Upper and lower bounds of some graph operations w.r.t. Sombor index

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To my parents and grandparents.

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Humna Arshad Butt

Abstract

A topological index or a connectivity index is a mathematical measure, which is a numerical value that correspond to the chemical structure of a finite graph. Topological indices are isomorphism invariant and is also useful in fields like chemical graph theory, molecular topology and mathematical chemistry. They are an important tool in the study of QSAR (quantitative structure-activity relationships) and QSPR (quantitative structure-property relationships) where chemical structures are associated with other properties of molecules. Recently, due to increasing scope in chemistry, they have become more important.

The concept of Sombor index which is a topological index based on degrees, was given by Ivan Gutman in the field of chemical graph theory. The upper bounds as well as the lower bounds of Sombor index of graphs have been already calculated. In this research work, we computed the upper and lower bounds of some graph operations w.r.t. Sombor index.

Introduction

In 18th century, Euler solved the Königsberg's bridge problem which lead to new branch of mathematics called graph theory. Graph theory is considered as a field of modern mathematics and is applied due to its assorted applications in fields like chemistry, biology, biochemistry, electrical engineering and computers applications, computer science, genetics, industry, communication science, business, engineering, linguistics, sociology, physics, social sciences and in psychology. It is an advance field used to address the problems that are difficult to handle with other branches such as calculus or algebra. It is also interlinked with other branches of mathematics, that is, matrics representation, group theory, topology and probability.

A topological index or a connectivity index is a mathematical measure, which is a numerical value that correspond to the chemical structure of a finite graph. Topological indices are isomorphism invariant and is also useful in fields like chemical graph theory, molecular topology and mathematical chemistry. They are an important tool in the study of QSAR (quantitative structure-activity relationships) and QSPR (quantitative structure-property relationships) where chemical structures are associated with other properties of molecules.

In 1947, the study of topological indices is being started by the introduction of a distance based topological index, the Wiener index. It assembles the relation between the physico-chemistry and the structure of molecular graphs of alkanes. The study shows that there is a strong relation between the chemical structures of drugs and compounds, that is, the boiling and melting points and their structures. In the literature, there are a lot of topological indices and they have been thoroughly studied.Among these indices, degree-based topological indices are considered to be most important indices. The general formula [10] for topological indices is

$$\mathbb{TI} = \mathbb{TI}(\mathbb{G}) = \sum_{\mathbb{e}_{\mathbb{ij}} \in \mathbb{E}(\mathbb{G})} \mathbb{F}(deg(\mathbb{v}_{\mathbb{i}}), deg(\mathbb{v}_{\mathbb{j}}))$$

where $\mathbb{F}(\mathbb{w}, \mathbb{z})$ is a function with symmetric property, that is, $\mathbb{F}(\mathbb{w}, \mathbb{z}) = \mathbb{F}(\mathbb{z}, \mathbb{w})$.

In 1947, H. Wiener [8] introduced first degree based topological index known as Wiener index which is defined as the summation of all distances between the distinct pair of vertices of a finite graph. In 1998, Ivan Gutman [9] introduced Szeged index which is the generalization of Wiener index. After that, the Padmakar-Ivan or PI index [15] was given by Ivan Gutman and Padmakar V. Khadikar.

There are several many degree-based topological indices. The topological indices that are based on degrees were given by Gutman and Trinajstić [11] in 1974. These indices were then named as the first and second Zagreb indices. The Randić connectivity index [16] was then given by Milan Randić in 1976 which is the most investigated degree-based topological index. Later this concept was generalized by Li and Gutman in 2006. Later in 2009, Zhou and Trinajstić [1] worked on the index known as sum-connectivity index. This concept was generalized by them in 2010. Recently , Gutman gave the concept of Sombor index [10] in the field of chemical graph theory.

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Chapter 1

Introduction to the graph theory

In this chapter, we define some basic definitions which will be useful later in our work. We also define different terminologies used in graph theory.

1.1 Introduction to graph

Graph is generally defined as visual representation of the data, which present data in an organized manner. Mathematically, we define graph as $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$, where the set $\mathbb{V}(\mathbb{G}) \neq \phi$ denotes the set contains all the vertices and the edge set $\mathbb{E}(\mathbb{G})$, includes all the edges in the graph $\mathbb{G}$. Mainly graphs are divided into two caregories, that is, directed and undirected graphs. In directed graphs, the edges are directed and they show a one way relation, that is, the edges are traversed in only one direction whereas undirected graphs does not contain such edges. Throughout this work, we will consider undirected graphs.

Furthermore, the elements in vertex set $\mathbb{V}(\mathbb{G})$ depicts the order of the graph $\mathbb{G}$ while edge set $\mathbb{E}(\mathbb{G})$ gives us the size of the graph. If we take the vertices from $\mathbb{V}(\mathbb{G})$, say $\mathbb{a}$ and $\mathbb{b}$, then the vertex $\mathbb{a}$ is adjacent to the vertex $\mathbb{b}$, if there is an edge between them. We denote this edge as $\mathbb{ab}$ or $\mathbb{ba}$. The edge $\mathbb{e}$ is known to be incident to the vertex $\mathbb{a}$ and $\mathbb{b}$ if $\mathbb{e} = \mathbb{ab}$, where the vertices $\mathbb{a}$ and $\mathbb{b}$ are called the endpoints of $\mathbb{e}$. An edge $\mathbb{e}$ forms a loop in the graph $\mathbb{G}$ if $\mathbb{e} = \mathbb{aa}$, where $\mathbb{a} \in \mathbb{V}(\mathbb{G})$. However, two edges, say $\mathbb{e}_{\mathbb{1}}$ and $\mathbb{e}_2$, forms multi-edges if $\mathbb{e}_{\mathbb{1}} = \mathbb{ab} = \mathbb{e}_2$, where $\mathbb{a}, \mathbb{b} \in \mathbb{V}(\mathbb{G})$. The degree of $\mathbb{a} \in \mathbb{V}(\mathbb{G})$ is the total count of edges that are connected to $\mathbb{a}$ and it is represented by $deg(\mathbb{a})$. If $\mathbb{a} \in \mathbb{V}(\mathbb{G})$ has a degree 1, then it is called a pendant vertex and if the degree is zero then it is known as an isolated vertex.

A simple graph, say $\mathbb{G}$, is a graph if it neither have a loop, that is, the edge with same starting and ending point, nor multi-edges. If it has any multi-edge or the parallel edge, then the graph $\mathbb{G}$ becomes a multigraph. If the graph $\mathbb{G}$ is a multigraph and also it has loops then $\mathbb{G}$ becomes a Pseudograph.

If we are able to find some path between every two distinct vertices in $\mathbb{G}$, then it becomes a connected graph, or else $\mathbb{G}$ is considered to be a disconnected graph. If a graph is disconnected, this implies that it will have at least two components. A component is defined as the maximal connected subgraph of $\mathbb{G}$.

Figure 1.1: (a) Simple Graph. (b) Pseudograph.

1.2 Graph parameters and degrees

Definition 1.2.1. *The minimum degree of a graph* $\mathbb{G}$ *is denoted by* $\delta(\mathbb{G})$*. It is the smallest value of all the degrees of vertices in* $\mathbb{G}$*.*

Definition 1.2.2. *The maximum degree of a graph* $\mathbb{G}$ *is denoted by* $\mathbb{\Delta}(\mathbb{G})$*. It is the largest value of all the degrees of vertices in* $\mathbb{G}$*.*

From these two definitions, it is clear that for any $\mathbb{a} \in \mathbb{V}(\mathbb{G})$, $\delta(\mathbb{G}) \leq deg(\mathbb{a}) \leq \mathbb{\Delta}(\mathbb{G})$. Equality in this expression holds if each vertex of the graph under consideration has equal number of edges incident to them, that is, regular graph.
The most renowned finding concerning the sum of the degrees of vertices was given by Leonhard Euler in his research paper. This lemma was named as the Handshaking Lemma also known as the degree sum formula.

Definition 1.2.3 (Handshaking Lemma [3])**.** *The sum of all the degrees of the vertices in any graph* $\mathbb{G}$ *is always equal to double of its length. Mathematically, we express it as:*

$$2|\mathbb{E}(\mathbb{G})| = \sum_{\mathbb{a} \in \mathbb{V}(\mathbb{G})} deg(\mathbb{a}).$$

Corollary 1.2.4 (Vasudev [3])**.** *The sum of all vertex degrees is always an even number. This means there will be an even number of vertices with odd degrees.*

Corollary 1.2.5 (Vasudev [3])**.** *In a graph with* $\mathbb{n}$ *vertices, the greatest degree of any vertex will be* $n - 1$*.*

Definition 1.2.6. *If* $\mathbb{V}(\mathbb{G}') \subseteq \mathbb{V}(\mathbb{G})$ *and* $\mathbb{E}(\mathbb{G}') \subseteq \mathbb{E}(\mathbb{G})$*, then the graph* $\mathbb{G}'$ *is called subgraph of* $\mathbb{G}$*, represented by* $\mathbb{G}' \subseteq \mathbb{G}$*. If* $\mathbb{V}(\mathbb{G}') = \mathbb{V}(\mathbb{G})$ *then* $\mathbb{G}'$ *becomes a spanning subgraph .*

1.3 Some other graphs

Definition 1.3.1. *A walk in* $\mathbb{G}$ *is given by a sequence* $\mathbb{W} = \mathbb{a}_0 \mathbb{e}_1 \mathbb{a}_1 \mathbb{e}_2 ... \mathbb{a}_{\mathbb{r}-1} \mathbb{e}_{\mathbb{r}} \mathbb{a}_{\mathbb{r}}$. *The terms in* $\mathbb{W}$ *alternate between edges and vertices and endpoints of the edge* $\mathbb{e}_{\mathbb{j}}$ *are the vertices* $\mathbb{a}_{\mathbb{j}-1}$ *and* $\mathbb{a}_{\mathbb{j}}$*, that is,* $\mathbb{a}_{\mathbb{j}-1}\mathbb{a}_{\mathbb{j}} = \mathbb{e}_{\mathbb{j}}, 1 \leq \mathbb{j} \leq \mathbb{r}$. *In a walk, vertices and edges may repeat. A walk without any repetition in edges is called a trail.*

Definition 1.3.2. *A walk in which there is no repetition in vertices and edges, is known as path. Mathematically, a path in* $\mathbb{G}$ *is a sequence* $\mathbb{P} = \mathbb{a}_0 \mathbb{e}_1 \mathbb{a}_1 \mathbb{e}_2 ... \mathbb{a}_{\mathbb{r}-1} \mathbb{e}_{\mathbb{r}} \mathbb{a}_{\mathbb{r}}$*, the terms in* $\mathbb{P}$ *alternate between edges and vertices and endpoints of the edge* $\mathbb{e}_{\mathbb{j}}$ *are the vertices* $\mathbb{a}_{\mathbb{j}-1}$ *and* $\mathbb{a}_{\mathbb{j}}$ *and* $\mathbb{a}_{\mathbb{j}} \neq \mathbb{a}_{\mathbb{k}}$*, that is,* $\mathbb{a}_{\mathbb{j}-1}\mathbb{a}_{\mathbb{j}} = \mathbb{e}_{\mathbb{j}}, 1 \leq \mathbb{j}, \mathbb{k} \leq \mathbb{r}$. *Invariably, a path of order* $\mathbb{n}$ *is indicated as* $\mathbb{P}_{\mathbb{n}}$. *The length of* $\mathbb{P}$ *is equal to the total number of edges in* $\mathbb{P}$.

Proposition 1.3.3. *If a walk (respectively a trail, a path) starts with* $\mathbb{a} \in \mathbb{V}(\mathbb{G})$ *and terminates at* $\mathbb{b} \in \mathbb{V}(\mathbb{G})$*, then it is called an* $(\mathbb{a}, \mathbb{b})$*-walk (respectively* $(\mathbb{a}, \mathbb{b})$*-trail,* $(\mathbb{a}, \mathbb{b})$*-path).*

Theorem 1.3.4 (Derek [4])**.** *If* $\mathbb{a}$ *and* $\mathbb{b}$ *are the two different vertices of* $\mathbb{V}(\mathbb{G})$ *then every* $(\mathbb{a}, \mathbb{b})$*-walk contains an* $(\mathbb{a}, \mathbb{b})$*-path in it.*

Remarks 1.3.5. *(a) A walk or a trail is closed if it has the same vertex at start and end.*

(b) The length of a walk (respectively trail, path) is the total number of edges in that walk (respectively trail, path).

(c) A closed trail with at least one edge is known as a circuit.

Definition 1.3.6. *A cycle, say* $\mathbb{C}_{\mathbb{n}}$ *containing* $\mathbb{n}$ *vertices where* $\mathbb{n} \geq 3$*, is a simple graph where* $\mathbb{V}(\mathbb{G}) = \mathbb{E}(\mathbb{G})$*, is a circuit in which no other vertex except the initial vertex (which is also a final vertex) is repeated. Mathematically, a cycle is a trail* $\mathbb{a}_0 \mathbb{e}_1 \mathbb{a}_1 \mathbb{e}_2 ... \mathbb{a}_{\mathbb{r}-1} \mathbb{e}_{\mathbb{r}} \mathbb{a}_{\mathbb{r}}, \mathbb{1} \leq \mathbb{j} \leq \mathbb{r}$ *such that* $\mathbb{a}_0 = \mathbb{a}_{\mathbb{r}}$ *and* $\mathbb{a}_{\mathbb{j}} \neq \mathbb{a}_{\mathbb{k}}, 2 \leq \mathbb{j}, \mathbb{k} \leq \mathbb{r} - \mathbb{1}$.

If there is no cycle present in a graph, then such graph is called an acyclic graph. A connected graph which contains only one cycle in it, is known as a unicyclic graph. The circumference is defined as the largest length of a cycle in $\mathbb{G}$ whereas the smallest length of the cycle in $\mathbb{G}$ is called the girth of $\mathbb{G}$.

Proposition 1.3.7 (Doglas [5]). *If a closed walk of odd length, say $\mathbb{m}$, exists in $\mathbb{G}$, then it must include an odd cycle.*

Proposition 1.3.8 (Wilson [17]). *There is a cycle in $\mathbb{G}$ if each vertex in $\mathbb{G}$ has a degree of at least 2.*

Definition 1.3.9. *A simple graph, say $\mathbb{G}$, is defined as a complete graph if each pair of distinct vertices are connected by an edge. A complete graph with $\mathbb{n}$ vertices is represented by $\mathbb{K}_{\mathbb{n}}$.*

Remarks 1.3.10. *(a) Every complete graph of order $\mathbb{n}$ is $(\mathbb{n} - \mathbb{1})$-regular.*

(b) A complete graph is always a regular graph but not the other way around.

Definition 1.3.11. *An independent set of $\mathbb{G} = (\mathbb{V}(\mathbb{G}), \mathbb{E}(\mathbb{G}))$ is a collection of pairwise non-adjacent vertices in $\mathbb{G}$.*

Definition 1.3.12. *A graph, say $\mathbb{G}$, is known as bipartite if $\mathbb{V}(\mathbb{G})$ can be partitioned into two independent sets, that is, $\mathbb{V}'(\mathbb{G})$ and $\mathbb{V}''(\mathbb{G})$, such that one end of each edge is in one independent set while other end is in the other one. The graph $\mathbb{G}$ becomes a complete bipartite if every vertex present in one independent set is linked to each vertex present in another independent set.*

Theorem 1.3.13 (König [13]). *A graph $\mathbb{G}$ is bipartite if and only if there are no odd-length cycles, say $\mathbb{m}$, in it.*

Theorem 1.3.14 (Wilson [17]). *If there is a cycle present in $\mathbb{G}$, it will be of even length if the said graph $\mathbb{G}$ is bipartite.*

1.4 Operation on graphs

In this section, we are going to define some graph operations w.r.t. two graphs.

Definition 1.4.1. *Let* $\mathbb{G}'$ *and* $\mathbb{G}$ *with different vertex sets be the simple graphs then the cartesian product is expressed as* $\mathbb{G}' \times \mathbb{G}$*, where* $\mathbb{V}(\mathbb{G}' \times \mathbb{G}) = \mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ *and* $(\mathbb{a}_{\mathbb{1}}, \mathbb{b}_{\mathbb{1}})(\mathbb{a}_2, \mathbb{b}_2)$ *make an edge in* $\mathbb{G}' \times \mathbb{G}$ *if* $\mathbb{a}_{\mathbb{1}} = \mathbb{a}_2$ *and* $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$ *or* $\mathbb{b}_{\mathbb{1}} = \mathbb{b}_2$ *and* $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$.

If we have the graphs $\mathbb{G}_{\mathbb{1}}, \mathbb{G}_2, \mathbb{G}_{\mathbb{3}}, ..., \mathbb{G}_{\mathbb{r}}$, then the cartesian product $\mathbb{G}_{\mathbb{1}} \times \mathbb{G}_2 \times \mathbb{G}_{\mathbb{3}} \times ... \times \mathbb{G}_{\mathbb{r}}$ of these graphs is expressed by $\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}$. In case of $\mathbb{G}_{\mathbb{1}} = \mathbb{G}_2 = \mathbb{G}_{\mathbb{3}} = ... = \mathbb{G}_{\mathbb{r}}$, $\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}$ is then expressed by $\mathbb{G}^{\mathbb{r}}$.

Example 1.4.2. *Let us consider the graphs* $\mathbb{P}_3$ *and* $\mathbb{P}_2$*, where* $\mathbb{P}_3$ *and* $\mathbb{P}_2$ *are the paths of order 3 and 2, respectively. Here we will find the cartesian product* $\mathbb{P}_3 \times \mathbb{P}_2$ *as follows:*

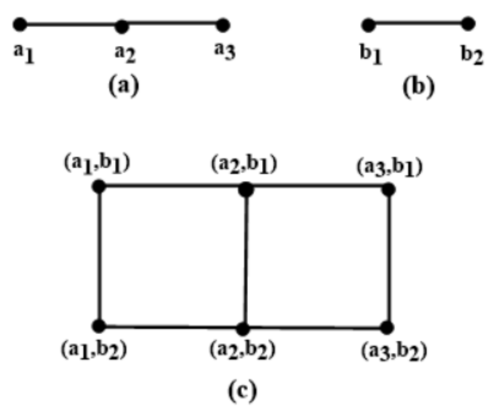

Figure 1.2: (a) $\mathbb{P}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{P}_3 \times \mathbb{P}_2$

Example 1.4.3. *Let us consider the graphs* $\mathbb{C}_3$ *and* $\mathbb{P}_2$*, where* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *are the cycle and path of order 3 and 2, respectively. Here we will find the cartesian product* $\mathbb{C}_3 \times \mathbb{P}_2$ *as follows:*

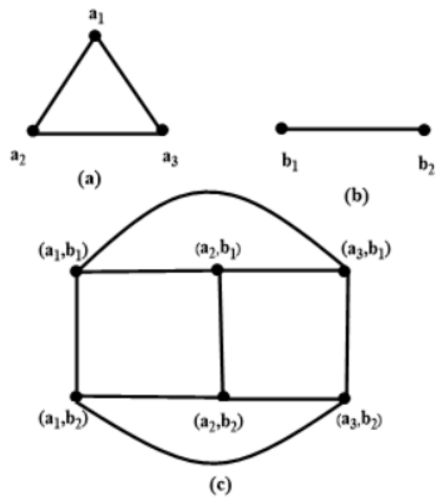

Figure 1.3: (a) $\mathbb{C}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{C}_3 \times \mathbb{P}_2$

Example 1.4.4. *Let us consider the graphs $\mathbb{C}_3$ and $\mathbb{C}_3$, where both are the cycles of order 3. Here we will find the cartesian product $\mathbb{C}_3 \times \mathbb{C}_3$ as follows:*

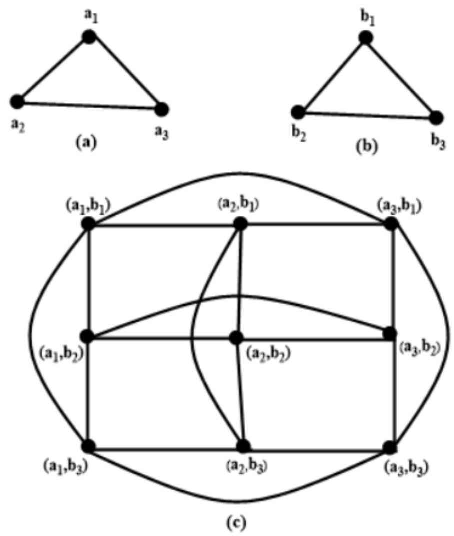

Figure 1.4: (a) $\mathbb{C}_3$ (b) $\mathbb{C}_3$ (c) $\mathbb{C}_3 \times \mathbb{C}_3$

Definition 1.4.5. *Let $\mathbb{G}'$ and $\mathbb{G}$ with dissociated vertex sets be the simple graphs then the lexicographic product of these graphs is expressed by $\mathbb{G}'[\mathbb{G}]$, where $\mathbb{V}(\mathbb{G}'[\mathbb{G}])$ is given by $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and $(\mathbb{a}_1, \mathbb{b}_1)(\mathbb{a}_2, \mathbb{b}_2)$ make an edge in $\mathbb{G}'[\mathbb{G}]$ when $\mathbb{a}_1\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ or $\mathbb{a}_1 = \mathbb{a}_2$ and $\mathbb{b}_1\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$.*

Example 1.4.6. *By using the graphs from example 1.4.2, we can find the lexicographic product of the graphs $\mathbb{P}_3$ and $\mathbb{P}_2$ as follow:*

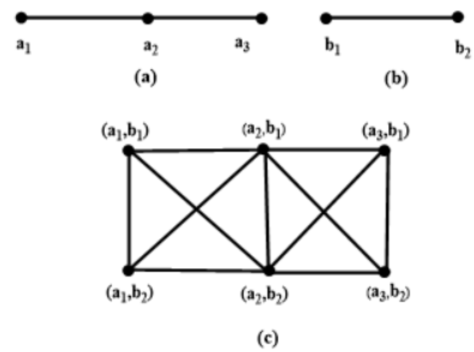

Figure 1.5: (a) $\mathbb{P}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{P}_3[\mathbb{P}_2]$

Example 1.4.7. *By using the graphs from example 1.4.3, we can find the lexicographic product of the graphs* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *as follow:*

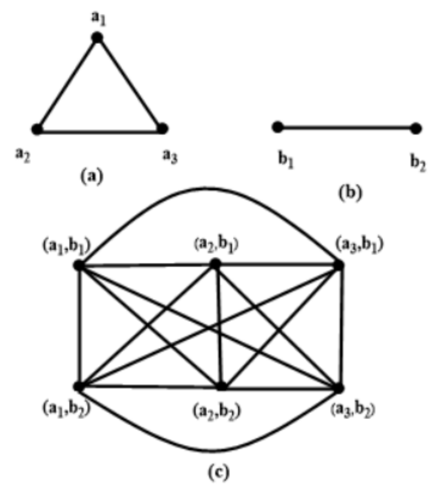

Figure 1.6: (a) $\mathbb{C}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{C}_3[\mathbb{P}_2]$

Example 1.4.8. *By using the graphs from example 1.4.4, we can find the lexicographic product of the graphs* $\mathbb{C}_3$ *and* $\mathbb{C}_3$ *as follow:*

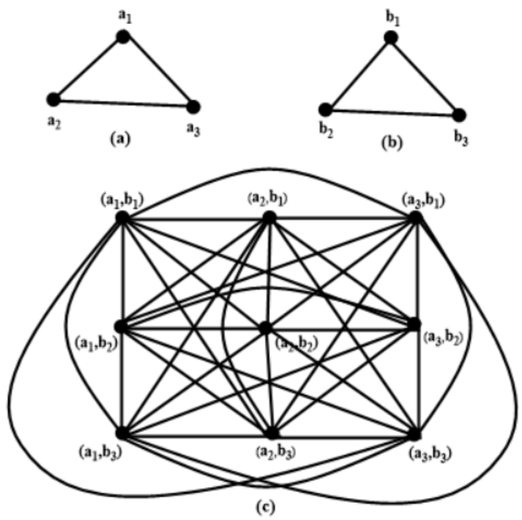

Figure 1.7: (a) $\mathbb{C}_3$ (b) $\mathbb{C}_3$ (c) $\mathbb{C}_3[\mathbb{C}_3]$

Definition 1.4.9. *Let $\mathbb{G}'$ and $\mathbb{G}$ with different vertex sets be the simple graphs then the tensor product of these graphs is expressed by $\mathbb{G}'\otimes\mathbb{G}$, where $\mathbb{V}(\mathbb{G}'\otimes\mathbb{G})$ is given by $\mathbb{V}(\mathbb{G}')\times\mathbb{V}(\mathbb{G})$ and $(\mathbb{a}_{\mathbb{1}},\mathbb{b}_{\mathbb{1}})(\mathbb{a}_2,\mathbb{b}_2)$ make an edge in $\mathbb{G}'\otimes\mathbb{G}$ whenever $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ and $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$.*

Example 1.4.10. *By using the graphs from example 1.4.2, we can find the tensor product of the graphs $\mathbb{P}_3$ and $\mathbb{P}_2$ as follow:*

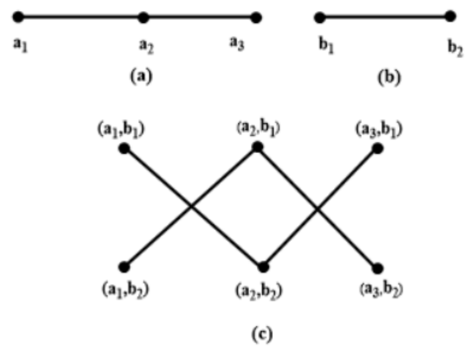

Figure 1.8: (a) $\mathbb{P}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{P}_3 \otimes \mathbb{P}_2$

Example 1.4.11. *By using the graphs from example 1.4.3, we can find the tensor product of the graphs* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *as follow:*

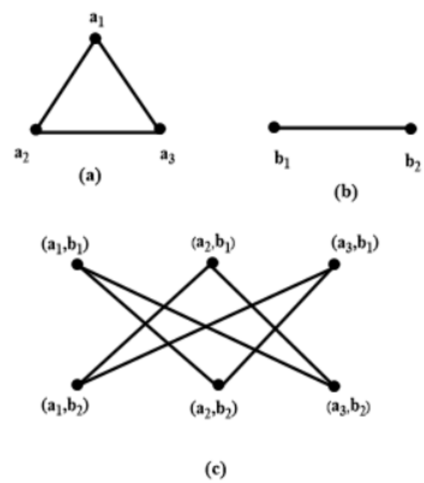

Figure 1.9: (a) $\mathbb{C}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{C}_3 \otimes \mathbb{P}_2$

Example 1.4.12. *By using the graphs from example 1.4.4, we can find the tensor product of the graphs* $\mathbb{C}_3$ *and* $\mathbb{C}_3$ *as follow:*

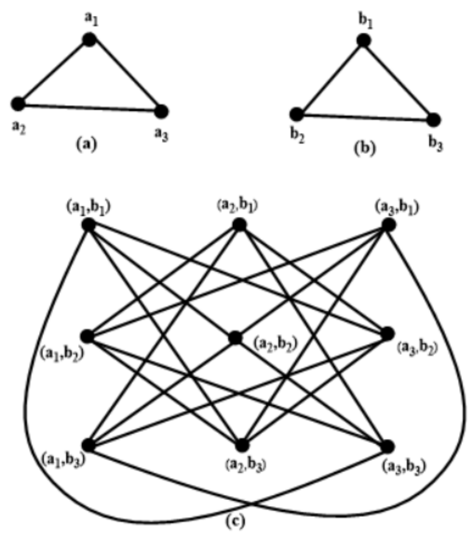

Figure 1.10: (a) $\mathbb{C}_3$ (b) $\mathbb{C}_3$ (c) $\mathbb{C}_3 \otimes \mathbb{C}_3$

Definition 1.4.13. *Let $\mathbb{G}'$ and $\mathbb{G}$ with different vertex sets be the simple graphs then the strong product of the these graphs is expressed by $\mathbb{G}' \boxtimes \mathbb{G}$, where $\mathbb{V}(\mathbb{G}' \boxtimes \mathbb{G})$ is given by $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and $(\mathbb{a}_{\mathbb{1}}, \mathbb{b}_{\mathbb{1}})(\mathbb{a}_2, \mathbb{b}_2)$ make an edge in $\mathbb{G}' \boxtimes \mathbb{G}$ if $\mathbb{a}_{\mathbb{1}} = \mathbb{a}_2$ and $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$ or $\mathbb{b}_{\mathbb{1}} = \mathbb{b}_2$ and $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ or $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ and $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$. Strong product can also be defined as the union of products such as cartesian and tensor.*

Example 1.4.14. *By using the graphs from example 1.4.2, we can find the strong product of the graphs $\mathbb{P}_3$ and $\mathbb{P}_2$ as follow:*

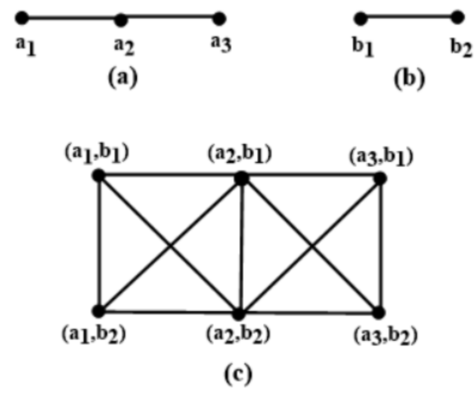

Figure 1.11: (a) $\mathbb{P}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{P}_3 \boxtimes \mathbb{P}_2$

Example 1.4.15. *By using the graphs from example 1.4.3, we can find the strong product of the graphs* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *as follow:*

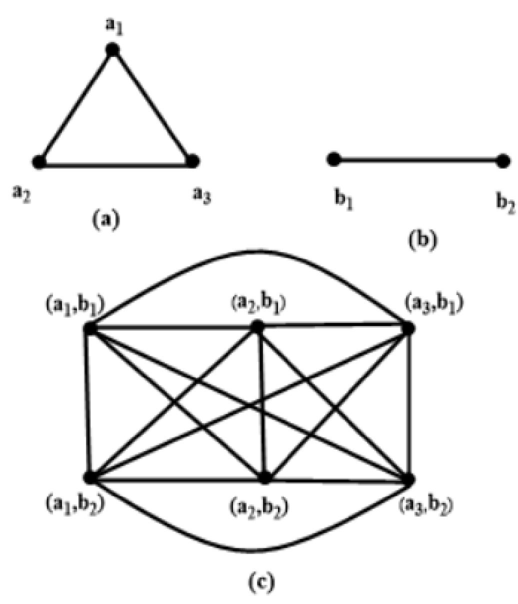

Figure 1.12: (a) $\mathbb{C}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{C}_3 \boxtimes \mathbb{P}_2$

Example 1.4.16. *By using the graphs from example 1.4.4, we can find the strong product of the graphs* $\mathbb{C}_3$ *and* $\mathbb{C}_3$ *as follow:*

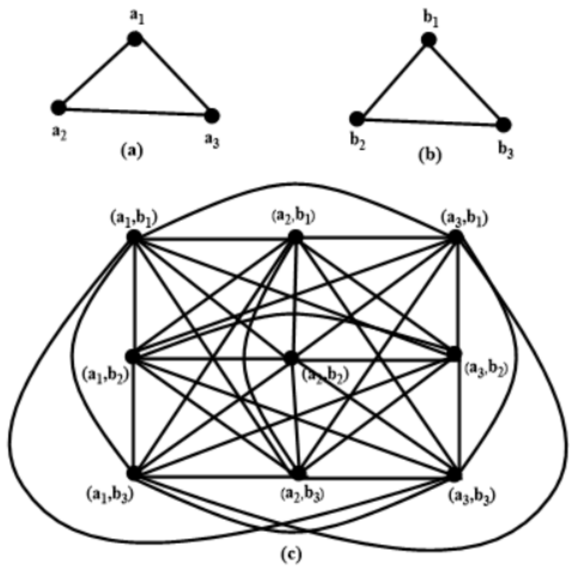

Figure 1.13: (a) $\mathbb{C}_3$ (b) $\mathbb{C}_3$ (c) $\mathbb{C}_3 \boxtimes \mathbb{C}_3$

Definition 1.4.17. *Let* $\mathbb{G}'$ *and* $\mathbb{G}$ *with different vertex sets be the simple graphs then the disjunction or co-normal product of the these graphs is expressed by* $\mathbb{G}' \vee \mathbb{G}$*, where* $\mathbb{V}(\mathbb{G}' \vee \mathbb{G})$ *is given by* $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ *and* $(\mathbb{a}_1, \mathbb{b}_1)(\mathbb{a}_2, \mathbb{b}_2)$ *make an edge in* $\mathbb{G}' \vee \mathbb{G}$ *whenever* $\mathbb{a}_1\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ *or* $\mathbb{b}_1\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$*.*

Example 1.4.18. *By using the graphs from example 1.4.2, we can find the disjunction of the graphs* $\mathbb{P}_3$ *and* $\mathbb{P}_2$ *as follow:*

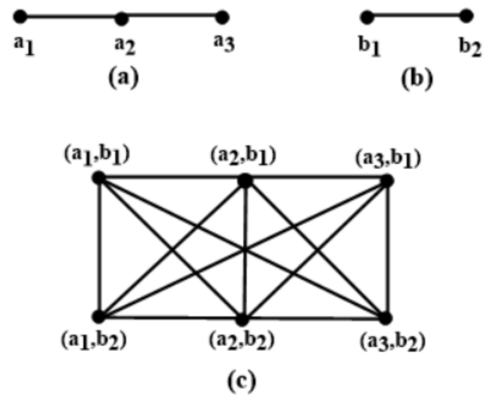

Figure 1.14: (a) $\mathbb{P}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{P}_3 \vee \mathbb{P}_2$

Example 1.4.19. *By using the graphs from example 1.4.3, we can find the disjunction of the graphs* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *as follow:*

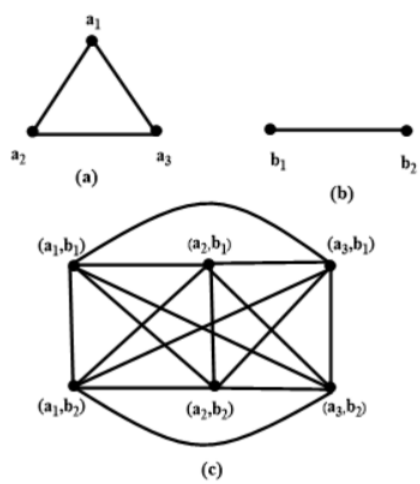

Figure 1.15: (a) $\mathbb{C}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{C}_3 \vee \mathbb{P}_2$

Example 1.4.20. *By using the graphs from example 1.4.4, we can find the disjunction of the graphs* $\mathbb{C}_3$ *and* $\mathbb{C}_3$ *as follow:*

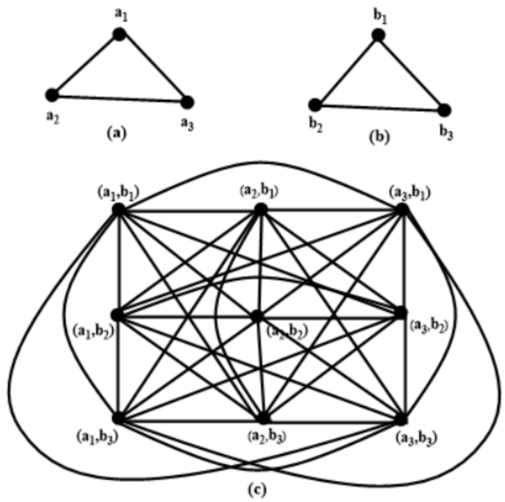

Figure 1.16: (a) $\mathbb{C}_3$ (b) $\mathbb{C}_3$ (c) $\mathbb{C}_3 \vee \mathbb{C}_3$

Definition 1.4.21. *Let $\mathbb{G}'$ and $\mathbb{G}$ with different vertex sets be the simple graphs then the symmetric difference of the these graphs is represented by $\mathbb{G}' \oplus \mathbb{G}$, where $\mathbb{V}(\mathbb{G}' \oplus \mathbb{G})$ is given by $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and $(\mathbb{a}_{\mathbb{1}}, \mathbb{b}_{\mathbb{1}})(\mathbb{a}_2, \mathbb{b}_2)$ make an edge in $\mathbb{G}' \oplus \mathbb{G}$ whenever $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ or $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$ but not both earlier statements at once.*

Example 1.4.22. *By using the graphs from example 1.4.2, we can find* $\mathbb{P}_3 \oplus \mathbb{P}_2$ *as follow:*

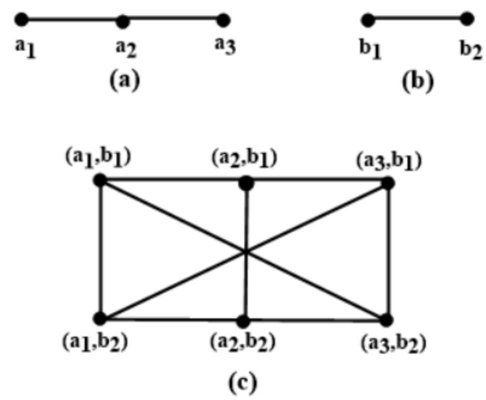

Figure 1.17: (a) $\mathbb{P}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{P}_3 \oplus \mathbb{P}_2$

Example 1.4.23. *By using the graphs from example 1.4.3, we can find* $\mathbb{C}_3 \oplus \mathbb{P}_2$ *as follow:*

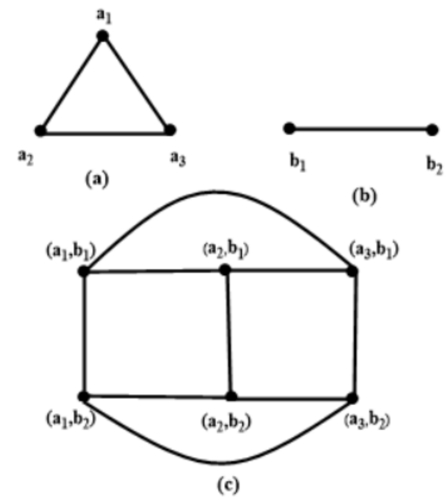

Figure 1.18: (a) $\mathbb{C}_3$ (b) $\mathbb{P}_2$ (c) $\mathbb{C}_3 \oplus \mathbb{P}_2$

Example 1.4.24. *By using the graphs from example 1.4.4, we can find* $\mathbb{C}_3 \oplus \mathbb{C}_3$ *as follow:*

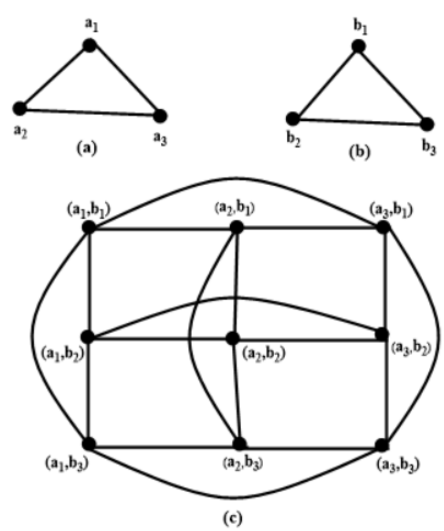

Figure 1.19: (a) $\mathbb{C}_3$ (b) $\mathbb{C}_3$ (c) $\mathbb{C}_3 \oplus \mathbb{C}_3$

Chapter 2

Degree-Based Topological Indices: Sombor Index

2.1 Topological indices

A topological index or a connectivity index is a mathematical measure [6, 7], which is a numerical value that correspond to the chemical structure of a finite graph. Topological indices are isomorphism invariant and is also useful in fields like chemical graph theory, molecular topology and mathematical chemistry. They are an important tool in the study of QSAR (quantitative structure-activity relationships) and QSPR (quantitative structure-property relationships) where chemical structures are associated with other properties of molecules.

In 1947, the study of topological indices is being started by the introduction of a distance based topological index, that is, the Wiener index. It assembles the relation between the physico-chemistry and the structure of molecular graphs of alkanes. The study shows that there is a strong relation between the chemical structures of drugs and compounds, that is, the boiling and melting points and their structures. In the literature, there are a lot of topological indices and they have been thoroughly studied. Among these indices, degree-based topological indices are considered to be most important indices. The general formula [10] for topological indices is

$$\mathbb{TI} = \mathbb{TI}(\mathbb{G}) = \sum_{\mathbb{e}_{\mathbb{ij}} \in \mathbb{E}(\mathbb{G})} \mathbb{F}(deg(\mathbb{v}_{\mathbb{i}}), deg(\mathbb{v}_{\mathbb{j}}))$$

where $\mathbb{F}(\mathbb{w}, \mathbb{z})$ is a function with symmetric property, that is, $\mathbb{F}(\mathbb{w}, \mathbb{z}) = \mathbb{F}(\mathbb{z}, \mathbb{w})$.

2.2 Distance based topological index

Initially, the topological indices that are based on distance were introduced. Some of them are discussed in this section.

The Wiener index

Wiener index is considered as the first index based on distance and it was given by Wiener [8] in 1947. Initially it was given the name path invariant but after some time it was named as Wiener index. The Wiener index of $\mathbb{G}$ is defined as the summation of all distances between the distinct pair of vertices of a finite graph. Wiener index is given by

$$\mathbb{W}(\mathbb{G}) = \sum_{\{\mathbb{a},\mathbb{b}\} \subseteq \mathbb{V}(\mathbb{G})} d(\mathbb{a}, \mathbb{b})$$

where $\mathbb{V}(\mathbb{G})$ is a vertex set and $d(\mathbb{a}, \mathbb{b})$ denotes the shortest distance in $\mathbb{G}$.

The Szeged Index

The Szeged index was made known by Gutman [9] in 1998. It is the generalization of Wiener index, that was given by Wiener. The Szeged index is determined as

$$\mathbb{S}\mathbb{z}(\mathbb{G}) = \sum_{\mathbb{e}\in\mathbb{E}(\mathbb{G})} \mathbb{n}_{\mathbb{1}}(\mathbb{e}|\mathbb{G})\mathbb{n}_2(\mathbb{e}|\mathbb{G}),$$

where $\mathbb{e}$ in the above expression represents an edge that attach the vertices $\mathbb{a}$ and $\mathbb{b}$ and $\mathbb{n}_{\mathbb{1}}(\mathbb{e}|\mathbb{G})$ displays the vertices count in $\mathbb{G}$ that are closer to $\mathbb{a}$ than to $\mathbb{b}$ and $\mathbb{n}_2(\mathbb{e}|\mathbb{G})$ displays the vertices count in $\mathbb{G}$ that are closer to $\mathbb{b}$ than to $\mathbb{a}$.

Padmakar-Ivan Index

The Padmakar-Ivan or PI index [15] was made known by Gutman and Padmakar V. Khadikar. It is also the generalization of Wiener index, given by Wiener. The PI index is summation all over the edges $\mathbb{a}\mathbb{b}$ in $\mathbb{G}$, where edges that are equidistant from both ends of $\mathbb{a}\mathbb{b}$ are not included. The PI index is mathematically represented as

$$\mathbb{P}\mathbb{I}(\mathbb{G}) = \sum_{\mathbb{e}\in\mathbb{E}(\mathbb{G})} \mathbb{n}_{\mathbb{e}\mathbb{a}}(\mathbb{e}|\mathbb{G}) + \mathbb{n}_{\mathbb{e}\mathbb{b}}(\mathbb{e}|\mathbb{G}),$$

where $\mathbb{e}$ represents an edge in $\mathbb{G}$ that attach the vertices $\mathbb{a}$ and $\mathbb{b}$ and $\mathbb{n}_{\mathbb{e}\mathbb{a}}(\mathbb{e}|\mathbb{G})$ presents the number of edges in $\mathbb{G}$ that are closer to $\mathbb{a}$ than to $\mathbb{b}$ and $\mathbb{n}_{\mathbb{e}\mathbb{b}}(\mathbb{e}|\mathbb{G})$ presents the number of edges in $\mathbb{G}$ that are closer to $\mathbb{b}$ than to $\mathbb{a}$.

2.3 Degree-based Topological Indices

There are several many topological indices that are based on degree. Some of these are being discussed in this section.

The Zagreb Indices

The topological indices that are based on degree were given by Gutman and Trinajsti´c [11] in 1974. These indices were then named as the first and second Zagreb indices. These indices are given by

$$\mathbb{M}_{\mathbb{1}}(\mathbb{G}) = \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G})} (deg_{\mathbb{G}}(\mathbb{a}))^2$$

$$\mathbb{M}_2(\mathbb{G}) = \sum_{\mathbb{a}\mathbb{b}\in\mathbb{E}(\mathbb{G})} (deg_{\mathbb{G}}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b}))$$

where $\mathbb{M}_{\mathbb{1}}(\mathbb{G})$ is known as the first Zagreb index and $\mathbb{M}_2(\mathbb{G})$ is second Zagreb index. After a while, the generalized Zagreb index was introduced, that is,

$$\mathbb{M}_{\alpha}(\mathbb{G}) = \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G})} (deg_{\mathbb{G}}(\mathbb{a}))^{\alpha}, \alpha \in \mathbb{R}.$$

In the first and generalised Zagreb indices, the sum is all over the vertices but in the second Zagreb index, summation is all over the edges.

Randić connectivity index

The Randić connectivity index [16] was introduced by Milan Randić in 1976. This degree-based topological index has received the most research. The Randić connectivity index is mathematically written as

$$\mathbb{R}(\mathbb{G}) = \sum_{\mathbb{a}\mathbb{b}\in\mathbb{E}(\mathbb{G})} \frac{1}{\sqrt{deg_{\mathbb{G}}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b})}}.$$

Later this concept was generalized by Li and Gutman in 2006. The generalized Randić connectivity index [19] is determined as

$$\mathbb{R}_{\alpha}(\mathbb{G}) = \sum_{\mathbb{a}\mathbb{b}\in\mathbb{E}(\mathbb{G})} (deg_{\mathbb{G}}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b}))^{\alpha}, \alpha \in \mathbb{R}.$$

where the sum is taken over the edges in $\mathbb{G}$. Then $\mathbb{R}_{-\frac{1}{2}}$ is recognized as Randić connectivity index.

The sum-connectivity index

The concept of sum-connectivity index [1] was made known by Zhou and Trinajstić in 2009. The sum-connectivity index is determined as

$$\chi(\mathbb{G}) = \sum_{\mathbb{ab} \in \mathbb{E}(\mathbb{G})} \frac{1}{\sqrt{deg_{\mathbb{G}}(\mathbb{a}) + deg_{\mathbb{G}}(\mathbb{b})}}.$$

Later this concept was generalized by Zhou and Trinajstić in 2010. The generalized sum connectivity index [2] is determined as

$$\mathbb{R}_{\alpha}(\mathbb{G}) = \sum_{\mathbb{ab} \in \mathbb{E}(\mathbb{G})} (deg_{\mathbb{G}}(\mathbb{a}) + deg_{\mathbb{G}}(\mathbb{b}))^{\alpha},$$

where $\alpha \in \mathbb{R}$ and the sum is defined over the edges in $\mathbb{G}$. Then $\chi_{-\frac{1}{2}}$ is recognized as classical sum connectivity index.

Sombor Index

Recently , Gutman gave the concept of Sombor index [10] in the field of chemical graph theory. It is also considered as a topological index based on degree and is given by

$$\mathbb{SO} = \mathbb{SO}(\mathbb{G}) = \sum_{\mathbb{e}_{\mathbb{ij}} \in \mathbb{E}(\mathbb{G})} \sqrt{deg_{\mathbb{G}}(\mathbb{v}_{\mathbb{i}})^2 + deg_{\mathbb{G}}(\mathbb{v}_{\mathbb{j}})^2}$$

where the sum is over the edges in $\mathbb{G}$. The reduced sombor index is given by

$$\mathbb{SO}_{red}(\mathbb{G}) = \sum_{\mathbb{e}_{\mathbb{ij}} \in \mathbb{E}(\mathbb{G})} \sqrt{(deg_{\mathbb{G}}(\mathbb{v}_{\mathbb{i}}) - 1)^2 + (deg_{\mathbb{G}}(\mathbb{v}_{\mathbb{j}}) - 1)^2}.$$

Some basic properties of Sombor index

In this part of chapter, we will go over some of the basic properties of the Sombor index [10, 12].

Theorem 2.3.1 (Gutman [10]). *Let $\mathbb{K}_{\mathbb{n}}$ be a complete graph with $\mathbb{n}$ vertices and $\bar{\mathbb{K}_{\mathbb{n}}}$ be the compliment of the graph $\mathbb{K}_{\mathbb{n}}$. Then for any graph $\mathbb{G}$,*

$$\mathbb{SO}(\bar{\mathbb{K}_{\mathbb{n}}}) \leq \mathbb{SO}(\mathbb{G}) \leq \mathbb{SO}(\mathbb{K}_{\mathbb{n}}).$$

Equality holds if and only if either $\bar{\mathbb{K}_{\mathbb{n}}}$ or $\mathbb{K}_{\mathbb{n}}$ is isomorphic to $\mathbb{G}$.

Theorem 2.3.2 (Gutman [10]). *Let a path with $\mathbb{n}$ vertices, say $\mathbb{P}_{\mathbb{n}}$ and $\mathbb{G}$ be any connected graph with $\mathbb{n}$ vertices. Then*

$$\mathbb{SO}(\mathbb{P}_{\mathbb{n}}) \leq \mathbb{SO}(\mathbb{G}) \leq \mathbb{SO}(\mathbb{K}_{\mathbb{n}}).$$

Equality holds if and only if either $\mathbb{P}_{\mathbb{n}}$ or $\mathbb{K}_{\mathbb{n}}$ is isomorphic to $\mathbb{G}$.

Theorem 2.3.3 (Gutman [10]). *Let $\mathbb{S}_{\mathbb{n}}$ be a star with $\mathbb{n}$ vertices and $\mathbb{T}$ be any tree with $\mathbb{n}$ vertices. Then*

$$\mathbb{SO}(\mathbb{P}_{\mathbb{n}}) \leq \mathbb{SO}(\mathbb{T}) \leq \mathbb{SO}(\mathbb{S}_{\mathbb{n}}).$$

Equality holds if and only if either $\mathbb{P}_{\mathbb{n}}$ or $\mathbb{S}_{\mathbb{n}}$ is isomorphic to $\mathbb{T}$.

Theorem 2.3.4 (Gutman [12]). *Let $\mathbb{M}_{\mathbb{1}}(\mathbb{G})$ be the first Zagreb index of $\mathbb{G}$ and $\mathbb{m}$ be the edges of $\mathbb{G}$. Then*

$$\mathbb{M}_{\mathbb{1}}(\mathbb{G}) < \mathbb{SO}(\mathbb{G}) \leq \frac{1}{\sqrt{2}}\mathbb{M}_{\mathbb{1}}(\mathbb{G}), \tag{2.1}$$

and

$$\mathbb{M}_{\mathbb{1}}(\mathbb{G}) - 2\mathbb{m} < \mathbb{SO}_{red}(\mathbb{G}) \leq \frac{1}{\sqrt{2}}[\mathbb{M}_{\mathbb{1}}(\mathbb{G}) - 2\mathbb{m}]. \tag{2.2}$$

Equality in above two expressions hold if and only if $\mathbb{G}$ or its components are regular.

2.4 The first Zagreb index of some graph operations

We will go through some explicit formulae for the first Zagreb index of some graph operations [14].

Some important results

Lemma 2.4.1 (Khalifeh [14]). *Let $\mathbb{G}'$ and $\mathbb{G}$ be the two simple graphs. Then*

(a)

$$\begin{aligned}
|\mathbb{V}(\mathbb{G}' \times \mathbb{G})| &= |\mathbb{V}(\mathbb{G}' \vee \mathbb{G})| = |\mathbb{V}(\mathbb{G}'[\mathbb{G}])| = |\mathbb{V}(\mathbb{G}' \oplus \mathbb{G})| = |\mathbb{V}(\mathbb{G}')||\mathbb{V}(\mathbb{G})|,\\
|\mathbb{E}(\mathbb{G}' \times \mathbb{G})| &= |\mathbb{E}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| + |\mathbb{V}(\mathbb{G}')||\mathbb{E}(\mathbb{G})|,\\
|\mathbb{E}(\mathbb{G}'[\mathbb{G}])| &= |\mathbb{E}(\mathbb{G}')||\mathbb{V}(\mathbb{G})|^2 + |\mathbb{V}(\mathbb{G}')||\mathbb{E}(\mathbb{G})|,\\
|\mathbb{E}(\mathbb{G}' \vee \mathbb{G})| &= |\mathbb{E}(\mathbb{G}')||\mathbb{V}(\mathbb{G})|^2 + |\mathbb{V}(\mathbb{G}')|^2|\mathbb{E}(\mathbb{G})| - 2|\mathbb{E}(\mathbb{G}')||\mathbb{E}(\mathbb{G})|,\\
|\mathbb{E}(\mathbb{G}' \oplus \mathbb{G})| &= |\mathbb{E}(\mathbb{G}')||\mathbb{V}(\mathbb{G})|^2 + |\mathbb{V}(\mathbb{G}')|^2|\mathbb{E}(\mathbb{G})| - 4|\mathbb{E}(\mathbb{G}')||\mathbb{E}(\mathbb{G})|.
\end{aligned}$$

(b) The cartesian product of graphs $\mathbb{G}'$ and $\mathbb{G}$ is said to be connected in case if both graphs are connected,

(c) If $(\mathbb{a}_{\mathbb{1}}, \mathbb{b}_{\mathbb{1}})$ and $(\mathbb{a}_2, \mathbb{b}_2)$ are vertices in $\mathbb{G}' \times \mathbb{G}$ then $deg_{\mathbb{G}'\times\mathbb{H}'}((\mathbb{a}_{\mathbb{1}}, \mathbb{b}_{\mathbb{1}}), (\mathbb{a}_2, \mathbb{b}_2)) = deg_{(\mathbb{G}')}(\mathbb{a}_{\mathbb{1}}, \mathbb{a}_2) + deg_{(\mathbb{G})}(\mathbb{b}_{\mathbb{1}}, \mathbb{b}_2)$,

(d) The graph operations such as cartesian, composition, tensor, co-normal, symmetric difference are associative and all these operations are commutative with the exception of composition.

(e) $deg_{\mathbb{G}'\times\mathbb{G}}(\mathbb{a}, \mathbb{b}) = deg_{\mathbb{G}'}(\mathbb{a}) + deg_{\mathbb{G}}(\mathbb{b})$,

(f) $deg_{\mathbb{G}'[\mathbb{G}]}(\mathbb{a}, \mathbb{b}) = |\mathbb{V}(\mathbb{G})|deg_{\mathbb{G}'}(\mathbb{a}) + deg_{\mathbb{G}}(\mathbb{b})$,

(g) $deg_{\mathbb{G}'\vee\mathbb{G}}(\mathbb{a}, \mathbb{b}) = |\mathbb{V}(\mathbb{G})|deg_{\mathbb{G}'}(\mathbb{a}) + |\mathbb{V}(\mathbb{G}')|deg_{\mathbb{G}}(\mathbb{b}) - deg_{\mathbb{G}'}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b})$,

(h) $deg_{\mathbb{G}'\oplus\mathbb{G}}(\mathbb{a},\mathbb{b}) = |\mathbb{V}(\mathbb{G})|deg_{\mathbb{G}'}(\mathbb{a}) + |\mathbb{V}(\mathbb{G}')|deg_{\mathbb{G}}(\mathbb{b}) - 2deg_{\mathbb{G}'}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b})$.

Theorem 2.4.2 (Khalifeh [14]). *Let* $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_{\mathbb{r}}$ *be graphs and let* $\mathbb{V}(\mathbb{G}_{\mathbb{i}})$ *and* $\mathbb{E}(\mathbb{G}_{\mathbb{i}})$, $1 \leq \mathbb{i} \leq \mathbb{r}$ *be the vertex and edge set respectively. Also* $|\mathbb{V}(\mathbb{G}_{\mathbb{i}})| = \mathbb{n}_{\mathbb{i}}$, $|\mathbb{E}(\mathbb{G}_{\mathbb{i}})| = \mathbb{m}_{\mathbb{i}}$, $1 \leq \mathbb{i} \leq \mathbb{r}$, *and* $\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}\right)\right| = \sum_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{m}_{\mathbb{i}} \prod_{\mathbb{l}=1,\mathbb{l}\neq\mathbb{i}}^{\mathbb{r}} \mathbb{n}_{\mathbb{l}}$. *Then*

$$\mathbb{M}_1\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}\right) = |\mathbb{V}| \sum_{\mathbb{i}=1}^{\mathbb{r}} \frac{\mathbb{M}_1(\mathbb{G}_{\mathbb{i}})}{\mathbb{n}_{\mathbb{i}}} + 4|\mathbb{V}| \sum_{\mathbb{i}\neq\mathbb{j},\mathbb{i},\mathbb{j}=1}^{\mathbb{r}} \frac{\mathbb{m}_{\mathbb{i}}\mathbb{m}_{\mathbb{j}}}{\mathbb{n}_{\mathbb{i}}\mathbb{n}_{\mathbb{j}}}.$$

Specifically, $\mathbb{M}_1(\mathbb{G}^{\mathbb{r}}) = \mathbb{r}\mathbb{n}_{\mathbb{G}}^{\mathbb{r}-2}(\mathbb{M}_1(\mathbb{G})\mathbb{n}_{\mathbb{G}} + 4(\mathbb{r}-1)\mathbb{m}_{\mathbb{G}}^2)$.

Theorem 2.4.3 (Khalifeh [14]). *Let* $\mathbb{G}'$ *and* $\mathbb{G}$ *be graphs. Then*

(a)

$$\mathbb{M}_1(\mathbb{G}'[\mathbb{G}]) = \mathbb{n}_{\mathbb{G}}^3\mathbb{M}_1(\mathbb{G}') + \mathbb{n}_{\mathbb{G}'}\mathbb{M}_1(\mathbb{G}) + 8\mathbb{n}_{\mathbb{G}}\mathbb{m}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}},$$

(b)

$$\mathbb{M}_1(\mathbb{G}' \vee \mathbb{G}) = (\mathbb{n}_{\mathbb{G}}^3 - 4\mathbb{m}_{\mathbb{G}}\mathbb{n}_{\mathbb{G}})\mathbb{M}_1(\mathbb{G}') + \mathbb{M}_1(\mathbb{G}')\mathbb{M}_1(\mathbb{G}) + \\ 8\mathbb{n}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}}\mathbb{m}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}} + (\mathbb{n}_{\mathbb{G}'}^3 - 4\mathbb{m}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}'})\mathbb{M}_1(\mathbb{G}),$$

(c)

$$\mathbb{M}_1(\mathbb{G}' \oplus \mathbb{G}) = (\mathbb{n}_{\mathbb{G}}^3 - 8\mathbb{m}_{\mathbb{G}}\mathbb{n}_{\mathbb{G}})\mathbb{M}_1(\mathbb{G}') + 4\mathbb{M}_1(\mathbb{G}')\mathbb{M}_1(\mathbb{G}) + \\ 8\mathbb{n}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}}\mathbb{m}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}} + (\mathbb{n}_{\mathbb{G}'}^3 - 8\mathbb{m}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}'})\mathbb{M}_1(\mathbb{G}).$$

Chapter 3

Upper and lower bounds of some graph operations w.r.t. Sombor index

In this chapter, the work is done on some graph operations. We calculate the upper and lower bounds of some graph operations w.r.t. Sombor index. Moreover, these results are then elaborated by the help of some examples on different graphs.

3.1 Cartesian product

Cartesian product of the graphs $\mathbb{G}'$ and $\mathbb{G}$ is symbolized as $\mathbb{G}' \times \mathbb{G}$, where the vertex set of the product is $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and ordered pairs $(\mathbb{a}_{\mathbb{1}}, \mathbb{b}_{\mathbb{1}})$ and $(\mathbb{a}_2, \mathbb{b}_2)$ make an edge if $\mathbb{a}_{\mathbb{1}} = \mathbb{a}_2$ and $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$ or $\mathbb{b}_{\mathbb{1}} = \mathbb{b}_2$ and $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$.
The order of cartesian product is $|\mathbb{V}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}}$, whereas the size is $|\mathbb{E}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| + |\mathbb{E}(\mathbb{G})||\mathbb{V}(\mathbb{G}')| = \mathbb{m}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}} + \mathbb{m}_{\mathbb{G}}\mathbb{n}_{\mathbb{G}'}$, where $\mathbb{n}_{\mathbb{G}'}$ and $\mathbb{n}_{(\mathbb{G})}$ presents the vertices count of $\mathbb{G}'$ and $\mathbb{G}$ and $\mathbb{m}_{\mathbb{G}'}$ and $\mathbb{m}_{(\mathbb{G})}$ denotes the edge count of the graphs $\mathbb{G}'$ and $\mathbb{G}$. Mathematically, The degree of a vertex $(\mathbb{a}, \mathbb{b})$ in $\mathbb{G}' \times \mathbb{G}$

is given by

$$deg_{\mathbb{G}'\times\mathbb{G}}((\mathbb{a},\mathbb{b})) = deg_{\mathbb{G}'}(\mathbb{a}) + deg_{\mathbb{G}}(\mathbb{b}). \tag{3.1}$$

The bounds for cartesian product w.r.t. Sombor index are calculated in the following theorem.

Theorem 3.1.1. *Let* $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_{\mathbb{r}}$ *be graphs with vertex and edges sets* $\mathbb{V}(\mathbb{G}_{\mathbb{i}})$ *and* $\mathbb{E}(\mathbb{G}_{\mathbb{i}})$*,* $1 \leq \mathbb{i} \leq \mathbb{r}$*, respectively. Also let* $|\mathbb{V}(\mathbb{G}_{\mathbb{i}})| = \mathbb{n}_{\mathbb{i}}$*,* $|\mathbb{E}(\mathbb{G}_{\mathbb{i}})| = \mathbb{m}_{\mathbb{i}}$*,* $1 \leq \mathbb{i} \leq \mathbb{r}$*, and* $\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}\right)\right| = \sum_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{m}_{\mathbb{i}} \prod_{\mathbb{l}=1,\mathbb{l}\neq\mathbb{i}}^{\mathbb{r}} \mathbb{n}_{\mathbb{l}}$*. Then*

$$\sqrt{2}\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}\right)\right| \sum_{\mathbb{i}=1}^{\mathbb{r}} \delta_{\mathbb{G}_{\mathbb{i}}} \leq \mathbb{SO}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}\right) \leq \sqrt{2}\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{r}} \mathbb{G}_{\mathbb{i}}\right)\right| \sum_{\mathbb{i}=1}^{\mathbb{r}} \Delta_{\mathbb{G}_{\mathbb{i}}}.$$

Equality holds in case of $\mathbb{G}_1, \mathbb{G}_2, \ldots, \mathbb{G}_{\mathbb{r}}$ *being regular graphs.*

Proof. We will prove by induction. For this, at first we will consider $\mathbb{r} = 2$. From the definition of Sombor index and equation (2.1), we have

$$\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \times \mathbb{G}_2) &= \sum_{(\mathbb{a},\mathbb{b})(\mathbb{c},\mathbb{d})\in\mathbb{E}(\mathbb{G}_1\times\mathbb{G}_2)} \sqrt{(deg_{\mathbb{G}_1\times\mathbb{G}_2}(\mathbb{a},\mathbb{b}))^2 + (deg_{\mathbb{G}_1\times\mathbb{G}_2}(\mathbb{c},\mathbb{d}))^2} \\
&= \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} \sqrt{\begin{matrix} deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_2}(\mathbb{b}) + 2deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) \\ +deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_2}(\mathbb{d}) + 2deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{d}) \end{matrix}} \\
&+ \sum_{\mathbb{b}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sqrt{\begin{matrix} deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_2}(\mathbb{b}) + 2deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) \\ +deg^2_{\mathbb{G}_1}(\mathbb{c}) + deg^2_{\mathbb{G}_2}(\mathbb{d}) + 2deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) \end{matrix}} \\
&= \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} \sqrt{\begin{matrix} 2deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_2}(\mathbb{d}) \\ +2deg_{\mathbb{G}_1}(\mathbb{a})(deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_2}(\mathbb{d})) \end{matrix}} \\
&+ \sum_{\mathbb{b}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sqrt{\begin{matrix} deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_1}(\mathbb{c}) + 2deg^2_{\mathbb{G}_2}(\mathbb{b}) \\ +2deg_{\mathbb{G}_2}(\mathbb{b})(deg_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{c})) \end{matrix}}.
\end{aligned}$$

Note that $deg_{\mathbb{G}_1}(\mathbb{a}), deg_{\mathbb{G}_1}(\mathbb{c}) \geq \delta_{\mathbb{G}_1}$ and $deg_{\mathbb{G}_2}(\mathbb{b}), deg_{\mathbb{G}_2}(\mathbb{d}) \geq \delta_{\mathbb{G}_2}$, in case of

$\mathbb{G}_1$ and $\mathbb{G}_2$ being regular graphs, equality holds.

$$\begin{aligned}
SO(\mathbb{G}_1 \times \mathbb{G}_2) &\geq \mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}\sqrt{2\delta^2_{\mathbb{G}_1} + \delta^2_{\mathbb{G}_2} + \delta^2_{\mathbb{G}_2} + 2\delta_{\mathbb{G}_1}(\delta_{\mathbb{G}_2} + \delta_{\mathbb{G}_2})} \\
&+ \mathbb{n}_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1}\sqrt{2\delta^2_{\mathbb{G}_2} + \delta^2_{\mathbb{G}_1} + \delta^2_{\mathbb{G}_1} + 2\delta_{\mathbb{G}_2}(\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_1})} \\
&= \mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}\sqrt{2\delta^2_{\mathbb{G}_1} + 2\delta^2_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}} \\
&+ \mathbb{n}_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1}\sqrt{2\delta^2_{\mathbb{G}_1} + 2\delta^2_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}} \\
&= \sqrt{2}(\mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2} + \mathbb{n}_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1})\sqrt{\delta^2_{\mathbb{G}_1} + \delta^2_{\mathbb{G}_2} + 2\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}} \\
&= \sqrt{2}(\mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2} + \mathbb{n}_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1})\sqrt{(\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_2})^2} \\
&= \sqrt{2}|\mathbb{E}(\mathbb{G}_1 \times \mathbb{G}_2)| \sum_{\mathbb{i}=1}^{2} \delta_{\mathbb{G}_\mathbb{i}}.
\end{aligned}$$

Assume, the result holds for $\mathbb{r} = \mathbb{q}$, that is,

$$\mathbb{SO}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}} \mathbb{G}_\mathbb{i}\right) \geq \sqrt{2}\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}} \mathbb{G}_\mathbb{i}\right)\right| \sum_{\mathbb{i}=1}^{\mathbb{q}} \delta_{\mathbb{G}_\mathbb{i}}.$$

Now we will prove it for $\mathbb{r} = \mathbb{q} + 1$,

$$\begin{aligned}
\mathbb{SO}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}+1} \mathbb{G}_\mathbb{i}\right) &= \mathbb{SO}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}} \mathbb{G}_\mathbb{i} \times \mathbb{G}_{\mathbb{q}+1}\right) \\
&\geq \sqrt{2}\sum_{\mathbb{i}=1}^{\mathbb{q}+1} \delta_\mathbb{i}\left(\sum_{\mathbb{i}=1}^{\mathbb{q}} \mathbb{m}_\mathbb{i} \prod_{\mathbb{l}=1, \mathbb{l}\neq\mathbb{i}}^{\mathbb{q}} \mathbb{n}_\mathbb{l}\right) + \sqrt{2}\sum_{\mathbb{i}=1}^{\mathbb{q}+1} \delta_\mathbb{i}\left(\mathbb{m}_{\mathbb{q}+1}\prod_{\mathbb{l}=1}^{\mathbb{q}} \mathbb{n}_\mathbb{l}\right) \\
&= \sqrt{2}\sum_{\mathbb{i}=1}^{\mathbb{q}+1} \delta_\mathbb{i}\left(\sum_{\mathbb{i}=1}^{\mathbb{q}+1} \mathbb{m}_\mathbb{i} \prod_{\mathbb{l}=1, \mathbb{l}\neq\mathbb{i}}^{\mathbb{q}+1} \mathbb{n}_\mathbb{l}\right) \\
&= \sqrt{2}\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}+1} \mathbb{G}_\mathbb{i}\right)\right| \sum_{\mathbb{i}=1}^{\mathbb{q}+1} \delta_\mathbb{i},
\end{aligned}$$

Analogously, we can prove the upper bound

$$\mathbb{SO}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}+1}\mathbb{G}_{\mathbb{i}}\right) \leq \sqrt{2}\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}+1}\mathbb{G}_{\mathbb{i}}\right)\right|\sum_{\mathbb{i}=1}^{\mathbb{q}+1}\mathbb{\Delta}_{\mathbb{i}},$$

where $\left|\mathbb{E}\left(\bigotimes_{\mathbb{i}=1}^{\mathbb{q}+1}\mathbb{G}_{\mathbb{i}}\right)\right| = \sum_{\mathbb{i}=1}^{\mathbb{q}+1}\mathbb{m}_{\mathbb{i}}\prod_{\mathbb{l}=1,\mathbb{l}\neq\mathbb{i}}^{\mathbb{q}+1}\mathbb{n}_{\mathbb{l}}$ and equality holds in case of $\mathbb{G}_{\mathbb{i}}$, $1 \leq \mathbb{i} \leq \mathbb{r}$ being regular graphs. □

Example 3.1.1. *By Example 1.4.2, we can see that the degrees of the vertices are* $2, 3, 2, 2, 3, 2$. *The greatest and lowest degrees of* $\mathbb{P}_3$ *and* $\mathbb{P}_2$ *are* 2, 1, *and* 1, 1, *respectively. So, the Sombor index of* $\mathbb{P}_3 \times \mathbb{P}_2$ *is* $\mathbb{SO}(\mathbb{P}_3 \times \mathbb{P}_2) = 4\sqrt{3^2+2^2} + 2\sqrt{2^2+2^2} + \sqrt{3^2+3^2} = 24.32$. *By the above result, we have* $19.79 \leq \mathbb{SO}(\mathbb{P}_3 \times \mathbb{P}_2) \leq 29.69$.

Example 3.1.2. *By Example 1.4.3, we can see that the degrees of all vertices in* $\mathbb{C}_3 \times \mathbb{P}_2$ *are* 3, *therefore it is a regular graph. Thus* $\mathbb{SO}(\mathbb{C}_3 \times \mathbb{P}_2) = 38.183$.

Example 3.1.3. *By Example 1.4.4, we can see that the degrees of all vertices in* $\mathbb{C}_3 \times \mathbb{C}_3$ *is* 4 *and it is a regular graph. Therefore* $\mathbb{SO}(\mathbb{C}_3 \times \mathbb{C}_3) = 101.82$.

3.2 Lexicographic product

Lexicographic product or composition of the graphs $\mathbb{G}'$ and $\mathbb{G}$ is given by $\mathbb{G}'[\mathbb{G}]$, where the vertex set of the product is $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and ordered pairs $(\mathbb{a}_1, \mathbb{b}_1)$ and $(\mathbb{a}_2, \mathbb{b}_2)$ make an edge when $\mathbb{a}_1\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$, or $\mathbb{a}_1 = \mathbb{a}_2$ and $\mathbb{b}_1\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$. The order of lexicographic product is $|\mathbb{V}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}}$ whereas the size is given by $|\mathbb{E}(\mathbb{G}')||\mathbb{V}(\mathbb{G})|^2 + |\mathbb{E}(\mathbb{G})||\mathbb{V}(\mathbb{G}')| = \mathbb{m}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}}^2 + \mathbb{m}_{\mathbb{G}}\mathbb{n}_{\mathbb{G}'}$. The degree of a vertex in $\mathbb{G}'[\mathbb{G}]$ is given by

$$deg_{\mathbb{G}'[\mathbb{G}]}((\mathbb{a}, \mathbb{b})) = \mathbb{n}_{\mathbb{G}}deg_{\mathbb{G}'}(\mathbb{a}) + deg_{\mathbb{G}}(\mathbb{b}). \tag{3.2}$$

The bounds for lexicographic product w.r.t. Sombor index are calculated in the following theorem.

Theorem 3.2.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be graphs with orders $|\mathbb{V}(\mathbb{G}_1)| = \mathbb{n}_{\mathbb{G}_1}$ and $|\mathbb{V}(\mathbb{G}_2)| = \mathbb{n}_{\mathbb{G}_2}$, respectively. Then*

$$\sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1[\mathbb{G}_2])| \leq \mathbb{SO}(\mathbb{G}_1[\mathbb{G}_2]) \leq \sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\Delta_{\mathbb{G}_1} + \Delta_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1[\mathbb{G}_2])|.$$

Equality holds in case if the graphs under consideration are regular graphs.

Proof. By the definition of Sombor index and equation (2.2), we have

$$\begin{aligned}
\mathbb{SO}(\mathbb{G}_1[\mathbb{G}_2]) &= \sum_{(\mathbb{a},\mathbb{b})(\mathbb{c},\mathbb{d})\in\mathbb{E}(\mathbb{G}_1[\mathbb{G}_2])} \sqrt{(deg_{\mathbb{G}_1[\mathbb{G}_2]}(\mathbb{a},\mathbb{b}))^2 + (deg_{\mathbb{G}_1[\mathbb{G}_2]}(\mathbb{c},\mathbb{d}))^2} \\
&= \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [\mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{a}) + deg_{\mathbb{G}_2}^2(\mathbb{b}) + 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{c}) + deg_{\mathbb{G}_2}^2(\mathbb{d}) + 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})]^{1/2} \\
&+ \sum_{\mathbb{b},\mathbb{d}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} [\mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{a}) + deg_{\mathbb{G}_2}^2(\mathbb{b}) + 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{c}) + deg_{\mathbb{G}_2}^2(\mathbb{d}) + 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})]^{1/2} \\
&= \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [2\mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{a}) + deg_{\mathbb{G}_2}^2(\mathbb{b}) + deg_{\mathbb{G}_2}^2(\mathbb{d}) \\
&+ 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a})(deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_2}(\mathbb{d}))]^{1/2} \\
&+ \sum_{\mathbb{b},\mathbb{d}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} [\mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{a}) + deg_{\mathbb{G}_2}^2(\mathbb{b}) + 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{c}) + deg_{\mathbb{G}_2}^2(\mathbb{d}) + 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})]^{1/2} \\
&\geq \mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}\sqrt{2\mathbb{n}_{\mathbb{G}_2}^2\delta_{\mathbb{G}_1}^2 + 2\delta_{\mathbb{G}_2}^2 + 4\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}} \\
&+ \mathbb{n}_{\mathbb{G}_2}^2\mathbb{m}_{\mathbb{G}_1}2\sqrt{\mathbb{n}_{\mathbb{G}_2}^2\delta_{\mathbb{G}_1}^2 + 2\delta_{\mathbb{G}_2}^2 + 4\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}} \\
&= \sqrt{2}\sqrt{\mathbb{n}_{\mathbb{G}_2}^2\delta_{\mathbb{G}_1}^2 + \delta_{\mathbb{G}_2}^2 + 2\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}}(\mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2} + \mathbb{n}_{\mathbb{G}_2}^2\mathbb{m}_{\mathbb{G}_1})
\end{aligned}$$

$$\begin{aligned}
&= \sqrt{2}\sqrt{(\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_2})^2}(\mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2} + \mathbb{m}_{\mathbb{G}_2}^2\mathbb{m}_{\mathbb{G}_1}) \\
&= \sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1[\mathbb{G}_2])|.
\end{aligned}$$

Analogously, we can derive

$$\mathbb{SO}(\mathbb{G}_1[\mathbb{G}_2]) \leq \sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\mathbb{\Delta}_{\mathbb{G}_1} + \mathbb{\Delta}_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1[\mathbb{G}_2])|.$$

Equality holds in case if the graphs under consideration are regular graphs.

□

Example 3.2.2. *By Example 1.4.6, we can see that the degree of the vertices are* $3, 5, 3, 3, 5, 3$ *. The greatest and lowest degrees of* $\mathbb{P}_3$ *and* $\mathbb{P}_2$ *are* 2*,* 1*, and* 1*,* 1*, respectively. So the Sombor index of* $\mathbb{P}_3[\mathbb{P}_2]$ *is* $\mathbb{SO}(\mathbb{P}_3[\mathbb{P}_2]) = 8\sqrt{5^2+3^2} + 2\sqrt{3^2+3^2} + \sqrt{5^2+5^2} = 62.20$*. By the above result, we have* $46.667 \leq \mathbb{SO}(\mathbb{P}_3[\mathbb{P}_2]) \leq 77.78$*.*

Example 3.2.3. *By Example 1.4.7, we can see that the degrees of all vertices in* $\mathbb{C}_3[\mathbb{P}_2]$ *is* 5*. Therefore, the Sombor index of* $\mathbb{C}_3[\mathbb{P}_2]$ *is* 106.06601*, where* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *are regular graphs.*

Example 3.2.4. *By Example 1.4.8, we can see that the degrees of all vertices in* $\mathbb{C}_3[\mathbb{C}_3]$ *are* 8*. Therefore, the Sombor index of* $\mathbb{C}_3[\mathbb{C}_3]$ *is* $\mathbb{SO}(\mathbb{C}_3[\mathbb{C}_3]) = 407.29$*.*

3.3 Tensor product

Tensor product of the graphs $\mathbb{G}'$ and $\mathbb{G}$ is given by $\mathbb{G}' \otimes \mathbb{G}$, where the vertex set of the product is $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and ordered pairs $(\mathbb{a}_1, \mathbb{b}_1)$ and $(\mathbb{a}_2, \mathbb{b}_2)$ make an edge in $\mathbb{G}' \otimes \mathbb{G}$ whenever $\mathbb{a}_1\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ and $\mathbb{b}_1\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$. The order of tensor product is $|\mathbb{V}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}}$ whereas the size is $2|\mathbb{E}(\mathbb{G}')||\mathbb{E}(\mathbb{G})| = 2\mathbb{m}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}}$. The degree of a vertex in $\mathbb{G}' \otimes \mathbb{G}$ is given by

$$deg_{\mathbb{G}'\otimes\mathbb{G}}((\mathbb{a}, \mathbb{b})) = deg_{\mathbb{G}'}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b}). \tag{3.3}$$

The bounds for tensor product w.r.t. Sombor index are calculated in the following theorem.

Theorem 3.3.1. *Let* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *be graphs with* $|\mathbb{V}(\mathbb{G}_1)| = \mathbb{n}_{\mathbb{G}_1}$ *and* $|\mathbb{V}(\mathbb{G}_2)| = \mathbb{n}_{\mathbb{G}_2}$, $|\mathbb{E}(\mathbb{G}_1)| = \mathbb{m}_{\mathbb{G}_1}$ *and* $|\mathbb{E}(\mathbb{G}_2)| = \mathbb{m}_{\mathbb{G}_2}$. *Then*

$$2\sqrt{2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}|\mathbb{E}(\mathbb{G}_1 \otimes \mathbb{G}_2)| \leq \mathbb{SO}(\mathbb{G}_1 \otimes \mathbb{G}_2) \leq 2\sqrt{2}\Delta_{\mathbb{G}_1}\Delta_{\mathbb{G}_2}|\mathbb{E}(\mathbb{G}_1 \otimes \mathbb{G}_2)|.$$

Equality holds in case if the graphs under consideration are regular graphs.

Proof. By the definition of Sombor index and equation (2.3), we have

$$\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \otimes \mathbb{G}_2) &= \sum_{(\mathbb{a},\mathbb{b})(\mathbb{c},\mathbb{d}) \in \mathbb{E}(\mathbb{G}_1 \otimes \mathbb{G}_2)} \sqrt{(deg_{\mathbb{G}_1 \otimes \mathbb{G}_2}(\mathbb{a},\mathbb{b}))^2 + (deg_{\mathbb{G}_1 \otimes \mathbb{G}_2}(\mathbb{c},\mathbb{d}))^2} \\
&= 2 \sum_{\mathbb{a}\mathbb{c} \in \mathbb{E}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d} \in \mathbb{E}(\mathbb{G}_2)} \sqrt{deg_{\mathbb{G}_1}(\mathbb{a})^2 deg_{\mathbb{G}_2}(\mathbb{b})^2 + deg_{\mathbb{G}_1}(\mathbb{c})^2 deg_{\mathbb{G}_2}(\mathbb{d})^2} \\
&\geq 2\mathbb{m}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}\sqrt{\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + \delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2}} \\
&= \sqrt{2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2}}|\mathbb{E}(\mathbb{G}_1 \otimes \mathbb{G}_2)| \\
&= \sqrt{2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}|\mathbb{E}(\mathbb{G}_1 \otimes \mathbb{G}_2)|.
\end{aligned}$$

Equality holds in case if the graphs under consideration are regular graphs. Analogously, we can compute the upper bound of Sombor index of $\mathbb{G}_1 \otimes \mathbb{G}_2$. □

Example 3.3.2. *By Example 1.4.10, we can see that the degrees of the vertices are* $1, 2, 1, 1, 2, 1$. *So the Sombor index of* $\mathbb{P}_3 \otimes \mathbb{P}_2$ *is* $4\sqrt{2^2 + 1^2} = 8.9$. *The greatest and lowest degrees of* $\mathbb{P}_3$ *and* $\mathbb{P}_2$ *are* 2, 1, *and* 1, 1, *respectively. By the result, we have* $5.65 \leq \mathbb{SO}(\mathbb{P}_3 \otimes \mathbb{P}_2) \leq 16$.

Example 3.3.3. *By Example 1.4.11, we can see that the degree of every vertex in* $\mathbb{C}_3 \otimes \mathbb{P}_2$ *is* 2. *Therefore, the Sombor index of* $\mathbb{C}_3 \otimes \mathbb{P}_2$ *is* $\mathbb{SO}(\mathbb{C}_3 \otimes \mathbb{P}_2) = 16.97$.

Example 3.3.4. *By Example 1.4.12, we can see that the degree of every vertex in* $\mathbb{C}_3 \otimes \mathbb{C}_3$ *is* 4. *Therefore, the Sombor index of* $\mathbb{C}_3 \otimes \mathbb{C}_3$ *is* $\mathbb{SO}(\mathbb{C}_3 \otimes \mathbb{C}_3) = 101.82$.

3.4 Strong product

Strong product or normal product or AND product of the graphs $\mathbb{G}'$ and $\mathbb{G}$ is given by $\mathbb{G}' \boxtimes \mathbb{G}$, where the vertex set of the product is $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and ordered pairs $(\mathbb{a}_{\mathbb{1}}, \mathbb{b}_{\mathbb{1}})$ and $(\mathbb{a}_2, \mathbb{b}_2)$ make an edge in $\mathbb{G}' \boxtimes \mathbb{G}$ if $\mathbb{a}_{\mathbb{1}} = \mathbb{a}_2$ and $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$ or $\mathbb{b}_{\mathbb{1}} = \mathbb{b}_2$ and $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ or $\mathbb{a}_{\mathbb{1}}\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ and $\mathbb{b}_{\mathbb{1}}\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$. The order of strong product is $|\mathbb{V}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}}$ whereas the size is $|\mathbb{V}(\mathbb{G}')||\mathbb{E}(\mathbb{G})| + |\mathbb{V}(\mathbb{G})||\mathbb{E}(\mathbb{G}')| + \mathbb{2}|\mathbb{E}(\mathbb{G}')||\mathbb{E}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}} + \mathbb{n}_{\mathbb{G}}\mathbb{m}_{\mathbb{G}'} + 2\mathbb{m}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}}$. The degree of a vertex in $\mathbb{G}' \boxtimes \mathbb{G}$ is given by

$$deg_{\mathbb{G}' \boxtimes \mathbb{G}}((\mathbb{a}, \mathbb{b})) = deg_{\mathbb{G}'}(\mathbb{a}) + deg_{\mathbb{G}}(\mathbb{b}) + deg_{\mathbb{G}'}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b}). \tag{3.4}$$

The bounds for strong product w.r.t. Sombor index are calculated in the following theorem.

Theorem 3.4.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be graphs with $|\mathbb{V}(\mathbb{G}_1)| = \mathbb{n}_{\mathbb{G}_1}$ and $|\mathbb{V}(\mathbb{G}_2)| = \mathbb{n}_{\mathbb{G}_2}$, $|\mathbb{E}(\mathbb{G}_1)| = \mathbb{m}_{\mathbb{G}_1}$ and $|\mathbb{E}(\mathbb{G}_1)| = \mathbb{m}_{\mathbb{G}_2}$. Then $\mathbb{P} \leq \mathbb{SO}(\mathbb{G}_1 \boxtimes \mathbb{G}_2) \leq \mathbb{Q}$, where*

$$\mathbb{P} = \sqrt{2}(\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_2} + \delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1 \boxtimes \mathbb{G}_2)|,$$
$$\mathbb{Q} = \sqrt{2}(\mathbb{\Delta}_{\mathbb{G}_1} + \mathbb{\Delta}_{\mathbb{G}_2} + \mathbb{\Delta}_{\mathbb{G}_1}\mathbb{\Delta}_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1 \boxtimes \mathbb{G}_2)|.$$

Equality holds in case if the graphs under consideration are regular graphs.

Proof. By the definition of Sombor index and equation (2.4), we have

$$
\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \boxtimes \mathbb{G}_2) &= \sum_{(\mathbb{a},\mathbb{b})(\mathbb{c},\mathbb{d})\in\mathbb{E}(\mathbb{G}_1\boxtimes\mathbb{G}_2)} \sqrt{(deg_{\mathbb{G}_1\boxtimes\mathbb{G}_2}(\mathbb{a},\mathbb{b}))^2 + (deg_{\mathbb{G}_1\boxtimes\mathbb{G}_2}(\mathbb{c},\mathbb{d}))^2} \\
&= \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b})^2 \\
&+ deg_{\mathbb{G}_1}(\mathbb{a})^2deg_{\mathbb{G}_2}(\mathbb{b})^2 + deg_{\mathbb{G}_1}(\mathbb{a})^2deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
&+ deg^2_{\mathbb{G}_1}(\mathbb{c}) + deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_2}(\mathbb{d}) \\
&+ deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) \\
&+ deg^2_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d})]^{1/2} + \sum_{\mathbb{b}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} [deg^2_{\mathbb{G}_1}(\mathbb{a}) \\
&+ deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ deg_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ deg_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{c}) + deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathscr{G}_1}(\mathbb{c})deg_{\mathscr{G}_2}(\mathbb{d}) \\
&+ deg^2_{\mathscr{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) \\
&+ deg^2_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d})]^{1/2} + 2\sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [deg^2_{\mathbb{G}_1}(\mathbb{a}) \\
&+ deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
&+ deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
&+ deg^2_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{c}) + deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) \\
&+ deg^2_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) \\
&+ deg^2_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d})]^{1/2} \\
\mathbb{SO}(\mathbb{G}_1 \boxtimes \mathbb{G}_2) &= \sum_{\mathbb{a}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [2deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_2}(\mathbb{d}) + 2deg_{\mathbb{G}_1}(\mathbb{a})(deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ deg_{\mathbb{G}_2}(\mathbb{d})) + 2deg^2_{\mathbb{G}_1}(\mathbb{a})(deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_2}(\mathbb{d})) + 2deg_{\mathbb{G}_1}(\mathbb{a})(deg^2_{\mathbb{G}_2}(\mathbb{b})
\end{aligned}
$$

$$
\begin{aligned}
&+ deg^2_{\mathbb{G}_2}(\mathbb{d})) + deg_{\mathbb{G}_1}(\mathbb{a})^2(deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_2}(\mathbb{d}))]^{1/2}\\
&+ \sum_{\mathbb{b}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} [2deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_1}(\mathbb{c}) + 2deg_{\mathbb{G}_2}(\mathbb{b})(deg_{\mathbb{G}_1}(\mathbb{a})\\
&+ deg_{\mathbb{G}_1}(\mathbb{c})) + 2deg^2_{\mathbb{G}_2}(\mathbb{b})(deg_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{c})) + 2deg_{\mathbb{G}_2}(\mathbb{b})(deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_1}(\mathbb{c}))\\
&+ deg^2_{\mathbb{G}_2}(\mathbb{b})(deg_{\mathbb{G}_1}(\mathbb{a})^2 + deg^2_{\mathbb{G}_1}(\mathbb{c}))]^{1/2} + 2\sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b})\\
&+ deg^2_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b})\\
&+ deg^2_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{a})deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg^2_{\mathbb{G}_1}(\mathbb{c})\\
&+ deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_2}(\mathbb{d})\\
&+ deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) + deg^2_{\mathbb{G}_1}(\mathbb{c})deg^2_{\mathbb{G}_2}(\mathbb{d}) + deg_{\mathbb{G}_1}(\mathbb{c})^2deg_{\mathbb{G}_2}(\mathbb{d})\\
&+ deg_{\mathbb{G}_1}(\mathbb{c})deg_{\mathbb{G}_2}(\mathbb{d})^2]^{1/2}\\
&\geq \mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}(2\delta^2_{\mathbb{G}_1} + 2\delta^2_{\mathbb{G}_2} + 2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2})^{1/2}\\
&+ \mathbb{n}_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1}(2\delta^2_{\mathbb{G}_1} + 2\delta^2_{\mathbb{G}_2} + 2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2})^{1/2}\\
&+ 2\mathbb{m}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}(2\delta^2_{\mathbb{G}_1} + 2\delta^2_{\mathbb{G}_2} + 2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2})^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \boxtimes \mathbb{G}_2) &\geq [2\delta^2_{\mathbb{G}_1} + 2\delta^2_{\mathbb{G}_2} + 2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 4\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2}]^{1/2}[\mathbb{n}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}\\
&\quad + \mathbb{n}_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1} + 2\mathbb{m}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}]\\
&\geq \sqrt{2}(\delta^2_{\mathbb{G}_1} + \delta^2_{\mathbb{G}_2} + \delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 2\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} + 2\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2}\\
&\quad + 2\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2})^{1/2}|\mathbb{E}(\mathbb{G}_1 \boxtimes \mathbb{G}_2)|\\
&\geq \sqrt{2}\sqrt{(\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_2} + \delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2})^2}|\mathbb{E}(\mathbb{G}_1 \boxtimes \mathbb{G}_2)|\\
&\geq \sqrt{2}(\delta_{\mathbb{G}_1} + \delta_{\mathbb{G}_2} + \delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1 \boxtimes \mathbb{G}_2)|.
\end{aligned}
$$

Analogously, we acquire

$$
\mathbb{SO}(\mathbb{G}_1 \boxtimes \mathbb{G}_2) \leq \sqrt{2}[\mathbb{\Delta}_{\mathbb{G}_1} + \mathbb{\Delta}_{\mathbb{G}_2} + \mathbb{\Delta}_{\mathbb{G}_1}\mathbb{\Delta}_{\mathbb{G}_2}]|\mathbb{E}(\mathbb{G}_1 \boxtimes \mathbb{G}_2)|.
$$

Equality holds in case if the graphs under consideration are regular graphs.

□

Example 3.4.2. *By Example 1.4.14, we can see that the degrees of the vertices are* $3, 5, 3, 3, 5, 3$. *The greatest and lowest degrees of* $\mathbb{P}_3$ *and* $\mathbb{P}_2$ *are* 2, 1 *and* 1, 1, *respectively. Therefore, the Sombor index of* $\mathbb{P}_3 \boxtimes \mathbb{P}_2$ *is* $8\sqrt{5^2+3^2} + 2\sqrt{3^2+3^2} + \sqrt{5^2+5^2} = 62.20$. *By the above result, we have* $46.66 \leq \mathbb{SO}(\mathbb{P}_3 \boxtimes \mathbb{P}_2) \leq 77.78$.

Example 3.4.3. *By Example 1.4.15, we can see that the degree of every vertex in* $\mathbb{C}_3 \boxtimes \mathbb{P}_2$ *is* 5. *Therefore, the Sombor index of* $\mathbb{C}_3 \boxtimes \mathbb{P}_2$ *is* 106.06601, *where* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *are regular graphs.*

Example 3.4.4. *By Example 1.4.16, we can see that the degree of every vertex in* $\mathbb{C}_3 \boxtimes \mathbb{C}_3$ *is* 8. *Therefore, the Sombor index of* $\mathbb{C}_3 \boxtimes \mathbb{C}_3$ *is* 407.29, *where* $\mathbb{C}_3$ *and* $\mathbb{C}_3$ *are regular graphs.*

3.5 Disjunction

Disjunction or co-normal product or OR product of the graphs $\mathbb{G}'$ and $\mathbb{G}$ is given by $\mathbb{G}' \vee \mathbb{G}$, where the vertex set of the product is $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and ordered pairs $(\mathbb{a}_1, \mathbb{b}_1)$ and $(\mathbb{a}_2, \mathbb{b}_2)$ make an edge in $\mathbb{G}' \vee \mathbb{G}$ whenever $\mathbb{a}_1\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ or $\mathbb{b}_1\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$. The order of disjunction is $|\mathbb{V}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}}$ whereas the size is $|\mathbb{V}(\mathbb{G}')|^2|\mathbb{E}(\mathbb{G})| + |\mathbb{V}(\mathbb{G})|^2|\mathbb{E}(\mathbb{G}')| - 2|\mathbb{E}(\mathbb{G}')||\mathbb{E}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}^2\mathbb{m}_{\mathbb{G}} + \mathbb{n}_{\mathbb{G}}^2\mathbb{m}_{\mathbb{G}'} - 2\mathbb{m}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}}$. The degree of a vertex in $\mathbb{G}' \vee \mathbb{G}$ is given by

$$deg_{\mathbb{G}' \vee \mathbb{G}}((\mathbb{a}, \mathbb{b})) = \mathbb{n}_{\mathbb{G}} deg_{\mathbb{G}'}(\mathbb{a}) + \mathbb{n}_{\mathbb{G}'} deg_{\mathbb{G}}(\mathbb{b}) - deg_{\mathbb{G}'}(\mathbb{a}) deg_{\mathbb{G}}(\mathbb{b}). \tag{3.5}$$

The bounds for co-normal product w.r.t. Sombor index are calculated in the following theorem.

Theorem 3.5.1. *Let* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *be graphs with* $|\mathbb{V}(\mathbb{G}_1)| = \mathbb{n}_{\mathbb{G}_1}$ *and* $|\mathbb{V}(\mathbb{G}_2)| = \mathbb{n}_{\mathbb{G}_2}$, $|\mathbb{E}(\mathbb{G}_1)| = \mathbb{m}_{\mathbb{G}_1}$ *and* $|\mathbb{E}(\mathbb{G}_2)| = \mathbb{m}_{\mathbb{G}_2}$. *Then* $\mathbb{P} \leq \mathbb{SO}(\mathbb{G}_1 \vee \mathbb{G}_2) \leq \mathbb{Q}$, *where*

$$\mathbb{P} = \sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - \delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1 \vee \mathbb{G}_2)|,$$
$$Q = \sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\mathbb{\Delta}_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\mathbb{\Delta}_{\mathbb{G}_2} - \mathbb{\Delta}_{\mathbb{G}_1}\mathbb{\Delta}_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1 \vee \mathbb{G}_2)|.$$

Equality holds in case if the graphs under consideration are regular graphs.

Proof. By the definition of Sombor index, we have

$$\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \vee \mathbb{G}_2) &= \sum_{(\mathbb{a},\mathbb{b})(\mathbb{c},\mathbb{d})\in\mathbb{E}(\mathbb{G}_1\vee\mathbb{G}_2)} [(deg_{\mathbb{G}_1\vee\mathbb{G}_2}(\mathbb{a},\mathbb{b}))^2 + (deg_{\mathbb{G}_1\vee\mathbb{G}_2}(\mathbb{c},\mathbb{d}))^2]^{1/2} \\
&= \sum_{\mathbb{a},\mathbb{c}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [\mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{a}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
&- \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_1}^2 deg_{\mathbb{G}_2}^2(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
&- \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b}) + deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
&- \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{c}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
&- \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}^2 deg_{\mathbb{G}_2}^2(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
&- \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}^2(\mathbb{d}) + deg_{\mathbb{G}_1}^2(\mathbb{c}) deg_{\mathbb{G}_2}^2(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 deg_{\mathbb{G}_2}(\mathbb{d}) \\
&- \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})^2]^{1/2} + \sum_{\mathbb{b},\mathbb{d}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} [\mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{a}) \\
&+ \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_1}^2 deg_{\mathbb{G}_2}^2(\mathbb{b}) \\
&+ \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b}) + deg_{\mathbb{G}_1}(\mathbb{a})^2 deg_{\mathbb{G}_2}^2(\mathbb{b}) \\
&- \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{c}) \\
&+ \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 deg_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}^2 deg_{\mathbb{G}_2}^2(\mathbb{d}) \\
&+ \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}^2(\mathbb{d}) + deg_{\mathbb{G}_1}^2(\mathbb{c}) deg_{\mathbb{G}_2}^2(\mathbb{d}) \\
&- \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}^2(\mathbb{d})]^{1/2} - 2 \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} \\
&[\mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{a}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
&+ \mathbb{n}_{\mathbb{G}_1}^2 deg_{\mathbb{G}_2}^2(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b})
\end{aligned}$$

$$\begin{aligned}
&+ deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}^2(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}^2 deg_{\mathbb{G}_1}^2(\mathbb{c}) \\
&+ \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}^2(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
&+ \mathbb{n}_{\mathbb{G}_1}^2 deg_{\mathbb{G}_2}^2(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}^2(\mathbb{d}) + deg_{\mathbb{G}_1}^2(\mathbb{c}) deg_{\mathbb{G}_2}^2(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 deg_{\mathbb{G}_2}(\mathbb{d}) \\
&- \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})^2]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \vee \mathbb{G}_2) = & \sum_{\mathbb{a},\mathbb{c}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [2\mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{d}) \\
& + 2\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a})(deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_2}(\mathbb{d})) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a})[deg_{\mathbb{G}_2}(\mathbb{b}) \\
& + deg_{\mathbb{G}_2}(\mathbb{d})] - 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a})[deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_2}(\mathbb{d})^2] + deg^2_{\mathbb{G}_1}(\mathbb{a})[deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& + deg_{\mathbb{G}_2}(\mathbb{d})^2]]^{1/2} + \sum_{\mathbb{b},\mathbb{d}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} [2\mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a})^2 \\
& + \mathbb{n}^2_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 + 2deg_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_2}(\mathbb{b})[deg_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{c})] \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b})[deg_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{c})] - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_2}(\mathbb{b})(deg^2_{\mathbb{G}_1}(\mathbb{a}) \\
& + deg^2_{\mathbb{G}_1}(\mathbb{c})) + deg^2_{\mathbb{G}_2}(\mathbb{b})(deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_1}(\mathbb{c}))]^{1/2} - 2 \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} \\
& [\mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
& + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& + deg^2_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& + \mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
& + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) \\
& + deg^2_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - \mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d})]^{1/2} \\
& \geq \mathbb{n}^2_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}[2\mathbb{n}^2_{\mathbb{G}_2}\delta^2_{\mathbb{G}_1} + 2\mathbb{n}^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 4\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} \\
& - 4\mathbb{n}_{\mathbb{G}_2}\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 4\mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2}]^{1/2} + \mathbb{n}^2_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1}[2\mathbb{n}^2_{\mathbb{G}_2}\delta^2_{\mathbb{G}_1} + 2\mathbb{n}^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} \\
& + 2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 4\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 4\mathbb{n}_{\mathbb{G}_2}\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 4\mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2}]^{1/2} \\
& - 2\mathbb{m}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}[2\mathbb{n}^2_{\mathbb{G}_2}\delta^2_{\mathbb{G}_1} + 2\mathbb{n}^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 2\delta^2_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2} + 4\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} \\
& - 4\mathbb{n}_{\mathbb{G}_2}\delta^2_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 4\mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_1}\delta^2_{\mathbb{G}_2}]^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \vee \mathbb{G}_2) \geq & \sqrt{2}\sqrt{[\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - \delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}]^2}[\mathbb{n}^2_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2} + \mathbb{n}^2_{\mathbb{G}_2}\mathbb{m}_{\mathbb{G}_1} - 2\mathbb{m}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}] \\
& = \sqrt{2}[\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - \delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}]|\mathbb{E}(\mathbb{G}_1 \vee \mathbb{G}_2)|.
\end{aligned}
$$

Analogously, we can get

$$
\mathbb{SO}(\mathbb{G}_1 \vee \mathbb{G}_2) \leq \sqrt{2}[\mathbb{n}_{\mathbb{G}_2}\Delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\Delta_{\mathbb{G}_2} - \Delta_{\mathbb{G}_1}\Delta_{\mathbb{G}_2}]|\mathbb{E}(\mathbb{G}_1 \vee \mathbb{G}_2)|.
$$

Equality holds in case if the graphs under consideration are regular graphs. $\square$

Example 3.5.2. *By Example 1.4.14, we can see that the degrees of the vertices are* $4, 5, 4, 4, 5, 4$. *The greatest and lowest degrees of* $\mathbb{P}_3$ *and* $\mathbb{P}_2$ *are* $2, 1$ *and* $1, 1$, *respectively. Therefore, the Sombor index of* $\mathbb{P}_3 \vee \mathbb{P}_2$ *is* $8\sqrt{5^2+4^2} + 4\sqrt{4^2+4^2} + \sqrt{5^2+5^2} = 80.923$. *By the result, we have* $73.53 \leq \mathbb{SO}(\mathbb{P}_3 \vee \mathbb{P}_2) \leq$.

Example 3.5.3. *By Example 1.4.15, we can see that the degree of every vertex in* $\mathbb{C}_3 \vee \mathbb{P}_2$ *is* 5. *Therefore, the Sombor index of* $\mathbb{C}_3 \vee \mathbb{P}_2$ *is* 106.06601, *where* $\mathbb{C}_3$ *and* $\mathbb{P}_2$ *are regular graphs.*

Example 3.5.4. *By Example 1.4.16, we can see that the degree of every vertex in* $\mathbb{C}_3 \vee \mathbb{C}_3$ *is* 8. *Therefore, the Sombor index of* $\mathbb{C}_3 \vee \mathbb{C}_3$ *is* 407.29, *where* $\mathbb{C}_3$ *and* $\mathbb{C}_3$ *are regular graphs.*

3.6 Symmetric difference

Symmetric difference of the graphs $\mathbb{G}'$ and $\mathbb{G}$ is given by $\mathbb{G}' \oplus \mathbb{G}$, where the vertex set of the product is $\mathbb{V}(\mathbb{G}') \times \mathbb{V}(\mathbb{G})$ and ordered pairs $(\mathbb{a}_1, \mathbb{b}_1)$ and $(\mathbb{a}_2, \mathbb{b}_2)$ make an edge in $\mathbb{G}' \oplus \mathbb{G}$, whenever $\mathbb{a}_1\mathbb{a}_2 \in \mathbb{E}(\mathbb{G}')$ or $\mathbb{b}_1\mathbb{b}_2 \in \mathbb{E}(\mathbb{G})$ but not both. The order of symmetric difference is $|\mathbb{V}(\mathbb{G}')||\mathbb{V}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}\mathbb{n}_{\mathbb{G}}$ whereas the size is $|\mathbb{V}(\mathbb{G}')|^2|\mathbb{E}(\mathbb{G})| + |\mathbb{V}(\mathbb{G})|^2|\mathbb{E}(\mathbb{G}')| - 4|\mathbb{E}(\mathbb{G}')||\mathbb{E}(\mathbb{G})| = \mathbb{n}_{\mathbb{G}'}^2\mathbb{m}_{\mathbb{G}} + \mathbb{n}_{\mathbb{G}}^2\mathbb{m}_{\mathbb{G}'} - 4\mathbb{m}_{\mathbb{G}'}\mathbb{m}_{\mathbb{G}}$. The degree of a vertex in $\mathbb{G}' \oplus \mathbb{G}$ is given by

$$deg_{\mathbb{G}'\oplus\mathbb{G}}((\mathbb{a}, \mathbb{b})) = \mathbb{n}_{\mathbb{G}} deg_{\mathbb{G}'}(\mathbb{a}) + \mathbb{n}_{\mathbb{G}'} deg_{\mathbb{G}}(\mathbb{b}) - 2deg_{\mathbb{G}'}(\mathbb{a})deg_{\mathbb{G}}(\mathbb{b}). \quad (3.6)$$

The bounds for symmetric difference w.r.t. Sombor index are calculated in the following theorem.

Theorem 3.6.1. *Let* $\mathbb{G}_1$ *and* $\mathbb{G}_2$ *be graphs with* $|\mathbb{V}(\mathbb{G}_1)| = \mathbb{n}_{\mathbb{G}_1}$ *and* $|\mathbb{V}(\mathbb{G}_2)| =$

$\mathbb{n}_{\mathbb{G}_2}$, $|\mathbb{E}(\mathbb{G}_1)| = \mathbb{m}_{\mathbb{G}_1}$ and $|\mathbb{E}(\mathbb{G}_2)| = \mathbb{m}_{\mathbb{G}_2}$. Then $\mathbb{P} \leq \mathbb{SO}(\mathbb{G}_1 \oplus \mathbb{G}_2) \leq \mathbb{Q}$, where

$$\mathbb{P} = \sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 2\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1 \vee \mathbb{G}_2)|,$$
$$Q = \sqrt{2}(\mathbb{n}_{\mathbb{G}_2}\mathbb{\Delta}_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\mathbb{\Delta}_{\mathbb{G}_2} - 2\mathbb{\Delta}_{\mathbb{G}_1}\mathbb{\Delta}_{\mathbb{G}_2})|\mathbb{E}(\mathbb{G}_1 \vee \mathbb{G}_2)|.$$

Equality holds in case if the graphs under consideration are regular graphs.

Proof. By the definition of Sombor index, we have

$$\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \oplus \mathbb{G}_2) = & \sum_{(\mathbb{a},\mathbb{b})(\mathbb{c},\mathbb{d}) \in \mathbb{E}(\mathbb{G}_1 \oplus \mathbb{G}_2)} [(deg_{\mathbb{G}_1 \oplus \mathbb{G}_2}(\mathbb{a},\mathbb{b}))^2 + (deg_{\mathbb{G}_1 \oplus \mathbb{G}_2}(\mathbb{c},\mathbb{d}))^2]^{1/2} \\
& = \sum_{\mathbb{a},\mathbb{c} \in \mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d} \in \mathbb{E}(\mathbb{G}_2)} [\mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
& - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) + 4 deg^2_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
& - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) + 4 deg^2_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 deg_{\mathbb{G}_2}(\mathbb{d}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})^2]^{1/2} + \sum_{\mathbb{b},\mathbb{d} \in \mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c} \in \mathbb{E}(\mathbb{G}_1)} [\mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) \\
& + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) + 4 deg_{\mathbb{G}_1}(\mathbb{a})^2 deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c})
\end{aligned}$$

$$+ \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 deg_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})$$

$$deg_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) + 4deg^2_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})$$

$$- 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d})]^{1/2} - 4 \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [\mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b})$$

$$- 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a})$$

$$deg^2_{\mathbb{G}_2}(\mathbb{b}) + 4deg^2_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b})$$

$$+ \mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})$$

$$+ \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) + 4deg^2_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d})$$

$$- 2\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 deg_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d})^2]^{1/2}$$

$$\begin{aligned}
\mathbb{SO}(\mathbb{G}_1 \oplus \mathbb{G}_2) = & \sum_{\mathbb{a},\mathbb{c}\in\mathbb{V}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [2\mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{d}) \\
& + 2\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a})(deg_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_2}(\mathbb{d})) - 4\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a})[deg_{\mathbb{G}_2}(\mathbb{b}) \\
& + deg_{\mathbb{G}_2}(\mathbb{d})] - 4\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a})[deg^2_{\mathbb{G}_2}(\mathbb{b}) + deg_{\mathbb{G}_2}(\mathbb{d})^2] + 4deg^2_{\mathbb{G}_1}(\mathbb{a})[deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& + deg_{\mathbb{G}_2}(\mathbb{d})^2]]^{1/2} + \sum_{\mathbb{b},\mathbb{d}\in\mathbb{V}(\mathbb{G}_2)} \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} [2\mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{a})^2 \\
& + \mathbb{n}^2_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c})^2 + 2deg_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_2}(\mathbb{b})[deg_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{c})] - 4\mathbb{n}_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& [deg_{\mathbb{G}_1}(\mathbb{a}) + deg_{\mathbb{G}_1}(\mathbb{c})] - 4\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_2}(\mathbb{b})[(deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_1}(\mathbb{c}))] + 4deg^2_{\mathbb{G}_2}(\mathbb{b}) \\
& [(deg^2_{\mathbb{G}_1}(\mathbb{a}) + deg^2_{\mathbb{G}_1}(\mathbb{c}))]^{1/2} - 4 \sum_{\mathbb{a}\mathbb{c}\in\mathbb{E}(\mathbb{G}_1)} \sum_{\mathbb{b}\mathbb{d}\in\mathbb{E}(\mathbb{G}_2)} [\mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) \\
& deg_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}_{\mathbb{G}_2}\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) + 4deg^2_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{a}) deg_{\mathbb{G}_2}(\mathbb{b}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{a}) deg^2_{\mathbb{G}_2}(\mathbb{b}) + \mathbb{n}^2_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
& - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}^2_{\mathbb{G}_1} deg^2_{\mathbb{G}_2}(\mathbb{d}) + \mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2} deg_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) + 4deg^2_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d}) - 2\mathbb{n}_{\mathbb{G}_2} deg^2_{\mathbb{G}_1}(\mathbb{c}) deg_{\mathbb{G}_2}(\mathbb{d}) \\
& - 2\mathbb{n}_{\mathbb{G}_1} deg_{\mathbb{G}_1}(\mathbb{c}) deg^2_{\mathbb{G}_2}(\mathbb{d})]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{n}_{\mathbb{G}_1}^2 \mathbb{m}_{\mathbb{G}_2}[2\mathbb{n}_{\mathbb{G}_2}^2 \delta_{\mathbb{G}_1}^2 + 2\mathbb{n}_{\mathbb{G}_1}^2 \delta_{\mathbb{G}_2}^2 + 8\delta_{\mathbb{G}_1}^2 \delta_{\mathbb{G}_2}^2 + 4\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} \\
&- 8\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}^2\delta_{\mathbb{G}_2} - 8\mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}^2]^{1/2} + \mathbb{n}_{\mathbb{G}_2}^2 \mathbb{m}_{\mathbb{G}_1}[2\mathbb{n}_{\mathbb{G}_2}^2 \delta_{\mathbb{G}_1}^2 + 2\mathbb{n}_{\mathbb{G}_1}^2 \delta_{\mathbb{G}_2}^2 \\
&+ 8\delta_{\mathbb{G}_1}^2 \delta_{\mathbb{G}_2}^2 + 4\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 8\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}^2\delta_{\mathbb{G}_2} - 8\mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}^2]^{1/2} \\
&- 4\mathbb{m}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}[2\mathbb{n}_{\mathbb{G}_2}^2 \delta_{\mathbb{G}_1}^2 + 2\mathbb{n}_{\mathbb{G}_1}^2 \delta_{\mathbb{G}_2}^2 + 8\delta_{\mathbb{G}_1}^2 \delta_{\mathbb{G}_2}^2 + 4\mathbb{n}_{\mathbb{G}_1}\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2} \\
&- 8\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1}^2\delta_{\mathbb{G}_2} - 8\mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}^2]^{1/2} \\
\mathbb{SO}(\mathbb{G}_1 \oplus \mathbb{G}_2) &\geq \sqrt{2}\sqrt{[\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 2\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}]^2}[\mathbb{n}_{\mathbb{G}_1}^2\mathbb{m}_{\mathbb{G}_2} + \mathbb{n}_{\mathbb{G}_2}^2\mathbb{m}_{\mathbb{G}_1} - 4\mathbb{m}_{\mathbb{G}_1}\mathbb{m}_{\mathbb{G}_2}] \\
&= \sqrt{2}[\mathbb{n}_{\mathbb{G}_2}\delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\delta_{\mathbb{G}_2} - 2\delta_{\mathbb{G}_1}\delta_{\mathbb{G}_2}]|\mathbb{E}(\mathbb{G}_1 \oplus \mathbb{G}_2)|.
\end{aligned}
$$

Analogously, we can get

$$
\mathbb{SO}(\mathbb{G}_1 \oplus \mathbb{G}_2) \leq \sqrt{2}[\mathbb{n}_{\mathbb{G}_2}\Delta_{\mathbb{G}_1} + \mathbb{n}_{\mathbb{G}_1}\Delta_{\mathbb{G}_2} - 2\Delta_{\mathbb{G}_1}\Delta_{\mathbb{G}_2}]|\mathbb{E}(\mathbb{G}_1 \oplus \mathbb{G}_2)|.
$$

Equality holds in case if the graphs under consideration are regular graphs.

□

Example 3.6.2. *By Example 1.4.22, we can see that the degree of every vertex in $\mathbb{P}_3 \oplus \mathbb{P}_2$ is* 3*. Therefore, the Sombor index of $\mathbb{P}_3 \oplus \mathbb{P}_2$ is* $9\sqrt{3^2 + 3^2} = 38.183$*, because $\mathbb{P}_3 \oplus \mathbb{P}_2$ is a regular graph.*

Example 3.6.3. *By Example 1.4.23, we can see that the degree of every vertex in $\mathbb{C}_3 \oplus \mathbb{P}_2$ is* 3*. Therefore, the Sombor index of $\mathbb{C}_3 \oplus \mathbb{P}_2$ is* $9\sqrt{3^2 + 3^2} = 38.183$*, because $\mathbb{C}_3 \oplus \mathbb{P}_2$ is a regular graph.*

Example 3.6.4. *By Example 1.4.24, we can see that the degree of every vertex in $\mathbb{C}_3 \oplus \mathbb{C}_3$ is* 4*. Therefore, the Sombor index of $\mathbb{C}_3 \oplus \mathbb{C}_3$ is* $18\sqrt{4^2 + 4^2} = 101.82$*, because $\mathbb{C}_3 \oplus \mathbb{P}_2$ is a regular graph.*

Chapter 4

Conclusion

The first chapter of this dissertation includes the fundamentals of graph theory. Different graphs are being discussed and further we have dealt with graph operations such as cartesian product, lexicographic product, tensor product, strong product, disjunction and symmetric difference.

In the second chapter, we discussed the literature review of topological indices. We briefly discussed the distance based topological indices, that is, Weiner index, Szeged index, PI index. Furthermore degree based topological indices such as first and second Zagreb index, Randić connectivity index, sum connectivity index and Sombor index are being discussed.

In chapter 3, we have computed the upper and lower bounds of some graph operations w.r.t. Sombor index. Moreover, these results are then elaborated by the help of some examples on different graphs.

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