Heisenberg Limited Phase Estimation using Quantum Optical Interferometry

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Dedication

I'd like to dedicate my work to my family, specifically to my parents, for their unwavering support in my academic endeavors.

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First and foremost, I would like to praise Allah the Almighty, the Most Gracious, and the Most Merciful for His blessing given to me during my study and in completing this thesis. May Allah's blessing goes to His final Prophet Muhammad (peace be up on him), his family and his companions.

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Abstract

Quantum optical interferometry has played a crucial role in precision measurements and quantum sensing, which laid the foundation of quantum metrology—the science of precision measurements using the laws of quantum mechanics. The precision of any measurement may be limited by experimental human error and/or ingenuity of measuring apparatus. However, beyond these limitations quantum mechanics imposes a fundamental restriction on metrology which can be expressed by Heisenberg Uncertainty principle. Fortunately, on the other hand, solution to this quantum-induced limitation is also provided by quantum theory using some non-classical concepts. Hence, quantum metrology deals with techniques to improve the measurement precision beyond the limit set by standard quantum fluctuations. In many physical situations, precision measurement process can be reduced to detection of a small phase shift by using optical interferometric set up.

This thesis is focused on studying the quantum optical phase-estimation and exploring the fundamental bounds on its ultimate precision in the perspective of classical and quantum measurement theories. Using the notion of Fisher information entropy, we find the so-called Cramér–Rao bounds on optical phase-estimation while using classical as well as quantum probes at our optical set up. We show that by using classical probe as initial state, the estimation error is estimated at the best as $\approx 1/\sqrt{\bar{n}}$. This limit is known as Short Noise Limit (SNL). Interestingly, it is shown that using quantum probes the ultimate precision limit surpasses SNL and approaches the value as $\approx 1/\bar{n}$, which is known as *Heisenberg-limit*. It is important to note that in the Heisenberg-limited phase-estimation, the accuracy is enhanced by a factor of $\sqrt{\bar{n}}$ with respect to SNL.

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Chapter 1

Introduction

> "Measurement is the first step that leads to control and eventually to improvement. If you can't measure something, you can't understand it. If you can't understand it, you can't control it. If you can't control it, you can't improve it".
>
> James Harrington

In the early twentieth century the theory currently known as quantum mechanics was developed by Max Planck, Albert Einstein, Louis de Broglie, Neil Bohr, Werner Heisenberg, Erwin Schrodinger, Max Born and Paul Dirac. It began with the plank's assumption of a single energy level of simple harmonic oscillators in the walls of the black body, which was needed to understand the black body radiation spectrum. Due to the expansion of this theory, Einstein was able to explain the concept of photons by explaining photoelectric effect and introduce the concept of photons through analogies with particles. Dirac next integrated each mode of the radiation field with a quantized simple harmonic oscillator to merge the wave and particle-like properties of light, which is the foundation of quantum optics. Quantum mechanics divides the universe into two portions, known as the system and the observer. Except at certain periods, the system and the observer do not interact. Measurement is an interaction that occurs at the given times. Quantum mechanics indicates all of the information about a system that an observer may receive through measurement. All information about unknown parameters is obtained by measuring the systems under investigation; the measurement process is one of the basic pillars of quantum optical metrology. The purpose of taking the measurement is to assign a value to a physical quantity and hence provide an estimate of it. A measure of uncertainty must accompany each experimental estimate, which is defined as "a parameter, associated with a measurement, that describes the dispersion of the value that might fairly be assigned to the measured quantity". Physical laws impose essential bounds on uncertainty.

Now the question arises is that is there any fundamental limits on uncertainty of unknown parameter? As early as 1940s, Cramer [18], Rao [3], and Frechet [4] were able to set fundamental bounds to decrease the variance of an arbitrary estimator with their work. This bound is named as Cramér–Rao lower bound that is closely correlated to the Fisher information, introduced by Fisher in 1920s[5]. Thus, in the theory of estimation the role of Fisher information is very important. Maximization of Fisher information over all possible quantum measurements gives us the quantum Fisher information[6, 7] that provides the quantum lower limits to the Cramér–Rao bounds[8, 9, 10] named as quantum Cramér–Rao bounds. The most critical scenario in the estimating process is referred to as phase estimation which is one of the most important applications of quantum metrology [17].

Over the previous many years, different optical interferometers were proposed to improve the phase sensitivity. They essentially differ in input states and measurement schemes that is needed to obtaining the phase information but the Mach–Zehnder interferometer is one of the most often used optical interferometers to determined the phase shift. The Mach–Zehnder interferometer is a very simple device for showing amplitude division interference. A light beam is split into two components by the first beam splitter, which are then recombined by the second beam splitter. The second beam splitter reflects the beam with a 0 to 100% efficiency depending on the relative phase shift achieved by the beam along the two paths. Phase sensitivity of the Mach-Zehnder Interferometer (MZI) is usually determined by the states that interconnect from the two input ports.The Cramér–Rao bound, which is the ultimate limit on phase sensitivity provided by Quantum Fisher Information [11] and solely depends on the input states, can be reached by making modifications to the measurement technique. The initial step in interferometric measurement is to choose a probe state, then perform interferometric transformations, measure an observable, and lastly, estimate the phase shift with smallest possible error, such as Shot Noise limit and Heisenberg limit.

In interferometry, several approaches are used to determine phase shift, but the main two are output intensity difference method and photon number parity[12] measurements. In the first approach, the difference between the output photons of the Mach-Zehnder interferometer is measured, after which the phase uncertainty can be obtained. Anihilation and creation operators of the field are defined as standard boson operators $\hat{a}$ and $\hat{a^\dagger}$, satisfy the standard commutation relation $[\hat{a}^\dagger, \hat{a}]= 1$ and the number operator $\hat{n}$ so that $\hat{n}\,|n\rangle = n\,|n\rangle$. With these operators, the photon number parity operator may be introduced as

$$\hat{\mathbf{P}} = (-1)^{\hat{n}}, \tag{1.0.1}$$

such that $\hat{\mathbf{P}}\,|n\rangle = (-1)^{\hat{n}}\,|n\rangle$. Numbers are given their parity as even (+1) or odd (-1) by parity operator. There are two eigenvalues for this operator, both of which are degenerate. Even while it is obvious that parity operator is Hermitian, it can also be demonstrated that photon number parity has no classical counterpart. In order to show this, examine the energies of the quantized field, which are discrete in comparison to the energies of a classical field, which are continuous.

We can use classical resources, if we want to get more sensitive phase-shift measurement which can only get us up to the Shot Noise Limit (SNL). In case of classical light as input[13] for Mach-Zehnder interferometer (MZI) the sensitivity of phase estimation can be achieved up to Short Noise limit (SNL)[14]

$$\Delta\phi = 1/\sqrt{\bar{n}}, \tag{1.0.2}$$

this is equal to the average number of photons in the laser field, $\bar{n}$. Accordingly, the SNL is the maximum sensitivity possible with classical light. The Heisenberg-limit (HL) is the maximum quantum sensitivity permitted by quantum mechanics when phase-shifts are caused by linear interactions. It is defined as

$$\Delta\phi_{HL} = 1/n. \tag{1.0.3}$$

Table 1.1: Comparison of Shot Noise limit and Heisenberg limit

Regime	Limits	Attainability
Shot Noise Limit	$\frac{1}{\sqrt{\bar{n}}}$	Remove all classical noise
Heisenberg limit	$\frac{1}{\bar{n}}$	Quantum resources

There are just a few states of light that have no classical counterpart, such as entangled states of light that are able to breach the Shot Noise Limit level of sensitivity. A non-classical state (Entangled state)– the biggest mystery of quantum mechanics – can surpass the sensitivity of an interferometer beyond the short noise limit even approach the Heisenberg-limit, $\Delta\phi = 1/\bar{n}$ [15, 16]

1.1 Thesis Structure

The current thesis is organized as follows:

Chapter 2: It introduces some fundamental concepts in theory of quantum optics that are crucial to our work. Section 2.1 describes the many states of light, including Fock states, coherent states, and N00N states, as well as their properties. Section 2.2 presents an optical beam splitter, phase shifter, and photon detector, as well as their classical and quantum mechanical behavior, as key instruments in quantum optics. Section 2.3 discusses two approaches to beam splitter transformations: the Schrodinger approach for state ket transformation and the Heisenberg approach for field operator transformation. Sections 2.4 and 2.5 describe the behavior of a beam splitter and a beam splitter transformation for a single photon, two photons, and coherent states.

Chapter 3: It is devoted to a review of the main concepts in the estimation theory. Section 3.1 introduces estimators, likelihood functions, and estimator properties. Sections 3.2 and 3.3 analyze the Cramér–Rao lower and upper bounds on estimating theory, which are dependent on classical and quantum Fisher information. Section

4.6 discusses phase estimation, a subset of estimation theory, explained by the Mach-Zehnder interferometer.

Chapter 4: It provides a brief review of Heisenberg-limited phase estimation. Section 4.1 discusses phase estimation techniques in interferometry, including output intensity difference phase estimation and parity measurement phase estimation. Section 4.2 discusses how the parity operator helps us in phase estimation theory. In section 4.3, we calculate the quantum Fisher information in optical interferometry. In section 4.7 and 4.8, we calculate the phase estimation through classical and non-classical light.

Chapter 5: It presents conclusion to our work.

Chapter 2

Fundamentals of Quantized Light and Optical Interferometry

In this chapter, we will introduce the fundamental ideas of quantum optics. A discussion of the quantum states of the light field and their properties will be presented. After that, we will talk about simple optical devices including phase shifters, detectors, and beam splitters. Finally, we will look at how an optical device can change the states of light.

2.1 Quantum States of Light

In this section, we will look at the several intriguing states of light and its characteristics. To characterize quantum states of light, we must adopt a quantum interpretation. A state of light is defined by a state vector $|\phi\rangle$, which in the pure state is referred to as "ket," and a density operator $\hat{\rho} = |\phi\rangle \langle\phi|$ in the mixed state. It contains all of the information about the physical system. The two fundamental states of physical systems in quantum optics are the fock state and the coherent state, which will be discussed in the following subsections. Following that, the N00N state will be introduced.

Definite Photon Number State

Definite photon number states are quantum states of light that contain a well-defined number of particles. These are highly non-classical and are eigen states of the number operator $\hat{n}$

$$\hat{n}\left|n\right\rangle = n\left|n\right\rangle, \qquad (n = 0, 1, 2, 3...) \tag{2.1.1}$$

where n is the eigenvalue of this operator and $\hat{n} = \hat{a}^{\dagger}\hat{a}$ is the number operator. We will now go over how the creation and annihilation operators interact with the Fock state $|n\rangle$ in more detail? Using Eq. 2.1.1, the following equations may be derived.

$$\begin{aligned} \hat{a}^{\dagger}\left|n\right\rangle &= C_{n+1}\left|n+1\right\rangle \\ \hat{a}\left|n\right\rangle &= C_{n}\left|n-1\right\rangle, \end{aligned} \tag{2.1.2}$$

where $C_{n+1} = \sqrt{n+1}$ and $C_n = \sqrt{n}$. In order to get the number state $|n\rangle$, the creation operator $\hat{a}^{\dagger}$ must be applied repeatedly to the ground state $|0\rangle$ as follows:

$$\left|n\right\rangle = \left[\frac{(\hat{a}^{\dagger})^{n}}{\sqrt{n!}}\right]\left|0\right\rangle. \tag{2.1.3}$$

The Fock states are orthonormal and complete:

$$\langle n^{'}|n\rangle = \delta_{n^{'},n}, \tag{2.1.4}$$

and

$$\sum_{n}\langle n|n\rangle = 1, \tag{2.1.5}$$

and together they make up the set $\sum_{n}\left|n\right\rangle\left\langle n\right| = \hat{I}$. As a result, in these states, an arbitrary state can be expanded. Theoretically, photon number states are interesting to work with, but they are incredibly difficult to create experimentally. Only photon number states containing 0, 1 and 2 photons may be efficiently generated. Higher photon number states cannot yet be produced simply, efficiently, or in a regulated manner.

Minimum Uncertainty State — The Coherent State

Coherent states are a significantly more realistic method of expressing an optical field presented as right eigen state of annihilation operator $\hat{a}$. The coherent state is the most appropriate model of radiation emitted by laser. If $|\alpha\rangle$ is a coherent state, the eigen value equation for the annihilation operator $\hat{a}$ may be written as follows.

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \tag{2.1.6}$$

where α represents the eigen value of annihilation operator $\hat{a}$ operating on the coherent state. Coherent state can also be written using the fock states since it provides a complete set of basis

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\,, \tag{2.1.7}$$

Since the number of photons in a coherent state is unknown, we may rise a question that what the probability of having a specific number of photons is

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-\bar{n}} \frac{\bar{n}^n}{n!}, \tag{2.1.8}$$

This means that for a coherent state the photon number distribution is poissonian, with $\bar{n} = |\alpha|^2$. A formalism may also be derived by considering the coherent state as a displaced vacuum state. By using Eq. 2.1.3 into Eq. 2.1.7. we get

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha\hat{a}^\dagger)^n}{n!} |n\rangle = e^{\alpha\hat{a}^\dagger - \frac{|\alpha|^2}{2}} |0\rangle\,, \tag{2.1.9}$$

It is possible to show that

$$e^{\alpha\hat{a}^\dagger - \frac{|\alpha|^2}{2}} |0\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} |0\rangle \tag{2.1.10}$$

If we use the following definition of the unitary displacement operator $D(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, we find

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle\,, \tag{2.1.11}$$

This indicates that the coherent state is generated by displacing vacuum state. Coherent states are the quantum mechanical states that are most similar to classical ones.

The N00N States

This part covers the statistical models of the state of concern in this thesis, known as the N00N state. The two-mode N00N state can be written as follows;

$$|N00N\rangle_{a_0,b_0} = \frac{1}{\sqrt{2}}(|N0\rangle_{a_0,b_0} + |0N\rangle_{a_0,b_0}), \tag{2.1.12}$$

This is a combination of two possibilities in which all N photons appear either in mode a_0 or mode b_0. This state differs from the N photons Fock state in that the N photons in this state are entangled with equal probability in two orthogonal modes a_0 and b_0, Whereas the N photons in the latter have a 100% chance of appearing in a single mode. It takes a lot of effort to create two mode N00N states, which we will use in Chapter 4 to determine the phase.

2.2 Basic Optical Devices

In this section we discuss the fundamental optical devices used to produce NOON states after introducing various essential states of light. These are optical beam splitters, phase shifters, and photon detectors. The action of a phase shifter (PS) is to change the phase of a certain mode and a photon detector is used to detect the output photons. Now, we will discuss the optical beam splitter as well as its classical and quantum mechanical behavior. Now, we will discuss the optical beam splitter as well as its classical and quantum mechanical behavior.

2.2.1 Optical Beam Splitter

The optical beam splitter is an essential part of many optical interferometers that splits an incoming beam of light into multiple components on detecting some measurable quantity. Additionally, it is used in the reverse process to mix components of the beam into a single one. The most commonly discussed beam splitter is the so-called 50/50 beam splitter. It is considered to be a mirror in which the reflecting layer is so transparent that only half of the incident light is reflected, the other half being transmitted, splitting an incoming light beam by equal intensity. In this thesis we always consider a beam splitter to be a semi-reflecting device. Before discussing the quantum mechanical behavior of beam splitter, it is worth noting how one handles a classical beam splitter and why this approach fails for most quantum states?

A Classical Treatment

Classically, a beam splitter is a device that divides an incoming light beam into two equal parts. As illustrated in Figure 2.1, a classical beam of complex amplitude E is applied directly to the input port of the lossless beam splitter. The beam splitter generates two orthogonal beams, E_t and E_r, where E_t is the transmitted beam and E_r is the reflected beam. The input-output relations are expressed as follows:

$$\begin{aligned} E_t &= t_1 E \\ E_r &= r_1 E \end{aligned} \tag{2.2.1}$$

where r_1 is the reflection coefficient and t_1 is the transmission coefficient. Since the

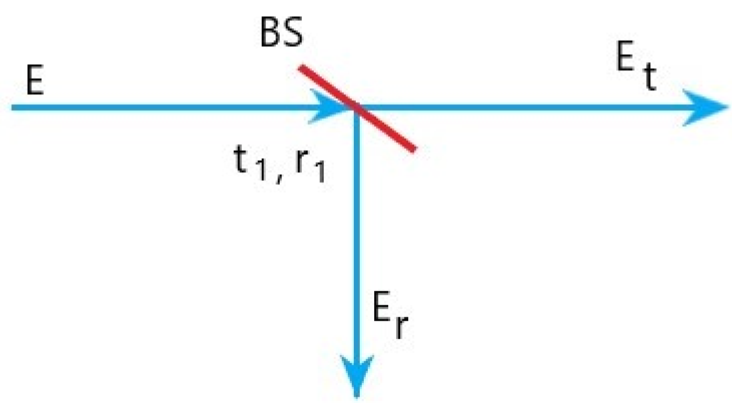

Figure 2.1: Classical Beam splitter

field intensity "I" is proportional to its amplitude, we may write

$$I \propto |E|^2. \tag{2.2.2}$$

The energy of the field should be conserved because we are employing a passive beam splitter. In order to keep the energy balance, it is essential that the incoming and outgoing fields are equal in strength

$$|E|^2 = |E_r|^2 + |E_t|^2, \tag{2.2.3}$$

which requires that

$$|r_1|^2 + |t_1|^2 = 1, \tag{2.2.4}$$

this is a condition for energy(field) conservation.

Fully Quantum Mechanical Model

However, a beam splitter in quantum optics does not split a light beam into two pieces since it reflects or transmits every photon equally. It is important to note that in the quantum theory of beam splitter, the first step is to quantize our system, which implies that classical field amplitudes will be replaced by some Heisenberg field operators

$$E \rightarrow \hat{a}_0, \quad E_t \rightarrow \hat{a}_f, \quad E_r \rightarrow \hat{b}_f. \tag{2.2.5}$$

Operators are Heisenberg in the sense that they have certain evolution parameters, as we have done in the many interpretations of Quantum Mechanics. It is important to note that the field in this case is evolving and operators are linked with it, therefore we referred as Heisenberg field operator. However, operators in each field must fulfill the commutation relations,

$$[\hat{a}_f^\dagger, \hat{a}_f] = [\hat{b}_f^\dagger, \hat{b}_f] = 1 \quad if \quad [\hat{a}_o^\dagger, \hat{a}_o] = 1. \tag{2.2.6}$$

Let us try to draw the comparable picture in Quantum Mechanics with the replacement of operators

$$\begin{aligned} \hat{a}_f &= t_1 \hat{a}_0 \\ \hat{b}_f &= r_1 \hat{a}_0. \end{aligned} \tag{2.2.7}$$

From above equation it is clear that

$$[\hat{a}_f^\dagger, \hat{a}_f] = |t_1|^2 \neq 1 \quad and \quad [\hat{b}_f^\dagger, \hat{b}_f] = |r_1|^2 \neq 1, \tag{2.2.8}$$

These are equal to one only if total field is transmitted or reflected. In short, Eq. (2.2.7) does not preserve commutation relations. Over here, we are dealing with the field operators so if we want to get equivalent intensity interpretation, what we need to do is? We should have the idea of field intensity if we are dealing with number of photons. Number of photon description in operator form can be taken in account by the means of number operators given as

$$\hat{a}_0^\dagger \hat{a}_0 = \hat{n}_{a_0}, \quad \hat{a}_f^\dagger \hat{a}_f = \hat{n}_{a_f} \quad and \quad \hat{b}_f^\dagger \hat{b}_f = \hat{n}_{b_f}, \tag{2.2.9}$$

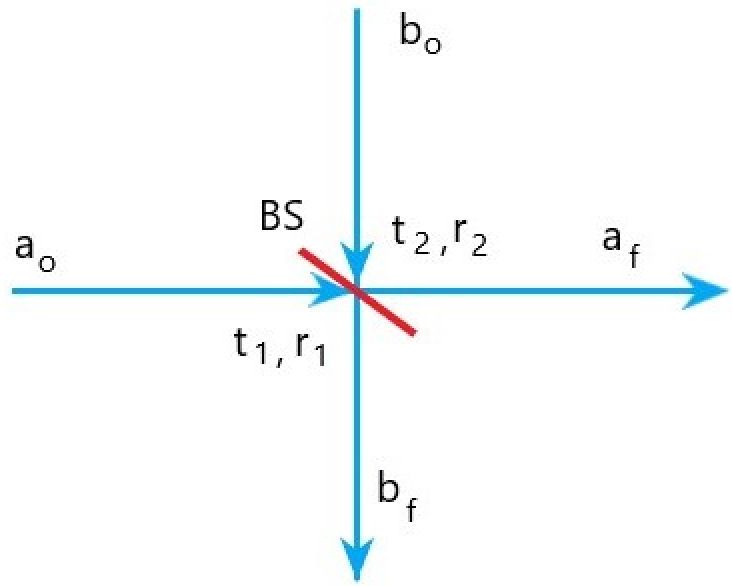

Figure 2.2: Quantum Beam splitter

so intensity can be written as

$$\hat{n}_{a_0} = \hat{n}_{a_f} + \hat{n}_{b_f}, \tag{2.2.10}$$

which demands that

$$|r_1|^2 + |t_1|^2 = 1, \tag{2.2.11}$$

This is trivial case that number of incoming photons are equal to number of outgoing photons. The above description of commutation relations implies that field quantization alone is not enough to understand the quantum behavior of beam splitter. In classical beam splitter, there is an unused port.

In Quantum theory vacuum fields are present even no field sent from the unused port of the beam splitter. These vacuum fluctuations lead to important physical effects. We need a complete description of the quantum beam splitter, for which we represent $\hat{b}_0$ as a field operator for vacant input mode. Now beam splitter transformation can have the form

$$\begin{aligned} \hat{a}_f &= t_1 \hat{a}_0 + r_2 \hat{b}_0 \\ \hat{b}_f &= r_1 \hat{a}_0 + t_2 \hat{b}_0, \end{aligned} \tag{2.2.12}$$

matrix form of Eq.(2.2.12) is

$$\begin{pmatrix} \hat{a}_f \\ \hat{b}_f \end{pmatrix} = \begin{pmatrix} t_1 & r_2 \\ r_1 & t_2 \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{b}_0 \end{pmatrix} \tag{2.2.13}$$

and

$$\begin{pmatrix} \hat{a}_f \\ \hat{b}_f \end{pmatrix} = \hat{U} \begin{pmatrix} \hat{a}_0 \\ \hat{b}_0 \end{pmatrix}. \tag{2.2.14}$$

For completeness of commutation relations, this transformation matrix U has to be unitary. The unitary requirement is

$$|t_1|^2 + |r_1|^2 = |t_2|^2 + |r_2|^2 = 1, \tag{2.2.15}$$

and along with

$$t_1^* r_2 + r_1^* t_2 = 0, \tag{2.2.16}$$

these requirements are equal to the conservation of energy(field) at the lossless beam splitter. In general beam splitter might have two different sets of transmissivity and reflectivity. Here we assume a symmetric beam splitter, for this $|t_1| = |t_2|$ and $|r_1| = |r_2|$, What does it mean is? that both sides have the same reflectivity and transmissivity. Eq (2.2.13) must satisfies the following relations given as

$$\begin{aligned} [\hat{a}_f^\dagger, \hat{a}_f] &= |t_1|^2 + |r_2|^2 = 1, \\ [\hat{b}_f^\dagger, \hat{b}_f] &= |r_1|^2 + |t_2|^2 = 1, \end{aligned} \tag{2.2.17}$$

if

$$[\hat{a}_0^\dagger, \hat{a}_0] = [\hat{b}_0^\dagger, \hat{b}_0] = 1, \tag{2.2.18}$$

so beam splitter transformation matrix can be written as

$$\hat{U} = \begin{pmatrix} \cos\theta & \iota \sin\theta \\ \iota \sin\theta & \cos\theta \end{pmatrix}, \tag{2.2.19}$$

Over here, $cos^2\theta$ is $|t_1|^2$,basically, is equal to transmissivity and $sin^2\theta$ is $|r_1|^2$ is equal to reflectivity of a beam splitter.

Now we choose a suitable angle, $\theta = \pi/4$, at which reflection and transmission coefficients are the same we call such type of beam splitter as 50:50 beam splitter, so unitary transformation matrix for 50:50 beam splitter can be written as

$$\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \iota \\ \iota & 1 \end{pmatrix}. \tag{2.2.20}$$

2.3 Beam splitter Transformation: Two different Approaches

In this section we will talk about the transformation of the beam splitter. A beam splitter can transform input state into the output state. A general input-output relation is given as

$$\begin{pmatrix} \hat{a}_f \\ \hat{b}_f \end{pmatrix} = \begin{pmatrix} cos\theta & \iota sin\theta \\ \iota sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{b}_0 \end{pmatrix}, \tag{2.3.1}$$

so we can write

$$\begin{aligned} \hat{a}_f &= \hat{a}_0 cos\theta + \iota \hat{b}_0 sin\theta \\ \hat{b}_f &= \iota \hat{a}_0 sin\theta + \hat{b}_0 cos\theta. \end{aligned} \tag{2.3.2}$$

In case of 50:50 beam splitter we can write

$$\begin{aligned} \hat{a}_f &= \frac{1}{\sqrt{2}}(\hat{a}_0 + \iota \hat{b}_0) \\ \hat{b}_f &= \frac{1}{\sqrt{2}}(\iota \hat{a}_0 + \hat{b}_0). \end{aligned} \tag{2.3.3}$$

Now we are interested to know, how these inputs are, actually, transforms as output? We can do this job in two different approaches, Heisenberg and Schrodinger approach.

2.3.1 The Schrodinger Approach: For State ket Transformations

In the Schrodinger approach, the state ket that represents a quantum system depends on the evolution parameter and we assume that states are evolving by these parameters. In the case of the beam splitter, the evolution parameter is θ, which affects the state of the system so we can write the state vector as

$$|\alpha, \theta\rangle \longrightarrow \hat{U}(\theta) |\alpha\rangle , \tag{2.3.4}$$

where $|\alpha, \theta\rangle$ is evolved state and $\hat{U}(\theta)$ is unitary evolution operator which can be expressed in term of Hamiltonian of system,

$$\hat{U}(\theta) = exp\left(\frac{-\iota \hat{H}\theta}{\hbar}\right), \tag{2.3.5}$$

The unitary property of $\hat{U}$ means that

$$\hat{U}^{\dagger}(\theta)\hat{U}(\theta) = 1. \tag{2.3.6}$$

So in this approach we, actually, need to know the hamiltonian of system. Since quantum operator $\hat{A}$ representing observable quantities, are all time-independent so we can write

$$\hat{A}(\theta) \longrightarrow \hat{A}(0). \tag{2.3.7}$$

When we talk about the transformation of states by a beam splitter we can understand that every state can be written in term of the field operator i.e. a number state $|n\rangle$ can be written as n time action of creation operator of this mode on vacuum state, which is

$$|n\rangle = \frac{(\hat{a}_0^{\dagger})^n}{\sqrt{n}} |0\rangle , \tag{2.3.8}$$

where $\hat{a}^{\dagger}$ is creation operator of input mode a. Likewise, a coherent state can be written as action of displacement operator on vacuum state, which is

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle , \tag{2.3.9}$$

where $\hat{D}(\alpha)$ is so-called displacement operator

$$\hat{D}(\alpha) = exp\left(\alpha\hat{a}_0^{\dagger} - \alpha^{*}\hat{a}_0\right). \tag{2.3.10}$$

Now the question is how can a beam splitter change the vacuum state? As a passive device, the beam splitter cannot produce or destroy any photons. The vacuum state at input remains the same at output due to the conservation of energy (field). As a result, we may say

$$|0,0\rangle \xrightarrow{BS} |0,0\rangle . \tag{2.3.11}$$

Considered a number state $|n, n^{'}\rangle$ that can be written in terms of n time action of creation operator on vacuum state. The unitary transformation matrix transforms the number state as

$$|n, n^{'}\rangle_{out} = \hat{U}(\theta) |n, n^{'}\rangle = \hat{U}(\theta)\frac{(\hat{a}_0^{\dagger})^n(\hat{b}_0^{\dagger})^{n^{'}}}{\sqrt{n!n^{'}!}} |0,0\rangle , \tag{2.3.12}$$

where we use the identity $\hat{U}^{\dagger}(\theta)\hat{U}(\theta)=1$, Due to the fact that the beam splitter does not affect the vacuum state, we may express the output state as follows:

$$|n,n^{'}\rangle_{out}=\frac{\left(\hat{a}_0^{\dagger}cos\theta+\iota\hat{b}_0^{\dagger}sin\theta\right)^{n}\left(\iota\hat{a}_0^{\dagger}sin\theta+\hat{b}_0^{\dagger}cos\theta\right)^{n^{'}}}{\sqrt{n!,n^{'}!}}\left|0,0\right\rangle. \tag{2.3.13}$$

we can solve this by using binomial expansion which is

$$\left(X+Y\right)^{n}=\sum_{k=0}^{n}\binom{n}{k}X^{n-k}\,Y^{k}. \tag{2.3.14}$$

This expansion yields

$$\begin{aligned}|n,n^{'}\rangle_{out}=&\sum_{k,k^{'}}\binom{n}{k}\binom{n^{'}}{k^{'}}\sqrt{\frac{(k+k^{'})!(n+n^{'}-k-k^{'})!}{n!n^{'}!}}\\&\times(cos)^{n^{'}+k-k^{'}}(\iota sin)^{n^{'}+k^{'}-k}\left|k^{'}+k,n+n^{'}-k-k^{'}\right\rangle.\end{aligned} \tag{2.3.15}$$

In order to understand how photons behave at the beam splitter? we may use above equation.

2.3.2 Heisenberg Approach: For Field Operator Transformations

In the Heisenberg approach, It is assumed that the state of the system remains constant but in contrast, the quantum operators are affected by evolution parameters θ. We might write state as follows:

$$|\alpha\rangle\longrightarrow|\alpha,\theta=0\rangle \tag{2.3.16}$$

and

$$\hat{A}(\theta)=exp\left(\frac{\iota\hat{H}\theta}{\hbar}\right)\hat{A}(0)exp\left(\frac{-\iota\hat{H}\theta}{\hbar}\right), \tag{2.3.17}$$

it is possible to write as follows

$$\hat{A}(\theta)=\hat{U}^{\dagger}(\theta)\hat{A}(0)\hat{U}(\theta), \tag{2.3.18}$$

where $\hat{U}(\theta)$ is unitary operator depending on Hamiltonian of system. As an alternative, we might use the Heisenberg equation of motion, which states that

$$\frac{d}{d\theta}\hat{A}=\frac{-\iota}{\hbar}[\hat{A}(0),\hat{H}]=\hat{A}(\theta). \tag{2.3.19}$$

Table 2.1: Comparison

Representation	State	Operators
Schrödinger	$\hat{U}(\theta)\left\|\alpha\right\rangle$	No change
Heisenberg	No change	$\hat{U}^{\dagger}(\theta)A(\theta)\hat{U}(\theta)$

We need to know the Hamiltonian of the system in both scenarios, this means we must build an effective Hamiltonian for the evolution operator first. Therefore, let us have a look at

$$\begin{aligned}\hat{a}_f &= \hat{a}_0 cos\theta + \iota\hat{b}_0 sin\theta \\ \hat{b}_f &= \iota\hat{a}_0 sin\theta + \hat{b}_0 cos\theta.\end{aligned} \tag{2.3.20}$$

If evolution parameter is zero or $\theta = 0$ then $\hat{a}_f = \hat{a}_0$ and $\hat{b}_f = \hat{b}_0$ when $\theta > 0$ the evolve states are

$$\begin{aligned}\hat{a}_f &= \hat{a}_0(\theta) \\ \hat{b}_f &= \hat{b}_0(\theta).\end{aligned} \tag{2.3.21}$$

Basically, Eq. (2.3.20) are evolve states and equal to differential equations

$$\frac{d}{d\theta}\hat{a}_o(\theta) = \iota\hat{b}_o(\theta), \qquad \frac{d}{d\theta}\hat{b}_0(\theta) = \iota\hat{a}_0(\theta), \tag{2.3.22}$$

These equations are like Heisenberg equations for $\hat{a}_0(\theta)$ and $\hat{b}_0(\theta)$ provided the evolution is given by the effective Hamiltonian

$$\hat{H} = -\hbar(\hat{a}_o^{\dagger}\hat{b}_0 + \hat{a}_0\hat{b}_0^{\dagger}). \tag{2.3.23}$$

It is now possible to express the evolution operator as

$$\hat{U}(\theta) = exp\left(\iota\theta(\hat{a}_0^{\dagger}\hat{b}_0 + \hat{a}_0\hat{b}_0^{\dagger})\right). \tag{2.3.24}$$

Now we want to see how evolution operator transforms the input field operators? According to Heisenberg approach the field operators can be transform as

$$\begin{bmatrix}\hat{a}_0(\theta) \\ \hat{b}_0(\theta)\end{bmatrix} = \hat{U}^{\dagger}(\theta)\begin{bmatrix}\hat{a}_0 \\ \hat{b}_0\end{bmatrix}\hat{U}(\theta), \tag{2.3.25}$$

where $\hat{U}(\theta)$ is unitary operator. We can solve this by using Baker-Hausdorff lemma which is

$$\begin{aligned} exp\left(\iota\theta\hat{H}\right)\hat{A}(0)exp\left(-\iota\theta\hat{H}\right) &= \hat{A}(0)+\iota\theta[\hat{H},\hat{A}]+\left(\tfrac{\iota^2\theta^2}{2!}\right)[\hat{H},[\hat{H},\hat{A}]] \\ &+....+\left(\tfrac{\iota^n\theta^n}{n!}\right)[\hat{H},[\hat{H},[\hat{H},...[\hat{H},A]...]]]+....., \end{aligned} \tag{2.3.26}$$

where $\hat{A}$ is arbitrary operator and $\hat{H}$ is Hamiltonian of system. By using this lemma in Eq (2.3.17) we can simply compute the transformation results which are equivalent to Eq (2.3.20).

2.4 Experiments with photons

2.4.1 Splitting a Single Photon State: Generate Entanglement

Now, we will discuss the simplest case when a single-photon is incident on the input port a_0 and no photon on port b_0. Since a single photon state is action of creation operator on vacuum state i.e.

$$|1\rangle = a_0^\dagger |0\rangle . \tag{2.4.1}$$

So we can write the initial state $|\psi\rangle$ in terms of field operators and vacuum state

$$|\psi\rangle_{in} = \hat{a}_0^\dagger |0,0\rangle . \tag{2.4.2}$$

As a result of the beam splitter transformation, it is straightforward to write $\hat{a}_0$ in terms of $\hat{a}_f$ and $\hat{b}_f$

$$\hat{a}_0^\dagger = \frac{1}{\sqrt{2}}\left(\hat{a}_f^\dagger + \iota\hat{b}_f^\dagger\right) . \tag{2.4.3}$$

The beam splitter transforms the input state as

$$|1,0\rangle = \hat{a}_0^\dagger |0,0\rangle \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(|1,0\rangle + \iota\,|0,1\rangle\right) . \tag{2.4.4}$$

This is a very interesting result we can see that the input state at the fundamental level had only a single photon. The output indicates that either we will have a single photon at port a_f and vacuum at b_f or we will have a vacuum at a_f and one photon at b_f.

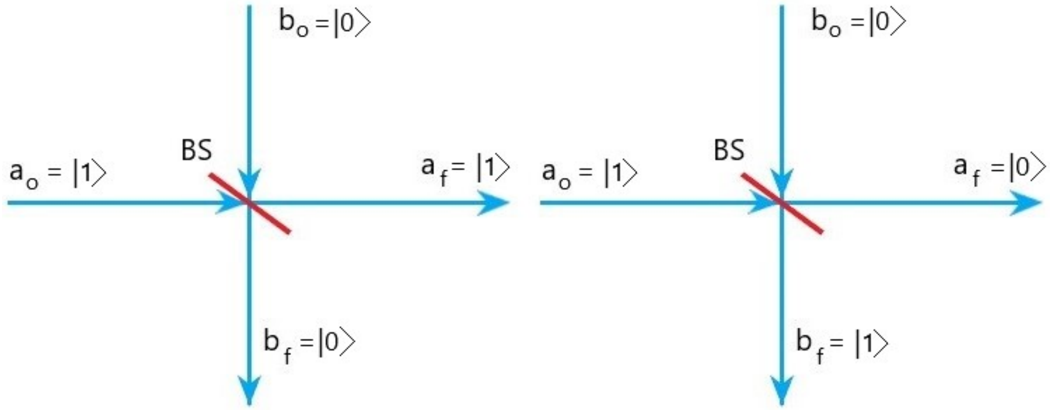

Figure 2.3: Single photon at Beam splitter

Here, the photon is shown to be a particle that may be transmitted or reflected but not split into two components. Since the mode of output state cannot be expressed in the tensor product, we may conclude that this is an entangled state.

2.4.2 Two-photon Interference: Hong–Ou–Mandel Experiment

In the case of two-photon on a beam splitter with one photon on input port a_0 and other at port b_0. We can rewrite this state in much the same way we did in the previous example so initial state $|\psi\rangle$ can has the form

$$|\psi\rangle = |1,1\rangle = \hat{a}_0^{\dagger}\hat{b}_0^{\dagger}|0,0\rangle\,. \tag{2.4.5}$$

Beam splitter transforms the state as

$$\begin{aligned}\hat{a}_0^{\dagger}\hat{b}_0^{\dagger}|0,0\rangle &\xrightarrow{BS} \left(\frac{\hat{a}_f^{\dagger}+\iota\hat{b}_f^{\dagger}}{\sqrt{2}}\right)\left(\frac{\iota\hat{a}_f^{\dagger}+\hat{b}_f^{\dagger}}{\sqrt{2}}\right)|0,0\rangle \\ &= \frac{\iota}{\sqrt{2}}\big(|2,0\rangle+|0,2\rangle\big)\,.\end{aligned} \tag{2.4.6}$$

Again, this is a really unusual outcome. One photon in either mode would have to be reflected into the opposite mode in order to generate the output state $|2,0\rangle$ or $|0,2\rangle$. Interestingly, if the beam splitter is 50:50, the output $|1,1\rangle$ is not achievable. This is because these results have two possibilities: either both photons are transmitted, as in

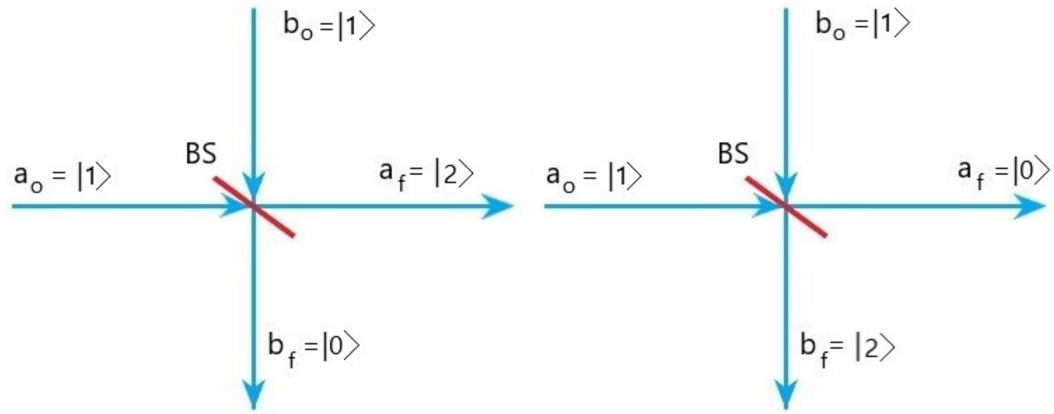

Figure 2.4: Two photons at Beam splitter

the case of $|1,1\rangle$ or both photons are reflected, as in the case of $\iota^2|1,1\rangle = -|1,1\rangle$. The outcome is that these probability amplitudes interact and cancel out in a destructive way. This is known as the Hong-Ou-Mandel effect and was first experimentally verified in 1987 [18]. Next, we will look at the case of a input coherent state.

2.4.3 Coherent State at a Beam Splitter

As we have discussed earlier, the coherent state $|\alpha\rangle$ is the most classical of pure single-mode field states and can be written as a superposition on the number state basis as

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle , \tag{2.4.7}$$

Moreover, we may write coherent state as action of the displacement operator on the vacuum state

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle , \tag{2.4.8}$$

Therefore, the coherent state is sometimes called the "displaced vacuum state" (DVS). So, let us say that a_0 is the input port for coherent state and b_0 is for vacuum.Because a coherent state is the outcome of a displacement operator operating on a vacuum, the initial state may be expressed as follows:

$$|\psi\rangle_{in} = \hat{D}_{a_0}(\alpha)|0,0\rangle . \tag{2.4.9}$$

We may take output state according to

$$
\begin{aligned}
|\alpha, 0\rangle &\xrightarrow{BS} exp\left[\frac{\alpha}{\sqrt{2}}\left(\hat{a}_f^\dagger + \iota \hat{b}_f^\dagger\right) - \frac{\alpha^*}{\sqrt{2}}\left(\hat{a}_f - \iota \hat{b}_f\right)\right]|0,0\rangle \\
&= exp\left[\left(\frac{\alpha}{\sqrt{2}}\right)\hat{a}_f^\dagger - \left(\frac{\alpha^*}{\sqrt{2}}\right)\hat{a}_f\right]|0\rangle + exp\left[\left(\frac{\iota\alpha}{\sqrt{2}}\right)\hat{b}_f^\dagger - \left(\frac{-\iota\alpha^*}{\sqrt{2}}\right)\hat{b}_f\right]|0\rangle \quad (2.4.10) \\
&= \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{a_f}\left|\frac{\iota\alpha}{\sqrt{2}}\right\rangle_{b_f}.
\end{aligned}
$$

This is also a really intriguing outcome. This is because the input state is coherent, and the intensity of input state amplitude is equal to 1. When we look at the output, we notice that it is a coherent state with intensity values of $\frac{|\alpha|^2}{2}$ at port a_f and $\frac{|\alpha|^2}{2}$ at port b_f respectively. To put it simply, 50% of the incoming intensity is transmitted, whereas 50% is reflected. In this case, the beam splitter behaves like a classical beam splitter, which divides the incoming beam into two halves. In the following chapters, we will use all these basic concepts for phase estimation with Mach-Zehnder interferometers.

Chapter 3

Quantum Measurement and Estimation Theory

Estimation theory is concerned with determining a quantitative value for essential parameter from a given data collection. Here quantum measurements are described in their most general form, in the formalism of positive operator-valued measures (POVM). We review the problem of estimating an unknown parameter from a given data. We discuss a widely used formalism in parameter estimation theory: The Cramer-Rao bound and Fisher information. Then we will extend our study of quantum metrology to cover all possible measurements, resulting in the concept of quantum Fisher information (QFI) and allowing us to go beyond the limitations of classical estimation theory. Finally, we provide phase estimation using a Mach-Zehnder interferometer, which is one of the most significant applications of quantum metrology.

3.1 Measurement Process

We begin with the measurement process which is one of the essential pillars of any physical system. The goal of the measurement process is to assign a value to a physical quantity, resulting in an estimate of it. Each experimental estimate must include a measure of uncertainty, defined as "a parameter associated with a measurement that shows the dispersion of the value that might appropriately be provided to the measurable quantity". Physical laws, such as the Heisenberg limit, set crucial bounds on uncertainty. Quantum mechanics is the most effective, predictive, and fundamental theory for understanding small scale phenomena. As a result, a theoretical model of the measurement process, as well as the final achievable precision limit, are required. Quantum physics, on the other hand, imposes fundamental limits on the precision of estimations, and quantum resources must be employed in order to meet such limitations.

The following are the building blocks of the estimation process:

i. State preparation

ii. Parameter encoding

iii. Measurement

iv. Result

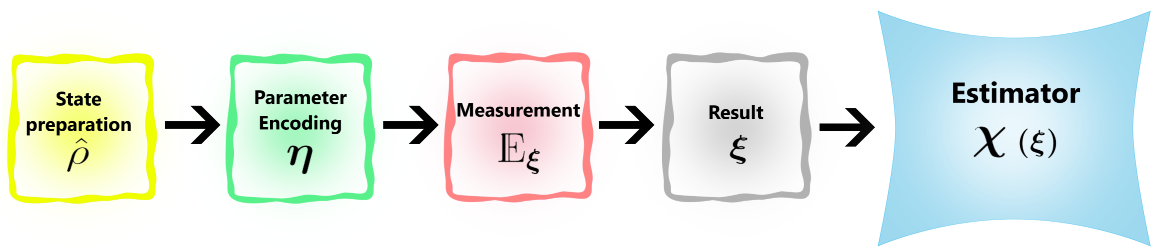

Figure 3.1: Starting with the setup of the probe state ρ, the estimation procedure proceeds with the encoding of the parameter η, followed by the readout measurement $\hat{\mathbb{E}}(\xi)$, and finally the mapping from the measurement outcomes to the parameter estimation using the estimator $\hat{\chi}(\xi)$.

In the first stage, the input (probe) state is prepared in such a way that it can be easily modified by making changes in unknown parameters. The system then interacts with the probe state with the help of a unitary transformation operator which changes the probe state to encode information about unknown parameters. Then, using a set of Hermitian operators with a non-zero probability known as positive operator valued measurement (POVM) $\hat{\mathbb{E}}(\xi)$, measurements are made and information about unknown parameters is extracted. Finally, an appropriate estimator gives an estimation of the unknown parameter based on measurement outcomes.

3.1.1 Likelihood Function and Estimator

The most extensive form of measurement in estimation theory[30] is positive-operator valued measure (POVM) which is a set of Hermitian operators $\hat{\mathbb{E}}$ with non-zero probability and satisfy $\sum_{\xi} \hat{\mathbb{E}}(\xi) = \mathbb{1}$ to obey the normalization condition. The most common scenario is having correlated subsystems represented by $\hat{\rho}$ and performing m correlated measurements expressed by $\hat{\mathbb{E}}$. So the conditional probability[29] to observe the resultξ for a given value η, called as the likelihood, is

$$P(\xi \mid \eta) = Tr[\hat{\mathbb{E}}(\xi)\hat{\rho}(\eta)]. \tag{3.1.1}$$

If the probe state $\hat{\rho}$ consists of k uncorrelated subsystems that are independent of one another, then

$$\hat{\rho} = \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)} \otimes \hat{\rho}^{(3)} \otimes ... \otimes \hat{\rho}^{(k)}, \tag{3.1.2}$$

If we perform a local operations, i.e. $\hat{\rho}(\eta) = \hat{\rho}^{(1)}(\eta) \otimes \hat{\rho}^{(2)}(\eta) \otimes \hat{\rho}^{(3)}(\eta) \otimes ... \otimes \hat{\rho}^{(k)}(\eta)$ and independent measurements, such that our estimator is given by $\hat{\mathbb{E}}(\xi) = \hat{\mathbb{E}}^{(1)}(\xi_1) \otimes \hat{\mathbb{E}}^{(2)}(\xi_2) \otimes \hat{\mathbb{E}}^{(3)}(\xi_3) \otimes ... \otimes \hat{\mathbb{E}}^{(k)}(\xi_k)$. It is straightforward to define the likelihood function as a product of probabilities based on a single measurement

$$P(\xi \mid \eta) = \prod_{i=1}^{k} P_i(\xi_i \mid \eta), \tag{3.1.3}$$

where $P_i(\xi_i \mid \eta) = Tr[\hat{\mathbb{E}}_i(\xi_i)\hat{\rho}_i(\eta)]$. For independent measurements, using the log-likelihood function is usually more convenient so

$$L(\xi \mid \eta) \equiv lnP(\xi \mid \eta) = \sum_{i=1}^{k} lnP_i(\xi_i \mid \eta). \tag{3.1.4}$$

Now we define an estimator as a mechanism for computing an estimate of an unknown parameter based on a given set of outcomes. It is just a function that correlates each set of data with an estimate of the unknown parameter. In fact, the estimator must be chosen properly so that it is near to the actual, unknown value of the parameter. The estimator can be characterized by its expectation value which is dependent on an unknown parameter.

$$\langle \hat{\chi} \rangle_{\eta} = \sum_{\xi} P(\xi \mid \eta)\hat{\chi}(\xi), \tag{3.1.5}$$

and its variance is

$$(\Delta\hat{\chi})^2_\eta = \sum_\xi P(\xi \mid \eta) \left[\hat{\chi}(\xi) - \langle\hat{\chi}\rangle_\eta\right]^2. \tag{3.1.6}$$

Now we will look at the characteristics of an estimator. It is obvious that there are good and bad choices of an estimator. If an estimator gives the least amount of uncertainty, it is considered to be "good". These are referred to as unbiased estimators. More in details:

Unbiased Estimator; An estimator is considered to be unbiased if its expectation value corresponds to the real value of unknown parameters

$$\langle\hat{\chi}(\xi)\rangle_\eta = \eta. \tag{3.1.7}$$

If estimators do not fulfill the above condition are supposed to be unbiased. In particular, for unbiased estimators we have $\frac{\partial\langle\hat{\chi}(\xi)\rangle_\eta}{\partial\phi} = 1$. while estimators that are unbiased for a certain range of the parameter η is said to be locally unbiased.

Consistent Estimator; A set of estimates $\chi(\xi_1), \chi(\xi_1, \xi_2), ..., \chi(\xi_1, \xi_2, ..., \xi_k)$ can be constructed by performing a set of measurements $\xi = \{\xi_1, \xi_2, \xi_3, ..., \xi_k\}$ in a sequence. If such a sequence converges in probability to ϕ, the estimator is considered to be consistent. A consistent estimator is locally unbiased in the limit of $k \to \infty$, that is

$$\lim_{k\to\infty} \langle\hat{\chi}(\xi)\rangle_\eta = \eta. \tag{3.1.8}$$

The ideas that have been described so far act as a foundation for estimation theory. We now go on to possibly one of the most significant tools in estimation theory: the Cramer-Rao bound (CRB). To properly define and understand this lower limit on estimation theory, we must first focus on a quantity that is very strongly connected to this bound, namely the Fisher information.

3.2 The Cramér-Rao Lower Bound and Classical Fisher Information

The Cramér-Rao lower bound express a lower bound on the variance of an arbitrary estimator. Considered a probability density function (PDF) for a set of the possible outcome $\xi = \{\xi_1, \xi_2, \xi_3, ..., \xi_k\}$ and the unknown parameter, η, is

$$\sum_{i=1}^{k} P_i(\xi_i \mid \eta) = 1, \tag{3.2.1}$$

By differentiating with respect to the unknown parameter η and then multiplying and dividing by $P_i(\xi_i \mid \eta)$, we obtain

$$\sum_{i=1}^{k} \left[\frac{\partial}{\partial \eta} L_i(\xi_i \mid \eta)\right] \; P_i(\xi_i \mid \eta) = 0, \tag{3.2.2}$$

where we have used the identities $\frac{1}{P_i(\xi_i|\eta)} \frac{\partial}{\partial \eta} P_i(\xi_i \mid \eta) = \frac{\partial}{\partial \eta} lnP_i(\xi_i \mid \eta)$ and $lnP_i(\xi_i \mid \eta) = L_i(\xi_i \mid \eta)$. The above equation is simply the expectation value of derivatives of the log-likelihood function which we may express as

$$\left\langle \frac{\partial(L_i(\xi_i \mid \eta))}{\partial \eta} \right\rangle = 0. \tag{3.2.3}$$

We will now examine the expectation value of an estimator which is $\langle \chi \rangle = \sum_{i=1}^{k} \hat{\chi} \; P_i(\xi_i \mid \eta)$, by differentiating with respect to unknown parameter η and then using identities $\frac{1}{P_i(\xi_i|\eta)} \frac{\partial}{\partial \eta} P_i(\xi_i \mid \eta) = \frac{\partial}{\partial \eta} lnP_i(\xi_i \mid \eta)$ and $lnP_i(\xi_i \mid \eta) = L_i(\xi_i \mid \eta)$, we obtain

$$\frac{\partial \langle \chi \rangle_\eta}{\partial \eta} = \left\langle \hat{\chi} \; \frac{\partial(L_i(\xi_i \mid \eta))}{\partial \eta} \right\rangle. \tag{3.2.4}$$

By combining the Eq. (3.2.3) and Eq. (3.2.4), we get $\left(\frac{\partial \langle \chi \rangle_\eta}{\partial \eta}\right)^2 = \left\langle \left(\hat{\chi} - \langle \hat{\chi} \rangle\right) \; \frac{\partial(L_i(\xi_i|\eta))}{\partial \eta} \right\rangle_\eta^2$, and then applying the Cauchy-Schwartz inequality $\langle X \rangle_\eta^2 \langle Y \rangle_\eta^2 \geq \langle XY \rangle_\eta^2$ where X and Y are real functions of the parameter ξ we obtain

$$\left\langle \left(\hat{\chi} - \langle \hat{\chi} \rangle\right)^2 \right\rangle_\eta \left\langle \left(\frac{\partial(L_i(\xi_i|\eta))}{\partial \eta}\right)^2 \right\rangle_\eta = \left(\frac{\partial \langle \chi \rangle_\eta}{\partial \eta}\right)^2, \tag{3.2.5}$$

Noting that $\left\langle \left(\hat{\chi} - \langle \hat{\chi} \rangle\right)^2 \right\rangle_\eta = \langle \Delta \chi \rangle_\eta^2$, we obtain the CRB, given formally

$$(\Delta \chi)_\eta^2 \geq \frac{\left(\frac{\partial \langle \eta \rangle}{\partial \eta}\right)^2}{FI(\eta)} = (\Delta \chi_{CR})_\eta^2, \tag{3.2.6}$$

where FI(η) denotes the classical Fisher information, given by

$$FI(\eta) \equiv \sum_{\xi} \frac{1}{P(\xi \mid \eta)} \left(\frac{\partial P(\xi \mid \eta)}{\partial \eta} \right)^2 = \left\langle \left(\frac{\partial L(\xi \mid \eta)}{\partial \eta} \right)^2 \right\rangle, \tag{3.2.7}$$

where sum extends over all possible outcomes ξ. The most general form of the Cramér-Rao lower bound is represented by Equation (3.2.6). The bound, on the other hand, is most useful when considering unbiased estimators for which $\frac{\partial \langle \hat{\chi}(\xi) \rangle_\eta}{\partial \eta} = 1$. Next, we will calculate an upper bound on parameter estimation, simply known as the quantum Cramér-Rao bound (qCRB).

3.3 Quantum Cramér-Rao Bound and Quantum Fisher Information

So far we have discussed Cramér-Rao bound which gives a lower bound on parameter estimation. Cramér-Rao lower bound depends on classical fisher information which is dependent on the choice of estimator employed. We will now examine the quantum Cramér-Rao upper bound on phase estimation, which is dependent on quantum fisher information. Maximizing the Fisher information over all POVMs is used to derive the quantum Cramer-Rao bound

$$FI_Q[\hat{\rho}(\eta)] \equiv \max_{\{\hat{\mathbb{E}}(\xi)\}} FI\ [\hat{\rho}(\eta), \hat{\mathbb{E}}(\xi)]. \tag{3.3.1}$$

It has been shown that this amount may be represented in the following way by Pezze' and colleagues [29]

$$FI_Q[\hat{\rho}(\eta)] = Tr\ [\hat{\rho}(\eta), \hat{L}_\eta^2]. \tag{3.3.2}$$

It is clear that quantum Fisher information does not depend on POVM and $\hat{L}_\eta$ in the above equation is known as the symmetric logarithmic derivative (SLD)[31] defined as equation

$$\frac{\partial \hat{\rho}(\eta)}{\partial \eta} = \frac{1}{2} \left(\hat{\rho}(\eta) \hat{L}_\eta + \hat{L}_\eta \hat{\rho}(\eta) \right), \tag{3.3.3}$$

and we define the quantum Cramér-Rao bound as

$$\langle \Delta \chi \rangle^2_{qCRB} \equiv \frac{\left(\frac{\partial \langle \eta \rangle}{\partial \eta} \right)^2}{FI_Q[\hat{\rho}(\eta)]}. \tag{3.3.4}$$

Quantum Fisher information is a maximization over all possible POVMs and qCRB is inversely proportional to QFI so it is clear to see how quantum Cramér-Rao bound serves as an upper bound on parameter estimation. Now we have the chain of inequalities

$$\left(\Delta\chi\right)^2_\eta \geq \left(\Delta\chi_{CRB}\right)^2_\eta \geq \left(\Delta\chi_{qCRB}\right)^2_\eta . \tag{3.3.5}$$

3.3.1 Quantum Fisher Information for Mixed State

In this section, we find an appropriate expression for quantum Fisher information in terms of complete basis $\{|n\rangle\}$ where the density operator is generally be written as $\hat{\rho}(\phi) = \sum_n p_n |n\rangle \langle n|$. In this basis quantum Fisher information can be written as

$$\begin{aligned} FI_Q[\hat{\rho}(\eta)] &= Tr[\hat{\rho}(\eta) \;\; \hat{L}^2_\eta] = Tr\left[\sum_l p_l |l\rangle\langle l| \;\; \hat{L}^2_\eta\right] \\ &= \sum_{l,l'} p_l \langle l'|l\rangle \langle l| \hat{L}^2_\eta |l'\rangle = \sum_{l,l'} p_l \,|\langle l| \hat{L}_\eta |l'\rangle|^2, \\ &= \sum_{l,l'} \frac{p_l + p_{l'}}{2} \times |\langle l| \hat{L}_\eta |l'\rangle|^2. \end{aligned} \tag{3.3.6}$$

where we have used identity $\sum_l |l\rangle\langle l| = 1$. To calculate the Quantum Fisher information we have to calculate the matrix element of SLD $\langle l| \hat{L}_\eta |l'\rangle$ for vectors $|l\rangle$ and $|l'\rangle$. By using Eq (3.3.3) we can write

$$\langle l| \partial_\eta \hat{\rho}_\eta |l'\rangle = \frac{\langle l| \hat{\rho}(\eta)\hat{L}_\eta |l'\rangle + \langle l| \hat{L}_\eta \hat{\rho}(\eta) |l'\rangle}{2}, \tag{3.3.7}$$

and

$$\langle l| \hat{L}_\eta |l'\rangle = \sum_{l,l'} \frac{2}{p_l + p_{l'}} \times \langle l| \partial_\eta \hat{\rho}(\eta) |l'\rangle , \tag{3.3.8}$$

Which makes Eq. 3.3.3

$$FI_Q[\hat{\rho}(\eta)] = \sum_{l,l'} \frac{2}{p_l + p_{l'}} \times |\langle l| \partial_\eta \hat{\rho}(\eta) |l'\rangle|^2. \tag{3.3.9}$$

We proceed further through the definition of the density operator $\hat{\rho}(\eta) = \sum_l p_l |l\rangle\langle l|$

$$\partial_\eta \hat{\rho}(\eta) = \sum_l (\partial_\eta p_l) |l\rangle\langle l| + \sum_l p_l |\partial_\eta l\rangle\langle l| + \sum_l p_l |l\rangle\langle \partial_\eta l| . \tag{3.3.10}$$

Which is simple product rule for derivative. Thus the matrix elements of Eq. 3.3.10 becomes

$$\langle l|\, \partial_\eta \hat{\rho}(\eta)\, |l^{'}\rangle = (\partial_\eta p_l)\delta_{l,l'} + (p_l - p_{l'})\, \langle \partial_\eta l|l^{'}\rangle\,, \tag{3.3.11}$$

where we have used $\partial_\eta \langle l|l^{'}\rangle = \langle \partial_\eta l|l^{'}\rangle + \langle l|\partial_\eta l^{'}\rangle = 1$. The Quantum Fisher information then becomes

$$FI_Q[\hat{\rho}(\eta)] = \sum_l \frac{(\partial_\eta p_l)^2}{p_l} + 2\sum_{l,l'} \frac{(p_l - p_{l'})^2}{p_l + p_{l'}} \times |\, \langle \partial_\eta l|l^{'}\rangle\, |^2. \tag{3.3.12}$$

3.3.2 Quantum Fisher Information for Pure State

In the case of a pure state, the density operator may be represented as $\hat{\rho}(\eta) = |\psi(\eta)\rangle \langle \psi(\eta)|$, and its derivatives and square of derivative can be written as

$$\begin{aligned} \frac{\partial}{\partial \eta}\hat{\rho}(\eta) &= \hat{\rho}(\eta)\left[\frac{\partial \hat{\rho}(\eta)}{\partial \eta}\right] + \left[\frac{\partial \hat{\rho}(\eta)}{\partial \eta}\right]\hat{\rho}(\eta), \\ \frac{\partial}{\partial \eta}\hat{\rho}^2(\eta) &= \hat{\rho}(\eta)\left[\frac{\partial \hat{\rho}(\eta)}{\partial \eta}\right] + \left[\frac{\partial \hat{\rho}(\eta)}{\partial \eta}\right]\hat{\rho}(\eta). \end{aligned} \tag{3.3.13}$$

Using these derivatives, SLD becomes

$$\begin{aligned} \hat{L}_\eta &= 2[\partial_\eta \hat{\rho}(\eta)] = 2[\partial_\eta\, |\psi(\eta)\rangle \langle \psi(\eta)|], \\ &= 2\, |\partial_\eta \psi(\eta)\rangle \langle \psi(\eta)| + 2\, |\psi(\eta)\rangle \langle \partial_\phi \psi(\eta)|\,. \end{aligned} \tag{3.3.14}$$

where in the last step, the η dependency of ψ is implicit for notational convenience. By using Eq. 3.3.6 into Fisher information formula, we obtain

$$\begin{aligned} FI_Q[\hat{\rho}(\eta)] &= Tr[\hat{\rho}(\eta)\hat{L}^2_\eta] = \langle \psi|\hat{L}^2|\psi\rangle\,, \\ &= 4\langle \partial_\eta \psi|\partial_\eta \psi\rangle - |\, \langle\langle \partial_\eta \psi|\psi\rangle\, |^2\rangle. \end{aligned} \tag{3.3.15}$$

This is Quantum Fisher Information (QFI) which we often used in Quantum Metrology. Following that, we will go over the most important scenario in the estimation process, referred to as phase estimation.

3.4 Phase Estimation

Phase estimation is one of the most important applications of quantum metrology. Many interferometers are used to compute phase shift, which is the difference in the length of an optical path between two modes, but the Mach–Zehnder interferometer is one of the most often used optical interferometers for phase estimation. Mach-Zehnder interferometer consist of optical instruments named beam splitters, mirrors, phase shifter and detectors. Now, we will talk about the action of the Mach-Zehnder Interferometer.

3.4.1 Mach-Zehnder Interferometer

The Mach–Zehnder interferometer is a particularly simple instrument for demonstrating interference via amplitude division. The first beam splitter divides a light beam into two components, which are subsequently recombined by the second beam splitter. The second beam splitter reflects the beam with an efficiency ranging from 0 to 100% depending on the relative phase shift obtained by the beam along the two pathways. Using a Mach–Zehnder Interferometer as an example in quantum mechanics is common since it illustrates the path-choice problem in a straightforward manner.

If a_0 and b_0 are two inputs operators immediately after the first beam splitter i.e. at the point b_1 and a_1, operators are given by

$$\begin{aligned} b_1 &= \frac{b + \iota a}{\sqrt{2}} \\ a_1 &= \frac{a + \iota b}{\sqrt{2}}, \end{aligned} \tag{3.4.1}$$

Mirrors provide a phase shift of π and phase shifter imparts a phase shift to the arm b_2 so the operator at point b_3 and a_3 have the form

$$\begin{aligned} b_3 &= -e^{-\iota\varphi}\,\frac{b + \iota a}{\sqrt{2}} \\ a_3 &= -\frac{a + \iota b}{\sqrt{2}}, \end{aligned} \tag{3.4.2}$$

In the end, the second beam splitter transforms these so that the output field at detectors are given as

$$\begin{aligned} a_f &= \frac{1}{2}\left[-(a + \iota b) - \iota e^{-\iota\varphi}(b + \iota a)\right] \\ b_f &= \frac{1}{2}\left[-e^{-\iota\varphi}(b + \iota a) - \iota(a + \iota b)\right]. \end{aligned} \tag{3.4.3}$$

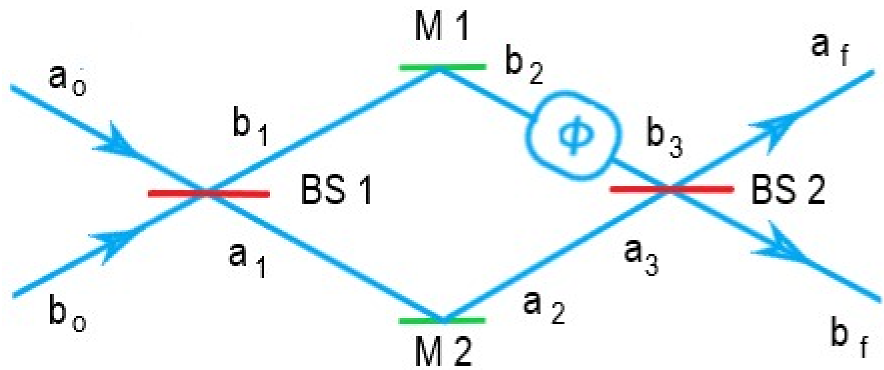

Figure 3.2: A standard Mach-Zehnder interferometer with a_0 and b_0 are two input ports. A phase shift is induced in the $b_2 - mode$ before the state reaches the second beam splitter.

In the absence of a phase shifter $\varphi = 0$ and the output field at detectors are

$$a_f = -\iota b, \qquad b_f = -\iota a. \tag{3.4.4}$$

If input field b is in a vacuum state the output field can be shown that

$$\begin{aligned} \langle a_f^\dagger a_f \rangle &= sin^2 \frac{\varphi}{2} \langle a_0^\dagger a_0 \rangle \\ \langle b_f^\dagger b_f \rangle &= cos^2 \frac{\varphi}{2} \langle b_0^\dagger b_0 \rangle \, . \end{aligned} \tag{3.4.5}$$

Chapter 4

Heisenberg-Limited Phase Estimation

In this chapter, we use the parity operator to investigate the estimation of an unknown parameter, phase. Here, we achieve phase sensitivity up to Shot Noise limit using the classical state of light as input in Mach-Zehnder interferometer and Heisenberg-limited phase estimation using N00N state.

4.1 Phase Estimation in Interferometry

Phase Estimation by Output Intensities Difference: First Method

First we introduce an operator $\hat{O}$ defined as difference of output photon of Mach-Zehnder interferometer which is given as

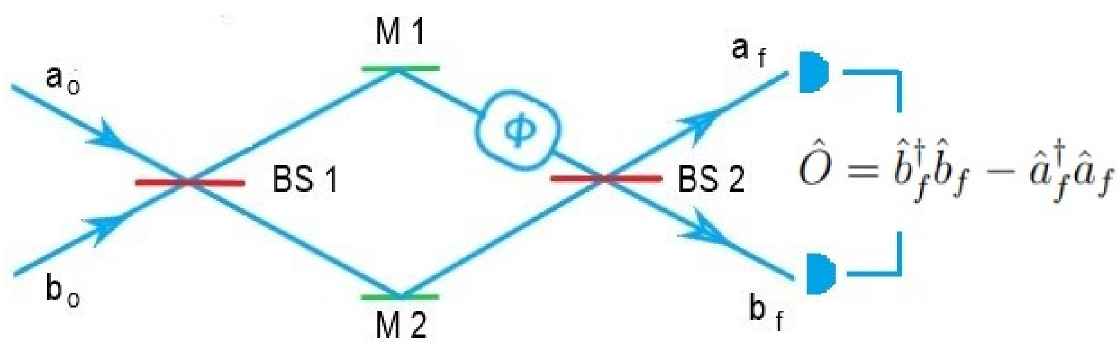

Figure 4.1: Phase Estimation by Output Mode Intensities

$$\hat{O} = \hat{b}_f^\dagger \hat{b}_f - \hat{a}_f^\dagger \hat{a}_f, \tag{4.1.1}$$

where $\hat{b}_f^\dagger \hat{b}_f$ and $\hat{a}_f^\dagger \hat{a}_f$ are the number operators on the output ports of the Mach-Zehnder interferometer. With output state, the mean value of operator $\hat{O}$ is

$$\langle \hat{O} \rangle = \langle \hat{b}_f^\dagger \hat{b}_f - \hat{a}_f^\dagger \hat{a}_f \rangle \,, \tag{4.1.2}$$

and expectation value of operator $\hat{O}^2$ is

$$\langle \hat{O}^2 \rangle = \langle (\hat{b}_f^\dagger \hat{b}_f - \hat{a}_f^\dagger \hat{a}_f)^2 \rangle \,. \tag{4.1.3}$$

We calculate phase uncertainty by using error propagation calculus given as

$$(\Delta\phi)^2 = \frac{(\Delta\hat{O})^2}{\partial_\phi \langle \hat{O} \rangle}, \tag{4.1.4}$$

where $\Delta\hat{O} = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}$. In the present method we choose an arbitrary input state and a beam splitter which transforms the input state. After this a phase shifter insert a phase shift to arm b of interferometer. When the interferometer output state is measured for photon counting, it leads directly to an equal expectation value over the output state. The phase uncertainty may be determined using equation (4.1.4).

It is quite straightforward to say that this technique is not effective for phase estimation when entangled state is the phase-dependent state of an interferometer, or any other state in which the photon number difference is more than 1.

Phase Estimation by using Parity Operator: Second Method

Many works [36, 37, 38, 39, 40, 41, 42, 43] the possibility of using $\hat{O}$ as a parity operator for output mode $\hat{P}$ has been explained. To explain photon counting measurements, we may use the parity operator p. The output photon number operator is defined as

$$\begin{aligned} \hat{n}_f(a) &= \hat{a}_f^\dagger \hat{a}_f. \\ \hat{n}_f(b) &= \hat{b}_f^\dagger \hat{b}_f. \end{aligned} \tag{4.1.5}$$

Suppose we work in Hilbert space in which basis vectors are represented by $\{|q\rangle\,, q \;\; \textit{an integer}\}$ and $|q\rangle$ is an Eigen ket of an operator $\hat{q}$ That is

$$\hat{q}\,|q\rangle = q\,|q\rangle\,. \tag{4.1.6}$$

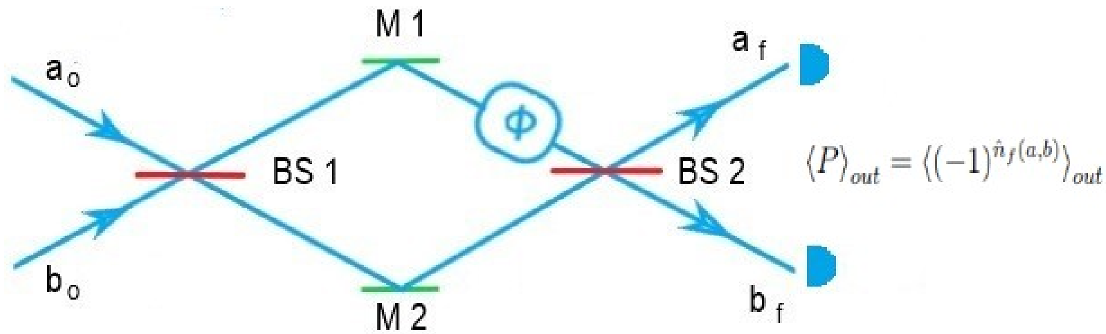

Figure 4.2: Phase Estimation by using Parity Operator

We can define the parity operator with in Hilbert space as

$$\hat{P} = (-1)^{\hat{n}_f(a,b)}, \tag{4.1.7}$$

such that

$$\hat{P}\,|q\rangle = (-1)^{\hat{n}_f(a,b)}\,|q\rangle\,, \tag{4.1.8}$$

and expectation value

$$\langle P\rangle_{out} = \langle(-1)^{\hat{n}_f(a,b)}\rangle_{out}\,. \tag{4.1.9}$$

We should take into account that $(-1)^{\hat{n}(a,b)}$ depends critically on precise measurement of $\hat{n}(a,b)$. Photon counting for parity operator measurement is useful normally for low photon numbers. Phase estimation is calculated by the phase uncertainty given in equation (4.1.4)

4.2 Phase Estimation through Parity Operator

In this section, we will discuss the use of a new detection observable named parity operator. This detection observable was first introduced by C. C. Gerry et al.[34] in 2010. A detection observable is considered to be optimal for a state, if the CRB reaches the qCRB. For any pure states that are path symmetric, parity detection provides maximum phase sensitivity at the qCRB. It is necessary to be used Eq. 3.2.7 to get the classical Fisher information, which in term of parameter ϕ is

$$FI(\phi) \equiv \sum_{\xi} \frac{1}{P(\xi \mid \phi)} \left(\frac{\partial P(\xi \mid \phi)}{\partial \phi} \right)^2, \tag{4.2.1}$$

where sum extends over all possible outcomes ξ. For parity, measurement outcomes for unknown parameter ϕ have only two values, + or - and we can write $p(+|\phi)+p(-|\phi) = 1$. The mean value of that operator may be represented as the sum of the possible eigenvalues weighted by the likelihood of that specific result, written as

$$\begin{aligned} \langle \hat{\mathbf{P}} \rangle &= \sum_i p(i|\phi)\lambda_i = p(+|\phi) - p(-|\phi), \\ &= 2p(+|\phi) - 1 = 1 - 2p(-|\phi). \end{aligned} \tag{4.2.2}$$

The variance can be calculated as

$$\begin{aligned} (\Delta\hat{\mathbf{P}})^2 &= \langle \hat{\mathbf{P}}^2 \rangle - \langle \hat{\mathbf{P}} \rangle^2 = 1 - \langle \hat{\mathbf{P}} \rangle^2, \\ &= 1 - \left(p(+|\phi) - p(-|\phi)\right)^2, \\ &= 1 - \left(p(+|\phi) + p(-|\phi)\right)^2 + 4p(+|\phi)p(-|\phi), \\ &= 4p(+|\phi)p(-|\phi). \end{aligned} \tag{4.2.3}$$

Finally, from Eq. (4.2.2), it follows that

$$\frac{\partial p(+|\phi)}{\partial \phi} = \frac{1}{2}\frac{\partial \langle \hat{\mathbf{P}} \rangle}{\partial \phi} - \frac{\partial p(-|\phi)}{\partial \phi}, \tag{4.2.4}$$

Combining Eq. (4.3.3) and (3.3.15) we find

$$\begin{aligned} FI(\phi) &= \sum_{\xi} \frac{1}{p(\xi|\phi)} \left(\frac{\partial(\xi|\phi)}{\partial \phi} \right)^2, \\ &= \frac{1}{(\Delta\hat{\mathbf{P}})^2} \left| \frac{\partial \langle \hat{\mathbf{P}} \rangle}{\partial \phi} \right|^2. \end{aligned} \tag{4.2.5}$$

making the CRB / qCRB

$$\begin{aligned} \Delta\Phi_{CRB/qCRB} &= \frac{\sqrt{1 - \langle \hat{\mathbf{P}} \rangle^2}}{|\partial_\phi \langle \hat{\mathbf{P}} \rangle|}, \\ &= \frac{1}{\sqrt{FI(\phi)}}. \end{aligned} \tag{4.2.6}$$

It indicates that the phase uncertainty produced by error propagation saturates the qCRB. This result will be used as we compute the qCRB in combination with the phase uncertainty acquired by error propagation of the parity operator and show that the results coincide perfectly. This outcome is especially useful in a number of the scenarios we will explore because the QFI is usually easier to compute from a computational standpoint.

4.3 Quantum Fisher Information in Optical Interferometry

We utilize the Schwinger representation of the SU(2) algebra with two sets of boson operators to describe a typical Mach-Zehnder interferometer[35], which is discussed in Appendix A. If you look at it from this perspective, the quantum mechanical beam splitter may be thought of as a spin around an imaginary, but chosen axis., i.e. the choice of a $\hat{J}_x$-operator performs a rotation around the x-axis, while the choice of a $\hat{J}_y$-operator performs a rotation around the y-axis. An induced phase shift, which is assumed to be in the b mode, is described by a rotation about the z-axis specified by the $\hat{J}_z$-operator. The state just before the second beam splitter is given as

$$|\psi(\phi)\rangle = e^{-\iota\phi\hat{J}_z} e^{-\iota\frac{\pi}{2}\hat{J}_x} |in\rangle . \tag{4.3.1}$$

where we assume the 50:50 beam splitter. This, in turn, makes the derivative

$$|\partial_\phi\psi(\phi)\rangle = -\iota e^{-\iota\phi\hat{J}_z}\hat{J}_z e^{-\iota\frac{\pi}{2}\hat{J}_x}, \tag{4.3.2}$$

leading to

$$\langle\partial_\phi\psi(\phi)|\psi(\phi)\rangle = \iota \langle in|\hat{J}_y|in\rangle , \tag{4.3.3}$$

and

$$\langle\partial_\phi\psi(\phi)|\partial_\phi\psi(\phi)\rangle = \langle in|\hat{J}_y^{\,2}|in\rangle , \tag{4.3.4}$$

where we have used the Baker-Hausdorff identity to make things easier

$$e^{\iota\frac{\pi}{2}\hat{J}_x}\hat{J}_z e^{-\iota\frac{\pi}{2}\hat{J}_x} = \hat{J}_y, \tag{4.3.5}$$

Combining Eq. (4.5.3) and (4.5.4) into Eq. (4.3.17) yields for the quantum Fisher information

$$
\begin{aligned}
FI_Q &= 4[\langle \partial_\phi \psi(\phi) | \partial_\phi \psi(\phi) \rangle - | \langle \partial_\phi \psi(\phi) | \psi(\phi) \rangle |^2], \\
&= 4 \left\langle (\Delta \hat{J}_y)^2 \right\rangle_{in}.
\end{aligned}
\tag{4.3.6}
$$

which is just the $\hat{J}_y$-operator's variance with regard to the $|in\rangle$ initial input state. It is worth noting that the quantum Fisher information in this situation is not affected by the phase ϕ, but rather by the initial state. Following that, we will show how classical and quantum mechanical states of light may be used to derive basic limitations on phase uncertainty.

4.3.1 Limits on the Phase Uncertainty

Let us first assume an interferometric setup with an input state given by $|in\rangle = |\alpha\rangle_{a_0} \otimes |0\rangle_{b_0}$, where $|\alpha\rangle$ is a coherent state, the most classical of pure single-mode field states. Assuming our beam splitters are $50:50$, that is, $\theta = \pi/2$, and following the convention of Yurke et. al [3.8.1], the output state after the interaction with interferometer is

$$
|output\rangle = \left| \frac{\alpha}{2}(1 + e^{\iota\phi}) \right\rangle_{a_0} \otimes \left| \frac{\iota\alpha}{2}(1 - e^{\iota\phi}) \right\rangle_{b_0}. \tag{4.3.7}
$$

Note that upon setting the phase $\phi = 0$ we end up with our initial input state $|in\rangle$, reflecting the unitarity of the transformation. More concisely written, we have performed the transformation

$$
|out\rangle = e^{-\iota\phi\hat{J}_y} |in\rangle. \tag{4.3.8}
$$

The intensities of the two output coherent states are given for the a_f and b_f-modes, respectively, as

$$
\begin{aligned}
I_{a_f} &= \langle \hat{a}_f^\dagger \hat{a}_f \rangle = \frac{|\alpha|^2}{2}(1 + cos\phi), \\
I_{b_f} &= \langle \hat{b}_f^\dagger \hat{b}_f \rangle = \frac{|\alpha|^2}{2}(1 - cos\phi).
\end{aligned}
\tag{4.3.9}
$$

We define the difference in intensities as

$$
\langle \hat{\omega} \rangle = I_{a_0} - I_{b_0} = |\alpha|^2 cos\phi, \tag{4.3.10}
$$

where the operator $\hat{\omega}$ is given by $\hat{\omega} = \hat{a}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}$. Finding the expectation value of the square of this operator is tedious but straight forward, and yields

$$\begin{aligned} \langle \hat{\omega}^2 \rangle &= \langle (\hat{a}_0^{\dagger}\hat{a}_0 - \hat{b}_0^{\dagger}\hat{b}_0)^2 \rangle \,, \\ & |\alpha|^2(1 - |\alpha|^2 cos^2\phi). \end{aligned} \tag{4.3.11}$$

We can define the phase uncertainty through error propagation calculus as

$$\begin{aligned} \Delta\phi &= \frac{\Delta\hat{\omega}}{|\partial_{\phi} \langle \hat{\omega} \rangle |}, \\ &= \frac{\sqrt{1 - \bar{n}cos^2\phi}}{\sqrt{\bar{n}}|sin\phi|}, \\ &\approxeq \frac{1}{\sqrt{\bar{n}}}. \qquad if \quad \phi = \frac{\pi}{2} \end{aligned} \tag{4.3.12}$$

This outcome puts a limit on the phase uncertainty achievable via classical-like light and is known as the Standard Quantum (or Shot Noise) Limit (SQL). Note, however, when our detection observable is simply taking the difference in intensities, the measurement is not optimized at the value $\phi = 0$; in fact, the phase uncertainty is infinite for this value of the phase. We formally define the SQL as

$$\Delta\phi_{SQL} = \frac{1}{\sqrt{\bar{n}}}. \tag{4.3.13}$$

It is worth showing out here the importance of using parity as a detection observable.For an arbitrary coherent state$|\beta\rangle$, we define the mean value of the parity operator now which is

$$\begin{aligned} \langle \hat{P} \rangle &= \langle \beta|\hat{P}|\beta \rangle \,, \\ &= \langle \beta| \{ e^{\frac{-|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} e^{\iota\pi\hat{n}} \,|n\rangle \}, \\ &= \langle \beta| \{ e^{\frac{-|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{(\beta e^{\iota\pi})^n}{\sqrt{n!}} \,|n\rangle \}, \\ &= \langle \beta| - \beta \rangle \,. \end{aligned} \tag{4.3.14}$$

Applying this result to the a- and b-modes of output state in Eq. (4.7.10) yields

$$\begin{aligned} \langle \hat{P} \rangle_a &= e^{-|\alpha|^2(1+cos\phi)}. \\ \langle \hat{P} \rangle_b &= e^{-|\alpha|^2(1-cos\phi)}. \end{aligned} \tag{4.3.15}$$

Which, when used related to the usual error propagation yields, a SNL optimized for small phase shifts when detection is made in the output mode b, that is

$$\Delta\phi_{\hat{P}b} = \frac{1}{\sqrt{\bar{n}}} \quad if \quad \phi = 0, \tag{4.3.16}$$

where we applied L'Hopital's rule for the limiting case of $\pi = 0$.

What about a reasonable bound on the phase uncertainty achieved when we use quantum states of light? Consider the maximal entangled state given by

$$|N00N\rangle = \frac{1}{2}[|N,0\rangle_{a_0,b_0} + e^{\iota\Phi n}|0,N\rangle_{a_0,b_0}]. \tag{4.3.17}$$

The photon number uncertainty is equal to the overall mean photon number N, since all photons can be found in either one mode or the other while the opposite mode will have zero photons. Thus the heuristic relation becomes $\Delta\phi N = 1$. Our understanding of the Heisenberg limit (HL) on phase uncertainty may be derived from this:

$$\Delta\phi_{HL} = \frac{1}{\bar{n}}. \tag{4.3.18}$$

Thus, the SNL and HL provide us our upper and lower bounds on phase uncertainty.

$$\begin{aligned} \Delta\phi_{SNL} &= \frac{1}{\sqrt{\bar{n}}}. \\ \Delta\phi_{HL} &= \frac{1}{\bar{n}}. \end{aligned} \tag{4.3.19}$$

Moreover, the Heisenberg-limit is an enhancement of the SNL by a factor of the SNL itself.

$$\Delta\phi_{SNL}^2 = \Delta\phi_{HL}. \tag{4.3.20}$$

We will look at these limits in more detail in the next sections, using both classical and quantum states of light as well as parity operator.

4.4 Phase Measurement with Classical (Coherent)Light

Now we will talk about the scenario where classical light is employed. Coherent state is classical state because a beam splitter does not creat entangled state for coherent light. For classical light we will get phase sensitivities up to the Shot Noise Limit (SNL) for the best choice of phase ϕ when studying intensity different measurement which is highest sensitivity attained by classical light. In this section we will illustrate a question that what will be the phase sensitivity when we use photon number parity?

Now, we will discuss the example in which coherent light is injected in the input port a and no photon on port b of interferometer so our input state can have the form

$$|in\rangle = |\alpha, 0\rangle_{a_0,b_0}, \tag{4.4.1}$$

where coherent state is defined as

$$|\alpha\rangle = e^{-|\frac{\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle . \tag{4.4.2}$$

Here we use $\hat{J}_x$ type beam splitter so the the first beam splitter transforms the initial state as

$$|in\rangle = |\alpha, 0\rangle_{a_0,b_0} \xrightarrow{BS\ 1} \left|\frac{\alpha}{2}, \frac{\iota\alpha}{2}\right\rangle_{a_1,b_1}, \tag{4.4.3}$$

interferometer arm b undergoes a phase change of ϕ as a result of $\hat{J}_z$ operator, so the state after the phase shift in b-mode is given as

$$\left|\frac{\alpha}{2}, \frac{\iota\alpha}{2}\right\rangle_{a_1,b_1} \xrightarrow{\phi-shift} \left|\frac{\alpha}{2}, \frac{\iota\alpha e^{\iota\phi}}{2}\right\rangle_{a_2,b_2}, \tag{4.4.4}$$

In the end, the output state at the detectors are given as

$$\left|\frac{\alpha}{2}, \frac{\iota\alpha e^{\iota\phi}}{2}\right\rangle_{a_2,b_2} \xrightarrow{BS,\ 2} \left|\frac{\alpha}{2}(1+e^{\iota\phi}), \frac{\iota\alpha(1-e^{\iota\phi})}{2}\right\rangle_{a_f,b_f}. \tag{4.4.5}$$

Now, we will discuss the phase uncertainty achieved by two different detection Schemes, taking intensity difference between two modes and parity operator measurement on one of the output mode.

4.4.1 Difference between Output Intensities

The expectation value of a number operator in a quantum state of light is directly related to their quantum field intensity[44].

$$\hat{I} \propto \langle\hat{n}\rangle . \tag{4.4.6}$$

As a result, the difference between the average photon number at the output of Mach-Zehnder interferometer may be expressed as

$$\Delta \hat{I} = \left\langle \hat{n}_{a_f} - \hat{n}_{b_f} \right\rangle . \tag{4.4.7}$$

This term is equivalent to the average value of the operator $2\hat{J}_z$ in SU(2) lie algebra. With interferometer output state, the mode intensities expectation values are derived as follows:

$$\begin{aligned} \left\langle \hat{n}_{a_f} \right\rangle &= \frac{\bar{n}_a}{2}(1 + cos\phi) \\ \left\langle \hat{n}_{b_f} \right\rangle &= \frac{\bar{n}_b}{2}(1 - cos\phi), \end{aligned} \tag{4.4.8}$$

where $\bar{n}_a = \bar{n}_b = |\alpha|^2$. The average value of operator $2\hat{J}_z$ and its square is given as

$$\begin{aligned} \langle 2\hat{J}_z \rangle &= |\alpha|^2 cos\phi, \\ \langle (2\hat{J}_z)^2 \rangle &= |\alpha|^4 cos^2\phi + |\alpha|^2. \end{aligned} \tag{4.4.9}$$

The phase uncertainty may be determined using the error propagation calculus.

$$\begin{aligned} \Delta\phi_{2\hat{J}_z} &= \frac{\Delta(2\hat{J}_z)}{\partial_\phi \langle 2\hat{J}_z \rangle}, \\ &= \frac{\sqrt{\langle (2\hat{J}_z)^2 \rangle - \langle 2\hat{J}_z \rangle^2}}{\partial_\phi \langle 2\hat{J}_z \rangle}, \\ &= \frac{1}{|\alpha|^2 sin^2\phi}. \end{aligned} \tag{4.4.10}$$

4.4.2 Parity-Based Detection

We now define the parity operator with regard to both interferometer output ports.

$$\hat{\mathbf{P}}_{a_f} = (-1)^{\hat{a}_f^\dagger \hat{a}_f}, \tag{4.4.11}$$

likewise

$$\hat{\mathbf{P}}_{b_f} = (-1)^{\hat{b}_f^\dagger \hat{b}_f}. \tag{4.4.12}$$

Therefore, we can conclude that from this the associated mean values and their first derivatives are

$$\langle \hat{\mathbf{P}}_{a_f(b_f)} \rangle = e^{-\bar{n}_0(1 \pm cos\phi)}, \tag{4.4.13}$$

$$\frac{\partial \langle \hat{\mathbf{P}}_{a_f(b_f)} \rangle}{\partial_\phi} = \pm \bar{n}_0 sin\phi \quad e^{-\bar{n}_0(1 \pm cos\phi)}. \tag{4.4.14}$$

As we discussed previously, the expected value of the square of the parity operator is $\langle \hat{\mathbf{P}}^2_{a_f(b_f)} \rangle = 1$ we can find phase uncertainty by use of error propagation calculus,

$$\Delta\phi_{\hat{\mathbf{P}}_{a_f}} = \frac{\sqrt{e^{4\bar{n}_0 cos^2\phi/2} - 1}}{\bar{n}_0 |sin\phi|}. \tag{4.4.15}$$

$$\Delta\phi_{\hat{\mathbf{P}}_{b_f}} = \frac{\sqrt{e^{4\bar{n}_0 sin^2\phi/2} - 1}}{\bar{n}_0 |sin\phi|}. \tag{4.4.16}$$

For both $\langle \hat{\mathbf{P}}_{a_f(b_f)} \rangle$, curves are displaced from one another by π phase shift. It means that the peak for occurs at $\phi = 0$ and peak for occurs at $\phi = \pi$. we can extend the above equations concerning their particular optimal phase values to find

$$\Delta\phi_{\hat{\mathbf{P}}_{a_f}} = \frac{1}{\sqrt{\bar{n}_0} |cos\phi|} \quad for\ \phi = \pi. \tag{4.4.17}$$

$$\Delta\phi_{\hat{\mathbf{P}}_{b_f}} = \frac{1}{\sqrt{\bar{n}_0} |cos\phi|} \quad for\ \phi = 0. \tag{4.4.18}$$

4.5 Phase Estimation with Maximally Entangled State: The N00N State

For Quantum Metrology, the N00N state has been extensively researched for its potential application[44, 45]. Maximal entangled state is define as the super position state in which one mode (mode a_0) contains N photons while other mode (mode b_0) contains no photon $|N, 0\rangle_{a_0,b_0}$ or no photon on a_0 mode and N photons on b_0 mode $|0, N\rangle_{a_0,b_0}$.

let us consider a different MZ interferometer configuration wherein we replace first beam splitter of MZ Interferometer by a magical device which transforms the input state as

$$|\psi_N\rangle_{BS\,1} = \frac{1}{\sqrt{2}} \left(|N, 0\rangle_{a_0,b_0} + e^{\iota\Phi_N} |0, N\rangle_{a_0,b_0}, \right) \tag{4.5.1}$$

where Φ_N is relative phase shift whose value may depends on method of state generation. The state cannot be expressed as a product of two distinct states, $|\psi\rangle_{N00N} \neq$

$|\psi_{a_0}\rangle \otimes |\psi_{b_0}\rangle$. By applying a phase shift operator $\hat{U}_{PS} = e^{\iota\phi}$ to b_0 mode which changes the state $|\psi_N\rangle$ as

$$|\psi_N(\phi)\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle_{a_0,b_0} + e^{\iota(N\phi+\Phi_N)}\,|0,N\rangle_{a_0,b_0}\right), \tag{4.5.2}$$

In this case, the emergence of $N\phi$ corresponds to a phase factor between the superposition elements. Finally, following the second beam splitter, the condition is as follows[4.2.1]:

$$|out,N\rangle_{MZI} = \frac{1}{\sqrt{2^{N+1}N!}}\ \left[(a_0^\dagger + \iota b_0^\dagger) + e^{\iota(N\phi+\Phi_N)}(b_0^\dagger + \iota a_0^\dagger)\right]\ |0,0\rangle_{a_0,b_0}. \tag{4.5.3}$$

For N=1 we obtained

$$|out,1\rangle_{MZI} = \frac{1}{2}\ \left[(1+\iota e^{\iota(\phi+\Phi_1)})\,|1,0\rangle_{a_0,b_0} + (\iota + e^{\iota(\phi+\Phi_1)})\,|0,1\rangle_{a_0,b_0}\right] \tag{4.5.4}$$

. Now, we calculate expectation value of operator $\hat{J}$ which is

$$\begin{aligned}\langle\hat{J}\rangle_{out} &= \langle\hat{n}_{b_0} - \hat{n}_{a_0}\rangle_{out}, \\ &= sin(\phi+\Phi_1).\end{aligned} \tag{4.5.5}$$

if $\Phi_1 = \frac{\pi}{2}$

$$\langle\hat{J}\rangle_{out} = sin\phi. \tag{4.5.6}$$

For N=2 we obtained

$$\begin{aligned}|out,2\rangle_{MZI} =& \frac{1}{4}\ [\sqrt{2}(1-e^{\iota(2\phi+\Phi_2)})\,|2,0\rangle_{a_0,b_0} + 2\iota(1+e^{\iota(2\phi+\Phi_2)}\,|1,1\rangle_{a_0,b_0} \\ &+ \sqrt{2}(\iota + e^{\iota(\phi+\Phi_2)})\,|0,2\rangle_{a_0,b_0}].\end{aligned} \tag{4.5.7}$$

Mean value for state $|out,2\rangle$ of operator $\hat{J}$ is

$$\langle\hat{J}\rangle = 0. \tag{4.5.8}$$

It is clear that expectation value does not depend upon the phase shift ϕ. There are no N00N states with $N > 1$ for which this technique gives an indication of phase. Now the question arises is that Is there a way to get phase information from NOON states? Dowling al. presented the Hermitian operator as follows:

$$\hat{\sum_N} = |N,0\rangle_{a_0,b_0}\langle 0,N| + |0,N\rangle_{a_0,b_0}\langle N,0|. \tag{4.5.9}$$

Now we are interested to find the expectation value of this operator for the state $|out, N\rangle$ which is

$$\left\langle \hat{\sum}_N \right\rangle_{out,N} = cosN\phi. \tag{4.5.10}$$

It is clear that expectation value depends upon the phase $N\phi$. By use of error propagation calculus we can find phase uncertainty which is [?]

$$\begin{aligned}(\Delta\phi)^2 &= \frac{\left\langle \hat{\sum}_N^2 \right\rangle - \left\langle \hat{\sum}_N \right\rangle^2}{\left(\partial_\phi \left\langle \hat{\sum}_N \right\rangle\right)^2}, \\ &= \frac{1 - cos^2 N\phi}{N^2 sin^2 N\phi}, \\ &= \frac{1}{N^2}.\end{aligned} \tag{4.5.11}$$

An enhancement above Standard Quantum Limit (classical limit) by a factor of $1/\sqrt{N}$ is seen in the above result, which is independent of phase shift ϕ.

Now we use parity measurement for the determination of Heisenberg-limited phase estimation. we considered the N=2 and parity operator just change the sign of middle term upon action of mode-b parity operator.

$$\begin{aligned}\hat{\mathbf{P}}_{b_f} |out, 2\rangle_{MZI} =& \frac{1}{4}\ [\sqrt{2}(1 - e^{\iota(2\phi+\Phi_2)}) |2, 0\rangle_{a_0,b_0} - 2\iota(1 + e^{\iota(2\phi+\Phi_2)}) |1, 1\rangle_{a_0,b_0} \\ &+ \sqrt{2}(\iota + e^{\iota(\phi+\Phi_2)}) |0, 2\rangle_{a_0,b_0}].\end{aligned} \tag{4.5.12}$$

Now we see that the mean value parity operator is

$$\langle \hat{\mathbf{P}} \rangle = cos(2\phi + \Phi_2), \tag{4.5.13}$$

It has a desired dependency on 2ϕ. For arbitrary N, we find

$$\langle \hat{\mathbf{P}} \rangle = \frac{i^N}{2} \left[e^{i(N\phi+\Phi_N)} + (-1)^N e^{-i(N\phi+\Phi_N)}\right], \tag{4.5.14}$$

or

$$\begin{aligned}\langle \hat{\mathbf{P}} \rangle &= (-1)^{N/2} cos(N\phi + \Phi_N) \quad For\ even\ N, \\ \langle \hat{\mathbf{P}} \rangle &= (-1)^{(N+1)/2} sin(N\phi + \Phi_N) \quad For\ odd\ N.\end{aligned} \tag{4.5.15}$$

It is clear that phase uncertainty is Heisenberg-limited

$$\Delta\phi = \frac{1}{N}. \tag{4.5.16}$$

Chapter 5

Summary and Conclusions

Over the past few years, several theoretical studies have been done in the field of quantum metrology. The purpose of this dissertation has been to make a small effort to learn about the small topic, phase estimation using quantum optical interferometry, of this giant temple field. I would like to conclude by summarizing the work presented in this dissertation, as following

In this thesis, we have investigated the problem of phase-estimation using quantum optical interferometry, which is an interesting implementation of quantum metrology in the area of quantum optics. In general, quantum metrology deals with the techniques to improve the measurement precision beyond the limits set by standard quantum fluctuations. In many physical situations, the process of precision measurement can be reduced to detection of a small phase shift by using optical interferometric set-up. In our study, we first have presented a quick review on the quantum mechanics of various optical devices, such as, beam-splitters, phase-shifters and interferometers. Then we have studied their response to different quantum mechanical input states with classical-like (coherent states) and non-classical (N00N state) nature.

Using these quantum optical set-ups, we have studied the quantum optical phase-estimation under various classical-like and non-classical probe states at the input. In this perspective, we have explored the fundamental bounds on ultimate precision limits in the phase estimation using the notion of Fisher information entropy. Such bounds are the so-called Cramér–Rao bounds which typical come out to be different for classical and quantum probes. We have shown that by using the coherent states as our initial probe state, the estimation error is computed as $\approx 1/\sqrt{\bar{n}}$, which is known as Short Noise Limit (SNL). However, for the case of N00N states (highly non-classical state), which is entangled state, the ultimate precision limit takes the value as $\approx 1/\bar{n}$, which is known as *Heisenberg-limit*. It is important to note that in the Heisenberg-limited phase-estimation, the accuracy is enhanced by a factor of $\sqrt{\bar{n}}$ with respect to SNL.

Appendix A

Brief Overview of the SU(2) Group

A.1 The Schwinger Realization of SU(2)

Suppose that a beam splitter has two incident modes, each of which is described by a set of boson operators, $\{\hat{a}_0, \hat{a}_0^\dagger\}$ for mode a_0 and $\{\hat{b}_0, \hat{b}_0^\dagger\}$ for mode b_0. These operators fulfill the commutation relations in the following manner

$$[\hat{a}_0, \hat{a}_0^\dagger] = [\hat{b}_0, \hat{b}_0^\dagger] = 1, \tag{A.1.1}$$

and

$$[\hat{a}_0, \hat{b}_0] = 0. \tag{A.1.2}$$

The Hermitian operator may be introduced as

$$\begin{aligned} \hat{J}_x &= \frac{1}{2}(\hat{a}_0^\dagger \hat{b}_0 + \hat{b}_0^\dagger \hat{a}_0), \\ \hat{J}_y &= -\frac{\iota}{2}(\hat{a}_0^\dagger \hat{b}_0 - \hat{b}_0^\dagger \hat{a}_0), \\ \hat{J}_z &= \frac{1}{2}(\hat{a}_0^\dagger \hat{a}_0 - \hat{b}_0^\dagger \hat{b}_0), \end{aligned} \tag{A.1.3}$$

and

$$\hat{J}_0 = \frac{1}{2}((\hat{a}_0^\dagger \hat{a}_0 + \hat{b}_0^\dagger \hat{b}_0). \tag{A.1.4}$$

Hermitian operators fulfilling the commutation relations of lie algebra of SU(2) include the following

$$[\hat{J}_i, \hat{J}_j] = \iota \hat{J}_k \epsilon_{i,j,k}, \tag{A.1.5}$$

where ϵ_{ijk} is the Levi-Civita symbol. An invariant that is compatible with all operators of the group, the Casimir invariant, may be derived as follows:

$$\begin{aligned} \hat{J}^2 &= \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = \hat{J}_0(\hat{J}_0 + 1), \\ &= \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2} + 1\right). \end{aligned} \tag{A.1.6}$$

It is fairly straightforward to see the action of angular momentum operators $\hat{J}_i$ on the quantized field state $|n\rangle_{a_0} \otimes |n^{'}\rangle_{b_0}$

$$\begin{aligned} \hat{J}_0 |n, n^{'}\rangle &= \frac{1}{2}(n + n^{'}) |n, n^{'}\rangle, \\ \hat{J}_z |n, n^{'}\rangle &= \frac{1}{2}(n - n^{'}) |n, n^{'}\rangle. \end{aligned} \tag{A.1.7}$$

As this is definitely the angular momentum algebra, it is straightforward to write a two-mode quantized field state $|n\rangle_{a_0} \otimes |n^{'}\rangle_{b_0}$ in terms of SU(2) multipelt states $|j, m\rangle$ so action of angular momentum operators on $|j, m\rangle$ state is

$$\begin{aligned} \hat{J}^2 |j, m\rangle &= j(j+1) |j, m\rangle, \\ \hat{J}_0 |j, m\rangle &= j |j, m\rangle, \\ \hat{J}_z |j, m\rangle &= m |j, m\rangle. \end{aligned} \tag{A.1.8}$$

Which informs us the values of j and m which are given as

$$j = \frac{1}{2}(n + n^{'}) \quad and \quad m = \frac{1}{2}(n - n^{'}). \tag{A.1.9}$$

Finally, we can write the multiple state as

$$|n, n^{'}\rangle \to |j, m\rangle = \left|\frac{n + n^{'}}{2}, \frac{n - n^{'}}{2}\right\rangle. \tag{A.1.10}$$

A.2 Beam Splitters Revisited using SU(2) Group

Next, we will briefly review the concepts of the lossless beam splitter in terms of SU(2) lie algebra that is essential for our work. It is assumed that the boson annihilation operators a_0 and b_0 represent the input ports of the beam splitter, whereas a_f and b_f represent the output ports. The scattering matrix combining them can have the form

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}, \end{pmatrix} \tag{A.2.1}$$

Such that

$$\begin{pmatrix} \hat{a}_f \\ \hat{b}_f \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{b}_0 \end{pmatrix}. \tag{A.2.2}$$

The scattering matrix must be unitary due to the conservation of boson operator commutation relations. The scattering matrix for $\hat{J}_x$ type is written as follows, based on Yurke et al. [32]

$$\hat{U} = \begin{pmatrix} cos\frac{\theta}{2} & -\iota sin\frac{\theta}{2} \\ -\iota sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{pmatrix}. \tag{A.2.3}$$

After that, we will examine how this scattering matrix transforms components of SU (2), $\hat{J} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$? For this we can see

$$\begin{aligned} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{output} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & -cos\theta \end{pmatrix} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{input}, \\ &= e^{\iota\theta\hat{J}_x} \begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}_{in} e^{-\iota\theta\hat{J}_x}. \end{aligned} \tag{A.2.4}$$

which can be checked by the use of baker-Hausdorff identity. that is, our input state is rotated by an angle θ about the x-axis. It should be noted that the analogy between the number state basis and the angular momentum states of the SU(2) group is purely formal [?].it arises because the Lie algebra of operators generating unitary transformations in two-dimensional space happen to be the same algebra of the operators generating rotations in a three-dimensional space. For this reason, these are often called 'quasi-spins', but have no direct physical interpretation in terms of any rotation in a real three-dimensional space [33]. Working in the Schrodinger picture, we can write the output state in terms of the input state as

$$|\psi\rangle_{out} = e^{-\iota\theta\hat{J}_x} |\psi\rangle_{in}. \tag{A.2.5}$$

A.3 Calculations in Interferometry using the SU(2) Group

Assuming our beam splitters are devices that rotate the state about the x-axis then the output state of MZI may be expressed as follows:

$$|out\rangle_{BS} = e^{\iota\frac{\pi}{2}\hat{J}_x} e^{-\iota\phi\hat{J}_z} e^{-\iota\frac{\pi}{2}\hat{J}_x}, |in\rangle_{BS} \tag{A.3.1}$$

where we use the Baker-Hausdorf relation to simplify

$$\begin{aligned} e^{\iota\frac{\pi}{2}\hat{J}_x} e^{-\iota\phi\hat{J}_z} e^{-\iota\frac{\pi}{2}\hat{J}_x} &= exp[-\iota\phi e^{\iota\frac{\pi}{2}\hat{J}_x} \hat{J}_z e^{-\iota\frac{\pi}{2}\hat{J}_x}], \\ &= e^{-\iota\phi\hat{J}_y}, \end{aligned} \tag{A.3.2}$$

Where $e^{\iota\frac{\pi}{2}\hat{J}_x} \hat{J}_z e^{-\iota\frac{\pi}{2}\hat{J}_x} = \hat{J}_y$. Assume that we have two arbitrary pure field states in our input ports, in the Schrodinger picture, we can write this as

$$\begin{aligned} |in\rangle &= |\psi^{(1)}\rangle_{a_0} \otimes |\psi^{(2)}\rangle_{b_0}, \\ &= \left(\textstyle\sum_{n=0}^{\infty} C_n^{(1)} |n\rangle\right) \otimes \left(\textstyle\sum_{n'=0}^{\infty} C_{n'}^{(2)} |n'\rangle\right), \\ &= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} C_n^{(1)} C_{n'}^{(2)} |n\rangle_{a_0} |n'\rangle_{b_0}, \\ &= \sum_{j=0,\frac{1}{2},\ldots}^{\infty} \sum_{m=j}^{-j} C_{j+m}^{(1)} C_{j-m}^{(2)} |j,m\rangle . \end{aligned} \tag{A.3.3}$$

Lastly, we go from number state to angular momentum. To describe our output state, we can use

$$\begin{aligned} |out\rangle &= e^{-\iota\phi\hat{J}_y} |in\rangle, \\ &= \sum_{j=0,\frac{1}{2},\ldots}^{\infty} \sum_{m=j}^{-j} C_{j+m}^{(1)} C_{j-m}^{(2)} e^{-\iota\phi\hat{J}_y} |j,m\rangle, \\ &= \sum_{j=0,\frac{1}{2},\ldots}^{\infty} \sum_{m=j}^{-j} \sum_{m'=j}^{-j} C_{j+m}^{(1)} C_{j-m}^{(2)} |j,m'\rangle \langle j,m'| e^{-\iota\phi\hat{J}_y} |j,m\rangle, \\ &= \sum_{j=0,\frac{1}{2},\ldots}^{\infty} \sum_{m=j}^{-j} \sum_{m'=j}^{-j} C_{j+m}^{(1)} C_{j-m}^{(2)} \langle j,m'| e^{-\iota\phi\hat{J}_y} |j,m\rangle |j,m'\rangle, \\ &= \sum_{j=0,\frac{1}{2},\ldots}^{\infty} \sum_{m=j}^{-j} \sum_{m'=j}^{-j} C_{j+m}^{(1)} C_{j-m}^{(2)} d_{m',m}^{j}(\phi) |j,m'\rangle, \end{aligned} \tag{A.3.4}$$

Where $d^j_{m',m}(\phi)$ is the Wigner-d matrix element and we have included a complete set of states to finish up the relation

$$\hat{I}_{m'} = \sum_{m'=j}^{-j} |j, m^{'}\rangle \langle j, m^{'}| \,. \tag{A.3.5}$$

Throughout this thesis, we often calculate the expectation value of a detection observable in the Heisenberg picture. In the case of output a_f-mode parity detection, this becomes

$$\begin{aligned} \langle \hat{\mathbf{P}} \rangle_{out} &= \langle e^{\iota\phi\hat{J}_y} \hat{\mathbf{P}} e^{-\iota\phi\hat{J}_y} \rangle_{in} \,, \\ &= \sum_{j,m'} \sum_{J,M} \mathit{\Gamma}^*_{J,M}(\phi) \mathit{\Gamma}_{j,m'}(\phi) \langle J, M | \hat{\mathbf{P}} | j, m^{'} \rangle \,, \end{aligned} \tag{A.3.6}$$

Where $\mathit{\Gamma}(\phi)$ is phase dependent state coffecient given as

$$\mathit{\Gamma}_{j,m'}(\phi) = \sum_{m=-j}^{j} C^{(1)}_{j+m} C^{(2)}_{j-m} d^j_{m,m'}(\phi). \tag{A.3.7}$$

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