Parallel propagating Electromagnetic Waves in Magnetized Quantum Electron Plasma

By

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Abstract

The quantum Vlasov equation has been derived with the use of quasi probability distribution named as the Wigner function and the electromagnetic Schrodinger equation. By using this quantum kinetic model the parallel propagating electromagnetic waves have been studied and the susceptibility tensor for R (right handed circularly polarized) and L (right handed circularly polarized wave) wave has been derived. The dispersion relation of R and L waves is obtained by using Fermi Dirac distribution at zero temperature. By comparing the curves obtained by plotting quantum dispersion relation of R and L wave with that of the classical R and L wave, it has been observed that the upper branch of the R wave in quantum plasma has slightly higher group velocity than that of the R wave in a classical plasma. While for the lower R wave branch the quantum mechanical group velocity is slower than the classical group velocity and the anomalous dispersion is observed. Moreover, the quantum mechanical L wave is almost the same as the classical L wave but minor corrections are observed at higher values of k which indicates the faster group velocity of quantum mechanical waves as compared to the classical L wave.

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Chapter 1

An Introduction to Quantum Plasma

1.1 What is a Plasma ?

A plasma is a multi-body system made up of numerous charged particles whose behavior is governed by quasi-neutrality and collective effects as mediated by electromagnetic forces.

1.1.1 Collective Behaviour in Plasmas

Collective behaviour refers to such motion of charge particles that are influenced not just by local conditions but also by the state of the plasma in distant regions. Since a plasma is composed of charged particles that can form local concentrations of positive or negative charge as they travel around, resulting in electric fields. Currents and, as a result, magnetic fields are generated by the movement of charges. The other charged particles in the plasma can no longer remain unaffected by these fields due to long range Coulomb's force. Thus the term "collective" refers to occurrences that are determined by the system's entire ensemble of particles. The behavior of neutral gases, on the other hand, is significantly affected by short-range interactions. It should be noted that plasmas are usually partially ionized, resulting in the presence of certain neutral atoms. In order to achieve plasma collective behaviour, electron and ion collision rates with the neutrals must be quite low. We can say that plasma exhibiting collective behaviour have minimum binary collisions in which each particle's velocity vector significantly change direction in a small spatial space and a short period. Instead, the more substantial trajectory changes in plasma are caused by the cumulative effect of several tiny scattering angle encounters [1].

1.1.2 Quasi neutrality and Debye Screening

The term "quasi neutrality" refers to the fact that charge separation may only occur across a limited distance, which in classical plasma is defined as the Debye length. The charged particles in plasma are highly conductive such that they can neutralize any potential induced in the plasma. For example, when a positive charge is introduced in the plasma it gets surrounded by a cloud of negative charges which insulate the positive charge and the potential caused by the positive charge is shielded out. This process of shielding external potential applied to plasma or external charges inserted into plasma is called Debye shielding and the blanket which shields the charges is called the Debye sphere [2].

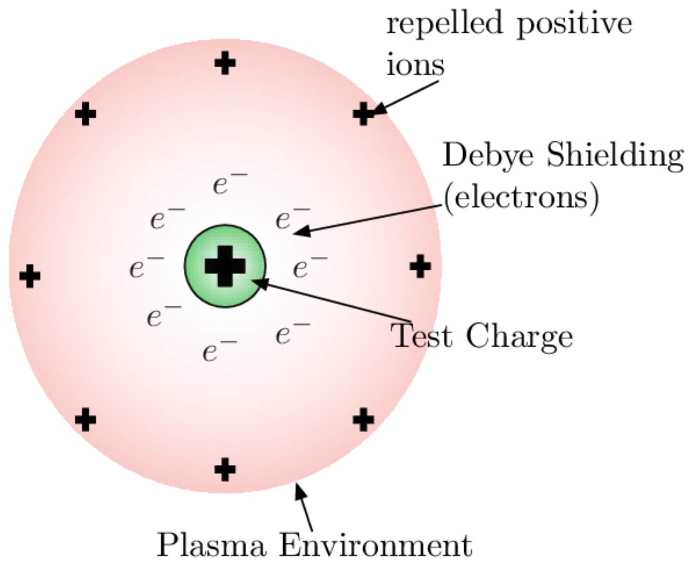

Figure 1.1: Debye sphere and Debye shielding [3]

The Debye length is represented by λ_D and it is given by the following expression.

$$\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{ne^2}\right)^{\frac{1}{2}},$$

where k_B is the Boltzmann's Constant, T_e is the temperature of electron, n is the number density and e being the charge on electron. The charge separation occurring at short distances (λ_D is small) compared to the dimensions of the system leads to quasi neutrality where $n_i = n_e$.

1.1.3 Temperature and Density basic Plasma parameters

Temperature

The temperature at the beginning of universe was so high that no atoms or molecules could have existed. As a result, the corresponding completely ionized gas was in the plasma state. So the Universe as a whole was a plasma, known as quark gluon plasma. As the temperature decreases, the capacity of the particles in a system to associate with one another increases in the same proportion. When a completely ionized gas is cooled down, a percentage of the positive and negative charges can unite to create atoms. We'd have a partly ionized plasma in this instance. As the temperature drops further the degree of ionization becomes insignificant and the system is said to be neutral. In the low-temperature plasma, the electron temperatures are in the range of electron volts, which is sufficient for ionization, and the heavy species temperature is often close to room temperature. The degree of plasma ionization is determined by the Saha equation that relates the electron temperature to the ionization energy.

$$\frac{n_i}{n_n} \approx 2.4 \times 10^{21} \frac{T^{2/3}}{n_i} e^{\frac{-U_i}{k_B T}} \tag{1.1}$$

Here n_i and n_n are the number densities per m^3 of atom and neutral atoms and U_i is the potential energy respectively.

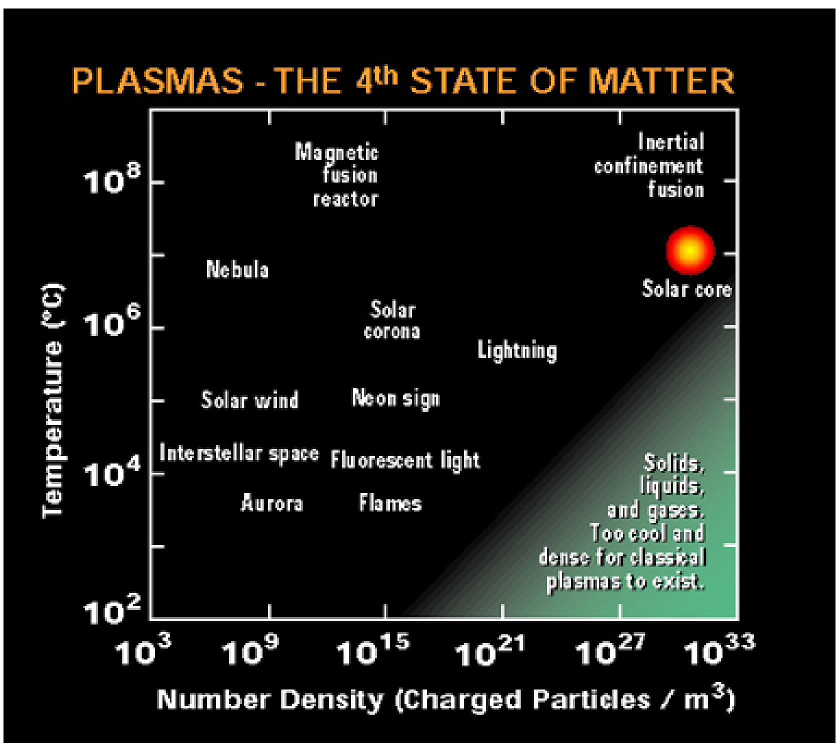

Figure 1.2: Profile of densities vs temperature for various plasmas [4]

As mentioned above, the drop in temperature increases the binding energy and ionization decreases. It is important to compare the thermal and interaction energy of the system in order to assign a meaning to the phrase "small temperature." A coupling parameter can be defined as the ratio of thermal and binding energy in this way. So for a gas to exist in plasma state, we may add the criteria of a sufficiently high thermal energy in contrast to the binding energy.

Plasmas are categorised as thermal or non-thermal based on the relative temperatures of electrons, ions, and neutrals. Electrons and heavy particles in thermal plasmas are at the same temperature, i.e. they are in thermal equilibrium. Ions and neutrals are at a considerably lower temperature in non-thermal plasmas, but electrons are much hotter.

Along with the temperature the density also plays an important role for justification of the existence of a plasma. The term "plasma density" often refers to the quantity of free electrons per unit volume [5, 6].

1.1.4 Creation of Plasma

For the generation of plasma in space objects such as in the Sun and stars, photo ionization is the most common process in which photons from sunlight are absorbed by an existing gas and electrons are released. Because the Sun and stars radiate continually, nearly all matter becomes ionised, and the plasma is considered to be completely ionised in such environments. For the production of plasma in the laboratory a gas is heated to a very high temperature which causes violent collisions between its atoms and molecules and the gas gets ionized [7].

1.2 Classical plasma

Classical plasma are characterized by high temperature and low densities. In classical plasma the De Broglie wave length is smaller than interatomic distance of the particles

Plasma Frequency

It is the oscillation frequency of the electrons in a neutralizing backdrop of positive ions, which are supposed to be immobile due to the enormous mass. These oscillations are caused by the Coulomb force, which pulls electrons back towards the excess positive charge when a section of the plasma is depleted of some electrons. The electrons will not just replace the positive region due to their inertia, but will instead go further away, re-creating an excess positive charge. This process produces undamped electron oscillations at the plasma frequency in the absence of collisions, given as [8]

$$\omega_p = (\frac{n_0 e^2}{\epsilon_0 m})^{\frac{1}{2}}$$

In the above equation, n_0 represents the density of electrons in the plasma, e being the charge on the electron and m is the mass of electron.

Thermal Velocity

The typical speed owing to random thermal motion is represented by thermal velocity, given as

$$v_T = \sqrt{\frac{k_B T}{m}}$$

Debye length

The Debye length is defined as ratio of thermal velocity and the plasma frequency i.e.

$$\lambda_D = \frac{v_T}{\omega_p},$$

or we can write

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{n_0 e^2}}$$

The Debye length represents the significant phenomena of electrostatic screening, in which an excess positive charge is immediately encircled by a cloud of electrons when introduced into the plasma. As a result of this phenomena the positive charges will be screened out for the particles locating at larger distances. So with the help of Debye screening we can define Quasi neutrality which says that charge separation only occur at the distance smaller than Debye length and it is filtered at larger distances.

Coupling parameter

The coupling parameter is defined as the ratio of interaction energy ($\frac{e^2}{\epsilon_0 d}$) and the thermal energy ($k_B T$).

$$\gamma_c = \frac{E_{int}}{E_k},$$

$$\gamma_c = \frac{e^2 n^{\frac{1}{3}}}{\epsilon_0 k_B T}. \tag{1.2}$$

When electrons have a sufficient amount of thermal energy, the interaction energy decreases due to a decrease in Coulombic interaction, resulting in a small coupling parameter which is known as a collision-less regime. In this regime binary collision are negligible and the plasma exhibit collective behaviour. The above expression indicates that at high temperature and low density plasma is said to be a collision-less classical plasma. The coupling parameter can be related to Debye length as follow:

$$\gamma_c = \left(\frac{1}{n\lambda_D^3}\right)^{\frac{1}{3}}$$

When the Debye sphere contains huge number of electrons Debye screening is supposed to be at a greater rate which makes the coupling parameter small hence the collisions decrease and it refers to a collision less classical plasma. On the other hand when the coupling parameters is high enough the plasma is considered as collisional plasma.

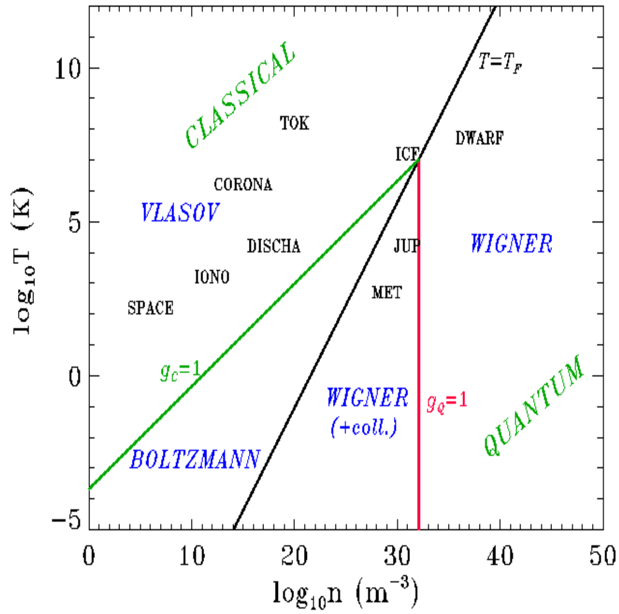

Figure 1.3: Density and temperature profile for classical and quantum plasma [9]

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1.3 Quantum Plasma

Quantum mechanics is generally required to describe a system that is so dense that the wave functions of distinct particles start to overlap. The thermal De Broglie wavelength of the particles forming the plasma, which represents the spatial extension of a particle's wave function owing to quantum uncertainty, may be used to quantify quantum effects. The thermal De Broglie wavelength is given by following expression:

$$\lambda_{DB} = \frac{\hbar}{m v_T}.$$

When the De Broglie wavelength is similar to or larger than the average inter particle distance, the quantum effects start playing significant role and the wave function associated with particles overlap. On the other hand for the classical regime the De Broglie wavelength is so short that the particles may be regarded as point like hence there is no overlapping of wave functions and no quantum interference.

1.3.1 Quantum Plasma Parameters

There are certain dimensionless parameters on the basis of which Quantum plasma is differentiated from classical plasma.

De Broglie wavelength and inter atomic particle distance

The primary criteria for a plasma to be treated quantum mechanically is that De Broglie wavelength must be comparable to inter atomic distance i.e,

$$n\lambda_{DB}^3 \geq 1,$$

where $n^{\frac{1}{3}}$ represents the inter atomic distance.

Thermal Velocity and Temperature

In a quantum plasma, the thermal velocity is described in terms of De Broglie wavelength. From the expression of De Broglie wavelength and the thermal velocity , expression for temperature is derived as given below:

$$T_{therm} = \frac{\hbar^2}{mk_B\lambda_{DB}^2}.$$

Fermi Energy and Fermi Temperature

In quantum systems as the energy levels are filled by following Pauli exclusion principle and the electrons fill the level up to Fermi level (above which all levels are empty and below Fermi level all states are filled). Hence the energy of highest occupied level at absolute zero temperature is termed as the Fermi energy given as: $E_F = k_BT_F$. The Fermi temperature is defined as

$$T_F = \frac{(3\pi^2 n)^{\frac{2}{3}}\hbar^2}{2mk_B}.$$

Degeneracy parameter

By using the above mentioned terms for temperature and the Fermi temperature a dimensionless parameter called degeneracy is defined as [9]

$$\frac{T_F}{T} = \frac{(3\pi^2 n\lambda^3)^{\frac{2}{3}}}{2}.$$

From the above equation we can write

$$\frac{T_F}{T} \propto n\lambda_{DB}^3$$

For **quantum** systems

$$\frac{T_F}{T}, \geq 1$$

and for **classical** systems

$$\frac{T_F}{T} \leq 1.$$

Debye Screening in Quantum plasma

For a quantum plasma, quasi neutrality and collective behaviour carries the same weight as in the classical plasma. The Debye length in a quantum plasma is defined as ratio of the Fermi velocity and the plasma frequency.

$$\lambda_{FD} = \frac{v_F}{\omega_p},$$

where λ_{FD} is the Fermi Debye length, v_F is the Fermi velocity and ω_p is the plasma frequency. By substituting the value for v_F and ω_p in the above expression, we get

$$\lambda_{FD} = \left(\frac{\epsilon_0 k_B T_F}{n_0 e^2}\right)^{\frac{1}{2}}.$$

This is the same expression as that for the classical plasma but here T is replaced by T_F, the Fermi temperature.

Coupling parameter

For the coupling parameter in quantum plasma the thermal energy is replaced by the Fermi energy and we can write it as

$$\gamma_Q = \frac{E_{int}}{E_F}$$

,

where

$$E_{int} = \frac{q^2 n^{\frac{1}{3}}}{\epsilon_0},$$

and

$$E_F = \frac{(3\pi^2 n)^{\frac{2}{3}} \hbar^2}{2m}.$$

So the quantum coupling parameter becomes

$$\gamma_Q = \frac{E_{int}}{E_F} = \frac{E_{int}}{k_B T_F} = \frac{2mq^2}{(3\pi^2 n)^{\frac{2}{3}} \epsilon_0 \hbar^2 n^{\frac{1}{3}}}.$$

As we have mentioned above that for a quantum plasma the Fermi temperature dominates over the thermal temperature. Therefore, in the above expression when the Fermi temperature increases it must decrease the coupling parameter or we can say that the coupling parameter should be less than one for a quantum plasma. Secondly, with an increase in the density, the coupling parameter should decrease which indicates that the quantum plasma is a weakly coupled plasma. For a system possessing high enough density one can assume that the system is highly collisional and the coupling parameter must be high but this does not seems to happen here because of Pauli Exclusion principle which forbids the occupation of same quantum state by two electrons. The electrons tend to fill all the levels below the Fermi level by following Pauli exclusion principle and this process leads to degeneracy. In this way, the electron-electron collision is reduced and the coupling parameter decreases.

1.4 Applications of Quantum Plasmas

1.4.1 Nanostructured Quantum Plasma

Metallic nanostructures offer a suitable setting for exploring quantum plasma dynamical characteristics. It is possible to obtain quantum characteristics in metallic structures by considering non interacting behaviour of electron. Treating the electron population as a plasma, globally neutralised by the lattice ions, provides a more realistic description for a quantum plasma. At room temperature and some suitable range of densities, quantum effects appear in nano-structures which can not be ignored. In conventional metals, lattice ions govern the properties of electrons in metals such as response to thermodynamic effects and band structure. However, in recent times nanostructures have been developed in such a way that there are no ionic lattice and the electron population is controlled by plasma processes. [10, 11]

Metal nanoparticle plasmonics is now a popular topic of research due to fundamental scientific interest and possible applications in spectroscopy and sensing optical nano-antennas, photochemistry, nonlinear optics etc.

1.4.2 Semiconductor Quantum plasma

Semiconductors have electrical properties between insulator and conductors. According to the band theory, semiconductors have comparatively small forbidden energy gap. When energy is provided to a semiconductor the electron goes from valence band to conduction band leaving behind a hole. This process of formation of electron-hole pair leads to the formation of semiconductor plasma.

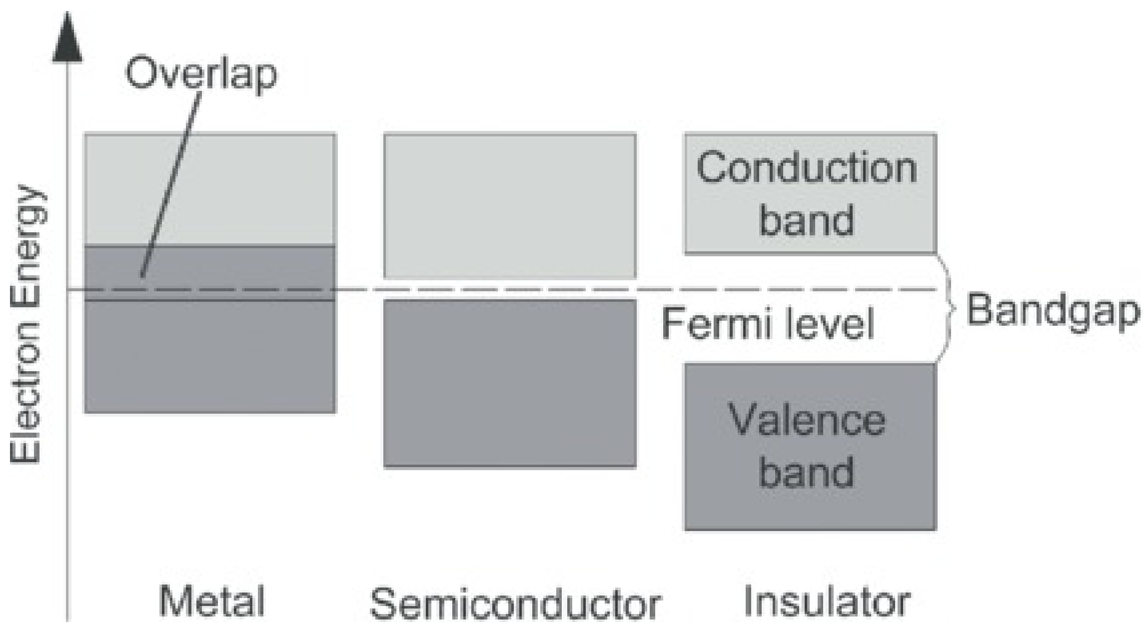

Figure 1.4: bands in semiconductors [13]

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In semiconductors, quantum effects become prominent when the De Broglie wavelength associated with an electron is equivalent to the size of a semiconductor. Quantum plasma has wide applications in the semiconductor devices. Despite the fact that semiconductors have a lower electron density than metals, the high degree of miniaturisation of today's electronic components allows the charge carrier (electrons and holes) De Broglie wavelength to be similar to the spatial variation of the doping profile. Semiconductors are utilised to investigate solid-state plasma in the quantum domain. Spintronics, nanotubes, Gunn oscillators, quantum wells, and quantum dots are a few examples of semiconductor system. In semiconductors, quantum effects arise when the dimensions of semiconductor plasma are comparable to the De Broglie wavelength. This occurs at high density and low temperature [12, 14, 15].

1.4.3 Quantum plasma in natural environments

Quantum plasmas may be found in some astronomical objects that are subjected to extreme temperature and density circumstances, such as white dwarf stars, where the density is 10 orders of magnitude more than that of a conventional solid. A white dwarf is a low or medium mass star with a mass less than roughly 8 times that of our Sun. White dwarfs are one of the densest forms of mass, only neutron stars and black holes being denser. A white dwarf may be as hot as a fusion plasma (108K) and yet act quantum-mechanically due to its enormous density.[16]

1.4.4 Degeneracy in White Dwarf

A white Dwarf is considered to be one of the final states of a Star. A star continuously fuse hydrogen into helium and when it run short of hydrogen it collapses and it transforms into a red giant. After this red giant forms, the core temperature of the star rises until it reaches a temperature high enough to fuse the helium produced by hydrogen fusion. It will eventually convert helium to carbon and other heavier elements. This process continues till the core is entirely made of iron, the most stable element. Now the fusion process ceases and the temperature of the core rises abruptly. Now the repulsive forces come into play and stops the gravitational collapse. This causes an explosion called supernova because of which the star's outer layers are blown away. The star's core, on the other hand, remains undamaged and evolves into a white dwarf or neutron star depends upon the mass of the star.[17, 19, 18] In a white dwarf, the fusion process ceases and there is no internal pressure, the gravity compacts the matter inward until the electrons in the atoms of the element that make up a white dwarf are squashed together. In ordinary matters the identical particles can occupy any energy level without any restriction but in white Dwarf particles are restricted by Pauli Exclusion principle which state that it is forbidden for identical electrons (those with the same "spin") to occupy the same energy level. The particles first fill the energy levels following the Pauli Exclusion Principle till the Fermi level, the highest occupied energy level. This is how degeneracy in a white dwarf appears. Now the gravity can no longer squeeze a degenerate star because quantum physics says that there is no more accessible space to be taken up. So, instead of internal fusion, quantum mechanical laws keep our white dwarf from collapsing completely.[20]

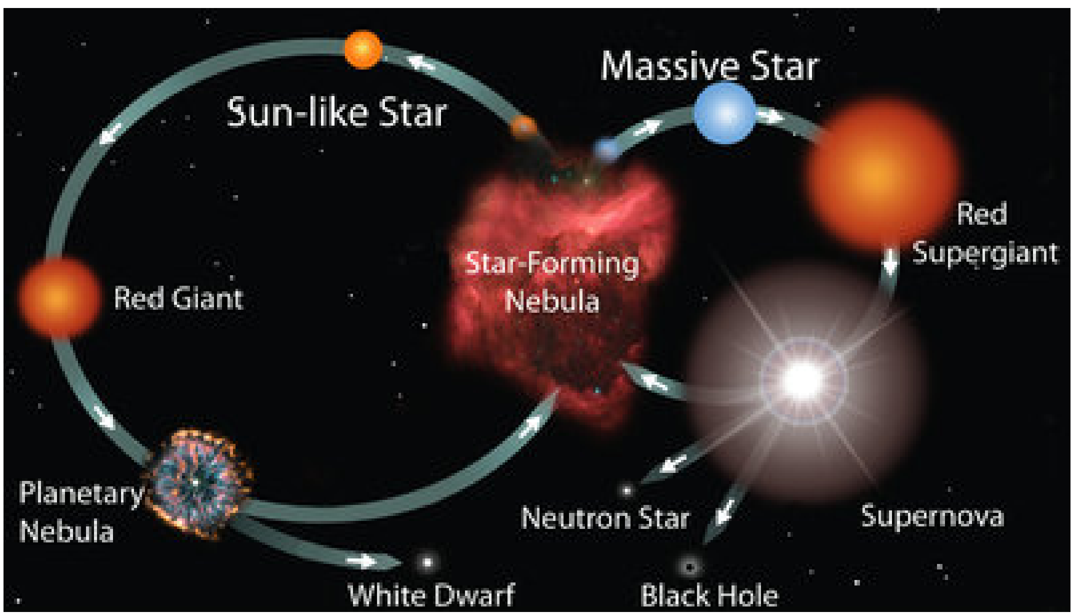

Figure 1.5: Life cycle of Star [21]

1.5 Waves in Plasma

As plasma is a dispersive medium it allows various waves to propagate through depending upon various conditions like frequency of the wave, density of plasma, ambient magnetic field etc. These various waves are discussed in brief in the following.

1.5.1 Longitudinal and Transverse Waves

The wave vector's orientation in relation to the oscillating electric field determines whether the wave is longitudinal or transverse. When $\vec{k}$ is parallel to $\vec{E_1}$ the corresponding waves are said to be longitudinal or electrostatic and when k is perpendicular to $\vec{E_1}$ the corresponding waves are said to be transverse or electromagnetic.

1.5.2 Parallel and Perpendicular Propagating Waves

The parallel or perpendicular propagation of a wave in plasma is decided according to the direction of the wave vector with respect to the unperturbed magnetic field. When $\vec{k}$ is parallel to the magnetic field $\vec{B_0}$, the wave is said to be a parallel propagating wave and when $\vec{k}$ is perpendicular to the magnetic field $\vec{B_0}$, the wave is said to be perpendicularly propagating wave.

1.5.3 Electrostatic and Electromagnetic Waves

The term electrostatic and electromagnetic for the waves in plasma is used by keeping in view the Maxwell's equation, given as

$$\nabla \times \vec{E_1} = -\frac{\partial \vec{B_1}}{\partial t},$$

$$i\vec{k} \times \vec{E_1} = -\frac{\partial \vec{B_1}}{\partial t}.$$

A wave is said to be electrostatic if the perturbed magnetic field is zero which indicates that that there is no orientation of $\vec{k}$ perpendicular to $\vec{E_1}$ rather it is parallel to $\vec{E_1}$ and the electrostatic wave is longitudinal. Furthermore an electrostatic wave in plasma can be parallel or perpendicularly propagating depending on the direction of $\vec{k}$ with respect to $\vec{B_0}$. If the perturbed magnetic field is not zero then the wave is said to be electromagnetic and the wave vector $\vec{k}$ will be perpendicular to $\vec{E_1}$ which indicates the transverse nature of an electromagnetic wave in plasma.

1.6 Parallel Propagating Electromagnetic Waves

1.6.1 R wave

R-wave is a parallel propagating electromagnetic wave which is also right handed circularly polarized wave. The dispersion relation is given as

$$\frac{c^2k^2}{\omega^2} = \frac{\frac{\omega_p^2}{\omega^2}}{1 - \frac{\omega_c}{\omega}}.$$

These waves encounter resonance at ω_c which means that these waves transfer their energy to the electrons as their plane of polarization is same as the direction of gyration of electrons. The cutoff for these waves occur at a higher frequency and there is a no propagation region between the cutoff and the resonance frequency. A whistler mode generates when the frequency of the R wave is less than the cyclotron frequency of electrons. As the frequency of the wave increases the phase velocity increases. So when a lightning flashing occurs in southern hemisphere, waves of higher frequency with higher phase velocity reached earlier than the low frequency waves with lower phase velocity at the northern hemisphere by travelling along the magnetic field lines. In this way a descending tone can be heard if these waves are converted into audio signals and this is why these are called whistler waves.

L waves

L waves are parallel propagating electromagnetic waves which are left handed circularly polarized. The dispersion relation for the L waves is given as

$$\frac{c^2k^2}{\omega^2} = \frac{\frac{\omega_p^2}{\omega^2}}{1 + \frac{\omega_c}{\omega}}$$

This wave has no resonance frequency as its plane of polarization is in opposite direction with respect to the gyration of electrons. However, if we include ion dynamics then this wave will resonate with the ions at ion cyclotron frequency. This wave behaves like an ordinary wave but the cutoff frequency is different

4 warnings from that of O-wave i.e., ω_p. For the lower branch of R wave the

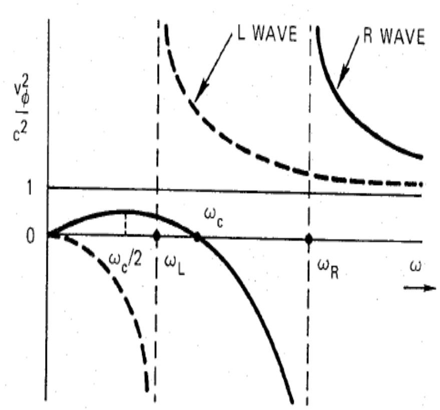

Figure 1.6: Graphical representation of R and L waves

wave vector becomes infinite at $\omega = \omega_c$ and the resonance occurs.The waves transfer their energy to the electrons as their plane of polarization is same as the direction of gyration of electrons. For the lower branch of L waves, there's no resonance encountered as L waves gyration is in opposite manner For the upper branches of R and L waves the cut off occurs at ω_R and ω_L.

Chapter 2

Background

2.1 Plasma Modelling

There are different ways to analyze a plasma system. The most important are:

1. Fluid Model
2. Kinetic Model

In the following, these models are described in detail.

2.1.1 Fluid Model

The macroscopic behaviour of plasma is described by Fluid model in which average velocities of the particles are considered. The individual identity of a particle is ignored in the fluid model, and plasma is defined by flow velocity, particle density, and temperature. All of these quantities are described as a functions of time t and position r. The fluid model successfully describes majority of the plasma phenomena, but in some circumstances where the velocity of a particular plasma specie is required, this model fails. The equations that characterize the plasma behaviour in the fluid model are described below.

Equation of Continuity

The conservation of the particles in the plasma is described by the equation of continuity given as

$$\frac{\partial n_\beta}{\partial t} + \nabla.(n_\beta v_\beta) = 0,$$

where β refers to species i.e., electrons and ions.

The Force Equation

The force equation in the fluid model is given as

$$m_\beta n_\beta \left(\frac{\partial u_\beta}{\partial t} + (u_\beta.\nabla)u_\beta \right) = q_\beta n_\beta \left(\vec{E} + u_\beta \times \vec{B} \right) - \nabla p_\beta$$

On the left hand side the term in the parenthesis is called the convective derivative and the term on the right hand side in the parenthesis is the Lorentz force where $\vec{E}$ and $\vec{B}$ are the electric and magnetic field respectively and ∇p_β represents the change in pressure which arises due to random motion of particles in the fluid. These equations along with the Maxwell's equations describe the dynamics of a plasma system.

2.2 Classical Kinetic Theory

Plasma physics is usually concerned with the phenomena that is connected to statistical mechanics dynamical processes. The features and structure of the basic kinetic equations regulating the dynamical behaviour of plasma are thus extremely important to investigate. The Boltzmann's equation is commonly used to examine the dynamical behaviour of a system of N-interacting particles. This equation is derived by using a distribution function which describes the behaviour of plasma on microscopic level. It gives the probability of number of particles in a given volume element d^3v

$$n = \int f(\vec{r}, \vec{v}, t) d^3v$$

Consider an ensemble of N charged particles in a plasma. The density of the charged particles in an ensemble is given by the distribution function while the dynamics of the motion of charged particle is found out by taking time derivative of the distribution function. The resulting equation is termed as Boltzmann's equation. The time derivative of the distribution function which depends upon position, velocity and time is given as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{r}}.\frac{d\vec{r}}{dt} + \frac{\partial f}{\partial \vec{v}}.\frac{d\vec{v}}{dt}.$$

With the use of Lorentz force expression the above equation gets modified as follow.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v}.\frac{\partial f}{\partial \vec{r}} + \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B})\frac{\partial f}{\partial \vec{v}}. \tag{2.1}$$

By setting the L.H.S of the above equation equals to zero it reduces to the Vlasov equation which says when the interaction between the particles are absent (collisionless plasma) the density of the particles remains constant. The Vlasov equation gives information about trajectories of charged particles by averaging over all the micro states. This equation is valid till the plasma has collective effects and the binary collisions are absent. Whenever the collisions arise the Vlasov equation cannot be use and then the Boltzmann's equation turns to Fokker Planks equation. The Vlasov equation is used to describe the dynamics of a plasma in phase space when the collective interactions dominate over binary interactions. In plasma physics, the Vlasov equation has a wide range of applications. It may be used to investigate linear waves, resonance effects that can not be represented by fluid theory for example, cyclotron and Landau damping in plasma [25, 26]. The term in the parenthesis in equation (2.1) can be solved with the aid of Maxwell's equations.

2.2.1 Limitation of the Vlasov Equation

The Vlasov equation describes the dynamics of a plasma in phase space when the collective interaction dominates over binary interactions. If the binary collisions are large enough the Vlasov equation can not be used because due to these collisions the mean free path becomes small as compared to Debye length.

2.3 Classical Kinetic theory formalism

2.3.1 The Vlasov equation

We'll start with the Vlasov equation i.e.,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{v}} = 0.$$

In order to simplify the Vlasov equation we linearize it. The product of two perturbed quantities can be neglected as the amplitude of the oscillations is very small. Separating the variables into two parts equilibrium part and perturbed part.

$$\vec{E} = \vec{E}_0 + \vec{E}_1,$$

$$f = f_0 + f_1,$$

and

$$\vec{B} = \vec{B}_0 + \vec{B}_1,$$

where 0 subscript is for the equilibrium values and 1 subscript is for the perturbed values.

For a homogeneous magnetized plasma

$$\vec{E}_0 = 0,$$

$$\nabla f_0 = \frac{\partial f_0}{\partial t} = 0.$$

Substituting these in the Vlasov equation, we get

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{r}} + \frac{e}{m}(\vec{E}_1 + \frac{1}{c}\vec{v} \times \vec{B}_1) \cdot \frac{\partial f_0}{\partial \vec{v}} + \frac{e}{mc}(\vec{v} \times \vec{B}_0) \cdot \frac{\partial f_0}{\partial \vec{v}} + \frac{e}{mc}(\vec{v} \times \vec{B}_0) \cdot \frac{\partial f_1}{\partial \vec{v}} = 0. \tag{2.2}$$

2.3.2 Solution of zeroth order terms of the Vlasov equation

The first zeroth order term of the Vlasov equation is given as

$$(\vec{v} \times \vec{B}_0) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0. \tag{2.3}$$

Let

$$\vec{B} = B_0 \hat{z},$$

so we can write

$$B_0(\vec{v} \times z) \cdot \frac{\partial f_0}{\partial \vec{v}} = 0.$$

This is the scalar triple product which can be termed as the volume of a parallelopiped and can be solved in the following way.

$$\begin{vmatrix} v_x & v_y & v_z \\ 0 & 0 & 1 \\ \frac{\partial f_0}{\partial v_x} & \frac{\partial f_0}{\partial v_y} & \frac{\partial f_0}{\partial v_z} \end{vmatrix} = v_y \frac{\partial f_0}{\partial v_x} - v_x \frac{\partial f_0}{\partial v_y} = 0,$$

For an anisotropic plasma (direction dependent) in a uniform magnetic field acting along the z axis, the velocity component along the z axis is given by $v_\parallel$ and across the z axis is $v_\perp$. Therefore, the velocity components in the cylindrical coordinates are given as

$$\vec{v} = (v_x, v_y, v_z) = (v_\perp \cos\phi, v_\perp \sin\phi, v_\parallel).$$

Since

$$f_0 = f_0(v_\perp, v_\parallel, \phi),$$

we can write

$$\frac{\partial f_0}{\partial v_x} = \frac{\partial f_0}{\partial v_\perp}\frac{\partial v_\perp}{\partial v_x} + \frac{\partial f_0}{\partial \phi}\frac{\partial \phi}{\partial v_x}. \tag{2.4}$$

Since

$$v_\perp^2 = v_x^2 + v_y^2, \tag{2.5}$$

so by differentiating w.r.t x, we get

$$2v_\perp\Big(\frac{\partial v_\perp}{\partial v_x}\Big) = 2v_x,$$

or

$$\frac{\partial v_\perp}{\partial v_x} = \frac{v_x}{v_\perp}.$$

Using the value for v_x in the above expression we get

$$\frac{\partial v_\perp}{\partial v_x} = \cos\phi. \tag{2.6}$$

Now by differentiating the following equation w.r.t x

$$\tan\phi = \frac{v_y}{v_x}, \tag{2.7}$$

we get

$$\sec^2\phi\left(\frac{\partial \phi}{\partial v_x}\right) = -\frac{v_y}{v_x^2}.$$

By using the value for v_y and v_x in the above equation, we get

$$\frac{\partial \phi}{\partial v_x} = -\frac{\sin\phi}{v_\perp}. \tag{2.8}$$

Utilizing Eq. (2.10) and Eq. (2.13) in Eq. (2.8), we get

$$\frac{\partial f_0}{\partial v_x} = \cos\phi \, \frac{\partial f_0}{\partial v_\perp} - \frac{\sin\phi}{v_\perp} \, \frac{\partial f_0}{\partial \phi}. \tag{2.9}$$

Now by differentiating Eq. (2.9) w.r.t y, we get

$$2v_\perp \left(\frac{\partial v_\perp}{\partial v_y} \right) = 2v_y,$$

or

$$\frac{\partial v_\perp}{\partial v_y} = \frac{v_y}{v_\perp}.$$

Putting the value for v_y in the above expression, we get

$$\frac{\partial v_\perp}{\partial v_y} = \sin\phi. \tag{2.10}$$

Now differentiate equation Eq. (2.11) w.r.t y to get

$$\sec^2\phi \left(\frac{\partial \phi}{\partial v_y} \right) = \frac{1}{v_x}.$$

Putting the value for v_x in the above equation, we finally get

$$\frac{\partial \phi}{\partial v_y} = \frac{1}{v_\perp \cos\phi \sec^2\phi}, \tag{2.11}$$

or

$$\frac{\partial \phi}{\partial v_y} = \frac{\cos\phi}{v_\perp}.$$

Using these values in the following Eq. (2.16), we get

$$\frac{\partial f_0}{\partial v_y} = \frac{\partial f_0}{\partial v_\perp}\frac{\partial v_\perp}{\partial v_y} + \frac{\partial f_0}{\partial \phi}\frac{\partial \phi}{\partial v_y}, \tag{2.12}$$

$$\frac{\partial f_0}{\partial v_y} = \sin\phi\frac{\partial f_0}{\partial v_\perp} + \frac{\cos\phi}{v_\perp}\frac{\partial f_0}{\partial \phi}. \tag{2.13}$$

Now by putting Eq. (2.17) and Eq. (2.13) in equation given below

$$v_y\frac{\partial f_0}{\partial v_x} - v_x\frac{\partial f_0}{\partial v_y} = 0$$

$$-v_\perp\cos\phi\left(\sin\phi\frac{\partial f_0}{\partial v_\perp} + \frac{\cos\phi}{v_\perp}\frac{\partial f_0}{\partial \phi}\right) + v_\perp\sin\phi\left(\cos\phi\frac{\partial f_0}{\partial v_\perp} - \frac{\sin\phi}{v_\perp}\frac{\partial f_0}{\partial \phi}\right) = 0$$

or

$$-(\cos^2\phi + \sin^2\phi)\frac{\partial f_0}{\partial \phi} = 0$$

As we know $\cos^2\phi + \sin^2 = 1$ so $\frac{\partial f_0}{\partial \phi} = 0$ which indicates that there is no dependence of the equilibrium distribution function on ϕ angle. Now the linearized part of Eq. (2.6) can be written as

$$\frac{\partial f_1}{\partial t} + \vec{v}.\frac{\partial f_1}{\partial \vec{r}} + \frac{e}{m}(\vec{E}_1 + \frac{1}{c}\vec{v}\times\vec{B}_1).\frac{\partial f_0}{\partial \vec{v}} + \frac{e}{mc}(\vec{v}\times\mathbf{B}_0).\frac{\partial f_1}{\partial \vec{v}} = 0. \tag{2.14}$$

The last term can be written as

$$\frac{e}{mc}(\vec{v}\times\vec{B}_0)\cdot\frac{\partial f_1}{\partial \vec{v}} = -\Omega\frac{\partial f_1}{\partial \phi},$$

where $\Omega = \frac{eB_0}{mc}$ is the cyclotron frequency and ϕ is the azimuthal angle. Now Eq. (2.18), can be written as

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{r}} + \frac{e}{m}(\vec{E}_1 + \frac{1}{c}\vec{v}\times\vec{B}_1)\cdot\frac{\partial f_0}{\partial \vec{v}} - \Omega\frac{\partial f_1}{\partial \phi} = 0, \tag{2.15}$$

or

$$\frac{\partial f_1}{\partial \phi} - \frac{1}{\Omega}\left(\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{r}}\right) = \frac{e}{m\Omega}(\vec{E}_1 + \frac{1}{c}\vec{v}\times\vec{B}_1)\cdot\frac{\partial f_0}{\partial \vec{v}} = 0. \tag{2.16}$$

Since f_1 is a slowly varying quantity so we will try to deal with each term of the Vlasov equation separately. Applying Fourier and Laplace transform to the terms in parenthesis in Eq. (2.20) we get

$$\mathcal{L}(\frac{\partial f_1}{\partial t}) = F(s),$$

$$\mathcal{L}(\frac{\partial f_1}{\partial t}) = \int_0^\infty \frac{\partial f_1}{\partial t} e^{-st} dt,$$

$$\mathcal{L}\left(\frac{\partial f_1}{\partial t}\right) = e^{-st} f_1 \Big|_0^\infty + s\int_0^\infty e^{-st} f_1 dt.$$

As we know that $\int_0^\infty e^{-st} f_1 dt = \mathcal{L}(f_1)$, therefore

$$\mathcal{L}\left(\frac{\partial f_1}{\partial t}\right) = e^{-0} f_1(0) + s\mathcal{L}(f_1),$$

or

$$\mathcal{L}\left(\frac{\partial f_1}{\partial t}\right) = s,$$

where

$$s = -i\omega.$$

Similarly, the 2nd term in parenthesis in Eq. (2.20), can be solved by Fourier Transform and it gives

$$\mathcal{F}\left(\frac{\partial f_1}{\partial x}\right) = ikf_1.$$

Now we can write Eq.(2.20) as

$$\frac{\partial f_1}{\partial \phi} - \frac{1}{\Omega}\bigg(- i\omega + i\vec{k}\cdot\vec{v}\bigg) = \frac{e}{m\Omega}\bigg(\vec{E}_1 + \frac{1}{c(\vec{v}\times\vec{B}_1)}\bigg)\cdot\frac{\partial f_0}{\partial \vec{v}} = 0. \tag{2.17}$$

This is an inhomogeneous differential equation. We can solve this equation by considering f_1 to be a single valued function which is periodic in ϕ . We first solve the homogeneous part of the equation which is written as

$$\frac{\partial G}{\partial \phi} - \left(\frac{-i\omega + i\vec{k}\cdot\vec{v}}{\Omega}\right) = 0. \tag{2.18}$$

The solution of the above equation can be written as

$$G(\phi') = \exp\left[\frac{-i}{\Omega}\int_{\phi}^{\phi'}(\omega - \vec{k}\cdot\vec{v''})d\phi''\right]. \tag{2.19}$$

In cylindrical coordinates, we can write

$$\vec{k} = (k_\perp, 0, k_\parallel),$$

and

$$\vec{v} = (v_\perp\cos\phi, v_\perp\sin\phi, v_\parallel).$$

Hence

$$\vec{k}\cdot\vec{v} = k_\perp v_\perp\cos\phi + k_\parallel v_\parallel.$$

Substituting for $\vec{k}\cdot\vec{v}$ in Eq. (2.23), and integrating over ϕ'', we get

$$G(\phi') = \exp\frac{-i}{\Omega}\left[\frac{(\omega - k_\parallel v_\parallel)(\phi - \phi') - k_\perp v_\perp(\sin(\phi) - \sin(\phi'))}{\Omega}\right]. \tag{2.20}$$

The solution of the inhomogeneous equation is written as

$$f_1 = \int \frac{\mathbf{G}(\phi')\Phi(\phi')}{\Omega} d\phi', \tag{2.21}$$

where

$$\Phi(\phi') = \frac{e}{m}(\vec{E}_1 + \frac{1}{c}\vec{v} \times \vec{B}_1) \cdot \frac{\partial f_0}{\partial \vec{v}}.$$

Substituting for $G(\phi')$ and $\Phi(\phi')$ in Eq. (2.25), we get

$$f_1 = \exp\left[\frac{-i}{\Omega}((\omega - k_\parallel v_\parallel)(\phi - \phi') - \imath k_\perp v_\perp(\sin(\phi) - \sin(\phi))\right] \times \left[\frac{e}{m\Omega}(\vec{E}_1 + \frac{\vec{v} \times \vec{B}_1}{c}).\frac{\partial f_0}{\partial \vec{v}}\right] d\phi. \tag{2.22}$$

As we know from the Maxwell's equation

$$\nabla \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t}, \tag{2.23}$$

so by applying Fourier and Laplace transform, we get

$$\vec{B}_1 = \frac{c(\vec{k} \times \vec{E}_1)}{\omega}.$$

By putting the value of $\vec{B}_1$ in Eq. (2.26), we get

$$f_1 = \frac{1}{\Omega}\int \exp\left[\frac{-i}{\Omega}((\omega - k_\parallel v_\parallel)(\phi - \phi') - k_\perp v_\perp(\sin(\phi) - \sin(\phi')))\right] \times \left[\frac{e}{m}\left(\vec{E}_1 + \frac{\vec{v} \times (\vec{k} \times \vec{E}_1)}{\omega}\right) \cdot \frac{\partial f_0}{\partial \vec{v}}\right] d\phi. \tag{2.24}$$

From the Maxwell's equation

$$\nabla \times \vec{B}_1 = \frac{4\pi}{c^2}\vec{J}_1 + \frac{1}{c^2}\frac{\partial \vec{E}_1}{\partial t}, \tag{2.25}$$

where $\vec{J}_1$ is the current density. Take curl on both sides of Eq. (2.27) to get

$$\nabla \times (\nabla \times \vec{E}_1) = -\frac{\partial(\nabla \times \vec{B}_1)}{\partial t},$$

or we can write

$$\nabla(\nabla \cdot \vec{E}_1) - E_1(\nabla \cdot \nabla) = -\frac{\partial(\nabla \times \vec{B}_1)}{\partial t}. \tag{2.26}$$

Substituting the value of $\nabla = ik$ and $\nabla \times \vec{B}_1$ from Eq. (2.29), the above Eq. will become

$$i\vec{k}(i\vec{k} \cdot \vec{E}_1) - \vec{E}_1(i^2\vec{k}.\vec{k} = -\left[\frac{4\pi}{c^2}\frac{\partial \vec{J}_1}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{E}_1}{\partial t^2}\right].$$

Since

$$\frac{\partial}{\partial t} = -i\omega$$

we can write the above equation as

$$-c^2\vec{k}(\vec{k}.\vec{E}_1) + c^2k^2\vec{E}_1 = \omega^2\vec{E}_1 + 4\pi i\omega\vec{J}_1,$$

or

$$(\omega^2 - c^2k^2)\vec{E}_1 + c^2\vec{k}(\vec{k} \cdot \vec{E}_1) + 4\pi i\omega\vec{J}_1 = 0. \tag{2.27}$$

The current density $\vec{J}$ can be written as

$$\vec{J} = ne\vec{v}.$$

or

$$\vec{J} = \sigma \cdot \vec{E}$$

where σ is the conductivity, n is the number density, e is the charge on an electron and $\vec{v}$ is the velocity. After linearization, we get

$$\vec{J_1} = ne\vec{v_1},$$

where

$$\vec{v_1} = \int f_1 \, \vec{v} \, d^3\vec{v}.$$

If there are many species in plasma such as ion, electron, positron, negative ion then current density relation modifies as given below

$$\vec{J}_{1\alpha} = \sum_{\alpha} q_\alpha n_{0\alpha} \int f_{1\alpha} v_\alpha d^3 v. \tag{2.28}$$

For electron plasma, the above equation becomes

$$\vec{J_1} = e \, n_0 \int f_1 \, \vec{v} \, d^3 v, \tag{2.29}$$

where d^3v is the volume element in cylindrical coordinates given as

$$\int d^3 v = \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_0^{2\pi} d\phi.$$

Using Eq.(2.27) in Eq.(2.33), we get

$$\vec{J_1} = e \, n_0 \int \vec{v} \, d^3 v \left(\frac{-i}{\Omega}\right) \int \exp\left[\frac{1}{\Omega}(\omega - k_\parallel v_\parallel)(\phi - \phi') - k_\perp v_\perp(\sin(\phi) - \sin(\phi'))\right] \times \left[\frac{e}{m}\left(\vec{E}_1 - \frac{\vec{v} \times (\vec{k} \times \vec{E}_1)}{\omega}\right) \cdot \frac{\partial f_0}{\partial \vec{v}}\right] d\phi. \tag{2.30}$$

Use Eq. (2.34) in Eq. (2.31), to get

$$(\omega^2-c^2k^2)\vec{E}_1+c^2\vec{k}(\vec{k}\cdot\vec{E}_1)+\frac{4\pi i\omega e^2\, n_0}{m\Omega}\int \vec{v}\, d^3v \int \exp\left[\frac{-i}{\Omega}((\omega-k_\parallel v_\parallel)(\phi-\phi')-k_\perp v_\perp(\sin(\phi)-\sin(\phi'))\right] \times \left[\left(\vec{E}_1+\frac{\vec{v}\times(\vec{k}\times\vec{E}_1)}{\omega}\right)\cdot\frac{\partial f_0}{\partial\vec{v}}\right]d\phi = 0. \quad (2.31)$$

The last term in Eq. (2.35) can be solved as follow

$$\begin{aligned}\left(\vec{E}_1+\frac{\vec{v}\times\vec{B}_1}{c}\right)\cdot\frac{\partial f_0}{\partial\vec{v}} &= \left[\vec{E}_1+\frac{1}{\omega}\left(\vec{v}\times(\vec{k}\times\vec{E}_1)\right)\right]\cdot\frac{\partial f_0}{\partial\vec{v}} \\ &= \left[\vec{E}_1\cdot\frac{\partial f_0}{\partial\vec{v}}+\frac{1}{\omega}\left[\vec{k}(\vec{v}\cdot\vec{E}_1)-\vec{E}_1(\vec{k}\cdot\vec{v})\right]\cdot\frac{\partial f_0}{\partial\vec{v}}\right] \\ &= \left[\vec{E}_1\cdot\frac{\partial f_0}{\partial\vec{v}}-\left(\frac{\vec{k}\cdot\vec{v}}{\omega}\right)\vec{E}_1\cdot\frac{\partial f_0}{\partial\vec{v}}+\left(\frac{\vec{k}}{\omega}\right)(\vec{v}\cdot\vec{E}_1)\cdot\frac{\partial f_0}{\partial\vec{v}}\right] \\ &= \left[\left(1-\frac{\vec{k}\cdot\vec{v}}{\omega}\right)\vec{E}_1\cdot\frac{\partial f_0}{\partial\vec{v}}+\left(\frac{\vec{k}\vec{v}}{\omega}\right)\cdot\vec{E}_1\cdot\frac{\partial f_0}{\partial\vec{v}}\right] \qquad (2.32) \\ &= \left[\left(1-\frac{\vec{k}\cdot\vec{v}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v}}{\omega}\right)\right]\cdot\vec{E}_1\cdot\frac{\partial f_0}{\partial\vec{v}} \\ &= \frac{\partial f_0}{\partial\vec{v}}\cdot\left[\left(1-\frac{\vec{k}\cdot\vec{v}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v}}{\omega}\right)\right]\cdot\vec{E}_1\end{aligned}$$

Putting the final expression of Eq. (2.36) in Eq. (2.35),

$$\begin{aligned}&(\omega^2-c^2k^2)\vec{E}_1+c^2\vec{k}(\vec{k}\cdot\vec{E}_1)+\frac{4\pi i\omega e^2\, n_0}{m\Omega}\int\vec{v}\, d^3v \\ &\quad\times\int\exp\left[\frac{-i}{\Omega}((\omega-k_\parallel v_\parallel)(\phi-\phi')-k_\perp v_\perp(\sin(\phi)-\sin(\phi')))\right]d\phi' \\ &\quad\times\frac{\partial f_0}{\partial\vec{v}}\cdot\left[\left(1-\frac{\vec{k}\cdot\vec{v}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v}}{\omega}\right)\right]\cdot\vec{E}_1 = 0 \qquad (2.33)\end{aligned}$$

or we can write

$$(\omega^2 - c^2k^2)\vec{E}_1 + c^2\vec{k}(\vec{k}\cdot\vec{E}_1) + 4\pi i\omega\overleftrightarrow{\sigma}\cdot\vec{E} = 0, \tag{2.34}$$

where

$$\overleftrightarrow{\sigma} = \frac{e^2 n_0}{m\Omega}\int \vec{v}\, d^3v \int \exp\left[\frac{-i}{\Omega}(\omega - k_\parallel v_\parallel)(\phi-\phi') - k_\perp v_\perp(\sin(\phi) - \sin(\phi'))\right] d\phi' \\ \times \frac{\partial f_0}{\partial \vec{v}}\cdot\left[\left(1 - \frac{\vec{k}\cdot\vec{v}}{\omega}\right)\tilde{I} + \left(\frac{\vec{k}\vec{v}}{\omega}\right)\right], \tag{2.35}$$

is called the conductivity tensor.

$$(\omega^2-c^2k^2)\tilde{I}+c^2k^2+\frac{4\pi i\omega e^2\, n_0}{m\Omega}\int \vec{v}\, d^3v\int \exp\left[\frac{1}{\Omega}(\omega-k_\parallel v_\parallel)(\phi-\phi')-k_\perp v_\perp(\sin(\phi)-\sin(\phi'))\right]d\phi' \\ \times\frac{\partial f_0}{\partial\vec{v}}\cdot\left(1-\frac{\vec{k}\cdot\vec{v}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v}}{\omega}\right)\cdot\vec{E}_1 = 0 \tag{2.36}$$

In Eq. (2.40), $\tilde{I}$ is an identity matrix given as

$$\tilde{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By changing variables as

$$\alpha = \phi - \phi'$$

$$d\phi' = -d\alpha$$

$$\vec{v'} = (v_\perp\cos(\phi-\alpha), v_\perp\sin(\phi-\alpha), v_\parallel)$$

The Eq. (2.40) becomes

$$(\omega^2-c^2k^2)\tilde{I}+c^2k^2-\frac{4\pi i\omega e^2\, n_0}{m\Omega}\int \vec{v}\, d^3v \int \exp\left[\frac{-i}{\Omega}((\omega-kv_{\|})\alpha-k_\perp v_\perp(\sin(\phi)-\sin(\phi-\alpha))\right]d\alpha$$
$$\times\frac{\partial f_0}{\partial\vec{v'}}\cdot\left[\left(1-\frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\,\vec{v'}}{\omega}\right)\right]\cdot\vec{E}_1=0. \quad (2.37)$$

The above equation can be written as

$$\overleftrightarrow{R}\cdot\vec{E}=0$$

where

$$\overleftrightarrow{R}=(\omega^2-c^2k^2)\tilde{I}+c^2k^2-\frac{4\pi i\omega e^2\, n_0}{m\Omega}\int \vec{v}\, d^3v \int \exp\left[\frac{-i}{\Omega}(\omega-k_{\|}v_{\|})\alpha-k_\perp v_\perp(\sin(\phi)-\sin(\phi-\alpha))\right]d\alpha$$
$$\times\frac{\partial f_0}{\partial\vec{v'}}\cdot\left[\left(1-\frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v'}}{\omega}\right)\right] \quad (2.38)$$

or we can write

$$\overleftrightarrow{R}=(\omega^2-c^2k^2)\tilde{I}+c^2k^2-4\pi i\omega\overleftrightarrow{\sigma},$$

where R is a dyadic and can be written as a square 3 by 3 matrix.

$$\begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}=0.$$

2.4 Dispersion relation for parallel propagating electromagnetic waves

In parallel propagating electromagnetic waves in plasma, the wave vector $\vec{k}$ is kept in parallel direction to magnetic field which is along z axis and electric field is in x and y direction i.e.,

$$\vec{k} = k\hat{z},$$

$$\vec{B} = B_0\hat{z},$$

and

$$\vec{E}_1 = E_x\hat{x} + E_y\hat{y}.$$

For parallel propagation consider the following matrices

$$\begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ 0 \end{bmatrix} = 0$$

By multiplying above matrix we get

$$R_{xx}E_x + R_{xy}E_y = 0,$$

and

$$R_{yx}E_x + R_{yy}E_y = 0.$$

So, in order to find the dispersion relation for R and L waves we will need R_{xx}, R_{xy}, R_{yx} and R_{yy} components of $\overleftrightarrow{R}$. For parallel propagation we put $k_\perp = 0$ and $k_\parallel = k$

in Eq. (2.41) to get

$$\overleftrightarrow{R} = (\omega^2 - c^2k^2)\tilde{I} + c^2k^2 - \frac{4\pi i\omega e^2\, n_0}{m\Omega}\int \vec{v}\, d^3v \int \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \times \frac{\partial f_0}{\partial \vec{v'}} \cdot \left[\left(1 - \frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I} + \left(\frac{\vec{k}\,\vec{v'}}{\omega}\right)\right]. \quad (2.39)$$

Simplifying the last term in parenthesis in Eq. (2.43) for parallel propagation, we get

$$\frac{\partial f_0}{\partial \vec{v'}} \cdot \left[\left(1 - \frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I} + \left(\frac{\vec{k}\vec{v'}}{\omega}\right)\right] = \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\cos(\phi - \alpha)\hat{x} + \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\sin(\phi - \alpha)\hat{y} + \frac{\partial f_0}{\partial v_\parallel}\hat{z} \quad (2.40)$$

$$\overleftrightarrow{R} = (\omega^2 - c^2k^2)\tilde{I} + c^2k^2 - \frac{4\pi i\omega e^2 n_0}{m\Omega}\int \vec{v}\, d^3v \int \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha$$

$$\times\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\cos(\phi - \alpha)\hat{x} + \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\sin(\phi - \alpha)\hat{y} + \frac{\partial f_0}{\partial v_\parallel}\hat{z}\Bigg] \quad . \quad (2.41)$$

Writing down the expression for σ

$$4\pi i\omega \overleftrightarrow{\sigma} = \frac{4\pi i\omega e^2 n_0}{m\Omega}\int \vec{v}\, d^3v \int \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\cos(\phi - \alpha)\hat{x} + \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\sin(\phi - \alpha)\hat{y} + \frac{\partial f_0}{\partial v_\parallel}\hat{z}\Bigg]. \quad (2.42)$$

The components of σ are

$$4\pi i\omega\sigma_{xx} = i\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_0^{2\pi} \cos\phi\cos(\phi-\alpha)d\phi \int_{\pm\infty}^0 \exp\left[\frac{-i}{\Omega}(\omega-kv_\parallel)\alpha\right]d\alpha$$
$$\times\left[\left(1-\frac{k\,v_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.43)$$

$$4\pi i\omega\sigma_{xy} = i\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_0^{2\pi} \cos\phi\sin(\phi-\alpha)d\phi \int_{\pm\infty}^0 \exp\left[\frac{-i}{\Omega}(\omega-kv_\parallel)\alpha\right]d\alpha$$
$$\times\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.44)$$

$$4\pi i\omega\sigma_{yy} = i\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_0^{2\pi} \sin\phi\sin(\phi-\alpha)d\phi \int_{\pm\infty}^0 \exp\left[\frac{-i}{\Omega}(\omega-kv_\parallel)\alpha\right]d\alpha$$
$$\times\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.45)$$

$$4\pi i\omega\sigma_{zz} = i\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} \frac{\partial f_0}{\partial v_\parallel}dv_\parallel \int_0^{2\pi} d\phi \int_{\pm\infty}^0 \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right]d\alpha\,. \quad (2.46)$$

σ_{zz} plays no role in the parallel propagation of electromagnetic waves but it is significantly important for the parallel propagation of electrostatic waves (Langmuir waves). The solution of ϕ-integral in the above components is given as follow

$$\int_0^{2\pi}\cos\phi\cos(\phi-\alpha)d\phi = \pi\cos(\alpha),$$

$$\int_0^{2\pi}\cos\phi\sin(\phi-\alpha)d\phi = -\pi\sin(\alpha),$$

$$\int_0^{2\pi}\sin\phi\cos(\phi-\alpha)d\phi = \pi\sin(\alpha),$$

and

$$\int_0^{2\pi} \sin\phi \sin(\phi - \alpha) d\phi = \pi \cos(\alpha).$$

After using these results for ϕ integration in the components of σ, we have

$$4\pi i\omega\sigma_{xx} = i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^{\infty} v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_{\pm\infty}^{0} \cos\alpha \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \\ \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.47)$$

$$4\pi i\omega\sigma_{xy} = -i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^{\infty} v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_{\pm\infty}^{0} \sin\alpha \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \\ \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.48)$$

$$4\pi i\omega\sigma_{yx} = i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^{\infty} v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_{\pm\infty}^{0} \sin\alpha \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \\ \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.49)$$

$$4\pi i\omega\sigma_{yy} = i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^{\infty} v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_{\pm\infty}^{0} \cos\alpha \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \\ \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]. \quad (2.50)$$

From the above equation it can be seen that

$$\sigma_{xx} = \sigma_{yy},$$

and

$$\sigma_{xy} = -\sigma_{yx}.$$

Therefore,

$$4\pi i\omega\sigma_{xx} = 4\pi i\omega\sigma_{yy} = i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_{\pm\infty}^{0} \cos\alpha \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.51)$$

and

$$4\pi i\omega\sigma_{yx} = -4\pi i\omega\sigma_{xy} = -i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \int_{\pm\infty}^{0} \sin\alpha \exp\left[\frac{-i}{\Omega}(\omega - kv_\parallel)\alpha\right] d\alpha \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]. \quad (2.52)$$

Perform α integration to get

$$4\pi i\omega\sigma_{xx} = 4\pi i\omega\sigma_{yy} = \pi\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \frac{\Omega(\omega - kv)}{[\Omega^2 - (\omega - kv)^2]} \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \quad (2.53)$$

and

$$4\pi i\omega\sigma_{yx} = -4\pi i\omega\sigma_{xy} = -i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \frac{\Omega^2}{[\Omega^2 - (\omega - kv)^2]} \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]. \quad (2.54)$$

Using the equation

$$\overleftrightarrow{R} = (\omega^2 - c^2k^2)\tilde{I} + c^2k^2 - 4\pi i\omega\overleftrightarrow{\sigma},$$

we can write

$$R_{xx} = R_{yy} = \omega^2 - c^2k^2 - 4\pi i\omega\sigma_{xx} \quad (2.55)$$

and

$$R_{yx} = -R_{xy} = -4\pi i\omega\sigma_{xy}. \tag{2.56}$$

Using Eq. (2.58) and Eq. (2.59) in Eq. (2.60) and Eq. (2.61) respectively, we have

$$R_{xx} = R_{yy} = \omega^2 - c^2k^2 - \pi\omega\frac{\omega_p^2}{\Omega}\int_0^{\infty} v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \frac{\Omega(\omega - kv)}{[\Omega^2 - (\omega - kv)^2]} \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right], \tag{2.57}$$

and

$$R_{yx} = -R_{xy} = -i\pi\omega\frac{\omega_p^2}{\Omega}\int_0^{\infty} v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \frac{\Omega^2}{[\Omega^2 - (\omega - kv)^2]} \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]. \tag{2.58}$$

As $R_{xx} = R_{yy}$ and $R_{yx} = -R_{xy}$ the matrice for parallel propagating electromagnetic wave can be written as

$$\begin{vmatrix} R_{xx} & -R_{xy} \\ R_{xy} & R_{xX} \end{vmatrix} = 0.$$

It gives

$$R_{xx} \pm iR_{xy} = 0 \tag{2.59}$$

By using the values for R_{xx} and R_{xy}, we get

$$\omega^2 - c^2k^2 - \pi\omega\omega_p^2 \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{+\infty} dv_\parallel \frac{1}{[(\omega - kv)^2 \pm \Omega^2]} \times \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right] = 0. \quad (2.60)$$

This is the general dispersion relation for R-wave (right handed circularly polarized wave) and L-wave (left handed circularly polarized wave). The upper sign is for R-wave and lower sign is for L-wave.

Chapter 3

Quantum Kinetics

3.1 Quantum Kinetics

Following the success of the classical theory of non-equilibrium physics, it was natural to give an identical theory for quantum systems within the late 1920s and early 1930s. In classical kinetics, the Vlasov equation is used to analyze the dynamics of a plasma system by using a phase space distribution $f(x, p)$. In a quantum plasma, the classical Vlasov equation cannot be adopted due to Heisenberg's uncertainty principle which says that position and momentum cannot be find simultaneously or in other words the observable do not commute. So in quantum systems the particles trajectories are smeared out. It is necessary to formulate such model that can connect classical kinetics to quantum kinetics. The use of quasi-distributions is extremely useful for linking classical plasma physics to the evolution of non-equilibrium quantum systems. There are mainly two ways of connecting classical kinetic theory and quantum kinetic theory. Firstly utilising ensemble averages of observable to understand the quasi-distribution function is a straightforward analogue to the classical situation. Secondly, the quantum Liouville equation for the density operator gives rise to the quasi-distribution evolution, which is a quantum version of the Vlasov or Boltzmann equation. A quantum kinetic theory for the quasi-distribution function may be beneficial for adapting classical numerical codes to the quantum realm. Of course, there are an endless number of ways to design a quasi-distribution function, as long as some basic requirements are met. However, the literature emphasizes a few quasi-distribution functions like the most well-known Wigner distribution, over others [27].

3.2 Wigner Function

In classical physics a state of a system is defined by a specific point in a 6N-dimensional phase space for momentum "p" and position "q". Because classical physics has no uncertainty principle, it is acceptable to know a particle's momentum and position at the very same time but in quantum mechanics it is impossible to do so due to the uncertainty principle. Therefore, in the formulation of quantum mechanics, probability densities are used, one for the position-based wave function and one for the momentum-based wave function.

$$P(x) = |\psi(x)|^2$$

$$P(k) = |\phi(k)|^2$$

The Fourier transform connects the two functions, and we used $\vec{p} = \hbar\vec{k}$ to link them.

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ \psi(x) \ exp^{-\iota kx}$$

We require a single valued function which could offer probability in both position and momentum. The Wigner function was created specifically for this purpose. Eugene Wigner proposed it in 1932 to investigate quantum corrections to classical statistical mechanics. The objective was to connect Schrodinger's equation's wave function to a probability distribution. It should also be able to provide operator's correct expectation values. A probability distribution in phase space P(x,p) is desired that is positive everywhere and such that

$$\int\int dx\ dp\ P(x,y)\ A(x,y)$$

gives expectation value for operator A(x,y). Due to Heisenberg's uncertainty principle it is not possible to find such a probability distribution although the Wigner function approaches these requirements, but it does not satisfy all the requirements of probability distribution function. For example, the Wigner function carries negative in areas of phase space which have no physical value.

3.2.1 Weyl Transformation

The Wigner function is an attempt to construct a new quantum mechanics formalism based on the idea of phase space. To develop such a formalism, a mapping between functions in the quantum phase space formulation and Hilbert space operators in the Schrodinger picture is necessary. This mapping is given by Weyl transformation $\tilde{A}(x,p)$ of an operator $\hat{A}(\hat{x},\hat{p})$ defined as

$$\tilde{A}(x,p) = \int dy\ e^{\frac{-\iota\vec{p}\cdot\vec{y}}{\hbar}} \left\langle x+\frac{y}{2}\left|\hat{A}(\hat{x},\hat{p})\right|x-\frac{y}{2}\right\rangle. \tag{3.1}$$

This transformation replaces an operator with a function. The product of two operators $\hat{A}$ and $\hat{B}$ has trace given by

$$Tr[\hat{A}\hat{B}] = \frac{1}{2\pi\hbar}\int\int dx\ dp\ \tilde{A}(x,y)\ \tilde{B}(x,y). \tag{3.2}$$

This is the key property of Weyl-transformation and can be proved with a simple formalism.

3.2.2 Formalism for Wigner function

The Wigner function is obtained by Weyl transform of density operator. The density operator in quantum mechanics gives the physical state of the system. A pure quantum state defines density operator as

$$\hat{\rho} = |\psi\rangle\langle\psi| \, .$$

In position basis it can be expressed as

$$\langle x |\hat{\rho}| x' \rangle = \psi(x)\psi^*(x').$$

The density matrix has the property of being normalised, i.e., $Tr[\hat{\rho}] = 1$.

$$Tr[\hat{\rho}] = \sum_n \langle n |\hat{\rho}| n \rangle = \sum_n \langle n |\psi\rangle\langle\psi| n \rangle = \sum_n \langle\psi |n\rangle\langle n| \psi\rangle = 1$$

The expectation value of an operator $\hat{A}$ is obtained from $\hat{\rho}$ as

$$\langle\hat{A}\rangle = Tr[\hat{\rho}\hat{A}] = Tr[|\psi\rangle\langle\psi| \hat{A}] = \sum_n \left\langle n |\psi\rangle\langle\psi| \hat{A} \; |n \right\rangle = \sum_n \langle\psi|\hat{A} \; |n\rangle \, \langle n|\psi\rangle$$

$$\langle\hat{A}\rangle = Tr[\hat{\rho}\hat{A}] = \sum_n \langle\psi|\hat{A} \; |\psi\rangle$$

Using Eq. (3.2), we get

$$\langle\hat{A}\rangle = Tr[\hat{\rho}\hat{A}] = \frac{1}{2\pi\hbar} \int\int dx \; dp \; \tilde{\rho}\tilde{A}$$

Now its time to define Wigner function, which is given as

$$f(x,p) = \frac{\tilde{\rho}}{2\pi\hbar} = \frac{1}{2\pi\hbar} \int dy \; e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} \; \psi\left(x + \frac{y}{2}\right) \psi^*\left(x - \frac{y}{2}\right). \tag{3.3}$$

3.3 Hamiltonian of a Charged Particle Moving in an Electric and Magnetic Field

The force experienced by a charged particle travelling in a region containing an electric and magnetic field is described by the Lorentz force Law.

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}]. \tag{3.4}$$

Electric and Magnetic fields are expressed in terms of scalar and vector potentials as

$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{3.5}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}. \tag{3.6}$$

Now substituting Eq. (3.5) and Eq. (3.6) in Eq. (3.4), we get

$$\vec{F} = q \left[-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \vec{\nabla} \times \vec{A} \right] \tag{3.7}$$

Eq. (3.7) gives the time dependent nature of the potential energy. Now simplifying the calculation as

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{v} \times \vec{\nabla} \times \vec{A} \;=\; \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \\ v_x & v_y & v_z \\ \\ \left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right) & \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right) & \left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right) \end{vmatrix}$$

In order to make the calculations simple we consider the z-component i.e.,

$$(\vec{v} \times \vec{\nabla} \times \vec{A})_z \;=\; v_x \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - v_y \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right). \tag{3.8}$$

By using z-component of Eq. (3.7) in Eq. (3.8), we get

$$F_z = q \left[-\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} + \left(\vec{v} \times \vec{\nabla} \times \vec{A}\right)_z\right],$$

$$F_z = q \left[-\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} + v_x \frac{\partial A_x}{\partial z} + v_y \frac{\partial A_y}{\partial z} - v_x \frac{\partial A_z}{\partial x} - v_y \frac{\partial A_z}{\partial y}\right].$$

Adding and subtracting $v_z \dfrac{\partial A_z}{\partial z}$ on the R.H.S, we get

$$F_z = q \left[-\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} + v_x \frac{\partial A_x}{\partial z} + v_y \frac{\partial A_y}{\partial z} + v_z \frac{\partial A_z}{\partial z} - v_x \frac{\partial A_z}{\partial x} - v_y \frac{\partial A_z}{\partial y} - v_z \frac{\partial A_z}{\partial z}\right],$$

or we can write

$$F_z = q \left[-\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} + \frac{\partial}{\partial z}\left(\vec{v}.\vec{A}\right) - v_x \frac{\partial A_z}{\partial x} - v_y \frac{\partial A_z}{\partial y} - v_z \frac{\partial A_z}{\partial z}\right].$$

As we know that $\quad \vec{A} = \vec{A}(\vec{r}, t) = \vec{A}(x, y, z, t)$

$$F_z = q \left[-\frac{\partial \phi}{\partial z} + \frac{\partial}{\partial z}\left(\vec{v}.\vec{A}\right) - \left(\frac{dA_z}{dt}\right)\right].$$

$$F_z = q \left[\frac{\partial}{\partial z}\left(\vec{v}.\vec{A} - \phi\right) - \frac{dA_z}{dt}\right]. \tag{3.9}$$

Now consider

$$\frac{\partial}{\partial v_z}(\vec{v}\cdot\vec{A}) = \frac{\partial}{\partial v_z}(v_x A_x + v_y A_y + v_z A_z),$$

$$\frac{\partial}{\partial v_z}(\vec{v}\cdot\vec{A}) \;=\; A_z.$$

Since we know $\phi = \phi(\vec{r}, t)$

$$\frac{\partial}{\partial v_z}(\vec{v}\cdot\vec{A} - \phi) \;=\; A_z$$

Substituting A_z in Eq. (3.9),

$$F_z = q \left[\frac{\partial}{\partial z}\left(\vec{v}.\vec{A} - \phi\right) - \frac{d}{dt}\left(\frac{\partial}{\partial v_z}(\vec{v}.\vec{A} - \phi)\right)\right]$$

$$F_z = \left[\frac{\partial}{\partial z} q\left(\vec{v}\cdot\vec{A} - \phi\right) - \frac{d}{dt}\left(\frac{\partial}{\partial v_z} q\left(\vec{v}\cdot\vec{A} - \phi\right)\right)\right] \tag{3.10}$$

Eq. (3.10) is termed as generalized force equation, where

$$U(\vec{r}, t) \;=\; q\left(\vec{v}\cdot\vec{A} - \phi\right). \tag{3.11}$$

So, Lagrangian for our system can be written as

$$L(\vec{r}, \dot{\vec{r}}) \;=\; \frac{1}{2}mv^2 + q\left(\vec{v}\cdot\vec{A} - \phi\right). \tag{3.12}$$

Finally, constructing Hamiltonian for our system.

$$H(\vec{r}, \vec{p}, t) = \vec{v}\cdot\vec{p} - L(\vec{r}, \vec{v}, t). \tag{3.13}$$

The canonical momentum of system is given as

$$\vec{p} = \frac{\partial L}{\partial \vec{v}}.$$

By using Eq. (3.12), we get

$$\vec{p} = \frac{\partial}{\partial \vec{v}} \left(\frac{1}{2} m v^2 + q \left(\vec{v} \cdot \vec{A} - \phi \right) \right)$$

or

$$\vec{p} = m\vec{v} + q\vec{A} \tag{3.14}$$

Putting Eq. (3.14) and Eq. (3.12) in Eq. (3.13), we get

$$H = \vec{v} \cdot \left(m\vec{v} + q\vec{A} \right) - \left[\frac{1}{2} m v^2 + q \left(\vec{v} \cdot \vec{A} - \phi \right) \right],$$

$$H = \left(m\vec{v} \cdot \vec{v} + q\ \vec{v} \cdot \vec{A} \right) - \frac{1}{2} m v^2 - q \left(\vec{v} \cdot \vec{A} - \phi \right),$$

$$H = m v^2 + q\ \vec{v} \cdot \vec{A} - \frac{1}{2} m v^2 - q\ \vec{v} \cdot \vec{A} + q\ \phi,$$

$$H = \frac{1}{2} m v^2 + q\ \phi. \tag{3.15}$$

From Eq. (3.24), we can write $\vec{v}$ as

$$\vec{v} = \frac{\vec{p} - q\vec{A}}{m}.$$

By using this value in Eq. (3.15), we get

$$H = \frac{1}{2} m \frac{\left(\vec{p} - q\vec{A} \right)^2}{m^2} + q\ \phi$$

$$\boxed{H = \frac{1}{2m} \left(\vec{p} - q\vec{A} \right)^2 + q\ \phi}. \tag{3.16}$$

The above equation is all about the Hamiltonian of a charged particle moving in an electric and magnetic field.

3.4 Derivation for Quantum Vlasov equation

3.4.1 Schrodinger Equation

In classical mechanics, the Schrodinger equation serves as a counterpart to Newton's laws of energy conservation, predicting the behaviour of a dynamic system in the future. It is a wave equation in terms of the wave function that predicts the probability of occurrences or outcomes analytically and accurately. The Schrodinger equation for a particle of mass m and charge q moving in an electromagnetic field can be given as

$$\frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial r} - \frac{q}{c}A_i\right)^2 + q\phi\right]\psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{dt}. \tag{3.17}$$

Replacing the wave function $\psi(\vec{r},t)$ with $\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)$, we get

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{q^2}{2mc^2}A^2\left(\vec{r}+\frac{\vec{y}}{2}\right) - \frac{1}{m}\left(\frac{\hbar}{i}\frac{\partial}{\partial r}\right)\left(\frac{q}{c}\vec{A}\left(\vec{r}+\frac{\vec{y}}{2}\right)\right) + q\phi\right]\psi\left(\vec{r}+\frac{\vec{y}}{2}\right) = i\hbar\frac{\partial\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)}{dt},$$

$$\left[-\frac{\hbar^2}{2mi\hbar}\frac{\partial^2}{\partial r^2} + \frac{q^2}{2i\hbar mc^2}A^2\left(\vec{r}+\frac{\vec{y}}{2}\right) - \frac{1}{i\hbar m}\left(\frac{\hbar}{i}\frac{\partial}{\partial r}\right)\left(\frac{q}{c}\vec{A}\left(\vec{r}+\frac{\vec{y}}{2}\right)\right) + \frac{q\phi}{i\hbar}\right]\psi\left(\vec{r}+\frac{\vec{y}}{2}\right) = \frac{\partial\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)}{dt},$$

$$\frac{\partial\psi(\vec{r}+\frac{\vec{y}}{2})}{dt} = \Bigg[\frac{i\hbar}{2m}\frac{\partial^2\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)}{\partial r^2} - \frac{iq^2}{2\hbar mc^2}A^2(\vec{r}+\frac{\vec{y}}{2})\psi(\vec{r}+\frac{\vec{y}}{2})$$

$$+\frac{q}{mc}\left[\frac{\partial}{\partial r}\right]\left(\vec{A}\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\right) - \frac{iq\phi\left(\vec{r}+\frac{\vec{y}}{2}\right)}{\hbar}\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\Bigg]. \quad (3.18)$$

Taking complex conjugate of Eq. (3.18), we get

$$\frac{\partial\psi^*(\vec{r}-\frac{\vec{y}}{2})}{dt} = \Bigg[-\frac{i\hbar}{2m}\frac{\partial^2\psi\left(\vec{r}-\frac{\vec{y}}{2}\right)}{\partial r^2} + \frac{iq^2}{2\hbar mc^2}A^2(\vec{r}-\frac{\vec{y}}{2})\psi^*(\vec{r}-\frac{\vec{y}}{2})$$

$$+\frac{q}{mc}\frac{\partial}{\partial r}\left(\vec{A}\left(\vec{r}-\frac{\vec{y}}{2}\right)\psi\left(\vec{r}-\frac{\vec{y}}{2}\right)\right) + \frac{iq\phi\left(\vec{r}-\frac{\vec{y}}{2}\right)}{\hbar}\psi\left(\vec{r}-\frac{\vec{y}}{2}\right)\Bigg]. \quad (3.19)$$

3.4.2 Wigner function and its Evolution

Taking time derivative of Wigner function from Eq. (3.3), we get [31],

$$\frac{\partial f}{\partial t} = \frac{1}{(2\pi\hbar)^3}\frac{\partial}{\partial t}\left[\int d^3y\ e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}+\frac{\vec{y}}{2}\right)\right],$$

or

$$\frac{\partial f}{\partial t} = \frac{1}{(2\pi\hbar)^3}\int d^3y\ e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left[\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\frac{\partial\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)}{\partial t} + \psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)\frac{\partial\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)}{\partial t}\right]$$

Using Eq. (3.18) and Eq. (3.19), this becomes

$$\frac{\partial f}{\partial t} = \frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\Bigg[\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\Bigg\{\frac{-i\hbar}{2m}\frac{\partial^2\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial r^2}+\frac{iq^2}{2\hbar mc^2}A^2(\vec{r}-\frac{\vec{y}}{2})\psi^*(\vec{r}-\frac{\vec{y}}{2})$$
$$+\frac{q}{mc}\frac{\partial}{\partial r}\Bigg(\vec{A}\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\Bigg)+\frac{iq\phi(\vec{r}-\frac{\vec{y}}{2})}{\hbar}\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\Bigg\}\Bigg]$$
$$+\frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\Bigg[\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\Bigg\{\frac{i\hbar}{2m}\frac{\partial^2\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial r^2}-\frac{iq^2}{2\hbar mc^2}A^2(\vec{r}+\frac{\vec{y}}{2})\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)$$
$$+\frac{q}{mc}\frac{\partial}{\partial r}\Bigg(\vec{A}\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\Bigg)-\frac{iq\phi(\vec{r}+\frac{\vec{y}}{2})}{\hbar}\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\Bigg\}\Bigg].$$

By rearranging this equation and combining similar terms, we get

$$\frac{\partial f}{\partial t} = \frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{i\hbar}{2m}\right)\left[\psi^*(r-\frac{y}{2})\frac{\partial^2\psi(\vec{r}+\frac{\vec{y}}{2})}{\partial r^2}-\psi(r+\frac{y}{2})\frac{\partial^2\psi^*(\vec{r}-\frac{\vec{y}}{2})}{\partial r^2}\right]$$
$$+\frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{iq^2}{2\hbar mc^2}\right)\left[A^2(\vec{r}-\frac{\vec{y}}{2})-A^2(\vec{r}+\frac{\vec{y}}{2})\right]\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)$$
$$+\frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{q}{mc}\right)\left[\frac{\partial\vec{A}\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial r}+\frac{\partial\vec{A}\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial r}\right]\psi\Big(r+\frac{y}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)$$
$$+\frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{iq}{\hbar}\right)\left[\phi\Big(\vec{r}-\frac{\vec{y}}{2}\Big)-\phi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\right]\psi\Big(r+\frac{y}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)$$
$$+\frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{q}{mc}\right)\left[\vec{A}\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\frac{\partial\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial r}+\vec{A}\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\frac{\partial\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial r}\right].$$

So, there are five integrals in the above expression and we will solve each of them separately.

$$\frac{\partial f}{\partial t} = I_1 + I_2 + I_3 + I_4 + I_5.$$

Solving Integral - I_1

$$I_1 = \frac{1}{(2\pi\hbar)^3}\int d^3y\, e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{i\hbar}{2m}\right)\left[\psi^*(r-\frac{y}{2})\frac{\partial^2\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial r^2}-\psi(r+\frac{y}{2})\frac{\partial^2\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial r^2}\right]$$

Using the identity

$$\partial_x \psi\Big(x \pm \frac{\vec{y}}{2}\Big) = \pm 2\partial_y \psi\Big(x \pm \frac{\vec{y}}{2}\Big)$$

This implies

$$\partial_r \psi\Big(\vec{r} \pm \frac{\vec{y}}{2}\Big) = \pm 2\partial_y \psi\Big(\vec{r} \pm \frac{\vec{y}}{2}\Big) \tag{3.20}$$

$$\partial_r^2 \psi\Big(\vec{r} \pm \frac{\vec{y}}{2}\Big) = \pm 2\partial_{\vec{r}}\partial_{\vec{y}} \psi\Big(\vec{r} \pm \frac{\vec{y}}{2}\Big) \tag{3.21}$$

Multiplying Eq. (3.22) with $\psi^*\Big(\vec{r} - \frac{\vec{y}}{2}\Big)$ and $\psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big)$ separately will give us

$$\psi^*\Big(\vec{r} - \frac{\vec{y}}{2}\Big)\partial_r^2 \psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big) = +2\partial_{\vec{r}}\partial_{\vec{y}} \psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r} - \frac{\vec{y}}{2}\Big) \tag{3.22}$$

$$\psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big)\partial_r^2 \psi\Big(\vec{r} - \frac{\vec{y}}{2}\Big) = -2\partial_{\vec{r}}\partial_{\vec{y}} \psi\Big(\vec{r} - \frac{\vec{y}}{2}\Big)\psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big). \tag{3.23}$$

Subtract Eq. (3.23) and Eq. (3.24) to get

$$\psi^*\Big(\vec{r} - \frac{\vec{y}}{2}\Big)\partial_r^2 \psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big) - \psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big)\partial_r^2 \psi\Big(\vec{r} - \frac{\vec{y}}{2}\Big) = 2\partial_{\vec{r}}\partial_{\vec{y}}\bigg(\psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r} - \frac{\vec{y}}{2}\Big)\bigg) \tag{3.24}$$

Now we can write I_1 as

$$I_1 = \frac{2}{(2\pi\hbar)^3}\int d^3y\ \bigg(\frac{i\hbar}{2m}\bigg)\partial_{\vec{r}}\partial_{\vec{y}}\bigg(\psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r} - \frac{\vec{y}}{2}\Big)\bigg)e^{\frac{-i\vec{p}.\vec{y}}{\hbar}},$$

and

$$I_1 = \frac{1}{(2\pi\hbar)^3}\bigg(\frac{i\hbar}{m}\bigg)\partial_{\vec{r}}\int d^3y\ \partial_{\vec{y}}^3\bigg(\psi\Big(\vec{r} + \frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r} - \frac{\vec{y}}{2}\Big)\bigg)e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}. \tag{3.25}$$

Integrating by parts

$$I_1 = \frac{1}{(2\pi\hbar)^3}\left(\frac{i\hbar}{m}\right)\partial_{\vec{r}}\left[e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}\int d^3y\ \partial^3_{\vec{y}}\left(\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)\right)\right]$$
$$-\int d^3y\ \left(\frac{d^3}{d^3\vec{y}}e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}\right)\ \cdot\ \int d^3y\ \partial^3_{\vec{y}}\left(\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)\right)$$

$$\implies I_1 = \frac{1}{(2\pi\hbar)^3}\left(\frac{i\hbar}{m}\right)\partial_{\vec{r}}\left[e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}\left(\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)\right)\Big|_{-\infty}^{\infty}\right.$$
$$\left.-\int d^3y\ \left(\frac{-i\vec{p}}{\hbar}\right)e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}\ \cdot\ \left(\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)\right)\right]$$

$$I_1 = \frac{1}{(2\pi\hbar)^3}\left(\frac{i^2\vec{p}}{m}\right)\ \cdot\ \frac{\partial}{\partial\vec{r}}\int d^3y\ e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}\left(\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)\right)$$
$$I_1 = \left(\frac{-\vec{p}}{m}\right)\ \cdot\ \frac{\partial}{\partial\vec{r}}\ \frac{1}{(2\pi\hbar)^3}\int d^3y e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}\left(\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)\right)$$

Making use of Eq. (3.3), we can write

$$\boxed{I_1 = -\frac{\vec{p}}{m}\cdot\frac{\partial f}{\partial\vec{r}}} \tag{3.26}$$

Solving integral - I_2

$$I_2 = \frac{1}{(2\pi\hbar)^3}\int d^3y\ e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}\left(\frac{iq^2}{2\hbar mc^2}\right)\left[A^2(\vec{r}-\frac{\vec{y}}{2}) - A^2(\vec{r}+\frac{\vec{y}}{2})\right]\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)$$

Let's have a look at some mathematical analysis before we tackle the second integral. Consider the Taylor's series for an arbitrary function $\phi\left(\vec{r}+\frac{\vec{y}}{2}\right)$

$$\phi\left(\vec{r}+\frac{\vec{y}}{2}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\ \frac{\partial^n\ \phi(\vec{r})}{\partial^n\vec{r}}\ \left(\frac{y}{2}\right)^n$$

Multiplying both sides by $e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}$

$$\phi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}.\vec{y}}{\hbar}} = \sum_{n=0}^{\infty}\frac{1}{n!}\ \frac{\partial^n\ \phi(\vec{r})}{\partial^n\vec{r}}\ \left(\frac{y}{2}\right)^n e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}$$

Multiplying and diving by $\frac{-i}{\hbar}$ on the R.H.s of the above equation, we get

$$\phi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}.\vec{y}}{\hbar}} = \sum_{n=0}^{\infty}\frac{1}{n!}\ \frac{\partial^n\ \phi(\vec{r})}{\partial^n\vec{r}}\ \left(\frac{-i}{\hbar}\right)^n\left(\frac{-\hbar}{i}\right)^n\ \left(\frac{y}{2}\right)^n e^{\frac{-i\vec{p}.\vec{y}}{\hbar}}$$

$$\phi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}.\vec{y}}{\hbar}} = \sum_{n=0}^{\infty}\frac{1}{n!}\ \frac{\partial^n\ \phi(\vec{r})}{\partial^n\vec{r}}\ \left(\frac{-\hbar}{2i}\right)^n\left(\frac{-iy}{\hbar}\right)^n\ e^{\frac{-i\vec{p}\vec{y}}{\hbar}}$$

where

$$\frac{\partial^n}{\partial p^n}e^{\frac{-i\vec{p}.\vec{y}}{\hbar}} = \left(\frac{-iy}{\hbar}\right)^n e^{\frac{-i\vec{p}\vec{y}}{\hbar}},$$

$$\phi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} = \sum_{n=0}^{\infty}\frac{1}{n!}\ \frac{\partial^n\ \phi(\vec{r})}{\partial^n\vec{r}}\ \left(\frac{-\hbar}{2i}\right)^n\frac{\partial^n}{\partial p^n}e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

$$\phi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\vec{y}}{\hbar}} = \phi(\vec{r})\ \left[\sum_{n=0}^{\infty}\frac{1}{n!}\ \left(\frac{-\hbar}{2i}\right)^n\frac{\overleftarrow{\partial}^n}{\partial^n\vec{r}}\ \frac{\overrightarrow{\partial}^n}{\partial p^n}\right]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

It can be seen that the series in square brackets is actually the Taylor series of an exponential function.

$$\phi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} = \phi(\vec{r})\exp\left(\frac{-\hbar}{2i}\ \frac{\overleftarrow{\partial}^n}{\partial^n\vec{r}}\ \frac{\overrightarrow{\partial}^n}{\partial p^n}\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

where $\exp\left(\frac{-\hbar}{2i}\ \frac{\overleftarrow{\partial}^n}{\partial^n\vec{r}}\ \frac{\overrightarrow{\partial}^n}{\partial p^n}\right)$ is considered as an operator, with $\frac{\overleftarrow{\partial}}{\partial\vec{r}}$ acting to the left on $\phi(\vec{r})$ and $\frac{\overrightarrow{\partial}}{\partial\vec{p}}$ acting to the right on $e^{\frac{-i\vec{p}\vec{y}}{\hbar}}$. It may be written in a more general form as

$$\phi\Big(\vec{r}\pm\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\vec{y}}{\hbar}} = \phi(\vec{r})\exp\left(\mp\frac{\hbar}{2i}\ \frac{\overleftarrow{\partial}^n}{\partial^n\vec{r}}\ \frac{\overrightarrow{\partial}^n}{\partial p^n}\right)\ \exp\frac{-i\vec{p}\cdot\vec{y}}{\hbar} \qquad (3.27)$$

Solving I_2 and consider

$$\left(A^2(\vec{r}-\frac{\vec{y}}{2})-A^2(\vec{r}+\frac{\vec{y}}{2})\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}=A^2(\vec{r})\left[\exp\left(\frac{\hbar}{2i}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)-\exp\left(\frac{-\hbar}{2i}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\right]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

or

$$\left(A^2(\vec{r}-\frac{\vec{y}}{2})-A^2(\vec{r}+\frac{\vec{y}}{2})\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}=-A^2(\vec{r})\left[\exp\left(\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)-\exp\left(\frac{-i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\right]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

By using Euler's formula, we can write $e^{ix}-e^{-ix}=2i\sin(x)$. So the above expression in parenthesis can be written as

$$\left(A^2(\vec{r}-\frac{\vec{y}}{2})-A^2(\vec{r}+\frac{\vec{y}}{2})\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}=-2iA^2(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} \tag{3.28}$$

So,

$$I_2=\left(\frac{iq^2}{2\hbar mc^2}\right)\frac{1}{(2\pi\hbar)^3}\int\left[\left(A^2(\vec{r}-\frac{\vec{y}}{2})-A^2(\vec{r}+\frac{\vec{y}}{2})\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\right]\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)d^3y$$

$$I_2=\left(\frac{iq^2}{2\hbar mc^2}\right)\frac{1}{(2\pi\hbar)^3}\int-2iA^2(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial r}\frac{\overrightarrow{\partial}}{\partial p}\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)d^3y$$

$$I_2=\left(-\frac{i^2q^2}{\hbar mc^2}\right)A^2(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\frac{1}{(2\pi\hbar)^3}\int e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)d^3y$$

$$\boxed{I_2=\left(\frac{q^2}{\hbar mc^2}\right)A^2(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f} \tag{3.29}$$

Solving integral - I_3

$$I_3 = \frac{1}{(2\pi\hbar)^3}\int d^3y\ e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{q}{mc}\right)\left[\frac{\partial\vec{A}\left(\vec{r}+\frac{\vec{y}}{2}\right)}{\partial r}+\frac{\partial\vec{A}\left(\vec{r}-\frac{\vec{y}}{2}\right)}{\partial r}\right]\psi\left(r+\frac{y}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)$$

Using Coulomb Gauge which says that the divergence of the magnetic vector potential must be zero i.e.,

$$\nabla\cdot\vec{A} = 0,$$

where

$$\nabla\cdot\vec{A} = \frac{\partial\vec{A}}{\partial\vec{r}} = 0.$$

so

$$\boxed{I_3 = 0}$$

Solving integral - I_4

$$I_4 = \frac{1}{(2\pi\hbar)^3}\int d^3y\ e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{iq}{\hbar}\right)\left[\phi\left(\vec{r}-\frac{\vec{y}}{2}\right)-\phi\left(\vec{r}+\frac{\vec{y}}{2}\right)\right]\psi\left(r+\frac{y}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)$$

Using Eq. (3.28) and Eq. (3.29),

$$\left(\phi(\vec{r}-\frac{\vec{y}}{2})-\phi(\vec{r}+\frac{\vec{y}}{2})\right)e^{\frac{-i\vec{p}\vec{y}}{\hbar}} = -2i\ \phi(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

$$\implies I_4 = \left(\frac{iq}{\hbar}\right)\frac{1}{(2\pi\hbar)^3}\int d^3y\left[-2i\phi(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\right]\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)$$

$$I_4 = \left(-\frac{2i^2q}{\hbar}\right)\phi(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\frac{1}{(2\pi\hbar)^3}\int d^3ye^{\frac{-i\vec{p}\vec{y}}{\hbar}}\psi\left(\vec{r}+\frac{\vec{y}}{2}\right)\psi^*\left(\vec{r}-\frac{\vec{y}}{2}\right)$$

$$\boxed{I_4 = \left(\frac{2q}{\hbar}\right)\phi(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f} \tag{3.30}$$

Solving integral -I_5

$$I_5 = \frac{1}{(2\pi\hbar)^3}\int d^3y \; e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\left(\frac{q}{mc}\right)\Bigg[\vec{A}\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\frac{\partial\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial r} + \vec{A}\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\frac{\partial\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial r}\Bigg]$$

$$I_5 = \frac{1}{(2\pi\hbar)^3}\int d^3y \;\left(\frac{q}{mc}\right)\Bigg[\vec{A}\Big(\vec{r}-\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\frac{\partial\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial r} + \vec{A}\Big(\vec{r}+\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\vec{y}}{\hbar}}\;\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\frac{\partial\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial r}\Bigg]$$

Here using Eq.(3.28),

$$A(\vec{r}-\frac{\vec{y}}{2})e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} = A(\vec{r})\exp\left(\frac{\hbar}{2i}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

$$A(\vec{r}+\frac{\vec{y}}{2})e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} = A(\vec{r})\exp\left(\frac{-\hbar}{2i}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

Using Eular's formula, $e^{ix} = \cos(x) + i\sin(x)$, we can write above expressions as

$$A(\vec{r}-\frac{\vec{y}}{2})e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} = A(\vec{r})\left[\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right) + i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\right]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

Likewise

$$A(\vec{r}+\frac{\vec{y}}{2})e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}} = A(\vec{r})\left[\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right) - i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\right]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

Using the above expressions, I_5 simplifies to:

$$I_5 = \frac{1}{(2\pi\hbar)^3}\left(\frac{q}{mc}\right)$$
$$\times \int d^3y\Bigg[A(\vec{r})\Bigg\{\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right) + i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\Bigg\}e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\frac{\partial\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial r}$$
$$+ A(\vec{r})\Bigg\{\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right) - i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\Bigg\}e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\ \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\frac{\partial\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial r}\Bigg]$$

$$I_5 = \frac{1}{(2\pi\hbar)^3}\left(\frac{q}{mc}\right)$$
$$\times \int d^3yA(\vec{r})\Bigg[\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\Bigg\{\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\frac{\partial\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial\vec{r}} + \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\frac{\partial\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial\vec{r}}\Bigg\}$$
$$- i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\Bigg\{\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\frac{\partial\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)}{\partial\vec{r}} - \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\frac{\partial\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)}{\partial\vec{r}}\Bigg\}\Bigg]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

Use the product rule of derivative to simplify the first integral, whereas Eq. (3.21) in the second integral becomes

$$I_5 = \frac{1}{(2\pi\hbar)^3}\left(\frac{q}{mc}\right)\int d^3yA(\vec{r})\ \Bigg[\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\ \frac{\partial}{\partial\vec{r}}\Bigg\{\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\Bigg\}$$
$$- i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\Bigg\{ - 2\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\partial^3_{\vec{y}}\ \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big) - 2\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\partial^3_{\vec{y}}\ \psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\Bigg\}\Bigg]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

$$I_5 = \frac{1}{(2\pi\hbar)^3}\left(\frac{q}{mc}\right)\int d^3yA(\vec{r})\ \Bigg[\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\ \frac{\partial}{\partial\vec{r}}\Bigg\{\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\Bigg\}$$
$$+ 2i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\Bigg\{\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\partial^3_{\vec{y}}\ \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big) + \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\partial^3_{\vec{y}}\ \psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\Bigg\}\Bigg]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

$$I_5 = \frac{1}{(2\pi\hbar)^3}\left(\frac{q}{mc}\right)\int d^3y A(\vec{r})\left[\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\frac{\partial}{\partial\vec{r}}\Big\{\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\Big\}\right.$$
$$\left. + 2i\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\partial^3_{\vec{y}}\Big\{\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\ \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\Big\}\right]e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}$$

$$I_5 = \left(\frac{q}{mc}\right)\left[A(\vec{r})\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\frac{\partial}{\partial\vec{r}}\Big\{\frac{1}{(2\pi\hbar)^3}\int d^3y\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\Big\}\right.$$
$$\left. + 2iA(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\Big\{\frac{1}{(2\pi\hbar)^3}\int d^3y\ \partial^3_{\vec{y}}\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\ \psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)e^{\frac{-i\vec{p}\cdot\vec{y}}{\hbar}}\Big\}\right]$$

The 1st integral can be solved simply by using Eq. (3.20) and 2nd integral can be solved by using integration by parts.

$$I_5 = \left(\frac{q}{mc}\right)\left[A(\vec{r})\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\frac{\partial f}{\partial\vec{r}}\right.$$
$$\left. + 2iA(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\left(\frac{i\vec{p}}{\hbar}\right)\frac{1}{(2\pi\hbar)^3}\int d^3y e^{\frac{-i\vec{p}\vec{y}}{\hbar}}\psi\Big(\vec{r}+\frac{\vec{y}}{2}\Big)\psi^*\Big(\vec{r}-\frac{\vec{y}}{2}\Big)\right]$$

$$\boxed{I_5 = \frac{q}{mc}A(\vec{r})\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\frac{\partial f}{\partial\vec{r}} - \left(\frac{2}{\hbar}\right)\frac{q}{mc}\vec{p}\cdot\vec{A}(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f} \tag{3.31}$$

Combining I_1, I_2, I_3, I_4 and I_5 we finally get

$$\frac{\partial f}{\partial t} = -\frac{\vec{p}}{m}\cdot\frac{\partial f}{\partial\vec{r}} + \left(\frac{q^2}{\hbar mc^2}\right)A^2(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f + 0 + \left(\frac{2q}{\hbar}\right)\phi(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f$$
$$+\frac{q}{mc}A(\vec{r})\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)\frac{\partial f}{\partial\vec{r}} - \left(\frac{2}{\hbar}\right)\frac{q}{mc}\vec{p}\cdot\vec{A}(\vec{r})\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f$$

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= -\frac{\vec{p}}{m}\cdot \ \frac{\partial f}{\partial \vec{r}} + \frac{q}{mc}A(\vec{r})\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)\cdot\frac{\partial f}{\partial \vec{r}} \\
&\qquad + \left[\left(\frac{q^2}{\hbar m c^2}\right)A^2(\vec{r}) - \left(\frac{2}{\hbar}\right)\frac{q}{mc}\vec{p}\cdot\vec{A}(\vec{r}) + \left(\frac{2q}{\hbar}\right)\phi(\vec{r})\right]\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)f
\end{aligned}
$$

Now introduce two new terms which will not actually effect the expression

$$
\frac{\vec{p}}{m}\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)\cdot\frac{\partial f}{\partial \vec{r}} = \ \frac{\vec{p}}{m}\cdot\frac{\partial f}{\partial \vec{r}}
$$

and

$$
\frac{p^2}{\hbar m}\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)f = 0
$$

Both of these term can be proved just by expanding the Taylor's series of operators.

Now we can write the above expression as

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= -\frac{\vec{p}}{m}\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)\cdot\frac{\partial f}{\partial \vec{r}} + \frac{q}{mc}A(\vec{r})\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)\cdot\frac{\partial f}{\partial \vec{r}} + \frac{p^2}{\hbar m}\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)f \\
&\qquad + \left[\left(\frac{q^2}{\hbar m c^2}\right)A^2(\vec{r}) - \left(\frac{2}{\hbar}\right)\frac{q}{mc}\vec{p}\cdot\vec{A}(\vec{r}) + \left(\frac{2q}{\hbar}\right)\phi(\vec{r})\right]\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)f
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{m}\left(-\vec{p} + \frac{q}{c}\vec{A}\right)\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)\cdot\frac{\partial f}{\partial \vec{r}} \\
&\qquad + \left(\frac{2}{\hbar}\right)\left[\frac{1}{2m}\left(p - \frac{q}{c}\vec{A}\right)^2 + q\ \phi(\vec{r})\right]\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right)f
\end{aligned}
\tag{3.32}
$$

This is the Quantum Vlasov equation.

3.5 Transformation of the Quantum Vlasov Equation from $(\vec{r}, \vec{p}, t)$ space to $(\vec{r}, \vec{v}, t)$ space

The quantum Vlasov equation in $(\vec{r}, \vec{p}, t)$ space with canonical momentum $\vec{p}$ can be represented in $(\vec{r}, \vec{v}, t)$ space with ordinary velocity $\vec{v}$ with help of following transformation for canonical momentum

$$\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}(\vec{r}) \tag{3.33}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial t} \cdot \frac{\partial}{\partial \vec{v}}. \tag{3.34}$$

$$\frac{\partial}{\partial r_i} \to \frac{\partial}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i} \cdot \frac{\partial}{\partial \vec{v}} \tag{3.35}$$

$$\frac{\partial}{\partial p_i} \to \frac{1}{m_s}\frac{\partial}{\partial \vec{v}} \tag{3.36}$$

Now we can transform all the terms in Eq. (3.33) with the help of these transformation equations.

<u>Transforming the term on the L.H.S</u>

$$\frac{\partial f_s}{\partial t} \to \frac{\partial f_s}{\partial t} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial t} \cdot \frac{\partial f_s}{\partial \vec{v}} \tag{3.37}$$

<u>Transforming 1st term on the R.H.S</u>

$$\begin{aligned}
-\frac{1}{m_s}\left(\vec{p} - \frac{q_s}{c}\vec{A}(r)\right)\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right) \cdot \frac{\partial f_s}{\partial \vec{r}} \\
= -\frac{\vec{p}}{m_s}\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right) \cdot \frac{\partial f_s}{\partial \vec{r}} + \left(\frac{q_s}{m_s c}\vec{A}(r)\right)\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right) \cdot \frac{\partial f_s}{\partial \vec{r}} \\
= -\frac{\vec{p}}{m_s} \cdot \frac{\partial f_s}{\partial \vec{r}} + \left(\frac{q_s}{m_s c}\vec{A}(r)\right)\cos\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{p}}\right) \cdot \frac{\partial f_s}{\partial \vec{r}}
\end{aligned}$$

Transformation of above equation gives

$$\to -\frac{1}{m_s}\left(m_s\vec{v}+\frac{q_s}{c}\vec{A}(r)\right)\cdot\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right)$$
$$+\left(\frac{q_s}{m_s c}\vec{A}(r)\right)\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\cdot\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right),$$

or we can write

$$= -\frac{1}{m_s}\left(m_s\vec{v}\right)\cdot\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right)-\frac{1}{m_s}\left(\frac{q_s}{c}\vec{A}(r)\right)\cdot\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right)$$
$$+\left(\frac{q_s}{m_s c}\vec{A}(r)\right)\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\cdot\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right).$$

Using tensor notation for dot product, we get

$$= -\vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}}+\frac{q_s}{m_s c}\,v_i\,\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}-\frac{1}{m_s}\left(\frac{q_s}{c}\vec{A}(r)\right)\cdot\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right)$$
$$+\frac{q_s}{m_s c}\,A_i\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right),$$

$$= -\vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}}+\frac{q_s}{m_s c}\,v_i\,\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}+\frac{q_s}{m_s c}\,A_i\left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)-1\right]\left(\frac{\partial f_s}{\partial r_i}-\frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right).$$

(3.38)

Transforming 2nd term on the R.H.S

$$\frac{2}{\hbar}\left(\frac{1}{2m_s}\left(\vec{p}-\frac{q}{c}\vec{A}(r)\right)^2+q_s\phi(r)\right)\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f_s$$

$$=\frac{2}{\hbar}\left(\frac{1}{2m_s}\left((\vec{p})^2+\left(\frac{q}{c}\vec{A}(r)\right)^2-2\vec{p}.\left(\frac{q}{c}\vec{A}(r)\right)\right)+q_s\phi(r)\right)\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f_s$$

we know that when $\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)$ applied to $\vec{p}$, it will give zero. So

$$=\frac{2}{\hbar}\left(\frac{1}{2m_s}\left(\left(\frac{q}{c}\vec{A}(r)\right)^2-2\vec{p}\cdot\left(\frac{q}{c}\vec{A}(r)\right)\right)+q_s\phi(r)\right)\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f_s$$

$$=\frac{2}{\hbar}\frac{1}{2m_s}\left[\left(\frac{q}{c}\vec{A}(r)\right)^2\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f_s-2\,\vec{p}.\left(\frac{q}{c}\vec{A}(r)\right)\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f_s\right]$$
$$+\frac{2}{\hbar}\,q_s\,\phi(r)\sin\left(\frac{\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{p}}\right)f_s$$

Transformation of this expression gives

$$\rightarrow\frac{2}{\hbar}\frac{1}{2m_s}\left(\frac{q}{c}\vec{A}(r)\right)^2\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$
$$-\frac{2}{\hbar}\frac{1}{2m_s}\,2\left(m_s\vec{v}+\frac{q_s}{c}\vec{A}(r)\right)\cdot\left(\frac{q}{c}\vec{A}(r)\right)\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$
$$+\frac{2}{\hbar}\,q_s\,\phi(r)\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$

Rearranging first and third term, we get

$$= -\frac{2q}{m_s c\hbar}\left(m_s\vec{v}+\frac{q_s}{c}\vec{A}(r)\right)\cdot\vec{A}(r)\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$
$$+\frac{2}{\hbar}\left(\frac{q^2}{2m_s c^2}A^2+q_s\ \phi(r)\right)\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s,$$

Using tensor notation for dot product gives us

$$= -\frac{2q}{m_s c\hbar}\left(m_s v_i+\frac{q_s}{c}A_i\right)\ A_i\ \sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$
$$+\frac{2}{\hbar}\left(\frac{q^2}{2m_s c^2}A^2+q_s\ \phi(r)\right)\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$

Add and subtract the following terms in the above expression.

$$\frac{2q_s}{m_s c\hbar}\left(m_s v_i+\frac{q_s}{c}A_i\right)\ A_i\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$

and

$$+\frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}A^2+q_s\ \phi(r)\right)\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s.$$

After simplification, we get

$$= \frac{2q_s}{m_s c\hbar}\left(m_s v_i+\frac{q_s}{c}A_i\right)A_i\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s-\frac{2q_s}{m_s c\hbar}\left(m_s v_i+\frac{q_s}{c}A_i\right)A_i\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$
$$-\frac{2q_s}{m_s c\hbar}\left(m_s v_i+\frac{q_s}{c}A_i\right)A_i\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$
$$+\frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}A^2+q_s\ \phi(r)\right)\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s-\frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}A^2+q_s\ \phi(r)\right)\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$
$$+\frac{2}{\hbar}\left(\frac{q^2}{2m_s c^2}A^2+q_s\ \phi(r)\right)\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)f_s$$

After combining first term with the third and the fifth term with the sixth, we get

$$
\begin{aligned}
= &-\frac{2q_s}{m_s c\hbar}\ \left(m_s v_i + \frac{q_s}{c}A_i\right)\ A_i \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) f_s \\
&\qquad -\frac{2q_s}{m_s c\hbar}\ \left(m_s v_i + \frac{q_s}{c}A_i\right)\ A_i \left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right) f_s \\
&+\frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}A^2 + q_s\ \phi(r)\right)\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) f_s \\
&\qquad +\frac{2}{\hbar}\ \left(\frac{q^2}{2m_s c^2}A^2 + q_s\ \phi(r)\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right) f_s
\end{aligned}
$$

Expanding the resulting terms the expression becomes

$$
\begin{aligned}
= &-\frac{q_s}{m_s c}\ v_i\ \frac{\partial A_i}{\partial\vec{r}}\cdot\frac{\partial f_s}{\partial\vec{v}}\ -\frac{q_s^2}{m_s^2 c^2}\ A_i\ \frac{\partial A_i}{\partial\vec{r}}\cdot\frac{\partial f_s}{\partial\vec{v}}\ +\frac{q_s^2}{m_s^2 c^2}\ A_i\ \frac{\partial A_i}{\partial\vec{r}}\cdot\frac{\partial f_s}{\partial\vec{v}}\ +\frac{q_s}{m_s}\frac{\partial\phi(r)}{\partial\vec{r}}\cdot\frac{\partial f_s}{\partial\vec{v}} \\
&-\frac{2q_s}{m_s c\hbar}\ \left(m_s v_i + \frac{q_s}{c}A_i\right)\ A_i \left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right) f_s \\
&\qquad +\frac{2}{\hbar}\ \left(\frac{q^2}{2m_s c^2}A^2 + q_s\ \phi(r)\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right) f_s
\end{aligned}
$$

and finally we get

$$
\begin{aligned}
= &-\frac{q_s}{m_s c}\ v_i\ \frac{\partial A_i}{\partial\vec{r}}\cdot\frac{\partial f_s}{\partial\vec{v}}\ +\frac{q_s}{m_s}\frac{\partial\phi(r)}{\partial\vec{r}}\cdot\frac{\partial f_s}{\partial\vec{v}} \\
&-\frac{2q_s}{m_s c\hbar}\ \left(m_s v_i + \frac{q_s}{c}A_i\right)\ A_i \left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right) f_s \\
&+\frac{2}{\hbar}\ \left(\frac{q^2}{2m_s c^2}A^2 + q_s\ \phi(r)\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right) f_s. \qquad (3.39)
\end{aligned}
$$

Using Eq. (3.43), Eq. (3.39) and Eq. (3.40) in Eq. (3.33), we achieve

$$
\begin{aligned}
&\frac{\partial f_s}{\partial t} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial t}\cdot\frac{\partial f_s}{\partial \vec{v}} = \\
&\qquad - \vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s c}\, v_i\, \frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}} - \frac{q_s}{m_s c}\, v_i\, \frac{\partial A_i}{\partial \vec{r}}\cdot\frac{\partial f_s}{\partial \vec{v}} + \frac{q_s}{m_s}\frac{\partial \phi(r)}{\partial \vec{r}}\cdot\frac{\partial f_s}{\partial \vec{v}} \\
&\qquad + \frac{q_s}{m_s c}\, A_i \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial f_s}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right) \\
&\qquad - \frac{2q_s}{m_s c\hbar}\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s \\
&\qquad + \frac{2}{\hbar}\left(\frac{q^2}{2m_s c^2}A^2 + q_s\, \phi(r)\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s
\end{aligned}
$$

After using tensor notation, we can write

$$
\begin{aligned}
&\frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s}\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{\partial \phi(r)}{\partial \vec{r}}\right)\cdot\frac{\partial f_s}{\partial \vec{v}} + \frac{q_s}{m_s c}\, v_i \left[\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right]\frac{\partial f_s}{\partial v_j} \\
&\qquad = \frac{q_s}{m_s c}\, A_i \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial f_s}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i}\cdot\frac{\partial f_s}{\partial \vec{v}}\right) \\
&\qquad - \frac{2q_s}{m_s c\hbar}\left(m_s v_i + \frac{q_s}{c}A_i\right) A_i \left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s \\
&\qquad + \frac{2}{\hbar}\left(\frac{q^2}{2m_s c^2}A^2 + q_s\, \phi(r)\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s,
\end{aligned}
$$

where $\left[\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right]$ is the kth components of $\left(\nabla \times \vec{A}\right)$, therefore

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s}\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{\partial \phi(r)}{\partial \vec{r}}\right) \cdot \frac{\partial f_s}{\partial \vec{v}} + \frac{q_s}{m_s c}\, v_i \left[\left(\nabla \times \vec{A}\right)_k\right] \frac{\partial f_s}{\partial v_j}$$
$$= \frac{q_s}{m_s c}\, A_i \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial f_s}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i} \cdot \frac{\partial f_s}{\partial \vec{v}}\right)$$
$$-\frac{2q_s}{m_s c\hbar}\left(m_s v_i + \frac{q_s}{c}A_i\right) A_i \left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s$$
$$+ \frac{2}{\hbar}\left(\frac{q^2}{2m_s c^2}A^2 + q_s\, \phi(r)\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s.$$

We can write the scalar triple product in tensor notation as

$$v_i\, B_k\, \frac{\partial f_s}{\partial v_j} = \left(\frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial f_s}{\partial \vec{v}}$$

Now the final expression will become [33],

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s}\left[\vec{E} + \left(\frac{\vec{v}}{c} \times \vec{B}\right)\right] \cdot \frac{\partial f_s}{\partial \vec{v}}$$

$$= \frac{q_s}{m_s c}\, A_i \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial f_s}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i} \cdot \frac{\partial f_s}{\partial \vec{v}}\right)$$
$$-\frac{2q_s}{m_s c\hbar}\left(m_s v_i + \frac{q_s}{c}A_i\right) A_i \left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s$$
$$+ \frac{2}{\hbar}\left(\frac{q^2}{2m_s c^2}A^2 + q_s\, \phi(r)\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right) f_s. \quad (3.40)$$

3.6 Linearization of Quantum Vlasov Equation

In order to study the linear response of a physical quantity say ϕ it is expressed as

$$\phi = \phi_0 + \phi_1$$

where ϕ_0 refers to the unperturbed quantity and $\phi_1(<< \phi_0)$ measures perturbation. Study of linear response leads us to small perturbation which is sinusoidal and can be written as

$$\phi(\vec{r}, t) = \phi(\vec{k}, \omega)\ e^{\iota(\vec{k}.\vec{r} - \omega t)} \tag{3.41}$$

where $\phi(\vec{k}, \omega)$ is the Fourier-Laplace transform of $\phi(\vec{r}, t)$.
After linearizing L.H.S of Eq. (3.40), we get

$$= \frac{\partial f_{s1}}{\partial t} + \vec{v} \cdot \frac{\partial f_{s1}}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\frac{\vec{v}}{c} \times \vec{B_0}\right) \cdot \frac{\partial f_{s1}}{\partial \vec{v}} + \frac{q_s}{m_s}\left(\vec{E_1} + \frac{\vec{v}}{c} \times \vec{B_1}\right) \cdot \frac{\partial f_{s0}}{\partial \vec{v}} \tag{3.42}$$

Now the first term on R.H.S of Eq. (3.40) will become

$$\frac{q_s}{m_s c}\, A_i \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial f_s}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial \vec{A}}{\partial r_i} \cdot \frac{\partial f_s}{\partial \vec{v}}\right)$$

In tensor notation we have

$$= \frac{q_s}{m_s c} A_i \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial f_s}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial A_j}{\partial r_i}\frac{\partial f_s}{\partial v_j}\right)$$

$$\to \frac{q_s}{m_s c} (A_{i0} + A_{i1}) \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right] \times \left(\frac{\partial (f_{s0} + f_{s1})}{\partial r_i} - \frac{q_s}{m_s c}\frac{\partial (A_{j0} + A_{j1})}{\partial r_i}\frac{\partial (f_{s0} + f_{s1})}{\partial v_j}\right)$$

$$= \frac{q_s}{m_s c} (A_{i0} + A_{i1}) \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial (f_{s0} + f_{s1})}{\partial r_i}\right) - (\frac{q_s}{m_s c})^2 (A_{i0} + A_{i1}) \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\left(\frac{\partial (A_{j0} + A_{j1})}{\partial r_i}\frac{\partial (f_{s0} + f_{s1})}{\partial v_j}\right)$$

As f_{s0} is homogeneous, so

$$= \frac{q_s}{m_s c} (A_{i0} + A_{i1}) \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right] \frac{\partial f_{s1}}{\partial r_i} - (\frac{q_s}{m_s c})^2 (A_{i0} + A_{i1}) \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right] \frac{\partial (A_{j0} + A_{j1})}{\partial r_i} \frac{\partial (f_{s0} + f_{s1})}{\partial v_j}$$

By using Taylor's series, we can expand $\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)$, which gives us second ordered derivatives. as we know A_{i0} is linear term in $\vec{r}$ i.e. $\vec{A}_0 = \vec{B}_0 \times \vec{r}/2$.

As we are taking linear term so the first term will vanish. Here we have to go just with 2nd term.

$$= -(\frac{q_s}{m_s c})^2 (A_{i0}+A_{i1}) \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right] \frac{\partial (A_{j0} + A_{j1})}{\partial r_i} \frac{\partial (f_{s0} + f_{s1})}{\partial v_j}$$

Here again A_{i0} is zero for the same reason and linearizing rest expression gives

$$= -(\frac{q_s}{m_s c})^2 \, A_{i1} \, \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - 1\right] \, \frac{\partial A_{j0}}{\partial r_i} \, \frac{\partial f_{s0}}{\partial v_j}$$

$$= -(\frac{q_s}{m_s c})^2 \, \frac{\partial A_{j0}}{\partial r_i} \, A_{i1} \, \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - 1\right] \, \frac{\partial f_{s0}}{\partial v_j}$$

Writing $(\vec{A_0} = \vec{B_0} \times \vec{r}/2)$ in tensor notation of cross product , we have

$$\vec{A_0} = \frac{1}{2} \, \epsilon_{ijk} \, B_{0i} \, r_j \, \hat{e_k}$$

Where $\quad \hat{e_k} = \hat{e_1}, \; \hat{e_2}, \; \hat{e_3}$

Since

$$\vec{B} = B_0 \, \hat{e_3}$$

$$\vec{A_0} = \frac{1}{2}(-B_0 r_2 \, \hat{e_1} + B_0 r_1 \, \hat{e_2})$$

$$\frac{\partial \vec{A_0}}{\partial r_2} = -\frac{B_0}{2} \, \hat{e_1} \quad \Rightarrow \quad \frac{\partial A_{01}}{\partial r_2} = -\frac{B_0}{2}$$

$$\frac{\partial \vec{A_0}}{\partial r_1} = \frac{B_0}{2} \, \hat{e_2} \quad \Rightarrow \quad \frac{\partial A_{02}}{\partial r_1} = \frac{B_0}{2}$$

$$\Rightarrow \qquad \frac{\partial A_{0j}}{\partial r_i} = -\frac{B_0}{2} \, \hat{e_1} + \frac{B_0}{2} \, \hat{e_2}$$

$$\Rightarrow \qquad \frac{\partial A_{0j}}{\partial r_i} \, A_{1i} \; = \; \left(-\frac{B_0}{2} \, \hat{e_1} + \frac{B_0}{2} \, \hat{e_2}\right) \, A_{1i} \; = \; \frac{1}{2}(-B_0 \, A_{1i} \, \hat{e_1} + B_0 \, A_{1i} \, \hat{e_2})$$

$$\Rightarrow \qquad \frac{\partial A_{0j}}{\partial r_i}\, A_{1i} \;=\; \frac{1}{2}\,(\vec{B}_0{\times}\vec{A}_1)_j$$

$$= -(\frac{q_s}{m_s c})^2\; \frac{1}{2}\; (\vec{B}_0 \times \vec{A}_1)_j\;\; \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - 1\right]\; \frac{\partial f_{s0}}{\partial v_j} \tag{3.43}$$

We will now linearize second term on R.H.S in the same way

$$\frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}A^2 + q_s\phi\right)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right] f_s$$

$$\rightarrow \frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}\; A_i\; A_i + q_s\phi\right)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right] f_s$$

$$= \frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}\; (A_{i0} + A_{i1})\; (A_{i0} + A_{i1}) + q_s(\phi_0 + \phi_1)\right)$$
$$\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right] (f_{s0} + f_{s1})$$

$$= \frac{2}{\hbar}\left(\frac{q_s^2}{2m_s c^2}(A_{i0}A_{i0} + A_{i1}A_{i1} + 2A_{i0}A_{i1})\right)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right] (f_{s0}{+}f_{s1})$$
$$+\frac{2}{\hbar}q_s(\phi_0{+}\phi_1)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right] (f_{s0}{+}f_{s1})$$

Expanding the Taylor's series of $\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)$,
then $\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right]$ give the third ordered derivative which can be neglected as A_{i0} and ϕ_0 are linear terms in $\vec{r}$. So, $A_{i0}\ A_{i0}$ in first expression and ϕ_0 in that of second will be zero for this reason.

Also the term $A_{i1}\ A_{i1}$ in first expression will be zero for being non-linear term.

$$= \frac{2\ q_s^2}{m_s c^2 \hbar}\ (\ A_{i0}\ A_{i1}) \left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right](f_{s0} + f_{s1})$$
$$+ \frac{2\ q_s}{\hbar}\ \phi_1 \left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right](f_{s0} + f_{s1})$$

$$= \frac{2\ q_s^2}{m_s c^2 \hbar}\ (\ A_{i0}\ A_{i1}) \left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right]\ f_{s0}$$
$$+ \frac{2\ q_s}{\hbar}\ \phi_1 \left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right]\ f_{s0} \quad (3.44)$$

The third term on R.H.S, liearized as

$$-\frac{2q_s}{m_s c\hbar}\left(m_s v_i + \frac{q_s}{c}A_i\right) A_i \left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right] f_s$$

$$\rightarrow -\frac{2q_s}{m_s c\hbar}\left(m_s v_i + \frac{q_s}{c}(A_{i0} + A_{i1})\right)\ (A_{i0} + A_{i1})$$
$$\times \left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right](f_{s0} + f_{s1})$$

Expanding the Taylor's series of $\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)$,
then $\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial \vec{r}}\frac{\overrightarrow{\partial}}{\partial \vec{v}}\right)\right]$ gives the third ordered derivatives so we neglect

them because A_{i0} is linear in $\vec{r}$. So, A_{i0} will be zero.

$$= -\frac{2q_s}{m_s c\hbar}\left(m_s v_i + \frac{q_s}{c}(A_{i0}+A_{i1})\right)\ A_{i1}\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right](f_{s0}+f_{s1})$$

$$\begin{aligned}= &-\frac{2q_s}{m_s c\hbar}(m_s v_i)\quad A_{i1}\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right](f_{s0}+f_{s1}) \\ &-\frac{2q_s}{m_s c\hbar}\left(\frac{q_s}{c}(A_{i0}+A_{i1})\right)\ A_{i1}\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right](f_{s0}+f_{s1})\end{aligned}$$

$$\begin{aligned}= &-\frac{2q_s}{c\hbar}\ v_i\ A_{i1}\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right]\ f_{s0} \\ &-\frac{2\ q_s^2}{m_s c^2\hbar}\ A_{i0}\ A_{i1}\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right]\ f_{s0} \quad (3.45)\end{aligned}$$

After Putting Eq. (3.43), Eq. (3.6) and Eq. (3.45) in Eq. (3.41) and simplifying the expression, we can write linearized quantum Vlasov equation as

$$\begin{aligned}&\frac{\partial f_{s1}}{\partial t} + \vec{v}\cdot\frac{\partial f_{s1}}{\partial\vec{r}} + \frac{q_s}{m_s}\left(\frac{\vec{v}}{c}\times\vec{B_0}\right)\cdot\frac{\partial f_{s1}}{\partial\vec{v}} + \frac{q_s}{m_s} \\ = &-\left(\vec{E_1}\frac{\vec{v}}{c}\times\vec{B_1}\right)\cdot\frac{\partial f_{s0}}{\partial\vec{v}} - (\frac{q_s}{m_s c})^2\ \frac{1}{2}(\vec{B_0}\times\vec{A_1})_j\left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - 1\right]\frac{\partial f_{s0}}{\partial v_j} \\ &-\frac{2\ q_s}{\hbar}\ \left(\frac{v_i}{c}\ A_{i1} - \phi_1\right)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right]\ f_{s0} \qquad (3.46)\end{aligned}$$

3.7 Derivation of Susceptibility Tenor

Suppose the Taylor's expansion of the function of kind given below

$$e^{\iota(\vec{k}.\vec{r})}\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)h(\vec{v}) = e^{\iota(\vec{k}.\vec{r})}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n!}\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)^{2n} h(\vec{v})$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n!}\left(\frac{\hbar}{2m_s}\right)^{2n}\left(\frac{\partial^{2n}}{\partial r^{2n}}\quad e^{\iota(\vec{k}.\vec{r})}\right)\left(\frac{\partial^{2n}}{\partial v^{2n}}\quad h(\vec{v})\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n!}\left(\frac{\hbar}{2m_s}\right)^{2n}(\iota\vec{k})^{2n}\, e^{\iota(\vec{k}.\vec{r})}\left(\frac{\partial^{2n}}{\partial v^{2n}}\quad h(\vec{v})\right)$$

Since $(-1)^n.(-\iota)^{2n} = 1$, So

$$= e^{\iota(\vec{k}.\vec{r})}\quad\sum_{n=0}^{\infty}\frac{1}{2n!}\left(\frac{\hbar\,\vec{k}}{2m_s}\right)^{2n}\left(\frac{\partial^{2n}}{\partial v^{2n}}\quad h(\vec{v})\right)$$

This is simply the addition of Taylor's expansion of $h\left(\vec{v}-\frac{\hbar\,\vec{k}}{2m_s}\right)$ and $h\left(\vec{v}+\frac{\hbar\,\vec{k}}{2m_s}\right)$

$$\Longrightarrow\qquad = \frac{e^{\iota(\vec{k}.\vec{r})}}{2}\quad\left(h\left(\vec{v}-\frac{\hbar\,\vec{k}}{2m_s}\right)+h\left(\vec{v}+\frac{\hbar\,\vec{k}}{2m_s}\right)\right)$$

Finally we get

$$e^{\iota(\vec{k}.\vec{r})}\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)h(\vec{v}) = \frac{e^{\iota(\vec{k}.\vec{r})}}{2}\left(h\left(\vec{v}-\frac{\hbar\,\vec{k}}{2m_s}\right)+h\left(\vec{v}+\frac{\hbar\,\vec{k}}{2m_s}\right)\right) \quad (3.47)$$

In the similar way, we can get

$$e^{\iota(\vec{k}.\vec{r})}\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)h(\vec{v}) = \frac{e^{\iota(\vec{k}.\vec{r})}}{2\iota}\left(h\left(\vec{v}-\frac{\hbar\,\vec{k}}{2m_s}\right)-h\left(\vec{v}+\frac{\hbar\,\vec{k}}{2m_s}\right)\right) \quad (3.48)$$

Using Eq.(3.48) and Eq. (3.49) given above we can simplify the linearized quantum Vlasov equation [34]

$$\frac{\partial f_{s1}}{\partial t} + \vec{v}\cdot\frac{\partial f_{s1}}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\frac{\vec{v}}{c}\times\vec{B_0}\right)\cdot\frac{\partial f_{s1}}{\partial \vec{v}}$$
$$= -\frac{q_s}{m_s}\left(\vec{E_1}+\frac{\vec{v}}{c}\times\vec{B_1}\right)\cdot\frac{\partial f_{s0}}{\partial \vec{v}}$$
$$-\left(\frac{q_s}{m_s c}\right)^2 (\vec{B_0}\times\vec{A_1})_j \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)-1\right]\frac{\partial f_{s0}}{\partial v_j}$$
$$-\frac{2\,q_s}{\hbar}\left(\frac{v_i}{c}\,A_{i1}-\,\phi_1\right)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)-\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right] f_{s0} \tag{3.49}$$

To study linear response, as we have discussed in details at start of section 3.6 using Eq. (3.42)

$$\phi_1(\vec{r},t) = \phi_1(\vec{k},\omega)\ e^{\iota(\vec{k}.\vec{r}-\omega t)} \tag{3.50}$$

In the similar way we can write

$$\vec{A_1}(\vec{r},t) = \vec{A_1}(\vec{k},\omega)\ e^{\iota(\vec{k}.\vec{r}-\omega t)} \tag{3.51}$$

So, simplifying 2nd term on R.H.S of Eq. (3.50) gives

$$-\left(\frac{q_s}{m_s c}\right)^2\frac{\partial}{\partial v_i}\left((\vec{B_0}\times\vec{A_1})_i\left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)-1\right] f_{s0}\right)$$

Using Eq.(3.48), it gives

$$= -\left(\frac{q_s}{m_s c}\right)^2\frac{\partial}{\partial v_i}\left(\frac{(\vec{B_0}\times\vec{A_1})_i}{2}\left(f_{s0}\left(\vec{v}-\frac{\hbar\vec{k}}{2m_s}\right)+f_{s0}\left(\vec{v}+\frac{\hbar\vec{k}}{2m_s}\right)\right)-(\vec{B_0}\times\vec{A_1})_i f_{s0}(\vec{v})\right)$$

$$= -\left(\frac{q_s}{m_s c}\right)^2 \left(\frac{(\vec{B_0}\times\vec{A_1})_i}{2}\right) \frac{\partial}{\partial v_i} \left(f_{s0}\left(\vec{v} - \frac{\hbar\,\vec{k}}{2m_s}\right) + f_{s0}\left(\vec{v} + \frac{\hbar\,\vec{k}}{2m_s}\right) - 2\ f_{s0}(\vec{v})\right)$$

Putting $\left(f_{s0}\left(\vec{v} - \frac{\hbar\,\vec{k}}{2m_s}\right) + f_{s0}\left(\vec{v} + \frac{\hbar\,\vec{k}}{2m_s}\right) - 2\ f_{s0}(\vec{v})\right) = F_s^+(\vec{v},\vec{k})$

$$-\left(\frac{q_s}{m_s c}\right)^2 \frac{\partial}{\partial v_i} \left((\vec{B_0}\times\vec{A_1})_i \left[\cos\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - 1\right] f_{s0}\right)$$
$$= -\left(\frac{q_s}{m_s c}\right)^2 \frac{(\vec{B_0}\times\vec{A_1})}{2}\cdot\frac{\partial F_s^+(\vec{v},\vec{k})}{\partial\vec{v}} \tag{3.52}$$

Similarly the third term of Eq. (3.50), gives

$$-\frac{2\ q_s}{\hbar}\left(\frac{v_i}{c}A_{i1} - \phi_1\right)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right] f_{s0}$$

Using Eq. (3.51) and Eq. (3.52),

$$= -\frac{2\ q_s}{\hbar}\left[\left(\frac{v_i}{c}A_{i1} - \phi_1\right)\left(\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) - \left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right) f_{s0}\right]$$
$$+\frac{2\ q_s}{\hbar}\left[\left(\frac{v_i}{c}A_{i1} - \phi_1\right)\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right) f_{s0}\right]$$

Simplifying by using Eq. (3.49), (3.51) and Eq. (3.52), we have

$$= -\frac{2\ q_s}{\hbar}\left[\frac{1}{2\iota}\left(\frac{v_i}{c}A_{i1} - \phi_1\right)\left(f_{s0}\left(\vec{v} - \frac{\hbar\vec{k}}{2m_s}\right) - f_{s0}\left(\vec{v} + \frac{\hbar\vec{k}}{2m_s}\right)\right)\right]$$
$$+\frac{2\ q_s}{\hbar}\left[\iota\vec{k}\left(\frac{v_i}{c}A_{i1} - \phi_1\right)\left(\frac{\hbar}{2m_s}\right)\frac{\partial}{\partial\vec{v}}\ f_{s0}(\vec{v})\right]$$

$$= \frac{\iota\ q_s}{\hbar}\left(\frac{v_i}{c}A_{i1} - \phi_1\right)\left[f_{s0}\left(\vec{v} - \frac{\hbar\vec{k}}{2m_s}\right) - f_{s0}\left(\vec{v} + \frac{\hbar\vec{k}}{2m_s}\right) + \vec{k}\left(\frac{\hbar}{m_s}\right)\frac{\partial\ f_{s0}(\vec{v})}{\partial\vec{v}}\right]$$

Putting $\left(f_{s0}\left(\vec{v}-\frac{\hbar\,\vec{k}}{2m_s}\right)-f_{s0}\left(\vec{v}\ +\ \frac{\hbar\,\vec{k}}{2m_s}\right)-\vec{k}\left(\frac{\hbar}{m_s}\right)\frac{\partial\ f_{s0}(\vec{v})}{\partial\vec{v}}\right)=F_s^{+}(\vec{v},\vec{k})$

$$-\frac{2\ q_s}{\hbar}\ \left(\frac{v_i}{c}A_{i1}-\phi_1\right)\left[\sin\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)-\left(\frac{\hbar}{2m_s}\frac{\overleftarrow{\partial}}{\partial\vec{r}}\frac{\overrightarrow{\partial}}{\partial\vec{v}}\right)\right]\ f_{s0}$$

$$=\ -\ \frac{\iota\ q_s}{\hbar}\ \left(\frac{\vec{v}}{c}\cdot\vec{A}_1-\phi_1\right)\ F_s^{-}(\vec{k},\vec{v}) \qquad (3.53)$$

Now using Maxwell's equations

$$\vec{E}_1\ =-\ \nabla\ \phi_1 \qquad (3.54)$$

and

$$\nabla\times\ \vec{E}_1=-\frac{1}{c}\frac{\partial\vec{B}_1}{\partial t} \qquad (3.55)$$

to express $\vec{A}_1$ and ϕ_1 in terms of $\vec{E}_1$ in $(\vec{k},\omega)$ space with the help of Coulomb gauge $\vec{k}.\vec{A}_1$.

Firstly,

$$\vec{E}_1\ =-\ \nabla\ \phi_1$$

Since

$$\phi_1(\vec{r},t)=\phi_1(\vec{k},\omega)\ e^{\iota(\vec{k}.\vec{r}-\omega t)}$$

Therefore

$$\vec{E}_1\ =-\iota\vec{k}\ \phi_1$$

$$\phi_1=-\ \frac{1}{\iota\vec{k}}\cdot\vec{E}_1$$

Multiply by $\vec{k}$ up and down

$$\phi_1=-\ \frac{\vec{k}}{\vec{k}}\ \frac{1}{\iota\vec{k}}\cdot\vec{E}_1$$

$$\phi_1=\ \frac{\iota\ \vec{k}}{k^2}\cdot\vec{E}_1 \qquad (3.56)$$

In order to find the expression for perturbed quantity $\vec{A}_1$ we make use of Eq. (3.56),

$$\nabla \times \ \vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{B}_1}{\partial t}$$

We know that

$$\vec{E}_1(\vec{r},t) = \vec{E}_1(\vec{k},\omega)\ e^{\iota(\vec{k}.\vec{r}-\omega t)}$$

and

$$\vec{B}_1(\vec{r},t) = \vec{B}_1(\vec{k},\omega)\ e^{\iota(\vec{k}.\vec{r}-\omega t)}$$

So,

$$\iota\vec{k} \times \ \vec{E}_1 = -\frac{-\ \iota\ \omega}{c}\vec{B}_1$$

$$\vec{B}_1 = \frac{c}{\omega}\ \vec{k} \times \ \vec{E}_1 \tag{3.57}$$

Now using Eq. (3.58),

$$\vec{B}_1 = \frac{c}{\omega}\ \vec{k} \times \ \vec{E}_1$$

Taking cross product with ∇ on both sides

$$\nabla \times \vec{B}_1 = \ \nabla \times \frac{c}{\omega}\ \vec{k} \times \ \vec{E}_1$$

$$\nabla \times \vec{B}_1 = \ \frac{c}{\omega}\ \nabla \times \ \vec{k} \times \ \vec{E}_1$$

Put

$$\vec{B}_1 \ = \ \nabla \times \vec{A}_1$$

$$\nabla \times \nabla \times \vec{A}_1 = \ \frac{c}{\omega}\ \nabla \times \ \vec{k} \times \ \vec{E}_1$$

$$\iota\vec{k} \times \iota\vec{k} \times \vec{A}_1 = \ \frac{c}{\omega}\ \iota\vec{k} \times \ \vec{k} \times \ \vec{E}_1$$

Using cross triple product formula for expansion of above expression

$$\iota^2 \ [\ \vec{k}\ (\vec{k}\cdot\vec{A}_1)\ -\ k^2\ \vec{A}_1] = \ \frac{c}{\omega}\ \iota\ [\ \vec{k}\ (\vec{k}\cdot\vec{E}_1)\ -\ k^2\ \vec{E}_1\ ]$$

Using Coulomb gauge i.e. $\vec{k}\cdot\vec{A}_1 = 0$

$$-\ [\ 0 -\ k^2\ \vec{A}_1] = \ \frac{c}{\omega}\ \iota\ [\ \vec{k}\ (\vec{k}\cdot\vec{E}_1)\ -\ k^2\ \vec{E}_1\ ]$$

$$\vec{A}_1 = \ \frac{c}{\omega}\ \iota\ \left[\ \frac{\vec{k}}{k^2}\ (\vec{k}\cdot\vec{E}_1)\ -\ \frac{k^2}{k^2}\ \vec{E}_1\ \right]$$

$$\vec{A}_1 = \ \frac{c}{\omega\ \iota}\ \left[\ \vec{E}_1\ -\ \frac{\vec{k}}{k^2}\ (\vec{k}\cdot\vec{E}_1)\ \right]$$

$$\vec{A}_1 = \ \frac{c}{\omega\ \iota}\ \left[\ \hat{I}\ -\ \frac{\vec{k}\ \vec{k}}{k^2}\ \right]\cdot\vec{E}_1 \tag{3.58}$$

Using values from Eq. (3.53) to Eq. (3.59), the linearized quantum Vlasov equation becomes

$$\frac{\partial f_{s1}}{\partial t}+\vec{v}\cdot\frac{\partial f_{s1}}{\partial\vec{r}}+\frac{q_s}{m_s}\left(\frac{\vec{v}}{c}\times\vec{B}_0\right)\cdot\frac{\partial f_{s1}}{\partial\vec{v}} = -\frac{q_s}{m_s}\left(\vec{E}_1+\frac{\vec{v}}{c}\times\vec{B}_1\right)\cdot\frac{\partial f_{s0}}{\partial\vec{v}}-\left(\frac{q_s}{m_s c}\right)^2\ \frac{(\vec{B}_0\times\vec{A}_1)}{2}\cdot\frac{\partial F_s^+(\vec{v},\vec{k})}{\partial\vec{v}}$$
$$-\ \frac{\iota\ q_s}{\hbar}\ \left(\frac{\vec{v}}{c}\cdot\vec{A}_1-\phi_1\right)\ F_s^-(\vec{k},\vec{v})$$

$$
\begin{aligned}
\frac{\partial f_{s1}}{\partial t} &+ \vec{v}\cdot\frac{\partial f_{s1}}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\frac{\vec{v}}{c}\times\vec{B_0}\right)\cdot\frac{\partial f_{s1}}{\partial \vec{v}} \\
&= -\frac{q_s}{m_s}\left(\vec{E_1}+\frac{\vec{v}}{c}\times\frac{c}{\omega}\ \vec{k}\times\ \vec{E_1}\right).\frac{\partial f_{s0}}{\partial \vec{v}} \\
&\quad -\frac{1}{2}\left(\frac{q_s}{m_s c}\right)^2\ \left(\vec{B_0}\times\left(\frac{c}{\omega\ \iota}\ \left[\ \hat{I}\ -\ \frac{\vec{k}\ \vec{k}}{k^2}\ \right]\ .\ \vec{E_1}\right)\right)\ .\ \frac{\partial F_s^{+}(\vec{v},\vec{k})}{\partial \vec{v}} \\
&\quad -\ \frac{\iota\ q_s}{\hbar}\ \left(\frac{\vec{v}}{c}.\frac{c}{\omega\ \iota}\ \left[\ \hat{I}\ -\ \frac{\vec{k}\ \vec{k}}{k^2}\ \right]\ .\ \vec{E_1}-\ \frac{\iota\ \vec{k}}{k^2}\ .\ \vec{E_1}\ \right)\ F_s^{-}(\vec{k},\vec{v})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_{s1}}{\partial t} &+ \vec{v}\cdot\frac{\partial f_{s1}}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\frac{\vec{v}}{c}\times\vec{B_0}\right)\cdot\frac{\partial f_{s1}}{\partial \vec{v}} \\
&= -\frac{q_s}{m_s}\ \frac{\partial f_{s0}}{\partial \vec{v}}\cdot\left(\vec{E_1}+\frac{1}{\omega}\ \left(\vec{k}\ (\vec{v}\cdot\vec{E_1})\ -\ (\vec{v}\cdot\vec{k})\ \vec{E_1}\right)\right) \\
&\quad -\frac{1}{2}\left(\frac{q_s}{m_s c}\right)^2\ \left(\frac{\partial\ F_s^{+}(\vec{v},\vec{k})}{\partial \vec{v}}\ \times\ \vec{B_0}\right)\cdot\left(\frac{c}{\omega\ \iota}\ \left[\ \hat{I}\ -\ \frac{\vec{k}\ \vec{k}}{k^2}\ \right]\cdot\vec{E_1}\right) \\
&\quad -\ \frac{\iota\ q_s}{\hbar}\ \left(\frac{\vec{v}}{\iota\ \omega}\cdot\left[\ \hat{I}\ -\ \frac{\vec{k}\ \vec{k}}{k^2}\ \right]\cdot\vec{E_1}+\ \frac{\vec{k}}{\iota\ k^2}\cdot\vec{E_1}\ \right)\ F_s^{-}(\vec{k},\vec{v})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f_{s1}}{\partial t} &+ \vec{v}\cdot\frac{\partial f_{s1}}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\frac{\vec{v}}{c}\times\vec{B_0}\right)\cdot\frac{\partial f_{s1}}{\partial \vec{v}} \\
&= -\frac{q_s}{m_s}\ \frac{\partial f_{s0}}{\partial \vec{v}}\cdot\left(\ \vec{E_1}\ -\ \frac{\vec{v}\cdot\vec{k}}{\omega}\ \vec{E_1}+\frac{\vec{k}\ \vec{v}}{\omega}\cdot\vec{E_1}\right) \\
&\quad -\ \frac{1}{2}\ \frac{c}{\omega\ \iota}\left(\frac{q_s}{m_s c}\right)^2\ \left(\frac{\partial\ F_s^{+}(\vec{v},\vec{k})}{\partial \vec{v}}\ \times\ \vec{B_0}\cdot\left[\ \hat{I}\ -\ \frac{\vec{k}\ \vec{k}}{k^2}\ \right]\cdot\vec{E_1}\right) \\
&\quad -\ \frac{q_s}{\omega\ \hbar}\ \left(\left[\ \vec{v}\ -\ \frac{\vec{v}\cdot\vec{k}}{k^2}\ \vec{k}\ \right]\ +\ \frac{\omega\ \vec{k}}{k^2}\ \right)\ F_s^{-}(\vec{k},\vec{v})\cdot\vec{E_1}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial f_{s1}}{\partial t} + \vec{v}\cdot\frac{\partial f_{s1}}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\frac{\vec{v}}{c}\times\vec{B_0}\right)\cdot\frac{\partial f_{s1}}{\partial \vec{v}} \\
&\qquad = -\frac{q_s}{m_s}\ \frac{\partial f_{s0}}{\partial \vec{v}}\ .\ \left(\left(1\ -\ \frac{\vec{v}\cdot\vec{k}}{\omega}\right)\hat{I} + \frac{\vec{k}\ \vec{v}}{\omega}\right)\cdot\vec{E_1} \\
&\qquad - \frac{1}{2}\left(\frac{q_s}{m_s c}\right)^2\ \left(\ \frac{\partial\ F_s^+(\vec{v},\vec{k})}{\partial \vec{v}}\ \times\ \vec{B_0}\right)\ .\ \left(\frac{c}{\omega\ \iota}\ \left[\ \hat{I}\ -\ \frac{\vec{k}\ \vec{k}}{k^2}\ \right]\ .\ \vec{E_1}\right) \\
&\qquad\qquad -\ \frac{q_s}{\omega\ \hbar}\ \left(\ \vec{v}\ +\ \frac{\omega-\vec{v}\ .\ \vec{k}}{k^2}\ \vec{k}\right)\ F_s^-(\vec{k},\vec{v})\ .\ \vec{E_1}
\end{aligned}
\tag{3.59}
$$

Since, $\qquad f_{s1} = f_{s1}(\vec{r}, \vec{v}, t)$

It can be seen that L.H.S of Eq. (3.60) is total time derivative of $f_{s1}(\vec{r}, \vec{v}, t)$

So,

$$\begin{aligned} \frac{df_{s1}}{dt} \; = \; & -\frac{q_s}{m_s} \, \frac{\partial f_{s0}}{\partial \vec{v}} \; . \; \left(\left(1 \; - \; \frac{\vec{v} \; . \; \vec{k}}{\omega} \right) \hat{I} + \frac{\vec{k} \; \vec{v}}{\omega} \right) \; . \; \vec{E_1}(\vec{r} \; , t) \\ & - \frac{1}{2} \left(\frac{q_s}{m_s c} \right)^2 \left(\frac{\partial \; F_s^{+}(\vec{v}, \vec{k})}{\partial \vec{v}} \; \times \; \vec{B_0} \right) \; . \; \left(\frac{c}{\omega \; \iota} \left[\hat{I} \; - \; \frac{\vec{k} \; \vec{k}}{k^2} \right] \; . \; \vec{E_1}(\vec{r} \; , t) \right) \\ & - \; \frac{q_s}{\omega \; \hbar} \left(\vec{v} \; + \; \frac{\omega - \vec{v} \; . \; \vec{k}}{k^2} \; \vec{k} \right) \; F_s^{-}(\vec{k}, \vec{v}) \; . \; \vec{E_1}(\vec{r} \; , t) \end{aligned}$$

In order to calculate the perturbation, integrate the above expression over time, take upper limit to be t and lower limit of integration to be $-\infty$ which indicates that we are to start from the time when there was no perturbation.

$$\begin{aligned} \int df_{s1} \; = \; & -\int_{-\infty}^{t} \frac{q_s}{m_s} \, \frac{\partial f_{s0}}{\partial \vec{v'}} \; . \; \left(\left(1 \; - \; \frac{\vec{v'} \; . \; \vec{k}}{\omega} \right) \hat{I} + \frac{\vec{k} \; \vec{v'}}{\omega} \right) \; . \; \vec{E_1}(\vec{r'} \; , t') \;\; dt' \\ & -\int_{-\infty}^{t} \frac{1}{2} \left(\frac{q_s}{m_s c} \right)^2 \left(\frac{\partial \; F_s^{+}(\vec{v'}, \vec{k})}{\partial \vec{v}} \; \times \; \vec{B_0} \right) \; . \; \left(\frac{c}{\omega \; \iota} \left[\hat{I} \; - \; \frac{\vec{k} \; \vec{k}}{k^2} \right] \; . \; \vec{E_1}(\vec{r'} \; , t') \right) \;\; dt' \\ & - \int_{-\infty}^{t} \frac{q_s}{\omega \; \hbar} \left(\vec{v'} \; + \; \frac{\omega - \vec{v'} \; . \; \vec{k}}{k^2} \; \vec{k} \right) \; F_s^{-}(\vec{k}, \vec{v'}) \; . \; \vec{E_1}(\vec{r'} \; , t') \;\; dt' \end{aligned}$$

$\vec{E_1}(\vec{r'} \; , t')$ can be written as

$$\vec{E_1}(\vec{r'} \; , t') = \vec{E_1}(\vec{r} \; , t) \; e^{\iota \left(\vec{k} . (\vec{r'} - \vec{r}) \; - \; \omega (t' \; - \; t) \right)}$$

$$f_{s1}(\vec{r},\vec{v},t) \;=\; -\frac{q_s}{m_s}\int_{-\infty}^{t}\frac{\partial f_{s0}}{\partial\vec{v'}}\cdot\left(\left(1\;-\;\frac{\vec{v'}\cdot\vec{k}}{\omega}\right)\hat{I}+\frac{\vec{k}\;\vec{v'}}{\omega}\right)\,e^{\iota\left(\vec{k}\cdot(\vec{r'}-\vec{r})\;-\;\omega(t'\;-\;t)\right)}\;dt'\cdot\vec{E_1}(\vec{r}\,,t)$$
$$-\frac{c}{2\;\iota\;\omega}\left(\frac{q_s}{m_s c}\right)^2\int_{-\infty}^{t}\left(\frac{\partial\;F_s^{+}(\vec{v'},\vec{k})}{\partial\vec{v}}\;\times\;\vec{B_0}\right)\cdot\left(\left[\hat{I}\;-\;\frac{\vec{k}\;\vec{k}}{k^2}\right]\;e^{\iota\left(\vec{k}\cdot(\vec{r'}-\vec{r})\;-\;\omega(t'\;-\;t)\right)}\right)\;dt'\cdot\vec{E_1}(\vec{r}\,,t)$$
$$-\;\frac{q_s}{\omega\;\hbar}\int_{-\infty}^{t}\left(\vec{v'}\;+\;\frac{\omega-\vec{v'}\cdot\vec{k}}{k^2}\;\vec{k}\right)\;F_s^{-}(\vec{k},\vec{v'})\,e^{\iota\left(\vec{k}\cdot(\vec{r'}-\vec{r})\;-\;\omega(t'\;-\;t)\right)}\;dt'\;.\;\vec{E_1}(\vec{r}\,,t) \tag{3.60}$$

Where we define

$$\beta = \left(\vec{k}.(\vec{r'}-\vec{r})\;-\;\omega(t'\;-\;t)\right) \tag{3.61}$$

Making change of variables we have

$$\tau = t - t' \tag{3.62}$$

$$\Rightarrow\; d\tau = -dt'$$

Limits of integration change accordingly as

$$t' = -\infty \quad\rightarrow\quad \tau = \infty \qquad and \qquad t' = t \quad\rightarrow\quad \tau = 0$$

Also

$$\vec{v} = (v_x, v_y, v_z) = (v_\perp\cos\theta, v_\perp\sin\theta, v_\parallel) \tag{3.63}$$

If we have particle of species "s" gyrating with cyclotron frequency ω_{cs}, then after time interval τ the phase changes and velocity becomes

$$\vec{v'} = \left(v_\perp\cos\left(\omega_{cs}\tau+\theta\right), v_\perp\sin\left(\omega_{cs}\tau+\theta\right), v_\parallel\right) \tag{3.64}$$

The above equation can be written in terms of net displacement as

$$\frac{d\vec{r''}}{d\tau} = \Big(v_\perp \cos{(\omega_{cs}\tau + \theta)}, v_\perp \sin{(\omega_{cs}\tau + \theta)}, v_\parallel\Big)$$

Integrating both sides

$$\int_{r}^{r'} d\vec{r''} \ = \int_{\tau}^{0} \Big(v_\perp \cos{(\omega_{cs}\tau' + \theta)}, v_\perp \sin{(\omega_{cs}\tau' + \theta)}, v_\parallel\Big) \quad d\tau'$$

$$\vec{r'} - \vec{r} \ = \left|\left(\frac{v_\perp}{\omega_{cs}} \cos{(\omega_{cs}\tau' + \theta)}, \ -\frac{v_\perp}{\omega_{cs}} \sin{(\omega_{cs}\tau' + \theta)}, \ v_\parallel \ \tau'\right)\right|_{\tau}^{0}$$

$$\vec{r'} - \vec{r} \ = \left(\frac{v_\perp}{\omega_{cs}} \left(\cos{(\theta)} - \cos{(\omega_{cs}\tau + \theta)}\right), -\frac{v_\perp}{\omega_{cs}} \left(\sin{(\theta)} - \sin{(\omega_{cs}\tau + \theta)}\right), -v_\parallel \ \tau\right)$$

$$\vec{r'} - \vec{r} \ = \left(-\frac{v_\perp}{\omega_{cs}} \left(\cos{(\omega_{cs}\tau + \theta)} - \cos{(\theta)}\right), \frac{v_\perp}{\omega_{cs}} \left(\sin{(\omega_{cs}\tau + \theta)} - \sin{(\theta)}\right), -v_\parallel \ \tau\right) \tag{3.65}$$

After applying change of variables the Eq. (3.61) can be written as

$$\begin{aligned} f_{s1}(\vec{r}, \vec{v}, t) \ = \ & -\frac{q_s}{m_s} \int_0^\infty \frac{\partial f_{s0}}{\partial \vec{v'}} \cdot \left(\left(1 \ - \ \frac{\vec{v'} \cdot \vec{k}}{\omega}\right) \hat{I} + \frac{\vec{k} \ \vec{v'}}{\omega}\right) \ e^{\iota\beta} \ d\tau \cdot \vec{E_1} - \frac{c}{2 \ \iota \ \omega} \ \left(\frac{q_s}{m_s c}\right)^2 \\ & \times \ \int_0^\infty \left(\frac{\partial \ F_s^+(\vec{v'}, \vec{k})}{\partial \vec{v}} \ \times \ \vec{B_0}\right) \ e^{\iota\beta} \ d\tau \cdot \left(\hat{I} \ - \ \frac{\vec{k} \ \vec{k}}{k^2}\right) \cdot \vec{E_1} \\ & - \ \frac{q_s}{\omega \ \hbar} \int_0^\infty \left(\vec{v'} \ + \ \frac{\omega - \vec{v'} \ . \ \vec{k}}{k^2} \ \vec{k}\right) \ F_s^-(\vec{k}, \vec{v'}) \ e^{\iota\beta} \ \ d\tau \cdot \ \vec{E_1} \end{aligned} \tag{3.66}$$

In cylindrical coordinates we have

$$\vec{v} = (v_\perp \cos\theta, v_\perp \sin\theta, v_\parallel) \tag{3.67}$$

$$\vec{v'} = (v_\perp \cos(\omega_c\tau + \theta), v_\perp \sin(\omega_c\tau + \theta), v_\parallel) \tag{3.68}$$

Now we derive the expression for susceptibility for deriving it we use Maxwell's equation

$$\nabla \times \vec{B}_1 = \frac{4\pi}{c^2}\vec{J}_1 + \frac{1}{c^2}\frac{\partial \vec{E}_1}{\partial t} \tag{3.69}$$

Current density $\vec{J}$ can be expressed in terms of polarazibility as

$$\vec{J} = \frac{\partial \vec{P}}{\partial t}$$

Using the above equation we get

$$\nabla \times \vec{B}_1 = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\left(\vec{P}_1 + \frac{1}{4\pi}\vec{E}_1\right) \tag{3.70}$$

The term in the parenthesis is described as Electric displacement $\vec{D}_1$

The Eq. (3.71) now modifies as

$$\nabla \times \vec{B}_1 = \frac{4\pi}{c^2}\frac{\partial \vec{D}_1}{\partial t} \tag{3.71}$$

Comparing Eq. (3.70) and Eq. (3.72), and using $\vec{D}_1 = \tilde{\epsilon} \cdot \vec{E}_1$

where $\tilde{\epsilon}$ is dielectric constant.

$$4\pi\vec{J}_1 = -i\omega 4\pi\vec{D}_1 + i\omega\vec{E}_1 \tag{3.72}$$

or

$$4\pi\vec{J}_1 = -i\omega\left(4\pi\tilde{\epsilon}(\omega) - 1\right).\vec{E}_1 \tag{3.73}$$

In order to simplify the term in parenthesis we make use of the following equations

$$\vec{D}_1 = \epsilon_0\vec{E}_1 + \epsilon_0\chi\vec{E}_1$$

$$\vec{D_1} = \epsilon_0 \left(1 + \chi\right) \cdot \vec{E_1}$$
$$\chi = 4\pi\tilde{\epsilon}(\omega) - 1 \tag{3.74}$$

Putting the above expression $\tilde{\epsilon}(\omega)$ in Eq. (3.74),

$$4\pi\vec{J_1} = -i\omega\left(4\pi\tilde{\epsilon}(\omega) - 1\right) \cdot \vec{E_1} = -i\omega\chi \cdot \vec{E_1}$$
$$\vec{J_1} = \frac{-i\omega\chi \cdot \vec{E_1}}{4\pi} \tag{3.75}$$

Also current density can be expressed in terms of number density

$$\vec{J_1} = n_1 q \vec{v}$$

where n_1 being the perturbed number density can be expressed as

$$n_1 = n_0 \int f_1(\vec{r}, \vec{v}, t) d^3v$$

Finally we write current density $\vec{J_1}$ as

$$\vec{J_1} = q \int \vec{v} f_1(\vec{r}, \vec{v}, t) d^3v \tag{3.76}$$

Comparing Eq. (3.76) and Eq. (3.77) we have

$$\frac{-i\omega\overleftrightarrow{\chi} \cdot \vec{E_1}}{4\pi} = q \int v f_1(\vec{r}, \vec{v}, t) d^3v$$

Now putting $f_1(\vec{r},\vec{v},t)$ from Eq. (3.67) in the above equation

$$\overleftrightarrow{\chi}\cdot\vec{E}_1 = \frac{4\pi q^2}{im\omega}\int d^3v\int_0^\infty \vec{v}\frac{\partial f_0}{\partial\vec{v'}}\cdot\left[\left(1-\frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v'}}{\omega}\right)\right]e^{i\alpha}d\tau\cdot\vec{E}_1$$
$$-\frac{2\pi n_0qc}{\omega^2}\left(\frac{q}{mc}\right)^2\int d^3v\int_0^\infty\left(\frac{\partial T^+(\vec{v'},\vec{k})}{\partial\vec{v'}}\times\vec{B}_0\right)\cdot\left[\tilde{I}-\frac{\vec{k}\vec{k}}{k^2}\right]e^{i\alpha}d\tau\cdot\vec{E}_1$$
$$+\frac{4\pi n_0q^2}{i\omega^2\hbar}\int d^3v\int_0^\infty\left[\vec{v'}+\left(\frac{\omega-\vec{k}\cdot\vec{v'}}{k^2}\right)\vec{k}\right]e^{i\alpha}d\tau T^-(\vec{v'},\vec{k})\cdot\vec{E}_1 \tag{3.77}$$

By comparing the both sides we get the expression for susceptibility $\overleftrightarrow{\chi}$ as follow

$$\overleftrightarrow{\chi} = \frac{4\pi q^2}{im\omega}\int d^3v\int_0^\infty \vec{v}\frac{\partial f_0}{\partial\vec{v'}}\cdot\left[\left(1-\frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v'}}{\omega}\right)\right]e^{i\alpha}d\tau$$
$$-\frac{2\pi n_0qc}{\omega^2}\left(\frac{q}{mc}\right)^2\int d^3v\int_0^\infty\left(\frac{\partial T^+(\vec{v'},\vec{k})}{\partial\vec{v'}}\times\vec{B}_0\right)\cdot\left[\tilde{I}-\frac{\vec{k}\vec{k}}{k^2}\right]e^{i\alpha}d\tau$$
$$+\frac{4\pi n_0q^2}{i\omega^2\hbar}\int d^3v\int_0^\infty\left[\vec{v'}+\left(\frac{\omega-\vec{k}\cdot\vec{v'}}{k^2}\right)\vec{k}\right]e^{i\alpha}d\tau T^-(\vec{v'},\vec{k}) \tag{3.78}$$
$$\overleftrightarrow{\chi} = \overleftrightarrow{\chi}_1+\overleftrightarrow{\chi}_2+\overleftrightarrow{\chi}_3$$

$$\overleftrightarrow{\chi_1} = \frac{4\pi q^2}{im\omega}\int d^3v\int_0^\infty \vec{v}\frac{\partial f_0}{\partial\vec{v'}}\cdot\left[\left(1-\frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I}+\left(\frac{\vec{k}\vec{v'}}{\omega}\right)\right]e^{i\alpha}d\tau \tag{3.79}$$

$$\overleftrightarrow{\chi}_2 = -\frac{2\pi n_0qc}{\omega^2}\left(\frac{q}{mc}\right)^2\int \vec{v}d^3v\int_0^\infty\left(\frac{\partial T^+(\vec{v'},\vec{k})}{\partial\vec{v'}}\times\vec{B}_0\right)\cdot\left[\tilde{I}-\frac{\vec{k}\vec{k}}{k^2}\right]e^{i\alpha}d\tau \tag{3.80}$$

$$\overleftrightarrow{\chi}_3 = \frac{4\pi n_0q^2}{i\omega^2\hbar}\int \vec{v}d^3v\int_0^\infty\left[\vec{v'}+\left(\frac{\omega-\vec{k}\cdot\vec{v'}}{k^2}\right)\vec{k}\right]e^{i\alpha}d\tau T^-(\vec{v'},\vec{k}) \tag{3.81}$$

3.8 Parallel propagating electromagnetic waves

In Section. (2.4), Using kinetic theory the dispersion relation for parallel propagating electromagnetic waves in classical regime has been derived. In this section we will derive the dispersion relation for parallel propagating electromagnetic waves following the quantum approach. For parallel propagating electromagnetic wave choosing magnetic field along z axis and the wave vector is also in the same direction.

$$\vec{B} = B_0 \hat{z}$$

$$\vec{k} = k\hat{z}$$

3.9 Calculations for χ_1

$$\overleftrightarrow{\chi_1} = \frac{4\pi q^2}{im\omega} \int d^3v \int_0^\infty \vec{v} \frac{\partial f_0}{\partial \vec{v'}} \cdot \left[\left(1 - \frac{\vec{k} \cdot \vec{v'}}{\omega} \right) \tilde{I} + \left(\frac{\vec{k}\vec{v'}}{\omega} \right) \right] e^{i\alpha} d\tau \tag{3.82}$$

In cylindrical coordinates we have

$$\int d^3v = \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} d\theta \tag{3.83}$$

The term in parenthesis and $\exp^{i\alpha}$ in Eq. (3.83) is solved for parallel propagating Em waves using cylindrical coordinates and we get

$$e^{i\alpha} = e^{i(\omega - kv_\parallel)\tau} \tag{3.84}$$

$$\frac{\partial f_0}{\partial \vec{v'}} \cdot \left[\left(1 - \frac{\vec{k}\cdot\vec{v'}}{\omega}\right)\tilde{I} + \left(\frac{\vec{k}\vec{v'}}{\omega}\right)\right] = \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{\vec{k}v_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\cos(\omega_{cs}\tau + \theta)\hat{x}$$
$$+ \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{\vec{k}v_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]\sin(\omega_{cs}\tau + \theta)\hat{y} + \frac{\partial f_0}{\partial v_\parallel}\hat{z} \quad (3.85)$$

Making use of Eq.(3.64), Eq.(3.84), Eq. (3.85) and Eq. (3.85) in Eq. (3.83) we get the following components for $\overleftrightarrow{\chi_1}$

$$\overleftrightarrow{\chi_1}_{xx} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \cos\theta\cos(\omega_{cs}\tau + \theta)d\theta$$
$$\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right] \quad (3.86)$$

$$\overleftrightarrow{\chi_1}_{xy} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \cos\theta\sin(\omega_{cs}\tau + \theta)d\theta$$
$$\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right] \quad (3.87)$$

$$\overleftrightarrow{\chi_1}_{xz} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty \frac{\partial f_0}{\partial v_\parallel}dv_\parallel \int_0^{2\pi} \cos\theta d\theta \quad (3.88)$$

$$\overleftrightarrow{\chi_1}_{yx} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \sin\theta\cos(\omega_{cs}\tau + \theta)d\theta$$
$$\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right] \quad (3.89)$$

$$\overleftrightarrow{\chi_1}_{yy} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \sin\theta\sin(\omega_{cs}\tau + \theta)d\theta$$
$$\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right] \quad (3.90)$$

$$\overleftrightarrow{\chi 1}_{yz} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty {v_\perp}^2 dv_\perp\int_{-\infty}^\infty \frac{\partial f_0}{\partial v_\parallel}dv_\parallel\int_0^{2\pi}\sin\theta d\theta \quad (3.91)$$

$$\overleftrightarrow{\chi 1}_{zx} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty v_\parallel dv_\parallel\int_0^{2\pi}\cos(\omega_{cs}\tau+\theta)d\theta \left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right] \quad (3.92)$$

$$\overleftrightarrow{\chi 1}_{zy} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty v_\parallel dv_\parallel\int_0^{2\pi}\sin(\omega_{cs}\tau+\theta)d\theta \left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right] \quad (3.93)$$

$$\overleftrightarrow{\chi 1}_{zz} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty v_\parallel\frac{\partial f_0}{\partial v_\parallel}dv_\parallel\int_0^{2\pi} d\theta \quad (3.94)$$

In all the above components it has been seen that there is no quantum term or quantum correction it is purely classical.

3.9.1 Calculations for XX component of χ

$$\overleftrightarrow{\chi 1}_{xx} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty {v_\perp}^2 dv_\perp\int_{-\infty}^\infty dv_\parallel\int_0^{2\pi}\cos\theta\cos(\omega_{cs}\tau+\theta)d\theta \left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]$$

Theta Integration

$$\int_0^{2\pi}\cos\theta\cos(\omega_{cs}\tau+\theta)d\theta = \pi\cos(\omega_{cs}\tau) = \pi\frac{e^{i\omega_{cs}\tau}+e^{-i\omega_{cs}\tau}}{2}$$

Tau Integration

Combining the results for theta integration with tau it gives

$$\frac{\pi}{2}\left[\int_0^\infty e^{i(\omega-kv_\parallel+\omega_{cs})}d\tau+\int_0^\infty e^{i(\omega-kv_\parallel-\omega_{cs})}d\tau\right]=\frac{\pi}{2}\left[\frac{i}{\omega-kv_\parallel+\omega_{cs}}+\frac{i}{\omega-kv_\parallel-\omega_{cs}}\right]$$

Perpendicular Integration

The perpendicular integration is being solved by integration by parts and using the fact that distribution gets vanish at the end points we have

$$\frac{1}{n_0}\int_0^\infty {v_\perp}^2\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]dv_\perp=\left(1-\frac{kv_\parallel}{\omega}\right)\int_0^\infty\frac{{v_\perp}^2}{n_0}\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp^3}{n_0\omega}\frac{\partial f_0}{\partial v_\parallel}\right)dv_\perp$$

$$\int_0^\infty\frac{{v_\perp}^2}{n_0}\frac{\partial f_0}{\partial v_\perp}=\frac{1}{n_0}\left[{v_\perp}^2\int_0^\infty\frac{\partial f_0}{\partial v_\perp}dv_\perp-\int\int_0^\infty\frac{\partial f_0}{\partial v_\perp}dv_\perp\frac{d{v_\perp}^2}{dv_\perp}\right]=\frac{1}{n_0}\int_0^\infty 2v_\perp f_0 dv_\perp=-2G_{s1}$$

$$\int_0^\infty\left(\frac{kv_\perp^3}{n_0\omega}\frac{\partial f_0}{\partial v_\parallel}\right)dv_\perp=\frac{k}{n_0\omega}\frac{\partial}{\partial v_\parallel}\int_0^\infty {v_\perp}^3 f_0 dv_\perp=\frac{k}{\omega}\frac{\partial}{\partial v_\parallel}G_{s3}$$

$$\frac{1}{n_0}\int_0^\infty {v_\perp}^2\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]dv_\perp=-2\left(1-\frac{kv_\parallel}{\omega}\right)G_{s1}+\frac{k}{\omega}\frac{\partial}{\partial v_\parallel}G_{s3}$$

In the above functions we have defined the function $G_{sn}(v_\parallel)$ as

$$G_{sn}(v_\parallel)=\frac{1}{n_0}\int_0^\infty v_\perp^n f_0(v_\perp^2,v_\parallel)dv_\perp$$

Now going back to the equation of $\overleftrightarrow{\chi_1}_{xx}$ and putting the results for the above integration in the respective places.

$$\overleftrightarrow{\chi_1}_{xx}=\frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^\infty\left[\frac{\omega-kv_\parallel}{\omega-kv_\parallel+\omega_{cs}}\right]G_{s1}dv_\parallel-\frac{\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^\infty\left[\frac{\omega-kv_\parallel}{\omega-kv_\parallel-\omega_{cs}}\right]G_{s1}dv_\parallel+$$
$$\frac{k\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_\parallel}\int_{-\infty}^\infty\left[\frac{G_{s3}}{\omega-kv_\parallel+\omega_{cs}}\right]dv_\parallel+\frac{k\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_\parallel}\int_{-\infty}^\infty\left[\frac{G_{s3}}{\omega-kv_\parallel-\omega_{cs}}\right]dv_\parallel \quad (3.95)$$

In Eq. (3.95), there appears four integral which cen be solved by making use of integration by parts , as first and seocnd term in Eq. (3.95) seems to be quite similar just with a difference of signs so they are simplified on the same pattern, likewise third and fourth terms carry the same terms but with different signs so they will be solved on the same pattern. Let denote the first term as integral I and 3rd term as integral II.

Integral I

$$\frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{\omega-kv_\parallel}{\omega-kv_\parallel+\omega_{cs}}\right]G_{s1}dv_\parallel=\frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty}G_{s1}dv_\parallel+\frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{G_{s1}}{\omega-kv_\parallel+\omega_{cs}}\right]dv_\parallel \tag{3.96}$$

Integral II

$$\frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{\omega-kv_\parallel}{\omega-kv_\parallel-\omega_{cs}}\right]G_{s1}dv_\parallel=\frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty}G_{s1}dv_\parallel-\frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{G_{s1}}{\omega-kv_\parallel-\omega_{cs}}\right]dv_\parallel \tag{3.97}$$

Integral III

$$\frac{k\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_\parallel}\int_{-\infty}^{\infty}\left[\frac{G_{s3}}{\omega-kv_\parallel+\omega_{cs}}\right]dv_\parallel=-\frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega-kv_\parallel+\omega_{cs})^2} \tag{3.98}$$

Integral IV

$$\frac{k\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_\parallel}\int_{-\infty}^{\infty}\left[\frac{G_{s3}}{\omega-kv_\parallel-\omega_{cs}}\right]dv_\parallel=-\frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega-kv_\parallel-\omega_{cs})^2} \tag{3.99}$$

After combining the results for the all the integrations performed above we have

$$\overleftrightarrow{\chi_1}_{xx}=\frac{-{\omega_p}^2}{\omega^2}+\frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{G_{s1}}{\omega-kv_\parallel+\omega_{cs}}\right]dv_\parallel-\frac{{\omega_p}^2}{\omega^2}$$
$$-\frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{G_{s1}}{\omega-kv_\parallel-\omega_{cs}}\right]dv_\parallel-\frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega-kv_\parallel+\omega_{cs})^2}-\frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega-kv_\parallel-\omega_{cs})^2} \tag{3.100}$$

3.9.2 Calculations for XY Component

$$\overleftrightarrow{\chi_1}_{xy} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty {v_\perp}^2 dv_\perp\int_{-\infty}^\infty dv_\parallel\int_0^{2\pi}\cos\theta\sin(\omega_{cs}\tau+\theta)d\theta$$
$$\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]$$

<u>Theta Integration</u>

$$\int_0^{2\pi}\cos\theta\sin(\omega_{cs}\tau+\theta)d\theta = \pi\sin(\omega_{cs}\tau) = \pi\frac{e^{i\omega_{cs}\tau}-e^{-i\omega_{cs}\tau}}{2i}$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^\infty e^{i(\omega-kv_\parallel+\omega_{cs})}d\tau - \int_0^\infty e^{i(\omega-kv_\parallel-\omega_{cs})}d\tau\right] = \frac{\pi}{2}\left[\frac{1}{\omega-kv_\parallel+\omega_{cs}}-\frac{1}{\omega-kv_\parallel-\omega_{cs}}\right]$$

Now inserting the results of perpendicular integration.

<u>Perpendicular Integration</u>

The perpendicular integration is same as we have done in calculations for XX component. So making use of Eq. (3.95) we can write XY component as follow

$$\frac{1}{n_0}\int_0^\infty {v_\perp}^2\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]dv_\perp = -2\left(1-\frac{kv_\parallel}{\omega}\right)G_{s1}+\frac{k}{\omega}\frac{\partial}{\partial v_\parallel}G_{s3}$$

$$\overleftrightarrow{\chi_1}_{xy} = \frac{i\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^\infty\left[\frac{\omega-kv_\parallel}{\omega-kv_\parallel+\omega_{cs}}-\frac{\omega-kv_\parallel}{\omega-kv_\parallel-\omega_{cs}}\right]G_{s1}dv_\parallel$$
$$-\frac{ik\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_\parallel}\int_{-\infty}^\infty\left[\frac{1}{\omega-kv_\parallel+\omega_{cs}}-\frac{1}{\omega-kv_\parallel-\omega_{cs}}\right]G_{s3}dv_\parallel \quad (3.101)$$

In Eq. (3.102) the terms in the parenthesis are exactly same as in Eq. (3.96) so they can be simplified on the same pattern as we have done before in solving integrals[I-IV], now making use of those integral here we get the XY component as

$$\overleftrightarrow{\chi_1}_{xy} = \left[\frac{i\pi{\omega_p}^2}{\omega^2} - \frac{i\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\right] + i\left[\frac{-\pi{\omega_p}^2}{\omega^2} - \frac{\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\right] + \frac{ik^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel - \frac{ik^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel - \omega_{cs})^2}dv_\parallel$$

$$\overleftrightarrow{\chi_1}_{xy} = -i\left[\frac{-\pi{\omega_p}^2}{\omega^2} + \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right] + i\left[\frac{-\pi{\omega_p}^2}{\omega^2} - \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel - \omega_{cs})^2}dv_\parallel\right] \tag{3.102}$$

3.9.3 Calculations for XZ component

$$\overleftrightarrow{\chi_1}_{xz} = \frac{{\omega_p}^2}{in_0\omega}\int_0^{\infty}e^{i(\omega - kv_\parallel)\tau}d\tau\int_0^{\infty}{v_\perp}^2dv_\perp\int_{-\infty}^{\infty}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel\int_0^{2\pi}\cos\theta d\theta$$

Theta integration

$$\int_0^{2\pi}\cos\theta d\theta = \sin(\theta)\Big|_0^{2\pi} = 0 \tag{3.103}$$

$$\overleftrightarrow{\chi_1}_{xz} = 0$$

3.9.4 Calculations for YX component

$$\overleftrightarrow{\chi_1}_{yx} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty {v_\perp}^2 dv_\perp\int_{-\infty}^\infty dv_\parallel\int_0^{2\pi}\sin\theta\cos(\omega_{cs}\tau+\theta)d\theta \left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]$$

Theta Integration

$$\int_0^{2\pi}\sin\theta\cos(\omega_{cs}\tau+\theta)d\theta = -\pi\sin(\omega_{cs}\tau) = -\pi\left(\frac{e^{i\omega_{cs}\tau}-e^{-i\omega_{cs}\tau}}{2i}\right)$$

The theta integration gives the same results as for XY component but with a minus sign here, Remaining calculations are exactly same as done before.

Tau Integration

$$\frac{\pi}{2}\left[\int_0^\infty e^{i(\omega-kv_\parallel+\omega_{cs})}d\tau-\int_0^\infty e^{i(\omega-kv_\parallel-\omega_{cs})}d\tau\right] = \frac{-\pi}{2i}\left[\frac{i}{\omega-kv_\parallel+\omega_{cs}}-\frac{i}{\omega-kv_\parallel-\omega_{cs}}\right]$$

Perpendicular Integration

$$\frac{1}{n_0}\int_0^\infty {v_\perp}^2\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]dv_\perp = -2\left(1-\frac{kv_\parallel}{\omega}\right)G_{s1}+\frac{k}{\omega}\frac{\partial}{\partial v_\parallel}G_{s3}$$

$$\overleftrightarrow{\chi_1}_{yx} = \frac{-i\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^\infty\left[\frac{1}{\omega-kv_\parallel+\omega_{cs}}-\frac{1}{\omega-kv_\parallel-\omega_{cs}}\right]\left(\omega-kv_\parallel\right)G_{s1}dv_\parallel + \frac{ik\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_\parallel}\int_{-\infty}^\infty\left[\frac{1}{\omega-kv_\parallel+\omega_{cs}}-\frac{1}{\omega-kv_\parallel-\omega_{cs}}\right]G_{s3}dv_\parallel \tag{3.104}$$

In order to simplify the terms given in Eq. (3.105) we again make use of Integrals solved in Sec. (3.9.1), we can write the YX component as

$$\overleftrightarrow{\chi_1}_{yx} = i\left[\frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty} G_{s1} dv_\parallel + \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega - kv_\parallel + \omega_{cs}} dv_\parallel - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2} dv_\parallel\right]$$
$$-i\left[\frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty} G_{s1} dv_\parallel - \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega - kv_\parallel - \omega_{cs}} dv_\parallel - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel - \omega_{cs})^2} dv_\parallel\right] \tag{3.105}$$

It can be seen that

$$\overleftrightarrow{\chi_1}_{xy} = -\overleftrightarrow{\chi_1}_{xy}$$

3.9.5 Calculations for YY component

The expression for YY component is given as

$$\overleftrightarrow{\chi_1}_{yy} = \frac{{\omega_p}^2}{in_0\omega}\int_0^{\infty} e^{i(\omega - kv_\parallel)\tau} d\tau \int_0^{\infty} {v_\perp}^2 dv_\perp \int_{-\infty}^{\infty} dv_\parallel \int_0^{2\pi} \sin\theta \sin(\omega_{cs}\tau + \theta) d\theta$$
$$\left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]$$

On inspecting the expression for YY component it can be seen that it has only theta terms different from that of XX component, remaining all terms are same as in XX. So we will solve for theta integration and other terms will be simplified in the same manner as done in XX.

Theta Integration

$$\int_0^{2\pi} \sin\theta \sin(\omega_{cs}\tau + \theta) d\theta = \pi\cos(\omega_{cs}\tau) = \pi\frac{e^{i\omega_{cs}\tau} + e^{-i\omega_{cs}\tau}}{2}$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^{\infty} e^{i(\omega - kv_{\parallel} + \omega_{cs})} d\tau + \int_0^{\infty} e^{i(\omega - kv_{\parallel} - \omega_{cs})} d\tau\right] = \frac{\pi}{2}\left[\frac{i}{\omega - kv_{\parallel} + \omega_{cs}} + \frac{i}{\omega - kv_{\parallel} - \omega_{cs}}\right]$$

Perpendicular Integration

$$\frac{1}{n_0}\int_0^{\infty} {v_{\perp}}^2\left[\left(1 - \frac{kv_{\parallel}}{\omega}\right)\frac{\partial f_0}{\partial v_{\perp}} + \left(\frac{kv_{\perp}}{\omega}\frac{\partial f_0}{\partial v_{\parallel}}\right)\right]dv_{\perp} = -2\left(1 - \frac{kv_{\parallel}}{\omega}\right)G_{s1} + \frac{k}{\omega}\frac{\partial}{\partial v_{\parallel}}G_{s3}$$

In the above functions we have defined the function G_{sn} as given below

$$G_{sn} = \frac{1}{n_0}\int_0^{\infty} v_{\perp}^n f_0(v_{\perp} v_{\parallel}) dv_{\perp}$$

Now going back to the equation of $\overleftrightarrow{\chi_1}_{yy}$ and putting the results for the above integration in the respective places.

$$\overleftrightarrow{\chi_1}_{yy} = \frac{-\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{\omega - kv_{\parallel}}{\omega - kv_{\parallel} + \omega_{cs}}\right]G_{s1}dv_{\parallel} - \frac{\pi{\omega_p}^2}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{\omega - kv_{\parallel}}{\omega - kv_{\parallel} - \omega_{cs}}\right]G_{s1}dv_{\parallel} + \frac{k\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_{\parallel}}\int_{-\infty}^{\infty}\left[\frac{G_{s3}}{\omega - kv_{\parallel} + \omega_{cs}}\right]dv_{\parallel} + \frac{k\pi{\omega_p}^2}{2\omega^2}\frac{\partial}{\partial v_{\parallel}}\int_{-\infty}^{\infty}\left[\frac{G_{s3}}{\omega - kv_{\parallel} - \omega_{cs}}\right]dv_{\parallel} \quad (3.106)$$

Eq. (3.106) is exactly same as Eq. (3.95) In Eq. (3.95) we have done simplifications so picking the result from four integral given in that section we can write YY component as

$$\overleftrightarrow{\chi_1}_{yy} = \frac{-{\omega_p}^2}{\omega^2} + \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{G_{s1}}{\omega - kv_{\parallel} + \omega_{cs}}\right]dv_{\parallel} - \frac{{\omega_p}^2}{\omega^2} - \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\left[\frac{G_{s1}}{\omega - kv_{\parallel} - \omega_{cs}}\right]dv_{\parallel} - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_{\parallel} + \omega_{cs})^2} - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_{\parallel} - \omega_{cs})^2} \quad (3.107)$$

The above expression indicates that

$$\overleftrightarrow{\chi_1}_{yy} = \overleftrightarrow{\chi_1}_{xx}$$

3.9.6 Calculations for YZ component

$$\overleftrightarrow{\chi_1}_{yz} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty \frac{\partial f_0}{\partial v_\parallel}dv_\parallel \int_0^{2\pi}\sin\theta d\theta$$

Theta integration

$$\int_0^{2\pi}\sin\theta d\theta = -\cos(\theta)\Big|_0^{2\pi} = 0 \tag{3.108}$$

$$\overleftrightarrow{\chi_1}_{yz} = 0$$

3.9.7 Calculations for ZX component

$$\overleftrightarrow{\chi_1}_{zx} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi}\cos(\omega_{cs}\tau + \theta)d\theta \left[\left(1 - \frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp} + \left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]$$

Theta Integration

$$\int_0^{2\pi}\cos(\omega_{cs}\tau + \theta)d\theta = \sin(\omega_{cs}\tau + \theta)\Big|_0^{2\pi} \tag{3.109}$$

$$\overleftrightarrow{\chi_1}_{zx} = 0$$

$$\overleftrightarrow{\chi_1}_{xz} = 0$$

3.9.8 Calculations for ZX component

$$\overleftrightarrow{\chi_1}_{zx} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty v_\parallel dv_\parallel\int_0^{2\pi}\cos(\omega_{cs}\tau+\theta)d\theta$$
$$\left[\left(1-\frac{kv_\parallel}{\omega}\right)\frac{\partial f_0}{\partial v_\perp}+\left(\frac{kv_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}\right)\right]$$

Theta Integration

$$\int_0^{2\pi}\cos(\omega_{cs}\tau+\theta)d\theta = \sin(\omega_{cs}\tau+\theta)\Big|_0^{2\pi} \tag{3.110}$$

$$\overleftrightarrow{\chi_1}_{zx} = 0$$

Calculation for ZZ component

$$\overleftrightarrow{\chi_1}_{zz} = \frac{{\omega_p}^2}{in_0\omega}\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^\infty {v_\perp}^2 dv_\perp\int_{-\infty}^\infty v_\parallel\frac{\partial f_0}{\partial v_\parallel}dv_\parallel\int_0^{2\pi}d\theta$$

Theta Integration

$$\int_0^{2\pi}d\theta = 2\pi$$

Tau Integration

$$\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau =$$

$$\int_0^\infty e^{i(\omega-kv_\parallel)\tau}d\tau = \frac{1}{-i(\omega-kv_\parallel)}$$

Parallel integration

$$\int_{-\infty}^{\infty}\frac{v_\parallel}{(\omega-kv_\parallel)}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel=\frac{v_\parallel}{(\omega-kv_\parallel)}\int_{-\infty}^{\infty}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel-\int\int_{-\infty}^{\infty}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel\left(\frac{d}{dv_\parallel}\frac{v_\parallel}{(\omega-kv_\parallel)}\right)$$

$$\int_{-\infty}^{\infty}\frac{v_\parallel}{(\omega-kv_\parallel)}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel=-\int_{-\infty}^{\infty}f_0\frac{\omega}{(\omega-kv_\parallel)^2}dv_\parallel \tag{3.111}$$

Putting the results for the integrations performed above in the expression for $\overleftrightarrow{\chi_1}_{zz}$

$$\overleftrightarrow{\chi_1}_{zz}=-2\pi{\omega_p}^2\int_{-\infty}^{\infty}\frac{G_{s1}}{(\omega-kv_\parallel)^2}$$

Although we don't need ZZ component for parallel propagating Em waves, yet it can be useful while solving for the dispersion relation for parallel propagating electrostatic. We will not make use of it here. Now combining the Results of all the components of $\overleftrightarrow{\chi_1}$ we get

$$\begin{aligned}\overleftrightarrow{\chi_1}=&\left[\frac{-{\omega_p}^2}{2\omega^2}+\frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega-kv_\parallel+\omega_{cs}}dv_\parallel-\frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega-kv_\parallel+\omega_{cs})^2}dv_\parallel\right]\begin{bmatrix}1&i&0\\-i&1&0\\0&0&0\end{bmatrix}\\
&+\left[\frac{-{\omega_p}^2}{2\omega^2}-\frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}}{\omega-kv_\parallel-\omega_{cs}}dv_\parallel-\frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega-kv_\parallel-\omega_{cs})^2}dv_\parallel\right]\begin{bmatrix}1&i&0\\-i&1&0\\0&0&0\end{bmatrix}\\
&+\left(-2\pi{\omega_p}^2\int_{-\infty}^{\infty}\frac{G_{s1}}{(\omega-kv_\parallel)^2}\right)\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\end{aligned} \tag{3.112}$$

In Eq. (3.112) we have inserted the results for the components derived above.These results are classical there is no quantum correction involved. Now we proceed for deriving the components $\overleftrightarrow{\chi_2}$ and $\overleftrightarrow{\chi_3}$.

3.10 Calculations for χ_2

$$\overleftrightarrow{\chi}_2 = -\frac{2\pi n_0 qc}{n_0\omega^2}\left(\frac{q}{mc}\right)^2 \int d^3v \int_0^\infty \left(\frac{\partial T^+(\vec{v'},\vec{k})}{\partial \vec{v'}} \times \vec{B}_0\right) \cdot \left[\tilde{I} - \frac{\vec{k}\vec{k}}{k^2}\right] e^{i\alpha} d\tau \tag{3.113}$$

Firs of all solving the term in parenthesis in Eq. (3.113) for parallel propagating electromagnetic waves configuration we get

$$\left(\frac{\partial T^+(\vec{v'},\vec{k})}{\partial \vec{v'}} \times \vec{B}_0\right) \cdot \left[\tilde{I} - \frac{\vec{k}\vec{k}}{k^2}\right] = \vec{B}_0\Bigg[\frac{\partial f_0\left(\vec{v}+\frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v}-\frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}} \sin(\omega_{cs}\tau+\theta)\hat{x} - \frac{\partial f_0\left(\vec{v}+\frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v}-\frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}} \cos(\omega_{cs}\tau+\theta)\hat{y}\Bigg] \tag{3.114}$$

Putting Eq. (3.114) in Eq. (3.113) we get

$$\overleftrightarrow{\chi}_2 = -\frac{4\pi n_0 qc}{2n_0\omega^2}\left(\frac{q}{mc}\right)^2 B_0 \int \vec{v} d^3v \int_0^\infty e^{i\alpha}\Bigg[\frac{\partial f_0\left(\vec{v}+\frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v}-\frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}} \sin(\omega_{cs}\tau+\theta)\hat{x} - \frac{\partial f_0\left(\vec{v}+\frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v}-\frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}} \cos(\omega_{cs}\tau+\theta)\hat{y}\Bigg] d\tau \tag{3.115}$$

Now writing down the components for $\overleftarrow{\chi_2}$ as

$$\overleftrightarrow{\chi_2}_{xx} = -\frac{\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right) B_0 \int_0^\infty e^{i(\omega - kv_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \cos\theta \sin(\omega_{cs}\tau+\theta) d\theta \cdot \left[\frac{\partial f_0\left(\vec{v}+\frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v}-\frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}}\right] \tag{3.116}$$

$$\overleftrightarrow{\chi_2}_{xy} = \frac{\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right)B_0 \int_0^\infty e^{i(\omega - kv_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \cos\theta\cos(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[\frac{\partial f_0\left(\vec{v} + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v} - \frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}}\right] \quad (3.117)$$

$$\overleftrightarrow{\chi_2}_{yy} = \frac{\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right)B_0 \int_0^\infty e^{i(\omega - kv_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \sin\theta\cos(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[\frac{\partial f_0\left(\vec{v} + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v} - \frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}}\right] \quad (3.118)$$

$$\overleftrightarrow{\chi_2}_{yx} = \frac{-\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right)B_0 \int_0^\infty e^{i(\omega - kv_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \sin\theta\sin(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[\frac{\partial f_0\left(\vec{v} + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v} - \frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}}\right]$$

3.10.1 Calculations for XX component

$$\overleftrightarrow{\chi_2}_{xx} = -\frac{\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right)B_0 \int_0^\infty e^{i(\omega - kv_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \cos\theta\sin(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[\frac{\partial f_0\left(\vec{v} + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v} - \frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_\perp}}\right] \quad (3.119)$$

Theta Integration

$$\int_0^{2\pi} \cos\theta \sin(\omega_{cs}\tau + \theta) d\theta = \pi \sin(\omega_{cs}\tau) = \pi \frac{e^{i\omega_{cs}\tau} - e^{-i\omega_{cs}\tau}}{2i}$$

Tau Integration

$$\frac{\pi}{2}\left[\int_0^{\infty} e^{i(\omega - kv_{\parallel} + \omega_{cs})} d\tau - \int_0^{\infty} e^{i(\omega - kv_{\parallel} - \omega_{cs})} d\tau\right] = \frac{\pi}{2}\left[\frac{1}{\omega - kv_{\parallel} + \omega_{cs}} - \frac{1}{\omega - kv_{\parallel} - \omega_{cs}}\right]$$

Perpendicular Integration

$$\begin{aligned}
&\int_0^{\infty} {v_{\perp}}^2 \left[\frac{\partial f_0\left(\vec{v} + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v} - \frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial \vec{v_{\perp}}}\right] dv_{\perp} \\
&= -2\int_0^{\infty} v_{\perp}\left[f_0\left(\vec{v} + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v} - \frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})\right] dv_{\perp} \\
&= -2\left[G_{s1}\left(v_{\parallel} + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_{\parallel} - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_{\parallel})\right]
\end{aligned} \tag{3.120}$$

where we have used

$$G_{sn}(v_{\parallel} + \frac{\hbar k}{2m}) = \frac{1}{n_0}\int_0^{\infty} v_{\perp}^n f_0\left(v_{\perp}^2, v_{\parallel} + \frac{\hbar k}{2m}\right) dv_{\perp} \tag{3.121}$$

The Eq. (3.120) indicates some quantum corrections in the parallel direction given by a factor of $\frac{\hbar k}{2m}$. These corrections give an indication of shift of velocities. These correction will alter the alter the results from that of classical ones. Using the results for the integrations performed above in expression for χ_{2xx}

$$\overleftrightarrow{\chi_2}_{xx} = \frac{\pi\omega_p^2\omega_{cs}}{2cn_0\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel+\frac{\hbar k}{2m}\right)+G_{s1}\left(v_\parallel-\frac{\hbar k}{2m}\right)-2G_{s1}(v_\parallel)}{\omega-kv_\parallel+\omega_{cs}}dv_\parallel$$
$$-\frac{\pi\omega_p^2\omega_{cs}}{2cn_0\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel+\frac{\hbar k}{2m}\right)+G_{s1}\left(v_\parallel-\frac{\hbar k}{2m}\right)-2G_{s1}(v_\parallel)}{\omega-kv_\parallel-\omega_{cs}}dv_\parallel \quad (3.122)$$

3.10.2 Calculations for XY component

$$\overleftrightarrow{\chi_2}_{xy} = \frac{\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right)B_0\int_0^{\infty}e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^{\infty}{v_\perp}^2dv_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}\cos\theta\cos(\omega_{cs}\tau+\theta)d\theta$$
$$\cdot\left[\frac{\partial f_0\left(\vec{v}+\frac{\hbar\vec{k}}{2m}\right)+f_0\left(\vec{v}-\frac{\hbar\vec{k}}{2m}\right)-2f_0(\vec{v})}{\partial\vec{v_\perp}}\right] \quad (3.123)$$

Theta Integration

$$\int_0^{2\pi}\cos\theta\cos(\omega_{cs}\tau+\theta)d\theta = \pi\cos(\omega_{cs}\tau) = \pi\frac{e^{i\omega_{cs}\tau}+e^{-i\omega_{cs}\tau}}{2}$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^{\infty}e^{i(\omega-kv_\parallel+\omega_{cs})}d\tau+\int_0^{\infty}e^{i(\omega-kv_\parallel-\omega_{cs})}d\tau\right] = \frac{\pi}{2}\left[\frac{i}{\omega-kv_\parallel+\omega_{cs}}+\frac{i}{\omega-kv_\parallel-\omega_{cs}}\right]$$

Perpendicular Integration The perpendicular integration is same as we have done in Sec. (3.10.1). Using the results from there we have After putting the results the above integrations in Eq. (3.123) we get

$$\overleftrightarrow{\chi_2}_{xy} = -i\Bigg[\frac{\pi\omega_p^2\omega_{cs}}{2cn_0\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\Bigg]$$
$$+i\Bigg[-\frac{\pi\omega_p^2\omega_{cs}}{2cn_0\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel\Bigg] \quad (3.124)$$

3.10.3 Calculations for YY component

$$\overleftrightarrow{\chi_2}_{yy} = \frac{\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right)B_0\int_0^{\infty}e^{i(\omega - kv_\parallel)\tau}d\tau\int_0^{\infty}v_\perp{}^2dv_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}\sin\theta\cos(\omega_{cs}\tau+\theta)d\theta$$
$$\cdot\Bigg[\frac{\partial f_0\left(\vec{v} + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(\vec{v} - \frac{\hbar\vec{k}}{2m}\right) - 2f_0(\vec{v})}{\partial\vec{v_\perp}}\Bigg] \quad (3.125)$$

$$\int_0^{2\pi}\sin\theta\cos(\omega_{cs}\tau + \theta)d\theta = -\pi\sin(\omega_{cs}\tau) = -\pi\left(\frac{e^{i\omega_{cs}\tau} - e^{-i\omega_{cs}\tau}}{2i}\right)$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^{\infty}e^{i(\omega - kv_\parallel + \omega_{cs})}d\tau - \int_0^{\infty}e^{i(\omega - kv_\parallel - \omega_{cs})}d\tau\right] = \frac{-\pi}{2}\left[\frac{1}{\omega - kv_\parallel + \omega_{cs}} - \frac{1}{\omega - kv_\parallel - \omega_{cs}}\right]$$

We are not going to write the expression for perpendicular integration here,because it has been mentioned before in Sec. (3.10.1) so we use it simply and the expression becomes

$$\overleftrightarrow{\chi_2}_{yy} = \frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel \quad (3.126)$$

3.10.4 Calculations for χ_{2yx}

$$\overleftrightarrow{\chi_2}_{yx} = \frac{-\omega_p^2}{2n_0\omega^2}\left(\frac{q}{mc}\right)B_0\int_0^{\infty}e^{i(\omega-kv_\parallel)\tau}d\tau\int_0^{\infty}{v_\perp}^2dv_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}\sin\theta\sin(\omega_{cs}\tau+\theta)d\theta \cdot\left[\frac{\partial f_0\left(\vec{v}+\frac{\hbar\vec{k}}{2m}\right)+f_0\left(\vec{v}-\frac{\hbar\vec{k}}{2m}\right)-2f_0(\vec{v})}{\partial\vec{v_\perp}}\right] \quad (3.127)$$

Theta Integration

$$\int_0^{2\pi}\sin\theta\sin(\omega_{cs}\tau+\theta)d\theta = \pi\cos(\omega_{cs}\tau) = \pi\frac{e^{i\omega_{cs}\tau}+e^{-i\omega_{cs}\tau}}{2}$$

$$\frac{\pi}{2}\left[\int_0^{\infty}e^{i(\omega-kv_\parallel+\omega_{cs})}d\tau + \int_0^{\infty}e^{i(\omega-kv_\parallel-\omega_{cs})}d\tau\right] = \frac{\pi}{2}\left[\frac{i}{\omega-kv_\parallel+\omega_{cs}} + \frac{i}{\omega-kv_\parallel-\omega_{cs}}\right]$$

$$\overleftrightarrow{\chi_2}_{yx} = i\left[\frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\right] - i\left[-\frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel\right] \quad (3.128)$$

$$\overleftrightarrow{\chi_2} = \left[\frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\right]\begin{bmatrix}1 & -i & 0\\ i & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$+\left[-\frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel\right]\begin{bmatrix}1 & i & 0\\ -i & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

(3.129)

3.11 Calculations for χ_3

$$\overleftrightarrow{\chi}_3 = \frac{4\pi n_0 q^2 m_s}{i n_0 \omega^2 \hbar m_s}\int \vec{v} d^3v \int_0^{\infty}\left[\vec{v'} + \left(\frac{\omega - \vec{k}\cdot\vec{v'}}{k^2}\right)\vec{k}\right]e^{i\alpha}d\tau T^-(\vec{v'},\vec{k}) \tag{3.130}$$

The components for $\overleftarrow{\chi_3}$ are given as

$$\overleftrightarrow{\chi_3}_{xx} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar}\int_0^{\infty} e^{i(\omega - kv_\parallel)\tau}d\tau\int_0^{\infty}{v_\perp}^3 dv_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}\cos\theta\cos(\omega_{cs}\tau+\theta)d\theta$$
$$.\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \tag{3.131}$$

$$\overleftrightarrow{\chi_3}_{xy} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar}\int_0^{\infty} e^{i(\omega - kv_\parallel)\tau}d\tau\int_0^{\infty}{v_\perp}^3 dv_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}\cos\theta\sin(\omega_{cs}\tau+\theta)d\theta$$
$$.\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \tag{3.132}$$

$$\overleftrightarrow{\chi_1}_{xz} = \frac{{\omega_p}^2 m_s}{i n_0 \omega \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi} \cos\theta d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]$$
$$+ \frac{{\omega_p}^2 m_s}{i n_0 \omega^2} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty \left(\frac{\omega - k v_\parallel}{k^2}\right)\vec{k} dv_\parallel \int_0^{2\pi} \cos\theta d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \quad (3.133)$$

$$\overleftrightarrow{\chi_3}_{yy} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^3 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \sin\theta \sin(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \quad (3.134)$$

$$\overleftrightarrow{\chi_3}_{yx} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^3 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \sin\theta \cos(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \quad (3.135)$$

$$\overleftrightarrow{\chi_3}_{yz} = \frac{{\omega_p}^2 m_s}{i n_0 \omega \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi} \sin\theta d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]$$
$$+ \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty \left(\frac{\omega - k v_\parallel}{k^2}\right)\vec{k} dv_\parallel \int_0^{2\pi} \sin\theta d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \quad (3.136)$$

$$\overleftrightarrow{\chi_3}_{zx} = \frac{{\omega_p}^2 m_s}{i n_0 \omega \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi} \cos(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m} \right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m} \right) - \frac{\hbar k}{m} \frac{\partial f_0}{\partial v_\parallel} \right] \quad (3.137)$$

$$\overleftrightarrow{\chi_3}_{zy} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi} \sin(\omega_{cs}\tau + \theta) d\theta$$
$$\left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m} \right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m} \right) - \frac{\hbar k}{m} \frac{\partial f_0}{\partial v_\parallel} \right] \quad (3.138)$$

$$\overleftrightarrow{\chi_3}_{zz} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty v_\parallel \left[\frac{k v_\parallel + \omega - k v_\parallel}{k} \right] dv_\parallel \int_0^{2\pi} d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m} \right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m} \right) - \frac{\hbar k}{m} \frac{\partial f_0}{\partial v_\parallel} \right] \quad (3.139)$$

3.11.1 Calculations for XX component

$$\overleftrightarrow{\chi_3}_{xx} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^3 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \cos\theta \cos(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m} \right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m_s} \right) - \frac{\hbar k}{m} \frac{\partial f_0}{\partial v_\parallel} \right] \quad (3.140)$$

Theta Integration

$$\int_0^{2\pi} \cos\theta \cos(\omega_{cs}\tau + \theta) d\theta = \pi \cos(\omega_{cs}\tau) = \pi \frac{e^{i\omega_{cs}\tau} + e^{-i\omega_{cs}\tau}}{2}$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^\infty e^{i(\omega-kv_\parallel+\omega_{cs})}d\tau+\int_0^\infty e^{i(\omega-kv_\parallel-\omega_{cs})}d\tau\right]=\frac{\pi}{2}\left[\frac{i}{\omega-kv_\parallel+\omega_{cs}}+\frac{i}{\omega-kv_\parallel-\omega_{cs}}\right]$$

$$\overleftrightarrow{\chi_3}_{xx}=\frac{{\omega_p}^2 m_s\pi}{2n_0\omega^2\hbar}\int_0^\infty {v_\perp}^3 dv_\perp\int_{-\infty}^\infty dv_\parallel.\left[f_0\left(v_\parallel+\frac{\hbar k}{2m}\right)-f_0\left(v_\parallel-\frac{\hbar k}{2m_s}\right)-\frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]$$
$$.\left[\frac{1}{\omega-kv_\parallel+\omega_{cs}}+\frac{1}{\omega-kv_\parallel-\omega_{cs}}\right]$$

Now simplifying the terms in parenthesis and performing parallel and perpendicular integrations we get

$$\frac{{\omega_p}^2 m_s\pi}{2n_0\omega^2\hbar}\int_{-\infty}^\infty dv_\parallel\int_0^\infty\left[\frac{v_\perp^3}{\omega-kv_\parallel+\omega_{cs}}\right]\left[f_0\left(v_\parallel+\frac{\hbar k}{2m}\right)-f_0\left(v_\parallel-\frac{\hbar k}{m}\right)-\frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]dv_\perp=$$
$$\frac{{\omega_p}^2 m_s\pi}{2n_0\omega^2\hbar}\left[\int_{-\infty}^\infty\frac{G_{s3}}{\omega-kv_\parallel+\omega_{cs}}dv_\parallel-\int_{-\infty}^\infty\frac{G_{s3}}{\omega-kv_\parallel+\omega_{cs}}dv_\parallel+\frac{\hbar k^2}{m}\int_{-\infty}^\infty\frac{G_{s3}}{(\omega-kv_\parallel+\omega_{cs})^2}dv_\parallel\right] \tag{3.141}$$

Likewise,

$$\frac{{\omega_p}^2 m_s\pi}{2n_0\omega^2\hbar}\int_{-\infty}^\infty dv_\parallel\int_0^\infty\left[\frac{v_\perp^3}{\omega-kv_\parallel-\omega_{cs}}\right]\left[f_0\left(v_\parallel+\frac{\hbar k}{2m}\right)-f_0\left(v_\parallel-\frac{\hbar k}{m}\right)-\frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]dv_\perp=$$
$$\frac{{\omega_p}^2 m_s\pi}{2n_0\omega^2\hbar}\left[\int_{-\infty}^\infty\frac{G_{s3}}{\omega-kv_\parallel-\omega_{cs}}dv_\parallel-\int_{-\infty}^\infty\frac{G_{s3}}{\omega-kv_\parallel-\omega_{cs}}dv_\parallel+\frac{\hbar k^2}{m}\int_{-\infty}^\infty\frac{G_{s3}}{(\omega-kv_\parallel-\omega_{cs})^2}dv_\parallel\right] \tag{3.142}$$

Finally χ_{3xx} becomes

$$\overleftrightarrow{\chi_3}_{xx} = \frac{{\omega_p}^2 m_s \pi}{2n_0\omega^2\hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right]$$
$$+\frac{{\omega_p}^2 m_s \pi}{2n_0\omega^2\hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel + \int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel - \omega_{cs})^2}dv_\parallel\right] \tag{3.143}$$

3.11.2 Calculations for XY component

$$\overleftrightarrow{\chi_3}_{xy} = \frac{{\omega_p}^2 m_s}{in_0\omega^2\hbar}\int_0^{\infty} e^{i(\omega - kv_\parallel)\tau}d\tau\int_0^{\infty}{v_\perp}^3 dv_\perp\int_{-\infty}^{\infty}dv_\parallel\int_0^{2\pi}\cos\theta\sin(\omega_{cs}\tau + \theta)d\theta$$
$$\cdot\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \tag{3.144}$$

Theta Integration

$$\int_0^{2\pi}\cos\theta\sin(\omega_{cs}\tau + \theta)d\theta = \pi\sin(\omega_{cs}\tau) = \pi\frac{e^{i\omega_{cs}\tau} - e^{-i\omega_{cs}\tau}}{2i}$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^{\infty} e^{i(\omega - kv_\parallel + \omega_{cs})}d\tau - \int_0^{\infty} e^{i(\omega - kv_\parallel - \omega_{cs})}d\tau\right] = \frac{\pi}{2}\left[\frac{1}{\omega - kv_\parallel + \omega_{cs}} - \frac{1}{\omega - kv_\parallel - \omega_{cs}}\right]$$

Parallel and Perpendicular Integration

$$\overleftrightarrow{\chi_3}_{xy} = \frac{-i{\omega_p}^2 m_s \pi}{2n_0\omega^2\hbar}\int_0^\infty {v_\perp}^3 dv_\perp \int_{-\infty}^\infty dv_\parallel . \left[f_0\left(v_\parallel + \frac{\hbar k}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar k}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] . \left[\frac{1}{\omega - kv_\parallel + \omega_{cs}} - \frac{1}{\omega - kv_\parallel - \omega_{cs}}\right]$$

Simplifying the above equation by following the same procedure as we have done while solving XX component we get

$$\overleftrightarrow{\chi_3}_{xy} = -i\left[\frac{{\omega_p}^2 m_s\pi}{2n_0\omega^2\hbar}\left(\int_{-\infty}^\infty \frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \int_{-\infty}^\infty \frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^\infty \frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right]\right.$$
$$\left. + i\left[\frac{{\omega_p}^2 m_s\pi}{2n_0\omega^2\hbar}\left(\int_{-\infty}^\infty \frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \int_{-\infty}^\infty \frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^\infty \frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right)\right] \quad (3.145)$$

Calculations for XZ component

$$\overleftrightarrow{\chi_3}_{xz} = \frac{{\omega_p}^2 m_s}{in_0\omega\hbar}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi}\cos\theta d\theta . \left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]$$
$$+ \frac{{\omega_p}^2 m_s}{in_0\omega^2}\int_0^\infty e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^\infty {v_\perp}^2 dv_\perp \int_{-\infty}^\infty \left(\frac{\omega - kv_\parallel}{k^2}\right)\vec{k}dv_\parallel \int_0^{2\pi}\cos\theta d\theta . \left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \quad (3.146)$$

Theta integration

$$\int_0^{2\pi}\cos\theta d\theta = \sin(\theta)\Big|_0^{2\pi} = 0 \quad (3.147)$$

As the theta integration is zero so χ_{3xz} component gets vanish

$$\overleftrightarrow{\chi_1}_{xz} = 0$$

3.11.3 Calculations for YY component

$$\overleftrightarrow{\chi_3}_{yy} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty {v_\perp}^3 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} \sin\theta \sin(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m_s}\right) - \frac{\hbar k}{m} \frac{\partial f_0}{\partial v_\parallel} \right] \tag{3.148}$$

Theta Integration

$$\int_0^{2\pi} \sin\theta \sin(\omega_{cs}\tau + \theta) d\theta = \pi \cos(\omega_{cs}\tau) = \pi \frac{e^{i\omega_{cs}\tau} + e^{-i\omega_{cs}\tau}}{2}$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^\infty e^{i(\omega - k v_\parallel + \omega_{cs})} d\tau + \int_0^\infty e^{i(\omega - k v_\parallel - \omega_{cs})} d\tau \right] = \frac{\pi}{2}\left[\frac{i}{\omega - k v_\parallel + \omega_{cs}} + \frac{i}{\omega - k v_\parallel - \omega_{cs}} \right]$$

From the above integration it has been observed that the expressions which differentiate the above component from the XX component is theta and tau integration. The theta and tau integrations are solved above it has been seen that the results are same as we obtained in solving the theta and tau integration of XX component so here we can write

$$\overleftrightarrow{\chi_3}_{yy} = \overleftrightarrow{\chi_3}_{xx} \tag{3.149}$$

$$\overleftrightarrow{\chi_3}_{yy} = \frac{{\omega_p}^2 m_s \pi}{2 n_0 \omega^2 \hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right]$$
$$+\frac{{\omega_p}^2 m_s \pi}{2 n_0 \omega^2 \hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel - \int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right] \tag{3.150}$$

Calculations for YX component

$$\overleftrightarrow{\chi_3}_{yx} = \frac{{\omega_p}^2 m_s}{i n_0 \omega^2 \hbar}\int_0^{\infty} e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^{\infty} {v_\perp}^3 dv_\perp \int_{-\infty}^{\infty} dv_\parallel \int_0^{2\pi}\sin\theta\cos(\omega_{cs}\tau + \theta)d\theta$$
$$\cdot\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \tag{3.151}$$

Theta Integration

$$\int_0^{2\pi}\sin\theta\cos(\omega_{cs}\tau + \theta)d\theta = -\pi\sin(\omega_{cs}\tau) = -\pi\left(\frac{e^{i\omega_{cs}\tau} - e^{-i\omega_{cs}\tau}}{2i}\right)$$

Tau Integration

Combining the results for theta integration with that of tau integral it gives

$$\frac{\pi}{2}\left[\int_0^{\infty} e^{i(\omega - kv_\parallel + \omega_{cs})}d\tau - \int_0^{\infty} e^{i(\omega - kv_\parallel - \omega_{cs})}d\tau\right] = \frac{-\pi}{2}\left[\frac{1}{\omega - kv_\parallel + \omega_{cs}} - \frac{1}{\omega - kv_\parallel - \omega_{cs}}\right]$$

Parallel and Perpendicular Integration

$$\overleftrightarrow{\chi_3}_{yx} = \frac{i{\omega_p}^2 m_s \pi}{2 n_0 \omega^2 \hbar}\int_0^{\infty} {v_\perp}^3 dv_\perp \int_{-\infty}^{\infty} dv_\parallel . \left[f_0\left(v_\parallel + \frac{\hbar k}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar k}{2m_s}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]$$
$$\cdot\left[\frac{1}{\omega - kv_\parallel + \omega_{cs}} - \frac{1}{\omega - kv_\parallel - \omega_{cs}}\right]$$

As the terms in the parenthesis are same as terms in XX component of χ_3. So we use Eq. (3.141) and Eq. (3.141) here and finally get

$$\overleftrightarrow{\chi_3}_{yx} = i\left[\frac{{\omega_p}^2 m_s \pi}{2n_0\omega^2\hbar}\left(\int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right]\right.$$
$$\left. -i\left[\frac{{\omega_p}^2 m_s \pi}{2n_0\omega^2\hbar}\left(\int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel - \int_{-\infty}^{\infty}\frac{G_{s3}}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel - \omega_{cs})^2}dv_\parallel\right)\right]\right] \tag{3.152}$$

3.11.4 Calculations for YZ component

$$\begin{aligned}
\overleftrightarrow{\chi_3}_{yz} = {} & \frac{{\omega_p}^2 m_s}{in_0\omega\hbar}\int_0^{\infty} e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^{\infty} {v_\perp}^2 dv_\perp \int_{-\infty}^{\infty} v_\parallel dv_\parallel \int_0^{2\pi}\sin\theta d\theta \\
& \cdot\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right] \\
& + \frac{{\omega_p}^2 m_s}{in_0\omega^2\hbar}\int_0^{\infty} e^{i(\omega - kv_\parallel)\tau}d\tau \int_0^{\infty} {v_\perp}^2 dv_\perp \int_{-\infty}^{\infty}\left(\frac{\omega - kv_\parallel}{k^2}\right)\vec{k}dv_\parallel \int_0^{2\pi}\sin\theta d\theta \\
& \cdot\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]
\end{aligned} \tag{3.153}$$

In the above expression theta integration gives the zero result which due to which the the whole expression gets vanish

Theta integration

$$\int_0^{2\pi}\sin\theta d\theta = -\cos(\theta)\Big|_0^{2\pi} = 0 \tag{3.154}$$

$$\overleftrightarrow{\chi_3}_{yz} = 0$$

3.11.5 Calculations for ZX component

$$\overleftrightarrow{\chi_3}_{zx} = \frac{{\omega_p}^2 m_s}{i n_0 \omega \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi} \cos(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m} \frac{\partial f_0}{\partial v_\parallel} \right] \tag{3.155}$$

Theta Integration

$$\int_0^{2\pi} \cos(\omega_{cs}\tau + \theta) d\theta = \sin(\omega_{cs}\tau + \theta)\Big|_0^{2\pi} \tag{3.156}$$
$$\overleftrightarrow{\chi_3}_{zx} = 0$$

3.11.6 Calculations for ZY

$$\overleftrightarrow{\chi_3}_{zx} = \frac{{\omega_p}^2 m_s}{i n_0 \omega \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi} \sin(\omega_{cs}\tau + \theta) d\theta$$
$$\cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m} \frac{\partial f_0}{\partial v_\parallel} \right] \tag{3.157}$$

Theta Integration

$$\int_0^{2\pi} \sin(\omega_{cs}\tau + \theta) d\theta = -\cos(\omega_{cs}\tau + \theta)\Big|_0^{2\pi} = 0 \tag{3.158}$$
$$\overleftrightarrow{\chi_3}_{zy} = 0$$

3.11.7 Calculations for ZZ component

$$\overleftrightarrow{\chi_3}_{zz} = \frac{{\omega_p}^2 m_s}{i n_0 k \omega \hbar} \int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty v_\parallel dv_\parallel \int_0^{2\pi} d\theta \cdot \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel} \right]$$

Theta Integration

$$\int_0^{2\pi} d\theta = 2\pi$$

Tau Integration

$$\int_0^\infty e^{i(\omega - k v_\parallel)\tau} d\tau = \frac{1}{-i(\omega - k v_\parallel)}$$

Simplification for Parallel Integration

$$\int_{-\infty}^\infty \frac{v_\parallel}{(\omega - k v_\parallel)} \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) + f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel} \right] dv_\parallel = \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) + f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) \right] \int_{-\infty}^\infty \frac{v_\parallel}{(\omega - k v_\parallel)} dv_\parallel$$

$$\frac{v_\parallel}{(\omega - k v_\parallel)} = \frac{k v_\parallel}{k(\omega - k v_\parallel)} = \frac{-k v_\parallel}{-k(\omega - k v_\parallel)} = \frac{\omega - k v_\parallel - \omega}{-k(\omega - k v_\parallel)}$$

The parallel integration gives here

$$\int_{-\infty}^\infty \frac{v_\parallel}{(\omega - k v_\parallel)} \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) + f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) \right] dv_\parallel = \frac{\omega}{k} \int_{-\infty}^\infty \left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) \right] dv_\parallel + \frac{\omega}{k} \int_{-\infty}^\infty \frac{\left[f_0\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right) \right]}{\omega - k v_\parallel} dv_\parallel \tag{3.159}$$

Similarly the parallel integration for the term given below give

$$\frac{\hbar k}{m}\int_{-\infty}^{\infty}\left(\frac{v_\parallel}{\omega - kv_\parallel}\frac{\partial f_0}{\partial v_\parallel}\right)dv_\parallel = \frac{v_\parallel}{(\omega - kv_\parallel)}\int_{-\infty}^{\infty}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel - \int\int_{-\infty}^{\infty}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel\left(\frac{d}{dv_\parallel}\frac{v_\parallel}{(\omega - kv_\parallel)}\right)$$

$$= \frac{v_\parallel}{(\omega - kv_\parallel)}\left|(f_0)\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f_0\left(\frac{(\omega - kv_\parallel)\frac{dv_\parallel}{dv_\parallel} - v_\parallel\frac{d}{dv_\parallel}(\omega - kv_\parallel)}{(\omega - kv_\parallel)^2}\right)$$

$$\frac{\hbar k}{m}\int_{-\infty}^{\infty}\frac{v_\parallel}{(\omega - kv_\parallel)}\frac{\partial f_0}{\partial v_\parallel}dv_\parallel = -\frac{\hbar k^2}{km}\int_{-\infty}^{\infty} f_0\frac{\omega}{(\omega - kv_\parallel)^2}dv_\parallel \tag{3.160}$$

Now combining Eq.(3.159) and Eq. (3.160)

$$\int_{-\infty}^{\infty}\frac{v_\parallel}{(\omega - kv_\parallel)}\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) + f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right) - \frac{\hbar k}{m}\frac{\partial f_0}{\partial v_\parallel}\right]dv_\parallel =$$

$$\frac{\omega}{k}\left[\int_{-\infty}^{\infty}\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)\right]dv_\parallel + \int_{-\infty}^{\infty}\frac{\left[f_0\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - f_0\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)\right]}{\omega - kv_\parallel}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{f_0}{(\omega - kv_\parallel)^2}dv_\parallel\right] \tag{3.161}$$

now combining the parallel integration results the perpendicular integration we get

$$\frac{\omega}{k}\left[\int_{-\infty}^{\infty} G_{s1}\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right)dv_\parallel - \int_{-\infty}^{\infty} G_{s1}\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)dv_\parallel + \int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - G_{s1}\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)}{\omega - kv_\parallel}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s1}(v_\parallel)}{(\omega - kv_\parallel)^2}dv_\parallel\right]$$

$$\overleftrightarrow{\chi_3}_{zz} = \frac{2\pi {\omega_p}^2 m_s}{k^2\hbar}\Bigg[\int_{-\infty}^{\infty} G_{s1}\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right)dv_\parallel - \int_{-\infty}^{\infty} G_{s1}\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)dv_\parallel$$

$$+\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - G_{s1}\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)}{\omega - kv_\parallel}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s1}(v_\parallel)}{(\omega - kv_\parallel)^2}dv_\parallel\Bigg]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.162}$$

By adding all the components, we get χ_3

$$\overleftrightarrow{\chi_3} = \frac{{\omega_p}^2 m_s \pi}{2n_0\omega^2\hbar}\Bigg[\int_{-\infty}^{\infty}\frac{G_{s3}\left(v_\parallel + \frac{\hbar k}{2m}\right) - G_{s3}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel$$

$$+\frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\Bigg]\begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{{\omega_p}^2 m_s \pi}{2n_0\omega^2\hbar}\Bigg[\int_{-\infty}^{\infty}\frac{G_{s3}\left(v_\parallel + \frac{\hbar k}{2m}\right) - G_{s3}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel$$

$$+\frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}(v_\parallel)}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\Bigg]\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2\pi {\omega_p}^2 m_s}{k^2\hbar}\Bigg[\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - G_{s1}\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)}{\omega - kv_\parallel}dv_\parallel$$

$$+\frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s1}(v_\parallel)}{(\omega - kv_\parallel)^2}dv_\parallel\Bigg]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.163}$$

Now Combining the results for χ by adding χ_1 , χ_2 and χ_3. it gives

$$\chi = \left[\frac{-{\omega_p}^2}{2\omega^2} + \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}(v_\parallel)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}(v_\parallel)}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right]\begin{bmatrix}1 & i & 0\\ -i & 1 & 0\\ 0 & 0 & 0\end{bmatrix} + \left[\frac{{\omega_p}^2}{2\omega^2}\right.$$

$$\left. - \frac{\pi{\omega_p}^2\omega_{cs}}{\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}(v_\parallel)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel - \frac{k^2\pi{\omega_p}^2}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s3}(v_\parallel)}{(\omega - kv_\parallel - \omega_{cs})^2}dv_\parallel\right]\begin{bmatrix}1 & i & 0\\ -i & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$+ \left(- 2\pi{\omega_p}^2\int_{-\infty}^{\infty}\frac{G_{s1}(v_\parallel)}{(\omega - kv_\parallel)^2}\right)\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$+ \left[\frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\right]\begin{bmatrix}1 & -i & 0\\ i & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$+ \left[- \frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right) - 2G_{s1}(v_\parallel)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel\right]\begin{bmatrix}1 & i & 0\\ -i & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$+\frac{{\omega_p}^2m_s\pi}{2\omega^2\hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s3}\left(v_\parallel + \frac{\hbar k}{2m}\right) - G_{s3}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right]\begin{bmatrix}1 & -i & 0\\ i & 1 & 0\\ 0 & 0 & 1\end{bmatrix} +$$

$$\frac{{\omega_p}^2m_s\pi}{2n_0\omega^2\hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s3}\left(v_\parallel + \frac{\hbar k}{2m}\right) - G_{s3}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s3}(v_\parallel)}{(\omega - kv_\parallel + \omega_{cs})^2}dv_\parallel\right]\begin{bmatrix}1 & i & 0\\ -i & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$+\frac{2\pi{\omega_p}^2m_s}{k^2\hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - G_{s1}\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)}{\omega - kv_\parallel}dv_\parallel + \frac{\hbar k^2}{m}\int_{-\infty}^{\infty}\frac{G_{s1}(v_\parallel)}{(\omega - kv_\parallel)^2}dv_\parallel\right]\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \quad (3.164)$$

The above equation gets modified as follow

$$\overleftrightarrow{\chi} = \Bigg[\frac{-{\omega_p}^2}{2\omega^2} - \frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\Big(v_{\parallel}+\frac{\hbar k}{2m}\Big)+G_{s1}\Big(v_{\parallel}-\frac{\hbar k}{2m}\Big)}{\omega-kv_{\parallel}-\omega_{cs}}dv_{\parallel}$$
$$+\frac{{\omega_p}^2 m_s\pi}{2\omega^2\hbar}\Bigg[\int_{-\infty}^{\infty}\frac{G_{s3}\Big(v_{\parallel}+\frac{\hbar k}{2m}\Big)-G_{s3}\Big(v_{\parallel}-\frac{\hbar k}{2m}\Big)}{\omega-kv_{\parallel}-\omega_{cs}}dv_{\parallel}\Bigg]\overleftrightarrow{U}$$
$$+\Bigg[\frac{-{\omega_p}^2}{2\omega^2} + \frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\Big(v_{\parallel}+\frac{\hbar k}{2m}\Big)+G_{s1}\Big(v_{\parallel}-\frac{\hbar k}{2m}\Big)}{\omega-kv_{\parallel}+\omega_{cs}}dv_{\parallel}$$
$$+\frac{{\omega_p}^2 m_s\pi}{2\omega^2\hbar}\Bigg(\int_{-\infty}^{\infty}\frac{G_{s3}\Big(v_{\parallel}+\frac{\hbar k}{2m}\Big)-G_{s3}\Big(v_{\parallel}-\frac{\hbar k}{2m}\Big)}{\omega-kv_{\parallel}+\omega_{cs}}dv_{\parallel}\Bigg)\Bigg]\overleftrightarrow{U^*}$$
$$+\frac{2\pi{\omega_p}^2 m_s}{k^2\hbar}\Bigg[\int_{-\infty}^{\infty}\frac{G_{s1}\Big(v_{\parallel}+\frac{\hbar\vec{k}}{2m}\Big)-G_{s1}\Big(v_{\parallel}-\frac{\hbar\vec{k}}{2m}\Big)}{\omega-kv_{\parallel}}dv_{\parallel}\Bigg]\overleftrightarrow{T} \tag{3.165}$$

The Eq. (3.165) is a major equation to proceed further for the dispersion relation of parallel propagating waves it contains some terms in which there is no quantum contribution while some terms contain a factor of $\hbar$ [35, 36, 37]. The first term in the parenthesis corresponds to R wave while second term refers to L waves. while third term is a zz component which is used to see the dispersion in parallel propagating electrostatic waves.

$$\overleftrightarrow{\chi} = \chi_{\perp R}\overleftrightarrow{U} + \chi_{\perp L}\overleftrightarrow{U*} + \chi_{\parallel}\overleftrightarrow{T}$$

Where $\chi_{\perp R}$ is susceptibility for R wave and $\chi_{\perp L}$ is susceptibility for L wave while $\overleftrightarrow{U}$, $\overleftrightarrow{U^*}$ and $\overleftrightarrow{T}$ are the matrices

$$\chi_{\perp R} = \left[\frac{-\omega_p{}^2}{2\omega^2} - \frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel\right.$$
$$\left. + \frac{\omega_p{}^2 m_s \pi}{2\omega^2\hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s3}\left(v_\parallel + \frac{\hbar k}{2m}\right) - G_{s3}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel - \omega_{cs}}dv_\parallel\right]\right], \quad (3.166)$$

$$\chi_{\perp L} = \left[\frac{-\omega_p{}^2}{2\omega^2} + \frac{\pi\omega_p^2\omega_{cs}}{2\omega^2}\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\right.$$
$$\left. + \frac{\omega_p{}^2 m_s \pi}{2\omega^2\hbar}\left(\int_{-\infty}^{\infty}\frac{G_{s3}\left(v_\parallel + \frac{\hbar k}{2m}\right) - G_{s3}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel + \omega_{cs}}dv_\parallel\right)\right], \quad (3.167)$$

and

$$\chi_\parallel = \frac{2\pi\omega_p{}^2 m_s}{k^2\hbar}\left[\int_{-\infty}^{\infty}\frac{G_{s1}\left(v_\parallel + \frac{\hbar\vec{k}}{2m}\right) - G_{s1}\left(v_\parallel - \frac{\hbar\vec{k}}{2m}\right)}{\omega - kv_\parallel}dv_\parallel\right] \quad (3.168)$$

Also We define $\vec{U}$ and $\vec{T}$ as follow

$$\vec{U} = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{U}* = \begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{T} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.12 Calculations for general dispersion relation of R and L in quantum kinetic theory

$$\epsilon_L = 1 - \frac{c^2k^2}{\omega^2} + \left[\frac{-{\omega_p}^2}{\omega^2} - \frac{\pi\omega_p^2\omega_{cs}}{\omega^2} \int_{-\infty}^{\infty} \frac{G_{s1}\left(v_\parallel + \frac{\hbar k}{2m}\right) + G_{s1}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel + \omega_{cs}} dv_\parallel \right.$$
$$\left. + \frac{{\omega_p}^2 m_s \pi}{\omega^2 \hbar} \left[\int_{-\infty}^{\infty} \frac{G_{s3}\left(v_\parallel + \frac{\hbar k}{2m}\right) - G_{s3}\left(v_\parallel - \frac{\hbar k}{2m}\right)}{\omega - kv_\parallel + \omega_{cs}} dv_\parallel \right] \right. \quad (3.169)$$

$$\epsilon_\parallel = 1 + \frac{2\pi {\omega_p}^2 m_s}{k^2 \hbar} \left[\int_{-\infty}^{\infty} \frac{G_{s1}\left(v_\parallel + \frac{\hbar \vec{k}}{2m}\right) - G_{s1}\left(v_\parallel - \frac{\hbar \vec{k}}{2m}\right)}{\omega - kv_\parallel} dv_\parallel \right] \quad (3.170)$$

3.13 Application of distribution

3.13.1 Zero temperature Fermi distribution

Because we're talking about quantum electron plasma, ion dynamics aren't taken into account. We know that electrons obey Pauli's exclusion principle since they are Fermions. The Fermi Distribution is used to determine the probability of an electron occupying a specific energy level. At zero temperature, where thermal velocities are substantially smaller than Fermi velocities, we utilise this distribution

$$f_{0e} = \frac{2m_e^3}{(2\pi\hbar)^3}, \tag{3.171}$$

3.13.2 Dispersion Relation for R wave using Fermi distribution at zero temperature

Making use of the distribution given by Eq. (3.171) in Eq. (3.169) we get the dispersion relation for R wave as

$$\begin{aligned}
&\omega^2 - c^2k^2 - \frac{3\omega_p^2}{8} - \frac{3\omega_p^2}{2}\frac{\omega_{cs}(\omega-\omega_{cs})}{k^2v_F^2} - \frac{9\omega_p^2}{8}\frac{(\omega-\omega_{cs})^2}{k^2v_F^2} - \frac{3\omega_p^2}{32}\left(\frac{\hbar k}{mv_F}\right)\\
&-\frac{3\omega_p^2}{8}\frac{\omega_{cs}}{kv_F}\left(\frac{\omega^2+(\frac{\hbar k^2}{2m})^2-k^2v_F^2-\omega_{cs}^2}{k^2v_F^2}\right)\left[ln\left(\frac{\omega+\frac{\hbar k^2}{2m}-kv_F-\omega_{cs}}{\omega^2+kv_F+\frac{\hbar k^2}{2m}-\omega_{cs}}\right)+ln\left(\frac{\omega-\frac{\hbar k^2}{2m}-kv_F-\omega_{cs}}{\omega^2-\frac{\hbar k^2}{2m}+kv_F-\omega_{cs}}\right)\right]\\
&-\frac{3\omega_p^2}{16}\left(\frac{mv_F}{\hbar k}\right)\left(\frac{\omega^2+(\frac{\hbar k^2}{2m})^2-k^2v_F^2-\omega_{cs}^2}{k^2v_F^2}\right)^2\left[ln\left(\frac{\omega+\frac{\hbar k^2}{2m}-kv_F-\omega_{cs}}{\omega^2+kv_F+\frac{\hbar k^2}{2m}-\omega_{cs}}\right)+ln\left(\frac{\omega-\frac{\hbar k^2}{2m}-kv_F-\omega_{cs}}{\omega^2-\frac{\hbar k^2}{2m}+kv_F-\omega_{cs}}\right)\right]
\end{aligned} \tag{3.172}$$

For L waves

The use of Fermi Dirac distribution given by Eq. (3.171) in Eq. (3.170) gives

$$\omega^2 - c^2k^2 - \frac{3\omega_p^2}{8} + \frac{3\omega_p^2}{2}\frac{\omega_{cs}(\omega+\omega_{cs})}{k^2v_F^2} - \frac{9\omega_p^2}{8}\frac{(\omega+\omega_{cs})^2}{k^2v_F^2} - \frac{3\omega_p^2}{32}\left(\frac{\hbar k}{mv_F}\right)$$
$$+\frac{3\omega_p^2}{8}\frac{\omega_{cs}}{kv_F}\left(\frac{\omega^2+(\frac{\hbar k^2}{2m})^2-k^2v_F^2+\omega_{cs}^2}{k^2v_F^2}\right)\left[ln\left(\frac{\omega+\frac{\hbar k^2}{2m}-kv_F+\omega_{cs}}{\omega^2+kv_F+\frac{\hbar k^2}{2m}+\omega_{cs}}\right)+ln\left(\frac{\omega-\frac{\hbar k^2}{2m}-kv_F+\omega_{cs}}{\omega^2-\frac{\hbar k^2}{2m}+kv_F+\omega_{cs}}\right)\right]$$
$$-\frac{3\omega_p^2}{16}\left(\frac{mv_F}{\hbar k}\right)\left(\frac{\omega^2+(\frac{\hbar k^2}{2m})^2-k^2v_F^2+\omega_{cs}^2}{k^2v_F^2}\right)^2\left[ln\left(\frac{\omega+\frac{\hbar k^2}{2m}-kv_F+\omega_{cs}}{\omega^2+kv_F+\frac{\hbar k^2}{2m}+\omega_{cs}}\right)+ln\left(\frac{\omega-\frac{\hbar k^2}{2m}-kv_F+\omega_{cs}}{\omega^2-\frac{\hbar k^2}{2m}+kv_F+\omega_{cs}}\right)\right]$$

(3.173)

Chapter 4

Results and discussion

By using classical kinetic theory the generalized dispersion relation for parallel propagating electromagnetic waves in an anisotropic environment has been studied in chap 2. The classical kinetic theory wasn't sufficient to deal with the quantum effects that appear at high density and low temperature. So a kinetic model is built to deal with these effects in a plasma. In chap 3, we have studied the Quantum kinetic model for electromagnetic waves in plasma. This model was built by using Wigner function in conjunction with Schrodinger equation which gave us three equations for susceptibility which contains classical as well as quantum effects. The quantum effects are accounted for the corrections given by $\frac{\hbar k^2}{2m}$. These corrections actually refer to the shift of velocities in quantum level. Further, we derived the dispersion relation for R and L wave using Fermi Dirac distribution at zero temperature. At this temperature the thermal effects play no role. Our main purpose is to compare the classical results for R and L wave with those obtained using quantum kinetic model.

4.1 Numerical analysis

In numerical analysis we present the values used for various parameters in the dispersion relation. The quantum effects become prominent at densities $n > 10^{30}$, so we choose density $n = 2 \times 10^{35}$ and the corresponding magnetic field at such a high density (White Dwarf) is about 10^{12} Tesla. The Fermi velocity v_F is taken to be 0.7c and $\frac{\hbar\omega_p^2}{mc^2} = 0.067$ and the ratio of electron's cyclotron frequency to plasma frequency $\frac{\omega_c}{\omega_p}$ is taken to be 0.75. These values are chosen in such a way that they preserve the quantum effects as well they must satisfy the the conditions for quasi neutrality and collective behaviour in a quantum plasma.

For the collective behaviour to exist the wavelength should be greater than inter particle distance i.e $\lambda >> n_0^{-\frac{1}{3}}$ where λ is the wavelength and n_0 represents the density. In present case, this condition can be satisfied by taking $\frac{kc}{\omega_p} << 43.6$. For the quasi neutrality to exist, it is necessary that the wavelength in collective motion should be larger than the Debye length given by $\frac{\omega_p}{v_F}$. So in our case this is justified by taking $\frac{kc}{\omega_p} << \frac{c}{v_F}$, where $\frac{c}{v_F} = 1.4$. So we will have to take this range for k in order to preserve quasi neutrality. Now we will discuss the results for R and L wave, first for classical regime and then for quantum regime for the same values of the parameters as mentioned above and will compare both of the results. Since we have derived the dispersion relation for R and L wave using quantum kinetic model, we cannot conclude anything about quantum correction without comparing it with the classical results. So we take classical results as reference and then make a comparison with the quantum results.

4.2 Classical R wave

The standard dispersion relation for classical R wave is given as

$$\frac{c^2k^2}{\omega^2} = \frac{\frac{\omega_p^2}{\omega^2}}{1 - \frac{\omega_c}{\omega}}.$$

We plot this dispersion relation for $\frac{\omega_c}{\omega_p} = 0.75$ by choosing the coordinates in such a way that we take $\frac{\omega}{\omega_p}$ on y axis and $\frac{ck}{\omega_p}$ on x axis

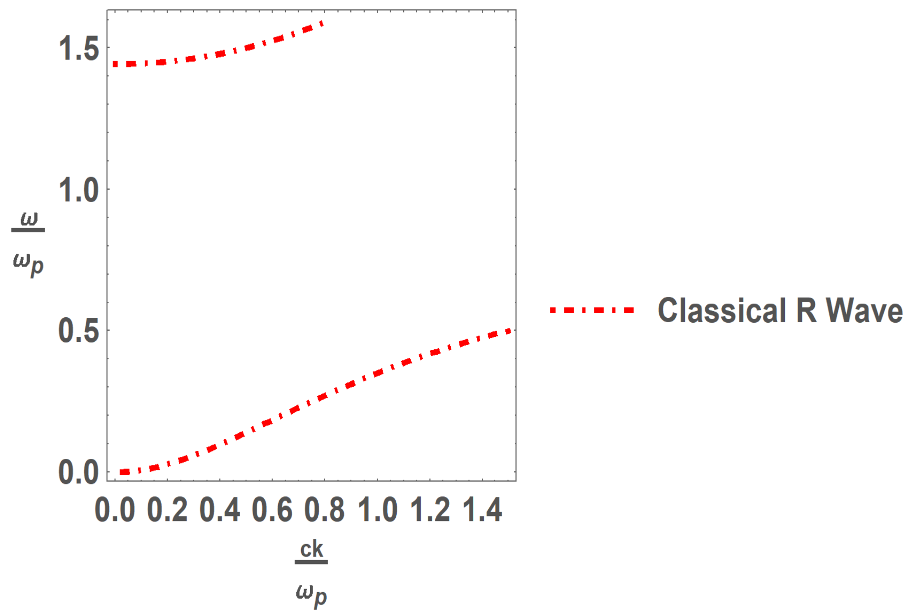

Figure 4.1: Classical R wave plot

4.3 Quantum R wave

The dispersion relation for quantum R wave is given by Eq. (3.180). We normalize this dispersion relation and plot $\frac{\omega}{\omega_p}$ versus $\frac{ck}{\omega_p}$. The density is chosen to be $n_0 = 2 \times 10^{35}\ m^{-3}$, the ratio of cyclotron frequency and plasma frequency is $\frac{\omega_c}{\omega_p} = 0.75$ and $\frac{\hbar\omega_p}{mc^2} = 0.067$.

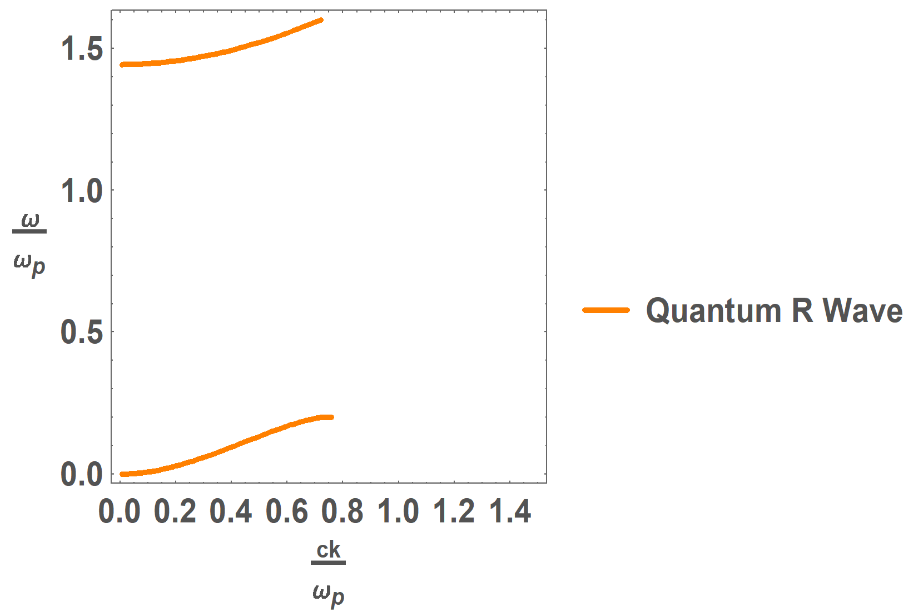

Figure 4.2: Quantum R wave plot

4.4 Comparison between classical R wave and quantum R wave

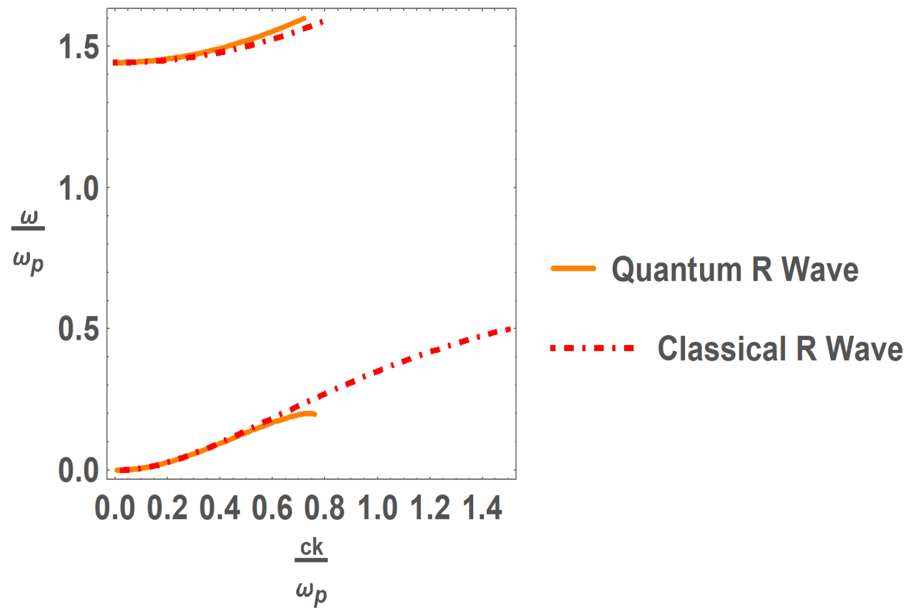

Figure 4.3: Comparison between Classical R wave and Quantum R wave

In this plot classical R waves are represented by purple color and the green color represents the quantum R wave. It has been observed that for the upper branch, group velocity of quantum R wave is larger than the classical R wave. For the lower branch, the quantum R wave has smaller group velocity as compared to classical R wave. We conclude that for larger wavelength (smaller k) the classical and quantum R wave have same curves whereas for shorter wavelength (larger k) quantum effects are observed.

4.5 Classical L wave

The dispersion relation for classical L wave is given as

$$\frac{c^2k^2}{\omega^2} = \frac{\frac{\omega_p^2}{\omega^2}}{1+\frac{\omega_c}{\omega}}.$$

This dispersion relation is plotted for $\frac{\omega_c}{\omega_p} = 0.75$

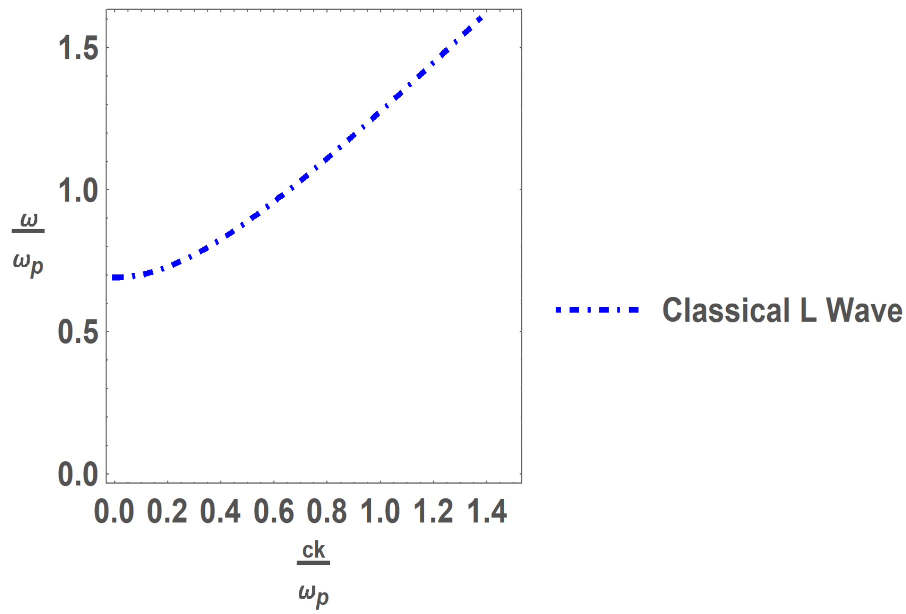

Figure 4.4: Classical L wave

4.6 Quantum L wave

The dispersion relation for L wave given by Eq. (3.181) is plotted in Fig. 4.5 for the same values as we have used for R wave.

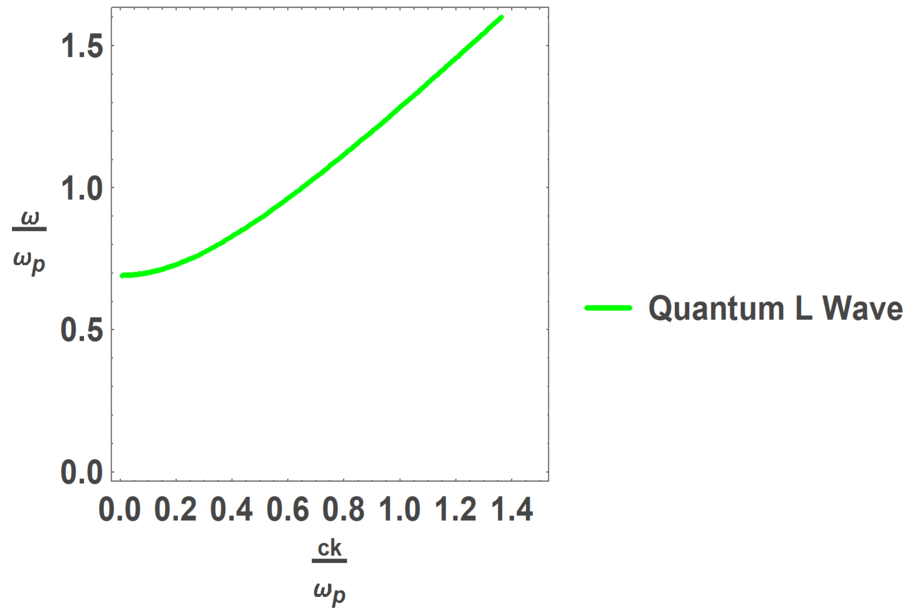

Figure 4.5: Quantum L wave

4.7 Comparison between Classical L wave and Quantum L wave

A comparison between classical L wave and quantum L wave is presented in Fig. 4.6 where classical L wave is represented by dotted line (Black) and Quantum L wave is represented by solid line (orange). It can be seen that for longer wavelengths

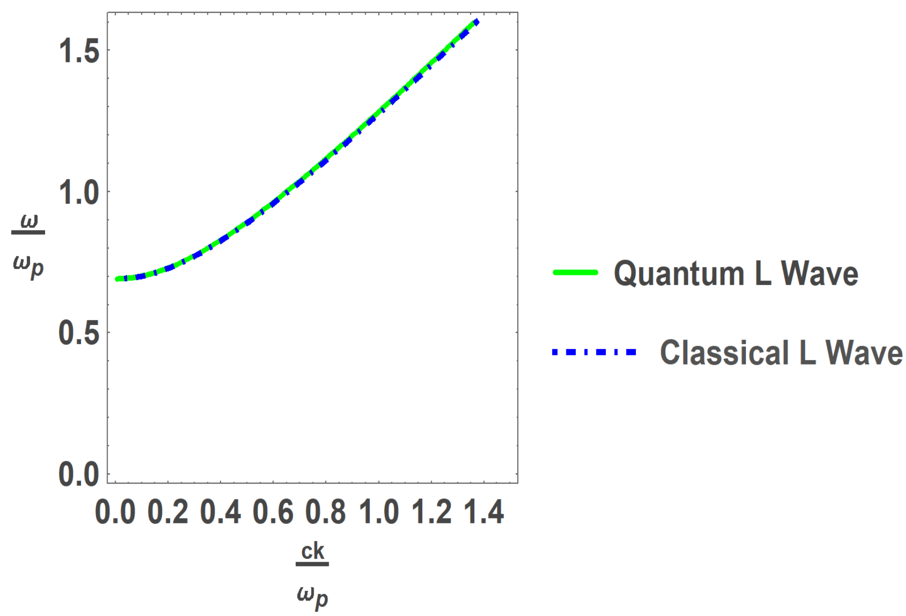

Figure 4.6: Comparison between classical L wave and quantum L wave

classical and quantum L waves are same but at shorter wavelength there are minor corrections which indicate the higher group velocity of quantum L wave as compared to classical L wave

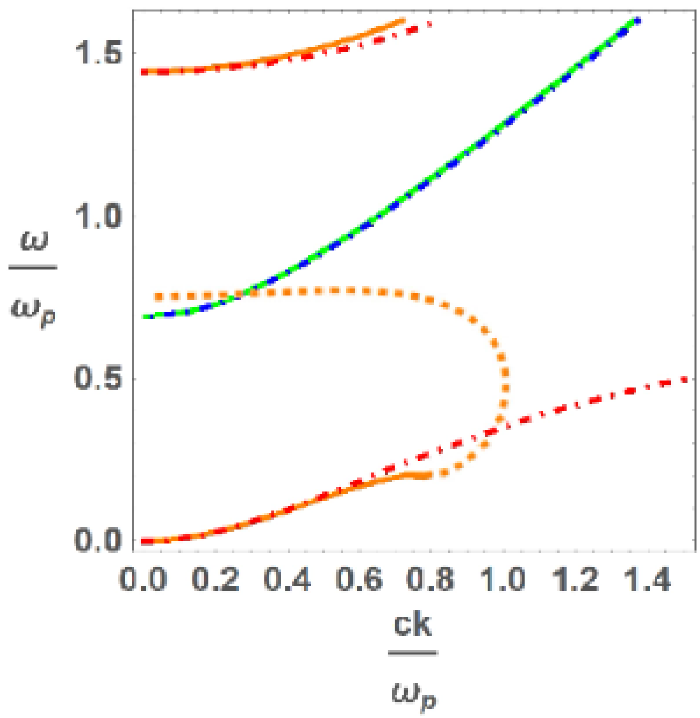

Figure 4.7: Comparison for Classical and Quantum R and L waves

4.8 Conclusion

We observe minor corrections for the upper branch of R wave which is interpreted in terms of slightly higher group than classical R wave. For upper branch the quantum mechanical group velocity is faster than classical group velocity whereas for lower branch of R wave a region of anomalous dispersion is observed at shorter wavelength. On the other hand for L wave we observe that there are very small corrections as compared to R wave. So we conclude that quantum mechanical effects are not prominent for L-wave. However, for R wave we observe significantaly important quantum mechanical effects which restrict the propagation of R-wave in shorter wave length region and an anomalous dispersion is observed.

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