Rayleigh-Lamb spectrum of an incompressible orthotropic wave guide

by

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Dedication

To My Parents

Thank you for your unconditional support with my studies. I am honored to have you as my parents.

Acknowledgements

First and foremost, I thank Allah (S.W.T) for endowing me with health, patience, and knowledge to complete this work.

I would like to express my gratitude to my supervisor Dr. Faiz Ahmad. I cannot thank him enough for his extensive support and encouragement at every stage of this study. His expertise in Continuum Mechanics and Wave Propagation made this thesis possible. He deserves the utmost thanks for his patience, guidance, enthusiasm, and support.

Many friends have helped me stay sane through these difficult years. Their support and care helped me overcome setbacks and stay focused on my studies. I greatly value their friendship and I deeply appreciate their belief in me.

Most importantly, I thank my family for their enduring love and support. I thank my parents for their faith in me and allowing me to be as ambitious as I wanted. It was under their watchful eye that I gained so much drive and an ability to tackle challenges head on.

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Abstract

This dissertation deals with elastic waves in orthotropic incompressible materials and the Rayleigh-Lamb spectrum of an elastic plate. The propagation of elastic plane waves and the Rayleigh equation for orthotropic incompressible elastic plate is discussed.

Some earlier work dealing with wave propagation in an incompressible material is reviewed [15]. An error in [15] is pointed out.

Also work of Ogden and Vinh [17] concerning the existence of a Rayleigh wave in an incompressible orthotropic half space is discussed. A dispersion relation for an orthotropic wave guide is derived. Rayleigh-Lamb spectrum for the wave-guide is analyzed and the shape of the dispersion curves is mathematically described using a technique developed by Ahmad [1].

Contents

Chapter 1

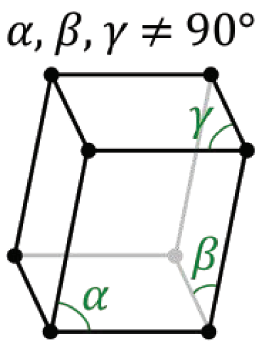
Introduction

A waveguide is a structure which guides waves, such as electromagnetic waves or sound waves. There are different types of wave-guides for each type of wave. The original and most common meaning is a hollow conductive metal pipe used to carry high frequency radio waves, particularly microwaves. Wave-guides differ in their geometry which can confine energy in one dimension such as in slab wave-guides or two dimensions as in fiber or channel wave-guides. The first structure for guiding waves was proposed by J. J. Thompson in 1893, and was first experimentally tested by O. J. Lodge in 1894.

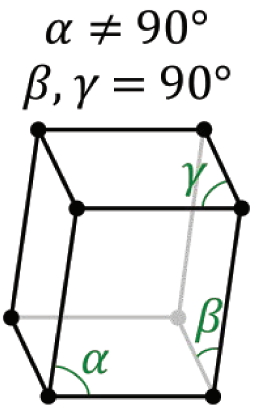
Propagation of guided Rayleigh-Lamb waves in a plate is of interest in seismology, electrical devices, ultrasonic material characterization and ultrasonic nondestructive evaluation of defects. Guided waves in an elastic wave guide are also of interest in medicine where a bone acts as a cylindrical wave guide. Two recent examples of such studies are found in ([24], [16]).

The symmetry axes of an object are lines about which it can be rotated through some angle which brings the object to a new orientation which appears identical to its starting position. The symmetry planes of an object are imaginary mirrors in which it can be reflected while appearing unchanged. When a mirror is placed on a line of symmetry of a two dimensional shape and looked at from either side, the shapes look identical. In other words, each half of the shape is a mirror image of the other half. In a similar way, when a plane cuts a 3-D shape in two so that each half is a mirror image of the other half, the plane is called a plane of symmetry.

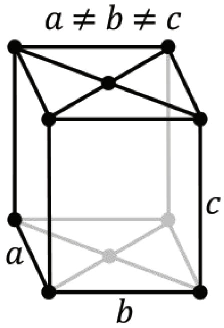
Elastic materials can be divided into eight classes on the basis of their symmetries. Let **a,b,c** denote the sides of a unit cell of a crystal and let α, β, γ denote the angles between these vectors so that γ is the angle between **a** and **b** etc. For an isotropic material every line is an axis of symmetry and every plane is a plane of symmetry. The crystal structure of the rest of the seven crystal systems for a triclinic, a monoclinic, an orthorhombic, a trigonal, a tetragonal, a hexagonal and a cubic system is as follows [26].



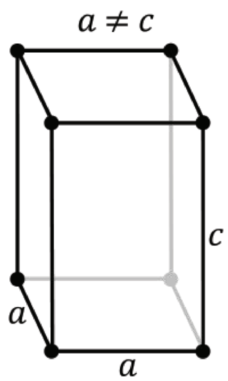
(a) Triclinic



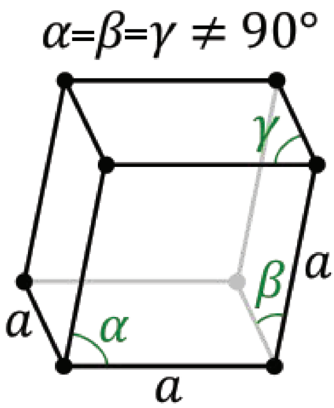
(b) Monoclinic



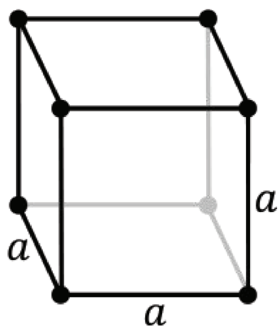
(c) Orthorhombic



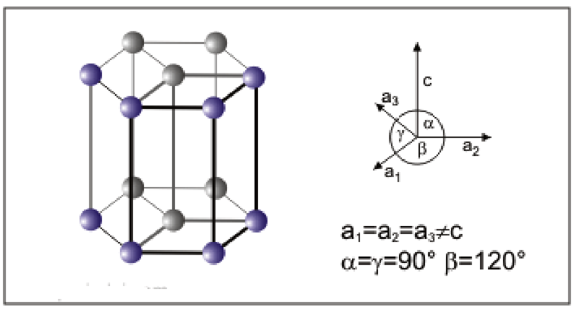
(d) Tetragonal



(e) Trigonal



(f) Cubic



(g) Hexagonal

Crystal systems	**No. of axes of symmetry**	**No. of planes of symmetry**	**No. of ind. comp. of $\mathbf{C_{ijkl}}$**
Triclinic	No axis of symmetry	No plane of symmetry	21
Monoclinic	One 2-fold axis of rotation	One symmetry plane	13
Orthorhombic	3 mutually perpendicular 2-fold rotation axes	Three coordinate planes are symmetry planes	9
Trigonal	One 3-fold rotation axis	Three symmetry planes	7
Tetragonal	One 4-fold rotation axis	Five symmetry planes	7
Hexagonal	One 6-fold rotation axis	Any plane containing x_3 axis	5
Cubic	Three 2-fold and four 3-fold rotation axes	Nine planes of symmetry	3
Isotropic	Any axis is an axis of symmetry	Any plane is a plane of symmetry	2

Table 1.1: **Description of crystal systems.**

The generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1.1}$$

is the constitutive relation for linear anisotropic elasticity, where all indices take values $1, 2, 3$ and summation over repeated indices is assumed. In Eq.(1.1) σ_{ij} are components of the stress tensor, ϵ_{kl} are components of the strain tensor and C_{ijkl} are the components of the fourth-rank elasticity tensor. Symmetries of C_{ijkl} depend upon the symmetry of the strain and stress tensors. Thus it must have the following symmetries

$$C_{ijkl} = C_{jikl} = C_{ijlk}.$$

The elasticity tensor has $3^4 = 81$ components, but due to the above symmetries, the number of independent components reduces to 36. Voigt (1910) introduced a two index notation as follows

$11 \leftrightarrow 1$,$22 \leftrightarrow 2$,$33 \leftrightarrow 3$,$23 = 32 \leftrightarrow 4$,$13 = 31 \leftrightarrow 5$,$12 = 21 \leftrightarrow 6$.

This means $C_{1111} = C_{11}, C_{2312} = C_{64}, C_{1221} = C_{66}$ etc.
Now equation (1.1) can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} \tag{1.2}$$

or

$$\sigma = \mathbf{C}\epsilon.$$

The number of independent components are further reduced from a consideration of strain energy due to the thermodynamics requirement. Ting [20] gave this condition in the following manner. The strain energy W per unit volume of a material is

$$W = \int \sigma_{ij} d\epsilon_{ij} = \int C_{ijkl}\epsilon_{kl} d\epsilon_{ij}. \tag{1.3}$$

This integral must be independent of the path traversed by ϵ_{ij} from 0 to ϵ_{pq}. We consider a loading from 0 to ϵ_{pq} and unloading from ϵ_{pq} to 0. By taking the difference of these two paths of the W's we obtain energy. By successive repetition of this process we gain infinite energy, but physically this is not possible for real materials. Therefore the integral in eq. (1.3) is independent of the path taken by ϵ_{ij} and W is the function of final strain ϵ_{pq} only. Therefore

$$dW = C_{ijkl}\epsilon_{kl} d\epsilon_{ij}.$$

But

$$dW = \frac{\partial W}{\partial \epsilon_{ij}} d\epsilon_{ij}.$$

Thus

$$\frac{\partial W}{\partial \epsilon_{ij}} d\epsilon_{ij} = C_{ijkl}\epsilon_{kl} d\epsilon_{ij}.$$

Since $d\epsilon_{ij}$ is arbitrary, we have

$$\frac{\partial W}{\partial \epsilon_{ij}} = C_{ijkl}\epsilon_{kl},$$

differentiating it w.r.t ϵ_{kl}, we find

$$\frac{\partial^2 W}{\partial \epsilon_{kl}\epsilon_{ij}} = C_{ijkl}. \tag{1.4}$$

Similarly

$$\frac{\partial^2 W}{\partial \epsilon_{ij}\epsilon_{kl}} = C_{klij}. \tag{1.5}$$

Now we know from calculus that $\frac{\partial^2 W}{\partial \epsilon_{kl}\epsilon_{ij}} = \frac{\partial^2 W}{\partial \epsilon_{ij}\epsilon_{kl}}$. Thus from Eqs. (1.4) and (1.5), we have

$$C_{ijkl} = C_{klij},$$

which is the required condition for the integral in eq. (1.3) to be independent of the loading path. Because of $C_{ijkl} = C_{klij}$, the number of independent elastic constants are reduced from 36 to 21, and we have a representation of (1.2) as follows

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} \tag{1.6}$$

A line is called an n-fold axis of symmetry A_n, of an elastic material if it leaves the material invariant when we rotate it about A_n through an angle $\frac{2\pi}{n}$. If the material is invariant w.r.t inversion through some point then that point is called a center of symmetry. If the material is invariant w.r.t reflection in some plane, then that plane is called a plane of symmetry.

If an axis of symmetry A_n exists and we choose x_3-axis parallel to A_n, then the rotation matrix corresponds to a rotation of an angle θ about this axis is

$$\mathbf{a}_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.7}$$

The invariance of a material w.r.t a rotation through θ, about x_3-axis gives

$$C_{ijkl} = a_{ip}a_{jq}a_{kr}a_{ls}C_{pqrs} \tag{1.8}$$

The propagation and reflection of elastic waves in anisotropic materials is a topic of research that finds application in the areas of nondestructive evaluation, design and seismology. For compressible materials a number of papers have appeared in the technical literature both in the experimental ([8], [25]) and in theoretical [7] aspects to complement older theoretical papers, for example ([10], [19]), and the classical book on the subject by Fedorov [9].

Nair and Sotiropoulos [15] examined the propagation of elastic plane waves in orthotropic incompressible materials under plain strain conditions in a plane of symmetry. The slowness surface is obtained by aligning a material axis of symmetry with the direction of minimum wave speed. The assumption of incompressibility and orthotropy are applicable to several materials as, e.g. polymer kratons, thermoplastic elastomers, rubber composites when low frequency waves are considered to justify the assumption of material homogeneity, etc.

Rayleigh (surface) waves were first studied by Rayleigh [11] for compressible isotropic elastic solid. The extension of surface wave analysis to anisotropic elastic materials has been the subject of many studies.

Rayleigh waves in incompressible orthotropic elastic materials were examined by Destrade [4]. Destrade used the method of first integrals proposed by Mozhaev [12] and found a form of the secular equation. He used this to prove that a Rayleigh wave exists and is unique in these materials for all values of the relevant material constants. However, the form of the secular equation obtained by use of Mozhaev's method necessarily admits spurious solution. Thus the analysis of Destrade requires some modification.

Ogden and Vinh [17] obtained a formula for the Rayleigh wave speed in an incompressible orthotropic elastic material. They obtained a secular equation that does not admit spurious solution.

Ahmad analyzed the Rayleigh-Lamb spectrum in [2] and explained the shapes of the curves. He also derived an explicit solution of the Rayleigh-Lamb equation which is valid over a major part of the spectrum [1].

The plan of this dissertation is as follows:
Chapter 2 consists of basic definitions and preliminaries. In Chapter 3, we discuss the propagation of elastic plane waves in orthotropic incompressible materials which are examined under plane strain conditions in a plane of symmetry. We examine the analysis of this problem made by Nair and Sotiropoulos [15] and point out an error in their calculations. Chapter 4 is concerned about finding a dispersion relation for a wave guide consisting of an infinite plate of an orthotropic material. We analyze its spectrum and find that it has several features common with that of an isotropic material. Also in the limit of long wave length, the dispersion relation transforms to the equation for the velocity of a Rayleigh wave propagating in an orthotropic half space [5].

Chapter 2

Preliminaries

2.1 Introduction

This chapter is devoted to some basic definitions and mathematical preliminaries. Here we examine constants that characterize the elastic properties of crystals. These are the components of a fourth-rank tensor. Also the relationships between the stresses and the strains in an anisotropic elastic material and some contracted notations are presented in this chapter. Moreover a brief introduction and some basic definitions about wave motion and types of wave are included in this chapter. The following definitions and concepts will be used throughout this dissertation.

2.2 Basic definitions and concepts

In physics, elasticity is the physical property of a material that returns to its original shape after the stress (e.g. external forces) that deforms it, is removed. The relative amount of deformation is called the strain.

By definition, a medium is called elastic if it returns to its initial state after the removal of external forces. This returns to its initial state due to internal stress. The elastic regime is characterized by a linear relationship between stress and strain, and is called linear elasticity. The classic example is a metal spring. This idea was first stated by Robert Hooke in 1675 as a Latin anagram "ceiiinosssttuu" whose solution he published in 1678 as "Ut tensio, sic vis" which means "As the extension, so the force."

This linear relationship is called Hooke's law. The classic model of linear elasticity is the perfect spring. Although the general proportionality constant between stress and strain in three dimensions is a 4th order tensor, when considering simple situations of higher symmetry such as a rod in one dimensional loading, the relationship may often be reduced to applications of Hooke's law.

The generalized Hook's law is defined as

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \tag{2.1}$$

where σ_{ij} is the stress tensor, ϵ_{kl} is the strain tensor and C_{ijkl} is the fourth-rank elasticity tensor which is the coefficient of linearity. Because most materials are elastic only under relatively small deformations, several assumptions are used to linearize the theory. Most importantly, higher order terms are generally discarded based on the small deformation assumption. In certain special cases, such as when considering a rubbery material, these assumptions may not be permissible. However, in general, elasticity refers to the linearized theory of the continuum stresses and strains.

2.3 Elastic stiffness

The coefficients C_{ijkl}, that describe the most general linear relationship between the two second rank tensors σ_{ij} and ϵ_{kl}, are the components of a fourth-rank tensor called the *elastic stiffness* tensor.
The *elastic stiffness* tensor has $3^4 = 81$ components. But since σ_{ij} and ϵ_{kl} are symmetric tensors, so the elastic constants remain unchanged under a permutation of i and j or k and l.

$$C_{ijkl} = C_{jikl}; C_{ijkl} = C_{ijlk}. \tag{2.2}$$

Hook's law can be written in terms of displacement as

$$\sigma_{ij} = \frac{1}{2}C_{ijkl}\left[\frac{\partial u_k}{\partial u_l} + \frac{\partial u_l}{\partial u_k}\right].$$

Since $C_{ijkl} = C_{ijlk}$, so both terms in the above eq. are equal, which results in

$$\sigma_{ij} = C_{ijkl}\frac{\partial u_l}{\partial u_k}. \tag{2.3}$$

The elastic stiffness tensor satisfies these symmetry conditions

$$C_{ijkl} = C_{jikl}, \tag{2.4}$$

$$C_{ijkl} = C_{ijlk}, \tag{2.5}$$

$$C_{ijkl} = C_{klij}. \tag{2.6}$$

These three conditions are written as

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{2.7}$$

From the above symmetries, we are left with 21 independent elastic constants instead of 81.

2.4 Contracted notations

Introducing the Voigt contracted notation

$ij\,(or kl) \leftrightarrow \alpha\,(or \beta)$

11↔1

22↔2

33↔3

23 or 32↔4

13 or 31↔5

12 or 21↔6

The transformation of this table can be written in the following form [20].

$$\alpha = \begin{cases} i, & (i = j) \\ 9 - i - j, & (i \neq j) \end{cases}, \qquad \beta = \begin{cases} k, & (k = l) \\ 9 - k - l, & (k \neq l) \end{cases}. \tag{2.8}$$

2.5 Isotropy

A continuum is called isotropic if its properties are same in all directions. This will happen if the tensor C_{ijkl} is an isotropic tensor. Most metals (steel, aluminum) are isotropic materials. They respond in the same way in all directions.

An isotropic tensor is defined as a tensor whose components remain invariant under all transformations of the reference frame. $\lambda\delta_{ij}$ is a second-rank isotropic tensor, where λ is a scalar. $\delta_{ij}\delta_{kl}$, $\delta_{il}\delta_{jk}$, $\delta_{ik}\delta_{jl}$ all are isotropic tensors of fourth-rank.

In an isotropic medium properties of a material are identical in all directions. The elasticity tensor C_{ijkl} must be a linear combination of the above three tensors

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_1\delta_{ik}\delta_{jl} + \mu_2\delta_{il}\delta_{jk},$$

where λ, μ_1, μ_2 are constants.

As

$$C_{ijkl} = C_{jikl},$$

we have

$$\mu_1 = \mu_2 = \mu.$$

Thus

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),$$

$$C_{11} = C_{1111} = \lambda\left(1\right) + \mu\left(1 + 1\right),$$

or

$$C_{11} = \lambda + 2\mu.$$

Similarly

$$C_{22} = \lambda + 2\mu,$$

$$C_{33} = \lambda + 2\mu,$$

$$C_{12} = C_{1122} = \lambda\,(1) + \mu\,(0)\,,$$

which implies

$$C_{12} = \lambda,$$

and

$$C_{13} = \lambda = C_{23},$$

$$C_{14} = C_{1123} = \lambda\,(0) + \mu\,(0)\,,$$

which implies

$$C_{14} = 0.$$

Similarly

$$C_{15} = 0 = C_{16} = C_{24} = C_{25} = C_{26} = C_{34} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56},$$

$$C_{44} = C_{2323} = \lambda\,(0) + \mu\,(1+0)\,,$$

which implies $C_{44} = \mu$, and $C_{55} = \mu = C_{66}$.

Thus an isotropic medium is characterized by the following matrix representation of the elasticity tensor

$$C_{\alpha\beta} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{2.9}$$

For an isotropic medium there are two independent parameters in terms of which all 81 components can be expressed.

2.6 Anisotropic materials

Anisotropy is the property of being directionally dependent, as opposed to isotropy, which means homogeneity in all directions. In an anisotropic material, properties of a material depend on the direction; for example, wood, which is easier to split along its grain than against it.

2.7 Equation of motion

In equilibrium we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = 0, \tag{2.10}$$

where f_i are the body forces. If the continuum is not in equilibrium then

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{2.11}$$

where u_i is displacement. Now

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \qquad \epsilon_{kl} = \frac{1}{2}[\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}].$$

Since

$$C_{ijkl} = C_{ijlk},$$

therefore

$$\sigma_{ij} = C_{ijkl}\frac{\partial u_l}{\partial x_k}. \tag{2.12}$$

Thus, using eq. (2.12) in eq. (2.11) and neglecting the body forces, we get

$$C_{ijkl}\frac{\partial^2 u_l}{\partial x_j \partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{2.13}$$

This is the equation of motion.

2.8 Equations of linear elasticity

Here we follow [5]. In terms of the infinitesimal strain tensor ϵ_{ij} and the corresponding stress σ_{ij}, the constitutive law for a linear isotropic elastic material can be written

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \tag{2.14}$$

for a compressible material, whereas

$$\sigma_{ij} = 2\mu\epsilon_{ij} - p\delta_{ij} \tag{2.15}$$

for an incompressible material.
With the small-strain tensor components

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.16}$$

the equation of motion is

$$\sigma_{ij,j} = \rho\ddot{u}_i, \tag{2.17}$$

on substituting for σ_{ij} from Eqs. (2.14) and (2.15), the equations of motion in terms of displacement u_i are

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} = \rho\ddot{u}_i \tag{2.18}$$

for a compressible material, and

$$\mu(u_{i,jj} + u_{j,ij}) + p_{,i} = \rho \ddot{u}_i \tag{2.19}$$

with

$$u_{j,j} = 0 \tag{2.20}$$

for an incompressible material.

2.9 Analysis for an incompressible material

The equation of motion for an incompressible material has been derived by Dowaikh [5]. Here we follow his derivation.

Consider the current position $\mathbf{x}$ of the particle $\mathbf{X}$, with a small displacement u, such that

$$\bar{u} = \bar{x} - \bar{X}. \tag{2.21}$$

Then

$$\bar{x} = \bar{X} + \bar{u}. \tag{2.22}$$

Or, in cartesian components

$$\begin{aligned} \bar{x_1} &= \bar{X_1} + \bar{u_1}, \\ \bar{x_2} &= \bar{X_2} + \bar{u_2}, \\ \bar{x_3} &= \bar{X_3} + \bar{u_3}, \end{aligned} \tag{2.23}$$

where in general u_1, u_2 and u_3 depend on x_1, x_2, x_3 and t.

The stress tensor σ_{jj}, for an incompressible material is

$$\sigma_{ij} = 2\mu\epsilon_{ij} - p\delta_{ij}, \tag{2.24}$$

with the small-strain tensor components

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.25}$$

Using these two equations, we have

$$\begin{aligned} \sigma_{11} &= 2\mu u_{1,1} - p, \\ \sigma_{22} &= 2\mu u_{2,2} - p, \\ \sigma_{33} &= 2\mu u_{3,3} - p, \\ \sigma_{12} = \sigma_{21} &= \mu(u_{1,2} + u_{2,1}), \\ \sigma_{13} = \sigma_{31} &= \mu(u_{1,3} + u_{3,1}), \\ \sigma_{23} = \sigma_{32} &= \mu(u_{2,3} + u_{3,2}), \end{aligned} \tag{2.26}$$

with

$$u_{1,1} + u_{2,2} + u_{3,3} = 0, \tag{2.27}$$

by using eq. (2.20).

Next we take $u_3 \equiv 0$ and assume u_1, u_2 are independent of x_3. Then eq. (2.27) becomes

$$u_{1,1} + u_{2,2} = 0. \tag{2.28}$$

Thus there exists a scalar function $\psi(x_1, x_2, t)$ such that

$$u_1 = \psi_{,2}, \qquad u_2 = -\psi_{,1}. \tag{2.29}$$

Also, Eqs. (2.26) become

$$\begin{aligned} \sigma_{11} &= 2\mu u_{1,1} - p, \\ \sigma_{22} &= 2\mu u_{2,2} - p, \\ \sigma_{33} &= -p, \\ \sigma_{12} = \sigma_{21} &= \mu(u_{1,2} + u_{2,1}), \end{aligned} \tag{2.30}$$

where p is independent of x_3. The equation of motion then simplify to

$$\begin{aligned} \sigma_{11,1} + \sigma_{21,2} &= \rho\ddot{u}_1, \\ \sigma_{12,1} + \sigma_{22,2} &= \rho\ddot{u}_2. \end{aligned} \tag{2.31}$$

From Eqs.(2.30), we have

$$\begin{aligned} \sigma_{11,1} &= 2\mu u_{1,11} - p_{,1}, \\ \sigma_{21,2} &= \mu(u_{1,22} + u_{2,12}), \\ \sigma_{12,1} &= \mu(u_{1,12} + u_{2,11}), \\ \sigma_{22,2} &= 2\mu u_{2,22} - p_{,2}, \end{aligned} \tag{2.32}$$

or, in terms of ψ

$$\begin{aligned} \sigma_{11,1} &= 2\mu\psi_{,112} - p_{,1}, \\ \sigma_{21,2} &= \mu(\psi_{,222} - \psi_{,112}), \\ \sigma_{12,1} &= \mu(\psi_{,122} - \psi_{,111}), \\ \sigma_{22,2} &= -(2\mu\psi_{,122} + p_{,2}). \end{aligned} \tag{2.33}$$

Using Eqs. (2.32) in (2.31), we have

$$\begin{aligned} 2\mu u_{1,11} + \mu(u_{1,22} + u_{2,12}) - p_{,1} &= \rho\ddot{u}_1, \\ 2\mu u_{2,22} + \mu(u_{1,12} + u_{2,11}) - p_{,2} &= \rho\ddot{u}_2, \end{aligned} \tag{2.34}$$

or, in terms of ψ

$$\mu\psi_{,112} + \mu\psi_{,222} - p_{,1} = \rho\ddot{\psi}_{,2},$$

$$\mu\psi_{,122} - \mu\psi_{,111} - p_{,2} = -\rho\ddot{\psi}_{,1}. \tag{2.35}$$

To eliminate p we must differentiate Eqs. (2.35) with respect to x_2 and x_1 respectively and the subtraction of the two equations yields

$$\mu(\psi_{,1111} + \psi_{,2222}) + 2\mu\psi_{,1122} - \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}) = 0. \tag{2.36}$$

We now choose axes so that u corresponds to a wave propagating along the x_1-axis, and we take $\psi(x_1, x_2, t)$ to have the form

$$\psi = f(x_2)\exp[iw(t - \frac{x_1}{c})]. \tag{2.37}$$

This represents a wave propagating with (a constant) wave speed c in the x_1-direction. Where ω is the angular frequency.
We also assume that the x_2-variation of ψ is of the form $\exp(-skx_2)$, where $k = \frac{\omega}{c}$ is the wave number. Then eq. (2.36) gives

$$\mu s^4 - (2\mu - \rho c^2)s^2 + \mu - \rho c^2 = 0. \tag{2.38}$$

Equation (2.38) is a quadratic equation for s^2. Suppose it has roots s_1^2 and s_2^2. Then

$$s_1^2 + s_2^2 = 2 - \frac{\rho c^2}{\mu}, \qquad s_1^2 s_2^2 = 1 - \frac{\rho c^2}{\mu}. \tag{2.39}$$

In fact, the roots s_1^2 , s_2^2 are 1 and $1 - \frac{\rho c^2}{\mu}$.
We also assume that p has the same time and spatial dependence as ψ. It follows from (2.35) that

$$ikp = k^2(\mu - \rho c^2)\psi_{,2} - \mu\psi_{,222}. \tag{2.40}$$

2.10 Hydrostatic pressure

The pressure exerted by a fluid at equilibrium at a given point within the fluid, due to the force of gravity is called the hydrostatic pressure . Hydrostatic pressure increases in proportion to depth measured from the surface because of the increasing weight of fluid exerting downward force from above.

2.11 Wave motion

A wave is a disturbance that travels through space and time, usually accompanied by the transfer of energy.

Waves travel and the wave motion transfers energy from one point to another, often with no permanent displacement of the particles of the medium i.e, with little or no associated mass transport. They consist, instead, of oscillations or vibrations around almost fixed locations. For example, a cork on rippling water will bob up and down, staying in about the same place while the wave itself moves onwards.

Consider a traveling transverse wave (which may be a pulse) on a string (the medium). Consider the string to have a single spatial dimension. Consider this wave as traveling in the x direction in space, e.g. let the positive x direction be to the right, and the negative x direction be to the left. This wave can then be described by the two-dimensional functions
$u(x,t) = F(x - vt)$ (wave form F traveling to the right)
$u(x,t) = G(x + vt)$ (wave form G traveling to the left)
where v is constant velocity of the wave.

There are two unit vectors associated with a wave. The propagation vector say $\mathbf{n}$ which specifies the direction in which a wave is traveling and the polarization vector say $\mathbf{p}$ which specifies the direction in which the displacement is taking place.

2.12 Plane Wave

In the physics of wave propagation, a plane wave is a constant-frequency wave whose wavefronts (surfaces of constant phase) are infinite parallel planes of constant peak-to-peak amplitude normal to the phase velocity vector.

It is not possible in practice to have a true plane wave; only a plane wave of infinite extent will propagate as a plane wave. However, many waves are approximately plane waves in a localized region of space. For example, a localized source such as an antenna produces a field that is approximately a plane wave far from the antenna in its far-field region. Similarly, if the length scales are much longer than the wave's wavelength, as is often the case for light in the field of optics, one can treat the waves as light rays which correspond locally to plane waves. A convenient representation of a plane harmonic wave is given by

$$u = A\mathbf{p}e^{i\eta}, \tag{2.41}$$

where

$$\eta = k(\mathbf{x}.\mathbf{n} - ct). \tag{2.42}$$

Eq. (2.41) describes a plane wave propagating with phase velocity c in a direction of the unit propagation vector $\mathbf{n}$. There are mainly two types of plane waves
1.Transvers waves.
2.Longitudinal waves.
A main distinction is between them is that, in transverse waves the disturbance occurs in a direction perpendicular (at right angles) to the motion of the wave and, in longitudinal

waves the disturbance is in the same direction as the wave propagation.

Plane waves are further classified into two types.

1.P-wave or longitudinal wave.

2.S-wave or transverse wave.

P-waves are type of elastic waves, also called seismic waves, that can travel through gases (as sound waves), solids and liquids, including the Earth. P-waves are produced by earthquakes and recorded by seismographs. The name P-wave stands either for primary wave, as it has the highest velocity and is therefore the first to be recorded, or pressure wave, [13] as it is formed from alternating compressions and rarefaction.

The velocity of a P-wave in a homogeneous isotropic medium is given by

$$v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

where μ is the shear modulus (modulus of rigidity, sometimes denoted as G and also called the second Lamé parameter), ρ is the density of the material through which the wave propagates, and λ is the first Lamé parameter.

In isotropic and homogeneous solids, the polarization of a P-wave is always longitudinal; thus, the particles in the solid have vibrations along or parallel to the travel direction of the wave energy.

A type of seismic wave, the S-wave, secondary wave, or shear wave (sometimes called an elastic S-wave) is one of the two main types of elastic body waves, so named because they move through the body of an object, unlike surface waves.

The S-wave moves as a shear or transverse wave, so motion is perpendicular to the direction of wave propagation. S-waves are like waves in a rope, as opposed to waves moving through a slinky, the P-wave. The wave moves through elastic media, and the main restoring force comes from shear effects. Its name, S for secondary, comes from the fact that it is the second direct arrival on an earthquake seismogram, after the compressional primary wave, or P-wave, because S-waves travel slower in rocks. The velocity of S-wave in an isotropic medium is given by

$$v_s = \sqrt{\frac{\mu}{\rho}}.$$

2.13 Surface wave

The criterion for surface waves is that the propagating disturbance decays exponentially with distance from the surface.

In physics, a surface wave is a mechanical wave that propagates along the interface between differing media, usually two fluids with different densities. In radio transmission, a ground wave is a surface wave that propagates close to the surface of the Earth.

2.14 Rayleigh wave

Rayleigh waves travel across surfaces, and are thus a type of surface wave. In perfectly homogenous, isotropic and infinite materials, Rayleigh waves would not be apparent. In seismology, Rayleigh waves (called "ground roll") are the most important type of surface waves [22]. Most obvious close to the surface of mediums, Rayleigh waves are made of longitudinal and traverse motion that decreases exponentially in amplitude as distance from the surface increases. Rayleigh waves are a type of surface acoustic wave that travels on solids. They are produced on the Earth by earthquakes, in which case they are also known as "ground roll", or by other sources of seismic energy such as ocean waves [14] an explosion or even a sledgehammer impact. They can also be produced in materials by many mechanisms, including piezo-electricity, and are frequently used in non-destructive testing for detecting defects. When guided in layers they are referred to as Lamb waves, Rayleigh-Lamb waves, or generalized Rayleigh waves.

2.15 Rayleigh-Lamb frequency Equations

For time-harmonic wave motion in plane strain of an elastic layer the equation relating frequency or phase velocity to the wave number can be derived on the basis of the principle of constructive interference. This approach was pursued by Tolstoy and Usdin [21].
In most treatments, however, the alternative approach is followed of employing expressions for the field variables representing a standing wave in the x_2 direction and a propagating wave in the x_1 direction. The expressions are then substituted into the boundary conditions to derive the frequency equation. This more straightforward approach will be followed here [3].
It is convenient to decompose the displacement field by the use of scalar and vector potentials. For motion in plane strain in the (x_1x_2)-plane, we have

$$u_3 \equiv 0, \qquad \frac{\partial}{\partial x_3}(\;\;) \equiv 0. \tag{2.43}$$

Eq. in (x_1x_2)-plane is then reduces to

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}, \tag{2.44}$$

$$u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}. \tag{2.45}$$

The relevant components of the stress tensor follow from Hook's law as

$$\sigma_{21} = \mu(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) = \mu(2\frac{\partial^2 \phi}{\partial x_1 \partial x_2} - \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}), \tag{2.46}$$

$$\sigma_{22} = \lambda(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}) + 2\mu\frac{\partial u_2}{\partial x_2} = \lambda(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}) + 2\mu(\frac{\partial^2 \phi}{\partial x_2^2} - \frac{\partial^2 \psi}{\partial x_1 \partial x_2}). \tag{2.47}$$

The potentials ϕ and ψ satisfy wave equations, which for plane strain are two-dimensional

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2}, \tag{2.48}$$

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = \frac{1}{c_T^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{2.49}$$

To investigate wave motion in the elastic layer, we consider solutions of (2.48) and (2.49) of the form (method of separation of variables)

$$\phi = \Phi(x_2)\exp[i(kx_1 - \omega t)], \tag{2.50}$$

$$\psi = \Psi(x_2)\exp[i(kx_1 - \omega t)]. \tag{2.51}$$

Substituting for ϕ and ψ from Eqs. (2.50) and (2.51) into Eqs. (2.48) and (2.49) respectively, the solutions of the resulting equations are obtained as

$$\Phi(x_2) = A_1\sin(px_2) + A_2\cos(px_2), \tag{2.52}$$

$$\Psi(x_2) = B_1\sin(qx_2) + B_2\cos(qx_2), \tag{2.53}$$

where

$$p^2 = \frac{\omega^2}{c_L^2} - k^2, \qquad q^2 = \frac{\omega^2}{c_T^2} - k^2. \tag{2.54}$$

In the expressions for the displacements and the stress components, which are obtained from (2.44)-(2.47), the term $\exp[i(kx_1 - \omega t)]$ appears as a multiplier. Since the exponential appears in all of the expressions it does not play a further role in determination of the frequency equation and it is therefore omitted in the sequel. Thus we write

$$u_1 = ik\Phi + \frac{d\Psi}{dx_2}, \tag{2.55}$$

$$u_2 = \frac{d\Phi}{dx_2} - ik\Psi, \tag{2.56}$$

$$\sigma_{21} = \mu(2ik\frac{d\Phi}{dx_2} + k^2\,\Psi + \frac{d^2\,\Psi}{dx_2^2}), \tag{2.57}$$

$$\sigma_{22} = \lambda(-k^2\Phi + \frac{d^2\Phi}{dx_2^2}) + 2\mu(\frac{d^2\Phi}{dx_2^2} - ik\frac{d\Psi}{dx_2}). \tag{2.58}$$

Inspection of Eqs. (2.55) and (2.56) shows that displacement components can be written in terms of elementary functions. For the displacements in the x_1-direction the motion is symmetric(antisymmetric) with regard to $x_2 = 0$, if u_1 contains cosines(sines). The displacements in the x_2-direction is symmetric(antisymmetric) if u_2 contains sines(cosines). The modes of wave propagation in the elastic layer may thus be split up into two systems of symmetric and anti symmetric modes, respectively:

Symmetric modes:

$$\Phi = A_2 \cos(px_2)$$

$$\Psi = B_1 \sin(qx_2)$$

$$u_1 = ikA_2 \cos(px_2) + qB_1 \cos(qx_2)$$

$$u_2 = -pA_2 \sin(px_2) - ikB_1 \sin(qx_2)$$

$$\sigma_{21} = \mu[-2ikpA_2 \sin(px_2) + (k^2 - q^2)B_1 \sin(qx_2)]$$

$$\sigma_{22} = -\lambda(k^2 + p^2)A_2 \cos(px_2) - 2\mu[p^2 A_2 \cos(px_2) + ikqB_1 \cos(qx_2)].$$

Antisymmetric modes:

$$\Phi = A_1 \sin(px_2)$$

$$\Psi = B_2 \cos(qx_2)$$

$$u_1 = ikA_1 \sin(px_2) - qB_2 \sin(qx_2)$$

$$u_2 = -pA_1 \cos(px_2) - ikB_2 \cos(qx_2)$$

$$\sigma_{21} = \mu[2ikpA_1 \cos(px_2) + (k^2 - q^2)B_2 \cos(qx_2)]$$

$$\sigma_{22} = -\lambda(k^2 + p^2)A_1 \sin(px_2) - 2\mu[p^2 A_1 \sin(px_2) - ikqB_2 \sin(qx_2)].$$

The frequency relation, i.e. the expression relating ω to the wave number k is now obtained from the boundary conditions. If the boundaries are free, we have at $x_2 = \pm h$

$$\sigma_{21} = \sigma_{22} \equiv 0.$$

For the symmetric modes the boundary conditions yield a system of two homogeneous equations for the constants A_2 and B_1. Similarly, for the antisymmetric modes two homogeneous equations for the constants A_1 and B_2 are obtained. Since the systems are homogeneous, the determinant of the coefficients must vanish, which yields the frequency equation. Thus, for the symmetric modes we find

$$\frac{(k^2 - q^2)\sin(qh)}{2ikp\sin(ph)} = -\frac{2\mu ikq\cos(qh)}{(\lambda k^2 + \lambda p^2 + 2\mu p^2)\cos(ph)}.$$

This equation can be written as

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{4k^2pq}{(q^2 - k^2)^2}. \tag{2.59}$$

For the antisymmetric modes the boundary conditions yield

$$\frac{(k^2 - q^2)\cos(qh)}{2ikp\cos(ph)} = -\frac{2\mu ikq\sin(qh)}{(\lambda k^2 + \lambda p^2 + 2\mu p^2)\sin(ph)},$$

or

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{(q^2 - k^2)^2}{4k^2pq}. \tag{2.60}$$

Eqs. (2.59) and (2.60) are the well-known Rayleigh-Lamb frequency equations.

Chapter 3

Wave motion in orthotropic incompressible materials

3.1 Introduction

This chapter is concerned with propagation of elastic plane waves in orthotropic incompressible materials which are examined under plane strain conditions in a plane of symmetry.

We review [15] in which Nair and Sotiropoulos discussed elastic waves in orthotropic incompressible materials. Here we obtain a formula in which the wave speed v is independent of propagation direction and finally give some classification of materials.

3.2 Harmonic waves in orthotropic incompressible media

Assume that for an orthotropic incompressible medium, we have three material axes of symmetry denoted by x_1, x_2 and x_3. Thus for an orthotropic material we have

$$C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{3.1}$$

Also the linear stress-strain relations for the material are

$$\begin{aligned}
\sigma_{11} &= -p + C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}\epsilon_{33}, \\
\sigma_{22} &= -p + C_{12}\epsilon_{11} + C_{22}\epsilon_{22} + C_{23}\epsilon_{33}, \\
\sigma_{33} &= -p + C_{13}\epsilon_{11} + C_{23}\epsilon_{22} + C_{33}\epsilon_{33}, \\
\sigma_{23} &= 2C_{44}\epsilon_{23}, \\
\sigma_{13} &= 2C_{55}\epsilon_{13}, \\
\sigma_{12} &= 2C_{66}\epsilon_{12},
\end{aligned} \tag{3.2}$$

where σ's, ϵ's and C's denote the stress, strain and elasticity constant components respectively and p is the hydrostatic pressure. Also we have

$$\epsilon_{ij} = \frac{(u_{i,j} + u_{j,i})}{2}, \qquad i, j = 1, 2, 3. \tag{3.3}$$

We assume a plane wave motion in (x_1, x_2) plane and incompressibility which implies,

$$\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0, \tag{3.4}$$

and

$$\epsilon_{11} + \epsilon_{22} = 0. \tag{3.5}$$

These conditions reduce the orthotropic stress-strain relations (3.2) to

$$\begin{aligned}
\sigma_{11} &= -p + (C_{11} - C_{12})\,\epsilon_{11}, \\
\sigma_{22} &= -p + (C_{12} - C_{22})\,\epsilon_{11}, \\
\sigma_{12} &= 2C_{66}\epsilon_{12}.
\end{aligned} \tag{3.6}$$

To get the positive strain energy function the following inequalities need to be true

$$C_{66} \geqslant 0, \qquad C_{11} + C_{22} - 2C_{12} \geqslant 0, . \tag{3.7}$$

These are obtained by taking $\epsilon_{11} = 0$ for the first and $\epsilon_{12} = 0$. From eq. (3.5) we have

$$u_{1,1} + u_{2,2} = 0, \tag{3.8}$$

which is satisfied by introducing a scalar function $\psi(x_1, x_2, t)$, such that

$$u_1 = \psi_{,2}, \qquad u_2 = -\psi_{,1}. \tag{3.9}$$

If the body forces are neglected the Equation of motion is given as

$$\sigma ij, j = \rho \ddot{u}_i. \tag{3.10}$$

Here we have

$$\sigma_{11,1} + \sigma_{12,2} = \rho \ddot{u}_1, \qquad \sigma_{12,1} + \sigma_{22,2} = \rho \ddot{u}_2 \tag{3.11}$$

as equations of motion. Using Eqs. (3.6), (3.9) and (3.3) in the equations of motion

$$\sigma_{11,1} = -p_{,1} + (C_{11} - C_{12})\epsilon_{11,1},$$

$$\sigma_{11,1} = -p_{,1} + (C_{11} - C_{12})\psi_{,112},$$

$$\sigma_{12,2} = 2C_{66}\epsilon_{12,2},$$

$$\sigma_{12,2} = C_{66}\left(\psi_{,222} - \psi_{,112}\right).$$

So the 1st equation of motion become

$$-p_{,1} + (C_{11} - C_{12} - C_{66}) + C_{66}\psi_{,222} = \rho\ddot{\psi}_{,2}. \tag{3.12}$$

Similarly

$$\sigma_{12,1} = C_{66}\left(\psi_{,122} - \psi_{,111}\right),$$

$$\sigma_{22,2} = -p_{,2} + \left(C_{12} - C_{22}\right)\psi_{,122},$$

$$p_{,2} + \left(C_{22} - C_{12} - C_{66}\right)\psi_{,122} + C_{66}\psi_{,111} = \rho\ddot{\psi}_{,1}. \tag{3.13}$$

To eliminate p, take partial derivative of eq. (3.12) and eq. (3.13) w.r.t x_2 and x_1 respectively,

$$-p_{,12} + \left(C_{11} - C_{12} - C_{66}\right)\psi_{,1122} + C_{66}\psi_{,2222} = \rho\ddot{\psi}_{,22}, \tag{3.14}$$

$$p_{,12} + \left(C_{22} - C_{12} - C_{66}\right)\psi_{,1122} + C_{66}\psi_{,1111} = \rho\ddot{\psi}_{,11}. \tag{3.15}$$

Adding the above two equations

$$\left(C_{11} - 2C_{12} + C_{22}\right)\psi_{,1122} - 2C_{66}\psi_{,1122} + C_{66}\left(\psi_{,1111} + \psi_{,2222}\right) = \rho\Delta\ddot{\psi}, \tag{3.16}$$

$$C_{66}\Delta^2\psi + \left(C_{11} - 2C_{12} + C_{22} - 4C_{66}\right)\psi_{,1122} = \rho\Delta\ddot{\psi},$$

$$C_{66}\Delta^2\psi + 4C_{66}\left(\frac{C_{11} + C_{22} - 2C_{12}}{4C_{66}} - 1\right)\psi_{,1122} = \rho\Delta\ddot{\psi}.$$

Thus the scalar function ψ satisfies

$$\gamma[\Delta^2\psi + 4\beta\psi_{,1122}] = \rho\Delta\ddot{\psi}, \tag{3.17}$$

where,

$$\gamma = C_{66}, \qquad \beta = \frac{C_{11} + C_{22} - 2C_{12}}{4C_{66}} - 1. \tag{3.18}$$

The inequalities given in (3.7) imply

$$\gamma \geqslant 0, \qquad \beta \geqslant -1 \tag{3.19}$$

In [15] Nair and Sotiropoulos use a new material coordinate system $(x_1^{'}, x_2^{'})$, obtained by the rotation about an angle of $\frac{\pi}{4}$, in an effort to get β positive for the range $-1 \leqslant \beta \leqslant 0$. However we shall show that such an effort is futile and one ended up with the same

equations as those one started with.
The matrix of rotation is given as

$$X' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.20}$$

The laplace operator Δ' in the new coordinates system becomes

$$\Delta' = \frac{1}{2}\left[\frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2}\right], \tag{3.21}$$

which is identical to Δ. In fact, in this paper the authors used the same components of elasticity tensor as they have used before transformation and only considered the change in Δ and the mixed derivative occurring in eq. (3.17). So we have,

$$4\psi_{,1122} = \left(\Delta'\right)^2 \psi' - 4\psi'_{,1122}. \tag{3.22}$$

Thus, eq. (3.17) in the new coordinate system becomes

$$\gamma\left(\Delta'\right)^2 \psi' + \gamma\beta\left[\left(\Delta'\right)^2 \psi' - 4\psi'_{,1122}\right] = \rho\Delta\ddot{\psi}',$$

$$\gamma\left(1+\beta\right)\left[\left(\Delta'\right)^2 \psi' + 4\beta'\psi'_{,1122}\right] = \rho\Delta\ddot{\psi}',$$

$$\gamma'\left[\left(\Delta'\right)^2 \psi' + 4\beta'\psi'_{,1122}\right] = \rho\Delta\ddot{\psi}', \tag{3.23}$$

where

$$\gamma' = \gamma\left(1+\beta\right), \qquad \beta' = -\beta/\left(1+\beta\right). \tag{3.24}$$

By using the values of γ and β given in Eqs. (3.18), we have

$$\delta = \gamma' = C_{11} + C_{22} - 2C_{12}, \qquad \beta' = \frac{4C_{66}}{C_{11} + C_{22} - 2C_{12}} - 1. \tag{3.25}$$

Thus the authors concluded that β will be positive by proper relabeling of the axes. Since the components of the elasticity tensor also change under the coordinate transformation, we claim that we have to use these new transformed components in the governing eq. (3.17) as well. The transformed components are given as

$$C'_{11} = \frac{1}{4}C_{11} + \frac{1}{4}C_{22} + \frac{1}{2}C_{12} + C_{66}, \tag{3.26}$$

$$C'_{22} = \frac{1}{4}C_{11} + \frac{1}{4}C_{22} + \frac{1}{2}C_{12} + C_{66}, \tag{3.27}$$

$$C'_{12} = \frac{1}{4}C_{11} + \frac{1}{4}C_{22} + \frac{1}{2}C_{12} - C_{66}, \tag{3.28}$$

$$C'_{66} = \frac{1}{4}C_{11} + \frac{1}{4}C_{22} - \frac{1}{2}C_{12}. \tag{3.29}$$

Now the orthotropic stress-strain relations are

$$\sigma'_{11} = -p + \left(C'_{11} - C'_{12}\right)\epsilon'_{11},$$

$$\sigma'_{22} = -p + \left(C'_{12} - C'_{22}\right)\epsilon'_{11},$$

$$\sigma'_{12} = 2C'_{66}\epsilon'_{12}. \tag{3.30}$$

Using the values of new components we get

$$\sigma'_{11} = -p + 2C_{66}\epsilon'_{11},$$

$$\sigma'_{22} = -p - 2C_{66}\epsilon'_{11},$$

$$\sigma'_{12} = \frac{1}{2}\left(C_{11} + C_{22} - 2C_{12}\right)\epsilon'_{12}. \tag{3.31}$$

Thus, the equations of motion become

$$-p_{,1} + 2C_{66}\psi'_{,112} + \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\left[\psi'_{,222} - \psi'_{,112}\right] = \rho\ddot{\psi}'_{,2}$$

$$-p_{,1} + \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\psi'_{,222} + \left[2C_{66} - \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\right]\psi'_{,112} = \rho\ddot{\psi}'_{,2} \tag{3.32}$$

$$p_{,2} + \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\psi'_{,111} + \left[2C_{66} - \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\right]\psi'_{,122} = \rho\ddot{\psi}'_{,1}. \tag{3.33}$$

To eliminate p, take partial derivative of eq. (3.32) and (3.33) w.r.t x'_2 and x'_1 respectively

$$-p_{,12} + \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\psi'_{,2222} + \left[2C_{66} - \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\right]\psi'_{,1122} = \rho\ddot{\psi}'_{,22}, \tag{3.34}$$

$$p_{,12} + \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\psi'_{,1111} + \left[2C_{66} - \frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\right]\psi'_{,1122} = \rho\ddot{\psi}'_{,11}. \tag{3.35}$$

Adding the above two Eqs.

$$\frac{1}{4}\left(C_{11} + C_{22} - 2C_{12}\right)\left[\psi'_{,1111} + \psi'_{,2222}\right] + \left[4C_{66} - \frac{1}{2}\left(C_{11} + C_{22} - 2C_{12}\right)\right]\psi'_{,1122} = \rho\Delta'\ddot{\psi}'$$

$$\left(C_{11} + C_{22} - 2C_{12}\right)\left(\Delta'\right)^2\psi' + \left[4C_{66} - \left(C_{11} + C_{22} - 2C_{12}\right)\right]\psi'_{,1122} = \rho\Delta'\ddot{\psi}'$$

$$\gamma'\left[\left(\Delta'\right)^2\psi' + 4\beta'\psi'_{,1122}\right] = \rho\Delta\ddot{\psi}'. \tag{3.36}$$

which is the same as eq. (3.23), γ' and β' are also the same. Thus, there is no use of the transformed coordinate system for the range $-1 \leqslant \beta \leqslant 0$. β could be negative. For time-harmonic plane waves ψ can be expressed as

$$\psi = A\exp\left[ik\left(x_1\cos\theta + x_2\sin\theta - ct\right)\right], \tag{3.37}$$

where A is the potential amplitude. Since by eq. (3.16), we have

$$\gamma\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = \rho\left(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}\right). \tag{3.38}$$

Material	C_{11}	C_{12}	C_{22}	C_{66}
Isotropic	$\lambda + 2\mu$	λ	$\lambda + 2\mu$	μ
Transversely Isotropic	C_{11}	C_{12}	C_{11}	$\frac{C_{11}-C_{12}}{2}$
Trigonal	C_{11}	C_{12}	C_{11}	$\frac{C_{11}-C_{12}}{2}$

Table 3.1: **Group of materials** for which $\beta - \gamma = 0$, where C's are the components of corresponding elasticity tensor.

Substituting ψ into this Eq. yields

$$\gamma \cos^4\theta + \gamma \sin^4\theta + 2\beta \cos^2\theta \sin^2\theta = \rho c^2$$

$$\gamma \left[\left(\cos^2\theta + \sin^2\theta \right)^2 - 2\cos^2\theta \sin^2\theta \right] + 2\beta \cos^2\theta \sin^2\theta = \rho c^2$$

$$\gamma \left[1 - 2\cos^2\theta \sin^2\theta \right] + 2\beta \cos^2\theta \sin^2\theta = \rho c^2$$

$$\gamma + 2\left(\beta - \gamma \right) \cos^2\theta \sin^2\theta = \rho c^2$$

$$\gamma + \frac{\beta - \gamma}{2} \sin^2 2\theta = \rho c^2$$

$$\gamma + \frac{\beta - \gamma}{2} \left(1 - \cos 4\theta \right) = \rho c^2,$$

and hence,

$$\frac{\beta}{4} + \frac{3\gamma}{4} - \frac{\beta - \gamma}{4} \cos^4\theta = \rho c^2. \tag{3.39}$$

Here we have,

$$\gamma \equiv C_{66} > 0, \qquad \delta \equiv C_{11} + C_{22} - 2C_{12} > 0. \tag{3.40}$$

And

$$2\beta = \delta - 2\gamma, \qquad \Longrightarrow \qquad \beta - \gamma = \frac{\delta}{2} - 2\gamma \tag{3.41}$$

For materials with $(\beta - \gamma) = 0$, the speed c is independent of the direction of propagation vector. The materials which lie in this aforesaid group are given in Tab.3.1.

Material	$C_{11}(10^{10}\frac{N}{m^2})$	$C_{12}(10^{10}\frac{N}{m^2})$	$C_{22}(10^{10}\frac{N}{m^2})$	$C_{66}(10^{10}\frac{N}{m^2})$	$\rho(10^3\frac{Kg}{m^3})$
Iodic Acid	3.01	1.61	5.8	1.58	4.64
Barium Sodium niobate	25.9	10.4	24.7	7.6	5.30

Table 3.2: Elastic stiffness constants

For orthotropic materials $(\beta - \gamma)$ can be negative in some cases, for example in case of Iodic Acid (HIO_3) and also for Barium Sodium Niobate $(Ba_2NaNb_5O_15)$. Here in Tab.3.2 the values of stiffness constants for these materials are given [6].

If $(\beta - \gamma)$ is positive, the wave speed c is maximum when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$
and minimum when $\theta = 0, \frac{\pi}{2}$.
The case would be opposite if $(\beta - \gamma)$ is negative.

Chapter 4

Wave propagation in an incompressible orthotropic elastic plate

This chapter is concerned with finding a dispersion relation for the propagation of a harmonic wave in an orthotropic plate. In the limit of long wave length the dispersion relation reduces to the equation for the Rayleigh wave in an orthotropic half space. For this purpose we review the paper by Ogden and Vinh [17] who have found an explicit formula for the Rayleigh wave speed.

We shall analyze this dispersion relation by using the technique developed by Ahmad [1] for the case of Rayleigh-Lamb equation for an isotropic material. He obtained an explicit solution of the Rayleigh-Lamb equation which can be used to plot the dispersion curves and the dispersion relation for the Rayleigh-Lamb modes in an elastic plate can be replaced by this simpler equation, which admits exact solution. Here we use the same approach for an incompressible orthotropic material.

4.1 Dispersion relation for the plate

In this section we obtain a dispersion relation for an incompressible orthotropic wave guide. The basic equations and notations have been presented in section (3.2) for describing motion in this material. Assume that the x_2-axis is normal to the surface of the plate. The origin is chosen in the middle so that the planes $x_2 = \pm h$ form the boundaries. The x_1-axis is chosen in the direction of propagation of waves. The problem is independent of x_3-coordinate. The strain energy function is positive, provided the inequalities given in (3.7) are satisfied

$$\gamma \equiv C_{66} > 0, \qquad \delta \equiv C_{11} + C_{22} - 2C_{12} > 0. \tag{4.1}$$

Since by eq. (3.16) we have

$$\gamma\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = \rho\left(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}\right) \tag{4.2}$$

where

$$2\beta = \delta - 2\gamma. \tag{4.3}$$

If the boundaries are free, we have at $x_2 = \pm h$

$$\sigma_{12} = \sigma_{22} = 0. \tag{4.4}$$

We have

$$\begin{aligned} &\gamma\left(\psi_{,22} - \psi_{,11}\right) = 0 \\ &\gamma\left(\psi_{,222} - \psi_{,112}\right) + \delta\psi_{,112} - \rho\ddot{\psi}_{,2} = 0 \qquad at \quad x_2 = \pm h \end{aligned} \tag{4.5}$$

Now consider harmonic waves propagating in the x_1 direction. We write ψ in the form

$$\psi\left(x_1, x_2, t\right) = \phi\left(y\right)\exp\left[ik\left(x_1 - ct\right)\right], \tag{4.6}$$

where k is the wave number, $y = kx_2$, c is the wave speed and the function ϕ is to be determined. Substitute eq. (4.6) in eq. (4.2)

$$k^4\gamma\phi - 2k^4\beta\phi^{''} + k^4\gamma\phi^{''''} = c^2\rho\left[\phi k^4 - k^4\phi^{''}\right].$$

Hence we get

$$\gamma\phi^{''''} - \left(2\beta - \rho c^2\right)\phi^{''} + \left(\gamma - \rho c^2\right)\phi = 0, \tag{4.7}$$

and the boundary conditions given in (4.5) yields

$$\phi^{''}\left(\pm h\right) + \phi\left(\pm h\right) = 0 \quad \text{Since} \quad \gamma \neq 0, \tag{4.8}$$

$$\gamma\phi^{'''}\left(\pm h\right) + \left(\gamma - \delta + \rho c^2\right)\phi^{'}\left(\pm h\right) = 0. \tag{4.9}$$

To find ϕ, we have to solve eq. (4.7) with the boundary conditions given in (4.9). Assume that the general solution for $\phi\left(y\right)$ that satisfies these boundary conditions is

$$\phi\left(y\right) = A\cos(s_1 y) + B\cos(s_2 y), \tag{4.10}$$

where A and B are constants, while s_1 and s_2 are the solutions of the equation

$$\gamma s^4 + \left(2\beta - \rho c^2\right)s^2 + \left(\gamma - \rho c^2\right) = 0, \tag{4.11}$$

which is obtained by putting

$$\psi\left(x_1, x_2, t\right) = A\cos(skx_2)\exp\left[ik\left(x_1 - ct\right)\right]$$

in eq. (4.2). Thus, we can write ψ in the form

$$\psi\left(x_1, x_2, t\right) = \left[A\cos\left(s_1 kx_2\right) + B\cos\left(s_2 kx_2\right)\right]\exp\left[ik\left(x_1 - ct\right)\right]. \tag{4.12}$$

Required partial derivatives of ψ are

$$\psi_{,1} = ik\left[A\cos(s_1kx_2) + B\cos(s_2kx_2)\right]\exp\left[ik\left(x_1 - ct\right)\right]$$

$$\psi_{,11} = -k^2\left[A\cos(s_1kx_2) + B\cos(s_2kx_2)\right]\exp\left[ik\left(x_1 - ct\right)\right]$$

$$\psi_{,112} = -k^2\left[-s_1kA\sin(s_1kx_2) - s_2kB\sin(s_2kx_2)\right]\exp\left[ik\left(x_1 - ct\right)\right]$$

$$\psi_{,2} = \left[-s_1kA\sin(s_1kx_2) - s_2kB\sin(s_2kx_2)\right]\exp\left[ik\left(x_1 - ct\right)\right]$$

$$\psi_{,22} = \left[-s_1^2k^2A\cos(s_1kx_2) - s_2^2k^2B\cos(s_2kx_2)\right]\exp\left[ik\left(x_1 - ct\right)\right]$$

$$\psi_{,222} = \left[s_1^3k^3A\sin(s_1kx_2) + s_2^3k^3B\sin(s_2kx_2)\right]\exp\left[ik\left(x_1 - ct\right)\right].$$

We have 1st boundary condition as

$$\gamma\left(\psi_{,22} - \psi_{,11}\right) = 0, \qquad at \;\; x_2 = \pm h \tag{4.13}$$

Substitution of (4.12) into this boundary condition say at $x_2 = h$ yields

$$\gamma\left[-s_1^2k^2A\cos(s_1kh) - s_2^2k^2B\cos(s_2kh) + k^2A\cos(s_1kh) + k^2B\cos(s_2kxh)\right] = 0$$

$$\left(1 - s_1^2\right)A\cos(s_1kh) + \left(1 - s_2^2\right)B\cos(s_2kh) = 0. \tag{4.14}$$

The second boundary eq. yields

$$\left[\gamma s_1^3 - \gamma s_1 + \delta s_1 - \rho c^2 s_1\right]A\sin(s_1kh) + \left[\gamma s_2^3 - \gamma s_2 + \delta s_2 - \rho c^2 s_2\right]B\sin(s_2kh) = 0. \tag{4.15}$$

Thus, we have system of Eqs. (4.14) and (4.15) and for a non-trivial solution the determinant must vanishes, hence

$$\begin{aligned}&\left(1 - s_1^2\right)\left[\gamma s_2^3 - \gamma s_2 + \delta s_2 - \rho c^2 s_2\right]\cos(s_1kh)\sin(s_2kh)\\ &= \left(1 - s_2^2\right)\left[\gamma s_1^3 - \gamma s_1 + \delta s_1 - \rho c^2 s_1\right]\cos(s_2kh)\sin(s_1kh).\end{aligned} \tag{4.16}$$

This yields

$$\frac{\tan(s_2kh)}{\tan(s_1kh)} = \frac{s_1\left(1 - s_2^2\right)\left[\gamma s_1^2 - \gamma + \delta - \rho c^2\right]}{s_2\left(1 - s_1^2\right)\left[\gamma s_2^2 - \gamma + \delta - \rho c^2\right]}. \tag{4.17}$$

From eq. (4.11) it follows that

$$s_1^2 + s_2^2 = -\frac{2\beta - \rho c^2}{\gamma}, \qquad s_1^2 s_2^2 = \frac{\gamma - \rho c^2}{\gamma}. \tag{4.18}$$

Since we have

$$s_1^2 + s_2^2 = -\frac{2\beta - \rho c^2}{\gamma},$$

or we can write

$$s_1^2 = -s_2^2 - \frac{\delta - 2\gamma - \rho c^2}{\gamma}, \qquad 2\beta = \delta - 2\gamma.$$

Using this value of s_1^2 we get

$$\gamma s_1^2 - \gamma + \delta - \rho c^2 = \gamma(1 - s_2^2).$$

Similarly

$$\gamma s_2^2 - \gamma + \delta - \rho c^2 = \gamma(1 - s_1^2).$$

Hence (4.17) becomes

$$\frac{\tan(s_2 kh)}{\tan(s_1 kh)} = \frac{s_1(1 - s_2^2)^2}{s_2(1 - s_1^2)^2}. \tag{4.19}$$

This is the Rayleigh-Lamb equation for an incompressible orthotropic plate. Dispersion curves are obtained numerically.

Now we discuss the two cases which are based on the nature of the roots s_1^2 and s_2^2.

4.2 Case-1:

Let s_1^2 and s_2^2 be the roots of eq. (4.11). Suppose

$$\left(\gamma - \rho c^2\right) < 0,$$

or

$$c^2 > \frac{\gamma}{\rho}.$$

Then

$$s_1^2 s_2^2 < 0.$$

Therefore they cannot be complex conjugate of each other. Hence one of them will be positive and the other is negative. Since we have

$$s^4 + \left(\frac{2\beta}{\gamma} - \frac{\rho c^2}{\gamma}\right) s^2 + \left(1 - \frac{\rho c^2}{\gamma}\right) = 0$$

$$s^4 + \left(\frac{2\beta}{\gamma} - \frac{\rho c^2}{\gamma}\right) s^2 - \left(\frac{\rho c^2}{\gamma} - 1\right) = 0. \tag{4.20}$$

s_1^2 and s_2^2 being the roots of the above quadratic equation, are given as

$$s_1^2 = \frac{-E - \Delta}{2}, \qquad s_2^2 = \frac{-E + \Delta}{2}, \tag{4.21}$$

where

$$E = \left(\frac{2\beta}{\gamma} - \frac{\rho c^2}{\gamma}\right),$$

and

$$\Delta = \sqrt{\left(\frac{2\beta}{\gamma} - \frac{\rho c^2}{\gamma}\right)^2 + 4\frac{\rho c^2}{\gamma} - 4}. \tag{4.22}$$

Suppose $s_1^2 < 0$ and $s_2^2 > 0$.

Thus, replacing s_1^2 by ir_1 in the dispersion relation (4.19), where $r_1 = \sqrt{\frac{E+\Delta}{2}}$

$$\frac{\tan(s_2 u)}{i\tanh(r_1 u)} = \frac{ir_1(1 - s_2^2)^2}{s_2(1 - (-r_1^2)^2}.$$

Hence

$$\frac{\tan(s_2 u)}{\tanh(r_1 u)} = -\frac{r_1\left(1 - s_2^2\right)^2}{s_2\left(1 - \left(-r_1^2\right)\right)^2} \tag{4.23}$$

where $u = kh$. The function $\tanh(x)$ is an increasing function bounded above by 1, therefor the approximation

$$\tanh(ur_1) \cong 1 \tag{4.24}$$

can be used for $u >> 1$. Equation (4.23) is now replaced by

$$\tan(us_2) = -\frac{r_1\left(1 - s_2^2\right)^2}{s_2\left(1 - \left(-r_1^2\right)\right)^2}. \tag{4.25}$$

Further eq. (4.25) yields

$$\tan(us_2 - n\pi) + \Omega = 0, \tag{4.26}$$

where

$$\Omega = \frac{r_1\left(1 - s_2^2\right)^2}{s_2\left(1 - \left(-r_1^2\right)\right)^2} \tag{4.27}$$

eq. (4.26) can be used to plot the Rayleigh-Lamb spectrum in an orthotropic incompressible material.

In Fig.(4.1) we have plotted the dispersion curves (4.23) and (4.25) for the orthotropic material Iodic Acid (HIO_3), whose elastic stiffness constants are given in Table (3.2).

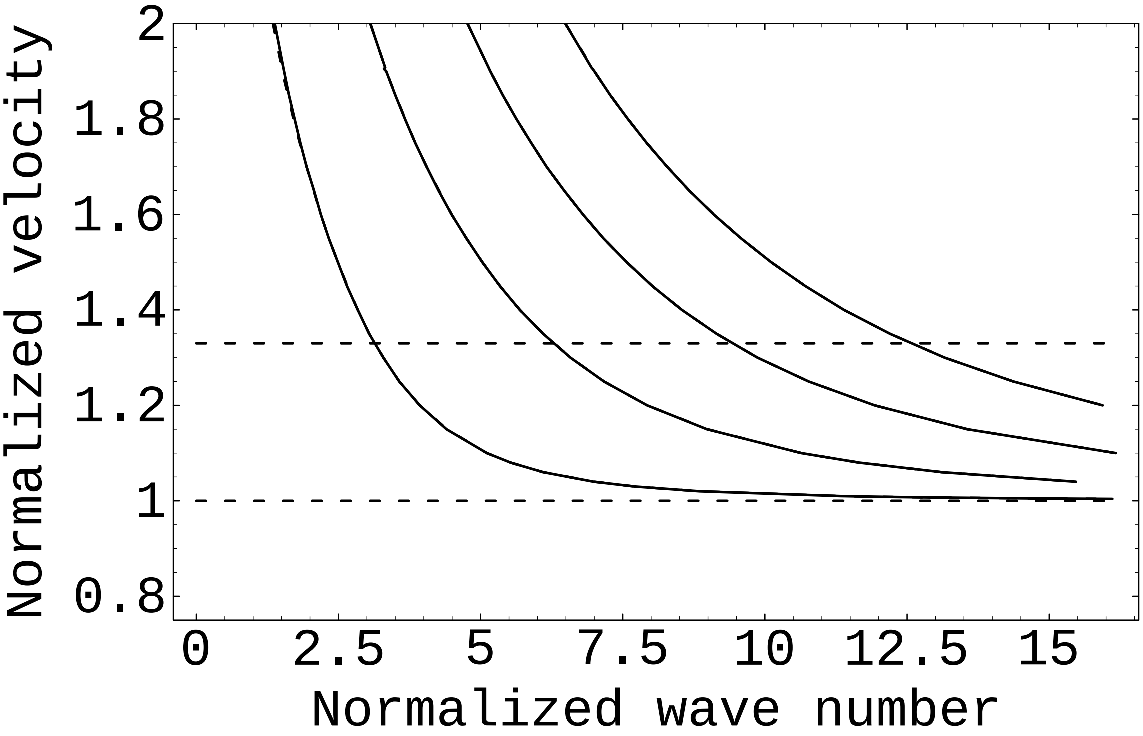


Figure 4.1: Calculated dispersion curves for the first four symmetric modes of Iodic acid (HIO_3), as functions of the normalized wave number. Solid curves represent exact solution (4.23) and the dashed curves represents approximate solution (4.25)

On the horizontal axis the normalized wave number represents hk. where h is the half of the width of the elastic plate and k is the wave number of the harmonic wave propagating along the wave guide. The vertical axis represents the normalized velocity $\frac{c}{c_T}$ where $c_T = \sqrt{\frac{C_{66}}{\rho}}$. The line $c = c_T$ is asymptote to all the modes when $u \longrightarrow \infty$. If we compare the spectrum with the spectrum for an isotropic material [2] we note that the velocity c_T for the orthotropic incompressible material is the counter part of C_t, the velocity of transverse waves in an isotropic material.

4.3 Analysis of the spectrum

When $\rho c^2 > \gamma$, we have $s_1^2 < 0$ and $s_2^2 > 0$. Let $s_1^2 = -r_1^2$ so that $r_1^2 > 0$. The dispersion relation (4.19) becomes

$$\frac{\tan(s_2 u)}{\tanh(r_1 u)} = -\frac{r_1 \left(1 - s_2^2\right)^2}{s_2 \left(1 + r_1^2\right)^2} \tag{4.28}$$

and

$$r_1^2 s_2^2 = \frac{\rho c_2}{\gamma} - 1, \qquad r_1^2 - s_2^2 = \frac{2\beta - \rho c^2}{\gamma}. \tag{4.29}$$

If u is large, $\tanh(ur_1) \longrightarrow 1$ and we have the approximate formula

$$\tan(s_2 u) = -\frac{r_1 \left(1 - s_2^2\right)^2}{s_2 \left(1 + r_1^2\right)^2} \tag{4.30}$$

For any value of c, the above equation has infinitely many roots for u, therefore the spectrum contains infinitely many dispersion curves.

When $\rho c^2 \longrightarrow \gamma$ from above, we see from (4.29) that $r_1^2 s_2^2 \longrightarrow 0$ which implies $s_2 \longrightarrow 0$. Let $s_2 = \varepsilon$. Since we have the approximate formula as

$$\tan(s_2 u) = -\frac{r_1 \left(1 - s_2^2\right)^2}{s_2 \left(1 + r_1^2\right)^2} \tag{4.31}$$

If $s_2 = \varepsilon << 1$, then to first order in ε the above equation has the solution

$$\varepsilon u_n = n\pi - \arctan \frac{r_1}{\varepsilon (1 + r_1^2)^2} \tag{4.32}$$

$$\varepsilon u_n \simeq n\pi - \frac{\pi}{2} \tag{4.33}$$

$$\varepsilon u_n \simeq \frac{(2n-1)\pi}{2}. \tag{4.34}$$

Thus

$$u_n \simeq \frac{(2n-1)\pi}{2\varepsilon}. \tag{4.35}$$

The above result indicates that every mode will approach the line $c_T = \sqrt{\frac{\gamma}{\rho}}$ from above in the limit of large wave numbers. The line $c_T = \sqrt{\frac{\gamma}{\rho}}$ is an asymptote to all modes.

The line $s_2 = 1$ has also a special significance, because in that case

$$\tan u = 0$$

which has the solution

$$u_n = n\pi.$$

This indicates that the spacing between successive modes $s_1, s_2, ...$ etc.when they intersect the line $s_2 = 1$ is exactly π. Now

$$s_2^2 = -\frac{E + \Delta}{2} \tag{4.36}$$

Let $p = \frac{2\beta}{\gamma}$ and $q^2 = \frac{\rho c^2}{\gamma}$, then (4.36) becomes

$$2s_2^2 = -p + q^2 + \sqrt{(p-q^2)^2 + 4q^2 - 4} \tag{4.37}$$

If $s_2^2 = 1$, then

$$2s_2^2 + p - q^2 = \sqrt{(p-q^2)^2 + 4q^2 - 4} \tag{4.38}$$

Taking square of both sides and simplification gives

$$q^2 = 1 + \frac{p}{2},$$

or

$$q^2 = 1 + \frac{\beta}{\gamma}. \tag{4.39}$$

For the example considered in this chapter

$$\delta = C_{11} + C_{22} - 2C_{12} = 5.59 \times 10^{10}\frac{N}{m^2}, \qquad \gamma = C_{66} = 1.58 \times 10^{10}\frac{N}{m^2}. \tag{4.40}$$

Hence

$$q = \sqrt{\frac{5.59}{3.16}} = 1.33003. \tag{4.41}$$

Thus the case $s_2 = 1$ corresponds to the velocity

$$\frac{\rho c^2}{\gamma} = q^2 = 1.769 \tag{4.42}$$

Or

$$c = \sqrt{\frac{1.769 \times 1.58 \times 10^{10}}{4.64 \times 10^3}} \tag{4.43}$$

$$c = 2454.33 \qquad msec^{-1}. \tag{4.44}$$

The line corresponding to this velocity has been indicated by c_L in Fig (4.1).

Next we consider the slope of the n-th mode as it intersects c_L. For this purpose, we have to examine

$$\frac{dc}{du}\,|_{u=u_n} \tag{4.45}$$

Since r_1 and s_2 both depend on c and c is a function of u, the right side of (4.30) is indeed a function of u. Since $u = u_n$ corresponds to $s_2 = 1$, the derivative of $\frac{r_1(1-s_2^2)^2}{s_2(1+r_1^2)^2}$ will vanish at $s_2 = 1$ because of the presence of $(1-s_2^2)^2$ as a factor. Also at $u = u_n$

$$\frac{d\tan(us_2)}{du} = \sec^2(us_2)[s_2 + u\frac{ds_2}{du}]$$

$$\frac{d\tan(us_2)}{du} = \sec^2(u_n)[1 + u_n\frac{ds_2}{du}]$$

$$\frac{d\tan(us_2)}{du} = 1 + n\pi\frac{ds_2}{du}\,|_{u=n\pi} \tag{4.46}$$

To find $\frac{ds_2}{du}$, differentiate (4.37) and then put $s_2 = 1$, $q = 1 + \frac{\beta}{\gamma}$

$$4s_2\frac{ds_2}{du} = [2q + \frac{-4q(p-q^2) + 8q}{2\sqrt{(p-q^2)^2 + 4q^2 - 4}}]\frac{dq}{du},$$

use (4.37) with $s_2^2 = 1$, the above expression becomes

$$4\frac{ds_2}{du}\mid_{u=n\pi} = [2q + \frac{-4q(p-q^2) + 8q}{2(2+p-q^2)}]\frac{dq}{du},$$

$$4\frac{ds_2}{du} = 2q[1 + \frac{-(p-q^2)+2}{2+p-q^2}]\frac{dq}{du}\mid_{u=n\pi},$$

$$4\frac{ds_2}{du} = 2\sqrt{\frac{2+p}{2}}[1 + \frac{2-(\frac{p}{2}-1)}{2+\frac{p}{2}-1}]\frac{dq}{du}\mid_{u=n\pi},$$

$$4\frac{ds_2}{du} = 2\sqrt{\frac{2+p}{2}}[1 + \frac{6-p}{2+p}]\frac{dq}{du}\mid_{u=n\pi},$$

$$4\frac{ds_2}{du} = 2\sqrt{\frac{2+p}{2}}(\frac{8}{2+p})\frac{dq}{du}\mid_{u=n\pi},$$

$$4\frac{ds_2}{du} = \sqrt{\frac{2}{2+p}}(8)\frac{dq}{du}\mid_{u=n\pi},$$

$$4\frac{ds_2}{du} = g(p)\frac{dq}{du}\mid_{u=n\pi},$$

where

$$g(p) = \sqrt{\frac{2}{2+p}}(8).$$

Thus eq. (4.46) gives

$$1 + \frac{n\pi}{4}g(p)\frac{dq}{du}\mid_{u=n\pi} = 0.$$

Finally we obtain

$$\frac{dq}{du}\mid_{u=n\pi} = \frac{-4}{n\pi g(p)}.$$

For the example considered in this chapter

$$p = \frac{2\beta}{\gamma} = 1.538,$$

and

$$g(p) = 6.0143,$$

we have

$$\frac{dq}{du}\mid_{u=n\pi} = -\frac{0.2116}{n}.$$

Thus slope of every mode is negative and progressively decreases in magnitude. In the limit $n \longrightarrow \infty$, it becomes zero.

4.4 Case-2:

If s_1^2 and s_2^2 are the roots of eq. (4.11) and $\left(\gamma - \rho c^2\right) > 0$, then both of them would be negative. Hence if both the radicals on the right side in equation (4.17) are imaginary and limit $u \longrightarrow \infty$. Let $s_1^2 = -r_1^2$ and $s_2^2 = -r_2^2$, where r_1 and r_2 are positive. Then equation (4.19) takes the form

$$1 = \frac{r_1(1 + r_2^2)^2}{r_2(1 + r_1^2)^2}, \tag{4.47}$$

and

$$r_1^2 + r_2^2 = \frac{2\beta - \rho c^2}{\gamma}, \qquad r_1^2 r_2^2 = \frac{\gamma - \rho c^2}{\gamma}. \tag{4.48}$$

On simplifying (4.47) becomes

$$r_1(1 + r_2^2)^2 = r_2(1 + r_1^2)^2,$$

$$r_1(1 + 2r_2^2 + r_2^4) = r_2(1 + 2r_1^2 + r_1^4),$$

$$(r_1 - r_2) + 2r_1r_2(r_2 - r_1) + r_1r_2(r_2^3 - r_1^3) = 0,$$

$$(r_1 - r_2) + 2r_1r_2(r_2 - r_1) + r_1r_2(r_2 - r1)(r_2^2 + r_1r_2 + r_1^2) = 0. \tag{4.49}$$

After removal of the factor $(r_2 - r_1)$, this yields

$$-1 + 2r_1r_2 + r_1r_2(r_2^2 + r_1r_2 + r_1^2) = 0,$$

$$-1 + r_1r_2[2 + r_1^2 + r_2^2] + r_1^2r_2^2 = 0, \tag{4.50}$$

substituting the values of $r_1^2 + r_2^2$ and $r_1^2r_2^2$ in the above eq.

$$-1 + \sqrt{\frac{\gamma - \rho c^2}{\gamma}}[2 + \frac{2\beta - \rho c^2}{\gamma}] + 1 - \frac{\rho c^2}{\gamma} = 0. \tag{4.51}$$

since $2\beta = \delta - 2\gamma$, the above eq. becomes

$$(\delta - \rho c^2)\sqrt{1 - \frac{\rho c^2}{\gamma}} - \rho c^2 = 0. \tag{4.52}$$

eq. (4.52) is the same Rayleigh equation discussed by Ogden and Vinh [17] for an incompressible orthotropic half space.

4.5 Conclusion

There are several materials to which the assumptions of incompressibility and orthotropy are applicable e.g. polymer kratons, thermoplastic elastomers, rubber composites etc.

We have obtained a dispersion relation for the Rayleigh-Lamb modes in an incompressible orthotropic elastic plate. An infinite number of modes exist and the qualitative picture is similar to the spectrum of antisymmetric modes in an isotropic elastic plate [23]. However the lowest mode A_0, present in Fig.1 on page 909 of the above reference does not appear in the present case.

In [18] Ogden and Vinh have studied Rayleigh wave in a *compressible* orthotropic material. It should be passible to utilize their formulation of the problem to discuss Rayleigh-Lamb spectrum of the compressible orthotropic plate.

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