Generalized Mellin Transform

by

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MS THESIS WORK

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to

My Beloved Parents

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Abstract

Many generalization of fractional operator have been introduced. In this thesis we study one of these generalization called ξ-fractional operator. We introduce the generalized Mellin transform called ξ-Mellin transform for the generalized fractional operators. We discuss some properties of ξ-Mellin transform. ξ-Mellin transform of various fractional integral and derivative have been established.

Contents

Chapter 1

Introduction to fractional calculus

In the start of the chapter we review some spacial functions that will be helpful in upcoming chapters. Next we discuss literature about fractional calculus. Some important fractional operators Riemann-Liouville (RL), Caputo, Hadamard, Hadamard-type and Weyl are studied. Also we discuss some important properties of these operator. After this we review the generalization of these fractional operator called ξ-fractional operators and their properties.

1.0.1 Euler's gamma function

Euler's gamma function was introduced by Leonhard Euler to generalize the factorial (the product of a positive integer less than or equal to a given positive integer) to non integer values. It plays an essential role in field of mathematics and was also studied by Legendre, Gauss, Gudermann, Liouville and many others. It belongs to the category of special transcendental functions. It also appears in various areas of mathematics such as integration, number theory, hyper-geometric series etc.

Definition 1.0.1. *The Euler's gamma function* $\Gamma :]0, \infty[\to \Re$ *is defined as*

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \exp(-u) du, \qquad \alpha > 0, \tag{1.1}$$

where u *is a dummy variable.*

Here are some properties of Gamma function such as

$$\Gamma(1) = 1,$$
$$\Gamma(1+\alpha) = \alpha\Gamma(\alpha).$$

We can identify gamma function for negative values of α as

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}, \quad \alpha > -1, \ \alpha \neq 0.$$

For different values of α we get $\Gamma(2) = 1!, \Gamma(3) = 2!, \Gamma(4) = 3!, \cdots \Gamma(\alpha) = (\alpha-1)!$ which shows that gamma function is the generalization of factorial function.

1.0.2 Beta function

Now we discuss the spacial type of function related to gamma function which occurs in computation of many definite integrals. The Beta function $B(m_1, m_2)$ was developed by Legendre, Whittaker and Watson in 1990. Beta function is known as the Eulerian integral of first kind and it can be defined as:

$$B(m_1, m_2) = \int_0^1 u^{m_1-1}(1-u)^{m_2-1} du. \tag{1.2}$$

Relation of gamma function with beta function for positive m_1, m_2 is

$$B(m_1, m_2) = \frac{\Gamma(m_1)\Gamma(m_2)}{\Gamma(m_1+m_2)}. \tag{1.3}$$

1.0.3 Mittag-Leffler function

In 1903, first time classical Mittag-Leffler function $E_\alpha(u)$ was introduced by Magnus Gosta Mittag-Leffler in the form of special function as

$$E_\alpha(u) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma(\alpha m+1)}, \qquad \alpha \in \mathbb{C}, \text{ such that } \operatorname{Re}(\alpha) > 0. \tag{1.4}$$

After that Wiman [3] introduced the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(u)$ as

$$E_{\alpha,\beta}(u) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma(\alpha m+\beta)}, \qquad \alpha \in \mathbb{C}, \text{ such that } \operatorname{Re}(\alpha) > 0. \tag{1.5}$$

1.1 Fractional operators

In this section we discuss some important definitions and properties of fractional operators. First useful definition of fractional integral and derivative was introduced by Riemann and Liouville. After that many definitions of fractional integral and derivative were introduced by different mathematician [3]. For instant Caputo and Hadamard operators are more papular in literature.

Lemma 1.1.1. *Let $f : [a, b] \to \Re$ be a continous function and let $F : [a, b] \to \Re$, be defined by*

$$F(u) = \int_a^u f(s)ds.$$

Then, F is differentiable and

$$F' = f. \tag{1.6}$$

Notations

(a) We denote differential operator $\frac{d}{du}$ by D that maps a differentiable function on to its derivative, i.e

$$Df(u) = \frac{df}{du} = f'(u).$$

(b) We denote integral operator by I_a that maps a integrable function f defined on $[a, b]$, onto its primitive centered at a, i.e.

$$I_a f(u) = \int_a^u f(s)ds.$$

(c) For nth order differential and integral operator we use the symbols D^n and I_a^n respectively, where $n \in \mathbb{N}$. i.e. $D^n := DD^{n-1}$ and $I_a^n := I_a I_a^{n-1}$ for $n \geq 2$. Eq (1.6) can be written in our notation as

$$DI_a f = f. \tag{1.7}$$

Similarly,

$$D^2 I_a^2 = D(DI_a)I_a f, \qquad \because D^{m+n} = D^m D^n.$$

Making use of (1.7), we have

$$D^n I_a^n f = f, \qquad \text{for} \;\; n \in \mathbb{N}.$$

Lemma 1.1.2. *For $u \in [a, b]$ and $m \in \mathbb{N}$, we have a Riemann integrable function f defined on $[a, b]$*

$$I_a^m f(u) = \frac{1}{(m-1)!} \int_a^u (u-s)^{m-1} f(s) ds.$$

Proof. We would write

$$I_a f(u) = \int_a^u f(s) ds,$$

and the 2nd iterate becomes

$$I_a^2 f(u) = \int_a^u \int_a^{s_2} f(s_1) ds_1 ds_2.$$

By interchanging the order of integration, we get

$$I_a^2 f(u) = \int_a^u \int_{s_1}^u f(s_1) ds_2 ds_1.$$

Because $f(s_1)$ is independent of s_2, therefore we get

$$I_a^2 f(u) = \int_a^u f(s_1) \int_{s_1}^u ds_2 ds_1 = \int_a^u (u-s) f(s) ds.$$

Similarly, we can prove that

$$I_a^3 f(u) = \frac{1}{2} \int_a^u (u-s)^2 f(s) ds,$$

and by integrating m-times, we get

$$I_a^m f(u) = \frac{1}{(m-1)!} \int_a^u (u-s)^{m-1} f(s) ds.$$

Now replacing m by $-m$, we get

$$_aD_u^m f(u) = I_a^{-m} f(x) = \frac{1}{\Gamma(-m)} \int_a^u \frac{f(s)}{(u-s)^{m+1}} ds.$$

Above expression is the definition of fractional derivative. But for $m > -1$ this is an improper integral for $(s-u) \to 0$ and integral diverges for $m \geq 0$. And the improper integral converges for $m \in (-1, 0)$. Integral also converges for negitive values of m. Hence it is fractional integral. If b be the lower limit then we can write the above expression as,

$$D_b^m fu = \frac{1}{\Gamma(-m)} \int_u^b \frac{f(s)}{(u-s)^{m+1}} ds.$$

□

Lemma 1.1.3. *Let $f \in C^\beta[a, b]$ with $\alpha, \beta \in \mathbb{N}$ s.t $\alpha > \beta$. Then*

$$D^\beta f = D^\alpha I_a^{\alpha-\beta} f. \tag{1.8}$$

Proof. Since

$$DI_a f(u) = f(u),$$

it implies

$$D^{\alpha-\beta} I_a^{\alpha-\beta} f(u) = f(u).$$

Appling operator D, β-times on both sides, we get

$$D^\beta D^{\alpha-\beta} I_a^{\alpha-\beta} f(u) = D^\beta f(u)$$

$$D^{\alpha-\beta+\beta} I_a^{\alpha-\beta} f(u) = D^\beta f(x), \qquad \because \quad D^{\alpha+\beta} = D^\alpha D^\beta$$

$$D^\alpha I_a^{\alpha-\beta} f(u) = D^\beta f(u).$$

If β is replaced by any $m > 0$, relation (1.8) is valid for particular class of functions unless $\alpha - \beta > 0$. And this leads to the definition of RL-differential operator. □

1.1.1 Riemann-Liouville (RL) fractional integral

Definition 1.1.4. *An integrable function f defined on $L_1[a, b]$. For $\alpha > 0$, the left/right-sided RL-fractional integral of a function f are defined as*

$${}^{RL}I_{a+}^\alpha f(u) = \frac{1}{\Gamma(\alpha)} \int_a^u (u-s)^{\alpha-1} f(s) ds, \tag{1.9}$$

$${}^{RL}I_{b-}^\alpha f(u) = \frac{1}{\Gamma(\alpha)} \int_u^b (s-u)^{\alpha-1} f(s) ds. \tag{1.10}$$

Theorem 1.1.1. *Let $\alpha_1, \alpha_2 \geq 0$ and $\phi \in L_1[a,b]$. Then, almost every where*

$$I_a^{\alpha_1} I_a^{\alpha_2} \phi = I_a^{\alpha_1+\alpha_2} \phi, \tag{1.11}$$

hold on $[a,b]$. If $\phi \in C[a,b]$ or $\alpha_1 + \alpha_2 \geq 1$, then (1.11) holds every where on $[a,b]$.

Proof. We have

$$I_a^{\alpha_1} I_a^{\alpha_2} \phi(u) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u (u-t)^{\alpha_1 - 1} \int_a^t (t-\tau)^{\alpha_2 - 1} \phi(\tau) d\tau dt.$$

By Fubin's theorem, we have

$$\begin{aligned} I_a^{\alpha_1} I_a^{\alpha_2} \phi(u) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u \int_\tau^u (u-t)^{\alpha_1 - 1} (t-\tau)^{\alpha_2 - 1} \phi(\tau) dt d\tau \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u \phi(\tau) \int_\tau^u (u-t)^{\alpha_1 - 1} (t-\tau)^{\alpha_2 - 1} dt d\tau. \end{aligned}$$

The substitution $t = \tau + s(u-\tau) \Rightarrow dt = ds(x-\tau)$ yields

$$\begin{aligned} I_a^{\alpha_1} I_a^{\alpha_2} \phi(u) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u \phi(\tau) \int_0^1 [(u-\tau)(1-s)]^{\alpha_1 - 1} [s(u-\tau)]^{\alpha_1 - 1} (u-\tau) ds d\tau \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^u \phi(\tau)(u-\tau)^{\alpha_1+\alpha_2-1} \int_0^1 (1-s)^{\alpha_1 - 1} s^{\alpha_2 - 1} ds d\tau. \end{aligned}$$

In view of (1.3), we get

$$\begin{aligned} I_a^{\alpha_1} I_a^{\alpha_2} \phi(u) &= \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^u \phi(\tau)(u-\tau)^{\alpha_1+\alpha_2-1} d\tau \\ &= I_a^{\alpha_1+\alpha_2} \phi(u). \end{aligned}$$

□

Corollary 1.1.5. *By the assumption of above theorem we have*

$$I_a^{\alpha_1} I_a^{\alpha_2} \phi = I_a^{\alpha_2} I_a^{\alpha_1} \phi. \tag{1.12}$$

Example. 1: Let $f(u) = (u-v)^\eta$ for some $\eta > -1$ and $m > 0$, then

$$I_c^m f(u) = \frac{\Gamma(\eta+1)}{\Gamma(m+\eta+1)} (u-c)^{m+\eta}. \tag{1.13}$$

Proof. From eq (1.21), we have

$$\begin{aligned}
I_c^m f(u) &= \frac{1}{\Gamma(m)} \int_c^u f(s)(u-s)^{m-1} ds \\
&= \frac{1}{\Gamma(m)} \int_c^u (s-c)^\eta (u-s)^{m-1} ds \\
&= \frac{1}{\Gamma(m)} \int_c^u (s-c)^\eta (u-c-s+c)^{m-1} ds \\
&= \frac{1}{\Gamma(m)} \int_c^u (s-c)^\eta (u-c)^{m-1} \left(1-\frac{s-c}{u-c}\right)^{m-1} ds.
\end{aligned}$$

Using change of variable $\dfrac{s-c}{u-c} = v$

$$\begin{aligned}
I_c^m f(u) &= \frac{1}{\Gamma(m)} \int_c^u (v(u-c))^\eta (u-c)^{m-1} (1-v)^{m-1} (u-c) dv \\
&= \frac{1}{\Gamma(m)} \int_c^u v^\eta (u-c)^{m+\eta-1+1} (1-v)^{m-1} dv \\
&= \frac{(u-c)^{m+\eta}}{\Gamma(m)} \int_c^u v^\eta (1-v)^{m-1} dv \\
&= \frac{(u-c)^{m+\eta}}{\Gamma(m)} . \frac{\Gamma(\eta+1)\Gamma(m)}{\Gamma(m+\eta+1)} \\
&= \frac{\Gamma(\eta+1)}{\Gamma(m+\eta+1)} (u-c)^{m+\eta}.
\end{aligned}$$

□

Example. 2: We compute fractional integral I_0^α of a function $f(u) = \exp(\eta u)$, where $\eta, u > 0$.

For $\alpha \in N$ we have $I_0^\alpha f(u) = \eta^{-\alpha} \exp(\eta u)$ with $\eta > 0$. For $j \notin \mathbb{N}$ we use the series of exponantial function

$$\begin{aligned}
I_0^\alpha f(u) &= I_0^\alpha [\exp(\eta u)] \\
&= I_0^\alpha \left[\sum_{j=0}^{\infty} \frac{\eta^j u^j}{j!} \right] \\
&= \sum_{j=0}^{\infty} \frac{\eta^j}{j!} I_0^\alpha [u^j].
\end{aligned}$$

Making use of (1.13), we have

$$
\begin{aligned}
I_0^\alpha f(u) &= \sum_{j=0}^{\infty} \frac{\eta^j}{\Gamma(j+1)} \, \frac{\Gamma(j+1)u^{j+\alpha}}{\Gamma(j+\alpha+1)} \\
&= \sum_{j=0}^{\infty} \frac{\eta^j u^{j+\alpha}}{\Gamma(j+\alpha+1)} \\
&= \eta^{-\alpha} \sum \frac{(\eta u)^{j+\alpha}}{\Gamma(j+\alpha+1)}.
\end{aligned}
$$

Example. 3: We compute fractional integral I_0^α of a function $f(u) = \sin(\lambda u)$, where $\lambda, \alpha > 0$.

Using the series expension of $\sin(u)$, we have

$$
\begin{aligned}
f(u) = \sin(\lambda u) &= \lambda u - \frac{(\lambda u)^3}{3!} + \frac{(\lambda u)^5}{5!} \cdots \\
&= \sum_{j=0}^{\infty} \frac{(\lambda u)^{2j+1}}{(2j+1)!}.
\end{aligned}
$$

Applying I_0^α on both sides

$$
\begin{aligned}
I_0^\alpha f(u) &= I_0^\alpha \sum_{j=0}^{\infty} \frac{(\lambda u)^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty} \frac{\lambda^{2j+1}}{(2j+1)!} \; I_0^\alpha \; u^{2j+1}.
\end{aligned}
$$

Using (1.13), we get

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{\lambda^{2j+1}\Gamma(2j+1+1)}{(2j+1)!\Gamma(2j+1+1+\alpha)} (u)^{2j+1+\alpha} \\
&= \sum \frac{\lambda^{2j+1}\Gamma(2j+2)}{(2j+1)!\Gamma(2j+2+\alpha)} (u)^{2j+1+\alpha} \\
&= \sum \frac{\lambda^{2j+1}(2j+2)}{(2j+1)!\Gamma(2j+2+\alpha)} (u)^{2j+1+\alpha} \\
&= \frac{1}{\lambda^\alpha} \sum \frac{(\lambda u)^{2j+\alpha+1}}{\Gamma(2j+\alpha+1)}.
\end{aligned}
$$

Theorem 1.1.2. *Let f be a continous function defined on $[a, b]$, and f_n be uniformly convergent sequence. Then we have*

$$
\left(I_a^\alpha \lim_{n\to\infty} f_n(u) \right) = \left(\lim_{n\to\infty} I_a^\alpha f_n(u) \right),
$$

i.e sequence of function converges uniformly.

Proof. We know that f be the limit of f_n. Consider

$$\begin{aligned}|I_a^\alpha f_n(u) - I_a^\alpha f(u)| &= \left|\frac{1}{\Gamma(\alpha)}\int_a^u f_n(s)(u-s)^{\alpha-1}ds - \frac{1}{\Gamma(\alpha)}\int_a^u f(s)(u-s)^{\alpha-1}ds\right| \\ &= \left|\frac{1}{\Gamma(\alpha)}\int_a^u (f_n(s)-f(s))(u-s)^{\alpha-1}ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^u |f_n(s)-f(s)|(u-s)^{\alpha-1}ds \\ &\leq \frac{1}{\Gamma(\alpha)}||f_n-f||\int_a^u (u-s)^{\alpha-1}ds\end{aligned}$$

$$\begin{aligned}|I_a^\alpha f_n(u) - I_a^\alpha f(u)| &\leq \frac{1}{\Gamma(\alpha)}||f_n-f||_\infty\frac{(u-a)^\alpha}{\alpha} \\ &\leq \frac{1}{\Gamma(\alpha+1)}||f_n-f||_\infty(b-a)^\alpha.\end{aligned}$$

Expression on left side approches to zero as $n \to \infty$. □

1.1.2 Riemann-Liouville (RL) fractional derivative

Definition 1.1.6. *For $k-1 < \alpha \leq k$, $k = \lceil \alpha \rceil$. The left and right sided RL-fractional derivative of a function f can be defined with the help of (1.8), as*

$${}^{RL}D_{a+}^\alpha f(u) = \frac{1}{\Gamma(k-\alpha)}\left(\frac{d}{du}\right)^k\int_a^u (u-s)^{k-\alpha-1}f(s)ds, \tag{1.14}$$

$${}^{RL}D_{b-}^\alpha f(u) = (-1)^k\frac{1}{\Gamma(k-\alpha)}\left(\frac{d}{du}\right)^k\int_u^b (s-u)^{k-\alpha-1}f(s)ds. \tag{1.15}$$

Lemma 1.1.7. *For $k-1 < \alpha_1 < k$, $k = \lceil \alpha_1 \rceil$ and $\alpha_2 \in \mathbb{N}$ such that $\alpha_2 > \alpha_1$, then we have*

$$D_a^{\alpha_1} = D^{\alpha_2}I_a^{\alpha_2-\alpha_1}. \tag{1.16}$$

Proof. Since $\alpha_2 \geq \lceil \alpha_1 \rceil$, thus

$$\begin{aligned}D^{\alpha_2}I_a^{\alpha_2-\alpha_1} &= D^{\lceil \alpha_1 \rceil}D^{\alpha_2-\lceil \alpha_1 \rceil}I_a^{\alpha_2-\lceil \alpha_1 \rceil}I_a^{\lceil \alpha_1 \rceil-\alpha_1} \\ &= D^{\lceil \alpha_1 \rceil}I_a^{\lceil \alpha_1 \rceil-\alpha_1} \\ &= D_a^{\alpha_1}.\end{aligned}$$

$\square$

Lemma 1.1.8. *The differential operator $D_a^\alpha f$ exists nearly all over the interval $[a,b]$ for $f \in A^1[a,b]$ and $\alpha \in [0,1]$. Also $D_a^\alpha f \in L_p$ for $p \in \left[1, \frac{1}{h}\right]$ and*

$$D_a^\alpha f(u) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(u-a)^\alpha} + \int_a^u f^{'}(s)(u-s)^{-\alpha} ds \right).$$

Proof. By using Eq (1.16), we have

$$\begin{aligned} D_a^\alpha f(u) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_a^u f(s)(u-s)^{-\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_a^u \left(f(a) + \int_a^u f^{'}(v) dv \right) (u-s)^{-\alpha} ds \end{aligned}$$

$$\begin{aligned} D_a^\alpha f(u) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_a^t \left(f(a) \frac{ds}{(u-s)^{-\alpha}} + \int_a^u \int_a^s f^{'}(v)(u-s)^{-\alpha} dv ds \right) \\ &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(u-a)^\alpha} + \frac{d}{du} \int_a^u \int_a^s f^{'}(v)(u-s)^{-\alpha} dv ds \right). \end{aligned}$$

By Fubini's theorem, we have

$$D_a^\alpha f(u) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(u-a)^\alpha} + \frac{d}{du} \int_a^u f^{'}(v) \frac{(u-v)^{1-\alpha}}{1-\alpha} dv \right).$$

$\square$

Example. 4: Let $f(u) = (u-c)^\lambda$ for $\lambda > -1$ and $k > 0$,

then by (1.16) and (1.13), we have

$$\begin{aligned} D_a^k f(u) &= D^{\lceil k \rceil} J_a^{\lceil k \rceil - k} f(u) \\ &= D^{\lceil k \rceil} \frac{\Gamma(\lambda+1)}{\Gamma(\lceil k \rceil - k + \lambda + 1)} (u-c)^{\lceil k \rceil - k + \lambda} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lceil k \rceil - k + \lambda + 1)} D^{\lceil k \rceil} (u-c)^{\lceil k \rceil - k + \lambda}. \end{aligned}$$

The expression on right side vanish when $k - \lambda \in \mathbb{N}$ because the degree $\lceil k \rceil - (k - \lambda)$ of polynomial is less than the order of derivative $\lceil k \rceil$. That is

$$D_a^k (u-c)^{k-n} = 0 \qquad \forall\ k > 0, n \in \{1, 2, \cdots \lceil k \rceil\}.$$

And when $k - \lambda \notin \mathbb{N}$, we get

$$D_a^k (u-c)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-k)} (u-c)^{\lambda-k}.$$

1.1.3 The Caputo differential operator

There are certain impediments observed in utilizing the RL-differential operator for presenting the present reality. RL-derivative of constant is not zero. So in 1967 another definition is presented by Caputo to solve this problem.

Definition 1.1.9. *Assume $\alpha > 0$, and $k = \lceil \alpha \rceil$, then*

$$\begin{aligned} {}^{C}D_a^{\alpha} f(u) &= I_a^{k-\alpha} D^k f(u) \\ &= \int_a^u \frac{(u-s)^{k-\alpha-1}}{\Gamma(k-\alpha)} D^k f(s) ds. \end{aligned}$$

The operator ${}^{C}D_a^{\alpha}$ is said to be α order Caputo differential operator.

1.1.4 Hadamard fractional integral and derivative

The Hadamard fractional integral was introduced by Jacques Hadamard in 1892 and stated as:

Definition 1.1.10. *For $\alpha > 0$ and the function $f \in L_p[a, b]$. Then the Hadamard fractional integral operator is defined as*

$$ {}^{H}I_a^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_a^u \left(\ln \frac{u}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \qquad u > a, \tag{1.17}$$

and if $f \in AC^n[a, b]$ then the Hadamard derivative is defined as

$$ {}^{H}D_a^{\alpha} f(u) = \frac{1}{\Gamma(k-\alpha)} \left(u \frac{d}{du}\right)^k \int_a^u \left(\ln \frac{u}{s}\right)^{k-\alpha-1} f(s) \frac{ds}{s}, \tag{1.18}$$

where $k-1 < \alpha \leq k$, $k = \lceil \alpha \rceil$. Butzer et al. [19] added a simple modification in the definition of Hadamard operators, to introduce a new concept known as Hadamard-type fractional operator.

1.1.5 Hadamard-type fractional calculus

Definition 1.1.11. *For $u \in [a, b]$ and $k - 1 < \alpha \leq k$, left Hadamard-type fractional integral and derivative of order $\alpha > 0$ is defined as*

$$ {}^{HT}I_{a,c}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_a^u \left(\frac{u}{s}\right)^c \left(\ln \frac{u}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \tag{1.19}$$

$$ {}^{HT}D_a^{\alpha} f(u) = \frac{x^{-c}}{\Gamma(k-\alpha)} \left(u \frac{d}{du}\right)^k x^c \int_a^u \left(\frac{u}{s}\right)^c \left(\ln \frac{u}{s}\right)^{k-\alpha-1} f(s) \frac{ds}{s}. \tag{1.20}$$

1.2 Fractional integral and derivative operator with respect to another function ξ

In this section we will study the generalization of classical fractional operators. For the sake of simplicity we introduce the notation $D_\xi = \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)$ for differential operator. We have discussed some important properties which will be useful for upcoming chapters.

Definition 1.2.1. *Let $\alpha > 0$, an integrable function f defined over $[a,b]$(finite or infinite) and $\xi \in C^1[a,b]$ an strictly increasing function s.t $\xi'(u) \neq 0$ for all $u \in [a,b]$. Then the RL-fractional integrals of a function f with respect to another function ξ are defined as*

$$^{RL}I^{\alpha}_{a+,\xi}f(u) = \frac{1}{\Gamma(\alpha)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{\alpha-1}f(s)ds, \tag{1.21}$$

$$^{RL}I^{\alpha}_{b-,\xi}f(u) = \frac{1}{\Gamma(\alpha)}\int_u^b \xi'(s)(\xi(s)-\xi(u))^{\alpha-1}f(s)ds. \tag{1.22}$$

And for $k-1 < \alpha \leq k$ the RL-fractional derivative of a function f with respect to a function ξ are defined as

$$^{RL}D^{\alpha}_{a+,\xi}f(u) = \frac{1}{\Gamma(k-\alpha)}\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k \int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\alpha-1}f(s)ds, \tag{1.23}$$

$$^{RL}D^{\alpha}_{b-,\xi}f(u) = \frac{1}{\Gamma(k-\alpha)}\left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^k \int_u^b \xi'(s)(\xi(s)-\xi(u))^{k-\alpha-1}f(s)ds. \tag{1.24}$$

If we put $\xi(s) = s$ in (1.21) - (1.24) *it will become standard RL-fractional integral and derivative. And if we put $\xi(u) = \log(u)$ then it will become Hadamard frational integral and derivative respectively.*

Definition 1.2.2. *[11] Let $\alpha > 0, [a,b]$ be infinite or finite interval, $k = \lfloor \alpha \rfloor + 1$ and f, ξ be two functions from $C^1[a,b]$ where $\xi(u)$ is an increasing function s.t $\xi'(u) \neq 0$ for all $u \in [a,b]$. Then the Caputo fractional derivative of a function f with respect to* *another function ξ are defined as*

$$ {}^{C}D^{\alpha}_{a^+,\xi}f(u) = I^{k-\alpha}_{a^+,\xi}\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k f(u), \tag{1.25}$$

$$ {}^{C}D^{\alpha}_{b^-,\xi}f(u) = I^{k-\alpha}_{b^-,\xi}\left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^k f(u). \tag{1.26}$$

We may write (1.25) *and* (1.26) *as*

$$ {}^{C}D^{\alpha}_{a^+,\xi}f(u) = I^{k-\alpha}_{a^+,\xi}(D_\xi)^k f(u), \tag{1.27}$$

$$ {}^{C}D^{\alpha}_{b^-,\xi}f(u) = I^{k-\alpha}_{b^-,\xi}(-D_\xi)^k f(u). \tag{1.28}$$

Definition 1.2.3. *[12] Let $\alpha, t \in C$ with $Re(\alpha) > 0$, an integrable function f defined over $[a,b]$(finite or infinite) and $\xi \in C^1[a,b]$ an increasing function s.t $\xi'(u) \neq 0$ for all $u \in [a,b]$. Then the Hadamard-type fractional integral of a function f with respect to another function ξ is defined as*

$$ {}^{H}I^{\alpha,t}_{a^+,\xi}f(u) = \frac{1}{\Gamma(\alpha)}\int_a^u \left(\frac{\xi(s)}{\xi'(u)}\right)^t \left(log\frac{\xi(u)}{\xi(s)}\right)^{\alpha-1} f(s)\frac{\xi'(s)}{\xi(s)}ds. \tag{1.29}$$

Definition 1.2.4. *[20] Let f and g be piecewise continuous functions of ξ-exponential order, defined on finite interval $[0,S]$. Then, the ξ-convolution of f and g is the function $f * g_\xi$ defined by*

$$ f(u) * g_\xi(u) = \int_0^\infty f(s) g\left[\xi^{-1}\left(\frac{\xi(u)}{\xi(s)}\right)\right]\xi'(s)\frac{ds}{\xi(s)}, \tag{1.30}$$

$$ f_\xi(u)\ o\ g(u) = \int_0^\infty f[\xi^{-1}(\xi(u)\xi(s))]g(s)\xi'(s)ds. \tag{1.31}$$

Theorem 1.2.1 (**Convolution property**). *Let f and g be piecewise continuous functions of ξ-exponential order, defined on finite interval $[0,S]$. Then*

$$ f(u) * g_\xi(u) = f_\xi(u) * g(u). \tag{1.32}$$

Proof. Making substitution $\xi^{-1}\left(\frac{\xi(u)}{\xi(s)}\right) = v$ for $v \longrightarrow \infty$ as $s \longrightarrow 0$ and $v \longrightarrow 0$ as $s \longrightarrow \infty$ in the definition (1.30) we get,

$$ \begin{aligned} f(u) * g_\xi(u) &= \int_0^\infty f\left(\xi^{-1}\left(\frac{\xi(u)}{\xi(v)}\right)\right) g(v)\frac{\xi'(v)}{\xi(v)}dv \\ &= g(u) * f_\xi(u) \end{aligned} $$

□

1.3 Function spaces

In this section we will review some important function spaces that will be used in the next chapters.

Definition 1.3.1. *The space* $L_p(a,b),(1 \leq p \leq \infty)$ *consists of all Lebesgue complex-valued measurable functions* $k:[a,b] \longrightarrow C$ *for which* $\|k\|_p < \infty$*, where*

$$\|k\|_p = \left(\int_a^b |k(u)|^p du\right)^{\frac{1}{p}}, \qquad 1 \leq p \leq \infty,$$

and

$$\|k\|_\infty = ess \sup_{a \leq u < b} |k(u)|.$$

Definition 1.3.2. *Let* $-\infty \leq a < b \leq \infty$ *and* $m \in N_0$*. We denote by* $C^m[a,b]$ *a space of funtions* k *which are* m *times continuously differentiable on* $[a,b]$ *with the norm*

$$\|k\|_{C^m} = \sum_{q=0}^{m} \|k^q\|_C = \sum_{q=0}^{m} \max_{u \in [a,b]} |k^q(u)|,$$

for $m = 0$*,* $C^0[a,b] = C[a,b]$*.*

Definition 1.3.3. *The space* $X^p_{c,\xi}(a,b)(c \in \Re; 1 \leq p < \infty)$*, consists of all Lebesgue measurable functions* $k:[a,b] \longrightarrow \Re$ *for which* $\|k\|_{X^p_{c,\xi}} < \infty$*, with norm defined as*

$$\|k\|_{X^p_{c,\xi}} = \left(\int_a^b |\xi(u)^c k(u)|^p \frac{\xi'(u)}{\xi(u)} du\right)^{\frac{1}{p}}, \tag{1.33}$$

and

$$\|k\|_{X^\infty_{c,\xi}} = ess \sup_{a \leq u < b} [\xi(u)^c |k(u)|].$$

And the space $X^p_{c,\xi}(a,b)$ *coincides with the* $L_p(a,b)$ *space, when* $c = \frac{1}{p}$*.*

1.4 Weyl fractional integral and derivative with respect to another function

Definition 1.4.1. *For $Re(\alpha) \in (0,1)$ and ξ be strictly increasing function with $\xi^{'}(u) \neq 0$. Then the Weyl fractional integral of $f(u)$ with respect to a function $\xi(u)$ is*

$$W_{\xi}^{-\alpha}[f(u)] = \frac{1}{\Gamma(\alpha)} \int_{u}^{\infty} (\xi(s) - \xi(u))^{\alpha-1} \xi'(s) f(s) ds, \qquad u > 0. \tag{1.34}$$

Above result can be rewritten as the Weyl transform of $f(u)$ with respect to ξ, defined by

$$W_{\xi}^{-\alpha}[f(u)] = F_{\xi}(u-\alpha) = \frac{1}{\Gamma(\alpha)} \int_{u}^{\infty} (\xi(s) - \xi(u))^{\alpha-1} f(s) \xi'(s) ds.$$

Example. 5: If $f(u) = e^{(-a\xi(u))}, Re(a) > 0$, we have

$$W_{\xi}^{-\alpha}[e^{(-a\xi(t))}] = \frac{e^{-a\xi(x)}}{a^{\alpha}}. \tag{1.35}$$

Using equation (1.34), we have.

$$W_{\xi}^{-\alpha}[e^{(-a\xi(u))}] = \frac{1}{\Gamma(\alpha)} \int_{u}^{\infty} (\xi(s) - \xi(u))^{\alpha-1} e^{(-a\xi(t))} \xi'(s) ds.$$

Using change of variable $\xi(s) - \xi(u) = y$,

$$W_{\xi}^{-\alpha}[e^{(-a\xi(u))}] = \frac{e^{-a\xi(u)}}{\Gamma(\alpha)} \int_{0}^{\infty} y^{\alpha-1} e^{-ay} dy.$$

Now substitusing $ay = s$

$$\begin{aligned} W_{\xi}^{-\alpha}[e^{(-a\xi(u))}] &= \frac{e^{-a\xi(u)}}{a^{\alpha}} \frac{1}{\Gamma(\alpha)} \int_{u}^{\infty} s^{\alpha-1} e^{-s} ds \\ &= \frac{e^{-a\xi(u)}}{a^{\alpha}}. \end{aligned}$$

Similarly, we can solve for a function $f(u) = \xi(u)^{-\mu}$ by using defination (1.34) and making change of variable $v = \frac{\xi(u)}{\xi(s)}$, and properties of the beta function.

$$\begin{aligned} W_{\xi}^{-\alpha}[\xi(u)^{-\mu}] &= \frac{1}{\Gamma(\alpha)} \int_{u}^{\infty} (\xi(s) - \xi(u))^{\alpha-1} \xi(s)^{-\mu} \xi'(s) ds \\ &= \frac{(\xi(u))^{\alpha-\mu}}{\Gamma(\alpha)} \int_{0}^{1} (v)^{\mu-\alpha-1} (1-v)^{\alpha-1} dv \\ &= \frac{\Gamma(\mu-\alpha)}{\Gamma(\mu)} (\xi(u))^{\alpha-\mu}, \qquad 0 < Re(\alpha) < Re(\mu). \end{aligned}$$

Example. 6: For $\alpha > 0$, we have

$$W_{\xi}^{-\alpha}[\sin\ a\xi(u)] = a^{-\alpha}\sin\left(a\xi(u) + \frac{\pi\alpha}{2}\right),$$
$$W_{\xi}^{-\alpha}[\cos\ a\xi(u)] = a^{-\alpha}\cos\left(a\xi(u) + \frac{\pi\alpha}{2}\right).$$

We know that from equation (1.35),

$$\begin{aligned}
W_{\xi}^{-\alpha}[e^{-ia\xi(u)}] &= \frac{e^{-ia\xi(u)}}{(-ia)^{\alpha}} \\
&= \frac{e^{-ia\xi(u)}}{(a)^{\alpha}}\left(Cos\frac{\alpha\pi}{2} - iSin\frac{\alpha\pi}{2}\right) \\
&= a^{-\alpha}[\cos\ a\xi(u)\cos\frac{\alpha\pi}{2} - i\sin\frac{\alpha\pi}{2}\cos\ a\xi(u) \\
&\quad - i\cos\frac{\alpha\pi}{2}Sin\ a\xi(u) - \sin\ a\xi(u)\sin\frac{\alpha\pi}{2}] \\
&= a^{-\alpha}\left(\cos\left(a\xi(u) + \frac{\alpha\pi}{2}\right)\right) - i\left(\sin\left(a\xi(u) + \frac{\alpha\pi}{2}\right)\right) \\
W_{\xi}^{-\alpha}[\cos\ a\xi(u) - i\sin\ a\xi(u)] &= a^{-\alpha}\cos\left(a\xi(u) + \frac{\alpha\pi}{2}\right) - ia^{-\alpha}\sin\left(a\xi(u) + \frac{\alpha\pi}{2}\right) \\
W_{\xi}^{-\alpha}[\cos\ a\xi(u)] &= a^{-\alpha}\cos\left(a\xi(u) + \frac{\alpha\pi}{2}\right) \\
W_{\xi}^{-\alpha}[\sin\ a\xi(u)] &= a^{-\alpha}\sin\left(a\xi(u) + \frac{\alpha\pi}{2}\right).
\end{aligned}$$

Theorem 1.4.1. *For $\alpha_1, \alpha_2 > 0$, we have exponent law for ξ-type Weyl fractional operator as*

$$W_{\xi,-\infty}^{-\alpha_1}[W_{\xi,-\infty}^{-\alpha_2}f(u)] = W_{\xi,-\infty}^{-(\alpha_1+\alpha_2)}[f(u)].$$

Proof. Consider

$$\begin{aligned}
W_{\xi,-\infty}^{-\alpha_1}[W_{\xi,-\infty}^{-\alpha_2}f(u)] = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}(\xi(u) - \xi(s))^{\alpha_1-1}\int_{-\infty}^{s}(\xi(s) - \xi(\tau))^{\alpha_2-1} \\
\times \xi'(s)\xi'(\tau)f(\tau)d\tau ds.
\end{aligned}$$

By Fubini's theorem and Dirichlet formula, we have

$$\begin{aligned}
W^{-\alpha_1}_{\xi,-\infty}[W^{-\alpha_2}_{\xi,-\infty}f(u)] &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}\int_{\tau}^{u}(\xi(u)-\xi(s))^{\alpha_1-1}(\xi(s)-\xi(\tau))^{\alpha_1-1} \\
&\times \xi'(s)\xi'(\tau)f(\tau)dsd\tau \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}\xi'(\tau)f(\tau)\int_{\tau}^{u}(\xi(u)-\xi(s))^{\alpha_1-1} \\
&\times (\xi(s)-\xi(\tau))^{\alpha_2-1}\xi'(s)dsd\tau \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}\xi'(\tau)f(\tau)\int_{\tau}^{u}(\xi(u)-\xi(\tau)+\xi(\tau)-\xi(s))^{\alpha_1-1} \\
&\times (\xi(s)-\xi(\tau))^{\alpha_2-1}\xi'(s)dsd\tau \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}\xi'(\tau)f(\tau)\int_{\tau}^{u}(\xi(u)-\xi(\tau))^{\alpha_1-1} \\
&\times \left(1-\frac{\xi(s)-\xi(\tau)}{\xi(u)-\xi(\tau)}\right)^{\alpha_1-1}(\xi(s)-\xi(\tau))^{\alpha_2-1}\xi'(s)dsd\tau.
\end{aligned}$$

Making the change of variable $v=\frac{\xi(s)-\xi(\tau)}{\xi(u)-\xi(\tau)}$, we get

$$\begin{aligned}
W^{-\alpha_1}_{\xi,-\infty}[W^{-\alpha_2}_{\xi,-\infty}f(u)] &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}\xi'(\tau)f(\tau)\int_{0}^{1}(\xi(u)-\xi(\tau))^{\alpha_1-1}(1-v)^{\alpha_1-1} \\
&\times (\xi(u)-\xi(\tau))^{\alpha_2-1}v^{\alpha_2-1}(\xi(u)-\xi(\tau))dvd\tau \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}\xi'(\tau)f(\tau)\int_{0}^{1}(\xi(u)-\xi(\tau))^{\alpha_1+\alpha_2-1} \\
&\times (1-v)^{\alpha_1-1}v^{\alpha_2-1}dvd\tau \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{-\infty}^{u}\xi'(\tau)f(\tau)(\xi(u)-\xi(\tau))^{\alpha_1+\alpha_2-1}d\tau\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \\
&= \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{-\infty}^{u}\xi'(\tau)f(\tau)(\xi(u)-\xi(\tau))^{\alpha_1+\alpha_2-1}d\tau \\
&= W_{\xi}^{-(\alpha_1+\alpha_2)}[f(u)].
\end{aligned}$$

□

Derivative of Weyl fractional integral

Lemma 1.4.2. *For $\alpha>0$, and f be integrable function defined on $\Re_+$. We have*

$$D_\xi[W_\xi^{-\alpha}f(u)]=W_\xi^{-\alpha}[D_\xi f(u)]$$

Proof. From definition (1.34)

$$W_\xi^{-\alpha}[f(u)] = \frac{1}{\Gamma(\alpha)} \int_0^\infty (\xi(s) - \xi(u))^{\alpha-1} f(s)\xi'(s)ds.$$

Using change of variable $\xi(s) - \xi(u) = \xi(v)$

$$W_\xi^{-\alpha}[f(u)] = \frac{1}{\Gamma(\alpha)} \int_0^\infty \xi(v)^{\alpha-1} f(\xi^{-1}(\xi(u) + \xi(v)))\xi'(v)dv.$$

Now applying $D_\xi = \frac{1}{\xi(u)}\frac{d}{du}$, we get

$$\begin{aligned} D_\xi[W_\xi^{-\alpha} f(u)] &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \xi(v)^{\alpha-1} \frac{1}{\xi(u)}\frac{d}{du} f(\xi^{-1}(\xi(u) + \xi(v)))\xi'(v)dv \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \xi(v)^{\alpha-1} D_\xi f(\xi^{-1}(\xi(u) + \xi(v)))\xi'(v)dv \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty (\xi(s) - \xi(u))^{\alpha-1} D_\xi f(s)\xi'(s)ds \\ &= W_\xi^{-\alpha}[D_\xi f(u)]. \end{aligned}$$

□

Weyl Fractional derivative with respect to another function

Definition 1.4.3. *[14] For a positive number β with $k > \beta$, where k is a smallest integer. We have $k - \beta = \alpha > 0$ and ξ be strictly increasing function with $\xi'(u) \neq 0$. Then the Weyl fractional derivative of a function f with respect to another function ξ is defined by*

$$\begin{aligned} W_\xi^\beta[f(u)] &= E_\xi^k\, W_\xi^{-(k-\beta)}[f(u)] \\ &= \frac{(-1)^k}{\Gamma(k-\eta)} \left(\frac{1}{\xi(u)}\frac{d}{du}\right)^k \int_u^\infty (\xi(s) - \xi(u))^{k-\beta-1} f(s)\xi'(s)ds, \end{aligned} \tag{1.36}$$

where $E_\xi = -D_\xi$.

Example. 7: For $f(u) = e^{(-c\xi(u))}$ with $c > 0$, we have

$$W_\xi^\beta[e^{-c\xi(u)}] = c^\beta e^{-c\xi(u)}.$$

By Definition 1.4.3 and making use of (1.35), we have

$$\begin{aligned}
W_\xi^\beta[e^{-c\xi(u)}] &= E_\xi^k[W_\xi^{-\alpha} e^{-c\xi(u)}] \\
&= E_\xi^k[c^{-\alpha} e^{-c\xi(u)}] \\
&= c^{-\alpha}\left(-\frac{1}{\xi(u)}\frac{d}{du}\right)^k e^{-c\xi(u)} \\
&= c^{-\alpha}\left(\frac{1}{\xi(u)}\frac{d}{du}\right)^{k-1}\left(\frac{1}{\xi(u)}\frac{d}{du} e^{-c\xi(u)}\right) \\
&= c^{-\alpha}\left(-\frac{1}{\xi(u)}\frac{d}{du}\right)^{k-1}\left(-\frac{1}{\xi(u)} e^{-c\xi(u)} - c\xi'(u)\right) \\
&= c^{-\alpha}\left(-\frac{1}{\xi(u)}\frac{d}{du}\right)^{k-1} c\, e^{-c\xi(u)}.
\end{aligned}$$

Repeating the same process $(k-1)$-times, we get

$$\begin{aligned}
W_\xi^\beta[e^{-c\xi(u)}] &= c^{-\alpha+k} e^{-c\xi(u)} \\
&= c^\beta e^{-c\xi(u)}, \qquad\qquad \because \beta = k - \alpha.
\end{aligned}$$

Replacing β by $-\alpha$ we get (1.35).

Similarly, we can prove following result as we proved for Weyl fractional integral of a function with respect to another function.

$$\begin{aligned}
W_\xi^\beta(\xi(u))^{-\mu} &= \frac{\Gamma(\beta+\mu)}{\Gamma(\mu)}\xi(u)^{-(\beta+\mu)}, \\
W_\xi^\beta[\cos\ a\xi(u)] &= c^\beta \cos\left(a\xi(u) - \frac{\beta\pi}{2}\right), \\
W_\xi^\beta[\sin\ a\xi(u)] &= c^\beta \sin\left(a\xi(u) - \frac{\beta\pi}{2}\right),
\end{aligned}$$

given that $\alpha, \beta \in [0,1]$.

Chapter 2

Caputo fractional derivative with respect to another function

Many generalization have been introduced for classical fractional operator by different mathematician. One of these generalization is ξ-fractional operator. In [2] Samko introduced RL-fractional integral and derivative with respect to another function. In [11] Almeida use the idea to devlop the defination of Caputo fractional derivative with respect to another function. In this chapter we have reviwed a paper on Caputo fractional derivative of a function with respect to another function by Ricardo Almeida [11]. We have reviewed the definition of ξ-Caputo derivative . Also we have discussed some basic properties of this opertor. As we know that the classical definition of Caputo fractional derivative is obtained from RL-fractional derivative. Similarly the definition of ξ-Caputo fractional derivative is also obtained from RL-fractional derivative with respect to another function. To do this, we will use the Definition 1.2.1 of generalized RL-fractional derivative. The definition of ξ-Caputo fractional derivative is as follow:

For $\beta \in \Re_+$, $k = [\beta]+1$, I be the interval that may be finite or infinite. $f, \xi \in C^k[a, b]$ functions such that ξ is increasing and $\xi'(s) \neq 0$, for all $s \in I$. The left/right ξ-Caputo fractional derivative of f of order β is given by

$$
{}^C D^{\beta}_{a^+,\xi} f(u) = I^{k-\beta}_{a^+,\xi} \left(\frac{1}{\xi'(u)} \frac{d}{du} \right)^k f(u),
$$

$$
{}^C D^{\beta}_{b^-,\xi} f(u) = I^{k-\beta}_{b^-,\xi} \left(-\frac{1}{\xi'(u)} \frac{d}{du} \right)^k f(u).
$$

For the sake of simplicity we use the symbol

$$f_{\xi}^{[k]} f(u) := \left(\frac{1}{\xi'(u)} \frac{d}{du} \right)^k f(u),$$

when $\beta = k$, the above definition becomes

$$\begin{aligned} {}^C D_{a^+,\xi}^{\alpha} f(u) &= I_{a^+,\xi}^{k-k} f_{\xi}^{[k]}(u) = f_{\xi}^{[k]}(u), \\ {}^C D_{b^-,\xi}^{\alpha} f(u) &= (-1)^k I_{b^-,\xi}^{k-k} f_{\xi}^{[k]}(u) = (-1)^k f_{\xi}^{[k]}(s). \end{aligned}$$

And if $\beta \notin \mathbb{N}$, then

$$ {}^C D_{a^+,\xi}^{\alpha} f(u) = \frac{1}{\Gamma(k-\beta)} \int_a^s \xi'(s)\xi(u) - \xi(s))^{k-\beta-1} f_{\xi}^{[k]}(s) ds, \tag{2.1}$$

$$ {}^C D_{b^-,\xi}^{\beta} f(u) = \frac{1}{\Gamma(k-\beta)} \int_u^b \xi'(s)\xi(s) - \xi(u))^{k-\beta-1} (-1)^k f_{\xi}^{[k]}(s) ds. \tag{2.2}$$

For the case $\beta \in (0, 1)$, we have

$$\begin{aligned} {}^C D_{a^+,\xi}^{\beta} f(u) &= \frac{1}{\Gamma(k-\beta)} \int_a^u \xi(u) - \xi(s))^{-\beta} f'(s) ds, \\ {}^C D_{b^-,\xi}^{\beta} f(u) &= \frac{-1}{\Gamma(k-\beta)} \int_u^b \xi(s) - \xi(u))^{-\beta} f'(s) ds. \end{aligned}$$

Now we study the case $\beta > 0$, for ξ-Caputo fractional derivative. We solve the result for only left fractional derivative, method for right side will be same with necessery changing.

Theorem 2.0.1. *[11] If $f, \xi \in C^{k+1}[a, b]$, then for all $\beta > 0$,*

$$ {}^C D_{a^+,\xi}^{\beta} f(u) = \frac{(\xi(u) - \xi(a))^{k-\beta}}{\Gamma(k+1-\beta)} f_{\xi}^{[k]}(a) + \frac{1}{\Gamma(k+1-\beta)} \int_a^u (\xi(u) - \xi(s))^{k-\beta} \frac{d}{ds} f_{\xi}^{[k]}(s) ds,$$

and

$$\begin{aligned} {}^C D_{b^-,\xi}^{\beta} f(u) = (-1)^k \frac{(\xi(b) - \xi(u))^{k-\beta}}{\Gamma(k+1-\beta)} f_{\xi}^{[k]}(b) - \frac{1}{\Gamma(k+1-\beta)} \int_u^b (\xi(s) - \xi(u))^{k-\beta} \\ \times (-1)^k \frac{d}{ds} f_{\xi}^{[k]}(s) ds. \end{aligned}$$

Proof. By definition (2.1), we have

$$ {}^C D_{a^+,\xi}^{\beta} f(u) = \frac{1}{\Gamma(k+1-\beta)} \int_a^u \xi'(s)(\xi(u) - \xi(s))^{k-\beta-1} f_{\xi}^{[k]}(s) ds.$$

Using integration by parts

$$
\begin{aligned}
{}^{C}D^{\beta}_{a^+,\xi}f(u) &= \frac{1}{\Gamma(k-\beta)}[-f^{[k]}_{\xi}\int_a^u -\xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}ds \\
&- \int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}\frac{d}{ds}f^{[k]}_{\xi}ds] \\
&= \frac{1}{\Gamma(k-\beta)}\left[-f^{[k]}_{\xi}\frac{(\xi(u)-\xi(s))^{k-\beta}}{k-\beta}\right]_a^u \\
&+ \frac{1}{\Gamma(k-\beta)}\int_a^u \frac{(\xi(u)-\xi(s))^{k-\beta}}{k-\beta}\frac{d}{ds}f^{[k]}_{\xi}ds \\
&= \frac{1}{\Gamma(k+1-\beta)}[-f^{[k]}(u)(\xi(u)-\xi(u))^{k-\beta} - (-f^{[k]}_{\xi}(a)(\xi(u)-\xi(a))^{k-\beta})] \\
&+ \frac{1}{\Gamma(k-\beta+1)}\int_a^u (\xi(u)-\xi(s))^{k-\beta}\frac{d}{ds}f^{[k]}_{\xi}ds
\end{aligned}
$$

$$
= \frac{(\xi(u)-\xi(a))^{k-\beta}}{\Gamma(k-\beta+1)}f^{[k]}_{\xi}(a) + \frac{1}{\Gamma(k-\beta+1)}\int_a^u (\xi(u)-\xi(s))^{k-\beta}\frac{d}{ds}f^{[k]}_{\xi}ds.
$$

□

In this part of theorem

Consider (2.1),

$$
{}^{C}D^{\beta}_{a^+,\xi}f(u) = I^{k-\beta}_{a^+,\xi}\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k f(u).
$$

If $k=\beta$ for $\beta \in \mathbb{N}$, we get

$$
\begin{aligned}
f^{[k]}_{\xi}f(u) &= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k f(u) \\
{}^{C}D^{\beta}_{a^+,\xi}f(u) &= f^{[k]}_{\xi}f(u).
\end{aligned}
$$

When $\beta \notin \mathbb{N}$, then

$$
{}^{C}D^{\beta}_{a^+,\xi}f(u) = \frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}f^{[k]}_{\xi}(s)ds.
$$

When $\beta \to (k-1)^+$, we get

$$\begin{aligned}
\lim_{\beta \longrightarrow (k-1)^+} {}^C D^{\beta}_{a^+,\xi} f(u) &= \frac{1}{\Gamma(\beta-1-\beta)} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-k+1-1} f^{[k]}_{\xi}(s)ds \\
&= \frac{1}{\Gamma(-1)} \int_a^u \xi'(s)(\xi(u)-\xi(s))^0 f^{[k]}_{\xi}(s)ds \\
&= \int_a^u \xi'(s) \frac{1}{\xi'(s)} \frac{d}{ds} f^{[k-1]}_{\xi}(s)ds \\
&= \int_a^u \frac{d}{ds} f^{[k-1]}_{\xi}(s)ds \\
&= f^{[k-1]}_{\xi}(u) - f^{[k-1]}_{\xi}(a).
\end{aligned}$$

When $\beta \to k^-$,

$$\begin{aligned}
\lim_{\beta \longrightarrow k^-} {}^C D^{\beta}_{a^+,\xi} f(u) &= I^{(k-k)}_{a^+,\xi} f^{[k]}_{\xi} f(u) \\
&= f^{[k]}_{\xi} f(u).
\end{aligned}$$

Consider $\|.\| : C[a,b] \longrightarrow \Re$,

$$\|f\|_C = \max_{u \in [a,b]} |f(u)|.$$

And $\|.\|_{C^{[k]}_{\xi}} : C^k[a,b] \longrightarrow \Re$,

$$\|f\|_{C^{[k]}_{\xi}} = \sum_{n=0}^{n} \|f^{[k]}_{\xi}\|_C.$$

Theorem 2.0.2. *[11] For $\beta > 0$, ξ-Caputo fractional derivative are bounded operator*

$$\begin{aligned}
\|{}^C D^{\beta}_{a^+,\xi} f\|_C \leq K \|f\|_{C^{[k]}_{\xi}}, \\
\|{}^C D^{\beta}_{b^-,\xi} f\|_C \leq K \|f\|_{C^{[k]}_{\xi}},
\end{aligned}$$

where $K = \dfrac{(\xi(b)-\xi(a))^{k-\beta}}{\Gamma(k+1-\beta)}$.

Proof. Since we know that

$$|f^{[k]}_{\xi}(u)| \leq \|f\|_{C^{[k]}_{\xi}}, \qquad u \in [a,b]. \tag{2.3}$$

From (2.1), we have

$$
\begin{aligned}
\|{}^C D^{\beta}_{a^+,\xi}\|_C &= \|\frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} f_{\xi}^{[k]}(s)ds\| \\
&\leq \frac{1}{\Gamma(k-\beta)}\max_{u\in[a,b]}\int_a^u |\xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} f_{\xi}^{[k]}(s)|ds \\
&\leq \frac{1}{\Gamma(k-\beta)}\max_{u\in[a,b]}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} |f_{\xi}^{[k]}(s)|ds.
\end{aligned}
$$

Using (2.3), we get

$$
\begin{aligned}
|{}^C D^{\beta}_{a^+,\xi} f(u)| &\leq \frac{1}{\Gamma(k-\beta)}\max_{u\in[a,b]}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} \|f\|_{C_{\xi}^{[k]}} ds \\
&= \frac{\|f\|_{C_{\xi}^{[k]}}}{\Gamma(k-\beta)}\max_{u\in[a,b]}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} ds \\
&= \frac{\|f\|_{C_{\xi}^{[k]}}}{\Gamma(k-\beta)}\frac{(\xi(b)-\xi(a))^{k-\beta}}{k-\beta} \\
&= \frac{\|f\|_{C_{\xi}^{[k]}}}{\Gamma(k-\beta+1)}(\xi(b)-\xi(a))^{k-\beta} \\
&= K\|f\|_{C_{\xi}^{[k]}}.
\end{aligned}
$$

Also we can conclude from above

$$
{}^C D^{\beta}_{a^+,\xi} f(b) = {}^C D^{\beta,\xi}_{b-} f(a) = 0.
$$

□

Theorem 2.0.3. *[11]For $\beta > 0$, and $f \in C^k[a,b]$, we have*

$$
\begin{aligned}
{}^C D^{\beta}_{a^+,\xi} f(u) &= D^{\beta}_{a^+,\xi}\left[f(u) - \sum_{n=0}^{k-1}\frac{1}{n!}(\xi(u)-\xi(a))^n f_{\xi}^{[n]}(a)\right], \\
{}^C D^{\beta}_{b^-,\xi} f(u) &= D^{\beta}_{b^-,\xi}\left[f(u) - \sum_{n=0}^{k-1}\frac{1}{n!}(\xi(u)-\xi(b))^n f_{\xi}^{[n]}(b)\right].
\end{aligned}
$$

Proof. By Definition, we know that

$$
D^{\beta}_{a^+,\xi} f(u) = \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k \frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} f(s)ds.
$$

Right hand side gives

$$\Gamma(k-\beta).D^{\beta}_{a^+,\xi}\left[f(u)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n f^{[n]}_{\xi}(a)\right]$$
$$=\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}\left[f(s)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(s)-\xi(a))^n\right]ds.$$

Now using Integration by parts on above expression

$$\Gamma(k-\beta).D^{\beta}_{a^+,\xi}\left[f(u)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n f^{[n]}_{\xi}(a)\right]$$
$$=\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k[f(s)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(s)-\xi(a))^n\left(-\int_a^u -\xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}ds\right)$$
$$+\int_a^u\left[\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}ds\frac{d}{ds}\left[f(s)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}\right]\right]ds]$$
$$=\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k[-\left[f(s)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(s)-\xi(a))^n ds\right]\frac{(\xi(u)-\xi(s))^{k-\beta}}{k-\beta}|_a^u$$
$$+\int_a^u\frac{(\xi(u)-\xi(s))^{k-\beta}}{k-\beta}\left[f^{[1]}_{\xi}(s)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}n(\xi(s)-\xi(a))^{n-1}\xi'(s)\right]ds].$$

When we put upper limit second integral vanishes and first one will vanish for lower limit. So first term vanish completely and we are left with

$$\Gamma(k-\beta).D^{\beta}_{a^+,\xi}\left[f(u)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n f^{[n]}_{\xi}(a)\right]$$
$$=\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k\int_a^u\frac{\xi'(s)(\xi(u)-\xi(s))^{k-\beta}}{k-\beta}\left[f^{[1]}_{\xi}(s)-\sum_{n=1}^{k-1}\frac{f^{[n]}_{\xi}(a)}{(n-1)!}(\xi(s)-\xi(a))^{n-1}\right]ds.$$

Now applying derivative with respect to u,

$$\Gamma(k-\beta).D^{\beta}_{a^+,\xi}\left[f(u)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n f^{[n]}_{\xi}(a)\right]$$
$$=\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{k-1}\frac{1}{\xi'(u)}\int_a^u\frac{(k-\beta)\xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}}{k-\beta}\xi'(u)$$
$$\times\left[f^{[1]}_{\xi}(s)-\sum_{n=1}^{k-1}\frac{f^{[n]}_{\xi}(a)}{(n-1)!}(\xi(s)-\xi(a))^{n-1}\right]ds.$$

Repeating this procedure, we get

$$\begin{aligned}
&D^{\beta}_{a^+,\xi}\left[f(u)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n f^{[n]}_{\xi}(a)\right]\\
&=\frac{1}{\Gamma(k-\beta)}\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{k-2}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}\\
&\times\left[f^{[2]}_{\xi}(s)-\sum_{n=2}^{k-1}\frac{f^{[n]}_{\xi}(a)}{(n-2)!}(\xi(s)-\xi(a))^{n-2}\right]ds.
\end{aligned}$$

Repeating (k-3) times, we get

$$\begin{aligned}
&D^{\beta}_{a^+,\xi}\left[f(u)-\sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n f^{[n]}_{\xi}(a)\right]\\
&=\frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}f^{[k]}_{\xi}(s)ds\\
&=\ {}^{C}D^{\beta}_{a^+,\xi}f(u).
\end{aligned}$$

Thus of for all $n=0,......k-1$, $f^{[n]}_{\xi}(a)=0$ then ${}^{C}D^{\beta}_{a^+,\xi}f(u)=D^{\beta}_{a^+,\xi}f(u)$. □

Lemma 2.0.1. *[11]For $\eta\in\Re$, and*

$$f(u)=(\xi(u)-\xi(a))^{\eta-1},$$
$$I(u)=(\xi(b)-\xi(u))^{\eta-1},$$

where $\eta>k$, then for $\beta>0$, we have

$${}^{C}D^{\beta}_{a^+,\xi}f(u)=\frac{\Gamma(\eta)}{\Gamma(\eta-\beta)}(\xi(u)-\xi(a))^{\eta-\beta-1}, \tag{2.4}$$
$${}^{C}D^{\beta}_{b^-,\xi}I(u)=\frac{\Gamma(\eta)}{\Gamma(\eta-\beta)}(\xi(b)-\xi(u))^{\eta-\beta-1}. \tag{2.5}$$

Proof. Since

$$\begin{aligned}
f(u)&=(\xi(u)-\xi(a))^{\eta-1}\\
f^{[1]}_{\xi}(u)&=\frac{1}{\xi'(u)}\frac{d}{du}(\xi(u)-\xi(a))^{\eta-1}=\frac{1}{\xi'(u)}(\eta-1)(\xi(u)-\xi(a))^{\eta-2}\xi'(u)\\
f^{[1]}_{\xi}(u)&=(\eta-1)(\xi(u)-\xi(a))^{\eta-2}.
\end{aligned}$$

Agian differentiating, we get

$$f_{\xi}^{[2]}(u) = (\eta-2)(\eta-1)(\xi(u)-\xi(a))^{\eta-3}.$$

Repeating the process $(n-2)$-times, we get

$$\begin{aligned}
f_{\xi}^{[k]}(u) &= (\eta-n)\cdots(\eta-1)(\xi(u)-\xi(a))^{\eta-k-1} \\
&= \frac{(\eta-1)(\eta-2)\cdots(\eta-k)(\eta-k-1)!}{(\eta-k-1)!}(\xi(u)-\xi(a))^{\eta-k-1} \\
&= \frac{(\eta-1)!}{(\eta-k-1)!}(\xi(u)-\xi(a))^{\eta-k-1} = \frac{\Gamma(\eta)}{\Gamma(\eta-k)}(\xi(u)-\xi(a))^{\eta-k-1}.
\end{aligned}$$

Now consider (2.1), and making use of above result, we get

$$\begin{aligned}
{}^{C}D_{a^+,\xi}^{\beta} f(u) &= \frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} f_{\xi}^{[k]}(s)ds \\
&= \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}(\xi(u)-\xi(a))^{\eta-k-1}ds \\
&= \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}\int_a^u \xi'(s)(\xi(u)-\xi(a)+\xi(a)-\xi(s))^{k-\beta-1} \\
&\times(\xi(s)-\xi(a))^{\eta-k-1}ds \\
&= \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}\int_a^u \xi'(s)(\xi(u)-\xi(a)-(\xi(s)-\xi(a))^{k-\beta-1} \\
&\times(\xi(s)-\xi(a))^{\eta-k-1}ds \\
&= \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}(\xi(u)-\xi(a))^{k-\beta-1}\int_a^u \xi'(s)\left[1-\frac{\xi(s)-\xi(a)}{\xi(u)-\xi(a)}\right]^{k-\beta-1} \\
&\times(\xi(s)-\xi(a))^{\eta-k-1}ds.
\end{aligned}$$

Multiplying and Dividing by $(\xi(u)-\xi(a))^{\eta-k-1}$,

$$\begin{aligned}
{}^{C}D_{a^+,\xi}^{\beta} f(u) = &\frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}(\xi(u)-\xi(a))^{k-\beta-1}(\xi(u)-\xi(a))^{\eta-k-1} \\
&\times\int_a^u \xi'(s)\left[1-\frac{\xi(s)-\xi(a)}{\xi(u)-\xi(a)}\right]^{k-\beta-1}\times\left[\frac{\xi(s)-\xi(a)}{\xi(u)-\xi(a)}\right]^{\eta-k-1} ds.
\end{aligned}$$

Making the change of variable $v = (\xi(s)-\xi(a))/(\xi(u)-\xi(a))$, implies that

$(\xi(u)-\xi(a))du = \xi'(s)ds$, we get

$$
\begin{aligned}
{}^{C}D^{\beta}_{a^+,\xi}f(u) &= \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}(\xi(u)-\xi(a))^{\eta-\beta-2}\int_0^1 (1-v)^{k-\beta-1}(v)^{\eta-k-1} \\
&\times (\xi(u)-\xi(a))dv \\
&= \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}(\xi(u)-\xi(a))^{\eta-\beta-1}\int_0^1 (v)^{\eta-k-1}(1-v)^{k-\beta-1}dv.
\end{aligned}
$$

Using eq (1.2), we get

$$
{}^{C}D^{\eta}_{a^+,\xi}f(u) = \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}(\xi(u)-\xi(a))^{\eta-\beta-1}B(\eta-k,k-\beta).
$$

Now using (1.3), we obtain

$$
\begin{aligned}
{}^{C}D^{\beta}_{a^+,\xi}f(u) &= \frac{\Gamma(\eta)}{\Gamma(k-\beta)\Gamma(\eta-k)}\frac{\Gamma(\eta-k)\Gamma(k-\beta)}{\Gamma(\eta-k+k-\beta)}(\xi(u)-\xi(a)^{\eta-\beta-1} \\
&= \frac{\Gamma(\eta)}{\Gamma(\eta-\beta)}(\xi(u)-\xi(a))^{\eta-\beta-1}.
\end{aligned}
$$

□

Example. [11] Consider $f(u) = (\xi(u)-\xi(0))^2$ with $a=0$ and $\eta=3$, using (2.4) in given function we get

$$
\begin{aligned}
{}^{C}D^{\beta}_{a^+,\xi}f(u) &= \frac{\Gamma(3)}{\Gamma(3-\beta)}(\xi(u)-\xi(0))^{3-\beta-1} \\
&= \frac{(2)}{\Gamma(3-\beta)}(\xi(u)-\xi(0))^{2-\beta}.
\end{aligned}
$$

When $\beta=1$, we get

$$
\begin{aligned}
{}^{C}D^{1}_{0^+,\xi}f(u) &= \frac{2}{\Gamma(3-1)}(\xi(u)-\xi(0))^{2-1} = \frac{2}{\Gamma(2)}(\xi(u)-\xi(0)) \\
&= 2(\xi(u)-\xi(0)).
\end{aligned}
$$

In particular, $k \leq n \in \mathbb{N}$, we have

$$
\begin{aligned}
{}^{C}D^{\beta}_{a^+,\xi}(\xi(u)-\xi(a))^{n} &= \frac{n!}{\Gamma(n+1-\beta)}(\xi(u)-\xi(a))^{n-\beta}, \\
{}^{C}D^{\beta}_{b^-,\xi}(\xi(b)-\xi(u))^{n} &= \frac{n!}{\Gamma(n+1-\beta)}(\xi(b)-\xi(u))^{n-\beta}.
\end{aligned}
$$

On the other hand $k > n \in \mathbb{N}_0$, we have

$$ {}^{C}D^{\beta}_{a^+,\xi}(\xi(u)-\xi(a))^n = {}^{C}D^{\beta}_{b^-,\xi}(\xi(b)-\xi(u))^n = 0. \tag{2.6} $$

Since $D^k_\xi(\xi(u)-\xi(a))^n = D^k_\xi(\xi(b)-\xi(u))^n = 0$.

Lemma 2.0.2. *[11] For $\omega \in \Re$, $\beta > 0$ and we have a function*

$$ f(u) = E_\beta(\omega(\xi(u)-\xi(a))^\beta), $$

where E_β is the Mittage-Lefler function, then

$$ {}^{C}D^{\beta}_{a^+,\xi} f(u) = \omega f(u). $$

Proof. Consider function and using (1.4), we have

$$ f(u) = E_\beta(\omega(\xi(u)-\xi(a))^\beta) = \sum_{n=0}^{\infty} \frac{(\omega(\xi(u)-\xi(a))^\beta)^n}{\Gamma(\beta n+1)}, $$

$$ {}^{C}D^{\beta}_{a^+,\xi} f(t) = \sum_{n=0}^{\infty} \frac{(\omega)^n}{\Gamma(\beta n+1)} \, {}^{C}D^{\beta}_{a^+,\xi}(\xi(u)-\xi(a))^{\beta n}. $$

By (2.4) for $\eta - 1 = \beta n$,

$$ \begin{aligned} {}^{C}D^{\beta}_{a^+,\xi} f(u) &= \sum_{n=1}^{\infty} \frac{(\omega)^n}{\Gamma(\beta n+1)} \frac{\Gamma(\beta n+1)}{\Gamma(\beta n+1-\beta)}(\xi(u)-\xi(a))^{\beta n-\beta} \\ &= \omega \sum_{n=1}^{\infty} \frac{(\omega)^{n-1}}{\Gamma(\beta(n-1)+1)}(\xi(u)-\xi(a))^{\beta(n-1)} \\ &= \omega \sum_{n=1}^{\infty} \frac{(\omega(\xi(u)-\xi(a))^\beta)^{n-1}}{\Gamma(\beta(n-1)+1)} = \omega f(u). \end{aligned} $$

Now consider the case $\omega = 1$, $a = 0$, we have

$$ f(u) = E_\beta(\xi(u)-\xi(0))^\beta. $$

Derivative for above function becomes

$$ {}^{C}D^{\beta}_{a^+,\xi} f(u) = f(u). $$

For $\beta = 1$, we get

$$ {}^{C}D^{1}_{a^+,\xi} f(u) = \exp(\xi(u)-\xi(0)). $$

□

Theorem 2.0.4. *[11] For $\beta > 0$ and $f \in C^k[a,b]$, we have*

$$I^{\beta}_{a^+,\xi}\ {}^C D^{\beta}_{a^+,\xi} f(u) = f(u) - \sum_{n=0}^{k-1} \frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n,$$
$$I^{\beta}_{b^-,\xi}\ {}^C D^{\beta}_{b^+,\xi} f(u) = f(u) - \sum_{n=0}^{k-1} (-1)^n \frac{f^{[n]}_{\xi}(b)}{n!}(\xi(b)-\xi(u))^n.$$

Proof. By definition (1.25), we have

$$\begin{aligned} {}^C D^{\beta}_{a^+,\xi} f(u) &= I^{k-\beta}_{a^+,\xi}\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^k f(u) = I^{k-\beta,\xi}_{a+} f^{[k]}_{\xi}(u) \\ I^{\beta}_{a^+,\xi}\ {}^C D^{\beta}_{a^+,\xi} f(u) &= I^{\beta}_{a^+,\xi} I^{k-\beta}_{a^+,\xi} f^{[k]}_{\xi}(u). \end{aligned}$$

Now by semigroup property and applying integration by parts, we get

$$\begin{aligned} I^{\beta}_{a^+,\xi}\ {}^C D^{\beta}_{a^+,\xi} f(u) &= I^{k}_{a^+,\xi} f^{[k]}_{\xi}(u) \\ &= \frac{1}{\Gamma(k)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-1} f^{[k]}_{\xi}(s)ds \\ &= \frac{1}{\Gamma(k)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-1}\frac{1}{\xi'(s)}\frac{d}{ds} f^{[k-1]}_{\xi}(s)ds \\ &= \frac{1}{\Gamma(k)}\int_a^u (\xi(u)-\xi(s))^{k-1}\frac{d}{ds} f^{[k-1]}_{\xi}(s)ds \\ &= \frac{1}{\Gamma(k)}[(\xi(u)-\xi(s))^{k-1}\int_a^u \frac{d}{ds} f^{[k-1]}_{\xi}(s)ds \\ &- \int_a^u \left(\frac{d}{ds}(\xi(u)-\xi(s)\right)^{k-1}\int \frac{d}{ds} f^{[k-1]}_{\xi}(s)ds)ds] \\ &= \frac{1}{\Gamma(k)}[(\xi(u)-\xi(s))^{k-1} f^{[k-1]}_{\xi}(s)|_a^u \\ &- \int_a^u ((k-1)(\xi(u)-\xi(s))^{k-2}(-\xi'(s)) f^{[k-1]}_{\xi}(s))ds] \\ &= \frac{1}{\Gamma(k)}[-(\xi(u)-\xi(a))^{k-1} f^{[k-1]}_{\xi}(a) \\ &+ \int_a^u ((k-1)(\xi(u)-\xi(s))^{k-2}\xi'(s)\frac{1}{\xi'(s)}\frac{d}{ds} f^{[k-2]}_{\xi}(s))ds] \\ &= -\frac{1}{\Gamma(k)}(\xi(x)-\xi(a))^{k-1} f^{[k-1]}_{\xi}(a) + \frac{(k-1)}{(k-1)(k-2)!} \\ &\times \int_a^u (\xi(u)-\xi(s))^{k-2}\frac{d}{ds} f^{[k-2]}_{\xi}(s)ds \end{aligned}$$

$$I^{\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi}f(u) = \frac{1}{(k-2)!}\int_a^u (\xi(u)-\xi(s))^{k-2}\frac{d}{ds}f^{[k-2]}_{\xi}(s)ds$$
$$-\frac{1}{(k-1)!}f^{[k-1]}_{\xi}(a)(\xi(u)-\xi(a))^{k-1}.$$

Repeating same procedure, we get

$$\begin{aligned}
I^{\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi}f(u) &= \frac{1}{(k-2)!}\int_a^u (\xi(u)-\xi(s))^{k-2}\frac{d}{ds}f^{[k-2]}_{\xi}(s)ds \\
&\quad - \sum_{n=k-2}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n, \\
&= \cdots \\
&= \int_a^u \frac{d}{ds}f(s)ds - \sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n \\
&= f(u)-f(a) - \sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n
\end{aligned}$$

$$I^{\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi}f(u) = f(u) - \sum_{n=0}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n.$$

For the particular case $\beta \in (0,1)$, we have

$$I^{\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi}f(u) = f(u) - f(a).$$

Taylor formula can be obtained from above theorem

$$f(u) = \sum_{n=1}^{k-1}\frac{f^{[n]}_{\xi}(a)}{n!}(\xi(u)-\xi(a))^n + I^{\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi}f(u).$$

□

Theorem 2.0.5. *[11]For $\beta > 0$ and $f \in C^1[a,b]$, we have*

$${}^{C}D^{\beta}_{a^+,\xi}I^{\beta}_{a^+,\xi}f(u) = f(u)$$
$${}^{C}D^{\beta}_{b^-,\xi}I^{\beta}_{b^-,\xi}f(u) = f(u).$$

Proof. Using definition (2.1) and let $F(u) = I^{\beta}_{a^+,\xi} f(u)$, we have

$$^{C}D^{\beta}_{a^+,\xi} I^{\beta}_{a^+,\xi} f(u) = \frac{1}{\Gamma(k-\beta)} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} F^{[k]}_{\xi}(s) ds. \tag{2.7}$$

Consider

$$\begin{aligned}
F(u) &= \frac{1}{\Gamma(\beta)} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{\beta-1} f(s) ds \\
F^{[1]}_{\xi}(u) &= \frac{1}{\Gamma(\beta)} \left(\frac{1}{\xi'(u)} \frac{d}{du} \right) \int_a^u \xi'(s)(\xi(u)-\xi(s))^{\beta-1} f(s) ds \\
F^{[1]}_{\xi}(u) &= \frac{1}{\Gamma(\beta)} \frac{1}{\xi'(u)} \int_a^u \xi'(s)(\beta-1)(\xi(u)-\xi(s))^{\beta-2} \xi'(u) f(s) ds \\
F^{[1]}_{\xi}(u) &= \frac{(\beta-1)}{\Gamma(\beta)} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{\beta-2} f(s) ds \\
F^{[2]}_{\xi}(u) &= \frac{(\beta-1)(\beta-2)}{\Gamma(\beta)} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{\beta-3} f(s) ds \\
&= \cdots \\
F^{[k-1]}_{\xi}(u) &= \frac{(\beta-1)(\beta-2)\cdots(\beta-k+1)}{(\beta-1)(\beta-2)\cdots(\beta-k+1)(\beta-k)!} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{k} f(s) ds \\
&= \frac{1}{(\beta-k)!} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{\beta-k} f(s) ds.
\end{aligned}$$

Now Integrating by parts

$$\begin{aligned}
F^{[k-1]}_{\xi}(u) &= \frac{1}{(\beta-k)!} [-f(s) \int_a^u -\xi'(s)(\xi(u)-\xi(s))^{\beta-k} ds \\
&+ \int_a^u \left(\frac{d}{ds} f(s) \int_a^u -\xi'(s)(\xi(u)-\xi(s))^{\beta-k} ds \right) ds] \\
&= \frac{1}{\Gamma(\beta-k+1)} [-f(s) \frac{(\xi(u)-\xi(s))^{\beta-k+1}}{(\beta-k+1)} |_a^u \\
&+ \int_a^u f'(s) \frac{(\xi(u)-\xi(s))^{\beta-k+1}}{(\beta-k+1)} ds] \\
&= \frac{1}{\Gamma(\beta-k+1)} [f(a) \frac{(\xi(u)-\xi(a))^{\beta-k+1}}{(\beta-k+1)} \\
&+ \int_a^u f'(s) \frac{(\xi(u)-\xi(s))^{\beta-k+1}}{(\beta-k+1)} f'(s) ds]
\end{aligned}$$

$$F_{\xi}^{[k-1]}(u) = \left[\frac{f(a)}{\Gamma(\beta-k+1)}\frac{(\beta-k+1)(\xi(u)-\xi(a))^{\beta-k}}{\xi'(u)(\beta-k+1)}\xi'(u)\right]$$
$$+\frac{1}{\Gamma(\beta-k+1)}\frac{(\beta-k+1)}{\xi'(u)(\beta-k+1)}\int_a^u (\xi(u)-\xi(a))^{\beta-k}\xi'(u)f'(s)ds$$
$$= \frac{f(a)}{\Gamma(\beta-k+1)}(\xi(u)-\xi(a))^{\beta-k} + \frac{1}{\Gamma(\beta-k+1)}\int_a^u (\xi(u)-\xi(a))^{\beta-k}f'(s)ds.$$

Using above result in (2.7), we get

$$^{C}D^{\beta}_{a^+,\xi}I^{\beta}_{a^+,\xi}f(u) = \frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}\frac{f(a)}{\Gamma(\beta-k+1)}(\xi(u)-\xi(a))^{\beta-k}$$
$$+\frac{1}{\Gamma(\beta-k+1)}\int_a^u (\xi(u)-\xi(a))^{\beta-k}f'(s)dsds$$
$$= \frac{f(a)}{\Gamma(k-\beta)\Gamma(\beta-k+1)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}$$
$$\times(\xi(u)-\xi(a))^{\beta-k}ds\frac{1}{\Gamma(k-\beta)\Gamma(\beta-k+1)}$$
$$\times\int_a^u\int_a^s \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}(\xi(s)-\xi(y))^{\beta-k}f'(y)dyds.$$

Using Dirichlet's formula

$$^{C}D^{\beta}_{a^+,\xi}I^{\beta}_{a^+,\xi}f(u) = \frac{f(a)(\xi(u)-\xi(a))^{k-\beta-1}}{\Gamma(k-\beta)\Gamma(\beta-k+1)}\int_a^u \xi'(s)\left(1-\frac{\xi(s)-\xi(a)}{\xi(u)-\xi(a)}\right)^{k-\beta-1}$$
$$\times(\xi(u)-\xi(a))^{\beta-k}ds\frac{1}{\Gamma(k-\beta)\Gamma(\beta-k+1)}$$
$$\times\int_a^u\int_s^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}(\xi(s)-\xi(y))^{\beta-k}f'(y)dsdy.$$

Using the change of variable $v=\xi(s)-\xi(a)/\xi(u)-\xi(a)$,

$$^{C}D^{\beta}_{a^+,\xi}I^{\beta}_{a^+,\xi}f(u) = \frac{f(a)}{\Gamma(k-\beta)\Gamma(\beta-k+1)}\int_a^u (1-v)^{k-\beta-1}(v)^{\beta-k}dv$$
$$\times\frac{1}{\Gamma(k-\beta)\Gamma(\beta-k+1)}\int_a^t\int_s^t \xi'(y)(\xi(t)-\xi(y))^{k-\beta-1}$$
$$\times(\xi(y)-\xi(s))^{\beta-k}f'(s)dyds$$
$$^{C}D^{\beta}_{a^+,\xi}I^{\beta}_{a^+,\xi}f(u) = \frac{f(a)}{\Gamma(k-\beta)\Gamma(\beta-k+1)}\Gamma(k-\beta)\Gamma(\beta-k+1)$$
$$\times\frac{1}{\Gamma(k-\beta)\Gamma(\beta-k+1)}\int_a^u \Gamma(k-\beta)\Gamma(\beta-k+1)f'(s)ds$$
$$= f(u).$$

□

Theorem 2.0.6. *[11]For $\beta > 0$ if $f_1, f_2 \in C^k[a,b]$, then*

$$^{C}D^{\beta}_{a^+,\xi} f_1(u) = \ ^{C}D^{\beta}_{a^+,\xi} f_2(u) \Leftrightarrow f_1(u) = f_2(u) + \sum_{n=0}^{k-1} c_n(\xi(u) - \xi(a))^n,$$

where c_n is constant.

Proof. If $^{C}D^{\beta}_{a^+,\xi} f_1(u) = \ ^{C}D^{\beta}_{a^+,\xi} f_2(u)$, that is

$$^{C}D^{\beta}_{a^+,\xi}(f_1(u) - f_2(u)) = 0.$$

Applying left integral operator to both sides

$$I^{\beta}_{a^+,\xi} \ ^{C}D^{\beta}_{a^+,\xi}(f_1(u) - f_2(u)) = I^{\beta}_{a^+,\xi}(0).$$
$$I^{\beta}_{a^+,\xi} \ ^{C}D^{\beta}_{a^+,\xi}(f_1(u) - f_2(u)) = 0.$$

Using theorem (2.0.4), we get

$$f_1(u) - f_2(u) - \sum_{n=0}^{k-1} \frac{(f_1 - f_2)^{[n]}_{\xi}(a)}{n!}(\xi(u) - \xi(a))^n = 0$$
$$f_1(u) = f_2(u) + \sum_{n=0}^{k-1} \frac{(f_1 - f_2)^{[n]}_{\xi}(a)}{n!}(\xi(u) - \xi(a))^n.$$

Taking $c_n = \dfrac{(f_1 - f_2)^{[n]}_{\xi}(a)}{n!}$,

$$f_1(u) = f_2(u) + \sum_{n=0}^{k-1} c_n(\xi(u) - \xi(a))^n.$$

Now conversly suppose

$$f_1(u) = f_2(u) + \sum_{n=0}^{k-1} c_n(\xi(u) - \xi(a))^n.$$

Applying left ξ-Caputo fractional derivative of order β on both sides

$$^{C}D^{\beta}_{a^+,\xi} f_1(u) = \ ^{C}D^{\beta}_{a^+,\xi} f_2(u) + \ ^{C}D^{\beta}_{a^+,\xi} \sum_{n=0}^{k-1} c_n(\xi(u) - \xi(a))^n$$
$$^{C}D^{\beta}_{a^+,\xi} f_1(u) = \ ^{C}D^{\beta}_{a^+,\xi} f_2(u) + \sum_{n=0}^{k-1} c_n(^{C}D^{\beta}_{a^+,\xi})(\xi(u) - \xi(a))^n.$$

Using (2.6), we get

$$\begin{aligned} {}^{C}D^{\beta}_{a^+,\xi} f_1(u) &= {}^{C}D^{\beta}_{a^+,\xi} f_2(u) \\ {}^{C}D^{\beta}_{a^+,\xi} f_1(u) &= {}^{C}D^{\beta}_{a^+,\xi} f_2(u). \end{aligned}$$

□

Theorem 2.0.7. *[11] For $\beta > 0$ and $f \in C^{m+k}[a,b]$, for all $n, m \in \mathbb{N}$, we have*

$$(I^{\beta}_{a^+,\xi})^n ({}^{C}D^{\beta}_{a^+,\xi})^m f(u) = \frac{({}^{C}D^{\beta}_{a^+,\xi})^m f(z)}{\Gamma(\beta n + 1)} (\xi(u) - \xi(a))^{n\beta}.$$

For some $z \in (a, u)$.

Proof. By semigroup property for fractional integral

$$\begin{aligned} (I^{\beta}_{a^+,\xi})^n &= I^{\beta}_{a^+,\xi} . I^{\beta}_{a^+,\xi} \cdots I^{\beta}_{a^+,\xi} \\ &= I^{\beta+\beta+\beta+\cdots+\xi}_{a+} = I^{\beta n}_{a^+,\xi}. \end{aligned}$$

Now consider

$$\begin{aligned} (I^{\beta}_{a^+,\xi})^n ({}^{C}D^{\beta}_{a^+,\xi})^m f(u) &= I^{\beta n}_{a^+,\xi} ({}^{C}D^{\beta}_{a^+,\xi})^m f(u) \\ &= \frac{1}{\Gamma(\beta n)} \int_a^u \xi'(s)(\xi(u) - \xi(s))^{\beta n - 1} ({}^{C}D^{\beta}_{a^+,\xi})^m f(s) ds. \end{aligned}$$

Making use of mean value theorom for integral, we get

$$\begin{aligned} (I^{\beta}_{a^+,\xi})^n ({}^{C}D^{\beta}_{a^+,\xi})^m f(u) &= \frac{1}{\Gamma(\beta n)} ({}^{C}D^{\beta}_{a^+,\xi})^m f(z) \int_a^u \xi'(s)(\xi(u) - \xi(s))^{\beta n - 1} ds \\ &= \frac{1}{\Gamma(\beta n + 1)} ({}^{C}D^{\beta}_{a^+,\xi})^m f(z) (\xi(u) - \xi(s))^{\beta n}. \end{aligned}$$

□

Theorem 2.0.8. *[11] For $\beta > 0$ and $f \in C^{m+k}[a,b]$, for all $m \in N$, we have*

$${}^{C}D^{\beta}_{a^+,\xi} \left(\frac{1}{\xi'(u)} \frac{d}{du} \right)^m f(u) = {}^{C}D^{\beta+m}_{a^+,\xi} f(u).$$

Proof. From definition (2.1), we have

$$
\begin{aligned}
{}^{C}D^{\beta}_{a^+,\xi}\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m f(u) &= \frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} \\
&\times \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m f^{[k]}_{\xi}(s)ds \\
&= \frac{1}{\Gamma((k+m)-(\beta+m))}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{(k+m-\beta-m-1)} \\
&\times f^{[k+m]}_{\xi}(s)ds \\
&= {}^{C}D^{\beta+m}_{a^+,\xi}f(u).
\end{aligned}
$$

□

In general we know

$$\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^{C}D^{\beta}_{a^+,\xi}f(u) \neq {}^{C}D^{\beta+m}_{a^+,\xi}f(u).$$

For $\xi(u)=u$, result does not hold. From Theorom 2.0.8, we define $\eta=\beta-(k-1)\in(0,1)$, we have

$${}^{C}D^{\beta}_{a^+,\xi}f(u) = {}^{C}D^{\eta+m}_{a^+,\xi}f^{[k-1]}_{\xi}(u).$$

Theorem 2.0.9. *[11] For $\beta>0$ and $f\in C^{m+k}[a,b]$, for all $m\in\mathbb{N}$, we have*

$$\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^{C}D^{\beta}_{a^+,\xi}f(u) = {}^{C}D^{\beta+m}_{a^+,\xi}f(u) + \sum_{n=0}^{m-1}\frac{(\xi(u)-\xi(a))^{n+k-\beta-m}}{\Gamma(n+k-\beta-m+1)}f^{[n+k]}_{\xi}(a).$$

Proof. From (2.1), we have

$${}^{C}D^{\beta+m}_{a^+,\xi}f(u) = \frac{1}{\Gamma(k-\beta)}\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}f^{[k+m]}_{\xi}(s)ds.$$

Integrating by parts, we get

$$
\begin{aligned}
\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^{C}D^{\beta}_{a^+,\xi}f(u) &= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m [\frac{(\xi(u)-\xi(a))^{k-\beta}}{\Gamma(k-\beta+1)}f^{[k]}_{\xi}(a) + \frac{1}{\Gamma(k-\alpha+1)} \\
&\times \int_a^t (\xi(u)-\xi(s))^{k-\beta}\frac{d}{ds}f^{[k]}_{\xi}(s)ds].
\end{aligned}
$$

Multiplying and dividing by $\xi'(s)$, we have

$$\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^C D^{\beta}_{a^+,\xi} f(u) = \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{m-1} [\frac{(\xi(u)-\xi(a))^{k-\beta-1}}{\Gamma(k-\beta)} f^{[k]}_{\xi}(a) + \frac{1}{\Gamma(k-\beta)} \times \int_a^u \xi'(s)(\xi(u)-\xi(a))^{k-\beta-1} \frac{1}{\xi'(s)}\frac{d}{ds} f^{[k]}_{\xi}(s)ds]$$

$$\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^C D^{\beta}_{a^+,\xi} f(u) = \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{m-1} [\frac{(\xi(u)-\xi(s))^{k-\beta-1}}{\Gamma(k-\beta)} f^{[k]}_{\xi}(a) + \int_a^u \frac{\xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}}{\Gamma(k-\beta)} f^{[k+1]}_{\xi}(s)ds].$$

Integrating again

$$\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^C D^{\beta}_{a^+,\xi} f(u) = \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{m-2} [\sum_{k=0}^{1} \frac{(\xi(u)-\xi(s))^{n+k-\beta-2}}{\Gamma(n+k-\beta-1)} f^{[n+k]}_{\xi}(a) + \frac{1}{\Gamma(k-\beta)} \int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} f^{[k+2]}_{\xi}(s)ds].$$

Repeating this process $(m-2)$-times, we get the result

$$\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^C D^{\beta}_{a^+,\xi} f(u) = \sum_{k=0}^{m-1} \frac{(\xi(u)-\xi(a))^{n+k-\beta-m}}{\Gamma(n+k-\beta-m+1)} f^{[n+k]}_{\xi}(a) + {}^C D^{\beta}_{a^+,\xi} f(u).$$

□

$f^{[k]}_{\xi}(a) = 0$ for all $k = n, n+1, ..., n+m-1$, in above theorem, we get

$$\left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^m {}^C D^{\beta}_{a^+,\xi} f(u) = {}^C D^{\beta+m}_{a^+,\xi} f(u).$$

Theorem 2.0.10. *[11] Let $\beta, \eta > 0$ be such that there exist some $n \in \mathbb{N}$, with $\eta, \beta+\eta \in [n-1, n]$. Then for $f \in C^n[a,b]$ the following hold*

$${}^C D^{\beta}_{a^+,\xi} \; {}^C D^{\eta}_{a^+,\xi} f(u) = {}^C D^{\beta+\eta}_{a^+,\xi} f(u).$$

Proof. By assumption $\beta+\eta = n$, for $\beta \in [n-1, n]$, we can write $[\eta] = n-1 = \beta+\eta-1$

$${}^C D^{\beta}_{a^+,\xi} f(u) \; {}^C D^{\eta}_{a^+,\xi} f(u) = {}^C D^{\beta}_{a^+,\xi} \; I^{\beta+\eta-\eta}_{a^+,\xi} f^{[\beta+\eta]}_{\xi}.$$

Using Theorem 2.0.5, we have

$$ {}^{C}D^{\beta}_{a^+,\xi} f(u)\ {}^{C}D^{\eta}_{a^+,\xi} f(u) = f_{\xi}^{[\beta+\eta]} = {}^{C}D^{\beta+\eta}_{a^+,\xi} f(u). $$

Now if $\beta + \eta < n$, then for $\beta \in (0,1)$ and $[\eta] = [\beta+\eta] = k-1$, so by Theorem 2.0.3, we have

$$ {}^{C}D^{\beta}_{a^+,\xi} f(u) = D^{\beta}_{a^+,\xi}\left[f(u) - \sum_{k=0}^{k-1} \frac{1}{k!}(\xi(u)-\xi(a))^k f_{\xi}^{[k]}(a) \right] $$

$$ {}^{C}D^{\beta}_{a^+,\xi}\ {}^{C}D^{\eta}_{a^+,\xi} f(u) = D^{\beta}_{a^+,\xi}\left[{}^{C}D^{\eta}_{a^+,\xi} f(u) - \sum_{k=0}^{k-1} \frac{1}{k!}(\xi(u)-\xi(a))^k\ {}^{C}D^{\eta}_{a^+,\xi} f_{\xi}^{[k]}(a) \right]. $$

Since ${}^{C}D^{\eta}_{a^+,\xi} f(a) = 0$, we get

$$ {}^{C}D^{\beta}_{a^+,\xi}\ {}^{C}D^{\eta}_{a^+,\xi} f(u) = D^{\beta}_{a^+,\xi}\ {}^{C}D^{\eta}_{a^+,\xi} f(u). $$

Since we know that $D^{\beta}_{a^+,\xi} = \left(\frac{1}{\xi'(u)}\frac{d}{du}\right) I^{1-\beta}_{a^+,\xi}$ and ${}^{C}D^{\eta}_{a^+,\xi} f(u) = I^{[\eta]+1-\eta}_{a^+,\xi} f_{\xi}^{[\eta+1]}(u)$, therefore we have

$$ \begin{aligned} {}^{C}D^{\beta}_{a^+,\xi}\ {}^{C}D^{\eta}_{a^+,\xi} f(u) &= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right) I^{1-\beta}_{a^+,\xi} I^{[\eta]+1-\eta}_{a^+,\xi} f_{\xi}^{[\eta+1]}(u) \\ &= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right) I^{1,\xi}_{a+} I^{-\beta,\xi}_{a+} I^{[\beta+\eta]+1-\eta,\xi}_{a+} f_{\xi}^{[\beta+\eta]+1}(u) \\ &= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right) I^{1,\xi}_{a+} I^{[\beta+\eta]+1-(\beta+\eta),\xi}_{a+} f_{\xi}^{[\beta+\eta]+1}(u) \\ &= {}^{C}D^{\beta+\eta}_{a^+,\xi} f(u). \end{aligned} $$

□

Theorem 2.0.11. *[11] If $\beta > 0$ and $f \in C^n[a,b]$, then*

$$ {}^{C}D^{k-\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi} f(u) = {}^{C}D^{k}_{a^+,\xi} f(u). $$

Proof. Since we know that ${}^{C}D^{\beta}_{a^+,\xi} f(a) = 0$, so by Theorem 2.0.3, we get

$$ {}^{C}D^{k-\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi} f(u) = D^{k-\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi} f(u) $$

$$
\begin{aligned}
{}^{C}D^{k-\beta}_{a^+,\xi}\ {}^{C}D^{\beta}_{a^+,\xi} f(u) &= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{k-[\beta]} I^{\beta-[\beta],\xi}_{a+} I^{[\beta]+1-\beta}_{a+} f^{[\beta]+1}_{\xi}(u) \\
&= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{k-[\beta]-1} \left(\frac{1}{\xi'(u)}\frac{d}{du}\right) I^{1,\xi}_{a+} f^{[\beta]+1}_{\xi}(u) \\
&= \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{k-[\beta]-1} \left(\frac{1}{\xi'(u)}\frac{d}{du}\right)^{[\beta]+1} f(u) \\
&= {}^{C}D^{k,\xi}_{a+} f(u).
\end{aligned}
$$

□

Theorem 2.0.12. *[11] For $\beta > 0$, $g \in C[a,b]$, and $h \in C^k[a,b]$, then we have*

$$
\int_a^b g(u)\ {}^{C}D^{\beta}_{a^+,\xi} h(u)du = \int_a^b D^{\beta}_{b^-,\xi}\left(\frac{g(u)}{\xi'(u)}\right) h(t)\xi'(u)du + \left[\sum_{n=0}^{k-1}\left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^{n} I^{k-\beta}_{b^-,\xi} h^{[k-n-1]}_{\xi}(u)\right]_a^b .
$$

Proof. By (2.1), we have

$$
\begin{aligned}
\int_a^b g(u)\ {}^{C}D^{\beta}_{a^+,\xi} h(u)du &= \frac{1}{\Gamma(k-\beta)}\int_a^b g(u)\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1} h^{[k]}_{\xi}(s)dsdu \\
&= \frac{1}{\Gamma(k-\beta)}\int_a^b g(u)\int_a^u \xi'(s)(\xi(u)-\xi(s))^{k-\beta-1}\frac{1}{\xi'(s)} \\
&\times \frac{d}{ds}\ h^{[k-1]}_{\xi}(s)dsdu \\
&= \frac{1}{\Gamma(k-\beta)}\int_a^b \int_a^u g(u)(\xi(u)-\xi(s))^{k-\beta-1}\frac{d}{ds}\ h^{[k-1]}_{\xi}(s)dsdu.
\end{aligned}
$$

Using drichlet formula, we get

$$
\begin{aligned}
\int_a^b g(u)\ {}^{C}D^{\beta}_{a^+,\xi} h(u)du &= \frac{1}{\Gamma(k-\beta)}\int_a^b \int_u^b g(u)(\xi(u)-\xi(s))^{k-\beta-1}\frac{d}{ds}\ h^{[k-1]}_{\xi}(s)dsdu \\
&= \frac{1}{\Gamma(k-\beta)}\int_a^b \int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\frac{d}{du}\ h^{[k-1]}_{\xi}(u)du.
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
\int_a^b g(u)\ {}^{C}D^{\beta}_{a^+,\xi}h(u)du &= \frac{1}{\Gamma(k-\beta)}\left[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\ h_\xi^{[k-1]}(u)\right]_a^b \\
&- \frac{1}{\Gamma(k-\beta)}\int_a^b \frac{d}{du}\left[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}d\right]\ h_\xi^{[k-1]}(u)du \\
&= \frac{1}{\Gamma(k-\beta)}\left[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\ h_\xi^{[k-1]}(u)\right]_a^b \\
&+ \frac{1}{\Gamma(k-\beta)}\int_a^b \left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)\left[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\right] \\
&\times \frac{d}{du}\ h_\xi^{[k-2]}(u)du.
\end{aligned}
$$

Again integrating by parts

$$
\begin{aligned}
\int_a^b g(u)\ {}^{C}D^{\beta}_{a^+,\xi}h(u)du &= \frac{1}{\Gamma(k-\beta)}\left[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\ h_\xi^{[k-1]}(u)\right]_a^b \\
&+ \frac{1}{\Gamma(k-\beta)}\left[-\frac{1}{\xi'(u)}\frac{d}{du}\left[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\right] h_\xi^{[k-2]}\right]_a^b \\
&- \frac{1}{\Gamma(k-\beta)}\int_a^b \frac{d}{du}\left[-\frac{1}{\xi'(u)}\frac{d}{du}\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\right] \\
&\times h_\xi^{[k-2]}(u)du \\
&= \sum_{n=0}^{1}\left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^n \frac{1}{\Gamma(k-\beta)}[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds \\
&\times h_\xi^{[k-n-1]}(u)]_a^b + \frac{1}{\Gamma(k-\beta)}\int_a^b\left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^2 \\
&\times \left[\int_u^b f(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\right]\frac{d}{du}\ h_\xi^{[k-3]}(u)du.
\end{aligned}
$$

Repeating process $(k-2)-$ times, we get

$$
\begin{aligned}
\int_a^b g(u)\ {}^{C}D^{\beta}_{a^+,\xi}h(u)du &= \sum_{n=0}^{k-1}\left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^n \frac{1}{\Gamma(k-\beta)}[\int_u^b g(s)(\xi(s)-\xi(u))^{k-\beta-1}ds \\
&\times h_\xi^{[k-n-1]}(u)]_a^b + \frac{1}{\Gamma(k-\beta)}\int_a^b\left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^n \\
&\times \left[\int_u^b f(s)(\xi(s)-\xi(u))^{k-\beta-1}ds\right]\frac{d}{du}\ h(u)\xi'(u)du
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b g(u)\ {}^C D^{\beta}_{a^+,\xi} h(u)du &= \sum_{n=0}^{k-1} \left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^n \frac{1}{\Gamma(k-\beta)}[\int_u^b \xi'(s)(\xi(s)-\xi(u))^{k-\beta-1} \\
&\times \frac{g(s)}{\xi'(s)} ds\ h_{\xi}^{[k-n-1]}(u)]_a^b + \frac{1}{\Gamma(k-\beta)} \int_a^b \left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^n \\
&\times \left[\int_u^b \xi'(s)(\xi(s)-\xi(u))^{k-\beta-1} \frac{f(s)}{\xi'(s)} ds\right] \frac{d}{du}\ h(u)\xi'(u)du \\
&= \sum_{n=0}^{k-1} \left(-\frac{1}{\xi'(u)}\frac{d}{du}\right)^n I^{k-\beta}_{b^-,\xi} \left(\frac{g(u)}{\xi'(u)}\right) ds\ h_{\xi}^{[k-n-1]}(u)]_a^b \\
&+ \int_a^b D^{\beta}_{b^-,\xi} \left(\frac{g(u)}{\xi'(u)}\right)\ h(u)\xi'(u)du.
\end{aligned}
$$

□

Chapter 3

Generalized Mellin transform

3.1 Introduction

The idea of integral transform arises from the Fourier integral formula. Integral transform are use to solve initial value problem and intial-boundry value problems for linear differential equation. Real life problems mostly involve time. And is taken as infinite in the domain. For this purpose mostly used integral transforms are Laplace, Fourier and Mellin transform.

Mellin transform is closely connected to Laplace and Fourier transformation. The formula for Mellin and inverse Mellin transform are derived from Fourier transform. First time Riemann used the transformation to study the zeta function in his memor. Cahen [13] further extend this work. It was the R. H. Mellin who first gave the systematic formulation of the Mellin transform and its inverse.

Mellin transform is very useful in many areas of engineering and physics. Problems regarding number theory, mathematical statistics, and the theory of asymptotic expansions can be solved using Mellin transform. There are many application of Mellin transformation that the computational solution of a potential problem in a Wedge-shaped region function is most famous, resolution of linear differential equation in electrical engineering is another important application.

So much work have been done on this field by different mathematician. In this chapter we have introduce the generalization of Mellin transform and discuss some properties. Also we apply this transform on some generalized fractional operators.

3.2 Mellin transform

Since we know that f satisfies the Dirichlet conditions in $(-a, a)$, therefore complex Fourier series can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(in\pi x/a), \tag{3.1}$$

where

$$c_n = \frac{1}{2a}\int_{-a}^{a} f(\eta)\exp(-in\pi\eta/a)d\eta. \tag{3.2}$$

The function f is periodic of period 2π. Since we know that Fourier series is valid for the problems over finite interval and Fourier integral are valid for the problems over $(-\infty, \infty)$. Now we want to find the integral representation of (3.1) by letting $a \to \infty$. Let $k_n = \frac{n\pi}{a}$ then $\delta k = k_n - k_{n-1} = \frac{\pi}{a}$ and using (3.2) in (3.1), we get

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \delta k \left[\int_{-a}^{a} f(\eta)\exp(-i\eta k_n)d\eta\right]\exp(ixk_n). \tag{3.3}$$

k_n becomes k and δk becomes dk when $a \to \infty$. So the summition can be replaced by integral in (3.3), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\eta)\exp(-i\eta k)d\eta\right]\exp(ixk)dk. \tag{3.4}$$

From above defination (3.4), we get the formula for Fourier transform and it's inverse, which can be written as

$$F(g(t)) = G(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ist}g(t)dt, \tag{3.5}$$

$$F^{-1}(G(s)) = g(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist}G(s)ds. \tag{3.6}$$

Using the change of variable $e^t = y$ and $is = u - v$ (where u is constant) $\implies e^t dt = dy$ and $ids = -dv$ in (3.5) and (3.6), we get

$$G(iv - iu) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} y^{v-u-1}g(\log(y))dy, \tag{3.7}$$

$$g(\log(y))) = \frac{1}{\sqrt{2\pi}}\int_{u-i\infty}^{u+i\infty} y^{u-v}G(iv - iu)dv. \tag{3.8}$$

Now we substitute $\frac{1}{2\pi}y^{-u}g(\log(y)) = f(y)$ and $G(iv - iu) = \tilde{f}(v)$ to define Mellin and inverse Mellin transform as

$$Mf(y) = \tilde{f}(v) = \int_0^\infty y^{v-1}f(y)dy, \tag{3.9}$$

$$M^{-1}\tilde{f}(y) = f(v) = \frac{1}{2i\pi}\int_{u-i\infty}^{u+i\infty} y^{-v}f(v)dv, \tag{3.10}$$

where $f(y)$ defined on $(0, \infty)$ and Mellin variable v is complex number in general.

3.2.1 Basic properties of Mellin transform

In the following; we are going to summarize some basic properties of classical Mellin transform.

Theorem 3.2.1. *Let f Lebesgue integerable function over* $\Re_+$*. Some basic properties of Mellin transform of function are as follow.*

(a) $M[f(au)] = a^{-p}\tilde{f}(p), \quad a > 0,$ (Scaling property).

(b) $M[u^a f(u)] = \tilde{f}(p + a),$ (Shifting property).

(c) $M[f(u^a)] = \frac{1}{a}\tilde{f}\left(\frac{p}{a}\right),$

(d) $M[f'(u)] = -(p-1)\tilde{f}(p-1),$

(e) $M[u^n f^n(u)] = (-1)^n \frac{\Gamma(n+p)}{\Gamma(p)}\tilde{f}(p),$

(f) $M\left[\int_0^u f(s)ds\right] = -\frac{1}{p}\tilde{f}(p+1),$

(g) $M[(uf')^n(u)] = (-1)^n p^n \tilde{f}(p),$

(h) $M[f(u) * g(u)] = \tilde{f}(p)\tilde{g}(p),$ (Convolution type).

Application of Mellin transform

Mellin transform is very useful in solving differential equations. In this section we discuss one of the application of Mellin transform.

Example [22] We solve the Laplace equation in polar coordinates (r, θ) to find the potential $u(r, \theta)$ in an infinite wedge. Consider

$$r^2 u_{rr}(r,\theta) + r u_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0, \tag{3.11}$$

in an infinite wedge $0 < r < \infty, -\alpha < \theta < \alpha, \alpha \in (0, \pi/2)$, with the boundary conditions

$$\left\{ \begin{array}{ll} u(r,\alpha) = u_+(r), & \text{for } 0 \leq r < \infty \\ u(r,-\alpha) = u_-(r), & \text{for } 0 \leq r < \infty \end{array} \right\} \tag{3.12}$$

$u(r, \theta) \longrightarrow 0$ as $r \longrightarrow \infty$, $\forall\ \theta \in (-\alpha, \alpha)$.

Applying Mellin transform on (3.11) with respect to the variable r and using property (e) of Theorem 3.2.1, we get

$$\begin{aligned} M[r^2 u_{rr}(r,\theta)] + M[r u_r(r,\theta)] + M[u_{\theta\theta}(r,\theta)] &= 0 \\ s(s+1)\tilde{u}(s,\theta) + (-s)\tilde{u}(s,\theta) + \tilde{u}_{\theta\theta}(s,\theta) &= 0 \\ \frac{\partial^2 \tilde{u}(s,\theta)}{\partial \theta^2} + s^2 \tilde{u}(s,\theta) &= 0. \end{aligned}$$

Solution of above differential equation is

$$\tilde{u}(s,\theta) = C(s)\cos(\theta s) + D(s)\sin(\theta s). \tag{3.13}$$

Using boundary conditions (3.12) and solving, we get

$$C(s) = \frac{\tilde{u}_+(s) + \tilde{u}_-(s)}{2\cos\ \alpha s}, \qquad D(s) = \frac{\tilde{u}_+(s) - \tilde{u}_-(s)}{2\sin\ \alpha s}.$$

Putting values of C and D in (3.13), we get

$$\tilde{u}(s,\theta) = \tilde{u}_+(s)\frac{\sin(\alpha+\theta)s}{\sin 2\alpha s} + \tilde{u}_-(s)\frac{\sin(\alpha-\theta)s}{\sin 2\alpha s}.$$

Now making substitution $\tilde{g}(s,\theta) = \dfrac{\sin\theta s}{\sin 2\alpha s}$, we have

$$\tilde{u}(s,\theta) = \tilde{u}_+(s)\tilde{g}(s,\alpha+\theta) + \tilde{u}_-(s)\tilde{g}(s,\alpha-\theta),$$

with $v = \pi/2\alpha,\ v \in (-1,1)$. By means of formula, namely

$$\tilde{g}(r,\theta) = M\left[\frac{v}{\pi}\frac{r^v sin\ v\theta}{1+2r^v\cos\ v\theta + r^{2v}}\right] = \frac{\sin\theta s}{\sin 2\alpha s},$$

and with the help of the convolution theorem of the Mellin transform we obtain the solution of our problem after a simple calculation:

$$u^*(r,\theta) = u_+ Vg(.,\alpha+\theta) + u_- Vg(.,\alpha-\theta)$$

$$= \frac{vr^v\cos\ vs}{\pi}[\int_0^\infty \frac{s^{n-1}u_+(\theta)}{s^{2v}-2(rs)^v\sin\ v\theta + r^{2v}}ds$$
$$+\int_0^\infty \frac{s^{n-1}u_-(\theta)}{s^{2v}+2(rs)^v\sin\ v\theta + r^{2v}}ds], \quad -1<v<1.$$

3.3 Generalized Mellin transform

In this section we introduce the generalization of classical Mellin transform called ξ-Mellin transform. We discuss some basic properties of this transformation and will use on some fractional operator. The idea of this generalization comes from Fahd etal. [20].

Definition 3.3.1. *The generalized Mellin transform with reapect to a function ξ of a real valued function $f(u)$ on $(0,\infty)$ is defined as*

$$M_\xi[f(u)] = \tilde{f}_\xi(p) = \int_0^\infty \xi^{p-1} f(u)\xi'(u)du, \quad p>0, \tag{3.14}$$

where ξ is an increasing function. We also use the notation $M_\xi[f(u),p]$ for $M_\xi[f(u)]$ whenever emphisis on Mellin variable is needed.

3.3.1 Properties of generalized Mellin transform

As we studied some properties of Mellin transform in the previous section. Now we will prove those properties in generalized case of Mellin transform defined in (3.14).

Lemma 3.3.2. *(Shifting property) Let f be Lebesgue integerable function over $\Re_+$ and ξ be an increasing function s.t $\xi(0) = 0$. Then we have*

$$M_\xi\{u^a f(u)\} = \tilde{f}_\xi(a+p). \tag{3.15}$$

Proof. By using the Defination 3.3.1, we have

$$\begin{aligned} M_\xi\{u^a f(u)\} &= \int_0^\infty \xi^{p-1}\xi^a f(s)\xi'(s)ds \\ &= \int_0^\infty \xi^{a+p-1} f(s)\xi'(s)ds \\ &= \int_0^\infty \xi^{(a+p)-1} f(s)\xi'(s)ds \\ &= \tilde{f}_\xi(a+p). \end{aligned}$$

□

Lemma 3.3.3. *The generalized Mellin transforms of derivatives of a function defined on (0, ∞) with $\xi(u)^{p-1}f(u) \to 0$ as $u = 0$ and $u \to \infty$ is*

$$M_\xi\{D_\xi^n f(u)\} = \frac{\Gamma(1-p+n)}{\Gamma(1-p)}\tilde{f}(p-n). \tag{3.16}$$

Proof. By the Definition 3.3.1,

$$M_\xi\{D_\xi f(u)\} = \int_0^\infty \xi^{p-1} D_\xi f(u)\xi'(u)du.$$

Using $D_\xi = \frac{1}{\xi'(u)}\frac{d}{du}$ and applying integration by parts, we get

$$M_\xi\{D_\xi f(u)\} = -(p-1)\int_0^\infty \xi^{(p-1)-1} f(u)\xi'(u)du$$

$$M_\xi\{D_\xi f(u)\} = -(p-1)\tilde{f}(p-1). \tag{3.17}$$

In a similar way we can prove for $n = 2$,

$$M_\xi\{D^2_\xi(u)\} = \int_0^\infty \xi^{p-1} D^2 f(u)\xi'(u)du$$
$$= \int_0^\infty \xi^{p-1} \frac{1}{\xi'(u)}\frac{d}{du} f^{'}(u)\xi'(u)du$$

$$\begin{aligned}
M_\xi\{D^2_\xi(u)\} &= \int_0^\infty \xi^{p-1} \frac{d}{du} f^{'}(u)du \\
&= \xi^{p-1} f^{'}|_0^\infty - \int_0^\infty (p-1)\xi^{p-2}\xi'(u)f^{'}(u)du \\
&= -(p-1)\int_0^\infty \xi^{p-2}\frac{1}{\xi'(u)}\frac{d}{du} f(u)\xi'(u)du \\
&= -(p-1)\int_0^\infty \xi^{p-2}\frac{d}{du} f(u)du.
\end{aligned}$$

Applying Integration by parts, we get

$$\begin{aligned}
M_\xi\{D^2_\xi(u)\} &= (p-1)(p-2)\int_0^\infty \xi^{p-3} f(u)\xi'(u)du \\
&= (p-1)(p-2)\tilde{f}(p-2).
\end{aligned}$$

Repeating this process $(n-2)$-times, we have

$$M_\xi\{D^n_\xi(u)\} = (-1)^n(p-1)(p-2)\cdots(p-n)\tilde{f}(p-n).$$

Now consider

$$\begin{aligned}
M_\xi\{D^n_\xi(u)\} &= (-1)^n(p-1)(p-2)(p-3)\cdots(p-n)\tilde{f}(p-n) \\
&= (-1)^n(-1)^n(1-p)(2-p)(3-p)\cdots(n-p)\tilde{f}(p-n) \\
&= (n-p)(n-p-1)(n-p-2)\cdots(3-p)(2-p)(1-p)\tilde{f}(p-n) \\
&= (n-p)(n-p-1)(n-p-2)\cdots(3-p)(2-p)(1-p)\frac{\Gamma(1-p)}{\Gamma(1-p)}\tilde{f}(p-n) \\
&= \frac{\Gamma(1-p+n)}{\Gamma(1-p)}\tilde{f}(p-n).
\end{aligned}$$

□

Lemma 3.3.4. *For a function f defined on $(0,\infty)$ with $\xi(u)$ increasing function s.t $\xi(0)=0$ and $D_\xi = \frac{1}{\xi'(u)}\frac{d}{du}$, we have*

$$M_\xi\{\xi(u)^n D_\xi^n f(u)\} = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p), \tag{3.18}$$

provided that $\xi^{p-k}f^k(u)$ vanish at $u=0$ and as $u \longrightarrow \infty$ for $k=0,1,2,3,...(n-1)$.

Proof. By Definition 3.3.1, we have

$$M_\xi\{\xi(u)D_\xi f(u)\} = \int_0^\infty \xi^p \frac{d}{du} f(u)du.$$

Applying integration by parts, we get

$$M_\xi\{\xi(u)D_\xi f(u)\} = -p\int_0^\infty \xi^{p-1} f(u)du \tag{3.19}$$

$$= -p\tilde{f}(p). \tag{3.20}$$

Similar argument can be used to prove (3.18). □

Lemma 3.3.5. *The generalized Mellin transform of differential operator of a function f defined on $(0,\infty)$ with ξ increasing function s.t $\xi(0)=0$ and $D_\xi = \frac{1}{\xi'(u)}\frac{d}{du}$ is*

$$M_\xi[(\xi(u)D_\xi)^n f(u)] = (-1)^n p^n \tilde{f}_\xi(p). \tag{3.21}$$

Proof. Consider

$$M_\xi[(\xi(u)D_\xi)^2 f(u)] = M_\xi[\xi(u)D_\xi(\xi(u)D_\xi)(u)].$$

Puting $g(u) = \xi(u)D_\xi f(u)$ in above and using equation (3.20),

$$\begin{aligned} M_\xi[(\xi(u)D_\xi)^2 f(u)] &= M_\xi[\xi(u)D_\xi g(u)] \\ &= -pM_\xi[g(u)] \\ &= -pM_\xi[\xi(u)D_\xi f(u)] \\ &= (-1)^2 p^2 M_\xi[f(u)]. \end{aligned}$$

Now for $n=3$,

$$M_\xi[(\xi(u)D_\xi)^3 f(u)] = M_\xi[(\xi(u)D_\xi)(\xi(u)D_\xi)^2 f(u)].$$

By using equation (3.20) and above result, we get

$$M_\xi[(\xi(u)D_\xi)^3 f(u)] = (-1)^3 p^3 M_\xi[f(u)].$$

Repeating this process for $(n-1)$-times, we can prove

$$M_\xi[(\xi(u)D_\xi)^n f(u)] = (-1)^n p^n M_\xi[f(u)].$$

□

Lemma 3.3.6. *Let f be Lebesgue integerable function over $\Re_+$ and $\xi(u)$ be an increasing function s.t $\xi(0) = 0$. The generalized Mellin transform of integral is*

$$M_\xi[I_n f(u)] = M_\xi\left[\int_0^u I_{n-1} f(s)\xi'(s)ds\right] = \frac{\Gamma(1-p-n)}{\Gamma(1-p)} M_\xi[f(u); p+n], \tag{3.22}$$

where I_n represents the nth repeated integral.

Proof. Taking

$$F(u) = \int_0^u f(s)\xi'(s)ds,$$

so that $F'(u) = f(u)\xi'(u)$ which implies $\frac{1}{\xi'(u)}F'(u) = D_\xi F(u) = f(u)$ with $F(0) = 0$. Now using equation (3.17) in above definition

$$M_\xi[D_\xi F(u); p] = -(p-1)M_\xi[F(u); p-1].$$

Now replacing p with $p+1$,

$$\begin{aligned} M_\xi[D_\xi F(u); p+1] &= -pM_\xi[F(u)] \\ M_\xi\left[\int_0^u f(s)\xi'(s)ds\right] &= -\frac{1}{p}M_\xi[f(u); p+1]. \end{aligned}$$

Putting $n = 2$ in left side of equation (3.22). And making use of above result, we get

$$\begin{aligned} M_\xi[I_2 f(u)] &= M_\xi\left[\int_0^u If(s)\xi'(s)ds\right] \\ &= -\frac{1}{p}M_\xi[If(u); p+1] \\ &= -\frac{1}{p}M_\xi\left[\int_0^u f(s)\xi'(s)ds, p+1\right] \\ &= \frac{(-1)^2}{p(p+1)}M_\xi[f(u); p+2]. \end{aligned}$$

Repeating this process for (n-2)-times, we get

$$M_\xi[I_\xi^n f(u)] = \frac{(-1)^n}{(p)(p+1)(p+2)(p+n-1)} M_\xi[f(u);p+n].$$

Now consider

$$(p)(p+1)(p+2)\cdots(p+n-1).$$

Substituting $p+n-1=-k$ in above expression

$$\begin{aligned}
&(-k)(-k-1)(-k-2)\cdots(-k-n+1) \\
&= (-1)^n(k)(k+1)(k+2)\cdots(k+n-1) \\
&= \frac{(-1)^n(k+n-1)(k+n)\cdots(k+1)(k)\Gamma(k)}{\Gamma(k)} \\
&= \frac{(-1)^n\Gamma(k+n)}{\Gamma(k)} \\
&= \frac{(-1)^n\Gamma(1-n-p+n)}{\Gamma(1-p-n)} = \frac{(-1)^n\Gamma(1-p)}{\Gamma(1-p-n)}.
\end{aligned}$$

Using this in above result, we have

$$M_\xi[I_\xi^n f(u)] = \frac{\Gamma(1-p-n)}{\Gamma(1-p)} M_\xi[f(u);p+n].$$

□

Lemma 3.3.7. *(Convolution type theorem) If f and g Lebesgue integerable functions defined on (0, ∞) with increasing function ξ. Then the generalized Mellin transform of their covolution is*

$$M_\xi[f(u) * g_\xi(u)] = \tilde{f}_\xi(p) * \tilde{g}_\xi(p), \tag{3.23}$$

$$M_\xi[f_\xi(u)\ o\ g(u)] = \tilde{f}_\xi(p)\tilde{g}_\xi(1-p). \tag{3.24}$$

Proof. We prove this theorm by using Defination (3.14) and (1.30). By using Fubini's theorm and the Dirichlet technique, we have

$$\begin{aligned}
M_\xi[f(u) * g_\xi(u)] &= \int_0^\infty \xi^{p-1}(u) \left[\int_0^\infty f(s) g\left(\xi^{-1}\left(\frac{\xi(u)}{\xi(s)}\right)\right)\frac{\xi'(s)}{\xi(s)}ds\right]\xi'(u)du \\
&= \int_0^\infty f(s)\frac{\xi'(s)}{\xi(s)} \int_0^\infty \xi^{p-1}(u) g\left(\xi^{-1}\left(\frac{\xi(u)}{\xi(s)}\right)\right)\xi'(u)du)ds.
\end{aligned}$$

Using the change of variable $v = \xi^{-1}\left(\dfrac{\xi(u)}{\xi(s)}\right)$,

$$
\begin{aligned}
M_\xi[f(u) * g_\xi(u)] &= \int_0^\infty (f(s)\frac{\xi'(s)}{\xi(s)} \int_0^\infty \xi(v)^{p-1}\xi(s)^{p-1} g(t)\xi'(v)\xi(s)dv)ds \\
&= \int_0^\infty f(s)\xi^{p-1}(s)\xi'(s)ds \int_0^\infty \xi^{p-1}(v)g(v)\xi'(v)dv \\
&= \tilde{f}_\xi(p) * \tilde{g}_\xi(p).
\end{aligned}
$$

Similarly by using (3.14) and (1.31), we have

$$
\begin{aligned}
M_\xi[f_\xi(u)\ o\ g\xi(u)] &= \int_0^\infty \xi^{p-1}(u) \int_0^\infty f(\xi^{-1}(\xi(u)\xi(s)))g(s)\xi'(s)ds\xi'(u)du \\
&= \int_0^\infty g(s)\xi'(s)\left[\int_0^\infty \xi^{p-1}(u)f(\xi^{-1}(\xi(u)\xi(s)))\xi'(u)du\right]ds.
\end{aligned}
$$

Using the change of variable $v = \xi^{-1}(\xi(u)\xi(s))$,

$$
\begin{aligned}
M_\xi[f_\xi(u)\ o\ g\xi(u)] &= \int_0^\infty g(s)\xi'(s) \int_0^\infty \frac{\xi(v)^{p-1}}{\xi(s)^{p-1}} f(v)\frac{\xi'(v)}{\xi(s)}dvds \\
&= \int_0^\infty g(s)\xi(s)^{-p}\xi'(s)ds \int_0^\infty (\xi(v))^{p-1}f(v)\xi'(v)dv \\
&= f_\xi(p)g_\xi(1-p).
\end{aligned}
$$

□

3.4 The generalized Mellin transform of fractional integrals and derivatives

Now we compute Mellin transforms of left and right sided RL-integrals and derivatives with respect to another function.

Theorem 3.4.1. *If $\alpha \in C$, $Re(\alpha) > 0$ with $f(u) \in X^1_{p+\alpha}(\Re_+)$ and ξ is an increasing function. Then*

$$
M_\xi\{{}^{RL}I^\alpha_{a^+,\xi}f(u)\} = \frac{\Gamma(1-p-\alpha)}{\Gamma(1-p)}\tilde{f}_\xi(p+a), \quad Re(p+\alpha) < 1, \quad u > a, \tag{3.25}
$$

$$
M_\xi\{{}^{RL}I^\alpha_{b^-,\xi}f(u)\} = \frac{\Gamma(p)}{\Gamma(p+\alpha)}\tilde{f}_\xi(p+a), \qquad Re(p) > 0, \quad u < b, \tag{3.26}
$$

provided $\tilde{f}_\xi(p+\alpha)$ exists for $p \in C$.

Proof. Using definitions of fractional integral with respect to functions (1.21), Fubini's theorem and the Dirichlet technique, we have

$$\begin{aligned} M_\xi\{{}^{RL}I^\alpha_{a^+,\xi}f(u)\} &= \frac{1}{\Gamma(\alpha)}\int_0^\infty \xi^{p-1}\xi'(u)\int_a^u f(s)\xi'(s)(\xi(u)\xi(s))^{\alpha-1}dsdu \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty f(s)\xi'(s)\int_s^\infty \xi^{p-1}\xi'(u)(\xi(u)\xi(s))^{\alpha-1}duds. \end{aligned}$$

Using the change of variable $v = \dfrac{\xi(s)}{\xi(u)}$, and relation of Beta function with Gamma function, we have

$$\begin{aligned} M_\xi\{{}^{RL}I^\alpha_{a^+,\xi}f(u)\} &= \frac{1}{\Gamma(\alpha)}\int_0^\infty f(s)(\xi(s))^{p+\alpha-1}\xi'(s)\int_0^1 v^{-p-\alpha}(1-v)^{\alpha-1}dvds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty f(s)(\xi(s))^{p+\alpha-1}\xi'(s)B(1-p-\alpha,\alpha)ds \\ &= \frac{\Gamma(1-p-\alpha)}{\Gamma(1-p)}\int_0^\infty (\xi(s))^{p+\alpha-1}f(s)\xi'(s)ds \\ &= \frac{\Gamma(1-p-\alpha)}{\Gamma(1-p)}\tilde{f}(p+a). \end{aligned}$$

Similarly we can prove (3.26). □

Theorem 3.4.2. *If $\alpha \in C$, $Re(\alpha) > 0$ with $f(u) \in X^1_{p+\alpha}(\Re_+)$ and ξ is an increasing function. Then*

$$M_\xi\{{}^{RL}D^\alpha_{a^+,\xi}f(u)\} = \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)}\tilde{f}(p-a), \quad Re(p) < 1, \quad u > a, \tag{3.27}$$

$$M_\xi\{{}^{RL}D^\alpha_{b^-,\xi}f(u)\} = \frac{\Gamma(p)}{\Gamma(p-\alpha)}\tilde{f}(p-a), \qquad Re(p-\alpha) > 0, \quad u < b, \tag{3.28}$$

provided $\tilde{f}_\xi(p-\alpha)$ exists for $p \in C$.

Proof. Apply Mellin transform on both sides of (1.23),

$$M_\xi\{{}^{RL}D^\alpha_{a^+,\xi}f(u)\} = M_\xi\{D^n_\xi {}^{RL}I^{n-\alpha}_{a^+,\xi}f(u)\}.$$

By virtue of property (3.16) and relation (3.25), we get

$$\begin{aligned} M_\xi\{^{RL}D^{\alpha}_{a^+,\xi}f(u)\} &= \frac{\Gamma(1-p+n)}{\Gamma(1-p)} M[^{RL}I^{n-\alpha}_{a^+}f(u);(p-n)] \\ &= \frac{\Gamma(1-p+n)}{\Gamma(1-p)} \frac{\Gamma(1-p+n-n+\alpha)}{\Gamma(1-p+n)} \tilde{f}_\xi(p-n+n-\alpha) \\ &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)} \tilde{f}_\xi(p-\alpha). \end{aligned}$$

Formula (3.28) can be proved in similar way. □

Theorem 3.4.3. *If $\alpha \in C$, $Re(\alpha) > 0$ with $f(u) \in X^1_{p+\alpha}(\Re_+)$ and ξ is an increasing function. Then*

$$M_\xi\{^{C}D^{\alpha}_{a^+,\xi}f(u)\} = \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)} \tilde{f}(p-a), \quad Re(p) < 1, \quad u > a, \tag{3.29}$$

$$M_\xi\{^{C}D^{\alpha}_{b^-,\xi}f(u)\} = \frac{\Gamma(p)}{\Gamma(p-\alpha)} \tilde{f}(p-a), \quad Re(p-\alpha) > 0, \quad u < b, \tag{3.30}$$

provided $\tilde{f}_\xi(p-\alpha)$ exists for $p \in C$.

Proof. Apply Mellin transform on both sides of (1.27),

$$M_\xi\{^{C}D^{\alpha}_{a^+,\xi}f(u)\} = M_\xi\{I^{n-\alpha}_{a^+,\xi}D^n_\xi f(u)\}.$$

By virtue of property (3.16) and relation (3.25), we get

$$\begin{aligned} M_\xi\{^{C}D^{\alpha}_{a^+,\xi}f(u)\} &= \frac{\Gamma(1-p-n+\alpha)}{\Gamma(1-p)} M[D^n_\xi f(u);(p+n-\alpha)] \\ &= \frac{\Gamma(1-p-n+\alpha)}{\Gamma(1-p)} \frac{\Gamma(1-p+n-n\alpha)}{\Gamma(1-p+n)} \tilde{f}_\xi(p-n+n-\alpha) \\ &= \frac{\Gamma(1-p+\alpha)}{\Gamma(1-p)} \tilde{f}_\xi(p-\alpha). \end{aligned}$$

Formula (3.30) can be proved in similar way. □

3.4.1 The generalized Mellin transform of Hadamard-type fractional integral

Firstly we express the Hadamard-type fractional integral with respect to function defined in (1.29) as Mellin convolution $K * f$ as

$${}^{H}I^{\alpha,\mu}_{a^+,\xi}f(u) = (K*f)(u) = \int_0^\infty K\left(\xi^{-1}\left(\frac{\xi(u)}{\xi(s)}\right)\right) f(s)\frac{\xi'(s)}{\xi(s)}ds, \tag{3.31}$$

where the function K is given by

$$K(u) = \left\{ \begin{array}{ll} 0, & \text{for } (0 < u < \xi^{-1}(1)) \\ \dfrac{\xi(u)^{-\mu}}{\Gamma(\alpha)}(\log \xi(u))^{\alpha-1}, & \text{for } (u > \xi^{-1}(1)) \end{array} \right\} \tag{3.32}$$

Lemma 3.4.1. *Let $\alpha > 0$, $\mu \in \mathbb{C}$ and $c \in \Re$ then $K \in X^1_{c,\xi}$ if $Re(\mu) > 0$.*

Proof. From equations (1.33) and (3.32), we have

$$\|K\|_{X^1_{c,\xi}} = \frac{1}{\Gamma(\alpha)} \int_{\xi^{-1}(1)}^{\infty} (\xi(u))^{c-\mu} (\log \xi(u))^{\alpha-1} \frac{\xi'(u)}{\xi(u)} du.$$

Using the change of variable $\log \xi(u) = v \Rightarrow \xi(u) = e^v$, we have

$$\|K\|_{X^1_{c,\xi}} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-[Re(\mu-c)]v} (v)^{\alpha-1} dv.$$

Integral is convergent for $\alpha > 0$ and $Re(\mu) > 0$. □

Lemma 3.4.2. *For $\alpha > 0$, $\mu, s \in \mathbb{C}$ and $Re(\mu - s) > 0$. Then*

$$(M_\xi K)(u) = (\mu - s)^{-\alpha}. \tag{3.33}$$

Proof. Applying defination of Mellin transform (3.14) and using (3.32). And making substitution $\xi(u) = e^v$ and $v(\mu - s) = \tau$, we have

$$\begin{aligned} (M_\xi K)(u) &= \frac{1}{\Gamma(\alpha)} \int_{\xi^{-1}(1)}^{\infty} (\xi(u))^{s-\mu-1} (\log \xi(u))^{\alpha-1} du \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\mu(\mu-s)} v^{\alpha-1} dv \\ &= \frac{(\mu - s)^{-\alpha}}{\Gamma(\alpha)} \int_0^{\infty} e^{-\tau} \tau^{\alpha-1} d\tau \\ &= (\mu - s)^{-\alpha}. \end{aligned}$$

By using the definition of gamma function lemma is proved for $\mu, s \in \Re$.

This result is also true for complex μ and s by analytic continuation when $Re(\mu - s) > 0$, and $Re(\mu + s) > 0$, respectively. □

Lemma 3.4.3. *For $\alpha > 0, \mu \in \mathbb{C}$ if $Re(\mu) > 0$ and $f \in X^p_{c,\xi}$, Then the Mellin transform of ${}^{HT}I^{\alpha,\mu}_{a+,\xi} f$ is given by*

$$M_\xi\{{}^{HT}I^{\alpha,\mu}_{a+,\xi} f(u)\} = (\mu - s)^{-\alpha} (M_\xi f)(u).$$

Proof. According to (3.31) the Hadamard-type operator ${}^{HT}I^{\alpha,\mu}_{a+,\xi}f$ is a Mellin convolation operator with the Kernel.

By lemma (3.4.4) then applying (3.23) to (3.31) and using (3.33), we obtain

$$\begin{aligned} M_\xi\{{}^{HT}I^{\alpha,\mu}_{a+,\xi}f(u)\} &= (M_\xi K)(u)(M_\xi f)(u) \\ &= (\mu - s)^{-\alpha}(M_\xi f)(u). \end{aligned}$$

□

3.4.2 The generalized Mellin transform of Weyl fractional integral and derivative with respect to function

Theorem 3.4.4. *Let* $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ *with* f *be integrable function defined on* $(-\infty, \infty)$ *and* ξ *is an increasing function then*

$$M_\xi\left[W_\xi^{-\alpha}[f(u)]\right] = \frac{\Gamma(p)}{\Gamma(p+\alpha)}\tilde{f}_\xi(p+\alpha). \tag{3.34}$$

Proof. Using the Definition (1.34)of Weyl fractional integral with respect to another function

$$\begin{aligned} W_\xi^{-\alpha}[f(u)] = F(u,\alpha) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty (\xi(s)-\xi(u))^{\alpha-1}f(s)\xi'(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty (\xi(s))^{\alpha-1}\left(1-\frac{\xi(u)}{\xi(s)}\right)^{\alpha-1} f(s)\xi'(s)ds. \end{aligned}$$

Putting $h(s) = \xi(s)^\alpha f(s)$ and $g\left(\xi^{-1}\left(\frac{\xi(u)}{\xi(s)}\right)\right) = \frac{1}{\Gamma(\alpha)}\left(1-\frac{\xi(u)}{\xi(s)}\right)^{\alpha-1} H\left(1-\frac{\xi(u)}{\xi(s)}\right)$ in above equation, we get

$$W_\xi^{-\alpha}[f(u)] = \int_0^\infty h(s)g\left(\xi^{-1}\left(\frac{\xi(u)}{\xi(s)}\right)\right)\frac{\xi'(s)}{\xi(s)}ds,$$

where $H\left(1-\frac{\xi(u)}{\xi(s)}\right)$ is the Heaviside unit step function. Now applying generalized Mellin transform, and making use of (3.23), we get

$$M_\xi\left[W_\xi^{-\alpha}[f(u)]\right] = \tilde{F}_\xi(p,\alpha) = \tilde{h}_\xi(p)\tilde{g}_\xi(p), \tag{3.35}$$

where $\tilde{h}_\xi(p) = M_\xi\{\xi(u)^a f(u)\} = \tilde{f}_\xi(p+\alpha)$ by (3.15), and also

$$\begin{aligned}\tilde{g}_\xi(p) &= M_\xi\left[\frac{1}{\Gamma(\alpha)}(1-\xi(u))^{\alpha-1}H(1-\xi(u))\right]\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 \xi(u)^{p-1}(1-\xi(u))^{\alpha-1}\xi'(u)du\\ &= \frac{B(p,\alpha)}{\Gamma(\alpha)} = \frac{\Gamma(p)\Gamma(\alpha)}{\Gamma(p+\alpha)\Gamma(\alpha)} = \frac{\Gamma(p)}{\Gamma(p+\alpha)}.\end{aligned}$$

So (3.35) becomes

$$M_\xi\left[W_\xi^{-\alpha}[f(u)]\right] = \frac{\Gamma(p)}{\Gamma(p+\alpha)}\tilde{f}_\xi(p+\alpha). \tag{3.36}$$

□

Theorem 3.4.5. *Let $\beta > 0$ be a positive real number and n is the smallest integer greater than β such that $n-1 < \beta < n$. The generalized Mellin transform of Weyl fractional derivative is*

$$M_\xi[W_\xi^\beta f(u)] = \frac{\Gamma(p)}{\Gamma(p-\beta)}\tilde{f}_\xi(p-\beta).$$

Proof. By the definition of Weyl fractional derivative (1.36) and using (3.16), we have

$$\begin{aligned}M_\xi[W_\xi^\beta \xi(u)] &= M_\xi[E_\xi^n W_\xi^{-(n-\beta)} f(u)]\\ &= (-1)^n M_\xi[D_\xi^n W_\xi^{-(n-\beta)} f(u)]\\ &= (-1)^{2n}\frac{\Gamma(p)}{\Gamma(p-n)} M_\xi[W_\xi^{-(n-\beta)} f(u), p-n].\end{aligned}$$

Using (3.34), we get

$$\begin{aligned}M_\xi[W_\xi^\beta \xi(u)] &= \frac{\Gamma(p)}{\Gamma(p-n)}.\frac{\Gamma(p-n)}{\Gamma(p-\beta)}\tilde{f}_\xi(p-\beta)\\ &= \frac{\Gamma(p)}{\Gamma(p-\beta)}\tilde{f}_\xi(p-\beta).\end{aligned}$$

□

Summary

In the begining we have discussed some spacial functions. We reviewed definitions of some important fractional operators and their generalized form. Also we have studied some properties of these fractional operators.

Next we have studied the Caputo fractional derivative with respect to another function and some important result releated to this operator.

The classical Mellin transform is used to solve classical differential equation and some fractional operator Riemann-Liouville, Caputo, Hadamard and Weyl fractional operator. But when the operator are in generalized form as in our case ξ -fractional operator are used to solve. For this purpose we also need the generalization of the integral transform. And here we introduced the generalization of Mellin transform named as ξ -Mellin transform. We discussed some basic properties of this generalized Mellin transform and convolution theorem.

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