Effects of Magnetic Field and Thermal Radiation on Bödewadt Flow of Nanofluid.

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Dedication

Thanks to Almighty Allah who gave me the strength and patient to finish this work

My dedication also goes to

My husband, Abdul Ahad Rao who encouraged and supported me as this research progressed.

My children, Shahan & Eshmal source of happiness and hope.

My mother and father, who advised me on the significance of

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LIST OF SYMBOLS

(r, θ,z)	*Cylindrical coordinates*
u,v	*Components of velocity in x-y directions respectively*
T	*Temperature*
t	*Time*
P	*Fluid pressure*
c_p	*Specific heat*
μ	*Viscosity*
ρ	*Density*
Nu	*Nusslet number*
h	*Convective heat transfer*
τ	*Shear stress*
q	*Heat flux*
D_B	*Brownian motion diffusion*
Φ	*Nanoparticles volume fraction*
C	*Stretching sheet constant*
T_f	*Fluid temperature*
T_∞	*Temperature at wall*
Ψ	*Stream function*
η	*Similarity variable*
C_f	*Skin friction coefficient*
θ_w	Temperature ratio
Rd	*Radiation parameter*

ABSTRACT

Rotating flow over a stationary disk is referred as Bödewadt flow. It appears in spin-up and impulsive spin-down to rest problems. The transient flow in these problems evolves into a quasi-steady regime that displays the properties of the Bödewadt flow. On the other hand, nanofluids are relatively new generation heat transfer fluids which show strong heat transfer even at very low concentrations of particles and are very helpful when resolving thermal system problems. Literature showed that many researchers have worked on nanofluids and Bödewadt flows. However, the effect of magnetic field along with thermal radiation has yet to be reported for Bödewadt flow of nanofluids. This dissertation provides a numerical solution for Bödewadt flow of nanofluid within magnetic field and thermal radiation, it also contains an introduction to nanofluids their mathematical models and numerical methods. Mathematical models are described in detail by using tensor operators in cylindrical coordinates. To find graphical and numerical solutions, a simple routine bvp4c of the MATLAB and finite difference scheme as Keller box method is implemented. Solution showed negligible effect of magnetic parameter on radial component of velocity. By increasing values of Prandtl number rate of heat transfer decreases. It is observed that skin friction coefficient and volume fraction of nanoparticles have non-linear relationship with each other. Rising magnetic forces further dampens the local Nusslet number. We observed a rapid rise in the rate of heat transfer as for specific Rd values, θ_w is increased, with an increase in function slope Rd values will be further increased. Stretching forces are balanced by magnetic forces. Results in both Graphical and tabular form are present in section of result and discussion.

CHAPTER 1: INTRODUCTION

This chapter includes some basic definitions. It also includes a detailed literature review relevant to the problems considered in respective chapters. Nanofluids and their applications in different areas are discussed in detail in this part of thesis. Review about nanofluids models with magnetic hydrodynamics effects are also studied in this section.

1.1 Nanofluids

Nano fluids are comparatively new generation heat transfer fluids which show large heat transfer even at very low particle concentrations. In 1959 Richard Feynman became the first man to provide the concept of nanotechnology. After that Taniguchi was the man who uses the word nanotechnology very first time in 1974. After Taniguchi in 1977 Deluxer gives basic idea about nanotechnology in MIT. First nanotechnology book was published in 1990, and first nanotechnology article was published in 1992 [1]. Nano fluids are useful in dealing with problems of thermal systems. Fluids having higher thermal conductivities can be used as a coolant, lubricants, and engine oils. Mahian et al. [2] described that nanofluid can be used as an optimal absorption of the solar energy. Because of the remarkable advancement in aerodynamics, there is greater demand of mechanisms having high heat dissipation like brake nanofluid.

These fluids are also used in microchips as a liquid cooling in processors of systems and in other electronics applications which use microfluidic application. Buongiorno and Hu [3] had worked on heat transfer through nanoparticles for nuclear reactor application. By research it is observed that use of nanofluid as a heat exchanger makes systems more energy efficient. Magnetic nanofluid may be used to direct the particles to a tumor with magnets up the bloodstream [4]. Buongiorno [4] studied the convective phenomena in nanofluids and found that of the seven slip mechanisms namely inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage and gravity settling only Brownian motion and thermophoresis diffusion of nanoparticles contribute to the massive increase of the absolute thermal conductivity of the liquids.

Afterwards Tiwari and Das [6] presented a mathematical model for nanofluids by accounting the solid volume fraction of two phase mixture. They used the work of Brinkman [7] for effective viscosity, Xuan and Li [8] for heat capacitance and Maxwell-Garnett’s model for determining effective thermal conductivity.

The search for renewable energy generation to solve the worldwide energy crisis has led in fresh study opportunities for researchers and technicians around the globe. Solar power is considered one of the best pollution-free sources of clean energy. Hunt was first who explored the concept of using small particles to harvest solar energy [9] in the 1970s. Unlike standard heat transfer liquids (for solar thermal programs) such as water, ethylene glycol and molten salts, nanofluids are not opaque to solar radiant energy. In reality, they can considerably absorb and disperse the solar radiation passing through them.

1.2 Applications of Nanofluid

Nanofluid has numerous applications in various fields some of them are mentioned here,

- Transportation (engine cooling)
- As a heat exchanger
- In medical equipment's cleaner
- In rotating machinery
- As gas turbine rotors
- As a cleaner in air cleaning machine
- Fuel cells
- Solar water
- As lubricants in drilling
- Electronic cooling

1.3 Applications of Rotating Flow

Rotational flow through a wide variety of research, technological and material applications is vitally important and provides design and simulation tools for a variety of products as a

- Jet engines
- Pumps
- Vacuum cleaners
- In geophysical flows
- As a break Spinning in textile industry

Theory behind engineering systems such as spinning plates, cones and cavities, and natural phenomena such as air and oceanic currents used to illustrate fundamental concepts of rotating fluid.

1.4 Fundamental Terminologies

1.4.1 Reynolds Number

Reynolds Number is used to approximate the behavior of fluid and to measure the transition from laminar to turbulent flow, it is in general the inertial force to viscous forces ratio

$$Re = \frac{\rho \upsilon L}{\mu} = \frac{\upsilon L}{\nu}$$

Here, L is a characteristic Length, v = μ/ρ is termed as kinematic viscosity, μ is the dynamic viscosity of the fluid and represents the density of fluid under consideration. If value of Reynolds number is small it means viscous forces are dominant and flow is laminar, contrary if value of Reynolds number is high then fluid flow is turbulent.

1.4.2 Prandtl Number

Prandtl number monitors the relative thickness of the momentum and thermal boundary layers as it comes to problems of heat transfer. This implies that when Pr is small the heat diffuses quickly compared to the velocity (momentum).

$$\text{Pr} = \frac{\mu/\rho}{k/\rho C_p}$$

Where k = thermal conductivity, C_p = specific heat capacity, μ = viscocity.

1.4.3 Nusslet Number

Nusslet number is also a dimensionless quantity and we use it to measure rate of heat transfer between moving Plate and solid body.

$$Nu = \frac{convective\ heat\ transfer}{conductive\ heat\ transfer} = \frac{h}{k/L} = \frac{hL}{k}$$

Where k = thermal conductivity, L = length, k = thermal conductivity.

1.4.4 Hartmann Number

Hartmann is an important parameter when Magnetic field is applied on Fluid flow externally.

Mathematical representation is like;

$$M = \sqrt{\frac{\sigma B^2}{\rho U}}$$

Where **σ** denotes electrical Conductivity, **B** denotes magnetic field intensity and U is horizontal velocity of plate.

1.4.5 Schmidt Number

Schmidt number is a kinematic viscosity ratio with the thermal diffusivity. Its mathematical expression is

$$Sc = \frac{Kinematic\ Viscosity}{Mass\ Diffusivity} = \frac{v}{D}$$

1.4.6 Brownian Motion

Brownian motion is defined as a motion of molecules/particles in fluid which is because of their interaction with other particles/molecules.

1.4.7 Thermophoresis

This terminology is defined as a response found in moving particle mixtures where different types of particles demonstrate different responses to the intensity of a gradient of temperature.

1.4.8 Rayleigh Number

It is defined as an instability parameter of layer of fluid which occurs because of difference in density and temperature**.**

1.4.9 (A) Newtonian Fluids

Newtonian fluid is defined as a fluid that maintains a linear relationship between the shear stress and the rate of deformation changes, i.e. organic solvents, and honey.

$$\tau = \mu \frac{du}{dy}$$

1.4.10 (B) Non-Newtonian Fluids

Fluid which couldn't maintain the linear relationship or having Non-Linear relationship between the shear stress and the rate of change in deformation is known as Non- Newtonian fluids.

$$\tau = \mu \left(\frac{du}{dy}\right)^n$$

1.5 Boundary Layer

Boundary layer refers to the fluid layer that included the boundary where the viscosity effect occurs. It can be categorized as laminar and turbulent boundary layer; Inflow laminar boundary layer flow is smooth, having less skin friction drag and is less stable, while in turbulent boundary layer smooth flow is disturbed and converted into turbulent flow.

1.5.1 No-slip Condition

When fluid particles interact with solid surface, attraction force between fluid molecules and solid particles is far more than fluid molecules itself in other words we can say adhesive forces is more than cohesive forces. This force difference takes the momentum of the fluid down to zero. The no slip condition is specified only for viscous flows.

1.5.2 Thermal Boundary Layer

Thermal Boundary Layer is the area where temperature gradients are the product of a heat exchange mechanism between the fluid and the surface. In this sheet, the heat transfer operator changes from the temperature of the wall to the free stream U_∞ temperature.

1.6 Mathematical models for nanofluid transportation

For nanofluid transportation following models is used for calculations and for getting solutions.

I. Non-homogenous Model
II. Homogenous Model

1.6.1 Non-Homogenous model

Consider the incompressible fluid flow of nanometer-sized metallic particles. The conservation of mass and momentum are given as

$$\nabla . V = 0 \tag{1.1}$$

$$\rho f \left(\frac{\partial V}{\partial t} + (V . \nabla)\mathrm{V}\right) = \nabla . \sigma \tag{1.2}$$

Where V is the velocity vector, ρ_f is the fluid density, $\sigma = -pI + T$ is the Cauchy stress tensor,

T= μA_1 is the extra stress tensor and $A_1 = \nabla V + (\nabla V)^T$ is the first Rivlin Ericksen tensors. By ignoring viscous dissipation, heat source/sink, Joule heating and thermal radiation energy equation can be written as

$$(\rho c)_f \left(\frac{\partial T}{\partial t} + (V . \nabla)T\right) = -\nabla . q + h_s \nabla . j_s \tag{1.3}$$

Where $(\rho c)_f$ is the effective heat capacity of the base fluid, q is the heat flux, h_s is the specific enthalpy and j_s is the diffusive mass flux of nanoparticle. Buongiorno the heat flux q is given by;

$$q = -k\nabla T + h_s j_s; \ , h_s = c_p T \tag{1.4}$$

where “k” is the thermal conductivity. The diffuse mass flux is expressed as a product of Brownian motion and thermophoretic diffusion is stated as

$$j_s = j_{s,B} + j_{s,T} = -\rho s \, DB \, \nabla C - \rho s \, DT \, \frac{\nabla T}{T_\infty} \tag{1.5}$$

In which D_B is the Brownian diffusion coefficient given by the Einstein–Stokes's equation, and D_T is the thermophoretic diffusion coefficient. Now adding the expressions for q and j_s from eqns. (1.4) and (1.5) in Eq. (1.3) we obtain

$$(\rho c)_f \left(\frac{\partial T}{\partial t} + V.\nabla T\right) = k\nabla^2 T + (\rho c)_s \left(D_B \nabla C.\nabla T + \frac{\nabla T.\nabla T}{T_\infty} \right) \quad (1.6)$$

The equation for the conservation of nanoparticles without chemical reaction and dilute mixture can be compiled as

$$\frac{\partial C}{\partial t} + V.\nabla C = -\frac{1}{\rho s} \nabla . j_s \quad (1.7)$$

$$\frac{\partial C}{\partial t} + V.\nabla C = \nabla.\left(D_B \nabla C + D_T \frac{\nabla T}{T_\infty}\right) \quad (1.8)$$

1.6.2 Homogenous model

For steady incompressible flow of nanofluid the continuity, momentum and energy equations are given below:

$$\nabla . V = 0 \quad (1.9)$$

$$\rho_{nf}(V.\nabla)V = \nabla p + \mu_{nf}\nabla^2 V \quad (1.10)$$

$$(V.\nabla)T = \alpha_{nf}\nabla^2 T \quad (1.11)$$

Where $\boldsymbol{\rho_{nf}}$ is the density of the nanofluid, μ_{nf} is dynamic viscosity of the nanofluid and $\boldsymbol{\alpha_{nf}}$ is thermal diffusivity of the nanofluid which are define

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}} \qquad \rho_{nf} = (1-\phi)\,\rho_f + \phi\,\rho_s \qquad \alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}} \quad (1.12)$$

$$(\rho c_p)_{nf} = (1-\phi)(\rho c_p)_f + \phi(\rho c_p)_s \qquad \frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\phi(k_f + 2k_s)}{(k_s + 2k_f) + \phi(k_f + 2k_s)} \quad (1.13)$$

In which $\boldsymbol{\phi}$ expressed the nanoparticle volume fraction, $\rho_f \; and \; \rho_s$ and are the densities of the base fluid and nanoparticle material, k_f and k_s are the thermal conductivities of the base fluid and nanoparticle material respectively, $(\rho c_p)_{nf}$ is the effective heat capacity of the nanofluid and k_{nf} is the thermal conductivity of the nanofluid.

1.7 Problem Statement

The transient flows involving spin-up and spin-down problems evolves into a quasi-steady regime that exhibits the properties of the Bödewadt flow, literature review showed that there's not enough work have been done on Bödewadt Flow by taking Nanofluid, therefore study the effects of Magnetic field and Thermal radiation on flow and heat transfer in Bödewadt flow of Nanofluid.

1.8 Research Objectives

- To model the boundary layer flow of nanofluid over a rotating disk.
- To examine the effects of magnetic parameters and thermal radiation on Bödewadt flow of nanofluid.
- To find the numerical solution by using Keller box method and MATLAB® built-in bvp4c.

CHAPTER 2: LITERATURE REVIEW

Blasius conducted an important work into the convective boundary layer flows in an ambient fluid with uniform free stream over a flat plate at zero incident angles [13]. He analytically solved in the form of power series. On the other hand, Skadius [14] discussed his results on a flow over moving plate. Crane [15] expanded their work for a stretch sheet and regarded the incompressible flow induced by stretching a flat elastic sheet with a velocity that varies linearly to the distance from a fixed point.

It has been shown that the point of concern shows the two-dimensional Navier-Stokes equations with a closed form solution. [16] Take into account effects of the porous medium on the Bödewadt flow of the magnetic nanofluid through a uniformly heated stationary disk when geothermal viscosity occurs.

Ram et al. implemented the time-dependent, tri-dimensional nano suspension boundary layer flow [17] considering the effects of heat transfer in order to examine the important features for the unstable convective flow. Joshi et al. [18] observed a systematic study relating to the time-dependent boundary layer flow of a magnetic nanofluid over a spinning disk with the combined effects of thermal radiation and variable viscosity.

Anderson et al. [19] have investigated the axi-symmetric power-law fluid flow in the boundary layer consistently. [20] Turkyilmazoglu has obtained the solution of the steady laminar flow of an incompressible viscous electrically conducting fluid by rotation of a disk in the presence of a magnetic field. [21] The influences of the magnetic field, the electrical heating, the viscous dissipation effect, the Hall Effect generated by the electrical potential and the current caused by the ion slip were investigated in the steady flow resulting from the rotating disc. Ariel [22] presented the solution of the steady flow of an incompressible fluid moving through a spinning disk with the help of an external uniform magnetic field. In the presence of a magnetic field, the focus of Attia [23] was on the unstable flow of the viscous fluid.

Afterwards the concept of injection and suction is introduced. An interesting model was developed by Riaz and Khan [24] for the couple stress fluid flow near a stagnation point due to the rotation of a non-aligned disk. Hayat et al [25] have attempted an analysis of the boundary layer flow of a couple stress fluids passing through a continuous moving surface with the effect of heat transfer in the fluid flow. Devakar et al. [26] proposed analytical solutions for evaluating fluid flow behavior along with slip boundary conditions to find the exact solutions of different types of flows such as Couette, Poiseuille and generalized Couette flows for the couple stress fluid model.

A lot of other new methods for solving ODEs and PDEs can be found in literature. A new integral transformation method has recently been proposed and applied successfully on many physical problems [27, 28]

2.1 Bödewadt Flow

Bödewadt boundary layer flow occurs due to a rotating flow over a stationary disk.

Reverse flow is observed for the revolving flow over a stationary disk, in a result of the radial pressure gradient being balanced by the centrifugal force so that the fluid drawn to the axis of rotation is swept Upwards.

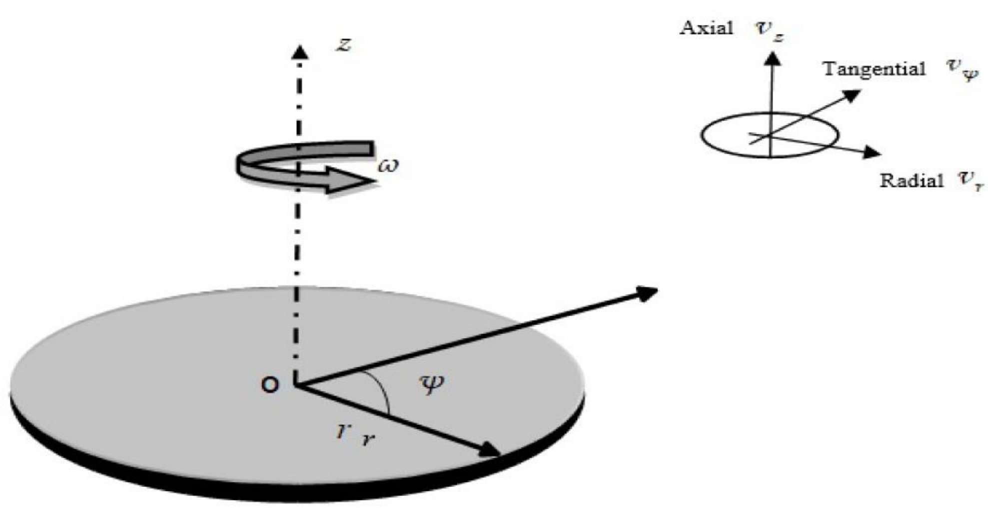


Figure 2-1: Physical model and coordinate system

2.2 Numerical Methods

2.2.1 Keller box Method

In Keller Box method we follow these steps;

- In First step we reduced differential equations into first order by applying some Suitable substitutions.
- After that we apply central difference scheme method and convert those first Order ordinary differential equations into difference Equations.
- Those difference equations are then linearized by applying Newton’s method.

- As a result, we obtained equations in matrix-vector form and obtain results by block-tridiagonal elimination method

MATLAB® is a high-performance, scientific programming language. It combines calculation, simulation and computing into a convenient-to-use environment where common mathematical expression communicates problems and solutions. Main application includes: Math and computing.

In our problem we solve in Keller box method then solved those equations and block-tridiagonal matrix in MATLAB bvp4c computing technique. And compare results by using both techniques.

CHAPTER 3 MATHEMATICAL MODELING AND NUMERICAL METHODOLOGY

The governing equations of fluid flow motion were drawn from simple concepts of mass, momentum, and energy conservation. Via the rheological properties of the fluid the interaction between shear stress and strain rate may be of various types. Here are the simple equations of boundary layers that will be included in the later chapter of this study.

3.1 Conservation Laws

3.1.1 Conservation Law of Mass

Physical Principle Mass can neither be created nor destroyed

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0 \tag{3.1}$$

ρ is the fluid density, t is the time, ∇ is the divergence and V = [u,v,w] is the velocity vector. The above equation (3.1.1) is called continuity equation.

3.1.2 Conservation Law of Momentum

Physical Principle: Force = time rate of change of momentum

As a fluid moves it experiences two kinds of forces

- **i) Body forces**
  Like gravity, electromagnetic force or any other forces.

- **ii) Surface forces**
  Pressure and shear stresses acting on the surface of body.

Mathematical formulation of momentum equation is

$$\rho \frac{DV}{Dt} = -\nabla P + \nabla.\tau + \rho g \tag{3.2}$$

$\frac{DV}{Dt}$ Is the total derivative, P is the pressure, τ is the shear stress and $\boldsymbol{g}$ is the gravitational force vector.

The total derivative can also be represented as:

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V.\nabla)V \tag{3.3}$$

By substituting in equation we get;

$$\rho\left(\frac{\partial V}{\partial t} + (V.\nabla)V\right) = -\nabla P + \nabla.\tau + \rho g \tag{3.4}$$

3.1.3 Conservation Law of Energy

Conservation law of energy states that the rate of increase of energy of fluid particle is equal to net rate of heat added to the fluid particle plus net rate of work done on the fluid particle.

Physical Principle:

Energy can neither be created nor destroyed. Mathematical representation is;

$$\rho c_p \frac{DT}{Dt} = -\nabla.q + \phi \tag{3.5}$$

c_p is the specific heat and q heat flux of the fluid flow, where viscous dissipation is $\emptyset$.

3.2 Magneto hydrodynamic (MHD) Flow

This illustrates how the conductive fluid flows affect a magnetic field. When we analyze a fluid flow of MHD the equation of momentum becomes:

$$\rho\left(\frac{\partial V}{\partial t} + (V.\nabla)V\right) = -\nabla P + \nabla.\tau + J \times B \tag{3.6}$$

In the following equation $J \times B$ is a Lorentz force, J is current density B is a magnetic field and its value is $(0,0,B_o)$.

$$j = \sigma E' \tag{3.7}$$

$$E' = E + (V \times B) \tag{3.8}$$

When fluid moves at velocity V with regard to external magnetic and electrical field, external electrical field is zero i.e. E = 0. So we left with

$$E' = (V \times B) \tag{3.9}$$

We can represent as;

$$J \times B = \sigma(V \times B) \times B \tag{3.10}$$

$$V \times B = \begin{vmatrix} i & j & k \\ u & v & w \\ 0 & 0 & B_o \end{vmatrix} = ivB_o - juB_o \tag{3.11}$$

$$(V \times B) \times B = \begin{vmatrix} i & j & k \\ uB_o & -uB_o & 0 \\ 0 & 0 & B_o \end{vmatrix} = -iuB_o{}^2 - jvB_o{}^2 \tag{3.12}$$

3.3 Tensor Operations

Commonly known operations for vectors "dot" and "Cross" have simple notations and formulas to understand we can write a vector **v** as a sum $\sum_i \boldsymbol{\delta_i v_i}$ and also can manipulate the unit vectors $\boldsymbol{\delta_i}$ here we solve by parallel procedure. We can write Tenors $\boldsymbol{\tau}$ as a sum of $\sum_i \sum_j \boldsymbol{\delta_i \delta_j \tau_{ij}}$ same as we can also manipulate unit deltas $\boldsymbol{\delta_i \delta_j}$ **,** so we can develop expressions for common operations "dot" and "cross" for tensors.

3.3.1 Expansion of a Tensor in Terms of its Components

We expand a tensor in terms of its components by multiplying each component by appropriate unit vector but in case of tensor (second-order) that associates a scalar with each ordered pair of coordinate directions in the following sense:

$$\tau = \delta_i\delta_j\tau_{11} + \delta_i\delta_j\tau_{12} + \delta_i\delta_j\tau_{13}$$

$$+ \delta_i\delta_j\tau_{21} + \delta_i\delta_j\tau_{22} + \delta_i\delta_j\tau_{23}$$

$$+ \delta_i\delta_j\tau_{31} + \delta_i\delta_j\tau_{32} + \delta_i\delta_j\tau_{33}$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_j\delta_j\tau_{ij} \tag{3.13}$$

3.3.2 Differential operations in Cylindrical Coordinate

We can write spatial derivatives of a unit vectors $\boldsymbol{\delta_r}, \boldsymbol{\delta_\theta}$ ***and*** $\boldsymbol{\delta_z}$ it would be better to interpret these derivatives geometrically by changing as the location of point as it changes

$$\frac{\partial}{\partial r}\delta_r = 0 \qquad \frac{\partial}{\partial r}\delta_\theta = 0 \qquad \frac{\partial}{\partial z}\delta_z = 0$$

$$\frac{\partial}{\partial \theta}\delta_r = \delta_\theta \qquad \frac{\partial}{\partial \theta}\delta_\theta = -\delta_\theta \qquad \frac{\partial}{\partial \theta}\delta_z = 0 \tag{3.14}$$

We can use definition of operator **∇** and derivative operator to obtain mathematical representation of in cylindrical coordinates

$$\nabla = \delta_x \frac{\partial}{\partial x} + \delta_y \frac{\partial}{\partial y} + \delta_z \frac{\partial}{\partial z} \tag{3.15}$$

$$= (\delta_r \cos\theta - \delta_\theta \sin\theta)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$+ (\delta_r \sin\theta + \delta_\theta \cos\theta)\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right) + \delta_z \frac{\partial}{\partial z} \tag{3.16}$$

After multiplication and simplifications of operators we get,

$$\nabla = \delta_r \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z} \tag{3.17}$$

Here are mathematical expressions for $(\nabla . v)$ and ∇v in cylindrical coordinates [19];

$$(\nabla . v) = \left(\left\{\delta_r . \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z}\right\} . \{\delta_r v_r + \delta_\theta v_\theta + \delta_z v_z\}\right) \tag{3.18}$$

$$(\nabla . v) = \left(\delta_r . \frac{\partial}{\partial r}\delta_r v_r\right) + \left(\delta_r . \frac{\partial}{\partial r}\delta_\theta v_\theta\right) + \left(\delta_r . \frac{\partial}{\partial r}\delta_z v_\theta\right)$$

$$+ \left(\delta_\theta . \frac{1}{r}\frac{\partial}{\partial \theta}\delta_r v_r\right) + \left(\delta_\theta . \frac{1}{r}\frac{\partial}{\partial \theta}\delta_\theta v_\theta\right) + \left(\delta_\theta . \frac{1}{r}\frac{\partial}{\partial \theta}\delta_z v_z\right)$$

$$+ \left(\delta_z . \frac{\partial}{\partial z}\delta_r v_r\right) + \left(\delta_z . \frac{\partial}{\partial z}\delta_\theta v_\theta\right) + \left(\delta_z . \frac{\partial}{\partial z}\delta_z v_z\right)$$

Now we evaluate the derivatives of the unit vectors this implies that;

$$(\nabla . v) = (\delta_r . \delta_r)\frac{\partial v_r}{\partial r} + (\delta_r . \delta_\theta)\frac{\partial v_\theta}{\partial r} + (\delta_r . \delta_z)\frac{\partial v_\theta}{\partial r}$$
$$+(\delta_\theta . \delta_r)\frac{1}{r}\frac{\partial v_r}{\partial \theta} + (\delta_\theta . \delta_\theta)\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + (\delta_\theta . \delta_z)\frac{1}{r}\frac{\partial v_z}{\partial \theta}$$
$$+(\delta_z . \delta_r)\frac{\partial v_r}{\partial z} + (\delta_z . \delta_\theta)\frac{\partial v_\theta}{\partial z} + (\delta_z . \delta_z)\frac{\partial v_z}{\partial z} \quad (3.19)$$

Since $(\delta_r . \delta_r) = 1, (\delta_r . \delta_\theta) = 0$ afterwards we get;

$$(\nabla . v) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \quad (3.20)$$

In any material, total stress tensor is the sum of viscous stress tensor and hydrostatic pressure p. in a fluid material, value of elastic stress is zero because by definition it cannot have static shear stress.

$$\rho((v.\nabla)v) = \nabla.\sigma \quad (3.21)$$

$$\sigma_{ij} = -PI + \tau \quad (3.22)$$
$$\tau = \mu A_1 \quad (3.23)$$

Where A_1 is a 1^{st} Rivlin Tensor.

$$A_1 = (\nabla v) + (\nabla v)^T \quad (3.24)$$

Solving right hand side of the equation we get;

$$(\nabla v) = (\{\delta_r \frac{\partial}{\partial r} + \delta_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \delta_z \frac{\partial}{\partial z}\}\{\delta_r v_r + \delta_\theta v_\theta + \delta_z v_z\})$$
$$= (\delta_r\delta_r)\frac{\partial v_r}{\partial r} + (\delta_r\delta_\theta)\frac{\partial v_\theta}{\partial r} + (\delta_r\delta_z)\frac{\partial v_z}{\partial r} + (\delta_\theta\delta_r)\frac{1}{r}\frac{\partial v_r}{\partial \theta} + (\delta_\theta\delta_\theta)\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$
$$+ (\delta_\theta\delta_z)\frac{1}{r}\frac{\partial v_z}{\partial \theta} + (\delta_\theta\delta_\theta)\frac{v_r}{r} - (\delta_\theta\delta_r)\frac{v_\theta}{r}$$
$$+(\delta_z\delta_r)\frac{\partial v_r}{\partial z} + (\delta_z\delta_\theta)\frac{\partial v_\theta}{\partial z} + (\delta_z\delta_z)\frac{\partial v_z}{\partial z} \quad (3.25)$$

$$= \delta_r\delta_r \frac{\partial v_r}{\partial r} + \delta_r\delta_\theta \frac{\partial v_\theta}{\partial r} + \delta_r\delta_z \frac{\partial v_z}{\partial r} + \delta_\theta\delta_r \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \delta_\theta\delta_\theta \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$+ (\delta_\theta\delta_z)\frac{1}{r}\frac{\partial v_z}{\partial \theta} + (\delta_z\delta_r)\frac{\partial v_r}{\partial z} + (\delta_z\delta_\theta)\frac{\partial v_\theta}{\partial z} + (\delta_z\delta_z)\frac{\partial v_z}{\partial z} \quad (3.26)$$

We can write this equation in tensor matrix form as well;

$$(\nabla v) = \begin{bmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ \frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} & \frac{\partial v_z}{\partial r} \\ \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_\theta}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad (3.27)$$

$$(\nabla v)^T = \begin{bmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad (3.28)$$

Now from equation (3.22), (3.23),(3.24) we get ;

$$\tau = \mu\big((\nabla v) + (\nabla v)^T\big) \quad (3.28)$$

$$\sigma_{ij} = -PI + \mu\big((\nabla v) + (\nabla v)^T\big) \quad (3.29)$$

$$\rho((v.\nabla)v) = -PI + \mu\big((\nabla v) + (\nabla v)^T\big) \quad (3.30)$$

By adding (∇v) and $(\nabla v)^T$ we get matrix of τ;

$$\begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_z \\ \tau_{zr} & \tau_{\theta z} & \tau_{zz} \end{bmatrix} \quad (3.23)$$

Where values of entries by addition of two above mentioned matrices are;

$$\tau_{rr} = 2\frac{\partial v_r}{\partial r} \qquad \tau_{r\theta} = \tau_{\theta r} = r\frac{\partial}{\partial r}(\frac{v_\theta}{r}) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

$$\tau_{\theta\theta} = 2(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}) \qquad \tau_{\theta z} = \tau_{z\theta} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}$$

$$\tau_{zz} = 2\frac{\partial z}{\partial r} \qquad \tau_{rz} = \tau_{zr} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}$$

Now we have to calculate value of $\boldsymbol{\nabla.\sigma}$ which is possible if we separately calculate all shear stresses components along (r,$\boldsymbol{\theta}$, z) cylindrical coordinates.

Left hand side of the equation is;

$$\nabla.\sigma = -\nabla P + \mu\nabla^2 v \tag{3.32}$$

$$\nabla.\tau|_r = [\nabla^2 v]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,(rv_r)\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \tag{3.33}$$

$$\nabla.\tau|_\theta = [\nabla^2 v]_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,(rv_\theta)\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \tag{3.34}$$

$$\nabla.\tau|_z = [\nabla^2 \mathrm{v}]_z = \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial \mathrm{v_z}}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 \mathrm{v_z}}{\partial \Theta^2} + \frac{\partial^2 \mathrm{v_z}}{\partial \mathrm{z}^2} \tag{3.35}$$

So from equation (3.22) we are going to solve right hand side which is also well known equation of Newtonian fluid having constant density (ρ) and constant viscosity (μ)

$$\rho\big((v.\nabla)v\big) = -\nabla P + \mu\nabla^2 v \tag{3.36}$$

$$(v.\nabla) = (\delta_r v_r + \delta_\theta v_\theta + \delta_z v_z).\left(\delta_r\frac{\partial}{\partial r} + \delta_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \delta_z\frac{\partial}{\partial z}\right) \tag{3.37}$$

$$(v.\nabla) = \left((\delta_r.\delta_r)v_r\frac{\partial}{\partial r} + (\delta_\theta.\delta_\theta)v_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + (\delta_z.\delta_z)v_z\frac{\partial}{\partial z}\right) \tag{3.38}$$

$$(v.\nabla) = \left(v_r\frac{\partial}{\partial r} + v_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}\right) \tag{3.39}$$

Now calculate the value of $(\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v}$ same as $\boldsymbol{\tau}$ in (r,$\boldsymbol{\theta}$,z) components by using scalar triple product rule we found out this mathematical expression is a true representation of equation (3.21).

$$(v.\nabla)\, v_r = v_r\left(\frac{\partial v_r}{\partial r}\right) + \left(\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} - \frac{{v_\theta}^2}{r}\right) + v_z\frac{\partial v_r}{\partial z}$$

$$(v.\nabla)\, v_\theta = v_r\left(\frac{\partial v_\theta}{\partial r}\right) + \left(\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r}\right) + v_z\frac{\partial v_\theta}{\partial z}$$

$$(v.\nabla)\, v_z = v_r\left(\frac{\partial v_z}{\partial r}\right) + \left(\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}\right) + v_z\frac{\partial v_z}{\partial z} \tag{3.40}$$

In matrix form we can also write this expression as in which 1st we take $\hat{\boldsymbol{r}}$ as scaler and $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{z}})$ as vector quantity and solved the expression by using product rule, again take $\hat{\boldsymbol{\theta}}$ as scaler and treat remaing two $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{z}})$ as vector product and solve by using product rule and in last take $\hat{\boldsymbol{z}}$ as a scaler and solved rest of two by using by same product rule.

$$(\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v} = \begin{bmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ \boldsymbol{v_r}\left(\frac{\partial \boldsymbol{v_r}}{\partial \boldsymbol{r}}\right) & \boldsymbol{v_r}\left(\frac{\partial \boldsymbol{v_\theta}}{\partial \boldsymbol{r}}\right) & \boldsymbol{v_r}\left(\frac{\partial \boldsymbol{v_z}}{\partial \boldsymbol{r}}\right) \\ \boldsymbol{v_\theta}\left(\frac{\boldsymbol{1}}{\boldsymbol{r}}\frac{\partial \boldsymbol{v_r}}{\partial \boldsymbol{\theta}} - \frac{\boldsymbol{v_\theta}}{\boldsymbol{r}}\right) & \boldsymbol{v_\theta}\left(\frac{\boldsymbol{1}}{\boldsymbol{r}}\frac{\partial \boldsymbol{v_\theta}}{\partial \boldsymbol{\theta}} + \frac{\boldsymbol{v_r}}{\boldsymbol{r}}\right) & \boldsymbol{v_z}\frac{\partial \boldsymbol{v_\theta}}{\partial \boldsymbol{z}} \\ \boldsymbol{v_r}\left(\frac{\partial \boldsymbol{v_z}}{\partial \boldsymbol{r}}\right) & \boldsymbol{v_\theta}\left(\frac{\boldsymbol{1}}{\boldsymbol{r}}\frac{\partial \boldsymbol{v_z}}{\partial \boldsymbol{\theta}}\right) & \boldsymbol{v_z}\frac{\partial \boldsymbol{v_z}}{\partial \boldsymbol{z}} \end{bmatrix}$$

As we have all expressions so we can write equation (3.21) which is also known as equation of

Motion with constant density in cylindrical coordinates (r, $\boldsymbol{\theta}$, z) as;

$$\rho((v.\nabla)v) = \nabla.\sigma$$

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\left(\frac{\partial v_r}{\partial r}\right)+\left(\frac{v_\theta}{r}\frac{\partial v_r}{\partial \Theta}-\frac{{v_\theta}^2}{r}\right)+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\ (rv_r)\right)+\frac{1}{\mathrm{r}^2}\frac{\partial^2 v_r}{\partial \Theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \Theta}\right]-\frac{\partial p}{\partial r}+\rho g_r \tag{3.41}$$

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\left(\frac{\partial v_\theta}{\partial r}\right)+\left(\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \Theta}+\frac{v_r v_\theta}{r}\right)+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\ (rv_\theta)\right)+\frac{1}{\mathrm{r}^2}\frac{\partial^2 v_\theta}{\partial \Theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \Theta}\right]-\frac{1}{r}\frac{\partial p}{\partial r}+\rho g_r \tag{3.42}$$

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\left(\frac{\partial v_z}{\partial r}\right)+\left(\frac{v_\theta}{r}\frac{\partial v_z}{\partial \Theta}\right)+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \Theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]-\frac{\partial p}{\partial r}+\rho g_r \tag{3.43}$$

CHAPTER 4: RESULTS AND DISCUSSION

4.1 Introduction

In this section we analyze numerically how magnetic field and thermal radiation effects on Bödewadt flow of Nanofluid. Three different Nanoparticles Copper-Cu and silver– Ag, copper oxide – CuO with Base fluid water (H_2O) are taken for the study of flow.

Appropriate and conventional transformations are applied to the equations, resulting in a strong non-linear differential system that is solved by MATLAB® Keller Box method.

4.1.1 Problem Formulation

Physical problem is formulated in Cylindrical coordinates (r,θ,z) and The rotational fluid flow is created far from the disk surface as a rigid body with a constant angular velocity Ω. The fluid is exposed to a continuous magnetic field (B) while neglecting the influence of electrical fields. Consider the incompressible three-dimensional flow of nanofluid over a circular disk situated at z=0. For axial symmetry purposes the derivatives around the polar coordinate will be excluded.

The flow field is defined by the Vector u whose radial, tangential and axial velocity components are (u,v,w) respectively. Considering Newtonian and viscous fluid with pressure (p), density(ρ) temperature of fluid is represented by T in a way the surface of disk is kept at a uniform temperature(T_w). Assumption implemented here is fluid physical properties like, viscosity (μ), effective heat capacity (C_p), thermal conductivity coefficient (κ) and density (ρ) are all constant parameters.

The fluid consists of three separate nanoparticles namely Copper-*Cu* and silver– *Ag*, copper oxide – *CuO*. Base fluid is supposed to be water (H_2O).

Table 4-1: Thermo-physical properties of Base Fluid and Nanoparticles

Properties	Base Fluid (water)	CuO	Cu	Ag
Κ(W/ mK)	0.613	76.5	400	429
P(kg/m^3)	997.1	6320	8940	10500
$c_p(J/kg\ K)$	4179	531.5	385	233

4.2 Equations of Motion in cylindrical coordinates:

Let us consider an incompressible, steady flow of Newtonian fluid over a stretching sheet. By using most familiar Homogenous Model known as a Tiwari and Das Model [6] also known as a Homogenous model for nanofluid, governing Navier-Stokes and Energy Equations are then given below;

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{4.1}$$

$$(\rho c_p)_{nf}\left(u\frac{\partial u}{\partial r}-\frac{v^2}{r}+\frac{w\partial u}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial r}+\mu_{nf}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)-\ u\sigma_{nf}\beta_{\circ}^2 \tag{4.2}$$

$$(\rho c_p)_{nf}\left(u\frac{\partial v}{\partial r}+\frac{uv}{r}+\frac{w\partial v}{\partial z}\right)$$
$$=\mu_{nf}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right)-v\sigma_{nf}\beta_{\circ}^2 \tag{4.3}$$

$$(\rho c_p)_{nf}\left(u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+\frac{w\partial w}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial r}+\mu_{nf}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right) \tag{4.4}$$

$$\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)=\alpha_{nf}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1\partial T}{r\,\partial r}+\frac{\partial^2 T}{\partial z^2}\right)-\frac{1}{(\rho c_{p)}{}_{nf}}\frac{\partial q_r}{\partial z} \tag{4.5}$$

Here ρ_{nf} is the density of the nanofluid , $\mu_{\boldsymbol{nf}}$ is dynamic viscosity of the nanofluid and $\boldsymbol{\alpha_{nf}}$ represents thermal diffusivity of the nanofluid whereas magnetic field is kept constant along z-axis.

The boundary conditions are:

$$u=sr,\qquad v=0,\qquad w=0,\qquad T=T_W,\qquad at\quad z=0,$$

$$u=0,\qquad v=r\Omega\,,\qquad T=T_\infty,\qquad \text{As}\qquad z\to\infty, \tag{4.6}$$

Here is Mathematical representation of $\boldsymbol{\mu_{nf}}$ dynamic viscosity, $\boldsymbol{\alpha_{nf}}$thermal diffusivity, $\boldsymbol{\rho_{nf}}$ density of nanofluid along with $(\boldsymbol{\rho c_p})_{\boldsymbol{nf}}$ effective heat capacity and $\boldsymbol{k_{nf}}$ thermal conductivity of nanofluid. kf and ks are thermal conductivities of base fluids and nanoparticles [29].

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}} \qquad \alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}} \qquad \rho_{nf} = (1-\phi)\rho_f + \phi\rho_s \tag{4.7}$$

$$(\rho c_p)_{nf} = (1-\phi)(\rho c_p)_f + \phi(\rho c_p)_s \qquad \frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\phi(k_f - k_s)}{(k_s + 2k_f) + \phi(k_f - k_s)} \tag{4.8}$$

The radial pressure gradient being balanced by the centrifugal force so that the fluid is drawn to the axis of rotation is swept upwards by applying these Von Karman similarity transformations [30].

$$(u, v, w) = (r\,\Omega F(\eta), \quad r\Omega G(\eta), \ \sqrt{\Omega\nu}H(\eta)), \tag{4.9}$$

$$(p, T) = \left(p_\infty - \rho\nu P(\eta)\right), \qquad T_\infty + (T_w - T_\infty)\theta(\eta), \tag{4.10}$$

$$\eta = \sqrt{\frac{\Omega}{\nu}}\ z, \qquad \frac{1}{\rho_{nf}}\frac{\partial p}{\partial r} = r\Omega^2 \qquad C = \frac{s}{\Omega} \tag{4.11}$$

By the implementation of famous Von Karman similarity transformations, system of equations is reduced to the set ordinary differential equations (ODE's) along with Boundary Conditions. We obtained set of following ODE's having parameters as $M, Pr, Rd, \boldsymbol{\phi}, \boldsymbol{\theta_w}$ in Equations will be solved by using MATLAB® and Keller Box .

As we obtained system of 2nd ordered equations so in 1st step we have to reduce our system of ODE's into first order by inserting these variables.

$$f' = u, \qquad G' = v \ , \qquad \theta' = y \tag{4.12}$$

By using these variables our system of Equation is transformed into;

$$H' + 2F = 0 \tag{4.13}$$

$$Cf\text{u}' - F^2 + G^2 - \text{Hu}' - 1 - \text{MF} = 0, \tag{4.14}$$

$$Cg\text{v}' - 2\text{FG} - \text{Hv}' - \text{MG} = 0, \tag{4.15}$$

$$Ct\left[\left(k_f + Rd(1 + Tw\theta)^{3}\right)y' + 3Rd(1 + Tw\theta)^2 Twy^2\right] - P_r\text{H}\theta y' = 0 \tag{4.16}$$

Where $Tw = (\theta w - 1)$, considering all above mentioned assumptions in cylindrical coordinates (r, θ, z) and applying homogenous Model for Nanofluids, we can represent our System of Equations of Mass, energy and Momentum as;

4.2.1 Continuity

$$H' + 2F = 0 \tag{4.17}$$

4.2.2 Momentum

$$\frac{1}{(1-\varphi)^{2.5}(1 - \varphi + \varphi\frac{\rho_s}{\rho_f})}\text{F}'' - F^2 + G^2 - \text{HF}' - 1 - \text{MF} = 0 \tag{4.18}$$

$$\frac{1}{(1-\varphi)^{2.5}(1 - \varphi + \varphi\frac{\rho_s}{\rho_f})}\text{G}'' - 2\text{FG} - \text{HG}' - \text{MG} = 0, \tag{4.19}$$

4.2.3 Energy

$$\frac{1}{(1-\varphi)^{2.5}\left(1 - \varphi + \varphi\frac{(\rho c_p)_s}{(\rho c_p)_f}\right)}\left[\left(\frac{k_{nf}}{k_f} + Rd(1 + (\theta_w - 1)\theta)^3\right)\theta'' + \right] \tag{4.20}$$

$$3Rd(1 + (\theta_w - 1)\theta)^2(\theta_w - 1)\theta'^2 - P_r\text{H}\theta' = 0$$

$$F(0) = C,\quad G(0) = 0,\quad H(0) = 0,\quad \theta(0) = 1,\quad at\ \eta = 0$$
$$F'(\infty) = 0,\quad G(\infty) = 1,\quad \theta(\infty) = 0\ as\ \ \eta \to \infty \tag{4.21}$$

In the following ordinary equation $\text{P}_\text{r} = \frac{(\mu c_p)_f}{k_f}$ is a Prandtl number of the base fluid, $M = \frac{\sigma\beta_\circ^2}{\rho_{nf}}$ is Magnetic parameter, $\text{C} = \frac{s}{\Omega}$ is a stretching strength parameter and $\theta w = \frac{Tw}{T\infty}$is ratio between wall temperature and ambient temperature.

Mathematical representation of skin friction coefficient is

$$C_f = \frac{\sqrt{\tau_r^2 + \tau_\theta^2}}{\rho(r\Omega)^2} \tag{4.22}$$

In which τ_r , τ_θ are radial and tangential stresses;

$$\tau_r = r\Omega\sqrt{\frac{\Omega r^2}{v_f}}\frac{\mu_f}{(1-\varphi)^{2.5}}\, F'(0) \qquad \tau_r = r\Omega\sqrt{\frac{\Omega r^2}{v_f}}\frac{\mu_f}{(1-\varphi)^{2.5}}\, G'(0) \tag{4.23}$$

$$\tau_r = \mu\frac{\partial u}{\partial z}\Big|_{z=0} \qquad \tau_\theta = \mu\frac{\partial v}{\partial z}\Big|_{z=0} \tag{4.24}$$

By [43] applying transformations from equation (4.11) we found out

$$c_f = \sqrt{\frac{v_f}{\Omega r^2}}\frac{1}{(1-\varphi)^{2.5}}\left(F'(0)^2 + G'(0)^2\right)^{1/2} \tag{4.25}$$

By using Transformations from equation (4.11) we get this expression;

Mathematical representation of Local Nusslet number is;

$$N_u = \frac{rq_w}{k_f(T_w - T_\infty)} \tag{4.26}$$

Here q_w is the heat flux [34], [40] from the disk and defined as;

$$q_w = -\frac{k_{nf}}{k_f}\frac{\partial T}{\partial z}\Big|_{z=0} \tag{4.27}$$

After [35] using variables from equation (4.11), (4.19-20), we find out

$$N_u = -\left(\frac{k_{nf}}{k_f} + Rd\theta_w^3\right)\theta'(0) \tag{4.28}$$

Rd is a radiation parameter;

$$\mathrm{Rd} = \frac{16\sigma^* T^3}{3k^* k_f} \tag{4.29}$$

4.3 Numerical Results and Discussion

We have solved in our problem Coupled Non-Linear Equations from (4.16) to (4.19) For the validation of our computational results, we did compare calculations of $F'(0), G'(0), H(\infty)$ for different values of stretching parameter C with results obtained by Turkyilmazoglu [29] see Table (4.2) by taking case of regular fluid ($\phi = 0$) an efficient association can be observed with already existing calculated values.

Table 4-2: Results of parameters for various values of C stretching parameter

C	F'(0)	G'(0)	H(∞)
0	-0.941971 (-0.941971)	0.772885 (0.772885)	1.349421 (1.349421)
1	-1.865469 (-1.865469)	0.685170 (0.685170)	-0.469350 (-0.469350)
5	-13.46623 (-13.46623)	1.200076 (1.200076)	-3.215196 (-3.215196)
10	-37.36036 (-37.36036)	1.678167 (1.678167)	-4.700370 (-4.700370)
20	-105.1535 (-105.1535)	2.366484 (2.366484)	-6.702932 (-6.702932)

Figs. 4-1 – 4-4 shows the result of stretching parameter on Bödewadt flow having fixed volume fraction of Nanoparticles. We can observe that oscillatory nature of radial, azimuthal and tangential velocity distributions decreased by enhancing of stretching parameter. Thickness of Boundary layer significantly decreases with an expansion in radial stretching. By increasing radial stretching outward flow accelerated in near wall region see Fig 4-2. It is also observed that with the increase in amount of stretching parameter in Fig 4-3 the profile reduces. However, Fig 4-4 shows the effects of stretching parameter C on the thermal boundary layer for three different nanoparticles mentioned in Table 4-1. As show in the profile of H increasing C increase the downwards flow which increase the convective flow towards rotating disk as a result thermal boundary layer is reduced.

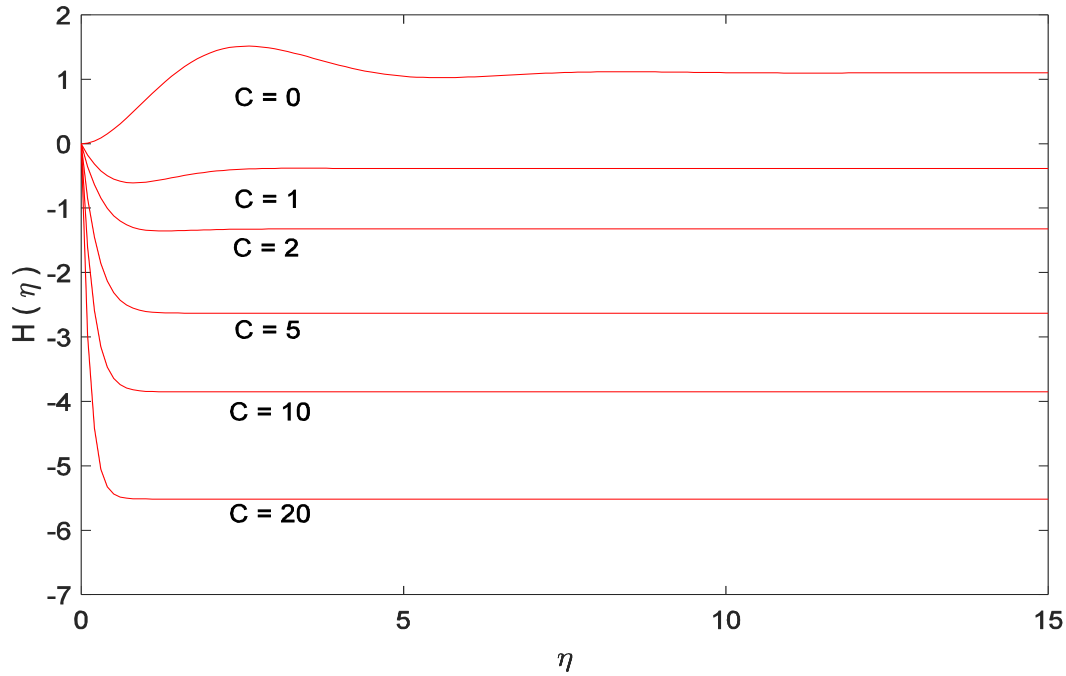


Figure 4-1: Effect of C on H(η)

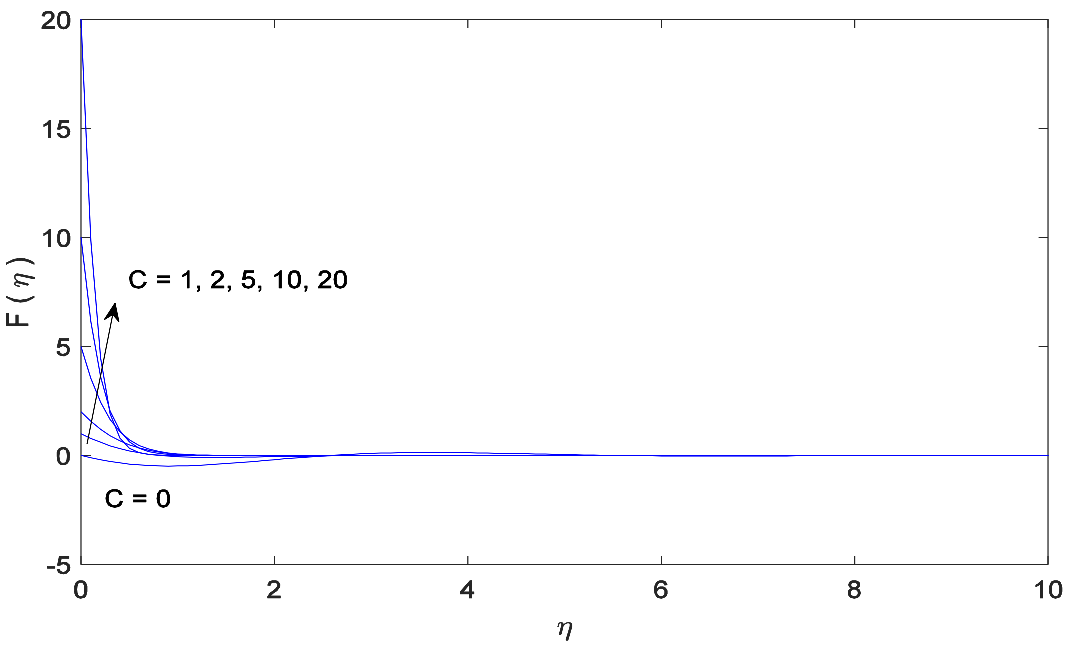


Figure 4-2: Effects of C on F(η)

Figure 4.1 effect of C on $H(\eta)$

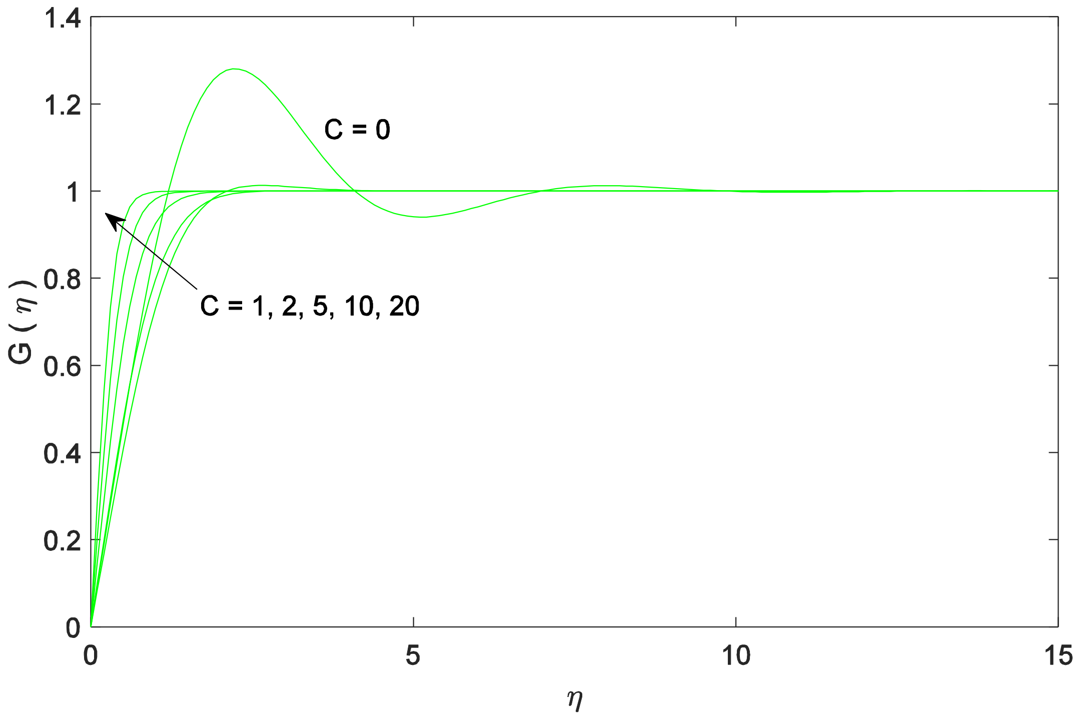


Figure 4-3: Effects of C on G(η)

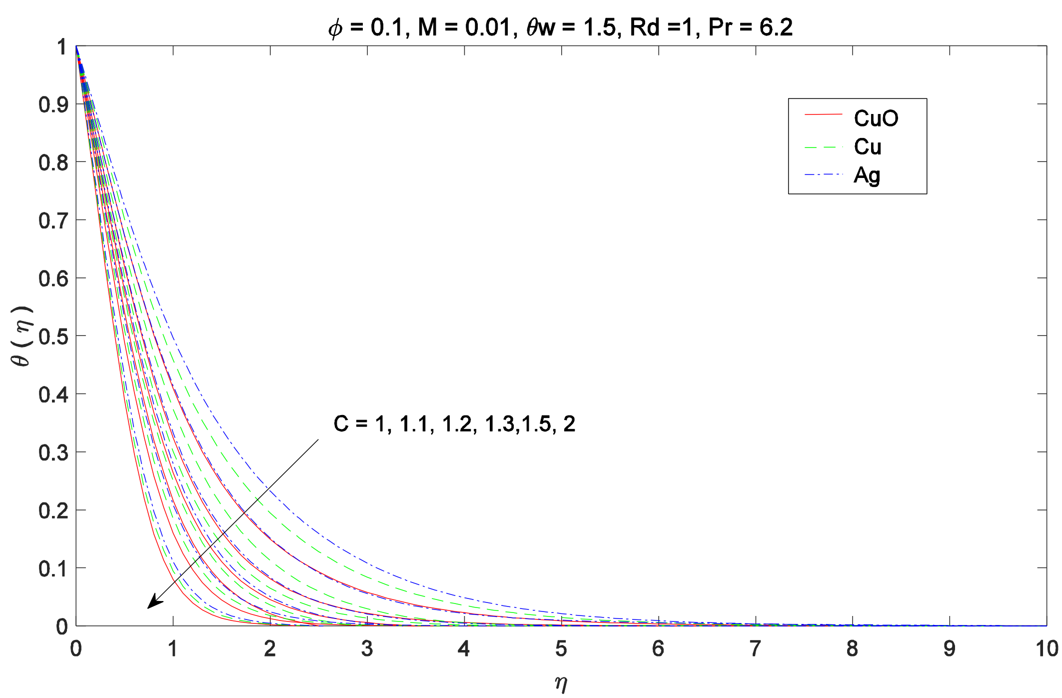


Figure 4-4: Effect of C on θ(η)

Table 4-3: Numerical values of skin friction coefficient Cf for Ag-Water with different values of parameters.

M	**Pr**	**Rd**	**C**	θ_w	φ	**bvp4c**	**Time (s)**	**Keller Box**	**Time (s)**
0.01	6.2	1	1	1.5	0.1	3.1777	0.836	3.1770	16.876
0.02						3.1872	0.913	3.1864	15.776
0.03						3.1966	1.007	3.1958	15.986
0.04						3.2059	1.244	3.2051	14.678
0.01	6.2	1	1	1.5	0	1.9909	3.375	1.9930	9.592
					0.01	2.1124	3.434	2.1121	9.768
					0.05	2.5836	4.272	2.5819	9.063
					0.1	3.1777	3.468	3.1770	9.272
					0.2	4.4901	3.552	4.4900	9.068
0.01	6.2	1	1	1.5	0.1	3.1777	2.494	3.1770	9.214
			1.5			4.5579	2.310	4.5579	8.649
			2			6.2847	2.608	6.2861	9.111
			2.5			8.2977	3.481	8.2987	9.049
			3			10.5526	3.718	10.5567	10.677
			3.5			13.1288	3.532	13.0332	9.530

Table 4-4: Numerical values of Nusslet number Nu $= -(\frac{\mathrm{knf}}{\mathrm{kf}} + \mathrm{Rd}\theta_w^3)\theta'(0)$ for Ag-Water with different values of parameters.

M	**Pr**	**Rd**	**C**	θ_w	φ	**bvp4c**	**Bvp4c Time(s)**	**Keller Box**	**Keller Box Time (s)**
0.01	5	1	1	1.5	0.1	0.5569	2.380	2.0812	9.350
	6.2					0.7071	2.607	2.6603	9.366
	7					0.8113	2.776	3.0388	12.887
	10					1.1608	3.516	4.3475	8.505
	13					1.4596	3.165	5.4664	9.417
0.01	6.2	1	1	1.1	0.1	0.7071	2.957	2.6014	8.505
				1.5		2.3149	3.497	2.6603	9.350
				2		0.1724	2.799	2.4932	9.176
				2.5			3.516	2.2535	9.430

0.01	*6.2*	*1*	*1*	*1.5*	*0.1*	*0.7071*	2.223	*2.6603*	*18.788*
0.02						*0.6823*	2.253	*2.5623*	*12.011*
0.03						*0.6467*	2.715	*2.3983*	*11.073*
0.04						*0.5903*	1.190	*1.9399*	*17.139*

Figs 4-5 to 4-8 displays the effect of nanoparticle volume fraction ϕ on velocity and thermal profiles. Fig 4-5 to 4-7 shows that increasing ϕ suppress the boundary layer thickness as a result thermal boundary layer increases which is giver in Fig. 4-8. In Fig 4-5 result depicts axial velocity profiles for various values of volume fraction of nanoparticles. In fig we can see values corresponding to three types of Nanoparticles. As per expected $H(\eta)$ is negative because of downward flow in vertical direction by stretching of disk. Increase in concentration of nanoparticles axial velocity tends to decrease and gave maximum absolute value for CuO and gave minimum absolute value for Ag nanoparticles. Fig 4-6 shows pattern of volume fraction of Nanoparticles ϕ on radial component of velocity$F(\eta)$. The profile of velocity indicates that direction of flow is radially outward in near wall area. We can also conclude from this velocity profile that in radial direction volume fraction ϕ is inversely proportional to velocity boundary layer. Plot in Fig 4-8 are temperature profiles corresponding to volume fraction ϕ it also indicate with Ag nanoparticle boundary layer thickness is larger than by using (CuO, Cu).

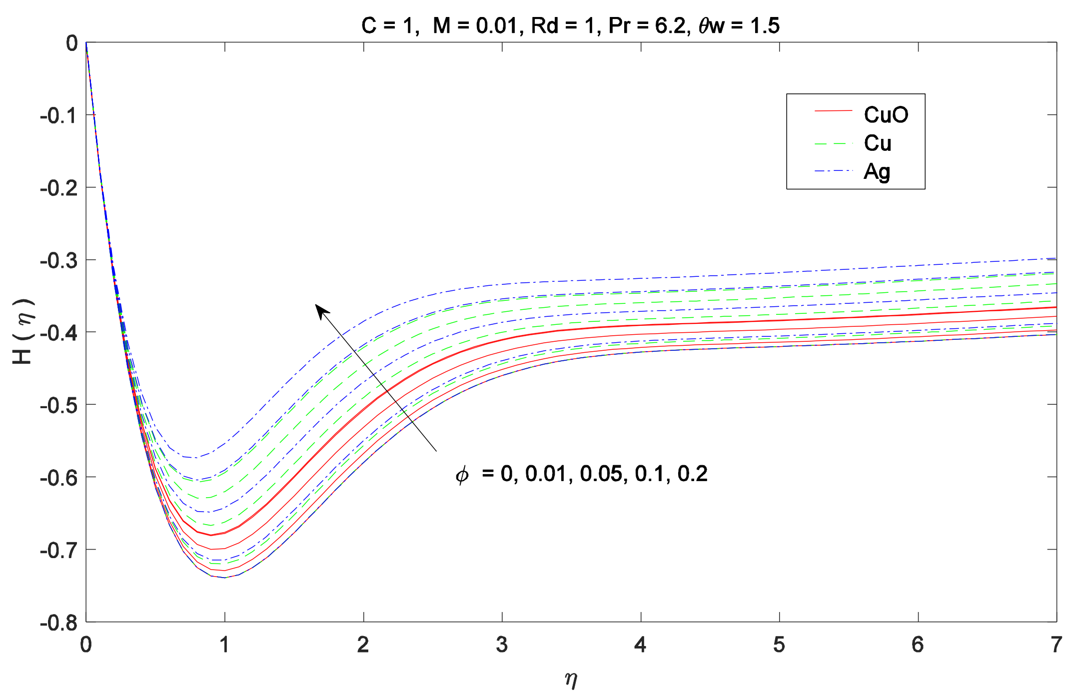


Figure 4-5: Effects of ϕ on H(η)

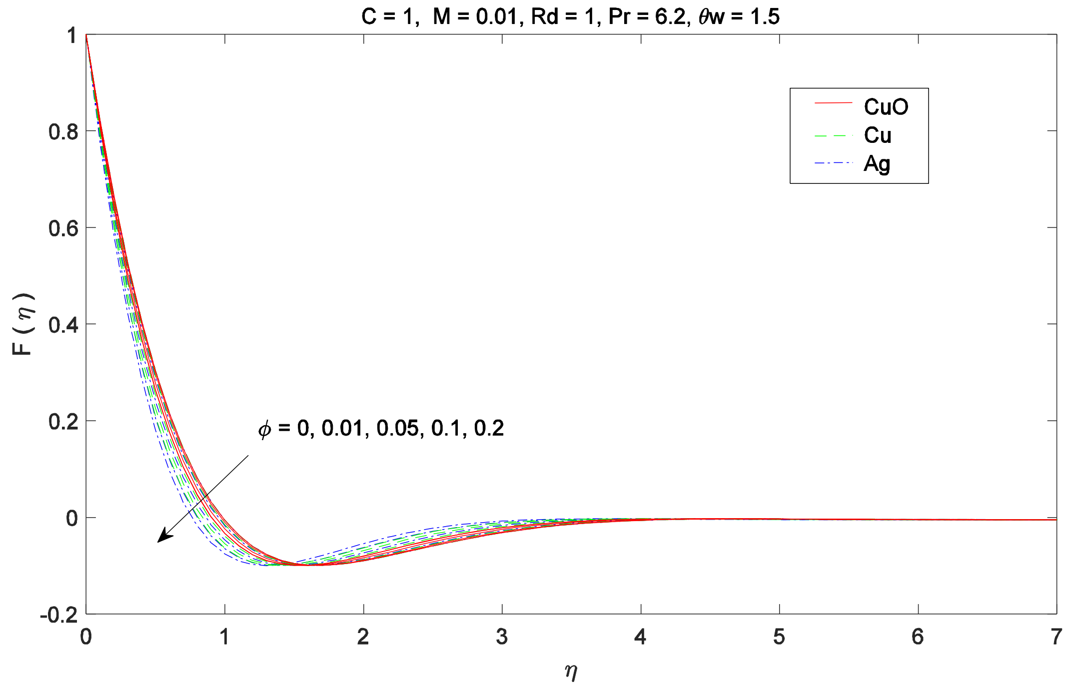


Figure 4-6: Effects of ϕ on F(η)

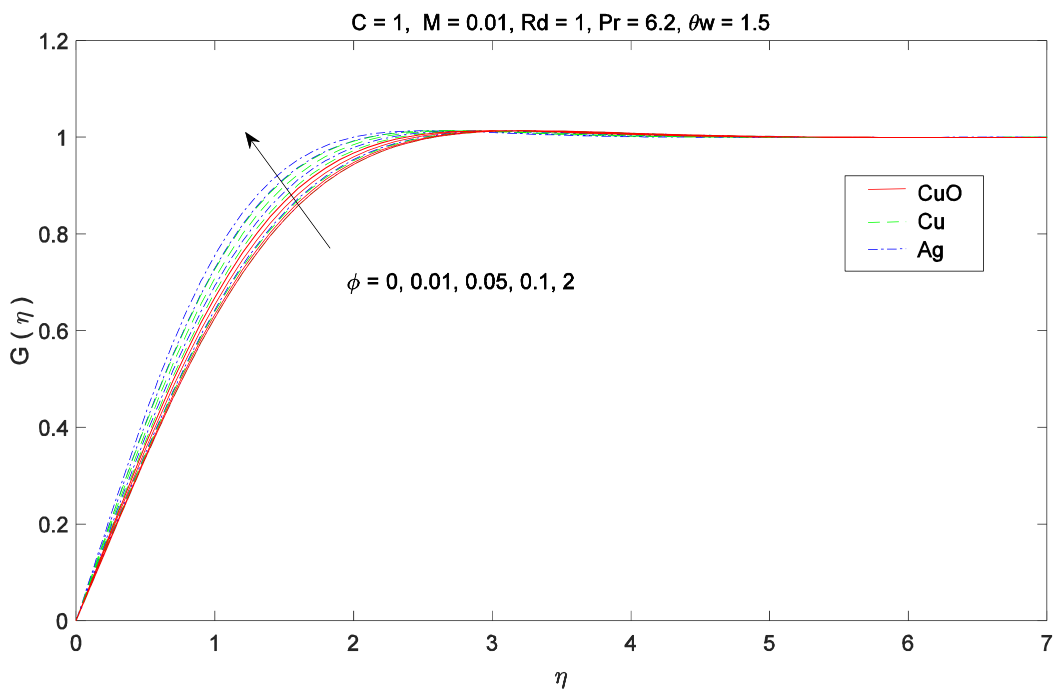


Figure 4-7: Effects of ϕ on G(η)

From Fig 4-9 to Fig 4-14 are the results of behavior of velocity profiles $F(\eta), G(\eta), H(\eta)$, and temperature profile $\theta(\eta)$ when fluid flow is exposed to magnetic field with all three nanoparticles

CuO, Cu and Ag considered in base fluid water respectively. These plots clearly represent decrease of velocity profiles and increase in thermal profile by changing in values of magnetic field. As mention before, Figs. 4-9 to 4-14 depicts the effects of Hartmann number on velocity and thermal boundary layers. There is a minimal effect on radial and tangential velocity component however axial component is significantly effected in far field. In the absence of magnetic field, the overall direction of the flow is towards the stretching disk, however introduction of magnetic field changes the flow direction at certain distance from the disk. In the near wall region, the flow is downward i.e. towards stretching disk and in far field region the direction is towards far field. This phenomenon can be seen in streamlines Figs. 4-12 and 4-13. Due to reduction in H thermal boundary layer increases as shown in Fig. 4-14.

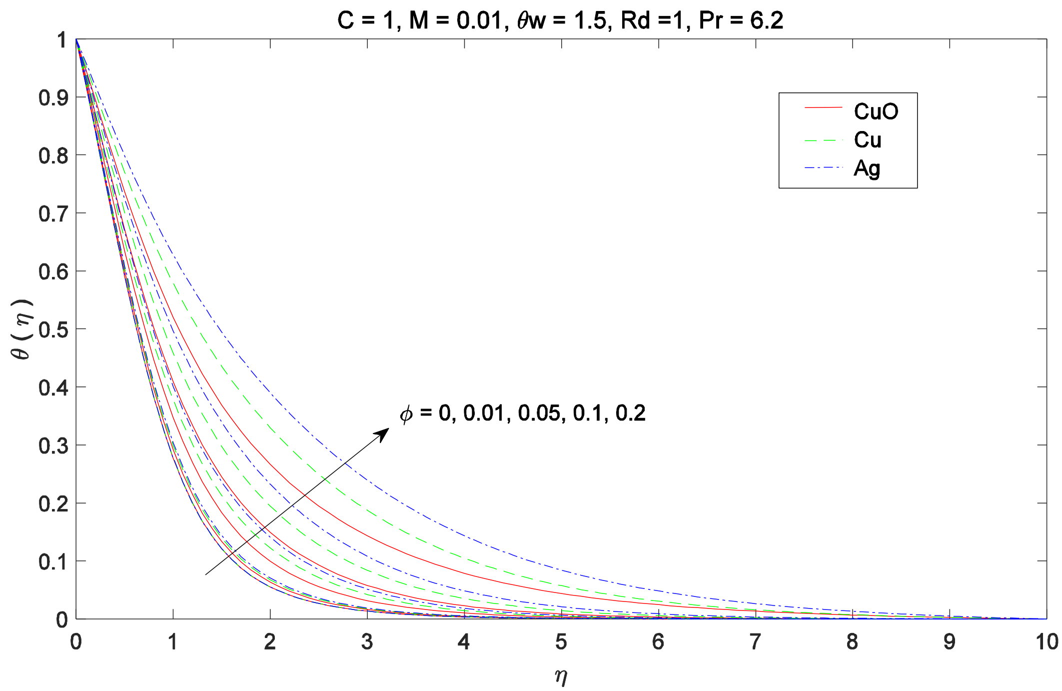


Figure 4-8: Effects of on ϕ on θ (η)

In Fig 4-15 it is clearly observed that with increment in Prandtl number temperature profile for all three nanoparticles CuO, Cu, Ag reduces.

Fig 4-16 illustrated when values of thermal Radiation is increased temperature profile changes its behavior. According to Cortell [30], wall slope of temperature must be constant to finite value when radiation parameter Rd is increased for linear radiation case, but when material/fluid is exposed to non-linear radiation sharp increase in temperature is observed, this can be seen in Fig. 4-16.

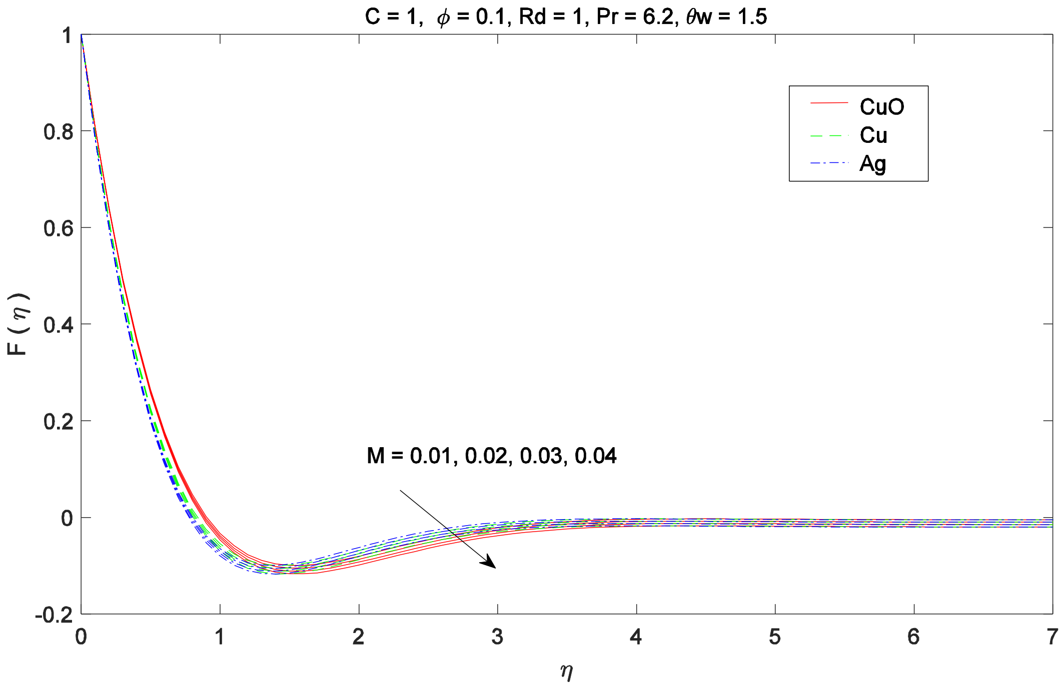


Figure 4-9: Effects of M on F(η)

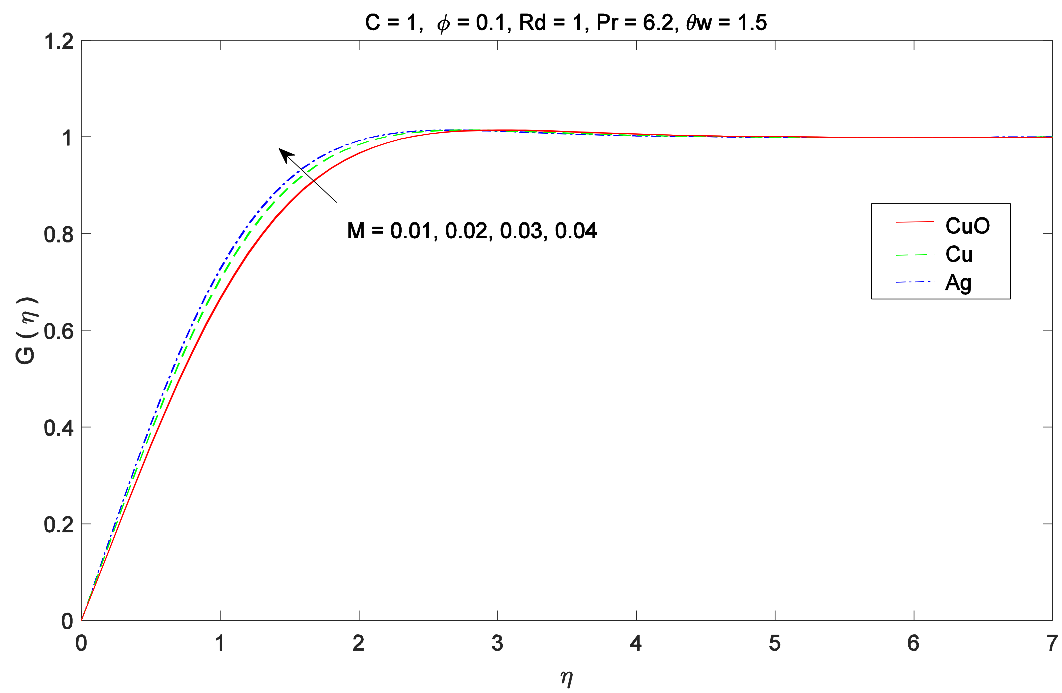


Figure 4-10: Effects of M on $G(\eta)$.

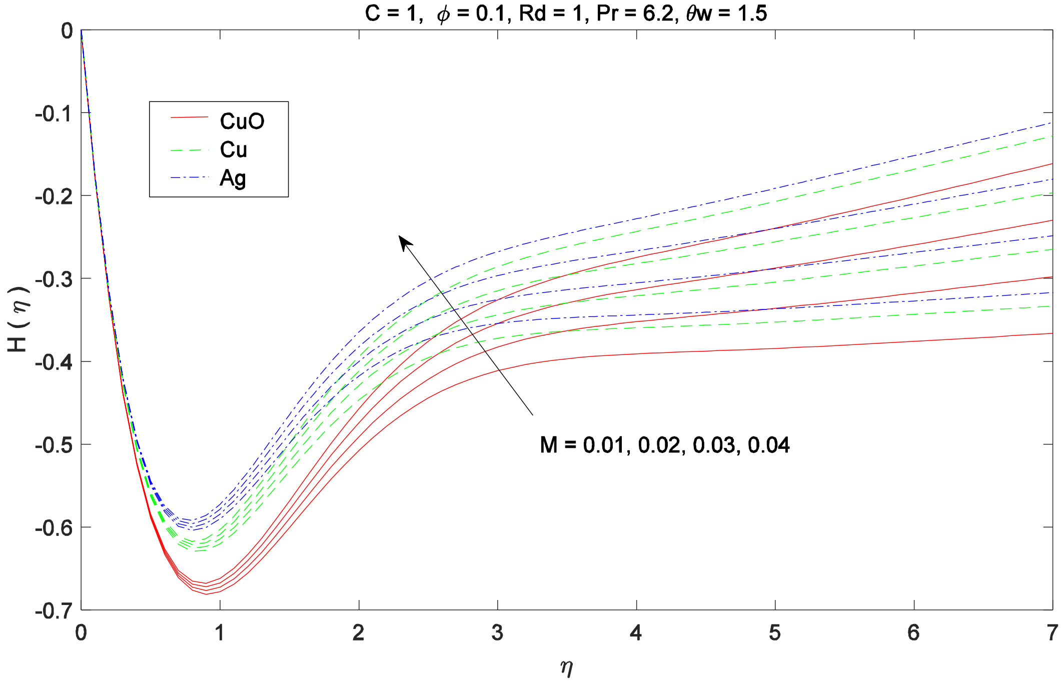


Figure 4-11: Effects of M on $H(\eta)$

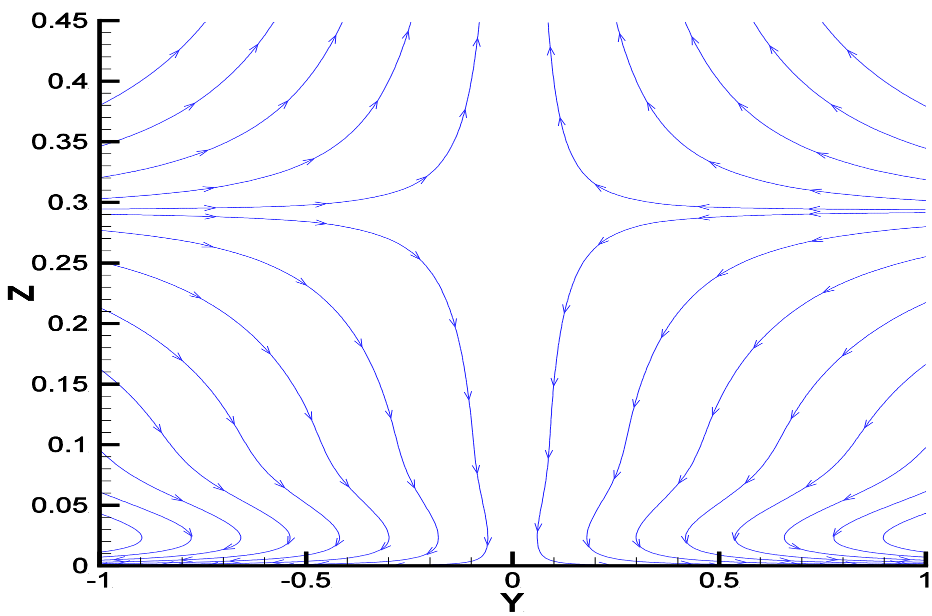


Figure 4-12: 2D streamlines of Rotating Fluid

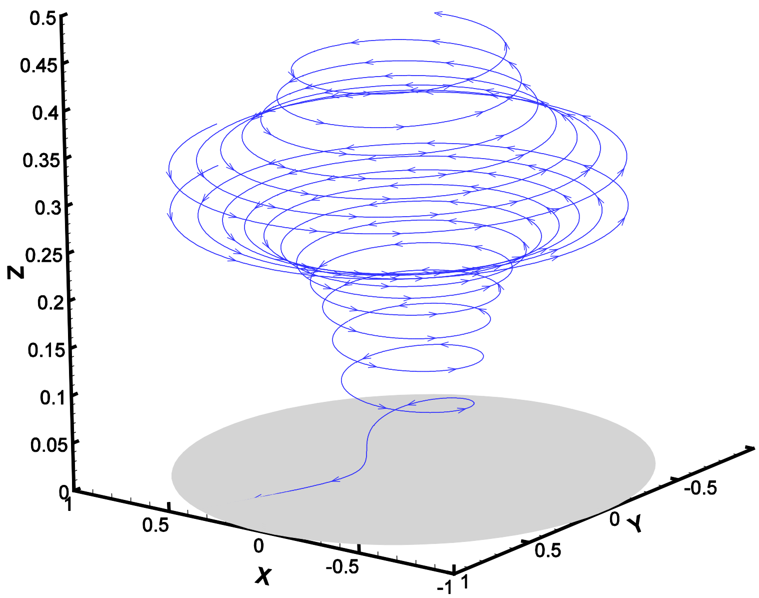


Figure 4-13: 3D streamlines of Rotating Fluid

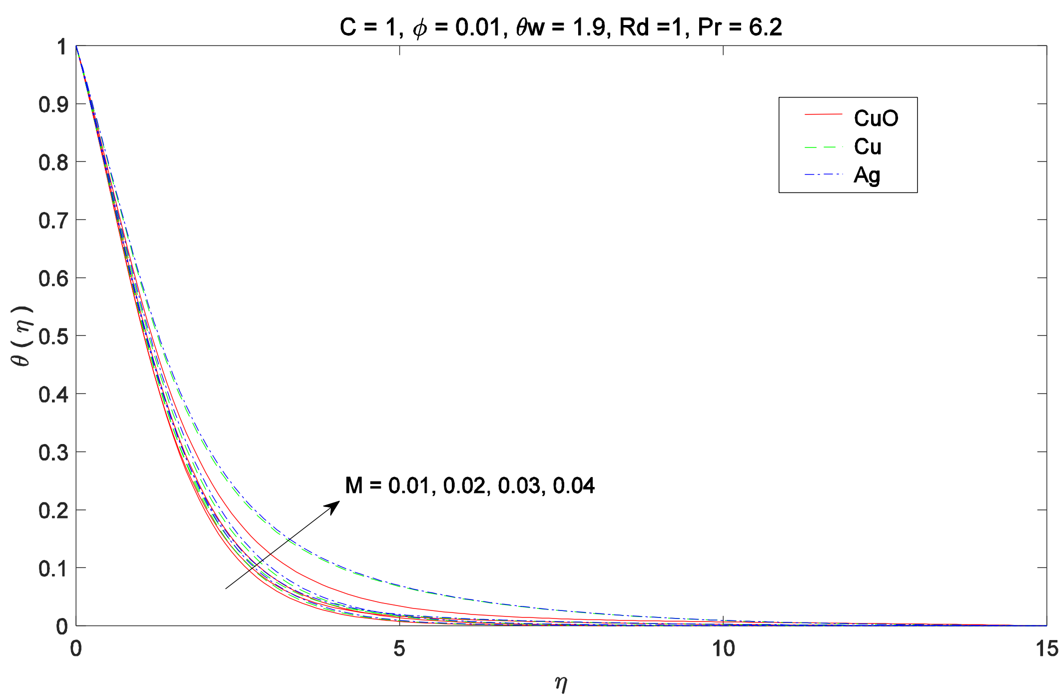


Figure 4-14: Effects of M on $\theta(\eta)$

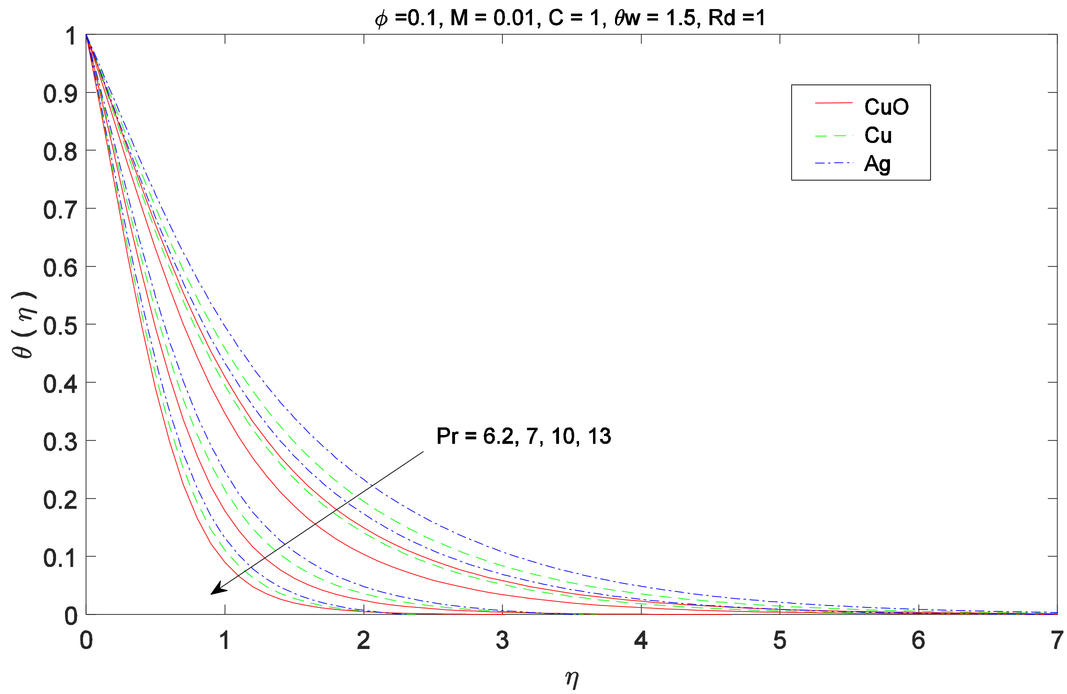


Figure 4-15: Effects of Pr on $\theta(\eta)$

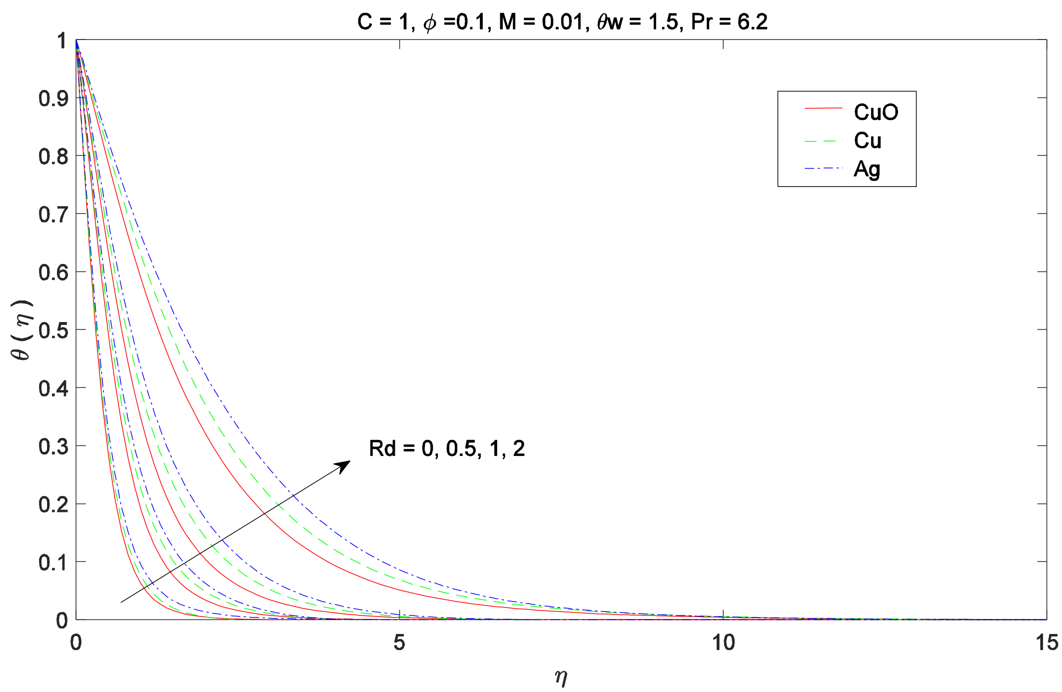


Figure 4-16: Effects of Rd on $\theta(\eta)$

Plot in Fig 4-17 indicates how temperature profile changes when values of θ_w is increased. Because of this, the rise in the wall to the ambient temperature ratio parameter θ_w tends to decrease the temperature gradient close to the surface. More specifically, the temperature variable concavity varies in $[0, \infty]$ when θ_w is large enough. [29]As value of θ_w increases it means there's a big difference between wall and ambient temperature which results a thick boundary layer. After calculations we got results that are comparable with the results published in [30].

In Fig 4-18 we can see skin friction C_f is plotted against stretching parameter C, it shows how within increasing values of C skin friction coefficient value is increased for all 3 nanoparticles (CuO, Cu, Ag).

Fig 4-19 depicts that how stretching sheet parameter affects Nusslet number. It is observed that by increase in value of stretching parameter value of Nusslet number for all three nanoparticles (CuO, Cu, Ag) increased. They are directly proportional to each other.

Result obtained in fig 4-20 shows that increase in concentration of volume fraction Φ of nanoparticles on C_f here we observe clealy that if we increase Nanoparticles concentration Φ value of C_f increases.

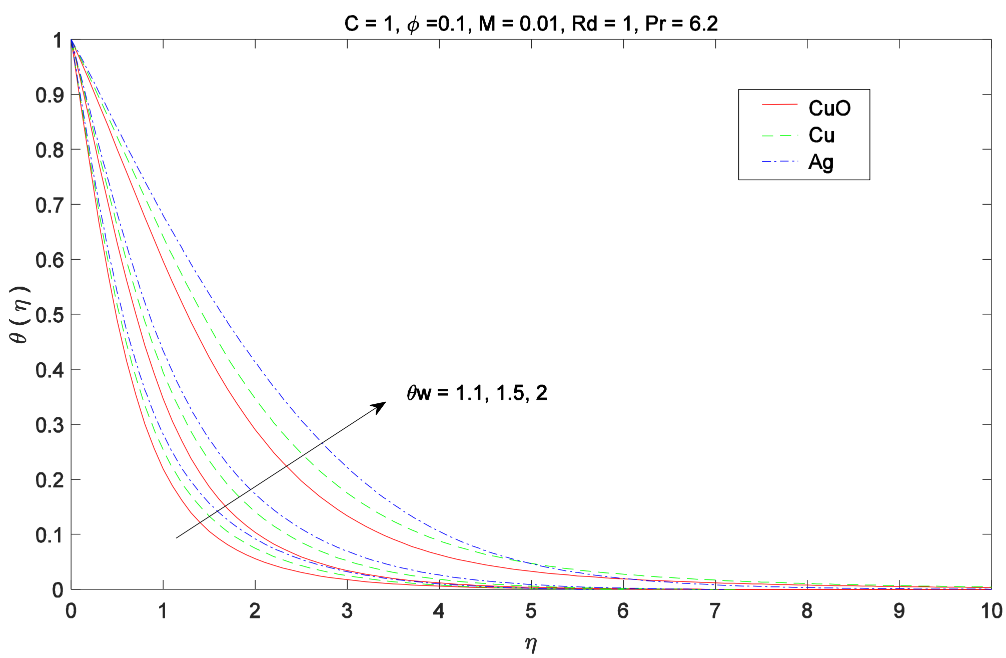


Figure 4-17: Effects of θ_w on $\theta(\eta)$

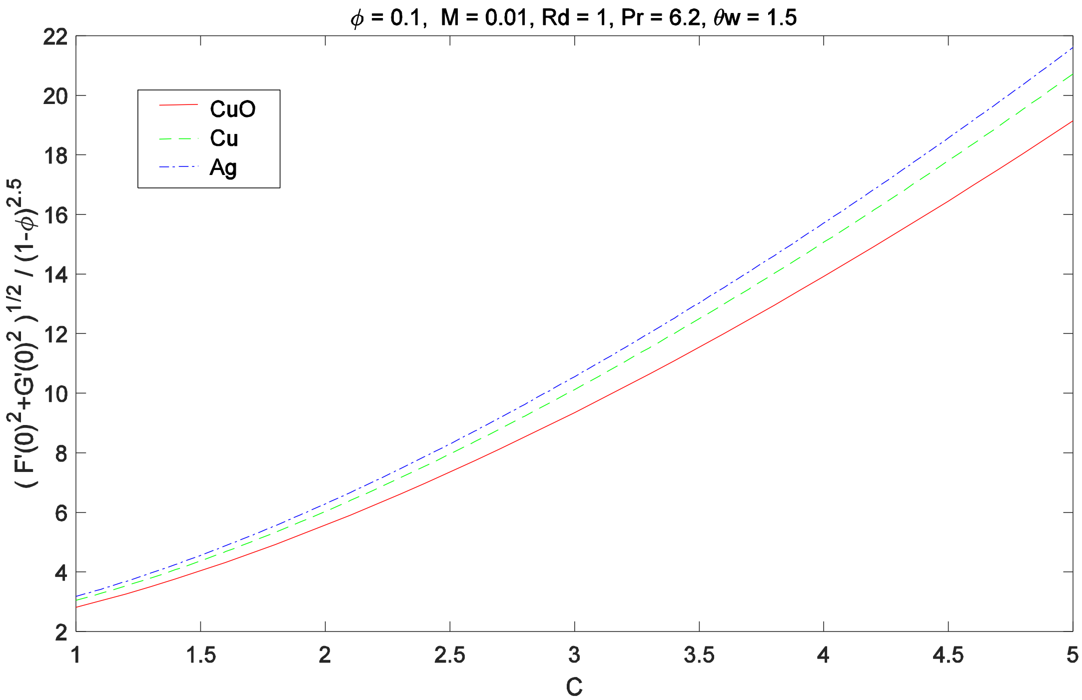


Figure 4-18: Effects of C on C_f

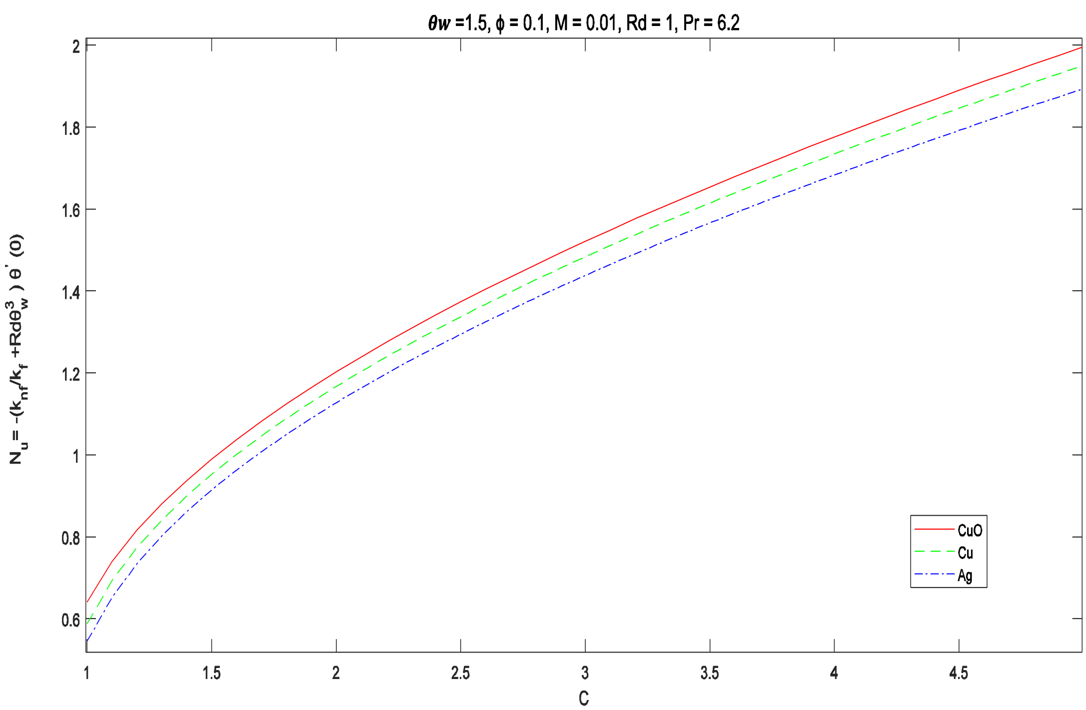


Figure 4-19: Effects of C on Nu

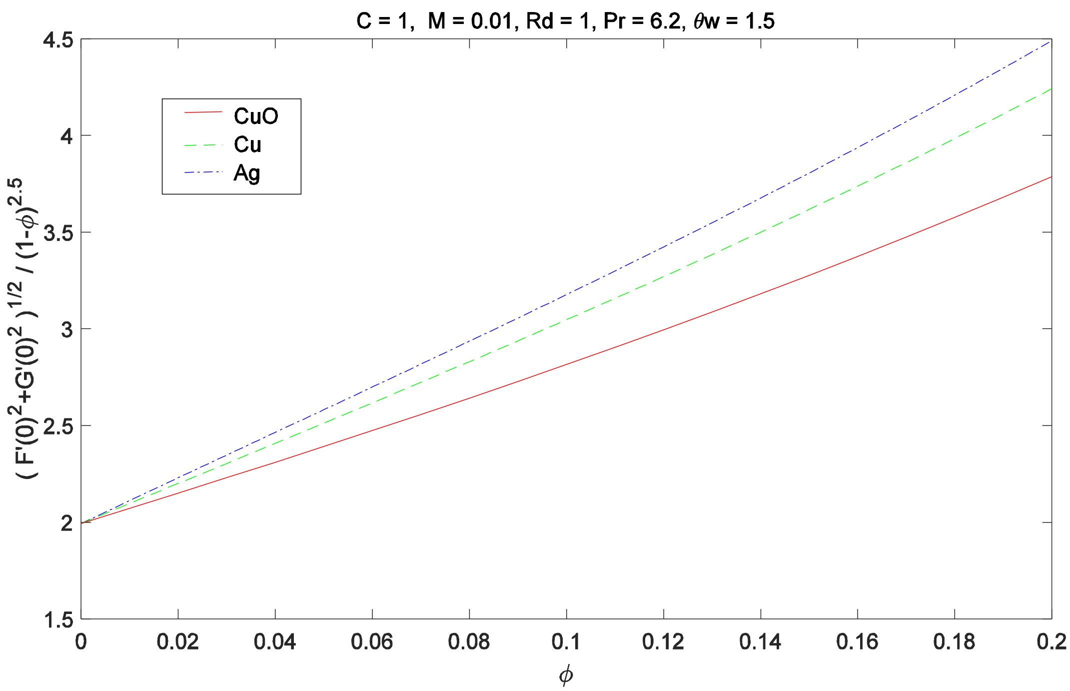


Figure 4-20: Effects of ϕ on C_f

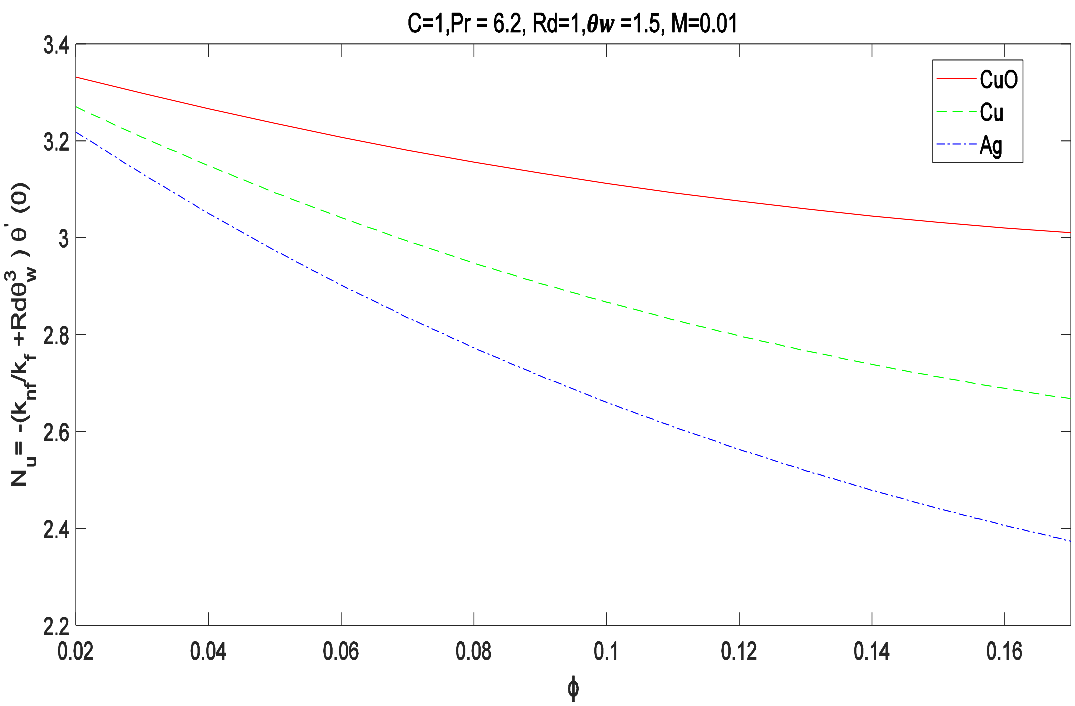


Figure 4-21: Effects of ϕ on Nu

In fig 4-22 it is observed that by increasing value of φ value of Nu decreases as a nonlinear profile. Result in fig 4-22 shows that by increasing value of magnetic field the values of skin friction increases linearly within given range. Behavior of results shows that for Ag value of $C_f$is highest and for CuO value of C_f is lowest. In Fig 4-23 effects of magnetic field on Nusselt is shown, which is a nonlinear behavior. It can be seen that increasing M reduce the Nusselt number.

Figs 4-24 and 4-25 depicts that how Rd and θw affects Nusslet number. It is observed that by increasing these parameters we observe mix behavior in Nusselt number. For $Rd = 0$, Nusselt number become invariant which is logical. Increasing temperature ratio overall tend to reduce Nusselt number. Similarly increasing radiation parameter also tends to reduce Nusselt number as can been seen in Fig 4-25.

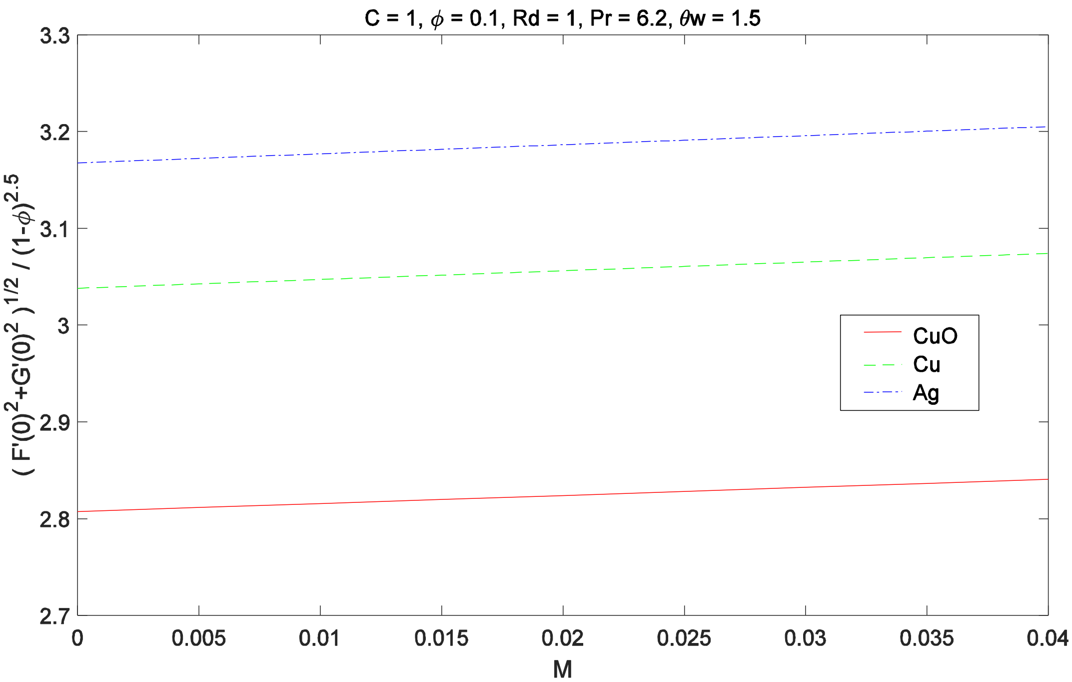


Figure 4-22: Effects of M on C_f

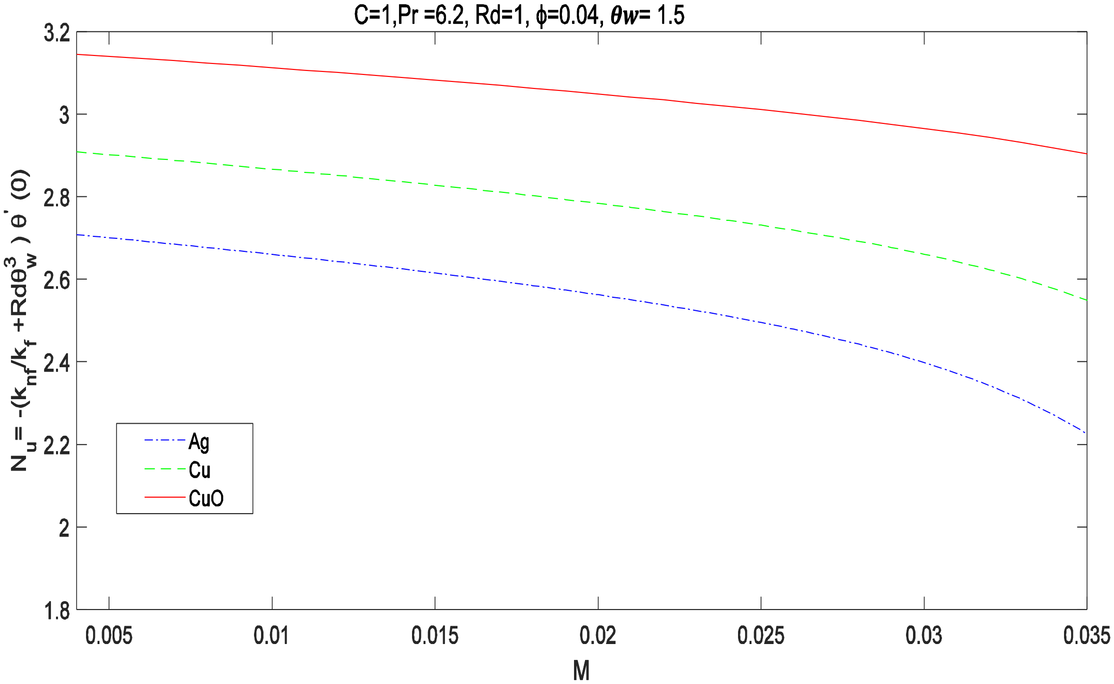


Figure 4-23: Effects of M on Nu

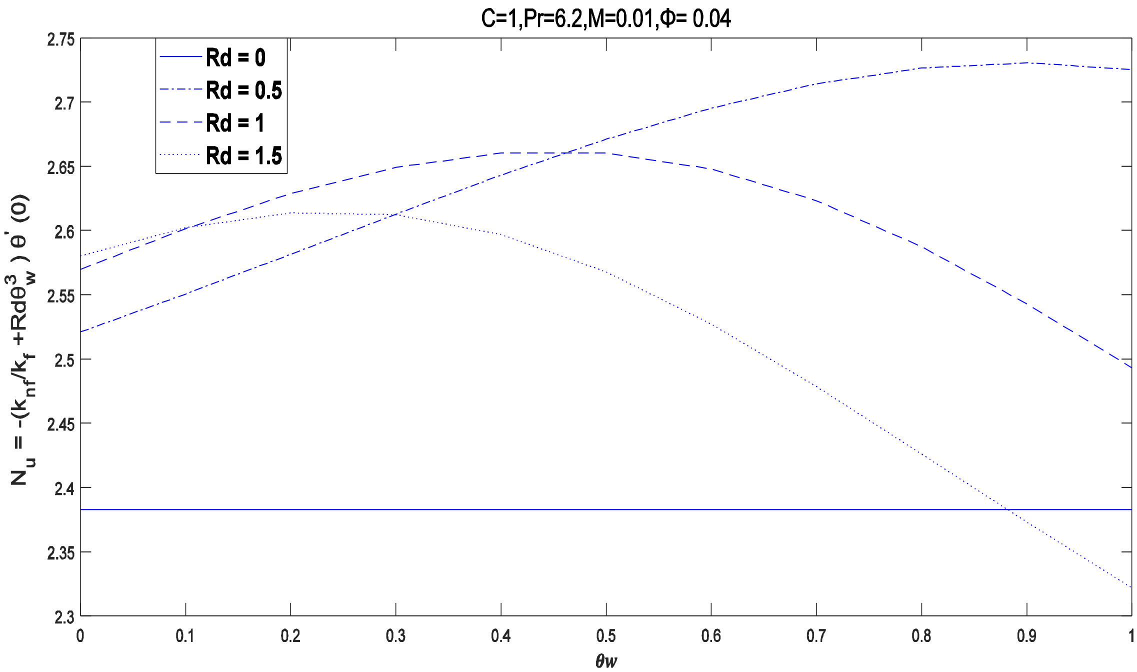
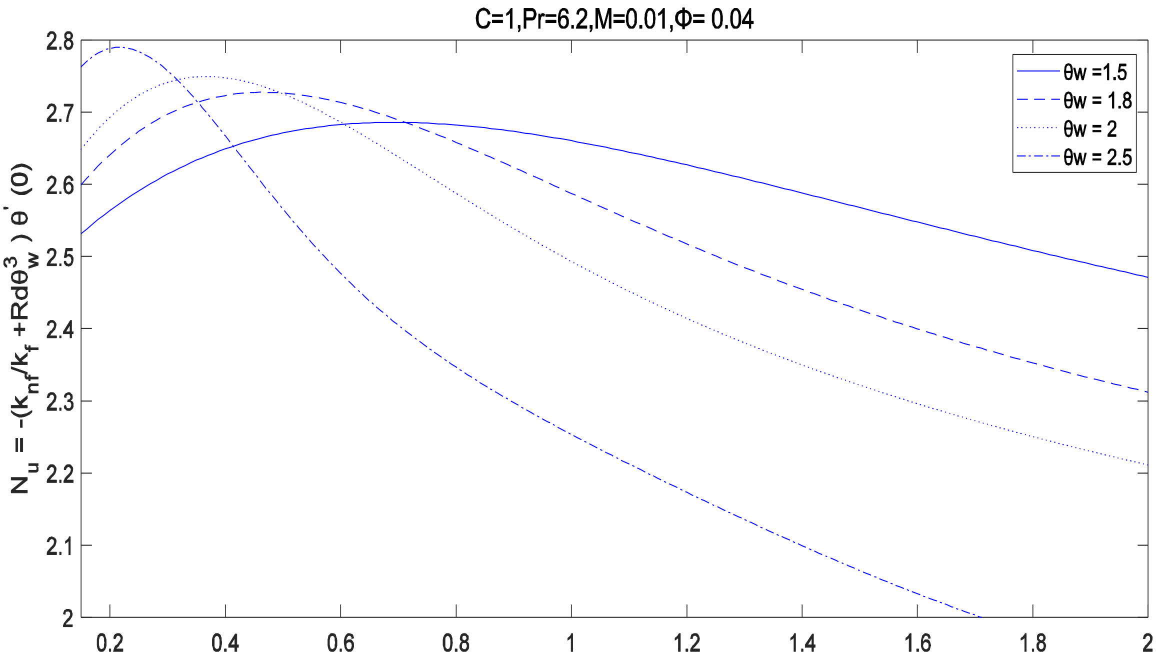


Figure 4-24: Effects of θ_w on Nu



Rd

Figure 4-25: Effects of Rd on Nu

CHAPTER 5: CONCLUSIONS AND FUTURE RECOMMENDATIONS

5.1 Analysis of Results

We analyzed our results with [46] in which effects of Hartman number M, nanoparticle volume fraction ϕ, temperature ratio parameter θ_w and radiation parameter Rd on flow and heat transfer of fluid over stretching sheet were published afterwards we check their normalized values with our results which are for rotating fluids and observed same trends for increasing temperature ration which can be seen in Table 5-1 and Fig. 5-1, which sort of validate our results. By comparing we can see that the value of Nusselt number increases in case of rotating fluid

Table 5-1: Analysis of $\boldsymbol{\theta w}$ value with [46]

M	θ_w	Rd	ϕ	Nu T.Hayat.et.al	Nu Present	Normalized T.Hayat.et.al	Normalized Present
0.1	1.1	0.1	0.03	0.7408	2.4499	1	1
	1.2			0.7441	2.4677	1.0044	1.0073
	1.3			0.7478	2.4877	1.0093	1.0154
	1.4			0.7518	2.510	1.10148	1.01245

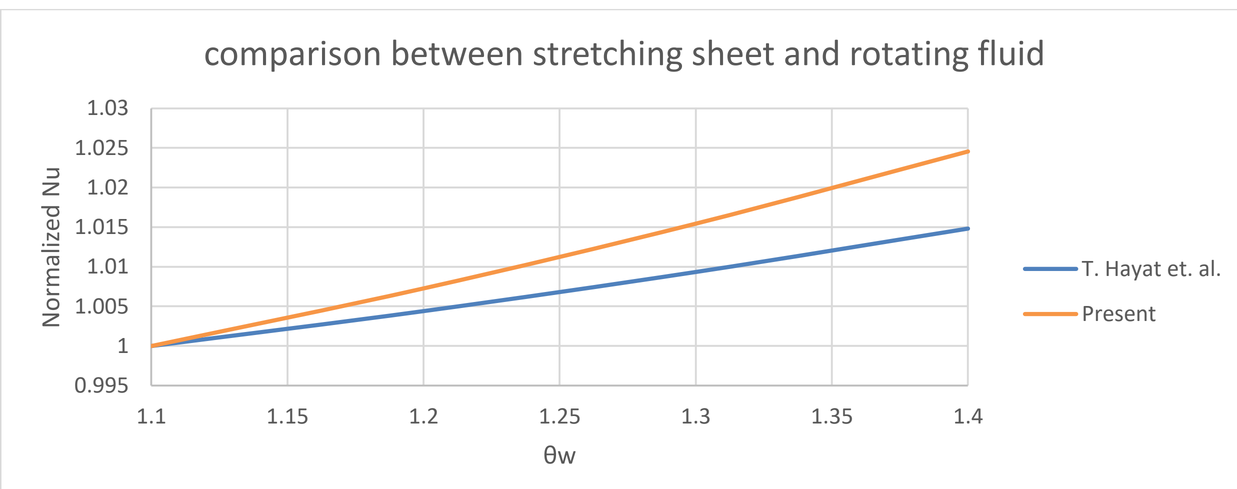


Figure 5-1: Result b/w stretching sheet & rotating fluid

In table 5-2 and Fig. 5-2 the effects of radiation parameter on Nusselt number were compared and found the results in agreement.

Table 5-2: Comparison of Rd values with [46]

M	θ_w	**Rd**	**Φ**	**Nu T.Hayat.et.al**	**Nu Present**	**Normalized T.Hayat.et.al**	**Normalized Present**
0.1	1.1	0.0.5	0.03	0.7333	2.410	1	1
		0.1		0.7404	2.4499	1.01031	1.016556
		0.15		0.7480	2.4876	1.020101	1.032199
		0.2		0.7548	2.5231	1.029375	1.046929

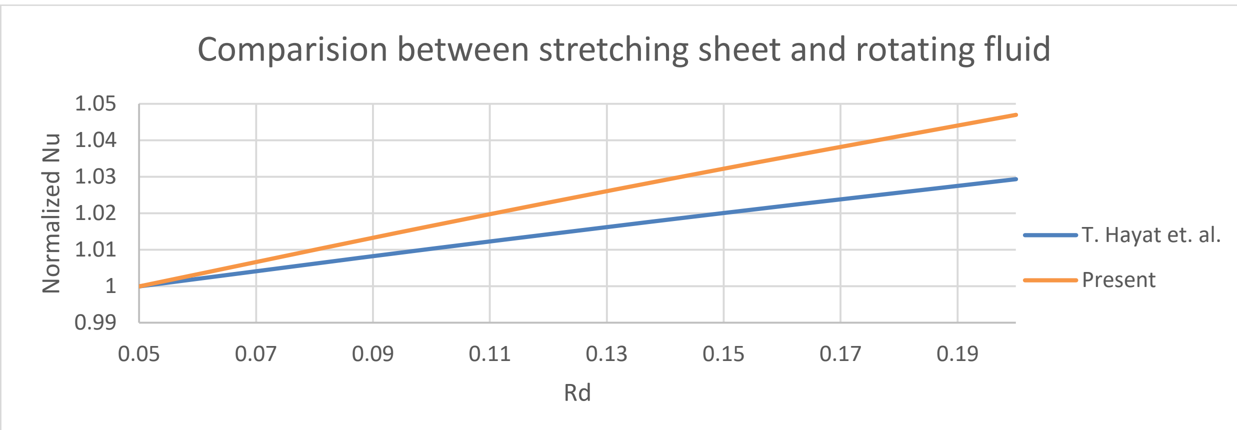


Figure 5-2: Comparison Nu between stretching sheet and rotating fluid by increasing Rd

5.2 Concluding Remarks

- Velocities in axial as well as in radial direction reduces as per the concentration of nanoparticles ϕ particles volume fraction.
- Stretching forces being balanced by magnetic forces and fluid tends to move in upward direction.
- When magnetic field is applied on nanofluid magnetic effects are insignificant on tangential component $G(\eta)$ of velocity.
- Heat transfer rate is higher by using CuO Nanoparticles with base fluid water than by using Ag nanoparticles with same base fluid.

- Stretching in radial direction tends to decrease the boundary layer thickness.
- Skin friction coefficient is directly proportional to both nanoparticles volume fraction and magnetic field.
- Local Nusslet number reduces by increasing magnetic field strength.
- We note a mixed behavior in the heat transfer rate as $\boldsymbol{\theta_w}$ and Rd are increased.
- Overall Nusselt number reduces by increasing θw and Rd.

5.3 Future Recommendations

- This work can be done by taking unsteady flow.
- Hybrid nanofluid transport model can also be applied on the current problem.
- Same problem can also be solved by taking Cattaneo–Chritov heat flux model.
- Various Non-Newtonian fluids model such as Maxwell fluid model, Jaffery fluid model, Bingham plastic model, Bird Carreau model, Cross Power law model, Herschel-Bulkey model, second –grade model etc. can be used to make flow analysis for current problem.
- Non-linear and linear stretching effects can also be observed and calculated by applying same kind of methodology.

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