Depth and Stanley Depth of Edge Ideals Associated with Quotient of Tadpole Graph and it's Square Graph

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MS THESIS WORK

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I dedicate this thesis to my caring parents and family members for their advice, trust and prayers because they always understood.
Thank You!

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Durr-e-sameen Majid

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Abstract

This dissertation aims to find a geometric invariant as well as the algebraic invariant of the graphs. The desire is to obtain required invariants such as depth and Stanley depth by using edge ideals as module over a polynomial ring as well as the quotient of polynomial ring by monomial ideals. For this purpose, square free monomial ideals are sought and critically reviewed.
In this thesis, we find the exact values of depth and Stanley depth of quotient of the polynomial ring by the edge ideal associated with a Tadpole graph. It is shown that the values of these two invariants coincide. We also find the tight bounds for Sdepth and Depth of quotient of the polynomial ring by the edge ideal corresponding to square of a Tadpole graph.

Introduction

In 1982, Richard P. Stanley gave a conjecture [21] in which he proposed a relationship between geometric invariant Stanley depth(Sdepth) and homological invariant depth. The idea of using techniques and methods of Commutative Algebra for sorting out the problems of Combinatorics has became a running trend and current focus of research after the contribution of Stanley. He proposed the concept of linking these two fields (Commutative Algebra and Combinatorics) by using monomial ideals. The problems in Combinatorics is ciphered typically in a form of square free monomial ideals since then, the theory of square free monomials boosted up as a very interesting research area in Commutative Algebra. For any two monomial ideals $\mathcal{I} \subset \mathcal{J}$, Stanley's conjecture was proven for some multi-graded modules of the type $\Upsilon = \mathcal{J}/\mathcal{I}$ see [1] [13] [2] [16]. But Dual et. al repudiated by giving a counter example; see [6]. In this thesis, the exact values of depth and Stanley depth for the factor of edge ideal associated with a Tadpole graph and its square graph are computed.
This thesis contain four chapters.

Chapter 1 incorporates the preliminaries in which elementary concepts, definitions and core results from Commutative Algebra are being stated. Moreover, It includes the definitions, notations and primary concepts of Ring Theory and Module Theory. A precise recap of Graph Theory is given at the end of the chapter.

In chapter 2, a brief introduction of monomial ideals and related concepts are given including colon ideals, graded rings, regular sequences and Krull dimension etc.

Chapter 3 starts with introduction of Stanley depth along with the definition of exact sequences succeeding the method of computing Stanley depth in detail with example. Summing up the chapter with previous published results and well known theorems.

In the last, the edge ideals associated with Tadpole graph and square of Tadpole graph are presented in chapter 4. By using mathematical induction and depth Lemma on short exact sequences, exact values for depth and Stanley depth of above stated two graphs are computed.

Chapter 1

Preliminaries

The following chapter comprise of fundamental definitions and worthy results of Abstract Algebra in order to build a strong background of a reader for advance concepts that will later stated in forthcoming chapters.

1.1 Ring and Module Theory

Definition 1.1.1. A nonempty set defined over two operations addition$(+)$ and multiplication$(\cdot)$ is called a ring $\mathcal{R}$ which satisfy three axioms as given below:

1. $\mathcal{R}$ is an abelian group with respect to $+$,
2. Multiplication in $\mathcal{R}$ is a associative,
3. Distributive over multiplication with respect to addition. That is for all $\mathfrak{t}, \mathfrak{i}, \mathfrak{s} \in \mathcal{R}$
   - $t \cdot (\mathfrak{i} + \mathfrak{s}) = (\mathfrak{t} \cdot \mathfrak{i}) + (\mathfrak{t} \cdot \mathfrak{s})$ (Left distributive),
   - $(\mathfrak{i} + \mathfrak{s}) \cdot \mathfrak{t} = (\mathfrak{i} \cdot \mathfrak{t}) + (\mathfrak{s} \cdot \mathfrak{t})$ (Right distributive).

All over this thesis, a ring is possessing a multiplicative identity 1 which is also called unity of $\mathcal{R}$.

Definition 1.1.2. A ring $\mathcal{R}$ is said to be commutative ring. If all elements $\mathfrak{i}, \mathfrak{s} \in \mathcal{R}$ commute with respect to multiplication. That is

$$\mathfrak{i} \cdot \mathfrak{s} = \mathfrak{s} \cdot \mathfrak{i}.$$

It must be remembered that throughout this thesis, we are dealing with commutative rings with unity.

Corollary 1.1.3 ([7])**.** Let $\mathcal{R}$ be a ring. For elements $\mathfrak{s}, \mathfrak{q} \in \mathcal{R}$, we have

1. $\mathfrak{s} \cdot 0 = 0 \cdot \mathfrak{s} = 0$,
2. $\mathfrak{s}(-\mathfrak{q}) = (-\mathfrak{s})\mathfrak{q} = -(\mathfrak{s}\mathfrak{q})$,
3. $(-\mathfrak{s})(-\mathfrak{q}) = +(\mathfrak{s}\mathfrak{q})$.

Example 1. 1. All of $\mathbb{Z}$ (Integers), $\mathbb{C}$ (Complex Numbers) and $\mathbb{R}$ (Real Numbers) are commutative rings with number 1 as unity.

2. The set of metrics $\mathbb{M}_n(\mathbb{R})$ and $\mathbb{M}_n(\mathbb{C})$ with the standard matrix addition and multiplication are non commutative rings with 1 unless $n = 1$.

Definition 1.1.4. Let $\mathcal{I}$ be a subset of $\mathcal{R}$ then $\mathcal{I}$ is called an ideal if it is a subgroup of $\mathcal{R}$ with respect to addition and it satisfies the following condition:

- For each $\mathfrak{s} \in \mathcal{R}$ and $\mathfrak{q} \in \mathcal{I}$, the product $\mathfrak{s}\mathfrak{q} \in \mathcal{I}$.

Definition 1.1.5. An ideal that is generated by only one element is said to be principal.

Definition 1.1.6. An improper ideal $\mathcal{K}$ of $\mathcal{R}$ is called maximal if no such proper ideal exist that lies in between $\mathcal{K}$ and $\mathcal{R}$. Alternatively, for an ideal $\mathcal{J} \subset \mathcal{R}$, whenever $\mathcal{K} \subset \mathcal{J} \subset \mathcal{R}$ infer that either $\mathcal{K} = \mathcal{J}$ or $\mathcal{J} = \mathcal{R}$.

Proposition 1.1.1. Let $\mathcal{R}$ be the ring and $\mathcal{I}$ is an ideal of $\mathcal{R}$, then the set $\mathcal{Q} = \{\mathfrak{y}+\mathcal{I} : \mathfrak{y} \in \mathcal{R}\}$ form a ring structure under two operations $+$ and $\cdot$ defined by

1. $(\mathfrak{y} + \mathcal{I}) + (\mathfrak{s} + \mathcal{I}) = (\mathfrak{y} + \mathfrak{s}) + \mathcal{I}$,
2. $(\mathfrak{y} + \mathcal{I})(\mathfrak{s} + \mathcal{I}) = (\mathfrak{y}\mathfrak{s}) + \mathcal{I}$.

This ring is called factor ring also known as quotient ring and it is denoted by $\mathcal{R}/\mathcal{I}$.

Definition 1.1.7. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be rings. A mapping

$$\beta : \mathcal{R}_1 \to \mathcal{R}_2$$

satisfying the following conditions is said to be an isomorphism

- For all $\mathfrak{t}, \mathfrak{s} \in \mathcal{R}_1$, β is homomorphism if
  1. $\beta(\mathfrak{t} + \mathfrak{s}) = \beta(\mathfrak{t}) + \beta(\mathfrak{s})$,
  2. $\beta(\mathfrak{t}\mathfrak{s}) = \beta(\mathfrak{t})\beta(\mathfrak{s})$.
- The mapping is surjective.
- The mapping is injective.

Definition 1.1.8. An ideal $\mathfrak{J}$ is called prime if for any $q_1, q_2 \in \mathcal{R}$ whenever $q_1 q_2 \in \mathfrak{J}$ implies that either $q_1 \in \mathfrak{J}$ or $q_2 \in \mathfrak{J}$.

Example 2. $n^*\mathbb{Z}$ is a prime in the ring of integers when n^* is a prime number.

Theorem 1.1.1 ([7])**.** Let $\mathcal{K}$ be an ideal in a ring $\mathcal{R}$ with $1 \neq 0$, then the two statements written below are equivalent

- $\mathcal{R}/\mathcal{K}$ is a field,
- $\mathcal{K}$ is maximal ideal.

Theorem 1.1.2 ([7])**.** Let $\mathfrak{J}$ be an ideal in a ring $\mathcal{R}$ with $1 \neq 0$, then the two statements written below are equivalent

- $\mathcal{R}/\mathfrak{J}$ is an integral domain,
- $\mathfrak{J}$ is prime ideal.

Definition 1.1.9. A ring $\mathcal{R}$ is said to be local if it contains a single maximal ideal.

Example 3. 1. Z_9 is a local ring as the ideal generated by $(3) = \{0, 3, 6\}$ is unique maximal ideal in Z_9.

2. Any field $\mathbb{K}$ is a local ring.

Definition 1.1.10. For a commutative additive semi group L. A ring $\mathcal{R}$ is L-graded along with a decomposition(as additive groups)

$$\mathcal{R} = \bigoplus_{q \in L} \mathcal{R}_q,$$

such that $\mathcal{R}_q \mathcal{R}_s \subset \mathcal{R}_{q+s}$ for all $q, s \in L$. Then for $\acute{t} \in \mathcal{R}$ can be written uniquely as

$$\acute{t} = \sum_{q \in L} t_q,$$

where $t_q \in \mathcal{R}_q$ and almost all $t_q = 0$ and t_q is known to be the q^{th} homogeneous component. If $\acute{t} \in \mathcal{R}_q$, then $\acute{t}$ is called homogeneous of degree q.

Definition 1.1.11. The polynomial ring denoted by $\mathcal{R} = \mathcal{R}[\varsigma]$ is a ring that consist of elements of the type

$$q_n \varsigma^n + q_{n-1} \varsigma^{n-1} + \cdots + q_1 \varsigma + q_0,$$

where $n \geq 0$ and $q_i \in \mathcal{R}$ with operations defined as component to component addition and multiplication by

1. $\sum_{i=0}^{k} q_i\varsigma^i + \sum_{i=0}^{k} s_i\varsigma^i = \sum_{i=0}^{k}(q_i + s_i)\varsigma^i$,

2. $\sum_{i=0}^{j} q_i\varsigma^i \times \sum_{i=0}^{k} s_i\varsigma^i = \sum_{n=0}^{j+k}\sum_{i=0}^{n}(q_i s_{n-i})\varsigma^n$.

The ring of polynomial in two variables ς_1, ς_2 is represented as $\mathcal{R}[\varsigma_1, \varsigma_2] = \mathcal{R}[\varsigma_1][\varsigma_2]$. Where coefficients are in $\mathcal{R}$. Similarly, inductively we can define

$$\mathcal{R}[\varsigma_1, \cdots, \varsigma_n] = \mathcal{R}[\varsigma_1, \varsigma_2 \cdots, \varsigma_{n-1}][\varsigma_n]$$

which is a polynomial ring in n variables where coefficients are coming from $\mathcal{R}$.

Definition 1.1.12. Let $\mathcal{R}$ be a ring and Υ be an abelian group with respect to addition. Then Υ is said to be an $\mathcal{R}$-module, if we can define a mapping

$$\cdot : \mathcal{R} \times \Upsilon \to \Upsilon$$

by

$$\cdot(\mathfrak{r}, \mathfrak{q}) = \mathfrak{r}\mathfrak{q}$$

for every $\mathfrak{q}, \mathfrak{q}_1, \mathfrak{q}_2 \in \Upsilon$ and for each $\mathfrak{r}, \mathfrak{r}_1, \mathfrak{r}_2 \in \mathcal{R}$ satisfying the following axioms:

1. $(\mathfrak{r}_1 + \mathfrak{r}_2)\mathfrak{q} = \mathfrak{r}_1\mathfrak{q} + \mathfrak{r}_2\mathfrak{q}$,

2. $\mathfrak{r}(\mathfrak{q}_1 + \mathfrak{q}_2) = \mathfrak{r}\mathfrak{q}_1 + \mathfrak{r}\mathfrak{q}_2$,

3. $\mathfrak{r}(\mathfrak{q}_1\mathfrak{q}_2) = (\mathfrak{r}\mathfrak{q}_1)\mathfrak{q}_2$,

4. $1 \cdot \mathfrak{q} = \mathfrak{q}$.

Definition 1.1.13. Let Υ be an $\mathcal{R}$-module. An element $\mathfrak{s} \neq 0$ in $\mathcal{R}$ is called regular on a module Υ also called Υ-regular, if it is a nonzero divisor. That is, whenever $\mathfrak{q} \in \Upsilon$ and $\mathfrak{s}\mathfrak{q} = 0$, implies $\mathfrak{q} = 0$.

Example 4. Let $\mathcal{I} = (\varsigma_2^2, \varsigma_2\varsigma_3)$ be an ideal in a polynomial ring $\mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2, \varsigma_3]$. Then ς_1 is a regular element on a module $\Upsilon = \mathcal{S}/\mathcal{I}$.

Definition 1.1.14. Let Υ be an $\mathcal{R}$-module. Then the annihilator of module Υ is an ideal of $\mathcal{R}$ denoted by $\mathrm{Ann}_{\mathcal{R}}(\Upsilon)$ and defined as

$$\mathrm{Ann}_{\mathcal{R}}(\Upsilon) = \{\mathfrak{s} \in \mathcal{R} : \mathfrak{s}\mathfrak{q} = 0_{\Upsilon} \text{ for all } \mathfrak{q} \in \Upsilon\}.$$

Definition 1.1.15. The Krull dimension of a ring $\mathcal{R}$ is the maximum length of all the chains of prime ideals defined by inclusion relation.

Definition 1.1.16. Let $\mathcal{R}$ be a ring and Υ be an $\mathcal{R}$-module. The Krull dimension of a module Υ is same as the Krull dimension of the residue class ring of $\mathcal{R}$.
i.e.

$$\dim_{\mathcal{R}}\Upsilon := \dim\{\mathcal{R}/(\mathrm{Ann}_{\mathcal{R}}(\Upsilon))\}.$$

Example 5. Let $\mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4]$ and ideal $\mathcal{K} = (\varsigma_1^2, \varsigma_2\varsigma_3, \varsigma_3\varsigma_4)$, then the collection of all associated primes of $\mathcal{S}/\mathcal{K}$ is

$$\mathrm{Ass}(\mathcal{K}) = \{(\varsigma_1, \varsigma_2, \varsigma_3), (\varsigma_1, \varsigma_2, \varsigma_4), (\varsigma_1, \varsigma_3), (\varsigma_1, \varsigma_3, \varsigma_4)\}$$

So

$$\dim(\mathcal{S}/\mathcal{K}) = \mathrm{Sup}\{\dim(\mathbb{K}[\varsigma_4]), \dim(\mathbb{K}[\varsigma_2, \varsigma_4]), \dim(\mathbb{K}[\varsigma_3]), \dim(\mathbb{K}[\varsigma_2])\}.$$

Hence

$$\dim(\mathcal{S}/\mathcal{K}) = \mathrm{Sup}\{1, 2, 1, 1\}.$$

Thus

$$\dim(\mathcal{S}/\mathcal{K}) = 2.$$

Definition 1.1.17. Let $\mathcal{R}$ be a ring and Υ be an $\mathcal{R}$-module. Then Υ is called Noetherian if every ascending chain of $\mathcal{R}$-submodules of Υ become static. A ring $\mathcal{R}$ is called Noetherian if considered as being an $\mathcal{R}$-module.

Proposition 1.1.2. *A ring $\mathcal{R}$ is Noetherian over itself then these three statements are equivalent.*

1. *Every nonempty collection of ideals of a ring possess a maximal element.*
2. *Each increasing chain of ideals is stationary.*
3. *All the ideals of $\mathcal{R}$ are finitely generated.*

Example 6.
1. Set of Real Numbers $\mathbb{R}$ and Rational Numbers $\mathbb{Q}$ are Noetherian modules over themselves.
2. Every PID is a Noetherian ring.

Definition 1.1.18. Let $\mathcal{J}$ be an ideal of a noetherian ring $\mathcal{R}$ and Υ be a finitely generated module over $\mathcal{R}$. Then $\mathcal{J}$ is said to be associated prime ideal of a module Υ if there exist an element $q \in \Upsilon$ such that

$$\mathcal{J} = \mathrm{Ann}(\mathfrak{q}) = \{\mathfrak{s} \in \mathcal{R} : \mathfrak{s}\mathfrak{q} = 0\}.$$

$\mathrm{Ass}(\Upsilon)$ denotes the collection of all associated prime ideal of Υ.

Example 7. Let $\mathcal{S} = K[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4]$ and $\mathcal{J} = (\varsigma_1\varsigma_2^3\varsigma_4, \varsigma_2^2\varsigma_3\varsigma_4^2)$, then

$$\mathcal{J} = (\varsigma_2^2) \cap (\varsigma_4) \cap (\varsigma_1, \varsigma_3) \cap (\varsigma_4, \varsigma_3),$$

is unique irredundant primary decomposition of $\mathcal{I}$ and set of minimal prime ideals are

$$\text{Ass}(\mathcal{J}) = \{(\varsigma_4), (\varsigma_2), (\varsigma_1, \varsigma_3)\}.$$

Example 8. Let $\mathcal{S} = K[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4]$ and $\mathcal{J} = (\varsigma_1\varsigma_2, \varsigma_3\varsigma_4, \varsigma_2\varsigma_3)$, then

$$\mathcal{J} = (\varsigma_1, \varsigma_2, \varsigma_4) \cap (\varsigma_1, \varsigma_2, \varsigma_3) \cap (\varsigma_1, \varsigma_3) \cap (\varsigma_2, \varsigma_3, \varsigma_4) \cap (\varsigma_1, \varsigma_3, \varsigma_4) \cap (\varsigma_2, \varsigma_4) \cap (\varsigma_2, \varsigma_3).$$

is unique irredundant primary decomposition of $\mathcal{I}$ and set of minimal prime ideals are

$$\text{Ass}(\mathcal{J}) = \{(\varsigma_1, \varsigma_3), (\varsigma_2, \varsigma_3), (\varsigma_2, \varsigma_4)\}.$$

1.2 Graph Theory

In 18^{th} century, Leonhard Euler has initiated a branch of Discrete Combinatorial Mathematics though a famous problem [22] and commenced its formal development by the middle of 19^{th} century. Graph theory has committed a remarkable growth during the last 70 years.
Graph theory is a rich field of abstract techniques that allow modeling of different types of problems including the construction of topological models, data analysis of operational research, network and circuit illustrations. Graphs represents many digital programs which make advance communication possible.
Over the last decennium, algebraist got concerned to study the properties of finite simple graphs by using the concept of monomial ideals. To build connection between these two areas of mathematics, algebraist borrowed the methods and strategies of Commutative Algebra and applied on Combinatorics. For this purpose, they encoded the problems into monomial ideals by using the edges of a finite simple graphs. Villarreal, Froberg, Simis, Vasconcelos were among the starting explorer of this field. Later, in the connection between Commutative Algebra and Combinatorics, Square free monomial ideals were being used by Stanley.
The main focus of this chapter is to provide knowledge to the reader about graph theory. This knowledge will help the reader to study further sections and to apply it to have worthy results.

Definition 1.2.1. A graph $\mathcal{G}$ is a mathematical structure represented as an ordered pair $(\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$, with $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ are vertex set and edge set of a graph respectively.

Definition 1.2.2. A graph is planer if there exists an embedding in the plane. More clearly, when it is drawn on paper then the edges must not cross each other.

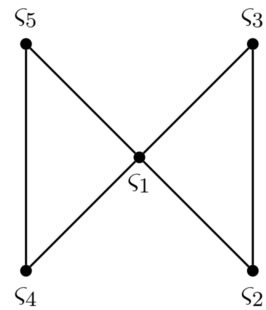

Figure 1.1: Planer graph

Definition 1.2.3. An undirected graph is called simple if there does not exist any loop or more then one edge between every pair of vertices.

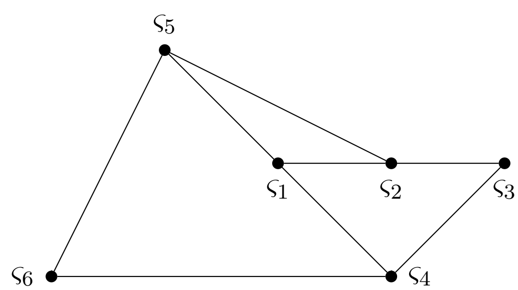

Figure 1.2: Simple graph

Definition 1.2.4. Let $\mathcal{G}$ be a graph with the vertex set $\mathcal{V}(\mathcal{G})$. The length of a shortest or minimal path that connects any two vertices $v_1, v_2 \in \mathcal{V}(\mathcal{G})$ is called distance between v_1 and v_2.

Definition 1.2.5. Let $\mathcal{G}$ be a graph. The maximum distance between the pair of vertices from vertex set $\mathcal{V}(\mathcal{G})$ is called diameter of $\mathcal{G}$ denoted by $\mathrm{diam}(\mathcal{G})$.

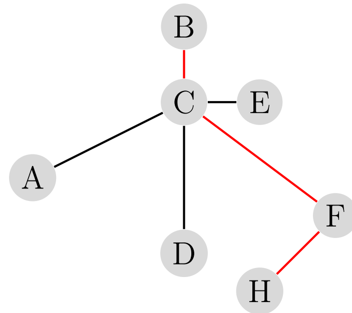

Figure 1.3: Graph $\mathcal{G}$

Here diameter of graph $\mathcal{G}$ is 3.

Definition 1.2.6. The degree of any vertex in a graph $\mathcal{G}$ is the count of edges incident to that vertex.

Definition 1.2.7. Let $\mathcal{G}$ be a graph with vertex set $\mathcal{V}(\mathcal{G}) = \{\varsigma_1, \varsigma_2, \cdots, \varsigma_n\}$ and edge set $\mathcal{E}(\mathcal{G})$. A square free monomial ideal in a polynomial ring $\mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2. \cdots, \varsigma_n]$ which is generated by the elements related to the edges of the graph $\mathcal{G}$ is called an edge ideal. That is

$$\mathcal{I}(\mathcal{G}) = (\varsigma_j \varsigma_k | \{\varsigma_j, \varsigma_k\} \in \mathcal{E}(\mathcal{G})) \subset \mathcal{S}.$$

Definition 1.2.8. A path is a list of vertices with the property that each vertex is connected to the vertex written next to it in the list. A path with $\mathfrak{r}$ vertices is denoted by $P_{\mathfrak{r}}$.

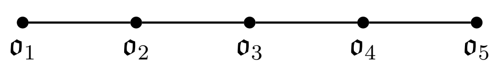

Figure 1.4: P_5

Definition 1.2.9. A nontrivial closed path is called a cycle. $C_{\mathfrak{r}}$ is the symbolic representation of cycle graph with $\mathfrak{r}$ vertices.

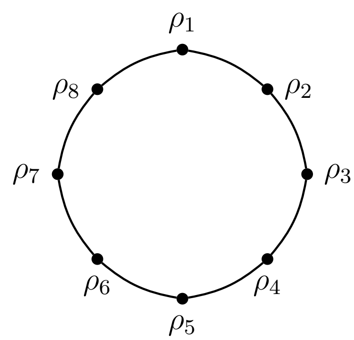

Figure 1.5: C_8

Definition 1.2.10. A tree is a graph $\mathcal{T}$ in which every two vertices from the vertex set $\mathcal{V}(\mathcal{T})$ is joined by a unique path.

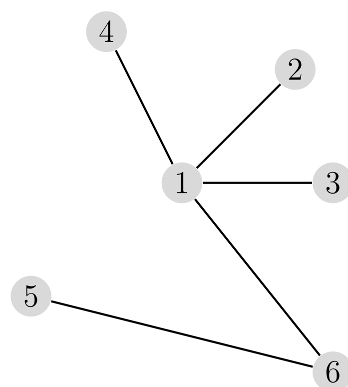

Figure 1.6: A labeled tree with 6 vertices and 5 edges.

Definition 1.2.11. A star graph is a tree with 1 internal vertex and $\mathfrak{r}$ outer vertices incident on it. It is denoted as $S_{\mathfrak{r}}$.

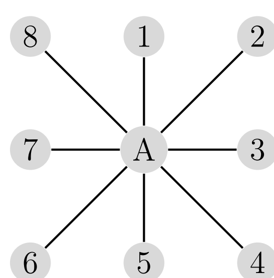

Figure 1.7: A labeled star graph S_8 with 9 vertices and 8 edges.

Definition 1.2.12. The square of a path graph with $\mathfrak{r}$ vertices is obtained by connecting every pair of vertices by an edge at the distance of two in the path and is denoted by $P_{\mathfrak{r}}^2$.

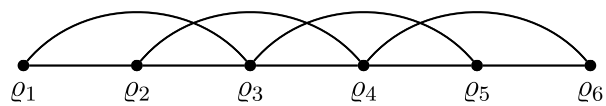

Figure 1.8: P_6^2

Definition 1.2.13. The square of a cycle graph with $\mathfrak{r}$ vertices is obtained by connecting every pair of vertices by an edge at the distance of two in the cycle and is denoted by $C_{\mathfrak{r}}^2$.

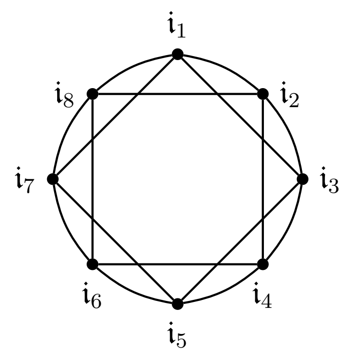

Figure 1.9: C_8^2

Definition 1.2.14. The l^{th} power of a path graph with $\mathfrak{r}$ vertices is the graph in which two vertices are linked by an edge when their distance in path graph is at most l and is denoted by $P_{\mathfrak{r}}^l$.

Definition 1.2.15. The l^{th} power of a cycle graph with $\mathfrak{r}$ vertices is the graph in which two vertices are tie up by an edge when their distance is at most l in the cycle and is denoted by $C_{\mathfrak{r}}^l$.

Chapter 2

Monomial Ideals

This chapter is devoted to monomial ideals which includes basic concepts and related properties. This section is extremely helpful to create a dictionary for further working in coming chapters.

Definition 2.0.1. An ideal $\mathcal{J} \subset \mathcal{S}$ which is generated by monomials of $\mathcal{S}$ is called monomial ideal.

Example 9. The ideal $\mathcal{J} = (\varsigma_1^3, \varsigma_3^2, \varsigma_4, \varsigma_5\varsigma_6)$ is the monomial ideal in $\mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2, \varsigma_3.\varsigma_4, \varsigma_5, \varsigma_6]$.

Definition 2.0.2. For a monomial ideal $\mathcal{J} \subset \mathcal{S}$, the $\mathbb{K}$-basis of the factor ring of $\mathcal{J}$ is the set of all those monomials that does not belong to $\mathcal{J}$.

Definition 2.0.3. Let $\mathcal{S} = \mathbb{K}[\varsigma_1, \cdots, \varsigma_n]$ be the ring of polynomial with n variables and $\mathbb{R}_+^n$ contains n-tuple of the form $\mathfrak{b} = (\mathfrak{b}_1, \cdots, \mathfrak{b}_n) \in \mathbb{R}_+^n$ and here $\mathbb{Z}_+^n = \mathbb{R}^n \cap \mathbb{Z}_+^n$. A product of the form $v = \varsigma_1^{\mathfrak{b}_1} \cdots \varsigma_n^{\mathfrak{b}_n}$ with $\mathfrak{b}_j \in \mathbb{Z}_+$ is called a monomial. We write $v = \varsigma^{\mathfrak{b}}$, where $\mathfrak{b} = (\mathfrak{b}_1, \cdots, \mathfrak{b}_n) \in \mathbb{Z}_+^n$ and we have

$$\varsigma^{\mathfrak{b}}\varsigma^{\chi} = \varsigma^{\mathfrak{b}+\chi}.$$

Mon($\mathcal{S}$) is the $\mathbb{K}$-basis for $\mathcal{S}$ defined as the collection of all monomials of $\mathcal{S}$. More precisely, any polynomial $g \in \mathcal{S}$ is uniquely written as a linear combination of monomials. That is

$$g = \sum_{v \in \mathrm{Mon}(\mathcal{S})} \mathfrak{b}_v v,$$

with $\mathfrak{b}_v \in \mathbb{K}$. Here support of g denoted by supp(g) is represented by the set $\{v \in \mathrm{Mon}(\mathcal{S}) : \mathfrak{b}_v \neq 0\}$. Similarly, for a monomial v in Mon$\mathcal{S}$ we can write support of v as

$$\mathrm{supp}(v) = \{\varsigma_i : \varsigma_i | v\}.$$

Definition 2.0.4. A monomial of the form $\varsigma^b = \varsigma_1^{b_1}\varsigma_2^{b_2} \cdots \varsigma_n^{b_n}$ is said to be square free monomial if every b_j as a component of $b = (b_1, b_2, \cdots, b_n)$ is either 1 or 0.

Definition 2.0.5. An ideal $\mathcal{J}$ in $\mathcal{S}$ that is generated by square free monomials is called square free monomial ideal.

Example 10. Let $\mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6]$ be a polynomial ring then the ideal $\mathcal{J} = (\varsigma_1\varsigma_2\varsigma_3, \varsigma_2\varsigma_3\varsigma_4, \varsigma_3\varsigma_4\varsigma_5)$ is a square free ideal in $\mathcal{S}$.

Definition 2.0.6. Let $\mathcal{J}_1, \mathcal{J}_2 \subset \mathcal{S}$ be two ideals. The ideal quotient of $\mathcal{J}_1$ by $\mathcal{J}_2$ is defined as

$$\mathcal{J}_1 : \mathcal{J}_2 = \{\mu \in S : \mu\lambda \in \mathcal{J}_1 \text{ for all } \lambda \in \mathcal{J}_2\}.$$

Proposition 2.0.1 ([10])**.** Let $\mathcal{J}_1, \mathcal{J}_2$ be two ideals of $\mathcal{S}$. Then

$$\mathcal{J}_1 : \mathcal{J}_2 = \bigcap_{w \in \mathfrak{G}(\mathcal{J}_2)} \mathcal{J}_1 : (w).$$

Furthermore, the set $\{x/gcd(x, w) : x \in \mathfrak{G}(\mathcal{J}_1)\}$ generates $\mathcal{J}_1 : (w)$. The ideal quotient is also known as colon ideal.

Example 11. Let $\mathcal{I} = (8)$ and $\mathcal{K} = (16)$ be two ideals of integers $\mathbb{Z}$. Then

$$\mathcal{K} : \mathcal{I} = \{\vartheta \in \mathbb{Z} : \vartheta(8) \subseteq (16)\}.$$

So

$$\mathcal{K} : \mathcal{I} = (2).$$

Example 12. Let $\mathcal{R} = \mathcal{R}[\varsigma]$ be a ring of polynomial in one variable. $\mathcal{I} = (\varsigma^3)$ *and* $\mathcal{K} = (\varsigma^7)$ *be two ideals in* $\mathcal{R}$*. Then*

$$\mathcal{I} : \mathcal{K} = \{\varphi \in \mathbb{Z} : \varphi(\varsigma^3) \subseteq (\varsigma^7)\}.$$

So

$$\mathcal{I} : \mathcal{K} = (\varsigma^4).$$

Definition 2.0.7. Let $\mathcal{S}$ be the N-graded ring such that

$$\mathcal{S} = \bigoplus_{p \in N} \mathcal{S}_p,$$

such that $\mathcal{S}_p\mathcal{S}_r \subseteq \mathcal{S}_{p+r}$, for all non negative integers p and r. An element $\vartheta \neq 0$ in $\mathcal{S}_p$ is called homogeneous of degree p.

Definition 2.0.8. Let Υ be an $\mathcal{S}$-module. Υ is said to be N-graded module if

$$\Upsilon = \bigoplus_{r \in N} \Upsilon_r,$$

and $\mathcal{S}_p\Upsilon_r \subset \Upsilon_{p+r}$ for each $p, r \in N$.

Example 13. Every graded ring is graded module over itself.

Definition 2.0.9. Let the ring $\mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2, ..., \varsigma_n]$ and $a = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$ with $a_i \in \mathbb{Z}$, then $s \in S$ is said to be homogeneous with degree a if s has a representation of αY^a, where $\alpha \in \mathbb{K}$, $Y = \varsigma_1\varsigma_2...\varsigma_n$ and $Y^a = \varsigma_1^{a_1}\varsigma_2^{a_2}...\varsigma_n^{a_n}$.
An $\mathcal{S}$-module is called $\mathbb{Z}^n$-graded if

$$\Upsilon = \bigoplus_{a \in Z^n} \Upsilon_a,$$

and $\mathcal{S}_\alpha \Upsilon_\beta \subset \Upsilon_{\alpha+\beta}$ for each $\alpha, \beta \in \mathbb{Z}^n$.

Definition 2.0.10. Let Υ be an $\mathcal{R}$-module. A sequence of elements $a_1, \ldots, a_r$ in $\mathcal{R}$ is said to be an Υ-regular if the following two conditions are satisfied. That is

1. The module $\Upsilon/(a_1, ..., a_r)\Upsilon \neq 0$,
2. a_i is a non-zero divisor on a factor $\Upsilon/(a_1, \cdots, a_{i-1})\Upsilon$, for each $i = 1, \cdots, r$.

Example 14. $\varsigma_1, \varsigma_2, \cdots, \varsigma_n$ is a regular sequence in a ring of polynomial $\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n]$, where $\mathbb{K}$ is a field.

Definition 2.0.11. Let $\mathcal{S}$ is a local Noetherian ring with the maximal ideal m and Υ is a finitely generated $\mathcal{S}$-module. Then all maximal regular sequences of the form $\varsigma_1, ..., \varsigma_r$ for Υ, where all ς_i belongs to m, have the common length r called depth of Υ. It is represented by depth(Υ).

Corollary 2.0.12 ([10])**.** Let $\Upsilon(\neq 0)$ be any finitely generated graded module over a local noetherian ring $\mathcal{S}$. Then depth of a module is related with its Krull dimension. That is,

$$\text{depth}(\Upsilon) \leq \dim(\Upsilon).$$

Chapter 3

Depth and Stanley Depth of Modules

3.1 Introduction

Richard P. Stanley is famous for his contribution to Combinatorics as he established a relationship to Algebra and Geometry. He proposed a conjecture which relates two important kinds of simplicial complexes, i.e paritionable and Cohen Mecaulay complexes. In 1982, Stanley defined a geometric invariant of a graded module over a commutative graded ring [21]. His conjecture exploit the connection between the geometric invariant Stanley depth with the algebraic invariant which is called depth.
This particular work is about paritionable Cohen Macaulay simplicial complexes. This amazing concept then garbed attention of algebraist in early twenties when Herzog and popescu gave some remarkable results after studying this conjecture deeply. Afterwards, many articles have been published in which this conjecture was proved for various cases. But later, Dual et. al has turned the tables by giving a counter example of such type of module for which the conjecture was not satisfied [6]. This field gets a whole new direction when Herzog, Vladoiu and Zheng developed a method in which they gave a recipe to compute Stanley depth of the type of module $\Upsilon = \mathcal{J}/\mathcal{I}$, where $\mathcal{I} \subset \mathcal{J}$ are monomial ideals in $\mathcal{S}$ [9]. Meanwhile, Rinaldo [19] encoded this method into an algorithm which is then executed in computer software WinCoCoA [5].
This chapter contains some known results and conjecture about depth and Stanley depth along with the definitions which will help in coming chapter.

Definition 3.1.1. Let $\mathcal{S} = \mathbb{K}[\varsigma_1, ..., \varsigma_n]$, Υ be a finitely generated $\mathbb{Z}^n$-graded $\mathcal{S}$-module. For $t \in \Upsilon$, $t\mathbb{K}[Y]$ be the $\mathbb{K}$-subspace of a module Υ generated by the elements of the form tv for $v \in \mathbb{K}[Y]$ where $Y \subset \{\varsigma_1, ..., \varsigma_n\}$. $t\mathbb{K}[Y]$ is called Stanley space. The dimension of this $\mathbb{K}$-subspace is $|Y|$ if it is a free $\mathbb{K}[Y]$-module.
A Stanley decomposition of Υ as a $\mathbb{K}$-vector space is written as a finite direct sum of Stanley spaces

$$\chi : \Upsilon = \bigoplus_{j=1}^{s} t_j\mathbb{K}[Y_j].$$

The Stanley depth of a decomposition χ is the minimum of all the cardinalities of Y_j's. That is

$$\text{sdepth}(\chi) = \min\{|Y_j| : j = 1, ..., s\}.$$

and the Stanley depth of module Υ is

$$\text{sdepth}_{\mathcal{S}}(\Upsilon) = \max \{ \text{ sdepth}(\chi) : \chi \text{ is a Stanley decomposition of } \Upsilon\}.$$

Conjecture 3.1.1 ([21])**.** Let $\mathcal{S}$ be the ring and Υ be any $\mathbb{Z}^n$-graded $\mathcal{S}$-module, then Stanley proposed a conjecture that

$$\text{depth}(\Upsilon) \leq \text{sdepth}(\Upsilon).$$

Definition 3.1.2. An exact sequence is a sequence of homomorphism between objects either groups, rings or modules such that the kernel of one homomorphism equals to the image of the succeeding one. In particular, let $\mathcal{R}$ be the ring then the sequence of $\mathcal{R}$-homomorphism and $\mathcal{R}$-module

$$\cdots \longrightarrow \zeta_{j-1} \xrightarrow{e_j} \zeta_j \xrightarrow{e_{j+1}} \zeta_{j+1} \xrightarrow{e_{j+2}} \zeta_{j+2} \longrightarrow \cdots$$

If $\text{Ker}(e_{j+1}) = \text{Im}(e_j)$. The sequence is exact if it is exact at every j^{th} level.

Definition 3.1.3. A sequence of the form

$$0 \longrightarrow \zeta_0 \xrightarrow{e_1} \zeta_1 \xrightarrow{e_2} \zeta_2 \longrightarrow 0.$$

is said to be short exact sequence if e_1 is injective and e_2 is surjective with $\text{Im}(e_1) = \text{Ker}(e_2)$.

3.2 Method of computing Stanley depth

A recipe of computing the Stanley depth was introduced by Herzog et al. by using partially ordered sets [9]. Consider a square free monomial ideal $\mathcal{I}$ and let $\mathcal{I}$ is minimally generated by the set $\mathfrak{G}(\mathcal{I}) = \{v_1, \cdots, v_m\}$. Choose $h = (1, \cdots, 1)$, then the characteristic poset corresponding to $\mathcal{I}$ is defined to be

$$\mathfrak{L}_{\mathcal{I}}^{(1,\cdots,1)} = \{\eta \subset [n] : \eta \text{ contains supp}(v_j) \text{ for some } j\},$$

where

$$\text{supp}(v_j) = \{i : \varsigma_i | v_j\} \subseteq [n] := \{1, \cdots, n\}.$$

For every $\beta, \mathcal{I} \in \mathfrak{L}_{\mathcal{I}}^{(1,...,1)}$ with $\beta \subseteq \mathcal{I}$, define the interval $[\beta, \mathcal{I}]$ to be $\{\eta \in \mathfrak{L}_{\mathcal{I}}^{(1,...,1)} : \beta \subseteq \eta \subseteq \mathcal{I}\}$.
Let $\mathfrak{L} : \mathfrak{L}_{\mathcal{I}}^{(1,...,1)} = \bigcup_{j=1}^{k} [\xi_j, \eta_j]$ be a partition of $\mathfrak{L}_{\mathcal{I}}^{(1,...,1)}$, and for each j, define $w(j) \in$

$\{0,1\}^n$, be a n-tuple in which each entry at n^{th} place is either 0 or 1 such that $\mathrm{supp}(v^{w(j)}) = \eta_j$, then there is a Stanley decomposition of an ideal which is defined as

$$\chi(\mathfrak{L}) : \mathcal{I} = \bigoplus_{i=1}^{r} v^{w(j)} \mathbb{K}[\{\varsigma_k | k \in \xi_j\}].$$

Clearly, $\mathrm{sdepth}\chi(\mathfrak{L}) = \min\{|\xi_1|, \cdots, |\xi_r|\}$, and

$$\mathrm{sdepth}(\mathcal{I}) = \max\{\mathrm{sdepth}\chi(\mathfrak{L}) : \mathfrak{L} \text{ is a partition of } \mathfrak{L}_{\mathcal{I}}^{(1,\dots,1)}\}.$$

Example 15. For an ideal $\mathcal{I} = (\varsigma_1\varsigma_3, \varsigma_2\varsigma_4, \varsigma_1\varsigma_4\varsigma_5) \subset \mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5]$. Choose $h = (1, 1, 1, 1, 1)$

$$\begin{aligned}\mathfrak{L}_{\mathcal{I}}^{(1,1,1,1,1)} = \{&(1,0,1,0,0), (0,1,0,1,0), (1,1,1,0,0), (,1,0,1,0), (1,0,0,1,1),\\ &(1,0,1,0,1), (1,0,1,1,0), (0,1,1,1,0), (0,1,0,1,1), (1,1,1,1,0),\\ &(1,1,1,0,1), (1,1,0,1,1), (1,0,1,1,1), (0,1,1,1,1), (1,1,1,1,1)\}.\end{aligned}$$

Consider the following non overlapping intervals
Let $\beta = (1,1,0,1,0)$ and $\gamma = (1,1,1,1,1)$
then the interval, $[\beta,\gamma] = \{(1,1,1,1,0), (1,1,0,1,1), (1,1,0,1,0), (1,1,1,1,1)\}$.
Let $\beta = (1,1,1,0,0)$ and $\gamma = (1,1,1,0,1)$
then the interval, $[\beta,\gamma] = \{(1,1,1,0,0), (1,1,1,0,1)\}$.
Let $\beta = (1,0,1,0,0)$ and $\gamma = (1,0,1,1,1)$
then the interval, $[\beta,\gamma] = \{(1,0,1,0,0), (1,0,1,1,1), (1,0,1,0,1), (1,0,1,1,0)\}$.
Let $\beta = (0,1,0,1,0)$ and $\gamma = (0,1,1,1,1)$
then the interval, $[\beta,\gamma] = \{(0,1,0,1,0), (0,1,1,1,1), (0,1,1,1,0), (0,1,0,1,1)\}$.
Let $\beta = (1,0,0,1,1)$ and $\gamma = (1,0,0,1,1)$
then the interval, $[\beta,\gamma] = \{(1,0,0,1,1)\}$.
$\mathfrak{L}_{\mathcal{I}}^{(1,1,1,1,1)}$ is covered with these partitions. Hence

$$\begin{aligned}\chi_1 : \mathcal{I} = {}&[(0,1,0,1,0), (0,1,1,1,1)] \bigcup [(1,1,0,1,0), (1,1,1,1,1)] \bigcup\\ &[(1,0,1,0,0), (1,0,1,1,1)] \bigcup [(1,0,0,1,1), (1,0,0,1,1)] \bigcup\\ &[(1,1,1,0,0), (1,1,1,0,1)].\end{aligned}$$

So

$$\begin{aligned}\mathcal{I} = {}&\varsigma_2\varsigma_4\mathbb{K}[\varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5] \bigoplus \varsigma_1\varsigma_2\varsigma_4\mathbb{K}[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5] \bigoplus \varsigma_1\varsigma_3\\ &\mathbb{K}[\varsigma_1, \varsigma_3, \varsigma_4, \varsigma_5] \bigoplus \varsigma_1\varsigma_4\varsigma_5\mathbb{K}[\varsigma_1, \varsigma_4, \varsigma_5] \bigoplus \varsigma_1\varsigma_2\varsigma_3\mathbb{K}[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_5].\end{aligned}$$

Hence

$$\mathrm{sdepth}(\chi_1) = 3.$$

In a similar manner, we can construct different decompositions of $\mathcal{I}$. That is,

$$\chi_2 : \mathcal{I} = [(1,0,1,0,0),(1,1,1,1,1)] \bigcup [(0,1,0,1,0),(0,1,1,1,1)] \bigcup [(0,1,1,1,0),(0,1,1,1,1)] \bigcup [(1,1,0,1,0),(1,1,0,1,1)] \bigcup [(1,0,0,1,1),(1,0,0,1,1)] \bigcup [(0,1,0,1,1),(0,1,0,1,1)].$$

So

$$\mathcal{I} = \varsigma_1\varsigma_3\mathbb{K}[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5] \bigoplus \varsigma_2\varsigma_4\mathbb{K}[\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5] \bigoplus \varsigma_2\varsigma_3\varsigma_4 \mathbb{K}[\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5] \bigoplus \varsigma_1\varsigma_2\varsigma_4\mathbb{K}[\varsigma_1,\varsigma_2,\varsigma_4,\varsigma_5] \bigoplus \varsigma_1\varsigma_4\varsigma_5\mathbb{K}[\varsigma_1,\varsigma_4,\varsigma_5] \bigoplus \varsigma_2\varsigma_4\varsigma_5\mathbb{K}[\varsigma_2,\varsigma_4,\varsigma_5].$$

Hence

$$\operatorname{sdepth}(\chi_2) = 3.$$

And

$$\chi_3 : \mathcal{I} = [(1,0,1,1,0),(1,1,1,1,1)] \bigcup [(1,0,0,1,1),(1,0,0,1,1)] \bigcup [(1,1,0,1,0),(1,1,0,1,1)] \bigcup [(1,1,1,0,0),(1,1,1,0,1)] \bigcup [(1,0,1,0,0),(1,0,1,0,0)] \bigcup [(0,1,0,1,0),(0,1,1,1,1)] \bigcup [(1,0,1,0,1),(1,0,1,0,1)].$$

So

$$\mathcal{I} = \varsigma_1\varsigma_3\varsigma_4\mathbb{K}[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5] \bigoplus \varsigma_1\varsigma_4\varsigma_5\mathbb{K}[\varsigma_1,\varsigma_4,\varsigma_5] \bigoplus \varsigma_1\varsigma_2\varsigma_3 \mathbb{K}[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_5] \bigoplus \varsigma_1\varsigma_2\varsigma_4\mathbb{K}[\varsigma_1,\varsigma_2,\varsigma_4,\varsigma_5] \bigoplus \varsigma_1\varsigma_3\mathbb{K}[\varsigma_1,\varsigma_3] \bigoplus \varsigma_2\varsigma_4\mathbb{K}[\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5] \bigoplus \varsigma_1\varsigma_3\varsigma_5\mathbb{K}[\varsigma_1,\varsigma_3,\varsigma_5]$$

Hence

$$\operatorname{sdepth}(\mathfrak{D}_3) = 2.$$

Therefore

$$\operatorname{sdepth}(\mathcal{S}/\mathcal{K}) \geq \max\{3,3,2\}.$$

$$\operatorname{sdepth}(\mathcal{S}/\mathcal{K}) \geq 3.$$

Example 16. Let $\mathcal{S} = \mathbb{K}[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5]$ and an ideal

$$\mathcal{K} = (\varsigma_1\varsigma_2,\varsigma_1\varsigma_3,\varsigma_2\varsigma_3,\varsigma_2\varsigma_4,\varsigma_3\varsigma_4,\varsigma_3\varsigma_5,\varsigma_4\varsigma_5,\varsigma_4\varsigma_1,\varsigma_4\varsigma_2,\varsigma_5\varsigma_1,\varsigma_5\varsigma_2).$$

Choose $h = (1,1,1,1,1)$. The characteristic poset corresponding to $\mathcal{K}$ is

$$\begin{aligned}\mathfrak{L}_{\mathcal{K}}^{(1,1,1,1,1)} = \{&(1,1,0,0,0),(1,0,1,0,0),(0,1,1,0,0),(0,1,0,0,1),(0,0,1,1,0),\\ &(0,0,1,0,1),(0,0,0,1,1),(1,0,0,1,0),(1,0,0,0,1),(0,1,0,0,1),\\ &(1,1,1,0,0),(1,1,0,1,0),(1,1,0,0,1),(1,0,1,0,1),(1,0,0,1,1),\\ &(0,1,0,1,1),(0,0,1,1,1),(0,1,1,1,0),(0,1,1,0,1),(1,0,1,1,0),\\ &(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1),(1,0,1,1,1),(0,1,1,1,1),\\ &(1,1,1,1,1)\}.\end{aligned}$$

Then the characteristic poset associated to $\mathcal{S}/\mathcal{K}$ becomes

$$\begin{aligned}\mathfrak{L}_{\mathcal{S}/\mathcal{K}}^{(1,1,1,1,1)} = \{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,1),(0,0,0,1,0),\\ &(0,0,0,0,1)\}.\end{aligned}$$

Consider the following non overlapping intervals
Let $\beta = (1,0,0,0,0)$ and $\gamma = (1,0,0,0,0)$
then the interval, $[\beta,\gamma] = \{(1,0,0,0,0)\}$.
Let $\beta = (0,1,0,0,0)$ and $\gamma = (0,1,0,0,1)$
then the interval, $[\beta,\gamma] = \{(0,1,0,0,0)\}$.
Let $\beta = (0,0,1,0,0)$ and $\gamma = (0,0,1,0,0)$
then the interval, $[\beta,\gamma] = \{(0,0,1,0,0)\}$.
Let $\beta = (0,0,0,1,0)$ and $\gamma = (0,0,0,1,0)$
then the interval, $[\beta,\gamma] = \{(0,0,0,1,0)\}$.
Let $\beta = (0,0,0,0,0)$ and $\gamma = (0,0,0,0,1)$
then the interval, $[\beta,\gamma] = \{(0,0,0,0,0),(0,0,0,0,1)\}$.
$\mathfrak{L}_{\mathcal{S}/\mathcal{K}}^{(1,1,1,1,1)}$ is covered with these partitions. Hence

$$\begin{aligned}\chi^* : \mathcal{S}/\mathcal{K} = &[(1,0,0,0,0),(1,0,0,0,0)]\bigcup[(0,1,0,0,0),(0,1,0,0,0)]\bigcup\\ &[(0,0,1,0,0),(0,0,1,0,0)]\bigcup[(0,0,0,1,0),(0,0,0,1,0)]\bigcup\\ &[(0,0,0,0,0),(0,0,0,0,1)].\end{aligned}$$

So

$$\mathcal{S}/\mathcal{K} = \varsigma_1\mathbb{K}[\varsigma_1]\bigoplus\varsigma_2\mathbb{K}[\varsigma_2]\bigoplus\varsigma_3\mathbb{K}[\varsigma_4]\bigoplus\mathbb{K}[\varsigma_5].$$

Hence

$$\operatorname{sdepth}(\chi^*) = 1.$$

Similarly, we have another decompositions

$$\begin{aligned}\chi^{**} : \mathcal{S}/\mathcal{K} = &[(0,0,0,0,1),(0,0,0,0,1)]\bigcup[(0,1,0,0,0),(0,1,0,0,0)]\bigcup\\ &[(0,0,1,0,0),(0,0,1,0,0)]\bigcup[(0,0,0,1,0),(0,0,0,1,0)]\bigcup\\ &[(0,0,0,0,0),(1,0,0,0,0)].\end{aligned}$$

So

$$\mathcal{S}/\mathcal{K} = \mathbb{K}[\varsigma_1] \bigoplus \varsigma_2\mathbb{K}[\varsigma_2] \bigoplus \varsigma_3\mathbb{K}[\varsigma_4] \bigoplus \varsigma_5\mathbb{K}[\varsigma_5].$$

Hence

$$\text{sdepth}(\chi^{**}) = 1.$$

$$\begin{aligned}\chi^{***} : \mathcal{S}/\mathcal{K} = & [(0,0,0,0,1),(0,0,0,0,1)] \bigcup [(0,1,0,0,0),(0,1,0,0,0)] \bigcup \\ & [(0,0,1,0,0),(0,0,1,0,0)] \bigcup [(0,0,0,1,0),(0,0,0,1,0)] \bigcup \\ & [(1,0,0,0,0),(1,0,0,0,0)] \bigcup \\ & [(0,0,0,0,0),(0,0,0,0,0)].\end{aligned}$$

So

$$\mathcal{S}/\mathcal{K} = \varsigma_1\mathbb{K}[\varsigma_1] \bigoplus \varsigma_2\mathbb{K}[\varsigma_2] \bigoplus \varsigma_3\mathbb{K}[\varsigma_4] \bigoplus \varsigma_5\mathbb{K}[\varsigma_5] \bigoplus \mathbb{K}.$$

Hence

$$\text{sdepth}(\chi^{***}) = 0.$$

Therefore

$$\text{sdepth}(\mathcal{S}/\mathcal{K}) \geq \max\{1,1,0\}.$$

$$\text{sdepth}(\mathcal{S}/\mathcal{K}) \geq 1.$$

Example 17. Let $\mathcal{S} = \mathbb{K}[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4]$ and an ideal $\mathcal{K} = (\varsigma_1\varsigma_2,\varsigma_1\varsigma_3,\varsigma_2\varsigma_3,\varsigma_2\varsigma_4,\varsigma_3\varsigma_4)$. Choose $h = (1,1,1,1,1)$. The characteristic poset corresponding to $\mathcal{K}$ is

$$\begin{aligned}\mathfrak{L}_{\mathcal{K}}^{(1,1,1,1,1)} = \{ & (1,1,0,0),(1,0,1,0),(0,1,1,0),(0,1,0,1),(1,1,1,0), \\ & (1,1,0,1),(1,0,1,1),(0,1,1,1),(1,1,1,1),(0,0,1,1)\}.\end{aligned}$$

Then the characteristic poset associated to $\mathcal{S}/\mathcal{K}$ becomes

$$\begin{aligned}\mathfrak{L}_{\mathcal{S}/\mathcal{K}}^{(1,1,1,1,1)} = \{ & (0,0,0,0,0),(0,0,0,1),(0,1,0,0),(0,0,1,0),(1,0,0,0), \\ & (1,0,0,1)\}.\end{aligned}$$

Consider the following non overlapping intervals
Let $\beta = (0,0,0,0)$ and $\gamma = (1,0,0,1)$
then the interval, $[\beta,\gamma] = \{(0,0,0,0),(1,0,0,0),(0,0,0,1),(1,0,0,1)\}$.
Let $\beta = (0,0,1,0)$ and $\gamma = (0,0,1,0)$
then the interval, $[\beta,\gamma] = \{(0,0,1,0)\}$.
Let $\beta = (0,1,0,0)$ and $\gamma = (0,1,0,0)$

then the interval, $[\beta, \gamma] = \{(0,1,0,0)\}$.
Hence, $\mathfrak{L}_{\mathcal{S}/\mathcal{K}}^{(1,1,1,1,1)}$ is covered with these partitions.

$$\chi' : \mathcal{S}/\mathcal{K} = [(0,0,0,0),(1,0,0,1)] \bigcup [(0,1,0,0),(0,1,0,0)] \bigcup$$
$$[(0,0,1,0),(0,0,1,0)].$$

So

$$\mathcal{S}/\mathcal{K} = \mathbb{K}[\varsigma_1, \varsigma_4] \bigoplus \varsigma_2\mathbb{K}[\varsigma_2] \bigoplus \varsigma_3\mathbb{K}[\varsigma_3].$$

Hence

$$\text{sdepth}(\chi') = 1.$$

Similarly, another decomposition is

$$\chi'' : \mathcal{S}/\mathcal{K} = [(0,0,0,0),(0,0,0,1)] \bigcup [(1,0,0,0),(1,0,0,1)] \bigcup$$
$$[(0,0,1,0),(0,0,1,0)] \bigcup [(0,1,0,0),(0,1,0,0)].$$

So

$$\mathcal{S}/\mathcal{K} = \mathbb{K}[\varsigma_4] \bigoplus \varsigma_1\mathbb{K}[\varsigma_1, \varsigma_4] \bigoplus \varsigma_3\mathbb{K}[\varsigma_3] \bigoplus \varsigma_2\mathbb{K}[\varsigma_2].$$

Hence

$$\text{sdepth}(\chi'') = 1.$$

Therefore

$$\text{sdepth}(\mathcal{S}/\mathcal{K}) \geq \max\{1, 1\}.$$
$$\text{sdepth}(\mathcal{S}/\mathcal{K}) \geq 1.$$

3.3 Some known results and bounds for Stanley depth

This chapter provides a brief literature review. In the first part, we collected some bounds and values for depth and Sdepth of different classes of modules. In the second part, we identify some graphs for which these two invariants(depth and Stanley depth) have been calculated. The basic objective of this chapter is to motivate the reader to build a ground in order to understand further concepts in a smooth way.

Lemma 3.3.1. (Depth Lemma)
Let ζ_0, ζ_1 and ζ_2 be finitely generated modules over a Noetherian local ring.

$$0 \longrightarrow \zeta_0 \longrightarrow \zeta_1 \longrightarrow \zeta_2 \longrightarrow 0.$$

Then the above short exact sequence of modules satisfies the following three conditions

1. $\mathrm{depth}(\zeta_1) \geq \min\{\mathrm{depth}(\zeta_0), \mathrm{depth}(\zeta_2)\}$,
2. $\mathrm{depth}(\zeta_2) \geq \min\{\mathrm{depth}(\zeta_0) - 1, \mathrm{depth}(\zeta_1)\}$,
3. $\mathrm{depth}(\zeta_0) \geq \min\{\mathrm{depth}(\zeta_1), \mathrm{depth}(\zeta_2) + 1\}$.

Lemma 3.3.2 ([20])**.** Let ζ_0, ζ_1 and ζ_2 be $\mathbb{Z}^n$-graded $\mathcal{S}$-modules and the short exact sequence of the type

$$0 \longrightarrow \zeta_o \longrightarrow \zeta_1 \longrightarrow \zeta_2 \longrightarrow 0.$$

Then we have the following inequality

$$\mathrm{sdepth}(\zeta_1) \geq \min\{\mathrm{sdepth}(\zeta_0), \mathrm{sdepth}(\zeta_2)\}.$$

Theorem 3.3.3 ([1, Theorem 2.2])**.** Let $\mathcal{S}$ be a ring and an ideal $\digamma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_n) \subset \mathcal{S}$. Then

$$\mathrm{sdepth}(\digamma) = \lceil \frac{n}{2} \rceil.$$

Theorem 3.3.4 ([9])**.** Let $\mathcal{S} = \mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n]$ and $\mathcal{J}$ be the monomial ideal which is generated by minimal s elements then the lower bound for Stanley depth of module $\mathcal{S}/\mathcal{J}$ is

$$\mathrm{sdepth}(\mathcal{S}/\mathcal{J}) \geq n - s.$$

Theorem 3.3.5 ([23, Theorem 2.3])**.** Let $\mathcal{J}$ be a monomial ideal in $\mathcal{S}$ which is generated by minimal s number of elements then lower bound for Stanley depth of $\mathcal{J}$ is

$$\mathrm{sdepth}(\mathcal{J}) \geq n - \lfloor \frac{s}{2} \rfloor.$$

Theorem 3.3.6 ([20, Proposition 1.3])**.** Let $\mathcal{J} \subset \mathcal{S}$ be a monomial ideal in the ring of polynomials then

$$\mathrm{sdepth}_{\mathcal{S}}(\mathcal{J} : u) \geq \mathrm{sdepth}_{\mathcal{S}}(\mathcal{J}),$$

where monomial $u \notin \mathcal{J}$.

Corollary 3.3.1 ([3, proposition 2.7]). For a monomial ideal $\mathcal{J}$ in $\mathcal{S}$, we have

$$\operatorname{sdepth}_{\mathcal{S}}(\mathcal{S}/(\mathcal{J}:u)) \geq \operatorname{sdepth}_{\mathcal{S}}(\mathcal{S}/\mathcal{J}),$$

for monomial $u \notin \mathcal{J}$.

Lemma 3.3.7 ([20]). Let $\mathcal{S} = \mathcal{R}[\varsigma_{n+1}]$ be the ring of polynomial over $\mathcal{R}$ in the variable ς_{n+1} and $\mathcal{K} \subset \mathcal{J}$ be two monomial ideals of $\mathcal{R}$. Then

$$\operatorname{sdepth}(\mathcal{J}\mathcal{S}/\mathcal{K}\mathcal{S}) = \operatorname{sdepth}(\mathcal{J}/\mathcal{K}) + 1,$$
$$\operatorname{depth}(\mathcal{J}\mathcal{S}/\mathcal{K}\mathcal{S}) = \operatorname{depth}(\mathcal{J}/\mathcal{K}) + 1.$$

Theorem 3.3.8 ([12]). Let $\mathcal{K} \subset \mathcal{J} \subset \mathcal{S}$ be two monomial ideals in a polynomial ring $\mathcal{S}$, then

$$\operatorname{sdepth}(\sqrt{\mathcal{K}}/\sqrt{\mathcal{J}}) \geq \operatorname{sdepth}(\mathcal{K}/\mathcal{J}).$$

Theorem 3.3.9 ([8, Theorems 3.1 and 4.18]). Let $\Gamma = \mathcal{I}(\mathcal{G})$ be an edge ideal associated to a connected graph $\mathcal{G}$ and if $\Delta = \operatorname{diam}(\mathcal{G})$, then we have

$$\operatorname{depth}(\mathcal{S}/\Gamma), \operatorname{sdepth}(\mathcal{S}/\Gamma) \geq \lceil \frac{\Delta + 1}{3} \rceil.$$

Theorem 3.3.10 ([20, Theorem 3.1]). Let $\mathcal{J}' \subset \mathcal{R}' = \mathbb{K}[\varsigma_1, \cdots, \varsigma_r]$, $\mathcal{J}'' \subset \mathcal{R}'' = \mathbb{K}[z_1, \cdots, z_s]$ be two monomial ideals and $\mathcal{S} = \mathbb{K}[\varsigma_1, \cdots, \varsigma_r, z_1, \cdots, z_s]$. Then

$$\operatorname{sdepth}_{\mathcal{S}}\mathcal{S}/(\mathcal{J}'\mathcal{S} + \mathcal{J}''\mathcal{S}) \geq \operatorname{sdepth}_{\mathcal{S}'}\mathcal{R}'/\mathcal{J}' + \operatorname{sdepth}_{\mathcal{S}''}\mathcal{R}''/\mathcal{J}''.$$

Theorem 3.3.11 ([3, Proposition 1.1]). If $\mathcal{J}' \subset \mathcal{R}' = \mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_r]$ and $\mathcal{J}'' \subset \mathcal{R}'' = \mathbb{K}[\varsigma_{r+1}, \cdots, \varsigma_n]$ are the monomial ideals, for $1 \leq r < n$, then

$$\operatorname{depth}_{\mathcal{S}}(\mathcal{S}/(\mathcal{J}'\mathcal{S} + \mathcal{J}''\mathcal{S})) = \operatorname{depth}_{\mathcal{S}'}(\mathcal{R}'/\mathcal{J}') + \operatorname{depth}_{\mathcal{S}''}(\mathcal{R}''/\mathcal{J}'').$$

Proving Stanley's conjecture for a variety of classes of modules is an important task for the researchers so far and they have proved the truthfulness of this inequality for different cases. In the following two results we get the positive answer.

Theorem 3.3.12 ([17]). The Stanley's inequality is satisfied for intersections of three prime ideals.

Theorem 3.3.13 ([18]). The inequality given by Stanley is satisfied for intersections of four prime ideals.

We now need to transition to the second part of this section. The following results exploits active research related to edge ideals associated with different graphs. We introduce these bounds and values for constructing a ground that will facilitate the reader to make connection with the approach in the next section.

Lemma 3.3.14 ([15, Lemma 2.8]). Let $\mathfrak{Q}_1 = \mathcal{I}(P_\mathfrak{r})$ denotes the edge ideal corresponding to path graph then for $\mathfrak{r} \geq 2$, we have

$$\operatorname{depth}(\mathcal{S}/\mathfrak{Q}_1) = \lceil \frac{\mathfrak{r}}{3} \rceil.$$

Lemma 3.3.15 ([24, Lemma 4]). Let $\mathfrak{Q}_1 = \mathcal{I}(P_\mathfrak{r})$ denotes the edge ideal corresponding to path graph then for $\mathfrak{r} \geq 2$, we have

$$\operatorname{sdepth}(\mathcal{S}/\mathfrak{Q}_1) = \lceil \frac{\mathfrak{r}}{3} \rceil.$$

Proposition 3.3.1 ([4, Proposition 1.3]). Let $\mathfrak{Q}_2 = \mathcal{I}(C_\mathfrak{r})$ denotes the edge ideal corresponding to cycle graph then for $\mathfrak{r} \geq 3$, we have

$$\operatorname{depth}(\mathcal{S}/\mathfrak{Q}_2) = \lceil \frac{\mathfrak{r}-1}{3} \rceil.$$

Theorem 3.3.16 ([4, Theorem 1.9]). Let $\mathfrak{Q}_2 = \mathcal{I}(C_\mathfrak{r})$ denotes the edge ideal corresponding to cycle graph then for $\mathfrak{r} \geq 3$, we have

1. $\operatorname{sdepth}(\mathcal{S}/\mathfrak{Q}_2) = \lceil \frac{\mathfrak{r}-1}{3} \rceil$, if $\mathfrak{r} \equiv 0, 2 (\bmod\ 3)$.
2. $\operatorname{sdepth}(\mathcal{S}/\mathfrak{Q}_2) \leq \lceil \frac{\mathfrak{r}-1}{3} \rceil$, if $\mathfrak{r} \equiv 1 (\bmod\ 3)$.

Now we give some results for depth and Sdepth of edge ideal associated with two interesting graphs that will frequently appear within next section.

Theorem 3.3.17 ([11, Theorem 3.8]). Let $\mathfrak{Q}^* = \mathcal{I}(P_\mathfrak{r}^l)$ denotes the edge ideal corresponding to l^{th} power of path graph then for $\mathfrak{r} \geq 2$ and $l \in \mathbb{Z}^+ \cup \{0\}$, we have $\operatorname{depth}(\mathcal{S}/\mathfrak{Q}^*) = \lceil \frac{\mathfrak{r}}{2l+1} \rceil$.

Theorem 3.3.18 ([11, Theorem 3.14]). Let $\mathfrak{Q}^* = \mathcal{I}(P_\mathfrak{r}^l)$ denotes the edge ideal corresponding to l^{th} power of path graph then for $\mathfrak{r} \geq 2$ and $l \in \mathbb{Z}^+ \cup \{0\}$, we have $\operatorname{sdepth}(\mathcal{S}/\mathfrak{Q}^*) = \lceil \frac{\mathfrak{r}}{2l+1} \rceil$.

Theorem 3.3.19 ([11, Theorem 4.5]). Let $\mathfrak{Q}^{**} = \mathcal{I}(C_\mathfrak{r}^l)$ denotes the edge ideal corresponding to l^{th} power of cycle graph then for $\mathfrak{r} \geq 3$. Then
for $\mathfrak{r} \leq 2l+1$, $\operatorname{depth}(\mathcal{S}/\mathfrak{Q}^{**}) = 1$,
for $\mathfrak{r} \geq 2l+2$, $\operatorname{depth}(\mathcal{S}/\mathfrak{Q}^{**}) \geq \lceil \frac{\mathfrak{r}-l}{2l+1} \rceil$. .

Theorem 3.3.20 ([11, Theorem 4.7]). Let $\mathfrak{Q}^{**} = \mathcal{I}(C_\mathfrak{r}^l)$ denotes the edge ideal corresponding to l^{th} power of cycle graph then for $\mathfrak{r} \geq 3$. Then
for $\mathfrak{r} \leq 2l+1$, $\operatorname{sdepth}(\mathcal{S}/\mathfrak{Q}^{**}) = 1$,
for $\mathfrak{r} \geq 2l+2$, $\operatorname{sdepth}(\mathcal{S}/\mathfrak{Q}^{**}) \geq \lceil \frac{\mathfrak{r}-l}{2l+1} \rceil$.

Chapter 4

Depth and Stanley Depth of Tadpole Graph

4.1 Depth and Stanley Depth of quotient of a Tadpole Graph

In this section, we find the accurate values for depth and Sdepth of the factor of edge ideal corresponding to a Tadpole graph. For this graph, the values of these two invariants coincide.
Let $n \geq 3$ and $m \geq 1$. Throughout this section, we set

$$\Omega = \Omega_{n,m} := \mathbb{K}[\varsigma_1, \cdots, \varsigma_{n-1}, \varsigma_n, \varsigma_{n+1}, \cdots, \varsigma_{n+m-1}, \varsigma_{n+m}].$$

Definition 4.1.1. A Tadpole graph is a planner graph which consist of path graph on m and cycle graph on n vertices vertices connected with the an edge. Total number of vertices and edges of a graph is $n+m$ and it is symbolically represented by $T_{n,m}$. The ideal $\daleth = \daleth(T_{n,m})$ denotes the edge ideal associated with Tadpole graph.

$$\daleth(T_{n,m}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_n\varsigma_{n+1}, \cdots, \varsigma_{n+m-1}\varsigma_{n+m}).$$

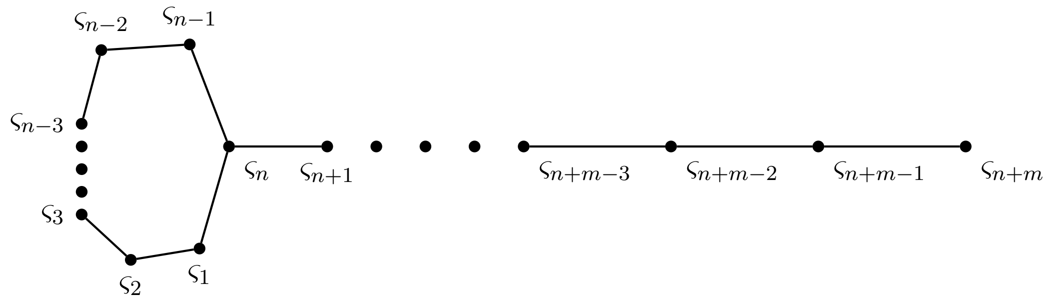

Figure 4.1: $T_{n,m}$ Tadpole graph.

Example 18. Consider a Tadpole graph with cycle on 5 vertices and path on 2 vertices. Then the associated edge ideal is

$$\daleth(T_{5,2}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \varsigma_3\varsigma_4, \varsigma_4\varsigma_5, \varsigma_5\varsigma_1, \varsigma_5\varsigma_6, \varsigma_6\varsigma_7).$$

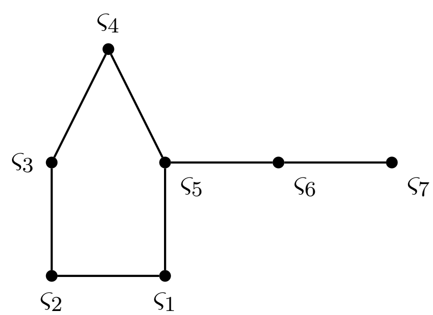

Figure 4.2: $T_{5,2}$ Tadpole graph.

Before general proof, we prove the results for initial cases.

Lemma 4.1.1. $\operatorname{depth}(\Omega/\daleth(T_{n,1})) = \operatorname{sdepth}(\Omega/\daleth(T_{n,1})) = \lceil \frac{n}{3} \rceil$.

Proof. Consider a short exact sequence

$$0 \longrightarrow \Omega/(\daleth : \varsigma_n) \xrightarrow{\varsigma_n} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_n) \longrightarrow 0.$$

Then by depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth) \geq \min\{\operatorname{depth}(\Omega/(\daleth : \varsigma_n)), \operatorname{depth}(S/(\daleth, \varsigma_n))\}.$$

$$(\daleth, \varsigma_n) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_n).$$

Here, we have

$$(\daleth, \varsigma_n) = (\daleth(P_{n-1}), \varsigma_n).$$

Similarly

$$(\daleth : \varsigma_n) = (\varsigma_3\varsigma_4, \cdots, \varsigma_{n-3}\varsigma_{n-4}, \varsigma_{n-2}, \varsigma_{n-1}, \varsigma_1, \varsigma_2, \varsigma_{n+1}).$$

Here $\Omega = \Omega_{n,1} = \mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}]$. So

$$\begin{aligned}\Omega/(\daleth, \varsigma_n) &= \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}]}{(\daleth, \varsigma_n)} \\ &= \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_n)} \\ &\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n-1}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1})}[\varsigma_{n+1}] \cong \frac{S_{n-1}}{\daleth(P_{n-1})}[\varsigma_{n+1}].\end{aligned}$$

By [9, Lemma 3.6], we get

$$\operatorname{depth}(\Omega/(\daleth, \varsigma_n)) = \operatorname{depth}(\frac{\Omega_{n-1}}{\daleth(P_{n-1})}) + 1.$$

Now by using [15, Lemma 2.8], we have

$$\operatorname{depth}(\Omega/(\daleth, \varsigma_n)) = \lceil\frac{n-1}{3}\rceil + 1 = \lceil\frac{n+2}{3}\rceil.$$

Similarly,

$$\Omega/(\daleth : \varsigma_n) = \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n-3}]}{\daleth(P_{n-3})}[\varsigma_n].$$

Again using [15, Lemma 4], we have

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_n)) = \operatorname{depth}(\frac{\Omega_{n-3}}{\daleth(P_{n-3})}) + 1.$$

So

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_n)) = \lceil\frac{n-3}{3}\rceil + 1 = \lceil\frac{n}{3}\rceil.$$

Hence

$$\operatorname{depth}(\Omega/\daleth) = \lceil\frac{n}{3}\rceil.$$

Now we can find Stanley depth by the analogous work as above. We replace Lemma 3.3.2 from depth lemma for applying on exact sequences. So, we have

$$\operatorname{sdepth}(\Omega/\daleth(T_{n,1})) \geq \min\{\operatorname{sdepth}(\Omega/(\daleth(T_{n,1}), \varsigma_n), \operatorname{sdepth}(\Omega/(\daleth(T_{n,1}) : \varsigma_n))\}.$$

$$\operatorname{sdepth}(\Omega/\daleth(T_{n,1}) \geq \lceil\frac{n}{3}\rceil.$$

As by Corollary 3.3.1, we know that

$$\mathrm{sdepth}(\Omega/\daleth) \leq \mathrm{sdepth}(\Omega/(\daleth : y)).$$

So

$$\mathrm{sdepth}(\Omega/\daleth(T_{n,1}) \leq \lceil \frac{n}{3} \rceil.$$

Hence

$$\mathrm{sdepth}(\Omega/\daleth(T_{n,1}) = \lceil \frac{n}{3} \rceil.$$

□

Lemma 4.1.2. $\mathrm{depth}(\Omega/\daleth(T_{n,2})) = \mathrm{sdepth}(\Omega/(\daleth(T_{n,2})) = \lceil \frac{n+2}{3} \rceil$.

Proof. Consider a short exact sequence

$$0 \longrightarrow S/(\daleth : \varsigma_{n+1}) \xrightarrow{\varsigma_{n+1}} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_{n+1}) \longrightarrow 0.$$

Then by depth lemma, we have

$$\mathrm{depth}(\Omega/I) \geq \min\{\mathrm{depth}(\Omega/(\daleth : \varsigma_{n+1}), \mathrm{depth}(\Omega/(\daleth, \varsigma_{n+1})\}.$$

So, we have

$$(\daleth, \varsigma_{n+1}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-1}\varsigma_n, \varsigma_1\varsigma_n, \varsigma_{n+1}).$$
$$(\daleth, \varsigma_{n+1}) = (\daleth(C_n), \varsigma_{n+1}).$$

Similarly

$$(\daleth : \varsigma_n) = (\varsigma_1\varsigma_2, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_n, \varsigma_{n+2}).$$

Here $\Omega = \Omega_{n,2} = \mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}]$. So

$$\begin{aligned}
\Omega/(\daleth, \varsigma_{n+1}) &= \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}]}{(\daleth, \varsigma_{n+1})} \\
&= \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_{n-1}, \varsigma_1\varsigma_n)} \\
&\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n]}{\daleth(C_n)}[\varsigma_{n+2}].
\end{aligned}$$

Hence by [9, lemma 3.6], we get

$$\mathrm{depth}(\Omega/(\daleth, \varsigma_{n+1})) = \mathrm{depth}(\frac{\Omega_n}{\daleth(C_n)}) + 1.$$

So by using [4, proposition 1.3], we have

$$\mathrm{depth}(\Omega/(\daleth, \varsigma_{n+1})) = \lceil \frac{n-1}{3} \rceil + 1 = \lceil \frac{n+2}{3} \rceil.$$

Similarly

$$\Omega/(\daleth : \varsigma_{n+1}) \cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n-1}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1})}[\varsigma_{n+1}].$$

Using [15, Lemma 2.8], we get

$$\mathrm{depth}(\Omega/(\daleth : \varsigma_{n+1})) = \mathrm{depth}(\frac{\Omega_{n-1}}{\daleth(P_{n-1})}) + 1.$$

So

$$\mathrm{depth}(\Omega/(\daleth : \varsigma_{n+1})) = \lceil \frac{n-1}{3} \rceil + 1 = \lceil \frac{n+2}{3} \rceil.$$

Hence

$$\mathrm{depth}(\Omega/\daleth) = \lceil \frac{n+2}{3} \rceil.$$

Similarly

$$\mathrm{sdepth}(\Omega/\daleth) = \lceil \frac{n+2}{3} \rceil.$$

□

Lemma 4.1.3. $\mathrm{depth}(\Omega/\daleth(T_{n,3})) = \mathrm{sdepth}(\Omega/\daleth(T_{n,3})) = \lceil \frac{n+2}{3} \rceil$.

Proof. Consider a short exact sequence

$$0 \longrightarrow \Omega/(\daleth : \varsigma_{n+2}) \xrightarrow{\varsigma_{n+2}} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_{n+2}) \longrightarrow 0.$$

Then by depth lemma, we have

$$\mathrm{depth}(\Omega/\daleth) \geq \min\{\mathrm{depth}(\Omega/(\daleth : \varsigma_{n+2})), \mathrm{depth}(\Omega/(\daleth, \varsigma_{n+2}))\}.$$

So we have

$$(\daleth, \varsigma_{n+2}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-1}\varsigma_n, \varsigma_{n+1}\varsigma_n).$$

Similarly

$$(\daleth : \varsigma_{n+2}) = (\varsigma_1\varsigma_2, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+3}).$$

Here $\Omega = \Omega_{n,3} = \mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}]$. So

$$\begin{aligned}
\Omega/(\daleth, \varsigma_{n+2}) &= \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}]}{(\daleth, \varsigma_{n+2})} \\
&= \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_{n-1}, \varsigma_{n+1}\varsigma_n)} \\
&\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n+1}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_{n+1})}[\varsigma_{n+3}].
\end{aligned}$$

Hence by [9, lemma 3.6] and [Lemma 4.1.3], we get

$$\operatorname{depth}(\Omega/(\daleth,\varsigma_{n+2}))=\operatorname{depth}(\frac{\Omega_{n,2}}{\daleth(T_{n,2})})+1.$$

So

$$\operatorname{depth}(\Omega/(\daleth,\varsigma_{n+2}))=\lceil\frac{n+2}{3}\rceil+1=\lceil\frac{n+5}{3}\rceil.$$

Similarly

$$\Omega/(\daleth:\varsigma_{n+2})=\frac{\mathbb{K}[\varsigma_1,\varsigma_2,\cdots,\varsigma_{n-1},\varsigma_n]}{(\varsigma_1\varsigma_2,\varsigma_2\varsigma_3,\cdots,\varsigma_n\varsigma_1)}[\varsigma_{n+2}].$$

By using [4, proposition 1.3] and [9, lemma 3.6], we get

$$\operatorname{depth}(\Omega/(\daleth:\varsigma_{n+2}))=\operatorname{depth}(\frac{\Omega_n}{\daleth(C_n)})+1.$$

So

$$\operatorname{depth}(\Omega/(\daleth:\varsigma_{n+2}))=\lceil\frac{n-1}{3}\rceil+1=\lceil\frac{n+2}{3}\rceil.$$

Thus

$$\operatorname{depth}(\Omega/\daleth)=\lceil\frac{n+2}{3}\rceil.$$

In a similar way, we get

$$\operatorname{sdepth}(\Omega/\daleth)=\lceil\frac{n+2}{3}\rceil.$$

□

Theorem 4.1.4. Let $m\geq 4$ and $n\geq 5$. If $n\equiv 2(\text{mod }3)$, then

$$\operatorname{depth}(\Omega/\daleth(T_{n,m}))=\lceil\frac{n+m}{3}\rceil.$$

Otherwise

$$\operatorname{depth}(\Omega/(\daleth(T_{n,m}))=\lceil\frac{n+m-1}{3}\rceil.$$

Proof. We will prove this by induction on m. Consider a short exact sequence

$$0\longrightarrow\Omega/(\daleth:\varsigma_{n+m-1})\xrightarrow{\varsigma_{n+m-1}}\Omega/\daleth\longrightarrow\Omega/(\daleth,\varsigma_{n+m-1})\longrightarrow 0.$$

By depth lemma, we know that

$$\operatorname{depth}(\Omega/\daleth)\geq\min\{\operatorname{depth}(\Omega/(\daleth:\varsigma_{n+m-1})),\operatorname{depth}(\Omega/(\daleth,\varsigma_{n+m-1}))\}.$$

For this, we have

$$\daleth(T_{n,m})=(\varsigma_1\varsigma_2,\varsigma_2\varsigma_3,\cdots,\varsigma_n\varsigma_1,\varsigma_n\varsigma_{n+1},\cdots,\varsigma_{n+m-1}\varsigma_{n+m}).$$

$$(\daleth, \varsigma_{n+m-1}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_n\varsigma_{n+1}, \cdots, \varsigma_{n+m-3}\varsigma_{n+m-2}, \varsigma_{n+m-1}).$$

$$(\daleth : \varsigma_{n+m-1}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_n\varsigma_{n+1}, \cdots, \varsigma_{n+m-4}\varsigma_{n+m-3}, \varsigma_{n+m-2}, \varsigma_{n+m}).$$

Here
$\Omega = \Omega_{n,m} = \mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \cdots, \varsigma_{n+m-1}, \varsigma_{n+m}]$.

$$\Omega/(\daleth, \varsigma_{n+m-1}) \cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n+m-2}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n+m-3}\varsigma_{n+m-2})}[\varsigma_{n+m}].$$

$$\text{depth}(\Omega/(\daleth, \varsigma_{n+m-1})) = \text{depth}(\frac{\Omega_{n,m-2}}{\daleth(T_{n,m-2})}) + 1.$$

Apply induction on m. Here we have two cases.
If $n \equiv 2(\text{mod } 3)$

$$\text{depth}(\Omega/(\daleth, \varsigma_{n+m-1})) = \lceil\frac{n + (m-2)}{3}\rceil + 1,$$

$$\text{depth}(\Omega/(\daleth, \varsigma_{n+m-1})) = \lceil\frac{n+m+1}{3}\rceil.$$

Otherwise

$$\text{depth}(\Omega/(\daleth, \varsigma_{n+m-1})) = \lceil\frac{n + (m-2) - 1}{3}\rceil + 1,$$

$$\text{depth}(\Omega/(\daleth, \varsigma_{n+m-1})) = \lceil\frac{n+m}{3}\rceil.$$

Now

$$\Omega/(\daleth : \varsigma_{n+m-1}) = \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n+m-3}]}{(\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n+m-4}\varsigma_{n+m-3})}[\varsigma_{n+m-1}].$$

So

$$\text{depth}(\Omega/(\daleth : \varsigma_{n+m-1})) = \text{depth}(\Omega_{n,m-3}/(\daleth(T_{n,m-3})) + 1.$$

Again apply induction on m.
if $n \equiv 2(\text{mod } 3)$

$$\text{depth}(\Omega/(\daleth : \varsigma_{n+m-1})) = \lceil\frac{n + (m-3)}{3}\rceil + 1.$$

$$\text{depth}(\Omega/(\daleth : \varsigma_{n+m-1})) = \lceil\frac{n+m}{3}\rceil.$$

Otherwise

$$\text{depth}(\Omega/(\daleth : \varsigma_{n+m-1})) = \lceil\frac{n + (m-3) - 1}{3}\rceil + 1.$$

$$\text{depth}(\Omega/(\daleth : \varsigma_{n+m-1})) = \lceil\frac{n+m-1}{3}\rceil.$$

Now for
Case 1: When $n \equiv 2(\text{mod } 3)$

$$\text{depth}(\Omega/(\daleth(T_{n,m})) \geq min\{\lceil\frac{n+m+1}{3}\rceil, \lceil\frac{n+m}{3}\rceil\}.$$

So by Corollary 3.3.1, we get

$$\operatorname{depth}(\Omega/(\daleth(T_{n,m})) = \lceil\frac{n+m}{3}\rceil.$$

Case 2: Otherwise

$$\operatorname{depth}(\Omega/(\daleth(T_{n,m})) \geq \min\{\lceil\frac{n+m-1}{3}\rceil, \lceil\frac{n+m}{3}\rceil\}.$$

So by Corollary 3.3.1, we get

$$\operatorname{depth}(\Omega/(\daleth(T_{n,m})) = \lceil\frac{n+m-1}{3}\rceil.$$

□

Theorem 4.1.5. Let $m \geq 4$ and $n \geq 5$. If $n \equiv 2 (\mathrm{mod}\ 3)$, then

$$\operatorname{sdepth}(\Omega/(\daleth(T_{n,m})) = \lceil\frac{n+m}{3}\rceil,$$

otherwise

$$\operatorname{sdepth}(\Omega/(\daleth(T_{n,m})) = \lceil\frac{n+m-1}{3}\rceil.$$

Proof. We will proof this by analogous work as Theorem 4.1.4 for finding Stanley depth of a graph. We replace Lemma 3.3.2 from Depth Lemma for applying on the exact sequences and the we get the required result by applying induction on m.
By Lemma 3.3.2, we have

$$\operatorname{sdepth}(\Omega/\daleth(T_{n,m})) \geq \min\{\operatorname{sdepth}(\Omega/(\daleth, \varsigma_{n+m-1})), \operatorname{sdepth}(\Omega/(\daleth : \varsigma_{n+m-1}))\}.$$

So, when $n \equiv 2 (\mathrm{mod}\ 3)$

$$\operatorname{sdepth}(\Omega/\daleth(T_{n,m})) = \lceil\frac{n+m}{3}\rceil.$$

Otherwise,

$$\operatorname{sdepth}(\Omega/(\daleth(T_{n,m})) = \lceil\frac{n+m-1}{3}\rceil.$$

□

4.2 Depth and Stanley Depth of Quotient of a Square of a Tadpole Graph

In the following section, we find the lower bound for depth and Sdepth of quotient of ideal associated with a square of a Tadpole graph.

Definition 4.2.1. Let $\mathcal{G}$ be a simple graph then the square of a graph is represented as $\mathcal{G}^2$ on the same set of vertices of $\mathcal{G}$, in which each pair of vertices having distance of 2 or less in $\mathcal{G}$ is linked by an edge. For $m \geq 1$ and $n \geq 3$, the edge ideal associated with square of a Tadpole graph is denoted by $(\daleth(T^2_{n,m})$.

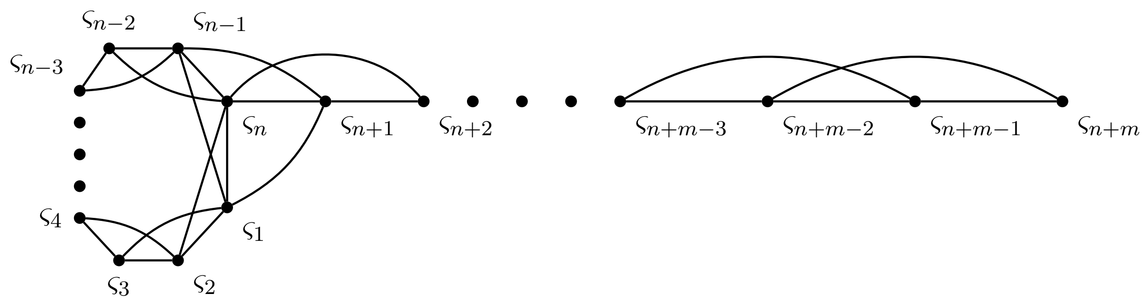

Figure 4.3: $T^2_{n,m}$ Square Tadpole graph.

Example 19. A square of a Tadpole graph with path of 4 vertices and cycle of 6 vertices, denoted by $T^2_{6,4}$. The associated edge ideal is

$$\daleth(T^2_{6,4}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_6\varsigma_1, \varsigma_6\varsigma_7, \cdots, \varsigma_1\varsigma_3, \varsigma_1\varsigma_5, \cdots, \varsigma_8\varsigma_{10}).$$

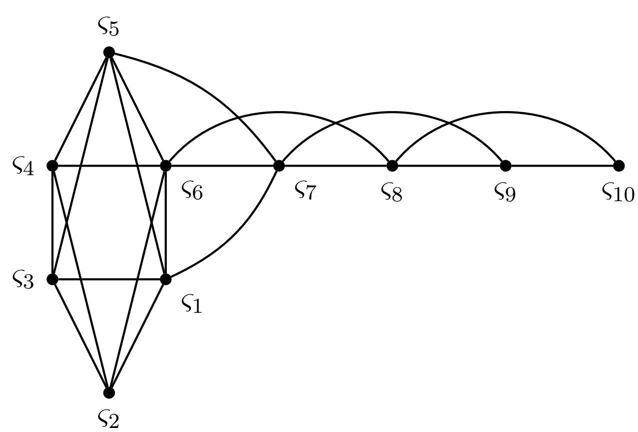

Figure 4.4: $T^2_{6,4}$ Square Tadpole graph.

Before general proof, we prove the results for initial cases.

Lemma 4.2.1. $\operatorname{depth}(\Omega/\daleth(T^2_{n,1})), \operatorname{sdepth}(\Omega/\daleth(T^2_{n,1})) \geq \lceil \frac{n}{5} \rceil$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega/(\daleth : \varsigma_{n-1}) \xrightarrow{\varsigma_{n-1}} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_{n-1}) \longrightarrow 0.$$

By depth lemma, we have

$$\operatorname{depth}(\Omega/(\daleth) \geq \min\{\operatorname{depth}(\Omega/(\daleth : \varsigma_{n-1})), \operatorname{depth}(\Omega/(\daleth, \varsigma_{n-1}))\}.$$

Here we have $\Omega = \Omega_{n,1} = [\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \varsigma_{n+1}]$ and

$$\daleth = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_{n-1}\varsigma_1, \varsigma_n\varsigma_{n+1}, \varsigma_{n-1}\varsigma_{n+1}, y_1\varsigma_{n+1}).$$

For this, we have

$$(\daleth : \varsigma_{n-1}) = (\daleth(P^2_{n-5}), \varsigma_1, \varsigma_{n-2}, \varsigma_{n-3}, \varsigma_n, \varsigma_{n+1}).$$

$$\Omega/(\daleth : \varsigma_{n-1}) \cong \frac{\mathbb{K}[\varsigma_2, \varsigma_3, \cdots, \varsigma_{n-4}]}{(\daleth(P^2_{n-5}), \varsigma_2, \varsigma_{n-2}, \varsigma_{n-3}, \varsigma_{n+1})} \cong \frac{\mathbb{K}[\varsigma_2, \varsigma_3, \cdots, \varsigma_{n-4}]}{\daleth(P^2_{n-5})}[\varsigma_{n-1}].$$

Here by [9, lemma 3.6] and [14, Theorem 3.1], we get

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_{n-1})) = \operatorname{depth}(\Omega_{n-5}/\daleth(P^2_{n-5})) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_{n-1})) = \lceil \frac{n-5}{5} \rceil + 1 = \lceil \frac{n}{5} \rceil.$$

Again construct a short exact sequence and let $\daleth^* := (\daleth(T^2_{n,1}), \varsigma_{n-1})$. Here we have,

$$(\daleth, \varsigma_{n-1}) = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_{n+1}\varsigma_1, \varsigma_{n-2}\varsigma_n, \varsigma_n\varsigma_2, \varsigma_{n-1}).$$

Consider the following short exact sequence

$$0 \longrightarrow \Omega/(\daleth^* : \varsigma_n) \xrightarrow{\varsigma_n} \Omega/\daleth^* \longrightarrow \Omega/(\daleth^*, \varsigma_n) \longrightarrow 0.$$

By using depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth^*) \geq \min\{\operatorname{depth}(\Omega/(\daleth^* : \varsigma_n)), \operatorname{depth}(\Omega/(\daleth^*, \varsigma_n))\}.$$

$$(\daleth^* : \varsigma_n) = (\daleth(P^2_{n-5}), \varsigma_1, \varsigma_2, \varsigma_{n-2}, \varsigma_{n+1}, \varsigma_{n-1}).$$

And

$$S/(\daleth^* : \varsigma_n) \cong \frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}[\varsigma_n].$$

Again using [14, Theorem 3.1], we get

$$\operatorname{depth}(\Omega/(\daleth^* : \varsigma_n)) = \operatorname{depth}(\frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth^* : \varsigma_n)) = \lceil \frac{n-5}{5} \rceil + 1 = \lceil \frac{n}{5} \rceil.$$

Now take

$$(\daleth^*, \varsigma_n) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_1\varsigma_{n+1}, \varsigma_{n-3}\varsigma_{n-2}, \varsigma_{n-1}, \varsigma_n).$$

Again construct a short exact sequence and let $\daleth^{**} := (\daleth^*, \varsigma_n)$. Consider the following short exact sequence

$$0 \longrightarrow \Omega/(\daleth^{**} : \varsigma_1) \xrightarrow{\varsigma_1} \Omega/\daleth^{**} \longrightarrow \Omega/(\daleth^{**}, \varsigma_1) \longrightarrow 0.$$

By using depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth^{**}) \geq \min\{\operatorname{depth}(\Omega/(\daleth^{**} : \varsigma_1)), \operatorname{depth}(\Omega/(\daleth^{**}, \varsigma_1))\}.$$

$$(\daleth^{**} : \varsigma_1) = (\daleth(P^2_{n-5}), \varsigma_2, \varsigma_3, \varsigma_n, \varsigma_{n-1}, \varsigma_{n+1}).$$

And

$$\Omega/(\daleth^{**} : \varsigma_1) \cong \frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}[\varsigma_1].$$

$$\operatorname{depth}(\Omega/(\daleth^{**} : \varsigma_1)) = depth(\frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth^{**} : \varsigma_1)) = \lceil \frac{n-5}{5} \rceil + 1 = \lceil \frac{n}{5} \rceil.$$

$$(\Omega/(\daleth^{**}, \varsigma_1)) \cong \frac{\Omega_{n-3}}{\daleth(P^2_{n-3})}[\varsigma_{n+1}].$$

By using [14] and [9, lemma 3.6], we get

$$\operatorname{depth}(\Omega/(\daleth^{**}, \varsigma_1)) = \operatorname{depth}(\frac{\Omega_{n-3}}{\daleth(P^2_{n-3})}) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth^{**}, \varsigma_1)) = \lceil \frac{n-3}{5} \rceil + 1 = \lceil \frac{n+2}{5} \rceil.$$

So

$$\operatorname{depth}(\Omega/\daleth^{**}) = \lceil \frac{n}{5} \rceil.$$

Similarly

$$\operatorname{depth}(\Omega/\daleth^*) \geq \{\lceil \frac{n}{5} \rceil, \lceil \frac{n}{5} \rceil\}.$$

Hence
$$\operatorname{depth}(\Omega/\daleth) = \lceil \frac{n}{5} \rceil.$$
Now for finding Stanley depth of a graph, we proceed with the same arguments as above. We replace Lemma 3.3.2 from depth lemma and Theorem 3.3.10 from Theorem 3.3.11. We get our required result.
i.e.
$$\operatorname{sdepth}(\Omega/\daleth) \geq \lceil \frac{n}{5} \rceil.$$
□

Lemma 4.2.2. $\operatorname{depth}(\Omega/\daleth(T^2_{n,2})), \operatorname{sdepth}(\Omega/\daleth(T^2_{n,2})) \geq \lceil \frac{n}{5} \rceil$.

Proof. Consider the short exact sequence
$$0 \longrightarrow \Omega/(\daleth : \varsigma_n) \xrightarrow{\varsigma_n} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_n) \longrightarrow 0.$$
By depth lemma, we have
$$\operatorname{depth}(\Omega/\daleth) \geq \min\{\operatorname{depth}(\Omega/(\daleth : \varsigma_n)), \operatorname{depth}(\Omega/(\daleth, \varsigma_n))\}.$$
Here we have $\Omega = \Omega_{n,2} = [\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}]$ and
$$\daleth = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_{n-1}\varsigma_1, \varsigma_n\varsigma_{n+1}, \varsigma_{n-1}\varsigma_{n+1}, x_1\varsigma_{n+1}, \varsigma_{n+1}\varsigma_{n+2}, \varsigma_n\varsigma_{n+2}).$$
For this, we have
$$(\daleth : \varsigma_n) = (\daleth(P^2_{n-5}), \varsigma_1, \varsigma_2, \varsigma_{n-1}, \varsigma_{n-2}, \varsigma_{n+1}, \varsigma_{n+2}).$$
$$\Omega/(\daleth : \varsigma_n) \cong \frac{\mathbb{K}[\varsigma_3, \varsigma_4, \cdots, \varsigma_{n-3}]}{(\daleth(P^2_{n-5}), \varsigma_1, \varsigma_2, \varsigma_{n-1}, \varsigma_{n-2}, \varsigma_{n+1}, \varsigma_{n+2})} \cong \frac{\mathbb{K}[\varsigma_3, \varsigma_4, \cdots, \varsigma_{n-3}]}{\daleth(P^2_{n-5})}[\varsigma_n].$$
Here by using [9, Lemma 3.6] and [14], we have
$$\operatorname{depth}(\Omega/(\daleth : \varsigma_n)) = \operatorname{depth}(\Omega_{n-5}/\daleth(P^2_{n-5})) + 1.$$
$$\operatorname{depth}(\Omega/(\daleth : \varsigma_n)) = \lceil \frac{n-5}{5} \rceil + 1 = \lceil \frac{n}{5} \rceil.$$
Again construct a short exact sequence and let $I' := (\daleth(T^2_{n,2}), \varsigma_n)$. Here, we have
$$((\daleth, \varsigma_n) = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_{n-1}\varsigma_1, \varsigma_{n+1}\varsigma_1, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_{n+1}\varsigma_{n+2}, \varsigma_n).$$

Consider the following short exact sequence
$$0 \longrightarrow \Omega/((\daleth^* : \varsigma_{n+1}) \xrightarrow{\varsigma_{n+1}} \Omega/(\daleth^* \longrightarrow \Omega/(\daleth^*, \varsigma_{n+1}) \longrightarrow 0.$$

By using depth lemma, we have

$$\mathrm{depth}(\Omega/\daleth^*) \geq \min\{\mathrm{depth}(\Omega/(\daleth^* : \varsigma_{n+1})), \mathrm{depth}(\Omega/(\daleth^*, \varsigma_{n+1}))\}.$$

$$(\daleth^* : \varsigma_{n+1}) = (\daleth(P^2_{n-3}), \varsigma_1, \varsigma_n, \varsigma_{n-1}, \varsigma_{n+2}).$$

And

$$\Omega/(\daleth^* : \varsigma_{n+1}) \cong \frac{\Omega_{n-3}}{\daleth(P^2_{n-3})}[\varsigma_{n+1}].$$

$$\mathrm{depth}(\Omega/(\daleth^* : \varsigma_{n+1})) = \mathrm{depth}(\frac{\Omega_{n-3}}{\daleth(P^2_{n-3})}) + 1.$$

$$\mathrm{depth}(\Omega/(\daleth^* : \varsigma_{n+1})) = \lceil \frac{n-3}{5} \rceil + 1 = \lceil \frac{n+2}{5} \rceil.$$

Now take

$$(\daleth^*, \varsigma_{n+1}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_{n-1}\varsigma_1, \varsigma_n, \varsigma_{n+1}).$$

Again construct a short exact sequence and let $\daleth^{**} := (\daleth^*, \varsigma_{n+1})$. Consider the following short exact sequence

$$0 \longrightarrow \Omega/(\daleth^{**} : \varsigma_1) \xrightarrow{\varsigma_1} \Omega/\daleth^{**} \longrightarrow \Omega/(\daleth^{**}, \varsigma_1) \longrightarrow 0.$$

By using depth lemma, we have

$$\mathrm{depth}(\Omega/\daleth^{**}) \geq \min\{\mathrm{depth}(\Omega/(\daleth^{**} : \varsigma_1)), \mathrm{depth}(\Omega/(\daleth^{**}, \varsigma_1))\}.$$

$$(\daleth^{**} : \varsigma_1) = (\daleth(P^2_{n-5}), \varsigma_2, \varsigma_3, \varsigma_n, \varsigma_{n-1}, \varsigma_{n+1}).$$

And

$$\Omega/(\daleth^{**} : \varsigma_1) \cong \frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}[\varsigma_1, \varsigma_{n+2}].$$

Now by [3], [9, Lemma 3.6] and [14], we get

$$\mathrm{depth}(\Omega/(\daleth^{**} : \varsigma_1)) = \mathrm{depth}(\frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}) + 2.$$

$$\mathrm{depth}(\Omega/(\daleth^{**} : \varsigma_1)) = \lceil \frac{n-5}{5} \rceil + 2 = \lceil \frac{n+5}{5} \rceil.$$

$$(\Omega/(\daleth^{**}, \varsigma_1)) \cong \frac{\Omega_{n-2}}{\daleth(P^2_{n-2})}[\varsigma_{n+2}].$$

$$\mathrm{depth}(\Omega/(\daleth^{**}, \varsigma_1)) = \mathrm{depth}(\frac{\Omega_{n-2}}{\daleth(P^2_{n-2})}) + 1.$$

$$\mathrm{depth}(\Omega/(\daleth^{**}, \varsigma_1)) = \lceil \frac{n-2}{5} \rceil + 1 = \lceil \frac{n+3}{5} \rceil.$$

So

$$\operatorname{depth}(\Omega/\daleth^{**}) \geq \lceil \frac{n+3}{5} \rceil.$$

Similarly

$$\operatorname{depth}(\Omega/\daleth^{*}) \geq \{\lceil \frac{n+3}{5} \rceil, \lceil \frac{n+2}{5} \rceil\}.$$

And

$$\operatorname{depth}(\Omega/\daleth) \geq \{\lceil \frac{n+2}{5} \rceil, \lceil \frac{n}{5} \rceil\}.$$

Hence

$$\operatorname{depth}(\Omega/\daleth) = \lceil \frac{n}{5} \rceil.$$

In a similar manner, we get

$$\operatorname{sdepth}(\Omega/\daleth) = \lceil \frac{n}{5} \rceil.$$

$\square$

Lemma 4.2.3. $\operatorname{depth}(\Omega/\daleth(T^2_{n,3})), \operatorname{sdepth}(\Omega/\daleth(T^2_{n,3})) \geq \lceil \frac{n+2}{5} \rceil$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega/(\daleth : \varsigma_{n+1}) \xrightarrow{\varsigma_{n+1}} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_{n+1}) \longrightarrow 0.$$

By depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth) \geq \min\{\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+1})), \operatorname{depth}(\Omega/(\daleth, \varsigma_{n+1}))\}.$$

Here we have $\Omega = \Omega_{n,3} = [\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}]$ and

$$\daleth = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_{n-1}\varsigma_1, \varsigma_n\varsigma_{n+1}, \varsigma_{n-1}\varsigma_{n+1}, x_1\varsigma_{n+1}, \varsigma_{n+1}\varsigma_{n+2}, \varsigma_n\varsigma_{n+2}, \varsigma_{n+1}\varsigma_{n+3}, \varsigma_{n+2}\varsigma_{n+3}).$$

For this, we have

$$\daleth : \varsigma_{n+1}) = (\daleth(P^2_{n-3})), \varsigma_1, \varsigma_n, \varsigma_{n-1}, \varsigma_{n+2}, \varsigma_{n+3}).$$

$$\Omega/(\daleth : \varsigma_{n+1}) \cong \frac{\mathbb{K}[\varsigma_2, \varsigma_3, \cdots, \varsigma_{n-3}]}{\daleth(P^2_{n-3}), \varsigma_1, \varsigma_n, \varsigma_{n-1}, \varsigma_{n+2}, \varsigma_{n+3})} \cong \frac{\mathbb{K}[\varsigma_2, \varsigma_3, \cdots, \varsigma_{n-3}]}{\daleth(P^2_{n-3})}[\varsigma_{n+1}].$$

Here we have

$$\operatorname{depth}(\Omega/\daleth : \varsigma_{n+1})) = \operatorname{depth}(\Omega_{n-3}/\daleth((P^2_{n-3})) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+1})) = \lceil \frac{n-3}{5} \rceil + 1 = \lceil \frac{n+2}{5} \rceil.$$

Again construct a short exact sequence and let $\daleth^* := (\daleth(T^2_{n,3}), \varsigma_{n+1})$. Here we have

$$(\daleth, \varsigma_{n+1}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, x_1\varsigma_3 \cdots, \varsigma_{n-1}\varsigma_1, \varsigma_n\varsigma_1, \varsigma_n\varsigma_{n+2}, \varsigma_n\varsigma_2, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}).$$

Consider the following short exact sequence

$$0 \longrightarrow \Omega/(\daleth^* : \varsigma_n) \xrightarrow{\varsigma_n} \Omega/\daleth^* \longrightarrow \Omega/(\daleth^*, \varsigma_n) \longrightarrow 0.$$

By using depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth^*) \geq \min\{\operatorname{depth}(\Omega/(\daleth^* : \varsigma_n)), \operatorname{depth}(\Omega/(\daleth^*, \varsigma_n))\}.$$

$$(\daleth^* : \varsigma_n) = (\daleth(P^2_{n-5}), \varsigma_1, \varsigma_2, \varsigma_{n-1}, \varsigma_{n-2}, \varsigma_{n+1}, \varsigma_{n+2}).$$

and

$$\Omega/(\daleth^* : \varsigma_n) \cong \frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}[\varsigma_{n+3}, \varsigma_n].$$

So by [3], [9, lemma 3.6] and [14], we have

$$\operatorname{depth}(\Omega/(\daleth^* : \varsigma_n)) = \operatorname{depth}(\frac{\Omega_{n-5}}{\daleth(P^2_{n-5})}) + 2.$$

$$\operatorname{depth}(\Omega/(\daleth^* : \varsigma_n)) = \lceil\frac{n-5}{5}\rceil + 2 = \lceil\frac{n+5}{5}\rceil.$$

Now take

$$(\daleth^*, \varsigma_n) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_{n-1}\varsigma_1, \varsigma_{n+2}\varsigma_{n+3}, \varsigma_n, \varsigma_{n+1}).$$

Again construct a short exact sequence and let $\daleth^{**} := (\daleth^*, \varsigma_n)$. Consider the following short exact sequence

$$0 \longrightarrow \Omega/(\daleth^{**} : \varsigma_1) \xrightarrow{\varsigma_1} \Omega/\daleth^{**} \longrightarrow \Omega/(\daleth^{**}, \varsigma_1) \longrightarrow 0.$$

By using depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth^{**}) \geq \min\{\operatorname{depth}(\Omega/(\daleth^{**} : \varsigma_1)), \operatorname{depth}(\Omega/(\daleth^{**}, \varsigma_1))\}.$$

$$(\daleth^{**} : \varsigma_1) = (\daleth(P^2_{n-4}), \varsigma_{n+2}\varsigma_{n+3}, \varsigma_2, \varsigma_3, \varsigma_n, \varsigma_{n+1}).$$

And

$$\operatorname{depth}(\Omega/(I'' : \varsigma_1)) = \operatorname{depth}(\frac{\Omega_{n-4}}{\daleth(P^2_{n-4})}) + 2.$$

$$\operatorname{depth}(\Omega/(\daleth^{**} : \varsigma_1)) = \lceil\frac{n-4}{5}\rceil + 2 = \lceil\frac{n+6}{5}\rceil.$$

Similarly

$$(\daleth^{**}, \varsigma_1) = (\varsigma_2\varsigma_3, \cdots, \varsigma_{n-2}\varsigma_{n-1}, \varsigma_{n+2}\varsigma_{n+3}, \varsigma_1, \varsigma_n, \varsigma_{n+1}).$$

$$\operatorname{depth}(\Omega/(\daleth^{**}, \varsigma_1)) = \operatorname{depth}(\frac{\Omega_{n-2}}{\daleth(P^2_{n-2})}) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth^{**}, \varsigma_1)) = \lceil \frac{n-2}{5} \rceil + 1 = \lceil \frac{n+3}{5} \rceil.$$

So

$$\operatorname{depth}(\Omega/\daleth^{**}) \geq \lceil \frac{n+3}{5} \rceil.$$

Similarly

$$\operatorname{depth}(\Omega/\daleth^{*}) \geq \{\lceil \frac{n+3}{5} \rceil, \lceil \frac{n+5}{5} \rceil\}.$$

And

$$\operatorname{depth}(\Omega/\daleth) \geq \{\lceil \frac{n+2}{5} \rceil, \lceil \frac{n+3}{5} \rceil\}.$$

Hence

$$\operatorname{depth}(\Omega/\daleth) = \lceil \frac{n+2}{5} \rceil.$$

Similarly, we have

$$\operatorname{sdepth}(\Omega/\daleth) = \lceil \frac{n+2}{5} \rceil.$$

□

Lemma 4.2.4. $\operatorname{depth}(\Omega/\daleth(T^2_{n,4})), \operatorname{sdepth}(\Omega/\daleth(T^2_{n,4})) \geq \lceil \frac{n+2}{5} \rceil$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega/(\daleth : \varsigma_{n+4}) \xrightarrow{\varsigma_{n+4}} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_{n+4}) \longrightarrow 0.$$

By depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth) \geq \min\{\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+4})), \operatorname{depth}(\Omega/(\daleth, \varsigma_{n+4}))\}.$$

Here we have $\Omega = \Omega_{n,4} = \mathbb{K}[\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}, \varsigma_{n+4}]$ and

$$\daleth = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_{n-1}\varsigma_1, \varsigma_n\varsigma_{n+1}, \cdots, \varsigma_{n+2}\varsigma_{n+3}, \varsigma_{n+2}\varsigma_{n+4}.\varsigma_{n+3}\varsigma_{n+4}).$$

For this, we have

$$(\daleth : \varsigma_{n+4}) = ((\daleth(T^2_{n,1})), \varsigma_{n+2}, \varsigma_{n+3}).$$

$$\begin{aligned} \Omega/(\daleth : \varsigma_{n+4}) &\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}]}{I(T^2_{n,1}), \varsigma_{n+2}, \varsigma_{n+3}}) \\ &\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}]}{I(T^2_{n,1})}[\varsigma_{n+4}]. \end{aligned}$$

Here we have

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+4})) = \operatorname{depth}(\Omega_{n,1}/\daleth((T_{n,1}^2)) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+4})) = \lceil \frac{n}{5} \rceil + 1 \geq \lceil \frac{n+5}{5} \rceil.$$

And

$$(\daleth, \varsigma_{n+4}) = ((\daleth(T_{n,3}^2)), \varsigma_{n+4}).$$

$$\operatorname{depth}(\Omega/(\daleth, \varsigma_{n+4})) = \operatorname{depth}(\frac{\Omega_{n,3}}{\daleth(T_{n,3}^2)}).$$

By using Lemma 4.2.3, we have

$$\operatorname{depth}(\Omega/(\daleth, \varsigma_{n+4})) = \lceil \frac{n+2}{5}.$$

So

$$\operatorname{depth}(\Omega/\daleth) \geq \{\lceil \frac{n+2}{5} \rceil, \lceil \frac{n+5}{5} \rceil\}.$$

Hence

$$\operatorname{depth}(\Omega/\daleth) \geq \lceil \frac{n+2}{5} \rceil.$$

Now we proceed with the same arguments, we get

$$\operatorname{sdepth}(\Omega/\daleth) \geq \lceil \frac{n+2}{5} \rceil.$$

$\square$

Lemma 4.2.5. $\operatorname{depth}(\Omega/\daleth(T_{n,5}^2)), \operatorname{sdepth}(\Omega/\daleth(T_{n,5}^2)) \geq \lceil \frac{n+3}{5} \rceil$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Omega/(\daleth : \varsigma_{n+3}) \xrightarrow{\varsigma_{n+3}} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_{n+3}) \longrightarrow 0.$$

By depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth) \geq \min\{\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+3})), \operatorname{depth}(\Omega/(\daleth, \varsigma_{n+3}))\}.$$

Here we have $\Omega = \Omega_{n,5} = [\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}, \varsigma_{n+4}, \varsigma_{n+5}]$ and

$$\daleth = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_{n-1}\varsigma_1, \varsigma_n\varsigma_{n+1}, \cdots, \varsigma_{n+2}\varsigma_{n+3},$$
$$\varsigma_{n+2}\varsigma_{n+4}.\varsigma_{n+3}\varsigma_{n+4}, \varsigma_{n+4}\varsigma_{n+5}, \varsigma_{n+3}\varsigma_{n+5}.$$

For this, we have

$$(\daleth : \varsigma_{n+3}) = ((\daleth(C_n^2)), \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+4}, \varsigma_{n+5}).$$

$$\begin{aligned}\Omega/(\daleth : \varsigma_{n+3}) &\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+3}, \varsigma_{n+4}, \varsigma_{n+5}]}{(I(C_n^2), \varsigma_{n+1}, \varsigma_{n+2}, \varsigma_{n+4}, \varsigma_{n+5})} \\ &\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_n]}{\daleth(C_n^2)}[\varsigma_{n+3}].\end{aligned}$$

Here by using [4, Proposition 1.3] and [9, Lemma 3.6], we have

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+3})) = \operatorname{depth}(\Omega_n/\daleth((C_n^2)) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth : \varsigma_{n+3})) \geq \lceil \frac{n-2}{5} \rceil + 1 \geq \lceil \frac{n+3}{5} \rceil.$$

And

$$(\daleth, \varsigma_{n+3}) = ((\daleth(T_{n,1}^2)), \varsigma_{n+2}\varsigma_{n+4}, \varsigma_{n+4}\varsigma_{n+5}, \varsigma_{n+3}).$$

Again construct a short exact sequence and let $\daleth^* := (\daleth, \varsigma_{n+3})$. Consider the following short exact sequence

$$0 \longrightarrow \Omega/(\daleth^* : \varsigma_{n+4}) \xrightarrow{\varsigma_{n+4}} \Omega/\daleth^* \longrightarrow \Omega/(\daleth^*, \varsigma_{n+4}) \longrightarrow 0.$$

By using depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth^*) \geq \min\{\operatorname{depth}(\Omega/(\daleth^* : \varsigma_{n+4})), \operatorname{depth}(\Omega/(\daleth^*, \varsigma_{n+4}))\}.$$

$$(\daleth^* : \varsigma_{n+4}) = (\daleth(T_{n,1}^2), \varsigma_{n+2}, \varsigma_{n+3}, \varsigma_{n+5}).$$

And

$$\Omega/(\daleth^* : \varsigma_{n+4}) \cong \frac{\Omega_{n,1}}{\daleth(T_{n,1}^2)}[\varsigma_{n+4}].$$

$$\operatorname{depth}(\Omega/(\daleth^* : \varsigma_{n+4})) = \operatorname{depth}(\frac{\Omega_{n,1}}{\daleth(T_{n,1}^2)}) + 1.$$

$$\operatorname{depth}(\Omega/(\daleth^* : \varsigma_{n+4})) = \lceil \frac{n}{5} \rceil + 1 = \lceil \frac{n+5}{5} \rceil.$$

Now

$$(\daleth^*, \varsigma_{n+4}) = (\daleth(T_{n,1}^2), \varsigma_{n+3}, \varsigma_{n+4}).$$

By using Lemma 4.2.1, we get

$$\operatorname{depth}(\Omega/(\daleth, \varsigma_{n+4})) = \operatorname{depth}(\frac{\Omega_{n,1}}{\daleth(T_{n,1}^2)}[\varsigma_{n+5}]).$$

$$\operatorname{depth}(\Omega/(\daleth^*, \varsigma_{n+4})) = \lceil \frac{n}{5} \rceil + 1.$$

So

$$\operatorname{depth}(\Omega/\daleth^*) \geq \{\lceil \frac{n+5}{5} \rceil, \lceil \frac{n+5}{5} \rceil\}.$$

Hence
$$\text{depth}(\Omega/\daleth^*) \geq \lceil \frac{n+5}{5} \rceil.$$
Thus
$$\text{depth}(\Omega/\daleth) \geq \{\lceil \frac{n+3}{5} \rceil, \lceil \frac{n+5}{5} \rceil\}.$$
Hence
$$\text{depth}(\Omega/\daleth) \geq \lceil \frac{n+3}{5} \rceil.$$
Similarly, we have
$$\text{sdepth}(\Omega/\daleth) \geq \lceil \frac{n+3}{5} \rceil.$$
□

Theorem 4.2.6. For $m \geq 6$ and $n \geq 9$, we have
$$\text{depth}(\Omega/\daleth(T^2_{n,m})) \geq \lceil \frac{n+m-2}{5} \rceil.$$

Proof. For $1 \leq m \leq 5$ and $3 \leq n \leq 8$, we can easily find exact values for depth and Stanley depth of the quotient of edge ideal associated of a square Tadpole graph from the CoCoA software [5]. We will prove by induction on m. Consider the short exact sequence,
$$0 \longrightarrow \Omega/(\daleth : \varsigma_{n+m-2}) \xrightarrow{\varsigma_{n+m-2}} \Omega/\daleth \longrightarrow \Omega/(\daleth, \varsigma_{n+m-2}) \longrightarrow 0.$$
By depth lemma, we have
$$\text{depth}(\Omega/\daleth) \geq min\{\text{depth}(\Omega/(\daleth : \varsigma_{n+m-2})), \text{depth}(\Omega/(\daleth, \varsigma_{n+m-2}))\}.$$
For this, we have
$$\daleth(T^2_{n,m}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_n\varsigma_1, \varsigma_n\varsigma_{n+1}, \cdots, \\ \varsigma_{m+n-1}\varsigma_{n+m}, \varsigma_1\varsigma_3, \varsigma_2\varsigma_4, \cdots, \varsigma_{n-1}\varsigma_{n+3}, \varsigma_1\varsigma_{n+3}, \cdots, y_{n+m-2\varsigma_{n+m}}).$$
$$(\daleth, \varsigma_{n+m-2}) = (\varsigma_1\varsigma_2, \varsigma_1\varsigma_3, \varsigma_2\varsigma_3, \cdots, \varsigma_{n+m-3}\varsigma_{n+m-1}, \varsigma_{n+m-1}\varsigma_{n+m}, \varsigma_{n+m-2}).$$
$$(\daleth : \varsigma_{n+m-2}) = (\varsigma_1\varsigma_2, \varsigma_2\varsigma_3, \cdots, \varsigma_{n+m-6}\varsigma_{n+m-5}, \varsigma_{n+m-4}, \varsigma_{n+m-3}, \varsigma_{n+m-1}, \varsigma_{n+m}).$$
Take
$$\Omega/(\daleth : \varsigma_{n+m-2}) \cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n+m}]}{(\daleth(T^2_{n,m-5}), \varsigma_{n+m-4}, \varsigma_{n+m-3}, \varsigma_{n+m-1}, \varsigma_{n+m})}$$
$$\cong \frac{\mathbb{K}[\varsigma_1, \varsigma_2, \cdots, \varsigma_{n+m-5}]}{\daleth(T^2_{n,m-5})}[\varsigma_{n+m-2}].$$

Here we have

$$\operatorname{depth}(\Omega/(\daleth:\varsigma_{n+m-2}))=\operatorname{depth}(\frac{\Omega_{n-5,m}}{\daleth(T^2_{n-5,m})})+1.$$

By applying induction, we get

$$\operatorname{depth}(\Omega/(\daleth:\varsigma_{n+m-2}))\geq\lceil\frac{n-5+m-2}{5}\rceil+1=\lceil\frac{n+m-2}{5}\rceil.$$

Again construct a short exact sequence and let $I':=(\daleth(T^2_{n,m}),\varsigma_{n+m-2})$.
Here we have

$$\daleth(T^2_{n,m}),\varsigma_{n+m-2})=(\varsigma_1\varsigma_2,\varsigma_1\varsigma_3,\varsigma_2\varsigma_4,\cdots,\varsigma_{n+m-3}\varsigma_{n+m-5},\varsigma_{n+m-3}$$
$$,\varsigma_{n+m-1},\varsigma_{n+m-1}\varsigma_{n+m},\varsigma_{n+m-2}).$$

$\daleth(T^2_{n,m}),\varsigma_{n+m-2})=(I(T^2_{n-3,m}),\varsigma_{n+m-1}\varsigma_{n+m-3},\varsigma_{n+m-1}\varsigma_{n+m},\varsigma_{n+m-2})$. Consider the following short exact sequence

$$0\longrightarrow\Omega/(\daleth^*:\varsigma_{n+m-1})\xrightarrow{\varsigma_{n+m-1}}\Omega/\daleth^*\longrightarrow\Omega/(\daleth^*,\varsigma_{n+m-1})\longrightarrow 0.$$

By using depth lemma, we have

$$\operatorname{depth}(\Omega/\daleth^*)\geq min\{\operatorname{depth}(\Omega/(\daleth^*:\varsigma_{n+m-1})),\operatorname{depth}(\Omega/(\daleth^*,\varsigma_{n+m-1}))\}.$$

$$(\daleth^*:\varsigma_{n+m-1})=(\daleth(T^2_{n-4,m}),\varsigma_{n+m-3},\varsigma_{n+m-2},\varsigma_{n+m}).$$

And

$$\Omega/(\daleth^*:\varsigma_{n+m-1})\cong\frac{\Omega_{n,m-4}}{\daleth(T^2_{n,m-4})}[\varsigma_{n+m-1}].$$

$$\operatorname{depth}(\Omega/(\daleth^*:\varsigma_{n+m-1}))=\operatorname{depth}(\frac{\Omega_{n,m-4}}{\daleth(T^2_{n,m-4})})+1.$$

$$\operatorname{depth}(\Omega/(\daleth^*:\varsigma_{n+m-1}))\geq\lceil\frac{n-4+m-2}{5}\rceil+1=\lceil\frac{n+m-1}{5}\rceil.$$

Now take

$$(\daleth^*,\varsigma_{n+m-1})=(\daleth(T^2_{n,m-3}),\varsigma_{n+m-2},\varsigma_{n+m-1}).$$

And then

$$(\Omega/(\daleth^*,\varsigma_{n+m-1}))\cong\frac{S_{n,m-3}}{\daleth(T^2_{n,m-3})}[\varsigma_{n+m}].$$

$$\operatorname{depth}(\Omega/(\daleth^*,\varsigma_{n+m-1}))=\operatorname{depth}(\frac{\Omega_{n,m-3}}{\daleth(T^2_{n,m-3})})+1.$$

$$\operatorname{depth}(\Omega/(\daleth^*,\varsigma_{n+m-1}))\geq\lceil\frac{n-3+m-2}{5}\rceil+1=\lceil\frac{n+m}{5}\rceil.$$

So

$$\text{depth}(\Omega/\daleth^{*}) = \lceil\frac{n+m-1}{5}\rceil.$$

Similarly

$$\text{depth}(\Omega/\daleth) \geq \{\lceil\frac{n+m-2}{5}\rceil, \lceil\frac{n+m-1}{5}\rceil\}.$$

Hence

$$\text{depth}(\Omega/\daleth) \geq \lceil\frac{n+m-2}{5}\rceil.$$

□

Theorem 4.2.7. For $m \geq 6$ and $n \geq 9$, we have

$$\text{sdepth}(\Omega/\daleth(T^{2}_{n,m})) \geq \lceil\frac{n+m-2}{5}\rceil.$$

Proof. Now for $m \geq 6$ and $n \geq 9$. We can do analogous work as Theorem 4.2.6 for finding Stanley depth of a graph. We replace Lemma 3.3.2 from Depth Lemma for applying on the exact sequences. We also use Corollary 3.3.1. So the required result can be deducted by applying induction on m.
Now by Lemma 3.3.2, we have

$$\text{sdepth}(\Omega/\daleth) \geq \min\{\text{sdepth}(\Omega/(\daleth, \varsigma_{n+m-2})), \text{sdepth}(\Omega/(\daleth : \varsigma_{n+m-2}))\}.$$

Hence

$$\text{sdepth}(\Omega/\daleth) \geq \lceil\frac{n+m-2}{5}\rceil.$$

□

4.3 Conclusion

We have calculated the exact values for depth and Sdepth of the factor of the edge ideals associated with Tadpole graph. We obtained quality results as we have computed exact values in contrast to the already existing bounds.

- *For a Tadpole graph $T_{n,m}$, we have diam$(T_{n,m}) = \Delta$*

$$\Delta = \lfloor \frac{n}{2} \rfloor + m.$$

- *By Fouli and Morey's formula [8], we have*

$$depth(\Omega/(\daleth(T_{n,m})), sdepth(\Omega/(\daleth(T_{n,m})) \geq \lceil \frac{\lfloor \frac{n}{2} \rfloor + m + 1}{3} \rceil.$$

Example 20. For $n = 122$ and $m = 400$. We have $\Delta(T_{144,400}) = 461$, therefore $\mathrm{depth}(\Omega/(\daleth(T_{n,m})) \geq 154$ by [8], whereas by our given result $\mathrm{depth}(\Omega/(\daleth(T_{n,m})) = 174$. Moreover, we can observe that the existing bounds deviate even more from the exact values for the larger values of n and m.

Similarly, we have calculated the lower bound for depth and Sdepth of the factor of the edge ideal associated with the square of a Tadpole graph. We have obtained the following sharp bound. That is for $n \geq 3$ and $m \geq 1$, we have

$$\mathrm{depth}(\Omega/\daleth(T^2_{n,m})), \mathrm{sdepth}(\Omega/\daleth(T^2_{n,m})) \geq \lceil \frac{n + m - 2}{5} \rceil.$$

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