Depth and Stanley Depth of the Quotient Ring of Edge Ideals Associated with Some Classes of Lobster Trees and Unicyclic Graphs

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MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Adnan Iqbal, Regn No. 00000321166 Titled: "**Depth and Stanley Depth of the Quotient Ring of Edge Ideals Associated with Some Classes of Lobster Trees and Unicyclic Graphs**" accepted in partial fulfillment of the requirements for the award of **MS** degree.

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I dedicate this thesis to my loving parents especially my grandfather, venerable supervisor, respectable teachers and fellows for their limitless support and encouragement.

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Abstract

In this Thesis, we discuss rings and modules, which are the fundamental algebraic structures of Abstract Algebra. Moreover, we discuss two algebraic invariants of a module which are Stanley depth and depth of a module. In addition, we discuss how the regular element property helps in determining the depth of a module. Then, we discuss some recent results of depth and Stanley depth of edge ideals associated with different graphs. Thereafter, we find the Stanley depth of some modules and edge ideals using the method of posets. Lastly, we compute the Stanley depth and depth of the quotient ring of edge ideals associated with different classes of graphs. These classes include some lobster trees and unicyclic graphs. Then, we show that the values of depth and Stanley depth are equal and can be stated in terms of n and m. In addition, we prove the Stanley's inequality for modules associated with these classes of graphs.

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Introduction

Richard P. Stanley is known for his contributions in Combinatorics and its connections to Algebra and Geometry, particularly in the theory of simplicial complexes. The relationship between Combinatorics and Commutative Algebra is studied using monomial ideals. Combinatorics problems are converted to monomial ideals, which are then solved using methods and techniques from Commutative Algebra. In [27], Stanley defined the Stanley depth of $\mathbb{Z}^q$-graded modules over a graded commutative ring. He used square-free monomial ideals to connect Combinatorics and Commutative Algebra.

According to Stanley conjecture, Stanley depth of a module is at least the depth of a module. Later in [8], Duval et al established that the Stanley conjecture is not valid for the modules of type $\aleph/\Upsilon$, where $\aleph$ is the polynomials ring in q variables and Υ is the monomial ideal of $\aleph$. However, finding classes that nonetheless meet the inequality is still a difficult task.

In general, there is no known procedure for calculating the Stanley depth of a module. However in [15], it was proposed that, when a $\mathbb{Z}^q$-graded $\aleph$-module $\daleth$ is of the type Υ_1/Υ_2, where Υ_2, Υ_1 are monomial ideals of $\aleph$ and $\Upsilon_2 \subset \Upsilon_1$, then the Stanley depth of $\daleth$ can be determined in finite number of steps using posets.

For the quotient ring of edge ideals associated with some lobster trees and a unicyclic graph, the precise values of Stanley depth and depth are determined in this Thesis. We calculate these values using the induction approach.

This Thesis has four chapters.

The essential principles, basic definitions, and findings of Abstract Algebra and Commutative Algebra are covered in chapter 1. It covers the types, fundamental attributes, standard operations, and primary decomposition of ideals. It also covers the exact sequences, graded rings, polarizaton of monomial ideals as well as other fundamentals of Module Theory.

Chapter 2 covers the fundamentals of Graph Theory as well as the most common graph types. The chapter wraps off with a view of lobster trees and uncylcic graphs.

Chapter 3 starts with the introduction of depth and its examples, then it moves on to the Stanley decomposition, Stanley depth of modules and the Stanley conjecture. In this chapter, the techniques for finding the Stanley depth of square-free monomial ideals are also covered.

In chapter 4, the edge ideals associated with some lobster trees and a unicylclic graph are introduced. The depth and Stanley depth of the quotient ring of edge ideals associated with some lobster trees and a unicyclic graph is then determined using the mathematical induction approach and the depth lemma on short exact sequences.

Chapter 1

Ring Theory and Module Theory

1.1 Introduction

Rings are algebraic structures (a nonempty set and a collection of operations on that set must meet the relevant axioms to form an algebraic structure) that enable the generalization of fields and it includes two binary operations such as addition and multiplication.

Problems and ideas from Algebraic Number Theory and Algebraic Geometry have influenced the development of rings. In the 19th century, it became a universal approach to obtain an integer solution of polynomial problems using the rings of higher degree algebraic numbers (an algebraic number x have degree q if it is the zero of some irreducible qth-degree polynomial with integer coefficients). An early attempt to prove Fermat's last theorem led to the emergence of the basic idea of a ring. In an attempt to find the algebraic numbers that are the solutions of

$$x^2 + 2 = 0, \tag{1.1}$$

which is similar to finding its factorization in the ring of integers of quadratic

field Q($\sqrt{-2}$). Similarly, for a positive integer p, the polynomial $c^p - b^p$ (which is relevant for solving the Fermat equation $a^p + b^p = c^p$) can be factored over the ring $\mathbb{Z}[\zeta_q]$, where ζ_q is a primitive q-th root of unity.
Later, in [9] and [14] significant contributions to the development of rings were made.

1.2 Rings, Fields and Integral domains

The concept of a group comes from a set of mappings or permutations of sets to itself. So far, we've just looked at sets with one binary operation. On the other hand, in rings, we have two binary operations.

Definition 1.2.1 Let $\aleph$ be a non-empty set. A map $* : \aleph \times \aleph \to \aleph$ is said to be a binary operation on $\aleph$, if for any $m, n \in \aleph$, we have $m * n \in \aleph$. In this situation, we may say that $\aleph$ is closed under $*$.

Example 1.2.2 For example, the sum of two real numbers is a real number, so $*\ (\zeta_1, \zeta_2)=\zeta_1 + \zeta_2$ is a binary operation on the set of real numbers.
The addition operation is a commutative and associative binary operation on $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$. But $+$ is not a binary operation on the set $\aleph=\{0,1\}$, because for $1 \in \aleph$, we have $1 + 1 = 2 \notin \aleph$.

Definition 1.2.3 Let $\aleph$ be a non empty set, $+$ and $\times$ are binary operations on $\aleph$. Then, the structure $(\aleph, +, \times)$ is a ring if it satisfies the below axioms.

1. $(\aleph, +)$ is an abelian group, that is for any $\zeta_1, \zeta_2, \zeta_3, \zeta \in \aleph$, we have

   - $\zeta_1+\zeta_2 \in \aleph$.
   - $(\zeta_1+\zeta_2)+\zeta_3=\zeta_1+(\zeta_2+\zeta_3)$.
   - $\zeta+0=\zeta=\zeta+0$.
   - $\zeta+(-\zeta)=(-\zeta)+\zeta=0$.
   - $\zeta_1+\zeta_2=\zeta_2+\zeta_1$.

2. $(\aleph, \times)$ is a semi group, i.e for all $\zeta_1, \zeta_2, \zeta_3 \in \aleph$, we have

   - $\zeta_1 \times \zeta_2 \in \aleph$.
   - $(\zeta_1 \times \zeta_2) \times \zeta_3=\zeta_1 \times (\zeta_2 \times \zeta_3)$.

3. Multiplication is distributive on both the left and right sides, i.e for all $\zeta_1, \zeta_2, \zeta_3 \in \aleph$, we have

   - $\zeta_1 \times (\zeta_2+\zeta_3)=(\zeta_1 \times \zeta_2)+(\zeta_1 \times \zeta_3)$.
   - $(\zeta_2+\zeta_3) \times \zeta_1= (\zeta_2 \times \zeta_1) + (\zeta_3 \times \zeta_1)$.

Definition 1.2.4 Let $(\aleph, +, \times)$ be a ring. If the multiplication of $\aleph$ is commutative, that is for any $\zeta_1, \zeta_2 \in \aleph$, we have $\zeta_1 \times \zeta_2 = \zeta_2 \times \zeta_1$, we call it a commutative ring.

Remark 1.2.5 If there exists $1 \in \aleph$, such that for all $\zeta \in \aleph$, we have $\zeta \times 1 = 1 \times \zeta = \zeta$, we say that $\aleph$ is a ring with identity.

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Example 1.2.6 1. Set of integers, set of rationals, set of reals, and the set of complex numbers are all commutative rings having the same identity that is 1.

2. There is no identity in the commutative ring $2\mathbb{Z}$.

3. $M_q(\mathbb{Z})$ and $M_q(\mathbb{C})$ are rings with 1 under standard matrix addition and multiplication, but they are not commutative unless $q = 1$.

4. Polynomials with real coefficients form a commutative ring with identity, which we refer to as $\mathbb{R}[x]$.

Following that, we write down some ring-related basics. To keep the assertions simple, we focus on commutative rings.

Definition 1.2.7 Let $\aleph$ be a ring and Υ be a non empty subset of $\aleph$. Then Υ is a subring of $\aleph$ iff $\forall\, \upsilon_1,\, \upsilon_2 \in \Upsilon$, we have

1. υ_1-$\upsilon_2 \in \Upsilon$,

2. $\upsilon_1{\times}\upsilon_2 \in \Upsilon$.

Example 1.2.8 1. $\mathbb{Z}$ and $\mathbb{Q}$ are subrings of $\mathbb{R}$.

2. $\mathbb{R}$ is a subring of $\mathbb{C}$ with elelments of the type $a + 0i$, for $a \in \mathbb{R}$.

3. For each $\zeta \in \mathbb{N}$, $\zeta\mathbb{Z}$=$\{\,\zeta k\,:\, k{\in}\mathbb{Z}\}$ is a subring of $\mathbb{Z}$.

Definition 1.2.9 Assume that $\aleph_1$ and $\aleph_2$ are rings. The direct product $\aleph_1 \times \aleph_2$ of rings is a ring, in which the binary operations are defined in terms of coordinates, such that for any $\zeta_1,\, \zeta_1' \in \aleph_1$ and $\zeta_2,\, \zeta_2' \in \aleph_2$, we have

$$(\zeta_1,\, \zeta_2) + (\zeta_1',\, \zeta_2') = (\zeta_1 + \zeta_1',\, \zeta_2 + \zeta_2')$$

and

$$(\zeta_1,\, \zeta_2) \times (\zeta_1',\, \zeta_2') = (\zeta_1 \times \zeta_1',\, \zeta_2 \times \zeta_2').$$

We show that all of the ring axioms hold for the direct product $\aleph = \aleph_1 \times \aleph_2$.

- Clearly $\aleph$ is closed under $+$ and $\times$, because $\aleph_1$ and $\aleph_2$ are closed under addition and multiplication.

- Here $+$ is associative on $\aleph$ due to the associativity of $+_1$ on $\aleph_1$ and $+_2$ on $\aleph_2$. The operation $+$ is also commutative due to the commutativity of $+_1$ on $\aleph_1$ and $+_2$ on $\aleph_2$.

- Multiplication $\times$ is associative on $\aleph$ due to the associativity of $\times_1$ on $\aleph_1$ and $\times_2$ on $\aleph_2$. The operation $\times$ is also commutative due to the commutativity of $\times_1$ on $\aleph_1$ and $\times_2$ on $\aleph_2$.

- The identity on $+$ is $(0_1,\, 0_2)$, where 0_1 is the identity of $+_1$ on $\aleph_1$ and 0_2 is the identity of $+_2$ on $\aleph_2$. While $(-\zeta_1,\, -\zeta_2)$ is the additive inverse of $(\zeta_1,\, \zeta_2)$. Furthermore, if $\aleph_1$ and $\aleph_2$ have the same identity, $\aleph_1 \times \aleph_2$ have the same identity that is $(1,\, 1)$.

- Left distributivity holds, because $(\zeta_1,\, \zeta_2) \times [(\zeta_1',\, \zeta_2')+(\zeta_3,\, \zeta_4)]=(\zeta_1,\, \zeta_2) \times (\zeta_1'+\zeta_3,\, \zeta_2'+\zeta_4)=([\zeta_1 \times \zeta_1']+[\zeta_1 \times \zeta_3],\, [\zeta_2 \times \zeta_2']+[\zeta_2 \times \zeta_4])=(\zeta_1,\, \zeta_2) \times (\zeta_1',\, \zeta_2')+(\zeta_1,\, \zeta_2) \times (\zeta_3,\, \zeta_4)$.

- Right distributivity holds, because $[(\zeta_1,\, \zeta_2) + (\zeta_1',\, \zeta_2')] \times (\zeta_3,\, \zeta_4)=(\zeta_1 + \zeta_1',\, \zeta_2+\zeta_2') \times (\zeta_3,\, \zeta_4)=([\zeta_1 \times \zeta_3]+[\zeta_1' \times \zeta_3],\, [\zeta_2 \times \zeta_4]+[\zeta_2' \times \zeta_4])=(\zeta_1,\, \zeta_2) \times (\zeta_3,\, \zeta_4)+(\zeta_1',\, \zeta_2') \times (\zeta_3,\, \zeta_4)$.

Example 1.2.10 Under the coordinatewise operations derived from $\mathbb{R}$, $\mathbb{R}^4$ and more generally $\mathbb{R}^q$ is a commutative ring with 1, for some $q > 2$.

Definition 1.2.11 If a structure $(V, +, \times)$ meets the below conditions (where $+$ and $\times$ are binary operations on V),

- $(V, +)$ is an abelian group.
- $(V \backslash \{0\}, \times)$ is an abelian group.
- The distributive laws hold.

then it is called a field.

Example 1.2.12 $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields. The commutative rings with identity that fails to be field are $\mathbb{Z}$ and polynomial ring $V[\zeta]$, here V can be $\mathbb{R}$, $\mathbb{C}$, or any other field.

Definition 1.2.13 Let $\aleph$ be a commutative ring. If for any $\zeta_1 \neq 0 \in \aleph$, there exists $\zeta_2 \neq 0 \in \aleph$ corresponding to ζ_1, such that $\zeta_1\zeta_2 = 0$, we call an element ζ_1 a zero divisor in $\aleph$.

Definition 1.2.14 If a commutative ring with identity has no zero divisors, it is called an integral domain. In integral domains, if for any ζ_1, ζ_2, $\zeta_3 \in \aleph$, we have $\zeta_1\zeta_2 = \zeta_1\zeta_3$ and $\zeta_1 \neq 0$, then $\zeta_2 = \zeta_3$.

Example 1.2.15 $\overline{2}$ and $\overline{3}$ are zero divisors in $\mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$.

Example 1.2.16

1. To show that any field is an integral domain. Assume that for any ζ_1, ζ_2 in field, we have $\zeta_1.\zeta_2 = 0$. If $\zeta_1 \neq 0$, then their exists ζ_1^{-1} in field, such that $0 = \zeta_1^{-1}.0 = \zeta_1^{-1}.(\zeta_1.\zeta_2) = (\zeta_1^{-1}.\zeta_1).\zeta_2 = 1.\zeta_2 = \zeta_2$ and similarly, if we reversed the roles of ζ_1 and ζ_2.
2. $\mathbb{Z}$ and $V[\zeta]$ are integral domains that aren't fields.

3. $\mathbb{R}^3$ is a commutative ring with identity that fails to be an integral domain with coordinatewise addition and multiplication, i.e for $(0,1,0)$, $(1,0,0) \in \mathbb{R}^3$, we have $(0,1,0)(1,0,0) = (0,0,0)$.

Theorem 1.2.17 An integral domain having a finite number of elements is a field.

Definition 1.2.18 Let $\aleph$ be a commutative ring with 1. Let for any $\zeta_1 \in \aleph$, if there exists $\zeta_2 \in \aleph$ corresponding to ζ_1, such that $\zeta_1\zeta_2 = 1$, then ζ_1 is a unit in $\aleph$. As a result, the units are those (necessarily non-zero) elements whose multiplicative inverses exist.

Example 1.2.19

1. Every non-zero element in a field is a unit.

2. In $\mathbb{Z}$, the units are $\{+1, -1\}$.

3. The non-zero constant polynomials are the units in the ring of real and complex polynomials.

Definition 1.2.20 Let $\aleph$ be a ring, if $\zeta^2 = \zeta$ for any $\zeta \in \aleph$, then ζ is called an idempotent element.

Definition 1.2.21 A ring is called a boolean ring if each of its element is idempotent. It is also a commutative ring.

1.3 Ring of polynomials

One of the oldest problems in mathematics is determining the roots of polynomials or solving algebraic equations. The elegant and functional notations we use now date back to the early 1400s. Equations were previously written in words. Leonardo Fibonacci achieved a near approximation of the

cubic equation $x^3+2x^2+cx=d$ in the early 1300s. René Decartes, a famous mathematician from the 16th century, invented Analytic Geometry, in which he converted the geometric problems into algebraic equations.
A polynomial ring is a sort of ring that is formed by set of polynomials with variables from another ring. Polynomial rings appear in many areas of Mathematics including Number Theory, Commutative Algebra, Ring Theory and Algebraic Geometry.

Definition 1.3.1 Let

$$\mathcal{P} := \eta_0 + \eta_1\zeta^1 + \cdots + \eta_q\zeta^q$$

be a polynomial with cofficients in V, where V be a commutative ring with unity (often a field) and q be an integer that is non-negative. Then, the set $\aleph = V[\zeta]$ of all polynomials with coefficients in V is a commutative ring with unity under the usual addition and multiplication of polynomials.

Theorem 1.3.2

1. $V[\zeta]$ is a commutative ring with 1, in which the multiplicative identity is the constant polynomial 1.
2. We say that $\mathcal{P} \in V[\zeta]$ has multiplicative inverse, or $\mathcal{P}$ is a unit if and only if $\mathcal{P}$ is a non-zero constant.
3. $V[\zeta]$ is not a field but is an integral domain, because for any polynomials $\mathcal{P}_1,\, \mathcal{P}_2 \in V[\zeta]$, we have $\mathcal{P}_1.\mathcal{P}_2 = 0$ implies $\mathcal{P}_1 = 0$ or $\mathcal{P}_2 = 0$.

Definition 1.3.3 Let $V[\zeta_1,\, \zeta_2] = V[\zeta_1][\zeta_2]$ be a polynomial ring in two variables $\zeta_1,\, \zeta_2$ with coefficients in V, then the polynomial ring in q variables $\zeta_1,\, \zeta_2,\, \ldots,\, \zeta_q$ is defined as $V[\zeta_1,\, \zeta_2,\, \ldots,\, \zeta_q] = V[\zeta_1,\, \ldots,\, \zeta_{q-1}][\zeta_q]$ with coefficients in V. This means that the polynomials in q variables with coefficients in V are now considered as the polynomials in only one variable ζ_q, in which the coefficients are the polynomials in $q-1$ variables.

Proposition 1.3.4 For any $\mathcal{P}_1$, $\mathcal{P}_2 \in V[\zeta]$, $\deg(\mathcal{P}_1\mathcal{P}_2) = \deg(\mathcal{P}_1) + \deg(\mathcal{P}_2)$.

Example 1.3.5 If V_1 is a subring of V_2, then $V_1[\zeta]$ is a subring of $V_2[\zeta]$.

1.3.1 Ring homomorphism

Let $\aleph_1$ and $\aleph_2$ be two rings. A map $\mathcal{L} : \aleph_1 \to \aleph_2$ is said to be a ring homomorphism, if it meets the following criteria, i.e for any ζ_1, $\zeta_2 \in \aleph_1$

- $\mathcal{L}(\zeta_1 + \zeta_2) = \mathcal{L}(\zeta_1) + \mathcal{L}(\zeta_2)$,
- $\mathcal{L}(\zeta_1\zeta_2) = \mathcal{L}(\zeta_1)\mathcal{L}(\zeta_2)$.

and if it is one-one and onto, it's called an isomorphism.

Example 1.3.6 Let $\aleph$=$\{\zeta_1 + \zeta_2\sqrt{2} : \zeta_1, \zeta_2 \in \mathbb{Z}\}$ be a ring, then the map $\mathcal{L}$ from $\aleph$ to itself defined as,

$$\mathcal{L}(\zeta_1 + \zeta_2\sqrt{2}) = \zeta_1 - \zeta_2\sqrt{2}$$

is a ring homomorphism.

Definition 1.3.7 If $\mathcal{L}$ is the above-mentioned ring homomorphism, then the kernal of $\mathcal{L}$ is the set of all those elements for which $\mathcal{L}(\zeta)$ =$0'$, where $0'$ is the additive identity of $\aleph_2$. Kernal and image of $\mathcal{L}$ are subrings of $\aleph_1$ and $\aleph_2$, respectively.

1.3.2 Ideals and Operations on Ideals

Definition 1.3.8 Let Υ be a non-empty subset of a ring $\aleph$, then it is said to be an ideal, if for any υ_1, $\upsilon_2 \in \Upsilon$ and $\zeta \in \aleph$, we have υ_1-$\upsilon_2 \in \Upsilon$ and $\upsilon_1\zeta$ $\in \Upsilon$.

Remark 1.3.9 Every ideal is a subring, but every subring is not necessary an ideal.

Example 1.3.10 $\mathbb{Z}$ is a ring of integers and $4\mathbb{Z}$ is a subring of $\mathbb{Z}$. The subring $4\mathbb{Z}$ is also an ideal of $\mathbb{Z}$.

Theorem 1.3.11 If we have two ideals Υ_1 and Υ_2 of a ring $\aleph$, then

1. $\Upsilon_1 \cap \Upsilon_2$ is an ideal of $\aleph$.

2. $\Upsilon_1 + \Upsilon_2 = \{\upsilon_1 + \upsilon_2 : \upsilon_1 \in \Upsilon_1, \upsilon_2 \in \Upsilon_2\}$ is an ideal of $\aleph$.

Remark 1.3.12

1. For any ideal Υ of $\aleph$, if $1 \in \Upsilon$, then $\Upsilon = \aleph$.

2. The only proper ideal of a field is $\{0\}$.

Definition 1.3.13 If for any $\zeta \in \aleph$, we have $\Upsilon = \zeta\aleph$, then the ideal Υ of $\aleph$ is called principal, and if every ideal is principal in $\aleph$, then the ring $\aleph$ is called principal.

Example 1.3.14 The ring of integers is an example of a principal ring.

Definition 1.3.15 The set $\aleph\upsilon = \{\zeta\upsilon : \forall \zeta \in \aleph\}$ is an ideal of $\aleph$.

Definition 1.3.16 Let Υ be an ideal of $\aleph$ and $\aleph_1$ be a subring of $\aleph$, then $\Upsilon \cap \aleph_1$ is an ideal of $\aleph$.

Definition 1.3.17 Let $\aleph$ be a ring and Υ be an ideal of $\aleph$. Then the set $\aleph/\Upsilon = \{\zeta + \Upsilon : \zeta \in \aleph\}$ of cosets of Υ in $\aleph$ is a ring. This ring is known as the quotient ring. For any ζ_1, $\zeta_2 \in \aleph$, the multiplication and addition are defined as,

$$(\zeta_1 + \Upsilon) + (\zeta_2 + \Upsilon) = \zeta_1 + \zeta_2 + \Upsilon.$$

$$(\zeta_1 + \Upsilon)(\zeta_2 + \Upsilon) = \zeta_1\zeta_2 + \Upsilon.$$

Remark 1.3.18 If $\aleph$ is a commutative ring with unity, then $\aleph/\Upsilon$ is a commutative ring with unity, where $0+\Upsilon$, $1+\Upsilon$ are additive and multiplicative identities of $\aleph/\Upsilon$, respectively.

Theorem 1.3.19 For an ideal Υ of ring $\aleph$, there is an epimorphism (onto homomorphism) given by, $\mathcal{L} : \aleph \to \aleph/\Upsilon$, with $\mathrm{Ker}\,(\mathcal{L}) = \Upsilon$.

Next there are the isomorphism theorems for the rings.

Theorem 1.3.20 (Isomorphism Theorems)

1. Let $\aleph_1$ and $\aleph_2$ be rings and $\mathcal{L} : \aleph_1 \to \aleph_2$ be a ring homomorphism. Then:

    - the Kernal of $\mathcal{L}$ is an ideal of $\aleph_1$,
    - the image of $\mathcal{L}$ is a subring of $\aleph_2$,
    - $\mathrm{Im}\,\mathcal{L} \cong \aleph_1/\ker\mathcal{L}$.

    If $\mathcal{L}$ is surjective, then $\aleph_2 \cong \aleph_1/\ker(\mathcal{L})$.

2. Assume that $\aleph$ be a ring, let $\aleph_1$ be a subring of $\aleph$ and Υ be an ideal of $\aleph$. Then:

    - $\aleph_1 + \Upsilon$ is a subring of $\aleph$,
    - $\aleph_1 \cap \Upsilon$ is an ideal of $\aleph_1$, and
    - $(\aleph_1 + \Upsilon)/\Upsilon \cong \aleph_1/(\aleph_1 \cap \Upsilon)$.

3. Let $\aleph_1$ be a ring. Υ_1 and Υ_2 are ideals of $\aleph_1$, such that $\Upsilon_1 \subseteq \Upsilon_2$. Then Υ_2/Υ_1 is an ideal of $\aleph/\Upsilon_1$ and

$$(\aleph_1/\Upsilon_1)/(\Upsilon_2/\Upsilon_1) \cong \aleph_1/\Upsilon_2.$$

Definition 1.3.21 Let $\aleph$ be a ring and Υ_1 be a proper ideal of $\aleph$, it is said to be a maximal ideal, if there are no other ideals contained between Υ_1 and $\aleph$, and if for any other ideal Υ_2 of $\aleph$, we have $\Upsilon_1 \subset \Upsilon_2$, then $\Upsilon_1 = \Upsilon_2$ or $\Upsilon_2 = \aleph$.

Definition 1.3.22 If the only ideals in a ring are $\{0\}$ and $\aleph$, then the ring $\aleph$ is called simple.

Example 1.3.23 The ideals (2) and (3) are the maximal ideals of $\mathbb{Z}$, but (4) is not maximal, because it is contained in (2).

Theorem 1.3.24 In the ring of integers, an ideal (ζ) is maximal iff ζ is prime.

Remark 1.3.25 Let Υ be an ideal of $\aleph$ and take $\zeta \in \aleph$, such that $\zeta \notin \Upsilon$ and (ζ) is an ideal of $\aleph$, then $\Upsilon + (\zeta) = \{\upsilon_1 + \zeta\eta : \eta \in \aleph, \, \upsilon_1 \in \Upsilon\}$ is an ideal of $\aleph$. It is called an ideal generated by $\Upsilon \cup (\zeta)$, denoted as (Υ, ζ).

Theorem 1.3.26 Let Υ be an ideal of a ring $\aleph$, then Υ is maximal iff $(\Upsilon, \zeta) = \aleph$, for any $\zeta \notin \Upsilon$.

Theorem 1.3.27 Let $\aleph$ be a commutative ring with unity and Υ be a proper ideal of $\aleph$. Then, Υ is maximal iff $\aleph/\Upsilon$ is a field.

Definition 1.3.28 An ideal Υ of $\aleph$ is said to be prime, if for any $\zeta_1, \zeta_2 \in \aleph$, we have $\zeta_1\zeta_2 \in \Upsilon$, then $\zeta_1 \in \Upsilon$ or $\zeta_2 \in \Upsilon$.

Theorem 1.3.29 Let $\aleph$ be a commutative ring and Υ be an ideal of $\aleph$, then Υ is prime iff $\aleph/\Upsilon$ is an integral domain.

Definition 1.3.30 If any ring $\aleph$ meets the following criteria, it is called a Neotherian ring.

1. There is a maximal element in every non-empty subset of ideals.

2. Every ascending chain $\Upsilon_0 \subseteq \Upsilon_1 \subseteq \Upsilon_2 \ldots$ of ideals of $\aleph$ stabilizes at some point, i.e $\Upsilon_n = \Upsilon_m \ \forall \ m \geq n$.

Theorem 1.3.31 If $\aleph$ is a Neotherian ring, then the polynomial ring $\aleph[\zeta]$ is also neotherian.

Proposition 1.3.32 If $\aleph_1$ is a Neotherian ring and $\aleph_2$ is homomorphic image of $\aleph_1$, then $\aleph_2$ is also Neotherian.

Definition 1.3.33 Let Υ be an ideal of $\aleph$, then the radical of Υ, denoted by $\sqrt{\Upsilon}$ is also an ideal of $\aleph$, defined in this way

$$\sqrt{\Upsilon} = \{\zeta \in \aleph \mid \zeta^q \in \Upsilon, \text{ for some } q > 0\}.$$

Example 1.3.34 Let $\aleph = V[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$ be a polynomial ring and $\Upsilon = (\zeta_1^2\zeta_2^2, \zeta_2^2\zeta_3^2, \zeta_3^2\zeta_4^2, \zeta_4^2\zeta_1^2)$ be an ideal of $\aleph$, then $\sqrt{\Upsilon} = \{\zeta_1\zeta_2, \zeta_2\zeta_3, \zeta_3\zeta_4, \zeta_4\zeta_1\}$.

Remark 1.3.35 $\sqrt{\Upsilon} = \Upsilon$, for all prime ideals Υ of $\aleph$.

Definition 1.3.36 The local ring is a ring with unique maximal ideal.

Example 1.3.37 The ideal $(2) = \{0, 2, 4, 6\}$ is maximal ideal in $\mathbb{Z}_8$, and it is also unique. Thus, $\mathbb{Z}_8$ is a local ring.

Definition 1.3.38 Let $\aleph$ be a commutative ring and Υ be a proper ideal of $\aleph$. Then Υ is said to be primary, if for any $\zeta_1, \zeta_2 \in \aleph$, we have $\zeta_1\zeta_2 \in \Upsilon$, then $\zeta_1 \in \Upsilon$ or $\zeta_2^q \in \Upsilon$, for some $q > 0$.

Definition 1.3.39 Let Υ_1 be a primary ideal and Υ_2 be a prime ideal of $\aleph$, if $\Upsilon_2 = \sqrt{\Upsilon_1}$, then Υ_1 is called Υ_2-primary.

1.3.3 Monomial ideal

Let V be a field and $\aleph = V[\zeta_1, \zeta_2, \ldots, \zeta_q]$ be a ring of polynomials over V. The term monomial refers to a product of this type $\zeta_1^{b_1} \ldots \zeta_q^{b_q}$, with $b_r \in \mathbb{N}$, this imply that $p = \zeta_1^{b_1} \ldots \zeta_q^{b_q}$ is a monomial. We let $p = \zeta^b$, where $b = (b_1, \ldots, b_q) \in \mathbb{N}^n$. The term monomial ideal refers to an ideal whose generating set is made up entirely of monomials. The set of all monomials that form the basis for any polynomial in $\aleph$, is denoted by Mon($\aleph$). If we take any polynomial $h \in \aleph$ and for $z_p \in V$, then this polynomial can be written as

$$h = \sum_{p \in Mon(\aleph)} z_p p,$$

where support of h is defined as

$$\text{supp}(h) = \{p \in Mon(\aleph) \,|\, z_p \neq 0\}.$$

Definition 1.3.40 If the components of b are 0 and 1, then the monomial ζ^b is called a square free monomial and the ideal generated by these monomials is called a sqaure free monomial ideal.

Proposition 1.3.41 If an ideal is generated by monomials, then it has a unique minimal set of monomials that generates the monomial ideal.

Example 1.3.42 The ideal $\Upsilon = (\zeta_1^3\zeta_2^3,\, \zeta_2^3\zeta_3^3,\, \zeta_2^4\zeta_3^3,\, \zeta_1^3\zeta_2^4)$ has a unique minimal set of generators, given by $G(\Upsilon)=\{\zeta_1^3\zeta_2^3,\, \zeta_2^3\zeta_3^3\}$.

Definition 1.3.43 For any monomial η, $\text{supp}(\eta) = \{j : \zeta_j | \eta\}$ and for an ideal Υ generated by monomials, $\text{supp}(\Upsilon) = \{j : \zeta_j|\eta,$ for some $\eta \in \mathcal{G}(\Upsilon)\}$, here $\mathcal{G}(\Upsilon)$ is the unique minimal set of monomial generators of Υ.

Proposition 1.3.44 Let Υ_1 and Υ_2 are monomial ideals of $\aleph$, then the intersection of these two monomial ideals is again a monomial ideal. The set $\{\text{lcm}(\zeta, \eta) : \zeta \in G(\Upsilon_1), \eta \in G(\Upsilon_2)\}$ is a generating set for $\Upsilon_1 \cap \Upsilon_2$.

Definition 1.3.45 Let Υ_1 and Υ_2 are monomial ideals of $\aleph$, then the colon ideal of Υ_1 with respect to Υ_2 is also an ideal of $\aleph$, given as,

$$(\Upsilon_1 : \Upsilon_2) = \bigcap_{\zeta' \in G(\Upsilon_2)} \Upsilon_1 : (\zeta').$$

Where,

$$(\Upsilon_1 : \zeta') = \{\zeta / \gcd(\zeta, \zeta') : \zeta \in G(\Upsilon_1)\}.$$

.

Example 1.3.46 Assume $\Upsilon = (\zeta_1\zeta_2, \zeta_2\zeta_3)$ and $G(\Upsilon) = \{\zeta_1\zeta_2, \zeta_2\zeta_3\}$, then

$$(\Upsilon : \zeta_3) = (\zeta_1\zeta_2, \zeta_2) = (\zeta_2).$$

1.3.4 Primary decomposition of monomial ideals

A classic pillar of ideal theory is the breakdown of an ideal into its fundamental ideals.

Definition 1.3.47 A monomial ideal Υ is said to be an irreducible ideal, if for any two ideals Υ_1 and Υ_2, we have

$$\Upsilon = \Upsilon_1 \cap \Upsilon_2.$$

Then, either $\Upsilon = \Upsilon_1$ or $\Upsilon = \Upsilon_2$.

Proposition 1.3.48 A monomial ideal Υ is generated by pure powers of variables, i.e $\Upsilon = (\zeta_{i_1}^{b_{i_1}}, \ldots, \zeta_{i_q}^{b_{i_q}})$ iff it is irreducible.

Definition 1.3.49 When an ideal is expressed as the intersection of primary ideals, it is referred to as a primary decomposition, i.e

$$\Upsilon = \bigcap_{b=1}^{q} \Upsilon_b,$$

where each Υ_b is a primary ideal.

Proposition 1.3.50 Primary decomposition is the process of splitting an ideal into irreducible ideals.

Remark 1.3.51 For decomposing a monomial ideal Υ, we write it in intersection of irreducible monomial ideals that are generated by pure power of variables.

Examples of monomial ideals are shown below.

Example 1.3.52 Let we have an ideal $\Upsilon = (\zeta\eta,\ \zeta^3 - \zeta^2\,,\ \zeta^2\eta - \zeta\eta\,)$, then

$$\begin{aligned}\Upsilon &= (\zeta,\, \zeta^2(\zeta-1),\, \zeta\eta(\zeta-1)\,) \cap (\eta,\, \zeta^2(\zeta-1),\, \zeta\eta(\zeta-1)\,)\\ &= (\zeta) \cap (\eta\,,\, \zeta^2(\zeta-1)\,)\\ &= (\zeta) \cap (\eta\,,\, \zeta^2\,) \cap (\zeta-1\,).\end{aligned}$$

So, ideal Υ is decomposed here.

1.4 Polarization

Polarization is used to convert the monomial ideals into square-free monomial ideals. Diverse types of polarization have been used in literature for various reasons in Algebra and Algebraic Combinatorics. One of the most useful properties of polarization is the chain of substitutions that transforms a given monomial ideal into a square-free monomial ideal, that can be expressed in terms of regular sequences. We can specify the depth of modules by using these regular sequences [20]. The advantage of this method is that numerous Combinatorics tools deal with square-free monomial ideals. The Facet Theory developed by authors in [10], [11] and [12], is another tool for analysing monomial ideals.

Definition 1.4.1 Let $\aleph = V[\zeta_1, \ldots, \zeta_q]$ be a polynomial ring over field V. Let $\Upsilon_1 = \zeta_1^{\eta_1}\zeta_2^{\eta_2}\ldots\zeta_q^{\eta_q}$ be a monomial in $\aleph$. Now, we characterize the polarization of Υ_1, which is a sqaure-free monomial,

$$P(\Upsilon_1) = \zeta_{1,1}\zeta_{1,2}...\zeta_{1,\eta_1}\zeta_{2,1}...\zeta_{2,\eta_2}...\zeta_{q,1}...\zeta_{q,\eta_q}$$

in the polynomial ring $\aleph' = V[\,\zeta_{i,j}|\, 1 \leq i \leq q,\, 1 \leq j \leq \eta_i\,]$. Let Υ be a monomial ideal, it is generated by monomials $\Upsilon_1, \Upsilon_2 \ldots, \Upsilon_q$, then the polarization of Υ is given as,

$$P(\Upsilon) = (P(\aleph_1),\, P(\aleph_2),\, P(\aleph_3),\, \ldots,\, P(\aleph_q)\,).$$

It is a sqaure free monomial ideal in the polynomial ring $\aleph'$.

Here is an illustration of how polarization works.

Example 1.4.2 Let $\Upsilon = (\zeta_1^2,\, \zeta_1\zeta_2,\, \zeta_2^3) \subseteq \aleph = V[\zeta_1,\, \zeta_2]$ be a monomial ideal in $\aleph$. Then, the polarization of Υ in the polynomial ring
$\aleph' = V[\zeta_{1,1}, \zeta_{1,2}, \zeta_{2,1}, \zeta_{2,2}, \zeta_{2,3}]$ is given by,

$$P(\Upsilon) = (\zeta_{1,1}\zeta_{1,2},\, \zeta_{1,1}\zeta_{2,1},\, \zeta_{2,1}\zeta_{2,2}\zeta_{2,3}).$$

Remark 1.4.3 Note that by recognizing each ζ_i by $\zeta_{i,1}$, one can say that $\aleph'$ is a polynomial extension of $\aleph$. Precisely, the number of variables in $\aleph'$ consistently rely on what we polarize. As long, we are keen on the polarization of finitely many monomial ideals, $\aleph'$ will be a finite polynomial ring.

Basic properties of polarization

Suppose that $\aleph = V[\zeta_1, \ldots, \zeta_q]$ be a polynomial ring over a field V. Υ_1 and Υ_2 are two monomial ideals of $\aleph$, then

1. $P(\Upsilon_1 + \Upsilon_2) = P(\Upsilon_1) + P(\Upsilon_2)$.

2. Let η_1 and η_2 be two monomials in $\aleph$, then $\eta_1|\eta_2$ if and only if $P(\eta_1)|P(\eta_2)$.

3. $P(\Upsilon_1 \cap \Upsilon_2) = P(\Upsilon_1) \cap P(\Upsilon_2)$.

Example 1.4.4 Let $\Upsilon = (\zeta_1^2, \zeta_2^3, \zeta_1\zeta_2)$ be an ideal of $\aleph$, then primary decomposition of Υ is given by,

$$\Upsilon = (\zeta_1, \zeta_2^3) \cap (\zeta_1^2, \zeta_2).$$

The polarization of Υ is given as

$$P(\Upsilon) = (\zeta_{1,1}\zeta_{1,2}, \zeta_{2,1}\zeta_{2,2}\zeta_{2,3}, \zeta_{1,1}\zeta_{2,1}).$$

$P(\Upsilon)$ is a sqaure free monomial ideal.

1.5 Module Theory

Definition 1.5.1 Let $\aleph$ be a commutative ring, then the $\aleph-$module $\beth$ is an additive abelian group together with the map

$$\cdot : \aleph \times \beth \rightarrow \beth$$

defined by $\cdot(\zeta, \eta) = \zeta\eta$, that meets the following four requirments i.e $\forall\, \zeta_1, \zeta_2 \in \aleph$ and $\eta_1, \eta_2 \in \beth$,

1. $\zeta(\eta_1 + \eta_2) = \zeta\eta_1 + \zeta\eta_2$,
2. $(\zeta_1 + \zeta_2)\eta = \zeta_1\eta + \zeta_2\eta$,
3. $(\zeta_1\zeta_2)\eta = \zeta_1(\zeta_2\eta)$,
4. $1\eta = \eta$.

Remark 1.5.2 Our above definition of left $\aleph$-module $\beth$ is based on a function from $\aleph \times \beth$ to $\beth$. If, we reverse the order of $\aleph$ and $\beth$ in this cartesian product and make further notational changes that seem natural, we obtain the definition of a right $\aleph$-module.
If $\aleph$ is a commutative ring and $\beth$ be a left $\aleph$-module, we can make $\beth$ into a right $\aleph$-module by defining $\zeta\eta = \eta\zeta\ \ \forall\, \zeta \in \aleph$ and $\eta \in \beth$. If $\aleph$ is not a commutative ring then axiom (2) in general will not hold with this definition, so every left $\aleph$-module is not right $\aleph$-module.
Module satisfying axiom 4 are called left unital modules and throughout this thesis we consider left unital modules. If $\aleph$ is commutative ring with unity, then every left unital $\aleph$-module is right $\aleph$-module, therefore it is enough to consider left unital $\aleph$-modules and we will call the unital left $\aleph$-module simply $\aleph$-module.

Example 1.5.3 Every additive abelian group $\beth$ may be regarded as a $\mathbb{Z}$-module by defining $\zeta\eta$, for any $\zeta \in \mathbb{Z}$ and $\eta \in \beth$, as follows:

1. If $\zeta > 0$, then $\zeta\eta$ is the sum of ζ copies of η,

2. If $\zeta = 0$, then $\zeta\eta = 0$,

3. If $\zeta < 0$, then $\zeta\eta$ is the sum of $-\zeta$ copies of η.

Where, $\mathbb{Z}$ is the ring of integers.

Example 1.5.4

1. Each ring $\aleph$ can be viewed as an $\aleph$ module, as follows: for abelian group, use $\aleph$ with its additive structure and use ring multiplication for the scalar multiplication of the ring $\aleph$ into the abelian group $(\aleph, +)$. Each of the four conditions in the definition of module $\aleph$ can be derived from the usual conditions in the definition of ring.

2. The smallest of all $\aleph$-modules is the module that has one element, namely an additive identity. We will denote this module by (0).

Example 1.5.5 Let $\beth$ be a vector space over a field V with $\dim(\beth) = q$ and $\aleph$ be the ring of $q \times q$ matrices with entries in V, then $\beth$ is a right $\aleph-$module, with scalar multiplication being the usual multiplication of a vector by a matrix.

Definition 1.5.6 Let $\beth$ be an $\aleph$-module. A subset $\beth_1$ of $\beth$ is said to be a submodule of $\beth$, if it is a subgroup of $\beth$, and for any $\zeta \in \aleph$ and $\eta \in \beth_1$, we have $\zeta\eta \in \beth_1$.

Example 1.5.7 1. Let $\aleph$ be a field, then the submodules of $\aleph$-module (that is to say, a vector space over $\aleph$) are precisely the familiar subspaces.

2. Let Υ be an ideal of $\aleph$ and $\beth$ be an $\aleph$-module, then the set $\Upsilon\beth$ consisting of all elements of the form $\upsilon_1\eta_1 + \cdots + \upsilon_n\eta_n$ ($\upsilon_i \in \Upsilon$ and $\eta_i \in \beth$) is a submodule of $\beth$.

Definition 1.5.8 Let $\aleph$ be an arbitrary ring and $\beth_1$ be a submodule of $\aleph$-module $\beth$. Since $\beth_1$ is a subgroup of the abelian group $\beth$, then the factor group $\beth/\beth_1$ is also defined. Here question arises, how does the scalar operation of $\aleph$ on $\beth$ provide the scalar operation of $\aleph$ on $\beth/\beth_1$. For this, let η_1 and η_2 be elements of $\beth$, belonging to the same coset modulo $\beth_1$. That is to say, $\eta_1 + \beth_1 = \eta_2 + \beth_1$ if and only if difference between η_1 and η_2 belongs to $\beth_1$. Then, for any $\zeta \in \aleph$ and $\eta_1 - \eta_2 \in \beth_1$, we have $\zeta(\eta_1 - \eta_2) = \zeta\eta_1 - \zeta\eta_2 \in \beth_1$ (because $\beth_1$ is a submodule of $\aleph$-module $\beth$), and $\zeta\eta_1$ and $\zeta\eta_2$ belongs to the same coset modulo $\beth_1$, i.e $\zeta\eta_1 + \beth_1 = \zeta\eta_2 + \beth_1$. Thus, we have a well-defined operation of $\aleph$ on $\beth/\beth_1$, given by $\zeta(\eta + \beth_1) = \zeta\eta + \beth_1$. Now, using this operation, we will verify the four conditions of module for $\beth/\beth_1$,

1. $\zeta[(\eta_1 + \beth_1) + (\eta_2 + \beth_2)] = \zeta[(\eta_1 + \eta_2) + \beth_1] = \zeta(\eta_1 + \eta_2) + \beth_1 = (\zeta\eta_1 + \zeta\eta_2) + \beth_1 = (\zeta\eta_1 + \beth_1) + (\zeta\eta_2 + \beth_1) = \zeta(\eta_1 + \beth_1) + \zeta(\eta_2 + \beth_1)$,

2. $(\zeta_1 + \zeta_2)(\eta + \beth_1) = (\zeta_1 + \zeta_2)\eta + \beth_1 = (\zeta_1\eta + \zeta_2\eta) + \beth_1 = (\zeta_1\eta + \beth_1) + (\zeta_2\eta + \beth_1) = \zeta_1(\eta + \beth_1) + \zeta_2(\eta + \beth_1)$,

3. $(\zeta_1\zeta_2)(\eta+\beth_1) = (\zeta_1\zeta_2)\eta+\beth_1 = \zeta_1(\zeta_2\eta)+\beth_1 = \zeta_1[(\zeta_2)\eta+\beth_1] = \zeta_1[\zeta_2(\eta+\beth_1)]$,

4. $1(\eta + \beth_1) = 1\eta + \beth_1 = \eta + \beth_1$.

The module $\beth/\beth_1$ is called the factor module of $\beth$ by $\beth_1$.

Definition 1.5.9 Let $\beth_1$ and $\beth_2$ be $\aleph$-modules. A function $\mathcal{L}$ from $\beth_1$ to $\beth_2$ is called an $\aleph$-homomorphism if and only if

- $\mathcal{L}(\eta_1 + \eta_2) = \mathcal{L}(\eta_1) + \mathcal{L}(\eta_2), \qquad \forall\, \eta_1,\, \eta_2 \in \beth_1.$
- $\mathcal{L}(\zeta\eta) = \zeta\mathcal{L}(\eta), \qquad \forall\, \zeta \in \aleph,\, \eta \in \beth_1.$

If $\mathcal{L}$ is (1-1) and onto, then it is called an $\aleph$-module isomorphism.

Examples 1.5.10

1. When $\aleph$ is a field, then the $\aleph$-homomorphisms are the familiar $\aleph$-linear transformations studied in linear algebra.
2. The 0 map from $\beth_1$ to $\beth_2$, which maps every element of $\beth_1$ to 0 is an example of $\aleph$-homomorphism.

Definition 1.5.11 Let $\mathcal{L}$ be an $\aleph$-homomorphism from an $\aleph$-module $\beth_1$ to $\aleph$-module $\beth_2$. The set of elements of $\beth_1$, for which we have $\mathcal{L}(\eta) = 0$, referred to as the kernel of $\mathcal{L}$. This set is denoted by $\mathrm{Ker}(\mathcal{L})$, and it is a submodule of $\beth_1$.

Definition 1.5.12 Let $\beth$ be an $\aleph$-module, then the annihilator of $\aleph$- module $\beth$ is given as, $\mathrm{Ann}(\beth) = \{\zeta \in \aleph \mid \zeta\beth = 0\}$. It is an ideal of $\aleph$.

Definition 1.5.13 Let $\aleph$ be a ring and $\{\beth_j | j \in \mathcal{A}\}$ be a family of $\aleph$-modules, indexed by an arbitrary nonempty index set $\mathcal{A}$. From this indexed family of modules, we can form an $\aleph$-module $\beth$ given as,

$$\beth = \prod_{j \in \mathcal{A}} \beth_j.$$

It is called the direct product of the modules $\beth_j$, whose elements are the collections $\{\eta_j\}$, where $\eta_j \in \beth_j$ and $j \in \mathcal{A}$. In $\beth$, the operation of addition is the componentwise addition and the scalar multiplication is distributed all over the components, i.e for any $\zeta \in \aleph$ and $\eta_j \in \beth_j$, we have $\zeta\{\eta_j\} = \{\zeta\eta_j\}$.

Definition 1.5.14 Let $\beth$ be defined as above in definition 1.5.13, then the subset $\beth_0$ of $\aleph$-module $\beth$ consisting of finitely many non zero elements, this subset is called the coproduct (or often direct sum) of $\{\beth_j | j \in \mathcal{A}\}$, and it is denoted by,

$$\bigoplus_{j \in \mathcal{A}} \beth_j$$

or

$$\coprod_{j \in \mathcal{A}} \beth_j.$$

Definition 1.5.15 Let $\aleph$ be a ring. Let $\beth_1$ be a right $\aleph$-module, $\beth_2$ be a left $\aleph$-module, and $\mathbb{P}$ be an abelian group. A map

$$\mathcal{L}_1 : \beth_1 \times \beth_2 \to \mathbb{P}$$

is said to be $\aleph$-balanced product, if for all $\eta'_1, \eta'_2 \in \beth_1$, $\eta_1, \eta_2 \in \beth_2$, and $\zeta \in \aleph$, the following holds

1. $\mathcal{L}_1(\eta', \eta_1 + \eta_2) = \mathcal{L}_1(\eta', \eta_1) + \mathcal{L}_1(\eta', \eta_2)$.

2. $\mathcal{L}_1(\eta'_1 + \eta'_2, \eta) = \mathcal{L}_1(\eta'_1, \eta) + \mathcal{L}_1(\eta'_2, \eta)$.

3. $\mathcal{L}_1(\eta' \cdot \zeta, \eta) = \mathcal{L}_1(\eta', \zeta \cdot \eta)$.

Definition 1.5.16 Let $\aleph$, $\beth_1$, $\beth_2$ and $\mathbb{P}$ are defined as above. Then, the abelian group $\beth_1 \otimes_\aleph \beth_2$ is called the tensor product of $\beth_1$ and $\beth_2$, together with an $\aleph$-balanced map

$$\mathcal{L}_2 : \beth_1 \times \beth_2 \to \beth_1 \otimes_\aleph \beth_2,$$

and for every abelian group $\mathbb{P}$ and $\aleph$-balanced map

$$\mathcal{L}_1 : \beth_1 \times \beth_2 \to \mathbb{P}$$

there is a unique abelian group homomorphism

$$\mathcal{L}_3 : \beth_1 \otimes_\aleph \beth_2 \to \mathbb{P},$$

such that

$$\mathcal{L}_1 = \mathcal{L}_3 \circ \mathcal{L}_2.$$

Definition 1.5.17 Let $\beth_1$ be a subset of $\aleph$-module $\beth$, if for any $\eta \in \beth$, there are $\zeta_1 \ldots \zeta_q \in \aleph$ and $\upsilon_1 \cdots \upsilon_q \in \beth_1$, such that

$$\eta = \zeta_1\upsilon_1 + \zeta_2\upsilon_2 \cdots + \zeta_q\upsilon_q,$$

then, we say that $\beth_1$ is the generating set of $\beth$ over ring $\aleph$.
More generally, for any subset $\beth_2$ of $\beth$, the $\aleph$-submodule of $\beth$ generated by $\beth_2$ is the submodule of this type, $\beth'' = \aleph\dot{\beth}_2$. This submodule consists off all limited sums of this sort $\zeta_1\upsilon'_1 + \zeta_2\upsilon'_2 \cdots + \zeta_q\upsilon'_q$, where $\zeta_q \in \aleph$ and $\upsilon'_q \in \beth_2$.

Definition 1.5.18 If $\beth_1$ and $\beth_2$ are submodules of $\aleph$-module $\beth$, then $\beth_1 + \beth_2$ is a submodule of $\beth$ consisting of all elements of the form $\upsilon_1\zeta_1 + \upsilon_2\zeta_2$, where $\upsilon_q \in \beth_q$ and $\zeta_q \in \aleph$. The submodule $\beth_1 + \beth_2$ is generated by $\beth_1 \bigcup \beth_2$.

Definition 1.5.19 If there exists a finite subset $\beth_1$ of $\beth$, such that $\beth$ is generated by $\beth_1$, then we say that $\beth$ is finitely generated. If $\beth_1 = \eta$ is a single element set, such that $\beth = \aleph \cdot \eta$, then we say that $\beth$ is cyclic with generator η.

Definition 1.5.20 If $\beth$ is finitely generated module, there exists a generating set of minimal cardinality. Such a generating set is called minimal generating set.

Definition 1.5.21 Assume that the subset $\beth_1$ generates $\aleph$-module $\beth$. Then, $\beth_1$ freely generates $\beth$, if for every $\eta \in \beth$, there are distinctive $\zeta_1, \ldots, \zeta_q \subseteq \aleph$ and $\upsilon_1, \ldots, \upsilon_q \subseteq \beth_1$, such that

$$\eta = \upsilon_1 \zeta_1 + \cdots + \upsilon_q \zeta_q.$$

Corollary 1.5.22 For any neotherian ring $\aleph$, we have

$$\dim \aleph[\zeta_1, \ldots, \zeta_q] = q + \dim \aleph.$$

For any field V, $\dim V[\zeta_1, \ldots, \zeta_q] = q$ and $\dim V[\zeta_1, \ldots] = \infty$.

1.5.1 Exact sequences

Definition 1.5.23 The sequence of $\aleph$-modules and $\aleph$-homomorphisms, given by

$$\cdots \longrightarrow \beth_{b-1} \xrightarrow{\Upsilon_b} \beth_b \xrightarrow{\Upsilon_{b+1}} \beth_{b+1} \xrightarrow{\Upsilon_{b+2}} \ldots$$

is said to be exact at $\beth_b$, if

$$\mathrm{Im}\,(\Upsilon_b) = \ker(\Upsilon_{b+1}).$$

If the sequence is exact at every $\beth_b$, then it is called an exact sequence. The particular sequence

$$0 \longrightarrow \beth_1 \xrightarrow{\Upsilon_1} \beth$$

is said to be exact at $\beth_1$ iff Υ_1 is one-one, and the sequence

$$\beth \xrightarrow{\Upsilon_2} \beth_2 \longrightarrow \; 0$$

is exact at $\beth_2$ iff Υ_2 is onto.

Proposition 1.5.24 This particular sequence

$$0 \longrightarrow \; \beth_1 \xrightarrow{\Upsilon_1,} \beth \xrightarrow{\Upsilon_2} \beth_2 \longrightarrow \; 0$$

is said to be short exact iff Υ_1 is one-one, Υ_2 is onto and

$$\mathrm{Im}(\Upsilon_1) = \ker(\Upsilon_2).$$

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1.5.2 Graded modules

Definition 1.5.25 Let $\mathbb{G}$ be a abelian-semi group with addition, then the ring $\aleph$ is said to be a $\mathbb{G}$-graded ring if it has a decomposition

$$\aleph = \bigoplus_{b \in \mathbb{G}} \aleph'_b,$$

such that $\aleph'_{b_1} \aleph'_{b_2} \subset \aleph'_{b_1+b_2}$, $\forall\, b_1,\, b_2 \in \mathbb{G}$. Then for $\zeta \in \aleph$, we can write ζ in a unique expression

$$\zeta = \sum_{b \in G} \zeta_b,$$

where $\zeta_b \in \aleph_b$ and almost all $\zeta_b = 0$ and ζ_b is called the p-th homogeneous component and If $\zeta = \zeta_b$, then ζ is called homogeneous of degree b.

Definition 1.5.26 Let $\aleph$ be a $\mathbb{G}$-graded ring and $\beth$ be an $\aleph$-module, then $\beth$ is said to be a $\mathbb{G}$-graded module if it has a decompostion

$$\beth = \bigoplus_{b \in \mathbb{G}} \beth_b,$$

such that $\aleph_{b_1} \beth_{b_2} \subset \beth_{b_1+b_2}$ $\forall\, b_1, b_2 \in \mathbb{G}$.

Example 1.5.27 Let $\aleph = V[\zeta_1, \ldots, \zeta_q]$ be a ring of polynomials over field V. For $b = (b_1, \cdots, b_q)$ in $\mathbb{Z}^q$, we set $\zeta^b = \zeta_1^{b_1} \cdots \zeta_q^{b_q}$ and if $P \in \aleph$ has the form $c\zeta^b$, where $c \in V$, it is called a homogeneous element of degree b, then the induced $\mathbb{Z}^q$-grading of polynomial ring $\aleph$ is given by

$$\aleph = \bigoplus_{b \in \mathbb{Z}^q} \aleph'_b,$$

where $\aleph'_b = \begin{cases} V\zeta^d, & \text{if } b \in \mathbb{Z}^q_+; \\ 0\,, & \text{otherwise.} \end{cases}$

Definition 1.5.28 An $\aleph$-module $\beth$ is said to be $\mathbb{Z}^q$-graded if

$$\beth = \bigoplus_{b \in \mathbb{Z}^q} \beth_b,$$

and $\aleph_b \beth_d \subset \beth_{b+d}$ $\forall$ $b,\, d \in \mathbb{Z}^q$.

Example 1.5.29

- Let $\aleph = V[\zeta]$ be a polynomial ring. When $b = 0$, we have $\aleph'_0 = V\zeta^0 = V$, if $b = 1$, we have $\aleph'_1 = V\zeta^1 = V\zeta$, and for $b = 2$, $\aleph'_2 = V\zeta^2$, then $V[\zeta]$ is $\mathbb{Z}$-graded with decomposition

$$\aleph = V \oplus V\zeta \oplus V\zeta^2 \oplus V\zeta^3 \oplus \cdots$$

- $V[\eta,\, \zeta]$ is $\mathbb{Z}$-graded, because we have following decomposition

$$V[\eta,\, \zeta] = V \oplus (V\eta + V\zeta) \oplus (V\eta^2 + V\eta\zeta + V\zeta^2) \oplus (V\eta^3 + V\eta^2\zeta + V\eta\zeta^2 + V\zeta^3) \oplus \cdots$$

Chapter 2

A brief overview of Graph Theory

2.1 Introduction

Mathematical and non-mathematical problems can be solved using diagrams. The diagrams involved in such problems represent a more or less subtle rewrite of the problems, and the quest to solve such problems gave rise to Graph Theory.

In recent years, there has been a surge of interest in utilizing combinatorial approaches to model algebraic problems. This technique of using a combinatorial object to describe the behavior of an algebraic problem has yielded impressive results. This approach has allowed us to characterize the algebraic properties of edge ideals in terms of the combinatorial behavior of associated graphs.

Since the introduction of edge rings, there have been many papers studying algebraic properties of the edge rings of graphs. Studying invariants of the rings, such as depth in [16] and [17] are examples of algebraic properties that have been investigated.

In this chapter, we discuss the fundamentals of Graph Theory. We also discuss different types of graphs which we will use later.

2.2 Graph fundamentals

Definition 2.2.1 A graph is made up of two sets, $\aleph_V$ and $\aleph_E$. The elements of $\aleph_V$ are the vertices of graph, while the elements of $\aleph_E$ are the edges of graph.

Definition 2.2.2 If the ends of an edge coincides, then the edge is called a loop.

Definition 2.2.3 A multiple edge is formed when two edges have same pair of end points.

Definition 2.2.4 If an edge connects two vertices such as η_1 and η_2, then they are considered adjacent, also η_1 and η_2 are said to be neighbours of each other.

If a point isn't an endpoint of an edge, it is considered isolated.

Definition 2.2.5 Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a graph on q vertices, $\aleph$ be a polynomial ring in q variables over a field V, then the edge ideal $I = I(\mathcal{H})$ of $\mathcal{H}$ be the ideal generated by all monomials of the type $\zeta_j\zeta_{j+1}$, where $\{\zeta_j, \zeta_{j+1}\} \in E(\mathcal{H})$.

Definition 2.2.6 The degree of a vertex in a graph is the total number of its neighbours and it is identified as $d_{\mathcal{H}}(\eta)$.

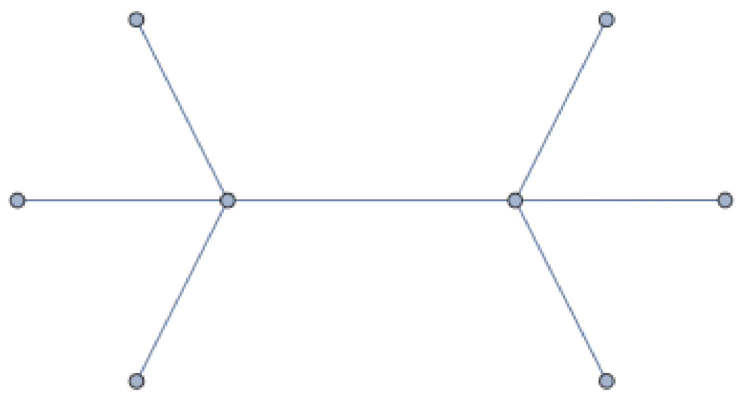

Figure 2.1: An example of simple graph without loop and multiple edges.

Figure 2.2: A graph with loop.

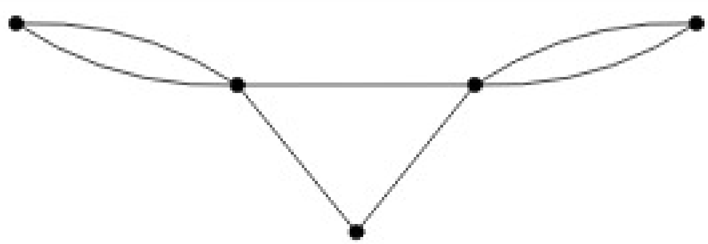

Figure 2.3: An example of graph with multiple edges.

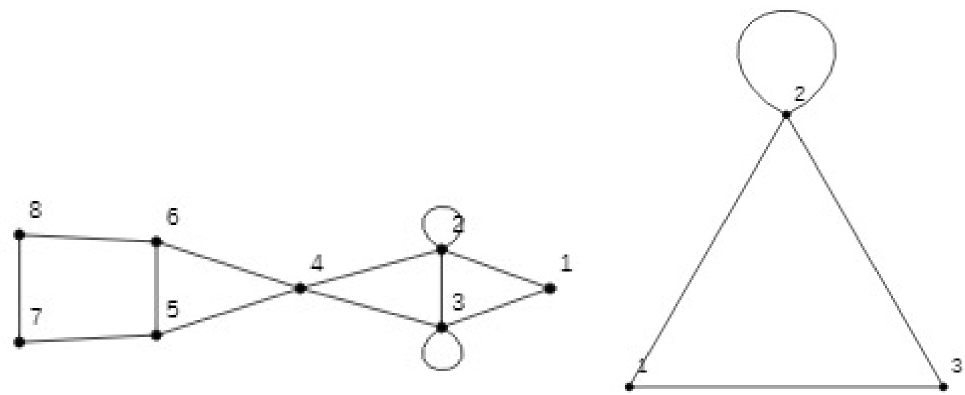

Figure 2.4: Graph on left with its subgraph on right.

Definition 2.2.7 The cardinallity of the vertex set of a graph determines its order and the cardinallity of the edge set of a graph determines its size.

Definition 2.2.8 For a graph $\mathcal{H} = (\aleph_V, \beth_E)$, If $\aleph_{V_1}$ is a subset of vertex set $\aleph_V$ and $\beth_{E_1}$ is a subset of edge set $\beth_E$, and any edge in $\beth_E$ has its end points in $\aleph_{V_1}$, then $\mathcal{H}_1 = (\aleph_{V_1}, \beth_{E_1})$ is a subgraph of $\mathcal{H}$.
For example see Fig 2.4.

Definition 2.2.9 Let $\mathcal{H}_1$ be a subgraph of $\mathcal{H}$, if all the vertices of $\mathcal{H}$ are in $\mathcal{H}_1$, then the subgraph $\mathcal{H}_1$ of $\mathcal{H}$ is called a spanning subgraph of $\mathcal{H}$, and for any vertices η_1, $\eta_2 \in \mathcal{H}_1$, if all edges between η_1 and η_2 belong to $\mathcal{H}_1$, then the subgraph $\mathcal{H}_1$ is said to be an induced subgraph of $\mathcal{H}$.
For example see Fig 2.5.

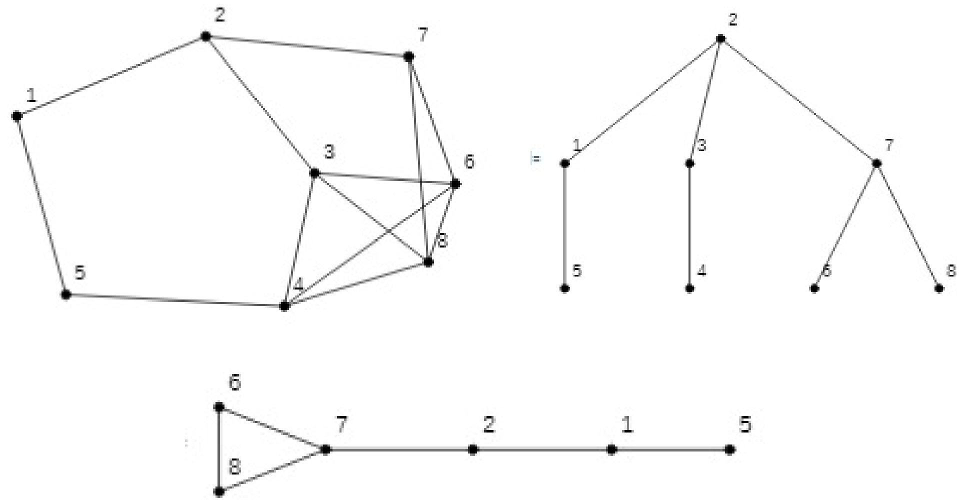

Figure 2.5: The top left graph with its spanning subgraph on top right and induced subgraph below.

Definition 2.2.10 A graph $P_q = (\aleph_V, \beth_E)$ comprising of vertex set $\{\eta_1, \eta_2, \ldots, \eta_q\}$ and an edge set $\{\eta_1\eta_2, \eta_2\eta_3, \ldots, \eta_{q-1}\eta_q\}$ is called a path graph, denoted by P_q and the length of a path is the number of edges it contains. On the other hand, a cycle is a graph with vertex and edge sets are given by $\{\eta_1, \eta_2, \ldots, \eta_q\}$ and $\{\eta_1\eta_2, \eta_2\eta_3, \ldots, \eta_{q-1}\eta_q, \eta_1\eta_q\}$.
For example see Fig 2.6.

Definition 2.2.11 A graph is said to be connected if there is a path with the vertex sequence $\{\eta_1\eta_2, \eta_2\eta_3, \ldots, \eta_{q-1}\eta_q\}$ for every pair of vertices.

Definition 2.2.12 A complete graph $\mathcal{H} = (\aleph_V, \beth_E)$ is a graph in which any two vertices in the vertex set $\aleph_V$ are adjacent.

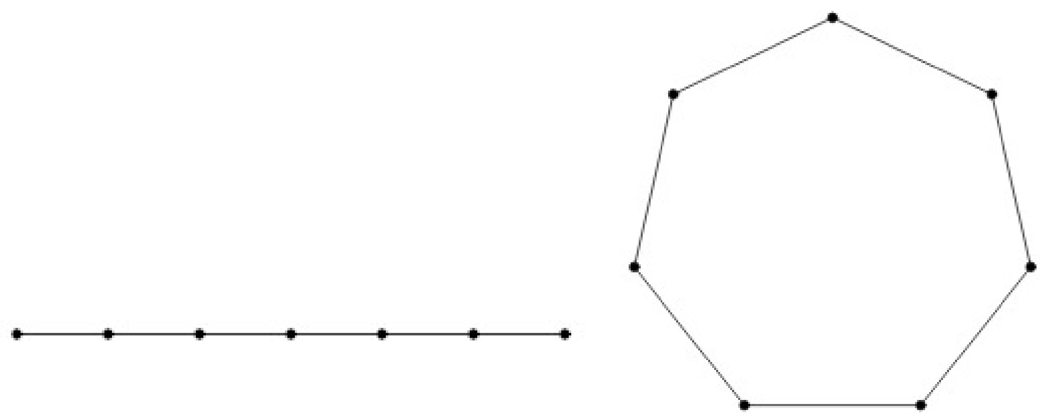

Figure 2.6: Path P_7 and cycle graph C_7.

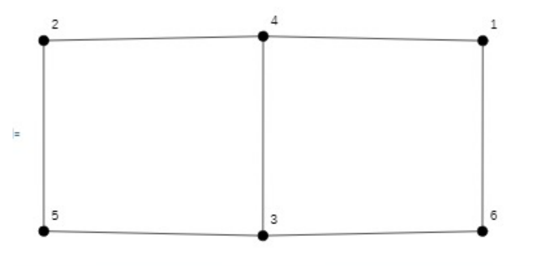

Figure 2.7: Bipartite graph.

Definition 2.2.13 For a graph $\mathcal{H} = (\aleph_V, \beth_E)$, if there is a subdivision of vertex set $\aleph_V$ given by $\aleph_V = \aleph_{V_1} \bigcup \aleph_{V_2}$, and for every edge $\eta_1\eta_2$ of $\aleph_V$, we have $\eta_1 \in \aleph_{V_1}$ and $\eta_2 \in \aleph_{V_2}$ or $\eta_1 \in \aleph_{V_2}$ and $\eta_2 \in \aleph_{V_1}$, then the graph is called bipartite graph.
For example, see Fig 2.7.

Definition 2.2.14 A regular graph is a graph in which every vertex has the same degree.
For example, see Fig 2.8.

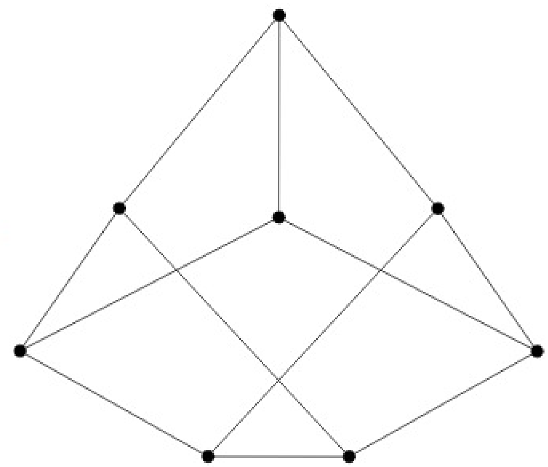

Figure 2.8: Franklin 3-regular graph.

Definition 2.2.15 The distance between two vertices in a graph is the number of edges in the shortest path.

Definition 2.2.16 The diameter of a graph is the maximum distance between the pair of vertices.

Definition 2.2.17

1. The eccentricity of a vertex η in a graph is the maximum distance between vertex η and any other vertex η' of graph.
2. The radius of a graph is the least of all eccentricities of its vertices.
3. The centre of a graph is a set of vertices with the least eccentricity.

Definition 2.2.18 A tree is a graph with no cycles and there exists a path connecting every pair of vertices.

Remark 2.2.19 Every tree of order q is $q-1$ in size.

Definition 2.2.20 A vertex with degree at least 2 is considered as an inner vertex. Similarly, a vertex of degree 1 is called a leaf or terminal vertex.

Definition 2.2.21 A caterpillar is a tree of order 3 or more in which a path graph is produced by deleting the leaves and their incident edges.
For example, see Fig 2.10.

Definition 2.2.22 A J-star is a tree with one internal vertex and $J - 1$-leaves.

Definition 2.2.23 The centre of every tree is made up of one or two adjacent vertices. Every longest path has a centre, which is the middle vertex or the middle two vertices.

2.3 Lobster trees and uniclyclic graphs

Definition 2.3.1 A lobster tree is a tree in which removing the leaf nodes leaves a caterpillar graph.
For example, see Fig 2.9.

Definition 2.3.2 A connected graph with precisely one cycle is called a unicyclic graph.
For example, see Fig 2.11.

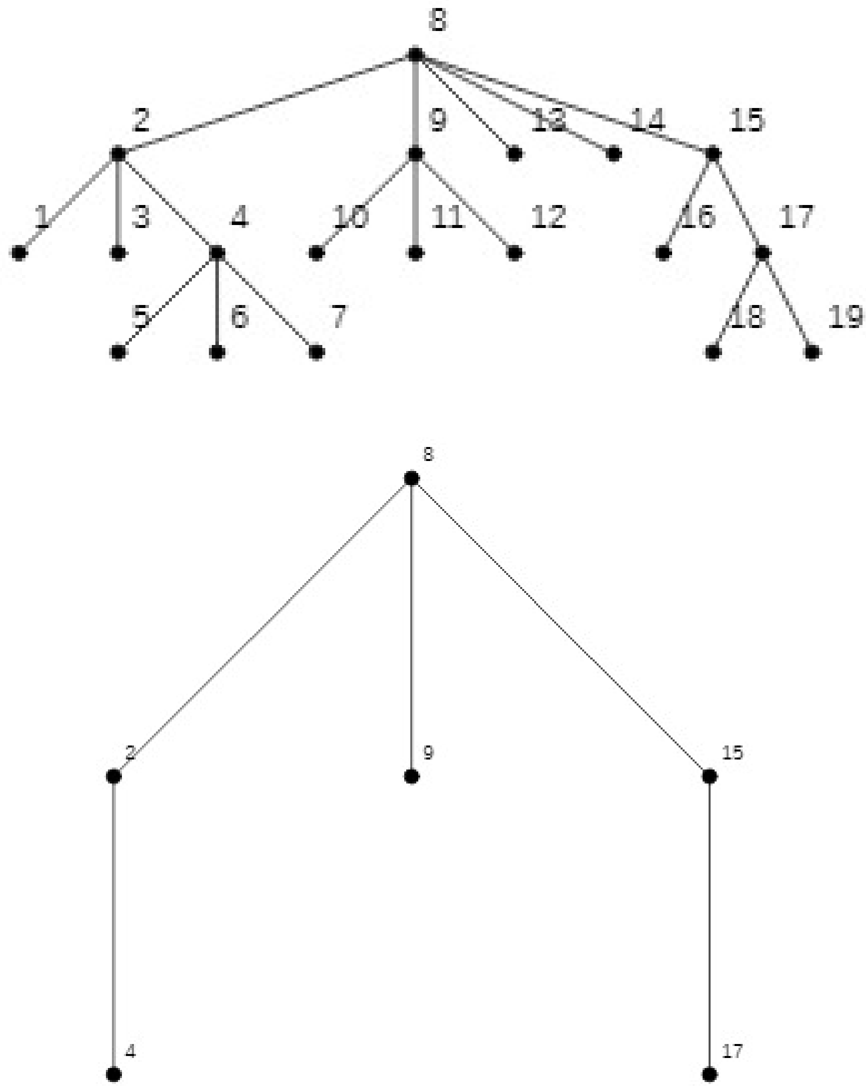

Figure 2.9: A lobster tree on the top, with its caterpillar subgraph below.

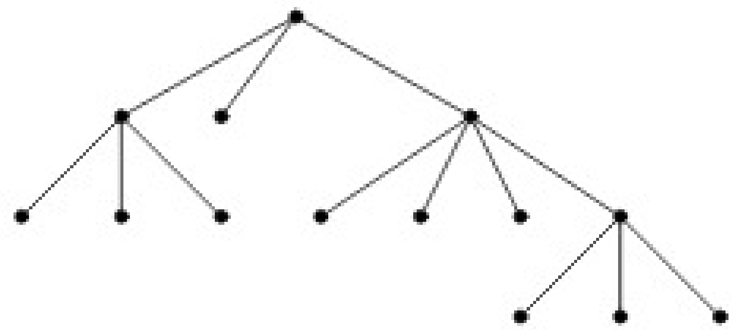

Figure 2.10: An example of caterpillar tree.

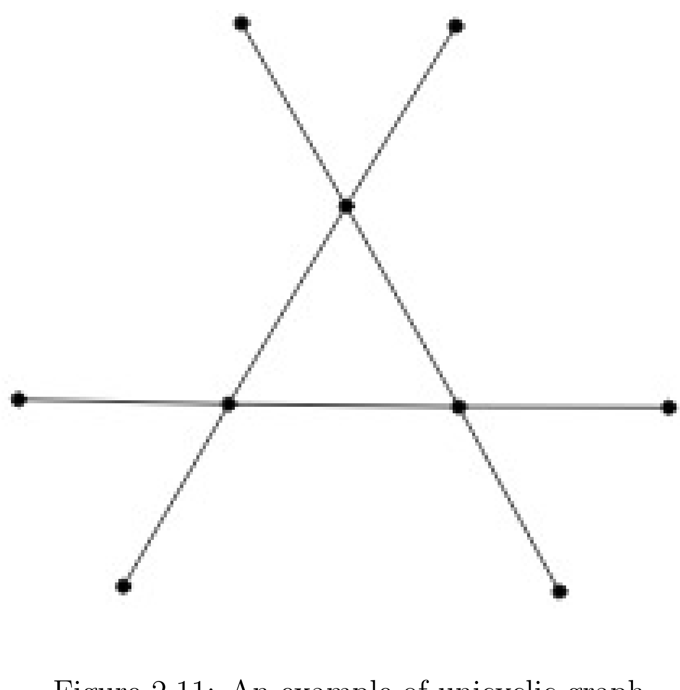

Figure 2.11: An example of unicyclic graph.

Chapter 3

Depth and Stanley depth

In this chapter, we discuss how the regular element property helps in determining the depth of modules. We discuss the Stanley depth([27]) of $\mathbb{Z}^q$-graded modules. We also discuss some recent results of the Stanley depth and depth of edge ideals associated with different graphs.

3.1 Depth

3.1.1 The regular element property

Definition 3.1.1 Let $\aleph$ be a commutative ring and $\beth$ be an $\aleph-$module, then for any subset Υ of $\aleph$, the set

$$\mathrm{Ann}_{\beth}(\Upsilon) = \{\eta \in \beth \mid \eta\Upsilon = 0\}$$

is called annihilator of Υ in $\beth$ and it is a submodule of $\beth$.

Definition 3.1.2 An element $\zeta \neq 0$ in $\aleph$ is said to be $\beth$-regular if $\mathrm{Ann}_{\beth}(\zeta) = 0$.

Definition 3.1.3 We say that $\aleph$ has regular element property if for each finitely presented $\aleph$-module $\beth$, and finitely generated ideal Υ of $\aleph$, we have $\mathrm{Ann}_{\beth}(\Upsilon) = 0$, then Υ contains a $\beth$-regular element.

Definition 3.1.4 Let $\beth$ be an $\aleph$-module, a sequence $\zeta = \zeta_1, \zeta_2, \ldots, \zeta_q \in \aleph$ is said to be $\beth$-regular sequence if

1. ζ_r is $\beth/(\zeta_1, \zeta_2, \ldots, \zeta_{r-1})\beth-$regular for each r.

2. $\beth/(\zeta_1, \ldots, \zeta_q)\beth \neq 0$.

Definition 3.1.5 Let $\aleph$ be a neotherian ring and $\beth$ be a finitely generated $\aleph$-module, then the depth of $\beth$ is the common length of all maximal $\beth$-regular sequences in the maximal ideal Υ of $\aleph$.

Definition 3.1.6 The depth of a local neotherian ring $\aleph$ as an $\aleph$-module is the utmost length of a regular sequence in the maximal ideal.

Theorem 3.1.7 Let $\beth$ be an $\aleph$-module, then sdepth$(\beth)$, depth$(\beth) \leq \dim \beth$, where $\dim \beth$ is the krull dimension of $\beth$.

Example 3.1.8 Let $\aleph = V[\zeta_1, \zeta_2]$ be a module over itself. Then ζ_1 is $\aleph-$regular because

$$\zeta_1 \cdot \zeta_r \neq 0,$$

for all $\zeta_r \neq 0 \in \aleph$. This indicates that

$$\text{Ann}_{\aleph}(\zeta_1) = 0$$

Hence ζ_1 is $\aleph$-regular. Also ζ_2 is $\beth = \aleph/(\zeta_1)\aleph$-regular becuase

$$\text{Ann}_{\beth}(\zeta_2) = 0,$$

where ideal $\Upsilon' = (\zeta_1)\aleph$ is the set of all possible finite sums of elements of form $\zeta'\eta'$, where $\zeta' \in (\zeta_1)$ and $\eta' \in \aleph$.
As a result $\zeta = \zeta_1, \zeta_2$ is $\aleph$-regular sequence in the maximal ideal (ζ_1, ζ_2), this gives us

$$\text{depth}(\aleph) = 2.$$

In general, if $\aleph = V[\zeta_1, \zeta_2, \cdots, \zeta_q]$ is a module over itself. Then the regular sequence $\zeta = \zeta_1, \zeta_2, \ldots, \zeta_q$ is maximal in the maximal ideal $\Upsilon = (\zeta_1, \zeta_2, \ldots, \zeta_q)$. As a conclusion, we have

$$\text{depth}(\aleph) = q.$$

Example 3.1.9 Let $\aleph = V[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$ be polynomial ring and $\beth = (\zeta_1, \zeta_2\zeta_3)/(\zeta_2\zeta_4)$ be an $\aleph$ module. The ideal $\Upsilon = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is maximal in $\aleph$ and the sequence $\zeta = \zeta_1, \zeta_3$ in Υ is $\beth-$regular, because ζ_1 is $\beth/(0)\beth = \beth-$regular and ζ_3 is $\beth/(\zeta_1)\beth$-regular, i.e

$$\text{Ann}(\zeta_1) = 0 = \text{Ann}(\zeta_3).$$

While,

$$\text{Ann}(\zeta_2) \neq 0 \neq \text{Ann}(\zeta_4).$$

Then, $\text{depth}(\beth) = 2$.

Example 3.1.10 Let $\aleph = V[\zeta_1, \zeta_2, \cdots, \zeta_q]$ be a polynomial ring and $\Upsilon = (\zeta_2^2, \zeta_2\zeta_1, \zeta_2\zeta_3, \zeta_2\zeta_4, \ldots, \zeta_2\zeta_q)$ is an ideal of $\aleph$. Let $\beth = \aleph/\Upsilon$ be an $\aleph-$module, then for each $\zeta_r \in \aleph$, we have

$$\text{Ann}(\zeta_r) \neq 0.$$

To verify this, let for ζ_1, we have $\zeta_1 \cdot (\zeta_2 + \Upsilon) = 0$, but $\zeta_1 \neq 0$. As a result, $\text{depth}(\beth) = 0$.

Lemma 3.1.11 ([4]) (Depth Lemma) Let we have a short exact sequence $0 \to \beth_1 \to \beth_2 \to \beth_3 \to 0$ of modules over a local ring $\aleph$, then

1. $\text{depth}(\beth_2) \geq \min\{\text{depth}(\beth_3), \text{depth}(\beth_1)\}$.
2. $\text{depth}(\beth_1) \geq \min\{\text{depth}(\beth_2), \text{depth}(\beth_3) + 1\}$.
3. $\text{depth}(\beth_3) \geq \min\{\text{depth}(\beth_1) - 1, \text{depth}(\beth_2)\}$.

3.2 Stanley decomposition and Stanley depth

Definition 3.2.1 Let $\aleph = V[\zeta_1, \ldots, \zeta_q]$ be a polynomial ring and $\beth$ be a finitely generated $\mathbb{Z}^q$-graded $\aleph$-module. Then, the V-subspace of $\beth$ denoted by $\eta V[D]$ is the subspace made up of all $\eta\upsilon$-type homogenous elements, where $\eta \in \beth$ is a homogeneous in terms of degree, υ represents a monomial in $V[D]$ and $D \subset \{\zeta_1, \zeta_2, \ldots, \zeta_q\}$. The space $\eta V[D]$ is called a Stanley space of dimension $|D|$ if it ia a free $V[D]$ module, here $|D|$ refers to the number of indeterminates in D. In Stanley decomposition, the V-vector space $\beth$ is decomposed into a finite direct sum of Stanley spaces as

$$\mathcal{A} : \beth = \bigoplus_{b=1}^{h} \eta_b V[D_b].$$

The Stanley depth of decomposition is given as

$$\operatorname{sdepth} \mathcal{A} = \min\{\, |D_b|\, , \, b = 1, \ldots, h\}.$$

The Stanley depth of $\beth$ is given by

$$\operatorname{sdepth}(\beth) = \max\{\operatorname{sdepth} \mathcal{A} \, : \, \mathcal{A} \text{ is a Stanley decomposition of } \beth\}.$$

In [27], Stanley proposed the following conjecture for $\mathbb{Z}^q$-graded $\aleph$-modules, given by

$$\operatorname{depth}(\beth) \leq \operatorname{sdepth}(\beth),$$

known as the Stanley conjecture. In [3], [2] and [22], the conjecture for $\aleph/\Upsilon$ was proved when $q \geq 3$, $q = 4$ and $q = 5$, respectively, where $\aleph$ is a ring of polynomials in q variables and Υ be an ideal of $\aleph$. But later In [8], it was proved that the Stanley conjecture does not hold for the modules of type $\aleph/\Upsilon$.

3.2.1 An approach for obtaining the Stanley depth of a monomial ideal.

In general, there was no method for computing the Stanley depth of modules. However, in [15] Herzog proposed that the Stanley depth of a module can be determined in finite number of steps using posets, when a $\mathbb{Z}^n$ graded R-module M is of the following type $M = \Upsilon_1/\Upsilon_2$, where $\Upsilon_2 \subset \Upsilon_1 \subset \aleph$ are monomial ideals. Let $\zeta^{\upsilon_1}, \zeta^{\upsilon_2}, \ldots, \zeta^{\upsilon_t}$ be the monomial set of generators of Υ_1 and $\eta^{\upsilon'_1}, \ldots, \eta^{\upsilon'_t}$ be the monomial set of generators of Υ_2. The monomial $\zeta_1^{\upsilon(1)}\zeta_2^{\upsilon(2)}\cdots\zeta_q^{\upsilon(q)}$ is denoted by ζ^{υ}.
Now we fix $\delta \in \mathbb{N}^q$ with the condition $\upsilon_i \leq \delta$ and $\upsilon'_j \leq \delta$, where $\leq$ stands for partial ordering in $\mathbb{N}^q$. The subposet $\mathcal{A}^{\delta}_{\Upsilon_1/\Upsilon_2}$ of $\mathbb{N}^q$ is defined as the characteristic poset with consideration to δ, given by

$$\mathcal{A}^{\delta}_{\Upsilon_1/\Upsilon_2} = \{\upsilon \in \mathbb{N}^q \ : \ \zeta^{\upsilon} \in \Upsilon_1/\Upsilon_2, \ \upsilon \leq \delta\}.$$

Since our goal is to determine the Stanley depth of a square free monomial ideal Υ_1, we choose $\Upsilon_2 = 0$ and $\delta = (1, \ldots, 1)$. For any $\upsilon, \omega \in \mathcal{A}_{\Upsilon^{(1,\cdots,1)}}$, we establish

$$[\upsilon\, , \omega] = \{\varphi \in \mathcal{A}^{(1,\ldots,1)}_{\Upsilon_1} \ : \ \upsilon \subseteq \varphi \subseteq \omega\}.$$

Partition of $\mathcal{A}^{(1,\ldots,1)}_{\Upsilon_1}$ is the following disjoint union of intervals

$$\mathcal{L} \ : \ \mathcal{A}^{(1,\ldots,1)}_{\Upsilon_1} = \cup_{b=1}^{m}[\varphi_b, \, \varphi'_b].$$

The Stanley decomposition of Υ_1 is induced by each partition of $\mathcal{A}^{(1,\ldots,1)}_{\Upsilon_1}$ into intervals and is given by

$$\mathcal{D}(\mathcal{L}) \ : \ \Upsilon_1 = \bigoplus_{b=1}^{h} \zeta^{\varphi(b)} \ V[\{\, \zeta_c \,|\, c \in \varphi'_b \,\}].$$

Appaerently, $\operatorname{sdepth} \mathcal{D}(\mathcal{L}) = \min\{\, |\varphi'_b| : b = 1, \ldots, h\}$ and

$$\operatorname{sdepth}(\Upsilon_1) = \max\{ \ \operatorname{sdepth} \mathcal{D}(\mathcal{L}) \ \}.$$

Remark 3.2.2 If the module is of type $\beth = \aleph/\Upsilon$, then the Stanley depth of $\beth$ can be determined using the same approach as above by replacing Υ_1 with $\aleph$ and Υ_2 with Υ. However, in some cases, finding all possible partitions of the characteristic partial order set for finding Stanley depth is impossible.

Example 3.2.3 Let $\Upsilon_1 = (\zeta_1\zeta_4,\, \zeta_3\zeta_4) \subset V[\zeta_1,\, \zeta_2,\, \zeta_3,\, \zeta_4]$ be a square-free monomial ideal and $\Upsilon_2 = 0$. We set $\zeta^{\upsilon_b} = \zeta_1^{\upsilon(1)_b}\zeta_2^{\upsilon(2)_b}\zeta_3^{\upsilon(3)_b}\zeta_4^{\upsilon(4)_b}$ and Υ_1 is generated by ζ^{υ_1}, ζ^{υ_2}, where
$\upsilon_1 = (\upsilon(1)_1, \upsilon(2)_1, \upsilon(3)_1, \upsilon(4)_1) = (1, 0, 0, 1)$ and
$\upsilon_2 = (\upsilon(1)_2, \upsilon(2)_2, \upsilon(3)_2, \upsilon(4)_2) = (0, 0, 1, 1)$. Here, we choose $\delta = (1, 1, 1, 1)$.
Then, the poset $\mathcal{A}_{\Upsilon_1}^{(1,\ldots,1)}$ is given by

$$\mathcal{A}_{\Upsilon_1}^{(1,\ldots,1)} = \{(1,0,0,1), (0,0,1,1), (1,1,0,1), (1,0,1,1), (0,1,1,1), (1,1,1,1)\}.$$

Partitions of $\mathcal{A}_{\Upsilon_1^{(1,\ldots,1)}}$ are given by:

$$\begin{aligned}\mathcal{L}_1 :\ & [(1,0,0,1), (1,0,0,1)] \bigcup [(0,0,1,1), (0,0,1,1)] \bigcup \\ & [(1,1,0,1), (1,1,0,1)] \bigcup [(1,0,1,1), (1,0,1,1)] \bigcup \\ & [(0,1,1,1), (0,1,1,1)] \bigcup [(1,1,1,1), (1,1,1,1)].\end{aligned}$$

$$\mathcal{L}_2 :\ [(1,0,0,1), (1,1,0,1)] \bigcup [(0,0,1,1), (1,0,1,1)] \bigcup [(0,1,1,1), (1,1,1,1)].$$

$$\mathcal{L}_3 :\ [(1,0,0,1), (1,0,1,1)] \bigcup [(0,0,1,1), (0,1,1,1)] \bigcup [(1,1,0,1), (1,1,1,1)].$$

The Stanley decomposition of these partitions are given as:

$$\begin{aligned}\mathcal{D}(\mathcal{L}_1) &:= \zeta_1\zeta_4 V[\zeta_1,\, \zeta_4] \oplus \zeta_3\zeta_4\, V[\zeta_3,\, \zeta_4] \oplus \zeta_1\zeta_2\zeta_4 V[\zeta_1,\, \zeta_2,\, \zeta_4] \\ &\quad \oplus \zeta_1\zeta_3\zeta_4 V[\zeta_1,\, \zeta_3,\, \zeta_4] \oplus \zeta_2\zeta_3\zeta_4 V[\zeta_2,\, \zeta_3,\, \zeta_4] \oplus \zeta_1\zeta_2\zeta_3\zeta_4 V[\zeta_1,\, \zeta_2,\, \zeta_3,\, \zeta_4].\end{aligned}$$

$$\mathcal{D}(\mathcal{L}_2) := \zeta_1\zeta_4 V[\zeta_1,\, \zeta_2,\, \zeta_4] \oplus \zeta_3\zeta_4 V[\zeta_1,\, \zeta_3,\, \zeta_4] \oplus \zeta_2\zeta_3\zeta_4 V[\zeta_1,\, \zeta_2,\, \zeta_3,\, \zeta_4].$$

$$\mathcal{D}(\mathcal{L}_3) := \zeta_1\zeta_4 V[\zeta_1,\, \zeta_3,\, \zeta_4] \oplus \zeta_3\zeta_4 V[\zeta_2,\, \zeta_3,\, \zeta_4] \oplus \zeta_1\zeta_2\zeta_4 V[\zeta_1,\, \zeta_2,\, \zeta_3,\, \zeta_4].$$

Then,

$$\begin{aligned}\text{sdepth}(\Upsilon_1) &\geq \max\{\text{sdepth}(\mathcal{D}(\mathcal{L}_1)),\ \text{sdepth}(\mathcal{D}(\mathcal{L}_2)),\ \text{sdepth}(\mathcal{D}(\mathcal{L}_3))\} \\ &= \max\{2, 3, 3\} \\ &= 3.\end{aligned}$$

Example 3.2.4 Let $\Upsilon_1 = (\zeta_1,\, \zeta_2,\, \zeta_3) \subset V[\zeta_1,\, \zeta_2,\, \zeta_3]$ be a monomial ideal and $\Upsilon_2 = 0$. We set $\zeta^{\upsilon_b} = \zeta_1^{\upsilon(1)_b}\zeta_2^{\upsilon(2)_b}\zeta_3^{\upsilon(3)_b}$ and Υ_1 is generated by ζ^{υ_1}, ζ^{υ_2}, ζ^{υ_3}, where $\upsilon_1 = (\upsilon(1)_1, \upsilon(2)_1, \upsilon(3)_1) = (1, 0, 0)$, $\upsilon_2' = (\upsilon(1)_2, \upsilon(2)_2, \upsilon(3)_2) = (0, 1, 0)$ and $\upsilon_3 = (\upsilon(1)_3, \upsilon(2)_3, \upsilon(3)_3) = (0, 0, 1)$. Here, we choose $\delta = (1, 1, 1, 1)$. The poset $\mathcal{A}_{\Upsilon_1}^{(1,\ldots,1)}$ is given by

$$\mathcal{A}_{\Upsilon_1}^{(1,\ldots,1)} = \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}.$$

Partitions of $\mathcal{A}_{\Upsilon_1^{(1,\ldots,1)}}$ are given by:

$$\begin{aligned}\mathcal{L}_1 :\ & [(1,0,0), (1,0,0)] \bigcup [(0,1,0), (0,1,0)] \bigcup [(0,0,1), (0,0,1)] \bigcup \\ & [(1,1,0), (1,1,0)] \bigcup [(1,0,1), (1,0,1)] \bigcup [(0,1,1), (0,1,1)] \bigcup \\ & [(1,1,1), (1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{L}_2 :\ & [(1,0,0), (1,0,1)] \bigcup [(0,1,0), (1,1,0)] \bigcup [(0,0,1), (0,1,1)] \bigcup \\ & [(1,1,1), (1,1,1)].\end{aligned}$$

$\mathcal{L}_3: \; [(1,0,0),(1,1,0)] \bigcup [(0,1,0),(0,1,1)] \bigcup [(0,0,1),(1,0,1)] \bigcup$
$[(1,1,1),(1,1,1)]$.

The Stanley decomposition of these partitions are given as:

$$\begin{aligned}\mathcal{D}(\mathcal{L}_1) \;&:=\; \zeta_1 V[\zeta_1] \oplus \zeta_2\, V[\zeta_2] \oplus \zeta_3 V[\zeta_3] \oplus \\ &\quad \zeta_1\zeta_2 V[\zeta_1,\, \zeta_2] \oplus \zeta_1\zeta_3 V[\zeta_1,\, \zeta_3] \oplus \zeta_2\zeta_3 V[\zeta_2,\, \zeta_3] \;\oplus\; \zeta_1\zeta_2\zeta_3 V[\zeta_1,\, \zeta_2,\, \zeta_3].\end{aligned}$$

$$\mathcal{D}(\mathcal{L}_2) \;:=\; \zeta_1 V[\zeta_1,\, \zeta_3\,] \oplus \zeta_2 V[\zeta_1,\, \zeta_2] \oplus \zeta_3 V[\zeta_2,\, \zeta_3] \oplus \zeta_1\zeta_2\zeta_3 V[\zeta_1,\, \zeta_2,\, \zeta_3].$$

$$\mathcal{D}(\mathcal{L}_3) \;:=\; \zeta_1 V[\zeta_1,\, \zeta_2\,] \oplus \zeta_2 V[\zeta_2,\, \zeta_3] \oplus \zeta_3 V[\zeta_1,\, \zeta_2] \oplus \zeta_1\zeta_2\zeta_3 V[\zeta_1,\, \zeta_2,\, \zeta_3].$$

Then

$$\begin{aligned}\operatorname{sdepth}(\Upsilon_1) \;&\geq\; \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{L}_1))\,,\; \operatorname{sdepth}(\mathcal{D}(\mathcal{L}_2))\,,\; \operatorname{sdepth}(\mathcal{D}(\mathcal{L}_3))\} \\ &=\; \max\{1,2,2\} \\ &=\; 2.\end{aligned}$$

The following example demonstrates how to calculate the Stanley depth of $\aleph/\Upsilon$.

Example 3.2.5 Let $\Upsilon = (\zeta_1\zeta_4,\, \zeta_3\zeta_4) \subset \; V[\zeta_1,\, \zeta_2,\, \zeta_3,\, \zeta_4]$. Here, we choose $\delta = (1,1,1,1,1)$. The poset $\mathcal{A}^{(1,\ldots,1)}_{\aleph/\Upsilon_1}$ is then given by

$$\begin{aligned}\mathcal{A}^{(1,\ldots,1)}_{\aleph/\Upsilon} = \{&(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1), \\ &(1,1,0,0),(1,0,1,0),(0,1,1,0),(0,1,0,1),(1,1,1,0)\}.\end{aligned}$$

Partitions of $\mathcal{A}_{\aleph/\Upsilon^{(1,\ldots,1)}}$ are provided by:

$\mathcal{L}_1: \; [(0,0,0,0),(1,0,0,0)] \bigcup [(0,1,0,0),(0,1,0,0)] \bigcup$
$[(0,0,1,0),(0,0,1,0)] \bigcup [(0,0,0,1),(0,0,0,1)] \bigcup$
$[(1,1,0,0),(1,1,0,0)] \bigcup [(1,0,1,0),(1,0,1,0)] \bigcup$
$[(0,1,1,0),(0,1,1,0)] \bigcup [(0,1,0,1),(0,1,0,1)] \bigcup$
$[(1,1,1,0),(1,1,1,0)]$.

$$\mathcal{L}_2: \ [(0,0,0,0),(1,1,0,0)] \bigcup [(1,0,0,0),(1,0,1,0)] \bigcup$$
$$[(0,1,0,0),(0,1,1,0)] \bigcup [(0,0,0,1),(0,1,0,1)] \bigcup$$
$$[(0,0,1,0),(1,1,1,0)] \, .$$

$$\mathcal{L}_3: \ [(0,0,0,0),(0,1,0,0)] \bigcup [(1,0,0,0),(1,1,0,0)] \bigcup$$
$$[(0,0,1,0),(1,0,1,0)] \bigcup [(0,0,0,1),(0,1,0,1)] \bigcup$$
$$[(0,1,1,0),(1,1,1,0)] \, .$$

The Stanley decomposition of these partitions is as follows:

$$\begin{aligned}
\mathcal{D}(\mathcal{L}_1) &:= V[\zeta_1] \oplus \zeta_2 V[\zeta_2] \oplus \zeta_3 V[\zeta_3] \oplus \zeta_4 V[\zeta_4] \oplus \zeta_1\zeta_2 V[\zeta_1,\zeta_2] \oplus \\
&\qquad \zeta_1\zeta_3 V[\zeta_1,\zeta_3] \oplus \zeta_2\zeta_3 V[\zeta_2,\, \zeta_3] \oplus \zeta_2\zeta_4 V[\zeta_2,\, \zeta_4] \oplus \zeta_1\zeta_2\zeta_3 V[\zeta_1,\, \zeta_2,\, \zeta_3]. \\
\mathcal{D}(\mathcal{L}_2) &:= V[\zeta_1,\zeta_2] \oplus \zeta_1 V[\zeta_1,\, \zeta_3] \oplus \zeta_2 V[\zeta_2,\, \zeta_3] \oplus \zeta_4 V[\zeta_2,\, \zeta_4] \oplus \zeta_3 V[\zeta_1,\, \zeta_2,\, \zeta_3]. \\
\mathcal{D}(\mathcal{L}_3) &:= V[\zeta_2] \oplus \zeta_1 V[\zeta_1,\, \zeta_2] \oplus \zeta_3 V[\zeta_1,\, \zeta_3] \oplus \zeta_4 V[\zeta_2,\, \zeta_4] \oplus \zeta_3\zeta_2 V[\zeta_1,\, \zeta_2,\, \zeta_3].
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{sdepth}(\aleph/\Upsilon) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{L}_1))\,,\, \operatorname{sdepth}(\mathcal{D}(\mathcal{L}_2)),\, \operatorname{sdepth}(\mathcal{D}(\mathcal{L}_3))\} \\
&= \max\{1,2,1\} \\
&= 2.
\end{aligned}$$

The following are some basic results of depth and Stanley depth that will be used in chapter 4.

Lemma 3.2.6 Let the sequence $0 \to \beth_1 \to \beth_2 \to \beth_3 \to 0$ of $\mathbb{Z}^q$-graded $\aleph$-modules is short exact. Then

$$\operatorname{sdepth}(\beth_2) \geq \min\{\operatorname{sdepth}(\beth_1),\, \operatorname{sdepth}(\beth_3)\}.$$

Lemma 3.2.7 ([19], Lemma 2.8) Let Υ be an edge ideal associated with P_q, where $q \geq 2$, then

$$\text{depth}(\aleph/\Upsilon(P_q)) = \lceil \frac{q}{3} \rceil.$$

Lemma 3.2.8 ([26], Lemma 4) Let Υ be an edge ideal associated with P_q and $q \geq 2$, then

$$\text{sdepth}(\aleph/\Upsilon(P_q)) = \lceil \frac{q}{3} \rceil.$$

Proposition 3.2.9 ([6], Proposition 1.3) Assume $q \geq 3$, then

$$\text{depth}(\aleph/\Upsilon(C_q)) = \lceil \frac{q-1}{3} \rceil.$$

Proposition 3.2.10 ([1]) Let Υ be an edge ideal of q-star, then $\text{sdepth}(\aleph/\Upsilon) = 1$ and $\text{depth}(\aleph/\Upsilon) = 1$.

Proposition 3.2.11 ([5], Proposition 2.7) Let $\Upsilon \subset \aleph$ be a monomial ideal and for any monomial $\zeta \notin \Upsilon$, we have

1. $\text{sdepth}_{\aleph}(\Upsilon : \zeta) \geq \text{sdepth}_{\aleph}(\Upsilon)$, ([23], Proposition 1.3)
2. $\text{depth}_{\aleph}(\aleph/(\Upsilon : \zeta)) \geq \text{depth}_{\aleph}(\aleph/\Upsilon)$, [24]
3. $\text{sdepth}_{\aleph}(\aleph/(\Upsilon : \zeta)) \geq \text{sdepth}_{\aleph}(\aleph/\Upsilon)$.

Lemma 3.2.12 [19] Let $\Upsilon \subset \aleph$ to be an ideal generated by monomials. Let $\aleph' = \aleph \otimes_K V[\zeta_{q+1}] = \aleph[\zeta_{q+1}]$, then $\text{depth}(\aleph'/\Upsilon'\aleph') = \text{depth}(\aleph/\Upsilon) + 1$ and $\text{sdepth}(\aleph'/\Upsilon'\aleph') = \text{sdepth}(\aleph/\Upsilon) + 1$.

Lemma 3.2.13 ([5], Proposition 1.1) Let $\Upsilon' \subset \aleph' = V[\zeta_1, \ldots, \zeta_r]$ and $\Upsilon'' \subset \aleph'' = V[\zeta_{r+1}, \ldots, \zeta_q]$ be the ideals generated by monomials, where $1 \leq r \leq q$, then

$$\text{depth}_{\aleph}(\aleph/(\Upsilon'\aleph + \Upsilon''\aleph)) = \text{depth}_{\aleph'}(\aleph'/\Upsilon') + \text{depth}_{\aleph''}(\aleph''/\Upsilon'').$$

Lemma 3.2.14 ([5], Proposition 1.1) Let $\Upsilon' \subset \aleph' = V[\zeta_1, \ldots, \zeta_r]$ and $\Upsilon'' \subset \aleph'' = V[\zeta_{r+1}, \ldots, \zeta_q]$ be the ideals generated by monomials, where $1 \leq r \leq q$, then $\operatorname{depth}_{\aleph}(\aleph'/\Upsilon' \otimes_K \aleph''/\Upsilon'') = \operatorname{depth}_{\aleph}(\aleph/(\Upsilon'\aleph + \Upsilon''\aleph)) = \operatorname{depth}_{\aleph'}(\aleph'/\Upsilon') + \operatorname{depth}_{\aleph''}(\aleph''/\Upsilon'')$.

Theorem 3.2.15 ([24], Theorem 3.1) Let $\Upsilon' \subset \aleph' = V[\zeta_1, \ldots, \zeta_r]$ and $\Upsilon'' \subset \aleph'' = V[\zeta_{r+1}, \ldots, \zeta_q]$ be the ideals generated by monomials, where $1 \leq r \leq q$, then

$$\operatorname{sdepth}_{\aleph}(\aleph/(\Upsilon'\aleph + \Upsilon''\aleph)) \geq \operatorname{sdepth}_{\aleph'}(\aleph'/\Upsilon') + \operatorname{depth}_{\aleph''}(\aleph''/\Upsilon'').$$

Lemma 3.2.16 Let $\Upsilon' \subset \aleph' = K[\zeta_1, \ldots, \zeta_r]$ and $\Upsilon'' \subset \aleph'' = K[\zeta_{r+1}, \ldots, \zeta_q]$ be the ideals generated by monomials, where $1 \leq r \leq q$, then

$$\operatorname{sdepth}_{\aleph}(\aleph'/\Upsilon' \otimes_K \aleph''/\Upsilon'') \geq \operatorname{sdepth}_{\aleph'}(\aleph'/\Upsilon') + \operatorname{sdepth}_{\aleph''}(\aleph''/\Upsilon'').$$

Proof: From ([24], Proposition 2.2.20), we have

$$\aleph'/\Upsilon' \otimes_K \aleph''/\Upsilon'' \cong \aleph/(\Upsilon'\aleph + \Upsilon''\aleph).$$

Then, by using Theorem 3.2.15 we get our required result.

Theorem 3.2.17 ([13], Theorem 3.1 and 4.18) Let $\mathcal{H}$ be a connected graph and $\Upsilon(\mathcal{H})$ be the edge ideal associated with $\mathcal{H}$. Let p be the diameter of graph, then

$$\operatorname{depth}(\aleph/\Upsilon), \operatorname{sdepth}(\aleph/\Upsilon) \geq \lceil \frac{p+1}{3} \rceil.$$

Lemma 3.2.18 ([15], Lemma 3.6) Let $\Upsilon \subset \aleph$ be an ideal and $\aleph' = \aleph[\zeta_{q+1}, \ldots, \zeta_{q+t}]$ be a polynomial ring, then

$$\operatorname{depth}(\aleph'/\Upsilon\aleph') = \operatorname{depth}(\aleph/\Upsilon\aleph) + t \quad \text{and} \quad \operatorname{sdepth}(\aleph'/\Upsilon\aleph') = \operatorname{sdepth}(\aleph/\Upsilon\aleph) + t.$$

Proposition 3.2.19 ([7]) Let $\Upsilon \subset \aleph = V[\zeta_1, \dots, \zeta_q]$ be a monomial ideal minimally generated by t elements, then

$$\operatorname{sdepth}(\aleph/\Upsilon) \geq q - t.$$

Theorem 3.2.20 ([25]) Let $\Upsilon \subset \aleph = V[\zeta_1, \dots, \zeta_q]$ be a monomial ideal minimally generated by t elements, then

$$\operatorname{sdepth}(\Upsilon) \geq q - \lfloor \frac{t}{2} \rfloor.$$

Theorem 3.2.21 ([21]) Let $\Upsilon \subset \aleph$ be a monomial ideal, it is minimally generated by t monomials. Then,

$$\operatorname{sdepth}(\Upsilon) \geq \max\{1,\, q - \lceil \frac{t}{2} \rceil\}.$$

Theorem 3.2.22 ([15]) Let $\Upsilon \subset \aleph$ be a monomial ideal, it is minimally generated by t monomials. Then,

$$\operatorname{sdepth}(\Upsilon) \geq \max\{1,\, q - t + 1\}.$$

Lemma 3.2.23 ([18]) Let $\Upsilon \subset \aleph$ be a monomial ideal that is square-free with $\operatorname{supp}(\Upsilon) = [q]$, let $z := \zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_r} \in \aleph/\Upsilon$, such that $\zeta_m z \in \Upsilon$, for all $m \in [q] \setminus \operatorname{supp}(z)$. Then, $\operatorname{sdepth}(\aleph/\Upsilon) \leq r$.

Chapter 4

Depth and Stanley Depth of the Quotient Ring of Edge Ideals Associated with Some Classes of Lobster Trees and Unicyclic Graphs

In this chapter, we compute depth and Stanley depth of the cyclic modules associated with lobster tree $P_{n,m,h}$, where $m = h$. We prove for this lobster tree, the depth and Stanley depth values are equal. In addition, the Stanley's inequality hold for these values. Furthermore, we use lobster tree results to find the values of depth and Stanley depth of the quotient ring of edge ideal associated with unicyclic graph $C_{n,m,h}$. We show that the depth and Stanley depth values for this unicyclic graph are equal. We also give some bounds for the dimension of cyclic modules $S_{n,m,h}/I_{n,m,h}$ and $S_{n,m,h}/I'_{n,m,h}$. In the end, we give some future directions.

Throughout this chapter, we assume
$S_{n,m,h} := K[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{mn}, z_1, z_2, \dots, z_{hmn}]$, where n is the number of vertices of path P_n and length of cycle C_n in $P_{(n,m,h)}$ and $C_{(n,m,h)}$ respectively. Here m is the number of vertices attached at each vertex x_i of P_n and h is the number of pendant vertices attached at each vertex y_j, where $j = 1, \dots, mn$. We also have $m = h$.

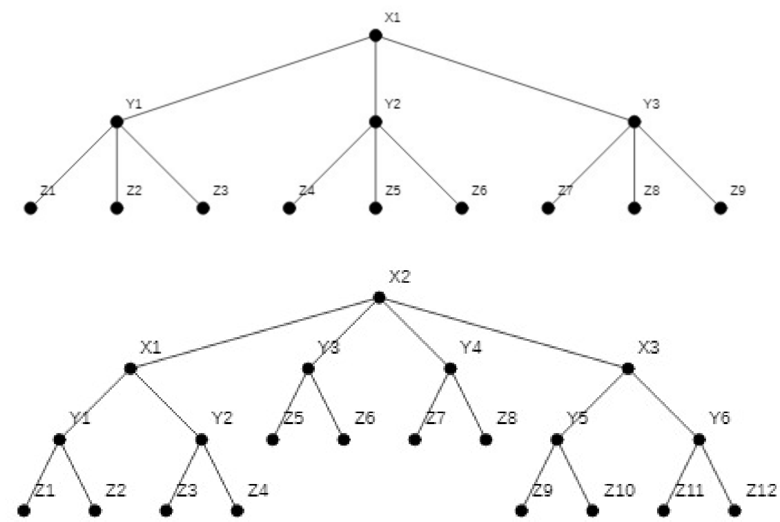

Figure 4.1: From top to bottom $P_{1,3,3}$ and $P_{3,2,2}$.

Definition 4.0.1 Let $n \geq 1$, $m \geq 2$, and $m = h$. Then P_n be a path on n vertices with vertex set $\{x_1, x_2, \cdots, x_n\}$. We construct a lobster tree by connecting m vertices to each vertex x_i of P_n and h pendant vertices to each vertex y_j, where $j = 1, \cdots, mn$ and $r = 1, \cdots, m^2n$. This lobster tree is denoted by $P_{n,m,h}$.
For example see Figure 4.1.

Definition 4.0.2 Let $n \geq 3$, $m \geq 2$, and $m = h$. Then C_n be an n-verticed cycle with $\{x_1, x_2, \cdots, x_n\}$ as the vertex set and the edge set is given as $\{x_1x_2, x_2x_3 \cdots, x_{n-1}x_n, x_1x_n\}$. We construct a unicyclic graph by connecting each vertex x_i of C_n with m vertices, and then connecting each vertex y_j with h pendant vertices, here $j = 1, \cdots, mn$ and $r = 1, \cdots, m^2n$. This unicyclic graph is denoted by $C_{n,m,h}$.
For example see Figure 4.2.

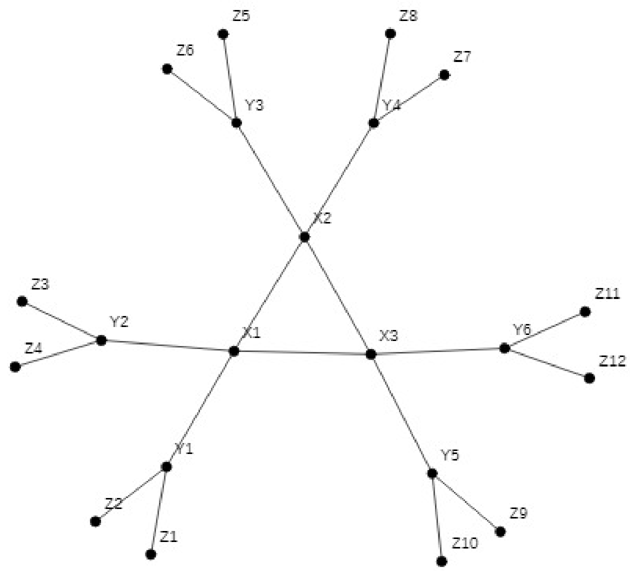

Figure 4.2: Unicyclic graph $C_{3,2,2}$.

Definition 4.0.3 Let $n \geq 1$, $m \geq 2$ and $m = h$. $P_{n,m,h}$ be a lobster tree on $[mn(m+1)+n]$ vertices, then the edge ideal $I_{n,m,h} = I(P_{n,m,h})$ is given by

$$I_{n,m,h} = (x_1y_1, x_1y_2, \ldots, x_1y_m, y_1z_1, y_1z_2, \ldots, y_1z_m, y_2z_{m+1}, y_2z_{m+2}, \ldots, y_2z_{2m}, \ldots, y_mz_{m^2-m+1}, \ldots, y_mz_{m^2}, \ldots, x_ix_{i+1}, x_{i+1}x_{i+2}, \ldots, x_{n-1}x_n, x_ny_{(n-1)m+1}, x_ny_{(n-1)m+2}, \ldots, x_ny_{mn}, y_{(n-1)m+1}z_{(n-1)m^2+1}, y_{(n-1)m+1}z_{(n-1)m^2+2}, \ldots, y_{(n-1)m+1}z_{(n-1)m^2+m}, y_{(n-1)m+2}z_{(n-1)m^2+m+1}, \ldots, y_{(n-1)m+2}z_{(n-1)m^2+2m}, \ldots, y_{mn}z_{m^2n-m+1}, y_{mn}z_{m^2n-m+2}, \ldots, y_{mn}z_{m^2n}).$$

Definition 4.0.4 Let $n \geq 3$, $m \geq 2$ and $m = h$. $C_{n,m,h}$ be a unicyclic graph on $[mn(m+1)+n]$ vertices , then the edge ideal $I'_{n,m,h} = I'(C_{n,m,h})$ is given by

$$I'_{n,m,h} = (I_{n,m,h}, x_1x_n). \tag{4.1}$$

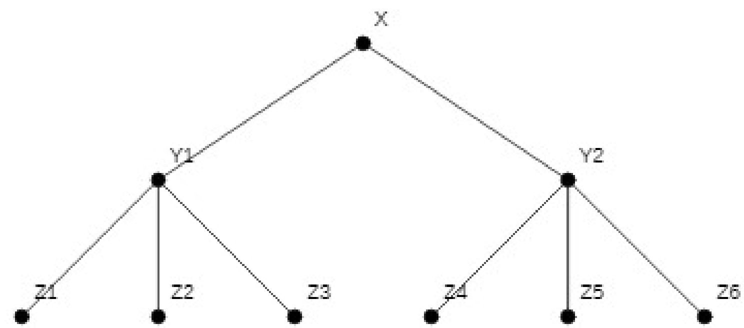

Figure 4.3: Lobster tree $P_{1,m-1,h}$ with $m = h = 3$.

Remark 4.0.5 $E(P_{n,m,h}) = E(C_{n,m,h}) \setminus \{n, 1\}$. Thus $|\mathcal{G}(I(P_{n,m,h}))| = mn(m+1) + n - 1$ and $|\mathcal{G}(I'(C_{n,m,h}))| = mn(m+1) + n$.

Remark 4.0.6 For $n \geq 1$, $m \geq 2$ and $m = h$, then $d(P_{n,m,h}) = n + 3$, where d is the diameter of lobster tree $P_{n,m,h}$.

4.1 Lobster tree-related results

In this section, we compute depth and Stanley depth of cyclic modules associated with lobster tree $P_{n,m,h}$, where $m = h$. We prove for this lobster tree, the depth and Stanley depth values are equal. In addition, the Stanley's inequality hold for these results. In the end, we give some bounds for the dimension of cyclic module $S_{n,m,h}/I_{n,m,h}$.

To prove our major result Theorem 4.1.3 for lobster tree $P_{n,m,h}$, we first prove a Lemma 4.1.1 and a Proposition 4.1.2. The following lemma will help in determining the depth and Stanley depth of cyclic module $S_{n,m,h}/I_{n,m,h}$, when $n > 1$ and $m \geq 2$.

Lemma 4.1.1 Let $n = 1$, $m \geq 3$ and $m = h$, then

$$\operatorname{depth}(S_{1,m-1,h}/I_{1,m-1,h}) = \operatorname{sdepth}(S_{1,m-1,h}/I_{1,m-1,h}) = m - 1.$$

Proof. We prove this by using induction on m. We consider the following two cases.

1. Let $m = 3$, see Fig 4.3. First, we will find the value of depth. Consider the short exact sequence

$$0 \longrightarrow S_{1,2,3}/(I_{1,2,3} : y_2) \xrightarrow{\cdot y_2} S_{1,2,3}/I_{1,2,3} \longrightarrow S_{1,2,3}/(I_{1,2,3}, y_2) \longrightarrow 0. \quad (4.2)$$

Here $(I_{1,2,3} : y_2) = (x_1, y_1z_1, y_1z_2, y_1z_3, z_4, z_5, z_6)$. Then, $S_{1,2,3}/(I_{1,2,3} : y_2) \cong K[y_1, z_1, z_2, z_3]/(y_1z_1, y_1z_2, y_1z_3) \otimes K[y_2]$. Using 3.2.10, we get $\operatorname{depth}(S_{1,2,3}/(I_{1,2,3} : y_2) = 1 + 1 = 2$. Also $(I_{1,2,3}, y_2) = (x_1y_1, y_1z_1, y_1z_2, y_1z_3, y_2)$, then $S_{1,2,3}/(I_{1,2,3}, y_2) \cong K[x_1, y_1, z_1, z_2, z_3]/(x_1y_1, y_1z_1, y_1z_2, y_1z_3) \otimes K[z_4, z_5, z_6]$. Thus Proposition 3.2.10 gives us $\operatorname{depth}(S_{1,2,3}/(I_{1,2,3} : y_2) = 1 + 3 = 4$. Using the depth Lemma on short exact sequence 4.2, we get $\operatorname{depth}(S_{1,2,3}/I_{1,2,3}) \geq 2$. For upper bound, since $y_2 \notin I_{1,2,3}$, from 3.2.11, we have $\operatorname{depth}(S_{1,2,3}/I_{1,2,3}) \leq \operatorname{depth}(S_{1,2,3}/(I_{1,2,3} : y_2)) = 2$. So,

$$\operatorname{depth}(S_{1,2,3}/I_{1,2,3}) = 2. \quad (4.3)$$

For Stanley depth, applying 3.2.6 and 3.2.11 on short exact sequence 4.2, we get $\operatorname{sdepth}(S_{1,2,3}/I_{1,2,3}) \geq 2$. For finding the upper bound, we assume $y_j = x_{1+j}$, where $n = 1$, $j = 1$, 2 and $z_r = x_{3+r}$, where $r = 1, \cdots, 6$. Consider $w_1 = x_{1+1}x_{1+2} \in (S_{1,2,3}/I_{1,2,3})$, but $x_lw_1 \in I_{1,2,3}$, for all $l \in [9] \setminus \operatorname{supp}(w_1)$, therefore by using Lemma 3.2.23, $\operatorname{sdepth}(S_{1,2,3}/I_{1,2,3}) \leq r = 2$. We get

$$\operatorname{sdepth}(S_{1,2,3}/I_{1,2,3}) = 2. \quad (4.4)$$

2. Let $m \geq 4$ and $m = h$. Then

$$\begin{aligned}(I_{1,m-1,h}) &= (x_1y_1, x_1y_2, \ldots, x_1y_{m-1}, y_1z_1, y_1z_2, \ldots, y_1z_m, y_2z_{m+1}, y_2z_{m+2}, \\ &\quad \ldots, y_2z_{2m}, \ldots, y_{m-1}z_{m(m-2)+1}, y_{m-1}z_{m(m-2)+2}, \ldots, y_{m-1}z_{m^2-m}).\end{aligned}$$

Consider following exact sequence

$$0 \longrightarrow S_{1,m-1,h}/I^* \xrightarrow{\cdot y_{m-1}} S_{1,m-1,h}/I_{1,m-1,h} \longrightarrow S_{1,m-1,h}/I^{**} \longrightarrow 0.$$

Here

$$I^* = (I_{1,m-1,n} : y_{m-1}) = (x_1, y_1z_1, y_1z_2, \dots, y_1z_m, \dots, y_{m-2}z_{m(m-3)+1}, \\ y_{m-2}z_{m(m-3)+2}, \dots, y_{m-2}z_{m^2-2m}, z_{m(m-2)+1}, \dots, z_{m^2-2m}),$$

this yield that,

$$S_{1,m-1,h}/(I_{1,m-1,h} : y_{m-1}) \cong K[y_1, z_1, \dots, z_m]/(y_1z_1, \dots, y_1z_m) \otimes \\ \cdots K[y_{m-2}, z_{m(m-3)+1}, \dots, z_{m(m-2)}]/(y_{m-2}z_{m(m-3)+1}, \dots, y_{m-2}z_{m(m-2)}) \\ \otimes K[y_{m-1}].$$

Using Proposition 3.2.11 and ([28], Theorem 2.2.21), we get

$$\operatorname{depth}(S_{1,m-1,h}/(I_{1,m-1,h} : y_{m-1}) = m-2+1 = m-1. \tag{4.5}$$

Also

$$I^{**} = (I_{1,m-1,h}, y_{m-1}) = (x_1y_1, x_1y_2, \dots, x_1y_{m-2}, y_1z_1, y_1z_2, \dots, y_1z_m, \dots, \\ y_{m-2}z_{m(m-3)+1}, \dots, y_{m-2}z_{m(m-2)}, y_{m-1}).$$

Then, $S_{1,m-1,h}/(I_{1,m-1,h}, y_{m-1}) \cong S_{1,m-2,h}/I_{1,m-2,h} \otimes K[z_{m(m-2)+1}, z_{m(m-2)+2}, \dots, z_{m(m-1)}]$. By using induction and ([28], Theorem 2.2.21), $\operatorname{depth}(S_{1,m-1,h}/(I_{1,m-1,h}, y_{m-1}) = m-2+m = 2m-2$. Then the depth Lemma is being applied to the exact sequence 4.1.1, we get $\operatorname{depth}(S_{1,m-1,h}/I_{1,m-1,h}) \geq m-1$. For upper bound, since $y_{m-1} \notin I_{1,m-1,h}$, and from 3.2.11, we have $\operatorname{depth}(S_{1,m-1,h}/I_{1,m-1,h}) \leq \operatorname{depth}(S_{1,m-1,h}/(I_{1,m-1,h} : y_{m-1}))$. By 4.5, $\operatorname{depth}(S_{1,m-1,h}/I_{1,m-1,h}) \leq m-1$. As a result,

$$\operatorname{depth}(S_{1,m-1,h}/I_{1,m-1,h}) = m-1.$$

For Stanley depth, applying 3.2.6 and 3.2.11 on short exact sequence 4.1.1 and by using induction, we get $\operatorname{sdepth}(S_{1,m-1,h}/I_{1,m-1,h}) \geq m-1$.
To determine the upper bound, we assume $y_j = x_{n+j}$, where $n = 1$, $j = 1, 2, \cdots, (m-1)n$ and $z_r = x_{n+(m-1)n+r}$, where $r = 1, \cdots, (m-1)mn$. Consider $w = x_{1+1}x_{1+2}, \cdots, x_{1+m-1} \in (S_{1,m-1,h}/I_{1,m-1,h})$, but $x_l w \in I_{1,m-1,h}$, for all $l \in [m^2]\setminus \operatorname{supp}(w)$, therefore by Lemma 3.2.23, $\operatorname{sdepth}(S_{1,m-1,h}/I_{1,m-1,h}) \leq r = m-1$. So,

$$\operatorname{sdepth}(S_{1,m-1,h}/(I_{1,m-1,h}) = m-1.$$

Using Lemma 4.1.1, we prove the following Proposition.

Proposition 4.1.2 Let $n = 1$, $m \geq 2$ and $m = h$, then

$$\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) = \operatorname{sdepth}(S_{1,m,h}/I_{1,m,h}) = m.$$

Proof. We prove this by considering the following cases.
1. Let $m = 2$. We start by determining the value of depth. Consider the short exact sequence

$$0 \longrightarrow S_{1,2,2}/(I_{1,2,2} : y_2) \xrightarrow{\cdot y_2} S_{1,2,2}/I_{1,2,2} \longrightarrow S_{1,2,2}/(I_{1,2,2}, y_2) \longrightarrow 0, \qquad (4.6)$$

Here $(I_{1,2,2} : y_2) = (x_1, y_1z_1, y_1z_2, z_3, z_4)$, so we have, $S_{1,2,2}/(I_{1,2,2} : y_2) \cong K[y_1, z_1, z_2]/(y_1z_1, y_1z_2) \otimes K[y_2]$. Using Lemma 3.2.7, we get
$\operatorname{depth}(S_{1,2,2}/(I_{1,2,2} : y_2) = 1 + 1 = 2$. Also $(I_{1,2,2}, y_2) = (x_1y_1, y_1z_1, y_1z_2, y_2)$. Then $S_{1,2,2}/(I_{1,2,2}, y_2) \cong K[x_1, y_1, z_1, z_2]/(x_1y_1, y_1z_1, y_1z_2) \otimes K[z_3, z_4]$.
By Proposition 3.2.10, we have $\operatorname{depth}(S_{1,2,2}/(I_{1,2,2}, y_2) = 1 + 2 = 3$. Using depth Lemma on short exact sequence 4.6, we get $\operatorname{depth}(S_{1,2,2}/I_{1,2,2}) \geq 2$. For upper bound, since $y_2 \notin I_{1,2,2}$, from 3.2.11, we have $\operatorname{depth}(S_{1,2,2}/I_{1,2,2}) \leq \operatorname{depth}(S_{1,2,2}/(I_{1,2,2} : y_2)) = 2$. So,

$$\operatorname{depth}(S_{1,2,2}/I_{1,2,2}) = 2.$$

For Stanley depth, applying 3.2.6 and 3.2.11 on short exact sequence 4.6, we get $\operatorname{sdepth}(S_{1,2,2}/I_{1,2,2}) \geq 2$. For finding the upper bound, we assume $y_j = x_1 + j$, where $j = 1,\ 2$ and $z_r = x_{1+m+r}$, where $r = 1, \cdots, m^2$. Consider $w = x_{1+1}x_{1+2} \in (S_{1,2,2}/I_{1,2,2})$, but $x_l w \in I_{1,2,2}$, for all $l \in [7]\backslash \operatorname{supp}(w)$, therefore by Lemma 3.2.23, $\operatorname{sdepth}(S_{1,2,2}/I_{1,2,2}) \leq r = 2$. Then

$$\operatorname{sdepth}(S_{1,2,2}/(I_{1,2,2}) = 2.$$

2. Let $m \geq 3$ and $m = h$. Then

$$\begin{aligned}(I_{1,m,h}) &= (x_1y_1, x_1y_2, \ldots, x_1y_m, y_1z_1, y_1z_2, \ldots, y_1z_m, y_2z_{m+1}, y_2z_{m+2}, \ldots, \\ &\quad y_2z_{2m}, \ldots, y_mz_{m(m-1)+1}, y_mz_{m(m-1)+2}, \ldots, y_mz_{m^2}).\end{aligned}$$

Consider following exact sequence

$$0 \longrightarrow S_{1,m,h}/(I_{1,m,h} : y_m) \xrightarrow{\cdot y_m} S_{1,m,h}/I_{1,m,h} \longrightarrow S_{1,m,h}/(I_{1,m,h}, y_m) \longrightarrow 0. \tag{4.7}$$

Here, $(I_{1,m,n} : y_m) = (x_1, y_1z_1, y_1z_2, \ldots, y_1z_m, y_2z_{m+1}, \ldots, y_2z_{2m}, \ldots, z_{m^2-m+1}, \ldots, z_{m^2})$. This implies,

$$S_{1,m,h}/(I_{1,m,h} : y_m) \cong K[y_1, z_1, \ldots, z_m]/(y_1z_1, \ldots, y_1z_m) \otimes \cdots \otimes$$
$$K[y_{m-1}, z_{m^2-2m+1}, \ldots, z_{m^2-m}]/(y_{m-1}z_{m^2-2m+1}, \ldots, y_1z_{m^2-m}) \otimes K[y_m].$$

Using Proposition 3.2.10 and ([28], Theorem 2.2.21), we get

$$\operatorname{depth}(S_{1,m,h}/(I_{1,m,h} : y_m)) = m - 1 + 1 = m. \tag{4.8}$$

Also

$$\begin{aligned}(I_{1,m,h}, y_m) = (&x_1y_1, x_1y_2, \ldots, x_1y_{m-1}, y_1z_1, y_1z_2, \ldots, y_1z_m, \ldots, \\ &y_{m-1}z_{m^2-2m+1}, \ldots, y_{m-1}z_{m^2-m}, y_m).\end{aligned}$$

Then $S_{1,m,h}/(I_{1,m,h}, y_m) \cong S_{1,m-1,h}/I_{1,m-1,h} \otimes K[z_{m(m-1)+1}, z_{m(m-1)+2}, \dots, z_{m^2}]$.

Using Lemma 4.1.1, we get $\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) = m - 1 + m = 2m - 1$. Then the depth lemma is being applied to exact sequence 4.7, we get $\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) \geq m$. For upper bound, since $y_m \notin I_{1,m,h}$, from Proposition 3.2.11, we have $\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) \leq \operatorname{depth}(S_{1,m,h}/(I_{1,m,h} : y_m))$. Therefore by using 4.8, $\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) \leq m$. As a result,

$$\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) = m.$$

For Stanley depth, applying 3.2.6 and 3.2.11 on short exact sequence 4.7 and by using Lemma 4.1.1, we get $\operatorname{sdepth}(S_{1,m,h}/I_{1,m,h}) \geq m$. To find the upper bound, we assume $y_j = x_{n+j}$, where $n = 1$, $j = 1, 2, \cdots, mn$ and $z_r = x_{n+mn+r}$, where $r = 1, \cdots, m^2 n$. Consider $w = x_{1+1}x_{1+2}, \cdots, x_{1+m} \in (S_{1,m,h}/I_{1,m,h})$, but $x_l w \in I_{1,m-1,h}$, for all $l \in [m^2+m+1] \setminus \operatorname{supp}(w)$, therefore by Lemma 3.2.23, $\operatorname{sdepth}(S_{1,m,h}/I_{1,m,h}) \leq r = m$. Hence,

$$\operatorname{sdepth}(S_{1,m,h}/(I_{1,m,h}) = m.$$

Using Proposition 4.1.2, we prove our key result for $P_{m,n,h}$, where $n > 1$, $m \geq 2$ and $m = h$.

Theorem 4.1.3 Let $n \geq 1$, $m \geq 2$ and $m = h$, then

$$\operatorname{depth}(S_{n,m,h}/I_{n,m,h}) = \operatorname{sdepth}(S_{n,m,h}/I_{n,m,h}) = mn.$$

Proof. To prove these results, we use induction on n. When $n = 1$, $m \geq 2$ and $m = h$, then the results hold by Lemma 4.1.2. Let $n = 2$, $m \geq 2$ and $m = h$, then

$$(I_{2,m,h}) = (x_1x_2, x_1y_1, x_1y_2, \dots, x_1y_m, y_1z_1, y_1z_2, \dots, y_1z_m, y_2z_{m+1}, y_2z_{m+2}, \dots,$$
$$y_2z_{2m}, \dots, y_mz_{m^2-m+1}, y_mz_{m^2-m+2}, \dots, y_mz_{m^2}, x_2y_{m+1}, x_2y_{m+2}, \dots, x_2y_{2m},$$
$$y_{m+1}z_{m^2+1}, y_{m+1}z_{m^2+2}, \dots, y_{m+1}z_{m^2+m}, y_{m+2}z_{m^2+m+1}, \dots, y_{m+2}z_{m^2+2m}, \dots,$$
$$y_{2m}z_{2m^2-m+1}, y_{2m}z_{2m^2-m+2}, \dots, y_{2m}z_{2m^2}).$$

Consider following exact sequence

$$0 \longrightarrow S_{2,m,h}/(I_{2,m,h} : y_{2m}) \xrightarrow{\cdot y_{2m}} S_{2,m,h}/I_{2,m,h} \longrightarrow S_{2,m,h}/(I_{2,m,h}, y_{2m}) \longrightarrow 0. \tag{4.9}$$

Here

$$(I_{2,m,h} : y_{2m}) = (I_{1,m,h}, x_2, y_{m+1}z_{m^2+1}, y_{m+1}z_{m^2+2}, \dots, y_{m+1}z_{m^2+m},$$
$$y_{m+2}z_{m^2+m+1}, y_{m+2}, \dots, y_{m+2}z_{m^2+2m}, \dots, y_{2m-1}z_{2m^2-2m+1}, y_{2m-1}z_{2m^2-2m+2},$$
$$\dots, y_{2m-1}z_{2m^2-m}, z_{2m^2-m+1}, \dots, z_{2m^2}).$$

It implies, $S_{2,m,h}/(I_{2,m,h} : y_{2m}) \cong S_{1,m,h}/I_{1,m,h} \otimes_K$
$K[y_{m+1}, z_{m^2+1}, \dots, z_{m^2+m}]/(y_{m+1}z_{m^2+1}, \dots, y_{m+1}z_{m^2+m}) \otimes_K \cdots \otimes_K$
$K[y_{2m-1}, z_{2m^2-2m+1}, \dots, z_{2m^2-m}]/(y_{2m-1}z_{2m^2-2m+1}, \dots, y_{2m-1}z_{2m^2-m}) \otimes_K$
$K[y_{2m}]$. Using Proposition 3.2.10, Lemma 4.1.2 and ([28], Theorem 2.2.21), we get

$$\operatorname{depth}(S_{2,m,h}/(I_{2,m,h} : y_{2m}) = m + (m-1) + 1 = 2m. \tag{4.10}$$

Furthermore

$$I^*_{2,m,h} = (I_{2,m,h}, y_{2m}) = (x_1x_2, x_1y_1, x_1y_2, \dots, x_1y_m, y_1z_1, y_1z_2, \dots, y_1z_m,$$
$$y_2z_{m+1}, y_2z_{m+2}, \dots, y_2z_{2m}, \dots, y_mz_{m^2-m+1}, y_mz_{m^2-m+2}, \dots, y_mz_{m^2}, x_2y_{m+1},$$
$$x_2y_{m+2}, \dots, x_2y_{2m-1}, y_{m+1}z_{m^2+1}, y_{m+1}z_{m^2+2}, \dots, y_{m+1}z_{m^2+m}, y_{m+2}z_{m^2+m+1},$$
$$\dots, y_{m+2}z_{m^2+2m}, \dots, y_{2m-1}z_{2m^2-2m+1}, y_{2m-1}z_{2m^2-2m+2}, \dots, y_{2m-1}z_{2m^2-m}, y_{2m}).$$

Now, consider following exact sequence

$$0 \longrightarrow S_{2,m,h}/(I^*_{2,m,h} : x_2) \xrightarrow{\cdot x_2} S_{2,m,h}/I^*_{2,m,h} \longrightarrow S_{2,m,h}/(I^*_{2,m,h}, x_2) \longrightarrow 0. \tag{4.11}$$

Here

$$(I^*_{2,m,n} : x_2) = (x_1, y_1z_1, y_1z_2, \dots, y_1z_m, y_2z_{m+1}, y_2z_{m+2}, \dots, y_2z_{2m}, \dots,$$
$$y_mz_{m^2-m+1}, y_mz_{m^2-m+2}, \dots, y_mz_{m^2}, y_{m+1}, y_{m+2}, \dots, y_{2m-1}, y_{2m}).$$

Then,

$$S_{2,m,h}/(I^*_{2,m,h} : x_2) \cong K[y_1, z_1, \dots, z_m]/(y_1z_1, \dots, y_1z_m) \otimes_K \dots$$
$$\otimes_K K[y_m, z_{m^2-m+1}, \dots, z_{m^2}]/(y_mz_{m^2-m+1}, \dots, y_mz_{m^2})$$
$$\otimes_K K[x_2, z_{m^2+1}, \dots, z_{2m^2}].$$

By using ([28], Theorem 2.2.21), we have

$$\text{depth}(S_{2,m,h}/(I^*_{2,m,h} : x_2) = \text{depth}(K[y_1, z_1, \dots, z_m]/(y_1z_1, \dots, y_1z_m)) + \cdots$$
$$+ \text{depth}(K[y_m, z_{m^2-m+1}, \dots, z_{m^2}]/(y_mz_{m^2-m+1}, \dots, y_mz_{m^2}))$$
$$+ \text{depth}(K[x_2, z_{m^2+1}, \dots, z_{2m}]).$$

By Proposition 3.2.10, we get $\text{depth}(S_{2,m,h}/I^*_{2,m,h} : x_2) = m + m^2 + 1$.

Moreover,

$$(I^*_{2,m,h}, x_2) = (I_{1,m,h}, y_{m+1}z_{m^2+1}, y_{m+1}z_{m^2+2}, \dots, y_{m+1}z_{m^2+m}, y_{m+2}z_{m^2+m+1}, \dots, y_{m+2}z_{m^2+2m}, \dots, y_{2m-1}z_{2m^2-2m+1}, y_{2m-1}z_{2m^2-2m+2}, \dots, y_{2m-1}z_{2m^2-m}, y_{2m}, x_2).$$

Then, $S_{2,m,h}/(I^*_{2,m,h}, x_2) \cong S_{1,m,h}/I_{1,m,h} \otimes_K$ $K[y_{m+1}, z_{m^2+1}, \dots, z_{m^2+m}]/(y_{m+1}z_{m^2+1}, \dots, y_{m+1}z_{m^2+m}) \otimes_K \cdots \otimes_K$ $K[y_{2m-1}, z_{2m^2-2m+1}, \dots, z_{2m^2-m}]/(y_{2m-1}z_{2m^2-2m+1}, \dots, y_{2m-1}z_{2m^2-m}) \otimes_K$ $K[z_{2m^2-m+1}, \dots, z_{2m^2}]$. Using ([28], Theorem 2.2.21), we get

$$\operatorname{depth}(S_{2,m,h}/(I^*_{2,m,h}, x_2) = \operatorname{depth}(S_{1,m,h}/I_{1,m,h}) + \operatorname{depth}(K[y_{m+1}, z_{m^2+1}, \dots, z_{m^2+m}]/(y_{m+1}z_{m^2+1}, \dots, y_{m+1}z_{m^2+m})) + \cdots + \operatorname{depth}(K[y_{2m-1}, z_{2m^2-2m+1}, \dots, z_{2m^2-m}]/(y_{2m-1}z_{2m^2-2m+1}, \dots, y_{2m-1}z_{2m^2-m}) + \operatorname{depth}(K[z_{2m^2-m+1}, \dots, z_{2m^2}]).$$

By using Proposition 4.1.2 and 3.2.10, we get $\operatorname{depth}(S_{2,m,h}/I^*_{2,m,h}, x_2) = m + (m-1) + m = 3m-1$. As $\operatorname{depth}(S_{2,m,h}/I^*_{2,m,h}, x_2) > \operatorname{depth}(S_{2,m,h}/I^*_{2,m,h} : x_2)$. Now by applying the depth Lemma on sequences 4.11 and 4.9, we get $\operatorname{depth}(S_{2,m,h}/I_{2,m,h}) \geq 2m$. For upper bound, since $y_{2m} \notin I_{2,m,h}$, from Proposition 3.2.11, $\operatorname{depth}(S_{2,m,h}/I_{2,m,h}) \leq \operatorname{depth}(S_{2,m,h}/(I_{2,m,h} : y_{2m}))$. Therefore by using equation 4.10, we have $\operatorname{depth}(S_{1,m,h}/I_{1,m,h}) \leq 2m$. As a result,

$$\operatorname{depth}(S_{2,m,h}/I_{2,m,h}) = 2m. \tag{4.12}$$

For Stanley depth, applying 3.2.6 and 3.2.11 on short exact sequence 4.11, 4.9 and by using Lemma 4.1.2, we get $\operatorname{sdepth}(S_{2,m,h}/I_{2,m,h}) \geq 2m$. To find the upper bound, we assume $y_j = x_{n+j}$, where $n = 2$, $j = 1, 2, \cdots, mn$ and $z_r = x_{n+mn+r}$, where $r = 1, \cdots, m^2n$. Consider $w = x_{2+1}x_{2+2}, \cdots, x_{2+2m} \in$

$(S_{2,m,h}/I_{2,m,h})$, but $x_l w \in I_{2,m,h}$, for all $l \in [2m(m+1)+2] \setminus \text{supp}(w)$, therefore by Lemma 3.2.23, $\text{sdepth}(S_{2,m,h}/I_{2,m,h}) \leq r = 2m$. Hence,

$$\text{sdepth}(S_{2,m,h}/(I_{2,m,h}) = 2m.$$

To prove our result in general, consider $n \geq 3$, $m \geq 2$ and $m = h$, then take the following exact sequence

$$0 \longrightarrow S_{n,m,h}/(I_{n,m,h} : y_{mn}) \xrightarrow{\cdot y_{mn}} S_{n,m,h}/I_{n,m,h} \longrightarrow S_{n,m,h}/(I_{n,m,h}, y_{mn}) \longrightarrow 0. \tag{4.13}$$

Over here

$$(I_{n,m,h} : y_{mn}) = (I_{n-1,m,h}, x_n, y_{(n-1)m+1}z_{(n-1)m^2+1}, y_{(n-1)m+1}z_{(n-1)m^2+2}, \dots,$$
$$y_{(n-1)m+1}z_{(n-1)m^2+m}, y_{(n-1)m+2}z_{(n-1)m^2+m+1}, \dots, y_{(n-1)m+2}z_{(n-1)m^2+2m}, \dots,$$
$$y_{mn-1}z_{m^2n-2m+1}, y_{mn-1}z_{m^2n-2m+2}, \dots, y_{mn-1}z_{m^2n-m}, z_{m^2n-m+1}, \dots, z_{m^2n}).$$

It means,

$$S_{n,m,h}/(I_{n,m,h} : y_{nm}) \cong S_{n-1,m,h}/I_{n-1,m,h} \otimes_K \frac{K[y_{nm-m+1}, z_{m^2n-m^2+1}, \dots, z_{nm^2-m^2+m}]}{(y_{mn-m+1}z_{nm^2-m^2+1}, \dots, y_{mn-m+1}z_{(m^2n-m^2+m)})} \otimes_K \cdots \otimes_K \frac{K[y_{mn-1}, z_{m^2n-2m+1}, \dots, z_{m^2n-m}]}{(y_{mn-1}z_{m^2n-2m+1}, \dots, y_{mn-1}z_{m^2n-m})} \otimes_K K[y_{mn}].$$

By ([28], Theorem 2.2.21) and Proposition 3.2.10, we get

$$\text{depth}(S_{n,m,h}/(I_{n,m,h} : y_{mn})) = \text{depth}(S_{n-1,m,h}/I_{n-1,m,h}) + (m-1) + 1. \tag{4.14}$$

Now using induction, we get

$$\text{depth}(S_{n,m,h}/(I_{n,m,h} : y_{mn}) = m(n-1) + (m-1) + 1 = mn. \tag{4.15}$$

Also

$$I^*_{n,m,h} = (I_{n,m,h}, y_{mn}) = (x_1y_1, x_1y_2, \ldots, x_1y_m, y_1z_1, y_1z_2, \ldots, y_1z_m, y_2z_{m+1}, y_2z_{m+2}, \ldots, y_2z_{2m}, \ldots, y_mz_{m^2-m+1}, \ldots, y_mz_{m^2}, \ldots, x_ix_{i+1}, x_{i+1}x_{i+2}, \ldots, x_{n-1}x_n, x_ny_{(n-1)m+1}, x_ny_{(n-1)m+2}, \ldots, x_ny_{mn-1}, y_{(n-1)m+1}z_{(n-1)m^2+1}, y_{(n-1)m+1}z_{(n-1)m^2+2}, \ldots, y_{(n-1)m+1}z_{(n-1)m^2+m}, y_{(n-1)m+2}z_{(n-1)m^2+m+1}, \ldots, y_{(n-1)m+2}z_{(n-1)m^2+2m}, \ldots, y_{mn-1}z_{m^2n-2m+1}, y_{mn-1}z_{m^2n-2m+2}, \ldots, y_{mn-1}z_{m^2n-m}, y_{mn}).$$

Again consider the following exact sequence

$$0 \longrightarrow S_{n,m,h}/(I^*_{n,m,h} : x_n) \xrightarrow{\cdot x_n} S_{n,m,h}/I^*_{n,m,h} \longrightarrow S_{n,m,h}/(I^*_{n,m,h}, x_n) \longrightarrow 0. \tag{4.16}$$

Here

$$(I^*_{n,m,h} : x_n) = (I_{n-2,m,h}, x_{n-1}, y_{(n-2)m+1}z_{m^2(n-2)+1}, y_{(n-2)m+1}z_{m^2(n-2)+2}, \ldots, y_{(n-2)m+1}z_{m^2(n-2)+m}, \ldots, y_{m(n-1)}z_{m^2(n-1)-m+1}, y_{m(n-1)}z_{m^2(n-1)-m+2}, \ldots, y_{m(n-1)}z_{m^2(n-1)}, y_{m(n-1)+1}, y_{m(n-1)+2}, \ldots, y_{mn-1}, y_{mn}).$$

This gives us,

$$S_{n,m,h}/(I^*_{n,m,h} : x_n) \cong S_{n-2,m,h}/I_{n-2,m,h} \otimes_K \frac{K[y_{m(n-2)+1}, z_{(n-2)m^2+1}, \ldots, z_{(n-2)m^2+m}]}{(y_{(n-2)m+1}z_{(n-2)m^2+1}, \ldots, y_{(n-2)m+1}z_{(n-2)m^2+m})} \otimes_K \cdots \otimes_K \frac{K[y_{m(n-1)}, z_{m^2(n-1)-m+1}, \ldots, z_{m^2(n-1)}]}{(y_{m(n-1)}z_{m^2(n-1)-m+1}, \ldots, y_{m(n-1)}z_{m^2(n-1)})} \otimes_K K[x_n, z_{m^2(n-1)+1}, \ldots, z_{m^2n}].$$

From ([28], Theorem 2.2.21), we have

$$\operatorname{depth}(S_{n,m,h}/(I^*_{n,m,h}:x_n) = \operatorname{depth}(S_{n-2,m,h}/I_{n-2,m,h}) + \operatorname{depth}(\frac{K[y_{(n-2)m+1}, z_{(n-2)m^2+1}, \dots, z_{(n-2)m^2+m}]}{(y_{(n-2)m+1}z_{(n-2)m^2+1}, \dots, y_{(n-2)m+1}z_{(n-2)m^2+m})}) + \dots + \operatorname{depth}(\frac{K[y_{m(n-1)}, z_{m^2(n-1)-m+1}, \dots, z_{m^2(n-1)}]}{(y_{m(n-1)}z_{m^2(n-1)-m+1}, \dots, y_{m(n-1)}z_{m^2(n-1)})}) + \operatorname{depth} K[x_n, z_{m^2(n-1)+1}, \dots, z_{m^2n}].$$

By induction and Proposition 3.2.10, we see that

$$\operatorname{depth}(S_{n,m,h}/I^*_{n,m,h}:x_n) = (n-2)m + m + m^2 + 1 = m(n-1) + m^2 + 1.$$

Furthermore,

$$(I^*_{n,m,h}, x_n) = (I_{n-1,m,h}, x_n, y_{(n-1)m+1}z_{(n-1)m^2+1}, y_{(n-1)m+1}z_{(n-1)m^2+2}, \dots, y_{(n-1)m+1}z_{(n-1)m^2+m}, y_{(n-1)m+2}z_{(n-1)m^2+m+1}, \dots, y_{(n-1)m+2}z_{(n-1)m^2+2m}, \dots, y_{mn-1}z_{m^2n-2m+1}, y_{mn-1}z_{m^2n-2m+2}, \dots, y_{mn-1}z_{m^2n-m}, y_{mn}).$$

Then,

$$S_{n,m,h}/(I^*_{n,m,h}, x_n) \cong S_{n-1,m,h}/I_{n-1,m,h} \otimes_K \frac{K[y_{(n-1)m+1}, z_{(n-1)m^2+1}, \dots, z_{(n-1)m^2+m}]}{(y_{(n-1)m+1}z_{(n-1)m^2+1}, \dots, y_{(n-1)m+1}z_{(n-1)m^2+m})} \otimes_K \cdots \otimes_K \frac{K[y_{mn-1}, z_{m^2n-2m+1}, \dots, z_{m^2n-m}]}{(y_{mn-1}z_{m^2n-2m+1}, \dots, y_{mn-1}z_{m^2n-m})} \otimes_K K[z_{m^2n-m+1}, \dots, z_{m^2n}].$$

From ([28], Theorem 2.2.21), we have

$$\operatorname{depth}(S_{n,m,h}/(I^*_{n,m,h}, x_n) = \operatorname{depth}(S_{n-1,m,h}/I_{n-1,m,h}) + \operatorname{depth}(\frac{K[y_{(n-1)m+1}, z_{(n-1)m^2+1}, \dots, z_{(n-1)m^2+m}]}{(y_{(n-1)m+1}z_{(n-1)m^2+1}, \dots, y_{(n-1)m+1}z_{(n-1)m^2+m})}) + \dots + \operatorname{depth}(\frac{K[y_{mn-1}, z_{m^2n-2m+1}, \dots, z_{m^2n-m}]}{(y_{mn-1}z_{m^2n-2m+1}, \dots, y_{mn-1}z_{m^2n-m)})}) + \operatorname{depth} K[z_{m^2n-m+1}, \dots, z_{m^2n}].$$

Again by induction and Proposition 3.2.10, we see that

$$\mathrm{depth}(S_{n,m,h}/I^*_{n,m,h}, x_n) = (n-1)m + (m-1) + m = m(n+1) - 1.$$

As $\mathrm{depth}(S_{n,m,h}/I^*_{n,m,h} : x_n) > \mathrm{depth}(S_{n,m,h}/I^*_{n,m,h}, x_n)$. Now applying the depth Lemma on sequence 4.16, we have $\mathrm{depth}(S_{n,m,h}/(I_{n,m,h}, y_{mn}) \geq m(n+1) - 1$. Again applying the depth Lemma on sequence 4.13, we have $\mathrm{depth}(S_{n,m,h}/I_{n,m,h}) \geq mn$. To find the upper bound, we use induction on n. For $n = 1$, 2 required result holds from Lemma 4.1.2 and equation 4.12. Since $y_{mn} \notin I_{n,m,h}$ and from 3.2.11, $\mathrm{depth}(S_{n,m,h}/I_{n,m,h}) \leq \mathrm{depth}(S_{n,m,h}/(I_{n,m,h} : y_{mn}))$. Equation 4.14 gives us

$$\mathrm{depth}(S_{n,m,h}/(I_{n,m,h} : y_{mn}) = \mathrm{depth}(S_{n-1,m,h}/I_{n-1,m,h}) + (m-1) + 1.$$

Using induction, we get

$$\mathrm{depth}(S_{n,m,h}/(I_{n,m,h} : y_{mn}) \leq m(n-1) + (m-1) + 1 = mn.$$

Therefore, $\mathrm{depth}(S_{n,m,h}/I_{n,m,h}) \leq mn$. As a result,

$$\mathrm{depth}(S_{n,m,h}/I_{n,m,h}) = mn.$$

To find the Stanley depth, we use 3.2.6 and 3.2.11 on short exact sequence 4.16, 4.13 and by using induction, we get $\mathrm{sdepth}(S_{n,m,h}/I_{n,m,h}) \geq mn$. For finding the upper bound , we assume $y_j = x_{n+j}$, where $n \geq 3$, $j = 1, 2, \cdots, mn$ and $z_r = x_{n+mn+r}$, where $r = 1, \cdots, m^2n$. Consider $w = x_{n+1}x_{n+2}, \ldots, x_{n+mn} \in (S_{n,m,h}/I_{n,m,h})$, but $x_l w \in I_{n,m,h}$, for all $l \in [mn(m+1)+n] \setminus \mathrm{supp}(w)$, therefore by Lemma 3.2.23, $\mathrm{sdepth}(S_{n,m,h}/I_{n,m,h}) \leq r = mn$. Finally,

$$\mathrm{sdepth}(S_{n,m,h}/(I_{n,m,h}) = mn.$$

Theorem 4.1.4 Let $n \geq 1$, $m \geq 2$ and $m = h$, then

$$\dim(S_{n,m,h}/I_{n,m,h}) \geq mn.$$

Proof: The required result follows from theorem 3.1.7 and 4.1.3.

Corollary 4.1.5 Stanley's inequality hold for cylic module $S_{n,m,h}/I_n(P_{n,m,h})$.

Remark 4.1.6

- For $n \geq 1$, $m \geq 2$ and $m = h$ we have $d(P_{n,m,h}) = n+3$. From Theorem 3.2.17, we have sdepth$(S_{n,m,h}/I_{n,m,h})$, depth$(S_{n,m,h}/I_{n,m,h})$ $\geq \lceil \frac{n+4}{3} \rceil$ and from our Theorem 4.1.3, we have sdepth$(S_{n,m,h}/I_{n,m,h})$, depth$(S_{n,m,h}/I_{n,m,h}) = mn$. For example, when $n = 6$ and $m = 5 = h$, then $d(P_{6,5,5}) = 9$. From Theorem 3.2.17, we have sdepth$(S_{6,5,5}/I_{6,5,5})$, depth$(S_{6,5,5}/I_{6,5,5}) \geq \lceil \frac{9+1}{3} \rceil = 4$, but from Theorem 4.1.3, we have sdepth$(S_{6,5,5}/I_{6,5,5})$, depth$(S_{6,5,5}/I_{6,5,5}) = 30$. Thus for any $n \geq 1$, $m \geq 2$ and $m = h$, Theorem 3.2.17, gives us a lower bound far away from exact value, but Theorem 4.1.3 gives us the exact values of depth and sdepth for cyclic module $S_{n,m,h}/I_{n,m,h}$.

- For $n = 1$, $m = 2 = h$ and $I_{1,2,2}$ be a monomial ideal minimally generated by 6 elements, then by Proposition 3.2.19 we have sdepth$(S_{1,2,2}/I_{1,2,2})$ $\geq 7-6 = 1$. Now from our Theorem 4.1.3, we have sdepth$(S_{1,2,2}/I_{1,2,2}) = 2$. Also when $n = 5$ and $m = 2 = h$, by Proposition 3.2.19 we have sdepth$(S_{5,2,2}/I_{5,2,2}) \geq 35 - 34 = 1$. Now from our Theorem 4.1.3, we have sdepth$(S_{5,2,2}/I_{5,2,2}) = 10$. As a conclusion, our Theorem 4.1.3 gives us exact values and Proposition 3.2.19 gives us lower bounds which are far away from the exact values.

4.2 The results of unicyclic graphs.

In this part, we use previous lobster tree results to determine the values of depth and Stanley depth of the quotient ring of edge ideal associated with a unicyclic graph $C_{n,m,h}$. We also show that the depth and Stanley depth values for this unicyclic graph are equal. As a result, for these values, Stanley's inequality holds.

Theorem 4.2.1 Let $n \geq 3$, $m \geq 2$ and $m = h$, then

$$\operatorname{depth}(S_{n,m,h}/I'_{n,m,h}) = \operatorname{sdepth}(S_{n,m,h}/I'_{n,m,h}) = mn.$$

Proof. Consider $n \geq 3$, $m \geq 2$ and $m = h$. Here $I'_{n,m,h} = (I_{n,m,h}, x_1x_n)$. Take the following exact sequence

$$0 \longrightarrow S_{n,m,h}/(I'_{n,m,h} : y_{mn}) \xrightarrow{\cdot y_{mn}} S_{n,m,h}/I'_{n,m,h} \longrightarrow S_{n,m,h}/(I'_{n,m,h}, y_{mn}) \longrightarrow 0. \tag{4.17}$$

Over here

$$(I'_{n,m,h} : y_{mn}) = (I_{n-1,m,h}, x_n, y_{(n-1)m+1}z_{(n-1)m^2+1}, y_{(n-1)m+1}z_{(n-1)m^2+2}, \dots,$$
$$y_{(n-1)m+1}z_{(n-1)m^2+m}, y_{(n-1)m+2}z_{(n-1)m^2+m+1}, \dots, y_{(n-1)m+2}z_{(n-1)m^2+2m}, \dots,$$
$$y_{mn-1}z_{m^2n-2m+1}, y_{mn-1}z_{m^2n-2m+2}, \dots, y_{mn-1}z_{m^2n-m}, z_{m^2n-m+1}, \dots, z_{m^2n}).$$

It gives us,

$$S_{n,m,h}/(I'_{n,m,h} : y_{nm}) \cong S_{n-1,m,h}/I_{n-1,m,h} \otimes_K$$
$$\frac{K[y_{(n-1)m+1}, z_{(n-1)m^2+1}, \dots, z_{(n-1)m^2+m}]}{(y_{(n-1)m+1}z_{(n-1)m^2+1}, \dots, y_{(n-1)m+1}z_{(n-1)m^2+m})} \otimes_K \cdots \otimes_K$$
$$\frac{K[y_{mn-1}, z_{m^2n-2m+1}, \dots, z_{m^2n-m}]}{(y_{mn-1}z_{m^2n-2m+1}, \dots, y_{mn-1}z_{m^2n-m})} \otimes_K K[y_{mn}].$$

By ([28], Theorem 2.2.21), we have

$$\begin{aligned}\operatorname{depth}(S_{n,m,h}/(I'_{n,m,h}:y_{mn}) &= \operatorname{depth}(S_{n-1,m,h}/I_{n-1,m,h})+\\ &\operatorname{depth}(\frac{K[y_{(n-1)m+1}, z_{(n-1)m^2+1}, \cdots, z_{(n-1)m^2+m}]}{(y_{(n-1)m+1}z_{(n-1)m^2+1}, \cdots, y_{(n-1)m+1}z_{(n-1)m^2+m})})+\cdots+\\ &\operatorname{depth}(\frac{K[y_{mn-1}, z_{m^2n-2m+1}, \ldots, z_{m^2n-m}]}{(y_{mn-1}z_{m^2n-2m+1}, \ldots, y_{mn-1}z_{m^2n-m})})+\operatorname{depth} K[y_{mn}].\end{aligned}$$

Using Proposition 3.2.10 and Theorem 4.1.3, we get

$$\operatorname{depth}(S_{n,m,h}/(I'_{n,m,h}:y_{mn}) = m(n-1)+(m-1)+1 = mn. \tag{4.18}$$

Also

$$\begin{aligned}I^{**}_{n,m,h} = (I'_{n,m,h}, y_{mn}) = (&x_1y_1, x_1y_2, \ldots, x_1y_m, y_1z_1, y_1z_2, \ldots, y_1z_m, y_2z_{m+1},\\ &y_2z_{m+2}, \ldots, y_2z_{2m}, \ldots, y_mz_{m^2-m+1}, \ldots, y_mz_{m^2}, \ldots, x_ix_{i+1}, x_{i+1}x_{i+2}, \ldots,\\ &x_{n-1}x_n, x_1x_n, x_ny_{(n-1)m+1}, x_ny_{(n-1)m+2}, \ldots, x_ny_{mn-1}, y_{(n-1)m+1}z_{(n-1)m^2+1},\\ &y_{(n-1)m+1}z_{(n-1)m^2+2}, \ldots, y_{(n-1)m+1}z_{(n-1)m^2+m}, y_{(n-1)m+2}z_{(n-1)m^2+m+1}, \ldots,\\ &y_{(n-1)m+2}z_{(n-1)m^2+2m}, \ldots, y_{mn-1}z_{m^2n-2m+1}, y_{mn-1}z_{m^2n-2m+2}, \ldots,\\ &y_{mn-1}z_{m^2n-m}, y_{mn}).\end{aligned}$$

Again, consider the following exact sequence

$$0 \longrightarrow S_{n,m,h}/(I^{**}_{n,m,h}:x_n) \xrightarrow{\cdot x_n} S_{n,m,h}/I^{**}_{n,m,h} \longrightarrow S_{n,m,h}/(I^{**}_{n,m,h}, x_n) \longrightarrow 0. \tag{4.19}$$

Here

$$\begin{aligned}(I^{**}_{n,m,h}:x_n) = (&y_1z_1, y_1z_2, \ldots, y_1z_m, y_2z_{m+1}, y_2z_{m+2}, \ldots, y_2z_{2m}, \ldots,\\ &y_mz_{m^2-m+1}, \ldots, y_mz_{m^2}, I_{n-3,m,h}, x_{n-1}, y_{(n-2)m+1}z_{(m^2(n-2)+1},\\ &y_{(n-2)m+1}z_{(m^2(n-2)+2}, \ldots, y_{(n-2)m+1}z_{m^2(n-2)+m}, \ldots, y_{m(n-1)}z_{m^2(n-1)-m+1},\\ &y_{m(n-1)}z_{m^2(n-1)-m+2}, \ldots, y_{m(n-1)}z_{m^2(n-1)}, y_{m(n-1)+1}, y_{m(n-1)+2}, \cdots, y_{mn-1}, y_{mn}).\end{aligned}$$

This gives us,

$$S_{n,m,h}/(I^{**}_{n,m,h} : x_n) \cong S_{n-3,m,h}/I_{n-3,m,h} \otimes_K \frac{K[y_1, z_1, \ldots, z_m]}{(y_1z_1, \ldots, y_1z_m)} \otimes_K \cdots \otimes_K$$
$$\frac{K[y_m, z_{m^2-m+1}, \ldots, z_{m^2}]}{(y_mz_{m^2-m+1}, \ldots, y_mz_{m^2})} \otimes_K \frac{K[y_{(n-2)m+1}, z_{(n-2)m^2+1}, \ldots, z_{(n-2)m^2+m}]}{(y_{(n-2)m+1}z_{(n-2)m^2+1}, \ldots, y_{(n-2)m+1}z_{(n-2)m^2+m})} \otimes_K$$
$$\cdots \otimes_K \frac{K[y_{m(n-1)}, z_{m^2(n-1)-m+1}, \ldots, z_{m^2(n-1)}]}{(y_{m(n-1)}z_{m^2(n-1)-m+1}, \ldots, y_{m(n-1)}z_{m^2(n-1)})} \otimes_K K[x_n, z_{m^2(n-1)+1}, \ldots, z_{m^2n}].$$

By using ([28], Theorem 2.2.21), we have

$$\begin{aligned}\operatorname{depth}(S_{n,m,h}/(I^{**}_{n,m,h} : x_n) &= \operatorname{depth}(S_{n-3,m,h}/I_{n-3,m,h}) + \\ \operatorname{depth}(\frac{K[y_1, z_1, \ldots, z_m]}{(y_1z_1, \ldots, y_1z_m)}) &+ \cdots + \operatorname{depth}(\frac{K[y_m, z_{m^2-m+1}, \ldots, z_{m^2}]}{(y_mz_{m^2-m+1}, \ldots, y_mz_{m^2})}) + \\ \operatorname{depth}(\frac{K[y_{(n-2)m+1}, z_{(n-2)m^2+1}, \ldots, z_{(n-2)m^2+m}]}{(y_{(n-2)m+1}z_{(n-2)m^2+1}, \ldots, y_{(n-2)m+1}z_{(n-2)m^2+m})}) &+ \cdots + \\ \operatorname{depth}(\frac{K[y_{m(n-1)}, z_{m^2(n-1)-m+1}, \ldots, z_{m^2(n-1)}]}{(y_{m(n-1)}z_{m^2(n-1)-m+1}, \ldots, y_{m(n-1)}z_{m^2(n-1)})}) &+ \\ &\operatorname{depth} K[x_n, z_{m^2(n-1)+1}, \ldots, z_{m^2n}].\end{aligned}$$

Using Theorem 4.1.3 and Proposition 3.2.10, we get

$$\operatorname{depth}(S_{n,m,h}/I^{**}_{n,m,h} : x_n) = (n-3)m + m + m + m^2 + 1 = m(n-1) + m^2 + 1.$$

Also,

$$\begin{aligned}(I^{**}_{n,m,h}, x_n) = (I_{n-1,m,h}, x_n, y_{(n-1)m+1}z_{(n-1)m^2+1}, y_{(n-1)m+1}z_{(n-1)m^2+2}, \ldots, \\ y_{(n-1)m+1}z_{(n-1)m^2+m}, y_{(n-1)m+2}z_{(n-1)m^2+m+1}, \ldots, y_{(n-1)m+2}z_{(n-1)m^2+2m}, \\ \ldots, y_{mn-1}z_{m^2n-2m+1}, y_{mn-1}z_{m^2n-2m+2}, \ldots, y_{mn-1}z_{m^2n-m}, y_{mn}).\end{aligned}$$

Then,

$$S_{n,m,h}/(I^{**}_{n,m,h}, x_n) \cong S_{n-1,m,h}/I_{n-1,m,h} \otimes_K \frac{K[y_{(n-1)m+1}, z_{(n-1)m^2+1}, \dots, z_{(n-1)m^2+m}]}{(y_{(n-1)m+1}z_{(n-1)m^2+1}, \dots, y_{(n-1)m+1}z_{(n-1)m^2+m})} \otimes_K \cdots \otimes_K \frac{K[y_{mn-1}, z_{m^2n-2m+1}, \dots, z_{m^2n-m}]}{(y_{mn-1}z_{m^2n-2m+1}, \dots, y_{mn-1}z_{m^2n-m})} \otimes_K K[z_{m^2n-m+1}, \dots, z_{m^2n}].$$

By using ([28], Theorem 2.2.21), we have

$$\operatorname{depth}(S_{n,m,h}/(I^{**}_{n,m,h}, x_n) = \operatorname{depth}(S_{n-1,m,h}/I_{n-1,m,h}) + \operatorname{depth}(\frac{K[y_{(n-1)m+1}, z_{(n-1)m^2+1}, \dots, z_{(n-1)m^2+m}]}{(y_{(n-1)m+1}z_{(n-1)m^2+1}, \dots, y_{(n-1)m+1}z_{(n-1)m^2+m})}) + \cdots + \operatorname{depth}(\frac{K[y_{mn-1}, z_{m^2n-2m+1}, \dots, z_{m^2n-m}]}{(y_{mn-1}z_{m^2n-2m+1}, \dots, y_{mn-1}z_{m^2n-m}))}) + \operatorname{depth} K[z_{m^2n-m+1}, \dots, z_{m^2n}].$$

Again by Theorem 4.1.3 and Proposition 3.2.10, we see that

$$\operatorname{depth}(S_{n,m,h}/I^{**}_{n,m,h}, x_n) = (n-1)m + (m-1) + m = m(n+1) - 1.$$

Since $\operatorname{depth}(S_{n,m,h}/I^{**}_{n,m,h} : x_n) > \operatorname{depth}(S_{n,m,h}/I^{**}_{n,m,h}, x_n)$. Now applying depth Lemma on sequence 4.19, we get $\operatorname{depth}(S_{n,m,h}/(I'_{n,m,h}, y_{mn}) \geq m(n+1) - 1$. Again applying depth Lemma on sequence 4.17, we have $\operatorname{depth}(S_{n,m,h}/I'_{n,m,h}) \geq mn$. For upper bound, since $y_{mn} \notin I_{n,m,h}$, from 3.2.11, we have $\operatorname{depth}(S_{n,m,h}/I'_{n,m,h}) \leq \operatorname{depth}(S_{n,m,h}/(I'_{n,m,h} : y_{mn}))$. Equation 4.18 gives us

$$\operatorname{depth}(S_{n,m,h}/(I'_{n,m,h} : y_{mn}) = \operatorname{depth}(S_{n-1,m,h}/I_{n-1,m,h}) + (m-1) + 1.$$

Using Theorem 4.1.3, we get

$$\operatorname{depth}(S_{n,m,h}/(I'_{n,m,h} : y_{mn}) = m(n-1) + (m-1) + 1 = mn.$$

Therefore, $\operatorname{depth}(S_{n,m,h}/I'_{n,m,h}) \leq mn$. As a result,

$$\operatorname{depth}(S_{n,m,h}/I'_{n,m,h}) = mn.$$

We find the Stanley depth by using 3.2.6 and 3.2.11 on short exact sequences 4.19, 4.17 and Theorem 4.1.3, we have $\operatorname{sdepth}(S_{n,m,h}/I'_{n,m,h}) \geq mn$. Now to find the upper bound, we assume $y_j = x_{n+j}$, where $n \geq 3$, $j = 1, 2, \cdots, mn$ and $z_r = x_{n+mn+r}$, where $r = 1, \cdots, m^2n$. Consider $w = x_{n+1}x_{n+2}, \cdots, x_{n+mn} \in (S_{n,m,h}/I'_{n,m,h})$, but $x_l w \in I'_{n,m,h}$, for all $l \in [mn(m+1)+n] \setminus \operatorname{supp}(w)$, therefore by Lemma 3.2.23, $\operatorname{sdepth}(S_{n,m,h}/I'_{n,m,h}) \leq r = mn$. Finally,

$$\operatorname{sdepth}(S_{n,m,h}/(I'_{n,m,h}) = mn.$$

Theorem 4.2.2 Let $n \geq 3$, $m \geq 2$ and $m = h$, then

$$\dim(S_{n,m,h}/I'_{n,m,h}) \geq mn.$$

Proof: The required result follows from Theorem 3.1.7 and 4.2.1.

Corollary 4.2.3 Stanley's inequality hold for cyclic module $S_{n,m,h}/I'(C_{n,m,h})$.

Remark 4.2.4 For $n = 6$ and $m = 3 = h$, we have diameter$(C_{6,3,3}) = 7$. From Theorem 3.2.17, we have $\operatorname{sdepth}(S_{6,3,3}/I'(C_{6,3,3}))$, $\operatorname{depth}(S_{6,3,3}/I'(C_{6,3,3})) \geq \lceil \frac{7+1}{3} \rceil = 3$, but from our Theorem 4.2.2 for unicyclic graph $C_{n,m,h}$ we have $\operatorname{sdepth}(S_{6,3,3}/I'_{6,3,3})$, $\operatorname{depth}(S_{6,3,3}/I'_{6,3,3}) = 18$. Thus for any $n \geq 3$, $m \geq 2$ and $m = h$, Theorem 3.2.17 gives us lower bounds far away from exact values, but our Theorem 4.2.2 gives us the exact values of depth and sdepth for cyclic module $S_{n,m,h}/I'_{n,m,h}$.

4.2.1 Future directions

- When $n \geq 1$ and $m \neq h$, determine the value of $\text{sdepth}(I_{n,m,h})$ and one can verify the Asia's question [24] and Herzog's conjecture [15] using these values.
- Determine the value of $\text{sdepth}(I'(C_{n,m,h}))$.
- Find the values of depth and sdepth for cyclic module $S_{n,m,h}/I^*_{n,m,h}$, where $I^*_{n,m,h}$ is the edge ideal associated with square of $P_{n,m,h}$.
- Let $n \geq 1$, $m \geq 2$ and $m = h$, then $\dim(S_{n,m,h}/I_{n,m,h}) = mn$?

Bibliography

[1] Alipour, A. Tehranian, A. (2017). Depth and Stanley Depth of Edge Ideals of Star Graphs, International Journal of Applied Mathematics and Statistics, 56, 63–69.

[2] Anwar, I., Popescu, D. (2007). Stanley conjecture in small embedding dimension. *Journal of Algebra, 318*(2), 1027-1031.

[3] Apel, J. (2003). On a conjecture of RP Stanley; part II—quotients modulo monomial ideals. *Journal of Algebraic Combinatorics, 17*(1), 57-74.

[4] Bruns, W., Herzog, H. J. (1998). Cohen-macaulay rings. *Cambridge university press*.

[5] Cimpoeas, M. (2012). *Several inequalities regarding Stanley depth*, *Romanian Journal of Mathematics and Computer Science*, **2** , 28–40.

[6] Cimpoeas, M. (2015). On the Stanley depth of edge ideals of line and cyclic graphs, Romanian Journal of Mathematics and Computer Science, 5 (2015), 70–75.

[7] Cimpoeas, M. (2009). Stanley depth of monomial ideals with small number of generators, Central European Journal of Mathematics, 7(4), 629 634.

[8] Duval, A. M., Goeckner, B., Klivans, C. J., Martin, J. L. (2016). A non-partitionable Cohen–Macaulay simplicial complex. *Advances in Mathematics, 299*, 381-395.

[9] Edwards, Harold M., (1977). Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory, New York: Springer.

[10] Faridi, S. (2002). The facet ideal of a simplicial complex, manuscripta math. 109, 159–174.

[11] Faridi, S. Cohen-Macaulay properties of square-free monomial ideals, preprint.

[12] Faridi, S. (2004) Simplicial trees are sequentially Cohen-Macaulay, J. Pure Appl. Algebra 190, 121–136.

[13] Fouli, L., Morey, S. (2015). A lower bound for depths of powers of edge ideals. *Journal of Algebraic Combinatorics, 42*(3), 829-848.

[14] Fraenkel, A. (1922). Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre. Math. Ann. 86, 230-237. Mathematische Annalen. 86. 230-237. 10.1007/BF01457986.

[15] Herzog, J., Vladoiu, M., Zheng, X. (2009). How to compute the Stanley depth of a monomial ideal. *Journal of Algebra, 322*(9), 3151-9.316

[16] Hibi, T., Higashitani, A., Kimura, K. and Augustine B. O'Keefe. (2011) Depth of edge rings arising from finite graphs. Proceedings of the American Mathematical Society, 139(11):3807–3813.

[17] Hibi, T., Higashitani, A., Kimura, K. and Augustine B. O'Keefe. Depth of initial ideals of normal edge rings. Communications in Algebra, 42(7):2908–2922.

[18] Iqbal, Z., Ishaq, M., Aamir, M., *Depth And Stanley Depth of the edge ideals of the edge ideals of sqaure paths and square cycles*, arXiv:1909.02610v1.

[19] Morey, S. (2010). Depths of powers of the edge ideal of a tree, Comm. Algebra, 38 (2010), 4042–4055.

[20] ¨Oberg, F. (1982). A study of graded extremal rings and of monomial rings, Math. Sc-and. 51, 22–34.

[21] Okazaki, R. (2011). A lower bound of Stanley depth of monomial ideals, J. Commut. Algebra,3 (1),83–88.

[22] Popescu, A. (2010). Special stanley decompositions. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 363-372.

[23] Popescu, D. (2009). An inequality between depth and Stanley depth. Bulletin math´ematique de la Soci´et´e des Sciences Math´ematiques de Roumanie, 377- 382.

[24] Rauf, A. (2010). Depth and Stanley depth of multigraded modules. *Communications in Algebra, 38*(2), 773-784.

[25] Shen, Y. (2009). Stanley depth of complete intersection monomial ideal and upperdiscrete partitions, J. Algebra 321, 12851292.

[26] Stefan, A. Stanley depth of powers of path ideal. Available from: http://arxiv.org/pdf/1409.6072.pdf.

[27] Stanley, R. P. (1982). Linear Diophantine equations and local cohomology. *Inventiones mathematicae, 68*(2), 175-193.

[28] Villarreal, R. H. (2001). Monomial algebras, Monographs and Textbooks in Pure and Applied Mathematics 238, Marcel Dekker, New York.