Quantum Information Approach to Composite Particles: Quantum Entanglement and Coherent States

by

Bushra Tabassum

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Supervised by

Dr. Shahid Iqbal

School of Natural Sciences

National University of Sciences and Technology

Islamabad, Pakistan

FORM TH-4

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Date: 29-08-2017

Signature (HoD): ____________________

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Dedicated to

My Loving Parents

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All praise is due to Allah (S.W.T), we praise him, seek his help and ask for His forgiveness. Words are bound to praise ALMIGHTY ALLAH, the Lord of the world, the Merciful and the Gracious who is the entire source of all knowledge and wisdom endowed to the mankind. Whose blessings are the cherish fruits of my thoughts and modest effort in the form of this dissertation.

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Abstract

This dissertation consists of quantum correlations in composite particles composed of pairs of elementary bosons or fermions. First we discuss the composite systems in the context of quantum information theory and then make thier connection with the composite two particle systems. The main focus is on systems of two distinguishable elementary fermions, as in the case of hydrogen atom. However, composite particles of other types have also been discussed. It has been found that such systems may exhibit their composite behavior depending on how strongly correlated they are, as measured by the amount of entanglement. The role of entanglement in the description of composite particles has been explored explicitly and various bounds and limitations have been discussed. Finally, we discuss the coherent states for composite particles composed of two or more distinguishable fermions or bosons as constituent and discussed their properties such as particle counting statistics by means of Mandel's Q-parameter.

Chapter 1

Introduction

Despite of the reality that the quantum theory is hundred years old now, in the last few decades some practical applications of the unique quantum mechanical phenomena have been found in the information processing and computation. Particularly, due to their distinctive computational capacities and faster information processing speeds than the conventional classical systems, there is a revolution in these fields. An efficient algorithm which bases on quantum computation and was found by Shor[1], perform factorization of composite numbers in fractions of seconds which are thought to take as much time as the life of universe if done using classical computers. Moreover, in the field of communication, quantum mechanics is successful in giving us a secure cryptography. The most important and unique feature of quantum mechanics is entanglement which help us explain different phenomena of quantum mechanics more specifically in quantum optics. it is very important and useful in the quantum information processing. There are already many of the questions has answered related to the entanglement or correlation between the particles but still so many are open many open for research, presently there are many criteria which are very helpful in testing the separability of a any quantum mechanical state. This thesis is related to the entanglement in composite bosons. Therefore, before discussing the entanglement one should know what bosons are and what is the difference between elementary and composite particle? The following sections have detail discussion about these questions.

1.1 Elementary particles and their composites

The idea of divisibility of matter into smaller particles and that fundamentalism is limited to certain primary, smallest particles in nature is as old as the natural philosophy. Here, we briefly discuss composite particle in terms of these fundamental particles. Molecules being made up of atoms, which are the basic units of all chemical element, are made up particles, divisible into Leptons and Quarks such as proton and neutron. At present, it is believed that two indivisible particles, the leptons and quarks are basically of six types which are used to constitute all the material of the universe. Leptons as well as quarks has their corresponding antiparticles: "A particle that has the same mass but opposite electrical charge and magnetic moment". Quarks exist in couple a collection from with other quark or anti-quark. we can not see in all alone, quarks combine with the other quarks and anti-quarks together to form particles named as hadrons. There are more than 200 types of hadron. Penta-quark hadrons have been synthetically prepared in laboratory.s been reported till date.

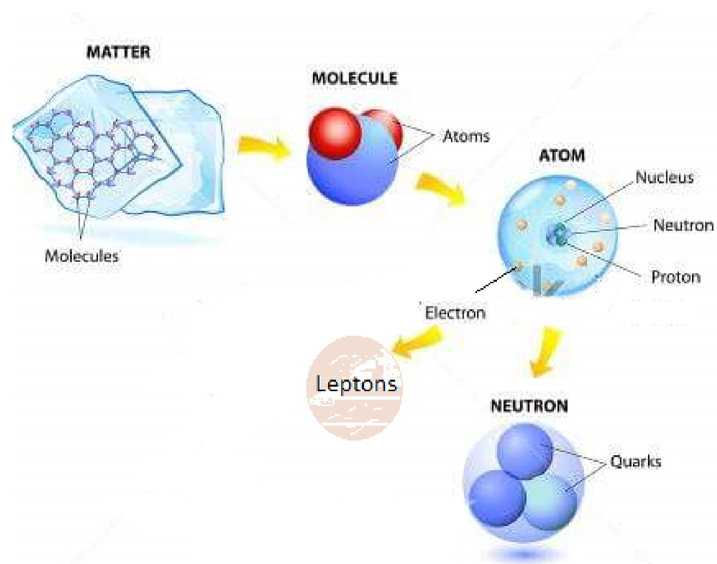

Figure 1.1: A description of material particles.

The six leptons are electron, muon, tau and their respective neutrinos. Up, down, charm,strange, top and bottom are the six quarks. The top quark is believed to be the heaviest particle nature has ever created. Ordinary matter contains up and down quarks which combines to form protons (two up quarks and a down quark) and neutrons (two down quarks and an up quark). A pentaquark is made up of up, down and strange quarks in the ratio of 2:2:1. The electric charge on quarks is integral multiple of one-third that of electron.

1.2 Quantum statistics of particles

For a quantum composite system containing two particles in it, the consideration of quantum nature of constituents is necessary. This results in categorization of particles into bosons and fermions. We aim at comprehending the statistics of these particles. Bosons are integral spin particles and comply with Bose-Einstein while fermions are half-integral spin with Fermi-Dirac statistics. In general, matter is composed of fermions. Bosons e.g. photons and Higgs particles are interactions carrier. The concept of identical particles is that all electrons are intrinsically the same and interchangeable. They may be in different states like spin up and down, low and high energy states but states should not be confused with intrinsic properties. Fermions have the property that no single state can accommodate two particles of the same sort (e.g., electrons). You will never see two fermions with all the quantum numbers the same. There is no such limitation on Bosons, infinitely many bosons can occupy same state.

The origin of spin-statistics theorem is relativistic quantum field theory, and it assign integral and half-integral spins to bosons and fermions respectively. Composite systems, may exhibit completely or partially bosonic or fermionic behavior depending upon the number of constituent particles e.g., a Helium-4 nucleus is a boson, however a Helium-3 nucleus is a fermion. Similarly atoms may be bosons or fermions e.g., a hydrogen atom is a boson, while deuterium is a fermion. Probing the subsystems of a composite system shows partially fermionic and partially bosonic behavior e.g., if we look too carefully at a Helium-4 nucleus we discover the fact that it is not a boson, but is a collection of fermions.

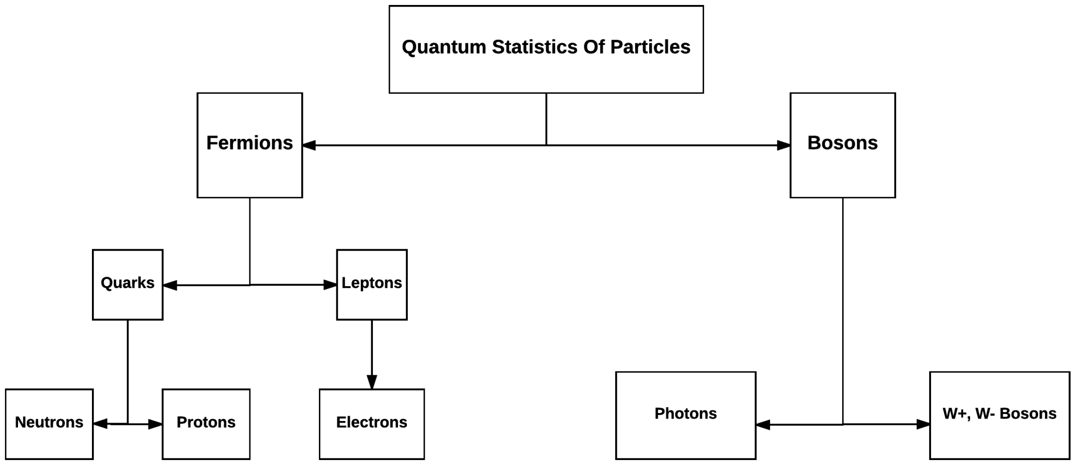

Figure 1.2: Classification of elementary particles.

1.2.1 Composite Particle

Any particle that is composed of two or more fundamental particles is known as composite particle. Composite particles may be either bosons or fermions depending on their constituent. In atomic nucleus protons and neutrons are composite particles. For the composite particles we have to extend the Pauli exclusion principle. If a particle is composed of two particles having spin 1/2 , after combing them we get either $\frac{1}{2}+\frac{1}{2}=1$ or $\frac{1}{2}-\frac{1}{2}=0$ so we get spin 0 or 1. In either case, the particle we get is a boson known as composite boson(co-boson) as we getting an integer spin. Now if we combine any odd number of fermions we get half integer spin and more precisely a composite fermion. Examples of composite fermions are helium-3 two electrons, two protons and on neutron. we can point out the different of the composite particle from elementary particle for instance, with the changes of the Pauli scattering through the exchanges of specific fermionic properties with which and their interaction correlated using different methods. System of composite particles can also follow the commutation relations but the results of those composite system does not satisfy the the result obtain from the commutation relation of bosons. This is due the the reflection of fermionic structure inside the composite system. An important phenomenon dealing the composite particles is entanglement discussed below.

The study of composite particles are owned by the field of many-body theories. A boundle of literature is present related to this subject, and it is unfortunate that the complexity of the problem usually escalates when we increase the number of the particles of the system. There are many methods to approach the problem and a popular way to deal with systems of many composite particles is to launch a program of bosonization. The term bosonization may be used in various different contexts, but here it specifically means a systematic transformation of a problem that deal with composite particles into a problem that involves only elementary bosons, a simplification which otherwise makes intractable problems unsolvable. This may be physically motivated by the Spin-Statistics theorem from relativistic quantum mechanics, from which we know the difference between integer spins and half-integer spins particles. A pair of strongly correlated fermions would outwardly appear to have integer spin, so long as its internal structure is not probed, and is therefore expected to exhibit boson-like behavior. For this reason, such systems are also sometimes conventionally called composite bosons, though the term is slightly misleading as not all composite systems of two fermions will necessarily exhibit bosonic behavior. For more on this subject, see [2, 3]. In this thesis, however, we will not be concerned with the explicit solution to many body problems. We are primarily interested in the study of how the correlations present in composite particles are responsible for various physical properties of the system. As such, it is necessary for us to retain the "compositeness" of our composite particles because it only makes sense to speak of correlations within a particle when you can subdivide said particle into partitions.

In the subsequent sections, we will primarily be dealing with systems of two correlated fermions or bosons. There are several reasons for this. The structure of composite particles quickly escalates as the number of particles increases. This makes it difficult to say anything general with regards to the correlations between the particles, so only the simplest of composite systems will be studied. Another reason is that quantum correlation is defined very well in the context of two correlated particles. The issue becomes much more controversial as the number of particles increases beyond two and this is very much still an open area of research. Considering only systems of two correlated fermions will make the issue of correlations something that is more easily quantifiable, a quality that will be exploited, once again, in the subsequent sections.

1.3 Quantum Entanglement

Quantum entanglement is the physical phenomenon which emerges from the description of two or more particles with reference to each other that is the quantum state associated with each particle and it can not be expressed independently but, we can define the quantum state, in which the complete details of the system are contained. In quantum physics, the connection between the particles can be explained in terms of entanglement and these particles keep connected in such ways so that their action operated on any one of them influences the whole system, independent of the distance between them. This phenomenon provoked Albert Einstein subsequently to name it "spooky action at a distance." According to the quantum physics and its rules we can say that any photon stay in all the states that are possible for the system when it is unobserved but, when a measurement is made, it must appear to be staying in only one state. Entanglement shows up when more than one particles, interact physically. A light beam incident on matter results in individual photons to get entangled. The photons can have a substantial separation, many miles or extensively more.

Entanglement helps us from various perspectives. It can not just perform tasks in quantum processing that are impossible without it additionally different tasks efficiently. It is one of the unique aspect of quantum mechanics [4, 5]. The advantage of entanglement is taken in execution of most of them in such an approach to exchange for something else. This is the reason for taking entanglement as a resource like energy. Entanglement, being the resource for the purpose of quantum processing of information, its evaluation in quantum states is vital. In quantum mechanically entangled composite systems, the constituent subsystems have a strong correlation to each other even when they are spatially isolated such that they do not interact. The composite system can be regarded as a definite pure state, however this definition is not valid for the constituent subsystems states. Making a measurement on any of the subsystems will influence measurement on other subsystems and this is the contradiction of local-realism, i.e. the quantum states of spatially isolated non-interacting particles are independent. For the first time, this phenomenon was discussed by Einstein, Podolsky and Rosen in their seminal paper in 1935 [6, 7, 8]. They analyzed the incompatible measurements made about one subsystem of the two-particle composite system, which interacted in the past but at the time of measurement they are spatially separated. The contradiction pointed out by Einstein, made a question on the completeness of quantum mechanics and this problem was fixed in their need for local-realism. In 1960s, John S. Bell worked on the EPR argument and showed the correlation between measurements of entangled state predicted by quantum mechanics are out of scope for what local-realism based theories explain [9, 10]. The inequalities derived by Bell and others [9, 10, 11] were experimentally tested for entangled photons and put a confirmation stamp on the predictions of quantum mechanics [12, 13].

Entanglement has been recognized to play a vital role in the advancement and comprehension of the theory of quantum mechanics. Entanglement is mostly associated with elementary particles. Entanglement has been perceived as a phenomenon of no viable significance since its first appearance in 1935 until the mid of the 90s. With advances in quantum information science, entanglement has been seen as a resource for quantum information processing and communications. The applications of this vital resource include quantum cryptography [15], dense coding [16], teleportation of a quantum state [17] and quantum algorithms that are faster than their classical counterparts [18, 19, 20].

1.4 Quantum information approach to composite particles

Dealing with composite quantum system here, a question that may arise is what the physics of such composite particles systems is. In the present work, the following view of this issue is adopted: the introduction of elementary fermions and bosons and their quantum correlations and extension to understanding composite particles. Primarily, pure states are considered which have only one type of correlation; Entanglement.

Entanglement

The main focus in this thesis is on the Entanglement in composite bosons and their coherent states. In this thesis, we focused on the entanglement in bosons and fermions and extracted their composite behavior. The best and simplest example of such composite bosons is the well-known hydrogen atom which comprises of an electron and a proton. Before presenting the topic of composite particles in detail, it is worthwhile to first introduce the basic objects that these particles are made of bosons and fermions.

When we are given a quantum state and a task that uses entanglement and we are using entanglement as a resource, the main question is about the efficiency of the process. The answer is not clear since different entangled states have different performance. There are numerous ways to deal with measurement of entanglement resource for a quantum state for this reason. Number state of harmonic oscillator and level of purity is utilized in this thesis for quantification of entanglement.

1.5 Coherent states of composite particles

The history of coherent states is as old as the development of wave mechanics by Erwin SchrÖdinger. In 1926, he made a successful attempt by building quantum mechanical states demonstrating dynamical behavior close to classical dynamics. He was successful in building such quantum mechanical states for the harmonic oscillator [21] having minimum uncertainty.

The overwhelming success Glauber coherent states in the fields of mathematical physics [22], is the main motivation for physicists to extend this concept to general systems beyond harmonic oscillator. One of the procedures concerning this was to generalize, fulfilling a set of requirements, any one of the coherent states definition given by Glauber, i.e., the generalization should preserve some properties of harmonic oscillator's coherent states. Ladder operators and their supplementary algebra of the system under consideration for construction of coherent states for the system is used in this kind of generalization techniques. The first breakthrough in this regard was the development of a formalism relating quantum and classical dynamics, by Klauder in 1963 [22, 23]. Afterward, Klauder and Sudarshan combined the generalized coherent states with Lie group algebra and Barut and Girardello introduced coherent states for non-compact groups [24] which are known as Barut-Girardello coherent states. The concept was generalized to all types of Lie groups by Perelomov [25] and those states are named after him. Klauder and Skagerstam organized the study on generalized coherent states in the form of a book [26]. which was the base for classification of literature was coherent states applications in different fields of mathematical physics. A direct method of construction of generalized coherent states for the degenerate spectrum of quantum mechanical systems, such as, hydrogen atom was given by Klauder in 1996 [27]. The uniqueness of this approach is that it is explicitly independent of the system's underlying algebra. This method was further widened by Gazeau and Klauder to the non-degenerate systems having continuous and discrete spectrum having a lower bound [28]. These states, referred to as GK coherent states are independent of algebra. The GK coherent states were derived for numerous Hamiltonians, some of which are Poschl-Teller potential [29], the pseudo-harmonic oscillator [30], the power-law potentials [31, 32], the triangular-well potential [33], the Morse potential [34, 35], and single mode periodic potential systems [36]. R. F. Fox [37], developed a technique for generalization of coherent states by approximating Gaussian function to describe the behavior of the coherent states. These states are referred to as Gaussian Klauder coherent states. These states exhibit some non-classical behavior This non-classicality of coherent states has vital role in quantum information and quantum communication such as quantum teleportation [38], quantum computation, quantum cryptography and interferometric measurements [39]. In this thesis, we have constructed generalized coherent states, based on Gazeau-Klauder formalism, and analyzed the dynamics of so developed wave-packets in both momentum and position space.

1.6 Outline of the thesis

The organization of this thesis is as under:

Chapter 2 deals with preliminary basic concepts which provide us the necessary background for the later work. This chapter includes a brief quantum mechanical description of particles. Here quantum states and their categories have been discussed. A brief description regarding entanglement in a composite system is discussed in terms of Schmidt decomposition.

In chapter 3, the concept of entanglement between the constituent particles is introduced. We related the composite behavior to quantum entanglement. With the help of the properties of creation and annihilation operators, we discuss the dependence of bosonic character on the strength of entanglement. We confine ourselves to a class of two-particle wave functions. In general, we showed that the level of entanglement between two fermions or bosons decides the bosonic / fermionic nature of composite particle.

Chapter 4 is dedicated to a detailed description of coherent states and its properties. It enables us to present a systematic analysis of coherent states of composite bosons having two distinguishable particles. With the help of a composite boson annihilation operator, we extract its eigenstate and commutator. Having the idea that composite particle is the combination of elementary particles, we determine the resemblance between this eigenstate and traditional coherent states through typical measures of non-classicality, in terms of Mandel's Q parameter. As coherent states of elementary bosons have Poissonian statistics, the Mandel's Q-parameter of composite particle can have a sub or super-Poissonian statistics.

In chapter 5 we present the conclusion of our whole work we have done here.

Chapter 2

Basic concept of quantum information

In physics, the quantum information is the information that is encoded in terms of the state of any quantum system. To study Quantum information theory it is the basic entity to know about quantum information, and using the engineering techniques one can manipulate known as the quantum information processing.

Quantum system displays the properties which are unknown for the classical ones, like superposition of the quantum states, interference, or tunneling. All these examples are related to the single particle effects which can easily observed in the quantum systems. But this is not the only is not always true in quantum systems.

In this chapter, we briefly discuss the states of the composite system and then knowing about the state of the system we explain the phenomena of entanglement in the context of quantum information that appears in composite systems. We will explain the relation between quantum entanglement and Schmidt decomposition.

2.1 Bipartite composite quantum system

In quantum mechanics composite systems are those systems which can decompose naturally into its subsystems, where every subsystem is proper quantum system. as we refereed decomposition as natural it implies that the decomposition is given in the obvious fashion according to given physical situation. Mostly, the subsystems are individually described by the distance between them which must be larger than the individual subsystem's size. A very common example is the string of ions as a composite system and every ion is a the subsystem. Talking about the single-particle system, quantum mechanics deal with many properties that are unknown for classical system, but these properties are not enough to create a difference between a quantum system and a classical system. In a composite system, many other properties appear that shows their quantum mechanical nature.

In the multipartite composite quantum system the Hilbert space H_s associated with the system which may contains many sub systems is given by the tensor product of all Hilbert spaces of the sub systems

$$H_s = H_1 \otimes H_2 \otimes H_3 \otimes H_4...$$

H_1 and H_2 are the Hilbert spaces associated with subsystems. In the following we shall focus on the bipartite quantum system describes by the Hilbert space H_s, which is the tensor product or the Hilbert space of two subsystems.

$$H = H_A \otimes H_B. \tag{2.1.1}$$

2.1.1 Separable and entangled states

In the discussion of composite systems if we talk about their states then we can say that the state of a composite system is said to be pure if the subsystem stays independent to each other. Let us consider a bipartite system, such composite system which have only two subsystems. This composite system have pure states $(\psi_i, i = 1, 2)$ in its subsystems, the state of the composite system is given by ψ_s

$$|\psi_s\rangle = |\psi_A\rangle \otimes |\psi_B\rangle. \tag{2.1.2}$$

Now consider an observable $\hat{e} \otimes \mathbb{1}$ that can execute any local measurement. $\hat{e}$ is the hermitian operator and acts on H_A where $\mathbb{1}$ is an identity operator operating on H_B. after measurements the state of the first subsystem stand out in terms of the eigenstate of operator $\hat{e}$, but there will be no change in the states of the second subsystem because identity operator brings no change. Later on if, we execute an other local measurement, on the second subsystem, its outcome appears independent to the result of the other measurement. Therefore,we can conclude that the results after measurements for the different subsystems are lacking the mutual correlation, it only depends on the states of the respective subsystem.

In general a pure state in $\mathbb{H}_s$ can be expressed as a superposition of all the pure states of the subsystems, given in equation (2.1.2). Lets take the idea from following example

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}(|\nu_A\rangle \otimes |\nu_B\rangle + |\varphi_A\rangle \otimes |\varphi_B\rangle), \tag{2.1.3}$$

$|\nu_i\rangle \neq |\varphi_i\rangle$ (i= 1,2).We can also check how the state of the composite system $|\psi_c\rangle$ looks like when we try to measure its subsystems individually? Therefore applying local measurement $\hat{O} \otimes \mathbb{1}$ where $\hat{O}$ is the operator related to the first subsystem, then the result of expectation value noticed in this experiment appears as

$$\langle \hat{O} \rangle = \langle \psi_c | \hat{O} \otimes \mathbb{1} | \psi_c \rangle,$$

after rearranging the above equation we can write it in term of trace,

$$\langle \hat{O} \rangle = Tr(\hat{O} \otimes \mathbb{1} |\psi_c\rangle\langle\psi_c|),$$

as we know that ψ_c has two subsystem A and B and operator $\hat{O}$ is associated with subsystem A and the identity operator is associated with system B, therefore we can write

$$\begin{aligned} \langle \hat{O} \rangle &= Tr_A(\hat{O} Tr_B \psi_c\rangle\langle\psi_c|), \\ &= Tr(\hat{O}\hat{\rho}_A), \end{aligned} \tag{2.1.4}$$

where $Tr_{A,B}$ are partial traces associated to the both subsystem, where

$$\rho_A = Tr_B |\psi_c\rangle\langle\psi_c|,$$

is the reduced density matrix related to the first subsystem. The equation (4.2.23) in the is true for the local operator $\hat{O}$, therefore the measurement result shows that the states of the subsystem independently is easily given in terms of reduce density operators ρ_A and ρ_B, where

$$\rho_B = Tr_A(|\psi_c\rangle\langle\psi_c|).$$

However, we cannot say that the state representing the composite system is equal to the tensor product of the states of two subsystem, i.e,

$$\rho_c = |\psi_c\rangle\langle\psi_c| \neq \rho_A \otimes \rho_B.$$

Furthermore, if we execute any local measurements on any of the subsystems, the state of the overall composite system get reduced completely, regardless of the operation whether we have performed the measurement of system A or B. Thus, the probabilities appearing out as of a measurement result on any subsystem get affected by earlier measurements which have been done on the subsystem. This experiment results that the measurement for the non interacting and possibly distant subsystems are completely correlated.

Having all the above discussion, we can say,

"states that can be written as a product of pure states, as in equation(2.1.2), are called separable or product states. On the contrary, if there are no local states then ψ_A belongs to H_A and ψ_A belongs to H_A then state of the system can be written as a product of both, "

$$\exists \quad |\psi_A\rangle \in H_A, |\psi_B\rangle \in H_B \quad |\psi_c\rangle = |\psi_A\rangle \otimes |\psi_B\rangle. \tag{2.1.5}$$

Then we can say that $|\psi_c\rangle$ is an entangled state.

Mixed States

Despite considering only pure states, if we discuss more generally, Quantum system can have such states which are not pure so they can be known as mixed. In fact, the mixed states are basically those states which can be faced most frequently in real experiments because it is hard to get a quantum system completely isolated from its surroundings.

Mixed product states are given by,

$$\rho_c = \rho_A \otimes \rho_B, \tag{2.1.6}$$

where ρ_A and ρ_B are density matrix are the density matrix of their respective subsystems and they are completely independent. If we have more than one state, then their sum will be,

$$\rho_c = \sum_i p_i(\rho_{Ai} \otimes \rho_{Bi}), \qquad (2.1.7)$$

having $p_i > 0$ and the probability amplitude $\sum_i p_i = 1$, but generally it yields correlated measurement results as we have local observable a and b and

$$Tr(\rho_c(a \otimes b)) \neq Tr(\rho_c(a \otimes \mathbb{1}))Tr(\rho_c(\mathbb{1} \otimes b)) = Tr_1\rho_1 a \quad Tr_2\rho_2 b. \qquad (2.1.8)$$

However, these correlations are considered classical and explain in term of classical probabilities p_i. Therefore, the states having the form of the equation(2.1.7) are known as separable mixed states.

Mixed entangled states are defined by the non-existence of a decomposition into the product states [14]:

"A mixed state ρ is entangled if there are no local states $\rho_i^{(1)}, \rho_i^{(2)}$ and non-negative probabilities p_i, such that ρ can be expressed as a mixture" so then:

$$\exists \quad \rho_i^{(1)}, \rho_i^{(2)}, p_i \geq 0 \quad \text{such that} \quad \rho = \sum_i p_i(\rho_i^{(1)} \otimes \rho_i^{(2)}). \qquad (2.1.9)$$

In contrast to the classical correlations, entanglement is a quantum correlations of the measurements on different subsystems which has no classical analogy [50, 49].

2.1.2 Separability criteria

The above definitions of separable and entangled states comes out to be very simple at a first sight. But if we check separability of a given state, it can turn out to be much more involved than one might expect. Separability for the case of pure state is defined through the existence of the decomposition of a state into product states, or for mixed states by a convex sum of tensor products. Therefore, if we want to show that the given state is separable, we have to look for such decompositions. Once a decomposition is found, then we know that given state is separable. But if we get failed to find decomposition, we can have two different reasons: either the state is actually separable. but reasonable decomposition could not be defined, or the state is entangled so there is no decomposition [54].

criteria.png

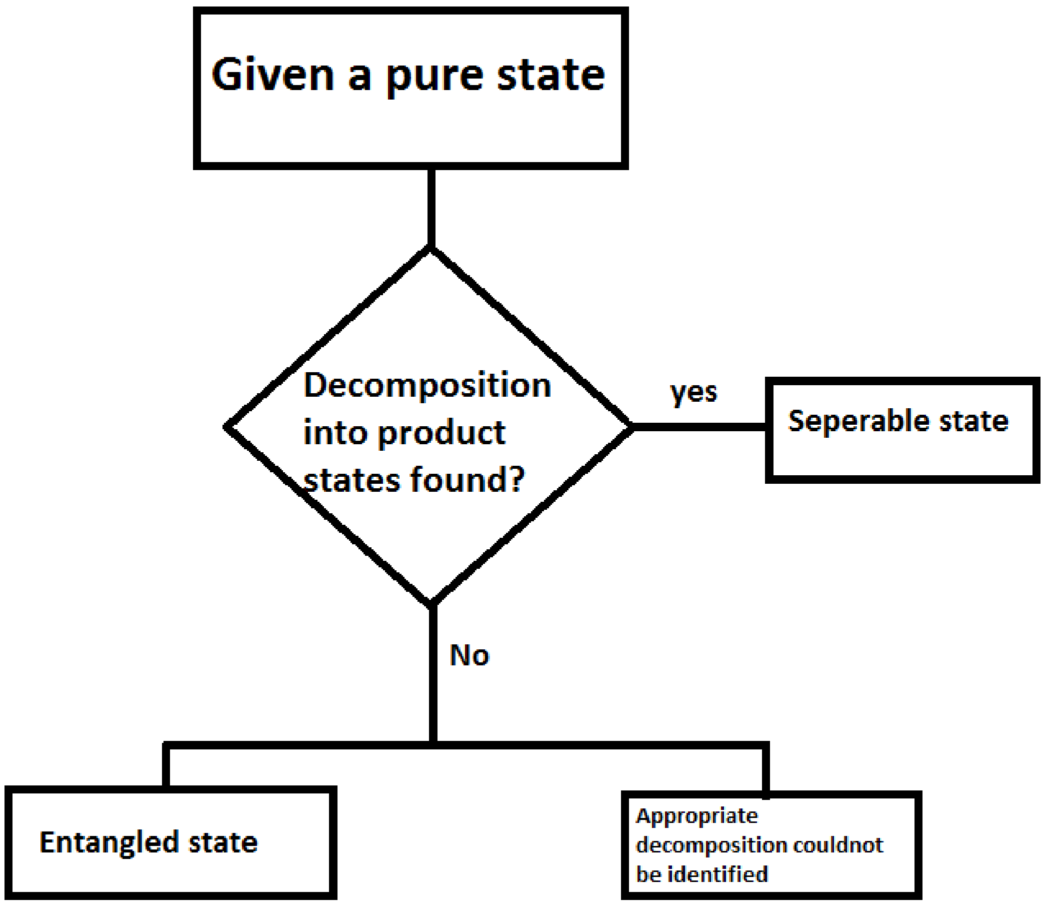

Figure 2.1: Separability criteria for pure states.

Due to this reason, there is a need for a simple criterion to distinguish entangled from separable states which do not require an explicit search. For pure states, there are criteria which differentiate separable and entangled states unambiguously, but in the case of mixed states, this criterion is applicable only for low dimensional system. For higher dimensional systems, this criterion can give only partial information, as it is not related to our work therefore will not discuss it in detail. our focus is basically on bipartite states. Before we discuss mixed states, we start out with the simpler case of pure states.

Pure State

Let us consider an example

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + 2|1\rangle}{\sqrt{5}}. \tag{2.1.10}$$

We can see that $|\psi\rangle$ here factorizes into local states, it is separable and can be rewritten as

$$|\psi\rangle = \frac{|00\rangle + 2|01\rangle + |10\rangle + 2|11\rangle}{\sqrt{10}}. \tag{2.1.11}$$

Here we can not clearly say that separability exists as we have less evidence here. we can express separability more clearly if we change the basis of state $|\psi\rangle$ into the basis $\{\frac{(|0\rangle+|1\rangle)}{\sqrt{2}}, \frac{(|0\rangle-|1\rangle)}{\sqrt{2}}\}$ of H_1 and $\{\frac{|0\rangle+2|1\rangle}{\sqrt{3}}, \frac{|0\rangle-2|1\rangle}{\sqrt{3}}\}$ of H_2 than in the basis of $\{|0\rangle, |1\rangle\}$. We can see that this observation is purely for the case of pure states because pure states has always such basis which helps them to express in terms of entanglement. If we managed to express some states in this way then this representation of the state in terms of those basis will be known by the term Schmidt decomposition.

2.1.3 Schmidt Decomposition:

To understand the phenomena of Schmidt decomposition le us consider a bipartite composite system in which the state of the system is the pure state. Let we have the state of the system $|\psi\rangle$ in the Hilbert space H which is given by the direct product of the Hilbert spaces of subsystems as mentioned in the equation(2.1.1)

$$H = H_A \otimes H_B,$$

H_A and H_B are the Hilbert spaces of the subsystems having two local basis $\{|\phi_a\rangle\}$ and $\{|\phi_b\rangle\}$ respectively. We can express the state of the system $|\psi\rangle$ in the terms of above mentioned basis

$$|\psi\rangle = \sum_{ab} d_{ab}(|\phi_a\rangle \otimes |\phi_b\rangle), \tag{2.1.12}$$

Where d_{ab} is the expansion coefficients, which represents the overlap of the state of system with the basis vector as given below,

$$d_{ab} = \langle\phi_a|\psi\rangle\langle\phi_b|\psi\rangle. \tag{2.1.13}$$

Now if we want to change the basis i.e. if

$$|\varphi_a\rangle = \nu|\phi_a\rangle, \quad |\varphi_b\rangle = \mu|\phi_b\rangle,$$

where ν and μ are any local arbitrary unitary transformations on H_1 and H_2 respectively. Therefore, the expansion coefficient d_{ab} changes as.

$$\tilde{d}_{ab} = \langle\varphi_a|\psi\rangle\langle\varphi_b|\psi\rangle,$$

substituting the values from above we can write the expansion coefficient as

$$\tilde{d}_{ab} = \langle\phi_a|\nu^\dagger|\psi\rangle\langle\phi_j|\mu^\dagger|\psi\rangle,$$

using the identities $\sum_p |\phi_p\rangle\langle\phi_p| = \mathbb{1}$ and $\sum_q |\phi_q\rangle\langle\phi_q| = \mathbb{1}$, we got the following form of expansion coefficient.

$$= \sum_{pq}\langle\varphi_a|\mu^\dagger|\phi_p\rangle\langle\varphi_b|\nu^\dagger|\phi_q\rangle\langle\phi_p|\psi\rangle\langle\phi_q|\psi\rangle, \tag{2.1.14}$$

$$= [udv]_{ab}.$$

we use these above identities, for the resolution of each subsystem. Unitary matrices are given by

$$\nu_{ap} = \langle\varphi_a|\nu^\dagger|\varphi_p\rangle, \quad \mu_{bq} = \langle\varphi_b|\mu^\dagger|\phi_q\rangle. \tag{2.1.15}$$

we can now express the state in terms of new basis as

$$|\psi\rangle = \sum_{ab}[udv]_{ab}|\varphi_a\rangle \otimes |\varphi_b\rangle. \tag{2.1.16}$$

For the purpose of extracting the Schmidt decomposition of the given $|\psi\rangle$, we can use understanding that the unitary transformation μ and ν must be present there for the expansion coefficient in its matrix representation so that $[\mu d\nu]$ collectively results a diagonal matrix. From this we can extract the information about singular value decomposition of expansion coefficient.this matrix have non negative real valuesS_i at its diagonal, these are called singular values. Therefore we can say that for each state $|\psi\rangle$, we can always find local basis $|\varphi_i^s\rangle$ and $|\phi_i^s\rangle$ in term of which equation(2.1.16) reduces to

$$|\psi\rangle = \sqrt{\lambda_i}|\varphi_i^s\rangle \otimes |\phi_j^s\rangle, \tag{2.1.17}$$

where λ_i is the Schmidt coefficients given by

$$\lambda_i = S_i^2, \tag{2.1.18}$$

the sum in the above equation depends on the dimensions of the smaller subsystem. Singular values are defined uniquely same Like eigenvalues of a matrix. Therefore, the Schmidt coefficients λ_i are unique for any state $|\psi\rangle$. We can extract the information related to the system entanglement from the factor of Schmidt coefficients.

Schmidt coefficients and separability

If the decomposed state$|\psi\rangle$ contains one non-zero Schmidt coefficient, then we can say that the state must be separable. On the other hand if there exist at least two non zero Schmidt coefficients, then the state $|\psi\rangle$ can not be explained in the form of the equation (2.1.2) that is we can not define it as separable. Hence, we can conclude that the pure state $|\psi\rangle$ is separable if there exist only one non-zero Schmidt coefficient.

As discussed above the Schmidt coefficients are very helpful in differentiating between entangled and separable states, therefore our main focus is how we can evaluate them. we can do this with the help of reduce density matrices, so then reduced density matrices are explicitly useful.

2.1.4 Entanglement and Schmidt number

Having the idea of Schmidt decomposition we can find out easily whether the state is separable of entangled, if we know about the purity of that quantum state. Purity of any normalized quantum state can be defined as the trace of the squared value of its density operator

$$P = Tr(\rho^2). \qquad \rho = |\psi\rangle\langle\psi| \tag{2.1.19}$$

The value of exists between $0 \leq P \leq 1$. If for any state we get value of $P = 1$ this means that the state is separable and all the other values of $P \geq 1$ the states appears to be entangled.

We can also define the purity in terms of Schmidt coefficient as

$$P = Tr(\rho^2) = \sum_i \lambda_i^2 \tag{2.1.20}$$

we can calculate the measure of the entanglement Schmidt numbers which defines the purity.Measurement of the entanglement can be define as the Schmidt number given by

$$\kappa \equiv \frac{1}{\sum_{n=0}^{\infty} \lambda_n^2}. \tag{2.1.21}$$

therefore from equation(2.1.20) we can conclude that the Schmidt numbers and the purity of a state are inversely proportional to each other.

$$\kappa \equiv \frac{1}{P}. \tag{2.1.22}$$

if $\kappa = 1$ this mean that there is only one term in Schmidt decomposition therefore the state will be separable for all the other values of $\kappa$the state of the composite system will appear to be entangled.

Chapter 3

Quantum Entaglement in Composite Bosons

3.1 Introduction

This chapter includes the discussion of composite particle representation in a bipartite system using the quantum information approach [46] and also with the idea of second quantization. Here We will show that the idea of quantum entanglement gives us the information all about the composite behavior. The measure of the entanglement between the sub-particles explains the deviation of the composite character from a pure bosonic character and also how closely the composite particle behaves under quantum entanglement. This phenomena entail some interesting ideas about the constituent particles that these particles are somehow bound by this phenomena. For the discussion of a composite particle and its behavior in a bipartite system the mechanical binding forces are actually not necessary because these forces usually helps us only as physical means when we try to apply the quantum correlations. As the correlations between the constituent particles can be find out in many different ways, since the representation of composite system is not bound to position or momentum space. Below in this chapter the underlying role of entanglement will be discussed on bases of the properties of the ladder operators associated with composite particles.

3.2 Composite two particle system

In this section, we comprehensively take a look at the composite system having a composite particle from the perspective of quantum information. More specifically we will show that the quantum entanglement gives the understanding about the beginning of the composite behavior. We see below that the measure of entanglement between the particles of composite system tells us that how closely a composite particle shows the properties of a boson. This phenomena speaks an interesting framework that the quantum entanglement of the continent particles are basically the boundary of the composite character. A clear picture about the correlation between the two particle system and the conditions which are important for a pair of particles to be treated as an ordinary boson will be discussed.

Let's consider a composite particle C in a bipartite composite system having Hilbert space H, which contains two sub particles A and B which are distinguishable. Both constituent particles can be bosons or fermions either. Hilbert space H_A be the Hilbert space of subsystem associated with particle A and H_B be the Hilbert space of other subsystem related with particle B. The Hilbert space H of the composite system is given by the tensor product of the subsystems the system is given by

$$H = H_A \otimes H_B. \tag{3.2.1}$$

And let Ψ_C be the wave function of two-particle system, given by

$$\Psi_C = \Psi_A \otimes \Psi_B, \tag{3.2.2}$$

where Ψ_A and Ψ_B are the states of particle A and B respectively. we follow the assumption of the both particles collectively behave as fermion or bosons depending upon their correlations. ideally the composite system should behave as bosons. however, we will find that how much its nature is deviated from ideal bosons. The state of the composite system can be written in terms of Schmidt decomposition as

$$\Psi_C = \sum_{i,j=0}^{\infty} C_{ij} |i\rangle_A \otimes |j\rangle_B. \tag{3.2.3}$$

where $|i\rangle_A$ and $|j\rangle_B$ are the states of the subsystems.

3.2.1 Second Quantization of composite particle

We can write an analogous expression for the composite two particle system using the vision of second quantization [45] in terms of ladder operators as

$$|\Psi_C\rangle = c^\dagger|0\rangle, \tag{3.2.4}$$

Hence, by comparing it with the equation(3.2.3) we can sy that this creation operator which is creating a particle in a composite system can also be the combination of two other creation operators which can create sub-particle in the relevant subsystem therefore we can write

$$c^\dagger = \sum_{i,j=0}^{\infty} C_{i,j} a_i^\dagger b_j^\dagger, \tag{3.2.5}$$

where $c_{i,j}$ is the probability amplitude of having particle A in i basis and particle B in j basis. $a_i^\dagger$ and $b_j^\dagger$ is the creation operators of particle A and particle B in the mode of $|i\rangle$ and $|j\rangle$. $|0\rangle$ is the vacuum state. In the perspective of entanglement theory we use the process of decomposition to calculate the probability amplitude therefore we can rewrite the state expressed above as

$$\Psi_C = \sum_{n=0}^{\infty} \sqrt{\lambda_n} a_n^\dagger b_n^\dagger |0\rangle = c^\dagger|0\rangle, \tag{3.2.6}$$

where basis n is the superposition of i and j and $\sqrt{\lambda_n}$ is the Schmidt coefficient which tells us about the probability of having both particles having in the same basis n.

The value of λ_n also provides the measure of entanglement as we have discussed in chapter 2. We can also write it in terms of the entanglement entropy $E = -\Sigma_n \lambda_n \log_2 \lambda_n$.In order to check the entanglement measurement, we count the average number of Schmidt modes that are involved actively. The Schmidt number κ provides us the following information

$$\kappa \equiv \frac{1}{\sum_{n=0}^{\infty} \lambda_n^2}. \tag{3.2.7}$$

We can rewriter the operator for composite in terms of the Schmidt coefficient as

$$\hat{c}^{\dagger} = \sum_{n=0}^{\infty} \sqrt{\lambda_n} \hat{a}^{\dagger} \hat{b}^{\dagger}, \tag{3.2.8}$$

The operator $\hat{c}^{\dagger}$ can be treated as the ladder operator for the composite particle and we can discuss its properties as well.

3.2.2 Properties ladder operator of composite particle

Being ladder operator $\hat{c}$ and $\hat{c}^{\dagger}$ satisfy the non-boson commutation relation. If A and B are bosons then the commutation relation results as

$$[\hat{c}, \hat{c}^{\dagger}] = 1 + \sum_{n=0}^{\infty} \lambda_n (a_n^{\dagger} a_n + b_n^{\dagger} b_n), \tag{3.2.9}$$

and if both particles are fermions then the result of commutation relation appears as

$$[\hat{c}, \hat{c}^{\dagger}] = 1 - \sum_{n=0}^{\infty} \lambda_n (a_n^{\dagger} a_n + b_n^{\dagger} b_n), \tag{3.2.10}$$

Collectively we can write the above relation as

$$[\hat{c}, \hat{c}^{\dagger}] = 1 + s\Delta, \tag{3.2.11}$$

where $s = +1$. If both A and B are bosonic and $s = -1$ if both are fermionc. The operator Δ is defined as

$$\Delta = \sum_{n=0}^{\infty} \lambda_n (a_n^{\dagger} a_n + b_n^{\dagger} b_n), \tag{3.2.12}$$

Δ is basically a matrix having non-zero element that depends on the involved states. Hence, the operator $\hat{c}$ and $\hat{c}^{\dagger}$ both are not pure bosonic operator.

Further examining the properties of $\hat{c}$ and $\hat{c}^{\dagger}$ we consider a system which contains composite particle so starting from the vacuum state we can construct the $|N\rangle$ particle states [52].

$$c^{\dagger}|0\rangle = \sqrt{\chi_1}|1\rangle.$$

let $\chi_1 = 1$ so, then

$$c^\dagger|1\rangle = \sqrt{2\chi_2}|2\rangle.$$

Therefore, N-particles state for composite particles will be given by:

$$|N\rangle = \chi_N^{-\frac{1}{2}} \frac{c^{\dagger N}}{\sqrt{N!}}|0\rangle, \tag{3.2.13}$$

Here $|N\rangle$ is a normalized state and this χ_N is the normalization constant. We can calculate this normalization constant by considering $\langle N|N\rangle = 1$. To test the operator c that how well it behaves like a pure bosoinc operator, we are required to check its action on the composite particle state $|N\rangle$ and check its result. Therefore,:

$$c|N\rangle = \alpha_N\sqrt{N}|N-1\rangle + |\epsilon_N\rangle, \tag{3.2.14}$$

where α_n is constant and $|\epsilon_n\rangle$ is another term which appears to be orthogonal to $|N-1\rangle$ i.e, $\langle\epsilon_n|N-1\rangle = 0$. It is basically correction term which should appear here because the state of composite particle $|N\rangle$ is only subset itself of the whole Hilbert space associated with composite system. It does no correlate to any state of the composite particle. Note that $\hat{c}^\dagger|N\rangle$ is given by

$$\hat{c}^\dagger|N\rangle = \alpha_{n+1}\sqrt{N+1}|N+1\rangle. \tag{3.2.15}$$

To calculate α_N by taking the projection of above equation , we get

$$\langle N|\hat{c}\hat{c}^\dagger|N\rangle = \alpha_{N+1}^2(N+1)\langle N+1|N+1\rangle. \tag{3.2.16}$$

Therefore,

$$\alpha_N = \sqrt{\frac{\chi_N}{\chi_{N-1}}}, \tag{3.2.17}$$

and using equation (3.2.17), and equation(3.2.14) we get $\langle\varepsilon_n|\varepsilon_n\rangle$ as,

$$\langle\varepsilon_N|\varepsilon_N\rangle = 1 - N\frac{\chi_N}{\chi_{N-1}} + (N-1)\frac{\chi_{N+1}}{N}, \tag{3.2.18}$$

In equation (3.2.14), the operator c is pure bosonic if it satisfies the following conditions:

$$\alpha_N \longrightarrow 1, \tag{3.2.19}$$

$$\langle\varepsilon_N|\varepsilon_N\rangle \longrightarrow 0. \tag{3.2.20}$$

Therefore, the condition mentioned in the equations (3.2.19) and (3.2.20) can be controlled by ratio of the normalization constant.

3.2.3 Normalization Constant for Composite system

An ideal composite boson can emerge in the limit $\frac{\chi_{N+1}}{\chi_N} \longrightarrow 1$ for our own comfort let us write

$$\chi_N = \begin{cases} \chi_N^F & \text{A, B are fermions,} \\ \\ \chi_N^B & \text{A, B are bosons.} \end{cases} \tag{3.2.21}$$

For allowed states of bosons and fermions, normalization constant is given by

$$\chi_N = \frac{1}{N!}\langle 0|c^N c^{\dagger N}|0\rangle. \tag{3.2.22}$$

When the creation operator of composite bosons acts on the ground state it gives

$$c^{\dagger N}|0\rangle = \sum_{P_N \geq p_{N-1} \geq ....P_1} \sqrt{\lambda_{p_1}\lambda_{p_2}...\lambda_{p_N}}\, F(P_1, P_2, ..., P_N)|P_1, P_2, P_3, ..., P_N\rangle, \tag{3.2.23}$$

$|P_J\rangle$ is the occupation number with A particle and B particle in the Schmidt mode P_j. The state $|P_1, P_2, P_3, ..., P_N\rangle$ have the weight factor as $F(P_1, P_2, ..., P_N)$. Now if $P_j's$ have the same terms then $P_1 = P_2 = ... = P_N$ and other terms are distinct, then this weight factor is $\frac{N!}{d!}d! = N!$. After taking the projection of equation (3.2.23) and using equation (3.2.22) we get

$$\chi_N^B = N! \sum_{P_N \geq p_{N-1} \geq ....P_1} \lambda_{p_1}\lambda_{p_2}....\lambda_{p_N}. \tag{3.2.24}$$

Similarly, for fermions

$$\chi_N^F = N! \sum_{P_N > p_{N-1} > ....P_1} \lambda_{p_1}\lambda_{p_2}....\lambda_{p_N}. \tag{3.2.25}$$

The solution of the summations expressed above can be complicated. If we have both constituent particles as fermions then we can calculate χ_N^F by methods discussed in Ref.[40].

Realistically for the case of two particle wave function we can consider χ^N in terms of some specified Schmidt eigenvalues, which allows the very close and exact form to our system. therefore:

$$\lambda_n = (1-x)x^n, \qquad \text{n=0,1,2,3,..,} \tag{3.2.26}$$

here x parameter is defined in the range $0 < x < 1$. it explains the rapid decrease of the normalization constant λ_n with n. To calculate the constant χ_N^B and χ_N^F we consider some assumptions

$P_1 = q_N$, $p_2 = q_N + q_{N-1}$, $P_N = q_N + q_{N-1} + .. + q_1$

and get

$$\chi_N^B = N!(1-x)^N \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} ... \sum_{q_N=0}^{\infty} x^{q_1+2q_2+3q_3+...+Nq_N}, \tag{3.2.27}$$

$$\chi_N^F = N!(1-x)^N \sum_{q_1=1}^{\infty} \sum_{q_2=1}^{\infty} ... \sum_{q_N=1}^{\infty} x^{q_1+2q_2+3q_3+...+Nq_N}. \tag{3.2.28}$$

Summations that appears in equations (3.2.27) and (3.2.28) are easily carried out as

$$\chi_N^B = \frac{N!(1-x)^N}{(1-x)(1-x^2)...(1-x^N)}, \tag{3.2.29}$$

$$\chi_N^F = \frac{N!x^{N(N-1)}(1-x)^N}{(1-x)(1-x^2)...(1-x^N)}. \tag{3.2.30}$$

So the normalization ratios are given by

$$\frac{\chi_{N+1}^B}{\chi_N^B} = \frac{(N+1)(1-x)}{1-x^{N+1}}, \tag{3.2.31}$$

,

$$\frac{\chi_{N+1}^F}{\chi_N^F} = x^N \frac{(N+1)(1-x)}{1-x^{N+1}}. \tag{3.2.32}$$

From equations (3.2.14) and (3.2.19) the normalization ratio determines the modification of Bose factor. The results in equations (3.2.31) and (3.2.32) shows that

$$\chi_{N+1}^B / \chi_N^B > 1,$$

and

$$\chi_{N+1}^F / \chi_N^F < 1. \tag{3.2.33}$$

We can understand the difference of the two types of constituents because the bosons are those particles which can stay together in the same state but the fermions behave opposite to them under the action of Pauli exclusion principle. Quantum statistics associated to the constituent particles appears to be less important when x approaches to one, because then the composite particle behaves as a pure boson.

3.2.4 Entanglement as a Measurement composite character

We can connect this normalization constant with quantum entanglement by using the definition of quantum number κ . For the Schmidt eigenvalues given by equation (3.2.26), Schmidt number K defined in equation (3.2.7) becomes:

$$\kappa = \frac{1+x}{1-x}. \tag{3.2.34}$$

κ is an increasing function and its value increases monotonically in the range of $0 < x < 1$. We can relate the degree of entanglement by the results of both χ^B_{N+1}/χ^B_N and χ^F_{N+1}/χ^F_N when we express the in terms of x because its directly gives us the measurement of entanglement in term of κ . When we increase κ, we notice that χ^B_{N+1}/χ^B_N and χ^F_{N+1}/χ^F_N tends to approach to one. Specifically, we can show that for $\kappa \gg N$.

$$\frac{\chi_{N+1}}{\chi_N} \approx 1 + sN/\kappa, \tag{3.2.35}$$

where s is defined above in equation(3.2.11). As κ contains the values that are analogous to the Schmidt modes. Summing up the work, we can analyses the beginning of the representation of the composite character for the composite systems. Keeping in our discussion the two-particle wave functions here, we provide the basic information about the composite system which tells us that the composite character is directly related to the correlation between the constituent element. Therefore, we can apply a composite representation to those particles which are strongly entangled.

Now consider a pair of fundamental fermions which makes a composite particle after combining them, then we can explain the above assumptions as follows:
“ For a composite particle in a pure state, let P be the purity of the reduced state for

any of the two fermions. To find the large entanglement between the two fermions, P will be small." We consider that the number of composite particles be N for the quantum state. Therefore the composite particles behaves like an ordinary bosons if they satisfy the following conditions,

$$NP \geq 1.$$

Therefore, according to the above hypothesis, we can get the idea about the quantity $1/P$ that it gives us the number of particles which we can can add in any pure state, without looking on to their composite behavior or before the interference of composite character with the independet ideal behaviour of constituents. The ratio χ_{N+1}/χ_N is consider as the quantifier of the bosonic character where χ_N is basically a normalization factor which appears here due to the presence of composite behavior which is different from the idea case. Ideally for pure bosons, $\chi_N = 1$ for all N [51]

Chapter 4

Choerent States of Composite Bosons

4.1 Introduction

The idea about the coherent state is not new, in the past thirty year a lot of progress have been made in the construction and development of coherent states. Still it conceive to be an important subject to create coherent states in a quantum system. As we know that the coherent states in the quantum field are the states of quantized electromagnetic field [41, 42]. The idea of the coherent states was first proposed in 1926 by Schrdinger, which was connected with the quantum mechanical states for harmonic oscillator. This chapter contains the detail discussion about the coherent states for a composite system. Here we discuss the basic algebra for coherent states and then we try to map this algebra on composite systems. coherent states posses some properties therefore after the creation of coherent states we check its properties. The last part of this chapter contains the discuss about Mandels Q-parameter, through which we examine the behavior of coherent states of composite system.

4.2 Basic concept of Coherent states

Before we move towards the discussion of the coherent states of composite Particles, first, we take review of pioneering work related to the coherent states for bosons more specifically the coherent states of harmonic oscillator. Furthermore, we discuss the basic idea and the important algebra relating to these states and then present the coherent states of composite bosons.

4.2.1 Coherent states of harmonic oscillator

Initially, the coherent states were established for the harmonic oscillator. Glauber explained the coherent electromagnetic field in the context of quantum optics. With the help of three different methods he described those coherent states. Instead moving towards the formal definition, first, we develop the essential mathematical background for the harmonic oscillator.

Algebraic structure for the harmonic oscillator

The Hamiltonian of the harmonic oscillator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{4.2.1}$$

We can express $\hat{H}$ in terms of some other operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{H} = \left(\hat{a}^\dagger\hat{a} + 1/2\right)\omega, \tag{4.2.2}$$

here we can say that ($\hat{a}$ and $\hat{a}^\dagger$) are directly related the position operator $\hat{x}$ and momentum operator ($\hat{p}$)

$$\hat{a} = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x} + i\hat{p}\right) \text{ and } \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x} + i\hat{p}\right). \tag{4.2.3}$$

here $\hat{a}^\dagger$ is the creation operator, $\hat{a}$ is the annihilation operator and $\hat{a}^\dagger\hat{a} = \hat{N}$ is the number operators [56], in above equation.

The commutation relations for operators $\{\hat{a}^\dagger, \hat{a}, \hat{N}\}\}$ are

$$[\hat{a}, \hat{a}^\dagger] = \hat{I}, \quad [\hat{a}, \hat{I}] = [\hat{a}^\dagger, \hat{I}] = 0. \tag{4.2.4}$$

also

$$[\hat{a}, \hat{N}] = \hat{a}, \quad [\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger. \tag{4.2.5}$$

Fock space

We can Express fock space with the help of Hilbert space that is comprised by the number eigenstates given by $\{|0\rangle, |1\rangle, |2\rangle, ...|N\rangle\}$ and we can say that these states are orthonormal i.e, $\langle N|N\acute{}\rangle = \delta_{n\acute{n}}$. The eigenvalue equation of the number operator follows

$$\hat{N}\,|N\rangle = N\,|N\rangle\,. \tag{4.2.6}$$

We can operate $\hat{a}^\dagger$ and $\hat{a}$ upon the number states as

$$\hat{a}^\dagger\,|N\rangle = \sqrt{N+1}\,|N+1\rangle\,, \tag{4.2.7}$$

also

$$\hat{a}\,|N\rangle = \sqrt{N}\,|N-1\rangle\,. \tag{4.2.8}$$

let us consider the condition

$$\hat{a}|0\rangle = 0. \tag{4.2.9}$$

We got to know that $|0\rangle$ is the ground state of the harmonic oscillator. We can obtain the Fock space $|N\rangle$ by applying creation operator $\hat{a}^\dagger$ on the vacuum state $|0\rangle$ repeatedly. The property of completeness is also satisfied by Fock states. Therefore,

$$\sum_N |N\rangle\,\langle N| = \hat{I}, \tag{4.2.10}$$

where $\hat{I}$ is the identity operator of n-dimensions.

4.2.2 Coherent states for bosons

Having the knowledge about basic algebra, the coherent states defined by the Glauber can easily built initiating from any of the three but analogous mathematical definitions [57]. We tried summarize all the three definitions briefly.

Definition 1: " The coherent states $|\alpha\rangle$ are the eigenstates of the harmonic oscillator annihilation operator $\hat{a}$, i.e.,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{4.2.11}$$

where α is a complex number."

Definition 2: " They can be generated by the operation of a displacement operator $\hat{D}(\beta)$ on the ground state $|0\rangle$ of the harmonic oscillator,

$$|\beta\rangle = \hat{D}(\beta)|0\rangle, \tag{4.2.12}$$

where the displacement operator $\hat{D}(\beta) = e^{\beta\hat{a}^{\dagger}-beta^{*}\hat{a}}$, here we know that $\hat{a}^{\dagger}$ are the creation operator."

Definition 3: " They are the quantum states minimizing uncertainties relationship, i.e.,

$$\Delta x \Delta p = \frac{1}{2}." \tag{4.2.13}$$

we can easily explain this by investigating the dispersions of the operators related to position and momentum in the context of coherent states as

$$\begin{aligned}(\Delta x)^2 &= \langle\beta|\hat{x}^2|\beta\rangle - \langle\beta|\hat{x}|\beta\rangle^2,\\ (\Delta p)^2 &= \langle\beta|\hat{p}^2|\beta\rangle - \langle\beta|\hat{p}|\beta\rangle^2,\end{aligned}$$

we can express these operators in terms of creation and annihilation operator as

$$\begin{aligned}\hat{x} &= \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^{\dagger}),\\ \hat{p} &= \frac{-i}{\sqrt{2}}(\hat{a}-\hat{a^{\dagger}}),\end{aligned}$$

respectively.

Fock space representation of coherent states

In order to find the coherent state $|\alpha\rangle$ we take the inner product of equation(4.2.11) with number state, $\langle N|$, i.e.,

$$\langle N|\hat{a}|\alpha\rangle = \alpha\langle n|\alpha\rangle. \tag{4.2.14}$$

One can find the hermitian adjoint of equation (4.2.7) as,

$$\langle N|\hat{a} = \sqrt{N+1}\langle N+1|, \tag{4.2.15}$$

which helps us to find recursion relation

$$\sqrt{N+1}\langle N+1|\alpha\rangle = \alpha\langle N|\alpha\rangle. \tag{4.2.16}$$

from equation(4.2.16) we can show that the scalar products $\langle N|\alpha\rangle$ can be written as

$$\langle N|\alpha\rangle = \frac{\alpha^N}{\sqrt{N!}}\langle 0|\alpha\rangle. \tag{4.2.17}$$

we can clearly see that the result of the scalar products expresses by means of space as the expansion coefficients of the state $|N\rangle$, i.e.,

$$|\alpha\rangle = \sum_N |N\rangle\langle N|\alpha\rangle = \langle 0|\alpha\rangle \sum_N \frac{\alpha^N}{\sqrt{N!}}|N\rangle. \tag{4.2.18}$$

The factor $\langle 0|\alpha\rangle$ appearing above can be fixed with the use of normalization condition, $\langle\alpha|\alpha\rangle = 1$, therefore

$$\langle 0|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right]. \tag{4.2.19}$$

So using above result we can express the state $|\alpha\rangle$, as

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_N \frac{z^N}{\sqrt{N!}}|N\rangle, \tag{4.2.20}$$

This equation defines coherent states for bosons [42].

Properties:

Coherent states required a set of properties to be satisfied for its existence. From which two properties are the basic on which required to beat the minimum criteria for coherent states and in fact these properties are applicable for all types of coherent states. One of these properties is the "continuity in parameter space", whereas the other one is completeness property. Based on the basic definitions explained above, we discussed these properties below.

Continuity: "The state vector $|\alpha\rangle$ is a continuous function of the continuous complex parameter α, that is,

$$\alpha \to \acute{\alpha} \Rightarrow |\alpha\rangle \to |\acute{\alpha}\rangle\,.\text{''} \tag{4.2.21}$$

Resolution of Unity: We can show that the identity operator is given by the integral multiple of the projection operators $|\alpha\rangle\langle\alpha|$ spanned over the complex plane this explains that these states gives us the resolution of unity under consideration of positive measure $d^2\alpha/\pi$, given by,

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \sum_{n=0}^{\infty} |N\rangle\langle N| = \hat{I}. \tag{4.2.22}$$

We can prove this equation by expanding $|\alpha\rangle$ in the energy eigenstates of harmonic oscillator, therefore, with the help of *definition 3*, and the identities we can write

$$\frac{1}{\pi}\int e^{-|\alpha|^2}(\alpha^*)^n(\alpha)^m d^2\alpha = N!\delta_{nm}. \tag{4.2.23}$$

We can see that the equation(4.2.22) turns out to be exactly similar with the expression of the resolution of unity.

Non-orthogonality: Another property that is given in the context of the *definition 3* is that in general the two coherent states are not orthogonal to eath other. The scalar product $\langle\alpha|\,\acute{\alpha}\rangle$ can be calculated more simply as

$$\begin{aligned}\langle\alpha|\,\acute{\alpha}\rangle &= e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\acute{\alpha}|^2}\sum_{n,m}\frac{(\alpha^*)^n(\acute{\alpha})^m}{\sqrt{n!m!}}\langle\alpha|\,\alpha\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\acute{\alpha}|^2+\alpha^*\acute{\alpha}} \neq 0. \end{aligned} \tag{4.2.24}$$

Note that

$$|\langle\alpha|\,\acute{\alpha}\rangle|^2 = \exp\left(-|\alpha-\acute{\alpha}|^2\right), \tag{4.2.25}$$

which represents that the coherent states appear to be approximately orthogonal when α and $\acute{\alpha}$ deviate from each another in any complex plane. Therefore we can say that this property tells us that that coherent states gives an over complete set.

4.3 Coherent states of composite boson

Earlier, in this chapter we have discussed a general formalism for the construction of coherent states and we discussed the coherent states of harmonic oscillator. In the previous chapter, we have discussed the algebra for composite bosons,in terms of entanglement. The creation operator for composite bosons is given in equation (3.2.8) and the number states is given in equation (3.2.13). Therefore, by the help of corresponding annihilation operator, we can generate its eigenstates.

4.3.1 Eigenstates of composite boson's annihilation operator

The annihilation operator for composite boson, the constituents of which are two bosons or fermions annihilation operators, is given by

$$\hat{c} = \sum_{n=0}^{\infty} \sqrt{\lambda_n} ba. \tag{4.3.1}$$

Following the definitions for coherent states used in above section, we define the coherent state as the eigenstates of annihilation operator $\hat{c}$, therefore we can write

$$\hat{c}|\gamma\rangle = \gamma|\gamma\rangle, \tag{4.3.2}$$

where γ is basically a complex parameter. For the derivation of an expression for coherent states $|\gamma\rangle$, we expand $|\gamma\rangle$ as a superposition of all $|n\rangle$ states. Therefore,

$$|\gamma\rangle = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{4.3.3}$$

Operating $\hat{c}$ on both sides of the above expression, we obtain

$$\hat{c}|\gamma\rangle = \sum_{n=0}^{\infty} c_n\hat{c}|n\rangle. \tag{4.3.4}$$

The action of the annihilation operator on the number state $|n\rangle$ is given by,

$$\hat{c}|n\rangle = \sum_{n=0}^{\infty} \alpha_n\sqrt{n}|n-1\rangle + |\epsilon_n\rangle, \tag{4.3.5}$$

where $\alpha_n = \sqrt{\frac{\chi_n}{\chi_{n-1}}}$ and χ_n is the normalization constant for composite boson given in equation (3.2.29) and equation (3.2.30) depending upon the constituent particles. Therefore, we can rewrite the above expression as

$$\hat{c}|\gamma\rangle = \sum_{n=0}^{\infty} c_n\alpha_n\sqrt{n}|n-1\rangle + \sum_{n=0}^{\infty} c_n|\epsilon_n\rangle. \tag{4.3.6}$$

Apply the above mentioned definition of coherent state on right side of above expression

$$\gamma\sum_{n=0}^{\infty} c_n|n\rangle = \sum_{n=1}^{\infty} c_n\alpha_n\sqrt{n}|n-1\rangle + \sum_{n=0}^{\infty} c_n|\epsilon_n\rangle. \tag{4.3.7}$$

Replacing n by $n-1$ in the left side of above equation and operating $\langle n-1|$ from the left side of above equation, we are left with

$$\gamma c_{n-1} = c_n \alpha_n \sqrt{n}, \tag{4.3.8}$$

note that $\langle n-1|\epsilon_n\rangle = 0$. After going through some simplification of above equation, we arrive at

$$c_n = \frac{\gamma^n}{\sqrt{\chi_n n!}} c_0. \tag{4.3.9}$$

Having above result we can write the equation (4.3.3) as

$$|\gamma\rangle = \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{\chi_n n!}} c_0 |n\rangle \tag{4.3.10}$$

c_0 is the normalization constant. We can easily calculate this constant by using the condition of normalization $\langle\gamma|\gamma\rangle = 1$. As

$$c_0 = \frac{1}{\sqrt{\mathcal{N}(|\gamma|^2)}}. \tag{4.3.11}$$

Where $\mathcal{N}(|\gamma|^2) = \sum_{n=0}^{\infty} \frac{(|\gamma|^2)^n}{\chi_n n!}$. Therefore, the required coherent state for composite boson is given by

$$|\gamma\rangle = \frac{1}{\sqrt{\mathcal{N}(|\gamma|^2)}} \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{\chi_n n!}} |n\rangle. \tag{4.3.12}$$

4.3.2 Properties of coherent states

Now we check that coherent states defined above, satisfy certain set of conditions which are required for any state to be called coherent states.

Orthognality

The projection of a coherent state on another, for composite bosons is given by

$$\langle\gamma|\gamma'\rangle = \frac{1}{\sqrt{\mathcal{N}(|\gamma|^2)\mathcal{N}(|\gamma'|^2)}} \sum_{n=0}^{\infty} \frac{\gamma\gamma'}{\chi_n n!}. \tag{4.3.13}$$

This shows that coherent states $|\gamma\rangle$ for composite bosons are normalized but these states are not orthogonal i.e. $\langle\gamma|\gamma'\rangle \neq 0$.

Resolution of unity

A very useful property due which one can easily understand the practical use of states as a basis in the Hilbert space is the resolution of unity. To investigate the overcompleteness property for coherent states of composite Bosons, we suppose that there exist a weight function $w(|\gamma|^2)$, such that

$$\int |\gamma\rangle\langle\gamma| d\mu = \hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{4.3.14}$$

where

$$d\mu = \frac{w(|\gamma|^2)}{\pi} d^2\gamma. \tag{4.3.15}$$

We use equation (4.3.12) in equation (4.3.14) and transformation the equation into polar coordinates, setting

$$\gamma = re^{\iota\theta}, \quad |\gamma|^2 = r^2, \quad d^2\gamma = rdrd\theta. \tag{4.3.16}$$

Therefore we have

$$\int_0^{\infty} r\frac{dr}{\pi} w(r^2) \sum_{n=0}^{\infty} \frac{r^{2n}}{\mathcal{N}(r^2)\chi_n n!} |n\rangle\langle n| \int_0^{2\pi} e^{\iota(n-m)\theta} = 1. \tag{4.3.17}$$

The angular integral gives us

$$\int_0^{2\pi} e^{i(n-m)\theta} = 2\pi\delta_{mn}. \tag{4.3.18}$$

The above expression reduces to radial integral equation. To find the weight function $w(r^2)$, we have to solve that integral,

$$\int_0^{\infty} 2rdrw(r^2) \sum_{n=0}^{\infty} \frac{r^{2n}}{\mathcal{N}(r^2)\chi_n n!} |n\rangle\langle n| = 1. \tag{4.3.19}$$

After doing some further change of variables by $r^2 = y$, the above result appears as,

$$\int_0^{\infty} \tilde{w}(y) y^n dy = \chi_n n!, \tag{4.3.20}$$

where, $\tilde{w}(y) = \frac{w(y)}{\mathcal{N}(y)}$. Equation (4.3.20), is basically inverse moment problem one can solve it by using well known Mellin transforms [53] or by using the Meijers G-function [55] . The weight function $w(|\gamma|^2)$ can also be determined by using Fourier transform technique.

4.3.3 Mandel's Q-parameter

The Mandel Q parameter is the measure of the deviation of occupation number distribution from Poissonian statistics. it is known as photon counting statistics. It is an easy way to characterize non-classical states depicting a sub-Poissonian nature, which clearly do not have classical analog. It is defined as the normalized variance of the boson distribution.

$$Q = \frac{(\Delta\hat{N})^2 - \langle\hat{N}\rangle}{\langle\hat{N}\rangle} = \frac{\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2 - \langle\hat{N}\rangle}{\langle\hat{N}\rangle}, \tag{4.3.21}$$

where

$$(\Delta\hat{N})^2 = \langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2 \tag{4.3.22}$$

with $\hat{N} = \hat{c}^\dagger\hat{c}$.

Quantum states can show different nature depending upon the different values of Q. The states display sub-Poissonian nature for which $-1 \leq Q \leq 0$ and those for which $Q \geq 0$ have exhibit super-Poissonian nature. Importantly Coherent states are the basis of non-classicality; they are characterized by $Q = 0$ also display a Poissonian nature. Depending on the comprising particles(fermions or bosons), we can show the nature of coherent states of composite bosons. Using the properties of composite boson annihilation operator $\hat{c}$. By substituting the vales in equation(4.3.21), we can calculate the Mandel's Q parameter for composite bosons,

$$\begin{aligned} Q &= \frac{\langle\hat{c}^\dagger\hat{c}\hat{c}^\dagger\hat{c}\rangle - \langle\hat{c}^\dagger\hat{c}\rangle^2}{\langle\hat{c}^\dagger\hat{c}\rangle} - 1 \\ &= \frac{|\gamma|^4 + |\gamma|^2\langle[\hat{c},\hat{c}^\dagger]\rangle - |\gamma|^4}{|\gamma|^2} - 1 \\ &= \langle[\hat{c},\hat{c}^\dagger]\rangle - 1. \end{aligned} \tag{4.3.23}$$

Therefore, we can say that the value of Q will be 0 when $\langle[\hat{c},\hat{c}^\dagger]\rangle = 1$. Which is the case for ideal bosons. The composite boson states having constituent particles as bosons can attain positive values that correspond to super-Poissonian nature. Those composite boson states which have fermions as their constituent particles can get negative values therefore, will display sub-Poissonian nature.

Chapter 5

Concluding Remarks

In this thesis, the role of entanglement in the description of the composite particles have been studied. The main focus of our work is on the composite particles made up of two distinguishable particles, both either bosons or fermions e.g., hydrogen atom. First we study bipartite quantum systems in the context of quantum information theory. Since it is well known that the constituent subsystems of bipartite systems may exhibit a special kind of correlation, namely entanglement, under certain circumstances. In our work we relate bi-fermionic or bi-bosonic composite particles with the bipartite systems and discuss their behavior in terms of entanglement. In this regard, we invoke the theory of second quantization to represent the composite particles by means of their annihilation and creation operators. These ideas are then connected with the construction of quantum mechanical states for the composite particles, such as, number states.

Using the commutation relations of annihilation and creation operator for composite particles, we derive some conditions for the composite character of particles in terms of the physical parameters of the system. These parameters are then connected with the Schmidt numbers (Schmidt numbers are the parameters that can be used to determine the extent of entanglement) for bipartite systems. This leads us to explain composite character of the particles in terms of entanglement.

Finally, we make use of the ladder operators of composite particles and associated algebra to construct the coherent states for the system under consideration. We construct these states as eigenstates of annihilation of composite particles. Then we discuss their basic properties. Furthermore, we analyze the particle counting statistics of our constructed coherent states by means of Mandel's Q-parameter. We found that the statistics of our coherent states is sub-Poissonian.

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