Generalization of Mean-Nonexpansive Semigroup and Fundamentally Nonexpansive Mapping

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Dedication

This piece of work is dedicated to my late Parents.

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All praises is for **Almighty Allah**, the most gracious and the most merciful, Who created this entire universe. I am highly grateful to **Almighty Allah** for showering His countless blessings upon me and giving me the ability and strength to complete this thesis successfully and blessing me more than I deserve.

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Abstract

The theory of fixed point is one of the most incredible asset of present day Mathematical Analysis. Theorem concerning the presence and properties of fixed point are known as fixed point theorems. Fixed point theory is the beautiful mixture of Analysis, Topology and Geometry which has many applications in various fields. In 19th century the study of fixed point theory was initiated by Poincare and in the 20th century this area developed by many mathematicians like Brouwer, Schauder, Kakutani, Banach, Kannan, Tarski and others.

In 1922 the concept of Banach Space was introduced by Stefen Banach and introduced a Fixed Point Theorem for contraction mapping. There are numerous generaliztaion of Banach contraction principle to unique fixed point of the mapping. Sehgal, Kannan, Caristi and Husain worked for some generalization of contraction mappings and proved number of the result for contraction mapping.

As the fundamental properties of contraction mapping do not extend to nonexpansive and mean nonexpansive mapping. Now, the study of nonexpansive and mean nonexpansive mapping is the main feature in recent development of fixed points. Contractive mappping, isometries and orthogonal projections are all nonexpansive mapping. In most of the cases we have studied fixed points for different types of mapppings. The objective of following work is to find the fixed point for mean-nonexpansive semigroup and fundamentally nonexpansive mapping.

Contents

Chapter 1

Preliminaries

1.1 Introduction

The most basic results of Functional Analysis and Fixed Point Theory are the foundation of our study. Over the years, there have been many efforts to generalize this theorem for various classes of topological spaces and Banach spaces. In a broad sense, by a fixed point theorem we would say a statement asserting that under definite conditions a self map M on U accepts one or more points such as $M(r) = r$. Many researchers have studied fixed point theory for a reason in its application to the idea of variational and linear inequalities, boundary value problem, the approximation theory, chemical reactions, nonlinear analysis, integral equations, the dynamic systems theory, mathematical economics, partial differential equations, economic theory and game theory.

There are number of the fixed point theorems which fulfills specific conditions for a compression type planning. In all of these one result looks at the sequence of iterates, i.e. due to contraction conditions, it becomes the Cauchy sequence and its limit is the fixed point of a defined mapping M.

Fixed point theory has a key role for boundary value problems and eigen value problems. For details one can see Williams [14], Cronin [9], Martin [32] ,Leggett [21], Smart [29] , Collatz [8], Kreyszig [20] and Cesari [6], etc. Brouwer [3] proved the fixed point theorems for Euclidean space which states that there is atleast one fixed point for a continuous mapping of the closed unit ball in n-dimensional Euclidean space.

Then Kellog, Birkhoff, Schwartz and Dunford proved Brouwer's theorems by using classical methods of analysis. Hopf and Alexendroff also proved this theorem by using mechanisam from algebraic topology. Then Brouwer fixed point theorem extended by Schaiider in this case the set S is the closed, bounded and convex subset of a normed space. After that, Schaiider' result extended by Tychonoff [24] from normed space to the locally convex space.

In 1922 the concept of Banach Space was introduced by Stefen Banach and introduced a Fixed Point Theorem for contraction mapping. There are numerous generaliztaion of Banach contraction principle to unique fixed point of the mapping. Sehgal [28], Kannan [18], Caristi [5] and Husain [17] worked for several generalizations of contraction mappings and proved number of results for contraction mapping.

As a fundamental properties of contraction mapping do not extend to the nonexpansive and mean nonexpansive mapping. Now, the study of nonexpansive and mean nonexpansive mapping is the main feature in recent development of fixed points. Contractive mappping, isometries and orthogonal projections are all nonexpansive mapping. Mean nonexpansive mapping first introduced in 2007 by Goebel and Japon Pineda [15] and many authors working on propeties of Mean nonexpansive mapping in different fields.

1.2 Metric Space

Let $\aleph$ be a nonempty set. We define metric and metric space as folllows :

Definition 1.2.1. *Let $\aleph$ be a nonempty set and defined a function ρ such that $\rho : \aleph \times \aleph \to \mathbf{R}$, then the pair $M = (\aleph, \rho)$ is said to be the metric space if the following condidtions are satisfied:*

1. *For all $\zeta_1, \zeta_2 \in \aleph, \rho(\zeta_1, \zeta_2) \geq 0$*
2. *For all $\zeta_1, \zeta_2 \in \aleph$ we have that $\rho(\zeta_1, \zeta_2) = 0 \Leftrightarrow \zeta_1 = \zeta_2$*
3. *For all $\zeta_1, \zeta_2 \in \aleph, \rho(\zeta_1, \zeta_2) = \rho(\zeta_2, \zeta_1)$ (Symmetry)*
4. *(Triangular inequality) For all $\zeta_1, \zeta_2, \zeta_3 \in \aleph$*

   we have that

$$\rho(\zeta_1, \zeta_3) \leq \rho(\zeta_1, \zeta_2) + \rho(\zeta_2, \zeta_3). \tag{1.1}$$

Example 1.2.2. *We can define the usual metric ρ on the set of real numbers $\mathbf{R}$ such that for all $\zeta_1, \zeta_2 \in \mathbf{R}$,*

$$\rho(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|. \tag{1.2}$$

Example 1.2.3. *The plane $\mathbf{R^2}$ with the usual distance (measured by using Pythagoras's theorem): $\rho((\xi_1, \zeta_1), (\xi_2, \zeta_2)) = \sqrt{(\xi_1 - \xi_2)^2 + (\zeta_1 - \zeta_2)^2}$.*

Definition 1.2.4. *Let $\aleph$ be the metric space and $\beth$ be the subset of $\aleph$, then $\beth$ is said to be open if for every ζ in $\beth$ there exists $\epsilon > 0$ such that $B_\epsilon(\zeta)$ is contained in $\beth$.*

Example 1.2.5. *The open interval $(3, 6)$ is an open set in real line $\mathbf{R}$.*

Remark 1. *The arbitrary union of open sets is an open set.*

Example 1.2.6. *The empty set ϕ and $\mathbf{R}$ both are open.*

Definition 1.2.7. *Let* $\aleph$ *be the metric space and the sequence* ξ_n *in* $\aleph$ *is said to converge to a point* ξ_0 *belongs to* $\aleph$ *if* $lim_{n\to\infty}\rho(\xi_n, \xi_0) = 0$. *In this case we can write* $\xi_n \to \xi_0$. *It can be easily proved that if* $\xi_n \to \xi_0$ *and* $\xi_n \to \zeta_0$, *then* $\xi_0 = \zeta_0$, *that is, a convergent sequence has the unique limit.*

Definition 1.2.8. *A sequence* ξ_n *in the metric space* $\aleph$ *is the Cauchy sequence if for evey* $\epsilon > 0$, *there is the* $N(\epsilon)$ *such that* $\rho(\xi_n, \xi_m) < \epsilon$ *whenever* $m, n \geq N(\epsilon)$.

Definition 1.2.9. *A metric space* $\aleph$ *is the complete metric space if every Cauchy sequence in* $\aleph$ *is convergent in* $\aleph$.

1.3 Normed Spaces

Definition 1.3.1. *A vector space* $\beth$ *with two binary operation addition and scalar multiplication satisfy the following axioms* $\xi_1, \xi_2, \xi_3 \in \beth$ *and* $\lambda, \mu \in \mathbf{F}$;

1. $\xi_1 + \xi_2 \in \beth$.

2. $\xi_1 + \xi_2 = \xi_2 + \xi_1$.

3. $(\xi_1 + \xi_2) + \xi_3 = \xi_1 + (\xi_2 + \xi_3)$.

4. *There is a zero vector in* $\beth$ *such that* $0 + \xi = \xi$ *for all* $\xi \in \beth$.

5. *For every* $\xi \in \beth$ *there is a additive inverse of* ξ *such that* $\xi + (-\xi) = 0$.

6. $\lambda\xi \in \beth$.

7. $\lambda(\xi_1 + \xi_2) = \lambda\xi_1 + \lambda\xi_2$.

8. $(\lambda + \mu)\xi = \lambda\xi + \mu\xi$.

9. $(\lambda\mu)\xi = \lambda(\mu\xi)$.

10. $1.\xi = \xi$.

Example 1.3.2. *Euclidean space $\mathbf{R^n}$ is a vector space over the field $\mathbf{R}$ and the set $\mathbf{C^n}$ is a vector space over the field $\mathbf{C}$.*

Example 1.3.3. *The polynomial space $P_n = \sum_{k=0}^{n} b_k x^k$ of all polynomials of degree n such thst $b_0, b_2, ..., b_n$ are the real numbers and n is the positive integer(called the degree of the polynomial).*

1. *An operation of vector adddition $+$ defined by,*
   $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$.

2. *An operation of scalar multiplication defined by,*
   $r(x_1, x_2, ..., x_n) = (rx_1, rx_2, ..., rx_n)$.

P_n is vector space with these operations.

Definition 1.3.4. *Let $\beth$ be the vector space over a field $\mathbf{K}$($\mathbf{R}$ or $\mathbf{C}$). A norm on $\beth$ is map $\|.\| : \beth \to [0, \infty)$ satisfy the following properties:*

1. $\|\xi\| \geq 0$

2. $\|\xi\| = 0$ *iff* $\xi = 0$

3. $\|\mu\xi\| = |\mu|\|\xi\|$ *for all* $\xi \in \beth$ *and* $\mu \in \mathbf{K}$

4. $\|\xi_1 + \xi_2\| \leq \|\xi_1\| + \|\xi_2\|$ *for all* $\xi_1, \xi_2 \in \beth$. *The ordered pair* $(\beth,\ \|.\|)$ *is said to be a normed space.*

Remark 2. *Every normed space is a metric space.*

A metric induced by norm is defined as

$$\rho(\xi_1, \xi_2) = \|\xi_1 - \xi_2\| \tag{1.3}$$

for all $\xi_1, \xi_2 \in \beth$.

Lemma 1.3.5. *A metric ρ induced by the norm on normed space $\beth$ satisfy the following condition:*

1. $\rho(\xi_1 + \lambda, \xi_2 + \lambda) = \rho(\xi_1, \xi_2)$.
2. $\rho(\lambda\xi_1, \lambda\xi_2) = |\lambda|\rho(\xi_1, \xi_2)$.

Example 1.3.6. *The Euclidean space $\mathbf{R^n}$ and unitary space $\mathbf{C^n}$ are the normed spaces with the norm defined by,*

$$\|x\| = (\sum_{i=1}^{n} |\xi_i|^2)^{1/2} \tag{1.4}$$

.

Example 1.3.7. *Space $C[a, b]$ is the normed space with a maximum norm given by,*

$$\|\xi\| = max_{t\in K}|\xi(t)| \tag{1.5}$$

where $K = [a, b] \subset \mathbf{R}$.

Definition 1.3.8. *A sequence ξ_k in the normed space $\beth$ is convergent if there is $\xi \in \beth$ such that,*

$$\|\xi_k - \xi\| \to 0 \tag{1.6}$$

as $k \to \infty$.

Definition 1.3.9. *Let $\beth$ be the normed space and the sequence ξ_k is Cauchy if,*

$$\|\xi_m - \xi_n\| \to 0 \tag{1.7}$$

as $m, n \to \infty$.

Definition 1.3.10. *A normed space* $\beth$ *is the complete space if every Cauchy sequence is convergent in* $\beth$*. A complete normed is a Banach space.*

Example 1.3.11. $C[a, b]$ *is the Banach space with a maximum norm given by,*

$$\|\xi\| = max_{t \in K} |\xi(t)| \tag{1.8}$$

where $K = [a, b] \subset R$*.*

1.4 Convexity

Definition 1.4.1. *A subset* $\aleph$ *of* $\mathbf{R^n}$ *is said to be a convex set if for any* $\xi_1, \xi_2 \in \aleph$ *and* $\zeta \in [0, 1]$ *,*

$$\zeta\xi_1 + (1 - \zeta)\xi_2 \in \aleph. \tag{1.9}$$

Definition 1.4.2. *A function* $M : \aleph \to \mathbf{R}$ *is the convex if for any* $\xi_1, \xi_2 \in \aleph$ *and* $\zeta \in [0, 1]$ *,*

$$M(\zeta\xi_1 + (1 - \zeta)\xi_2 \leq \zeta M(\xi_1) + (1 - \zeta)M(\xi_2). \tag{1.10}$$

If the inequality (1.10) *is strict whenever* $\xi_1 \neq \xi_2$ *and* $\mu \in (0, 1)$ *then* M *is strictly convex.*

1.5 Mapping and Fixed Points

Mapping is the way of assigning to each member in first set with the particular member of other set or with same set. For example, a mapping from the set of whole numbers, onto the set of even numbers. In mathematics mappings, map, and transformation are often used differently.

Some important mappings are isometries in geometry, operators in analysis, homeomorphisms in topology, homomorphisms in algebra, representations in group theory.

Definition 1.5.1. *Let* $\daleth \neq \emptyset$ *and* M *be a mapping from* $\daleth$ *to* $\daleth$*. A point* $\xi \in \daleth$ *is said to be a fixed point if* $M(\xi) = \xi$*.*

For example, 0 and 1 are the two fixed point for the mapping $\xi \to \xi^2$ of $\mathbf{R}$ into itself.

Example 1.5.2. *If* M *is defined on a set of natural numbers by*

$M(\xi) = \xi^2 - 3\xi + 4,$

then 2 *is a fixed point for* M*, because* $M(2) = 2$*.*

1.6 Asymptotically Regular Sequences and Maps

Definition 1.6.1. *[7] Let* M *be the mapping from metric space* $\daleth$ *to* $\daleth$ *a sequence* ξ_n *in* $\daleth$ *is asymptotically regular if* $lim_{n\to\infty}\rho(\xi_n, M\xi_n) = 0$*.*

Definition 1.6.2. *[7] A mapping* M *from the metric space* $\daleth$ *to* $\daleth$ *is said to be asymptotically regular at the point* ξ *in* $\daleth$ *if* $lim_{n\to\infty}\rho(M^n\xi, M^{n+1}\xi) = 0$*.*

Definition 1.6.3. *Let* $\aleph$ *be the Banach space. Let* H *and* K *be the two self mappings of the Banach space* $\aleph$*. We can say that the pair* H, K *be weakly commuting if* $|HK\xi - KH\xi| \leq |H\xi - K\xi|$*, for all* $\xi \in \aleph$*. But the converse is not true in general.*

Example 1.6.4. *Let* $\aleph = [0, 1]$ *be the usual metric. Then* H *and* K *are define as*

$H\xi = \frac{\xi}{2+\xi}$, $K\xi = \frac{\xi}{2}$ *for every* $\xi \in \aleph$*.*

Therefore for all $\xi \in \aleph$ $d(HK\xi, KH\xi) = \frac{\xi}{\xi+4} - \frac{\xi}{4+2\xi} = \frac{\xi^2}{(4+\xi)(4+2\xi)}$

$\leq \frac{\xi^2}{4+2\xi} = \frac{\xi}{2} - \frac{\xi}{2+\xi} = d(H\xi, K\xi)$*.*

So H *and* K *are weakly commuting.*

But $HK \neq KH$ *for any non-zero* ξ *in* $\aleph$*.*

1.7 Banach Fixed Point Theorem

In several branches of Analysis, Banach F.P.T plays vital role in research of existence and uniqueness fixed point theorems. Banach fixed point gives the magnificent sketch of the combine power of functional analytic methods and the importance of F.P.T in analysis. Banach F.P.T analyse the definite mappings of complete metric space $\beth$ to itself. Banach F.P.T provides the sufficient condition for the existence and uniqueness of that point which is mapped onto itself. Banach F.P.T also gives the approximation to the fixed point and error bound by using iterative process. This theorem also gives the effective method for achieving better and better approximation to the fixed point and the method is iterative.

Definition 1.7.1. *A mapping M from the metric space $\beth$ to $\beth$ is said to be the Lipschitz if $\exists\ L > 0$ such that,*

$$\rho(M\xi_1, M\xi_2) \leq L\rho(\xi_1, \xi_2) \tag{1.11}$$

for all $\xi_1, \xi_2 \in \beth$.

Definition 1.7.2. *Let $\beth$ be the metric space and the mapping M from $\beth$ to $\beth$ is said to be a contraction on $\beth$ if $\exists\ 0 < L < 1$ such that for all $\xi_1, \xi_2 \in \beth$,*

$$\rho(M\xi_1, M\xi_2) \leq L\rho(\xi_1, \xi_2). \tag{1.12}$$

This means that the ratio $\frac{\rho(M\xi_1, M\xi_2)}{\rho(\xi_1, \xi_2)}$ does not exceed a constant L which is strictly less than 1.

1.8 Continuous Operator

Definition 1.8.1. *Let $H = (H, \rho_1)$ and $K = (K, \rho_2)$ be two metric spaces. A mapping $M : H \to K$ is continuous at h_0, if for every $\epsilon > 0$, there exists δ, such that if $\rho_1(h_0, h) < \delta$, then $\rho_2(M(h_0), M(h)) < \epsilon$.*

Lemma 1.8.2. *If the mapping M is a continuous on the compact set $\beth$, then a mapping M is uniformly continuous on $\beth$.*

Example 1.8.3. *Suppose $M : \beth \to R$ is a continuous function where*

$$\beth = \{(\xi_1, \xi_2) : a \le \xi_1 \le b, c \le \xi_2 \le d\} \tag{1.13}$$

Since $\beth$ is the compact set and M is continuous, hence M is uniformly continuous.

Example 1.8.4. *Let $H = (0, 1]$ and $K = R$ be two sets with usual metric. The mapping M from H to K is defined as $M(h) = \frac{1}{h}$ which is continuous. M is not uniformly continuous because H is not compact.*

Theorem 1.8.1. *If the mapping M is the contraction on a metric space $\beth$, then the mapping M is continuous on $\beth$.*

Proof. Let $\epsilon > 0$ be given and ξ_1 be the any point in $\beth$. Since M is a contraction mapping, we have

$$\rho(M\xi_1, M\xi_2) \le L\rho(\xi_1, \xi_2),$$

for all $\xi_1, \xi_2 \in \beth$ and $L \in [0, 1)$. If $L = 0$, we have $\rho(M\xi_1, M\xi_2) = 0 < \epsilon$ for all $\xi \in \beth$, and M is continuous at ξ_1. Otherwise, for $L \ne 0$ let $\delta = \frac{\epsilon}{L}$ and ξ_2 be any other point in $\beth$ such that $\rho(\xi_1, \xi_2) < \delta$, we have

$$\rho(M\xi_1, M\xi_2) \le L\rho(\xi_1, \xi_2) < L.\delta = L.\epsilon/L < \epsilon.$$

Hence M is continuous at ξ_1 which is an arbitrary point therefore, the contraction map M is continuous everywhere. □

Theorem 1.8.2. *[1]* **(Banach F.P.T)**.
If M is the contraction mapping from the complete metric space $\beth$ to $\beth$. Then mapping M has the unique fixed point $\xi \in \beth$ (that is $M(\xi) = \xi$).

Proof. Choose any $\xi_0 \in \beth$, and define the sequence ξ_n, where
$\xi_1 = M\xi_0$, $\xi_2 = M\xi_1 = M^2\xi_0$, $\xi_3 = M\xi_2 = M^3\xi_0$, . . ., $\xi_n = M\xi_{n-1} = M^n\xi_0$

or

$$\xi_{n+1} = M(\xi_n), n = 0, 1, 2, ... \tag{1.14}$$

The proof strategy is to show that:

1. ξ_n is a Cauchy sequence.

2. its limit is fixed point of $\beth$.

3. And a fixed point ξ is the unique point.

Step 1: From (1.10) and (1.12)we have that

$$\begin{aligned} d(\xi_{m+1}, \xi_m) &= d(M(\xi_m), M(\xi_{m-1})) \\ &\leq Ld(\xi_m, \xi_{m-1}) \\ &= Ld(M(\xi_{m-1}), M(\xi_{m-2})) \\ &\leq L^2 d(\xi_{m-1}, \xi_{m-2}) \\ &. \\ &. \\ &. \\ &\leq L^m d(\xi_1, \xi_0). \end{aligned}$$

Hence by the triangle inequality we get (for $n \geq m$) that

$$\begin{aligned} d(\xi_m, \xi_n) &\leq d(\xi_m, \xi_{m+1}) + d(\xi_{m+1}, \xi_{m+2}) + ... + d(\xi_{n-1}, \xi_n) \\ &\leq (L^m + L^{m+1} + \ + L^{n-1}) d(\xi_1, \xi_0) \\ &= L^m (1 - L^{n-m})/(1 - L) d(\xi_0, \xi_1), \end{aligned}$$

Where in the last equality we have used the summation formula for a geometric series. Since $0 < L < 1$, we have $1 - L^{n-m} < 1$, and consequently

$$d(\xi_m, \xi_n) \leq (L^m)(d(\xi_1, \xi_0))/(1 - L). \tag{1.15}$$

Since $0 < L < 1$ and $d(\xi_0, \xi_1)$ are fixed, it is clear that we can make $d(\xi_m, \xi_n)$ as small as we required by choosing m sufficiently large. This shows that (ξ_n) is the Cauchy squence. Finally, the metric space $\daleth$ is complete, there exists an $\xi \in \daleth$ such that $\xi_n \to \xi$.

Step 2: To show that ξ is a fixed point, we consider the distance $d(\xi, M(\xi))$. From the triangle inequality and (1.10), we get

$$\begin{aligned} d(\xi, M(\xi)) &\leq d(\xi, \xi_m) + d(\xi_m, M(\xi)) \\ &= d(\xi, \xi_m) + d(M(\xi_{m-1}), M(\xi)) \\ &\leq d(\xi, \xi_m) + Ld(\xi_{m-1}, \xi) \ , \end{aligned}$$

and since $\xi_n \to \xi$. It is clear that we can make this distance as small as we please by choosing m sufficiently large. We conclude that

$$d(\xi, M(\xi)) = 0 \Rightarrow M(\xi) = \xi,$$

then $\xi \in \daleth$ is the fixed point of M.

Step 3: Suppose there are two fixed points $\xi = M(\xi)$ and $\xi_0 = M(\xi_0)$. Then from (1.10) it follows that

$$d(\xi, \xi_0) = d(M(\xi), M(\xi_0)) \leq Ld(\xi, \xi_0),$$

which implies $d(\xi, \xi_0) = 0$, since $0 < L < 1$. Hence $\xi = \xi_0$, and ξ is the unique fixed for contraction mapping M. □

Theorem 1.8.3. *[3] (Brouwer's Theorem) There is a fixed point in every continuous mapping from a unit ball of* $\boldsymbol{R}^n$ *into itself.*

Theorem 1.8.4. *[27] (Schauder's Theorem) Let* $\aleph$ *be the Banach space and* H *be the nonempty closed convex bounded subset of* $\aleph$*. Then every continuous compact mapping* $M : H \to H$ *has a fixed point.*

Theorem 1.8.5. *[31] (Tychonoff's Theorem) Let H be the nonempty compact convex subset of the locally convex topological linear space Y and $M : H \to H$ is a continuous mapping. Then M has the fixed point.*

Example 1.8.5. *Let M be the contraction mapping from $\mathbf{R}$ to $\mathbf{R}$ defined as $M(r) = \frac{r}{2}$ for all $r \in \mathbf{R}$. Then for the mapping M, 0 is a unique fixed point.*

Example 1.8.6. *A contraction mapping M from $\mathbf{R}$ to $\mathbf{R}$ is defined as $M(\xi) = \frac{3\xi}{4}+2$ for all $\xi \in \mathbf{R}$ then $\xi = 8$ is the unique fixed point.*

1.9 Common Fixed Points

A point $u \in U$ is the common fixed point for the pair of self mappings (H, K) on U if $H(u) = u = K(u)$.

Example 1.9.1. *Let H and K be the two self mapping on $Y = [1, 2]$ H and K are defined as $H(u) = u$ and $K(u) = u^2$ for all $u \in [1, 2]$. Then 1 is the common fixed point for H and K.*

Chapter 2

Fixed Point Results for Nonexpansive and Mean Nonexpansive Mapping

2.1 Expansive Mapping

Definition 2.1.1. *[33] Let $\aleph$ be a Banach space and $\beth$ be the nonemty subset of $\aleph$ and M is the mapping from $\beth$ to $\beth$ is a expansive map with a constant $C > 1$ such that,*

$$\rho(M\xi_1, M\xi_2) \geq C\rho(\xi_1, \xi_2) \tag{2.1}$$

for all $\xi_1, \xi_2 \in \beth$.

2.2 Nonexpansive Mapping

Definition 2.2.1. *[11] Let $\aleph$ be a Banach space and $\beth$ be the bounded, closed and convex subset of Banach $\aleph$ and the mapping M from $\beth$ to $\beth$ is said to be the non-expansive mapping with the constant $C > 1$ such that,*

$$\rho(M\xi_1, M\xi_2) \leq C\rho(\xi_1, \xi_2), \tag{2.2}$$

for all $\xi_1, \xi_2 \in \beth$.

The study of non-expansive mapping is the one of prominent characteristic work in fixed point theory. A number of the basic properties of a contraction mapping do not carry over to a non-expansive mapping. For example, the presence of the fixed point does not guarantee its uniqueness and a sequence of iterates need not to converge to the fixed point even in the case of compact space. Now it is important to look over either the non-expansive mapping has the fixed points or not. To ensure that the presence of fixed points for such mapping, other limitations should be made on the domain or on the mappings itself.

In a general Banach space there is no fixed point for non-expansive mapping. When we apply some additional conditions such as normal structure and uniform convexity on the given space then it is possible to have a fixed point for the non-expansive mapping. In 1967 the first existing result obtained by Belluce, Lawrence and Kirk [2] for nonexpansive mapping in Banach space. The number of results obtained by many authors on the generalization and extension of nonexpansive mappings. Some of the generalization and extention of nonexpansive mapping can be found in ([14], [12], [15] and [26]) and elsewhere. A class of new nonexpansive type mapping founded in 2008 by Suzuki [30] that is satisfying condition (C) and get some new important result for this type of mapping.

Example 2.2.2. *Translation, isometry and identity mapping these are the nonexpansive mapping.*

Definition 2.2.3. *[19] Let $\daleth$ be the metric space and the mapping M from metric sapce $\daleth$ to $\daleth$ is said to be generalized nonexpansive if,*
for all $\zeta_1, \zeta_2 \in \daleth$, $\rho(M\zeta_1, M\zeta_2) \leq h_1\rho(\zeta_1, M\zeta_1) + h_2\rho(\zeta_2, M\zeta_2) + h_3\rho(\zeta_1, M\zeta_2) + h_4\rho(\zeta_2, M\zeta_1) + h_5\rho(\zeta_1, \zeta_2)$.

Where $h_i \geq 0$, $i = 1, 2, 3, 4, 5$ and $\sum_{i=1}^{5} h_i \leq 1$.

In 1973 Hardy-Rogers [16] introduced the above type of mapping. Then many mathematician studied contractive conditions for single and multi-valued mapping for the generalization of nonexpansive mapping.

2.3 Mean Nonexpansive Mapping

Definition 2.3.1. *[34] Let $\aleph$ be a Banach space and $\beth$ be the closed, bounded and convex subset of the Banach space and the mapping M is mean-nonexpansive mapping from $\beth$ to $\beth$ if for all $\zeta_1, \zeta_2 \in \beth$,*

$$\|M\zeta_1 - M\zeta_2| \leq \beta_1\|\xi_1 - \zeta_2\| + \beta_2\|\zeta_1 - M\zeta_2| \tag{2.3}$$

$$\beta_1, \beta_2 \in [0, 1)\ , \beta_1 + \beta_2 \leq 1.$$

The class of mean nonexpannsive mapping is strictlly larger than the class of nonexpansive, there are some mean nonexpansive mappings such that neither M nor any iterate of M^n is nonexpansive.

Lemma 2.3.2. *Let M be a affine mapping from $\beth$ to $\beth$ is defined as for the multi index $\beta = (\beta_1, \beta_2)$ such that $c = \beta_1^2 + \beta_2 < 1$ and $d = \frac{\beta_1+\beta_2}{\beta_1} > 1$ is $(\beta_1, \beta_2)-$mean nonexpansive.*

Definition 2.3.3. *A mapping M is β-mean Lipschitzian mapping from metric space $\beth$ to $\beth$ with constant $L > 1$,*

$$\sum_{i=1}^{n} \beta_i \rho(M^i\xi_1, M^i\xi_2) \leq L\rho(\xi_1, \xi_2), \tag{2.4}$$

for all $\xi_1, \xi_2 \in \beth$.
Where $\beta = (\beta_1, \beta_2, ..., \beta_n)$ is multi index.
A mapping M is said to be mean Lipschitizian if there is a constan $L > 1$ and some multi-index β satisfying the inequlity (2.4).

Remark 3. *When the mapping M is the uniformly Lipschitzian then M is the β-mean Lipschitzian mapping with same constant L, for every multi-index β.*

2.4 Isometry Mapping

The distance of any two points in a first or original space is equal to the distance of their images in a second space when a metric space $\beth_1$ is mapped onto another space $\beth_2$ or onto itself $\beth_1$.
Condition for the isometry mapping M,

$$\rho(M\xi_1, M\xi_2) = \rho(\xi_1, \xi_2), \tag{2.5}$$

for all $\xi_1, \xi_2 \in \beth$.

Remark 4. *Rotation and translation are the isometries of the plane.*

2.5 Properties of Nonexpansive and Mean Nonexpansive Mapping

Definition 2.5.1. *Let $\aleph$ be a Banach space and $\beth$ be the convex subset of $\aleph$ and M is the mapping from $\beth$ to $\beth$ is affine if $M(\zeta\xi_1 + (1-\zeta)\xi_2) = \zeta M(\xi_1) + (1-\zeta)M(\xi_2)$ for all $\xi_1, \xi_2 \in \beth$ and $\zeta \in [0,1]$.*

Affine mappings plays an important role in the study of fixed point theory. In case of affine mappings, the fixed point property of nonexpansive mapping and mean nonexpansive mapping are identical.

Theorem 2.5.1. *[15] Let $\aleph$ be the Banach space and $\beth$ be the nonempty convex subset of $\aleph$ and $\beth$ verifies the F.P.P for β-mean nonexpansive mappings for all multi indics $\beta = (\beta_1, \beta_2, ..., \beta_n)$ with*

$$\beta_1 \geq \frac{1}{\sqrt[n-1]{2}} \tag{2.6}$$

If the multi-index has length $n = 2$, Theorem 2.5.1 applies when $\beta_1 \geq \frac{1}{2}$. By above Theorem 2.5.1 the following question can be uprised: Is it possible to weaken or eliminate the condition $\beta_1 \geq \frac{1}{\sqrt[n-1]{2}}$?

As a result, the following assumption emerges :
For nonexpansive and mean nonexpansive mapping, the fixed point property is equal for the closed, bounded, and convex subset of the Banach space. The proof of Theorem 2.5.1 strongly depends on forming of a auxiliary mapping M_β. Let $\beth$ be the convex subset, the mapping $M : \beth \to \beth$ and a multi index $\beta = (\beta_1, ..., \beta_n)$, this mapping is defined by $M_\beta(\xi) = \sum_{j=1}^{n} \beta_j M^j(\xi)$ for every $\xi \in \beth$. Then each fixed point of M is the fixed point of M_β, but the converse is not true in general. It proves that M_β is nonexpansive mappping when M is mean nonexpansive and the existence of the fixed point for M_β when $\beta_1 \geq \frac{1}{\sqrt[n-1]{2}}$ shows the existence of the fixed point for the original function M. It is worth pointing out that M_β may be nonexpansive even when M fails to be continuous ([13] Example 4).

Now we see the equivalence between Fixed Point Property (F.P.P) for affine nonexpansive mapping and F.P.P for affine mean-nonexpansive mapping. A mapping M and ξ is in the domain of M, then the $M-$orbit of ξ is defined as the sequence $\xi, M\xi, M^2\xi, M^3\xi, ...$ Let β be the multi index and define M_β as above. The fixed point for M_β does not shows generally the existence of the fixed point for M, even in a case of Lipschitzian mapping.

Example 2.5.2. *Let $\aleph$ is the Banach space, there exist the Lipschitzian mapping $M : \bar{B}_\xi \to S_\xi$, where $\bar{B}_\xi$ is the closed unit ball and S_ξ is the unit sphere of $\aleph$. Define $M : B_\xi \to B_\xi$ and $K = -M$. For $\beta = (1/2, 1/2)$, where M_β is a constant function $M_\beta(\xi) = 0$ for every $\xi \in B_y$, this implies that $Fix(M_\beta) = 0$. But Fix(M) is empty.*

Theorem 2.5.2. *[13] Let $\beth$ be a Topological vector space and Y be the convex subset of $\beth$, M be the mapping from Y to Y and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ is a multi index. Define M_β from V to V such that $M_\beta = \sum_{i=1}^{n} \beta_i M^i$ then the following condition are equivalent:*

1. $Fix(M) \neq \emptyset$.

2. *$Fix(M_\beta) \neq \emptyset$ there exist some $v \in V$ whose M-orbit is bounded and belongs to $Fix(M_\beta)$.*

Proof. For proof see [13]. □

Corollary 2.5.3. *[13] Let $\beth$ be a topological vector space and Y be the convex subset of $\beth$ M be a mapping from Y to Y, $\beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbf{R^n}$ where β is the multi index and $M_\beta = \sum_{j=1}^{n} \beta_j M^j$. If M and M_β commute then $Fix(M) = Fix(M_\beta)$.*

Proof. As we know that every fixed point of the mapping M is the fixed point of M_β here we have to show that for any fix point of M_β is the fix point of M. Let $y \in Fix(M_\beta)$ then for every $n \in N$,

$$M_\beta(M^n y) = M^n(M_\beta y) = M^n y.$$

This shows that the M orbit of y is hold in Fix(M_β) and this implies that $y \in Fix(M)$. □

Definition 2.5.4. *Frechet spaces are the special topological vector space and the generalization of Banach spaces are complete with respect to a metric induced by the norm. All Hilbert and Banach spaces are Frechet spaces. A Frechet space is a complete locally convex sapce over a field **R** or **C**.*

Theorem 2.5.3. *[13] Let F be the Frechet space and Y be the convex subset of F and M be the defined mapping from Y to Y. The following two conditions are equivalent:*

1. *When mappings are affine nonexpansive then Y has fixed point property.*

2. *If the mappings are affine $\beta-$mean nonexpansive then Y has fixed point property for every multi-index β.*

Proof. Case 1 is tirvial when 1 implies 2 because every nonexpansive mapping is β-mean nonexpansive. Now we have to show that 2 impies 1 for this assume that Y has fix point property for β-mean nonexpansive and M_β be a β-mean nonexpansive mapping from Y to Y and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ be the multi index for M_β and M is nonexpansive mapping from Y to Y. By assumption we know that $Fix(M_\beta) \neq \emptyset$ so $y \in Fix(M_\beta)$. As the mapping are affine so by affinity implies that the mapping M and M_β commute. By Corollary 2.5.2 $y \in Fix(M) = Fix(M_\beta)$. □

2.6 Certain Results for $\beta = (\beta_1, \beta_2)$

Goebel and Japon Pineda [15] studied the further class of (β, q)-nonexpansive maps. A mapping $M : U \to U$ is called (β, q)-nonexpansive if, for some $\beta = (\beta_1, \beta_2, ..., \beta_n)$ with $\sum_{k=1}^{n} \beta_k = 1$, $\beta_k \geq 0$ for all $k, \beta_1, \beta_n > 0$, and for some $q \in [1, \infty)$,

$$\sum_{k=1}^{n} \beta_k \|M^k \xi_1 - M^k \xi_2\|^q \leq \|\xi_1 - \xi_2\|^q, \ for\ all\ \xi_1, \xi_2 \in U. \tag{2.7}$$

For simplicity, we generally discuss the case when $n = 2$. That is, $M : U \to U$ is $((\beta_1, \beta_2), q)$- nonexpansive if for some $q \in [1, \infty)$, we have

$$\beta_1 \|M\xi_1 - M\xi_2\|^q + \beta_2 \|M^2\xi_1 - M^2\xi_2\|^q \leq \|\xi_1 - \xi_2\|^q. \tag{2.8}$$

$\forall\ \xi_1, \xi_2 \in U$ When $q = 1$, then M is a (β_1, β_2)-nonexpansive. If the multi-index β is not specified then the mapping M is mean nonexpansive mapping. It is simple to verify that each and every one of them is correct (β, q)-nonexpansive map for $q > 1$ is also β-nonexpansive, but the converse is true in general, there is the mapping which is β-nonexpansive that is not (β, q)-nonexpansive for any $q > 1$ (see [25] for details). It is also easy to see that, by the triangle inequality the mapping $M_\beta := \beta_1 M + \beta_2 M^2$ is nonexpansive if M is (β_1, β_2)-nonexpansive. As noted in [15], however, the nonexpansiveness of M_β is significantly weaker than the nonexpansiveness of M. For example, M_β being nonexpansive does not even guarantee continuity of M. When M is mean nonexpansive, Goebel and Japon Pineda [15] (and later Piasecki [25], Theorems 8.1 and 8.2) were able to use the nonexpansiveness of M_β to prove some interesting results about M, as sum up in a following theorem.

Theorem 2.6.1. *[25] Let S be the compact and convex subset of the Banach space $\aleph$ and $M : S \to S$ is $((\beta_1, \beta_2), q)$-nonexpansive for some $q \geq 1$. Then M has a approximate fixed point sequence, provided that $\beta_2^q \leq \beta_1$ (note that for $q = 1$, this inequality reduces to $\beta_1 \geq \frac{1}{2}$). Furthermore, if (B has a F.P.P for the nonexpansive mapping, then M has a fixed point if $\beta_2^q \leq \beta_1$.*

If $M_\beta = \beta_1 M + \beta_2 M^2$, and so we can write $M_\beta = (\beta_1 I + \beta_2 M) \circ M$, here I is a identity mapping. Here we will discuss properties of a mapping given by $m_\beta := M \circ (\beta_1 I + \beta_2 M)$. This clearly implies that if M is affine then $(\alpha_1 I + \alpha_2 M) \circ M = M \circ (\alpha_1 I + \alpha_2 M)$.

Definition 2.6.1. *A sequence $b = (b_n)$ of complex numbers (or more generally, elements of a Banach space) is called summable if,*

$$N = sup \left\{ \sum_{n \in E} |b_n| : S\ is\ finite\ set \right\} < \infty. \tag{2.9}$$

All summable sequences form a vector space, and N is a norm in this vector space. This vector space is complete, and it is called l_1.
For each summable sequence, the sequence of its partial sums (s_k),

$$s_k = \sum_{n=0}^{\infty} b_n \ , \ k = 0, 1, 2, ...$$

This is the Cauchy sequence and every Cauchy is convergent, so it has a limit. This limit is called the sum of the series,

$$\sum_{n=0}^{\infty} b_n. \tag{2.10}$$

Such series (whose terms form a summable sequence) are also called absolutely convergent.

Definition 2.6.2. *The space of square-summable (complex or real) sequences is the Banach space w.r.t the norm,*

$$\|y\| = (\sum_{j\geq 1} |y_j|^2)^{\frac{1}{2}}. \tag{2.11}$$

Definition 2.6.3. *A normed vector space is uniformly convex space if, for every* $\epsilon \in (0, 2]$ *there exist a some* $\delta > 0$ *such that for any two vectors* $\|\xi\| = 1$ *and* $\|\zeta\| = 1$*, the condition* $\|\xi - \zeta\| \geq \epsilon$ *implies that,*

$$\left\| \frac{\xi + \zeta}{2} \right\| \leq 1 - \delta.$$

Definition 2.6.4. *A Banach space* $\aleph$ *is said to satisfy Opial condition if for any sequence* ξ_n *in* $\aleph$ *such that* $\xi_n \to \xi_0$ *(weakly) it happens that for all* $\xi \in \aleph$, $\zeta \neq \xi_0$,

$$lim_{n\to\infty} inf\|\xi_n - \xi_0\| < lim_{n\to\infty} inf\|\xi_n - \zeta\|. \tag{2.12}$$

2.7 Demiclosedness Principle

Brouder's demiclosedness principle [4] is the important result for the nonexpansive mappings. Which states that let Y be the compact and convex subset of the uniformly convex Banach space $\aleph$ and M is the nonexpansive mapping from Y to $\aleph$. Then $I - T$ is demiclosed for each $\xi \in \aleph$ that is for any sequence $\xi_n \in Y$ and ξ_n converge weakly to ξ and $(I - T)\xi_n \to \zeta$ this implies that $(I - T)\xi = \zeta$. This principle has the key role for the study of asymptotic behaviour of nonexpansive mapping.

Theorem 2.7.1. *[12] Let* U *be the compact and convex subset of the Banach space* $\aleph$ *and* M *be the* (β_1, β_2)*-nonexpansive mapping from* U *to* U *then* u_n *and* v_n *are the sequences in* U *such that,*

$$\|M(\beta_1 u_n + \beta_2 v_n) - u_n\| \to 0 \text{ and } \|M^2(\beta_1 u_n + \beta_2 v_n) - v_n\| \to 0. \tag{2.13}$$

Particularly,

$$\|Mu_n - v_n\| \to 0 \text{ and } \|M(\beta_1 u_n + \beta_2 M v_n) - v_n\| \to 0. \tag{2.14}$$

Or u_n is a approximate fixed point sequence for $m_\beta := M \circ (\beta_1 I + \beta_2 M)$.

Theorem 2.7.2. *[12] Let $\aleph$ be the Banach space and U be the compact and convex subset of the Banach space $\aleph$ and M is the $((\beta_1, \beta_2), q)$-nonexpansive for some $q \geq 1$. Then u_n and v_n are the sequences satisfying (2.13) and (2.14) from the Theorem 2.7.1. and sequence u_n is the approximate fixed point sequence for $m_\beta := M \circ (\beta_1 I + \beta_2 M)$.*

Corollary 2.7.1. *Let U be the compact and convex subset of the Banach space $\aleph$ and $M : U \to U$ is $((\beta_1, \beta_2), q)$-nonexpansive. Then M_β has the approximate fixed point sequence w_n, where w_n is defined as,*

$$w_n := \beta_1 u_n + \beta_2 M u_n \tag{2.15}$$

From the Theorem 2.7.2 w_n is a approximat fixed point for $m_\beta := M \circ (\beta_1 I + \beta_2 M)$.

Corollary 2.7.2. *[12] Let U be the compact convex subset of the Banach space $\aleph$, if M is $((\beta_1, \beta_2), q)$-nonexpansive mapping from U to U and $(U^2, \|.\|_{\beta,q})$ admits a F.P.P for nonexpansive mapping then $\exists\ u, v \in U$ such that,*

$$M(\beta_1 u + \beta_2 v) = u \text{ and } M^2(\beta_1 u + \beta_2 v) = v \tag{2.16}$$

In particular,

$$Mu = v \text{ and } M(\beta_1 u + \beta_2 M u) = v \tag{2.17}$$

$\Rightarrow m_\beta$ has the fixed point.

Corollary 2.7.3. *[12] Let U be the compact convex subset of the Banach space $\aleph$, and M is (β_1, β_2)-nonexpansive mapping from U to U and suppose that $\aleph$ has the fixed point then $M_\beta = \beta_1 M + \beta_2 M^2$ has one fixed point v and V is defined as,*

$$v = \beta_1 u + \beta_2 M u \tag{2.18}$$

for some $u \in U$.

Now we discuss the results for (β, q)-nonexpansive mapping with the arbitrary length of β.

Theorem 2.7.3. *[12] Let $\aleph$ be the Banach space and U be the convex and compact subset of $\aleph$ and M is (β, q)-nonexpansive mapping from U to U, for $q \geq 1$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ each $\beta_i > 0$. Then $\exists$ sequence $u_m^i \in U$, for i = 1, 2, . . . , n such that,*

$$\|M(\beta_1 u_m^1 + \beta_2 u_m^2 + ... + \beta_n u_m^n) - u_m^1\| \to 0$$

$$\|M^2(\beta_1 u_m^1 + \beta_2 u_m^2 + ... + \beta_n u_m^n) - u_m^2\| \to 0$$

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$$\|M^n(\beta_1 u_m^1 + \beta_2 u_m^2 + ... + \beta_n u_m^n) - u_m^n\| \to 0$$

In particular,

$$\|M(\beta_1 u_m^1 + \beta_2 M u_m^1 + ... + \beta_n M^{n-1} u_m^1) - u_m^1\| \to 0.$$

Then $m_\beta = M \circ (\beta_1 I + \beta_2 M + ... + \beta_n M^{n-1})$ has the approximate fixed point sequence.

Theorem 2.7.4. *[12] Let $\aleph$ be a Banach space and U is a closed, bounded and convex subset of $\aleph$ and $(\aleph^n, \|.\|_{\beta,q})$ has the fixed point for the nonexpansive mapping. If M is a (β, q)-nonexpansive mapping for $q \geq 1$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ each $\beta_i > 0$ then*

$\exists\ u_1, u_2, ..., u_n \in U$ *such that*

$$M\overline{u} = u_1$$
$$M^2\overline{u} = u_2$$
$$.$$
$$.$$
$$.$$
$$M^n\overline{u} = u_n.$$

Where $\overline{u} = \beta_1 u_1 + \beta_2 u_2 + ... + \beta_n u_n$. *In particular,* $m_\beta u_1 = u_1$.

Corollary 2.7.4. *[12] Let* U *be the closed, bounded and convex subset of the Banach space* $\aleph$ *and* M *is the* (β, q)*-nonexpansive mapping for* $q \geq 1$ *and* $\beta = (\beta_1, \beta_2, ..., \beta_n)$ *each* $\beta_i > 0$. *Then* M_β *has the approximate fixed point sequence* w_n *and* w_n *is defined as,*

$$w_n := \beta_1 u_n + \beta_2 M u_n + ... + \beta_n M^{n-1} u_n. \tag{2.19}$$

By Theorem 2.7.3 w_n *is the approximate fixed point for* m_β. *Further, suppose that for nonexpansive mapping* $(\aleph^n,\ \|.\|_{\beta,q})$ *has the F.P.P, then* M_β *has a fixed point* w *and* w *is defined as,*

$$w := \beta_1 u + \beta_2 M u + ... + \beta_n M^{n-1} u. \tag{2.20}$$

By Theorem 2.7.4 u *is the fixed point of* m_β.

Chapter 3

Generalization of Mean-Nonexpansive Semigroup and Fundametally Nonexpansive Mapping

3.1 Mean Nonexpansive Semigroup

Definition 3.1.1. *Let $\aleph$ be the Banach space and U be the closed and convex subset of $\aleph$ has normal structure if every compact and convex subset E of U with $|E| > 1$ hold a point ξ such that sup$\{\|\xi - \zeta\| : \zeta \in E\} < \delta(E)$.*

Theorem 3.1.1. *[19] Let U be the nonempty closed and convex subset of Banach space $\aleph$, if U is the weakly compact and has normal structure, then U has a fixed point property.*

Remark 5. *Every compact and convex U subset of a Banach space $\aleph$ every time has the normal structure.*

Theorem 3.1.2. *[10] Let $\aleph$ be the Banach space and U the weakly compact and convex subset of $\aleph$ and M be the nonexpansive mapping from U to U such that each weakly compact convex subset of U having F.P.P for mapping M if the sequence of iterates of M is bounded, then a nonexpansive mapping M has a fixed point.*

Dotson and Mann [10] proved this theorem by taking $\aleph$ as a uniformly convex. Then Simeon Reich [26] prvoed the above theorem in two different ways,

1. By using iteration scheme of Mann [22].
2. By using nonlinear nonexpansive semigroup [4].

Theorem 3.1.3. *[26] Let $\aleph$ be the Banach space and S be the boundedly weakly compact and convex subset of $\aleph$. If M is the nonexpansive mapping from S to S and each bondedly weakly compact and convex subset of S hold a fixed point property for the nonexpansive mapping M. If T is a bounded sequence for some y_0 in S, then M has the fixed point.*

The above sequence T is defined as, let $B = \{b_{jk} : j, k \in N\}$ be a infinite matrix which satisfy the following properties,

$$b_{jk} \geq 0 \; for \; all \; j, k \in N$$
$$b_{jk} = 0 \; if k > j$$
$$\sum_{k=0}^{j} b_{jk} = 1 \; for \; all \; j \in N$$
$$lim_{j\to\infty} b_{jk} = 0 \; for \; all \; k \in N.$$

If $y_0 \in S$, then the sequence $T = \{y_j : j \in N\} \subset S$ which is defined as

$$y_j = b_{j0}y_0 + \sum_{k=1}^{j} b_{jk}My_{k-1}, j \in N. \tag{3.1}$$

Simeon Reich [26] used this above Mann [22] iteration scheme to prove Theorem 2.8.3. The proof of Theorem 2.8.3 shows the set of asymptotic centers of the sequence T with respect to S which is remained unchanged under the nonexpansive mapping M. This is a fixed point set for nonexpansive mapping M from S to S. If we take the space is uniformly convex then the mapping M also shows the set of asymptotic centers.

Let $y_1 \in S$ is a arbitrary element then we can defined this iteration process,

$$y_{k+1} = M(u_k), \tag{3.2}$$

where

$$u_k = \sum_{j=1}^{k} b_{jk} y_k.$$

Theorem 3.1.4. *Let u_k and y_k are the sequences if one them is converges then the other sequence is conveges to same point then the common limit of thsese sequences is the fixed point of M.*

Definition 3.1.2. *Let U be the boundedly weakly compact and convex subset $\exists$ the point $\xi \in U$ such that $limsup_{j\to\infty}|\xi - \zeta_j| = R < \infty$ this point ξ is said to be the asymptotic center of a sequence T with respect to U.*

Simeon Reich [26] also proved the Theorem 2.8.3 by using nonlinear nonexpansive semigroup that is given by Browder [4].

Definition 3.1.3. *[26] Let E be the subset of Banach space $\aleph$. A function $D : [0,\infty) \times E \to E$ is nonexpansive semigroup if satisfying following properties:*

1. $D(\xi_1 + \xi_2, y) = D(\xi_1, D(\xi_2, y))$, *for* $\xi_1, \xi_2 \geq 0$ *and* $y \in E$.
2. $|D(\xi, y_1) - D(\xi, y_2)| \leq |y_1 - y_2|$ *for* $\xi \geq 0$ *and* $y_1, y_2 \in E$.
3. *D(0 , y) = y for $y \in E$.*

If for every $y \in E$ there is $T(y)$ such that $|D(\xi, Y)| \leq T(y)$ for all $\xi \geq 0$ then D is said to be a bounded semigroup for each $y \in E$ and y_0 is said to be the fixed point if $d(\xi, y_0) = y_0$ for all $\xi \geq 0$.

Theorem 3.1.5. *[26] Let $\aleph$ be the Banach space and U be the boundedly weakly compact convex suabset of $\aleph$ and for nonexpansive mapping each subset of U hold the* *common fixed point proprty. If $D : [0, \infty) \times U \to U$ is the nonexpansive semigroup, if D is bounded then D has the fixed point.*

Definition 3.1.4. *Let $\aleph$ a Banach space and U be the nonempty subset of $\aleph$ define a function $K : [0, \infty) \times U \to U$ mean-nonexpansive semigroup if the following properties are satisfied:*

$$\beta_1, \beta_2 \in (0, 1)$$

$$K(\beta_1 t_1 + \beta_2 t_2,\ \xi) = K(\beta_1 t_1,\ K(\beta_2 t_2,\ \xi))$$

for

$$t_1, t_2 \geq 0,\ \xi \in U,$$

$$|K(\beta t,\ \xi_1) - K(\beta t,\ \xi_2)| \leq |(\xi_1 - \xi_2)|,$$

$\beta \in (0, 1)$ for $t \geq 0$ and $\xi_1, \xi_2 \in U$,

$$K(\alpha(0),\ \xi) = \xi,$$

or

$$K(0,\ \xi) = \xi,\ for\ \xi \in Y.$$

A mean-nonexpansive semigroup K is called bounded if for each ξ in $\aleph$ there is $M(\xi)$ such that,

$$|K(\beta t,\ \xi)| \leq M(\xi),$$

for all $t \geq 0$ and $\beta \in (0,\ 1)$.

Theorem 3.1.6. *Let $\aleph$ be a Banach space and U be the bounded weekly compact convex set of t $\aleph$. Suppose each weakly compact and convex subset of U hold the common F.P.P for mean-nonexpansive mapping and defined a function K such that $K : [0\ ,\ \infty) \times U \to U$ is a mean-nonexpansive semigroup. If K is buonded,then K has a fixed point.*

Proof. Let ξ_0 is the fix point in U and ζ be the other point in U and T is finite

$$T = lim_{t\to\infty} sup|\zeta - K(\beta t,\ \xi_0)| < \infty. \tag{3.3}$$

We defined an orbit which is bounded $\Big[K(\beta t\ ,\ \xi_0) : t \geq 0,\ \beta \in (0\ ,1)\Big]$,
let

$$F = \Big[z \in U : lim_{t\to\infty} sup|z - K(\beta t\ ,\ \xi_0)| \leq T\Big], \tag{3.4}$$

where $F \neq \Phi$ and F is a closed and convex subset of U.
If $z \in F$, $t_0 \geq 0$, $\epsilon \geq 0$ and $\beta_1, \beta_2 \in (0\ ,1)$ and t is large enough, then we have
$|K(\beta_1 t_0\ ,\ z) - K(\beta_2 t\ ,\ \xi_0)|$

$$\begin{aligned}
&= |K(\beta_1 t_0\ ,\ z) - K(\beta_1 t_0\ ,\ K(\beta_2 t - \beta_1 t_0\ ,\ \xi_0))| \\
&\leq |z - K(\beta_2 t - \beta_1 t_0\ ,\ \xi_0)| \\
&\leq T \\
&< T + \epsilon\ .
\end{aligned}$$

$\Rightarrow \quad K(\beta t_0, z) \in F.$

Thus K is invariant under the commuting family of mean-nonexpansive mapping $\Big[K(\alpha t, \bullet) : t \geq 0, \alpha \in (0, 1)\Big]$.
This completes the proof. □

3.2 Fundamentally Nonexpansive Mapping

Definition 3.2.1. *Let $\aleph$ be a Banach space and U be the nonempyt subset of $\aleph$ and the mapping M from U to U is said to be Fundamentally nonexpansive if,*

$$|M^2\xi_1 - M\xi_2| \leq |M\xi_1 - \xi_2|\ , \tag{3.5}$$

for all $\xi_1, \xi_2 \in U$.

Theorem 3.2.1. *Let U be the nonempty subset of the Banach space $\aleph$ and U is the bounded weakly compact and convex subset of $\aleph$ and suppose that each weakly compact convex subset of U hold common F.P.P for fundamentally non-expansive maping and a mapping $M : U \to U$ is fundamentally nonexpansive and a sequence ζ_n be the bounded sequence in U for some ζ_0 in U. Then M has a fixed point.*

Proof. Let $\zeta \in U$ and the set

$$T = lim_{t\to\infty} sup|\zeta - \zeta_n|. \tag{3.6}$$

T is finite because the sequence ζ_n is bounded. Suppose $B = (b_{nk} : n, k \in N)$ is a infinite matrix satisfy the following properties

$$b_{nk} \geq 0 \ \forall \ n, k \in N,$$

$$b_{nk} = 0 \ if \ k > n,$$

$$\sum_{k=0}^{n} b_{nk} = 1 \ \forall \ n \in N,$$

$$lim_{n\to\infty} b_{nk} = 0 \ \forall \ k \ \in N.$$

If $\zeta_0 \in U$ then the sequence ζ_n can be defined as

$$\zeta_n = b_{n_0}\zeta_0 + \sum_{n=0}^{\infty} b_{nk} M_{\zeta_{k-1}}, \tag{3.7}$$

and $n \in N$.

This iteration is due to Mann [22] and T is finite because the sequence is bounded. Let

$$K = [z \in U : lim_{n\to\infty} sup|z - \zeta_n| \leq T], \tag{3.8}$$

$K \neq \phi$, $K < \infty$

$\Rightarrow K$ is the closed and convex subset of U.

Then by fundamentally non expansive for any $\zeta_1, \zeta_2 \in Y$

$$\|M^2\zeta_1 - M\zeta_2\| \leq \|M\zeta_1 - \zeta_2\|, \tag{3.9}$$

for any $z \in K$

$$\|M^2 z - M\zeta_n\| \leq \|Mz - \zeta_n\|, \tag{3.10}$$

$$\begin{aligned}
&\leq \|Mz - (b_{n_0}\zeta_0 + \sum_{k=1}^{n} b_{n_k} M\zeta_{k-1})\| \\
&\leq b_{n_0}|Mz - \zeta_0| + \sum_{k=1}^{n} b_{n-k}|Mz - M_{\zeta_k - 1}| \\
&\leq b_{n_0}|Mz - \zeta_0| + \sum_{k=1}^{n} b_{n_k}|z - \zeta_{k-1}|
\end{aligned}$$

for any $\epsilon > 0 \ \exists \ m(\epsilon) \in N$ such that

$$|z - \zeta_k| \leq T, \tag{3.11}$$

$$|z - \zeta_k| \leq T + \epsilon,$$

then we get $n > m + 1$

$$|M^2 z - M\zeta_n| \leq b_{n_0}|Mz - \zeta_0| + \sum_{k=1}^{m} b_{n_k}|z - \zeta_{k-1}| + \sum_{k=m+2}^{n} b_{n_k}(T + \epsilon) \tag{3.12}$$

$$\begin{aligned}
&\leq b_{n_0}|Mz - \zeta_0| + \sum_{k=1}^{m} b_{n_k}|z - \zeta_{k-1}| + T + \epsilon \\
&= h(n) + T + \epsilon,
\end{aligned}$$

where $lim_{n\to\infty} h(n) = 0$

$\Rightarrow M^2 z \in K.$ □

Corollary 3.2.2. *Let $\aleph$ be the Banach space and U be the bounded weakly compact and convex subset of $\aleph$ and M is the mean-nonexpansive mapping from U to U then the mapping M has an approximate fixed point sequence in U.*
Mathematically, for any arbitrary sequence $\zeta_n \in U$

$$lim_{n\to\infty}\|M\zeta_n - \zeta_n\| = 0. \tag{3.13}$$

Theorem 3.2.2. *Let $\aleph$ be the Banach space and U be the bounded weakly compact and convex subset of $\aleph$ and each weakly compact convex subset of U hold the F.P.P and a mapping M from U to U for any $\zeta_1, \zeta_2 \in U$,*

$$\|M\zeta_1 - M\zeta_2\| \leq \beta_1\|\zeta_1 - \zeta_2\| + \beta_2\|\zeta_1 - M\zeta_2\| \tag{3.14}$$

$\beta_1, \beta_2 > 0$, $\beta_1 + \beta_2 \leq 1$ each $\beta_i \in (0, 1)$.
Then the mapping M has the fixed point.

Proof. As the mapping M is a mean-nonexpansive mapping by using corollary 3.2.2 every mean-nonexpansive mapping has a approximate fixed point sequence in bounded weakly compact convex subset of the Banach space. Let ζ_n be a approximate fixed point sequence in U then

$$lim_{n\to\infty}\|M\zeta_n - \zeta_n\| = 0 \tag{3.15}$$

Since U is the weakly compact convex subset of a Banach space $\aleph$.
$\Rightarrow$ Every convergent sequence ζ_n of U has convergent subsequence $\zeta_{n_k} \subset \zeta_n$.
As ζ_n is convergent in U then the subsequence ζ_{n_k} is weakly convergent to $\zeta_0 \in U$ now we have to show that ζ_0 is a fixed point for mapping M for this $\zeta_0 = M\zeta_0$ we will proof this by taking a contradiction $\zeta_0 \neq M\zeta_0$,
then

$$lim_{n\to\infty}inf\|\zeta_n - \zeta_0\| < lim_{n\to\infty}inf\|\zeta_n - M\zeta_0\| \tag{3.16}$$

$$= lim_{n\to\infty}inf\|M\zeta_n - M\zeta_0\|,$$

since M is a mean-nonexpansive mapping so by definition we have

$$lim_{n\to\infty}inf\|M\zeta_n - M\zeta_0\| \leq lim_{n\to\infty}\Big[\beta_1\|\zeta_n - \zeta_0\| + \beta_2\|\zeta_n - M\zeta_0\|\Big]$$
$$< lim_{n\to\infty}inf\Big[\beta_1\|\zeta_n - \zeta_0\| + (1-\beta_1)\|\zeta_n - M\zeta_0\|\Big]$$

$$\Rightarrow lim_{n\to\infty}inf\|\zeta_n - M\zeta_0\| \leq lim_{n\to\infty}inf\|\zeta_n - \zeta_0\|,$$

this is a contradiction to eq.(3.16)

Therefore ζ_0 is the fixed point of mapping M. □

Chapter 4

Summary

This chapter finishes up the thesis by expressing and summarizing the inferences and findings. The knowledge assists the reader to understand the essence of Fixed Point Theorems. From the above mentioned results of nonexpansive mapping and mean nonexpansive mapping, we see that in most of the cases we have studied Fixed Points for different types of mapppings. The objective of following work is to find the Fixed Point for Fundamentally Nonexpansive Mapping and Mean Nonexpansive Semigroup.

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