Generalized Dispersion Relation For Electron Bernstein Waves in non-Maxwellian Magnetized anisotropic Plasma

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Abstract

A generalized dielectric constant for electron Bernstein waves using non-Maxwellian distribution function is derived in a collisionless, uniform magnetized plasma. Using Neumann's series expansion for the product of Bessel's function, we can derive the dispersion relations for both kappa distribution and generalized(r,q) distribution in a straight forward manner. The dispersion relation now become dependent on spectral indices κ and (r,q) for kappa and generalized (r,q) distribution respectively. Our results show how the non-Maxwellian dispersion curve deviates from the Maxwellian depending upon the values of spectral indices chosen. It may be noted that (r,q) distribution is reduced to Kappa distribution at $r = 0$ and $q = \kappa + 1$ which is further reducible to Maxwellian distribution for $\kappa \to \infty$.

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Chapter 1

Introduction

Naturally, the existence of matter is in four states: plasma, gas , liquids and solids. In solids, atoms are tightly arranged together in regular patterns. Atoms have strong intermolecular bonding and solids hold their shape. When energy is given to the solid then these atoms break their bonds which hold them together or in shape. There is a phase transition occur from solids to the liquid phase. In liquids, atoms are close together with no regular arrangements and liquid don't hold their shape. When more energy is provided to the system then the atoms spread and move freely. The phase transition occurs from liquid to gas. When even more energy is given then electrons from the atoms free themselves and form a gas which is ionized. In quasi-neutral condition, ions, electrons and neutral atoms constitute the ultimate state of matter called plasma. There is 95 percent of plasma in the universe. There are various types of plasmas.[1]

1.1 Types of Plasma

1.1.1 Unmagnetized Plasma

Unmagnetized plasma is generally isotropic. There is no equilibrium magnetic field. The main parameter contributes to inhomogeneity is plasma density. Unmagnetized plasma supports high-frequency electromagnetic waves and sound waves. In cold plasma, at plasma frequency sound waves became an oscillation of it and electromagnetic waves do not propagate below the plasma frequency. [2]

1.1.2 Magnetized Plasma

Addition of magnetic field effects to the essence of plasma waves which adds new events of anisotropy and new kind of transverse waves exists only in magnetized plasma which are Alfven waves. Finite Larmour orbit effects due to thermal motions about the magnetic field lines. As the field strength increases Larmour orbits become more tightly wound. There are no thermal effects present in cold plasma and many kinds of waves form which are introduced by addition of magnetic field and varies greatly with magnetic field and angle of propagation. Magnetic field not just only introduce a new cause of inhomogeneity but gradient may also appear in different directions.[3][4].

1.1.3 Thermal Plasma

When thermal effects are introduced in cold plasma effects then two new phenomena forms. One is Acoustic waves and the other is kinetic phenomena. Acoustic wave phenomena is due to various kinds of sound waves. Kinetic phenomena is due to this fact that in a thermal distribution or the distribution which is nearly thermal, then there are particles which are moving at phase velocity. Such particles resonantly interact with waves and this is because of their long contact time with waves. These interactions may lead towards either collision-free damping or growth of the wave. When it is coupled with effects of magnetic field, finite Larmour orbit effects leads towards new kinds of instabilities.

1.2 Waves in Plasma

There are many types of waves in plasma, depending upon the direction of propagation with respect to electric field and magnetic field. The waves exist in plasma are perpendicular, parallel, longitudinal, transverse, electrostatic and electromagnetic.

Parallel propagating waves $\rightarrow k \parallel B_0$, Perpendicular propagating waves $\rightarrow k \perp B_0$ Longitudinal waves $\rightarrow k \parallel E_1$, Transverse waves $\rightarrow k \perp E_1$ Electrostatic waves $\rightarrow B_1 = 0$ and $k \parallel E_1$, Electromagnetic waves $B_1 = 0$ and $k \parallel E_1$,

where E_0, B_0 the ambient electric field and magnetic field and E_1 is perturbed electric field and B_1 is perturbed magnetic fields and **k** is the propagation vectors of the wave. Following terminology is usually used in plasma dynamics.[5] The wave we are studying is Electron Bernstein waves.

1.2.1 Plasma Oscillation

Plasma has a property to restore charge neutrality. If we have a uniform plasma which is made up of electrons and ions, the mass of ions is very large compared to the mass of electrons, so ions can be considered stationary. When electrons are displaced from their mean position by any means, electric field will be developed between the stationery ions and the displaced electrons. Under the influence of this field, the electrons will move towards the stationery ions. Electrons do not stop at their equilibrium position because of inertia and starts oscillations about their mean position. The frequency with which electrons will oscillate is known as plasma frequency. The plasma frequency will be given by the following relation

$$\omega_{pe} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}}, \tag{1.1}$$

where n_0 is the number density of plasma, e is the charge of an electron, m_e is the mass of an electron and ϵ_0 is the permittivity of free space. The number density of plasma is directly depending on the plasma frequency. Higher the number density of plasma greater will be the the plasma frequency.

1.2.2 Electron Plasma waves

Consider the thermal motion of electrons, plasma oscillations propagate with thermal velocity and carry information about the oscillating region. These are called electron plasma waves with the following dispersion relation,[6]

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{th}^2, \tag{1.2}$$

where ω is the wave frequency, ω_{pe} is the plasma frequency, k and v_{th} is the thermal velocity of electrons. When we consider the thermal motion of electrons, the wave frequency depends on the wavenumber k and group velocity $\frac{d\omega}{dk} \neq 0$. Information carried by wave travel from one region to another region with a group velocity, so group velocity v_g can not exceed from the speed of light c. In figure(1), the slope of the tangent at any point P, gives us group velocity v_g and slope of any point, P on the curve drawn from origin gives us phase velocity v_ϕ. From graph, we also see that the slope of $\sqrt{\frac{3}{2}} v_{the}$ is also greater than the slope of the tangent at any point P. So the above equation holds only when

$$v_\phi > \sqrt{\frac{3}{2}} v_{the} > v_g. \tag{1.3}$$

For large k, $v_g \approx v_{the}$ and for small k, $v_g < v_{the}$.

1.2.3 Electrostatic electron waves perpendicular to B

We are dealing with high-frequency response particles like electrons. Ions are massive so they are considered stationary and create a uniform background of the positive charge. The electron oscillation perpendicular to $\boldsymbol{B}_0$ means direction of propagating vector k is perpendicular to $\boldsymbol{B}_0$. Since we are considering electrostatic case so $\boldsymbol{B}_1 = 0$ and k parallel to $\boldsymbol{E}_1$

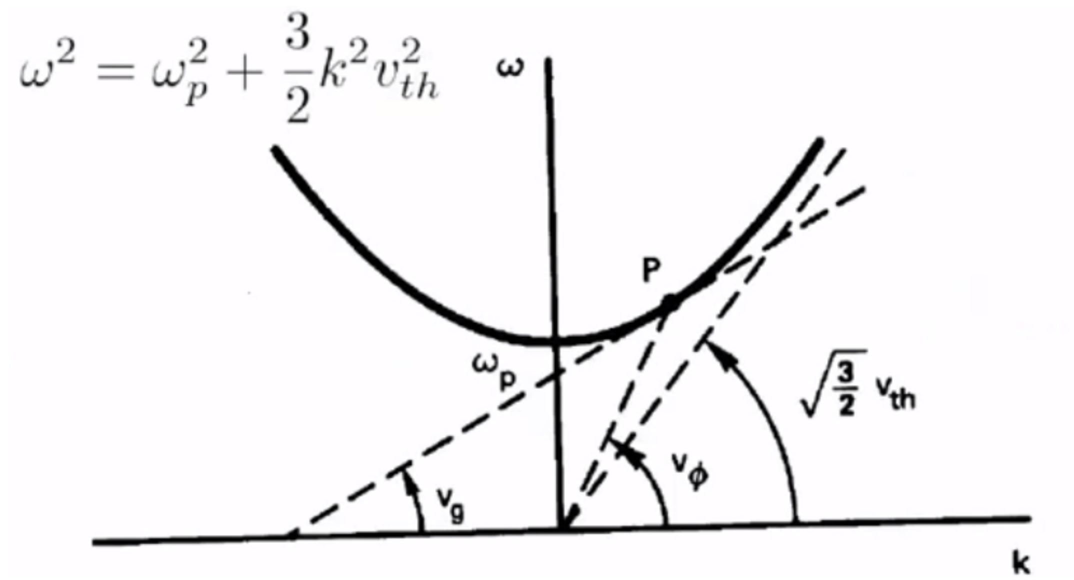

Figure 1.1: Dispersion relation for electron plasma waves.

so these are longitudinal plane wave propagating perpendicular to B_0. Under the following assumptions $E_0 = 0$, $v_0 = 0$, $k_B T_e = 0$, $\nabla n_0 = 0 = \frac{\partial n_0}{\partial t}$, we got the dispersion relation for electrostatic electron waves perpendicular to B_0,

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 = \omega_h^2, \tag{1.4}$$

where ω_{pe} is the plasma frequency and ω_{ce} is the cyclotron frequency of electrons.

These can be defined as

$$\omega_{pe} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}}, \tag{1.5}$$

and

$$\omega_{ce} = \frac{B_0 e}{m_e}. \tag{1.6}$$

Upper hybrid frequency is shown as " ω_h". It is a "hybrid" frequency because it is a mixture of plasma frequency and cyclotron frequency. Electrostatic electron oscillations which are perpendicular to B_0 have an upper hybrid frequency and those which are moving along B_0 have only plasma frequency. Magnetic field exerts a force on electrostatic electron waves which are propagating perpendicular to magnetic field B_0, and changes their direction into an elliptical path, instead of oscillating along a straight line. When the electrons are displaced from their mean position, the electric field will be developed in such direction that it opposes the motion of electrons, but initially magnetic forces are stronger as compared to the electric force and motion of particles is governed due to magnetic forces. When the speed of particles increases Lorentz force also increases. As the motion of particles in against the electric field so they lose energy. Lorentz force and electrostatic force are acting on particles which are propagating perpendicular to magnetic field. This additional Lorentz force gives an increase in the frequency.

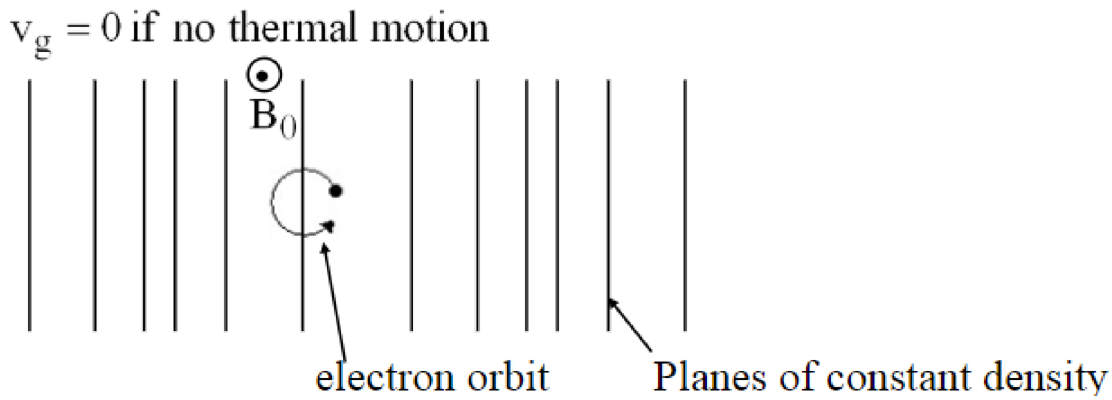

Figure 1.2: Motion of electrons in upper hybrid oscillation.

1.2.4 Electrostatic Ion wave

The frequency of electrostatic electron wave is very large as compared to both the plasma and cyclotron frequency but the response of ions in a field is very small due to the large mass. Therefore electrostatic ion waves are the lower frequency case. Here we discuss ions oscillation almost perpendicular to $\boldsymbol{B}_0$. Mean direction of propagation vector k is almost perpendicular to $\boldsymbol{B}_0$. Here we let $E_0 = 0$, $v_0 = 0$, $k_B T_i = 0$, $\nabla n_0 = 0 = \frac{\partial n_0}{\partial t}$. For nearly perpendicular propagation of a wave, for small but non-zero k_z the electron can oscillate in z-direction but due to large inertial effects ions cannot oscillate along with $\boldsymbol{B}_0$, so we can set $k_z \approx 0$ for the ions. The dispersion relation for electrostatic ion waves perpendicular to $\boldsymbol{B}_0$ is given as

$$\omega^2 = \omega_{ci}^2 + k^2 v_s^2, \tag{1.7}$$

where ω_{ci} is the ions cyclotron frequency, given as

$$\omega_{ci} = \frac{e B_0}{M} \tag{1.8}$$

and v_s is the ions acoustic speed, given as

$$v_s = \frac{k_b T_e}{M}. \tag{1.9}$$

The ions moving parallel to $\boldsymbol{B}_0$ have frequency $\omega^2 = k^2 v_s^2$, it is same as electrostatic ion wave parallel $\boldsymbol{B}_0$ and the additional cyclotron frequency is due to Lorentz force.

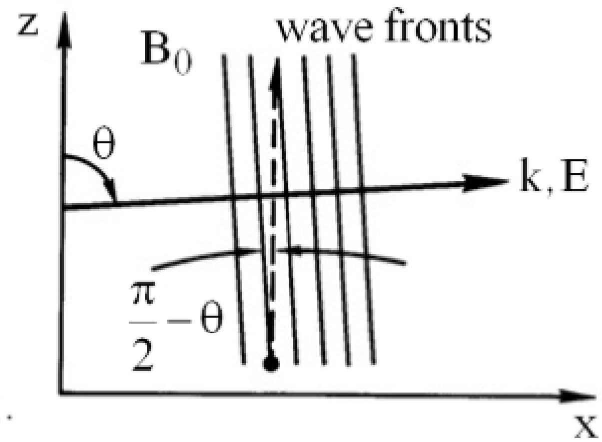

Figure 1.3: Geometry of an electrostatic ion cyclotron wave propagating nearly at right angles to B_0.

1.3 Different theoretical approaches

There are different theories to understand the plasma wave depending upon the study of interest and which theory is more efficient to describe the phenomena under study. Here we are discussing two theories of plasma which are more general and most of the researchers use to study the dynamics of plasma. First one is the fluid theory and the second is the kinetic theory.

1.3.1 Fluid theory

Plasma is made up of different types of particles having different masses and charges. It is hard to study the dynamics of each particle. To overcome this difficulty, we consider each species as a fluid. By considering species as fluid identity of individual particles are neglected so one can study the collective behavior of each species. In fluid theory we average over velocity, density and temperature. Fluid species behaves as a continuous medium, such that all amounts depend on time and position. We obtain fluid equation by taking moments of Boltzmann equation and average over velocity space.

1.3.2 Kinetic theory

Most of the plasma phenomena are accurately described by the fluid theory but for some phenomena fluid theory is inadequate, to deal with those phenomena we use velocity distribution $f(\vec{v})$ for each species, this treatment is called the Kinetic theory. The distribution function depends upon the seven independent variables, three for the position, three for velocity and one for time $f(\vec{r}, \vec{v}, t)$. We get more information about the plasma, when we use kinetic theory instead of fluid theory. The distribution function $f(\vec{r}, \vec{v}, t)$ gives us information about particles per unit volume at position r and time t with velocity component between v_x and $v_x + dv_x$, v_y and $v_y + dv_y$ and v_z and $v_z + dv_z$ is

$$f(x, y, z, v_x, v_y, v_z, t)dv_x dv_y dv_z. \tag{1.10}$$

By integrating all over viable velocities, we obtain the density of particles in given volume

$$n(\vec{r}, t) = \int_{-\infty}^{\infty} f(\vec{r}, \vec{v}, t)d^3v. \tag{1.11}$$

1.3.3 Example of different distribution functions

To study the properties of plasma, we use different distribution functions, it depends upon the environment of plasma. Here, we study two types of distribution functions one is the Maxwellian-Boltzmann distribution and the other one is a non-Maxwellian distribution function.

1.3.4 Maxwellian Boltzmann distribution

We use Maxwellian-Boltzmann distribution for uniform and isotropic plasma, and it is independent of "t". It is a classical distribution function, so one can find any number of particles in any state. When we treat with classical plasma we use Maxwellian-Boltzmann distribution.[7] It can be defined as

$$f_{0s} = n_{0s}\Big(\frac{1}{v_{th}\sqrt{\pi}}\Big)^3 e^{\left(-\frac{v_s^2}{v_{ths}^2}\right)}. \tag{1.12}$$

Therefore, the velocity distribution of charged particles is described by the Maxwell Boltzmann velocity distribution function. The Maxwellian-Boltzmann distribution is applicable for low density or high temperature plasma environment.[8][9]

1.3.5 Non-Maxwellian distribution function

Maxwellian is not an idealistic distribution under all circumstances. Other non-Maxwellian distributions are kappa or generalized (r, q) are better suited with many environs in space plasma such as planetary magnetospheres, astrophysical plasmas and the solar wind. In such cases, distribution function posses a tail that repreasents high energetic and fast-moving particles.[10] Space plasma frequently contain particles with high energy. Such nonthermal

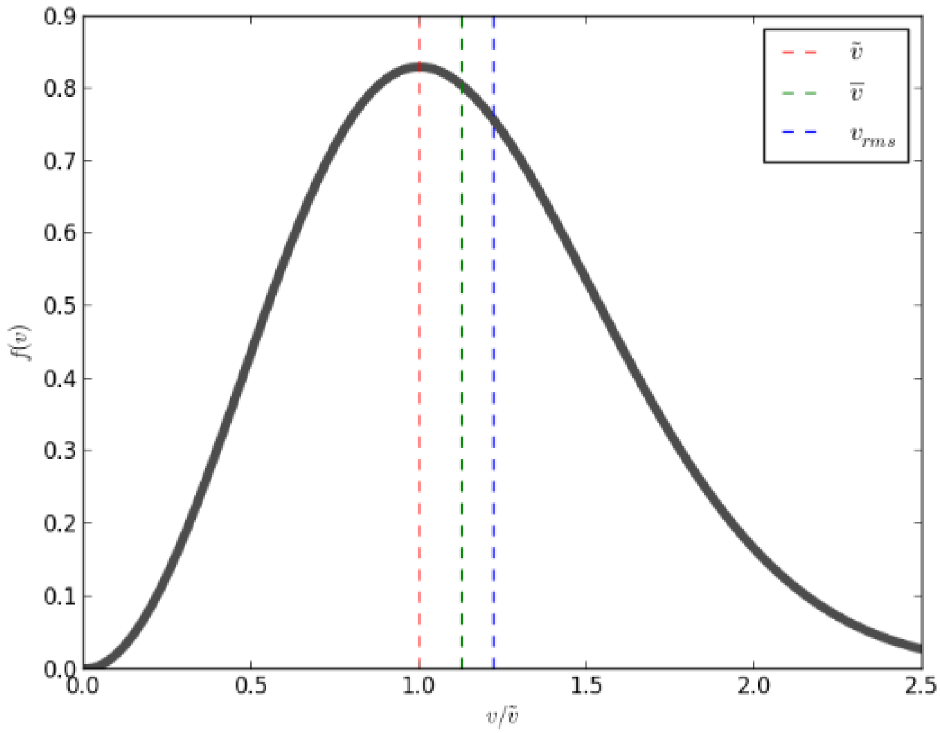

Figure 1.4: Plots of the Maxwellian distribution, red dotted line shows the most probable speed, green dotted line shows the average speed, blue dotted line shows the root mean square velocity.

distributions with excessive abundance of fast-paced particles can be preferable suited for suprathermal velocities by generalized(r,q) distribution or kappa distribution.[11] [12] [13]. Anisotropic KD is defined as

$$f_\kappa = \frac{1}{\pi^{\frac{3}{2}}\theta_{\perp\alpha}^2\theta_{\parallel\alpha}}\Big(\frac{\Gamma(\kappa+1)}{\kappa^{\frac{3}{2}}\Gamma(\kappa-\frac{1}{2})}\Big)\Big[1+\frac{v_\parallel^2}{\kappa\theta_{\parallel\alpha}}+\frac{v_\perp^2}{\kappa\theta_{\perp\alpha}^2}\Big]^{-\kappa-1}, \tag{1.13}$$

where κ is a spectral index. The thermal speed θ is related to particle temperature T by

$$\theta_{\parallel\alpha}^2 = \Big(\frac{2\kappa-3}{\kappa}\Big)v_{\parallel\alpha}^2, \tag{1.14}$$

$$\theta_{T\perp\alpha}^2 = \Big(\frac{2\kappa-3}{\kappa}\Big)v_{T\perp\alpha}, \tag{1.15}$$

with

$$v_{T(\parallel,\perp)\alpha}^2 = \frac{T_{\parallel,\perp}}{m}, \tag{1.16}$$

where Γ is the gamma function, and f_κ has been normalized so that $f_k d^3v = 1$. Here it is worth mentioning that the value of κ must be greater than $\frac{3}{2}$ which is dictated by the condition of normalization and definition of temperature for the distribution function. [14] the KD approaches towards the Maxwellian distribution [15], So KD is a generalization of Maxwellian distribution.[16][17][18] There is another three dimensional anisotropic distribution called generalized (r, q) distribution. It can be defined as

$$f_{(r,q)} = \Big(\frac{3}{4\pi\psi_{\perp\alpha}^2\psi_{\parallel\alpha}}\Big)\times\Big[\frac{\Gamma(q)}{(q-1)^{\frac{3}{2+2r}}\Gamma\Big(q-\frac{3}{2+2r}\Big)\Gamma\Big(1+\frac{3}{2+2r}\Big)}\Big]\times\Big[1+\frac{1}{(q-1)}\Big(\frac{v_\parallel^2}{\psi_{\parallel\alpha}^2}+\frac{v_\perp^2}{\psi_{\perp_\alpha}^2}\Big)^{r+1}\Big]^{-q}, \tag{1.17}$$

where q and r are the spectral indices;[**?**] [20]

the thermal speed ψ is

$$\psi_{\perp\alpha}^2 = \Big[\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma\Big(q-\frac{3}{2+2r}\Big)\Gamma\Big(\frac{3}{2+2r}\Big)}{\Gamma\Big(q-\frac{5}{2+2r}\Big)\Gamma\Big(\frac{5}{2+2r}\Big)}\Big]v_{T(\perp\alpha)}^2, \tag{1.18}$$

with

$$v_{T(\parallel,\perp)\alpha}^2 = \frac{T_{\parallel,\perp}}{m}. \tag{1.19}$$

The spectral indices r,q are constrained to $q > 1$ and $q(1 + r) > \frac{5}{2}$ the condition which emerges from the normalization and definition of temperature. $f_{(r,q)}$ has been normalized so that $\int f_{(r,q)} d^3v = 1$. This distribution reduces to kappa distribution function if $r = 0$ and $q = \kappa + 1$ and to Mawellian if $r = 0$ and $q \to \infty$.

1.4 Bernstein Waves

Electrostatic waves propagating at right angles to B_0 at harmonics of cyclotron frequency are called Bernstein waves. This important electrostatic mode called cannot be predicted by fluid theory. In fluid theory, as we average over Larmour orbits so these waves are lost. Bernstein waves depend on the cyclotron motion of particles around magnetic field lines. Bernstein modes spread between the harmonics of ω_c in frequency ranges. These waves are a function of temperature, field strength and density. The cyclotron waves with a frequency near cyclotron frequencies of any species $\omega = n\omega_{cs}$ are especially interesting where $n = 1, 2...$ and s for any specie.

1.4.1 Electron Bernstein Waves

Electrostatic electron cyclotron waves and Bernstein waves can be found by considering perpendicular propagation into the plasma. First consider high-frequency waves in which ions do not move.

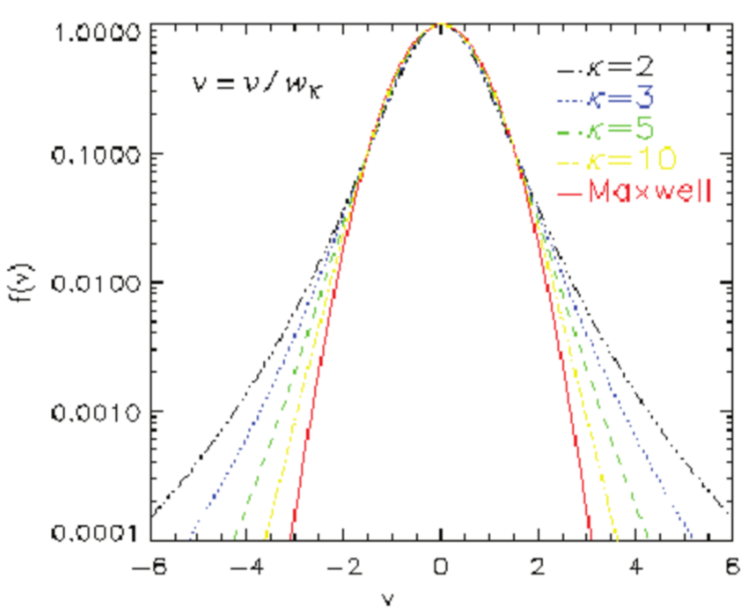

Figure 1.5: Plots of the Kappa distribution, Lower values of kappa gives superthermal tail as values of kappa increases it moves towards Maxwellian.

1.5 Plasma Environment

1.5.1 Solar wind

The external portion of the sun's atmosphere, the solar corona is composed of very hot plasma, a gas having kinetic energy with free electrons and positive ions. When these gases move away from the sun then internal pressure of gases becomes larger than the weight of upper plasma. These particles flow like a wind in the entire solar system with very high velocity. The flow of these particles with very high velocity is known as the solar wind.[21][22]

1.5.2 Fast and Slow solar wind

The solar wind is observed to exist in two modes termed as slow solar wind and fast solar wind, though their differences extend well beyond their speeds. In near Earth-space the slow solar wind is observed to have a velocity of 300-500 Km/sec, a temperature of $\sim 10^5 K$ and a composition that is a close match to corona. By contrast, the fast solar wind has a typical velocity of 750 Km/sec a temperature of $8 * 10^5 K$ and it really matches the composition of sun's photosphere. The slow solar wind is twice as dense and more variable in nature than the fast solar wind.

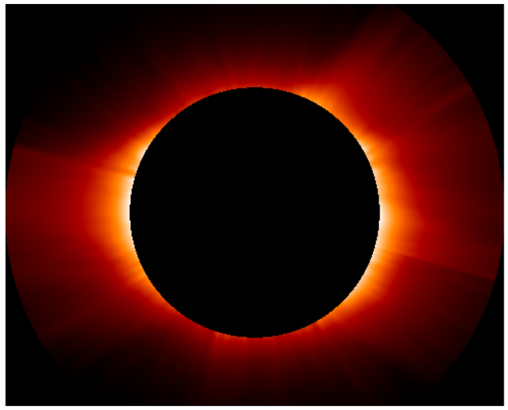

Figure 1.6: solar wind.

1.6 Motivation

The Maxwellian velocity distribution is not an idealistic one under many circumstances in laboratory and space plasmas. When lower hybrid waves are applied for cuurrent drive Tokamak edge plasma shows non-Maxwellian behaviour. [23]. In this kind of cases, distributions in velocity space like power-law shows high or suprathermal energy tails in the plasmas. Those nonthermal distributions have an excessive abundance of fast-moving particles can be effectively fitted by generalized Lorentzian or kappa distribution than by Maxwellian for superthermal velocities. High temperature and high density are required in order to raise the reaction rate in Tokamaks in the core region. Heating of wave is used for that purpose.[24] However high plasma density obstructs such kind of heating of wave for example electron cyclotron resonance heating (ECRH) beacuse of its penetration limit of density. On another side, EBW's are extremely backed for current drive and heating in Spherical Tokamaks as these waves are safe from density restrictions and can that's why attain plasmas of unlimited densities at the higher, fundamental and lower harmonics of the Electron Cyclotron(EC) frequency. Preffered heating source of plasma is EBW's. EBW's are a valuable analyzing mechanism for spherical tokamaks and it can be deployed to determine local electron temperature if its intensities are detected and its birth position is known. EBW's are undamped, In a plasma with no magnetic field, the propagating electrostatic waves show collision-free LD. It shows that the transition from magnetized to a non-magnetized situation is discontinuous. Sukhorukov proposed an empirical decode for magnetic field perpendicular disturbances when the external field is weak. Valentini et al have numerically analyzed the evolution of time of waves which are electrostatic and perpendicular propagating to the magnetic field which is at the background.

Chapter 2

Mathematical Model

2.1 Electromagnetic dispersion relation

The Vlasov Equation for collision less plasma [25] is given as,

$$\frac{\partial f_\alpha(r,v,t)}{\partial t} + v \cdot \nabla f_\alpha(r,v,t) + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla} f(r,v,t) = 0 \tag{2.1}$$

The Maxwell's equations [26] [27] are,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \sum_\alpha q_\alpha \int f_\alpha d^3 v, \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2.3}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2.4}$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \sum_\alpha q_\alpha \int v f_\alpha d^3 v. \tag{2.5}$$

Let

$$f_\alpha(\vec{r},\vec{v},t) = f_{\alpha 0}(\vec{r},\vec{v}) + f_{\alpha 1}(\vec{r},\vec{v},t) \tag{2.6}$$

$$\vec{B} = \vec{B}_0 + \vec{B}_1 \tag{2.7}$$

$$\vec{E} = \vec{E}_0 + \vec{E}_1 \tag{2.8}$$

where E_1 and B_1 are perturb electric and magneic fields and B_0 is taken uniform. After linearization, eq(2.2) will become

$$\imath k \cdot \vec{E}_1 = \frac{1}{\epsilon_0} \sum_\alpha q_\alpha n_\alpha \int f_{\alpha 0} d^3 v \tag{2.9}$$

$$\frac{1}{\mu_0}\vec{k}\times\vec{B}_1 = -\omega(\epsilon_0\vec{E}_1 + \frac{\iota}{\omega}\sum_\alpha n_\alpha q_\alpha \int f_{\alpha 0} d^3 v) \tag{2.10}$$

$$\vec{B}_1 = \frac{1}{\omega}(\vec{k}\times\vec{E}_1) \tag{2.11}$$

$$\frac{\partial f_{\alpha 1}}{\partial t} + \vec{v}\cdot\nabla_r \vec{f}_{\alpha 1} + \frac{q_\alpha}{m_\alpha}(\vec{v}\times\vec{B}_0)\cdot\nabla_v \vec{f}_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}(\vec{E_1} + \vec{v}\times\vec{B}_1)\cdot\vec{\nabla} f_{\alpha 0} \tag{2.12}$$

$$f_1 = f_1(r(t), v(t), t) \tag{2.13}$$

Taking the time derivative of distribution function. By using Chain rule

$$\frac{d}{dt} f_1(r(t), v(t), t) = \frac{d f_1}{dt} + \vec{\nabla} f_1 \cdot \frac{dr}{dt} + \frac{d f_1}{dv}\cdot\frac{dv}{dt} \tag{2.14}$$

$$a = \frac{dv}{dt} = \frac{q}{m}(\vec{v}\times\vec{B}) \tag{2.15}$$

$$\frac{d}{dt} f_1(\vec{r}, \vec{v}, t) = \frac{\partial f_1}{\partial t} + \vec{\nabla}\cdot\vec{v} + \frac{\partial f_1}{\partial v}\frac{q}{m}(\vec{v}\times\vec{B}_0) \tag{2.16}$$

Compairing equation (2.9) and equation (2.13)

$$\frac{d}{dt} f_1(\vec{r}, \vec{v}, t) = -\frac{q}{m}(\vec{E}_1 + \vec{v}\times\vec{B}_1)\cdot\frac{\partial f_0}{\partial v} \tag{2.17}$$

Integrating

$$f_1(\vec{r}(t), \vec{v}(t), t) = -\frac{q}{m}\int_{-\infty}^{t} dt(\vec{E}_1 + \vec{v}\times\vec{B}_1)\cdot\vec{\nabla}_v f_{\alpha 0} \tag{2.18}$$

Put equation (2.11) in equation (2.18)

$$f_{\alpha 1} = -\frac{q}{m}\int_{-\infty}^{t} dt'[E_1(\vec{r}, t) + \vec{v}(t)\times\frac{1}{\omega}(\vec{k}\times E_1(\vec{r}, t))]\cdot\vec{\nabla}_v f_{\alpha 0} \tag{2.19}$$

Let

$$X = [E_1(\vec{r}, t) + \vec{v}(t)\times\frac{1}{\omega}(\vec{k}\times E_1(\vec{r}, t))]\cdot\vec{\nabla}_v f_{\alpha 0} \tag{2.20}$$

Wave propagation in x-z plane.

$$\vec{k} = k_x\hat{x} + k_z\hat{z} \tag{2.21}$$

The motion of charged particles in uniform magnetic field.

$$\frac{dr'}{dt'} = \vec{v'} \tag{2.22}$$

$$a = \frac{d\vec{v'}}{dt'} = \frac{q}{m}(\vec{v}\times\vec{B}_0) \tag{2.23}$$

Now at $t = t'$ $r = r'$ In cylindrical coordinates

$$\vec{v'} = \vec{v} = (v_\perp cos\theta, v_\perp sin\theta, v_\shortparallel) \tag{2.24}$$

From equation (2.23)

$$\frac{dv'_x}{dt'} = \frac{q}{m} v'_y B_0 \tag{2.25}$$

$$\frac{dv'_y}{dt'} = -\frac{q}{m} v'_x B_0 \tag{2.26}$$

where as $\omega_c = \frac{qB_0}{m}$

$$v'_x(t') = v_\perp cos[\theta - \omega_c(t' - t)]\frac{\omega_c}{\omega_c} \tag{2.27}$$

$$v'_y(t') = v_\perp sin[\theta - \omega_c(t' - t)] \tag{2.28}$$

$$v'_z(t') = v_\shortparallel \tag{2.29}$$

now $\frac{dx'}{dt'} = v'_x(t')$

$$\int_t^{t'} \frac{dx'}{dt'} dt' = \int v_\perp cos[\theta - \omega_c(t' - t)]dt' \tag{2.30}$$

$$x'(t') - x(t) = \frac{v_\perp}{\omega_c}[sin\theta - sin\theta - \omega(t' - t)] \tag{2.31}$$

$$y'(t') - y(t) = \frac{v_\perp}{\omega_c}[cos\theta - cos\theta - \omega_c(t' - t)] \tag{2.32}$$

$$z'(t') - z(t) = v_z(t' - t) \tag{2.33}$$

Using the identity

$$a \times (b \times c) = (a \cdot c)b - (b \cdot a)c \tag{2.34}$$

$$\vec{v} \times (\vec{k} \times \vec{E}_1) = (\vec{v} \cdot \vec{E}_1)k - (\vec{k} \cdot \vec{v})E_1 \tag{2.35}$$

Put this in equation (2.20)

$$X = -\frac{q_\alpha}{m_\alpha}[E(r, t) + \frac{1}{\omega}(\vec{v} \cdot \vec{E}_1)k - \frac{(\vec{k} \cdot \vec{v})}{\omega} E_1] \cdot \vec{\nabla}_v f_{\alpha 0} \tag{2.36}$$

$$X = -\frac{q_\alpha}{m_\alpha}[(1 - \frac{\vec{k} \cdot \vec{v}}{\omega})E_1(\vec{r}, t) + \frac{1}{\omega}(\vec{v}(t) \cdot \vec{E}_1(\vec{r}, t))] \cdot \vec{\nabla}_v f_{\alpha 0} \tag{2.37}$$

Put this in equation (2.19)

$$f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha} \int_{-\infty}^{t} dt'[(1 - \frac{(\vec{k} \cdot \vec{v})}{\omega})E_1(\vec{r}, t') + \frac{1}{\omega}(\vec{v}(t') \cdot \vec{E}_1(\vec{r}, t'))k] \cdot \vec{\nabla}_{v'} f_{\alpha 0} \tag{2.38}$$

$$v_\perp^2 = v_x^2 + v_y^2 \tag{2.39}$$

$$E_1(\vec{r},t) = Eexp[\imath\vec{k}\cdot(\vec{r'}-\vec{r}) - \omega(t'-t)] \tag{2.40}$$

$$exp[\imath k\cdot(\vec{r'}-\vec{r}) - \omega(t'-t)] = exp[\imath k_x(x'-x) + k_z(z'-z) - \omega(t'-t)] \tag{2.41}$$

Using equation (2.31) and equation (2.33)

$$E_1(\vec{r},t) = exp[\frac{\imath k_x v_\perp}{\omega_c}[sin\theta - sin(\theta - \omega_c(t'-t))] + (k_z v_z - \omega)(t'-t)] \tag{2.42}$$

Put values in equation (2.38)

$$f_1 = -\frac{q}{m}\int_{-\infty}^{t} dt[(1-\frac{(\vec{k}\cdot\vec{v})}{\omega}E) + \frac{1}{\omega}(\vec{v}(t')\cdot\vec{E})k]exp\imath[\frac{k_x v_\perp}{\omega_c}[sin\theta - sin(\theta-\omega_c(t'-t))] \\ + (k_z v_z - \omega)(t'-t)\cdot\vec{\nabla}_{v'}f_0 \tag{2.43}$$

Let $\tau = t' - t$ and $d\tau = dt'$ when $t' \to t$ then $\tau \to 0$ when $t' \to t$ then $\tau \to -\infty$

$$f_1 = -\frac{q}{m}\int_{-\infty}^{0} d\tau[(1-\frac{(\vec{k}\cdot\vec{v})}{\omega}E) + \frac{1}{\omega}(\vec{v}(t')\cdot\vec{E})k]exp\imath[\frac{k_x v_\perp}{\omega_c}[sin\theta - sin(\theta-\omega_c(t'-t))] \\ + (k_z v_z - \omega)(t'-t)\cdot\vec{\nabla}_{v'}f_0 \tag{2.44}$$

From generating function of Bessel function

$$e^{\imath x sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{\imath n\theta} \tag{2.45}$$

Now taking the exponential part of above equation

$$X_1 = exp\imath[\frac{k_x v_\perp}{\omega_c}[sin\theta - sin(\theta-\omega_c\tau)] + (k_z v_z - \omega)\tau] \tag{2.46}$$

$$X_1 = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n(\frac{k_x v_\perp}{\omega_c})J_m(\frac{k_x v_\perp}{\omega_c})e^{-\imath n(\theta-\omega_c)\tau}e^{\imath m\theta}e^{\imath(k_z v_z - \omega)\tau} \tag{2.47}$$

$$f_1 = -\frac{q}{m}\int_{-\infty}^{0} d\tau[(1-\frac{(\vec{k}\cdot\vec{v})}{\omega}E) + \frac{1}{\omega}(\vec{v}(t')\cdot\vec{E})k] \\ \cdot\vec{\nabla}_v f_0 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n(\frac{k_x v_\perp}{\omega_c})J_m(\frac{k_x v_\perp}{\omega_c})e^{-\imath n(\theta-\omega_c)\tau}e^{\imath m\theta}e^{\imath(k_z v_z - \omega)\tau} \tag{2.48}$$

$$X_2 = [(1-\frac{(\vec{k}\cdot\vec{v})}{\omega})E + \frac{1}{\omega}(\vec{v}\cdot\vec{E})k]\cdot\vec{\nabla}_v f_0 \tag{2.49}$$

$$X_2 = \frac{\partial f_0}{\partial v'_x}[(1-\frac{k_z v'_z}{\omega})E_x + \frac{k_x}{\omega}(v'_y E_y + v'_z E_z)] + \frac{\partial f_0}{\partial v'_y}[1-\frac{k_x v'_x + k_z v'_z}{\omega}]E_y + \frac{\partial f_0}{\partial v'_z}[(1-\frac{k_x v'_x}{\omega})E_z + \frac{k_z}{\omega}(v'_x E_x + v'_y E_y)] \quad (2.50)$$

$$\frac{\partial f_0}{\partial v'_x} = \frac{\partial f_0}{\partial v_\perp}\frac{\partial v_\perp}{\partial v'_x} \quad (2.51)$$

$$v_\perp^2 = v_x^2 + v_y^2 \quad (2.52)$$

Taking derivative w.r.t v'_x

$$\frac{\partial v_\perp}{\partial v'_x} = \frac{v'_x}{v_\perp} \quad (2.53)$$

equation (2.51) becomes

$$\frac{\partial f_0}{\partial v'_x} = \frac{v'_x}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \quad (2.54)$$

Taking derivative w.r.t v'_y

$$\frac{\partial v_\perp}{\partial v'_y} = \frac{v'_y}{v_\perp} \quad (2.55)$$

$$\frac{\partial f_0}{\partial v'_y} = \frac{v'_y}{v_\perp}\frac{\partial f_0}{\partial v_\perp} \quad (2.56)$$

Similarly $\frac{\partial f_0}{\partial v'_z} = \frac{\partial f_0}{\partial v_z} = \frac{\partial f_0}{\partial v_\parallel}$ Equation (2.50)becomes

$$X_2 = [\frac{\partial f_0}{\partial v_\perp}(1-\frac{k_z v_z}{\omega}) + \frac{\partial f_0}{\partial v_z}\frac{k_z v_\perp}{\omega}](E_x\frac{v'_x}{v_\perp} + E_y\frac{v'_y}{v_\perp}) + [\frac{\partial f_0}{\partial v_\perp} - \frac{\partial f_0}{\partial v_z}\frac{k_x v_\perp}{\omega}](E_z\frac{v'_x}{v_\perp}) + \frac{\partial f_0}{\partial v_z}E_z \quad (2.57)$$

From equation (2.27) and equation (2.28)

$$\frac{v'_x}{v_\perp} = cos(\theta - \omega_c\tau) = \frac{e^{\iota(\theta-\omega_c\tau)} + e^{-\iota(\theta-\omega_c\tau)}}{2} \quad (2.58)$$

$$\frac{v'_y}{v_\perp} = sin(\theta - \omega_c\tau) = \frac{e^{\iota(\theta-\omega_c\tau)} + e^{-\iota(\theta-\omega_c\tau)}}{2\iota} \quad (2.59)$$

$$X_2 = [(1 - \frac{k_z v_z}{\omega})\frac{\partial f_0}{\partial v_\perp} + \frac{k_z v_\perp}{\omega}\frac{\partial f_0}{\partial v_x}](E_x\frac{e^{\iota(\theta-\omega_c\tau)+e^{-\iota(\theta-\omega_c\tau)}}}{2} + E_y\frac{e^{\iota(\theta-\omega_c\tau)} - e^{-\iota(\theta-\omega_c\tau)}}{2\iota}) + [\frac{k_x v_z}{\omega}\frac{\partial f_0}{\partial v_\perp} - \frac{k_x v_\perp}{\omega}\frac{\partial f_0}{\partial v_z}](\frac{e^{\iota(\theta-\omega_c\tau)} + e^{-\iota(\theta-\omega_c\tau)}}{2})E_z + \frac{\partial f_0}{\partial v_z}E_z \quad (2.60)$$

Let

$$U = [(1 - \frac{k_z v_z}{\omega})\frac{\partial f_0}{\partial v_\perp} + \frac{k_z v_\perp}{\omega}\frac{\partial f_0}{\partial v_z}] \tag{2.61}$$

$$W = [\frac{k_x v_z}{\omega}\frac{\partial f_0}{\partial v_z}] \tag{2.62}$$

$$a = \frac{k_x v_\perp}{\omega_c} \tag{2.63}$$

and

$$\omega_c = \frac{qB_0}{m} \tag{2.64}$$

$$X_2 = \frac{UE_x}{2}[e^{\imath}(\theta - \omega_c\tau) + e^{-\imath(\theta-\omega_c\tau)}] - \frac{\imath UE_y}{2}[e^{\imath(\theta-\omega_c\tau)} - e^{-\imath(\theta-\omega_c\tau)}] + \frac{W}{2}[e^{\imath(\theta-\omega_c\tau)} + e^{-\imath(\theta-\omega_c\tau)}] + \frac{\partial f_0}{\partial v_z}E_z \tag{2.65}$$

$$\begin{aligned} f_1 = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{0} d\tau & \\ \Bigg[\frac{UE_x}{2}[&e^{-\imath(n-1)\theta}e^{\imath(k_z v_z-\omega-(n-1)\omega_c)\tau}J_n(a) + \\ &e^{-\imath(n+1)\theta}e^{\imath(k_z v_z-\omega-(n+1)\omega_c)\tau}J_n(a)] \\ -\frac{\imath UE_y}{2}[&e^{-\imath(n-1)\theta}e^{\imath(k_z v_z-\omega-(n-1)\omega_c)\tau}J_n(a) + \\ &e^{-\imath(n+1)\theta}e^{\imath(k_z v_z-\omega-(n+1)\omega_c)\tau}J_n(a)] \\ +\frac{WE_z}{2}[&e^{-\imath(n-1)\theta}e^{\imath(k_z v_z-\omega-(n-1)\omega_c)\tau}J_n(a) + \\ &e^{-\imath(n+1)\theta}e^{\imath(k_z v_z-\omega-(n+1)\omega_c)\tau}J_n(a)]\Bigg] J_m e^{\imath m\theta} \\ &+ \frac{\partial f_0}{\partial v_z}E_z e^{-\imath n\theta}e^{\imath(k_z v_z-\omega-n\omega_c)}J_n(a) \end{aligned} \tag{2.66}$$

and $n \rightarrow n+1$ then $J_n(a) \rightarrow J_{n-1}$

$$f_1 = -\frac{q}{m} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{0} d\tau$$
$$\Bigg[\frac{UE_x}{2}[J_{n-1}(a) + J_{n+1}(a)] + \imath \frac{UE_y}{2}[J_{n-1}(a) + J_{n+1}(a)] + \frac{WE_z}{2}[J_{n-1}(a) + J_{n+1}(a)] + \frac{\partial f_0}{\partial v_x} E_z J_n(a) \Bigg] J_m(a) e^{\imath m\theta} e^{-\imath n\theta} e^{\imath(k_z v_z - \omega - n\omega_c)\tau} \tag{2.67}$$

From the Bessel function recurrence relation

$$\frac{J_{n-1}(a) + J_{n+1}(a)}{2} = \frac{n}{a} J_n(a) \tag{2.68}$$

$$\frac{J_{n-1}(a) - J_{n+1}(a)}{2} = \frac{dJ_n(a)}{da} = J_{n'}(a) \tag{2.69}$$

Equation (2.67) becomes

$$f_1 = -\frac{q}{m} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Bigg[U\frac{n}{a} J_n(a) E_x + \imath U J_{n'}(a) E_y + [W\frac{n}{a} J_n(a) + \frac{\partial f_0}{\partial v_y} J_n(a)] E_z \Bigg] \frac{J_m(a) e^{\imath(m-n)\theta}}{\imath(k_z v_z - \omega - n\omega_c)} \tag{2.70}$$

Put equation (2.11) in equation (2.10)

$$\frac{1}{\mu_0}(\vec{k} \times (\vec{k} \times \vec{E}_1)) = -\omega^2(\epsilon_0 E_1 + \frac{\imath}{\omega} \sum_{\alpha} n_\alpha q_\alpha \int v f_{\alpha 1} d^3 v) \tag{2.71}$$

Separating the equation in components

$$\epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z = 0 \tag{2.72}$$

$$\epsilon_{yx} E_x + \epsilon_{yy} E_y + \epsilon_{yz} E_z = 0 \tag{2.73}$$

$$\epsilon_{zx} E_x + \epsilon_{zy} E_y + \epsilon_{zz} E_z = 0 \tag{2.74}$$

$$\bar{\epsilon} \cdot \bar{E} = 0 \tag{2.75}$$

$$\vec{\vec{\epsilon}} = \tag{2.76}$$

$$-\vec{k} \times (\vec{k} \times \vec{E}_1) = -(k_\perp \hat{x} + k_\parallel \hat{z}) \times [(k_\perp \hat{x} + k_\parallel \hat{z}) \times (E_x \hat{x} + E_y \hat{y} + E_z \hat{z})] \tag{2.77}$$

$$-\vec{k} \times (\vec{k} \times \vec{E}_1) = (-k_\perp k_\parallel E_z + k_\parallel^2 E_x)\hat{x} + (k_\perp^2 E_y + k_\parallel^2 E_y \hat{y}) + (k_\perp^2 E_z - k_\perp k_\parallel E_x)\hat{z} \tag{2.78}$$

Equation (2.71) becomes

$$= \frac{\omega^2}{c^2}(E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) + \imath \frac{\omega}{c^2 \epsilon_0} \sum_\alpha n_\alpha q_\alpha \int (v_\perp cos\theta \hat{x} + v_\perp sin\theta \hat{y} + v_z \hat{z})[-\frac{q}{m} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[U \frac{n}{a} J_n(a) E_x + \imath U J_{n'}(a) E_y + [W \frac{n}{a} J_n(a) + \frac{\partial f_0}{\partial v_y} J_n(a)] E_z \right] \frac{J_m(a) e^{\imath(m-n)\theta}}{\imath(k_z v_z - \omega - n\omega_c)}] \tag{2.79}$$

Now calculating $\int v f_{\alpha 1} d^3 v$

A) $v_\perp cos\theta \hat{x}$

$$\int d^3 v f_1 v_\perp cos\theta \hat{x} = \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} cos\theta \hat{x} \tag{2.80}$$

$$= -\frac{q}{m} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^\infty dv_\parallel \int_0^{2\pi} cos\theta e^{\imath(m-n)\theta} d\theta \left[U \frac{n}{a} J_n(a) E_x + \imath U J_{n'}(a) E_y + [W \frac{n}{a} J_n(a) + \frac{\partial f_0}{\partial v_y} J_n(a)] E_z \right] \frac{J_m(a)}{\imath(k_z v_z - \omega - n\omega_c)}]\hat{x} \tag{2.81}$$

$$\int_0^{2\pi} cos\theta e^{\imath(m-n)\theta} d\theta = \int_0^{2\pi} co(m-n)\theta cos\theta + \imath sin(m-n)\theta sin\theta d\theta \tag{2.82}$$

$$\int_0^{2\pi} cosaxcosbxdx = \tag{2.83}$$

$\int_0^{2\pi} sinaxcosbxdx = 0$ for all integers a and b.

$$\int_0^{2\pi} cos\theta e^{\mp \imath\theta} = \pi \tag{2.84}$$

$m - n = \pm 1$ if and only if $cos(m-n)\theta cos\theta d\theta = \pi$

$$\sum_{m=-\infty}^{\infty} \int_0^{2\pi} cos\theta e^{\imath(m-n)\theta} J_m(a) = \int_0^{2\pi} cos\theta e^{\imath\theta} d\theta J_{n+1}(a) + \int_0^{2\pi} cos\theta e^{-\imath\theta} d\theta J_{n-1}(a) \quad (2.85)$$

$$= 2\pi(\frac{J_{n-1}(a) + J_{n+1}(a)}{2}) = 2\pi(\frac{n}{a})J_n(a) \quad (2.86)$$

$$\int_0^{2\pi} d\theta(\frac{n}{a})J_n(a) \quad (2.87)$$

So equation (2.81) becomes

$$\int d^3 v f_1 v_\perp cos\theta\hat{x} = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\int d^3 v v_\perp \left[U\frac{n}{a}J_n(a)E_x + \imath U J_{n'}(a)E_y + [W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_y}J_n(a)]E_z\right]\frac{(\frac{n}{a})J_n(a)}{\imath(k_z v_z - \omega - n\omega_c)}\hat{x} \quad (2.88)$$

Similarly **B)** $v_\perp sin\theta\hat{y}$

$$\int d^3 v f_1 v_\perp sin\theta\hat{y} = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\int d^3 v v_\perp \left[U\frac{n}{a}J_n(a)E_x + \imath U J_{n'}(a)E_y + [W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_y}J_n(a)]E_z\right]\cdot\frac{-J_{n'}(a)}{(k_z v_z - \omega - n\omega_c)}\hat{y} \quad (2.89)$$

C) $v_\parallel\hat{z}$

$$\int d^3 v f_1 v_\parallel\hat{z} = -\frac{q}{m}\sum_{n=-\infty}^{\infty}\int d^3 v v_\parallel \left[U\frac{n}{a}J_n(a)E_x + \imath U J_{n'}(a)E_y + [W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_y}J_n(a)]E_z\right]\frac{J_n(a)}{\imath(k_z v_z - \omega - n\omega_c)}\hat{z} \quad (2.90)$$

Equation (2.17) becomes

$$(\frac{-k_\perp k_\parallel E_z + k_\parallel^2 E_x}{\omega^2} - E_x)\hat{x} + (\frac{k_\perp^2 E_y + k_\parallel^2 E_y}{\omega^2}c^2 - E_y)\hat{y} + (\frac{k_\perp^2 E_z - k_\perp k_\parallel E_x}{\omega^2}c^2 - E_z) = -\sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v \Bigg[U\frac{n}{a}J_n(a)E_x + \imath U J_{n'}(a)E_y + [W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_y}J_n(a)]E_z \Bigg] \quad (2.91)$$

1) x-component

$$\left(1 - \frac{k_\parallel^2 c^2}{\omega^2} - \sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (U\frac{n}{a}J_n(a)) \frac{v_\perp(\frac{n}{a})J_n(a)}{k_z v_z - \omega - n\omega_c}\right)E_x + \left(-\sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (\imath U J_n'(a)) \frac{v_\perp(\frac{n}{a})J_n(a)}{(k_z v_z - \omega - n\omega_c)}\right)E_y + \left(\frac{k_\perp k_\parallel c^2}{\omega^2} - \sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_z}J_n(a)) \frac{v_\perp(\frac{n}{a})J_n(a)}{k_z v_z - \omega - n\omega_c}\right)E_z \quad (2.92)$$

2)y-component

$$\left(-\sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (U\frac{n}{a}J_n(a)) \frac{-\imath J_n'(a)}{k_z v_z - \omega - n\omega_c}\right)E_x + \left(1 - \frac{(k_\perp^2 + k_\parallel^2)}{\omega^2}c^2 - \sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (\imath U J_n'(a)) \frac{-\imath v_\perp J_n'(a)}{k_z v_z - \omega - n\omega_c}\right)E_y + \left(-\sum_\alpha \frac{n_\alpha q_\alpha^2}{m\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (W(\frac{n}{a})J_n(a) + \frac{\partial f_0}{\partial v_z}J_n()a) \frac{-\imath v_\perp J_n'(a)}{k_z v_z - \omega - n\omega_c}\right)E_z \quad (2.93)$$

$$\left(1 + \frac{k_\perp k_\parallel c^2}{\omega^2} - \sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (U\frac{n}{a}J_n(a)) \frac{v_\parallel J_n(a)}{k_z v_z - \omega - n\omega_c}\right)E_x + \left(-\sum_\alpha \frac{n_\alpha q_\alpha^2}{n\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (U\frac{n}{a}J_n'(a)) \frac{v_\parallel J_n(a)}{k_z v_z - \omega - n\omega_c}\right)E_y + \left(1 - \frac{k_\perp^2 c^2}{\omega^2} - \sum_\alpha \frac{n_\alpha q_\alpha^2}{m\omega\epsilon_0} \sum_{n=-\infty}^{\infty} \int d^3v (W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_z}J_n(a)) \frac{v_\parallel J_n(a)}{k_z v_z - \omega - n\omega_c}\right)E_z \quad (2.94)$$

Now by comparing E_x E_y E_z terms from the above equation

$$\epsilon_{xx} = 1 - \frac{k_{\parallel}^2 c^2}{\omega^2} - \sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m\omega\epsilon_0}\int d^3v v_{\perp} U\frac{[(\frac{n}{a})J_n(a)]^2}{k_z v_z - \omega - n\omega_c} \quad (2.95)$$

$$\epsilon_{xy} = -\sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_0}\int d^3v v_{\perp}(\imath U J_n'(a))\frac{(\frac{n}{a})J_n(a)}{k_z v_z - \omega - n\omega_c} \quad (2.96)$$

$$\epsilon_{xz} = \frac{k_{\perp}k_{\parallel}c^2}{\omega^2} - \sum_{\alpha}\sum_{n=-\infty}^{\infty}\int d^3v v_{\perp}(W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_z}J_n(a))\frac{\frac{n}{a}J_n(a)}{(k_z v_z - \omega - n\omega_c)} \quad (2.97)$$

$$\epsilon_{yx} = -\sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_0}\int d^3v v_{\perp}(U\frac{n}{a}J_n(a))\frac{-\imath J_n'(a)}{k_z v_z - \omega - n\omega_c} \quad (2.98)$$

$$\epsilon_{yy} = 1 - \frac{(k_{\perp}^2 + k_{\parallel}^2 c^2)}{\omega^2} - \sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_0}\int d^3v v_{\perp}(U J_n'(a)\frac{J_n'(a)}{k_z v_z - \omega - n\omega_c} \quad (2.99)$$

$$\epsilon_{yz} = -\sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_0}\int d^3v v_{\perp}(W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_z}J_n(a))\frac{-\imath J_n'(a)}{k_z v_z - \omega - n\omega_c} \quad (2.100)$$

$$\epsilon_{zx} = 1 + \frac{k_{\perp}k_{\parallel}c^2}{\omega^2} - \sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_0}\int d^3v v_{\parallel}(U\frac{n}{a}J_n(a))\frac{J_n(a)}{k_z v_z - \omega - n\omega_c} \quad (2.101)$$

$$\epsilon_{zy} = -\sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_0}\int d^3v v_{\parallel}(\imath U J_n(a))\frac{J_n(a)}{k_z v_z - \omega - n\omega_c} \quad (2.102)$$

$$\epsilon_{zz} = 1 - \frac{k_{\perp}^2 c^2}{\omega^2} - \sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{n_\alpha q_\alpha^2}{m_\alpha\omega\epsilon_0}\int d^3v v_{\parallel}(W\frac{n}{a}J_n(a) + \frac{\partial f_0}{\partial v_z}J_n(a))\frac{J_n(a)}{k_z v_z - \omega - n\omega_c} \quad (2.103)$$

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{A} - \sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{\omega_p^2}{\omega}\int d^3v\frac{S_{ij}}{k_z v_z - \omega - n\omega_c} \quad (2.104)$$

$$S_{ij} = \begin{bmatrix} v_{\perp}(\frac{n}{a})^2 J_n^2(a)U & \imath v_{\perp}(\frac{n}{a})J_n(a)J_n'(a)U & v_{\perp}(\frac{n}{a})J_n^2(a)(W(\frac{n}{a}) + \frac{\partial f_0}{\partial v_{\parallel}}) \\ -\imath v_{\perp}(\frac{n}{a})J_n(a)J_n'(a)U & v_{\perp}U J_n'^2(a) & -\imath v_{\perp}J_n(a)J_n'(a)(W(\frac{n}{a}) + \frac{\partial f_0}{\partial v_z}) \\ v_{\parallel}(\frac{n}{a})J_n^2(a)U & \imath v_{\parallel}J_n(a)J_n'(a)U & v_{\parallel}J_n^2(a)(W(\frac{n}{a}) + \frac{\partial f_0}{\partial v_{\parallel}}) \end{bmatrix}$$

$$\overleftrightarrow{A} = \begin{bmatrix} 1 - \frac{k_{\parallel}^2 c^2}{\omega^2} & 0 & \frac{k_{\parallel}k_{\perp}c^2}{\omega^2} \\ 0 & 1 - \frac{(k_{\perp}^2 + k_{\parallel}^2)c^2}{\omega^2} & 0 \\ 1 + \frac{k_{\parallel}k_{\perp}c^2}{\omega^2} & 0 & 1 - \frac{k_{\perp}^2 c^2}{\omega^2} \end{bmatrix} \quad (2.105)$$

As we have assumed

$$U = [(1 - \frac{k_z v_z}{\omega})\frac{\partial f_0}{\partial v_\perp} + \frac{k_z v_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel}] \tag{2.106}$$

$$W = \frac{k_x v_z}{\omega}\frac{\partial f_0}{\partial v_\perp} - \frac{k_x v_\perp}{\omega}\frac{\partial f_0}{\partial v_\parallel} \tag{2.107}$$

Maxwellian distribution

$$f_0(v) = \left(\frac{m}{2\pi K_B T}\right)^{\frac{3}{2}} exp\left[\frac{-m(v_\perp^2 + v_\parallel^2)}{2K_B T}\right] \tag{2.108}$$

$$\frac{\partial f_0}{\partial v_\perp} = \left(\frac{-m v_\perp}{K_B T}\right) f_0(v) \tag{2.109}$$

$$\frac{\partial f_0}{\partial v_\parallel} = \left(\frac{-m v_\parallel}{K_B T}\right) f_0(v) \tag{2.110}$$

Plasma dispersion function

$$Z(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - z} dx \tag{2.111}$$

$$Z'(z) = -2[1 + 2Z(z)] \tag{2.112}$$

$$\int_0^\infty x e^{-px^2} J_n^2(ax) dx = \frac{1}{2p^2} e^{(\frac{-a^2}{2p^2})} I_n(\frac{a^2}{2p^2}) \tag{2.113}$$

$$\frac{d(c)}{dx} = \int_0^\infty 2x^2 e^{-p^2x^2} J_n(ax) J_n'(ax) dx = \frac{1}{2p^2}\left(\frac{-a}{p^2}\right) exp\left(\frac{-a^2}{2p^2}\right) I_n(\frac{a^2}{2p^2}) + \frac{1}{2p^2} exp\left(\frac{-a^2}{2p^2}\right) I_n(\frac{a^2}{2p^2}) \tag{2.114}$$

$$\int_0^\infty x^2 e^{-px^2} J_n(ax) J_n'(ax) dx = \frac{a}{4p^4} e^{\frac{-a^2}{2p^2}} \left[I_n'\left(\frac{a^2}{2p^2}\right) - I_n\left(\frac{a^2}{2p^2}\right)\right] \tag{2.115}$$

$$\int_0^\infty x^3 e^{\frac{-x^2}{2p}} J_n'^2(x) dx = b e^{-b}\left[n^2 I_n(b) - 2b^2(I_n'(b) - I_n(b))\right] \tag{2.116}$$

S_{xx} Components:

$$I_1 = \int d^3v \frac{S_{xy}}{k_z v_z - \omega - n\omega_c} = \int d^3v \frac{v_\perp (\frac{n}{a})^2 J_n^2(a) U}{k_z v_z - \omega - n\omega_c} \tag{2.117}$$

Using equation (2.113)

$$I_1 = \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty 2\pi v_\perp dv_\perp \frac{v_\perp (\frac{n}{a})^2 J_n^2(a)}{k_z v_z - \omega - n\omega_c}\left(\frac{-m v_\perp}{K_B T}\right) n_0 \left(\frac{m}{2\pi K_B T}\right)^{\frac{3}{2}} exp\left[\frac{-m(v_\perp^2 + v_\parallel^2)}{2K_B T}\right. \tag{2.118}$$

Integration over $v_\perp$

$$v_\perp dv_\perp \left(\frac{n}{\frac{k_\perp v_\perp}{\omega_c}}\right) J_n^2(\frac{k_x v_\perp}{\omega_c})(\frac{-mv_\perp}{K_B T})exp[\frac{-mv_\perp^2}{2K_B T}] = n^2(\frac{\omega_c}{k_x})^2(\frac{-m}{K_B T})$$

$$\int_0^\infty v_\perp dv_\perp J_n^2(\frac{k_x v_\perp}{\omega})exp[\frac{-mv_\perp^2}{2K_B T}] \quad (2.119)$$

From equation (2.113)

$$= -n^2(\frac{\omega}{k_x})^2 exp[-(\frac{k_x^2 K_B T}{m\omega_c^2})] I_n(\frac{k_x^2 K_B T}{\omega_c^2 m}) \quad (2.120)$$

$$= -n^2(\frac{\omega}{k_x})^2 exp(-b) I_n(b) \quad (2.121)$$

Performing $v_\shortparallel$ integration

$$\int_{-\infty}^\infty dv_\shortparallel \frac{exp[\frac{-mv_\shortparallel^2}{2K_B T}]}{(k_z v_z - (\omega + n\omega_c))} = \frac{1}{k_z}\int_{-\infty}^\infty d(\frac{v_\shortparallel}{v_{th}}) \frac{exp(\frac{-v_\shortparallel^2}{v_{th}^2})}{(\frac{v_\shortparallel}{v_{th}} - (\frac{\omega + n\omega_c}{k_z v_{th}}))} \quad (2.122)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi} k_z}\int_{-\infty}^\infty dx \frac{exp(-x^2)}{(x - (\frac{\omega + n\omega_c}{k_z v_{th}}))} = \frac{\sqrt{\pi}}{k_z} Z(\frac{\omega + n\omega_c}{k_z v_{th}}) \quad (2.123)$$

$$\frac{\sqrt{\pi}}{k_z} Z(\xi_n) \quad (2.124)$$

where as

$$x = \frac{v_\shortparallel}{v_{th}} = v_\shortparallel(\frac{m}{2K_B T})^{\frac{1}{2}} \quad (2.125)$$

SO by putting respective values of integrals in I_1 we have

$$I_1 = -n^2(\frac{\omega_c}{k_x})^2 2\pi exp(-b) I_n(b) n_0 (\frac{m}{2\pi K_B T})^{\frac{3}{2}} \frac{\sqrt{\pi}}{k_z} Z(\xi_n) \quad (2.126)$$

$$= -n_0 \frac{1}{k_z v_{th}} \frac{n^2}{b} exp(-b) I_n(b) Z(\xi_n) \quad (2.127)$$

So using I_1 in equation

$$\epsilon_{xx} = 1 - \frac{k_\shortparallel^2 c^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \frac{e^{-b}}{b} \frac{\omega}{k_z v_{th}} \sum_{n=-\infty}^\infty n^2 I_n(b) Z(\xi_n) \quad (2.128)$$

$$1 - \frac{k_{\shortparallel}^2 c^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\frac{e^{-b}}{b}\xi_0 \sum_{n=-\infty} n^2 I_n(b) Z'(\xi_n) \tag{2.129}$$

$$b = \frac{k_x^2 v_{th}^2}{2\omega_c^2} \tag{2.130}$$

Similarly, the other components can be solved using (2.114)(2.115)and (2.116)

$$\epsilon_{yy} = 1 - \frac{(k_\perp^2 + k_{\shortparallel}^2)}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\xi_0 e^{-b} \sum_{n=-\infty}^{\infty} [\frac{n^2}{b} I_n(b) Z(\xi_n) - 2I_n'(b) - I_n(b) Z(\xi_n)] \tag{2.131}$$

$$\epsilon_{zz} = 1 - \frac{k_\perp^2 c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\xi_0 exp(-b) \sum_{n=-\infty}^{\infty} I_n(b)\xi_n Z'(\xi_n) \tag{2.132}$$

$$\epsilon_{xy} = -\epsilon_{yx} = \imath \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\xi_0 \sum_{n=-\infty}^{\infty} e^{-b}[I_n'(b) - I_n(b)] \tag{2.133}$$

$$\epsilon_{xz} = \epsilon_{zx} = \frac{k_\perp k_{\shortparallel} c^2}{\omega^2} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}(2b)^{\frac{-1}{2}}\xi_0 \sum_{n=-\infty}^{\infty} nexp(-b) I_n(b) Z'(\xi_n) \tag{2.134}$$

$$\epsilon_{yz} = -\epsilon_{zy} = -\imath \sum_\alpha \sum_{n=-\infty}^{\infty} \frac{\omega_{p\alpha}^2}{\omega^2}(\pm\sqrt{\frac{b}{2}}) e^{-b}\xi_0 [I_n(b) - I_n'(b)] Z'(\xi_n) \tag{2.135}$$

2.1.1 Propagation of waves perpendicular to magnetic field:

$k_{\shortparallel} = 0$, $\vec{k} = k_\perp \bar{x}$

$$\epsilon_{xx} = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\frac{e^{-b}}{b}\frac{\omega}{k_z v_{th}} \sum_{n=-\infty}^{\infty} n^2 I_n(b) Z(\xi_n) \tag{2.136}$$

$$\xi_0 Z(\xi_n) = \frac{\omega}{k_z v_{th}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx \frac{exp(-x^2)}{\left(x - \left(\frac{\omega + n\omega_c}{k_z v_{th}}\right)\right)} = \frac{-\omega}{\omega + n\omega_c} \tag{2.137}$$

$$\epsilon_{xx} = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\frac{e^{-b}}{b} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(b)}{\omega + n\omega_c} \tag{2.138}$$

Similarly the other terms can be simplified as

Dispersion relation

For Bernstein waves

$$1 = \sum_{\alpha} \sum_{n=1}^{\infty} \omega_{p\alpha}^2 \frac{e^{-b}}{b} \left[\frac{2n^2 I_n(b)}{\omega^2 - n^2\omega_{ce}^2} \right] \tag{2.139}$$

where as

$$\sum_{n=-\infty}^{\infty} \left[\frac{n^2}{\omega + n\omega_{ce}} \right] = \sum_{n=1}^{\infty} \left[\frac{2n^2\omega}{\omega^2 - n^2\omega_{ce}^2} \right] \tag{2.140}$$

$I_n(b) = I_{-n}(b)$

Chapter 3

Generalized dispersion relation for electron Bernstein waves in non-Maxwellian anisotrpic magnetized Plasma

3.1 Electrostatic dispersion relation

$$\partial_t f_\alpha + \vec{v} \cdot \vec{\nabla}_r f_\alpha + \vec{a}_\alpha \cdot \vec{\nabla}_v f_\alpha = 0 \tag{3.1}$$

$$\vec{a}_\alpha = \frac{q_\alpha}{m_\alpha}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$$

put this value in equation (3.1)

$$\partial_t f_\alpha + \vec{v} \cdot \vec{\nabla}_r f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) \cdot \vec{\nabla}_v f_\alpha = 0 \tag{3.2}$$

Linearized Vlasov Equation

$$f = f_0 + f_1$$
$$\vec{E} = \vec{E}_0 + \vec{E}_1$$
$$\vec{B} = \vec{B}_0 + \vec{B}_1$$

Putting these values in equation (2.2)

$$\partial_t (f_0 + f_1) + \vec{v} \cdot \vec{\nabla}_r (f_0 + f_1) + \frac{q_\alpha}{m_\alpha}\left(\vec{E}_0 + \vec{E}_1 + \frac{\vec{v}\times(\vec{B}_0 + \vec{B}_1)}{c}\right) \cdot \vec{\nabla}_v (f_0 + f_1) = 0$$

First order perturbation [28]

$$\partial_t f_{\alpha 1} + \vec{v}\cdot\partial_x f_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(\vec{E}_0 + \frac{\vec{v}\times\vec{B}_0}{c}\right)\cdot\partial_v f_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \frac{\vec{v}\times\vec{B}_1}{c}\right)\cdot\partial_v f_{\alpha 0} = 0 \quad (3.3)$$

$$\vec{E}_0 = 0$$

$$\vec{B}_1 = 0$$

Put these values in equation (2.3)

$$\partial_t f_{\alpha 1} + \vec{v}\cdot\partial_x f_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(\frac{\vec{v}\times\vec{B}_0}{c}\right)\cdot\partial_v f_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \frac{\vec{v}\times\vec{B}_1}{c}\right)\cdot\partial_v f_{\alpha 0} = 0 \quad (3.4)$$

Applying Fourier Laplace transformation

$$s f_{\alpha 1} + \imath\vec{k}\cdot\vec{v} f_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(\frac{\vec{v}\times\vec{B}_0}{c}\right)\cdot\partial_v f_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \frac{\vec{v}\times\vec{B}_1}{c}\right)\cdot\partial_v f_{\alpha 0} = 0 \quad (3.5)$$

As we know that

$$\frac{q_\alpha}{m_\alpha}\left(\frac{\vec{v}\times\vec{B}_0}{c}\right)\cdot\partial_v f_{\alpha 1} = -\omega_{c\alpha}\partial_\phi f_{\alpha 1}$$

put this in equation (3.5)

$$s f_{\alpha 1} + \imath\vec{k}\cdot\vec{v} f_{\alpha 1} - \omega_{c\alpha}\partial_\phi f_{\alpha 1} + \frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \frac{\vec{v}\times\vec{B}_1}{c}\right)\cdot\partial_v f_{\alpha 0} = 0 \quad (3.6)$$

Let

$$\frac{q_\alpha}{m_\alpha}\left(\vec{E}_1 + \frac{\vec{v}\times\vec{B}_1}{c}\right)\cdot\partial_v f_{\alpha 0} = \Phi(\phi)$$

Put this value in equation(3.6)

$$(s + \imath\vec{k}\cdot\vec{v}) f_{\alpha 1} - \omega_{c\alpha}\partial_\phi f_{\alpha 1} + \Phi(\phi) = 0$$

Rearranging it

$$\partial_\phi f_{\alpha 1} - \frac{(s + \imath\vec{k}\cdot\vec{v})}{\omega_{c\alpha}} f_{\alpha 1} = \frac{\Phi(\phi)}{\omega_{c\alpha}} \quad (3.7)$$

Solution for the homogeneous part

$$\frac{\partial G}{\partial\phi} - \frac{(s + \imath\vec{k}\cdot\vec{v})}{\omega_{c\alpha}} G = 0$$

$$\frac{\partial G}{\partial \phi} = \frac{(s + \imath\vec{k} \cdot \vec{v})}{\omega_{c\alpha}} G$$

$$\frac{dG}{G} = \frac{(s + \imath\vec{k} \cdot \vec{v})}{\omega_{c\alpha}} d\phi$$

Integrating

$$\int \frac{dG}{G} = \int_{\phi'}^{\phi} \frac{(s + \imath\vec{k} \cdot \vec{v})}{\omega_{c\alpha}} d\phi$$

$$\ln G = \int_{\phi'}^{\phi} \frac{(s + \imath(k_{\parallel} v_{\parallel} + k_{\perp} v_{\perp}) cos\phi)}{\omega_{c\alpha}} d\phi$$

Solution of this equation is

$$G(\phi') = exp[\frac{1}{\omega_{c\alpha}}[(s + \imath k_{\parallel} v_{\parallel})(\phi - \phi') + \imath k_{\perp} v_{\perp}(sin\phi - sin\phi')]] \tag{3.8}$$

Solution for the non-homogeneous part

$$\frac{q_\alpha}{m_\alpha} \left(\vec{E}_1 + \frac{\vec{v} \times \vec{B}_1}{c} \right) \cdot \partial_v f_{\alpha 0} = \Phi(\phi)$$

$$B_1 = 0$$

$$\frac{q_\alpha}{m_\alpha} \left(\vec{E}_1 \right) \cdot \frac{\partial f_{\alpha 0}}{\partial v} = \Phi(\phi)$$

$$\vec{E}_1 = -\vec{\nabla} \phi_1$$

$$-\frac{q_\alpha}{m_\alpha} \left(\nabla \phi_1 \right) \cdot \frac{\partial f_{\alpha 0}}{\partial v} = \Phi(\phi')$$

$$\Phi(\phi') = -\frac{\imath q_\alpha \phi_1}{m_\alpha} [k_{\perp} cos\phi' \frac{\partial f_{\alpha 0}}{\partial v_{\perp}} + k_{\parallel} \frac{\partial f_{\alpha 0}}{\partial v_{\parallel}}] \tag{3.9}$$

Solution of equation (3.7) is

$$f_{\alpha 1} = \int_{\pm\infty}^{\phi} \frac{G(\phi')\Phi(\phi')}{\omega_{c\alpha}} d\phi' \tag{3.10}$$

putting the values of (3.8) and (3.9) in equation (3.10)

$$f_{\alpha 1} = \int_{\pm\infty}^{\phi} [-\frac{\imath q_\alpha \phi_1}{m_\alpha} [k_{\perp} cos\phi' \frac{\partial f_{\alpha 0}}{\partial v_{\perp}} + k_{\parallel} \frac{\partial f_{\alpha 0}}{\partial v_{\parallel}}]][exp[\frac{1}{\omega_{c\alpha}}[(s + \imath k_{\parallel} v_{\parallel})(\phi - \phi') + \imath k_{\perp} v_{\perp}(sin\phi - sin\phi')]]] \tag{3.11}$$

Let

$$g(\phi') = [\frac{1}{\omega_{c\alpha}}[(s + \imath k_{\shortparallel} v_{\shortparallel})(\phi - \phi') + \imath k_{\perp} v_{\perp}(sin\phi - sin\phi')]] \tag{3.12}$$

Taking derivative of equation (3.12) w.r.t ϕ'

$$\omega_{c\alpha}\frac{dg(\phi')}{d\phi'} = -(s + \imath k_{\shortparallel} v_{\shortparallel}) - \imath k_{\perp} v_{\perp} cos\phi'$$

$$cos\phi' = \frac{\imath\omega_{c\alpha}}{k_{\perp}v_{\perp}}\frac{dg(\phi')}{d\phi'} + \frac{\imath(s + \imath k_{\shortparallel} v_{\shortparallel})}{k_{\perp}v_{\perp}} \tag{3.13}$$

Put values of equation (3.12) and (3.13) in equation (3.11)

$$f_{\alpha 1} = \int_{\pm\infty}^{\phi} [-\frac{\imath q_{\alpha}\phi_1}{m_{\alpha}}[\frac{\imath\omega_{c\alpha}}{v_{\perp}}\frac{dg(\phi')}{d\phi'} + \frac{\imath(s + \imath k_{\shortparallel} v_{\shortparallel})}{v_{\perp}}]\frac{\partial f_{\alpha 0}}{\partial v_{\perp}} + k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}] \ \cdot exp[g(\phi')]\frac{d\phi'}{\omega_{c\alpha}} \tag{3.14}$$

$$f_{\alpha 1} = -\frac{\imath q_{\alpha}\phi_1}{m_{\alpha}\omega_{c\alpha}}[\frac{\imath\omega_{c\alpha}}{v_{\perp}}\frac{\partial f_{\alpha 0}}{\partial v_{\perp}}\int_{\pm\infty}^{\phi}\frac{dg(\phi')}{d\phi'} \cdot exp[g(\phi')]d\phi' + \frac{\imath(s + \imath k_{\shortparallel} v_{\shortparallel})}{v_{\perp}}\frac{\partial f_{\alpha 0}}{\partial v_{\perp}}$$
$$\int_{\pm\infty}^{\phi} exp[g(\phi')]d\phi' + k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}]\int_{\pm\infty}^{\phi} exp[g(\phi')]d\phi' \tag{3.15}$$

$$\int_{\pm\infty}^{\phi}\frac{dg(\phi')}{d\phi'}exp[g(\phi')]d\phi' = \sum_{n,m} J_n(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})J_m(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})exp[\imath(n-m)\phi] \tag{3.16}$$

$$\int_{\pm\infty}^{\phi} exp[g(\phi')]d\phi' = \sum_{n,m} J_n(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})J_m(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})exp[\imath(n-m)\phi]\frac{-\omega_{c\alpha}}{s + \imath k_{\shortparallel} v_{\shortparallel} + \imath m\omega_{c\alpha}} \tag{3.17}$$

put values of equation (3.17) and (3.16) in equation (3.15)

$$f_{\alpha 1} = \frac{-\imath q_{\alpha}\phi_1}{m_{\alpha}\omega_{c\alpha}}[\frac{\imath\omega_{c\alpha}}{v_{\perp}}\frac{\partial f_{\alpha 0}}{\partial v_{\perp}}\sum_{n,m} J_n(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})J_m(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})exp[\imath(n-m)\phi] + \frac{\imath(s + \imath k_{\shortparallel} v_{\shortparallel})}{v_{\perp}}\frac{\partial f_{\alpha 0}}{\partial v_{\perp}}$$
$$\sum_{n,m} J_n(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})J_m(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})exp[\imath(n-m)\phi]\frac{-\omega_{c\alpha}}{s + \imath k_{\shortparallel} v_{\shortparallel} + \imath m\omega_{c\alpha}}$$
$$+ k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}\sum_{n,m} J_n(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})J_m(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})exp[\imath(n-m)\phi] \tag{3.18}$$

Simplifying

$$f_{\alpha 1} = \frac{\imath q_\alpha \phi_1}{m_\alpha}[\frac{\frac{m\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp} + k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}}{s + \imath k_{\shortparallel} v_{\shortparallel} + \imath m\omega_{c\alpha}}]\sum_{n,m} J_n(\frac{k_\perp v_\perp}{\omega_{c\alpha}})J_m(\frac{k_\perp v_\perp}{\omega_{c\alpha}})exp[\imath(n-m)\phi] \tag{3.19}$$

Consider the Poisson's equation

$$\vec{\nabla}\cdot\vec{E}_1 = 4\pi\sum_\alpha q_\alpha n_\alpha \int f_{\alpha 1} dv \tag{3.20}$$

put equation (3.19) in equation (3.20)

$$\vec{\nabla}\cdot\vec{E}_1 = 4\pi\sum_\alpha q_\alpha n_\alpha \int \frac{\imath q_\alpha \phi_1}{m_\alpha}[\frac{\frac{m\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp} + k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}}{s + \imath k_{\shortparallel} v_{\shortparallel} + \imath m\omega_{c\alpha}}]\sum_{n,m} J_n(\frac{k_\perp v_\perp}{\omega_{c\alpha}})J_m(\frac{k_\perp v_\perp}{\omega_{c\alpha}})exp[\imath(n-m)\phi]$$

$$\vec{E}_1 = -\vec{\nabla}\phi_1 \tag{3.21}$$

$$-\nabla^2\phi_1 = 4\pi\sum_\alpha q_\alpha n_\alpha \int \frac{\imath q_\alpha \phi_1}{m_\alpha}[\frac{\frac{m\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp} + k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}}{s + \imath k_{\shortparallel} v_{\shortparallel} + \imath m\omega_{c\alpha}}]\sum_{n,m} J_n(\frac{k_\perp v_\perp}{\omega_{c\alpha}})J_m(\frac{k_\perp v_\perp}{\omega_{c\alpha}})exp[\imath(n-m)\phi]$$

$\int_0^{2\Pi} exp[\imath(n-m)\phi]d\phi$ exist only if n=m

$$k^2\phi_1 - \imath\omega_{p\alpha}^2\phi_1\int[\frac{\frac{m\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp} + k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}}{s + \imath k_{\shortparallel} v_{\shortparallel} + \imath m\omega_{c\alpha}}]J_n^2(\frac{k_\perp v_\perp}{\omega_{c\alpha}})dv = 0$$

$$\epsilon(k,\omega) = 1 - \frac{\imath\omega_{p\alpha}^2}{k^2}\int\frac{\imath q_\alpha \phi_1}{m_\alpha}[\frac{\frac{m\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp} + k_{\shortparallel}\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}}}{s + \imath k_{\shortparallel} v_{\shortparallel} + \imath m\omega_{c\alpha}}]J_n^2(\frac{k_\perp v_\perp}{\omega_{c\alpha}})dv = 0 \tag{3.22}$$

[29]

3.2 Kappa distribution function

$$f_k = \frac{1}{\pi^{\frac{3}{2}}\theta_{\perp\alpha}^2\theta_{\shortparallel\alpha}}(\frac{\Gamma(\kappa+1)}{\kappa^{\frac{3}{2}}\Gamma(\kappa-\frac{1}{2})})[1 + \frac{v_\perp^2}{\kappa\theta_{\perp\alpha}^2} + \frac{v_{\shortparallel}^2}{\kappa\theta_{\perp\alpha}^2}]^{-\kappa-1} \tag{3.23}$$

Put this value in equation (3.22)

$$\frac{\partial f\alpha}{\partial v_\perp} = \frac{1}{\pi^{\frac{3}{2}}\theta_{\perp\alpha}^2\theta_{\shortparallel\alpha}}(\frac{\Gamma(\kappa+1)}{\kappa^{\frac{3}{2}}\Gamma(\kappa-\frac{1}{2})})[1 + \frac{v_\perp^2}{\kappa\theta_{\perp\alpha}^2} + \frac{v_{\shortparallel}^2}{\kappa\theta_{\perp\alpha}^2}]^{-\kappa-2}(-\kappa-1)\frac{2v_\perp}{\kappa\theta_{\perp\alpha}^2} \tag{3.24}$$

$$\frac{\partial f\alpha}{\partial v_{\shortparallel}} = \frac{1}{\pi^{\frac{3}{2}}\theta_{\perp\alpha}^2\theta_{\shortparallel\alpha}}(\frac{\Gamma(\kappa+1)}{\kappa^{\frac{3}{2}}\Gamma(\kappa-\frac{1}{2})})[1+\frac{v_\perp^2}{\kappa\theta_{\perp\alpha}^2}+\frac{v_{\shortparallel}^2}{\kappa\theta_{\perp\alpha}^2}]^{-\kappa-2}(-\kappa-1)\frac{2v_{\shortparallel}}{\kappa\theta_{\shortparallel\alpha}^2} \tag{3.25}$$

Put (3.24) and (3.25) in equation(3.22)

$$\epsilon(k,\omega) = 1-\sum_\alpha\sum_n \frac{\imath\omega_{p\alpha}^2}{k^2}\frac{1}{\pi^{\frac{3}{2}}\theta_{\perp\alpha}^2\theta_{\shortparallel\alpha}}(\frac{\Gamma(\kappa+1)}{\kappa\frac{3}{2}\Gamma(\kappa-\frac{1}{2})})(-\kappa-1)\frac{2}{\kappa}$$

$$\int\frac{[\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}][1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}}+\frac{v_\perp^2}{\kappa\theta_{\perp\alpha}^2}]^{-\kappa-2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}J_n^2(\frac{k\perp v\perp}{\omega_{c\alpha}})dv \tag{3.26}$$

$$dv = v_\perp dv_\perp d\phi dv_{\shortparallel}$$

$$\epsilon(k,\omega) = 1+\sum_\alpha\sum_{-\infty}^{\infty}\frac{4\imath\omega_{p\alpha}^2}{\pi^{\frac{1}{2}k^2\theta_{\perp\alpha}^2\theta_{\shortparallel\alpha}}}\cdot\frac{(\kappa+1)\Gamma(\kappa+1)}{\kappa^{\frac{5}{2}\Gamma(\kappa-\frac{1}{2})}}\int_{-\infty}^{\infty}[\frac{\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]dv_{\shortparallel}$$

$$\int_0^\infty v_\perp[1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}^2}+\frac{v_\perp^2}{\kappa\theta_{\perp\alpha}^2}]^{-\kappa-2}J_n^2(\frac{k_\perp v_\perp}{\omega_{c\alpha}})dv_\perp \tag{3.27}$$

Using the Neumann's series expansion for the product of Bessel function

$$J_n^2(x) = \frac{x^{2n}}{n!2^{2n}}\sum_{m=0}^{\infty}\frac{C_{n,m}}{(n+m)!}\frac{x^{2m}}{m!2^{2m}} \tag{3.28}$$

$$C_{n,m} = \frac{(-1)^m[2(n+m)!]n!}{(2n+m)!(n+m)!} \tag{3.29}$$

$$\int_0^\infty v_\perp[1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}^2}+\frac{v_\perp^2}{\kappa\theta_{\perp\alpha}^2}]^{-\kappa-2}dv_\perp = \sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}\frac{1}{2^{2(n+m)}}[(\frac{k_\perp^2}{2^2\omega_{c\alpha}^2})^{n+m}\cdot(1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}^2})^{-\kappa-1+n+m}$$

$$\frac{(\kappa\theta_{\perp\alpha}^2)^{1+m+n}\Gamma(1+\kappa-m-n)}{2\Gamma(\kappa+2)}] \tag{3.30}$$

$\because\Gamma(n+1) = n!$

$$= \sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}[(\frac{k_\perp^2}{2^2\omega_{c\alpha}})^{n+m}\cdot(1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}^2})^{-\kappa-1+n+m}\frac{(\kappa\theta_{\perp\alpha}^2)^{1+m+n}\Gamma(1+\kappa-m-n)}{2\Gamma(\kappa+2)}] \tag{3.31}$$

put this value in equation (3.26)

$$\epsilon(k,\omega) = 1+\sum_\alpha\sum_{-\infty}^{\infty}\frac{4\imath\omega_{p\alpha}^2}{\pi^{\frac{1}{2}k^2\theta_{\perp\alpha}^2\theta_{\shortparallel\alpha}}}\cdot\frac{(\kappa+1)\Gamma(\kappa+1)}{\kappa^{\frac{5}{2}\Gamma(\kappa-\frac{1}{2})}}\int_{-\infty}^{\infty}[\frac{\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]dv_{\shortparallel}$$

$$\sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}[(\frac{k_\perp^2}{2^2\omega_{c\alpha}})^{n+m}\cdot(1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}^2})^{-\kappa-1+n+m}\frac{(\kappa\theta_{\perp\alpha}^2)^{1+m+n}\Gamma(1+\kappa-m-n)}{2\Gamma(\kappa+2)}] \tag{3.32}$$

Taking

$$\sum_{n=-\infty}^{\infty}[\frac{\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]=[\sum_{n=-\infty}^{1}+\sum_{n=1}^{\infty}][\frac{\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]$$

$n=-n$

$$=\sum_{n=1}^{\infty}[\frac{\frac{-n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}-\imath n\omega_{c\alpha}}]+\sum_{n=1}^{\infty}[\frac{\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]$$

For perpendicular propagation $k_{\shortparallel}=0$

$$=\sum_{n=1}^{\infty}[\frac{\frac{-n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}-\imath n\omega_{c\alpha}}+\frac{\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]$$

$\because s=-\imath\omega$

$$\sum_{n=-\infty}^{\infty}[\frac{\frac{n\omega_{c\alpha}}{\theta_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\theta_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]=\frac{2\imath\omega_{c\alpha}^2}{\theta_{\perp\alpha}^2}\sum_{n=1}^{\infty}[\frac{n^2}{\omega^2-n^2\omega_{c\alpha}^2}] \tag{3.33}$$

put this value in equation (3.31)

$$\epsilon(k,\omega)=1-\sum_{\alpha}\sum_{n=1}^{\infty}\frac{4\omega_{p\alpha}^2}{\pi^{\frac{1}{2}}k^2\theta_{\perp\alpha}^2\theta_{\shortparallel\alpha}}\frac{(\kappa+1)\Gamma(\kappa+1)}{\kappa^{\frac{3}{2}}\Gamma(\kappa-\frac{1}{2})}\sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}(\frac{k_{\perp}^2}{2^2\omega_{c\alpha}^2})^{n+m}$$
$$(\frac{(\kappa\theta_{\perp\alpha}^2)\Gamma(\kappa+1-m-n)}{\Gamma(\kappa+2)})(\frac{n^2\omega_{c\alpha}^2}{\omega^2-n^2\omega_{c\alpha}^2})\int_{-\infty}^{\infty}(1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}^2})^{-\kappa-1+n+m} \tag{3.34}$$

$$\int_{-\infty}^{\infty}(1+\frac{v_{\shortparallel}^2}{\kappa\theta_{\shortparallel\alpha}^2})^{-\kappa-1+n+m}=\frac{\theta_{\shortparallel\alpha}(\pi\kappa)^{\frac{1}{2}}\Gamma(\kappa+\frac{1}{2}-m-n)}{\Gamma(\kappa+1-m-n)} \tag{3.35}$$

put equation(3.34) in equation (3.33)

$$\epsilon(k,\omega)=1-\sum_{n=1}^{\infty}\frac{4\omega_{p\alpha}^2}{k^2\theta_{\perp\alpha}^2}\frac{(\kappa+1)\Gamma(\kappa+1)}{\Gamma(\kappa-\frac{1}{2})}\sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}(\frac{k^2\theta_{\perp\alpha}^2}{2^2\omega_{c\alpha}^2})^{n+m}$$
$$\frac{\kappa^{n+m-1}\Gamma(\kappa+\frac{1}{2}-m-n)}{\Gamma(\kappa-\frac{1}{2})}(\frac{n^2\omega_{c\alpha}^2}{\omega^2-n^2\omega_{c\alpha}^2}) \tag{3.36}$$

Dielectric constant for Bernstein waves is zero.

$$1=\sum_{n=1}^{\infty}\frac{4\omega_{p\alpha}^2}{k^2\theta_{\perp\alpha}^2}\frac{(\kappa+1)\Gamma(\kappa+1)}{\Gamma(\kappa-\frac{1}{2})}\sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}(\frac{k^2\theta_{\perp\alpha}^2}{2^2\omega_{c\alpha}^2})^{n+m}\frac{\kappa^{n+m-1}\Gamma(\kappa+\frac{1}{2}-m-n)}{\Gamma(\kappa-\frac{1}{2})}(\frac{n^2\omega_{c\alpha}^2}{\omega^2-n^2\omega_{c\alpha}^2}) \tag{3.37}$$

This is the dispersion relation. As we know that

$$\theta_{\perp\alpha}^2 = (\frac{2\kappa - 3}{\kappa})v_{T\perp\alpha}^2 \tag{3.38}$$

put equation (3.37) in equation(3.36)

$$1 = \sum_{n=1}^{\infty} \frac{4\omega_{p\alpha}^2}{k_\perp^2(\frac{2\kappa-3}{\kappa})v_{T\perp\alpha}^2} \sum_{m=0}^{\infty} \frac{C_{n,m}}{m!n!}(\frac{k_\perp^2}{2^2\omega_{c\alpha}^2}(\frac{2\kappa - 3}{\kappa}v_{T\perp\alpha}^2))^{n+m} (\frac{n^2\omega_{c\alpha}^2}{\omega^2 - n^2\omega_{c\alpha}^2})\frac{\kappa^{n+m-1}\Gamma(\kappa + \frac{1}{2} - m - n)}{\Gamma(\kappa - \frac{1}{2})} \tag{3.39}$$

$$b = \frac{k_\perp^2 v_{T\perp\alpha}^2}{2\omega_{c\alpha}^2}, k_D^2 = \frac{2\omega_{p\alpha}^2}{v^2\perp\alpha}$$

put this value in equation (3.38)

$$1 = \sum_{n=1}^{\infty} \frac{2k_D^2}{k_\perp^2(\frac{2\kappa-3}{\kappa})} \sum_{m=0}^{\infty} \frac{C_{n,m}}{m!n!}(\frac{b}{2}(\frac{2\kappa - 3}{\kappa}))^{n+m}(\frac{n^2\omega_{c\alpha}^2}{\omega^2 - n^2\omega_{c\alpha}^2})\frac{\kappa^{n+m-1}\Gamma(\kappa + \frac{1}{2} - m - n)}{\Gamma(\kappa - \frac{1}{2})} \tag{3.40}$$

$$b_K = b(\frac{2\kappa - 3}{2\kappa}), k_{DK}^2 = k_D^2(\frac{2\kappa}{2\kappa - 3})$$

put these values in equation (3.39)

$$1 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{k_D^2}{k_\perp^2}\frac{C_{n,m}}{m!n!}(\frac{b_k}{2})^{n+m}\frac{\kappa^{n+m-1}\Gamma(\kappa + \frac{1}{2} - m - n)}{\Gamma(\kappa - \frac{1}{2})}(\frac{n^2\omega_{c\alpha}^2}{\omega^2 - n^2\omega_{c\alpha}^2}) \tag{3.41}$$

3.3 Generalized (r,q) distribution

3.3.1 Generalized dielectric constant

$$\epsilon(k,\omega) = 1 - \frac{\imath\omega_{p\alpha}^2}{k^2}\int \frac{\imath q_\alpha \phi_1}{m_\alpha}[\frac{\frac{m\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp} + k_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel}}{s + \imath k_\parallel v_\parallel + \imath m\omega_{c\alpha}}]J_n^2(\frac{k_\perp v_\perp}{\omega_{c\alpha}})dv \tag{3.42}$$

$$f(r,q) = (\frac{3}{4\pi\psi_{\perp\alpha}^2\psi_{\shortparallel\alpha}})[\frac{\Gamma(q)}{(q-1)^{\frac{3}{3}+2r}\Gamma(q - \frac{3}{2+2r})\Gamma)(1 + \frac{3}{2+2r})}][1 + \frac{1}{(q-1)}(\frac{v_\shortparallel^2}{\psi_{\shortparallel\alpha}} + \frac{v_\perp^2}{\psi_{\perp\alpha}^2})^{r+1}]^{-q} \tag{3.43}$$

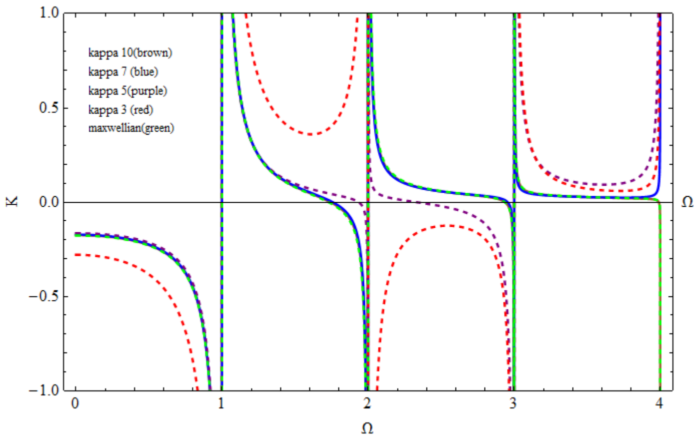

Figure 3.1: Dispersion plots of electron Bernstein waves for different values of kappa with b=0.5, Kappa 10 shows dashed brown line,kappa 7 shows dashed blue line, kappa 5 shows the dashed purple line, Kappa 3 shows the dashed red line while Maxwellian is shown by the green line

$$\frac{\partial f_{\alpha 0}}{\partial v_{\perp}} = -\left(\frac{3}{4\pi\psi_{\perp\alpha}^2\psi_{\shortparallel\alpha}}\right)\left[\frac{\Gamma(q)}{(q-1)^{\frac{3}{3}+2r}\Gamma(q-\frac{3}{2+2r})\Gamma)(1+\frac{3}{2+2r})}\right]\left[1+\frac{1}{(q-1)}\left(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}\right)^{r+1}\right]^{-q-1} \frac{q(r+1)}{q-1}\left(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}\right)^r\frac{2v_{\perp}}{\psi_{\perp\alpha}^2} \quad (3.44)$$

$$\frac{\partial f_{\alpha 0}}{\partial v_{\shortparallel}} = -\left(\frac{3}{4\pi\psi_{\perp\alpha}^2\psi_{\shortparallel\alpha}}\right)\left[\frac{\Gamma(q)}{(q-1)^{\frac{3}{3}+2r}\Gamma(q-\frac{3}{2+2r})\Gamma)(1+\frac{3}{2+2r})}\right]\left[1+\frac{1}{(q-1)}\left(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}\right)^{r+1}\right]^{-q-1} \frac{q(r+1)}{q-1}\left(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}\right)^r\frac{2v_{\shortparallel}}{\psi_{\perp\alpha}^2} \quad (3.45)$$

put values of (3.43) and (3.44) in equation (3.42)

$$\epsilon(k,\omega) = 1 + \sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{\imath\omega_{p\alpha}^2}{k^2}\left(\frac{3}{4\pi\psi_{\perp\alpha}^2\psi_{\shortparallel\alpha}}\right)\left[\frac{\Gamma(q)}{(q-1)^{\frac{3}{2}+2r}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}\right]\frac{2q(r+1)}{(q-1)} \int\frac{\left[1+\frac{1}{q-1}\left(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}}\right)^{r+1}\right]^{-q-1}\left(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}\right)^r\left(\frac{n\omega_{c\alpha}}{\psi_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\psi_{\shortparallel\alpha}^2}\right)}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}J_n^2 \quad (3.46)$$

$$dv = v_{\perp}dv_{\perp}d\phi dv_{\shortparallel}$$

$$\epsilon(k,\omega) = 1 + \sum_{\alpha}\sum_{n=-\infty}^{\infty}\frac{\imath\omega_{p\alpha}^2}{k^2}(\frac{3}{\psi_{\perp\alpha}^2\psi_{\shortparallel\alpha}})[\frac{\Gamma(q)}{(q-1)^{\frac{3}{2}+2r}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}]\frac{q(r+1)}{(q-1)}$$

$$\int_{-\infty}^{\infty}[\frac{(\frac{n\omega_{c\alpha}}{\psi_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\psi_{\shortparallel\alpha}^2})}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}]dv_{\shortparallel}\int_0^{\infty}v_{\perp}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2})^r[1+\frac{1}{(q-1)(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi_{\perp}^2})^{r+1}}]^{-q-1}J_n^2(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})dv_{\perp} \tag{3.47}$$

For perpendicular propagation $k_{\shortparallel} = 0$

$$\sum_{n=-\infty}^{\infty}[\frac{\frac{n\omega_{c\alpha}}{\psi_{\perp\alpha}^2}+\frac{k_{\shortparallel}v_{\shortparallel}}{\psi_{\shortparallel\alpha}^2}}{s+\imath k_{\shortparallel}v_{\shortparallel}+\imath n\omega_{c\alpha}}] = \frac{2\imath\omega_{c\alpha}^2}{\psi_{\perp\alpha}^2}\sum_{n=1}^{\infty}[\frac{n^2}{\omega^2-n^2\omega_{c\alpha}^2}] \tag{3.48}$$

Taking

$$\int_0^{\infty}v_{\perp}J_n^2(\frac{k_{\perp}v_{\perp}}{\omega_{c\alpha}})(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2})^r[1+\frac{1}{(q-1)}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi\ {}_{\perp\alpha}}^2)^{r+1}]^{-q-1}dv_{\perp} \tag{3.49}$$

$$\sum_{m=0}^{\infty}\frac{C_{n,m}}{(n+m)!n!m!}(\frac{k_{\perp}^2}{2^2\omega_{c\alpha}^2})\int_0^{\infty}v_{\perp}^{2n+2m+1}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2})^r[1+\frac{1}{(q-1)}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel}^2}+\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2})^{r+1}]^{-q-1}dv_{\perp}$$

$$\sum_{m=0}^{\infty}\frac{C_{n,m}}{n!m!(n+m)!}(\frac{k_{\perp}^2}{2^2\omega_{c\alpha}^2})^{n+m}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2})^r[\frac{1}{(q-1)}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2})^{r+1}]^{-q-1}$$

$$\int_0^{\infty}v_{\perp}^{2n+2m+1}(1+\frac{\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}}{\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}})^r[(q-1)(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2})^{-r-1}+(1+\frac{\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}}{\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}})^{r+1}]^{-q-1}dv_{\perp} \tag{3.50}$$

Let

$$1+t = [1+\frac{\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}}{\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}}]^{r+1} \tag{3.51}$$

Taking derivative

$$dt = (r+1)[1+\frac{\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}}{\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}}]^r\frac{\frac{2v_{\perp}}{\psi_{\perp\alpha}^2}}{\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}}dv_{\perp}$$

$$\frac{(\frac{v_{\shortparallel}}{\psi_{\shortparallel\alpha}^2})\psi_{\perp\alpha}^2 dt}{2(r+1)} = [1+\frac{\frac{v_{\perp}^2}{\psi_{\perp\alpha}^2}}{\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}}]^r v_{\perp}dv_{\perp} \tag{3.52}$$

From equation (3.50)

$$v_{\perp}^{2n+2m} = (-1)^{n+m}[1-(1+t)^{\frac{1}{1+r}}]^{n+m}(\frac{v_{\shortparallel}^{2}}{\psi_{\shortparallel\alpha}^{2}})^{n+m}(\psi_{\perp\alpha}^{2})^{n+m} \tag{3.53}$$

$$[1-(1+t)^{\frac{1}{1+r}}]^{n+m} = \frac{(n+m)!}{P!(n+m-P)!}(1+t)^{\frac{n+m-P}{1+r}} \tag{3.54}$$

Equation (3.52) can be written as

$$v_{\perp}^{2n+2m} = \frac{(n+m)!}{P!(n+m-P)!}(1+t)^{\frac{n+m-P}{1+r}}(\frac{v_{\shortparallel}^{2}}{\psi_{\shortparallel\alpha}^{2}})^{n+m}(\psi_{\perp\alpha}^{2})^{n+m} \tag{3.55}$$

Put equation (3.51) and equation (3.54) in equation(3.49)

$$\sum_{m=0}^{\infty}\frac{C_{n,m}}{n!m!}(\frac{k_{\perp}^{2}}{2^{2}\omega_{c\alpha}^{2}})(\psi_{\perp\alpha}^{2})\frac{(q-1)^{q+1}}{2(r+1)}[\frac{v_{\shortparallel}^{2}}{\psi_{\shortparallel\alpha}^{2}}]^{n+m-q-qr}\psi_{\perp\alpha}^{2}\frac{1}{P!(n+m-P)!} \int_{0}^{\infty}\frac{(1+t)^{\frac{n+m-P}{1+r}}}{[(q-1)[\frac{v_{\shortparallel}^{2}}{\psi_{\shortparallel\alpha}^{2}}]^{-r-1}+1+t]^{q+1}}dt \tag{3.56}$$

$$\int_{0}^{\infty}t^{-q-\frac{P-m-n}{1+r}+q+1+\frac{P-m-n}{1+r}-1}(1+t)^{\frac{P-m-n}{1+r}-1}(1+t)^{\frac{P-m-n}{1+r}}[t+1-(q-1)(\frac{v_{\shortparallel}^{2}}{\psi_{\shortparallel\alpha}^{2}})^{-r-1}]^{q+1}$$
$$= \frac{\Gamma(q+\frac{P-m-n}{1+r})}{\Gamma(q+1+\frac{P-m-n}{1+r})}F_{1}[q+1,q+\frac{P-m-n}{1+r};q+1+\frac{P-m-n}{1+r};-(q-1)(\frac{v_{\shortparallel}^{2}}{\Psi_{\shortparallel\alpha}^{2}})^{-1-r}] \tag{3.57}$$

Now the Guass Hypergeometric function [30]

$$_{2}F_{1}[q+1,q+\frac{P-m-n}{1+r};q+1+\frac{P-m-n}{1+r};-(q-1)(\frac{{v_{\shortparallel}^{2}}^{2}}{\Psi_{\shortparallel\alpha}})^{-1-r}]$$
$$= \frac{\Gamma(q+1+\frac{P-m-n}{1+r})}{\Gamma(q+\frac{P-m-n}{1+r})}\int_{0}^{\infty}t^{-q-\frac{P-m-n}{1+r}+q+1+\frac{P-m-n}{1+r}-1}[1+t]^{\frac{P-m-n}{1+r}}[t+1+(q-1)(\frac{v_{\shortparallel}^{2}}{\Psi_{\shortparallel\alpha}^{2}})^{-r-1}]^{q+1} \tag{3.58}$$

$$\int_{0}^{\infty}t^{-q-\frac{P-m-n}{1+r}+q+1+\frac{P-m-n}{1+r}-1}[1+t]^{\frac{P-m-n}{1+r}}[t+1+(q-1)(\frac{v_{\shortparallel}^{2}}{\Psi_{\shortparallel\alpha}^{2}})^{-r-1}]^{q+1} = \frac{\Gamma(q+\frac{P-m-n}{1+r})}{\Gamma(q+1+\frac{P-m-n}{1+r})}$$
$$=_{2}F_{1}[q+1,q+\frac{P-m-n}{1+r};q+1+\frac{P-m-n}{1+r};-(q-1)(\frac{{v_{\shortparallel}^{2}}^{2}}{\Psi_{\shortparallel\alpha}})^{-1-r}] \tag{3.59}$$

Equation (3.55) become

$$= \frac{\psi_{\perp\alpha}^2}{2}\sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}(\frac{k_\perp^2\psi_\perp^2}{2^2\omega_{c\alpha}^2})^{n+m}\frac{1}{P!(n+m-P)!}\frac{(q-1)^{q+1}}{1+r}(\frac{\Gamma(q+\frac{P-m-n}{1+r})}{\Gamma(q+1+\frac{P-m-n}{1+r})})$$

$$(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2})^{n+m-q-qr}{}_2F_1[q+1,q+\frac{P-m-n}{1+r};q+1+\frac{P-m-n}{1+r};-(q-1)(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2})^{-1-r}] \quad (3.60)$$

Equation (3.46) become by putting equation (3.59)

$$\epsilon(k,\omega) = 1-\sum_\alpha\sum_{n=1}^{\infty}\frac{3\imath\omega_{p\alpha}^2}{k_\perp^2\psi_{\perp\alpha}^2\psi_{\shortparallel\alpha}}[\frac{\Gamma(q)}{(q-1)^{\frac{3}{2+2r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}]q(q-1)^q(\frac{n^2\omega_{c\alpha}^2}{\omega^2-n^2\omega_{c\alpha}^2})$$

$$\sum_{m=0}^{\infty}\frac{C_{n,m}}{m!n!}(\frac{k_\perp^2\psi_{\perp\alpha}^2}{2^2\omega_{c\alpha}^2})^{n+m}\sum_{P=0}^{n+m}\frac{(-1)^P}{P!(n+m-P)!})(\frac{1}{q+\frac{P-m-n}{1+r}})$$

$$\int_{-\infty}^{\infty}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel}^2})^{n+m-q-qr}{}_2F_1[q+1,q+\frac{P-m-n}{1+r};q+1+\frac{P-m-n}{1+r};-(q-1)(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel}^2})^{-1-r}] \quad (3.61)$$

Taking

$$\int_{-\infty}^{\infty}(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel}^2})^{n+m-q-qr}{}_2F_1[q+1,q+\frac{P-m-n}{1+r};q+1+\frac{P-m-n}{1+r};-(q-1)(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel}^2})^{-1-r}]dv_{\shortparallel} \quad (3.62)$$

Let

$$t = (q-1)[\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2}]^{-r-1}$$

$$\frac{v_{\shortparallel}^2}{\psi_{\shortparallel\alpha}^2} = [\frac{t}{q-1}]^{\frac{1}{2(-r-1)}} \quad (3.63)$$

$$dv_{\shortparallel} = \psi_{\shortparallel\alpha}[\frac{t}{q-1}]^{\frac{1}{2(-r-1)}-1}\frac{dt}{(q-1)} \quad (3.64)$$

Putting (3.62) and equation (3.63) in equation (3.61)

$$= \int_{-\infty}^{\infty} t^{\frac{-qr-q+n+m}{-r-1}}t^{\frac{1}{2(-r-1)}-1}[\frac{1}{(q-1)}]^{\frac{-qr-q+n+m}{-r-1}}[\frac{1}{(q-1)}]^{\frac{1}{2(-r-1)}-1+1}$$

$${}_2F_1[q+1,q+(\frac{P-m-n}{1+r});q+1+(\frac{P-m-n}{1+r});-t]dt \quad (3.65)$$

$$\int_0^{\infty} t^{\frac{2qr+2q-2n-2m-1}{2+2r}-1}(q-1)^{\frac{-2qr-2q+2n+2m+1}{2r+2}}{}_2F_1[q+1,q+(\frac{P-m-n}{1+r});q+1+(\frac{P-m-n}{1+r});-t]dt$$

Let

$$\alpha = \frac{2qr + 2q - 2n - 2m + 1}{2 + 2r} \tag{3.66}$$

Using the identity

$$\int_0^\infty t_2^{\alpha-1} F_1[a, b; c; -t]dt = \frac{\Gamma(c)\Gamma(\alpha)\Gamma(a-\alpha)\Gamma(b-\alpha)}{\Gamma(a)\Gamma(b)\Gamma(c-\alpha)} \tag{3.67}$$

$$\int_{-\infty}^\infty (\frac{v_{\shortparallel}^2}{\psi_{\shortparallel}^2})_2^{n+m-q-qr} F_1[q+1, q + \frac{P-m-n}{1+r}; q+1+\frac{P-m-n}{1+r}; -(q-1)(\frac{v_{\shortparallel}^2}{\psi_{\shortparallel}^2})^{-1-r}]dv_{\shortparallel}$$
$$= (\frac{\psi_{\shortparallel e}}{1+r})(q-1)^{\frac{-q+2n+2m+1}{2+2r}}[\frac{[\frac{q+qr-m-n+P}{1+r}]\Gamma(\frac{2qr+2q-2n-2m-1}{2+2r})\Gamma(\frac{3+2n+2m+2r}{2+2r})\Gamma(\frac{2P+1}{2+2r})}{\Gamma(q+1)\Gamma(\frac{3+2P+2r}{2+2r}}\Big)] \tag{3.68}$$

Putting equation (3.66) in equation (3.60)

$$\epsilon(k,\omega) = 1 - \sum_{n=1}^\infty \frac{3\omega_{pe}^2}{k_{\perp e\psi_{\perp e}}^2}(\frac{1}{(q-1)^{\frac{1-n-m}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})})(\frac{n^2\omega_{ce}^2}{\omega^2 - n^2\omega_{ce}^2})$$
$$\sum_{m=0}^\infty \frac{C_{n,m}}{n!m!}(\frac{k_\perp^2\psi_{\perp e}^2}{2^2\omega_{ce}^2})^{n+m}\sum_{P=0}^\infty \frac{(-1)^P}{P!(n+m-P)!}\frac{(1+r)\Gamma(\frac{2P+1}{2+2r})\Gamma(\frac{3+2n+2m+2r}{2+2r})\Gamma(q-(\frac{1+2n+2m}{2+2r}))}{(1+r)^2\Gamma(\frac{3+2P+2r}{2+2r})} \tag{3.69}$$

$\therefore\Gamma(1+\frac{2P+1}{2+2r}) = (\frac{2P+1}{2+2r})\Gamma(\frac{2P+1}{2+2r})$ $\therefore\Gamma(\frac{3+2P+2r}{2+2r}) = \Gamma(1+\frac{2P+1}{2+2r}) = (\frac{2P+1}{2+2r})\Gamma(\frac{2P+1}{2+2r})$ Put these values in equation (3.67)

$$\epsilon(k,\omega) = 1 - \sum_{n=1}^\infty \frac{3\omega_{pe}^2}{k_\perp^2\psi_{\perp e}^2}\sum_{P+0}^{n+m}\frac{(-1)^P}{P!(n+m-P)!}(\frac{2}{2P+1})$$
$$(\frac{n^2\omega_{ce}^2}{\omega^2-n^2\omega_{ce}^2})[\frac{\Gamma(\frac{3+2n+2m+2r}{2+2r})\Gamma[q-(\frac{1+2n+2m}{2+2r})]}{(q-1)^{\frac{1-n-m}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}] \tag{3.70}$$

Put value of$\psi_\perp^2 = [\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(\frac{3}{2+2r})}{\Gamma(q-\frac{5}{2+2r})\Gamma(\frac{5}{2+2r})}]\upsilon_{T\perp e}^2$

$$\epsilon(k,\omega) = 1 - \sum_{n=1}^{\infty} \frac{3\omega_{pe}^2}{k_\perp^2[\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(\frac{3}{2+2r})}{\Gamma(q-\frac{5}{2+2r})\Gamma(\frac{5}{2+2r})}]\upsilon_{T\perp e}^2} \sum_{m=0}^{\infty} \frac{C_{n,m}}{m!n!}(\frac{k_\perp^2}{2^2\omega_{ce}^2}$$
$$[\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(\frac{3}{2+2r})}{\Gamma(q-\frac{5}{2+2r})\Gamma(\frac{5}{2+2r})}]\upsilon_{T\perp e}^2)^{n+m} \sum_{P=0}^{n+m}[\frac{(-1)^P}{P!(n+m-P)!}](\frac{2}{2P+1})$$
$$(\frac{n^2\omega_{ce}^2}{\omega^2 - n^2\omega_{ce}^2})[\frac{\Gamma(\frac{3+2n+2m+2r}{2+2r})\Gamma(q-(\frac{1+2n+2m}{2+2r}))}{(q-1)^{\frac{1-n-m}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}] \quad (3.71)$$

$$\therefore b = \frac{k_\perp^2\upsilon_{T\perp e}^2}{2\omega_{ce}^2} \therefore k_D^2 = \frac{2\omega_{pe}^2}{\upsilon_{T\perp e}^2} \quad (3.72)$$

$$\epsilon(k,\omega) = 1 - \sum_{n=1}^{\infty} \frac{3k_D^2}{2k_\perp^2[\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}{\Gamma(q-\frac{5}{2+2r})\Gamma(\frac{5}{2+2r})}]} \sum_{P=0}^{n+m} \frac{(-1)^P}{P!(n+m-P)!}$$
$$\sum_{m=0}^{\infty} \frac{C_{n,m}}{m!n!}(\frac{b}{2}[\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(\frac{3}{2+2r})}{\Gamma(q-\frac{5}{2+2r})\Gamma(\frac{5}{2+2r})}])$$
$$\sum_{P=0}^{n+m} \frac{(-1)^P}{P!(n+m-P)!} \sum_{m=0}^{\infty} \frac{C_{n,m}}{m!n!}(\frac{b}{2[\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(\frac{3}{2+2r})}{\Gamma(q-\frac{5}{2+2r})\Gamma(\frac{5}{2+2r})}]})^{n+m}(\frac{2}{2P+1})$$
$$(\frac{n^2\omega_{ce}^2}{\omega^2 - n^2\omega_{ce}^2})[\frac{\Gamma(\frac{3+2n+2m+2r}{2+2r})\Gamma(q-(\frac{1+2n+2m}{2+2r}))}{(q-1)^{\frac{1-n-m}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}] \quad (3.73)$$

$$\therefore k_{D(r,q)}^2 = \frac{2k_D^2\Gamma(\frac{5}{2+2r})\Gamma(q-\frac{5}{2+2r})}{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(\frac{3}{2+2r})} \therefore b_{(r,q)} = b(\frac{3(q-1)^{\frac{-1}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(\frac{3}{2+2r})}{2\Gamma(\frac{5}{2+2r})\Gamma(q-\frac{5}{2+2r})}) \quad (3.74)$$

$$\epsilon(k,\omega) = 1 - \sum_{n=1}^{\infty} \frac{3}{2}k_{D(r,q)}^2 \sum_{m=0}^{\infty} \frac{C_{n,m}}{n!m!}(b_{(r,q)})^{n+m} \sum_{P=0}^{n+m}[\frac{(-1)^P}{P!(n+m-P)!}](\frac{n^2\omega_{ce}^2}{\omega^2 - n^2\omega_{ce}^2})(\frac{1}{2P+1})$$
$$(\frac{n^2\omega_{ce}^2}{\omega^2 - n^2\omega_{ce}^2})[\frac{\Gamma(\frac{3+2n+2m+2r}{2+2r})\Gamma(q-(\frac{1+2n+2m}{2+2r}))}{(q-1)^{\frac{1-n-m}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}] \quad (3.75)$$

The dielectric constant of Bernstein Waves is zero.

$$1 = \sum_{n=1}^{\infty}\frac{3}{2}k^2_{D(r,q)}\sum_{m=0}^{\infty}\frac{C_{n,m}}{n!m!}(b_{(r,q)})^{n+m}\sum_{P=0}^{n+m}[\frac{(-1)^P}{P!(n+m-P)!}](\frac{n^2\omega_{ce}^2}{\omega^2-n^2\omega_{ce}^2})(\frac{1}{2P+1}) \\ (\frac{n^2\omega_{ce}^2}{\omega^2-n^2\omega_{ce}^2})[\frac{\Gamma(\frac{3+2n+2m+2r}{2+2r})\Gamma(q-(\frac{1+2n+2m}{2+2r}))}{(q-1)^{\frac{1-n-m}{1+r}}\Gamma(q-\frac{3}{2+2r})\Gamma(1+\frac{3}{2+2r})}] \quad (3.76)$$

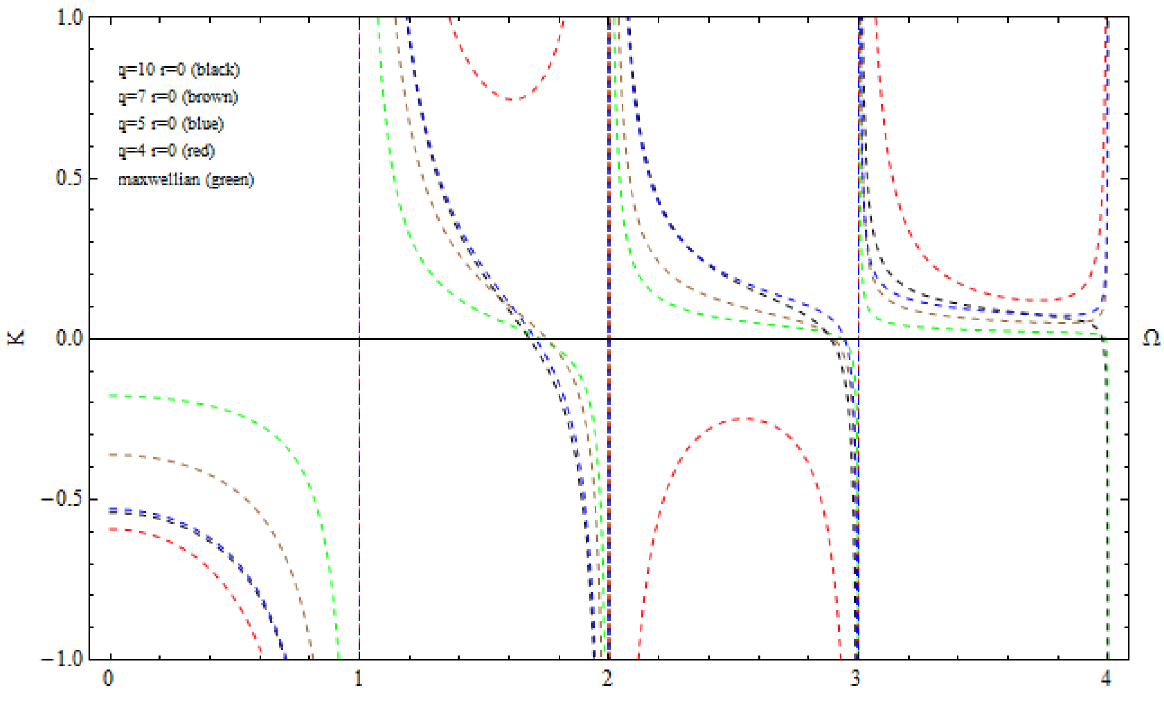

Figure 3.2: EBW's dispersion plots for different values of q with r=0 using the generalized (r,q) distribution. Dashed black line shows q=10 and r=0, Dashed brown line shows the q=7 and r=0, Dashed blue line shows the q=5 and r=0, Dashed red line shows the q=4 and r=0 while Maxwellian is shown by the green dashed line

3.3.2 Summary and Conclusion

The kinetic model is used to present the generalized dielectric constant in non-Maxwellian magnetized anisotropic plasma. We found out the dispersion relation for generalized (r,q) distribution function using Neumann's series expansion for the product of Bessel's function and compared the results with the kappa distribution in the limit $r = 0$, $q = \kappa + 1$ and to

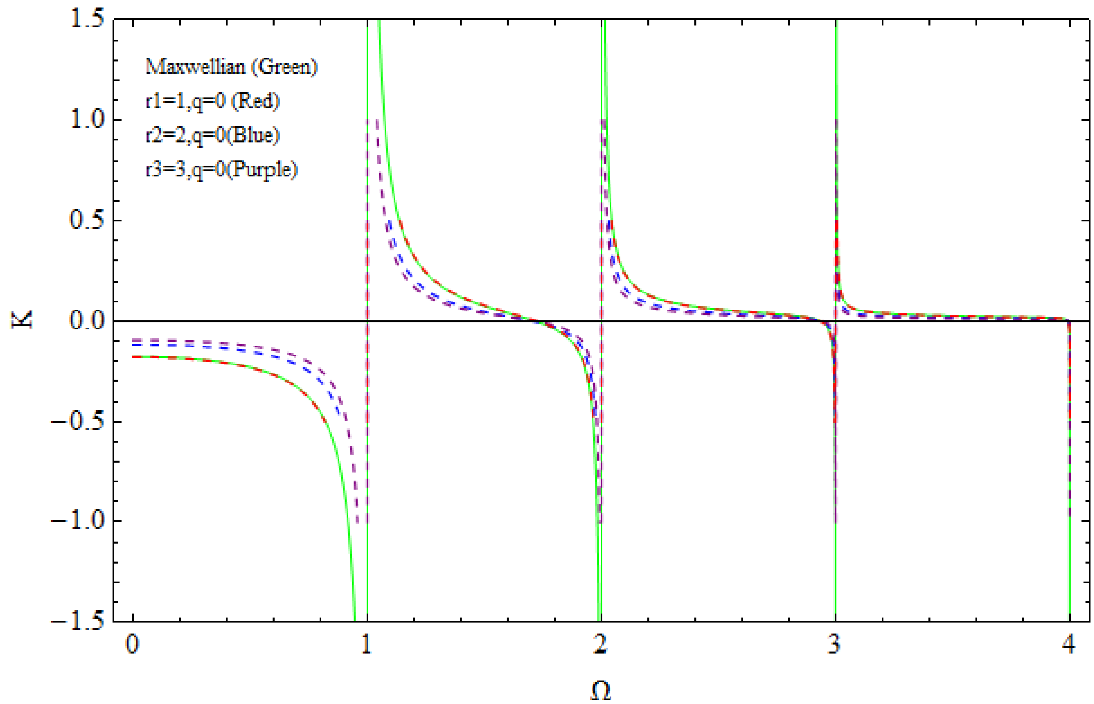

Figure 3.3: EBW's dispersion plots for different values of r with q=4 using the generalized (r,q) distribution. r=1 is shown by dashed red line, r=2 is shown by dashed blue line, r=3 is shown by dashed purple line whereas Maxwellian is shown by the dashed green line.

the Maxwellian for $\kappa \rightarrow \infty$. Plotting $K = \frac{k_{\perp}^2}{k_D^2}$ versus $\frac{\omega}{\omega_{ce}}$ Figure (3.1) shows the different modes of EBW's for $\kappa = 3, 5, 710$ taking $b = 0.5$. There are also result of dispersion relation of EBW's of Maxwellian distribution. For low values of kappa $\kappa = 3$ shows backward BW in frequency bands $\omega_{ce} - 2\omega_{ce}$. On the other side, for higher values of kappa, high energy particles contribute fewer and approached towards Maxwellian for $\kappa \rightarrow \infty$. Figure (3.2) shows fixing the value of r and changing q so by increasing the value of q dispersion curve move towards Maxwellian. In figure (3.3) and (3.4) keeping the value of q fixed and changing the value of r so by increasing the value of r the dispersion curve move away from the Maxwellian.

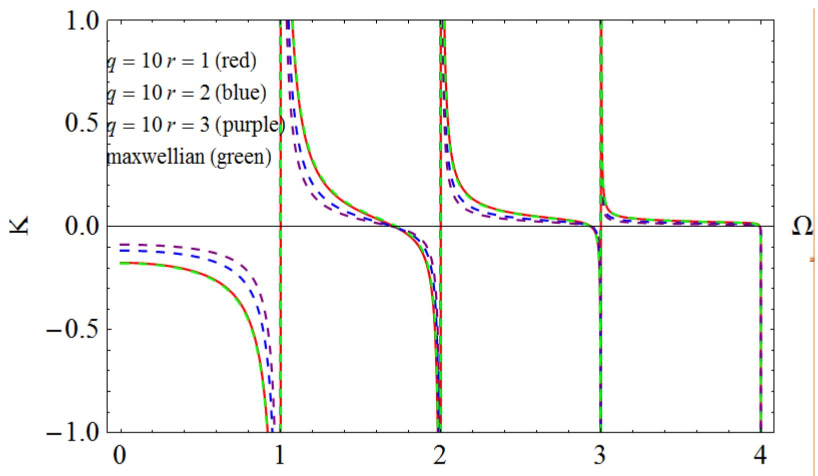

Figure 3.4: EBW's dispersion plots for fixed q=10 and different values of r by using the Grq distribution. r=1 shown by red solid line, r=2 shown by dashed blue line, r=3 shown by dashed purple line whereas Maxwellian is shown by the Dashed green line

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