1,1,0),(1,1,0,1,1),(1,0,1,1,1),(1,1,1,1,1),(1,1,1,0,0),(0,1,1,1,0), \\ &(0,1,1,0,1),(1,1,1,0,1),(0,1,1,1,1)\}. \end{aligned}$$

Partitions of $\mathcal{H}$ are given by

$$\begin{aligned} \mathcal{H}_1 : \ & [(1,0,0,1,0),(1,0,0,1,0)] \bigcup [(0,1,1,0,0),(0,1,1,0,0)] \bigcup \\ & [(1,0,1,0,1),(1,0,1,0,1)] \bigcup [(1,1,0,1,0),(1,1,0,1,0)] \bigcup \\ & [(1,0,0,1,1),(1,0,0,1,1)] \bigcup [(1,0,1,1,0),(1,0,1,1,0)] \bigcup \\ & [(1,1,1,0,0),(1,1,1,0,0)] \bigcup [(0,1,1,1,0),(0,1,1,1,0)] \bigcup \\ & [(0,1,1,0,1),(0,1,1,0,1)] \bigcup [(1,1,1,1,0),(1,1,1,1,0)] \bigcup \\ & [(1,1,1,0,1),(1,1,1,0,1)] \bigcup [(1,1,0,1,1),(1,1,0,1,1)] \bigcup \\ & [(1,0,1,1,1),(1,0,1,1,1)] \bigcup [(0,1,1,1,1),(0,1,1,1,1)] \bigcup \\ & [(1,1,1,1,1),(1,1,1,1,1)]. \end{aligned}$$

$$\mathcal{H}_2: \ [(1,0,0,1,0),(1,1,0,1,0)] \bigcup [(0,1,1,0,0),(0,1,1,1,0)] \bigcup$$
$$[(1,0,1,0,1),(1,0,1,1,1)] \bigcup [(1,0,0,1,1),(1,1,1,1,1)] \bigcup$$
$$[(1,0,1,1,0),(1,1,1,1,0)] \bigcup [(1,1,1,0,0),(1,1,1,0,1)] \bigcup$$
$$[(0,1,1,0,1),(0,1,1,1,1)].$$

$$\mathcal{H}_3: \ [(1,0,0,1,0),(1,1,0,1,1)] \bigcup [(0,1,1,0,0),(1,1,1,0,1)] \bigcup$$
$$[(1,0,1,0,1),(1,0,1,1,1)] \bigcup [(1,0,1,1,0),(1,1,1,1,0)] \bigcup$$
$$[(0,1,1,1,0),(1,1,1,1,1)].$$

And the corresponding Stanley decomposition is

$$\begin{aligned}
\mathcal{D}(\mathcal{H}_1) \ = \ & \psi_1\psi_4K[\psi_1,\psi_4] \oplus \psi_2\psi_3K[\psi_2,\psi_3] \oplus \psi_1\psi_3\psi_5K[\psi_1,\psi_3,\psi_5]\oplus \\
& \psi_1\psi_2\psi_4K[\psi_1,\psi_2,\psi_4] \oplus \psi_1\psi_4\psi_5K[\psi_1,\psi_4,\psi_5] \oplus \psi_1\psi_3\psi_4K[\psi_1,\psi_3,\psi_4]\oplus \\
& \psi_1\psi_2\psi_3K[\psi_1,\psi_2,\psi_3] \oplus \psi_2\psi_3\psi_4K[\psi_2,\psi_3,\psi_4] \oplus \psi_2\psi_3\psi_5K[\psi_2,\psi_3,\psi_5] \\
& \psi_1\psi_2\psi_3\psi_4K[\psi_1,\psi_2,\psi_3,\psi_4] \oplus \psi_1\psi_2\psi_3\psi_5K[\psi_1,\psi_2,\psi_3,\psi_5]\oplus \\
& \psi_1\psi_2\psi_4\psi_5K[\psi_1,\psi_2,\psi_4,\psi_5] \oplus \psi_1\psi_3\psi_4\psi_5K[\psi_1,\psi_3,\psi_4,\psi_5]\oplus \\
& \psi_2\psi_3\psi_4\psi_5K[\psi_2,\psi_3,\psi_4,\psi_5] \oplus \psi_1\psi_2\psi_3\psi_4\psi_5K[\psi_1,\psi_2,\psi_3,\psi_4,\psi_5].
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(\mathcal{H}_2) \ = \ & \psi_1\psi_4K[\psi_1,\psi_2,\psi_4] \oplus \psi_2\psi_3K[\psi_2,\psi_3,\psi_4]\oplus \\
& \psi_1\psi_3\psi_5K[\psi_1,\psi_3,\psi_4,\psi_5] \oplus \psi_1\psi_4\psi_5K[\psi_1,\psi_2,\psi_3,\psi_4,\psi_5]\oplus \\
& \psi_1\psi_3\psi_4K[\psi_1,\psi_2,\psi_3,\psi_4] \oplus \psi_1\psi_2\psi_3K[\psi_1,\psi_2,\psi_3,\psi_5]\oplus \\
& \psi_2\psi_3\psi_5K[\psi_2,\psi_3,\psi_4,\psi_5].
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(\mathcal{H}_3) \ = \ & \psi_1\psi_4K[\psi_1,\psi_2,\psi_4] \oplus \psi_2\psi_3K[\psi_1,\psi_2,\psi_3,\psi_5]\oplus \\
& \psi_1\psi_3\psi_5K[\psi_1,\psi_3,\psi_4,\psi_5] \oplus \psi_1\psi_3\psi_4K[\psi_1,\psi_2,\psi_3,\psi_4]\oplus \\
& \psi_2\psi_3\psi_4K[\psi_1,\psi_2,\psi_3,\psi_4,\psi_5].
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{sdepth}(U) \ & \geq \ \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{H}_1))\,,\, \operatorname{sdepth}(\mathcal{D}(\mathcal{H}_2))\,,\, \operatorname{sdepth}(\mathcal{D}(\mathcal{H}_3))\} \\
& = \ \max\{2,3,3\} \\
& = \ 3.
\end{aligned}$$

Now for $\mathcal{P}/V$, the poset $\mathcal{F} = \mathcal{F}^g_{\mathcal{P}/V}$ is given by

$$\begin{aligned}\mathcal{F} = \{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),\\ &(0,0,0,0,1),(1,1,0,0,0),(1,0,0,1,0),(1,0,0,0,1),(0,1,0,1,0),\\ &(0,1,0,0,1),(0,0,1,1,0),(0,0,1,0,1),(0,0,0,1,1),(1,1,0,0,1),\\ &(0,1,0,1,1),(0,0,1,1,1)\}.\end{aligned}$$

Partitions of $\mathcal{F}$ are given by

$$\begin{aligned}\mathcal{F}_1 :\ &[(0,0,0,0,0),(0,0,0,0,0)] \bigcup [(1,0,0,0,0),(1,0,0,0,0)] \bigcup\\ &[(0,1,0,0,0),(0,1,0,0,0)] \bigcup [(0,0,1,0,0),(0,0,1,0,0)] \bigcup\\ &[(0,0,0,1,0),(0,0,0,1,0)] \bigcup [(0,0,0,0,1),(0,0,0,0,1)] \bigcup\\ &[(1,1,0,0,0),(1,1,0,0,0)] \bigcup [(1,0,1,0,0),(1,0,1,0,0)] \bigcup\\ &[(1,0,0,0,1),(1,0,0,0,1)] \bigcup [(0,1,0,1,0),(0,1,0,1,0)] \bigcup\\ &[(0,1,0,1,1),(0,1,0,1,1)] \bigcup [(0,1,0,0,1),(0,1,0,0,1)] \bigcup\\ &[(0,0,1,1,0),(0,0,1,1,0)] \bigcup [(0,0,1,0,1),(0,0,1,0,1)] \bigcup\\ &[(0,0,0,1,1),(0,0,0,1,1)] \bigcup [(1,1,0,0,1),(1,1,0,0,1)] \bigcup\\ &[(0,0,1,1,1),(0,0,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{F}_2 :\ &[(0,0,0,0,0),(0,0,1,1,1)] \bigcup [(1,0,0,0,0),(1,1,0,0,1)] \bigcup\\ &[(0,1,0,0,0),(0,1,0,1,1)] \bigcup [(1,0,1,0,0),(1,0,1,0,0)].\end{aligned}$$

$$\begin{aligned}\mathcal{F}_3 :\ &[(0,0,0,0,0),(1,0,0,0,0)] \bigcup [(0,1,0,0,0),(0,1,0,1,1)] \bigcup\\ &[(0,0,1,0,0),(0,0,1,1,1)] \bigcup [(0,0,0,1,0),(0,0,0,1,1)] \bigcup\\ &[(0,0,0,0,1),(1,0,0,0,1)] \bigcup [(1,1,0,0,0),(1,1,0,0,1)] \bigcup\\ &[(1,0,1,0,0),(1,0,1,0,0)].\end{aligned}$$

And the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{F}_1) &= \psi_1 K[\psi_1] \oplus \psi_2 K[\psi_2] \oplus \psi_3 K[\psi_3] \oplus \psi_4 K[\psi_4] \oplus \psi_5 K[\psi_5] \oplus \psi_1\psi_2 K[\psi_1,\psi_2] \oplus \\ &\quad \psi_1\psi_3 K[\psi_1,\psi_3] \oplus \psi_1\psi_5 K[\psi_1,\psi_5] \oplus \psi_2\psi_4 K[\psi_2,\psi_4] \oplus \\ &\quad \psi_2\psi_4\psi_5 K[\psi_2,\psi_4,\psi_5] \oplus \psi_2\psi_5 K[\psi_2,\psi_5] \oplus \psi_3\psi_4 K[\psi_3,\psi_4] \oplus \\ &\quad \psi_3\psi_5 K[\psi_3,\psi_5] \oplus \psi_4\psi_5 K[\psi_4,\psi_5] \oplus \psi_1\psi_2\psi_5 K[\psi_1,\psi_2,\psi_5] \oplus \\ &\quad \psi_3\psi_4\psi_5 K[\psi_3,\psi_4,\psi_5].\end{aligned}$$

$$\mathcal{D}(\mathcal{F}_2) = K[\psi_3,\psi_4,\psi_5] \oplus \psi_1 K[\psi_1,\psi_2,\psi_5] \oplus \psi_2 K[\psi_2,\psi_4,\psi_5] \oplus \psi_1\psi_3 K[\psi_1,\psi_3].$$

$$\begin{aligned}\mathcal{D}(\mathcal{F}_3) &= K[\psi_1] \oplus \psi_2 K[\psi_2,\psi_4,\psi_5] \oplus \psi_3 K[\psi_3,\psi_4,\psi_5] \oplus \psi_4 K[\psi_4,\psi_5] \oplus \\ &\quad \psi_5 K[\psi_1,\psi_5] \oplus \psi_1\psi_2 K[\psi_1,\psi_2,\psi_5] \oplus \psi_1\psi_3 K[\psi_1,\psi_3].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(\mathcal{P}/V) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{F}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{F}_2)), \operatorname{sdepth}(\mathcal{D}(\mathcal{F}_3))\} \\ &= \max\{1,2,1\} \\ &= 2.\end{aligned}$$

Example 2.2.7. Consider $\mathcal{P} = K[\psi_1,\psi_2,\psi_3,\psi_4]$, $W = (\psi_1\psi_3, \psi_1\psi_4, \psi_2\psi_3, \psi_2\psi_4) \subset K[\psi_1,\psi_2,\psi_3,\psi_4]$ be a square-free monomial ideal and $J = 0$. Set $\gamma_1 = (1,0,1,0)$, $\gamma_2 = (1,0,0,1)$, $\gamma_3 = (0,1,1,0)$ and $\gamma_4 = (0,1,0,1)$. Thus V is generated by $\psi^{\gamma_1}, \psi^{\gamma_2}, \psi^{\gamma_3}, \psi^{\gamma_4}$ and choose $g = (1,1,1,1,1)$. The poset $\mathcal{H} = \mathcal{H}^g_{W/J}$ is given by

$$\mathcal{H} = \{(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(1,1,1,0),(1,1,0,1),(1,0,1,1),\\ (0,1,1,1),(1,1,1,1)\}.$$

Partitions of $\mathcal{H}$ are given by

$$\begin{aligned}\mathcal{H}_1:\ &[(1,0,1,0),(1,0,1,0)] \bigcup [(1,0,0,1),(1,0,0,1)] \bigcup [(0,1,1,0),(0,1,1,0)] \\ &\bigcup [(0,1,0,1),(0,1,0,1)] \bigcup [(1,1,1,0),(1,1,1,0)] \bigcup [(1,1,0,1),(1,1,0,1)] \\ &\bigcup [(1,0,1,1),(1,0,1,1)] \bigcup [(0,1,1,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{H}_2:\ &[(1,0,1,0),(1,0,1,1)] \bigcup [(1,0,0,1),(1,1,0,1)] \bigcup [(0,1,1,0),(1,1,1,0)] \\ &\bigcup [(0,1,0,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].\end{aligned}$$

$\mathcal{H}_3 : \; [(1,0,0,1),(1,0,1,1)] \bigcup [(1,0,1,0),(1,1,1,0)] \bigcup [(0,1,1,0),(0,1,1,1)]$
$\bigcup [(0,1,0,1),(1,1,1,1)].$

And the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{H}_1) &= \psi_1\psi_3K[\psi_1,\psi_3] \oplus \psi_1\psi_4K[\psi_1,\psi_4] \oplus \psi_2\psi_3K[\psi_2,\psi_3] \oplus \\ &\quad \psi_2\psi_4K[\psi_2,\psi_4] \oplus \psi_1\psi_2\psi_3K[\psi_1,\psi_2,\psi_3] \oplus \psi_1\psi_2\psi_4K[\psi_1,\psi_2,\psi_4] \oplus \\ &\quad \psi_1\psi_3\psi_4K[\psi_1,\psi_3,\psi_4] \oplus \psi_2\psi_3\psi_4K[\psi_2,\psi_3,\psi_4] \oplus \psi_1\psi_2\psi_3\psi_4K[\psi_1,\psi_2,\psi_3,\psi_4].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{H}_2) &= \psi_1\psi_3K[\psi_1,\psi_3,\psi_4] \oplus \psi_1\psi_4K[\psi_1,\psi_2,\psi_4] \oplus \\ &\quad \psi_2\psi_3K[\psi_1,\psi_2,\psi_3] \oplus \psi_2\psi_4K[\psi_2,\psi_3,\psi_4] \oplus \\ &\quad \psi_1\psi_2\psi_3\psi_4K[\psi_1,\psi_2,\psi_3,\psi_4].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{H}_3) &= \psi_1\psi_4K[\psi_1,\psi_3,\psi_4] \oplus \psi_1\psi_3K[\psi_1,\psi_2,\psi_3] \oplus \\ &\quad \psi_2\psi_3K[\psi_2,\psi_3,\psi_4] \oplus \psi_2\psi_4K[\psi_1,\psi_2,\psi_3,\psi_4].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(W) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{H}_1)) , \operatorname{sdepth}(\mathcal{D}(\mathcal{H}_2)) , \operatorname{sdepth}(\mathcal{D}(\mathcal{H}_3))\} \\ &= \max\{2,3,3\} \\ &= 3.\end{aligned}$$

Now for $\mathcal{P}/W$, the poset $\mathcal{F} = \mathcal{F}^g_{\mathcal{P}/W}$ is given by

$$\mathcal{F} = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,0,0),(0,0,1,1)\}.$$

Partitions of $\mathcal{F}$ are given by

$\mathcal{F}_1 : \; [(0,0,0,0),(0,0,0,0)] \bigcup [(1,0,0,0),(1,0,0,0)] \bigcup [(0,1,0,0),(0,1,0,0)]$
$\bigcup [(0,0,1,0),(0,0,1,0)] \bigcup [(0,0,0,1),(0,0,0,1)] \bigcup [(1,1,0,0),(1,1,0,0)]$
$\bigcup [(0,0,1,1),(0,0,1,1)].$

$\mathcal{F}_2 : \; [(0,0,0,0),(0,0,1,1)] \bigcup [(1,0,0,0),(1,1,0,0)].$

$$\mathcal{F}_3: \ [(0,0,0,0),(1,0,0,0)] \bigcup [(0,1,0,0),(1,1,0,0)] \bigcup [(0,0,1,0),(0,0,1,1)] \bigcup [(0,0,0,1),(0,0,0,1)].$$

And the corresponding Stanley decomposition is

$$\mathcal{D}(\mathcal{F}_1) := \psi_1 K[\psi_1] \oplus \psi_2 K[\psi_2] \oplus \psi_3 K[\psi_3] \oplus \psi_4 K[\psi_4] \oplus \psi_1\psi_2 K[\psi_1,\psi_2] \oplus \psi_3\psi_4 K[\psi_3,\psi_4].$$

$$\mathcal{D}(\mathcal{F}_2) := K[\psi_3,\psi_4] \oplus \psi_1 K[\psi_1,\psi_2].$$

$$\mathcal{D}(\mathcal{F}_3) := K[\psi_1] \oplus \psi_2 K[\psi_1,\psi_2] \oplus \psi_3 K[\psi_3,\psi_4] \oplus \psi_4 K[\psi_4].$$

Then

$$\begin{aligned} \operatorname{sdepth}(\mathcal{P}/W) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{F}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{F}_2)), \operatorname{sdepth}(\mathcal{D}(\mathcal{F}_3))\} \\ &= \max\{1,2,1\} \\ &= 2. \end{aligned}$$

2.3 Some elementary results on Stanley depth and depth of $\mathcal{P}$-modules

Let $\mathcal{P} = K[\delta_1, \delta_2, \ldots, \delta_n]$ be the polynomial ring in these variables over the field K, then we have different results for depth and Stanley depth of different types of modules.

Corollary 2.3.1. *[17] ([Corollary 1.3]). Let* $L \subset \mathcal{P}$ *be a monomial ideal. Then* $\operatorname{sdepth}(\mathcal{P}/L) \leq \operatorname{sdepth}(\mathcal{P}/(L : l))$ *for all monomials* $l \notin L$.

Corollary 2.3.2. *[18] ([Corollary 1.3]). Let* $L \subset \mathcal{P}$ *be a monomial ideal. Then* $\operatorname{depth}(\mathcal{P}/L) \leq \operatorname{depth}(\mathcal{P}/(L : l))$ *for all monomials* $l \notin L$.

Theorem 2.3.3. *[19] (Theorems 3.1 and 4.18]). Let* G *be a connected graph and* $I = I(G)$ *be the edge ideal of* G*. If* $d = diam(G)$*, then*

$$\operatorname{depth}(\mathcal{P}/I), \operatorname{sdepth}(\mathcal{P}/I) \geq \left\lceil \frac{d+1}{3} \right\rceil.$$

Theorem 2.3.4. *[20, Theorem 1.3] Let $i_1, \dots, i_m$ be some positive integers, then*

$$\operatorname{sdepth}((v_1^{i_1}, \dots, v_m^{i_m})) = \operatorname{sdepth}((v_1, \dots, v_m)) = \left\lceil \frac{m}{2} \right\rceil.$$

In particular, for any $1 \le n \le m$.

$$\operatorname{sdepth}((v_1^{i_1}, \dots, u_n^{i_n})) = m - n + \left\lceil \frac{n}{2} \right\rceil.$$

Proposition 2.3.5. *[17] For $L \subset \mathcal{M}$ and $\forall\ v \notin L$, $\operatorname{sdepth}_{\mathcal{M}}(L : v) \ge \operatorname{sdepth}_{\mathcal{M}}(L)$.*

Lemma 2.3.6. *[11, Lemma 3.6] Let K and L be two monomial ideals with $L \subset K$, suppose $\mathcal{P}' = \mathcal{P}[\delta_{n+1}]$, then*

$$\operatorname{depth}(K\mathcal{P}'/L\mathcal{P}') = \operatorname{depth}(K\mathcal{P}/L\mathcal{P}) + 1.$$

$$\operatorname{sdepth}(K\mathcal{P}'/L\mathcal{P}') = \operatorname{sdepth}(K\mathcal{P}/L\mathcal{P}) + 1.$$

Lemma 2.3.7. *[17] Assume that $K \subset \mathcal{P}' = K[\delta_1, \dots, \delta_r], L \subset \mathcal{P}'' = K[\delta_{r+1}, \dots, \delta_n]$ are monomial ideals, with $1 \le r \le n$, then*

$$\operatorname{depth}_{\mathcal{P}}(\mathcal{P}/(K\mathcal{P} + L\mathcal{P})) = \operatorname{depth}_{\mathcal{P}'}(\mathcal{P}'/K) + \operatorname{depth}_{\mathcal{P}''}(\mathcal{P}''/L).$$

Lemma 2.3.8. *[16] Consider a monomial ideal $K \subset \mathcal{P}$ and $\bar{\mathcal{P}} = \mathcal{P}[\delta_{n+1}, \dots, \delta_{n+r}]$ be a ring of polynomials, then*

$$\operatorname{depth}(\bar{\mathcal{P}}/K\bar{\mathcal{P}}) = \operatorname{depth}(\mathcal{P}/K\mathcal{P}) + r \quad \textit{and} \quad \operatorname{sdepth}(\bar{\mathcal{P}}/K\bar{\mathcal{P}}) = \operatorname{sdepth}(\mathcal{P}/K\mathcal{P}) + r.$$

Question 2.3.9. *[18] Let $I \subset \mathcal{P}$ be a monomial ideal. Does the following inequality hold*

$$\operatorname{sdepth}(I) \ge \operatorname{sdepth}(\mathcal{P}/I) + 1?$$

Herzog conjectured in [25], a weaker form of the above inequality:

Conjecture 2.3.10. *[25] Let $I \subset \mathcal{P}$ be a monomial ideal then*

$$\operatorname{sdepth}(I) \ge \operatorname{sdepth}(\mathcal{P}/I).$$

Proposition 2.3.11. *[16] Let $K \subset \mathcal{P}$ be a monomial ideal, then*

$$\text{sdepth}(K) \geq \max\{1, n - G(K) + 1\}.$$

Proposition 2.3.12. *[28] Let $K \subset \mathcal{P}$ be a monomial ideal, then*

$$\text{sdepth}(\mathcal{P}/K) \geq n - G(K).$$

Proposition 2.3.13. *Let $K \subset \mathcal{P}$ be a monomial ideal, minimally generated by m monomials. Then*

$$\text{sdepth}(K) \geq n - \left\lfloor \frac{m}{2} \right\rfloor.$$

Theorem 2.3.14. *[28] Let $K \subset \mathcal{P}$ be a monomial ideal which is not principal. Assume $K = pI'$, where $p \in \mathcal{P}$ is a monomial and $K' = (K : p)$. Then*

$\text{sdepth}(\mathcal{P}/K) = sdepth(\mathcal{P}/K')$. *And*

$\text{sdepth}(K) = sdepth(K')$.

Proposition 2.3.15. *[17, Proposition 1.1] Let $I \subset \mathcal{P}' = K[\delta_1, ..., \delta_r], J \subset \mathcal{P}'' = K[\delta_{r+1}, ..., \delta_n]$ be monomial ideals, where $1 \leq r < n$. Then we have the following inequalities:*

1. $\text{sdepth}_{\mathcal{P}}(I\mathcal{P} \cap J\mathcal{P}) \geq \text{sdepth}_{\mathcal{P}'}(I) + \text{sdepth}_{\mathcal{P}''}(J)$;

2. $\text{sdepth}_{\mathcal{P}}(\mathcal{P}/(I\mathcal{P} + J\mathcal{P})) \geq \text{sdepth}_{\mathcal{P}'}(\mathcal{P}'/I) + \text{sdepth}_{\mathcal{P}''}(\mathcal{P}''/J)$;

3. $\text{sdepth}_{\mathcal{P}}(\mathcal{P}/(I\mathcal{P} \cap J\mathcal{P})) - 1 = \text{depth}_{\mathcal{P}}(\mathcal{P}/(I\mathcal{P} + J\mathcal{P})) = depth_{\mathcal{P}'}(\mathcal{P}'/I) + depth_{\mathcal{P}''}(\mathcal{P}''/J)$.

Proposition 2.3.16. *[17, Theorem 1.3] Let $I \subset \mathcal{P}' = K[\delta_1, ..., \delta_r], J \subset \mathcal{P}'' = K[\delta_{r+1}, ..., \delta_n]$ be monomial ideals, where $1 \leq r < n$. Then we have the following inequalities:*

1. $\text{sdepth}_{\mathcal{P}}(I\mathcal{P}) \geq \text{sdepth}_{\mathcal{P}}(I\mathcal{P} + J\mathcal{P}) \geq min\{\text{sdepth}_{\mathcal{P}}(I\mathcal{P}), \text{sdepth}_{\mathcal{P}''}(J) + \text{sdepth}_{\mathcal{P}'}(\mathcal{P}'/I)\}$;

2. $\text{sdepth}_{\mathcal{P}}(\mathcal{P}/I\mathcal{P}) \geq \text{sdepth}_{\mathcal{P}}(\mathcal{P}/I\mathcal{P} \cap J\mathcal{P}) \geq min\{\text{sdepth}_{\mathcal{P}}(\mathcal{P}/I\mathcal{P}), \text{sdepth}_{\mathcal{P}''}(\mathcal{P}''/J) + \text{sdepth}_{\mathcal{P}'}(I)\}$.

Corollary 2.3.17. *[17, Corollary 1.6] Let* $I \subset \mathcal{P}' = K[\delta_1, ..., \delta_r]$ *be a monomial and* $J = (u_1, \ldots, u_m) \subset \mathcal{P}'' = K[\delta_{r+1}, \ldots, \delta_n]$ *be monomial ideal. Then:*

1. $\operatorname{sdepth}_{\mathcal{P}}(I\mathcal{P}) \geq \operatorname{sdepth}_{\mathcal{P}}(I\mathcal{P}+J\mathcal{P}) \geq \min\{\operatorname{sdepth}_{\mathcal{P}}(I\mathcal{P}), \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/I\mathcal{P}) - \lfloor \frac{m}{2} \rfloor\}$;

2. $\operatorname{sdepth}_{\mathcal{P}}(I\mathcal{P} \cap J\mathcal{P}) \geq \operatorname{sdepth}_{\mathcal{P}}(I\mathcal{P}) - \lfloor \frac{m}{2} \rfloor$;

3. $\operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/I\mathcal{P}) \geq \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/(I\mathcal{P} \cap J\mathcal{P})) \geq min\{\operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/I\mathcal{P}), \operatorname{sdepth}_{\mathcal{P}}(I\mathcal{P}) - m\}$;

4. $\operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/(I\mathcal{P} + J\mathcal{P})) \geq \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/I\mathcal{P}) - m$.

Corollary 2.3.18. *[17, Corollary 2.5] Let* $I \subset \mathcal{P}$ *be a monomial ideal and* $u \in \mathcal{P}$ *a monomial, then:*

1. $\operatorname{sdepth}_{\mathcal{P}}(I \cap (u)) \geq \operatorname{sdepth}_{\mathcal{P}}(I)$.

2. $\operatorname{sdepth}_{\mathcal{P}}(I, u) \geq min\{\operatorname{sdepth}_{\mathcal{P}}(I), \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/I)\}$.

3. $\operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/(I, u)) \geq \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/I) - 1$.

4. $\operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/(I \cap (u))) \geq \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/I)$.

Corollary 2.3.19. *[17, Corollary 2.11] Let* $k \geq 2$ *be an integer, and let* $J_j \subset \mathcal{P}$ *be some monomial ideals, where* $1 \leq j \leq k$. *Then:*

1. $\operatorname{sdepth}_{\mathcal{P}}(J_1 \cap \ldots \cap J_k) \geq \operatorname{sdepth}_{\mathcal{P}}(J_1) + \cdots + \operatorname{sdepth}_{\mathcal{P}}(J_k) - n(k-1)$.

2. $\operatorname{sdepth}_{\mathcal{P}}(J_1 + \cdots + J_k) \geq min\Big\{ \operatorname{sdepth}_{\mathcal{P}}(J_1), \operatorname{sdepth}_{\mathcal{P}}(J_2) + \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_1) - n, \ldots, \operatorname{sdepth}_{\mathcal{P}}(J_k) + \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_{k-1}) + \cdots + \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_1) - n(k-1)\Big\}$.

3. $\operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/(J_1 \cap \ldots \cap J_k)) \geq min\Big\{ \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_1), \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_2) + \operatorname{sdepth}_{\mathcal{P}}(J_1) - n, \ldots, \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_k) + \operatorname{sdepth}_{\mathcal{P}}(J_{k-1}) + \ldots, + \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_1) - n(k-1)\Big\}$.

4. $\operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/(J_1 + \cdots + J_k)) \geq \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_1) + \cdots + \operatorname{sdepth}_{\mathcal{P}}(\mathcal{P}/J_k) - n(k-1)$.

Theorem 2.3.20. *[23] For a monomial ideal J of $\mathcal{P}$ with $|G(J)|= t$, we have*

$$\operatorname{sdepth}(J) \leq max\Big\{1, n - \Big\lfloor \frac{t}{2} \Big\rfloor\Big\}.$$

Theorem 2.3.21. *[29] Let $J(P_n^k)$ be the edge ideal of the k-th power of a path P_n on n vertices. Where k is any positive integer. Let $t \geq 3$, then*

$$\operatorname{depth}(\mathcal{P}/J(P_t^k)) = \operatorname{sdepth}(\mathcal{P}/J(P_t^k)) = \Big\lceil \frac{t}{2k+1} \Big\rceil.$$

Theorem 2.3.22. *[29, Theorem 4.5] Let $J(C_t^k)$ be the edge ideal of the k-th power of a cycle C_t on t vertices. Where k is any positive integer.Let $t \geq 3$, then*

$$\operatorname{depth}(\mathcal{P}/J(C_t^k)) = 1, \qquad if\ \ t \leq 2k+1,$$

$$\operatorname{depth}(\mathcal{P}/J(C_t^k)) \geq \Big\lceil \frac{t-k}{2k+1} \Big\rceil, \qquad if\ \ t \geq 2k+2.$$

Theorem 2.3.23. *[29, Theorem 4.6] Let $J(C_t^k)$ be the edge ideal of the k-th power of a cycle C_t on t vertices. Where k is any positive integer. Let $t \geq 3$, then*

$$\operatorname{sdepth}(\mathcal{P}/J(C_t^k)) = 1, \qquad if\ \ t \leq 2k+1,$$

$$\operatorname{sdepth}(\mathcal{P}/J(C_t^k)) \geq \Big\lceil \frac{t-k}{2k+1} \Big\rceil, \qquad if\ \ t \geq 2k+2.$$

Theorem 2.3.24. *[29, Theorem 5.2] Let $t \geq 2$, then* $\operatorname{sdepth}(J(P_t^k)) \geq \left\lceil \frac{t}{2k+1} \right\rceil + 1$.

Theorem 2.3.25. *[29, Theorem 5.4] Let $t \geq 3$, then*

$$\operatorname{sdepth}(J(C_t^k)) \geq 2, \qquad if\ \ t \leq 2k+1;$$

$$\operatorname{sdepth}(J(C_t^k)) \geq \Big\lceil \frac{t-k}{2k+1} \Big\rceil + 1, \qquad if\ \ t \geq 2k+2.$$

Proposition 2.3.26. *[29, Theorem 5.3] Let $t \geq 2k+1$, then*

$$\operatorname{sdepth}(J(C_t^k)/J(P_t^k)) \geq \Big\lceil \frac{n+k+1}{2k+1} \Big\rceil.$$

Consider $\mathcal{P}_{t,m} = K[\cup_{j=1}^{m}\{y_{1j}, y_{2j}, \ldots, y_{tj}\}]$ be the polynomial ring in these variables over the field K. We denote $\mathcal{P}_t$ for $m = 0$. We have some results.

Lemma 2.3.27. *[30, Lemma 3.2] Let $I(P_{t,m})$ denotes the edge ideal of strong product of two paths of length t and m, respectively. Let $t \geq 1$ and $m = 2$, then*

$$\operatorname{depth}(\mathcal{P}_{t,2}/I(P_{t,2})) = \operatorname{sdepth}(\mathcal{P}_{t,2}/I(P_{t,2})) = \left\lceil \frac{t}{3} \right\rceil.$$

Proposition 2.3.28. *[30, Lemma 3.3] For $t \geq 1$ and $m = 3$, we have*

$$\operatorname{depth}(\mathcal{P}_{t,3}/I(P_{t,3})) = \operatorname{sdepth}(\mathcal{P}_{t,3}/I(P_{t,3})) = \left\lceil \frac{t}{3} \right\rceil.$$

Proposition 2.3.29. *[30, Theorem 3.4] Let $I(C_{t,m})$ denotes the edge ideal of strong product of path and cycle of length t and m respectively. Let $t \geq 3$ and $m = 2$, then*

$$\operatorname{depth}(\mathcal{P}_{t,2}/I(C_{t,2})) = \operatorname{sdepth}(\mathcal{P}_{t,2}/I(C_{t,2})) = \left\lceil \frac{t-1}{3} \right\rceil.$$

$$\left\lceil \frac{t-1}{3} \right\rceil \leq \operatorname{depth}(\mathcal{P}_{t,3}/I(C_{t,3})), \operatorname{sdepth}(\mathcal{P}_{t,3}/I(C_{t,3})) \leq \left\lceil \frac{t}{3} \right\rceil.$$

Proposition 2.3.30. *[30, Proposition 4.6] For $t \geq 3$ and $m = 2$, we have*

$$\operatorname{sdepth}(I(C_{t,2})/I(P_{t,2})) \geq \left\lceil \frac{t+2}{3} \right\rceil.$$

Proposition 2.3.31. *[31, Lemma 3.1] Let J_n denotes the edge ideal of line graph of ladder graph. Let $n \geq 2$, then*

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(\mathcal{P}_n/J_n), \operatorname{sdepth}(\mathcal{P}_n/J_n) \leq n-1.$$

Proposition 2.3.32. *[31, Theorem 3.6] Let K_n denotes the edge ideal of line graph of circular ladder graph. Let $n \geq 3$, then*

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(\mathcal{P}_n/K_n) \leq n-1.$$

And

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{sdepth}(\mathcal{P}_n/K_n) \leq n.$$

Proposition 2.3.33. *[31, Proposition 3.8] Let $n \geq 2$, we have that $\dim(\mathcal{P}_n/J_n) \geq n$.*

Proposition 2.3.34. *[31, Proposition 3.9] Let $n \geq 3$, we have that $\dim(\mathcal{P}_n/K_n) \geq n$.*

Theorem 2.3.35. *[27, Lemma 1.2] Let Q be a monomial primary ideal $\mathcal{P}$. Then*

$$\operatorname{depth}(\mathcal{P}/Q) = \operatorname{sdepth}(\mathcal{P}/Q) = \dim(\mathcal{P}/Q).$$

2.3.1 Computations in Commutative Algebra (CoCoA)

We sincerely appreciate the contribution of the computer algebra system CoCoA [32] for our experiments. It is very helpful for initial cases i.e., for small positive integers. It saves our time by verifying desired results very quickly. It can easily work up-to eighteen variables.

Chapter 3

Depth and Stanley depth of cyclic modules associated to some graphs

Let Δ be the graph, as shown in Figure 3.1, then n copies of Δ graph are shown in Figure 3.2 for i varies from 1 to n, where $n \geq 1$ and $m \geq 0$. Now the union of n copies of Δ graph is defined as follow

$$B_{n,m} = \bigcup_{i=1}^{n} A_{i,m}.$$

Clearly $|V(B_{n,m})|= 2nm + 3n + 1$. The graph $B_{n,m}$ has $2n$ and $2nm$ vertices of degree 2, $n-1$ vertices of degree $4m+6$ and two vertices of degree $2m+3$. So by using Lemma 1.4.8, we have $|E(B_{n,m}|= 4nm + 5n$. For examples of $B_{n,m}$ graph see Figures 3.3 and 3.4. For $m \geq 0$, if $n = 1$, then $B_{n,m}$ graph has diameter 1. And for $n \geq 2$, diameter is n.

We label the vertices of the $B_{n,m}$ graph by using five sets of variables $\{\delta_1, \delta_2, \ldots, \delta_n\}$, $\{\xi_1, \xi_2, \ldots, \xi_{n+1}\}$, $\{\omega_1, \omega_2, \ldots, \omega_n\}$, $\Big\{\{\mu_{11}, \mu_{12}, \ldots, \mu_{1m}\},\{\mu_{21}, \mu_{22},\ldots, \mu_{2m}\}, \ldots ,$ $\{\mu_{n1}, \mu_{n2}, \ldots, \mu_{nm}\}\Big\}$ and $\Big\{\{\nu_{11}, \nu_{12}, \ldots, \nu_{1m}\},\{\nu_{21}, \nu_{22}, \ldots, \nu_{2m}\}, \ldots, \{\nu_{n1}, \nu_{n2}, \ldots, \nu_{nm}\}\Big\}$ see figure 3.4. Let $S_{n,m} := K[\delta_1, \delta_2,\ldots, \delta_n, \xi_1, \xi_2,\ldots, \xi_{n+1}, \omega_1, \omega_2,\ldots, \omega_n, \mu_{11}, \mu_{12},\ldots, \mu_{1m}$ $, \mu_{21}, \mu_{22},\ldots, \mu_{2m} \ \ldots, \ \mu_{n1}, \mu_{n2},\ldots, \mu_{nm}, \nu_{11}, \nu_{12},\ldots, \nu_{1m}, \nu_{21}, \nu_{22},\ldots, \nu_{2m} \ \ldots, \ \nu_{n1}, \nu_{n2},\ldots$ $, \nu_{nm}]$ be the ring of polynomials in these variables over the field K. Then $I_{n,m}$ is square-free monomial ideal of $S_{n,m}$. Now with the labelling as shown in Figure 3.4, we have:

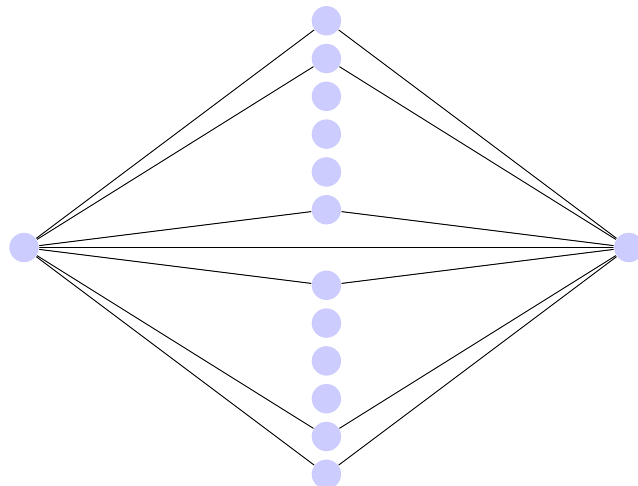

Figure 3.1: A (Δ) Graph

$\mathcal{M}(I_{n,m}) = \Big\{\{I(P_{n+1})\} \bigcup \bigcup\limits_{i=1}^{n}\{\delta_i\xi_i\} \bigcup \bigcup\limits_{i=1}^{n}\{\xi_i\omega_i\} \bigcup \bigcup\limits_{i=1}^{n}\{\delta_i\xi_{i+1}\} \bigcup \bigcup\limits_{i=1}^{n}\{\omega_i\xi_{i+1}\} \bigcup$
$\bigcup\limits_{i=1}^{n}\{\xi_i\mu_{i1}\} \bigcup \bigcup\limits_{i=1}^{n}\{\xi_i\mu_{i2}\} \bigcup \ ,\dots,\ \bigcup \bigcup\limits_{i=1}^{n}\{\xi_i\mu_{im}\} \bigcup \bigcup\limits_{i=1}^{n}\{\xi_i\nu_{i1}\} \bigcup \bigcup\limits_{i=1}^{n}\{\xi_i\nu_{i2}\} \bigcup \ ,\dots,\ \bigcup$
$\bigcup\limits_{i=1}^{n}\{\xi_i\nu_{im}\} \bigcup \bigcup\limits_{i=1}^{n}\{\xi_{i+1}\mu_{i1}\} \bigcup \bigcup\limits_{i=1}^{n}\{\xi_{i+1}\mu_{i2}\} \bigcup \ ,\dots,\ \bigcup \bigcup\limits_{i=1}^{n}\{\xi_{i+1}\mu_{im}\} \bigcup \bigcup\limits_{i=1}^{n}\{\xi_{i+1}\nu_{i1}\} \bigcup$
$\bigcup\limits_{i=1}^{n}\{\xi_{i+1}\nu_{i2}\} \bigcup \ ,\dots,\ \bigcup \bigcup\limits_{i=1}^{n}\{\xi_{i+1}\nu_{im}\}\Big\}$,
where $I(P_{n+1}) = \bigcup\limits_{i=1}^{n}\{\xi_i\xi_{i+1}\}$ is a path on $n+1$ vertices and $\mathcal{M}(I_{n,m})$ stands for the minimal set of monomial generators of monomial ideal $I_{n,m}$.

Let us consider a supergraph $D_{n,m}$ of the graph $B_{n,m}$. The vertex and edge sets of $D_{n,m}$ are $V(D_{n,m}) = V(B_{n,m}) \bigcup \{\delta_{n+1}, \omega_{n+1}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m}\}$ and $E(D_{n,m}) = E(B_{n,m}) \bigcup \{\delta_{n+1}\xi_{n+1}, \xi_{n+1}\omega_{n+1}, \xi_{n+1}\mu_{(n+1)1}, \xi_{n+1}\mu_{(n+1)2}, \dots, \xi_{n+1}\mu_{(n+1)m}, \xi_{n+1}\nu_{(n+1)1}, \xi_{n+1}\nu_{(n+1)2}, \dots, \xi_{n+1}\nu_{(n+1)m}\}$. For example of graph $D_{n,m}$, see Figure 3.5. We denote the edge ideal of graph $D_{n.m}$ with $I^*_{n,m}$, where $I^*_{n,m}$ is the monomial ideal of the polynomial ring $S^*_{n,m} = S_{n,m}[\delta_{n+1}, \omega_{n+1}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m}]$. The minimal set of monomial generators of $I^*_{n,m}$ is $\mathcal{M}(I^*_{n,m}) = \mathcal{M}(I_{n,m}) \bigcup \{\delta_{n+1}\xi_{n+1}, \xi_{n+1}\omega_{n+1}, \xi_{n+1}\mu_{(n+1)1}, \xi_{n+1}\mu_{(n+1)2}, \dots, \xi_{n+1}\mu_{(n+1)m}, \xi_{n+1}\nu_{(n+1)1}, \xi_{n+1}\nu_{(n+1)2}, \dots, \xi_{n+1}\nu_{(n+1)m}\}$.

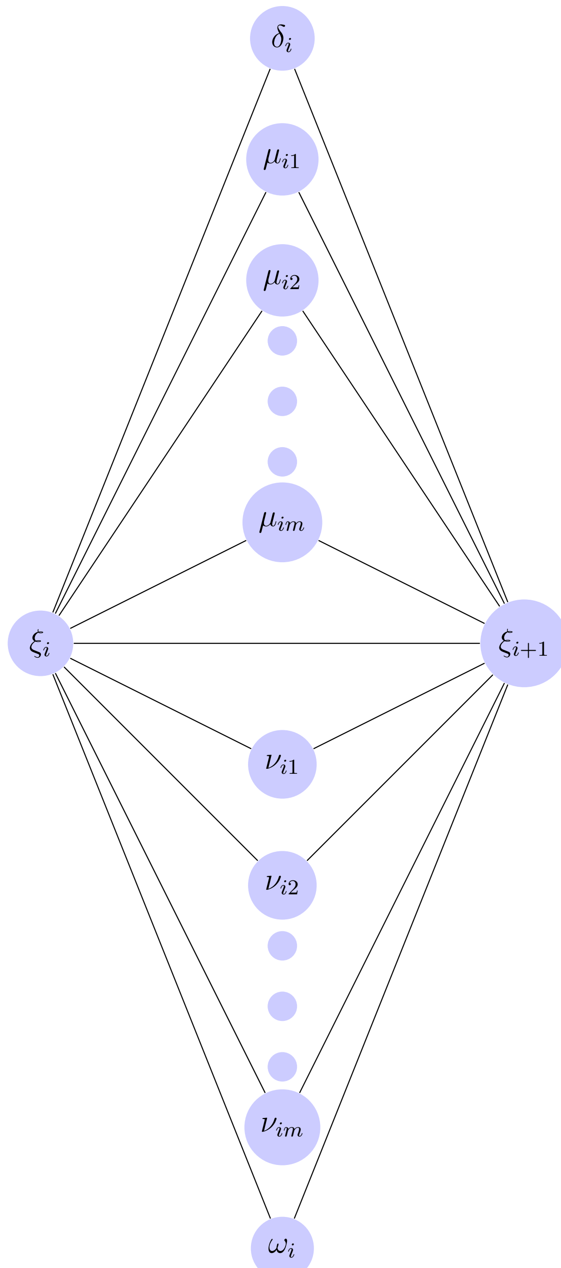

Figure 3.2: $(A_{i,m})$ Graph

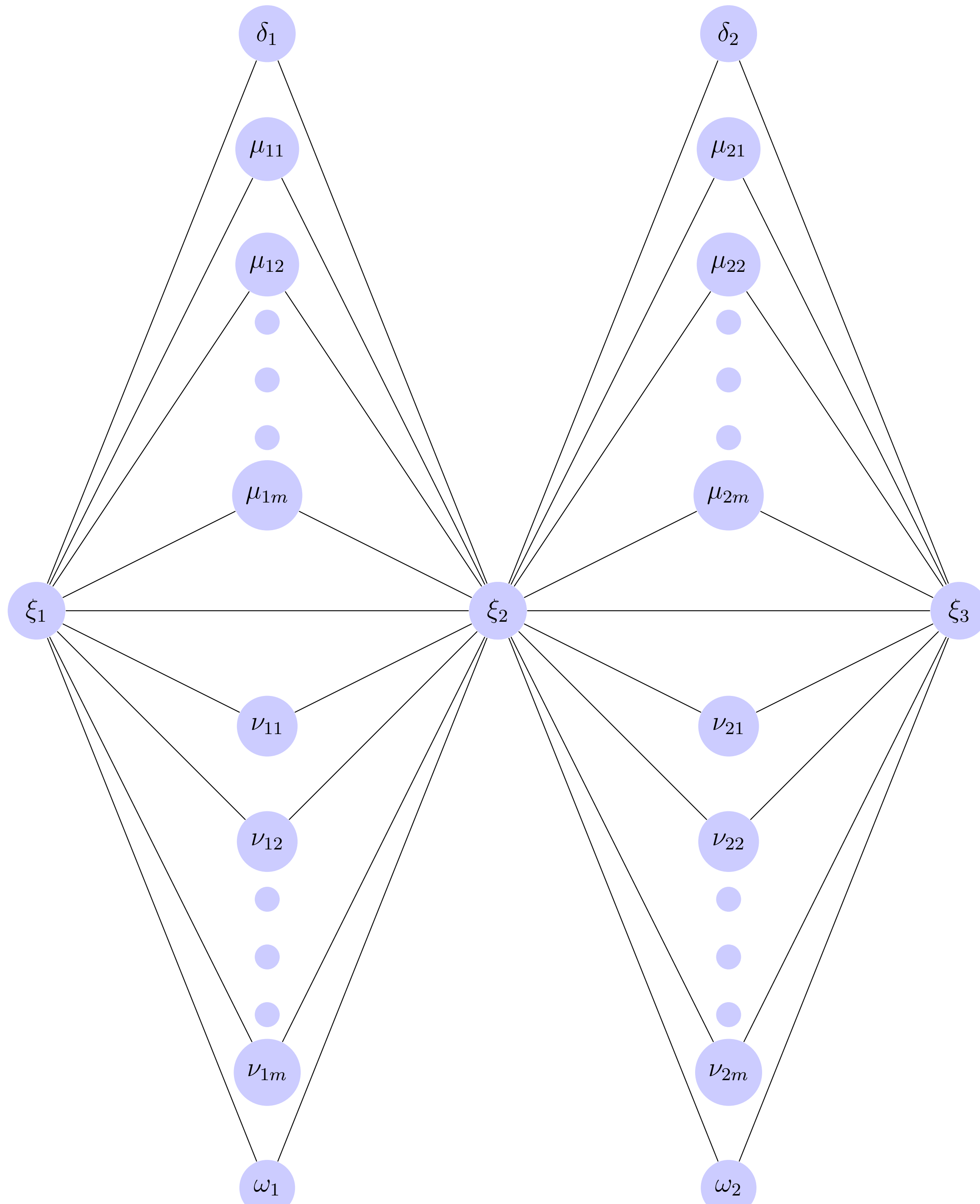

Figure 3.3: ($B_{2,m}$) Graph

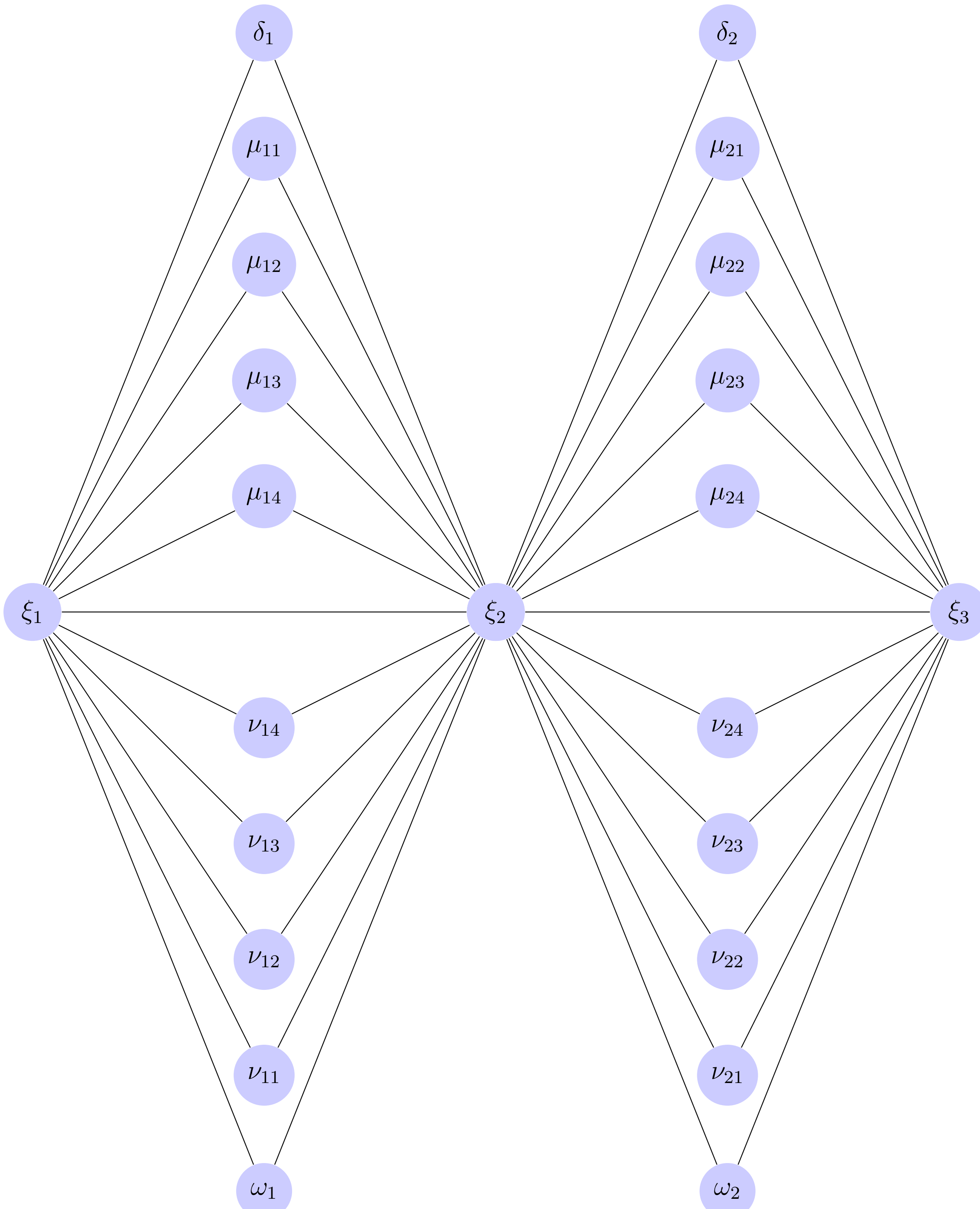

Figure 3.4: $(B_{2,6})$ Graph

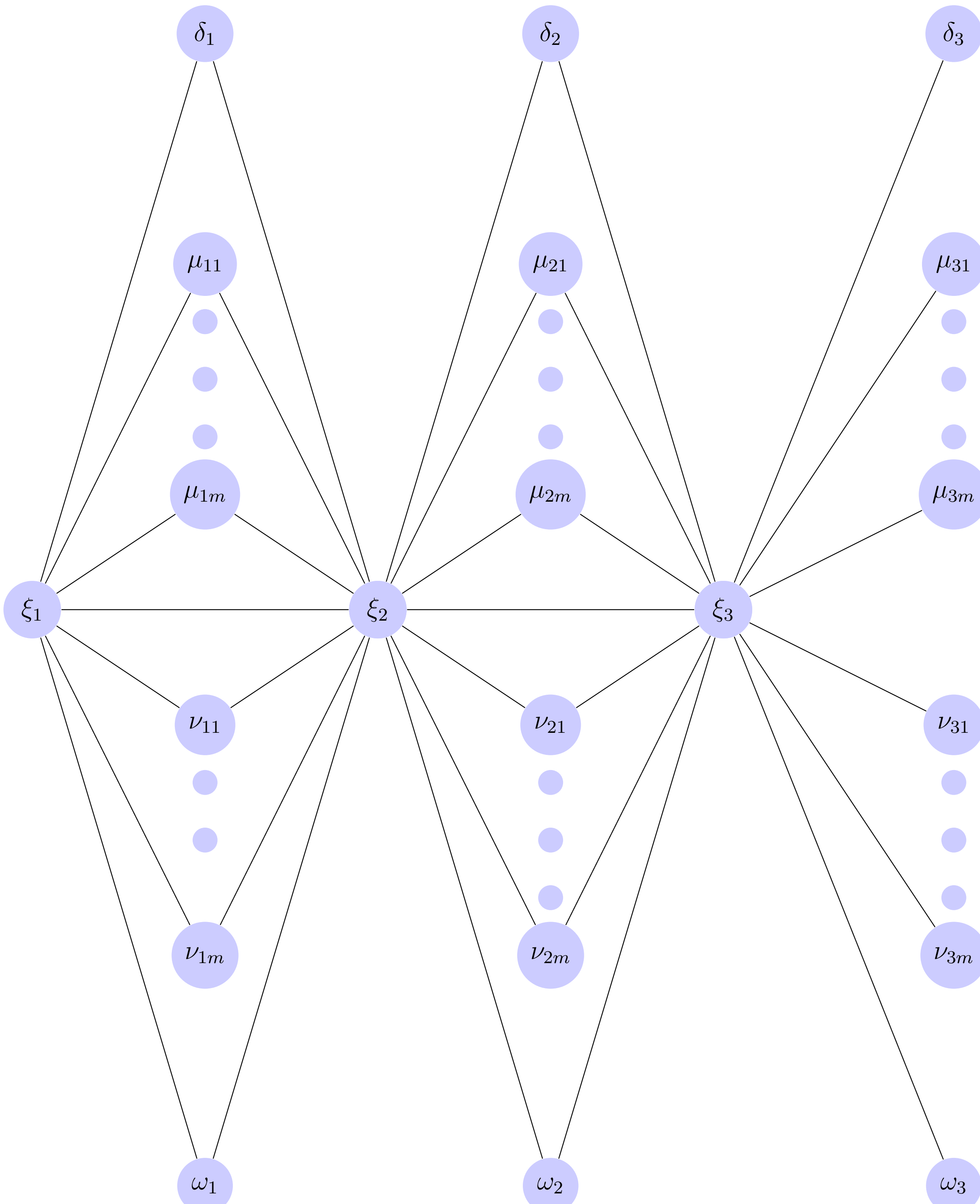

Figure 3.5: $(D_{2,m})$ Graph

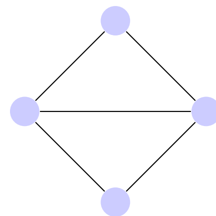

Figure 3.6: Kite Graph

3.1 Depth and Stanley depth of the cyclic module associated to the graph $B_{n,m}$

We find the value of depth and Stanley depth of the cyclic module $S_{n,m}/I_{n,m}$ associated to the graph $B_{n,m}$ when $n \equiv 1 (\mathrm{mod}\ 2)$, and give tight bounds when $n \equiv 0 (\mathrm{mod}\ 2)$. For this purpose, we first find depth and Stanley depth of the cyclic module $S^*_{n,m}/I^*_{n,m}$ associated to the super graph $D_{n,m}$. And we will use these results in our main proof. For $m = 0$ we denote $S_{n,m} = S_n$ and $S^*_{n,m} = S^*_n$, similarly, $I_{n,m} = I_n$ and $I^*_{n,m} = I^*_n$.

Proposition 3.1.1. *Let $n \geq 1$ and $m = 0$, then* $\mathrm{depth}(S^*_n/I^*_n) = \mathrm{sdepth}(S^*_n/I^*_n) = \lceil \frac{n+1}{2} \rceil$.

Proof. We will prove this by induction on n. When $1 \leq n \leq 4$, then by using CoCoA it is an easy exercise to see that the result holds. Suppose that $n \geq 5$. Consider the short exact sequence

$$0 \longrightarrow S^*_n/(I^*_n : \xi_{n+1}) \xrightarrow{\cdot \xi_{n+1}} S^*_n/I^*_n \longrightarrow S^*_n/(I^*_n, \xi_{n+1}) \longrightarrow 0. \tag{3.1}$$

By Depth Lemma

$$\mathrm{depth}(S^*_n/I^*_n) \geq \min\{\mathrm{depth}(S^*_n/(I^*_n : \xi_{n+1})), \mathrm{depth}(S^*_n/(I^*_n, \xi_{n+1}))\}.$$

$$(I^*_n : \xi_{n+1}) = (I^*_{n-2}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}).$$

We have $S^*_n/(I^*_n : \xi_{n+1}) \cong K[\delta_1, \delta_2, \dots, \delta_{n-1}, \xi_1, \xi_2, \dots, \xi_{n-1}, \xi_{n+1}, \omega_1, \omega_2, \dots, \omega_{n-1}]/I^*_{n-2} \cong (S^*_{n-2}/(I^*_{n-2}))[\xi_{n+1}]$. By induction on n and [16, Lemma 3.6],

$$\mathrm{depth}(S^*_n/(I^*_n : \xi_{n+1})) = \mathrm{depth}(S^*_{n-2}/(I^*_{n-2})) + 1 = \left\lceil \frac{n-2+1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil.$$

Now as

$$(I_n^*, \xi_{n+1}) = (I_{n-1}^*, \xi_{n+1}),$$

we obtain $S_n^*/(I_n^*, \xi_{n+1}) \cong K[\delta_1, \delta_2, \ldots, \delta_{n+1}, \xi_1, \xi_2, \ldots, \xi_n, \omega_1, \omega_2, \ldots, \omega_{n+1}]/I_{n-1}^*$ $\cong S_{n-1}^*/I_{n-1}^*[\delta_{n+1}, \omega_{n+1}]$. Thus by induction on n and [16, Lemma 3.6], $\operatorname{depth}(S_n^*/(I_n^*, \xi_{n+1})) = \operatorname{depth}(S_{n-1}^*/(I_{n-1}^*)) + 2 = \lceil \frac{n-1+1}{2} \rceil + 2 = \lceil \frac{n+4}{2} \rceil$. Also since $\operatorname{depth}(S_n^*/(I_n^*, \xi_{n+1})) \geq \operatorname{depth}(S_n^*/(I_n^* : \xi_{n+1}))$ by Depth Lemma $\operatorname{depth}(S_n^*/I_n^*) = \lceil \frac{n+1}{2} \rceil$. For Stanley depth applying Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.1, [16, Lemma 3.6] and induction on n. We have $\operatorname{sdepth}(S_n^*/(I_n^*)) \geq \lceil \frac{n+1}{2} \rceil$. For the upper bound since $\xi_{n+1} \notin I_n^*$ by Corollary 2.3.1, we get $\operatorname{sdepth}(S_n^*/I_n^*)$ $\leq \operatorname{sdepth}(S_n^*/(I_n^* : \xi_{n+1}))$. This implies that $\operatorname{sdepth}(S_n^*/I_n^*) \leq \lceil \frac{n+1}{2} \rceil$. Thus $\operatorname{sdepth}(S_n^*/I_n^*) = \lceil \frac{n+1}{2} \rceil$. □

Proposition 3.1.2. *Let* $n, m \geq 1$, *then* $\operatorname{depth}(S_{n,m}^*/I_{n,m}^*) = \operatorname{sdepth}(S_{n,m}^*/I_{n,m}^*) = \lceil \frac{n+1}{2} \rceil$.

Proof. If $n = 1$, observe the short exact sequence

$$0 \longrightarrow S_{1,m}^*/(I_{1,m}^* : \xi_2) \xrightarrow{\cdot \xi_2} S_{1,m}^*/I_{1,m}^* \longrightarrow S_{1,m}^*/(I_{1,m}^*, \xi_2) \longrightarrow 0, \tag{3.2}$$

applying Depth Lemma

$$\operatorname{depth}(S_{1,m}^*/I_{1,m}^*) \geq \min \Big\{ \operatorname{depth}(S_{1,m}^*/(I_{1,m}^* : \xi_2)), \operatorname{depth}(S_{1,m}^*/(I_{1,m}^*, \xi_2)) \Big\}.$$

$(I_{1,m}^* : \xi_2) = (\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{21}, \mu_{22}, \ldots, \mu_{2m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{21}, \nu_{22}, \ldots, \nu_{2m})$,
We have $S_{1,m}^*/(I_{1,m}^* : \xi_2) \cong K[\xi_2]$, since depth and sdepth of polynomial ring $K[\xi_2] = 1$. So

$$\operatorname{depth}(S_{1,m}^*/(I_{1,m}^* : \xi_2)) = 1.$$

Consider $(I_{1,m}^*, \xi_2) = (\delta_1\xi_1, \xi_1\omega_1, \xi_1\mu_{11}, \xi_1\mu_{12}, \ldots, \xi_1\mu_{1m}, \xi_1\nu_{11}, \xi_1\nu_{12}, \ldots, \xi_1\nu_{1m}, \xi_2)$ $= (g_{1,m}, \xi_2)$, where $g_{1,m} = (\delta_1\xi_1, \xi_1\omega_1, \xi_1\mu_{11}, \xi_1\mu_{12}, \ldots, \xi_1\mu_{1m}, \xi_1\nu_{11}, \xi_1\nu_{12}, \ldots, \xi_1\nu_{1m})$ is the edge ideal of Star graph. Since depth and sdepth of quotient module $A_{1,m}/g_{1,m}$ associated to Star graph is 1, where $A_{1,m} = K[\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}]$

is the polynomial ring in these variables over the field k.
We have $S^*_{1,m}/(I^*_{1,m},\xi_2) \cong K[\delta_1,\delta_2,\xi_1,\omega_1,\omega_2\mu_{11},\mu_{12},\ldots,\mu_{1m},\nu_{11},\nu_{12},\ldots,\nu_{1m}]/g_{1,m} \cong A_{1,m}/g_{1,m}[\delta_2,\omega_2]$. Hence by depth of star graph and [16, Lemma 3.6], we get

$$\operatorname{depth}(S^*_{1,m}/(I^*_{1,m},\xi_2)) = 1+2 = 3.$$

Since $\operatorname{depth}(S^*_{1,m}/(I^*_{1,m},\xi_2)) \geq \operatorname{depth}(S^*_{1,m}/(I^*_{1,m}:\xi_2))$, so using Depth Lemma

$$\operatorname{depth}(S^*_{1,m}/I^*_{1,m}) = 1 = \Big\lceil \frac{1+1}{2} \Big\rceil.$$

For Stanley depth if $n=1$ then by applying Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.2, [16, Lemma 3.6] , sdepth of polynomial ring and sdepth of quotient module associated to star graph, we have $\operatorname{sdepth}(S^*_{1,m}/(I^*_{1,m},\xi_2)) \geq 1$. For the upper bound since $\xi_2 \notin I^*_{1,m}$ by Corollary 2.3.1, we get $\operatorname{sdepth}(S^*_{1,m}/I^*_{1,m}) \leq \operatorname{sdepth}(S^*_{1,m}/(I^*_{1,m}:\xi_2))$. This implies $\operatorname{sdepth}(S^*_{1,m}/I^*_{1,m}) \leq 1$. Thus

$$\operatorname{sdepth}(S^*_{1,m}/I^*_{1,m}) = 1 = \Big\lceil \frac{1+1}{2} \Big\rceil.$$

For $n=2$. Considering the short exact sequence

$$0 \longrightarrow S^*_{2,m}/(I^*_{2,m}:\xi_3) \xrightarrow{\cdot\xi_3} S^*_{2,m}/I^*_{2,m} \longrightarrow S^*_{2,m}/(I^*_{2,m},\xi_3) \longrightarrow 0, \tag{3.3}$$

using Depth Lemma

$$\operatorname{depth}(S^*_{2,m}/I^*_{2,m}) \geq \min\Big\{ \operatorname{depth}(S^*_{2,m}/(I^*_{2,m}:\xi_3)), \operatorname{depth}(S^*_{2,m}/(I^*_{2,m},\xi_3))\Big\}.$$

$(I^*_{2,m}:\xi_3) = (g_{1,m},\delta_2,\delta_3,\xi_2,\omega_2,\omega_3,\mu_{21},\mu_{22},\ldots,\mu_{2m},\mu_{31},\ldots,\mu_{3m},\nu_{21},\nu_{22},\ldots,\nu_{2m},\nu_{31},$ $\nu_{32},\ldots,\nu_{3m})$.
We have $S^*_{2,m}/(I^*_{2,m}:\xi_3) \cong K[\delta_1,\xi_1,\xi_3,\omega_1,\mu_{11},\mu_{12},\ldots,\mu_{1m},\nu_{11},\nu_{12},\ldots,\nu_{1m}]/g_{1,m} \cong A_{1,m}/g_{1,m}[\xi_3]$. By [16, Lemma 3.6], $\operatorname{depth}(S^*_{2,m}/(I^*_{2,m}:\xi_3)) = \operatorname{depth}(A_{1,m}/(g_{1,m}))+1 = 2$.
Now by taking $(I^*_{2,m},\xi_3) \cong (I^*_{1,m},\xi_3)$, we have $S^*_{2,m}/(I^*_{2,m},\xi_3) \cong K[\delta_1,\delta_2,\delta_3,\xi_1,\xi_2,\omega_1,$ $\omega_2,\omega_3,\mu_{11},\mu_{12},\ldots,\mu_{1m},\mu_{21},\mu_{22},\ldots,\mu_{2m},\mu_{31},\mu_{32},\ldots,\mu_{3m},\nu_{11},\nu_{12},\ldots,\nu_{1m},\nu_{21},\nu_{22},\ldots,\nu_{2m},$ $\nu_{31},\nu_{32},\ldots,\nu_{3m}]/I^*_{1,m} \cong S^*_{1,m}/I^*_{1,m}[\delta_3,\omega_3,\mu_{31},\mu_{32},\ldots,\mu_{3m},\nu_{31},\nu_{32},\ldots,\nu_{3m}]$. Thus by induction on n and [16, Lemma 3.6], $\operatorname{depth}(S^*_{2,m}/(I^*_{2,m},\xi_3)) = 1+2m+2$.

Since $\operatorname{depth}(S^*_{2,m}/(I^*_{2,m},\xi_3)) \geq \operatorname{depth}(S^*_{2,m}/(I^*_{2,m}:\xi_3))$, using Depth Lemma

$$\operatorname{depth}(S^*_{2,m}/I^*_{2,m}) = 2 = \left\lceil\frac{n+1}{2}\right\rceil.$$

For Stanley depth applying Lemma 2.2.2 instead of Depth Lemma on the exact sequence 3.3, by induction on n, [16, Lemma 3.6] and sdepth of quotient module associated to star graph, we have $\operatorname{sdepth}(S^*_{2,m}/(I^*_{2,m},\xi_3)) \geq 2$. For the upper bound since $\xi_3 \notin I^*_{2,m}$ by Corollary 2.3.1, we get $\operatorname{sdepth}(S^*_{2,m}/I^*_{2,m}) \leq \operatorname{sdepth}(S^*_{2,m}/(I^*_{2,m}:\xi_3))$. This implies $\operatorname{sdepth}(S^*_{2,m}/I^*_{2,m}) \leq 2$. Hence

$$\operatorname{sdepth}(S^*_{2,m}/I^*_{2,m}) = 2 = \left\lceil\frac{n+1}{2}\right\rceil.$$

When $n \geq 3$, we have short exact sequence

$$0 \longrightarrow S^*_{n,m}/(I^*_{n,m}:\xi_{n+1}) \xrightarrow{\cdot\xi_{n+1}} S^*_{n,m}/I^*_{n,m} \longrightarrow S^*_{n,m}/(I^*_{n,m},\xi_{n+1}) \longrightarrow 0, \tag{3.4}$$

by Depth lemma, we have

$$\operatorname{depth}(S^*_{n,m}/I^*_{n,m}) \geq \min\Big\{\operatorname{depth}(S^*_{n,m}/(I^*_{n,m}:\xi_{n+1})), \operatorname{depth}(S^*_{n,m}/(I^*_{n,m},\xi_{n+1}))\Big\}.$$

$(I^*_{n,m}:\xi_{n+1}) = \Big(I^*_{(n-2),m}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots,$
$\mu_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m}\Big)$.
Now $S^*_{n,m}/(I^*_{n,m}:\xi_{n+1}) \cong K[\delta_1,\delta_2,\dots,\delta_{n-1},\xi_1,\xi_2,\dots,\xi_{n-1},\xi_{n+1},\omega_1,\omega_2,\dots,\omega_{n-1},\mu_{11}$
$,\mu_{12},\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m}\ \dots,\ \mu_{(n-1)1},\mu_{(n-1)2},\dots,\mu_{(n-1)m}\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},\dots$
$,\nu_{2m}\ \dots,\ \nu_{(n-1)1},\nu_{(n-1)2},\dots,\nu_{(n-1)m}]/I^*_{(n-2),m} \cong S^*_{n-2,m}/I^*_{n-2,m}[\xi_{n+1}]$. By induction on n and [16, Lemma 3.6], $\operatorname{depth}(S^*_{n,m}/(I^*_{n,m}:\xi_{n+1})) = \operatorname{depth}(S^*_{n-2,m}/(I^*_{n-2,m})) + 1 = \lceil\frac{n-2+1}{2}\rceil + 1 = \lceil\frac{n+1}{2}\rceil$.
Now as $(I^*_{n,m},\xi_{n+1}) \cong (I^*_{n-1,m},\xi_{n+1})$, we have $S^*_{n,m}/(I^*_{n,m},\xi_{n+1}) \cong K[\delta_1,\delta_2,\dots,\delta_{n+1},\xi_1$
$,\xi_2,\dots,\xi_n,\omega_1,\omega_2,\dots,\omega_{n+1},\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m}\ \dots\ \mu_{(n+1)1},\mu_{(n+1)2},\dots,$
$\mu_{(n+1)m},\nu_{11},\nu_{12},\dots\nu_{1m},\nu_{21},\nu_{22},\dots,\nu_{2m}\ \dots\ \nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m}]/I^*_{(n-1),m} \cong$
$S^*_{n-1,m}/I^*_{n-1,m}[\delta_{n+1},\omega_{n+1},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m}]$. Thus by induction on n and [16, Lemma 3.6], $\operatorname{depth}(S^*_{n,m}/(I^*_{n,m},\xi_{n+1})) =$
$\operatorname{depth}(S^*_{n-1,m}/(I^*_{n-1,m})) + 2m + 2 = \lceil\frac{n-1+1}{2}\rceil + 2m + 2 = \lceil\frac{n+4}{2}\rceil + 2m$. Since

$\operatorname{depth}(S^*_{n,m}/(I^*_{n,m}, \xi_{n+1})) \geq \operatorname{depth}(S^*_{n,m}/(I^*_{n,m} : \xi_{n+1}))$, so by Depth Lemma

$$\operatorname{depth}(S^*_{n,m}/I^*_{n,m}) = \left\lceil \frac{n+1}{2} \right\rceil.$$

For Stanley depth applying Lemma2.2.2 instead of Depth Lemma on the short exact sequence 3.4, [16, Lemma 3.6] and induction on n we obtain $\operatorname{sdepth}(S^*_{n,m}/(I^*_{n,m}, \xi_{n+1})) \geq \lceil \frac{n+1}{2} \rceil$. And for the upper bound, since $\xi_{n+1} \notin I^*_{n,m}$ by Corollary 2.3.1, we get $\operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) \leq \operatorname{sdepth}(S^*_{n,m}/(I^*_{n,m} : \xi_{n+1}))$. This implies that $\operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) \leq \lceil \frac{n+1}{2} \rceil$. Therefore

$$\operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) = \left\lceil \frac{n+1}{2} \right\rceil.$$

□

Remark 3.1.3. *Clearly for $n \geq 1$ and $m \geq 0$ diam $(D_{n,m}) = n+1$ by Theorem 2.3.3, we have*
$\operatorname{depth}(S^*_{n,m}/I^*_{n,m}), \operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) \geq \lceil \frac{n+2}{3} \rceil$. *While, our proposition 3.1.1 shows that* $\operatorname{depth}(S^*_{n,m}/I^*_{n,m}), \operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) = \lceil \frac{n+1}{2} \rceil$. *Thus we find a better result for depth and Stanley depth of this class of cyclic module.*

Proposition 3.1.4. *Let $n \geq 1$ and $m = 0$. If $n \equiv 0 \pmod 2$, then*

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(S_n/I_n), \operatorname{sdepth}(S_n/I_n) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, then

$$\operatorname{depth}(S_n/I_n) = \operatorname{sdepth}(S_n/I_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. If $1 \leq n \leq 5$, then by using CoCoA it is an uncomplicated exercise to see that the result satisfies. Assume that $n \geq 6$. Let us have a short exact sequence

$$0 \longrightarrow S_n/(I_n : \xi_{n+1}) \xrightarrow{\cdot \xi_{n+1}} S_n/I_n \longrightarrow S_n/(I_n, \xi_{n+1}) \longrightarrow 0, \tag{3.5}$$

using Depth lemma, we get

$$\operatorname{depth}(S_n/I_n) \geq \min\Big\{ \operatorname{depth}(S_n/(I_n : \xi_{n+1})), \operatorname{depth}(S_n/(I_n, \xi_{n+1}))\Big\}.$$

$$(I_n : \xi_{n+1}) = (I^*_{n-2}, \delta_n, \xi_n, \omega_n).$$

Now $S_n/(I_n : \xi_{n+1}) \cong K[\delta_1, \delta_2,\ldots, \delta_{n-1}, \xi_1, \xi_2,\ldots, \xi_{n-1}, \xi_{n+1}, \omega_1, \omega_2,\ldots, \omega_{n-1}]/I^*_{n-2}$ $\cong (S^*_{n-2}/(I^*_{n-2}))[\xi_{n+1}]$. So by proposition 3.1.1 and [16, Lemma 3.6],

$$\operatorname{depth}(S_n/(I_n : \xi_{n+1})) = \operatorname{depth}(S^*_{n-2}/(I^*_{n-2})) + 1 = \left\lceil \frac{n-2+1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil.$$

Also

$$\operatorname{sdepth}(S_n/(I_n : \xi_{n+1})) = \operatorname{sdepth}(S^*_{n-2}/(I^*_{n-2})) + 1 = \left\lceil \frac{n-2+1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil.$$

Now as

$$(I_n, \xi_{n+1}) \cong (I^*_{n-1}, \xi_{n+1}),$$

we have $S_n/(I_n, \xi_{n+1}) \cong K[\delta_1, \delta_2,\ldots, \delta_n, \xi_1, \xi_2,\ldots, \xi_n, \omega_1, \omega_2,\ldots, \omega_n]/I^*_{n-1} \cong S^*_{n-1}/I^*_{n-1}$. So by proposition, 3.1.1

$$\operatorname{depth}(S_n/(I_n, \xi_{n+1})) = \operatorname{depth}(S^*_{n-1}/(I^*_{n-1})) = \left\lceil \frac{n-1+1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$$

And

$$\operatorname{sdepth}(S_n/(I_n, \xi_{n+1})) = \operatorname{sdepth}(S^*_{n-1}/(I^*_{n-1})) = \left\lceil \frac{n-1+1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

Note that if $n \equiv 1 \pmod 2$, then $\operatorname{depth}(S_n/(I_n, \xi_{n+1})) = \operatorname{depth}(S_n/(I_n : \xi_{n+1}))$ so by Depth Lemma $\operatorname{depth}(S_n/I_n) = \lceil \frac{n+1}{2} \rceil$. And if $n \equiv 0 \pmod 2$, then $\operatorname{depth}(S_n/(I_n, \xi_{n+1}))$ $\leq \operatorname{depth}(S_n/(I_n : \xi_{n+1}))$, again by Depth Lemma $\operatorname{depth}(S_n/I_n) \geq \lceil \frac{n}{2} \rceil$. Same result holds for sdepth in both cases. For the upper bound, since $\xi_{n+1} \notin I_n$ by Corollary 2.3.2, we get $\operatorname{depth}(S_n/I_n) \leq \operatorname{depth}(S_n/(I_n : \xi_{n+1}))$, this shows $\operatorname{depth}(S_n/I_n) \leq \lceil \frac{n+1}{2} \rceil$. In a consequence

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(S_n/I_n) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

For Stanley depth if $n \equiv 1 \pmod 2$, then by applying Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.5, we get $\operatorname{sdepth}(S_n/(I_n) \geq \lceil \frac{n+1}{2} \rceil$. For the upper bound since $\xi_{n+1} \notin I_n$ by Corollary 2.3.1, we have $\operatorname{sdepth}(S_n/I_n) \leq$ $\operatorname{sdepth}(S_n/(I_n : \xi_{n+1}))$. This exhibits that $\operatorname{sdepth}(S_n/I_n) \leq \lceil \frac{n+1}{2} \rceil$. Therefore

$$\operatorname{sdepth}(S_n/I_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Moreover, if $n \equiv 0 \pmod 2$, then by applying Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.5, we have $\operatorname{sdepth}(S_n/(I_n)) \geq \lceil \frac{n}{2} \rceil$. Eventually

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{sdepth}(S_n/I_n) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

□

Remark 3.1.5. *In this case i.e., for $m = 0$ the B_n graph is equivalent to union of n copies of kite graph, where kite graph is shown as in figure* 3.6*.*

Remark 3.1.6. *Clearly diam$(B_n) = n$ by Theorem 2.3.3, we have* $\operatorname{depth}(S_n/I_n)$ $, \operatorname{sdepth}(S_n/I_n) \geq \lceil \frac{n+1}{3} \rceil$*. Our Proposition 3.1.4 shows that if $n \equiv 0 \pmod 2$, then* $\operatorname{depth}(S_n/I_n), \operatorname{sdepth}(S_n/I_n) \geq \lceil \frac{n}{2} \rceil$*. And if $n \equiv 1 \pmod 2$, then* $\operatorname{depth}(S_n/I_n)$, $\operatorname{sdepth}(S_n/I_n) = \lceil \frac{n+1}{2} \rceil$*. Thus in both cases we find a good results for depth and stanley depth of this type of cyclic modules.*

Proposition 3.1.7. *Let $n, m \geq 1$. If $n \equiv 0 \pmod 2$, then*

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m}) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, then

$$\operatorname{depth}(S_{n,m}/I_{n,m}) = \operatorname{sdepth}(S_{n,m}/I_{n,m}) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Assume that $n = 1$. Let us have a short exact sequence

$$0 \longrightarrow S_{1,m}/(I_{1,m} : \xi_2) \xrightarrow{\cdot \xi_2} S_{1,m}/I_{1,m} \longrightarrow S_{1,m}/(I_{1,m}, \xi_2) \longrightarrow 0, \tag{3.6}$$

by Depth lemma

$$\operatorname{depth}(S_{1,m}/I_{1,m}) \geq \min\Big\{ \operatorname{depth}(S_{1,m}/(I_{1,m} : \xi_2)), \operatorname{depth}(S_{1,m}/(I_{1,m}, \xi_2)) \Big\}.$$

$$(I_{1,m} : \xi_2) = (\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, , \nu_{11}, \nu_{12}, \dots, \nu_{1m}).$$

Also $S_{1,m}/(I_{1,m} : \xi_2) \cong K[\xi_2]$. Since $\operatorname{depth} K[\xi_2] = 1$, where $K[\xi_2]$ is the polynomial ring in variable ξ_2 over the field K. So $\operatorname{depth}(S_{1,m}/(I_{1,m} : \xi_2)) = 1$.

Now let $(I_{1,m}, \xi_2) = (\delta_1\xi_1, \xi_1\omega_1, \xi_1\mu_{11}, \xi_1\mu_{12}, \dots, \xi_1\mu_{1m}, \xi_1\nu_{11}, \xi_1\nu_{12}, \dots, \xi_1\nu_{1m}, \xi_2) = (g_{1,m}, \xi_2)$

where $g_{1,m} = (\delta_1\xi_1, \xi_1\omega_1, \xi_1\mu_{11}, \xi_1\mu_{12}, \dots, \xi_1\mu_{1m}, \xi_1\nu_{11}, \xi_1\nu_{12}, \dots, \xi_1\nu_{1m})$ is the edge ideal of Star graph. Since depth of quotient module $A_{1,m}/g_{1,m}$ associated to Star graph is 1, where $A_{1,m} = K[\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}]$ is the polynomial ring in these variables over the field k. We have $S_{1,m}/(I_{1,m}, \xi_2) \cong K[\delta_1, \xi_1, \omega_1, \mu_{11},$ $\mu_{12}, \dots, \mu_{1m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}]/g_{1,m} \cong A_{1,m}/g_{1,m}$. Thus by depth of quotient module of star graph $\text{depth}(S_{1,m}/(I_{1,m}, \xi_2)) = 1$. Since $\text{depth}(S_{1,m}/(I_{1,m}, \xi_2)) = \text{depth}(S_{1,m}/(I_{1,m} : \xi_2)$, so by Depth Lemma $\text{depth}(S_{1,m}/I_{1,m}) = 1 = \lceil \frac{n+1}{2} \rceil$.
Now as for as Stanley depth is concerned, applying Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.6 , sdepth of polynomial ring and sdepth of quotient module associated to star graph, we have $\text{sdepth}(S_{1,m}/(I_{1,m}, \xi_2)) \geq 1$. For the upper bound since $\xi_2 \notin I_{1,m}$ by Corollary 2.3.1, we get $\text{sdepth}(S_{1,m}/I_{1,m}) \leq \text{sdepth}(S_{1,m}/(I_{1,m} : \xi_2))$. This implies that $\text{sdepth}(S_{1,m}/I_{1,m}) \leq 1$. As a result

$$\text{sdepth}(S_{1,m}/I_{1,m}) = 1 = \left\lceil \frac{n+1}{2} \right\rceil.$$

When $n = 2$. By taking the short exact sequence

$$0 \longrightarrow S_{2,m}/(I_{2,m} : \xi_3) \xrightarrow{\cdot \xi_3} S_{2,m}/I_{2,m} \longrightarrow S_{2,m}/(I_{2,m}, \xi_3) \longrightarrow 0, \tag{3.7}$$

applying Depth lemma

$$\text{depth}(S_{2,m}/I_{2,m}) \geq \min\Big\{ \text{depth}(S_{2,m}/(I_{2,m} : \xi_3)), \text{depth}(S_{2,m}/(I_{2,m}, \xi_3)) \Big\}.$$

$$(I_{2,m} : \xi_3) = (g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}).$$

We have $S_{2,m}/(I_{2,m} : \xi_3) \cong K[\delta_1, \xi_1, \xi_3, \omega_1, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}]/g_{1,m} \cong A_{1,m}/g_{1,m}[\xi_3]$. By [16, Lemma 3.6], $\text{depth}(S^*_{2,m}/(I^*_{2,m} : \xi_3)) = \text{depth}(A_{1,m}/(g_{1,m}) + 1 = 2$. Now consider $(I_{2,m}, \xi_3) \cong (I^*_{1,m}, \xi_3)$, we have $S_{2,m}/(I_{2,m}, \xi_3) \cong K[\delta_1, \delta_2, \xi_1, \xi_2, \omega_1, \omega_2, \mu_{11}$ $, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}]/I^*_{1,m} \cong S^*_{1,m}/I^*_{1,m}$. Thus by Proposition 3.1.2, $\text{depth}(S_{2,m}/(I_{2,m}, \xi_3)) = 1$. Also by Depth Lemma $\text{depth}(S_{2,m}/I_{2,m}) \geq 1 = \lceil \frac{n}{2} \rceil$. And for the upper bound since $\xi_3 \notin I_{2,m}$ by Corollary 2.3.2, we get $\text{depth}(S_{2,m}/I_{2,m}) \leq \text{depth}(S_{2,m}/(I_{2,m} : \xi_3))$. This shows that $\text{depth}(S_{2,m}/I_{2,m}) \leq 2$. Thus

$$\left\lceil \frac{2}{2} \right\rceil \leq \text{depth}(S_{2,m}/I_{2,m}) \leq \left\lceil \frac{2+1}{2} \right\rceil.$$

For Stanley depth applying Lemma 2.2.2 as an alternative of Depth Lemma on the short exact sequence 3.7, Proposition 3.1.2, [16, Lemma 3.6] and sdepth of quotient module associated to star graph. We have $\operatorname{sdepth}(S_{2,m}/(I_{2,m}, \xi_3)) \geq 1$. For the upper bound since $\xi_3 \notin I_{2,m}$ by Corollary 2.3.1, we get $\operatorname{sdepth}(S_{2,m}/I_{2,m}) \leq \operatorname{sdepth}(S_{2,m}/(I_{2,m} : \xi_3))$. This shows that $\operatorname{sdepth}(S_{2,m}/I_{2,m}) \leq 2$. So

$$\left\lceil \frac{2}{2} \right\rceil \leq \operatorname{sdepth}(S_{2,m}/I_{2,m}) \leq \left\lceil \frac{2+1}{2} \right\rceil.$$

Finally for $n \geq 3$. consider the short exact sequence

$$0 \longrightarrow S_{n,m}/(I_{n,m} : \xi_{n+1}) \xrightarrow{\cdot \xi_{n+1}} S_{n,m}/I_{n,m} \longrightarrow S_{n,m}/(I_{n,m}, \xi_{n+1}) \longrightarrow 0, \tag{3.8}$$

by using Depth lemma

$$\operatorname{depth}(S_{n,m}/I_{n,m}) \geq \min\Big\{ \operatorname{depth}(S_{n,m}/(I_{n,m} : \xi_{n+1})), \operatorname{depth}(S_{n,m}/(I_{n,m}, \xi_{n+1})) \Big\}.$$

$(I_{n,m} : \xi_{n+1}) = (I^*_{(n-2),m}, \delta_n, \xi_n, \omega_n, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm})$.
Also we have $S_{n,m}/(I_{n,m} : \xi_{n+1}) \cong K[\delta_1, \delta_2, \dots, \delta_{n-1}, \xi_1, \xi_2, \dots, \xi_{n-1}, \xi_{n+1}, \omega_1, \omega_2, \dots,$ $\omega_{n-1}, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m} \; \dots \; \mu_{(n-1)1}, \mu_{(n-1)2}, \dots, \mu_{(n-1)m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m},$ $\nu_{21}, \nu_{22}, \dots, \nu_{2m} \; \dots \; \nu_{(n-1)1}, \nu_{(n-1)2}, \dots, \nu_{(n-1)m}]/I^*_{(n-2),m} \cong (S^*_{n-2,m}/(I^*_{n-2,m}))[\xi_{n+1}]$. By Proposition 3.1.2 and [16, Lemma 3.6],

$$\operatorname{depth}(S_{n,m}/(I_{n,m} : \xi_{n+1})) = \operatorname{depth}(S^*_{n-2,m}/(I^*_{n-2,m})) + 1 = \left\lceil \frac{n-2+1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil.$$

And

$$\operatorname{sdepth}(S_{n,m}/(I_{n,m} : \xi_{n+1})) = \operatorname{sdepth}(S^*_{n-2,m}/(I^*_{n-2,m})) + 1 = \left\lceil \frac{n-2+1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil.$$

Now as $(I_{n,m}, \xi_{n+1}) \cong (I^*_{n-1,m}, \xi_{n+1})$, we have $S_{n,m}/(I_{n,m}, \xi_{n+1}) \cong K[\delta_1, \delta_2, \dots, \delta_n, \xi_1, \xi_2,$ $\dots, \xi_n, \omega_1, \omega_2, \dots, \omega_n, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m} \; \dots \; \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \nu_{11}, \nu_{12}, \dots,$ $\nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m} \; \dots \; \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}]/I^*_{n-1,m} \cong S^*_{n-1,m}/I^*_{n-1,m}$. So by Proposition 3.1.2, $\operatorname{depth}(S_{n,m}/(I_{n,m}, \xi_{n+1})) = \operatorname{depth}(S^*_{n-1,m}/(I^*_{n-1,m})) = \lceil \frac{n-1+1}{2} \rceil = \lceil \frac{n}{2} \rceil$. And $\operatorname{sdepth}(S_{n,m}/(I_{n,m}, \xi_{n+1})) = \operatorname{sdepth}(S^*_{n-1,m}/(I^*_{n-1,m})) = \lceil \frac{n-1+1}{2} \rceil = \lceil \frac{n}{2} \rceil$.
Note that if $n \equiv 1 \pmod 2$, then $\operatorname{depth}(S^*_{n,m}/(I^*_{n,m}, \xi_{n+1})) = \operatorname{depth}(S^*_{n,m}/(I^*_{n,m} : \xi_{n+1}))$, so by Depth Lemma $\operatorname{depth}(S_{n,m}/I_{n,m}) = \lceil \frac{n+1}{2} \rceil$.

And if $n \equiv 0 \pmod 2$, then $\operatorname{depth}(S^*_{n,m}/(I^*_{n,m}, \xi_{n+1})) \leq \operatorname{depth}(S^*_{n,m}/(I^*_{n,m} : \xi_{n+1}))$. This implies that $\operatorname{depth}(S_{n,m}/I_{n,m}) \geq \lceil \frac{n}{2} \rceil$. For the upper bound since $\xi_{n+1} \notin I_{n,m}$ by Corollary 2.3.2, we get $\operatorname{depth}(S_{n,m}/I_{n,m}) \leq \operatorname{depth}(S_{n,m}/(I_{n,m} : \xi_{n+1}))$. This implies that $\operatorname{depth}(S_{n,m}/I_{n,m}) \leq \lceil \frac{n+1}{2} \rceil$. Hence

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(S_{n,m}/I_{n,m}) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

For Stanley depth if $n \equiv 1 \pmod 2$, then replacing Depth Lemma by Lemma 2.2.2 on the short exact sequence 3.8, we get $\operatorname{sdepth}(S_{n,m}/(I_{n,m}) \geq \lceil \frac{n+1}{2} \rceil$. For the upper bound since $\xi_{n+1} \notin I_{n,m}$ by Corollary2.3.1, we get $\operatorname{sdepth}(S_{n,m}/I_{n,m}) \leq \operatorname{sdepth}(S_{n,m}/(I_{n,m} : \xi_{n+1}))$. This implies that $\operatorname{sdepth}(S_{n,m}/I_{n,m}) \leq \lceil \frac{n+1}{2} \rceil$. Thus

$$\operatorname{sdepth}(S_{n,m}/I_{n,m}) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Moreover, if $n \equiv 0 \pmod 2$, then by applying Lemma 2.2.2 as an alternative of Depth Lemma on the short exact sequence 3.8, we have $\operatorname{sdepth}(S_{n,m}/(I_{n,m}) \geq \lceil \frac{n}{2} \rceil$. To conclude

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{sdepth}(S_{n,m}/I_{n,m}) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

□

Remark 3.1.8. *Apparently for $n = 1$ diam$(B_{1,m}) = 2$ and for $n \geq 2$ diam$(B_{n,m}) = n$. So by Theorem 2.3.3, for $n = 1$ we have* $\operatorname{depth}(S_{1,m}/I_{1,m}), \operatorname{sdepth}(S_{1,m}/I_{1,m}) \geq \lceil \frac{2+1}{3} \rceil$ *and for $n \geq 2$ we have* $\operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m}) \geq \lceil \frac{n+1}{3} \rceil$. *Our Proposition3.1.7 shows that if $n \equiv 0 \pmod 2$, then* $\operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m}) \geq \lceil \frac{n}{2} \rceil$. *And if $n \equiv 1 \pmod 2$, then* $\operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m}) = \lceil \frac{n+1}{2} \rceil$. *Thus in both cases we find a sharp results for depth and Stanley depth of this class of quotient module.*

If we fuse vertices ξ_1 and ξ_{n+1} in graph $B_{n,m}$, we get a new graph called $E_{n,m}$ graph. For example of this new graph, see Figure 3.7. Distinctly $|V(E_{n,m})|= 2nm + 3n$. Furthermore, The graph $E_{n,m}$ has $2n$ and $2nm$ vertices of degree 2 and n vertices of degree $4m + 6$ so by using Lemma 1.4.8, we have $|E(E_{n,m}|= 4nm + 5n$. Also for $m \geq 0$ and $n \geq 3$, $E_{n,m}$ graph has diameter $d = \lceil \frac{n+1}{2} \rceil$.

We label the vertices of the $E_{n,m}$ graph by using five sets of variables $\{\delta_1, \delta_2, \ldots, \delta_n\}$, $\{\xi_2, \ldots, \xi_n, y\}$, $\{\omega_1, \omega_2, \ldots, \omega_n\}$, $\Big\{\{\mu_{11}, \mu_{12}, \ldots, \mu_{1m}\}, \{\mu_{21}, \mu_{22}, \ldots, \mu_{2m}\}, \ldots, \{\mu_{n1}, \mu_{n2}, \ldots, \mu_{nm}\}\Big\}$ and $\Big\{\{\nu_{11}, \nu_{12}, \ldots \nu_{1m}\}, \{\nu_{21}, \nu_{22}, \ldots, \nu_{2m}\}, \ldots, \{\nu_{n1}, \nu_{n2}, \ldots, \nu_{nm}\Big\}$. Let $T_{n,m} := K[\delta_1, \delta_2, \ldots, \delta_n, y, \xi_2, \ldots, \xi_n, \omega_1, \omega_2, \ldots, \omega_n, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{21}, \mu_{22}, \ldots, \mu_{2m}, \ldots, \mu_{n1}, \mu_{n2}, \ldots, \mu_{nm}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{21}, \nu_{22}, \ldots, \nu_{2m} \ldots, \nu_{n1}, \nu_{n2}, \ldots, \nu_{nm}]$ be the ring of polynomial in these variables over the field K. Then $J_{n,m}$ is squarefree monomial ideal of $T_{n,m}$. Now with the labelling as shown in figure 3.7, we have:
$\mathcal{M}(J_{n,m}) = \Big\{ \bigcup_{i=2}^{n-1}\{\xi_i\xi_{i+1}\} \bigcup\{y\xi_2\} \bigcup\{y\xi_n\} \cup \bigcup_{i=2}^{n}\{\delta_i\xi_i\} \bigcup\{\delta_1 y\} \cup \bigcup_{i=2}^{n}\{\xi_i\omega_i\} \bigcup\{y\omega_1\} \bigcup \bigcup_{i=1}^{n-1}\{\delta_i\xi_{i+1}\} \bigcup\{\delta_n y\} \cup \bigcup_{i=1}^{n-1}\{\omega_i\xi_{i+1}\} \bigcup\{y\omega_n\} \cup \bigcup_{i=2}^{n}\{\xi_i\mu_{i1}\} \bigcup\{y\mu_{11}\} \cup \bigcup_{i=2}^{n}\{\xi_i\mu_{i2}\} \bigcup\{y\mu_{12}\} \bigcup, \ldots, \cup \bigcup_{i=1}^{n}\{\xi_i\mu_{im}\} \bigcup\{y\mu_{1m}\} \cup \bigcup_{i=1}^{n}\{\xi_i\nu_{i1}\} \bigcup\{y\nu_{11}\} \cup \bigcup_{i=1}^{n}\{\xi_i\nu_{i2}\} \bigcup\{y\nu_{12}\} \bigcup, \ldots, \bigcup \bigcup_{i=1}^{n}\{\xi_i\nu_{im}\} \bigcup\{y\nu_{1m}\} \cup \bigcup_{i=1}^{n-1}\{\xi_{i+1}\mu_{i1}\} \bigcup\{y\mu_{n1}\} \cup \bigcup_{i=1}^{n-1}\{\xi_{i+1}\mu_{i2}\} \bigcup\{y\mu_{n2}\} \bigcup, \ldots, \bigcup \bigcup_{i=1}^{n-1}\{\xi_{i+1}\mu_{im}\} \bigcup\{y\mu_{nm}\} \cup \bigcup_{i=1}^{n-1}\{\xi_{i+1}\nu_{i1}\} \bigcup\{y\nu_{n1}\} \cup \bigcup_{i=1}^{n-1}\{\xi_{i+1}\nu_{i2}\} \bigcup\{y\nu_{n2}\} \bigcup, \ldots, \bigcup \bigcup_{i=1}^{n-1}\{\xi_{i+1}\nu_{im}\} \bigcup\{y\nu_{nm}\}\Big\}$. Where $\mathcal{M}(J_{n,m})$ stands for the minimal set of monomial generators of monomial ideal $J_{n,m}$.

Let us first consider super graph $F_{n,m}$ of the graph $D_{n,m}$. The vertex and edge sets of $F_{n,m}$ are $V(F_{n,m}) = V(D_{n,m}) \bigcup\{a_1, b_1, q_1, q_2, \ldots, q_m, r_1, r_2, \ldots, r_m\}$ and$E(F_{n,m}) = E(D_{n,m}) \bigcup\{a_1\xi_{n+1}, \xi_1 q_1, \xi_1 q_2, \ldots, \xi_1 q_m, \xi_1 r_1, \xi_1 r_2, \ldots, \xi_1 r_m\}$. For example of graph $F_{n,m}$ see Figure 3.8. We denote the edge ideal of graph $F_{n.m}$ with $J^*_{n,m}$, where $J^*_{n,m}$ is the monomial ideal of the polynomial ring $T^*_{n,m} = S^*_{n,m}[a_1, b_1, q_1, q_2, \ldots, q_m, r_1, r_2, \ldots, r_m]$. The minimal set of monomial generators of $J^*_{n,m}$ is $\mathcal{M}(J^*_{n,m}) = \mathcal{M}(I^*_{n,m}) \bigcup\{a_1\xi_1, \xi_1 b_1\} \bigcup \bigcup_{i=1}^{m}\{\xi_1 q_{(n+2)i}\} \bigcup \bigcup_{i=1}^{m}\{\xi_1 r_{(n+2)i}\}$.

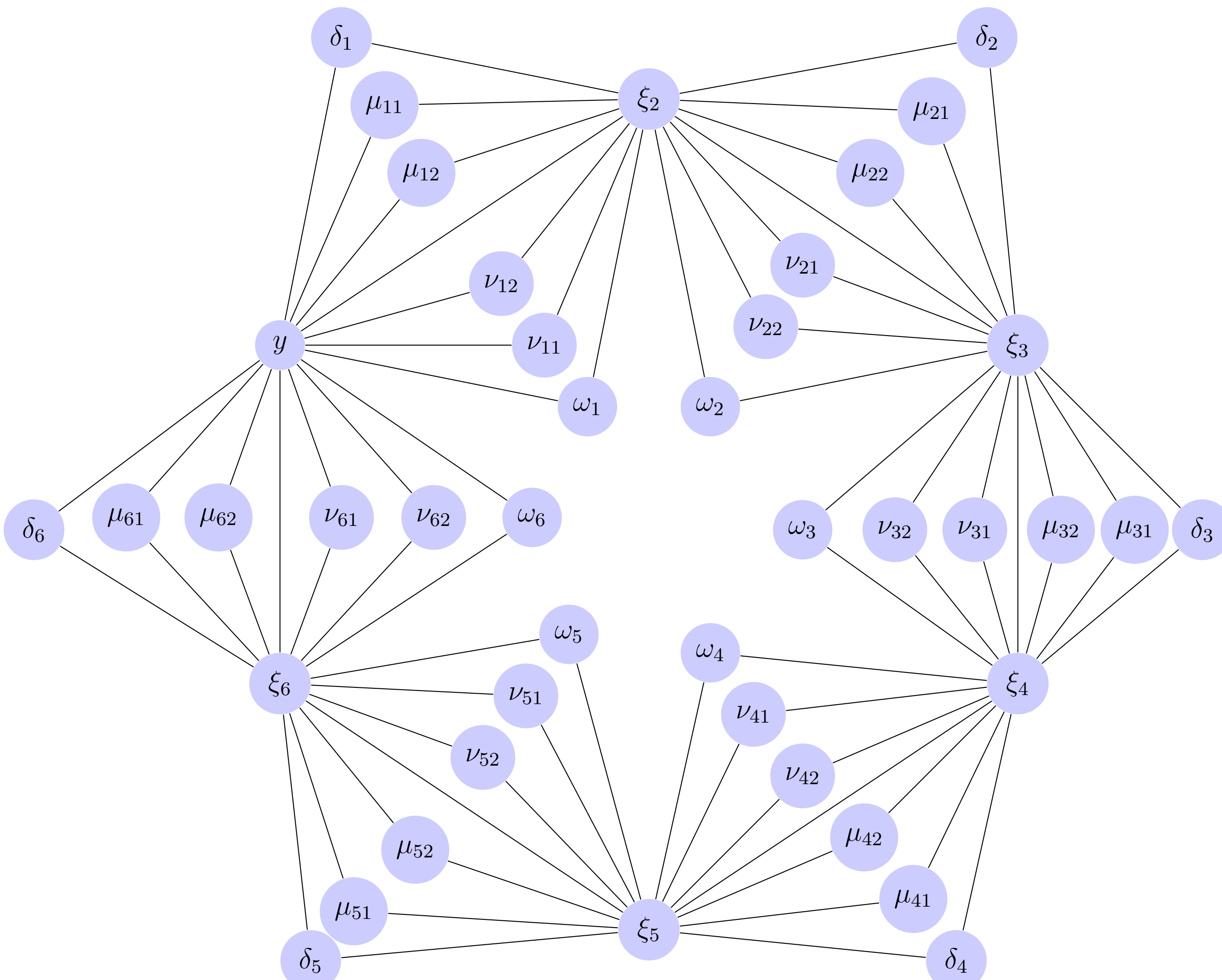

Figure 3.7: $(E_{6,2})$ Graph

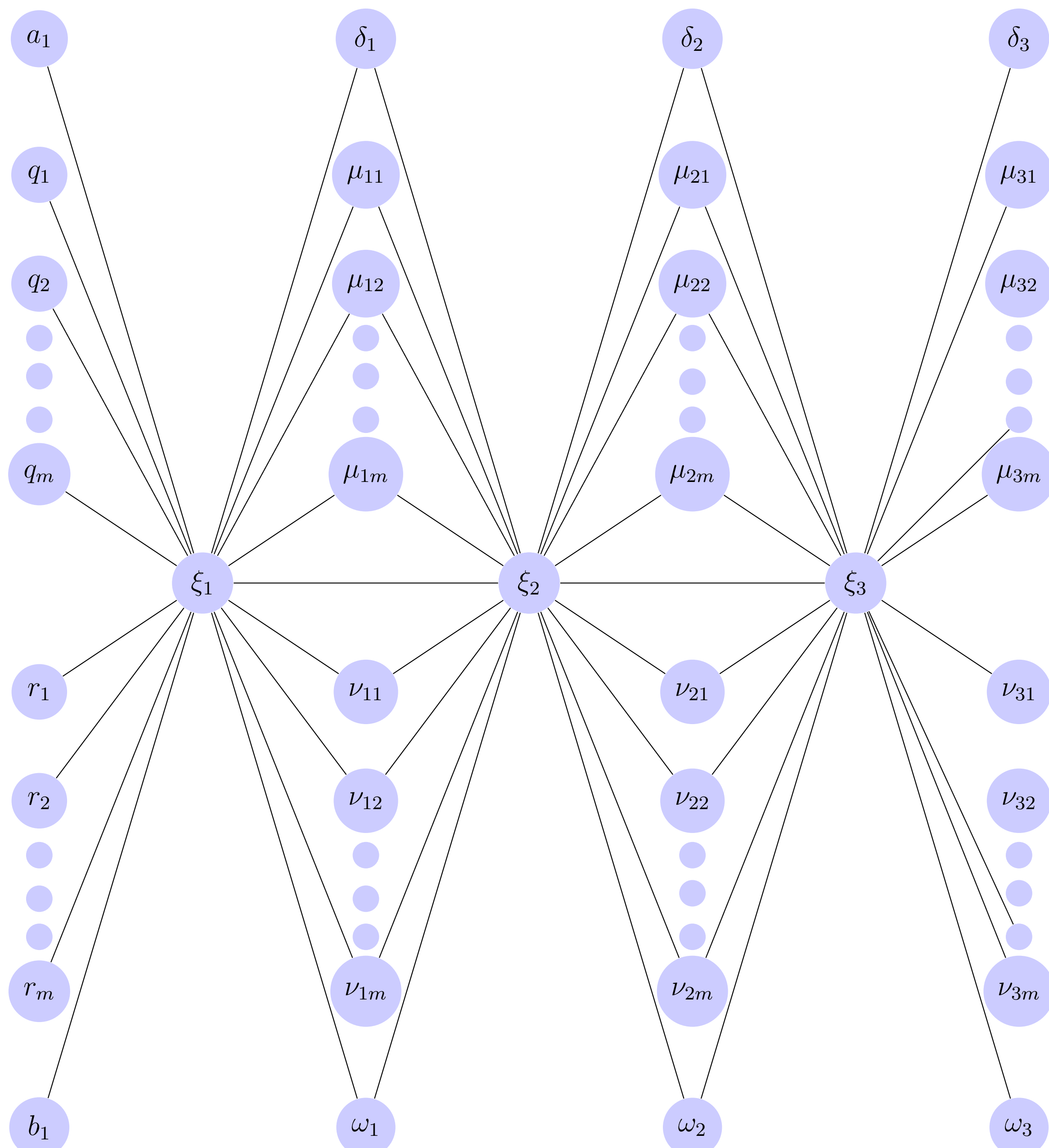

Figure 3.8: $(F_{2,m})$ Graph

3.2 Depth and Stanley depth of the cyclic module associated to the graph $E_{n,m}$

We find out the value of depth and Stanley depth of the cyclic module $T_{n,m}/J_{n,m}$ associated to the graph $E_{n,m}$. For this motive, we first find depth and Stanley depth of the cyclic module $T^*_{n,m}/J^*_{n,m}$ associated to the super graph $F_{n,m}$ of $D_{n,m}$ graph, where $D_{n,m}$ graph is a super graph of graph the $B_{n,m}$. These results will be used in our main proof.

Proposition 3.2.1. *Let $n \geq 1$ and $m \geq 0$. If $n \equiv 0 \pmod 2$, then*

$$\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = \operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) = \left\lceil \frac{n+2}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, then

$$\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = \operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) = 2m + \left\lceil \frac{n+5}{2} \right\rceil.$$

Proof. For $n = 1$. We have a short exact sequence

$$0 \longrightarrow T^*_{1,m}/(J^*_{1,m} : \xi_2) \xrightarrow{\cdot \xi_2} T^*_{1,m}/J^*_{1,m} \longrightarrow T^*_{1,m}/(J^*_{1,m}, \xi_2) \longrightarrow 0, \tag{3.9}$$

by Depth lemma

$$\operatorname{depth}(T^*_{1,m}/J^*_{1,m}) \geq \min\Big\{ \operatorname{depth}(T^*_{1,m}/(J^*_{1,m} : \xi_2)), \operatorname{depth}(T^*_{1,m}/(J^*_{1,m}, \xi_2)) \Big\}.$$

$(J^*_{1,m} : \xi_2) = (\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, , \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m})$.

We have $T^*_{1,m}/(J^*_{1,m} : \xi_2) \cong K[\xi_2, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]$. Since depth of polynomial ring $K[\xi_2, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m] = 2m+3$, so $\operatorname{depth}(T^*_{1,m}/(J^*_{1,m} : \xi_2)) = 2m + 3$.

Now let $(J^*_{1,m}, \xi_2) = (g^*_{1,m}, \xi_2)$,

where $g^*_{1,m} = (g_{1,m}, a_1\xi_1, \xi_1 b_1, \xi_1 q_1, \xi_1 q_2, \dots, \xi_1 q_m, \xi_1 r_1, \xi_1 r_2, \dots, \xi_1 r_m)$ is the edge ideal of Star graph. Since depth of quotient module $A^*_{1,m}/g^*_{1,m}$ associated to Star graph is 1, where $A^*_{1,m} = A_{1,m}[a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]$ is the polynomial ring in these

variables over the field k. We have $T^*_{1,m}/(J^*_{1,m},\xi_2) \cong K[\delta_1,\delta_2,\xi_1,\omega_1,\omega_2\mu_{11},\mu_{12},\ldots,$ $\mu_{1m},\mu_{21},\mu_{22},\ldots,\mu_{2m},\nu_{11},\nu_{12},\ldots,\nu_{1m},\nu_{21},\nu_{22},\ldots,\nu_{2m},a_1,b_1,q_1,q_2,\ldots,q_m,r_1,r_2,\ldots,r_m]$ $/g^*_{1,m} \cong A^*_{1,m}/g^*_{1,m}[\delta_2,\omega_2,\mu_{21},\mu_{22},\ldots,\mu_{2m},\nu_{21},\nu_{22},\ldots,\nu_{2m}]$. Thus by depth of star graph and [16, Lemma 3.6], $\operatorname{depth}(T^*_{1,m}/(J^*_{1,m},\xi_2)) = 1+2m+2 = 2m+3$. Also since $\operatorname{depth}(T^*_{1,m}/(J^*_{1,m},\xi_2)) = \operatorname{depth}(T^*_{1,m}/(J^*_{1,m}:\xi_2))$, by Depth Lemma $\operatorname{depth}(T^*_{1,m}/J^*_{1,m}) = 2m+3 = 2m+\lceil\frac{1+5}{2}\rceil$.
For Stanley depth applying Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.9, [16, Lemma 3.6], sdepth of polynomial ring in $2m+3$ variables and sdepth of quotient module associated to star graph. We have $\operatorname{sdepth}(T^*_{1,m}/(J^*_{1,m})) \geq 2m+3$. For the upper bound since $\xi_2 \notin J^*_{1,m}$ by Corollary 2.3.1, we get $\operatorname{sdepth}(T^*_{1,m}/J^*_{1,m}) \leq \operatorname{sdepth}(T^*_{1,m}/(J^*_{1,m}:\xi_2))$. This implies $\operatorname{sdepth}(T^*_{1,m}/J^*_{1,m}) \leq 2m+3$. Thus

$$\operatorname{sdepth}(T^*_{1,m}/J^*_{1,m}) = 2m+3 = 2m+\left\lceil\frac{1+5}{2}\right\rceil.$$

When $n=2$. We have a short exact sequence

$$0 \longrightarrow T^*_{2,m}/(J^*_{2,m}:\xi_3) \xrightarrow{\cdot\xi_3} T^*_{2,m}/J^*_{2,m} \longrightarrow T^*_{2,m}/(J^*_{2,m},\xi_3) \longrightarrow 0, \tag{3.10}$$

applying Depth lemma

$$\operatorname{depth}(T^*_{2,m}/J^*_{2,m}) \geq \min\Big\{\operatorname{depth}(T^*_{2,m}/(J^*_{2,m}:\xi_3)), \operatorname{depth}(T^*_{2,m}/(J^*_{2,m},\xi_3))\Big\}.$$

$(J^*_{2,m}:\xi_3) = (g^*_{1,m},\delta_2,\delta_3,\xi_2,\omega_2,\omega_3,\mu_{21},\mu_{22},\ldots,\mu_{2m},\mu_{31},\ldots,\mu_{3m},\nu_{21},\nu_{22},\ldots,\nu_{2m},\nu_{31},$ $\nu_{32},\ldots,\nu_{3m})$.
Also we have $T^*_{2,m}/(J^*_{2,m}:\xi_3) \cong K[\delta_1,\xi_1,\xi_3,\omega_1,\mu_{11},\mu_{12},\ldots,\mu_{1m},\nu_{11},\nu_{12},\ldots,\nu_{1m},a_1,b_1,$ $q_1,q_2,\ldots,q_m,r_1,r_2,\ldots,r_m]/g^*_{1,m} \cong A^*_{1,m}/g^*_{1,m}[\xi_3]$. Now by [16, Lemma 3.6],
$\operatorname{depth}(T^*_{2,m}/(J^*_{2,m}:\xi_3)) = \operatorname{depth}(A^*_{1,m}/(g^*_{1,m}))+1 = 2$.
As $(J^*_{2,m},\xi_3) \cong (J^*_{1,m},\xi_3)$, we get $T^*_{2,m}/(J^*_{2,m},\xi_3) \cong K[\delta_1,\delta_2,\delta_3,\xi_1,\xi_2,\omega_1,$ $\omega_2,\omega_3,\mu_{11},\mu_{12},\ldots,\mu_{1m},\mu_{21},\mu_{22},\ldots,\mu_{2m},\mu_{31},\mu_{32},\ldots,\mu_{3m},\nu_{11},\nu_{12},\ldots,\nu_{1m},\nu_{21},\nu_{22},\ldots,\nu_{2m},$ $\nu_{31},\nu_{32},\ldots,\nu_{3m},,a_1,b_1,q_1,q_2,\ldots,q_m,r_1,r_2,\ldots,r_m]/J^*_{1,m} \cong T^*_{1,m}/J^*_{1,m}[\delta_3,\omega_3,\mu_{31},\mu_{32},\ldots,$ $\mu_{3m},\nu_{31},\nu_{32},\ldots,\nu_{3m}]$. Hence by induction on n and [16, Lemma 3.6],
$\operatorname{depth}(T^*_{2,m}/(J^*_{2,m},\xi_3)) = 2m+3+2m+2$. Moreover, since $\operatorname{depth}(T^*_{2,m}/(J^*_{2,m},\xi_3)) \geq \operatorname{depth}(T^*_{2,m}/(J^*_{2,m}:\xi_3))$, using Depth Lemma $\operatorname{depth}(T^*_{2,m}/J^*_{2,m}) = 2 = \lceil\frac{2+2}{2}\rceil$.

As for as Stanley depth is concerned using Lemma 2.2.2 instead of Depth Lemma on the exact sequence 3.10, by induction on n [16, Lemma 3.6] and sdepth of quotient module associated to star graph, we have $\operatorname{sdepth}(T^*_{2,m}/(J^*_{2,m})) \geq 2$. For the upper bound since $\xi_3 \notin J^*_{2,m}$ by Corollary 2.3.1, we get $\operatorname{sdepth}(T^*_{2,m}/J^*_{2,m}) \leq \operatorname{sdepth}(T^*_{2,m}/(J^*_{2,m} : \xi_3))$. This implies that $\operatorname{sdepth}(T^*_{2,m}/J^*_{2,m}) \leq 2$. As a consequent $\operatorname{sdepth}(T^*_{2,m}/J^*_{2,m}) = 2 = \lceil \frac{2+2}{2} \rceil$. Now let us assume $n = 3$. And examine the short exact sequence

$$0 \longrightarrow T^*_{3,m}/(J^*_{n,m} : \xi_4) \xrightarrow{\cdot \xi_4} T^*_{4,m}/J^*_{4,m} \longrightarrow T^*_{3,m}/(J^*_{3,m}, \xi_4) \longrightarrow 0, \tag{3.11}$$

applying Depth lemma

$$\operatorname{depth}(T^*_{3,m}/J^*_{3,m}) \geq \min \Big\{ \operatorname{depth}(T^*_{3,m}/(J^*_{3,m} : \xi_4)), \operatorname{depth}(T^*_{3,m}/(J^*_{3,m}, \xi_4)) \Big\}.$$

$(J^*_{3,m} : \xi_4) = (J^*_{1,m}, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m})$.
We get $T^*_{3,m}/(J^*_{3,m} : \xi_4) \cong K[\delta_1, \delta_2, \xi_1, \xi_2, \xi_4, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]/J^*_{1,m} \cong T^*_{1,m}/J^*_{1,m}[\xi_4]$. By induction on n and [16, Lemma 3.6], $\operatorname{depth}(T^*_{3,m}/(J^*_{3,m} : \xi_4)) = \operatorname{depth}(T^*_{1,m}/(J^*_{1,m})) + 1$. And $\operatorname{sdepth}(T^*_{3,m}/J^*_{3,m} : \xi_4) = \operatorname{sdepth}(T^*_{1,m}/J^*_{1,m}) + 1 = 2m + 3 + 1 = 2m + 4$. Now as $(J^*_{3,m}, \xi_4) \cong (J^*_{2,m}, \xi_4)$, we have $T^*_{3,m}/(J^*_{3,m}, \xi_4) \cong K[\delta_1, \delta_2, \dots, \delta_4, \xi_1, \xi_2, \dots, \xi_3, \omega_1, \omega_2, \dots, \omega_4, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m} \ \dots \ \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m} \ \dots \ \nu_{41}, \nu_{42}, \dots, \nu_{4m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]/J^*_{2,m} \cong T^*_{2,m}/J^*_{2,m}[\delta_4, \omega_4, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{41}, \nu_{42}, \dots \nu_{4m}]$. Therefore by induction on n and [16, Lemma 3.6], $\operatorname{depth}(T^*_{3,m}/(J^*_{3,m}, \xi_4)) = \operatorname{depth}(T^*_{2,m}/(J^*_{2,m})) + 2m + 2 = 2 + 2m + 2 = 2m + 4$. As $\operatorname{depth}(T^*_{3,m}/(J^*_{3,m}, \xi_4)) = \operatorname{depth}(T^*_{3,m}/(J^*_{3,m} : \xi_4))$ and by Depth Lemma $\operatorname{depth}(T^*_{3,m}/J^*_{3,m}) = 2m + 4 = 2m + \lceil \frac{3+5}{2} \rceil$. For Stanley depth applying Lemma 2.2.2 as an alternative of Depth Lemma on the short exact sequence 3.11, [16, Lemma 3.6] and induction on n we have $\operatorname{sdepth}(T^*_{3,m}/(J^*_{3,m})) \geq 2m + 4$. For the upper bound as $\xi_4 \notin J^*_{3,m}$ by Corollary 2.3.1, we get
$\operatorname{sdepth}(T^*_{3,m}/J^*_{3,m}) \leq \operatorname{sdepth}(T^*_{3,m}/(J^*_{3,m} : \xi_4))$. This implies $\operatorname{sdepth}(T^*_{3,m}/J^*_{3,m}) \leq 2m + 4$. Thus

$$\operatorname{sdepth}(T^*_{3,m}/J^*_{3,m}) = 2m + 4 = 2m + \left\lceil \frac{n+5}{2} \right\rceil.$$

Now in general, for $n \geq 4$. Consider the short exact sequence

$$0 \longrightarrow T^*_{n,m}/(J^*_{n,m} : \xi_{n+1}) \xrightarrow{\cdot \xi_{n+1}} T^*_{n,m}/J^*_{n,m} \longrightarrow T^*_{n,m}/(J^*_{n,m}, \xi_{n+1}) \longrightarrow 0, \tag{3.12}$$

using Depth Lemma

$$\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) \geq \min\Big\{ \operatorname{depth}(T^*_{n,m}/(J^*_{n,m} : \xi_{n+1})), \operatorname{depth}(T^*_{n,m}/(J^*_{n,m}, \xi_{n+1}))\Big\}.$$

$(J^*_{n,m} : \xi_{n+1}) = (J^*_{(n-2),m}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m})$. We have $T^*_{n,m}/(J^*_{n,m} : \xi_{n+1}) \cong K[\delta_1, \delta_2, \dots, \delta_{n-1}, \xi_1, \xi_2, \dots, \xi_{n-1}, \xi_{n+1}, \omega_1, \omega_2, \dots, \omega_{n-1}, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m} \ \dots \ \mu_{(n-1)1}, \mu_{(n-1)2}, \dots, \mu_{(n-1)m} \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m} \ \dots \ \nu_{(n-1)1}, \nu_{(n-1)2}, \dots, \nu_{(n-1)m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]/J^*_{n-2,m} \cong (T^*_{n-2,m}/(J^*_{n-2,m}))[\xi_{n+1}]$. By induction on n and [16, Lemma 3.6], $\operatorname{depth}(T^*_{n,m}/(J^*_{n,m} : \xi_{n+1})) = \operatorname{depth}(T^*_{n-2,m}/(J^*_{n-2,m})) + 1$.

Note that if $n \equiv 0 \pmod 2$, then $n-2 \equiv 0 \pmod 2$. This implies $\operatorname{depth}(T^*_{n,m}/J^*_{n,m} : \xi_{n+1})) = \operatorname{depth}(T^*_{n-2,m}/J^*_{n-2,m}) + 1 = \lceil \frac{n-2+2}{2} \rceil + 1 = \lceil \frac{n+2}{2} \rceil$.

And if $n \equiv 1 \pmod 2$, then $n-2 \equiv 1 \pmod 2$. This implies $\operatorname{depth}(T^*_{n,m}/J^*_{n,m} : \xi_{n+1}) = \operatorname{depth}(T^*_{n-2,m}/J^*_{n-2,m}) + 1 = 2m + \lceil \frac{n-2+5}{2} \rceil + 1 = 2m + \lceil \frac{n+5}{2} \rceil$.

Now as $(J^*_{n,m}, \xi_{n+1}) \cong (J^*_{n-1,m}, \xi_{n+1})$, we have $T^*_{n,m}/(J^*_{n,m}, \xi_{n+1}) \cong K[\delta_1, \delta_2, \dots, \delta_{n+1}, \xi_1, \xi_2, \dots, \xi_n, \omega_1, \omega_2, \dots, \omega_{n+1}, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m} \ \dots \ \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m} \ \dots \ \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]/J^*_{(n-1),m} \cong T^*_{n-1,m}/J^*_{n-1,m}[\delta_{n+1}, \omega_{n+1}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots \nu_{(n+1)m}]$. Thus by induction on n and [16, Lemma 3.6], $\operatorname{depth}(T^*_{n,m}/(J^*_{n,m}, \xi_{n+1})) = \operatorname{depth}(T^*_{n-1,m}/(J^*_{n-1,m})) + 2m + 2$.

Now if $n \equiv 0 \pmod 2$, then $n-1 \equiv 1 \pmod 2$. This implies $\operatorname{depth}(T^*_{n,m}/(J^*_{n,m}, \xi_{n+1})) = \operatorname{sdepth}(T^*_{n-1,m}/J^*_{n-1,m}) + 2m + 2 = 2m + \lceil \frac{n-1+5}{2} \rceil + 2m + 2 = 4m + \lceil \frac{n+8}{2} \rceil$. And if $n \equiv 1 \pmod 2$, then $n-1 \equiv 0 \pmod 2$. This implies $\operatorname{depth}(T^*_{n,m}/(J^*_{n,m}, \xi_{n+1})) = \operatorname{depth}(T^*_{n-1,m}/J^*_{n-1,m}) + 2m + 2 = \lceil \frac{n-1+2}{2} \rceil + 2m + 2 = 2m + \lceil \frac{n+5}{2} \rceil$.

To sum up, if $n \equiv 0 \pmod 2$, then $\operatorname{depth}(T^*_{n,m}/(J^*_{n,m}, \xi_{n+1})) \geq \operatorname{depth}(T^*_{n,m}/(J^*_{n,m} : \xi_{n+1}))$, by Depth Lemma $\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = \lceil \frac{n+2}{2} \rceil$. And

if $n \equiv 1 \pmod 2$, then $\operatorname{depth}(T^*_{n,m}/(J^*_{n,m}, \xi_{n+1})) = \operatorname{depth}(T^*_{n,m}/(J^*_{n,m} : \xi_{n+1}))$, by Depth Lemma $\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = 2m + \lceil \frac{n+5}{2} \rceil$.

To examine Stanley depth, apply Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.12, [16, Lemma 3.6], and induction on n we have if $n \equiv 0 \pmod 2$, then $\operatorname{sdepth}(T^*_{n,m}/(J^*_{n,m})) \geq \lceil \frac{n+2}{2} \rceil$. And if $n \equiv 1 \pmod 2$, then $\operatorname{sdepth}(T^*_{n,m}/(J^*_{n,m}) \geq 2m + \lceil \frac{n+5}{2} \rceil$. For the upper bound since $\xi_{n+1} \notin I^*_{n,m}$ by Corollary 2.3.1, we get $\operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) \leq \operatorname{sdepth}(T^*_{n,m}/(J^*_{n,m} : \xi_{n+1}))$. This implies that if $n \equiv 0 \pmod 2$, then $\operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) \leq \lceil \frac{n+2}{2} \rceil$. Thus $\operatorname{sdepth}(T^*_{2,m}/J^*_{2,m}) = \lceil \frac{n+2}{2} \rceil$. And if $n \equiv 1 \pmod 2$, then $\operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) \leq 2m + \lceil \frac{n+5}{2} \rceil$. To conclude, $\operatorname{sdepth}(T^*_{2,m}/J^*_{2,m}) = 2m + \lceil \frac{n+5}{2} \rceil$. □

Remark 3.2.2. *Apparently* $diam(F_{n,m}) = n + 2$ *so by Theorem 2.3.3, we have* depth $(T^*_{n,m}/J^*_{n,m}), \operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) \geq \lceil \frac{n+3}{3} \rceil$. *Whereas, our Proposition 3.2.1 shows that if* $n \equiv 0 \pmod 2$ *then* $\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = \operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) = \lceil \frac{n+2}{2} \rceil$. *And if* $n \equiv 1 \pmod 2$, *then* $\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = \operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) = 2m + \lceil \frac{n+5}{2} \rceil$. *Thus we have better results for depth and stanley depth of this class of cyclic modules.*

Proposition 3.2.3. *Let* $n \geq 3$ *and* $m \geq 0$. *If* $n \equiv 0 \pmod 2$, *then*

$$\operatorname{depth}(T_{n,m}/J_{n,m}) = \operatorname{sdepth}(T_{n,m}/J_{n,m}) = \left\lceil \frac{n}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, *then*

$$\operatorname{depth}(T_{n,m}/J_{n,m}) = \operatorname{sdepth}(T_{n,m}/J_{n,m}) = 2m + \left\lceil \frac{n+3}{2} \right\rceil.$$

Proof. Firstly lets us have $n = 3$. Considering the short exact sequence

$$0 \longrightarrow T_{3,m}/(J_{3,m} : y) \xrightarrow{\cdot y} T_{3,m}/J_{3,m} \longrightarrow T_{3,m}/(J_{3,m}, y) \longrightarrow 0, \tag{3.13}$$

and Depth lemma, we have

$$\operatorname{depth}(T_{3,m}/J_{3,m}) \geq \min\Big\{ \operatorname{depth}(T_{3,m}/(J_{3,m} : y)), \operatorname{depth}(T_{3,m}/(J_{3,m}, y))\Big\}.$$

$(J_{3,m} : y) = (\delta_1, \delta_3, \xi_2, \xi_3, \omega_1, \omega_3, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})$.

We have $T_{3,m}/(J_{3,m} : y) \cong K[\delta_2, y, \omega_2, , \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}]$. Since Depth of polynomial ring $K[\delta_2, y, \omega_2, , \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}] = 2m + 3$, so

depth$(T_{3,m}/(J_{3,m}:y)) = 2m+3$.
Now let $(J_{3,m}, y) = (J^*_{1,m}, y)$, we have $T_{3,m}/(J_{3,m}, y) \cong K[\delta_1, \delta_2, \delta_3, \xi_2, \xi_3, \omega_1, \omega_2, \omega_3, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \dots, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \dots, \nu_{31}, \nu_{32}, \dots, \nu_{3m}]/J^*_{1,m} \cong T^*_{1,m}/J^*_{1,m}$. Thus by Proposition 3.2.1, depth$(T_{3,m}/(J_{3,m}, y)) = 2m+3$. Also since depth$(T_{3,m}/(J_{3,m}, y)) =$ depth$(T_{3,m}/(J_{3,m}:y)$, so by Depth Lemma depth$(T_{3,m}/J_{3,m}) = 2m+3 = 2m + \lceil \frac{3+3}{2} \rceil$.

For Stanley depth applying Lemma 2.2.2 instead of Depth Lemma on the short exact sequence 3.13, [16, Lemma 3.6], Proposition 3.2.1 and sdepth of polynomial ring in $2m+3$ variables. We have sdepth$(T_{3,m}/(J_{3,m})) \geq 2m+3$. For the upper bound since $y \notin J_{3,m}$ by Corollary 2.3.1, we get sdepth$(T_{3,m}/J_{3,m}) \leq$ sdepth$(T_{3,m}/(J_{3,m}:y))$. This implies that sdepth$(T_{3,m}/J_{3,m}) \leq 2m+3$. Thus

$$\text{sdepth}(T_{3,m}/J_{3,m}) = 2m+3 = 2m + \left\lceil \frac{3+3}{2} \right\rceil.$$

Secondly, if $n=4$. let us consider the short exact sequence

$$0 \longrightarrow T_{4,m}/(J_{4,m}:y) \xrightarrow{\cdot y} T_{4,m}/J_{4,m} \longrightarrow T_{4,m}/(J_{4,m}, y) \longrightarrow 0, \tag{3.14}$$

applying Depth lemma, we get

$$\text{depth}(T_{4,m}/J_{4,m}) \geq \min\Big\{ \text{depth}(T_{4,m}/(J_{4,m}:y)), \text{depth}(T_{4,m}/(J_{4,m}, y)) \Big\}.$$

$(J_{4,m}:y) = (g^*_{1,m}, \delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{41}, \dots, \mu_{4m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m})$. We have $T_{4,m}/(J_{4,m}:y) \cong K[\delta_2, \delta_3, y, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}\nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}]/g^*_{1,m} \cong (A^*_{1,m}/g^*_{1,m})[y]$. Now by [16, Lemma 3.6],

$$\text{depth}(T_{4,m}/(J_{4,m}:y)) = \text{depth}(A^*_{1,m}/g^*_{1,m}) + 1 = 2.$$

Further, let $(J_{4,m}, y) \cong (J^*_{2,m}, y)$, we get $T_{4,m}/(J_{4,m}, y) \cong K[\delta_1, \delta_2, \delta_3, \delta_4, \xi_2, \xi_3, \xi_4, \omega_1, \omega_2, \omega_3, \omega_4, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \mu_{41}, \mu_{42}, \dots, \mu_{4m}\nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m}]/J^*_{2,m} \cong T^*_{2,m}/J^*_{2,m}$. Thus by Proposition 3.2.1, depth$(T_{4,m}/(J_{4,m}, y)) = 3$. Also since depth$(T_{4,m}/(J_{4,m}, y)) >$ depth$(T_{4,m}/(J_{4,m}:y)$ so by Depth Lemma depth$(T_{4,m}/J_{4,m}) = 2 = \lceil \frac{4}{2} \rceil$.

Now for Stanley depth using Lemma 2.2.2 at the place of Depth Lemma on the exact sequence 3.14 , [16, Lemma 3.6] and Proposition 3.2.1. We get $\operatorname{sdepth}(T_{4,m}/(J_{4,m}) \geq 2$. For the upper bound since $y \notin J_{4,m}$ by Corollary 2.3.1, we have $\operatorname{sdepth}(T_{4,m}/J_{4,m}) \leq \operatorname{sdepth}(T_{4,m}/(J_{4,m} : y)$. This implies $\operatorname{sdepth}(T_{4,m}/J_{4,m}) \leq 2$. Thus

$$\operatorname{sdepth}(T_{4,m}/J_{4,m}) = 2 = \left\lceil \frac{4}{2} \right\rceil.$$

Thirdly, when $n = 5$. Examine the short exact sequence

$$0 \longrightarrow T_{5,m}/(J_{5,m} : y) \xrightarrow{\cdot y} T_{5,m}/J_{5,m} \longrightarrow T_{5,m}/(J_{5,m}, y) \longrightarrow 0, \tag{3.15}$$

and using Depth lemma

$$\operatorname{depth}(T_{5,m}/J_{5,m}) \geq \min\Big\{\operatorname{depth}(T_{5,m}/(J_{5,m} : y)), \operatorname{depth}(T_{5,m}/(J_{5,m}, y))\Big\}.$$

$(J_{5,m} : y) = (J^*_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{51}, \mu_{52}, \dots, \mu_{5m}, \nu_{11}, \nu_{12}$
$, \dots, \nu_{1m}, \nu_{51}, \nu_{52}, \dots, \nu_{5m})$.
We have $T_{5,m}/(J_{5,m} : y) \cong K[\delta_2, \delta_3, \delta_4, y, \xi_3, \xi_4, \omega_2, \omega_3, \omega_4 \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \dots, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \dots, \nu_{41}, \nu_{42}, \dots, \nu_{4m}]/J^*_{1,m} \cong (T^*_{1,m}/J^*_{1,m})[y]$. By Proposition 3.2.1 and [16, Lemma 3.6], $\operatorname{depth}(T_{5,m}/(J_{5,m} : y) = \operatorname{depth}(T^*_{1,m}/(J^*_{1,m})) + 1 = 2m + 3 + 1 = 2m + 4$.
Also as $(J_{5,m}, y) \cong (J^*_{3,m}, y)$ so we have $T_{5,m}/(J_{5,m}, y) \cong K[\delta_1, \delta_2, \dots, \delta_5, \xi_2, \dots, \xi_5, \omega_1, \omega_2, \dots, \omega_5, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m} \ \dots \ \mu_{51}, \mu_{52}, \dots, \mu_{5m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m} \ \dots \ \nu_{51}, \nu_{52}, \dots, \nu_{5m}]/J^*_{3,m} \cong T^*_{3,m}/J^*_{3,m}$. Hence by Proposition 3.2.1, $\operatorname{depth}(T_{5,m}/(J_{5,m}, y)) = \operatorname{depth}(T^*_{3,m}/(J^*_{3,m})) = 2m + 4$. As $\operatorname{depth}(T_{5,m}/(J_{5,m}, y)) =$ depth
$(T_{5,m}/(J_{5,m} : y))$ so by Depth Lemma $\operatorname{depth}(T_{5,m}/J_{5,m}) = 2m + 4 = 2m + \lceil \frac{5+3}{2} \rceil$.

For Stanley depth, replace Depth lemma by Lemma 2.2.2 on the exact sequence 3.15, [16, Lemma 3.6] and Proposition 3.2.1, we have $\operatorname{sdepth}(T_{5,m}/(J_{5,m})) \geq 2m+4$. And for the upper bound as we know $y \notin J_{5,m}$ so by Corollary 2.3.1, we get $\operatorname{sdepth}(T_{5,m}/J_{5,m}) \leq \operatorname{sdepth}(T_{5,m}/(J_{5,m} : y))$. This implies $\operatorname{sdepth}(T_{5,m}/J_{5,m}) \leq 2m + 4$. To conclude

$$\operatorname{sdepth}(T_{5,m}/J_{5,m}) = 2m + 4 = 2m + \left\lceil \frac{5+3}{2} \right\rceil.$$

Generally, for $n \geq 6$. We examine the short exact sequence

$$0 \longrightarrow T_{n,m}/(J_{n,m} : y) \xrightarrow{\cdot y} T_{n,m}/J_{n,m} \longrightarrow T_{n,m}/(J_{n,m}, y) \longrightarrow 0, \tag{3.16}$$

using Depth lemma

$$\text{depth}(T_{n,m}/J_{n,m}) \geq \min\Big\{ \text{depth}(T_{n,m}/(J_{n,m} : y)), \text{depth}(T_{n,m}/(J_{n,m}, y))\Big\}.$$

$(J_{n,m} : y) = (J^*_{(n-4),m}, \delta_1, \delta_n, \xi_2, \xi_n, \omega_1, \omega_n, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}$ $, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm})$.
We get $T_{n,m}/(J_{n,m} : y) \cong K[\delta_2, \dots, \delta_{n-1}, y, \xi_3, \dots, \xi_{n-1}, \omega_2, \dots, \omega_{n-1}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}$ $\dots\ \mu_{(n-1)1}, \mu_{(n-1)2}, \dots, \mu_{(n-1)m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}\ \dots\ \nu_{(n-1)1}, \nu_{(n-1)2}, \dots, \nu_{(n-1)m}]/$ $J_{n-4,m} \cong (T_{n-4,m}/J_{n-4,m})[y]$. By Proposition 3.2.1 and [16, Lemma 3.6], $\text{depth}(T_{n,m}/(J_{n,m} : y) = \text{depth}(T^*_{n-4,m}/(J^*_{n-4,m})) + 1$.

Now if $n \equiv 0 \pmod 2$, then $n-4 \equiv 0 \pmod 2$. This shows $\text{depth}(T_{n,m}/J_{n,m} : y) = \text{depth}(T^*_{n-4,m}/J^*_{n-4,m}) + 1 = \lceil \frac{n-4+2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$.
And if $n \equiv 1 \pmod 2$, then $n-4 \equiv 1 \pmod 2$. This implies $\text{depth}(T_{n,m}/J_{n,m} : y) = \text{depth}(T^*_{n-4,m}/J^*_{n-4,m}) + 1 = 2m + \lceil \frac{n-4+5}{2} \rceil + 1 = 2m + \lceil \frac{n+3}{2} \rceil$.

Now as $(J_{n,m}, y) \cong (J^*_{n-2,m}, y)$, we have $T_{n,m}/(J_{n,m}, y) \cong K[\delta_1, \delta_2, \dots, \delta_n, \xi_2, \dots, \xi_n,$ $\omega_1, \omega_2, \dots, \omega_n, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}\ \dots\ \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21},$ $\nu_{22}, \dots, \nu_{2m}\ \dots\ \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}]/J^*_{n-2,m} \cong T^*_{n-2,m}/J^*_{n-2,m}$. Thus by Proposition 3.2.1,

$\text{depth}(T_{n,m}/(J_{n,m}, y)) = \text{depth}(T^*_{n-2,m}/(J^*_{n-2,m}))$.

If $n \equiv 0 \pmod 2$, then $n-2 \equiv 1 \pmod 2$. This implies $\text{depth}(T_{n,m}/J_{n,m}, y) = \text{depth}(T^*_{n-2,m}/J^*_{n-2,m}) = \lceil \frac{n-2+2}{2} \rceil = \lceil \frac{n}{2} \rceil$. And if $n \equiv 1 \pmod 2$, then $n-2 \equiv 0 \pmod 2$. This shows $\text{depth}(T_{n,m}/J_{n,m}, y) = \text{depth}(T^*_{n-2,m}/J^*_{n-2,m}) = 2m + \lceil \frac{n-2+5}{2} \rceil = 2m + \lceil \frac{n+3}{2} \rceil$. Therefore if $n \equiv 0 \pmod 2$, then $\text{depth}(T_{n,m}/(J_{n,m}, y) = \text{depth}(T_{n,m}/(J_{n,m} : y)$ so by Depth Lemma $\text{depth}(T_{n,m}/J_{n,m}) = \lceil \frac{n}{2} \rceil$. Also if $n \equiv 1 \pmod 2$, then $\text{depth}(T_{n,m}/(J_{n,m}, y)$ $= \text{depth}(T_{n,m}/(J_{n,m} : y)$, again by Depth Lemma $\text{depth}(T_{n,m}/J_{n,m}) = 2m + \lceil \frac{n+3}{2} \rceil$.

To examine Stanley depth, apply Lemma 2.2.2 at the place of Depth Lemma on the short exact sequence 3.16, [16, Lemma 3.6] and Proposition 3.2.1, we have if $n \equiv 0 \pmod 2$, then $\text{sdepth}(T_{n,m}/(J_{n,m})) \geq \lceil \frac{n}{2} \rceil$. And if $n \equiv 1 \pmod 2$, then

$\text{sdepth}(T_{n,m}/(J_{n,m})) \geq 2m + \lceil \frac{n+3}{2} \rceil$. Now for the upper bound as we know that $y \notin J_{n,m}$ so by Corollary 2.3.1, we get $\text{sdepth}(T_{n,m}/J_{n,m}) \leq \text{sdepth}(T_{n,m}/(J_{n,m} : y)$. This implies If $n \equiv 0 \pmod 2$, then $\text{sdepth}(T_{n,m}/J_{n,m}) \leq \lceil \frac{n}{2} \rceil$. So $\text{sdepth}(T_{2,m}/J_{2,m}) = \lceil \frac{n}{2} \rceil$. And if $n \equiv 1 \pmod 2$, then $\text{sdepth}(T_{n,m}/J_{n,m}) \leq 2m + \lceil \frac{n+3}{2} \rceil$. To sum up, $\text{sdepth}(T_{n,m}/J_{n,m}) = 2m + \lceil \frac{n+3}{2} \rceil$. □

Remark 3.2.4. *As diam*$(E_{n,m}) = \lceil \frac{n+1}{2} \rceil$ *so by Theorem 2.3.3, we have* $\text{depth}(T_{n,m}/J_{n,m}), \text{sdepth}(T_{n,m}/J_{n,m}) \geq \lceil \frac{n+3}{6} \rceil$. *While, our Proposition 3.2.3 shows that if* $n \equiv 0 \pmod 2$, *then* $\text{depth}(T_{n,m}/J_{n,m}) = \text{sdepth}(T_{n,m}/J_{n,m}) = \lceil \frac{n}{2} \rceil$. *And if* $n \equiv 1 \pmod 2$, *then* $\text{depth}(T_{n,m}/J_{n,m}) = \text{sdepth}(T_{n,m}/J_{n,m}) = 2m + \lceil \frac{n+3}{2} \rceil$. *In a nutshell, we have better results for depth and stanley depth of this category of cyclic module.*

Chapter 4

Stanley depth of edge ideals

4.1 Stanley depth of edge ideal of graph $B_{n,m}$

In this section we will discuss Stanley depth of the edge ideal $I_{n,m}$ associated to the graph $B_{n,m}$. For this purpose, we first find Stanley depth of the edge ideal $I^*_{n,m}$ associated to the super graph $D_{n,m}$ and Stanley depth of edge ideal $g_{1,m}$ of star graph. We will use these results in our principal proof. Also for $m = 0$ we denote $S_{n,m} = S_n$ and $S^*_{n,m} = S^*_n$, similarly $I_{n,m} = I_n$ and $I^*_{n,m} = I^*_n$.

Lemma 4.1.1. *Let $m \geq 1$, then* $\operatorname{sdepth}(g_{1,m}) \geq m + 2$.

Proof. Since $\operatorname{sdepth}(I) \geq n - \lfloor \frac{m}{2} \rfloor$, where n is number of vertices and m is number of edges of associated graph of edge I. As vertices of star graph are $2m + 3$ and edges are $2m + 2$ so $\operatorname{sdepth}(g_{1,m}) \geq 2m + 3 - \lfloor \frac{2m+2}{2} \rfloor = 2m + 3 - m - 1 = m + 2$. Hence $\operatorname{sdepth}(g_{1,m}) \geq m + 2$. □

Proposition 4.1.2. *Let $n \geq 1$ and $m = 0$, then* $\operatorname{sdepth}(I^*_n) \geq \lceil \frac{n+7}{2} \rceil$.

Proof. We prove this result by induction on n. Firstly, for $n = 1$ the result can be easily verified by CoCoA. Now for $n = 2$. As $\xi_3 \notin I^*_2$, so we have

$$I^*_2 = I^*_2 \cap S^{**}_2 \bigoplus \xi_3 (I^*_2 : \xi_3) S^*_2,$$

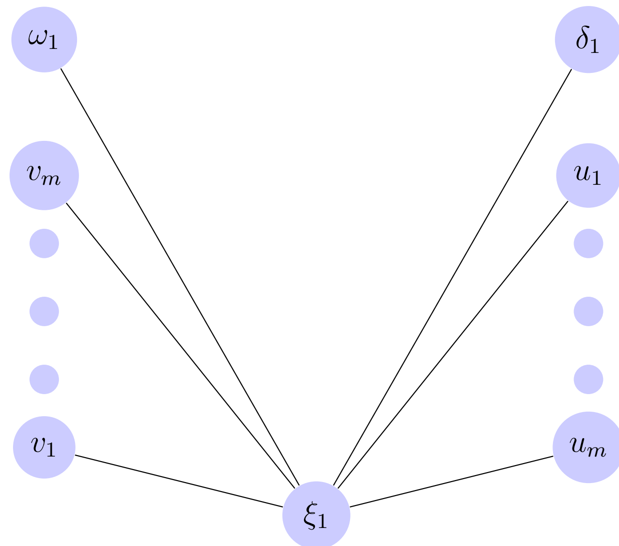

Figure 4.1: $g_{1,m}$ (Star Graph)

where $S_2^{**} = K[\delta_1, \delta_2, \delta_3, \xi_1, \xi_2, \omega_1, \omega_2, \omega_3]$. Now

$$I_2^* \cap S_2^{**} = (I_1^*)S_2^{**} \text{ and}$$

$$(I_2^* : \xi_3)S_2^* = (I(P_3), \delta_2, \delta_3, \xi_2, \omega_2, \omega_3)S_2^*.$$

$$\text{Thus}\,\text{sdepth}(I_2^*) \geq \min\Big\{\text{sdepth}(I_1^*)S_2^{**}, \text{sdepth}((I(P_3), \delta_2, \delta_3, \xi_2, \omega_2, \omega_3)S_2^*)\Big\}.$$

Now by [16, Lemma 3.6], we have

$$\text{sdepth}\Big((I(P_3), \delta_2, \delta_3, \xi_2, \omega_2, \omega_3)S_2^*\Big) = \text{sdepth}\Big((I(P_3), \delta_2, \delta_3, \xi_2, \omega_2, \omega_3)S_2^{**}\Big) + 1.$$

And by [17, Theorem 1.3],
$\text{sdepth}\Big((I(P_3), \delta_2, \delta_3, \xi_2, \omega_2, \omega_3)S_2^{**}\Big) \geq \min\Big\{\text{sdepth}\Big(I(P_3)T_3\Big) + 5, \text{sdepth}\Big((\delta_2, \delta_3, \xi_2$
$, \omega_2, \omega_3)\bar{S}_2\Big) + \text{sdepth}\Big(T_3/I(P_3)\Big)T_3\Big\},$
where $\bar{S}_2 = K[\delta_2, \delta_3, \xi_2, \omega_2, \omega_3]$. Now by [24, Theorems 3.14 and 5.2] and [21], we have

$$\text{sdepth}\Big((I(P_3), \delta_2, \delta_3, \xi_2, \omega_2, \omega_3)S_2^{**}\Big) \geq \min\{7, 3+1\} = 4. \text{ Thus}$$

$$\text{sdepth}\Big((I(P_3), \delta_2, \delta_3, \xi_2, \omega_2, \omega_3)S_2^*\Big) \geq 5.$$

Now by induction on n and [16, Lemma 3.6], we get

$$\text{sdepth}\Big((I_1^*)S_2^{**}\Big) = \text{sdepth}\Big((I_1^*)S_1^*\Big) + 2 = 6.$$

$$\text{Therefore sdepth}\, I_2^* \geq 5 = \left\lceil \frac{2+7}{2} \right\rceil.$$

Now for $n = 3$. Since $\xi_4 \notin I_3^*$, so we have

$$I_3^* = I_3^* \cap S_3^{**} \bigoplus \xi_4(I_3^* : \xi_4)S_3^*,$$

where $S_3^{**} = K[\delta_1,\ldots,\delta_4,\xi_1,\ldots,\xi_3,\omega_1,\ldots,\omega_4]$. Now

$$I_3^* \cap S_3^{**} = (I_2^*)S_3^{**} \text{ and}$$

$$(I_3^* : \xi_4)S_3^* = (I_1^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4)S_3^*$$

Thus

$$\text{sdepth}(I_3^*) \geq \min\Big\{\text{sdepth}(I_2^*)S_3^{**}, \text{sdepth}((I_1^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4)S_3^*)\Big\}.$$

By using [16, Lemma 3.6], we get

$$\text{sdepth}\left((I_1^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4)S_{3,m}^*\right) = \text{sdepth}\left(((I_1^*), \delta_3, \delta_4, \xi_3, \omega_3, \omega_4,)S_3^{**}\right) + 1,$$

and by [17, Theorem 1.3],

$$\text{sdepth}\left((I_1^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4)S_3^{**}\right) \geq$$
$$\min\Big\{\text{sdepth}\left((I_1^*)S_1^*\right) + 5, \text{sdepth}\left((\delta_3, \delta_4, \xi_3, \omega_3, \omega_4)\bar{S}_3\right) + \text{sdepth}\left(S_1^*/I_1^*\right)S_1^*\Big\},$$

where $\bar{S}_3 = K[\delta_3, \delta_4, \xi_3, \omega_3, \omega_4]$. Now by induction on n, [21], and Proposition 3.1.1, we have

$$\text{sdepth}\left((I_1^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4)S_3^{**}\right) \geq min\{9, 4\} = 4.$$

Hence

$$\text{sdepth}\Big((I_1^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4)S_3^*) \geq 5 = \left\lceil \frac{3+7}{2} \right\rceil.$$

Now by induction on n and [16, Lemma 3.6], we get

$$\text{sdepth}\left((I_2^*)S_3^{**}\right) = \text{sdepth}\left((I_2^*)S_2^*\right) + 2 \geq 7.$$

Thus

$$\text{sdepth}(I_3^*) \geq 5 = \left\lceil \frac{3+7}{2} \right\rceil.$$

Finally, for $n \geq 4$. Since $\xi_{n+1} \notin I_n^*$, thus we have

$$I_n^* = I_n^* \cap S_n^{**} \bigoplus \xi_{n+1}(I_n^* : \xi_{n+1})S_n^*,$$

where $S_n^{**} = K[\delta_1,\dots,\delta_{n+1},\xi_1,\dots,\xi_n,\omega_1,\dots,\omega_{n+1}]$. Now

$$I_n^* \cap S_n^{**} = (I_{n-1}^*)S_n^{**} \text{ and}$$

$$\Big(I_n^* : \xi_{n+1}\Big)S_n^* = \Big((I_{n-2}^*, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})S_n^*\Big)$$

Therefore

$$\operatorname{sdepth}(I_n^*) \geq \min\Big\{\operatorname{sdepth}(I_{n-1}^*)S_n^{**}, \operatorname{sdepth}((I_{n-2}^*, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})S_n^*)\Big\}.$$

Also by [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((I_{n-2}^*, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})S_n^*\Big) = \operatorname{sdepth}\Big((I_{n-2,m}^*, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})S_{n,m}^{**}\Big) + 1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((I_{n-2}^*, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})S_n^{**}\Big) \geq \min\Big\{\operatorname{sdepth}((I_{n-2}^*)S_{n-2}^*) + 5, \operatorname{sdepth}((\delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})\bar{S_n}) + \operatorname{sdepth}((S_{n-2}^*/I_{n-2}^*)S_{n-2}^*)\Big\},$$

where $\bar{S_n} = K[\delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}]$. Now by induction on n, [21], and Proposition 3.1.1, we have
$\operatorname{sdepth}\Big((I_{n-2}^*, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})S_n^{**}\Big) \geq \min\Big\{\lceil\frac{n+15}{2}\rceil, 3 + \lceil\frac{n-1}{2}\rceil\Big\} = \lceil\frac{n+5}{2}\rceil$. Thus

$$\operatorname{sdepth}\Big(((I_{n-2}^*), \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1})S_n^*\Big) \geq \Big\lceil\frac{n+7}{2}\Big\rceil.$$

Now by induction on n and [16, Lemma 3.6], we get
$\operatorname{sdepth}\Big((I_{n-1}^*)S_n^{**}\Big) = \operatorname{sdepth}\Big((I_{n-1}^*)S_{n-1}^*\Big) + 2 \geq \Big\lceil\frac{n+6}{2}\Big\rceil + 2 = \Big\lceil\frac{n+10}{2}\Big\rceil$. Therefore

$$\operatorname{sdepth}(I_n^*) \geq \Big\lceil\frac{n+7}{2}\Big\rceil.$$

□

Remark 4.1.3. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I_n^*) \geq 3n+3-\lfloor\frac{5n+2}{2}\rfloor$. *Thus our lower bound is sharper than this one.*

Proposition 4.1.4. *Assume that* $n, m \geq 1$, *then* $\operatorname{sdepth}(I_{n,m}^*) \geq \lceil\frac{4m+5}{2}\rceil + \lceil\frac{n-1}{2}\rceil + 1$.

Proof. The result will be proved by induction on n. Firstly, for $n = 1$. Since $\xi_2 \notin I_{1,m}^*$, thus we have

$$I_{1,m}^* = I_{1,m}^* \cap S_{1,m}^{**} \bigoplus \xi_2 \Big(I_{1,m}^* : \xi_2\Big) S_{1,m}^*,$$

where $S_{1,m}^{**} = K[\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}\nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}]$. Now

$$I_{1,m}^* \cap S_{1,m}^{**} = (g_{1,m}) S_{1,m}^{**} \text{ and}$$

$$\Big(I_{1,m}^* : \xi_2\Big) S_{1,m}^* = \Big(\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}\Big) S_{1,m}^*.$$

Therefore

$$\operatorname{sdepth}(I_{1,m}^*) \geq \min\Big\{ \operatorname{sdepth}(g_{1,m}) S_{1,m}^{**}, \operatorname{sdepth}(\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}) S_{1,m}^* \Big\}.$$

By using [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}) S_{1,m}^*\Big) = \operatorname{sdepth}\Big((\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}) S_{1,m}^{**}\Big) + 1,$$

and by [21],

$$\operatorname{sdepth}\Big((\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}) S_{1,m}^{**}\Big) = \Big\lceil \frac{4m+5}{2} \Big\rceil.$$

Hence

$$\text{sdepth}\Big((\delta_1,\delta_2,\xi_1,\omega_1,\omega_2,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},\dots,\nu_{2m})S^*_{1,m}\Big)=\left\lceil\frac{4m+5}{2}\right\rceil+1=\left\lceil\frac{4m+5}{2}\right\rceil+\left\lceil\frac{1-1}{2}\right\rceil+1.$$

Also by Lemma 4.1.1 and [16, Lemma 3.6], we get

$$\text{sdepth}\Big((g_{1,m})S^{**}_{1,m}\Big)=\text{sdepth}\Big((g_{1,m})A_{1,m}\Big)+2m+3\geq 3m+5.$$

Consequently

$$\text{sdepth}(I^*_{1,m})\geq\left\lceil\frac{4m+5}{2}\right\rceil+\left\lceil\frac{1-1}{2}\right\rceil+1.$$

Secondly, for $n=2$. As $\xi_3\notin I^*_{2,m}$, so we have

$$I^*_{2,m}=I^*_{2,m}\cap S^{**}_{2,m}\bigoplus\xi_3(I^*_{2,m}:\xi_3)S^*_{2,m},$$

where $S^{**}_{2,m}=K[\delta_1,\delta_2,\delta_3,\xi_1,\xi_2,\omega_1,\omega_2,\omega_3,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m},\mu_{31},\mu_{32},\dots,\mu_{3m}\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},\dots,\nu_{3m},\nu_{31},\nu_{32},\dots,\nu_{3m}]$. Further

$$I^*_{2,m}\cap S^{**}_{2,m}=(I^*_{1,m})S^{**}_{2,m}\text{ and}$$

$$\Big(I^*_{2,m}:\xi_3\Big)S^*_{2,m}=\Big(g_{1,m},\delta_2,\delta_3,\xi_2,\omega_2,\omega_3,\mu_{21},\mu_{22},\dots,\mu_{2m},\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{21},\nu_{22},\dots\nu_{2m},\nu_{31},\nu_{32},\dots,\nu_{3m}\Big)S^*_{2,m}$$

Therefore

$$\text{sdepth}(I^*_{2,m})\geq\min\Big\{\text{sdepth}(I^*_{1,m})S^{**}_{1,m},\text{sdepth}\Big(g_{1,m},\delta_2,\delta_3,\xi_2,\omega_2,\omega_3,\mu_{21},\mu_{22},\dots,\mu_{2m},\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{21},\nu_{22},\dots,\nu_{2m},\nu_{31},\nu_{32},\dots,\nu_{3m}\Big)S^*_{2,m}\Big\}.$$

Now [16, Lemma 3.6], we get

$$\text{sdepth}\Big((g_{1,m},\delta_2,\delta_3,\xi_2,\omega_2,\omega_3,\mu_{21},\mu_{22},\dots,\mu_{2m},\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{21},\nu_{22},\dots,\nu_{2m},\nu_{31},\nu_{32},\dots,\nu_{3m})S^*_{2,m}\Big)=\text{sdepth}\Big((g_{1,m},\delta_2,\delta_3,\xi_2,\omega_2,\omega_3,\mu_{21},\mu_{22},\dots,\mu_{2m},\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{21},\nu_{22},\dots,\nu_{2m},\nu_{31},\nu_{32},\dots,\nu_{3m})S^{**}_{2,m}\Big)+1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}((g_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{**}_{2,m}) \geq \min\Big\{ \operatorname{sdepth}((g_{1,m})A_{1,m}) + 4m + 5, \operatorname{sdepth}\Big((\delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{-}_{n,m}\Big) + \operatorname{sdepth}\Big((A_{1,m}/g_{1,m})A_{1,m}\Big)\Big\},$$

where $S^{-}_{2,m} = K[\delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}]$. Now using induction on n, [21], and sdepth of quotient module associated to star graph, we have

$$\operatorname{sdepth}\Big((g_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{**}_{2,m}\Big) \geq \min\Big\{5m + 7, \Big\lceil \frac{4m+5}{2} \Big\rceil + 1\Big\} = \Big\lceil \frac{4m+5}{2} \Big\rceil + 1.$$

Thus

$$\operatorname{sdepth}\Big((g_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{*}_{2,m}\Big) \geq \Big\lceil \frac{4m+5}{2} \Big\rceil + 1 + 1 = \Big\lceil \frac{4m+5}{2} \Big\rceil + \Big\lceil \frac{2-1}{2} \Big\rceil + 1.$$

Also by induction on n and [16, Lemma 3.6], we get

$$\operatorname{sdepth}\Big((I^*_{1,m})S^{**}_{2,m}\Big) = \operatorname{sdepth}\Big((I^*_{1,m})S^{*}_{1,m}\Big) + 2m + 2 = \Big\lceil \frac{4m+5}{2} \Big\rceil + \Big\lceil \frac{1-1}{2} \Big\rceil + 1 + 2m + 2.$$

To conclude

$$\operatorname{sdepth}(I^*_{2,m}) \geq \Big\lceil \frac{4m+5}{2} \Big\rceil + \Big\lceil \frac{2-1}{2} \Big\rceil + 1.$$

Thirdly, for $n = 3$. Since $\xi_4 \notin I^*_{3,m}$, thus we have

$$I^*_{3,m} = I^*_{3,m} \cap S^{**}_{3,m} \bigoplus \xi_4 (I^*_{3,m} : \xi_4) S^*_{3,m},$$

where $S^{**}_{3,m} = K[\delta_1, \dots, \delta_4, \xi_1, \dots, \xi_3, \omega_1, \dots, \omega_4, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \dots, \mu_{41}, \mu_{42}, \dots, \mu_{4m}\ \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \dots, \nu_{41}, \nu_{42}, \dots, \nu_{4m}]$. Moreover

$$I^*_{3,m} \cap S^{**}_{3,m} = (I^*_{2,m}) S^{**}_{3,m} \text{ and } \Big(I^*_{3,m} : \xi_4\Big) S^*_{3,m} = \Big((I^*_{1,m}, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m}) S^*_{3,m}\Big).$$

So

$$\operatorname{sdepth}(I_{3,m}^*) \geq \min\Big\{ \operatorname{sdepth}(I_{2,m}^*)S_{3,m}^{**}, \operatorname{sdepth}\Big((I_{1,m}^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S_{3,m}^*\Big)\Big\}.$$

Also by using [16, Lemma 3.6], we get

$$\operatorname{sdepth}\Big((I_{1,m}^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S_{3,m}^*\Big) = \operatorname{sdepth}\Big((I_{1,m}^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S_{3,m}^{**}\Big) + 1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((I_{1,m}^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S_{3,m}^{**}\Big) \geq \min\Big\{ \operatorname{sdepth}\Big((I_{1,m}^*)S_{1,m}^*\Big) + 4m+5, \operatorname{sdepth}\Big((\delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S_{3,m}^{-}\Big) + \operatorname{sdepth}\Big((S_{1,m}^*/I_{1,m}^*)S_{1,m}^*\Big)\Big\},$$

where $S_{3,m}^{-} = K[\delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m}]$. Now by induction on n, [21], and Proposition 3.1.2, we have

$$\operatorname{sdepth}\Big((I_{1,m}^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S_{3,m}^{**}\Big) \geq \min\Big\{\Big\lceil\frac{4m+5}{2}\Big\rceil + \Big\lceil\frac{n-3}{2}\Big\rceil + 1 + 4m + 5 = \Big\lceil\frac{4m+5}{2}\Big\rceil + \Big\lceil\frac{n-1}{2}\Big\rceil + 4m + 5, \Big\lceil\frac{4m+5}{2}\Big\rceil + \Big\lceil\frac{n-1}{2}\Big\rceil\Big\} = \Big\lceil\frac{4m+5}{2}\Big\rceil + \Big\lceil\frac{n-1}{2}\Big\rceil.$$

Thus

$$\operatorname{sdepth}\Big((I_{1,m}^*, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S_{3,m}^*\Big) \geq \Big\lceil\frac{4m+5}{2}\Big\rceil + 1 + 1 = \Big\lceil\frac{4m+5}{2}\Big\rceil + \Big\lceil\frac{3-1}{2}\Big\rceil + 1.$$

Now by induction on n and [16, Lemma 3.6], we get

$$\operatorname{sdepth}\Big((I_{2,m}^*)S_{3,m}^{**}\Big) = \operatorname{sdepth}\Big((I_{2,m}^*)S_{2,m}^*\Big) + 2m + 2 = \Big\lceil\frac{4m+5}{2}\Big\rceil + 2 + 2m + 2.$$

In conclusion

$$\operatorname{sdepth}(I^*_{3,m}) \geq \left\lceil \frac{4m+5}{2} \right\rceil + \left\lceil \frac{3-1}{2} \right\rceil + 1.$$

In general, for $n \geq 4$. As $\xi_{n+1} \notin I^*_{n,m}$, so we have

$$I^*_{n,m} = I^*_{n,m} \cap S^{**}_{n,m} \bigoplus \xi_{n+1}(I^*_{n,m} : \xi_n)S^*_{n,m},$$

where $S^{**}_{n,m} = K[\delta_1, \dots, \delta_{n+1}, \xi_1, \dots, \xi_n, \omega_1, \dots, \omega_{n+1}, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \dots, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \dots, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m}]$. Now

$$I^*_{n,m} \cap S^{**}_{n,m} = (I^*_{n-1,m})S^{**}_{n,m}, \text{ and}$$

$$\Big(I^*_{n,m} : \xi_{n+1}\Big)S^*_{n,m} = \Big((I^*_{n-2,m}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m})S^*_{n,m}\Big).$$

Thus

$$\operatorname{sdepth}(I^*_{n,m}) \geq \min\Big\{ \operatorname{sdepth}(I^*_{n-1,m})S^{**}_{n,m}, \operatorname{sdepth}\Big((I^*_{n-2,m}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m})S^*_{n,m}\Big)\Big\}.$$

By applying [16, Lemma 3.6], we have that

$$\operatorname{sdepth}\Big((I^*_{n-2,m}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m})S^*_{n,m}\Big) = \operatorname{sdepth}\Big((I^*_{n-2,m}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m})S^{**}_{n,m}\Big) + 1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((I^*_{n-2,m}, \delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, u_{(n+1)1}, u_{(n+1)2}, \dots, u_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, v_{(n+1)1}, v_{(n+1)2}, \dots, v_{(n+1)m})S^{**}_{n,m}\Big) \geq \min\Big\{ \operatorname{sdepth}\Big((I^*_{n-2,m})S^*_{n-2,m}\Big) + 4m + 5, \operatorname{sdepth}\Big((\delta_n, \delta_{n+1}, \xi_n, \omega_n, \omega_{n+1}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \mu_{(n+1)1}, \mu_{(n+1)2}, \dots, \mu_{(n+1)m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}, \nu_{(n+1)1}, \nu_{(n+1)2}, \dots, \nu_{(n+1)m})S^{-}_{n,m}\Big) + \operatorname{sdepth}\Big((S^*_{n-2,m}/I^*_{n-2,m})S^*_{n-2,m}\Big)\Big\},$$

where $S_{n,m}^{-}=K[\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},$ $\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m}]$. Now by induction on n, [21], and Proposition 3.1.2, we have

$$\operatorname{sdepth}\Big((I_{n-2,m}^{*},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},$$
$$\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})S_{n,m}^{**}\Big)\geq\min\Big\{\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil+$$
$$4m+5,\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil\Big\}=\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil.\text{ Thus}$$

$$\operatorname{sdepth}\Big((I_{n-2,m}^{*},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m}$$
$$,\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})S_{n,m}^{*}\Big)\geq\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil+1.$$

Now by induction on n and [16, Lemma 3.6], we get

$$\operatorname{sdepth}\Big((I_{n-1,m}^{*})S_{n,m}^{**}\Big)=\operatorname{sdepth}\Big((I_{n-1,m}^{*})S_{n-1,m}^{*}\Big)+2m+2=\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil$$
$$+1+2m+2.$$

In a nutshell

$$\operatorname{sdepth}(I_{n,m}^{*})\geq\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil+1.$$

□

Remark 4.1.5. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I_{n,m}^{*})\geq 2nm+3n+2m+3-\lfloor\frac{4mn+5n+2m+2}{2}\rfloor$. *Thus our lower bound is sharper than this one.*

Proposition 4.1.6. *Let* $n\geq 1$ *and* $m=0$. *If* $n=1$, *then* $\operatorname{sdepth}(I_1)=2$. *And if* $n\geq 2$, *then* $\operatorname{sdepth}(I_n)\geq\lceil\frac{n+5}{2}\rceil$.

Proof. For $n=1$. It can be easily checked by using CoCoA. Also for $n=2$, again by using CoCoA result holds. Now let us have $n=3$. Since $\xi_4\notin I_3$, so we have

$$I_3=I_3\cap S_3^{'}\bigoplus\xi_4(I_3:\xi_4)S_3,$$

where $S_3^{'}=K[\delta_1,\dots,\delta_3,\xi_1,\dots,\xi_3,\omega_1,\dots,\omega_3]$. Now

$$I_3\cap S_3^{'}=(I_2^{*})S_3^{'}\text{ and}$$

$$\Big(I_3 : \xi_4\Big) S_3 = \Big((I_1^*, \delta_3, \xi_3, \omega_3) S_3\Big)$$

Therefore

$$\text{sdepth}(I_3) \geq \min\Big\{\text{sdepth}(I_2^*)S_2^{'}, \text{sdepth}((I_1^*, \delta_2, \xi_3, \omega_3)S_3)\Big\}.$$

By [16, Lemma 3.6], we have

$$\text{sdepth}\Big((I_1^*, \delta_3, \xi_3, \omega_3)S_3\Big) = \text{sdepth}\Big((I_1^*, \delta_3, \xi_3, \omega_3)S_3^{'}\Big) + 1,$$

and by [17, Theorem 1.3],

$$\text{sdepth}\Big((I_1^*, \delta_3, \xi_3, \omega_3)S_3^{'}\Big) \geq \min\Big\{\text{sdepth}\Big((I_1^*)S_1^*\Big) + 3, \text{sdepth}\Big((\delta_3, \xi_3, \omega_3)S_3^{''}\Big) + \text{sdepth}\Big((S_1^*/I_1^*)S_1^*\Big)\Big\},$$

where $S_3^{''} = K[\delta_3, \xi_3, \omega_3]$. Now by Proposition 3.1.1, [21], And Proposition 4.1.2, we get

$$\text{sdepth}\Big((I_1^*, \delta_3, \xi_3, \omega_3)S_3^{'}\Big) \geq \min\Big\{\Big\lceil\frac{3+5}{2}\Big\rceil + 3, 2 + \Big\lceil\frac{3-1}{2}\Big\rceil\Big\} = 2 + \Big\lceil\frac{3-1}{2}\Big\rceil = \Big\lceil\frac{3+3}{2}\Big\rceil.$$

Therefore

$$\text{sdepth}\Big((I_1^*, \delta_3, \xi_3, \omega_3)S_3\Big) \geq \Big\lceil\frac{3+5}{2}\Big\rceil.$$

Now by Proposition 4.1.2, we get

$$\text{sdepth}\Big((I_2^*)S_3^{'}\Big) \cong \text{sdepth}\Big((I_2^*)S_2^*\Big) \geq \Big\lceil\frac{3+6}{2}\Big\rceil.$$

Hence

$$\text{sdepth}(I_3) \geq \Big\lceil\frac{3+5}{2}\Big\rceil.$$

Finally, when $n \geq 4$. As $\xi_{n+1} \notin I_n$, so

$$I_n = I_n \cap S_n^{'} \bigoplus \xi_{n+1}(I_n : \xi_{n+1})S_n,$$

where $S_n^{'} = K[\delta_1, \ldots, \delta_n, \xi_1, \ldots, \xi_n, \omega_1, \ldots, \omega_n]$. Now

$$I_n \cap S_n^{'} = (I_{n-1}^*)S_n^{'} \text{ and}$$

$$\Big(I_n:\xi_{n+1}\Big)S_n=\Big((I^*_{n-2},\delta_n,\xi_n,\omega_n)S_n\Big). \text{ Thus}$$

$$\operatorname{sdepth}(I_n)\geq\min\Big\{\operatorname{sdepth}(I^*_{n-1})S^{'}_n,\operatorname{sdepth}\Big((I^*_{n-2},\delta_n,\xi_n,\omega_n)S_n\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((I^*_{n-2},\delta_n,\xi_n,\omega_n)S_n\Big)=\operatorname{sdepth}\Big((I^*_{n-2},\delta_n,\xi_n,\omega_n)S^{'}_n\Big)+1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((I^*_{n-2},\delta_n,,\xi_n,\omega_n,z)S^{'}_n\Big)\geq$$
$$\min\Big\{\operatorname{sdepth}\Big((I^*_{n-2})S^*_{n-2}\Big)+3,\operatorname{sdepth}\Big((\delta_n,\xi_n,\omega_n)S^{''}_n\Big)+\operatorname{sdepth}\Big((S^*_{n-2}/I^*_{n-2})S^*_{n-2}\Big)\Big\},$$

where $S^{''}_n=K[\delta_n,\xi_n,\omega_n]$. Now by Proposition 3.1.2, [21], And Proposition 4.1.4, we have

$$\operatorname{sdepth}\Big((I^*_{n-2},\delta_n,\xi_n,\omega_n)S^{'}_n\Big)\geq\min\Big\{\Big\lceil\frac{n+5}{2}\Big\rceil+3,2+\Big\lceil\frac{n-1}{2}\Big\rceil\Big\}=2+\Big\lceil\frac{n-1}{2}\Big\rceil=\Big\lceil\frac{n+3}{2}\Big\rceil.$$

Thus

$$\operatorname{sdepth}\Big((I^*_{n-2},\delta_n,\xi_n,\omega_n)S_n\Big)\geq\Big\lceil\frac{n+5}{2}\Big\rceil.$$

Now by Proposition 4.1.4, we get

$$\operatorname{sdepth}\Big((I^*_{n-1})S^{'}_n\Big)=\operatorname{sdepth}\Big((I^*_{n-1})S^*_{n-1}\Big)\geq\Big\lceil\frac{n+6}{2}\Big\rceil.$$

To conclude

$$\operatorname{sdepth}(I_n)\geq\Big\lceil\frac{n+5}{2}\Big\rceil.$$

□

Remark 4.1.7. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I_n)\geq 3n+1-\lfloor\frac{5n}{2}\rfloor$. *Thus our lower bound is sharper than this one.*

Corollary 4.1.8. *For* $n=1$, $\operatorname{sdepth}(I_1)=\operatorname{sdepth}(S_1/I_1)+1$. *And for* $n\geq 2$, $\operatorname{sdepth}(I_n)>\operatorname{sdepth}(S_n/I_n)+1$.

Proposition 4.1.9. *If* $n=1$ *and* $m\geq 1$, *then* $\operatorname{sdepth}(I_{1,m})\geq m+2$.

Proof. Since $\xi_2 \notin I_{1,m}$, thus we have

$$I_{1,m} = I_{1,m} \cap S'_{1,m} \bigoplus \xi_2(I_{1,m} : \xi_2)S_{1,m},$$

where $S'_{1,m} = K[\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}]$. Now

$$I_{1,m} \cap S'_{1,m} = (g_{1,m})S'_{1,m} \text{ and}$$

$$(I_{1,m} : \xi_2)S_{1,m} = (\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, , \nu_{11}, \nu_{12}, \ldots, \nu_{1m})S_{1,m}.$$

Hence

$$\operatorname{sdepth}(I_{1,m}) \geq \min\Big\{ \operatorname{sdepth}(g_{1,m})S'_{1,m}, \operatorname{sdepth}\Big((\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m})S_{1,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m})S_{1,m}\Big) = \operatorname{sdepth}\Big((\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, , \nu_{11}, \nu_{12}, \ldots, \nu_{1m})S'_{1,m}\Big) + 1,$$

and by [21],

$$\operatorname{sdepth}\Big((\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m},)S'_{1,m}\Big) = \left\lceil \frac{2m+3}{2} \right\rceil.$$

Thus

$$\operatorname{sdepth}\Big((\delta_1, \xi_1, \omega_1, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m})S_{1,m}\Big) = \left\lceil \frac{2m+3}{2} \right\rceil + 1.$$

Now by Lemma 4.1.1 , we get

$$\operatorname{sdepth}\Big((g_{1,m})S'_{1,m}\Big) \cong \operatorname{sdepth}\Big((g_{1,m})A_{1,m}\Big) \geq m+2.$$

To sum up

$$\operatorname{sdepth}(I_{1,m}) \geq m+2.$$

□

Remark 4.1.10. *In this case we have no conclusion about Herzog's conjecture and Rauf's question.*

Proposition 4.1.11. *If $n \geq 2$ and $m \geq 1$, then* $\operatorname{sdepth}(I_{n,m}) \geq \lceil \frac{2m+3}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 1$.

Proof. Let $n = 2$. As $\xi_3 \notin I_{2,m}$, so we have

$$I_{2,m} = I_{2,m} \cap S'_{2,m} \bigoplus \xi_3(I_{2,m} : \xi_3)S_{2,m},$$

where $S'_{2,m} = K[\delta_1, \delta_2, \xi_1, \xi_2, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}]$. Now

$$I_{2,m} \cap S'_{2,m} = (I^*_{1,m})S'_{2,m} \text{ and}$$

$$\Big(I_{2,m} : \xi_3\Big)S_{2,m} = \Big((g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, , \nu_{21}, \nu_{22}, \dots, \nu_{2m})S_{2,m}\Big).$$

Thus

$$\operatorname{sdepth}(I_{2,m}) \geq \min\Big\{\operatorname{sdepth}(I^*_{1,m})S'_{1,m}, \operatorname{sdepth}\Big((g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m} , \nu_{21}, \nu_{22}, \dots, \nu_{2m})S_{2,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m})S_{2,m}\Big) = \operatorname{sdepth}\Big((g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, , \nu_{21}, \nu_{22}, \dots, \nu_{2m})S'_{2,m}\Big) + 1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m})S'_{2,m}\Big) \geq \min\Big\{\operatorname{sdepth}\Big((g_{1,m})A_{1,m}\Big) + 2m + 3, \operatorname{sdepth}\Big((\delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m})S''_{3,m}\Big) + \operatorname{sdepth}\Big((A_{1,m}/g_{1,m})A_{1,m}\Big)\Big\},$$

where $S^{''}_{2,m} = K[\delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}]$. Now by [21], sdepth of edge ideal and quotient module associated to star graph. We get

$$\operatorname{sdepth}\Big((g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m})S^{'}_{2,m}\Big) \geq \min\Big\{3m+5 \\ , \Big\lceil \frac{2m+3}{2} \Big\rceil + 1\Big\} = \Big\lceil \frac{2m+3}{2} \Big\rceil + 1.$$

Therefore

$$\operatorname{sdepth}\Big((g_{1,m}, \delta_2, \xi_2, \omega_2, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m})S_{2,m}\Big) \geq \Big\lceil \frac{2m+3}{2} \Big\rceil + 1 + 1 \\ = \Big\lceil \frac{2m+3}{2} \Big\rceil + \Big\lceil \frac{2-1}{2} \Big\rceil + 1.$$

Now by induction on n and [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((I^*_{1,m})S^{'}_{2,m}\Big) = \operatorname{sdepth}\Big((I^*_{1,m})S^*_{1,m}\Big) \geq \Big\lceil \frac{4m+5}{2} \Big\rceil + \Big\lceil \frac{1-1}{2} \Big\rceil + 1.$$

Hence

$$\operatorname{sdepth}(I_{2,m}) \geq \Big\lceil \frac{2m+3}{2} \Big\rceil + \Big\lceil \frac{2-1}{2} \Big\rceil + 1.$$

Now for $n = 3$. Since $\xi_4 \notin I_{3,m}$ this implies

$$I_{3,m} = I_{3,m} \cap S^{'}_{3,m} \bigoplus \xi_4(I_{3,m} : \xi_4)S_{3,m},$$

where $S^{'}_{3,m} = K[\delta_1, \dots, \delta_3, \xi_1, \dots, \xi_3, \omega_1, \dots, \omega_3, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \dots, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \dots, \nu_{31}, \nu_{32}, \dots, \nu_{3m}]$. Now

$$I_{3,m} \cap S^{'}_{3,m} = (I^*_{2,m})S^{'}_{3,m} \text{ and}$$

$$\Big(I_{3,m} : \xi_4\Big)S_{3,m} = \Big((I^*_{1,m}, \delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, , \nu_{31}, \nu_{32}, \dots, \nu_{3m})S_{3,m}\Big).$$

Thus

$$\operatorname{sdepth}(I_{3,m}) \geq \min\Big\{\operatorname{sdepth}(I^*_{2,m})S^{'}_{3,m}, \operatorname{sdepth}\Big((I^*_{1,m}, \delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m} \\ , \nu_{31}, \nu_{32}, \dots, \nu_{3m})S_{3,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\text{sdepth}\left((I^*_{1,m}, \delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})S_{3,m}\right)$$
$$= \text{sdepth}\left((I^*_{1,m}, \delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, , \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{'}_{3,m}\right) + 1,$$

and by [17, Theorem 1.3],

$$\text{sdepth}\left((I^*_{1,m}, \delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, , \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{'}_{3,m}\right) \geq$$
$$\min\Big\{ \text{sdepth}\left((I^*_{1,m})S^*_{1,m}\right) + 2m + 3, \text{sdepth}\left((\delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, , \nu_{31}, \nu_{32}, \dots\right.$$
$$\left., \nu_{3m})S^{''}_{3,m}\right) + \text{sdepth}\left((S^*_{1,m}/I^*_{1,m})S^*_{1,m}\right)\Big\},$$

where $S^{''}_{3,m} = K[\delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}]$. Now by Proposition 4.1.4, [21] and Proposition 3.1.7, we have

$$\text{sdepth}\left((I^*_{1,m}, \delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, , \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{'}_{3,m}\right) \geq \min\Big\{\left\lceil\frac{4m+5}{2}\right\rceil$$
$$+ 1 + 2m + 3, \left\lceil\frac{2m+3}{2}\right\rceil + \left\lceil\frac{3-1}{2}\right\rceil\Big\} = \left\lceil\frac{2m+3}{2}\right\rceil + \left\lceil\frac{3-1}{2}\right\rceil.$$

Consequently

$$\text{sdepth}\left((I^*_{1,m}, \delta_3, \xi_3, \omega_3, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, , \nu_{31}, \nu_{32}, \dots, \nu_{3m})S_{3,m}\right) \geq \left\lceil\frac{2m+3}{2}\right\rceil + \left\lceil\frac{3-1}{2}\right\rceil + 1.$$

Now by Proposition 4.1.4, we get

$$\text{sdepth}\left((I^*_{2,m})S^{'}_{3,m}\right) = \text{sdepth}\left((I^*_{2,m})S^*_{2,m}\right) \geq \left\lceil\frac{4m+5}{2}\right\rceil + 2.$$

To conclude

$$\text{sdepth}(I_{3,m}) \geq \left\lceil\frac{2m+3}{2}\right\rceil + \left\lceil\frac{3-1}{2}\right\rceil + 1.$$

Lastly, for $n \geq 4$. As $\xi_{n+1} \notin I_{n,m}$, thus we have

$$I_{n,m} = I_{n,m} \cap S^{'}_{n,m} \bigoplus \xi_{n+1}(I_{n,m} : \xi_{n+1})S_{n,m},$$

where $S^{'}_{n,m} = K[\delta_1, \dots, \delta_n, \xi_1, \dots, \xi_n, \omega_1, \dots, \omega_n, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \dots, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \dots, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm}]$. Now

$$I_{n,m} \cap S^{'}_{n,m} = (I^*_{n-1,m})S^{'}_{n,m} \text{ and}$$

$$\Big(I_{n,m}:\xi_{n+1}\Big)S_{n,m}=\Big((I^*_{n-2,m},\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},,\nu_{n1},\nu_{n2},\dots\nu_{nm})S_{n,m}\Big).$$

So

$$\operatorname{sdepth}(I_{n,m})\geq\min\Big\{\operatorname{sdepth}\Big((I^*_{n,m})S^{'}_{n,m}\Big),\operatorname{sdepth}\Big((I^*_{n-2,m},\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{n1},\nu_{n2},\dots,\nu_{nm})S_{n,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((I^*_{n-2,m},\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{n1},\nu_{n2},\dots,\nu_{nm},)S_{n,m}\Big)$$
$$=\operatorname{sdepth}\Big((I^*_{n-2,m},\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},,\nu_{n1},\nu_{n2},\dots,\nu_{nm})S^{'}_{n,m}\Big)+1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((I^*_{n-2,m},\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},,\nu_{n1},\nu_{n2},\dots,\nu_{nm})S^{'}_{n,m}\Big)\geq$$
$$\min\Big\{\operatorname{sdepth}\Big((I^*_{n-2,m})S^*_{n-2,m}\Big)+2m+3,\operatorname{sdepth}\Big((\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{n1},\nu_{n2},\dots,\nu_{nm})S^{''}_{n,m}\Big)+\operatorname{sdepth}\Big((S^*_{n-2,m}/I^*_{n-2,m})S^*_{n-2,m}\Big)\Big\},$$

where $S^{''}_{n,m}=K[\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{n1},\nu_{n2},\dots,\nu_{nm}]$. Now by Proposition 4.1.4, [21], and Proposition 3.1.7, we obtain

$$\operatorname{sdepth}\Big((I^*_{n-2,m},\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},,\nu_{n1},\nu_{n2},\dots,\nu_{nm})S^{'}_{n,m}\Big)\geq$$
$$\min\Big\{\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil+2m+3,\Big\lceil\frac{2m+3}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil\Big\}=\Big\lceil\frac{2m+3}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil.$$

Therefore

$$\operatorname{sdepth}\Big((I^*_{n-2,m},\delta_n,\xi_n,\omega_n,\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{n1},\nu_{n2},\dots,\nu_{nm})S_{n,m}\Big)\geq\Big\lceil\frac{2m+3}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil+1.$$

Now by proposition 4.1.4, we get

$$\operatorname{sdepth}\Big((I^*_{n-1,m})S^{'}_{n,m}\Big)=\operatorname{sdepth}\Big((I^*_{n-1,m})S^*_{n-1,m}\Big)\geq\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n-2}{2}\Big\rceil+1.$$

To sum up

$$\operatorname{sdepth}(I_{n,m})\geq\Big\lceil\frac{2m+3}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil+1.$$

□

Remark 4.1.12. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I_{n,m}) \geq 2nm+3n+1-\lfloor\frac{4mn+5n}{2}\rfloor$. *Thus our lower bound is finer than this lower bound.*

Corollary 4.1.13. *Let* $n \geq 2$ *and* $m \geq 1$, *then* $\operatorname{sdepth}(I_{n,m}) > \operatorname{sdepth}(S_{n,m}/I_{n,m}) + 1$.

4.2 Stanley depth of edge ideal of graph $E_{n,m}$

In this section Stanley depth of the edge ideal $J_{n,m}$ associated to the graph $E_{n,m}$ will be discussed. For this purpose, we first find Stanley depth of the edge ideal $J^*_{n,m}$ associated to the super graph $F_{n,m}$ of the graph $D_{n,m}$. And then we will use these results in our major proof.

Proposition 4.2.1. *Let* $n = 1$ *and* $m \geq 0$, *then*

$$\operatorname{sdepth}(J^*_{1,m}) \geq \left\lceil \frac{6m+7}{2} \right\rceil.$$

Let $n \geq 2$ *and* $m \geq 0$. *If* $n \equiv 0 \pmod 2$, *then*

$$\operatorname{sdepth}(J^*_{n,m}) \geq \left\lceil \frac{4m+5}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + 1.$$

And if $n \equiv 1 \pmod 2$, *then*

$$\operatorname{sdepth}(J^*_{n,m}) = \left\lceil \frac{8m+7}{2} \right\rceil + \left\lceil \frac{n+3}{2} \right\rceil.$$

Proof. We will prove this result by induction on n. Initially, for $n = 1$. Since $\xi_2 \notin J^*_{1,m}$, thus we have

$$J^*_{1,m} = J^*_{1,m} \cap T^{**}_{1,m} \bigoplus \xi_2 (J^*_{1,m} : \xi_2) T^*_{1,m},$$

where $T^{**}_{1,m} = K[\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m} \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]$. Now

$$J^*_{1,m} \cap T^{**}_{1,m} = (g^*_{1,m}) T^{**}_{1,m} \text{ and}$$

$$(J^*_{1,m} : \xi_2) T^*_{1,m} = (\delta_1, \delta_2, \xi_1, \omega_1, \omega_2, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m) T^*_{1,m}.$$

Therefore

$$\operatorname{sdepth}(J^*_{1,m}) \geq \min\Big\{ \operatorname{sdepth}\Big((g^*_{1,m})T^{**}_{1,m}\Big), \operatorname{sdepth}\Big((\delta_1,\delta_2,\xi_1,\omega_1,\omega_2,\mu_{11},\mu_{12},\dots,\mu_{1m},$$
$$\mu_{21},\mu_{22},\dots,\mu_{2m},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},\dots,\nu_{2m},a_1,b_1,q_1,q_2,\dots,q_m,r_1,r_2,\dots,r_m)T^*_{1,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big(\delta_1,\delta_2,\xi_1,\omega_1,\omega_2,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},$$
$$\dots,\nu_{2m},a_1,b_1,q_1,q_2,\dots,q_m,r_1,r_2,\dots,r_m\Big)T^*_{1,m} = \operatorname{sdepth}\Big((\delta_1,\delta_2,\xi_1,\omega_1,\omega_2,\mu_{11},\mu_{12},$$
$$\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},\dots,\nu_{2m},a_1,b_1,q_1,q_2,\dots,q_m,r_1,r_2,$$
$$\dots,r_m)T^{**}_{1,m}\Big)+1,$$

and by [21],

$$\operatorname{sdepth}\Big((\delta_1,\delta_2,\xi_1,\omega_1,\omega_2,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},$$
$$\dots,\nu_{2m},a_1,b_1,q_1,q_2,\dots,q_m,r_1,r_2,\dots,r_m)T^{**}_{1,m}\Big) = \left\lceil \frac{6m+5}{2} \right\rceil.$$

Consequently

$$\operatorname{sdepth}\Big((\delta_1,\delta_2,\xi_1,\omega_1,\omega_2,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{21},\mu_{22},\dots,\mu_{2m},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{21},\nu_{22},$$
$$\dots,\nu_{2m},a_1,b_1,q_1,q_2,\dots,q_m,r_1,r_2,\dots,r_m)T^*_{1,m}\Big) = \left\lceil \frac{6m+5}{2} \right\rceil + 1 = \left\lceil \frac{6m+7}{2} \right\rceil.$$

Now by Lemma 4.1.1 and [16, Lemma 3.6], we get

$$\operatorname{sdepth}\Big((g^*_{1,m})T^{**}_{1,m}\Big) = \operatorname{sdepth}\Big((g^*_{1,m})A^*_{1,m}\Big) + 2m+2 \geq 2m+3+2m+2 = 4m+5.$$

As a result

$$\operatorname{sdepth}(J^*_{1,m}) \geq \left\lceil \frac{6m+7}{2} \right\rceil.$$

Secondly, for $n=2$. As $\xi_3 \notin J^*_{2,m}$, so we obtain

$$J^*_{2,m} = J^*_{2,m} \cap T^{**}_{2,m} \bigoplus \xi_3(J^*_{2,m} : \xi_3)T^*_{2,m},$$

where $T^{**}_{2,m} = K[\delta_1, \delta_2, \delta_3, \xi_1, \xi_2, \omega_1, \omega_2, \omega_3, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{21}, \nu_{22}, \dots, \nu_{3m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]$.
Furthermore

$$J^*_{2,m} \cap T^{**}_{2,m} = (J^*_{1,m})T^{**}_{2,m} \text{ and}$$

$$\Big(J^*_{2,m} : \xi_3\Big)T^*_{2,m} = \Big((g^*_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})T^*_{2,m}\Big).$$

Consequently

$$\operatorname{sdepth}(J^*_{2,m}) \geq \min\Big\{ \operatorname{sdepth}\Big((J^*_{1,m})T^{**}_{2,m}\Big), \operatorname{sdepth}\Big((g^*_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})T^*_{2,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((g^*_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})T^*_{2,m}\Big) = \operatorname{sdepth}\Big((g^*_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})T^{**}_{2,m}\Big) + 1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((g^*_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})T^{**}_{2,m}\Big) \geq \min\Big\{ \operatorname{sdepth}\Big((g^*_{1,m})A^*_{1,m}\Big) + 4m + 5, \operatorname{sdepth}\Big((\delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{-}_{2,m}\Big) + \operatorname{sdepth}\Big((A^*_{1,m}/g^*_{1,m})A^*_{1,m}\Big)\Big\},$$

where $S^{-}_{2,m} = K[\delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}]$. Now by Proposition 4.1.1, [21] and sdepth of quotient module associated to star graph, we have

$$\operatorname{sdepth}\Big((g^*_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})S^{**}_{2,m}\Big) \geq \min\Big\{6m + 8, \Big\lceil \frac{4m+5}{2} \Big\rceil + 1\Big\} = \Big\lceil \frac{4m+5}{2} \Big\rceil + 1.$$

As a result

$$\text{sdepth}\Big((g^*_{1,m}, \delta_2, \delta_3, \xi_2, \omega_2, \omega_3, \mu_{21}, \mu_{22}, \dots, \mu_{2m}, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \nu_{21}, \nu_{22}, \dots, \nu_{2m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m})T^*_{2,m}\Big) \geq \left\lceil \frac{4m+5}{2} \right\rceil + 1 + 1 = \left\lceil \frac{4m+7}{2} \right\rceil + \left\lceil \frac{2}{2} \right\rceil.$$

Now by induction on n and [16, Lemma 3.6], we get

$$\text{sdepth}\Big((J^*_{1,m})T^{**}_{2,m}\Big) = \text{sdepth}\Big((J^*_{1,m})T^*_{1,m}\Big) + 2m + 2 \geq \left\lceil \frac{6m+7}{2} \right\rceil + 2m + 2 = \left\lceil \frac{10m+11}{2} \right\rceil.$$

To sum up

$$\text{sdepth}(J^*_{2,m}) \geq \left\lceil \frac{4m+7}{2} \right\rceil + \left\lceil \frac{2}{2} \right\rceil.$$

Thirdly, for $n = 3$. Since $\xi_4 \notin J^*_{3,m}$, thus we have

$$J^*_{3,m} = J^*_{3,m} \cap T^{**}_{3,m} \bigoplus \xi_4(J^*_{3,m} : \xi_4)T^*_{3,m},$$

where $T^{**}_{3,m} = K[\delta_1, \dots, \delta_4, \xi_1, \dots, \xi_3, \omega_1, \dots, \omega_4, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \dots, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \dots, \nu_{41}, \nu_{42}, \dots, \nu_{4m}, a_1, b_1, q_1, q_2, \dots, q_m, r_1, r_2, \dots, r_m]$. Now

$$J^*_{3,m} \cap T^{**}_{3,m} = (J^*_{2,m})T^{**}_{3,m} \text{ and}$$

$$\Big(J^*_{3,m} : \xi_4\Big)T^*_{3,m} = \Big((J^*_{1,m}, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m})T^*_{3,m}\Big).$$

As a result

$$\text{sdepth}(J^*_{3,m}) \geq \min\Big\{\text{sdepth}\Big((J^*_{2,m})T^{**}_{3,m}\Big), \text{sdepth}\Big((J^*_{1,m}, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m})T^*_{3,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we get

$$\text{sdepth}\Big((J^*_{1,m}, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m})T^*_{3,m}\Big) = \text{sdepth}\Big((J^*_{1,m}, \delta_3, \delta_4, \xi_3, \omega_3, \omega_4, \mu_{31}, \mu_{32}, \dots, \mu_{3m}, \mu_{41}, \mu_{42}, \dots, \mu_{4m}, \nu_{31}, \nu_{32}, \dots, \nu_{3m}, \nu_{41}, \nu_{42}, \dots, \nu_{4m})T^{**}_{3,m}\Big) + 1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((J^*_{1,m},\delta_3,\delta_4,\xi_3,\omega_3,\omega_4,\mu_{31},\mu_{32},\dots,\mu_{3m},\mu_{41},\mu_{42},\dots,\mu_{4m},\nu_{31},\nu_{32},\dots,\nu_{3m},$$
$$\nu_{41},\nu_{42},\dots,\nu_{4m})T^{**}_{3,m}\Big) \geq \min\Big\{\operatorname{sdepth}\Big((J^*_{1,m})T^*_{1,m}\Big)+4m+5, \operatorname{sdepth}\Big((\delta_3,\delta_4,\xi_3,\omega_3,\omega_4,$$
$$\mu_{31},\mu_{32},\dots,\mu_{3m},\mu_{41},\mu_{42},\dots,\mu_{4m},\nu_{31},\nu_{32},\dots,\nu_{3m},\nu_{41},\nu_{42},\dots,\nu_{4m})T^{-}_{3,m}\Big)$$
$$+\operatorname{sdepth}\Big((T^*_{1,m}/J^*_{1,m})T^*_{1,m}\Big)\Big\},$$

where $T^{-}_{3,m}=K[\delta_3,\delta_4,\xi_3,\omega_3,\omega_4,\mu_{31},\mu_{32},\dots,\mu_{3m},\mu_{41},\mu_{42},\dots,\mu_{4m},\nu_{31},\nu_{32},\dots,\nu_{3m},$ $\nu_{41},\nu_{42},\dots,\nu_{4m}]$. Now by induction on n, [21], and Proposition 3.2.1, we have

$$\operatorname{sdepth}\Big((J^*_{1,m},\delta_3,\delta_4,\xi_3,\omega_3,\omega_4,\mu_{31},\mu_{32},\dots,\mu_{3m},\mu_{41},\mu_{42},\dots,\mu_{4m},\nu_{31},\nu_{32},\dots,\nu_{3m},$$
$$\nu_{41},\nu_{42},\dots,\nu_{4m})T^{**}_{3,m}\Big) \geq \min\Big\{\Big\lceil\frac{6m+7}{2}\Big\rceil+4m+5=\Big\lceil\frac{14m+17}{2}\Big\rceil, \Big\lceil\frac{4m+5}{2}\Big\rceil+2m+3$$
$$=\Big\lceil\frac{8m+11}{2}\Big\rceil\Big\}=\Big\lceil\frac{8m+11}{2}\Big\rceil.$$

Consequently

$$\operatorname{sdepth}\Big((J^*_{1,m},\delta_3,\delta_4,\xi_3,\omega_3,\omega_4,\mu_{31},\mu_{32},\dots,\mu_{3m},\mu_{41},\mu_{42},\dots,\mu_{4m},\nu_{31},\nu_{32},\dots,\nu_{3m},\nu_{41},$$
$$\nu_{42},\dots,\nu_{4m})T^*_{3,m}\Big) \geq \Big\lceil\frac{8m+11}{2}\Big\rceil+1=\Big\lceil\frac{8m+13}{2}\Big\rceil.$$

Now by induction on n and [16, Lemma 3.6], we get

$$\operatorname{sdepth}\Big((J^*_{2,m})T^{**}_{3,m}\Big)=\operatorname{sdepth}\Big((J^*_{2,m})T^*_{2,m}\Big)+2m+2 \geq \Big\lceil\frac{4m+9}{2}\Big\rceil+2m+2=\Big\lceil\frac{8m+13}{2}\Big\rceil.$$

To sum up

$$\operatorname{sdepth}(J^*_{3,m}) \geq \Big\lceil\frac{8m+13}{2}\Big\rceil=\Big\lceil\frac{8m+7}{2}\Big\rceil+\Big\lceil\frac{3+3}{2}\Big\rceil.$$

Lastly, for $n \geq 4$. As we know that $\xi_{n+1} \notin J^*_{n,m}$, therefore we have

$$J^*_{n,m}=J^*_{n,m}\cap T^{**}_{n,m}\bigoplus \xi_{n+1}(J^*_{n,m}:\xi_n)T^*_{n,m},$$

where $T^{**}_{n,m}=K[\delta_1,\dots,\delta_{n+1},\xi_1,\dots,\xi_n,\omega_1,\dots,\omega_{n+1},\mu_{11},\mu_{12},\dots,\mu_{1m},\dots,\mu_{(n+1)1},$ $\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{11},\nu_{12},\dots,\nu_{1m},\dots,\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m},a_1,b_1,q_1,q_2,\dots,q_m,$ $r_1,r_2,\dots,r_m]$. Now

$$J^*_{n,m}\cap T^{**}_{n,m}=(J^*_{n-1,m})T^{**}_{n,m} \text{ and}$$

$$\left(J^*_{n,m}:\xi_{n+1}\right)T^*_{n,m}=\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^*_{n,m}\Big).$$

Thus

$$\operatorname{sdepth}(J^*_{n,m})\geq\min\Big\{\operatorname{sdepth}\Big((J^*_{n-1,m})T^{**}_{n,m}\Big),\operatorname{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})S^*_{n,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have that

$$\operatorname{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},u_{(n+1)1},u_{(n+1)2},\dots,u_{(n+1)m},\nu_{n1},\nu_{n2},\dots,\nu_{nm},v_{(n+1)1},v_{(n+1)2},\dots,v_{(n+1)m})T^*_{n,m}\Big)$$
$$=\operatorname{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^{**}_{n,m}\Big)+1,$$

and by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^{**}_{n,m}\Big)\geq\min\Big\{\operatorname{sdepth}\Big((J^*_{n-2,m})T^*_{n-2,m}\Big)+4m+5,\operatorname{sdepth}\Big((\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^{-}_{n,m}\Big)+\operatorname{sdepth}\Big((T^*_{n-2,m}/J^*_{n-2,m})T^*_{n-2,m}\Big)\Big\},$$

where $T^{-}_{n,m}=K[\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m}]$. Now by induction on n, [21], and Proposition 3.2.1, we have if $n\equiv 0\,(\mathrm{mod}\,2)$, then $n-2\equiv 0\,(\mathrm{mod}\,2)$. So

$$\operatorname{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},\nu_{n2},\dots\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^{**}_{n,m}\Big)\geq\min\Big\{\Big\lceil\frac{4m+7}{2}\Big\rceil+\Big\lceil\frac{n-2}{2}\Big\rceil+4m+5,\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n}{2}\Big\rceil\Big\}=\Big\lceil\frac{4m+5}{2}\Big\rceil+\Big\lceil\frac{n}{2}\Big\rceil.$$

Thus

$$\text{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},$$
$$\nu_{n1},\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^*_{n,m}\Big) \geq \left\lceil\frac{4m+5}{2}\right\rceil + \left\lceil\frac{n}{2}\right\rceil + 1.$$

If $n \equiv 1 \pmod 2$, then $n-2 \equiv 1 \pmod 2$. So

$$\text{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},$$
$$\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^{**}_{n,m}\Big) \geq \min\Big\{\left\lceil\frac{8m+7}{2}\right\rceil + \left\lceil\frac{n+1}{2}\right\rceil + 4m+5$$
$$,\left\lceil\frac{4m+5}{2}\right\rceil + 2m + \left\lceil\frac{n+3}{2}\right\rceil\Big\} = \left\lceil\frac{4m+5}{2}\right\rceil + \left\lceil\frac{n+3}{2}\right\rceil + 2m = \left\lceil\frac{8m+5}{2}\right\rceil + \left\lceil\frac{n+3}{2}\right\rceil.$$

As a result

$$\text{sdepth}\Big((J^*_{n-2,m},\delta_n,\delta_{n+1},\xi_n,\omega_n,\omega_{n+1},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\mu_{(n+1)1},\mu_{(n+1)2},\dots,\mu_{(n+1)m},\nu_{n1},$$
$$\nu_{n2},\dots,\nu_{nm},\nu_{(n+1)1},\nu_{(n+1)2},\dots,\nu_{(n+1)m})T^*_{n,m}\Big) \geq \left\lceil\frac{8m+5}{2}\right\rceil + \left\lceil\frac{n+3}{2}\right\rceil + 1$$
$$= \left\lceil\frac{8m+7}{2}\right\rceil + \left\lceil\frac{n+3}{2}\right\rceil.$$

Now by induction on n and [16, Lemma 3.6], if $n \equiv 0 \pmod 2$, then $n-1 \equiv 1 \pmod 2$. We get

$$\text{sdepth}\Big((J^*_{n-1,m})T^{**}_{n,m}\Big) = \text{sdepth}\Big((J^*_{n-1,m})T^*_{n-1,m}\Big) + 2m+2 \geq \left\lceil\frac{8m+7}{2}\right\rceil + \left\lceil\frac{n+3}{2}\right\rceil + 2m+2.$$

And if $n \equiv 1 \pmod 2$, then $n-1 \equiv 0 \pmod 2$. We get

$$\text{sdepth}\Big((J^*_{n-1,m})T^{**}_{n,m}\Big) = \text{sdepth}\Big((J^*_{n-1,m})T^*_{n-1,m}\Big) + 2m+2 \geq \left\lceil\frac{4m+7}{2}\right\rceil + \left\lceil\frac{n}{2}\right\rceil + 2m+2.$$

In a nutshell

if $n \equiv 0 \pmod 2$, then

$$\text{sdepth}(J^*_{n,m}) \geq \left\lceil\frac{4m+5}{2}\right\rceil + \left\lceil\frac{n}{2}\right\rceil + 1.$$

And if $n \equiv 1 \pmod 2$, then

$$\text{sdepth}(J^*_{n,m}) = \left\lceil\frac{8m+7}{2}\right\rceil + \left\lceil\frac{n+3}{2}\right\rceil.$$

□

Remark 4.2.2. *By [23, Theorem 2.3]* $\operatorname{sdepth}(J^*_{n,m}) \geq 2nm + 3n + 4m + 5 - \lfloor \frac{4mn+5n+3m+2}{2} \rfloor$. *Therefore our lower bound is finer than this one.*

Proposition 4.2.3. *Let* $n = 3$ *and* $m \geq 0$. *Then*

$$\operatorname{sdepth}(J_{3,m}) \geq \left\lceil \frac{4m+8}{2} \right\rceil.$$

For $n = 4$ *and* $m \geq 0$ *,*

$$\operatorname{sdepth}(J_{4,m}) \geq \left\lceil \frac{4m+9}{2} \right\rceil.$$

Finally, for $n \geq 5$ *and* $m \geq 0$. *If* $n \equiv 0 \pmod 2$, *then*

$$\operatorname{sdepth}(J_{n,m}) \geq \left\lceil \frac{4m+5}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, *then*

$$\operatorname{sdepth}(J_{n,m}) \geq \left\lceil \frac{8m+6}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Initially, for $n = 3$. Since $y \notin J_{3,m}$, thus we have

$$J_{3,m} = J_{3,m} \cap T'_{3,m} \bigoplus y(J_{3,m} : y)T_{3,m},$$

where $T'_{3,m} = K[\delta_1, \ldots, \delta_3, \xi_2, \xi_3, \omega_1, \ldots, \omega_3, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \ldots, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}\nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \ldots, \nu_{31}, \nu_{32}, \ldots, \nu_{3m}]$. Now

$$J_{3,m} \cap T'_{1,m} = (J^*_{1,m})T'_{1,m} \text{ and}$$

$$\Big(J_{3,m} : y\Big)T_{3,m} = \Big((\delta_1, \delta_3, \xi_2, \xi_3, \omega_1, \omega_3, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \ldots, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \ldots, \nu_{31}, \nu_{32}, \ldots, \nu_{3m})T_{3,m}\Big).$$

As a result

$$\operatorname{sdepth}(J_{3,m}) \geq \min\Big\{ \operatorname{sdepth}\Big((J^*_{1,m})T'_{3,m}\Big), \operatorname{sdepth}\Big((\delta_1, \delta_3, \xi_2, \xi_3, \omega_1, \omega_3, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \ldots, \mu_{31}, \mu_{32}, \ldots, \mu_{3m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \ldots, \nu_{31}, \nu_{32}, \ldots, \nu_{3m})T_{3,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\text{sdepth}\Big((\delta_1,\delta_3,\xi_2,\xi_3,\omega_1,\omega_3,\mu_{11},\mu_{12},\dots,\mu_{1m},\dots,\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{11},\nu_{12},\dots,\nu_{1m},\dots,\nu_{31},\nu_{32},\dots,\nu_{3m})T_{3,m}\Big) = \text{sdepth}\Big((\delta_1,\delta_3,\xi_2,\xi_3,\omega_1,\omega_3,\mu_{11},\mu_{12},\dots,\mu_{1m},\dots,\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{11},\nu_{12},\dots,\nu_{1m},\dots,\nu_{31},\nu_{32},\dots,\nu_{3m})T^{'}_{3,m}\Big)+1,$$

and by [21],

$$\text{sdepth}\Big((\delta_1,\delta_3,\xi_2,\xi_3,\omega_1,\omega_3,\mu_{11},\mu_{12},\dots,\mu_{1m},\dots,\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{11},\nu_{12},\dots,\nu_{1m},\dots,\nu_{31},\nu_{32},\dots,\nu_{3m})T^{'}_{3,m}\Big) = \Big\lceil\frac{4m+6}{2}\Big\rceil.$$

Thus

$$\text{sdepth}\Big((\delta_1,\delta_3,\xi_2,\xi_3,\omega_1,\omega_3,\mu_{11},\mu_{12},\dots,\mu_{1m},\dots,\mu_{31},\mu_{32},\dots,\mu_{3m},\nu_{11},\nu_{12},\dots,\nu_{1m},\dots,\nu_{31},\nu_{32},\dots,\nu_{3m})T_{3,m}\Big) = \Big\lceil\frac{4m+6}{2}\Big\rceil+1 = \Big\lceil\frac{4m+8}{2}\Big\rceil.$$

Now

$$\text{sdepth}\Big((J^*_{1,m})T^{'}_{3,m}\Big) = \text{sdepth}\Big((J^*_{1,m})T^*_{1,m}\Big) \geq 6m+7.$$

To sum up

$$\text{sdepth}(J_{3,m}) \geq \Big\lceil\frac{4m+8}{2}\Big\rceil.$$

Furthermore, let us have $n=4$. As $y \notin J_{4,m}$, therefore we have

$$J_{4,m} = J_{4,m}\cap T^{'}_{4,m} \bigoplus y(J_{4,m}:y)T_{4,m},$$

where $T^{'}_{4,m} = K[\delta_1,\dots,\delta_4,\xi_2,\dots,\xi_4,\omega_1,\dots,\omega_4,\mu_{11},\mu_{12},\dots,\mu_{1m},\dots,\mu_{41},\mu_{42},\dots,\mu_{4m},\nu_{11},\nu_{12},\dots,\nu_{1m},\dots,\nu_{41},\nu_{42},\dots,\nu_{4m}]$. Now

$$J_{4,m}\cap T^{'}_{4,m} = (J^*_{2,m})T^{'}_{4,m} \text{ and}$$

$$\Big(J_{4,m}:y\Big)T_{4,m} = \Big((g^*_{1,m},\delta_1,\delta_4,\xi_2,\xi_4,\omega_1,\omega_4,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{41},\mu_{42},\dots,\mu_{4m},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{41},\nu_{42},\dots,\nu_{4m})T_{4,m}\Big).$$

Thus

$$\operatorname{sdepth}(J_{4,m}) \geq \min\Big\{ \operatorname{sdepth}\Big((J^*_{2,m})T'_{4,m}\Big), \operatorname{sdepth}\Big((g^*_{1,m}, \delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})T_{4,m}\Big)\Big\}.$$

By [16, Lemma 3.6], we have

$$\operatorname{sdepth}\Big((g^*_{1,m}, \delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})T_{4,m}\Big) = \operatorname{sdepth}\Big((g^*_{1,m}, \delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})T'_{4,m}\Big) + 1.$$

And by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((g^*_{1,m}, \delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})T'_{4,m}\Big) \geq \min\Big\{ \operatorname{sdepth}\Big((g^*_{1,m})A^*_{1,m}\Big) + 4m + 6, \operatorname{sdepth}\Big((\delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})T^{-}_{4,m}\Big) + \operatorname{sdepth}\Big((A^*_{1,m}/g^*_{1,m})A^*_{1,m}\Big)\Big\},$$

where $T^{-}_{4,m} = K[\delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m}]$. Now by 4.1.1, [21], and sdepth of quotient module associated to star graph, we have

$$\operatorname{sdepth}\Big((g^*_{1,m}, \delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})S'_{2,m}\Big) \geq \min\Big\{6m+9, \Big\lceil \frac{4m+6}{2} \Big\rceil + 1\Big\} = \Big\lceil \frac{4m+6}{2} \Big\rceil + 1 = \Big\lceil \frac{4m+8}{2} \Big\rceil.$$

As a result

$$\operatorname{sdepth}\Big((g^*_{1,m}, \delta_1, \delta_4, \xi_2, \xi_4, \omega_1, \omega_4, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{41}, \mu_{42}, \ldots, \mu_{4m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{41}, \nu_{42}, \ldots, \nu_{4m})T_{4,m}\Big) \geq \Big\lceil \frac{4m+8}{2} \Big\rceil + 1 = \Big\lceil \frac{4m+10}{2} \Big\rceil.$$

Now by Proposition 4.2.1, we get

$$\operatorname{sdepth}\Big((J^*_{2,m})T'_{4,m}\Big) \cong \operatorname{sdepth}\Big((J^*_{2,m})T^*_{2,m}\Big) \geq \Big\lceil \frac{4m+9}{2} \Big\rceil.$$

To conclude

$$\text{sdepth}(J_{4,m}) \geq \left\lceil \frac{4m+9}{2} \right\rceil.$$

Next we have $n = 5$. Since $y \notin J_{5,m}$, therefore we have

$$J_{5,m} = J_{5,m} \cap T^{'}_{5,m} \bigoplus y(J_{5,m} : y)T_{5,m},$$

where $T^{'}_{5,m} = K[\delta_1, \ldots, \delta_5, \xi_2, \ldots, \xi_5, \omega_1, \ldots, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \ldots, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \ldots, \nu_{51}, \nu_{52}, \ldots, \nu_{5m}]$. Now

$$J_{5,m} \cap T^{'}_{5,m} \cong (J^{*}_{3,m})T^{'}_{5,m} \text{ and}$$

$$\Big(J_{5,m} : y\Big)T_{5,m} = \Big((J^{*}_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T_{5,m}\Big).$$

Consequently

$$\text{sdepth}(J_{5,m}) \geq \min\Big\{ \text{sdepth}\Big((J^{*}_{3,m})T^{'}_{5,m}\Big), \text{sdepth}\Big((J^{*}_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T_{5,m}\Big)\Big\}.$$

From [16, Lemma 3.6], we obtain

$$\text{sdepth}\Big((J^{*}_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T_{5,m}\Big) = \text{sdepth}\Big((J^{*}_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T^{'}_{5,m}\Big) + 1.$$

And by [17, Theorem 1.3],

$$\text{sdepth}\Big((J^{*}_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T^{'}_{5,m}\Big) \geq \min\Big\{ \text{sdepth}\Big((J^{*}_{1,m})T^{*}_{1,m}\Big) + 4m + 6, \text{sdepth}\Big((\delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T^{-}_{5,m}\Big) + \text{sdepth}\Big((T^{*}_{1,m}/J^{*}_{1,m})T^{*}_{1,m}\Big)\Big\},$$

where $T^{-}_{5,m} = K[\delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m}]$. Now by Propositions 3.2.1 , 4.2.1 and [21], we have

$$\operatorname{sdepth}\Big((J^*_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T^{'}_{5,m}\Big) \geq \min\Big\{\Big\lceil\frac{6m+7}{2}\Big\rceil + 4m+6 = \Big\lceil\frac{14m+19}{2}\Big\rceil, \Big\lceil\frac{4m+6}{2}\Big\rceil + 2m+3 = \Big\lceil\frac{8m+12}{2}\Big\rceil\Big\} = \Big\lceil\frac{8m+12}{2}\Big\rceil.$$

So

$$\operatorname{sdepth}\Big((J^*_{1,m}, \delta_1, \delta_5, \xi_2, \xi_5, \omega_1, \omega_5, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{51}, \mu_{52}, \ldots, \mu_{5m}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{51}, \nu_{52}, \ldots, \nu_{5m})T_{5,m}\Big) \geq \Big\lceil\frac{8m+12}{2}\Big\rceil + 1 = \Big\lceil\frac{8m+14}{2}\Big\rceil.$$

Now by Proposition 4.2.1, we get

$$\operatorname{sdepth}\Big((J^*_{3,m})T^{'}_{5,m}\Big) \cong \operatorname{sdepth}\Big((J^*_{3,m})T^*_{3,m}\Big) \geq \Big\lceil\frac{8m+13}{2}\Big\rceil.$$

To conclude

$$\operatorname{sdepth}(J_{5,m}) \geq \Big\lceil\frac{8m+13}{2}\Big\rceil = \Big\lceil\frac{8m+7}{2}\Big\rceil + \Big\lceil\frac{5+1}{2}\Big\rceil.$$

Ultimately, for $n \geq 6$. Since $y \notin J_{n,m}$, so we have

$$J_{n,m} = J_{n,m} \cap T^{'}_{n,m} \bigoplus y(J_{n,m} : y)T_{n,m},$$

where $T^{'}_{n,m} = K[\delta_1, \ldots, \delta_n, \xi_2, \ldots, \xi_n, \omega_1, \ldots, \omega_n, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \ldots, \mu_{n1}, \mu_{n2}, \ldots, \mu_{nm}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \ldots, \nu_{n1}, \nu_{n2}, \ldots, \nu_{nm}]$. Now

$$J_{n,m} \cap T^{'}_{n,m} = (J^*_{n-2,m})T^{'}_{n,m} \text{ and}$$

$$\Big((J_{n,m} : y)T_{n,m}\Big) = \Big((J^*_{n-4,m}, \delta_1, \delta_n, \xi_2, \xi_n, \omega_1, \omega_n, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{n1}, \mu_{n2}, \ldots, \mu_{nm}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{n1}, \nu_{n2}, \ldots, \nu_{nm})T_{n,m}\Big).$$

Thus

$$\operatorname{sdepth}(J_{n,m}) \geq \min\Big\{\operatorname{sdepth}\Big((J^*_{n-2,m})T^{'}_{n,m}\Big), \operatorname{sdepth}\Big((J^*_{n-4,m}, \delta_1, \delta_n, \xi_2, \xi_n, \omega_1, \omega_n, \mu_{11}, \mu_{12}, \ldots, \mu_{1m}, \mu_{n1}, \mu_{n2}, \ldots, \mu_{nm}, \nu_{11}, \nu_{12}, \ldots, \nu_{1m}, \nu_{n1}, \nu_{n2}, \ldots, \nu_{nm})T_{n,m}\Big)\Big\}.$$

From [16, Lemma 3.6], we get

$$\operatorname{sdepth}\Big((J^*_{n-4,m},\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm})T_{n,m}\Big) = \operatorname{sdepth}\Big((J^*_{n-4,m},\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm})T^{'}_{n,m}\Big)+1.$$

And by [17, Theorem 1.3],

$$\operatorname{sdepth}\Big((J^*_{n-4,m},\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm})T^{'}_{n,m}\Big) \geq \min\Big\{\operatorname{sdepth}\Big((J^*_{n-4,m})T^*_{n-4,m}\Big)+4m+6, \operatorname{sdepth}\Big((\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm})T^{-}_{n,m}\Big) + \operatorname{sdepth}\Big((T^*_{n-4,m}/J^*_{n-4,m})T^*_{n-4,m}\Big)\Big\},$$

where $T^{-}_{n,m}=K[\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm}]$. Now by [21], Propositions 3.2.1 and 4.2.1, we have
if $n\equiv 0 \pmod 2$, then $n-4\equiv 0 \pmod 2$. So

$$\operatorname{sdepth}\Big((J^*_{n-4,m},\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm})T^{'}_{n,m}\Big) \geq \min\Big\{\Big\lceil\frac{4m+7}{2}\Big\rceil+\Big\lceil\frac{n-4}{2}\Big\rceil+4m+6, \Big\lceil\frac{4m+6}{2}\Big\rceil+\Big\lceil\frac{n-2}{2}\Big\rceil\Big\} = \Big\lceil\frac{4m+6}{2}\Big\rceil+\Big\lceil\frac{n-2}{2}\Big\rceil.$$

To conclude

$$\operatorname{sdepth}\Big((J^*_{n-4,m},\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm})T_{n,m}\Big) \geq \Big\lceil\frac{4m+6}{2}\Big\rceil+\Big\lceil\frac{n-2}{2}\Big\rceil+1=\Big\lceil\frac{4m+6}{2}\Big\rceil+\Big\lceil\frac{n}{2}\Big\rceil.$$

And if $n\equiv 1 \pmod 2$, then $n-4\equiv 1 \pmod 2$. So

$$\operatorname{sdepth}\Big((J^*_{n-4,m},\delta_1,\delta_n,\xi_2,\xi_n,\omega_1,\omega_n,\mu_{11},\mu_{12},\dots,\mu_{1m},\mu_{n1},\mu_{n2},\dots,\mu_{nm},\nu_{11},\nu_{12},\dots,\nu_{1m},\nu_{n1},\nu_{n2},\dots,\nu_{nm})T^{'}_{n,m}\Big) \geq \min\Big\{\Big\lceil\frac{8m+7}{2}\Big\rceil+\Big\lceil\frac{n-1}{2}\Big\rceil+4m+6=\Big\lceil\frac{16m+7}{2}\Big\rceil+\Big\lceil\frac{n+11}{2}\Big\rceil, \Big\lceil\frac{4m+6}{2}\Big\rceil+2m+\Big\lceil\frac{n+1}{2}\Big\rceil\Big\} = \Big\lceil\frac{4m+6}{2}\Big\rceil+\Big\lceil\frac{n+1}{2}\Big\rceil+2m=\Big\lceil\frac{8m+6}{2}\Big\rceil+\Big\lceil\frac{n+1}{2}\Big\rceil.$$

Eventually

$$\operatorname{sdepth}\Big((J^*_{n-4,m}, \delta_1, \delta_n, \xi_2, \xi_n, \omega_1, \omega_n, \mu_{11}, \mu_{12}, \dots, \mu_{1m}, \mu_{n1}, \mu_{n2}, \dots, \mu_{nm}, \nu_{11}, \nu_{12}, \dots, \nu_{1m}, \nu_{n1}, \nu_{n2}, \dots, \nu_{nm})T_{n,m}\Big) \geq \left\lceil \frac{8m+6}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil + 1 = \left\lceil \frac{8m+8}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil.$$

Now by Proposition 4.2.1, if $n \equiv 0 \pmod 2$, then $n-2 \equiv 0 \pmod 2$. We get

$$\operatorname{sdepth}\Big((J^*_{n-2,m})T'_{n,m}\Big) \cong \operatorname{sdepth}\Big((J^*_{n-2,m})T^*_{n-2,m}\Big) \geq \left\lceil \frac{4m+7}{2} \right\rceil + \left\lceil \frac{n-2}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, then $n-2 \equiv 1 \pmod 2$. So we have

$$\operatorname{sdepth}\Big((J^*_{n-2,m})T'_{n,m}\Big) \cong \operatorname{sdepth}\Big((J^*_{n-2,m})T^*_{n-2,m}\Big) \geq \left\lceil \frac{8m+7}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil.$$

In a nutshell
if $n \equiv 0 \pmod 2$, then

$$\operatorname{sdepth}(J_{n,m}) \geq \left\lceil \frac{4m+5}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, then

$$\operatorname{sdepth}(J_{n,m}) \geq \left\lceil \frac{8m+7}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil.$$

□

Remark 4.2.4. *By [23, Theorem 2.3]* $\operatorname{sdepth}(J_{n,m}) \geq 2nm + 3n - \lfloor \frac{4mn+5}{2} \rfloor$. *Therefore our lower bound is better than this one.*

Corollary 4.2.5. *Let* $n = 3$ *and* $m \geq 0$*, then*

$$\operatorname{sdepth}(J_{n,m}) = \operatorname{sdepth}(T_{n,m}/J_{n,m}) + 1.$$

Let $n \geq 4$*, and* $m \geq 0$*. Then*

$$\operatorname{sdepth}(J_{n,m}) > \operatorname{sdepth}(T_{n,m}/J_{n,m}) + 1.$$

4.3 Bounds for the dimension of cyclic modules associated to graphs $B_{n,m}$ and $E_{n,m}$

Proposition 4.3.1. *For $n \geq 1$ and $m \geq 0$, we have that $dim(S_{n,m}/I_{n,m}) \geq 2nm + 2n$.*

Proof. Let $\alpha = \{\xi_1, \xi_2, \ldots, \xi_{n+1}\}$ be a subset of a vertex set $V(B_{n,m})$. The set α is a vertex cover because it covers all the edges. Now if we remove ξ_i for some $2 \leq i \leq n$ or ξ_1 or ξ_{n+1} from the set α, then the resulting set is not a vertex cover. Because the edges $\{\xi_i\delta_{i-1}, \xi_i\delta_i, \xi_i\omega_{i-1}, \xi_i\omega_i, \xi_i\mu_{(i-1)1}, \xi_i\mu_{(i-1)2}, \ldots, \xi_i\mu_{(i-1)m}, \xi_i\mu_{i1}, \xi_i\mu_{i2}, \ldots, \xi_i\mu_{im}, \xi_i\nu_{(i-1)1}, \xi_i\nu_{(i-1)2}, \ldots, \xi_i\nu_{(i-1)m}, \xi_i\nu_{i1}, \xi_i\nu_{i2}, \ldots, \xi_i\nu_{im}\}$, $\{\xi_1\delta_1, \xi_1\omega_1, \xi_1\mu_{11}, \xi_1\mu_{12}, \ldots, \xi_1\mu_{1m}, \xi_1\nu_{11}, \xi_1\nu_{12}, \ldots, \xi_1\nu_{1m}\}$, $\{\xi_{n+1}\delta_n, \xi_{n+1}\omega_n, \xi_{n+1}\mu_{n1}, \xi_{n+1}\mu_{n2}, \ldots, \xi_{n+1}\mu_{nm}, \xi_{n+1}\nu_{n1}, \xi_{n+1}\nu_{n2}, \ldots, \xi_{n+1}\nu_{nm}\}$ will not covered, respectively. This shows that the set α forms a minimal vertex cover of $I_{n,m}$. Thus we have height$(I_{n,m}) \leq n+1$. Since $S_{n,m}$ is a polynomial ring of dimension $2nm+3n+1$, which implies that $dim(S_{n,m}/I_{n,m}) \geq 2nm+3n+1-(n+1) = 2nm + 2n$. □

Proposition 4.3.2. *For $n \geq 3$ and $m \geq 0$, we have that $dim(T_{n,m}/J_{n,m}) \geq 2nm + 2n$.*

Proof. Assume that $\beta = \{y, \xi_2, \xi_3, \ldots, \xi_n\}$ be a subset of a vertex set $V(E_{n,m})$. The set β is a vertex cover because it covers all the edges. Now if we remove ξ_i for some $2 \leq i \leq n$ or y from set β then the resulting set is not a vertex cover. Because the sets of edges $\{\xi_i\delta_{i-1}, \xi_i\delta_i, \xi_i\omega_{i-1}, \xi_i\omega_i, \xi_i\mu_{(i-1)1}, \xi_i\mu_{(i-1)2}, \ldots, \xi_i\mu_{(i-1)m}, \xi_i\mu_{i1}, \xi_i\mu_{i2}, \ldots, \xi_i\mu_{im}, \xi_i\nu_{(i-1)1}, \xi_i\nu_{(i-1)2}, \ldots, \xi_i\nu_{(i-1)m}, \xi_i\nu_{i1}, \xi_i\nu_{i2}, \ldots, \xi_i\nu_{im}\}$ and $\{y\delta_1, y\omega_1, y\mu_{11}, y\mu_{12}, \ldots, y\mu_{1m}, y\nu_{11}, y\nu_{12}, \ldots, y\nu_{1m}, y\delta_n, y\omega_n, y\mu_{n1}, y\mu_{n2}, \ldots, y\mu_{nm}, y\nu_{n1}, y\nu_{n2}, \ldots, y\nu_{nm}\}$ will not covered, respectively. This exhibits that the set β forms a minimal vertex cover of $J_{n,m}$. Therefore, we have height$(J_{n,m}) \leq n$. As we know that $T_{n,m}$ is a polynomial ring of dimension $2nm + 3n$, which implies that $dim(T_{n,m}/J_{n,m}) \geq 2nm + 3n - n = 2nm + 2n$. □

Chapter 5

Conclusion

In this chapter we will justify our results by comparing them with the already known results in literature. Moreover, we will examine the Herzog's conjecture given in [25]. And, in some cases Rauf's Question presented in [18].

5.1 The graph $B_{n,m}$, and its super graph $D_{n,m}$ with edge ideals $I_{n,m}$ and $I^*_{n,m}$, respectively

Proposition 5.1.1. *Let $n \geq 1$ and $m = 0$, then* $\operatorname{depth}(S^*_n/I^*_n) = \operatorname{sdepth}(S^*_n/I^*_n) = \lceil \frac{n+1}{2} \rceil$.

Proposition 5.1.2. *Let $n, m \geq 1$, then* $\operatorname{depth}(S^*_{n,m}/I^*_{n,m}) = \operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) = \lceil \frac{n+1}{2} \rceil$.

Remark 5.1.3. *Clearly, for $n \geq 1$ and $m \geq 0$ diam$(D_{n,m}) = n+1$, then by Theorem 2.3.3 we have*
$\operatorname{depth}(S^*_{n,m}/I^*_{n,m}), \operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) \geq \lceil \frac{n+2}{3} \rceil$. *Our Proposition 3.1.2 shows that*
$\operatorname{depth}(S^*_{n,m}/I^*_{n,m}), \operatorname{sdepth}(S^*_{n,m}/I^*_{n,m}) = \lceil \frac{n+1}{2} \rceil$. *Thus we find a better result for depth and stanley depth of this class of edge ideal.*

Proposition 5.1.4. *Let $n \geq 1$ and $m = 0$. If $n \equiv 0 \pmod 2$, then*

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(S_n/I_n), \operatorname{sdepth}(S_n/I_n) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, *then*

$$\operatorname{depth}(S_n/I_n) = \operatorname{sdepth}(S_n/I_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Remark 5.1.5. *Clearly diam*$(B_n) = n$ *then by Theorem 2.3.3 we have* $\operatorname{depth}(S_n/I_n)$ $,\operatorname{sdepth}(S_n/I_n) \geq \lceil \frac{n+1}{3} \rceil$. *Our Proposition 3.1.4 shows that if* $n \equiv 0 \pmod 2$, *then* $\operatorname{depth}(S_n/I_n), \operatorname{sdepth}(S_n/I_n) \geq \lceil \frac{n}{2} \rceil$. *And if* $n \equiv 1 \pmod 2$, *then* $\operatorname{depth}(S_n/I_n)$, $\operatorname{sdepth}(S_n/I_n) = \lceil \frac{n+1}{2} \rceil$. *Thus in both cases we find a better results for depth and stanley depth of this type of cyclic modules.*

Proposition 5.1.6. *Let* $n, m \geq 1$. *If* $n \equiv 0 \pmod 2$, *then*

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m}) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, *then*

$$\operatorname{depth}(S_{n,m}/I_{n,m}) = \operatorname{sdepth}(S_{n,m}/I_{n,m}) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Remark 5.1.7. *Apparently for* $n = 1$ *diam*$(B_{1,m}) = 2$ *and for* $n \geq 2$ *diam*$(B_{n,m}) = n$. *So by Theorem 2.3.3 for* $n = 1$ *we have,* $\operatorname{depth}(S_{1,m}/I_{1,m}), \operatorname{sdepth}(S_{1,m}/I_{1,m}) \geq \lceil \frac{2+1}{3} \rceil$. *And for* $n \geq 2$, *we have* $\operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m}) \geq \lceil \frac{n+1}{3} \rceil$. *Our Proposition 3.1.7 shows that if* $n \equiv 0 \pmod 2$, *then* $\operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m})$ $\geq \lceil \frac{n}{2} \rceil$. *And if* $n \equiv 1 \pmod 2$, *then* $\operatorname{depth}(S_{n,m}/I_{n,m}), \operatorname{sdepth}(S_{n,m}/I_{n,m}) = \lceil \frac{n+1}{2} \rceil$. *Thus in both cases we find a sharp results for depth and stanley depth of this class of quotient module.*

Proposition 5.1.8. *Let* $n \geq 1$ *and* $m = 0$ *then* $\operatorname{sdepth}(I_n^*) \geq \lceil \frac{n+7}{2} \rceil$.

Remark 5.1.9. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I_n^*) \geq 3n + 3 - \lfloor \frac{5n+2}{2} \rfloor$. *Thus our lower bound is sharper than this one.*

Proposition 5.1.10. *Assume that* $n, m \geq 1$, *then* $\operatorname{sdepth}(I_{n,m}^*) \geq \lceil \frac{4m+5}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 1$.

Remark 5.1.11. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I_{n,m}^*) \geq 2nm + 3n + 2m + 3 - \lfloor \frac{4mn+5n+2m+2}{2} \rfloor$. *Thus our lower bound is sharper than this one.*

Proposition 5.1.12. *Let $n \geq 1$ and $m = 0$. If $n = 1$, then* $\operatorname{sdepth}(I_1) = 2$. *And if $n \geq 2$, then* $\operatorname{sdepth}(I_n) \geq \lceil \frac{n+5}{2} \rceil$.

Remark 5.1.13. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I_n) \geq 3n + 1 - \lfloor \frac{5n}{2} \rfloor$. *Thus our lower bound is sharper than this one.*

Corollary 5.1.14. *(Rauf's question) For $n = 1$ and $m = 0$* $\operatorname{sdepth}(I_1) = \operatorname{sdepth}(S_1/I_1) + 1$. *And for $n \geq 2$* $\operatorname{sdepth}(I_n) > \operatorname{sdepth}(S_n/I_n) + 1$.

Corollary 5.1.15. *(Herzog's conjecture) For $n \geq 1$ and $m = 0$* $\operatorname{sdepth}(I_n) > \operatorname{sdepth}(S_n/I_n)$.

Proposition 5.1.16. *If $n = 1$ and $m \geq 1$, then* $\operatorname{sdepth}(I_{1,m}) \geq m + 2$.

Remark 5.1.17. *In this case, we have no conclusion about Herzog's conjecture and Rauf's question as well.*

Proposition 5.1.18. *If $n \geq 2$ and $m \geq 1$, then* $\operatorname{sdepth}(I_{n,m}) \geq \lceil \frac{2m+3}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 1$.

Remark 5.1.19. *By [23, Theorem 2.3]* $\operatorname{sdepth}(I^*_{n,m}) \geq 2nm + 3n + 1 - \lfloor \frac{4mn+5n}{2} \rfloor$. *Thus our lower bound is stronger than this lower bound.*

Corollary 5.1.20. *(Rauf's question) Let $n \geq 2$ and $m \geq 1$, then* $\operatorname{sdepth}(I_{n,m}) > \operatorname{sdepth}(S_{n,m}/I_{n,m}) + 1$.

Corollary 5.1.21. *(Herzog's conjecture) Let $n \geq 2$ and $m \geq 1$, then* $\operatorname{sdepth}(I_{n,m}) > \operatorname{sdepth}(S_{n,m}/I_{n,m})$.

5.2 The graph $E_{n,m}$ and the super graph $F_{n,m}$ of the graph $D_{n,m}$ with edge ideals $J_{n,m}$ and $J^*_{n,m}$, respectively

Proposition 5.2.1. *Let $n \geq 1$ and $m \geq 0$. If $n \equiv 0 \pmod 2$, then*

$$\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = \operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) = \left\lceil \frac{n+2}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, then

$$\operatorname{depth}(T^*_{n,m}/J^*_{n,m}) = \operatorname{sdepth}(T^*_{n,m}/J^*_{n,m}) = 2m + \left\lceil \frac{n+5}{2} \right\rceil.$$

Remark 5.2.2. *Apparently* $diam(F_{n,m}) = n + 2$*, so by Theorem 2.3.3 we have* depth $(T^*_{n,m}/J^*_{n,m}), \mathrm{sdepth}(T^*_{n,m}/J^*_{n,m}) \geq \lceil \frac{n+3}{3} \rceil$*. Whereas our Proposition 3.2.1 shows that if* $n \equiv 0 \pmod 2$*, then* $\mathrm{depth}(T^*_{n,m}/J^*_{n,m}) = \mathrm{sdepth}(T^*_{n,m}/J^*_{n,m}) = \lceil \frac{n+2}{2} \rceil$*. And if* $n \equiv 1 \pmod 2$*, then* $\mathrm{depth}(T^*_{n,m}/J^*_{n,m}) = \mathrm{sdepth}(T^*_{n,m}/J^*_{n,m}) = 2m + \lceil \frac{n+5}{2} \rceil$*. Thus we have better results for depth and Stanley depth of this class of cyclic modules.*

Proposition 5.2.3. *Let* $n \geq 3$ *and* $m \geq 0$*. If* $n \equiv 0 \pmod 2$*, then*

$$\mathrm{depth}(T_{n,m}/J_{n,m}) = \mathrm{sdepth}(T_{n,m}/J_{n,m}) = \left\lceil \frac{n}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$*, then*

$$\mathrm{depth}(T_{n,m}/J_{n,m}) = \mathrm{sdepth}(T_{n,m}/J_{n,m}) = 2m + \left\lceil \frac{n+3}{2} \right\rceil.$$

Remark 5.2.4. *As* $diam(E_{n,m}) = \lceil \frac{n+1}{2} \rceil$ *so by Theorem 2.3.3 we have* $\mathrm{depth}(T_{n,m}/J_{n,m}), \mathrm{sdepth}(T_{n,m}/J_{n,m}) \geq \lceil \frac{n+3}{6} \rceil$*. While our Proposition 3.2.3 shows that if* $n \equiv 0 \pmod 2$*, then* $\mathrm{depth}(T_{n,m}/J_{n,m}) = \mathrm{sdepth}(T_{n,m}/J_{n,m}) = \lceil \frac{n}{2} \rceil$*. And if* $n \equiv 1 \pmod 2$*, then* $\mathrm{depth}(T_{n,m}/J_{n,m}) = \mathrm{sdepth}(T_{n,m}/J_{n,m}) = 2m + \lceil \frac{n+3}{2} \rceil$*. In a nutshell, we have good results for depth and Stanley depth of this type of cyclic module.*

Proposition 5.2.5. *Let* $n = 1$ *and* $m \geq 0$*, then*

$$\mathrm{sdepth}(J^*_{1,m}) \geq \left\lceil \frac{6m+7}{2} \right\rceil.$$

Let $n \geq 2$ *and* $m \geq 0$*. If* $n \equiv 0 \pmod 2$*, then*

$$\mathrm{sdepth}(J^*_{n,m}) \geq \left\lceil \frac{4m+5}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + 1.$$

And if $n \equiv 1 \pmod 2$*, then*

$$\mathrm{sdepth}(J^*_{n,m}) = \left\lceil \frac{8m+7}{2} \right\rceil + \left\lceil \frac{n+3}{2} \right\rceil.$$

Remark 5.2.6. *By [23, Theorem 2.3]* $\mathrm{sdepth}(J^*_{n,m}) \geq 2nm + 3n + 4m + 5 - \lfloor \frac{4mn+5n+3m+2}{2} \rfloor$*. Therefore our lower bound is finer than this one.*

Proposition 5.2.7. *Let $n = 3$ and $m \geq 0$. Then*

$$\mathrm{sdepth}(J_{3,m}) \geq \left\lceil \frac{4m+8}{2} \right\rceil.$$

For $n = 4$ and $m \geq 0$

$$\mathrm{sdepth}(J_{4,m}) \geq \left\lceil \frac{4m+9}{2} \right\rceil.$$

Finally, for $n \geq 5$ and $m \geq 0$. If $n \equiv 0 \pmod 2$, then

$$\mathrm{sdepth}(J_{n,m}) \geq \left\lceil \frac{4m+5}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil.$$

And if $n \equiv 1 \pmod 2$, then

$$\mathrm{sdepth}(J_{n,m}) \geq \left\lceil \frac{8m+6}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil.$$

Remark 5.2.8. *By [23, Theorem 2.3]* $\mathrm{sdepth}(J_{n,m}) \geq 2nm + 3n - \lfloor \frac{4mn+5}{2} \rfloor$*. Therefore our lower bound is better than this one.*

Corollary 5.2.9. *(Rauf's question) Let $n = 3$, and $m \geq 0$. Then*

$$\mathrm{sdepth}(J_{n,m}) = \mathrm{sdepth}(T_{n,m}/J_{n,m}) + 1.$$

Let $n \geq 4$, and $m \geq 0$. Then

$$\mathrm{sdepth}(J_{n,m}) > \mathrm{sdepth}(T_{n,m}/J_{n,m}) + 1.$$

Corollary 5.2.10. *(Herzog's conjecture) Let $n \geq 3$, and $m \geq 0$. Then*

$$\mathrm{sdepth}(J_{n,m}) > \mathrm{sdepth}(T_{n,m}/J_{n,m}).$$

5.3 Bounds for the dimension of cyclic modules associated to graphs $B_{n,m}$ and $E_{n,m}$

Proposition 5.3.1. *For $n \geq 1$ and $m \geq 0$, we have that dim$(S_{n,m}/I_{n,m}) \geq 2nm + 2n$.*

Proposition 5.3.2. *For $n \geq 3$ and $m \geq 0$, we have that dim$(T_{n,m}/J_{n,m}) \geq 2nm + 2n$.*

5.4 Open questions and conjecture

Question 5.4.1. *Let $n \geq 1$ and $m \geq 0$. For $n \equiv 0 \pmod 2$, is*

$$\operatorname{depth}(S_{n,m}/I_{n,m}) = \operatorname{sdepth}(S_{n,m}/I_{n,m}) = \left\lceil \frac{n+1}{2} \right\rceil?$$

Question 5.4.2. *For $n \geq 2$ and $m \geq 1$. Is* $\operatorname{sdepth}(I_{n,m}) \leq \lceil \frac{2m+3}{2} \rceil + \lceil \frac{n-1}{2} \rceil + 1$*?*

Question 5.4.3. *For $n = 3$ and $m \geq 0$. Is*

$$\operatorname{sdepth}(J_{3,m}) \leq \left\lceil \frac{4m+8}{2} \right\rceil?$$

For $n = 4$ and $m \geq 0$. Is

$$\operatorname{sdepth}(J_{4,m}) \leq \left\lceil \frac{4m+9}{2} \right\rceil?$$

Finally, for $n \geq 5$ and $m \geq 0$. When $n \equiv 0 \pmod 2$, is

$$\operatorname{sdepth}(J_{n,m}) \leq \left\lceil \frac{4m+5}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil?$$

And when $n \equiv 1 \pmod 2$. Is

$$\operatorname{sdepth}(J_{n,m}) \leq \left\lceil \frac{8m+6}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil?$$

Bibliography

[1] Edwards, Harold M., (1977). Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory, New York: Springer.

[2] Fraenkel, A. (1922). Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre. Math. Ann. 86, 230-237. Mathematische Annalen. 86. 230-237. 10.1007/BF01457986.

[3] The development of Ring Theory (http://www-gap.dcs.st-and.ac.uk/ history/HistTopics/Ring theory.html).

[4] Anderson, Dan, Naseer, M,(1993). Beck's Coloring of a Commutative Ring. Journal of Algebra. 159. 500-154.

[5] Sachs, H., Steibitz, M., and Wilson, R.J. (1988) "An Historical Note: Euler's Königsberg Letters." in Journal of Graph Theory 12 1: 133-39.

[6] Mckelvie, C.L.. (1975). William Hamilton Maxwell. 22. 10.1093/nq/22.10.450-a.

[7] Warburg, E.. (1925). Zur Erinnerung an Gustav Kirchhoff. The Science of Nature. 13. 205-212. 10.1007/BF01558883.

[8] Cayley, A., Halsted, G. (1895). Arthur Cayley. Science (New York, N.Y.). 1. 450-1. 10.1126/science.1.17.450.

[9] F.Harary, Graph theory, Addison-Wesley, Reading, MA,1969.

[10] Stanley, R. P. (1982). Linear Diophantine equations and local cohomology. *Inventiones Mathematicae, 68*(2), 175-193.

[11] Bruns, W., Herzog, H. J. (1998). Cohen-macaulay rings. *Cambridge university press*.

[12] Apel, J. (2003). On a conjecture of RP Stanley; part II—quotients modulo monomial ideals. *Journal of Algebraic Combinatorics, 17*(1), 57-74.

[13] Anwar, I., Popescu, D. (2007). Stanley conjecture in small embedding dimension. *Journal of Algebra, 318*(2), 1027-1031.

[14] Popescu, A. (2010). Special stanley decompositions. *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie,* 363-372.

[15] Duval, A. M., Goeckner, B., Klivans, C. J., Martin, J. L. (2016). A non-partitionable Cohen–Macaulay simplicial complex. *Advances in Mathematics, 299*, 381-395.

[16] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal*, J. Algebra, 322(9)(2009), 3151-3169.

[17] M. Cimpoeas, *Several inequalities regarding Stanley depth*, Romanian Journal of Mathematics and Computer Science, 2(2012), 28-40.

[18] A. Rauf, *Depth and Stanley depth of multigraded modules*, Comm. Algebra, 38(2)(2010), 773-784.

[19] L. Fouli, S. Morey, *A lower bound for depths of powers of edge ideals*, J. Algebra.Comb.,42(2015), 829-848.

[20] Cimpoeaş, M. (2008). Stanley depth of complete intersection monomial ideals. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 205-211.

[21] C. Biro, D. M. Howard, M. T. Keller, W. T. Trotter, S. J. Young, *Interval partitions and Stanley depth*, Journal of Combinatorial Theory, Series A, 117 (2010), 475-482

[22] M. Ishaq, *Upper bounds for the Stanley depth*, Comm. Algebra, 40(1)(2012), 87-97.

[23] R. Okazaki, *A lower bound of stanley depth of monomial ideals*, J.Comm. Algebra, 3(1)(2011), 83-88.

[24] Z. Iqbal, M.Ishaq, *Depth and Stanley depth of the edge ideals of the powers of paths and cycles*, , 27(3)(2019), 113-135.

[25] J. Herzog, *A survey on Stanley depth*, In Monomial ideals, computations and applications, vol. 2083 of Lecture Notes in Math. Springer, Heidelberg, 2013, pp. 3-45.

[26] M. T. Keller, S. J. Young, *Combinatorial reductions for the Stanley depth of I and S/I* , arXiv:1702.00781v1.

[27] D. Popescu, M.I. Qureshi, *Computing the Stanley depth*, J. Algebra, 323(10), 2943-2959.

[28] M. Cimpoeas, *Stanley depth of monomial ideals with small number of generators* , Cent. Eur. J. Math. , 7(3) , 629-634.

[29] Z. Iqbal, M.Ishaq, *Depth and Stanley depth of the edge ideals of the powers of paths and cycles*, Sciendo , 27(3),2019, 113–135.

[30] Z. Iqbal, M.Ishaq, M. Ahsan, *Depth And Stanley Depth of the edge ideals of the strong product of some graphs* , arXiv:1909.02610v1.

[31] M.Ishaq, Z. Iqbal, *Depth and Stanley depth of edge ideals associated to some line graphs*, AIMS Mathematics, 4(3): 686–698.

[32] CoCoATeam, CoCoA: *A system for doing Computations in Commutative Algebra*, Available at http://cocoa.dima.unige.it.