Hermite-Hadamard inequality for Hadamard-type fractional integral with respect to a function

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Abroo Batool

Abstract

In this work, we present a new Hermite-Hadamard inequality which is associated with Hadamard-type integral with respect to another function. This fractional integral is the generalization of fractional integral with respect to another function, Hadamard-type fractional integral, Hadamard fractional integral, Tempered fractional integral, Katugampola fractional integral and Riemann-Liouville fractional integral. The main significance of the inequality is that it contains Hermite-Hadamard inequalities for many fractional integrals as special cases. Generalized result of Hermite-Hadamard type inequality for fractional integral with respect to another function is also established.

Contents

Chapter 1

Basic Concepts

1.1 Introduction

Fractional calculus is a modern tool for researchers working in pure and applied mathematics. The Fractional calculus journey goes back to seventeen century, when L'Hospital raised a question to Leibniz that "what is the derivative of non-integer number". Leibniz response "an apparent paradox, from which one day useful consequences will be drawn,". First definition of fractional derivative was introduced by Lacroix in 1819. In 1823 Abel was the first who solved the tautochrone problem by using arbitrary order derivative. In 1834 J. Liouville worked on complementary functions, gave a reasonable definition of a fractional derivative and the point of existence of the right-sided and left-sided integrals and derivatives. He has a great contributions in fractional calculus. In 1841 D. F. Gregory gave the solution of heat equation. In 1846 P. Kelland assumed that the principal of permanence state for algebra is valid for all symbolic operation.

In 1847 B. Riemann gave the definition of fractional integration in which he added complementary function. In 1848 C. J. Hargreave generalized the Lebniz rule for arbitrary order derivative. In 1892 J. Hadamard introduced the new concept of fractional integrals and derivatives. Numerous mathematicians has contributed their work through out the 19th and 20th century in the advancement of fractional calculus. Few of them are W. Center(1850), H. R. Greer(1859), Z. Wastchenxo(1861), H. Holmgren(1865-67), A. K. Grunwald(1867), A. V. Letnikov(1868-72). A. Cayley(1880), H. Laurent(1884), P. A. Nekrassov(1888), A. Krug(1890), O. Heaviside(1893-99), G. Oltramare(1893), R. E. Mortiz(1902), H. Weyl(1917), H. T. Davis(1936), H. Kober(1940), A. S. Peters(1961), G. V. Welland, K. B. Oldham(1972), S. G. Samko, A. A. Kilbas, and O. I. Marichev(1993),

I. Podlubny(1999), N. Sudland(2000). For three centuries the theory of fractional derivative was only useful for mathematicians.
In literature there are different definitions of fractional operators. Among these, Riemann-Liouville, Caputo and Hadamard are most commonly studied [33, 10, 22]. Fractional calculus has wide applications in all fields of engineering and science [33, 35, 37, 42] such as electromagnetism, viscoelasticity, hydraulics, electromagnetic, biological population models [28, 30], optics, and signals processing. Fractional calculus is used to model the physical and engineering process and they are described by fractional differential equations. One of the recent and famous application of fractional calculus is describing the remembrance and ancestry properties of various materials and processes.

1.2 Some special functions of fractional calculus

In the first chapter we provide basic concept for the readers. We state fundamental theorem of calculus, definition of gamma function, beta function, Riemann-Liouville fractional integral and fractional derivative, Caputo fractional derivative, Hadamard fractional integral, Hadamard-type fractional integral and some of their basic properties.

1.2.1 Gamma function

Euler in 1729 discovered the gamma function. The gamma function plays an important role in fractional calculus. Gamma function is the generalization of factorial function to real and complex number argument.

Definition 1.2.1. [10] The function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(\omega) = \int_0^\infty s^{\omega-1} e^{-s} ds, \ \omega > 0,$$

is called Euler's gamma function.

Properties Following are some very common properties of gamma function.

(a) $\Gamma(w+1) = w\Gamma(w)$;

(b) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$;

(c) $\Gamma(w+1) = w!$, if w is non-negative integer.

Proof. (a) By definition of gamma function

$$\Gamma(w+1) = \int_0^\infty s^w e^{-s} ds.$$

Integrating by parts

$$\begin{aligned}\Gamma(w+1) =& s^w e^{-s}|_0^\infty - \int_0^\infty w s^{w-1} e^{-s} ds = 0 + \int_0^\infty w s^{w-1} e^{-s} ds \\ =& w \int_0^\infty s^{w-1} e^{-s} ds = w\Gamma(w).\end{aligned}$$

(b) Here substituting, $w = \frac{1}{2}$ in definition of gamma function and integrating, we have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty s^{\frac{1}{2}-1} e^{-s} ds = \int_0^\infty s^{\frac{-1}{2}} e^{-s} ds.$$

We evaluate the integral by substitution method. Let $s = u^2$

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) =& \int_0^\infty \frac{e^{-u^2}}{\sqrt{u}} 2u du = 2\int_o^\infty e^{-u^2} du \\ =& 2\int_0^\infty e^{-v^2} dv.\end{aligned}$$

Now,

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dudv.$$

We use transformation of rectangular coordinate to polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, $0 < \theta \leq \frac{\pi}{2}$, $dudv = Jdrd\theta$ where J is called the Jacobi matrix. Here in this case $J = r$. Thus

$$\begin{aligned}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 =& 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta \\ =& 4\left(\frac{\pi}{2} - 0\right) \int_0^\infty e^{-r^2} r dr.\end{aligned}$$

Let $z = r^2$, $dz = 2rdr$, then

$$\Gamma\left(\frac{1}{2}\right)^2 = 2\frac{\pi}{2}\int_0^\infty e^{-z} dz = \pi(e^{-\infty} + e^0) = \pi.$$

Taking square root on both side

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{2}.$$

□

(c) Since $\Gamma(1) = 1$, by part (a)

$$\Gamma(2) = \Gamma(1+1) = 1\ \Gamma(1) = 1.$$

Similarly,

$$\Gamma(3) = \Gamma(2+2) = 2\ \Gamma(2) = 2!.$$

In general

$$\Gamma(w+1) = (w)!.$$

Extension of domain of gamma function

The functional equation

$$\Gamma(j+1) = j\Gamma(j), \tag{1.2.1}$$

can be used to extend definition from $j > 0$ to all real number except $j \neq 0, -1, -2 \cdots$. From (1.2.1) we have,

$$\Gamma(j+1) = j\Gamma(j)\ or\ \Gamma(j) = \frac{\Gamma(j+1)}{j}.$$

Right hand side is define for $j + 1 > 0$, $j > -1$, $j \neq 0$ where $\Gamma(j)$ is define for $j > 0$. Now from (1.2.1)

$$\begin{aligned}\Gamma(j+2) =& (j+1)(j)\Gamma(j)\\ \Gamma(j) =& \frac{\Gamma(j+2)}{j(j+1)},\ j \neq 0, -1, -2.\end{aligned}$$

In general, repeated application of above procedure leads us to

$$\begin{aligned}\Gamma(j+n) =& (j+n-1)(j+n-2)\cdots(j-1)j\Gamma(j)\\ \Gamma(j) =& \frac{\Gamma(j+n)}{j(j+1)(j+2)\cdots(j+n-1)};\ j \neq 0, -1, -2\cdots\end{aligned}$$

Thus domain of $\Gamma(j)$ is extended for all real number except $j \neq 0, -1, -2 \cdots$

In mathematics, Euler integral of first kind is also called beta function, which is closely related to the gamma function and to the binomial co-efficient.

1.2.2 Beta function

Definition 1.2.2. [10] The beta function is defined as

$$B(\zeta, \mu) = \int_0^1 z^{(\zeta-1)}(1-z)^{(\mu-1)}dz,\ \zeta, \mu > 0.$$

1.2.3 Relation between beta and gamma functions

The relationship between gamma and beta functions can mathematically be expressed as,

$$B(\zeta,\mu) = \frac{\Gamma(\zeta)\Gamma(\mu)}{\Gamma(\zeta+\mu)},$$

where $B(\zeta,\mu)$ is two variable function and $\Gamma(y)$ is function of one independent variable.

Theorem 1.2.3. *If $h(z_1,z_2)$ is continuous on $R=[a_1,a_2]\times[b_1,b_2]$ then,*

$$\int_R\int h(z_1,z_2)dA = \int_{a_1}^{a_2}\int_{b_1}^{b_2} h(z_1,z_2)dz_2dz_1 = \int_{b_1}^{b_2}\int_{a_1}^{a_2} h(z_1,z_2)dz_1dz_2. \tag{1.2.2}$$

The integrals are called iterated integral.

Mittag-Leffler function is a generalization of exponential function. The Mittag-Leffler function for two variable is defined as

Definition 1.2.4. [33] Let $\alpha,\beta>0$, then Mittag-Leffler function be defined as

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\beta+\alpha k)},\ y\in\mathbb{R}.$$

Lemma 1.2.5. *(Leibniz rule)*
Let $h(y,t)$ be continuous and $\frac{\partial}{\partial y}$ be continuous in the domain of the $yt-$plane that contains the rectangular region R. $a\le y\le b$, $t_0\le t\le t_1$ and limit of integration $\eta(y)$ and $\xi(y)$. Then function, and having continuous derivative on $a\le y\le b$.

$$\frac{d}{dy}\int_{\eta(y)}^{\xi(y)} h(y,t)dt = h(y,\xi(y))\frac{d\xi}{dy} - f(y,\eta)\frac{d\eta}{dy} + \int_{\eta(y)}^{\xi(y)}\frac{\partial}{\partial y}h(y,t)dt.$$

1.2.4 Spaces

Definition 1.2.6. The space C[c,d] is the collection of all continuous real valued functions defined on [c,d], such that

$$||h(x)|| = \sup|h(x)|,\ for\ all\ h\in\ C[c,d]$$

or

$$||h(x)|| = \int_c^d |h(s)|ds,\ for\ all\ h\in C[c,d].$$

Definition 1.2.7. The space $L_p[c,d]$ $(1 \leqq p \leqq \infty)$ is the set of Lebesgue Measurable function on A (A=[c,d]) for which $||h||_p < \infty$, where

$$||h(y)||_p = \left(\int_c^d |h(s)|^p ds\right)^{\frac{1}{p}},\ 1 \leqq p < \infty$$

and

$$||h||_\infty = ess \sup_{c \leq y \leq d} |h(y)|.$$

Definition 1.2.8. [22] The space $X_c^p(k,l)$ $(c \in \mathbb{R})$ consists of those complex valued Lebesgue measurable functions h on (k,l) for which $||h||_{X_c^p} < \infty$, with

$$||h||_{X_c^p} = \left(\int_k^l |t^c h(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}},\ 1 \leqq p < \infty$$

and

$$||h||_{X_c^\infty} = ess \sup_{k \leq y \leq l} [y^c |h(y)|].$$

In particular when $c = \frac{1}{p}$ the space X_c^p is coincides with the $L_p(k,l)$ space: $X_{\frac{1}{p}}^p(k,l) = L_p(k,l)$.

Definition 1.2.9. [10] The space $A^m[c,d]$ is the set of functions h for which there exists a function $g \in L_1[c,d]$ almost everywhere such that

$$h^{n-1}(z) = h^{n-1}(c) + \int_c^z g(t)dt$$

defines the set of functions with absolutely continuous $(n-1)$ derivative.

1.3 Fractional integrals and derivatives

Integrals and derivatives play a significant role in mathematics. Integrals helps us either to obtain the area under the graph or to find the function whose derivative is integrated. Fractional integral can be determined by repeated integration. In contrast of previous section, the following concept seems rather natural.

1.3.1 Properties of differential and integral operators

Lemma 1.3.1. *[10] Let $h : [k,l] \to \mathbb{R}$ be a continuous function, and $H : [k,l] \to \mathbb{R}$ be defined as*

$$H(t) := \int_k^t h(s)ds.$$

Then, H is differentiable and $H'(t) = h$.

Remark 1.3.2. [10] Lemma (1.3.1) can also be read as

$$\mathcal{D}\mathcal{I}_c h(t) = h(t), \text{ where } D = \frac{d}{dt} \text{ and } \mathcal{I}_c h(t) = \int_a^t h(s)ds. \tag{1.3.1}$$

Now we state composition properties of differential and integral operators as follows. Repeated application of (1.3.1) gives

$$\mathcal{D}^2\mathcal{I}_c^2 h(t) = \mathcal{D}(\mathcal{D}\mathcal{I}_c(\mathcal{I}_c h(t))) = \mathcal{D}\mathcal{I}_c h(t) = h(t). \tag{1.3.2}$$

By using Eqn (1.3.2) we can deduce that

$$\mathcal{D}^3\mathcal{I}_c^2 h(t) = \mathcal{D}(\mathcal{D}^2\mathcal{I}_c^2 h(t)) = \mathcal{D}h(t),$$
$$\mathcal{D}^2\mathcal{I}_c^3 h(t) = \mathcal{D}^2\mathcal{I}_c^2(\mathcal{I}_c h(t)) = \mathcal{I}_c h(t).$$

Similarly we have

$$\mathcal{D}^3\mathcal{I}_c^3 h(t) = \mathcal{D}^2(\mathcal{D}\mathcal{I}_c(\mathcal{I}_c^2 f(x))) = \mathcal{D}^2\mathcal{I}_c^2 h(t) = \mathcal{D}(\mathcal{D}\mathcal{I}_c(\mathcal{I}_c h(t))) = \mathcal{D}\mathcal{I}_c h(t) = h(t). \tag{1.3.3}$$

By using Eqn (1.3.3)

$$\mathcal{D}^4\mathcal{I}_c^3 h(t) = \mathcal{D}(\mathcal{D}^3\mathcal{I}_c^3 h(t)) = \mathcal{D}h(t),$$
$$\mathcal{D}^3\mathcal{I}_c^4 h(t) = \mathcal{D}^3\mathcal{I}_c^3(\mathcal{I}_c h(t)) = \mathcal{I}_c h(t).$$

In general

$$\begin{aligned} \mathcal{D}^p\mathcal{I}_c^q h(t) &= \mathcal{D}^{p-q} h(t), \ p \geq q \\ \mathcal{D}^p\mathcal{I}_c^q h(t) &= \mathcal{I}^{q-p} h(t), \ p \leq q. \end{aligned} \tag{1.3.4}$$

From Remark (1.3.2) we can conclude that derivative is the left inverse of integral when $p = q$. The composition of Integral with itself obey the law of exponent that is $\mathcal{I}^p\mathcal{I}_c^q h(t) = \mathcal{I}^{p+q} h(t)$, Now we see the composition of integral $\mathcal{I}_c h(t) = \int_c^t h(s)ds$ with derivative $D = \frac{d}{ds}$. By Fundamental Theorem of Calculus we have

$$\mathcal{I}_c Dh(t) = \int_c^t Dh(s)ds = h(t) - h(a).$$

Using above relation

$$\mathcal{I}_c D^2 h(t) = \int_c^t D(Dh(s))ds = Dh(t) - h'(a).$$

Similarly

$$\mathcal{I}_c D^3 h(t) = \int_c^t D(D^2 h(s))ds = D^2 h(t) - h''(a).$$

$$\mathcal{I}_c^2 Dh(t) = \mathcal{I}_c(\mathcal{I}_c Dh(t)) = \mathcal{I}_c h(t) - h(c)(x-c).$$

In general

$$\begin{aligned} \mathcal{I}_c^p \mathcal{D}^q h(t) &= \mathcal{I}^{p-q} h(t) + extra\ term\ p \geq q, \\ \mathcal{I}_c^p \mathcal{D}^q h(t) &= \mathcal{D}^{p-q} h(t) + extra\ term\ p \leq q. \end{aligned} \tag{1.3.5}$$

Eqn (1.3.4) and Eqn (1.3.5) concide only when the extra term in Eqn (1.3.5) vanish this means that $h(c) = h'(c) = h''(c) = ... = 0$.

1.3.2 Riemann-Liouville fractional integral and derivative

Riemann-Liouville fractional integral is obtained by the Cauchy iterated formula.

Lemma 1.3.3. *[10, 22] Let h be Riemann integrable on $[p,q]$. Then, for $p \leq z \leq q$ and $m \in \mathbb{N}$ we have*

$$\mathcal{I}_p^m h(z) = \frac{1}{(m-1)!} \int_p^z (z-s)^{m-1} h(s) ds.$$

Proof. Let us start from the simple integral

$$\mathcal{I}_p h(z) = \int_p^z h(s) ds. \tag{1.3.6}$$

Iterating integral (1.3.6), and by using Theorem (1.2.3)

$$\begin{aligned} \mathcal{I}_p^2 h(z) = \mathcal{I}_p(\mathcal{I}_p h) z &= \int_p^z \int_p^{t_1} h(s) dt_1 dt_2 \\ &= \int_p^z \int_{t_1}^z h(t_1) dt_2 dt_1 \\ &= \int_p^z h(t_1)(z-t_1) dt_1 \\ &= \int_p^z h(t)(z-t) dt. \end{aligned}$$

The third iterate gives

$$\mathcal{I}_p^3 h(z) = \mathcal{I}_p(\mathcal{I}_p(\mathcal{I}_p h(z))) = \int_p^z \int_p^{t_1} \int_p^{t_2} h(t) dt dt_2 dt_1.$$

By using Theorem (1.2.3)

$$\begin{aligned}
\mathcal{I}_p^3 h(z) &= \int_p^z \int_p^{t_1} \int_t^{t_1} h(t) dt_2 dt dt_1 \\
&= \int_p^z \int_p^{t_1} h(t)(t_1 - t) dt dt_1 \\
&= \int_p^z \int_t^z h(t)(t_1 - t) dt_1 dt \\
&= \int_p^z h(t) \left(\frac{(z-t)^2}{2} - \frac{(t-t)^2}{2} \right) dt \\
&= \int_p^z h(t) \frac{(z-t)^2}{2!} dt.
\end{aligned}$$

Repeating the above process upto m-times we have,

$$\mathcal{I}_p^m h(z) = \frac{1}{(m-1)!} \int_p^z h(t)(z-t)^{m-1} dt. \tag{1.3.7}$$

The last integral is called Cauchy iterated integral formula. □

Using relation between gamma function and factorial function, we can define fractional integral. Replacing integer m with real $\nu > 0$ in Eqn (1.3.7). The integral (1.3.7) becomes fractional integral.

Definition 1.3.4. [10, 22] For a function $h : [p, q] \to \mathbb{R}$, the Riemann-Liouville fractional integral of order $\nu > 0$ is defined as

$$\mathcal{I}_p^\nu h(z) = \frac{1}{\Gamma(\nu)} \int_p^z (z-s)^{\nu-1} h(s) ds.$$

Lemma 1.3.5. *For $h(y) = y^\zeta$ we have*

$$\mathcal{I}_0^\nu h(y) = \frac{\Gamma(\zeta+1)}{\Gamma(\nu+\zeta+1)} y^{\nu+\zeta}. \tag{1.3.8}$$

Proof. By definition of Riemann-Liouville fractional integral

$$\begin{aligned}
\mathcal{I}_0^\nu h(y) =& \frac{1}{\Gamma(\nu)} \int_0^y (y-t)^{\nu-1} t^\zeta dt \\
\mathcal{I}_0^\nu y^\zeta =& \frac{1}{\Gamma(\nu)} \int_0^y y^{\nu-1} \left(1 - \frac{t}{y}\right)^{\nu-1} t^\zeta dt.
\end{aligned}$$

We evaluate the integral by substituting $v = \frac{t}{y}$.

$$\begin{aligned}
\mathcal{I}_0^\nu y^\zeta =& \frac{1}{\Gamma(\nu)} \int_0^1 y^{\nu-1}(1-v)^{\nu-1} y^\zeta v^\zeta dv \\
=& \frac{y^{\nu+\zeta+1-1}}{\Gamma(\nu)} \int_0^1 (1-v)^{\nu-1} v^\zeta du.
\end{aligned}$$

Since,

$$\int_0^1 (1-v)^{\nu-1} v^{\zeta} du = B(\nu, \zeta+1) \text{ and } B(\nu,\zeta) = \frac{\Gamma(\nu)\Gamma(\zeta)}{\Gamma(\nu+\zeta)}.$$

Therefore, we have

$$\begin{aligned}\mathcal{I}_0^{\nu} y^{\zeta} =& \frac{y^{\nu+\zeta} B(\nu,\zeta+1)}{\Gamma(\nu)} \\ =& \frac{\Gamma(\nu)\Gamma(\zeta+1) y^{\nu+\zeta}}{\Gamma(\nu)\Gamma(\nu+\zeta+1)} \\ =& \frac{\Gamma(\zeta+1) y^{\nu+\zeta}}{\Gamma(\nu+\zeta+1)}.\end{aligned}$$

□

Lemma (1.3.5) can be used to find fractional integrals of functions which can be expanded by Maclaurin series. As an example, here we will find the fractional integral of $\sin z$ for $z \in \mathbb{R}$.

Example 1.3.6. [10, 22] Find the fractional integral of $\sin z$. First we expand $\sin z$ into its maclaurin series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots, \tag{1.3.9}$$

$$= \sum_{q=0}^{\infty} \frac{(-1)^q z^{2q+1}}{(2q+1)!}, \tag{1.3.10}$$

where q is non-negative integer. Using Eqn (1.3.8) and property of gamma function we get

$$\begin{aligned}\mathcal{I}_0^{\nu} \sin z =& \sum_{q=0}^{\infty} \frac{(-1)^q \mathcal{I}_0^{\nu} z^{2q+1}}{(2q+1)!} \\ =& \sum_{q=0}^{\infty} \frac{(-1)^q \Gamma((2q+1)+1) z^{2q+1+\nu}}{(2q+1)!\Gamma(2q+2+\nu)} \\ =& \sum_{q=0}^{\infty} \frac{(-1)^q z^{2q+1+\nu}}{\Gamma(2q+2+\nu)}.\end{aligned}$$

As,

$$E_{2,2+\nu}(-z^2) = \sum_{q=0}^{\infty} \frac{(-z^2)^q}{\Gamma(2+\nu+2q)}.$$

Therefore

$$\begin{aligned}\mathcal{I}_0^{\nu} \sin z =& z^{1+\nu} \sum_{q=0}^{\infty} \frac{(-z^2)^q}{\Gamma(2+\nu+2q)} \\ =& z^{1+\nu} E_{2,2+\nu}(-z^2).\end{aligned}$$

In the following theorem we state and prove the semi-group property of Riemann-Liouville fractional integral

Theorem 1.3.7. *[10, 22] Let $\nu, \zeta \geq 0$ and $\psi \in L_1[p,q]$. Then*

$$\mathcal{I}_p^{\nu}\mathcal{I}_p^{\zeta}\psi(z) = \mathcal{I}_p^{\nu+\zeta}\psi(z).$$

If $\psi \in C[p,q]$ or $\nu + \zeta \geq q$, then the identity holds everywhere on $[p,q]$.

Proof.

$$\mathcal{I}_p^{\nu}\mathcal{I}_p^{\zeta}\psi(z) = \frac{1}{\Gamma(\nu)\Gamma(\zeta)}\int_p^{z_1}\int_p^{t_1}(z_1-t_1)^{\nu-1}(t_1-s)^{\zeta-1}\psi(s)dsdt_1.$$

Interchange the order of integration

$$\mathcal{I}_p^{\nu}\mathcal{I}_p^{\zeta}\psi(z) = \frac{1}{\Gamma(\nu)\Gamma(\zeta)}\int_p^{z_1}\int_s^{z_1}(z_1-t_1)^{\nu-1}(t_1-s)^{\zeta-1}\psi(s)dt_1ds.$$

By substituting $t_1 = s + z(z_1 - s)$

$$\begin{aligned}
\mathcal{I}_p^{\nu}\mathcal{I}_p^{\zeta}\psi(z) =&\frac{1}{\Gamma(\nu)\Gamma(\zeta)}\int_p^{z_1}\int_0^{1}(z_1-s-z(z_1-s))^{\nu-1}(s+z(z_1-s)-s)^{\zeta-1}(z_1-s)\psi(s)dzds\\
=&\frac{1}{\Gamma(\nu)\Gamma(\zeta)}\int_p^{z_1}\int_0^{1}(z_1-s)^{\nu-1}(1-z)^{\nu-1}z^{\zeta-1}(z_1-s)^{\zeta-1}(z_1-s)\psi(s)dzds\\
=&\frac{1}{\Gamma(\nu)\Gamma(\zeta)}\int_p^{z_1}\int_0^{1}(z_1-s)^{\nu+\zeta-1}(1-z)^{\nu-1}z^{\zeta-1}\psi(s)dzds\\
=&\frac{1}{\Gamma(\nu)\Gamma(\zeta)}\int_p^{z_1}(z_1-s)^{\nu+\zeta-1}\int_0^{1}(1-z)^{\nu-1}z^{\zeta-1}\psi(s)dzds,
\end{aligned}$$

where $B(\nu+\zeta) = \int_0^1(1-z)^{\nu-1}z^{\zeta-1}\psi(s)dz$. Therefore

$$\mathcal{I}_p^{\nu}\mathcal{I}_p^{\zeta}\psi(z) = \frac{B(\nu+\zeta)}{\Gamma(\nu)\Gamma(\zeta)}\int_p^{z}(z-s)^{\nu+\zeta-1}\psi(s)ds.$$

By using relation between beta and gamma functions

$$\begin{aligned}
\mathcal{I}_p^{\nu}\mathcal{I}_p^{\zeta}\psi(z) =&\frac{\Gamma(\nu)\Gamma(\zeta)}{\Gamma(\nu)\Gamma(\zeta)\Gamma(\nu+\zeta)}\int_p^{z_1}(z_1-s)^{\nu+\zeta-1}\psi(s)ds\\
=&\frac{1}{\Gamma(\nu+\zeta)}\int_p^{z_1}(z_1-s)^{\nu+\zeta-1}\psi(s)ds = \mathcal{I}_p^{\nu+\zeta}\psi(z).
\end{aligned}$$

□

Having completed the fundamental properties of Riemann-Liouville fractional integral, we are going to introduce the notation $\mathcal{D}_a^{\nu}$ which will denote the fractional derivative of a function of an arbitrary order of $\nu > 0$.

Definition 1.3.8. [10, 22] Let $\nu \in \mathbb{R}_+$ and $p = [\nu]$. The operator $\mathcal{D}_a^\nu$, is defined as

$$\mathcal{D}_a^\nu \psi(y) = \mathcal{D}^p \mathcal{I}_a^{p-\nu} \psi(y) = \left(\frac{d}{dy}\right)^p \left(\frac{1}{\Gamma(p-\nu)} \int_a^y (y-t)^{p-\nu-1} \psi(t) dt\right),$$

is called Riemann-Liouville fractional differential operator of order ν. In ordinary calculus Riemann-Liouville fractional derivative is the most essential extension.

Example 1.3.9. [10, 22] We will find the Riemann-Liouville fractional derivative of $\phi(y) = (y-c)^\zeta$, $\zeta \geq -1$ and $\nu \geq 0$. This can be done by using definition and evaluating the resulting integral. By definition of Riemann-Liouville fractional derivative

$$\mathcal{D}_c^\nu \phi(y) = \mathcal{D}^q \mathcal{I}_c^{q-\nu} \phi(y) = \mathcal{D}^q \mathcal{I}_c^{q-\nu} (y-c)^\zeta.$$

From Eqn (1.3.8) we have

$$\begin{aligned} \mathcal{D}_c^\nu \phi(y) =& \mathcal{D}^q \left[\frac{\Gamma(\zeta+1)}{\Gamma(\zeta+q-\nu+1)} (y-c)^{q-\nu+\zeta}\right] \\ =& \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+q-\nu+1)} \mathcal{D}^q (y-c)^{q-\nu+\zeta}. \end{aligned} \tag{1.3.11}$$

Case 1: If $\nu - \zeta \in \mathbb{N}$, the right hand side is the qth derivative of a classical polynomial of degree $q - (\nu - \zeta)$ and so the expression vanishes that is

$$\mathcal{D}^q (y-c)^{\nu-\zeta} = 0,$$

for all $\nu \geq 0$, $q \in 1, 2 \cdots [\nu]$.

Case 2: If $\nu - \zeta \neq \mathbb{N}$, here we generalize the integer-order derivative of a power function

$$\begin{aligned} \mathcal{D}(y-c)^r &= r(y-c)^{r-1} \\ \mathcal{D}^2(y-c)^r &= r(r-1)(y-c)^{r-2} \\ \mathcal{D}^3(y-c)^r &= r(r-1)(r-2)(y-c)^{r-3}. \end{aligned}$$

In general

$$\begin{aligned} \mathcal{D}^q(y-c)^r =& r(r-1)(r-2)\cdots(r-r-1)(y-c)^{r-q} \\ =& \frac{r(r-1)(r-2)\cdots(r-q-1)(r-q)!}{(r-q)!} (y-a)^{r-q} \\ =& \frac{(r)!}{(r-q)!} (y-c)^{r-q}. \end{aligned}$$

$$\mathcal{D}^q(y-c)^r = \frac{\Gamma(r+1)}{\Gamma(r-q+1)} (y-c)^{r-q}. \tag{1.3.12}$$

Equation (1.3.11) becomes

$$\mathcal{D}_c^\nu (y-a)^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+q-\nu+1)} \frac{\Gamma(\zeta+q-\nu+1)}{\Gamma(q+1-\nu+\zeta-q)} (y-c)^{q-\nu+\zeta-q}$$

$$\mathcal{D}_c^\nu (y-a)^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\nu+1)} (y-a)^{\zeta-\nu}. \tag{1.3.13}$$

Example 1.3.10. Now we will find the fractional derivative of $\sin z$.

First we expand $\sin z$ into its Maclaurin series:

$$\begin{aligned}\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots, \\ &= \sum_{q=0}^{\infty} \frac{(-1)^q z^{2q+1}}{(2q+1)!},\end{aligned}$$

where q is non-negative integers. Using Eqn (1.3.11) and property of gamma function we get

$$\begin{aligned}\mathcal{D}_0^\nu \sin z &= \sum_{q=0}^{\infty} \frac{(-1)^q \mathcal{D}_0^\nu z^{2q+1}}{(2q+1)!} \\ &= \sum_{q=0}^{\infty} \frac{(-1)^q \mathcal{D}^q \mathcal{I}_0^{q-\nu} z^{2q+1}}{(2q+1)!} \\ &= \sum_{q=0}^{\infty} \frac{(-1)^q \Gamma((2q+1)+1) \mathcal{D}^q z^{q-(\nu-2q-1)}}{(2q+1)! \Gamma(2q+2-\nu+q)}.\end{aligned}$$

Case 1: If $\nu - 2q - 1 \in \mathbb{N}$ then

$$\mathcal{D}_0^\nu \sin z = 0.$$

Case 2: If $\nu - 2q - 1 \notin \mathbb{N}$ then using Eqn (1.3.12)

$$\mathcal{D}_0^\nu \sin z = \sum_{q=0}^{\infty} \frac{(-1)^q z^{2q+1-\nu}}{\Gamma(2q+2-\nu)}.$$

As,

$$E_{2,2-\nu}(-z^2) = \sum_{q=0}^{\infty} \frac{(-z^2)^q}{\Gamma(2-\nu+2q)}.$$

Therefore

$$\begin{aligned}\mathcal{D}_0^\nu \sin z &= z^{1+\nu} \sum_{q=0}^{\infty} \frac{(-z^2)^q}{\Gamma(2+2q-\nu)}, \\ &= z^{1+\nu} E_{2,2-\nu}(-z^2).\end{aligned}$$

Next we come to show the relationship between Riemann-Liouville fractional integral with derivative and vice versa. Here we see that Riemann-Liouville fractional derivative is the left inverse of Riemann-Liouville fractional integral. Of course, we cannot claim that, it is the right inverse, because even it is not true in integer case.

Theorem 1.3.11. *[10, 22] Let $\nu > 0$. Then for every $h \in L_1[a, b]$*

$$\mathcal{D}_a^\nu \mathcal{I}_a^\nu h(y) = h(y)$$

holds almost everywhere.

Proof. By definition of Riemann-Liouville fractional derivative, semi group property of Riemann-Liouville fractional integral and by (1.3.4)

$$\begin{aligned}\mathcal{D}_a^\nu \mathcal{I}_a^\nu h(y) =& \mathcal{D}^m \mathcal{I}_a^{m-\nu} \mathcal{I}_a^\nu h(y) = \mathcal{D}^m \mathcal{I}_a^{m-\nu+\nu} h(y) \\ =& \mathcal{D}^m \mathcal{I}_a^\nu h(y) = h(y).\end{aligned}$$

□

Theorem 1.3.12. *[33] Assume that $\mu \geq 0$, $m = [\mu]$ and $h \in A^m[a, b]$. Then*

$$\mathcal{I}_a^\mu \mathcal{D}_a^\mu h(y) = h(y) - \sum_{j=1}^{k} \mathcal{D}_a^{\mu-j} h(y)|_{y=a} \frac{(y-a)^{\mu-j}}{\Gamma(\mu - j + 1)}.$$

Proof. By definition of Riemann-Liouville fractional integral

$$\mathcal{I}_a^\mu \mathcal{D}_a^\mu h(y) = \frac{1}{\Gamma(\mu)} \int_a^y (y-t)^{\mu-1} \mathcal{D}_a^\mu h(t) dt, \tag{1.3.14}$$

and

$$\frac{d}{dy} \left(\frac{1}{\Gamma(\mu+1)} \int_a^y (y-t)^\mu \mathcal{D}_a^\mu h(t) dt \right) = \frac{1}{\Gamma(\mu)} \int_a^y (y-t)^{\mu-1} \mathcal{D}_a^\mu h(t) dt. \tag{1.3.15}$$

Equation (1.3.15) is a consequence of Leibniz's Rule. Let us consider the left hand side of (1.3.15)

$$\begin{aligned}&\frac{d}{dy} \left(\frac{1}{\Gamma(\mu+1)} \int_a^y (y-t)^\mu \mathcal{D}_a^\mu h(t) dt \right) \\ =& \frac{1}{\Gamma(\mu+1)} (y-t)^\mu \mathcal{D}_a^\mu h(t)|_{t=y} \frac{d}{dy}(y) \\ &- \frac{1}{\Gamma(\mu+1)} (y-t)^\mu \mathcal{D}_a^\mu h(t)|_{t=a} \frac{d}{dy}(a) + \frac{1}{\Gamma(\mu+1)} \int_a^y \frac{\partial}{\partial y} (y-t)^\mu \mathcal{D}_a^\mu h(t) dt \\ =& \frac{\mu}{\Gamma(\mu+1)} \int_a^y (y-t)^{\mu-1} \mathcal{D}_a^\mu h(t) dt \\ =& \frac{1}{\Gamma(\mu)} \int_a^y (y-t)^{\mu-1} \mathcal{D}_a^\mu h(t) dt.\end{aligned}$$

On the other hand, by definition of Riemann-Liouville fractional derivative, repeatedly integrating and by Theorem (1.3.7) we have

$$\frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu}\mathcal{D}_a^{\mu}h(t)dt = \frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu}\mathcal{D}^{k}\mathcal{I}_a^{k-\mu}h(t)dt$$
$$= \frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu}\frac{d^k}{dy^k}\mathcal{I}_a^{k-\mu}h(t)dt.$$

Now we evaluate this by iterative method. Let $k=1$

$$\frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu}\frac{d}{dt}\mathcal{I}_a^{1-\mu}h(t)dt$$
$$= \frac{1}{\Gamma(\mu+1)}(y-t)^{\mu}\mathcal{I}_a^{1-\mu}h(t)|_a^y + \frac{\mu}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu-1}\mathcal{I}_a^{1-\mu}h(t)dt$$
$$= -\frac{1}{\Gamma(\mu+1)}(y-t)^{\mu}\mathcal{I}_a^{1-\mu}h(t)|_a + \frac{1}{\Gamma(\mu)}\int_a^y (y-t)^{\mu-1}\mathcal{I}_a^{1-\mu}h(t)dt.$$

For $k=2$

$$\frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu}\frac{d^2}{dt^2}\mathcal{I}_a^{2-\mu}h(t)dt$$
$$= \frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu}\frac{d}{dt}[\frac{d}{dt}\mathcal{I}_a^{2-\mu}h(t)dt]$$
$$= \left[\frac{1}{\Gamma(\mu+1)}(y-t)^{\mu}\frac{d}{dt}\mathcal{I}_a^{2-\mu}h(t)\right]_a^y + \frac{\mu}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu-1}\frac{d}{dt}\mathcal{I}_a^{2-\mu}h(t)dt.$$
$$= -\frac{1}{\Gamma(\mu+1)}(y-a)^{\mu}\frac{d}{dt}\mathcal{I}_a^{2-\mu}h(t)|_a + \frac{1}{\Gamma(\mu)}\int_a^y (y-t)^{\mu-1}\frac{d}{dt}\mathcal{I}_a^{1-\mu}h(t)dt$$
$$= -\frac{1}{\Gamma(\mu+1)}(y-a)^{\mu}\frac{d}{dt}\mathcal{I}_a^{2-\mu}h(t)|_a + \big[\frac{1}{\Gamma(\mu)}(y-t)^{\mu-1}\mathcal{I}_a^{2-\mu}h(t)\big]_a^y$$
$$+ \frac{\mu-1}{\Gamma(\mu)}\int_a^y (y-t)^{\mu-2}\mathcal{I}_a^{2-\mu}h(t)dt$$
$$= -\frac{1}{\Gamma(\mu+1)}(y-a)^{\mu}\frac{d}{dt}\mathcal{I}_a^{2-\mu}h(t)|_a - \frac{1}{\Gamma(\mu)}(y-a)^{\mu-1}\mathcal{I}_a^{2-\mu}h(t)|_a$$
$$+ \frac{1}{\Gamma(\mu-1)}\int_a^y (y-t)^{\mu-2}\mathcal{I}_a^{2-\mu}h(t)dt.$$
$$= \frac{1}{\Gamma(\mu-1)}\int_a^y (y-t)^{\mu-2}\mathcal{I}_a^{2-\mu}h(t)dt - \sum_{j=1}^{2}\frac{d^{k-j}}{dt^{k-j}}\mathcal{I}_a^{k-\mu}h(t)|a\frac{(y-a)^{\mu-j+1}}{\Gamma(\mu-j+2)}.$$

In general

$$\frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^{\mu}\frac{d^k}{dy^k}\mathcal{I}_a^{k-\mu}h(t)dt = \frac{1}{\Gamma(\mu-k+1)}\int_a^y (y-t)^{\mu-k}\mathcal{I}_a^{k-\mu}f(t)dt$$
$$- \sum_{j=1}^{n}\frac{d^{k-j}}{dt^{k-j}}\mathcal{I}_a^{k-\mu}h(t)|a\frac{(y-a)^{\mu-j+1}}{\Gamma(\mu-j+2)}$$
$$= \mathcal{I}_a^{\mu-k+1}[\mathcal{I}_a^{k-\mu}h(t)] - \sum_{j=1}^{n}\frac{d^{k-j}}{dt^{k-j}}\mathcal{I}_a^{k-\mu}h(t)|a\frac{(y-a)^{\mu-j+1}}{\Gamma(\mu-j+2)}.$$

$$\frac{1}{\Gamma(\mu+1)}\int_a^y (y-t)^\mu \frac{d^k}{dy^k}\mathcal{I}_a^{k-\mu}h(t)dt = \mathcal{I}_a^1 h(t) - \sum_{j=1}^{n}\frac{d^{k-j}}{dt^{k-j}}\mathcal{I}_a^{k-\mu}h(t)|a\frac{(y-a)^{\mu-j+1}}{\Gamma(\mu-j+2)}. \tag{1.3.16}$$

Combing (1.3.16) and (1.3.15) we obtain

$$\frac{1}{\Gamma(\mu)}\int_a^y (y-t)^{\mu-1}\mathcal{D}_a^\mu h(t)dt = \frac{d}{dy}\mathcal{I}_a^1 h(t) - \sum_{j=1}^{n}\frac{d^{k-j}}{dt^{k-j}}\mathcal{I}_a^{k-\mu}h(t)|a\frac{(y-a)^{\mu-j}}{\Gamma(\mu-j+1)}.$$

By Lemma (1.3.1)

$$\begin{aligned}\mathcal{I}_a^\mu\mathcal{D}_a^\mu h(y) =& h(t) - \sum_{j=1}^{n}\frac{d^{k-j}}{dt^{k-j}}\mathcal{I}_a^{k-\mu}h(t)|a\frac{(y-a)^{\mu-j}}{\Gamma(\mu-j+1)}.\\ =& h(t) - \sum_{j=1}^{n}\frac{d^{k-j}}{dt^{k-j}}\mathcal{I}_a^{k-\mu}h(t)|a\frac{(y-a)^{\mu-j}}{\Gamma(\mu-j+1)}.\end{aligned}$$

□

An important particular case of Theorem (1.3.12) is for $0<\mu<1$ then,

$$\mathcal{I}_a^\mu\mathcal{D}_a^\mu h(y) = h(y) - [\mathcal{D}_a^{\mu-1}h(t)]_a\frac{(y-a)^{\mu-1}}{\Gamma(\mu)}. \tag{1.3.17}$$

Above mentioned theorem is a particular case of more general property

$$\mathcal{I}_a^\mu\mathcal{D}_a^\zeta h(y) = \mathcal{I}_a^\mu\mathcal{D}_a^\zeta h(y) - \sum_{j=1}^{n}\mathcal{D}_a^{\zeta-j}h(y)|_{y=a}\frac{(y-a)^{\mu-j}}{\Gamma(\mu-j+1)}. \tag{1.3.18}$$

Having established the relation between Riemann-Liouville fractional integral with derivative, now we discuss the composition of Riemann-Liouville fractional derivative with itself.

Theorem 1.3.13. *[10, 22] Let $m-1\le\nu\le m$ and $n-1\le\zeta\le n$. Then*

$$\mathcal{D}_a^\nu\mathcal{D}_a^\zeta h(y) = \mathcal{D}_a^{\nu+\zeta}h(y) - \sum_{j=1}^{n}\mathcal{D}_a^{\zeta-j}h(y)|_{y=a}\frac{(y-a)^{-\nu-j}}{\Gamma(1-\nu-j)}. \tag{1.3.19}$$

Proof. By definition of Riemann-Liouville fractional derivative

$$\mathcal{D}_a^\nu\mathcal{D}_a^\zeta h(y) = \mathcal{D}^m\mathcal{I}_a^{m-\nu}\mathcal{D}_a^\zeta h(y) = \frac{d^m}{dy^m}[\mathcal{I}_a^{m-\nu}\mathcal{D}_a^\zeta h(y)].$$

By using (1.3.18) we obtain

$$\begin{aligned}\mathcal{D}_a^\nu\mathcal{D}_a^\zeta h(y) =& \frac{d^m}{dy^m}\left[\mathcal{I}_a^{m-\nu}\mathcal{D}_a^\zeta h(y) - \sum_{j=1}^{n}\mathcal{D}_a^{\zeta-j}h(y)|_{y=a}\frac{(y-a)^{m-\nu-j}}{\Gamma(1+m-\nu-j)}\right]\\ =& \mathcal{D}_a^{\nu+\zeta}h(y) - \sum_{j=1}^{n}\mathcal{D}_a^{\zeta-j}h(y)|_{y=a}\frac{(y-a)^{-\nu-j}}{\Gamma(1-\nu-j)}.\end{aligned}$$

□

Theorem (1.3.13) is also true for fractional derivative of different orders

$$\mathcal{D}_a^{\zeta}\mathcal{D}_a^{\nu}h(y) = \mathcal{D}_a^{\nu+\zeta}h(y) - \sum_{j=1}^{n} \mathcal{D}_a^{\nu-j}h(y)|_{y=a}\frac{(y-a)^{-\zeta-j}}{\Gamma(1-\zeta-j)}. \tag{1.3.20}$$

From relation given in Eqn (1.3.19) and (1.3.20) we conclude that in the general case, the Riemann-Liouville fractional derivative operators do not commute but only with the one exception that sum in right hand side of both equation vanish. In the following example we show that the composition property do not hold for Riemann-Liouville derivative.

Example 1.3.14. For $\nu = \frac{1}{2}$, $\zeta = \frac{1}{3}$ and $h(y) = y^{\frac{3}{2}}$, by using Eqn (1.3.11)

$$\mathcal{D}_0^{\frac{1}{2}}(y)^{\frac{3}{2}} = \frac{\Gamma(\frac{5}{2})}{\Gamma(m+2)}D^m y^{m+1} = \frac{\Gamma(\frac{5}{2})\Gamma(m+2)}{\Gamma(m+2)\Gamma(m+1-m)}y^{m+1-m} = \frac{3\sqrt{\pi}}{4}y,$$

$$\mathcal{D}_0^{\frac{1}{3}}(y)^{\frac{3}{2}} = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{13}{6}+m)}D^m y^{m+\frac{7}{6}} = \frac{\Gamma(\frac{5}{2})\Gamma(m+\frac{13}{6})}{\Gamma(m+\frac{13}{6})\Gamma(m+\frac{7}{6}-m)}y^{m+\frac{7}{6}-m} = \frac{3\sqrt{\pi}}{4\Gamma(\frac{7}{6})}y^{\frac{7}{6}} = \frac{9\sqrt{\pi}}{2\Gamma(\frac{1}{6})}y^{\frac{7}{6}}.$$

$$\mathcal{D}_0^{\frac{1}{3}}\mathcal{D}_0^{\frac{1}{2}}y^{\frac{3}{2}} = \frac{3\pi}{4\Gamma(\frac{2}{3})}y^{\frac{2}{3}}, \mathcal{D}_0^{\frac{1}{2}}\mathcal{D}_0^{\frac{1}{3}}y^{\frac{3}{2}} = \frac{9\pi}{8\Gamma(\frac{2}{3})}y^{\frac{2}{3}}.$$

$$\mathcal{D}_0^{\frac{1}{2}+\frac{1}{3}}y^{\frac{3}{2}} = \mathcal{D}_0^{\frac{5}{6}}y^{\frac{3}{2}} = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{3})}(y)^{\frac{3}{2}-\frac{5}{6}} = \frac{3\sqrt{\pi}}{4\Gamma(\frac{5}{3})}y^{\frac{2}{3}}.$$

Clearly we can seen that $\mathcal{D}_0^{\frac{1}{3}}\mathcal{D}_0^{\frac{1}{2}}h(y) \neq \mathcal{D}_0^{\frac{1}{3}+\frac{1}{2}}h(y)$ and also $\mathcal{D}_0^{\frac{1}{3}}\mathcal{D}_0^{\frac{1}{2}} \neq \mathcal{D}_0^{\frac{1}{2}}\mathcal{D}_0^{\frac{1}{3}}$.

1.3.3 Caputo derivative

Riemann-Liouville fractional derivative has a certain disadvantages when trying to model the real world phenomena with fractional differential equations. Function need not to be continuous when dealing with the Riemann-Liouville derivative. The derivative of the constant term with Riemann-Liouville fractional derivative is not equal to zero. We will now discuss the modified concept of fractional derivative, which is known as Caputo derivative. It has more advantage then the first one. One of them is that the derivative of a constant is zero.

Definition 1.3.15. [10, 22] Let $\nu > 0$ and $m = [\nu]$. Then we define the operator ${}_C\mathcal{D}_a^{\nu}$ as

$${}_C\mathcal{D}_a^{\nu}h(y) = \mathcal{I}_a^{m-\nu}\mathcal{D}^m h(y), \tag{1.3.21}$$

where $\mathcal{D}^m h \in L_1[a,b]$, is called Caputo derivative.

Lemma 1.3.16. *Let $\nu \geq 0$ and $m = [\nu]$. Assume that h is such that both ${}_C\mathcal{D}_a^{\nu}$, $\mathcal{D}_a^{\nu}$ exist. Then*

$${}_C\mathcal{D}_a^{\nu}h(y) = \mathcal{D}_a^{\nu}h(y) \tag{1.3.22}$$

holds if and only if h has k fold zero at a that is

$$\mathcal{D}^k h(y) = 0, \ \ for \ k = 0, 1, 2 \cdots m - 1.$$

Example 1.3.17. [10] We can find the Caputo derivative of $h(y) = y^\zeta$ as under

$$_C\mathcal{D}_a^\nu y^\zeta = \frac{\Gamma(\zeta + 1)}{\Gamma(\nu + \zeta + 1)} y^{\nu+\zeta}. \tag{1.3.23}$$

By definition of the Caputo fractional derivative

$$_C\mathcal{D}_0^\nu h(y) = \mathcal{I}_0^{m-\nu}\mathcal{D}^m h(y) = \frac{1}{\Gamma(m - \nu)} \int_0^y (y - t)^{m-\nu-1}\mathcal{D}^m t^\zeta dt. \tag{1.3.24}$$

Case 1: If $\zeta < m$, then $\mathcal{D}^m y^\zeta = 0$.
Case 2: If $\zeta \in \mathbb{N}$ and $m \leq \zeta$. Then we generalize the integer-order derivative of a power function

$$\begin{aligned}
\mathcal{D}y^q &= qy^{q-1} \\
\mathcal{D}^2 y^q &= q(q - 1)y^{q-2} \\
\mathcal{D}^3 y^q &= q(q - 1)(q - 2)y^{q-3}.
\end{aligned}$$

In general

$$\begin{aligned}
\mathcal{D}^p y^q =& q(q - 1)(q - 2) \cdots (q - p - 1)y^{q-p} \\
=& \frac{q(q - 1)(q - 2) \cdots (q - p - 1)(q - p)!}{(q - p)!} y^{q-p} \\
=& \frac{(q)!}{(q - p)!} y^{q-p} \\
\mathcal{D}^p y^q =& \frac{\Gamma(q + 1)}{\Gamma(q - p + 1)} y^{q-p}.
\end{aligned}$$

Equation (1.3.24) reduce to

$$\begin{aligned}
_C\mathcal{D}_0^\nu h(y) =& \frac{1}{\Gamma(m - \nu)} \int_0^y (y - t)^{m-\nu-1}\mathcal{D}^m t^\zeta dt. \\
=& \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta - m + 1)\Gamma(m - \nu)} \int_0^y (y - t)^{m-\nu-1} t^{\zeta-m} dt. \\
=& \frac{\Gamma(\zeta + 1)}{\Gamma(\zeta - m + 1)\Gamma(m - \nu)} \int_0^y \left(1 - \frac{t}{y}\right)^{m-\nu-1} y^{m-\nu-1} t^{\zeta-m} dt.
\end{aligned}$$

Let $z = \frac{t}{y}$

$$_C\mathcal{D}_0^\nu h(y) = \frac{\Gamma(\zeta + 1)y^{m-\nu}}{\Gamma(\zeta - m + 1)\Gamma(m - \nu)} \int_0^1 (1 - z)^{m-\nu-1} z^{\zeta-m} dz.$$

Since $\int_0^1 (1-z)^{m-\nu-1} z^{\zeta-m} dz = B(m-\nu, \zeta-m+1) = \frac{\Gamma(m-\nu)\Gamma(\zeta-m+1)}{\Gamma(m-\nu+\zeta-m+1)}$.
Thus

$$ {}_C\mathcal{D}_0^{\nu} y^{\zeta} = \frac{\Gamma(\zeta+1) y^{m-\nu}}{\Gamma(\zeta-\nu+1)}. $$

Example 1.3.18. To find the Caputo derivative of $\sin y$, we proceed as follows.

First we expand $\sin y$ into its Maclaurin series:

$$ \begin{aligned} \sin y =& y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} \cdots, \\ =& \sum_{q=0}^{\infty} \frac{(-1)^q y^{2q+1}}{(2q+1)!}, \end{aligned} $$

where q is non-negative integers. Using Eqn (1.3.12) and property of gamma function we get

$$ \begin{aligned} {}_C\mathcal{D}_0^{\nu} \sin y =& \mathcal{I}_0^{m-\nu} \sum_{q=0}^{\infty} \frac{(-1)^q \mathcal{D}^m y^{2q+1}}{(2q+1)!} \\ =& \sum_{q=0}^{\infty} \frac{(-1)^q \Gamma((2q+1)+1) \mathcal{I}_0^{m-\nu} y^{2q+1-m}}{(2q+1)! \Gamma(2q+2-m)} \\ =& \sum_{q=0}^{\infty} \frac{(-1)^q \mathcal{I}_0^{m-\nu} y^{2q+1-m}}{\Gamma(2q+2-m)} \\ =& \sum_{q=0}^{\infty} \frac{(-1)^q y^{2q+1-\nu}}{\Gamma(2q+2-\nu)}. \end{aligned} $$

As,

$$ E_{2,2-\nu}(-y^2) = \sum_{q=0}^{\infty} \frac{(-y^2)^q}{\Gamma(2-\nu+2q)}. $$

Thus

$$ \begin{aligned} {}_C\mathcal{D}_0^{\nu} \sin y &= y^{1+\nu} \sum_{q=0}^{\infty} \frac{(-y^2)^q}{\Gamma(2+2q-\nu)}, \\ &= y^{1+\nu} E_{2,2-\nu}(-y^2). \end{aligned} $$

In the following we discussed the composition of Caputo derivative with Riemann-Liouville fractional integral and vice versa.

Theorem 1.3.19. *[10, 22] If h is continuous and $\nu \geq 0$, then*

$$ {}_C\mathcal{D}_a^{\nu} \mathcal{I}_a^{\nu} h(z) = h(z). \tag{1.3.25} $$

Proof. Let $\phi = \mathcal{I}_a^{\nu} h(z)$, where $\mathcal{D}^k \phi(a) = 0$, then using Lemma (1.3.16) and Eqn (1.3.4) we have

$$ {}_C\mathcal{D}_a^{\nu}\mathcal{I}_a^{\nu} h(z) = {}_C\mathcal{D}_a^{\nu} h(z) = \mathcal{D}_a^{\nu} h(z) = \mathcal{D}^m \mathcal{I}_a^{m-\nu} h(z) = h(z). $$

□

Theorem 1.3.20. *[22] Assume that* $\nu \geq 0$, $m = [\nu]$ *and* $h \in A^m(a,b)$, *then*

$$ \mathcal{I}_a^{\nu}{}_C\mathcal{D}_a^{\nu} h(z) = h(z) - \sum_{k=0}^{m-1} \mathcal{D}^k \frac{h(a)}{k!}(z-a)^k. $$

Proof. By using fundamental theorem of calculus and iterative method we obtain

$$ \begin{aligned} \mathcal{I}_a \mathcal{D} h(z) = & \int_a^z \frac{d}{dz} h(z) dz = h(z) - h(a) \\ \mathcal{I}_a^2 \mathcal{D}^2 h(z) = & \mathcal{I}_a(\mathcal{I}_a \mathcal{D}(\mathcal{D} h(z))) \\ = & \mathcal{I}_a \left(\int_a^z \frac{d}{dz} \mathcal{D} h(z) dz \right) \\ = & \mathcal{I}_a \mathcal{D} h(z) - \mathcal{I}_a \mathcal{D} h(a) \\ = & \int_a^z \frac{d}{dz} h(z) dz - \mathcal{D} h(a) \int_a^z dz \\ = & h(z) - h(a) - \mathcal{D} h(a)(z-a) \\ \mathcal{I}_a^3 \mathcal{D}^3 h(z) = & \mathcal{I}_a(\mathcal{I}_a(\mathcal{I}_a \mathcal{D}(\mathcal{D}(\mathcal{D} h(z))))) \\ = & \mathcal{I}_a \left(\int_a^z \frac{d}{dz} \mathcal{D}^2 h(z) dz \right) \\ = & \mathcal{I}_a^2 \mathcal{D}^2 h(z) - \mathcal{I}_a^2 \mathcal{D}^2 h(a) \\ = & \mathcal{I}_a \mathcal{D} h(z) - \mathcal{I}_a \mathcal{D} h(a) - \mathcal{D}^2 h(a) \frac{(z-a)^2}{2} \\ = & h(z) - h(a) - \mathcal{D} h(a)(z-a) - \mathcal{D}^2 h(a) \frac{(z-a)^2}{2}. \end{aligned} $$

In general

$$ \mathcal{I}_a^m \mathcal{D}^m h(z) = h(z) - \sum_{k=0}^{m-1} \mathcal{D}^k h(a) \frac{(z-a)^k}{k!}. \tag{1.3.26} $$

By definition of Caputo derivative

$$ {}_C\mathcal{D}^n h(z) = \mathcal{I}_a^{m-n} \mathcal{D}^m h(z). \tag{1.3.27} $$

Apply $\mathcal{I}_a^n$ on both sides of Eqn (1.3.27), and using semi group property of Riemann-Liouville fractional integral we have

$$ \mathcal{I}_a^n {}_C\mathcal{D}^n h(z) = \mathcal{I}_a^n \mathcal{I}_a^{m-n} \mathcal{D}^m h(z) = \mathcal{I}_a^m \mathcal{D}^m h(z). \tag{1.3.28} $$

Use Eqn (1.3.28) into (1.3.26) we obtain

$$\mathcal{I}_a^n {}_{\mathcal{C}}\mathcal{D}^n h(z) = h(z) - \sum_{k=0}^{m-1} \mathcal{D}^k h(a) \frac{(z-a)^k}{k!}.$$

Replacing n with real α we have

$$\mathcal{I}_a^\nu {}_{\mathcal{C}}\mathcal{D}^\nu h(z) = h(z) - \sum_{k=0}^{m-1} \mathcal{D}^k h(a) \frac{(z-a)^k}{k!}.$$

□

In particular, if $0 < \nu \leq 1$ and $h(z) \in C[a,b]$ then $\mathcal{I}_a^\nu {}_{\mathcal{C}}\mathcal{D}^\nu h(z) = h(z) - h(a)$

1.3.4 Hadamard fractional integral and derivative

In 1892 Hadamard introduced the new fractional integral which involves logarithmic function. In this section we give the definition and some properties of Hadamard fractional integral and derivatives. The integration by part formula for integer order Hadamard calculus, is given by.

$$\int_a^y u_{\mathcal{H}}\mathcal{D}v \frac{ds}{s} = uv|_a^y - \int_a^y v_{\mathcal{H}}\mathcal{D}u \frac{ds}{s}, \ \textit{where} \ {}_{\mathcal{H}}\mathcal{D} = y\frac{d}{dy}. \tag{1.3.29}$$

Lemma 1.3.21. *Let h be Riemann integrable on $[k,l]$, then for $k \leq y \leq l$ we have,*

$$\mathcal{I}_k^n h(y) = \frac{1}{\Gamma(n)} \int_k^y \left(\log \frac{y}{s}\right)^{n-1} h(s) \frac{ds}{s}.$$

Proof. We start with the simple integral

$$\mathcal{I}_k h(y) = \int_k^y h(s) \frac{ds}{s}.$$

The second iterate of the integral is,

$$\mathcal{I}_k^2 h(y) = \mathcal{I}_k(\mathcal{I}_k h(y)) = \int_k^y \int_k^{y_1} h(s) \frac{ds}{s} \frac{dy_1}{y_1}.$$

We can simplify this integral by Theorem (1.2.3). Since $h(s)$ is not a function of y_1, it can be moved outside the inner integral, so

$$\begin{aligned}\mathcal{I}_k(\mathcal{I}_k h(y)) &= \int_k^y h(s) \int_s^y \frac{dy_1}{y_1} \frac{ds}{s} \\ &= \int_k^y h(s) \log y_1|_s^y \frac{ds}{s} \\ &= \int_k^y h(s) \log\left(\frac{y}{s}\right) \frac{ds}{s}.\end{aligned}$$

The third order integral is

$$\begin{aligned}\mathcal{I}_k^3 h(y) =&\mathcal{I}_k(\mathcal{I}_k(\mathcal{I}_k h(y)))\\ =&\frac{1}{\Gamma(\nu)}\int_k^y\int_k^{y_1}\int_k^{y_2} h(s)\frac{ds}{s}\frac{dy_1}{y_1}\frac{dy_2}{y_2}.\end{aligned}$$

Using the same procedure as above we can simplify this integral and up-to mth order as

$$\begin{aligned}\mathcal{I}_k(\mathcal{I}_k(\mathcal{I}_k h(y))) =&\frac{1}{\Gamma(\nu)}\int_k^y\int_k^{y_1} h(s)\int_s^{y_1}\frac{dy_2}{y_2}\frac{ds}{s}\frac{dy_1}{y_1}\\ =&\int_k^y\int_k^{y_1} h(s)\log y_2|_s^{y_1}\frac{ds}{s}\frac{dy_1}{y_1}\\ =&\int_k^y\int_k^{y_1} h(s)\log\left(\frac{y_1}{s}\right)\frac{ds}{s}\frac{dy_1}{y_1}\\ =&\int_k^y h(s)\int_s^y\log\left(\frac{y_1}{s}\right)\frac{dy_1}{y_1}\frac{ds}{s}\\ =&\frac{1}{2}\int_k^y h(s)\left(\log\frac{y_1}{s}\right)^2|_s^y\frac{ds}{s}\\ =&\frac{1}{2}\int_k^y h(s)\left(\log\frac{y}{s}\right)^2\frac{ds}{s}.\end{aligned}$$

Consequently, mth order integral is of the form

$$\mathcal{I}_k^m h(y) =\frac{1}{(m-1)!}\int_k^y\left(\log\frac{y}{s}\right)^{m-1}h(s)\frac{ds}{s} = \frac{1}{\Gamma(m)}\int_k^y\left(\log\frac{y}{s}\right)^{m-1}h(s)\frac{ds}{s}.$$

□

Here we defined Hadamard fractional integral and some of its properties.

Definition 1.3.22. [22] Let $0 \le c \le d < \infty$ be a finite or infinite interval of a half-axis $\mathbb{R}_+$. Then the left-sided integrals of fractional order $\mu > 0$ is defined as

$$_{\mathcal{H}}\mathcal{I}_c^\mu h(y) = \frac{1}{\Gamma(\mu)}\int_c^y\left(\log(\frac{y}{s})\right)^{\mu-1}h(s)\frac{ds}{s};\ \mu > 0, \qquad (1.3.30)$$

the $_{\mathcal{H}}\mathcal{I}_c^\mu$ is called the Hadamard fractional integral.

Example 1.3.23. [22] Let $h(y) = \left(\log(\frac{y}{k})\right)^\eta$.

Then by definition of Hadamard fractional integral

$$\begin{aligned}
{}_{\mathcal{H}}\mathcal{I}_k^{\mu} h(y) =& \frac{1}{\Gamma(\mu)} \int_k^y \left(\log \frac{y}{s}\right)^{\mu-1} h(s) \frac{ds}{s} \\
=& \frac{1}{\Gamma(\mu)} \int_k^y \left(\log \frac{y}{s}\right)^{\mu-1} \left(\log \frac{s}{k}\right)^{\eta} \frac{ds}{s} \\
=& \frac{1}{\Gamma(\mu)} \int_k^y (\log y - \log s)^{\mu-1} \left(\log \frac{s}{k}\right)^{\eta} \frac{ds}{s} \\
=& \frac{1}{\Gamma(\mu)} \int_k^y (\log y - \log k - (\log s - \log k))^{\mu-1} \left(\log \frac{s}{k}\right)^{\eta} \frac{ds}{s} \\
=& \frac{1}{\Gamma(\mu)} \int_k^y \left(\log \frac{y}{k} - \log \frac{s}{k}\right)^{\mu-1} \left(\log \frac{s}{k}\right)^{\eta} \frac{ds}{s} \\
=& \frac{1}{\Gamma(\mu)} \int_k^y \left(\log \frac{y}{k}\right)^{\mu-1} \left(1 - \frac{\log \frac{s}{k}}{\log \frac{y}{k}}\right)^{\mu-1} \left(\log \frac{s}{k}\right)^{\eta} \frac{ds}{s}.
\end{aligned}$$

We evaluate the integral by substituting $z = \frac{\log \frac{s}{k}}{\log \frac{y}{k}}$

$$\begin{aligned}
{}_{\mathcal{H}}\mathcal{I}_k^{\mu} \left(\log \frac{y}{k}\right)^{\eta} =& \frac{1}{\Gamma(\mu)} \int_0^1 \left(\log \frac{y}{k}\right)^{\mu-1} (1-z)^{\mu-1} z^{\eta} \left(\log \frac{y}{k}\right)^{\eta} \left(\log \frac{y}{k}\right) dz \\
=& \frac{\left(\log \frac{y}{k}\right)^{\mu+\eta}}{\Gamma(\mu)} \int_0^1 (1-z)^{\mu-1} z^{\eta} dy.
\end{aligned}$$

Since $B(\mu, \eta+1) = \int_0^1 (1-y)^{\mu-1} y^{\eta} dy$. Using the relationship between gamma and beta functions we obtain

$$ {}_{\mathcal{H}}\mathcal{I}_k^{\mu} \left(\log \frac{y}{k}\right)^{\eta} = \frac{\left(\log \frac{y}{k}\right)^{\mu+\eta}}{\Gamma(\mu)} \frac{\Gamma(\mu)\Gamma(\eta+1)}{\Gamma(\mu+\eta+1)} = \frac{\Gamma(\eta+1)}{\Gamma(\mu+\eta+1)} \left(\log \frac{y}{k}\right)^{\mu+\eta}. \tag{1.3.31}$$

The Hadamard fractional integral satisfies the semi-group property

Theorem 1.3.24. *[22] Let $\nu, \zeta > 0$, $1 \leq p \leq \infty$, $0 < k < l < \infty$ and $c \leq 0$ then for $h \in X_c^p(k, l)$*

$$ {}_{\mathcal{H}}\mathcal{I}_k^{\nu} {}_{\mathcal{H}}\mathcal{I}_k^{\zeta} f(y) = {}_{\mathcal{H}}\mathcal{I}_k^{\nu+\zeta} f(y). $$

Proof. By definition of Hadamard fractional integral

$$\begin{aligned}
{}_{\mathcal{H}}\mathcal{I}_k^{\nu} {}_{\mathcal{H}}\mathcal{I}_k^{\zeta} h(y) =& \frac{1}{\Gamma(\nu)} \int_k^y \left(\log \frac{y}{t}\right)^{\nu-1} \frac{dt}{t} \frac{1}{\Gamma(\zeta)} \int_k^t \left(\log \frac{t}{s}\right)^{\zeta-1} h(s) \frac{ds}{s} \\
=& \frac{1}{\Gamma(\nu)\Gamma(\zeta)} \int_k^y \int_k^t \left(\log \frac{y}{t}\right)^{\nu-1} \left(\log \frac{t}{s}\right)^{\zeta-1} h(s) \frac{ds}{s} \frac{dt}{t}.
\end{aligned}$$

Interchange the order of integration

$$ {}_{\mathcal{H}}\mathcal{I}_k^{\nu} {}_{\mathcal{H}}\mathcal{I}_k^{\zeta} h(y) = \frac{1}{\Gamma(\nu)\Gamma(\zeta)} \int_k^y h(s) \frac{ds}{s} \int_s^y \left(\log \frac{y}{t}\right)^{\nu-1} \left(\log \frac{t}{s}\right)^{\zeta-1} \frac{dt}{t}. \tag{1.3.32}$$

The inner integral can be evaluated as

$$\begin{aligned}\int_s^y \left(\log\frac{y}{t}\right)^{\nu-1}\left(\log\frac{t}{s}\right)^{\zeta-1}\frac{dt}{t} &= \int_s^y (\log y-\log t)^{\nu-1}\left(\log\frac{t}{s}\right)^{\zeta}\frac{dt}{t}\\ &= \int_s^y (\log y-\log s-(\log t-\log s)^{\nu-1}\left(\log\frac{t}{s}\right)^{\zeta-1}\frac{dt}{t}\\ &= \int_s^y \left(\log\frac{y}{s}\right)^{\nu-1}\left(1-\frac{\log\frac{t}{s}}{\log\frac{y}{s}}\right)^{\nu-1}\left(\log\frac{t}{s}\right)^{\zeta-1}\frac{dt}{t}.\end{aligned}$$

Let $y=\frac{\log\frac{t}{s}}{\log\frac{y}{s}}$, then

$$\begin{aligned}\int_s^y \left(\log\frac{y}{t}\right)^{\nu-1}\left(\log\frac{t}{s}\right)^{\zeta-1}\frac{dt}{t} &= \int_0^1 \left(\log\frac{y}{s}\right)^{\nu+\zeta-1}(1-y)^{\nu-1}(y)^{\zeta-1}dy\\ &= \left(\log\frac{y}{s}\right)^{\nu+\zeta-1}\int_0^1 (1-y)^{\nu-1}(y)^{\zeta-1}dy.\end{aligned}$$

By using definition of beta function and the relationship between gamma and beta functions we obtain

$$\int_s^y \left(\log\frac{y}{t}\right)^{\nu-1}\left(\log\frac{t}{s}\right)^{\zeta-1}\frac{dt}{t} = \frac{\Gamma(\nu)\Gamma(\zeta)}{\Gamma(\nu+\zeta)}\left(\log\frac{y}{s}\right)^{\nu+\zeta-1}.$$

Equation (1.3.32) will be

$$\begin{aligned}{}_{\mathcal{H}}\mathcal{I}_k^{\nu}{}_{\mathcal{H}}\mathcal{I}_k^{\zeta}h(y) &= \frac{1}{\Gamma(\nu)\Gamma(\zeta)}\frac{\Gamma(\zeta)\Gamma(\nu)}{\Gamma(\nu+\zeta)}\int_k^y \left(\log\frac{y}{s}\right)^{\nu+\zeta-1}h(s)\frac{ds}{s}\\ &= \frac{1}{\Gamma(\nu+\zeta)}\int_k^y \left(\log\frac{y}{s}\right)^{\nu+\zeta-1}h(s)\frac{ds}{s}\\ &= {}_{\mathcal{H}}\mathcal{I}_k^{\nu+\zeta}h(y).\end{aligned}$$

□

In order to introduce fractional differential operator in Hadamard sense, we proceed as follows.

Remark 1.3.25.

$$H(z) = \int_a^z \frac{h(s)}{s}dt.$$

Let H be differentiable, then using Lemma (1.3.1) we have

$$\begin{aligned}\frac{d}{dz}H(z) &= \frac{d}{dz}\int_a^z \frac{h(s)}{s}dt.\\ &= \frac{h(z)}{z}\\ z\frac{d}{dz}H(z) &= h(z)\end{aligned}$$

$$ {}_{\mathcal{H}}\mathcal{D}\mathcal{I}_a h(z) = h(z). \tag{1.3.33} $$

Where ${}_{\mathcal{H}}\mathcal{D} := z\frac{d}{dz}$. Then repeated application of Eqn (1.3.33) gives,

$$ \begin{aligned} {}_{\mathcal{H}}\mathcal{D}^2\mathcal{I}_a^2 h(z) =& {}_{\mathcal{H}}\mathcal{D}({}_{\mathcal{H}}\mathcal{D}\mathcal{I}_a(\mathcal{I}_a h(z))) \\ =& {}_{\mathcal{H}}\mathcal{D}\mathcal{I}_a h(z) = h(z) \\ {}_{\mathcal{H}}\mathcal{D}^3\mathcal{I}_a^3 h(z) =& {}_{\mathcal{H}}\mathcal{D}^2({}_{\mathcal{H}}\mathcal{D}\mathcal{I}_a(\mathcal{I}_a^2 h(z))) \\ =& -\mathcal{H}\mathcal{D}^2\mathcal{I}_a^2 h(z) \\ =& {}_{\mathcal{H}}\mathcal{D}({}_{\mathcal{H}}\mathcal{D}\mathcal{I}_a(\mathcal{I}_a h(z))) \\ =& {}_{\mathcal{H}}\mathcal{D}\mathcal{I}_a h(z) = h(z). \end{aligned} $$

In general

$$ {}_{\mathcal{H}}\mathcal{D}^n\mathcal{I}_a^n h(z) = h(z). \tag{1.3.34} $$

Definition 1.3.26. [22] The left sided fractional derivative of order $\nu > 0$ on $[a, b]$ is defined as

$$ \begin{aligned} \mathbf{D}_a^\nu h(y) =& {}_{\mathcal{H}}\mathcal{D}^q\mathcal{I}_a^{q-\nu} h(y) \\ =& \left(y\frac{d}{dy}\right)^q \frac{1}{\Gamma(q-\nu)} \int_a^y \left(\log\frac{y}{t}\right)^{q-\nu-1} f(t)\frac{dt}{t}, \end{aligned} $$

where $q - 1 < \nu < q$.

Example 1.3.27. [33, 22] If $\nu, \zeta > 0$ and $0 < k < l < \infty$, then

$$ \mathbf{D}_k^\nu \left(\log\frac{y}{k}\right)^\zeta = \frac{\Gamma(\zeta)}{\Gamma(\zeta+\nu)} \left(\log\frac{y}{k}\right)^{\zeta-\nu}. \tag{1.3.35} $$

Proof. By definition of Hadamard fractional derivative

$$ \begin{aligned} \mathbf{D}_k^\nu h(y) =& {}_{\mathcal{H}}\mathcal{D}^m {}_{\mathcal{H}}\mathcal{I}_{a_+}^{m-\nu} h(y) = \frac{{}_{\mathcal{H}}\mathcal{D}^m}{\Gamma(m-\nu)} \int_k^y \left(\log\frac{y}{t}\right)^{m-\nu-1} h(t)\frac{dt}{t} \\ \mathbf{D}_k^\nu \left(\log\frac{y}{k}\right)^\zeta =& {}_{\mathcal{H}}\mathcal{D}^m {}_{\mathcal{H}}\mathcal{I}_{a_+}^{m-\nu} \left(\log\frac{y}{k}\right)^\zeta = \frac{{}_{\mathcal{H}}\mathcal{D}^m}{\Gamma(m-\nu)} \int_k^y \left(\log\frac{y}{t}\right)^{m-\nu-1} \left(\log\frac{t}{k}\right)^\zeta \frac{dt}{t}. \end{aligned} $$

From (1.3.31) we have

$$ \mathbf{D}_k^\nu \left(\log\frac{y}{k}\right)^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(m-\nu+\zeta+1)} {}_{\mathcal{H}}\mathcal{D}^m \left(\log\frac{y}{t}\right)^{m-\nu+\zeta}. \tag{1.3.36} $$

Case:1 If $\nu - \zeta \in \mathbb{N}$, then ${}_{\mathcal{H}}\mathcal{D}^m \left(\log\frac{y}{k}\right)^{m-(\nu-\zeta)} = 0$ for all $\nu \geq 0$, $m \in 1, 2, \cdots, [\nu]$.

Case:2 If $\nu - \zeta \notin \mathbb{N}$, we find

$$
\begin{aligned}
{}_{\mathcal{H}}\mathcal{D}\left(\log\frac{y}{k}\right)^{q} =& y\frac{d}{dy}\left(\log\frac{y}{k}\right)^{q} = yq\left(\log\frac{y}{k}\right)^{q-1}\frac{k}{ay} = q\left(\log\frac{y}{k}\right)^{q-1} \\
{}_{\mathcal{H}}\mathcal{D}^{2}\left(\log\frac{y}{k}\right)^{q} =& y\frac{d}{dy}\left(y\frac{d}{dy}\left(\log\frac{y}{k}\right)^{q}\right) = y\frac{d}{dy}\left(yq\left(\log\frac{y}{k}\right)^{q-1}\frac{k}{ay}\right) = qy\frac{d}{dy}\left(\log\frac{y}{k}\right)^{q-1} \\
=& yq(q-1)\left(\log\frac{y}{k}\right)^{q-2}\frac{k}{ky} = q(q-1)\left(\log\frac{y}{k}\right)^{q-2}.
\end{aligned}
$$

In general,

$$
\begin{aligned}
{}_{\mathcal{H}}\mathcal{D}^{p}\left(\log\frac{y}{k}\right)^{q} =& q(q-1)\cdots(q-p-1)\left(\log\frac{y}{k}\right)^{q-p} \\
=& \frac{q(q-1)\cdots(q-p-1)(q-p)!}{(q-p)!}\left(\log\frac{y}{k}\right)^{q-p} \\
=& \frac{q!}{(q-p)!}\left(\log\frac{y}{k}\right)^{q-p} \\
=& \frac{\Gamma(q+1)}{\Gamma(q-p+1)}\left(\log\frac{y}{k}\right)^{q-p}.
\end{aligned}
$$

Eqn (1.3.36) reduce to

$$
\begin{aligned}
\mathbf{D}_{k}^{\nu}\left(\log\frac{y}{k}\right)^{\zeta} =& \frac{\Gamma(\zeta+1)\Gamma(m-\nu+\zeta+1)}{\Gamma(m-\nu+\zeta+1)\Gamma(m-\nu+\zeta-m+1)}\left(\log\frac{y}{k}\right)^{m-\nu+\zeta-m} \\
=& \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\nu+1)}\left(\log\frac{y}{k}\right)^{\zeta-\nu}.
\end{aligned} \tag{1.3.37}
$$

□

In the following we show that $\mathbf{D}_{k}^{\nu}$ is left inverse of ${}_{\mathcal{H}}\mathcal{I}_{k}^{\nu}$ and ${}_{\mathcal{H}}\mathcal{I}_{k}^{\nu}$ is not the left inverse of $\mathbf{D}_{k}^{\nu}$

Lemma 1.3.28. *[22] Let $\nu > 0$, if $0 < k < l < \infty$ and $1 \leq p < \infty$ then, for $h \in L^{p}(k,l)$*

$$\boldsymbol{D}_{k}^{\nu}{}_{\mathcal{H}}\mathcal{I}_{k}^{\nu}h(y) = h(y).$$

Proof. Now, by using definition of Hadamard fractional derivative, semi-group property of Hadamard fractional integral and Eqn (1.3.33), we get

$$
\begin{aligned}
\mathbf{D}_{k}^{\nu}{}_{\mathcal{H}}\mathcal{I}_{k}^{\nu}h(y) =& {}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}_{k}^{m-\nu}{}_{\mathcal{H}}\mathcal{I}_{k}^{\nu}h(y) \\
=& {}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}_{k}^{m}{}_{\mathcal{H}}\mathcal{I}_{k}^{\nu-\nu}h(y) \\
=& {}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}_{k}^{m}h(y) = h(y).
\end{aligned}
$$

□

Theorem 1.3.29. *[22] Let $\nu, \zeta > 0$ with $\zeta > \nu$, $0 < k < l < \infty$, then for $h \in L^p[k,l]$*

$$\boldsymbol{D}^{\nu}_{k\mathcal{H}}\mathcal{I}^{\zeta}_{k}h(y) = {}_{\mathcal{H}}\mathcal{I}^{\zeta-\nu}_{k}h(y).$$

Proof. By using definition of Hadamard fractional derivative, semi-group property of Hadamard fractional integral and Eqn (1.3.34), we obtain

$$\begin{aligned}\mathbf{D}^{\nu}_{k\mathcal{H}}\mathcal{I}^{\zeta}_{k}h(y) =& {}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}^{m-\nu}_{k}{}_{\mathcal{H}}\mathcal{I}^{\zeta}_{k}h(y)\\ =& {}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}^{m}_{k}{}_{\mathcal{H}}\mathcal{I}^{\zeta-\nu}_{k}h(y)\\ =& {}_{\mathcal{H}}\mathcal{I}^{\zeta-\nu}_{k}h(y).\end{aligned}$$

□

Theorem 1.3.30. *[22] Assume that $\mu > 0$, $m = [\mu]$ and $h \in A^m[a,b]$*

$${}_{\mathcal{H}}\mathcal{I}^{\mu}_{a}\boldsymbol{D}^{\mu}_{a}h(z) = h(z) - \sum_{q=1}^{n}\frac{\boldsymbol{D}^{n-q}_{a}\mathcal{I}^{n-\mu}_{a}h(z)|_{z=a}}{\Gamma(\mu-q+1)}\left(\log\frac{z}{a}\right)^{\mu-q}.$$

Proof. By definition of Hadamard fractional integral

$${}_{\mathcal{H}}\mathcal{I}^{\mu}_{a}\mathbf{D}^{\mu}_{a}h(z) = \frac{1}{\Gamma(\mu)}\int_a^z\left(\log\frac{z}{s}\right)^{\mu-1}\mathbf{D}^{\mu}_{a}h(s)\frac{ds}{s}. \tag{1.3.38}$$

For (1.3.38), first we prove,

$${}_{\mathcal{H}}\mathcal{D}\left[\frac{1}{\Gamma(\mu+1)}\int_a^z\left(\log\frac{z}{s}\right)^{\mu}\mathbf{D}^{\mu}_{a}h(s)\frac{ds}{s}\right] = \frac{1}{\Gamma(\mu)}\int_a^z\left(\log\frac{z}{s}\right)^{\mu-1}\mathbf{D}^{\mu}_{a}h(s)\frac{ds}{s}. \tag{1.3.39}$$

We solve the Eqn (1.3.39) by using Leibniz rule. Let us consider the left hand side of Eqn (1.3.39)

$$\begin{aligned}&{}_{\mathcal{H}}\mathcal{D}\left(\frac{1}{\Gamma(\mu+1)}\int_a^z\left(\log\frac{z}{s}\right)^{\mu}\mathbf{D}^{\mu}_{a}h(s)\frac{ds}{s}\right)\\ &=\left[\frac{(\log\frac{z}{s})^{\mu}}{\Gamma(\mu+1)}\mathbf{D}^{\mu}_{a}\frac{f(s)}{s}\right]_{s=z}\frac{d}{dz}(z) - \left[\frac{(\log\frac{z}{s})^{\mu}}{\Gamma(\mu+1)}\mathcal{D}^{\mu}_{a+}\frac{f(s)}{s}\right]_{s=a}\frac{d}{dz}(a)\\ &\quad+\frac{1}{\Gamma(\mu+1)}\int_a^z{}_{\mathcal{H}}\mathcal{D}\left(\log\frac{z}{s}\right)^{\mu}\mathbf{D}^{\mu}_{a}h(s)\frac{ds}{s}\\ &=\frac{\mu}{\Gamma(\mu+1)}\int_a^z z\left(\log\frac{z}{s}\right)^{\mu-1}\mathbf{D}^{\mu}_{a}\frac{h(s)}{zs}ds\\ &=\frac{1}{\Gamma(\mu)}\int_a^z\left(\log\frac{z}{s}\right)^{\mu-1}\mathbf{D}^{\mu}_{a}\frac{h(s)}{s}ds.\end{aligned}$$

On the other hand, by definition of Hadamard fractional derivative, repeated integration we have

$$\frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu} \mathbf{D}_a^{\mu} h(s)\frac{ds}{s} = \frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu} {}_{\mathcal{H}}\mathcal{D}_a^m {}_{\mathcal{H}}\mathcal{I}^{m-\mu} h(s)\frac{ds}{s}.$$

For $m = 1$, integration by parts formula give,

$$\begin{aligned}
&\frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu} s\frac{d}{ds} {}_{\mathcal{H}}\mathcal{I}_{a_+}^{1-\mu} h(s)\frac{ds}{s}\\
&\qquad = \frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu} {}_{\mathcal{H}}\mathcal{D} {}_{\mathcal{H}}\mathcal{I}_a^{1-\mu} h(s)\frac{ds}{s}\\
&\qquad = \left[\frac{1}{\mu+1}\left(\log\frac{z}{s}\right)^{\mu} {}_{\mathcal{H}}\mathcal{I}_{a+}^{1-\mu}\right]_a^z - \frac{1}{\Gamma(\mu+1)}\int_a^z {}_{\mathcal{H}}\mathcal{D}\left(\log\frac{z}{s}\right)^{\mu} {}_{\mathcal{H}}\mathcal{I}_a^{1-\mu}\frac{h(s)}{s}ds\\
&\qquad = -\frac{1}{\Gamma(\mu+1)}\left(\log\frac{z}{a}\right)^{\mu} {}_{\mathcal{H}}\mathcal{I}_a^{1-\mu} h(t)|_a + \frac{1}{\Gamma(\mu)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-1} {}_{\mathcal{H}}\mathcal{I}_a^{1-\mu}\frac{h(s)}{s}ds.
\end{aligned}$$

Similarly for $m = 2$

$$\begin{aligned}
&\frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu} {}_{\mathcal{H}}\mathcal{D}^2\mathcal{I}_{a_+}^{2-\mu} h(s)\frac{ds}{s}\\
&\quad = \frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu} {}_{\mathcal{H}}\mathcal{D}({}_{\mathcal{H}}\mathcal{D}{}_{\mathcal{H}}\mathcal{I}_a^{2-\mu} h(s))\frac{ds}{s}\\
&\quad = \frac{1}{\Gamma(\mu+1)}\left(\log\frac{z}{s}\right)^{\mu}({}_{\mathcal{H}}\mathcal{D}{}_{\mathcal{H}}\mathcal{I}_a^{2-\mu} h(s)|_a^z + \frac{1}{\Gamma(\mu)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-1}({}_{\mathcal{H}}\mathcal{D}{}_{\mathcal{H}}\mathcal{I}_a^{2-\mu} h(s))\frac{ds}{s}\\
&\quad = -\frac{1}{\Gamma(\mu+1)}\left(\log\frac{z}{a}\right)^{\mu}({}_{\mathcal{H}}\mathcal{D}{}_{\mathcal{H}}\mathcal{I}_a^{2-\mu} h(s))|_a - \frac{1}{\Gamma(\mu)}\left(\log\frac{z}{a}\right)^{\mu} {}_{\mathcal{H}}\mathcal{I}_a^{2-\mu} h(s)|_a\\
&\qquad + \frac{\mu-1}{\Gamma(\mu)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-2} {}_{\mathcal{H}}\mathcal{I}_a^{1-\mu}\frac{h(s)}{s}ds\\
&\quad = \frac{1}{\Gamma(\mu-1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-2} {}_{\mathcal{H}}\mathcal{I}_a^{1-\mu}\frac{h(s)}{s}ds - \sum_{q=1}^{2} {}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_a^{q-\mu} h(s)|a\frac{(\log\frac{z}{a})^{\mu-q+1}}{\Gamma(\mu-q+2)}.
\end{aligned}$$

In general

$$\begin{aligned}
&\frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu} \mathbf{D}_a^{\mu} f(s)\frac{ds}{s}\\
&\quad = \frac{1}{\Gamma(\mu-q+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-q} {}_{\mathcal{H}}\mathcal{I}_{a_+}^{q-\mu}\frac{h(s)}{s}ds - \sum_{q=1}^{n} {}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_{a_+}^{q-\mu} h(s)|a\frac{(\log\frac{z}{a})^{\mu-q+1}}{\Gamma(\mu-q+2)}\\
&\quad = \frac{1}{\Gamma(\mu-q+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-q} ({}_{\mathcal{H}}\mathcal{I}_a^{q-\mu}\frac{h(s)}{s})ds - \sum_{q=1}^{n} {}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_a^{q-\mu} h(s)|a\frac{(\log\frac{z}{a})^{\mu-q+1}}{\Gamma(\mu-q+2)}\\
&\quad = {}_{\mathcal{H}}\mathcal{I}_a^{\mu-q+1}\mathcal{I}_a^{q-\mu} h(s) - \sum_{q=1}^{n} {}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_a^{q-\mu} h(s)|a\frac{(\log\frac{z}{a})^{\mu-q+1}}{\Gamma(\mu-q+2)}.
\end{aligned}$$

$$\frac{1}{\Gamma(\mu+1)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu}\mathbf{D}_a^{\mu}h(s)\frac{ds}{s} = {}_{\mathcal{H}}\mathcal{I}_a h(s) - \sum_{q=1}^{n}{}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_a^{q-\mu}h(s)|a\frac{\left(\log\frac{z}{a}\right)^{\mu-q+1}}{\Gamma(\mu-q+2)}. \tag{1.3.40}$$

Combing (1.3.40) and (1.3.39) and by using (1.3.34), we have

$$\begin{aligned}
{}_{\mathcal{H}}\mathcal{D}({}_{\mathcal{H}}\mathcal{I}_a h(z)) - \sum_{q=1}^{n}{}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_a^{q-\mu}h(s)|a\frac{(\log\frac{z}{a})^{\mu-q+1}}{\Gamma(\mu-q+2)} &= \frac{1}{\Gamma(\mu)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-1}\mathcal{D}_{a+}^{\mu}h(s)\frac{ds}{s}\\
h(z) - \sum_{q=1}^{n}{}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_{a+}^{q-\mu}h(s)|a\frac{\left(\log\frac{z}{a}\right)^{\mu-q}}{\Gamma(\mu-q+1)} &= \frac{1}{\Gamma(\mu)}\int_a^z \left(\log\frac{z}{s}\right)^{\mu-1}\mathcal{D}_{a+}^{\mu}h(s)\frac{ds}{s}.
\end{aligned}$$

Equation (1.3.38) becomes

$${}_{\mathcal{H}}\mathcal{I}_a^{\mu}\mathbf{D}_a^{\mu}h(z) = h(z) - \sum_{q=1}^{n}{}_{\mathcal{H}}\mathcal{D}^{m-q}{}_{\mathcal{H}}\mathcal{I}_{a+}^{q-\mu}h(s)|a\frac{\left(\log\frac{z}{a}\right)^{\mu-q}}{\Gamma(\mu-q+1)}. \tag{1.3.41}$$

□

Having established the composition relations of Hadamard fractional integral with derivative and vice versa, now we will investigate the semi-group property of Hadamard fractional derivatives.

Theorem 1.3.31. *[22] Let $m-1\le\nu\le m$ and $n-1\le\zeta\le n$. Then*

$$\boldsymbol{D}_a^{\nu}\boldsymbol{D}_a^{\zeta}h(y) = \boldsymbol{D}_a^{\nu+\zeta}h(y) - \sum_{k=1}^{n}\boldsymbol{D}_a^{\zeta-k}h(y)|_{y=a}\frac{(y-a)^{-\nu-k}}{\Gamma(1-\nu-k)}. \tag{1.3.42}$$

Proof. By definition of Hadamard fractional derivative and using Eqn (1.3.41) we find,

$$\begin{aligned}
\mathbf{D}_a^{\nu}\mathbf{D}_a^{\zeta}h(y) &= {}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}_a^{m-\nu}\mathbf{D}_a^{\zeta}h(y) = {}_{\mathcal{H}}\mathcal{D}^{m}\left[{}_{\mathcal{H}}\mathcal{I}_a^{m-\nu}\mathbf{D}_a^{\zeta}h(y)\right]\\
&= {}_{\mathcal{H}}\mathcal{D}^{m}\left[{}_{\mathcal{H}}\mathcal{I}_a^{m-\nu}\mathbf{D}_a^{\zeta}h(y) - \sum_{k=1}^{n}\mathbf{D}_a^{\zeta-k}h(y)|_{y=a}\frac{(\log\frac{y}{a})^{m-\nu-k}}{\Gamma(1+m-\nu-k)}\right]
\end{aligned}$$

$$\mathbf{D}_a^{\nu}\mathbf{D}_a^{\zeta}h(y) = {}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}_a^{m-\nu}\mathbf{D}_a^{\zeta}h(y) - \sum_{k=1}^{n}\mathbf{D}_a^{\zeta-k}h(y)|_{y=a}\frac{{}_{\mathcal{H}}\mathcal{D}^{m}(\log\frac{y}{a})^{m-\nu-k}}{\Gamma(1+m-\nu-k)}. \tag{1.3.43}$$

Using definition of Hadamard fractional derivative we have

$${}_{\mathcal{H}}\mathcal{D}^{m}{}_{\mathcal{H}}\mathcal{I}_a^{m-\nu}\mathbf{D}_a^{\zeta}h(y) = \mathbf{D}_a^{\nu}\mathbf{D}_a^{\zeta}h(y) = \mathbf{D}_a^{\nu+\zeta}.$$

The mth integer order derivative is

$${}_{\mathcal{H}}\mathcal{D}^{m}\left(\log\frac{y}{a}\right)^{\Gamma(m-\nu-k+1)} = \frac{\Gamma(m-\nu-k)}{\Gamma(m-\nu-k-m+1)}\left(\log\frac{y}{a}\right)^{m-\nu-k-m}.$$

Now Eqn (1.3.43) becomes,

$$\mathbf{D}_a^{\nu}\mathbf{D}_a^{\zeta}h(y) = \mathbf{D}_a^{\nu+\zeta}h(y) - \sum_{k=1}^{n}\mathbf{D}_a^{\zeta-k}h(x)|_{x=a}\frac{{}_{\mathcal{H}}\mathcal{D}^{m}(\log\frac{x}{a})^{m-\nu-k}}{\Gamma(1+m-\nu-k)}.$$

□

1.3.5 The fractional integral with respect to another function

. In 1993 Samko et al. [39] defined the fractional integral which is known as Riemann-Liouville fractional integral with respect to a function.

Definition 1.3.32. [39] Let $0 \leq c \leq d < \infty$ and $\nu > 0$. The left and right sided fractional integrals with respect to another function ψ are defined as

$$\mathcal{I}_{c+}^{\nu,\psi} f(z) = \frac{1}{\Gamma(\nu)} \int_c^z (\psi(z) - \psi(s))^{\nu-1} f(s)\psi'(s)ds; \; z > a.$$

$$\mathcal{I}_{d-}^{\nu,\psi} f(z) = \frac{1}{\Gamma(\nu)} \int_z^d (\psi(s) - \psi(z))^{\nu-1} \psi'(s) f(s)ds; \; z < b.$$

1.3.6 Katugampola fractional integral

Fractional operators with respect to z^ρ was defined by Erdelyi in 1964. This operator was recently rediscovered by Katugampola [23] and studied by many researchers.

Definition 1.3.33. [23] Let $[c, d] \subset \mathbb{R}$. Then the so called left-and right-sided Katugampola fractional integrals of order $\nu > 0$ are defined as

$$\mathcal{I}_{c+}^{\nu,\rho} f(z) = \frac{1}{\Gamma(\nu)} \int_c^z \frac{(z^\rho - s^\rho)^{\nu-1}}{\rho^{\nu-1}} s^{\rho-1} f(s)ds; \; c < z < d,$$

and

$$\mathcal{I}_{d-}^{\nu,\rho} f(z) = \frac{1}{\Gamma(\nu)} \int_z^d \frac{(s^\rho - z^\rho)^{\nu-1}}{\rho^{\nu-1}} s^{\rho-1} f(s)ds; \; c < z < d.$$

1.3.7 Hadamard-type fractional integral

In 2001 Kilbas [21] introduced the Hadamard-type fractional integral.

Definition 1.3.34. [21] Let $0 \leq a \leq b < \infty$ and $\nu > 0$. The left and right-sided Hadamard-type fractional integrals are defined as

$$\mathcal{I}_{k+,c}^{\nu,H} f(z) = \frac{1}{\Gamma(\nu)} \int_k^z \left(\frac{s}{z}\right)^c \left(\log\left(\frac{z}{s}\right)\right)^{\nu-1} f(s)\frac{ds}{s}; \; k < z < l,$$

and

$$\mathcal{I}_{l-,c}^{\nu,H} f(z) = \frac{1}{\Gamma(\nu)} \int_z^l \left(\frac{z}{s}\right)^c \left(\log\left(\frac{s}{z}\right)\right)^{\nu-1} f(s)\frac{ds}{s}; \; k < z < l.$$

1.3.8 Tempered fractional integral

In 2015 Sabzikar et al. [41] introduced the tempered fractional integral.

Definition 1.3.35. [41] Let $z \in (a, b)$ and $c \in \mathbb{C}$. The right and left sided tempered fractional integral of order $\nu > 0$ are defined as

$$\mathcal{I}_{a+}^{\nu,c} f(z) = \frac{1}{\Gamma(\nu)} \int_a^z (z-s)^{\nu-1} e^{-c(z-s)} f(s) ds; \; a < z < b,$$

and

$$\mathcal{I}_{b-}^{\nu,c} f(z) = \frac{1}{\Gamma(\nu)} \int_a^z (s-z)^{\nu-1} e^{-c(s-z)} f(s) ds; \; a < z < b.$$

1.3.9 The Hadamard-type fractional integral with respect to another function

Fahad et al. [14] introduced the relation between Hadamard-type fractional calculus and Tempered fractional calculus, then he defined the fractional integral which cover both Tempered and Hadamard-type fractional calculus.

Definition 1.3.36. [14] Let $\nu > 0$ and $c \in \mathbb{C}$ where $z \in [a, b]$ and $\phi \in C^1[a, b]$ where $a < b$. ϕ is positive increasing function such that $\phi'(z) \neq 0$ for all $z \in [a, b]$. The left and right-sided Hadamard-type fractional integral of h with respect to ϕ are defined as

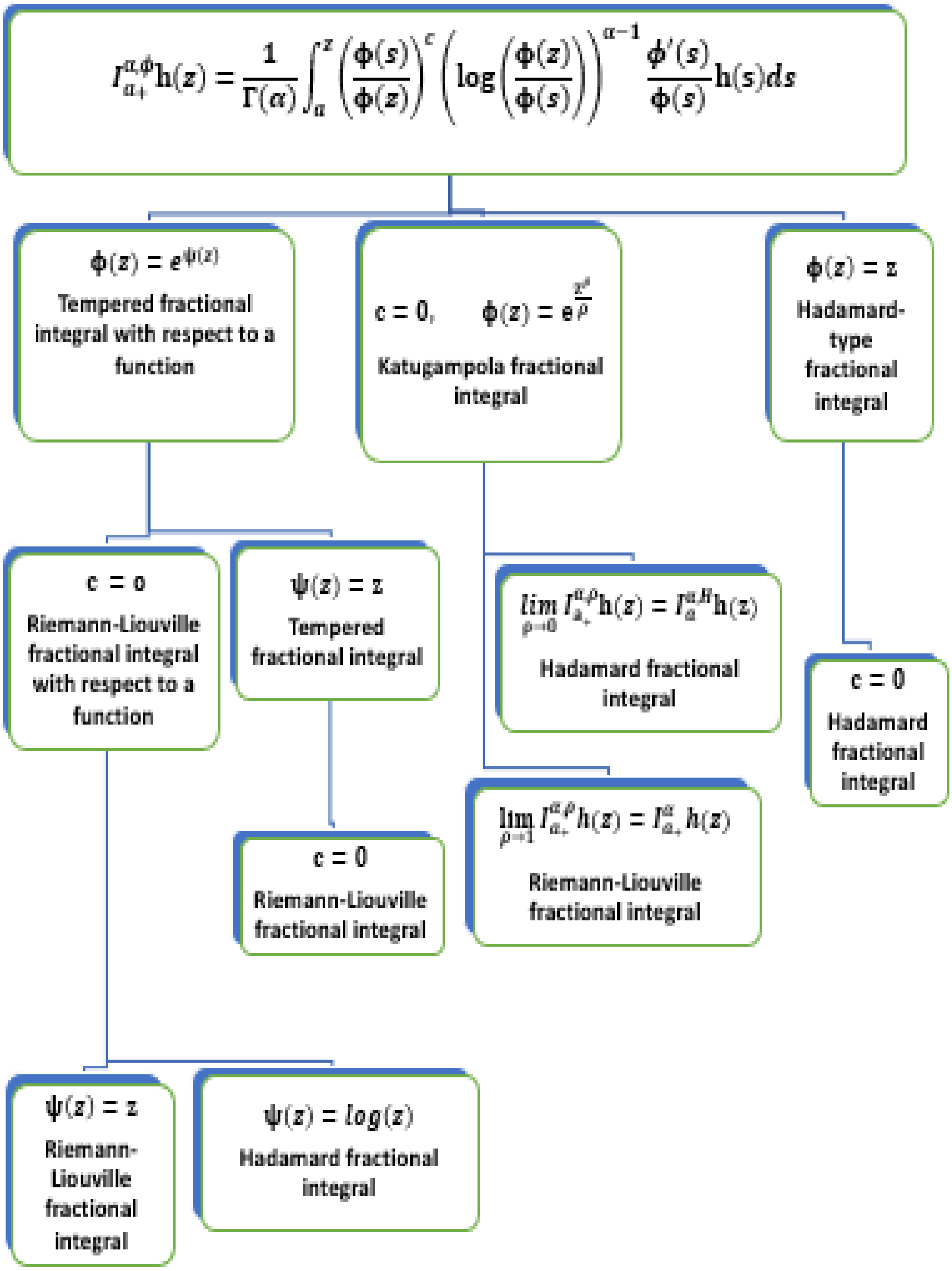
$$\mathcal{I}_{a_+}^{\nu,\phi} h(z) = \frac{1}{\Gamma(\nu)} \int_a^z \left(\frac{\phi(s)}{\phi(z)} \right)^c \left(\log \frac{\phi(z)}{\phi(s)} \right)^{\nu-1} \frac{\phi'(s)}{\phi(s)} h(s) ds. \tag{1.3.44}$$

$$\mathcal{I}_{b_-}^{\nu,\phi} f(z) = \frac{1}{\Gamma(\nu)} \int_z^b \left(\frac{\phi(z)}{\phi(s)} \right)^c \left(\log \frac{\phi(s)}{\phi(z)} \right)^{\nu-1} \frac{\phi'(s)}{\phi(s)} h(s) ds. \tag{1.3.45}$$

Remark 1.3.37. [14]

- For $c = 0$ and $\phi(z) = e^z$, we get the original Riemann-Liouville fractional integrals of order $\nu > 0$.
- For $\phi(z) = z$ we obtain the Hadamard-type fractional integrals.
- For $c = 0$ and $\phi(z) = z$, we obtain the classical Hadamard fractional integrals.
- For $c = 0$ and $\phi(z) = e^{\frac{z^\rho}{\rho}}$, we obtain the so-called Katugampola fractional integral.
- For $\phi(z) = e^z$ we recover the tempered fractional integrals.

- For $c = 0$, $\phi(z) = e^{\psi(z)}$ get the Riemann-Liouville fractional integral of a function with respect to another function ψ.



Theorem 1.3.38. *[14] Let $p, q \geq 0$ and $\phi \in C^1[p, q]$. Then*

$$\mathcal{I}_p^{\nu,\phi}\mathcal{I}_p^{\zeta,\phi}\psi(y) = \mathcal{I}_p^{\nu+\zeta,\phi}\psi(y).$$

Example 1.3.39. We will find the Hadamard-type fractional integral with respect to ψ of the function $h(y) = \psi(y)^{-c}\left(\log(\frac{\psi(y)}{\psi(a)})\right)^{\eta}$.

Proof. By definition of Hadamard fractional integral

$$\begin{aligned}
\mathcal{I}_a^{\nu,\phi}h(y) =& \frac{1}{\Gamma(\nu)}\int_a^y \left(\frac{\psi(s)}{\psi(y)}\right)^c \left(\log\frac{\psi(y)}{\psi(s)}\right)^{\nu-1}\frac{\psi'(s)}{\psi(s)}h(s)ds \\
=& \frac{\psi(y)^{-c}}{\Gamma(\nu)}\int_a^y \left(\log\frac{\psi(y)}{\psi(s)}\right)^{\nu-1}\frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(y)}{\psi(a)}\right)^{\eta}ds \\
=& \frac{\psi(y)^{-c}}{\Gamma(\nu)}\int_a^y (\log\psi(y) - \log\psi(a) - (\log\psi(s) - \log\psi(a)))^{\nu-1}\frac{\psi'(s)}{\psi(s)}\left(\log\frac{\psi(y)}{\psi(a)}\right)^{\eta}ds \\
=& \frac{\psi(y)^{-c}}{\Gamma(\nu)}\int_a^y \left(\log\frac{\psi(y)}{\psi(a)}\right)^{\nu-1}\left(1 - \frac{\log\frac{\psi(s)}{\psi(a)}}{\log\frac{\psi(y)}{\psi(a)}}\right)^{\nu-1}\left(\log\frac{\psi(s)}{\psi(a)}\right)^{\eta}\frac{ds}{s}.
\end{aligned}$$

We evaluate the integral by substituting method. Let $z = \frac{\log\frac{\psi(s)}{\psi(a)}}{\log\frac{\psi(y)}{\psi(a)}}$, then

$$\begin{aligned}
\mathcal{I}_a^{\nu,\phi}\left(\psi(y)^{-c}\left(\log\frac{\psi(y)}{\psi(a)}\right)^{\eta}\right) =& \frac{\psi(y)^{-c}}{\Gamma(\nu)}\int_0^1 \left(\log\frac{\psi(y)}{\psi(a)}\right)^{\nu-1}(1-z)^{\nu-1}z^{\eta}\left(\log\frac{\psi(y)}{\psi(a)}\right)^{\eta}(\log\frac{\psi(y)}{\psi(a)})dy \\
=& \frac{\psi(y)^{-c}\left(\log\frac{\psi(y)}{\psi(a)}\right)^{\nu+\eta}}{\Gamma(\nu)}\int_0^1 (1-z)^{\nu-1}z^{\eta}dz.
\end{aligned}$$

Since $B(\nu, \eta+1) = \int_0^1 (1-y)^{\nu-1}y^{\eta}dy$. Using the relationship between gamma and beta functions we obtain

$$\mathcal{I}_a^{\nu,\phi}\left(\psi(y)^{-c}\left(\log\frac{\psi(y)}{\psi(a)}\right)^{\eta}\right) = \frac{\psi(y)^{-c}(\log\frac{\psi(y)}{\psi(a)})^{\nu+\eta}}{\Gamma(\nu)}\frac{\Gamma(\nu)\Gamma(\eta+1)}{\Gamma(\nu+\eta+1)} = \frac{\Gamma(\eta+1)}{\Gamma(\nu+\eta+1)}\psi(y)^{-c}\left(\log\frac{\psi(y)}{\psi(a)}\right)^{\nu+\eta}. \tag{1.3.46}$$

□

Chapter 2

Classical Hermite-Hadamard inequality

Inequalities were demonstrated as most significant tools for the researcher in different fields of mathematics. Inequalities help us to analyzed the singularity and other properties of fractional differential equations. Integral inequalities have a fundamental role in both theoretical and applied mathematics. From the past few decade integral inequalities gain the attention of the researcher. Convexity plays an important role in establishing the inequalities. In 1881 C. Hermite and J. Hadamard discovered the classical Hermite-Hadamard inequality for convex function.

Various interesting results have been revealed by utilizing the concept of classical convexity. Where as Hermite-Hadamard inequality has strike the eye of many researchers. From classical Hermite-Hadamard inequality we can estimate the lower and upper integral average of any convex function. Numerous researcher established the Hermite-Hadamard inequality for different types of convex function some of them are Deng and Wang [11] for (α,m)-logarithmically convex functions, and Liao et al. [24, 25] for once and twice differentiable geometric-arithmetically s-convex functions.

M. A. Noor, G. Cristescu and M. U. Awan [29] developed the Hermite-Hadamard inequalities for twice differentiable s-convex functions. Du et al. [12] discussed some properties of Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized (α, m)-preinvex function. Considerable amount of work on Hermite-Hadamard inequality is also refer to Hwang, Y. Yeh, L. Tseng [7], Mehmet Zeki Sarikaya, Erhan Set and Hatice Yaldiz, Nagihan Basak [40], Ohud Almutairi, Adem Kiliman [3], M. Z. Sarikaya, H. Budak, F. Usta [38], Arran Fernandez, Pshtiwan Mohammed [15] and Hwang, Y. Yeh, L. Tseng [7].

2.1 Hermite-Hadamard inequality

Convex function Convex functions are well know functions in literature. The theories of convex functions and inequalities are closely interconnected. In past few decades it has been a subject of extensive research. Convex function is defined as follow: The function h is said to be convex on some interval I, if $z_1, z_2 \in I$ such that $z_1 < z_2$.

$$h((1-u)z_1 + uz_2) \leq (1-u)h(z_1) + uh(z_2),$$

for every $u \in [0,1]$.
Above definition holds for concave function in reverse direction.

Theorem 2.1.1. *[27] Let h be such that $h'' \geq 0$ on (k,l) and let $y_0 \in (k,l)$. Then for each $y \in (k,l)$*

$$h(y) \geq h(y_0) + h'(y_0)(y - y_0)$$

Proof. Let $y_0 \in (k,l)$ by Mean value theorem for second order derivative there exit c between y and y_0 such that,

$$h(y) = h(y_0) + h'(y_0)(y - y_0) + h''(c)\frac{(y-y_0)^2}{2}.$$

Since $h'' \geq 0$ on (k,l) and c is between y and y_0 then $h''(c) \geq 0$, we have

$$h(y) \geq h(y_0) + h'(y_0)(y - y_0).$$

□

Classical Hermite-Hadamard inequality for a convex function is defined as

Theorem 2.1.2. *[27] Let $h(z)$ be defined on $[k,l]$ with $h'' \geq 0$. Then*

$$h\left(\frac{k+l}{2}\right)(l-k) \leq \int_k^l h(z)dz \leq \frac{h(k)+h(l)}{2}(l-k). \tag{2.1.1}$$

Proof. First we prove the right hand side of inequality. For this, we let $z = (1-u)k + ul$, $u \in [0,1]$ and then by using definition of convex function. we obtain,

$$\begin{aligned}\int_k^l h(z)dz = (l-k)\int_0^1 h((1-u)k+ul)du \leq &(l-k)\int_0^1 (1-u)h(k) + uh(l)dt \\ =&(l-k)h(k)\int_0^1 (1-u)du + h(l)\int_0^1 udu \\ =&(l-k)\frac{h(k)+h(l)}{2}.\end{aligned}$$

$$\int_k^l h(z)dz \leq \frac{h(k)+h(l)}{2}(l-k). \tag{2.1.2}$$

For the left hand side inequality, the graph of convex function h is on or above the tangent line by Theorem (2.1.1). So its graph lies above the tangent line at $\left(\frac{k+l}{2}, h(\frac{k+l}{2})\right)$. That is,

$$h(z) \geq h\left(\frac{k+l}{2}\right) + h'\left(\frac{k+l}{2}\right)\left(z - \frac{k+l}{2}\right).$$

Integrating this with respect to z, we get

$$\int_k^l h(z)dz \geq \int_k^l h\left(\frac{k+l}{2}\right)dz + \int_k^l h'\left(\frac{k+l}{2}\right)\left(z - \frac{k+l}{2}\right)dz.$$

Since,

$$h'\left(\frac{k+l}{2}\right)\int_k^l \left(z - \frac{k+l}{2}\right)d_2 = 0.$$

Then

$$\int_k^l h(z)dz \geq h\left(\frac{k+l}{2}\right)\int_k^l dz \geq h\left(\frac{k+l}{2}\right)(l-k). \tag{2.1.3}$$

Combining (2.1.2) and (2.1.3), we have

$$h\left(\frac{k+l}{2}\right)(l-k) \leq \int_k^l h(z)dz \leq \frac{h(k)+h(l)}{2}(l-k).$$

□

Dragomir and Agrawal [9] use the right side of the classical Hermite-Hadamard inequality to establish different inequalities.

Lemma 2.1.3. *[9] Let $h : I \to \mathbb{R}$ be a differentiable mapping on I , $k, l \in [a, b]$ with $k < l$. If $h' \in L[k, l]$, then the following equality holds:*

$$\frac{h(k)+h(l)}{2} - \frac{1}{l-k}\int_k^l h(y)dy = \frac{l-k}{2}\int_0^1 (1-2s)h'(sk+(1-s)l)ds.$$

Theorem 2.1.4. *[9] Let $h : I \to \mathbb{R}$ be a differentiable mapping on I , $k, l \in I$ with $k < l$. If $|h'|$ l is convex on $[k, l]$, then the following inequality holds:*

$$\left|\frac{l-k}{2}\int_0^1 (1-2s)h'(sk+(1-s)l)ds\right| \leq \frac{(l-k)[|h'(k)|+|h'(l)|]}{8}.$$

Theorem 2.1.5. *[9] Let $h : I \to \mathbb{R}$ be a differentiable mapping on I, $k, l \in I$ with $k < l$, and let $q > 1$. If the mapping $|h'|^{\frac{q}{q-1}}$ convex on $[k, l]$, then the following inequality holds:*

$$\left|\frac{h(k)+h(l)}{2} - \frac{1}{l-k}\int_k^l h(y)dy\right| \leq \frac{(l-k)}{2(q+1)^{\frac{1}{q}}}\left[\frac{|h'(k)|^{\frac{q}{q-1}}+|h'(l)|^{\frac{q}{q-1}}}{2}\right]^{\frac{q-1}{q}}.$$

2.1.1 Application of Classical Hermite-Hadamard inequality

Hermite-Hadamard inequality has a wide application in pure and applied mathematics. If we apply HH-inequality For different convex function we can get Geometric, Logarithmic and Arithmetic mean inequality.

Proposition 2.1.6. *[31] Let $x_1, y_1 > 0$ then*

$$\sqrt{x_1 y_1} \leq \frac{y_1 - x_1}{\ln y_1 - \ln x_1} \leq \frac{x_1 + y_1}{2}.$$

Proof. Let $h(y_1) = e^{y_1}$ in Eqn (2.1.1) we have

$$e^{\frac{k+l}{2}}(l-k) \leq \int_k^l e^z dz \leq \frac{e^k + e^l}{2}.$$

As $x_1, y_1 > 0$, we may let $k = \ln x_1$ and $l = \ln y_1$ then

$$e^{\frac{\ln x_1 + \ln y_1}{2}}(\ln y_1 - \ln x_1) \leq \int_{\ln x_1}^{\ln y_1} e^z dz \leq \frac{e^{\ln x_1} + e^{\ln y_1}}{2}.$$

By simplifying this we get our desired result. □

The special means for arbitrary number that are also useful in proving new inequalities using the above theorems are defined as

Definition 2.1.7.

$$A(\omega, \eta) = \frac{\omega + \eta}{2}, \ \omega, \eta \in \mathbb{R}.$$

$$H(\omega, \eta) = \frac{2}{\frac{1}{xi} + \frac{1}{\eta}}, \ \omega, \eta \in \mathbb{R} - [0].$$

$$L(\omega, \eta) = \frac{\eta - \omega}{\ln|\eta| - \ln|\omega|}, \ |\eta| \neq |\omega|, \ \omega\eta \neq 0.$$

$$L_n(\omega, \eta) = \left[\frac{\eta^{n+1} - \omega^{n+1}}{(n+1)\eta - \omega}\right]^{\frac{1}{n}}, \ n \in \mathbb{Z} - [-1, 0], \ \omega \neq l.$$

If $h(x_1) = x_1^n$ where $m \in \mathbb{R}$ in Theorem (2.1.4) and Theorem (2.1.5), then we have following inequalities

Proposition 2.1.8. *Let $\omega, \eta \in \mathbb{R}$, ω, η and $m \in N$, $m > 2$, then*

$$|A(\omega^m + \eta^m) - L_m(\omega, \eta)| \leq \frac{m(\eta - \omega)}{4} A(|\omega|^{m-1}, |\eta|^{m-1}).$$

Proposition 2.1.9. *Let $\omega, \eta \in \mathbb{R}$, $\omega < \eta$ and $m \in N$, $m \geq 2$, then*

$$|A(\omega^m + \eta^m) - L_m(\omega, \eta)| \leq \frac{(\eta - \omega)}{2(q+1)^{\frac{1}{q}}} \left[A(|\omega|^{\frac{(m-1)q}{q-1}}, |\eta|^{\frac{(m-1)q}{q-1}})\right]^{\frac{q-1}{q}}.$$

If $h(z) = \frac{1}{z}$ where $m \in \mathbb{R}$ in Theorem (2.1.4) and Theorem (2.1.5), then we have following inequalities

Proposition 2.1.10. *Let $c, d \in \mathbb{R}$, $c < d$, and $0 \notin [c, d]$.*

$$\left|A(\omega^{-1} + \eta^{-1}) - L^{-1}(\omega, \eta)\right| \leq \frac{m(\eta - \omega)}{4} A(|\omega|^{-2}, |\eta|^{-2}).$$

Proposition 2.1.11. *Let $\omega, \eta \in \mathbb{R}$, $\omega < \eta$, and $0 \notin [c, d]$.*

$$\left|A(\omega^{-1} + \eta^{-1}) - L^{-1}(\omega, \eta)\right| \leq \frac{(\eta - \omega)}{2(q+1)^{\frac{1}{q}}} \left[A(|\omega|^{\frac{-2q}{q-1}}, |\eta|^{\frac{-2q}{q-1}})\right]^{\frac{q-1}{q}}.$$

Dragomir and Agrawal [9] also discussed error estimation of trapezoidal formula.

Definition 2.1.12. [9] Let $h : [k, l] \to \mathbb{R}$ is continuous on $[k, l]$ and d be a division of the interval $[k, l]$, that is $d : k = z_0 < z_l < \cdots < z_{n-1} < z_n = l$, then trapezoidal formula is defined as

$$T(h, d) = \sum_{i=0}^{n-1} \frac{h(z_i) + h(z_{i+1})}{2}(z_i + z_{i+1}).$$

If $h : [k, l] \to \mathbb{R}$ is twice differentiable on $[k, l]$ and $M = max_{t\in(k,l)}|h''(y)| < \infty$, then

$$\int_k^l h(y)dy = T(h, d) + E(h, d). \tag{2.1.4}$$

Where the approximation error $E(h, d)$ of the integral satisfies

$$|E(h, d)| = \frac{M}{12} \sum_{i=0}^{n-1} (y_{i+1} - y_i)^3$$

Proposition 2.1.13. *Let h be a differentiable mapping on I, $k, l \in I$ with $k < l$. If $|h'|$ is convex on $[k, l]$, then in (2.1.4), for every division d of $[k, l]$ we have*

$$|E(h, d)| \leq \frac{1}{8} \sum_{i=0}^{n-1} (z_{i+1} - z_i)^2 (|h'(z_i)| + |h'(z_{i+1})|) \leq \frac{max(|h'(k)|, |h'(l)|)}{4} \sum_{i=o}^{n-1} (z_{i+1} - z_i)^2$$

Chapter 3

Fractional Hadamard inequality

Fractional inequalities plays a fundamental role in the field of Mathematics. These inequalities also abetment the unique solution of boundary value problem. Considerable literature is available on the application of fractional inequalities. Fractional integral inequalities are helping hand to show the uniqueness of partial differential equation. Fractional integral inequalities has a different generalization in literature. Here we discussed some of the fractional integral inequalities.
Sarikaya et al. [40] established the Hermite-Hadamard inequality for Riemann-Liouville fractional integral.

3.1 Hermite-Hadamard inequality

Theorem 3.1.1. *[40] Let $h : [k, l] \to \mathbb{R}$ be a positive function and $h \in L_1[k, l]$. If h is convex function on $[k, l]$, then the following inequalities for Riemann-Liouville fractional integral holds:*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(l-k)^{\nu}}[\mathcal{I}^{\nu}_{k+}h(l) + \mathcal{I}^{\nu}_{l-}h(k)] \leq \frac{h(k)+h(l)}{2}, \tag{3.1.1}$$

with $\nu > 0$.

Proof. Since h is convex function on $[k, l]$, we have for $z_1, z_2 \in [k, l]$, with $u = \frac{1}{2}$

$$h((1-t)z_1 + tz_2) \leq (1-t)h(z_1) + th(z_2)$$
$$h\left(\frac{z_1+z_2}{2}\right) \leq \frac{h(z_1)+h(z_2)}{2}.$$

By substituting $z_1 = tk + (1-t)l$ and $z_2 = (1-t)k + tl$, we obtain

$$2h\left(\frac{k+l}{2}\right) \leq h\left(tk + (1-t)l\right) + h\left((1-t)k + tl\right).$$

Multiply $u^{\nu-1}$ on both side and then integrating with respect to u over $[0,1]$, we obtain

$$2h\left(\frac{k+l}{2}\right)\int_0^1 u^{\nu-1}du \leq \int_0^1 u^{\nu-1}h(uk+(1-u)l)du + \int_0^1 u^{\nu-1}h((1-u)k+ul)du$$

$$\frac{2}{\nu}h\left(\frac{k+l}{2}\right) \leq \int_0^1 u^{\nu-1}h(uk+(1-u)l)du + \int_0^1 u^{\nu-1}h((1-u)k+ul)du.$$

By substituting $w = uk + (1-u)l$ and $v = (1-u)k + ul$, we have

$$\begin{aligned}
&= \int_l^k \left(\frac{l-w}{l-k}\right)^{\nu-1} h(w)\frac{dw}{k-l} + \int_k^l \left(\frac{v-k}{l-k}\right)^{\nu-1} h(v)\frac{dv}{l-k}\\
&= \frac{\Gamma(\nu)}{(l-k)^\nu}\left(\frac{1}{\Gamma(\nu)}\int_k^l (l-w)^{\nu-1}h(w)dw + \frac{1}{\Gamma(\nu)}\int_k^l (v-k)^{\nu-1}h(v)dv\right)\\
&= \frac{\Gamma(\nu)}{(l-k)^\nu}(\mathcal{I}_{k+}^\nu h(l) + \mathcal{I}_{l-}^\nu h(k))
\end{aligned}$$

$$\frac{2}{\nu}h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\nu)}{(l-k)^\nu}[\mathcal{I}_{k+}^\nu h(l) + \mathcal{I}_{l-}^\nu h(k)]$$

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(l-k)^\nu}(\mathcal{I}_{k+}^\nu h(l) + \mathcal{I}_{l-}^\nu h(k)). \tag{3.1.2}$$

For the proof of second inequality. Since h is convex function, then for $u \in [0,1]$

$$h((1-u)k+ul) \leq (1-u)h(k) + uh(l)$$

and

$$h((1-u)z+uy) \leq (1-u)h(z) + uh(y).$$

By adding inequalities we have

$$h((1-u)k+ul) + h((1-u)z+uy) \leq (1-u)h(k) + uh(l) + (1-u)h(l) + uh(k).$$

Multiply $u^{\nu-1}$ on both side and integrating with respect to u over $[0,1]$, we get

$$\int_0^1 u^{\nu-1}h(uk+(1-u)l)du + \int_0^1 u^{\nu-1}h((1-u)k+ul)du \leq \int_0^1 u^{\nu-1}(h(k)+h(l)).$$

As we know that

$$\int_0^1 u^{\nu-1}h(uk+(1-u)l)du + \int_0^1 u^{\nu-1}h((1-u)k+ul)du = \frac{\Gamma(\nu)}{(l-k)^\nu}(\mathcal{I}_{k+}^\nu h(l) + \mathcal{I}_{l-}^\nu h(k)).$$

Then

$$\frac{\Gamma(\nu)}{(l-k)^\nu}(\mathcal{I}_{k+}^\nu h(l) + \mathcal{I}_{l-}^\nu h(k)) \leq \frac{h(k)+h(l)}{\nu}$$

$$\frac{\Gamma(\nu+1)}{2(l-k)^{\nu}}(\mathcal{I}^{\nu}_{k+}h(l)+\mathcal{I}^{\nu}_{l-}h(k)) \leq \frac{h(k)+h(l)}{2}. \tag{3.1.3}$$

Combining (3.1.2) and (3.1.3), we obtain

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(l-k)^{\nu}}[\mathcal{I}^{\nu}_{k+}h(l)+\mathcal{I}^{\nu}_{l-}h(k)] \leq \frac{h(k)+h(l)}{2}.$$

□

Example 3.1.2. For $h(x) = x^{2m}$, $m \in \mathbb{R}_+$ on the interval $[k, l] = [0, 1]$

By Lemma (1.3.5) we can find Riemann-Liouville fractional integral of a function is

$$\mathcal{I}^{\nu}_{0+}h(x) = \frac{\Gamma(2m+1)}{\Gamma(2m+\nu+1)}x^{2m+\nu}$$

For the right hand side

$$\mathcal{I}^{\nu}_{1-}h(0) = \frac{1}{\Gamma(\nu)}\int_0^1 s^{\nu+2m-1}ds = \frac{1}{(\nu+2m)\Gamma(\nu)}$$

Substituting this in Hermite-Hadamard inequality (3.1.1)

$$\left(\frac{1}{2}\right)^{2m} \leq \frac{\Gamma(\nu+1)}{2}\left[\frac{\Gamma(2m+1)}{\Gamma(2m+\nu+1)}+\frac{1}{(\nu+2m)\Gamma(\nu)}\right] \leq \frac{1}{2}.$$

Let $n = 1$, $\nu = \frac{1}{2}$ we get

$$0.25 \leq 0.36 \leq 0.5.$$

3.1.1 Hermite-Hadamard type inequality

In the following we discuss the Hermite-Hadamard type inequality for a convex function by considering integral identities which involve first order derivative of h.

Theorem 3.1.3. *[40] If $h : I \to \mathbb{R}$ is a convex function, then the following inequality holds*

$$h\left(\frac{k+l}{2}\right) \leq \frac{1}{l-k}\int_k^l h(z)dz \leq \frac{h(k)+h(l)}{2}.$$

Both inequalities hold in the reversed direction if h is concave.

Proof. Let $\alpha = 1$ in Eqn (3.1.1) and by using property $\Gamma(z+1) = z\Gamma(z)$, we obtain

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(1+1)}{2(l-k)}[\mathcal{I}_{k+}h(l)+\mathcal{I}_{l-}h(k)] \leq \frac{h(k)+h(l)}{2}$$

$$h\left(\frac{k+l}{2}\right) \leq \frac{1}{2(l-k)^{\alpha}}\left[\int_k^l h(z)dz+\int_k^l h(z)dz\right] \leq \frac{h(k)+h(l)}{2}$$

$$h\left(\frac{k+l}{2}\right) \leq \frac{1}{(l-k)^{\alpha}}\int_k^l h(z)dz \leq \frac{h(k)+h(l)}{2}.$$

□

Lemma 3.1.4. *[40] Let $h : I \to \mathbb{R}$ be a differentiable mapping. If $h' \in L_1[k,l]$ then the following equality holds:*

$$\frac{h(k)+h(l)}{2} - \frac{\Gamma(\nu+1)}{2(l-k)}[\mathcal{I}^{\nu}_{k+}h(l) + \mathcal{I}^{\nu}_{l-}h(k)] = \frac{l-k}{2}\int_0^1 [(1-t_1)^{\nu} - t_1^{\nu}]h'(t_1k+(1-t_1)l)dt_1.$$

Proof. Let us consider the right hand side

$$\int_0^1 [(1-t_1)^{\nu}-t_1^{\nu}]h'(t_1k+(1-t_1)l)dt_1 = \int_0^1 (1-t_1)^{\nu}h'(t_1k+(1-t_1)l) - \int_0^1 t_1^{\nu}h'(t_1k+(1-t_1)l). \tag{3.1.4}$$

We evaluate the integrals in Eqn (3.1.4) by integration by part, properties of gamma function and by the definition of Riemann-Liouville fractional integral

$$\begin{aligned}\int_0^1 (1-t_1)^{\nu}h'(t_1k+(1-t_1)l) =& \frac{(1-t_1)^{\nu}h(t_1k+(1-t_1)l)}{k-l}\Big|_0^1 - \nu\int_0^1 (1-t_1)^{\nu-1}\frac{h(t_1k+(1-t_1)l)}{k-l}dt_1 \\ =& \frac{h(l)}{l-k} - \frac{\nu\Gamma(\nu)}{k-l}\left[\frac{1}{\Gamma(\nu)}\int_0^1 (1-t_1)^{\nu-1}h(t_1k+(1-t_1)l)\right]dt_1 \\ =& \frac{h(l)}{l-k} - \frac{\Gamma(\nu+1)}{k-l}\left[\frac{1}{\Gamma(\nu)}\int_0^1 (1-t_1)^{\nu-1}h(t_1k+(1-t_1)l)\right]dt_1.\end{aligned}$$

By substituting $u = t_1k + (1-t_1)l$ we have

$$\begin{aligned} =& \frac{h(l)}{l-k} - \frac{\Gamma(\nu+1)}{k-l}\left[\frac{1}{\Gamma(\nu)}\int_l^k \left(\frac{u-a}{l-k}\right)^{\nu-1}\frac{h(u)}{k-l}du\right] \\ =& \frac{h(l)}{l-k} - \frac{\Gamma(\nu+1)}{(l-k)^{\nu+1}}\mathcal{I}^{\nu}_{l-}h(k).\end{aligned}$$

Let us consider

$$\int_0^1 t_1^{\nu}h'(t_1k+(1-t_1)l)dt_1 = \frac{t_1^{\nu}h(t_1k+(1-t_1)l)}{k-l}\Big|_0^1 - \frac{\nu}{k-l}\int_l^k t^{\nu-1}h(t_1k+(1-t_1)l)dt_1.$$

Again by substituting $u = t_1k + (1-t_1)l$

$$\begin{aligned}\int_0^1 t_1^{\nu}h'(t_1k+(1-t_1)l)dt_1 =& -\frac{h(k)}{l-k} - \frac{\nu}{k-l}\int_l^k \left(\frac{l-u}{l-k}\right)^{\nu-1} h(u)\frac{du}{k-l} \\ =& -\frac{h(k)}{l-k} + \frac{\Gamma(\nu+1)}{(l-k)^{\nu+1}}\left[\frac{1}{\Gamma(\nu)}\int_k^l (l-u)^{\nu-1}h(u)du\right] \\ =& -\frac{h(k)}{l-k} + \frac{\Gamma(\nu+1)}{(l-k)^{\nu+1}}\mathcal{I}^{\nu}_{k+}h(l).\end{aligned}$$

Eqn (3.1.4) will become

$$\int_0^1 [(1-t_1)^{\nu} - t_1^{\nu}]h'(t_1k+(1-t_1)f)dt_1 = \frac{h(k)+h(l)}{l-k} - \frac{\Gamma(\nu+1)}{(l-k)}[\mathcal{I}^{\nu}_{k+}h(l) + \mathcal{I}^{\nu}_{l-}h(k)].$$

Multiply by $\frac{l-k}{2}$ on both sides we obtain

$$\frac{l-k}{2}\int_0^1[(1-t_1)^\nu - t_1^\nu]h'(t_1k+(1-t_1)l)dt_1 = \frac{h(k)+h(l)}{2} - \frac{\Gamma(\nu+1)}{2(l-k)}[\mathcal{I}^\nu_{k+}h(l)+\mathcal{I}^\nu_{l-}h(k)].$$

□

Theorem 3.1.5. *[40] Let $h:[k,l]$ be a differentiable mapping. If $|h'|$ is convex then*

$$\left|\frac{h(k)+h(l)}{2} - \frac{\Gamma(\nu+1)}{2(l-k)^\nu}[\mathcal{I}^\nu_{k+}h(l)+\mathcal{I}^\nu_{l-}h(k)]\right| \leq \frac{l-k}{2(\nu+1)}\left(1-\frac{1}{2^\nu}\right)[|h'(k|+|h'(l)|)]$$

Proof. By using Lemma (3.1.4), and convexity of $|h'|$ we have

$$\begin{aligned}
&\left|\frac{h(k)+h(l)}{2} - \frac{\Gamma(\nu+1)}{2(l-k)^\nu}[\mathcal{I}^\nu_{k+}h(l)+\mathcal{I}^\nu_{l-}h(k)]\right| \\
&\leq \frac{l-k}{2}\int_0^1 |(1-t_1)^\nu - t_1^\nu||h'(t_1k+(1-t_1)l)|dt_1 \\
&|h'(t_1k+(1-t_1)l)| \leq t_1|h'(k)|+(1-t_1)|h'(l)| \\
&= \frac{l-k}{2}\int_0^1 |(1-t_1)^\nu - t_1^\nu|[t_1|h'(k)|+(1-t_1)|h'(l)|]dt_1 \\
&\mathcal{I}_1 = \int_0^{\frac{1}{2}}[(1-t_1)^\nu t_1 - t_1^{\nu+1}]|h'(k)|dt_1 + \int_0^{\frac{1}{2}}[(1-t_1)^{\nu+1} - t_1^\nu - t_1^{\nu+1}]|h'(l)|dt_1 \\
&\mathcal{I}_2 = -\int_{\frac{1}{2}}^1[(1-t_1)^\nu t_1 - t_1^{\nu+1}]|h'(k)|dt_1 - \int_{\frac{1}{2}}^1[(1-t_1)^{\nu+1} - t_1^\nu - t_1^{\nu+1}]|h'(l)|dt_1
\end{aligned}$$

Simplifying $\mathcal{I}_1$ and $\mathcal{I}_1$ and then add we get

$$\left[\frac{1}{\nu+2} - \frac{2\left(\frac{1}{2}\right)^{\nu+1}}{\nu+1} + \frac{1}{(\nu+2)(\nu+1)}\right][|h'(l)|+|h'(k)|] = \frac{1}{\nu+1}\left(1-\frac{1}{2^\nu}\right)[|h'(l)|+|h'(k)|].$$

Hence we get our desired result □

3.1.2 Fejér inequality

In the beginning of 20th century Fejér [13] proved the integral inequality which is weighted generalization of Hermite- Hadamard inequality and known as Fejér inequality. Iscan [18] established the Fejér inequality for the Riemann-Liouville fractional integral.

Lemma 3.1.6. *[18] If $h:[k,l]\to\mathbb{R}$ is integrable and symmetric to $\frac{k+l}{2}$ with $k<l$, then*

$$\mathcal{I}^\alpha_{k+}h(l) = \mathcal{I}^\alpha_{l-}h(k) = \frac{1}{2}[\mathcal{I}^\alpha_{k+}h(l)+\mathcal{I}^\alpha_{l-}h(k)].$$

Theorem 3.1.7. *[18] Let $h : [k,l] \to \mathbb{R}$ be convex function with $k < l$ and $h \in [k,l]$. If $g : [k,l] \to \mathbb{R}$ is non-negative, integrable and symmetric to $\frac{(k+l)}{2}$, then the following*

$$h\left(\frac{k+l}{2}\right)[\mathcal{I}^{\nu}_{k+}g(l) + \mathcal{I}^{\nu}_{l-}g(k)] \leq \mathcal{I}^{\nu}_{k+}hg(l) + \mathcal{I}^{\nu}_{l-}hg(k) \leq \frac{h(k)+h(l)}{2}[\mathcal{I}^{\nu}_{k+}g(l) + \mathcal{I}^{\nu}_{l-}g(k)].$$

Proof. Since g is symmetric to $\frac{k+l}{2}$, thus $g(k+l-u) = g(u)$, for all $u \in [k,l]$ as h is convex function on $[k,l]$, for all $u = \frac{1}{2}$. Therefore we have

$$2h\left(\frac{k+l}{2}\right) \leq h((1-u)k+ul) + h(uk+(1-u)l).$$

Multiply $u^{\nu-1}g((1-u)k+ul)$ on both sides and integrate over $u \in (0,1)$

$$2h\left(\frac{k+l}{2}\right)\int_0^1 u^{\nu-1}g((1-u)k+ul)du \leq \int_0^1 u^{\nu-1}g((1-u)k+ul)h((1-u)k+ul)du + \int_0^1 u^{\nu-1}g((1-u)k+ul)h(uk+(1-u)l)du.$$

Let $v = (1-u)k+ul$

$$\begin{aligned}
&\frac{2h\left(\frac{k+l}{2}\right)\Gamma(\nu)}{(l-k)^{\nu}}\left(\frac{1}{\Gamma(\nu)}\int_k^l (v-k)^{\nu-1}g(v)dv\right)\\
&\leq \frac{1}{(l-k)^{\nu}}\int_k^l (v-k)^{\nu-1}g(v)h(v)dv + \frac{1}{(l-k)^{\nu}}\int_k^l (v-k)^{\nu-1}g(v)h(k+l-v)dv\\
&\leq \frac{1}{(l-k)^{\nu}}\int_k^l (v-k)^{\nu-1}g(v)h(v)dv + \frac{1}{(l-k)^{\nu}}\int_k^l (l-v)^{\nu-1}g(k+l-v)h(v)dv\\
&= \frac{\Gamma(\nu)}{(l-k)^{\nu}}\left(\frac{1}{\Gamma(\nu)}\int_k^l (v-k)^{\nu-1}g(v)h(v)dv\right)\\
&\quad + \frac{\Gamma(\nu)}{(l-k)^{\nu}}\left(\frac{1}{\Gamma(\nu)}\int_k^l (l-v)^{\nu-1}g(v)h(v)dv\right).
\end{aligned}$$

By using above Lemma (3.1.6)

$$\frac{\Gamma(\nu)h(\frac{k+l}{2})}{(l-k)^{\nu}}[\mathcal{I}^{\nu}_{k+}g(l) + \mathcal{I}^{\nu}_{l-}g(k)] \leq \frac{\Gamma(\nu)}{(l-k)^{\nu}}[\mathcal{I}^{\nu}_{k+}gh(l) + \mathcal{I}^{\nu}_{l-}gh(k)]. \tag{3.1.5}$$

As h is convex for all $u \in (0,1)$

$$h((1-u)k+ul) + h(uk+(1-u)l) \leq h(k) + h(l).$$

Multiply $u^{\nu-1}g((1-u)k+ul)$ and integrate over $[0,1]$

$$\int_0^1 u^{\nu-1}g((1-u)k+ul)h((1-u)k+ul)du + \int_0^1 u^{\nu-1}g((1-u)k+ul)h(uk+(1-u)l)du \leq \int_0^1 u^{\nu-1}g((1-u)k+ul)[h(k)+h(l)]du.$$

Let $y = (1-u)k + ul$ and using Lemma (3.1.6)

$$\frac{\Gamma(\nu)}{(l-k)^\nu}\left(\frac{1}{\Gamma(\nu)}\int_k^l (y-k)^{\nu-1}g(y)h(y)dy + \frac{1}{\Gamma(\nu)}\int_k^l (l-y)^{\nu-1}g(y)h(y)dy\right)$$
$$\le \frac{\Gamma(\nu)}{(l-k)^\nu}\left(\frac{1}{\Gamma(\nu)}\int_k^l (y-k)^{\nu-1}g(y)[h(k)+h(l)]dy\right).$$

$$[\mathcal{I}^\nu_{k+}hg(l) + \mathcal{I}^\nu_{l-}hg(k)] \le [\mathcal{I}^\nu_{k+}h(l) + \mathcal{I}^\nu_{l-}h(k)]\frac{h(k)+h(l)}{2}. \tag{3.1.6}$$

Combining (3.1.5) and (3.1.6) we get the desired inequality. □

Corollary 3.1.8. *[18] For $\nu = 1$ in Theorem (3.1.7) we get Fejér inequality*

$$h\left(\frac{k+l}{2}\right)\int_k^l g(u)du \le \int_k^l h(u)g(u)du \le \frac{h(k)+h(l)}{2}\int_k^l g(u)du.$$

Corollary 3.1.9. *[18] For $g(x) = 1$ in Theorem (3.1.7) we get Hermite-Hadamard inequality for fractional integral*

$$h\left(\frac{k+l}{2}\right) \le \frac{\Gamma(\nu+1)}{2(l-k)^\nu}[\mathcal{I}^\nu_{k+}h(l) + \mathcal{I}^\nu_{l-}h(k)] \le \frac{h(k)+h(l)}{2}.$$

3.2 A survey of generalized Hermite-Hadamard fractional inequalities

The concept of Hermite-Hadamard inequality has been enhance various way in literature. In this section we discuss some of the few. Jleli and Samet [19] established the Hermite-Hadamard inequality via fractional integral with respect to another function given in Eqn (3.2.1).

Theorem 3.2.1. *[19] Let $h : I \to \mathbb{R}$ be a convex function on for $\alpha > 0$, then*

$$h\left(\frac{k+l}{2}\right) \le \frac{\Gamma(\alpha+1)}{4(\psi(l)-\psi(k))^\alpha}[\mathcal{I}^{\alpha,\psi}_{k+}H(l) + \mathcal{I}^{\alpha,\psi}_{l-}H(k)] \le \frac{h(k)+h(l)}{2}, \tag{3.2.1}$$

where $H(y) = h(y) + h^*(y)$ and $h^*(y) = h(k+l-y)$.

If $\psi(z) = z$ inequality (3.2.1) reduce to Hermite-Hadamard inequality for Riemann-Liouville fractional integral.

Corollary 3.2.2. *[19] If h is convex on $[k,l]$ and $\alpha > 0$, then*

$$h\left(\frac{k+l}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(l-k)^\alpha}[\mathcal{I}^\alpha_{k+}h(l) + \mathcal{I}^\alpha_{l-}h(k)] \le \frac{h(k)+h(l)}{2}.$$

If $\psi(z) = \ln(z)$, then inequality (3.2.1) reduce to Hermite-Hadamard inequality for the Hadamard fractional integral.

Corollary 3.2.3. *[19] If h is convex on $[k,l]$ and $\alpha > 0$ then*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4\left(\ln(\frac{l}{k})\right)^{\alpha}}[\mathcal{I}^{\alpha,H}_{k+}H(l) + \mathcal{I}^{\alpha,H}_{l-}H(k)] \leq \frac{h(k)+h(l)}{2}.$$

Theorem 3.2.4. *[19] If $h \in \mathbb{C}$ and $|h'|$ is convex on $[k,l]$ for $\alpha > 0$ then the following inequality holds:*

$$\left|\frac{h(k)+h(l)}{2} - \frac{\Gamma(\alpha+1)}{4[\psi(l)-\psi(k)]^{\alpha}}[\mathcal{I}^{\alpha,\psi}_{k+}H(l) + \mathcal{I}^{\alpha,\psi}_{l-}H(k)]\right| \leq \frac{\mathcal{I}^{\alpha}_{\psi}(k,l)}{4[\psi(k)-\psi(l)]^{\alpha}(l-k)}(|h'(k)|+|h'(l)|), \tag{3.2.2}$$

where $\mathcal{I}^{\alpha}_{\psi}(k,l) = \mathcal{L}^{\alpha}_{\psi}(l,l) + \mathcal{L}^{\alpha}_{\psi}(k,l) - \mathcal{L}^{\alpha}_{\psi}(l,k) - \mathcal{L}^{\alpha}_{\psi}(k,k)$.

If $\psi(x) = x$ in Eqn (3.2.2), we get Hermite-Hadamard type inequality for Riemann-Liouville fractional integral.

Corollary 3.2.5. *[19] If h is convex on $[k,l]$ and $\alpha > 0$, then*

$$\left|\frac{h(k)+h(l)}{2} - \frac{\Gamma(\alpha+1)}{2[l-k]^{\alpha}}[\mathcal{I}^{\alpha}_{k+}h(l) + \mathcal{I}^{\alpha}_{l-}h(k)]\right| \leq \frac{(l-k)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[h'(k)+h'(l)].$$

If $\psi(x) = \ln(x)$ in Eqn (3.2.2), we get Hermite-Hadamard inequality for Hadamard fractional integral.

Corollary 3.2.6. *[19] If $|h'|$ is convex on $[k,l]$ and $\alpha > 0$, then*

$$\left|\frac{h(k)+h(l)}{2} - \frac{\Gamma(\alpha+1)}{4[ln(\frac{l}{k})]^{\alpha}}[\mathcal{I}^{\alpha}_{k+}h(l) + \mathcal{I}^{\alpha}_{l-}h(k)]\right| \leq \frac{\mathcal{I}^{\alpha}_{ln}(k,l)}{4(\psi(l)-\psi(k))^{\alpha}(l-k)}|h'(k)|+|h'(l)|,$$

where $\mathcal{I}^{\alpha}_{ln}(k,l) = \mathcal{L}^{\alpha}_{ln}(l,l) + \mathcal{L}^{\alpha}_{ln}(k,l) - \mathcal{L}^{\alpha}_{ln}(l,k) - \mathcal{L}^{\alpha}_{ln}(k,k)$.

Raina's [34] introduced the function of bounded sequence of real (or complex) number.

Definition 3.2.7. [34] Let $\rho, \lambda > 0$ and $\sigma(m)$ be a bounded arbitrary sequence of real numbers where $m \in \mathbb{N}_{\circ}$ and $|y| < 1$.

$$\mathcal{Z}^{\sigma}_{\rho,\lambda}(y) = \mathcal{Z}^{\sigma(0),\sigma(1)\cdots}_{\rho,\lambda}(y) = \sum_{m=0}^{\infty}\frac{\sigma(m)}{\Gamma(\rho m+\lambda)}y^{m}.$$

Raina [34] also defined the following left and right-sided fractional integral operators.

Definition 3.2.8. [34] Let $\rho, \lambda > 0$ and $u \in \mathbb{R}$ then the fractional integral of left and right sided are defined as

$$\left(\mathcal{I}^{\sigma}_{\rho,\lambda,c+;u}\phi\right) y = \int_c^y (y-s)^{\lambda-1}\mathcal{Z}^{\sigma}_{\rho,\lambda}[u(y-s)^{\rho}]\phi(s)ds,$$

$$\left(\mathcal{I}^{\sigma}_{\rho,\lambda,d-;u}\phi\right) y = \int_y^d (s-y)^{\lambda-1}\mathcal{Z}^{\sigma}_{\rho,\lambda}[u(s-y)^{\rho}]\phi(s)ds.$$

Let $\lambda = \alpha$, $\sigma(0) = 1$ and $u = 0$ we get the classical Riemann-Liouville fractional integral. Set et al. [36] determine the Hermite-Hadamard type inequalities for above defined integrals as

Theorem 3.2.9. *[36] Let $\alpha, u \in \mathbb{R}$ and $h : [c,d] \to \mathbb{R}$ be a positive, twice differentiable function, and $h \in L(c,d)$. Also, let h'' be bounded on $[c,d]$. Then*

$$\frac{n}{2(d-c)^{\alpha}}\int_c^{\frac{c+d}{2}}\left(\frac{c+d}{2}-y\right)^2\left[(y-c)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(y-c)^{\rho}]+(l-y)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(y-c)^{\rho}]\right]dy$$
$$\leq \frac{1}{2(c-d)^{\alpha}}[(\tau^{\sigma}_{\rho,\alpha,c+;u}h)(d)+(\tau^{\sigma}_{\rho,\alpha,d-;u}h)(c)] - h\left(\frac{c+d}{2}\right)\mathcal{Z}^{\sigma}_{\rho,\alpha+1}[u(d-c)^{\rho}]$$
$$\leq \frac{N}{2(d-c)^{\alpha}}\int_c^{\frac{c+d}{2}}\left(\frac{c+d}{2}-y\right)^2[(y-c)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(y-c)^{\rho}]+(d-y)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(d-c)^{\rho}]]dy.$$

And

$$\frac{-N}{(d-c)^{\alpha}}\int_c^{\frac{c+d}{2}}(y-c)(d-y)[(y-c)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(y-c)^{\rho}]+(l-y)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(y-c)^{\rho}]]dy$$
$$\leq \frac{1}{2(d-c)^{\alpha}}[(\tau^{\sigma}_{\rho,\alpha,c+;u}h)(d)+(\tau^{\sigma}_{\rho,\alpha,d-;u}h)(c)] - \frac{h(c)+h(d)}{2}\mathcal{Z}^{\sigma}_{\rho,\alpha+1}[u(d-c)^{\rho}]$$
$$\leq \frac{-n}{2(d-c)^{\alpha}}\int_c^{\frac{c+d}{2}}(y-c)(d-y)[(y-c)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(y-c)^{\rho}]+(d-y)^{\alpha-1}\mathcal{Z}^{\sigma}_{\rho,\alpha}[u(d-c)^{\rho}]]dy.$$

Where $n = \inf_{y\in[c,d]} f''(y)$ and $N = \sup_{y\in[c,d]} h''(y)$.

Theorem 3.2.10. *[36] Let $\alpha, u \in \mathbb{R}_+$ and $h : [k,l] \to \mathbb{R}$ be a positive, differentiable function, and $h \in L(k,l)$. If $h\ (k+l-t) \geq h(t)$ for all $t \in [p, \frac{k+l}{2}]$, then we have*

$$h\left(\frac{k+l}{2}\right)\mathcal{Z}^{\rho}_{\sigma,\alpha+1}[u(l-k)^{\alpha}] \leq \frac{1}{2(l-k)^{\alpha}}[\mathcal{I}^{\sigma}_{\rho,\alpha,k+;u}h(l)+\mathcal{I}^{\sigma}_{\rho,\alpha,l-;u}h(k)]$$
$$\leq \frac{h(k)+h(l)}{2}\mathcal{Z}^{\sigma}_{\rho,\alpha+1}[u(l-k)^{\alpha}].$$

For $\sigma(0) = 1$ and $u = 0$ in Theorem (3.2.10) then we get following:

Corollary 3.2.11. *[36] Let $h : [k,l] \to \mathbb{R}$ be a positive and differentiable function and $h \in L(k,l)$. If $h'(k+l-z) \geq h(z)$ for all $z \in [k, \frac{k+l}{2}]$ then the following inequality holds.*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(l-k)^{\alpha}}[\mathcal{I}^{\alpha}_{k+}h(l) + \mathcal{I}^{\alpha}_{l-}h(k)] \leq \frac{h(k)+h(l)}{2}.$$

Chen and Katugampola [8] generalized the Hermite-Hadamard inequality for so called Katugampola fractional integral they also generalized result of Jelli [20]

Theorem 3.2.12. *[8] Let $\alpha, \rho > 0$ and $h : [p^{\rho}, q^{\rho}] \to \mathbb{R}$ be a positive function with $0 \leq p \leq q$ and $h \in X^{c}_{p}(p^{\rho}, q^{\rho})$. If h is convex function on $[p,q]$, then*

$$h\left(\frac{p^{\rho}+q^{\rho}}{2}\right) \leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(p^{\rho}-q^{\rho})^{\alpha}}[\mathcal{I}^{\alpha,\rho}_{p}h(q^{\rho}) + \mathcal{I}^{\alpha,\rho}_{q-}h(p^{\rho})] \leq \frac{h(p^{\rho})+h(q^{\rho})}{2}.$$

Theorem 3.2.13. *[8] Let $h : [c^{\rho}, d^{\rho}] \to \mathbb{R}$ be a differentiable mapping with $0 \leq c \leq d$. If h' is differentiable on (c^{ρ}, d^{ρ}), then we have*

$$\left|\frac{h(c^{\rho})+h(d^{\rho})}{2} - \frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2(d^{\rho}-c^{\rho})^{\alpha}}[\mathcal{I}^{\alpha,\rho}_{c+}h(d^{\rho}) + \mathcal{I}^{\alpha,\rho}_{d-}h(c^{\rho})]\right| \leq \frac{(d^{\rho}-c^{\rho})^{2}}{2(\alpha+1)(\alpha+2)}(\alpha+\frac{1}{2^{\alpha}}) \sup_{\xi\in[c^{\rho},d^{\rho}]} |h''(\xi)|.$$

Theorem 3.2.14. *[8] Let $h : [c^{\rho}, d^{\rho}] \to \mathbb{R}$ be a differentiable mapping. If $|h'|$ is convex on $[c^{\rho}, d^{\rho}]$, then the following inequality holds:*

$$\left|\frac{h(c^{\rho})+h(d^{\rho})}{2} - \frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2(d^{\rho}-c^{\rho})}[\mathcal{I}^{\alpha,\rho}_{c+}h(d^{\rho}) + \mathcal{I}^{\alpha}_{d-}h(c^{\rho})]\right| \leq \frac{(d^{\rho}-c^{\rho})}{2(\alpha+1)}[|h'(c^{\rho})|+|h'(d^{\rho})|].$$

Lemma 3.2.15. *Let $h : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ be a differentiable mapping on (a^{ρ}, b^{ρ}) with $0 \leq a \leq b$. Then the following equality holds, if the fractional integrals exist.*

$$\begin{aligned}\frac{h(c^{\rho})+h(d^{\rho})}{2} &- \frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2(d^{\rho}-c^{\rho})}[\mathcal{I}^{\alpha,\rho}_{c+}h(d^{\rho}) + \mathcal{I}^{\alpha}_{d-}h(c^{\rho})] \\ &= \frac{d^{\rho}-c^{\rho}}{2}\int_{0}^{1}|(1-t^{\rho})^{\alpha} - t^{\rho\alpha}|t^{\rho-1}h'(t^{\rho}c^{\rho}-(1-t^{\rho})d^{\rho})dt.\end{aligned}$$

By using Lemma (3.2.15) Chen and Katugampola [19] found the more strict inequality.

Theorem 3.2.16. *[8] Let $h : [c^{\rho}, d^{\rho}] \to \mathbb{R}$ be a differentiable mapping. If $|h'|$ is convex on $[c^{\rho}, d^{\rho}]$, then we have*

$$\left|\frac{h(c^{\rho})+h(d^{\rho})}{2} - \frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2(d^{\rho}-c^{\rho})^{\alpha}}[\mathcal{I}^{\alpha,\rho}_{c+}h(d^{\rho}) + \mathcal{I}^{\alpha,\rho}_{d-}h(c^{\rho})]\right| \leq \frac{d^{\rho}-c^{\rho}}{2\rho(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[|h'(c^{\rho})|+|h'(d^{\rho})|].$$

As Katugampola generalized the Riemann-Liouville fractional integral and Hadamard fractional integral, so the above result are also true for them.

Liu et al. [26] represented their work on Hermite-Hadamard and Hermite-Hadamard type inequality for the Riemann-Liouville fractional integral of a function with respect to another function. They also provided application to special means of real numbers.

Theorem 3.2.17. *[26] Let $h : [k, l] \to \mathbb{R}$ be a differentiable mapping. Also suppose that $|h'|$ is convex on $[k, l]$. $\psi(x)$ is monotonically increasing function having a continuous derivative and $\alpha \in (0, 1)$. Then we have*

$$\left| \frac{h(k) + h(l)}{2} - \frac{\Gamma(\alpha + 1)}{2(l - k)^{\alpha}} [\mathcal{I}^{\alpha}_{\psi^1(k)}(h \circ \psi)(\psi^{-1}(l)) + \mathcal{I}^{\alpha}_{\psi^1(l)}(h \circ \psi)(\psi^{-1}(k))] \right| \leq \frac{l - k}{2(\alpha + 1)} \left(1 - \frac{1}{2^{\alpha}}\right) [|h'(k)| + |h'(l)|].$$

Liu et al. [26] also give application to the special means of real numbers.

Proposition 3.2.18. *If $h(z) = z^2$, $\alpha = 1$, and $\psi(z) = z$ in Theorem (3.2.17), then*

$$\left| A(c^2, d^2) - L_2^2(c, d) \right| \leq \frac{d^2 - c^2}{4},$$

where $c, d \in \mathbb{R}+$, and $c < d$.

Proposition 3.2.19. *Let $c, d \in \mathbb{R}+$, $c < d$. If $h(z) = z^m$, $\alpha = 1$ and $\psi(z) = z$ in Theorem (3.2.17), Then*

$$|A(c^m, d^m) - L_m^m(c, d)| \leq \frac{d^m - c^m}{4}(mc^{m-1} + md^{m-1}),$$

where $c, d \in \mathbb{R}+$, and $c < d$.

Proposition 3.2.20. *If $h(z) = e^z$, $\alpha = 1$ and $\psi(z) = z$ in Theorem (3.2.17). Then*

$$\left| H(e^c, e^d) - L(e^c, e^d) \right| \leq \frac{d - c}{8}(e^c + e^d),$$

where $c, d \in \mathbb{R}+$, and $c < d$.

Proposition 3.2.21. *If $h(z) = \frac{1}{z}$, $\alpha = 1$ and $\psi(z) = z$ in Theorem (3.2.17). Then*

$$\left| H^{-1}(c, d) - L^{-1}(c, d) \right| \leq \frac{d - c}{8} \left(\frac{1}{c^2} + \frac{1}{d^2} \right),$$

where $c, d \in \mathbb{R}+$, and $c < d$.

Theorem 3.2.22. *[26] Let $\alpha \in (0, 1)$ and $h : [k, l] \to \mathbb{R}$ be a differentiable mapping. Also suppose that $|h'|$ is convex on $[k, l]$. $\psi(z)$ is an increasing and positive monotone function having a continuous derivative $\psi'(z)$ and $h \in L_1[k, f]$, the following inequality holds:*

$$\left| \frac{\Gamma(\alpha + 1)}{2(l - k)^{\alpha}} [\mathcal{I}^{\alpha}_{\psi^1(k)}(h \circ \psi)(\psi^{-1}(l)) + \mathcal{I}^{\alpha}_{\psi^1(l)}(h \circ \psi)(\psi^{-1}(k))] - h\left(\frac{l + k}{2}\right) \right| \leq \frac{|h(l) - h(k)|}{2} + \frac{l - k}{2(\alpha + 1)} \left(1 - \frac{1}{2^{\alpha}}\right) [|h'(k)| + |h'(l)|].$$

Proposition 3.2.23. *Let $c, d \in \mathbb{R}+$, $c < d$. If $h(z) = \frac{1}{z}$, $\alpha = 1$ and $\psi(z) = z$ in Theorem (3.2.22). Then*

$$\left|L^{-1}(c,d) - A^{-1}(c,d)\right| \leq \frac{d-c}{8}\left(4 + \frac{1}{c^2} + \frac{1}{d^2}\right).$$

Fernandez and Mohammed [15] used the Mittag-Leffler kernels to establish the Hermite-Hadamard inequalities for following fractional operators.

Definition 3.2.24. [1] Let h be a function that is differentiable with L^1 derivative on an interval $[k, l]$ and for any $y \in [k, l]$. The vth Atangana-Baleanu AB fractional derivative of $h(y)$ is defined for $0 < v < 1$.

$$^{\mathcal{ABR}}\mathcal{D}_{k+}^{v}h(y) = \frac{B(v)}{1-v}\frac{d}{dx}\int_k^x E_v\left(\frac{-v}{1-v}(x-\mu)^v\right)h(\mu)d\mu,$$

$$^{\mathcal{ABC}}\mathcal{D}_{k+}^{v}h(y) = \frac{B(v)}{1-v}\int_k^y E_v\left(\frac{v}{1-v}(y-\mu)^v\right)h'(\mu)d\mu,$$

where $E_v(z)$ is the standard $1-$parameter Mittag-Leffler function and $B(v)$ is a normalisation function that is real and positive and satisfies $B(0) = B(1) = 1$.

Definition 3.2.25. [1] Let $h \in L^1$ on an interval $[k, l]$, and for any $z \in [k, l]$, the vth AB fractional integral of $h(x)$ is defined as follows for $0 < v < 1$

$$^{\mathcal{AB}}\mathcal{I}_{k+}^{v}h(z) = \frac{v}{B(v)}\,^{\mathcal{RL}}\mathcal{I}_{k+}^{v}h(z) + \frac{1-v}{B(v)}h(z).$$

Definition 3.2.26. [32] Let $h \in L^1$ on an interval $[k, l]$ and for any $y \in [k, l]$, $\alpha, \beta > 0$ and $\gamma, \mu \in \mathbb{C}$ the Prabhakar fractional integral operator is defined as

$$\mathcal{I}_{k+}^{\alpha,\beta,\gamma,\mu}h(y) = \int_y^k (y-\mu)^{\beta-1}E_{\gamma,\alpha,\beta}(z(y-\mu)^\alpha)h(\mu)d\mu,$$

where $E_{\gamma,\alpha,\beta}(z)$ is the $3-$parameter Mittag-Leffler function.

Theorem 3.2.27. *[15] If $h : [k, l] \to \mathbb{R}$ is L^1 and convex, $\alpha \in (0, 1)$, then we have the following Hermite-Hadamard inequality for AB fractional integrals*

$$h\left(\frac{k+l}{2}\right) \leq \frac{B(\alpha)\Gamma(\alpha)}{2[(l-k)^\alpha + (1-\alpha)\Gamma(\alpha)]}[^{\mathcal{AB}}\mathcal{I}_{k+}^{\alpha}h(l) + {}^{\mathcal{AB}}\mathcal{I}_{l-}^{\alpha}h(k)] \leq \frac{h(k)+h(l)}{2}.$$

Theorem 3.2.28. *[15] If $h : [k, l] \to \mathbb{R}$ is L_1 and convex, and $\alpha \in (0, 1)$, then we have the following Hermite-Hadamard inequality for AB fractional derivatives*

$$\begin{aligned}\mathcal{T}_1(\alpha, l-k)h\left(\frac{k+l}{2}\right) &+ \mathcal{T}_2(\alpha, l-k)\frac{h(k)+h(l)}{2} \\ &\leq \frac{1-\alpha}{2B(\alpha)E_\alpha\left(\frac{-\alpha}{1-\alpha}(l-k)^\alpha\right)}[^{\mathcal{ABR}}\mathcal{D}_{k+}^{\alpha}h(l) + {}^{\mathcal{ABR}}\mathcal{D}_{l}^{\alpha}h(k)] \\ &\leq \mathcal{T}_2(\alpha, l-k)h\left(\frac{k+l}{2}\right) + \mathcal{T}_1(\alpha, l-k)\frac{h(k)+h(l)}{2},\end{aligned}$$

where the multipliers $\mathcal{T}_1(\alpha, q-p)$ and $\mathcal{T}_2(\alpha, l-k)$ sum to 1. Thus forming weighted averages on both ends of the inequality. Specifically, these multipliers are defined as follows:

$$\mathcal{T}_1(\alpha, l-k) = \frac{E_{2\alpha}((\frac{\alpha}{1-\alpha})^2(l-k)^{2\alpha})}{E_\alpha(\frac{-\alpha}{1-\alpha(l-k)\alpha})};$$
$$\mathcal{T}_2(\alpha, l-k) = \frac{\frac{-\alpha}{1-\alpha}(l-k)^\alpha E_{2\alpha,\alpha+1}((\frac{-\alpha}{1-\alpha})(l-k)^{2\alpha})}{E_\alpha(\frac{-\alpha}{1-\alpha}(l-k)^\alpha)}.$$

Theorem 3.2.29. *[15] If $h : [k, l] \to \mathbb{R}$ is L^1 and convex. The parameters $\alpha, \beta, \gamma, \mu$, are all real and positive, then we have the following Hermite-Hadamard inequality for Prabhakar fractional integrals:*

$$h\left(\frac{k+l}{2}\right) \leq \frac{{}^{\mathcal{P}}\mathcal{I}_{k+}^{\alpha,\beta,\gamma,\mu}h(l) + {}^{\mathcal{P}}\mathcal{I}_{l-}^{\alpha,\beta,\gamma,\mu}h(k)}{2(l-k)^\beta E_{\alpha,\beta+1}^{\gamma}(\mu(l-k)\alpha)} \leq \frac{h(k)+h(l)}{2}.$$

Gurbuz et al. [16] generalized the Hermite-Hadamard inequality via Caputo-Fabrizo integral.

Definition 3.2.30. [6, 2] Let $h \in H_1(k, l)$, $m < n$ and $\alpha \in [0, 1]$, then the definition of the left fractional derivative in the sense of Caputo and Fabrizio is

$$\mathcal{D}_k^{\alpha,*}h(y) = \frac{B(\alpha)}{1-\alpha}\int_k^y h'(u)e^{\frac{-\alpha(y-u)^\alpha}{1-\alpha}}du.$$

And related fractional integral is

$$\mathcal{I}_k^{\alpha,*}h(y) = \frac{1-\alpha}{B(\alpha)}h(y) + \frac{\alpha}{B(\alpha)}\int_k^y h(u)du.$$

The Hermite-Hadamard inequality for Caputo-Fabrizio fractional integral is stated as

Theorem 3.2.31. *[16] Let a function $h : [k, l] \subset \mathbb{R} \to \mathbb{R}$ be convex and $h \in L_1[k, l]$. If $\alpha \in [0, 1]$, then the following double inequality holds:*

$$h\left(\frac{k+l}{2}\right) \leq \frac{B(\alpha)}{\alpha(l-k)}\left[(\mathcal{I}_k^{\alpha,*}h)(a) + (\mathcal{I}_l^{\alpha,*}h)(a) - \frac{2}{(1-\alpha)}B(\alpha)h(a)\right] \leq \frac{h(k)+h(l)}{2},$$

where $a \in [k, l]$ and $B(\alpha) > 0$ is a normalization function.

Theorem 3.2.32. *[16] Let $h, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. If $hg \in L([k, l])$, then we have the following inequality:*

$$\frac{2B(\alpha)}{\alpha(l-k)}\left[\mathcal{I}_m^{\alpha,*}(hg)(a) + \mathcal{I}_n^{\alpha,*}(hg)(a) - \frac{2(1-\alpha)}{B(\alpha)}h(a)g(a)\right] \leq \frac{2}{3}M(k, l) + \frac{1}{3}N(k, l),$$

where $M(k, l) = h(k)g(k)+h(l)g(l)$, $N(k, l) = h(k)g(l)+h(l)g(k)$, and $a \in [k, l]$, $B(\alpha) > 0$ is a normalization function.

Theorem 3.2.33. *[16] Let $h, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. If $hg \in ([k,l])$, the set of integrable functions, then*

$$2h\left(\frac{k+l}{2}\right)g\left(\frac{k+l}{2}\right)-\frac{1}{l-k}[\mathcal{I}_k^{\alpha,*}(hg)(a)+\mathcal{I}_l^{\alpha,*}(hg)(a)]+\frac{1-\alpha}{\alpha(l-k)}h(k)g(k)$$
$$\leq \frac{2}{3}M(k,l)+\frac{4}{3}N(k,l),$$

where $M(k,l)$ and $N(k,l)$ are given in Theorem (3.2.32) *and $a \in [k,l]$, $B(\alpha) > 0$ is a normalization function.*

Theorem 3.2.34. *[16] Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable positive mapping and $|h'|$ be convex. If $h' \in L1[k,l]$ and $\alpha[0,1]$, the following inequality holds:*

$$\left|\frac{h(k)+h(l)}{2}+\frac{2(1-\alpha)}{\alpha(l-k)}h(k)-\frac{B(\alpha)}{\alpha(l-k)}[(\mathcal{I}_k^{\alpha,*}h)(a)+(\mathcal{I}_l^{\alpha,*}h)(a)]\right| \leq \frac{(l-k)(|h'(k)|+|h'(l)|)}{8},$$

where $a \in [k,l]$ and $B(\alpha) > 0$ is a normalization function.

Gurbuz et al. [16] also give applications to the special means.

Proposition 3.2.35. *Let $c, d \in \mathbb{R}+$, $c < d$. Then*

$$\left|A(c^2,d^2)-L_2^2(c,d)\right| \leq \frac{d-c}{4}[|c|+|d|],$$

provided $h(z) = z^2$, $\alpha = 1$ and $B(\alpha) = B(1) = 1$ in Theorem (3.2.34).

Proposition 3.2.36. *Let $c, d \in \mathbb{R}+$, $c < d$. Then*

$$\left|A(e^c,e^d)-L_2^2(e^c,e^d)\right| \leq \frac{d-c}{8}(e^c+e^d).$$

provided $h(z) = e^z$, $\alpha = 1$ and $B(\alpha) = B(1) = 1$ in Theorem (3.2.34).

Proposition 3.2.37. *Let $c, d \in \mathbb{R}+$, $c < d$. Then*

$$|A(c^m,d^m)-L_m^m(c,d)| \leq \frac{m(d-c)}{4}[|c^{m-1}|+|d^{m-1}|].$$

provided $h(z) = z^m$, $\alpha = 1$ and $B(\alpha) = B(1) = 1$ in Theorem (3.2.34).

Theorem 3.2.38. *[16] Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable positive mapping and $|h'|$ be convex on $[k,l]$ where $q > 1$. If $|h'| \in L_1[k,l]$ and $\alpha \in [0,1]$, the following inequality holds:*

$$\left|\frac{h(k)+h(l)}{2}+\frac{2(1-\alpha)}{\alpha(f-k)}h(k)-\frac{B(\alpha)}{\alpha(l-k)}[(\mathcal{I}_k^{\alpha,*}h)(a)+(\mathcal{I}_l^{\alpha,*}h)(a)]\right|$$
$$\leq \frac{l-k}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(|h'(k)|^q+|h'(l)|^q)^{\frac{1}{q}},$$

where $a \in [k,l]$ and $B(\alpha) > 0$ is a normalization function.

Sarikaya et al. [38] introduced the new definition for fractional integral and then established the Hermite-Hadamard type integral inequalities.

Definition 3.2.39. [38] Let $v : [k, l] \to R$ be an monotonically increasing function and $h, v \in L[k, l]$. The generalized Riemann-Liouville fractional integrals is defined as

$$\mathcal{I}^{\alpha,q}_{k+,v}h(z) = \frac{1}{\Gamma(\alpha)} \int_k^z (z - v)^{\alpha-1}(v(z) - v(u))^q h(u)du,$$

and

$$\mathcal{I}^{\alpha,q}_{l-,v}h(z) = \frac{1}{\Gamma(\alpha)} \int_z^l (u - z)^{\alpha-1}(v(u) - v(z))^q h(u)du.$$

Theorem 3.2.40. *[38] Let $h : [k, l] \to \mathbb{R}$ be a convex function and $u : [k, l] \to \mathbb{R}$ be an monotonically increasing function on (k, l), and $h, v \in L[k, l]$. Then H is also integrable and the following inequalities for fractional integral holds:*

$$\begin{aligned} h\left(\frac{k+l}{2}\right)[\mathcal{I}^{\alpha,q}_{k+,v}(1)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(1)(k)] &\leq \frac{1}{2}[\mathcal{I}^{\alpha,q}_{k+,v}H(l) + \mathcal{I}^{\alpha,q}_{l-,v}H(k)] \\ &\leq [\mathcal{I}^{\alpha,q}_{k+,v}(1)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(1)(k)]\frac{h(k) + h(l)}{2}. \end{aligned} \tag{3.2.3}$$

Corollary 3.2.41. *If $q = 0$, then Eqn (3.2.3) reduce to classical Hermite-Hadamard inequality.*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(l-k)^\alpha}[\mathcal{I}^{\alpha}_{k+}h(l) + \mathcal{I}^{\alpha}_{l-}h(k)] \leq \frac{h(k) + h(l)}{2}. \tag{3.2.4}$$

Corollary 3.2.42. *If $v(t) = t$, then we get following inequality for Riemann-Liouville fractional integral*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+q+1)}{4(l-k)^\alpha}[\mathcal{I}^{\alpha+q}_{k+,v}H(l) + \mathcal{I}^{\alpha+q}_{l-,v}H(k)] \leq \frac{h(k) + h(l)}{2}.$$

Lemma 3.2.43. *[38] Let $h : [k, l] \to \mathbb{R}$ be a differentiable function on (k, l) and $v : [k, l] \to \mathbb{R}$ be an monotonically increasing function. If $h', v \in L[k, l]$ then H is also differentiable and $H \in L[k, l]$, then following equality holds:*

$$\begin{aligned} [\mathcal{I}^{\alpha,q}_{k+,v}(1)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(1)(k)]\frac{h(k) + h(l)}{2} &- \frac{1}{2}[\mathcal{I}^{\alpha,q}_{k+,v}(H)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(H)(k)] \\ &= \frac{(l-k)^\alpha}{2\Gamma(\alpha)} \int_k^l G(z)F'(z)dz, \end{aligned}$$

where $H'(z) = h'(z) - h'(k + l - z)$ and
$G(z) = \int_0^{\frac{y-k}{l-k}} s^{\alpha-1}(v(b) - v(sl + (1-s)k))^q ds + \int_{\frac{y-k}{l-k}}^1 (1-s)^{\alpha-1}(v(sk + (1-s)l) - v(k))^q ds.$

Theorem 3.2.44. *[38] Let $h : [k,l] \to \mathbb{R}$ is a differentiable function and $v : [k,l] \to \mathbb{R}$ be an monotonically increasing function. Then H is also differentiable and $H \in L[k,l]$. If $|h'|$ is convex on $[k,l]$ and $h', v \in L[k,l]$ then the following inequality holds:*

$$\left|[\mathcal{I}^{\alpha,q}_{k+,v}(1)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(1)(k)]\frac{h(k)+h(l)}{2(l-k)^{\alpha}} - \frac{1}{2(l-k)^{\alpha}}[\mathcal{I}^{\alpha,q}_{k+,v}(H)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(H)(k)]\right|$$
$$\leq \frac{1}{2\Gamma(\alpha)}\int_k^l |G(z)|dz(|h'(k)|+|h'(l)|).$$

Corollary 3.2.45. *Let $h : [k,l] \to \mathbb{R}$ with $k < l$, if $|h'|$ is convex then*

$$\left|\frac{h(k)+h(l)}{2} - \frac{\Gamma(\alpha+1)}{2(l-k)^{\alpha}}[\mathcal{I}^{\alpha}_{k+}h(l) + \mathcal{I}^{\alpha}_{l-}h(k)]\right| \leq \frac{l-k}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[h'(k)+h'(l)],$$

provided $q = 0$ then $G(z) = \frac{(y-k)^{\alpha}+(l-y)^{\alpha}}{\alpha(l-k)^{\alpha}}$

Corollary 3.2.46. *Let $v : [k,l] \to \mathbb{R}$, if $|v|$ is convex on $[k,l]$ then the following inequality holds:*

$$\left|[\mathcal{I}^{\alpha,q}_{k+,v}(1)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(1)(k)]\frac{h(k)+h(l)}{2(l-k)^{\alpha}} - \frac{1}{2(l-k)^{\alpha}}[\mathcal{I}^{\alpha,q}_{k+,v}(H)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(H)(k)]\right|$$
$$\leq \frac{(l-k)}{\Gamma(\alpha+2)}[|v(k)|+|v(l)|](|h'(k)|+|h'(l)|),$$

provided $q = 1$ then $G(z) = |v(a)|\left[\frac{(z-k)^{\alpha}-(l-z)^{\alpha}}{(\alpha+1)(l-k)^{\alpha+1}} + \frac{2(y-k)^{\alpha}}{\alpha(l-k)^{\alpha}}\right] + |v(b)|\left[\frac{-(z-k)^{\alpha}+(l-z)^{\alpha}}{(\alpha+1)(l-k)^{\alpha+1}} + \frac{2(y-k)^{\alpha}}{\alpha(l-k)^{\alpha}}\right].$

Corollary 3.2.47. *If $v(t) = t$ then*

$$\left|\frac{2(l-k)^{\alpha+q}}{(\alpha+q)\Gamma(\alpha)}\frac{h(k)+h(l)}{2(l-k)^{\alpha}} - \frac{\Gamma(\alpha+q)}{2\Gamma(\alpha)(l-k)^{\alpha}}[\mathcal{I}^{\alpha,q}_{k+,v}(H)(l) + \mathcal{I}^{\alpha,q}_{l-,v}(H)(k)]\right|$$
$$\leq \frac{(l-k)^{\alpha+q}}{(\alpha+q)(\alpha+q+1)\Gamma(\alpha)}(|h'(k)|+|h'(l)|).$$

We generalized the result of [38] for Generalized Riemann-Liouville fractional integral with respect to a function .

3.3 Generalized Riemann-Liouville fractional integral with respect to a function

We introduce the generalize Riemann-Liouville fractional integral with respect to a function by using the same technique as Sarikaya et al. [38] introduced.

Definition 3.3.1. Let $v : [k,l] \to R$ be an monotonically increasing function on (k,l) and $h, v \in L[k,l]$ with $k < l$. The we define the generalized Riemann-Liouville fractional integral with respect to a function as:

$$\mathcal{I}_{k+,v}^{\alpha,\psi,q} h(z) = \frac{1}{\Gamma(\alpha)} \int_k^z (\psi(z) - \psi(u))^{\alpha-1} (v(z) - v(u))^q h(u) \psi'(u) du,$$

and

$$\mathcal{I}_{l-,v}^{\alpha,\psi,q} h(z) = \frac{1}{\Gamma(\alpha)} \int_z^l (\psi(u) - \psi(z))^{\alpha-1} (v(u) - v(z))^q h(u) \psi'(u) du.$$

Example 3.3.2. Let $v(u) = w(u)$ and $h(u) = 1$ then we get

$$\mathcal{I}_{k+,v}^{\alpha,\psi,q}(1)(z) = \frac{1}{\Gamma(\alpha)} \int_k^z (\psi(z) - \psi(u))^{\alpha+q-1} \psi'(u) du = \frac{(\psi(z) - \psi(k))^{\alpha+q}}{(\alpha+q)\Gamma(\alpha)}.$$

Similarly

$$\mathcal{I}_{l-,v}^{\alpha,\psi,q}(1)(z) = \frac{(\psi(l) - \psi(z))^{\alpha+q}}{(\alpha+q)\Gamma(\alpha)}.$$

We have to prove Hermite-Hadamard type inequality for new generalized fractional integral

Theorem 3.3.3. *Let $h : [k,l] \to \mathbb{R}$ be a convex function and $v : [k,l] \to \mathbb{R}$ be an monotonically increasing function on (k,l). $h, v \in L[k,l]$ with $k < l$ then H is also integrable, and*

$$\begin{aligned} h\left(\frac{k+l}{2}\right) [\mathcal{I}_{k+,v}^{\alpha,\psi,q}(1)(l) + \mathcal{I}_{l-,v}^{\alpha,\psi,q}(1)(k)] &\leq \frac{1}{2} [\mathcal{I}_{k+,v}^{\alpha,\psi,q} H(l) + \mathcal{I}_{l-,v}^{\alpha,\psi,q} H(k)] \\ &\leq [\mathcal{I}_{k+,v}^{\alpha,\psi,q}(1)(l) + \mathcal{I}_{l-,v}^{\alpha,\psi,q}(1)(k)] \frac{h(k) + h(l)}{2}. \end{aligned} \tag{3.3.1}$$

Proof. As h is a convex function, for $u_1 = \frac{1}{2}$. Let $x_1 = (u_1 k + (1-u_1)l)$ and $x_2 = ((1-u_1)k + u_1 l)$ where $x_1, x_2 \in [k,l]$ we have

$$2h\left(\frac{k+l}{2}\right) \leq h(u_1 k + (1-u_1)l) + h((1-u_1)k + u_1 l).$$

Multiplying $(\psi(l) - \psi((1-u_1)k + u_1 l))^{\alpha-1} (v(l) - v((1-u_1)k + u_1 l))^k \psi'((1-u_1)k + u_1 l)$, we have

$$2h\left(\frac{k+l}{2}\right) \int_0^1 (\psi(l) - \psi((1-u_1)k + u_1 l))^{\alpha-1} (v(l) - v((1-u_1)k + u_1 l))^k \psi'((1-u_1)k + u_1 l) du_1$$

$$\leq \int_0^1 (\psi(l) - \psi((1-u_1)k + u_1 l))^{\alpha-1} (v(l) - v((1-u_1)k + u_1 l))^k \psi'((1-u_1)k + u_1 l h(u_1 k + (1-u_1)l) d$$

$$+ \int_0^1 (\psi(l) - \psi((1-u_1)k + u_1 l))^{\alpha-1} (v(l) - v((1-u_1)k + u_1 l))^k \psi'((1-u_1)k + u_1 l) h((1-u_1)k + u_1 l) d$$

Let $z = (1-u_1)k + u_1 l$, then we have

$$2h\left(\frac{k+l}{2}\right)\mathcal{I}^{\alpha,\psi,q}_{k+,v}(1)(l) \leq \mathcal{I}^{\alpha,\psi,q}_{k+,v}(h^*)(l) + \mathcal{I}^{\alpha,\psi,q}_{k+,v}(h)(l). \tag{3.3.2}$$

Similarly multiplying $(\psi(u_1k + (1-u_1)l) - \psi(k))^{\alpha-1}(v(u_1k + (1-u_1)l) - v(k))^k\psi'((1-u_1)l + u_1k)$, and let $z_1 = (1-u_1)l + u_1k$ we have

$$2h\left(\frac{k+l}{2}\right)\mathcal{I}^{\alpha,\psi,q}_{l-,v}(1)(k) \leq \mathcal{I}^{\alpha,\psi,q}_{l-,v}(h^*)(k) + \mathcal{I}^{\alpha,\psi,q}_{l-,v}(h)(k). \tag{3.3.3}$$

Add (3.3.2) and (3.3.3)

$$2h\left(\frac{k+l}{2}\right)[\mathcal{I}^{\alpha,\psi,q}_{k+,v}(1)(l) + \mathcal{I}^{\alpha,\psi,q}_{l-,v}(1)(k)] \leq \mathcal{I}^{\alpha,\psi,q}_{l-,v}(H)(k) + \mathcal{I}^{\alpha,\psi,q}_{k+,v}(H)(l).$$

Where $H(k) = h(k) + h^*(k)$, for second inequality since h is convex then, $h((1-u_1)k + u_1l) + h(u_1k + (1-u_1)l) \leq h(k) + h(l)$. Multiplying $(\psi(l) - \psi((1-u_1)l + u_1k))^{\alpha-1}(v(l) - v((1-u_1)l + u_1k))^k\psi'((1-u_1)l + u_1k)$, we have

$$\begin{aligned}
&\int_0^1 (\psi(l) - \psi((1-u_1)l + u_1k))^{\alpha-1}(v(l) - v((1-u_1)l + u_1k))^k\psi'((1-u_1)l + u_1k)h((1-u_1)l + u_1k) \\
&+ \int_0^1 (\psi(l) - \psi((1-u_1)l + u_1k))^{\alpha-1}(v(l) - v((1-u_1)l + u_1k))^k\psi'((1-u_1)l + u_1k)h(u_1l + (1-u_1)k) \\
&\leq \int_0^1 (\psi(l) - \psi((1-u_1)l + u_1k))^{\alpha-1}(v(l) - v((1-u_1)l + u_1k))^k\psi'((1-u_1)l + u_1k)[h(k) + h(l)].
\end{aligned}$$

Let $y = (1-u_1)l + u_1k$, we get

$$\mathcal{I}^{\alpha,\psi,q}_{k+,v}(h)(l) + \mathcal{I}^{\alpha,\psi,q}_{k+,v}(h^*)(l) \leq \mathcal{I}^{\alpha,\psi,q}_{k+,v}(1)(l)[h(k) + h(l)].$$

$$\mathcal{I}^{\alpha,\psi,q}_{k+,v}(H)(l) \leq \mathcal{I}^{\alpha,\psi,q}_{k+,v}(1)(l)[h(k) + h(l)]. \tag{3.3.4}$$

Similarly multiplying $(\psi((1-u_1)k + u_1l) - \psi(k))^{\alpha-1}(v((1-u_1)k + u_1l) - v(k))^k\psi'((1-u_1)k + u_1l)$, and let $x_1 = (1-u_1)k + u_1l$

$$\begin{aligned}
&\int_k^l (\psi(x_1) - \psi(k))^{\alpha-1}(v(l) - v(x_1))^k\psi'(x_1)h(k + l - x_1)dx_1 \\
&+ \int_k^l (\psi(x-1) - \psi(k))^{\alpha-1}(v(l) - v(x_1))^k\psi'(x_1)h(x_1)dx_1 \\
&\leq \int_k^l (\psi(x_1) - \psi(k))^{\alpha-1}(v(l) - v(x_1))^k\psi'(x_1)dx_1[h(k) + h(l)]
\end{aligned}$$

$$\mathcal{I}^{\alpha,\psi,q}_{l-,v}(h)(k) + \mathcal{I}^{\alpha,\psi,q}_{l-,v}(h^*)(k) \leq \mathcal{I}^{\alpha,\psi,q}_{l-,v}(1)(k)[h(k) + h(l)]$$

$$\mathcal{I}_{l-,v}^{\alpha,\psi,q}(H)(k) \leq \mathcal{I}_{l-,v}^{\alpha,\psi,q}(1)(k)[h(k)+h(l)]. \tag{3.3.5}$$

Add (3.3.4) and (3.3.5)

$$\mathcal{I}_{l-,v}^{\alpha,\psi,q}(H)(k) + \mathcal{I}_{k+,v}^{\alpha,\psi,q}(H)(l) \leq \mathcal{I}_{l-,v}^{\alpha,\psi,q}(1)(k) + \mathcal{I}_{k+,v}^{\alpha,\psi,q}(1)(l)[h(k)+h(l)].$$

We get our desired result. □

Corollary 3.3.4. *If $q = 0$ then Eqn (3.3.1) reduce to Hermite-Hadamard inequality for fractional integral with respect to another function.*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4(\psi(l)-\psi(k))^{\alpha}}[\mathcal{I}_{k+}^{\alpha,\psi}H(l) + \mathcal{I}_{l-}^{\alpha,\psi}H(k)] \leq \frac{h(k)+h(l)}{2}. \tag{3.3.6}$$

Corollary 3.3.5. *If $v(t) = t$ and $\psi(t) = t$ then Eqn (3.3.1) reduce to Eqn (3.2.2)*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+q+1)}{4(l-k)^{\alpha}}[\mathcal{I}_{k+,v}^{\alpha+q}H(l) + \mathcal{I}_{l-,v}^{\alpha+q}H(k)] \leq \frac{h(k)+h(l)}{2}.$$

Chapter 4

Generalized Hermite-Hadamard inequality

Hadamard-type fractional integral with respect to a function cover both the Hadamard and tempered fractional calculus as a special case. It also generalize some integral such as Riemann-Liouville fractional integral, Hadamard fractional integral, Katugampola fractional integral, Riemann-Liouville fractional integral with respect to a function. As the Hermite-Hadamard inequality is the most well known inequality in literature, and due to its wide applications. we emphasize to establish the Hermite-Hadamard inequality for the Hadamard-type fractional integral with respect to a function which contain the Hermite-Hadamard inequality for different fractional integrals.

4.1 Main Results

In this section, we generalize Hermite-Hadamard inequality and Hermite-Hadamard type inequalities. Several special cases are also presented. For the sake of simplicity we assume $h : I := [k, l] \to \mathbb{R}$ is twice differentiable increasing convex functions and we define function g as $g(y) = \phi^p(y)h(y)$. Also, define $g^*(y) = g(k + l - y)$, $h^*(y) = h(k + l - y)$, $G(y) = h(y) + h^*(y)$ and $H(y) = g(y) + g^*(y)$. The substitution $t = \frac{s-k}{y-k}$ in (1.3.44) leads to

$$\mathcal{I}_{k_+}^{\alpha,\phi}h(y) = \frac{(y-k)}{\Gamma(\alpha)}\int_0^1 \left(\frac{\phi(ty+(1-t)k)}{\phi(y)}\right)^p \left(\log\frac{\phi(y)}{\phi(ty+(1-t)k)}\right)^{\alpha-1} \theta(t;k;l)h((1-t)k+tl)dt.$$

Similarly we introduce the transformation $t = \frac{s-y}{l-y}$, or $s = tl + (1-t)y$ in (1.3.45) and get

$$\mathcal{I}_{l_-}^{\alpha,\phi}h(y) = \frac{(l-y)}{\Gamma(\alpha)}\int_0^1 \left(\frac{\phi(y)}{\phi(tl+(1-t)y)}\right)^p \left(\log\frac{\phi(tl+(1-t)y)}{\phi(y)}\right)^{\alpha-1} \theta(t;k;l)h(tl+(1-t)y)dt.$$

In preceding equations, θ is defined as $\theta(t;k;l) = \frac{\phi'(ty+(1-t)k)}{\phi(ty+(1-t)k)}$. Setting $y = (1-t)k + tl$, $y = (tk + (1-t)l)$ and $t = \frac{1}{2}$ then definition of convex function we get

$$2g\left(\frac{k+l}{2}\right) \leq g((1-t)k + tl) + g(tk + (1-t)l). \tag{4.1.1}$$

Theorem 4.1.1. *Let $h : I \to \mathbb{R}$ be positive function and $h \in X^d_{\phi,c}(k,l)$ is convex on I, then*

$$\begin{aligned}
&\frac{2h(\frac{k+l}{2})\left(\log\frac{\phi(l)}{\phi(k)}\right)^\alpha}{(l-k)\alpha}\left(\phi^p(k) + \left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\right) \leq \frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^\alpha_{k+}G(l) + \mathcal{I}^\alpha_{l-}H(k)) \\
&\leq \frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^\alpha}{(l-k)\alpha}\left(\left((\frac{\phi(k)}{\phi(l)})^p + \phi^p(k)\right)h(k) + (1+\phi^p(k))\,h(l)\right).
\end{aligned} \tag{4.1.2}$$

Proof. Multiply both side of inequality (4.1.1) by $(\frac{1}{\phi(l)})^p(\log\frac{\phi(l)}{\phi((1-t)k+tl)})^{\alpha-1}\theta(t;k;l)$ and integrate over $t \in [0,1]$.

$$\begin{aligned}
&2\int_0^1 h\left(\frac{k+l}{2}\right)\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)dt \\
&\quad\leq \int_0^1\left(\frac{\phi((1-t)k+tl)}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)h((1-t)k+tl)dt \\
&\quad+\int_0^1\left(\frac{\phi((tk+(1-t)l))}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)h(tk+(1-t)l)dt \\
&=\mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{4.1.3}$$

Left hand side of inequality (4.1.3) can be simplified as

$$\begin{aligned}
&2\int_0^1 h\left(\frac{k+l}{2}\right)\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)dt \\
=&2h\left(\frac{k+l}{2}\right)\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\int_0^1\left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)dt \\
=&\frac{2}{(l-k)\alpha}\left(h\left(\frac{k+l}{2}\right)\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi(k)}\right)^\alpha\right).
\end{aligned}$$

The first integral on the right side of inequality (4.1.1) simplifies as

$$\mathcal{I}_1 = \int_0^1\left(\frac{\phi((1-t)k+tl)}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)h((1-t)k+tl)dt = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^\alpha_{k_+}h(l).$$

Since $g^*(y) = g(k+l-y)$. For $y = (1-t)k+tl$, $g^*((1-t)k+tl) = g(tk+(1-t)l)$ $\phi^p((1-t)k+tl)h^*((1-t)k+tl) = \phi^p(tk+(1-tl))h(tk+(1-t)l)$, and $\frac{\phi^p((1-t)k+tl)h^*((1-t)k+tl)}{\phi^p(tk+(1-t)l)} =$

$h(tk+(1-t)l)$. The second integral on the right side of inequality (4.1.1) becomes

$$\mathcal{I}_2 = \int_0^1 \left(\frac{\phi((1-t)k+tl)}{\phi(l)}\right)^p \left(\log \frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1} \theta(t;k;l)h^*(tl+(1-t)k)dt$$
$$= \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{k_+}h^*(l).$$

Therefore, inequality (4.1.1) reduces to

$$\frac{2h\left(\frac{k+l}{2}\right)\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha} \leq \frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l)). \tag{4.1.4}$$

Since h is convex functions, therefore

$$2h\left(\frac{k+l}{2}\right) \leq h((1-t)k+tl) + h(tk+(1-t)l). \tag{4.1.5}$$

On both sides of inequality (4.1.5), multiplying by $(\phi(k))^p \left(\log \frac{\phi((1-t)k+tl)}{\phi(k)}\right)^{\alpha-1} \theta(t;k;l)$ and integrate over $t \in [0,1]$

$$2\int_0^1 h\left(\frac{k+l}{2}\right)(\phi(k))^p\left(\log\frac{\phi(tl+(1-t)k)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)dt$$
$$\leq \int_0^1 (\phi(k))^p\left(\log\frac{\phi(tl+(1-t)k)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h((1-t)k+tl)dt$$
$$+ \int_0^1 (\phi(k))^p\left(\log\frac{\phi(tl+(1-t)k)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h(tk+(1-t)l)dt = I_3 + I_4. \tag{4.1.6}$$

Let left hand side of above inequality, we obtain

$$2\int_0^1 h\left(\frac{k+l}{2}\right)\phi^p(k)\left(\log\frac{\phi(tl+(1-t)k)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)dt = \frac{2}{\alpha(l-k)}\left(h(\frac{k+l}{2})\phi^p(k)\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}\right).$$

Now we solve the integral on right side of the inequality (4.1.6). Let $y=(1-t)k+tl$, then $h((1-t)k+tl) = \frac{g((1-t)k+tl)}{\phi^p((1-t)k+tl)}$. Therefore, integral $\mathcal{I}_3$ simplifies as

$$\mathcal{I}_3 = \frac{\Gamma(\alpha)}{l-k}\left(\frac{l-k}{\Gamma(\alpha)}\int_0^1\left(\frac{\phi(k)}{\phi((1-t)k+tl)}\right)^p\left(\log\frac{\phi((1-t)k+tl)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)g((1-t)k+tl)dt\right)$$
$$= \frac{\Gamma(\alpha)}{l-k}\mathcal{I}_{l-}g(k).$$

As $h^*(y) = h(k+l-y)$ therefore $\frac{g^*((1-t)k+tl)}{\phi^p((1-t)k+tl)} = h(tk+(1-t)l)$. The second integral in inequality (4.1.6) implies

$$\mathcal{I}_4 = \frac{\Gamma(\alpha)}{(l-k)}\left(\frac{(l-k)}{\Gamma(\alpha)}\int_0^1\left(\frac{\phi(k)}{\phi((1-t)k+tl)}\right)^p\left(\log\frac{\phi((1-t)k+tl)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)g^*((1-t)k+tl)dt\right)$$
$$= \frac{\Gamma(\alpha)}{l-k}\mathcal{I}_{l-}g^*(k)$$

$\mathcal{I}_4 = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}_{l-}g^*(k)$. Consequently, $\mathcal{I}_3 + \mathcal{I}_4 = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}_{l-}(g(k) + g^*(k)) = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}_{l-}H(k)$. So, inequality (4.1.6) becomes

$$\frac{2h(\frac{k+l}{2})\phi^p(k)\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha} \le \frac{\Gamma(\alpha)}{l-k}\mathcal{I}_{l-}H(k). \tag{4.1.7}$$

Adding together the inequality (4.1.4) and (4.1.7), we get

$$\frac{2h(\frac{k+l}{2})\phi^p(k)\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha} + \frac{2h(\frac{k+l}{2})\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha} \le \frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l)) + \frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{l-}H(k)).$$

$$\frac{2h(\frac{k+l}{2})\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\phi^p(k) + \left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\right) \le \frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l) + \mathcal{I}^{\alpha}_{l-}H(k)). \tag{4.1.8}$$

For the right side of the equality, the convexity of g allows us to write $g\left((1-t)k + tl\right) \le (1-t)g(k) + tg(l)$ and $g(tk + (1-t)l) \le tg(k) + (1-t)g(l)$. Adding preceding two inequalities, we get $g((1-t)k+tl)+g(tk+(1-t)l) \le (1-t)g(k)+tg(l)+tg(k)+(1-t)g(l)$.

$$g((1-t)k + tl) + g(tk + (1-t)l) \le g(k) + g(l). \tag{4.1.9}$$

Multiply inequality (4.1.9) by $\left(\frac{1}{\phi^p(l)}\right)\left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)$ on both sides and integrate over $t \in [0,1]$

$$\begin{aligned}
\mathcal{I}_5 + \mathcal{I}_6 =& \int_0^1 \left(\frac{\phi((1-t)k+tl)}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)h((1-t)k+tl)dt \\
&+ \int_0^1 \left(\frac{\phi(tk+(1-t)l)}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)h(tk+(1-t)l)dt \\
\le& \int_0^1 \left(\frac{1}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)k^p h(k)dt \\
&+ \int_0^1 \left(\frac{1}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)l^p h(l)dt = \mathcal{I}_7 + \mathcal{I}_8.
\end{aligned} \tag{4.1.10}$$

$$\begin{aligned}
\mathcal{I}_5 =& \frac{\Gamma(\alpha)}{(l-k)}\left(\frac{(l-k)}{\Gamma(\alpha)}\int_0^1 \left(\frac{\phi(1-t)k+tl}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1}\theta(t;k;l)h((1-t)k+tl)dt\right) \\
=& \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{k+}h(l).
\end{aligned}$$

The integral $\mathcal{I}_5$ simplifies as $\mathcal{I}_5 = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{k+}h(l)$. Since $g^*(y) = g(k+l-y)$. For $y = ((1-t)k + tl)$ $g^*((1-t)k+tl) = g(tk+(1-t)l)$, $\phi^p((1-t)k+tl)h^*((1-t)k+tl) =$

$\phi^p(tk+(1-tl))h(tk+(1-t)l)$,
$\frac{\phi^p((1-t)k+tl)h^*((1-t)k+tl)}{\phi^p(tk+(1-t)l)} = h(tk+(1-t)l)$. Therefore

$$\begin{aligned}\mathcal{I}_6 =& \int_0^1 \left(\frac{\phi(tk+(1-t)l)}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1} \theta(t;k;l)h(tk+(1-t)l)dt \\ =& \frac{\Gamma(\alpha)}{l-k}\left(\frac{(l-k)}{\Gamma(\alpha)}\int_0^1 \left(\frac{\phi((1-t)k+tl)}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi(s)}\right)^{\alpha-1} \frac{\phi'(s)}{\phi(s)} h^*((1-t)k+tl)ds\right) \\ =& \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{k+}h^*(l). \\ \mathcal{I}_5+\mathcal{I}_6 =& \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{k+}\left(h(l)+h^*(l)\right) = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{k+}\left(F(l)\right). \\ \mathcal{I}_7 =& \int_0^1 \left(\frac{\phi(k)}{\phi(l)}\right)^p \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha-1} \theta(t;k;l)h(k)dt = \left(\frac{\phi(k)}{\phi(l)}\right)^p \frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{\alpha(l-k)}h(l) \\ \mathcal{I}_8 =& \int_0^1 \left(\log\frac{\phi(l)}{\phi((1-t)k+tl)}\right)^{\alpha} \theta(t;k;l)h(l)dt = \frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{\alpha(l-k)}h(k)\end{aligned}$$

Evaluating and adding $\mathcal{I}_7$ and $\mathcal{I}_8$, we get $\mathcal{I}_7+\mathcal{I}_8 = \left(\log\frac{\phi(l)}{\phi(a)}\right)^{\alpha}\left((\frac{\phi(k)}{\phi(l)})^p h(k)+h(l)\right)$. Therefore inequality (4.1.10) reduce to

$$\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}F(l)) \leq \left((\frac{\phi(k)}{\phi(l)})^p h(k)+h(l)\right)\frac{\left(\log(\frac{\phi(l)}{\phi(k)})\right)^{\alpha}}{\alpha(l-k)}. \tag{4.1.11}$$

Since h is convex function, therefore

$$h((1-t)k+tl)+h(tk+(1-t)l) \leq h(k)+h(l).$$

Multiply both sides of the preceding inequality by $\phi^p(k)\left(\log\frac{\phi((1-t)k+tl)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)$ and integrate over $t\in[0,1]$

$$\begin{aligned}\mathcal{I}_9+\mathcal{I}_{10} =& \int_0^1 \left(\frac{\phi((1-t)k+tl)}{\phi(k)}\right)^p \left(\log\frac{\phi(tk-(1-t)l)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h((1-t)k+tl)dt \\ &+\int_0^1 \left(\frac{\phi(tk+(1-t)l)}{\phi(k)}\right)^p \left(\log\frac{\phi((1+t)k+tl)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h(tk+(1-t)l)dt \\ \leq& \int_0^1 \left(\log\frac{\phi(tk+(1-t)l)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h(k)dt \\ &+\int_0^1 \left(\frac{\phi(l)}{\phi(k)}\right)^p \left(\log\frac{\phi((1+t)k+tl)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h(l)dt \\ =& \mathcal{I}_{11}+\mathcal{I}_{12}.\end{aligned} \tag{4.1.12}$$

Since $g(y) = \phi^p(y)h(y)$, therefore $\mathcal{I}_9 = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{l-}g(k)$.

$$\begin{aligned}\mathcal{I}_9 =&\frac{\Gamma(\alpha)}{l-k}\left(\frac{(l-k)}{\Gamma(\alpha)}\int_0^1\left(\frac{\phi(k)}{\phi((1-t)k+tl)}\right)^p\left(\log\frac{\phi((1-t)k+tl)}{\phi(k)}\right)^{\alpha-1}g((1-t)k+tl)dt\right)\\ =&\frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{l-}g(k).\end{aligned}$$

As $h^*(y) = h(k+l-y)$. Hence $\frac{h^*((1-t)k+tl)}{\phi((1-t)k+tl)} = h(tk+(1-t)l)$. Thus $\mathcal{I}_{10} = \frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{l-}g^*(k)$. Consequently

$$\begin{aligned}\mathcal{I}_{10} =&\frac{\Gamma(\alpha)}{l-k}\left(\frac{(l-k)}{\Gamma(\alpha)}\int_0^1\phi^p(k)\left(\log\frac{\phi((1-t)k+tl)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h(tk+(1-t)l)dt\right)\\ =&\frac{\Gamma(\alpha)}{l-k}\left(\frac{(l-k)}{\Gamma(\alpha)}\int_0^1\left(\frac{\phi(k)}{\phi((1-t)k+tl)}\right)^p\left(\log\frac{\phi((1-t)k+tl)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h((1-t)k+tl)dt\right)\\ =&\frac{\Gamma(\alpha)}{l-k}\mathcal{I}^{\alpha}_{l-}g^*(k)\\ \mathcal{I}_9+\mathcal{I}_{10} =&\frac{\Gamma(\alpha)}{k-l}\mathcal{I}^{\alpha}_{l-}(g^*(k)+g(k)) = \frac{\Gamma(\alpha)}{k-l}\mathcal{I}^{\alpha}_{l-}H(k).\end{aligned}$$

$\mathcal{I}_9+\mathcal{I}_{10} = \frac{\Gamma(\alpha)}{k-l}\mathcal{I}^{\alpha}_{l-}(g^*(k)+g(k)) = \frac{\Gamma(\alpha)}{k-l}\mathcal{I}^{\alpha}_{l-}H(k)$. In similar way one can show that

$$\begin{aligned}\mathcal{I}_{11} =&\int_0^1\phi^p(k)\left(\log\frac{\phi(tk+(1-t)l)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h(k)dt\\ =&\frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}\phi^p(k)}{\alpha(l-k)}h(k)\\ \mathcal{I}_{12} =&\int_0^1\phi^p(k)\left(\log\frac{\phi(tk+(1-t)l)}{\phi(k)}\right)^{\alpha-1}\theta(t;k;l)h(l)dt = \phi^p(k)\frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{\alpha(l-k)}h(l).\end{aligned}$$

$\mathcal{I}_{11}+\mathcal{I}_{12} = \frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}\phi^p(k)}{\alpha(l-k)}(h(l)+h(k))$. Thus inequality (4.1.12) reduces to

$$\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{l-}H(k)) \leq (h(k)+h(l))\frac{\left(\log\frac{\phi(k)}{\phi(l)}\right)^{\alpha}\phi^p(k)}{\alpha(l-k)}. \tag{4.1.13}$$

Add inequalities (4.1.11) and (4.1.13) we obtain

$$\frac{\Gamma(\alpha+1)}{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}((\mathcal{I}^{\alpha}_{k+}G(l)+\mathcal{I}^{\alpha}_{l-}H(k))) \leq \left(\left(\frac{\phi(k)}{\phi(l)}\right)^c+\phi^p(k)\right)h(k)+(\phi^p(k)+1)\,h(l). \tag{4.1.14}$$

Combining (4.1.8) and (4.1.14), we get generalized Hermite-Hadamard inequality (4.1.2).

$$\begin{aligned}
&\frac{2h(\frac{k+l}{2})\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\phi^{p}(k)+\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^{p}\right)\le\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l)+\mathcal{I}^{\alpha}_{l-}H(k))\\
&\le\frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\left((\frac{\phi(k)}{\phi(l)})^{p}+\phi^{p}(k)\right)h(k)+(1+\phi^{p}(k))\,h(l)\right).
\end{aligned}\tag{4.1.15}$$

□

Now we present several special cases for the generalized Hermite-Hadamard inequality (4.1.2). For $p=0$ and $\phi(y)=e^{y}$ we must obtain the following inequality which involve Riemann-Liouville fractional operator.

Corollary 4.1.2. *If $p=0$ and $\phi(y)=e^{y}$. Then*

$$h\left(\frac{k+l}{2}\right)\le\frac{\Gamma(\alpha+1)}{2\,(l-k)^{\alpha}}(\mathcal{I}^{\alpha}_{k+}h(l)+\mathcal{I}^{\alpha}_{l-}h(k))\le\frac{h(k)+h(l)}{2}.\tag{4.1.16}$$

Proof. For $p=0$, we have $h(y)=g(y)$, implies that $h^{*}(y)=g^{*}(y)$ and $H(y)=G(y)$. For $\phi(y)=e^{y}$, $\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}=(l-k)^{\alpha}$. Therefore

$$\frac{2h(\frac{k+l}{2})\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\phi^{p}(k)+\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^{p}\right)=4h(\frac{k+l}{2})\frac{(l-k)^{\alpha}}{(l-k)\alpha}$$

$$\begin{aligned}
\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l)+\mathcal{I}^{\alpha}_{l-}H(k))=&\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}H(l)+\mathcal{I}^{\alpha}_{l-}H(k))\\
=&\frac{1}{\Gamma(\alpha)}\int_{k}^{l}(l-t)h(t)dt+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}(l-t)h(k+l-t)dt\\
&+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}(t-k)h(t)dt+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}(t-k)h(a+l-t)dt.
\end{aligned}\tag{4.1.17}$$

Simplify the integrals which contain $h(k+l-t)$ for this let $s=k+l-t$ we get

$$\begin{aligned}
\mathcal{I}^{\alpha}_{k+}H(l)+\mathcal{I}^{\alpha}_{l-}H(k)=&\frac{1}{\Gamma(\alpha)}\int_{k}^{l}(t-l)h(t)dt+\frac{1}{\Gamma(\alpha)}\int_{l}^{k}(k-t)h(t)dt\\
&+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}(s-l)h(s)ds+\frac{1}{\Gamma(\alpha)}\int_{l}^{k}(k-s)h(s)d\\
=&2\left(\mathcal{I}^{\alpha}_{k+}h(l)+\mathcal{I}^{\alpha}_{l-}h(k)\right).
\end{aligned}$$

Also, we observe that $\frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\left((\frac{\phi(k)}{\phi(l)})^{p}+\phi^{p}(k)\right)h(k)+(1+\phi^{p}(k))\,h(l)\right)=\frac{(l-k)^{\alpha}}{(l-k)\alpha}(2h(k)+2h(l))$. Therefore, inequality (4.1.16) follows immediately.

$$\frac{4h(\frac{k+l}{2})(l-k)^{\alpha}}{(l-k)\alpha}\le\frac{2\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}h(l)+\mathcal{I}^{\alpha}_{l-}h(k))\le\frac{(l-k)^{\alpha}}{(l-k)\alpha}(2h(k)+2h(l)).$$

□

For $p = 0$ and $\phi(y) = e^{\psi(y)}$, we obtain the following inequality from inequality (4.1.2) for fractional integral with respect to another function ψ.

Corollary 4.1.3. *If $p = 0$ and $\phi(y) = e^{\psi(y)}$. Then*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4(\psi(l)-\psi(k))^{\alpha}}(\mathcal{I}_{k+}^{\alpha,\psi}H(l) + \mathcal{I}_{l-}^{\alpha,\psi}H(k)) \leq \frac{h(k)+h(l)}{2}. \tag{4.1.18}$$

Proof. For $p = 0$ $h(y) = g(y)$, implies that $h^*(y) = g^*(y)$ as $g^*(y) = g(k+l-y)$, $G(y) = h(y) + h^*(y)$ and $H(y) = g(y) + g^*(y)$ this imply that $H(y) = G(y)$.
For $\phi(y) = e^{\psi(y)}$, $\left(\log \frac{\phi(l)}{\phi(k)}\right)^{\alpha} = (\psi(l) - \psi(k))^{\alpha}$.

$$\frac{2h(\frac{k+l}{2})\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\phi^p(k) + \left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\right) = 4h(\frac{k+l}{2})\frac{(\psi(l)-\psi(k))^{\alpha}}{(l-k)\alpha}$$

$$\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}_{k+}^{\alpha}G(l) + \mathcal{I}_{l-}^{\alpha}H(k)) = \frac{\Gamma(\alpha)}{l-k}(\mathcal{I}_{k+}^{\alpha}H(l) + \mathcal{I}_{l-}^{\alpha}H(k)).$$

$$\frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\left((\frac{\phi(k)}{\phi(l)})^p + \phi^p(k)\right)h(k) + (1+\phi^p(k))\,h(l)\right) = 2\frac{(\psi(l)-\psi(k))^{\alpha}}{(l-k)\alpha}(h(k)+h(l)).$$

$$4h\left(\frac{k+l}{2}\right)\frac{(\psi(l)-\psi(k))^{\alpha}}{(l-k)\alpha} \leq \frac{\Gamma(\alpha)}{l-k}(\mathcal{I}_{k+}^{\alpha}H(l) + \mathcal{I}_{l-}^{\alpha}H(k)) \leq 2\frac{(\psi(l)-\psi(k))^{\alpha}}{(l-k)\alpha}(h(k)+h(l)).$$

Multiply by $\frac{\alpha}{4(\psi(l)-\psi(k))^{\alpha}}$ we obtain

$$h(\frac{k+l}{2}) \leq \frac{\Gamma(\alpha+1)}{4(\psi(l)-\psi(k))^{\alpha}}(\mathcal{I}_{k+}^{\alpha}H(l) + \mathcal{I}_{l-}^{\alpha}H(k)) \leq \frac{h(k)+h(l)}{2}.$$

□

For $p = 0$ and $\phi(y) = y$ in (4.1.2) we get Hermite-Hadamard inequality for Hadamard fractional integral.

Corollary 4.1.4. *If $p = 0$ and $\phi(y) = y$. Then*

$$h\left(\frac{k+l}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4\left(\log(\frac{l}{k})\right)^{\alpha}}({}_{\mathcal{H}}\mathcal{I}_{k+}^{\alpha}H(l) + {}_{\mathcal{H}}\mathcal{I}_{l-}^{\alpha}H(k)) \leq \frac{h(k)+h(l)}{2}. \tag{4.1.19}$$

Proof. For $p = 0$ $h(y) = g(y)$, implies that $h^*(y) = g^*(y)$ as $g^*(y) = g(k+l-y)$, $G(y) = h(y)+h^*(y)$ and $H(y) = g(y)+g^*(y)$. this imply that $H(y) = G(y)$. For $\phi(y) = y$, $\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha} = \left(\log\frac{l}{k}\right)^{\alpha}$.

$$\frac{2h(\frac{k+l}{2})\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\phi^p(k) + \left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^p\right) = 4h(\frac{k+l}{2})\frac{\left(\log(\frac{l}{k})\right)^{\alpha}}{(l-k)\alpha}.$$

$$\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l)+\mathcal{I}^{\alpha}_{l-}H(k))=\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}H(l)+\mathcal{I}^{\alpha}_{l-}H(k)).$$

$$\frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\left((\frac{\phi(k)}{\phi(l)})^{p}+\phi^{p}(k)\right)h(k)+(1+\phi^{p}(k))\,h(l)\right)=2\frac{\left(\log\frac{l}{k}\right)^{\alpha}}{(l-k)\alpha}\left(h(k)+h(l)\right).$$

$$4h\left(\frac{k+l}{2}\right)\frac{\left(\log(\frac{l}{k})\right)^{\alpha}}{(l-k)\alpha}\leq\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}H(l)+\mathcal{I}^{\alpha}_{l-}H(k))\leq 2\frac{\left(\log\frac{l}{k}\right)^{\alpha}}{(l-k)\alpha}\left(h(k)+h(l)\right).$$

Multiply with $\frac{\alpha}{4\left(\log(\frac{l}{k})\right)^{\alpha}}$ we get

$$h\left(\frac{k+l}{2}\right)\leq\frac{\Gamma(\alpha+1)}{4\left(\log\frac{l}{k}\right)^{\alpha}}(\mathcal{I}^{\alpha}_{k+}H(l)+\mathcal{I}^{\alpha}_{l-}H(a))\leq\frac{h(k)+h(l)}{2}.$$

□

If we let $p=0$ and $\phi(y)=e^{\frac{y^{\rho}}{\rho}}$ in inequality (4.1.2) we obtain Hermite-Hadamard inequality for so called Katugampola fractional integral.

Corollary 4.1.5. *Let $p=0$ and $\phi(y)=e^{\frac{y^{\rho}}{\rho}}$. Then*

$$h\left(\frac{k^{\rho}+l^{\rho}}{2}\right)\leq\frac{\Gamma(\alpha+1)}{\frac{2^{\rho}}{\rho^{\alpha}}\left(l^{\rho}-k^{\rho}\right)^{\alpha}}(\mathcal{I}^{\alpha,\rho}_{k+}h(l)+\mathcal{I}^{\alpha,\rho}_{l-}h(k))\leq\frac{h(k^{\rho})+h(l^{\rho})}{2^{\rho}}.\tag{4.1.20}$$

Proof. For $p=0$ $h(y)=g(y)$, implies that $h^{*}(y)=g^{*}(y)$ as $g^{*}(y)=g(k+l-y)$, $G(y)=h(y)+h^{*}(y)$ and $H(y)=g(y)+g^{*}(y)$. this imply that $H(y)=G(y)$. For $\phi(y)=e^{\frac{y^{\rho}}{\rho}}$, $\left(\log\frac{\phi(l)}{\phi(a)}\right)^{\alpha}=\frac{(l^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}}$

$$\frac{2h(\frac{k+l}{2})\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\phi^{p}(k)+\left(\frac{\phi(\frac{k+l}{2})}{\phi(l)}\right)^{p}\right)=4^{\rho}h(\frac{k^{\rho}+l^{\rho}}{2^{\rho}})\frac{(l^{\rho}-k^{\rho})^{\alpha}}{\rho^{\alpha}(l-k)\alpha}.$$

$$\begin{aligned}\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l)+\mathcal{I}^{\alpha}_{l-}H(k))=&\frac{\Gamma(\alpha)}{l-k}(\frac{1}{\Gamma(\alpha)}\int_{k}^{l}\frac{(l^{\rho}-s^{\rho})^{\alpha-1}}{\rho^{\alpha-1}}s^{\rho-1}f(s^{\rho})ds\\&+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}\frac{(s^{\rho}-k^{\rho})^{\alpha-1}}{\rho^{\alpha-1}}s^{\rho-1}f(s^{\rho})ds)\\&+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}\frac{(l^{\rho}-s^{\rho})^{\alpha-1}}{\rho^{\alpha-1}}s^{\rho-1}h(k^{\rho}+l^{\rho}-s^{\rho})ds\\&+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}\frac{(s^{\rho}-k^{\rho})^{\alpha-1}}{\rho^{\alpha-1}}s^{\rho-1}h(k^{\rho}+l^{\rho}-s^{\rho})dt\\=&I_1+I_2.\end{aligned}\tag{4.1.21}$$

Evaluate I_2 by substituting $t^{\rho}=k^{\rho}+l^{\rho}-s^{\rho}$ we get

$$I_2=\frac{1}{\Gamma(\alpha)}\int_{k}^{l}\frac{(t^{\rho}-k^{\rho})^{\alpha-1}}{\rho^{\alpha-1}}t^{\rho-1}h(t)dt+\frac{1}{\Gamma(\alpha)}\int_{k}^{l}\frac{(l^{\rho}-t^{\rho})^{\alpha-1}}{\rho^{\alpha-1}}t^{\rho-1}h(t)dt.$$

Equation 4.1.21 becomes

$$\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}G(l)+\mathcal{I}^{\alpha}_{l-}H(a))=\frac{2^{\rho}\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}h(l^{\rho})+\mathcal{I}^{\alpha}_{l-}h(k^{\rho})).$$

$$\frac{\left(\log\frac{\phi(l)}{\phi(k)}\right)^{\alpha}}{(l-k)\alpha}\left(\left((\frac{\phi(k)}{\phi(l)})^{p}+\phi^{p}(k)\right)h(k)+(1+\phi^{p}(k))\,h(l)\right)=2^{\rho}\frac{(l^{\rho}-k^{\rho})^{\alpha}}{\rho^{\alpha}(l-k)\alpha}(h(k^{\rho})+h(l^{\rho})).$$

$$4^{\rho}h\left(\frac{k^{\rho}+l^{\rho}}{2}\right)\frac{(l^{\rho}-k^{\rho})^{\alpha}}{\rho^{\alpha}(l-k)\alpha}\leq\frac{\Gamma(\alpha)}{l-k}(\mathcal{I}^{\alpha}_{k+}h(l^{\rho})+\mathcal{I}^{\alpha}_{l-}h(k^{\rho}))\leq 2^{\rho}\frac{(l^{\rho}-k^{\rho})^{\alpha}}{\rho^{\alpha}(l-k)\alpha}(h(k^{\rho})+h(l^{\rho})).$$

$$h\left(\frac{k^{\rho}+l^{\rho}}{2^{\rho}}\right)\leq\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2^{\rho}(l^{\rho}-k^{\rho})^{\alpha}}(\mathcal{I}^{\alpha}_{k+}h(l^{\rho})+\mathcal{I}^{\alpha}_{l-}h(a^{\rho}))\leq\frac{h(k^{\rho})+h(l^{\rho})}{2^{\rho}}. \tag{4.1.22}$$

□

Next we observe that the Hermite-Hadamard inequality (4.1.2), reduces to Hermite-Hadamard inequality for Hadamard-type fractional integrals if $\phi(y)=y$.

Corollary 4.1.6. *For $\phi(y)=y$*

$$h\left(\frac{k+l}{2}\right)\left(k^{p}+\left(\frac{k+l}{2l}\right)^{p}\right)\leq\frac{\Gamma(\alpha+1)}{2\left(\log\frac{l}{k}\right)^{\alpha}}(\mathcal{I}^{\alpha,H}_{k+}G(l)+\mathcal{I}^{\alpha,H}_{l-}H(a))$$
$$\leq\left(\left(\left(\frac{k}{l}\right)^{p}+k^{p}\right)h(k)+(1+k^{p})h(l)\right).$$

The Hermite-Hadamard inequality (4.1.2), reduces to Hermite-Hadamard inequality for Tempered fractional integrals, if $\phi(y)=e^{y}$.

Corollary 4.1.7. *For $\phi(y)=e^{y}$,*

$$h\left(\frac{k+l}{2}\right)\left(e^{kp}+\left(\frac{e^{\frac{k+l}{2}}}{e^{l}}\right)^{p}\right)\leq\frac{\Gamma(\alpha+1)}{2\,(l-k)^{\alpha}}(\mathcal{I}^{\alpha,p}_{k+}G(l)+\mathcal{I}^{\alpha,p}_{l-}H(k))$$
$$\leq\frac{1}{2}\left(h(k)\left(\left(\frac{e^{k}}{e^{l}}\right)^{p}+e^{kp}\right)+(1+e^{kp})h(l)\right).$$

4.1.1 Hermite-Hadamard type inequality

First we prove a lemma which will be further used in our result.

Lemma 4.1.8. *Let $\nu>0, n\in\mathbb{N}$ and $h^{[n]}_{\psi}\in L_{1}[k,l]$. Then*

$$\mathcal{I}^{\nu+n,\psi}_{l-}h^{[n]}_{\psi}(k)-\mathcal{I}^{\nu+n,\psi}_{k+}h^{[n]}_{\psi}(l)=\sum_{i=1}^{n}\frac{(\psi(l)-\psi(k))^{\nu+n-i}}{\Gamma(\nu+n-i+1)}\left[(-1)^{i+1}h^{[n-i]}_{\psi}(l)+h^{[n-i]}_{\psi}(k)\right]$$
$$-\left[\mathcal{I}^{\nu,\psi}_{k+}h(l)+(-1)^{n+1}\mathcal{I}^{\nu,\psi}_{l-}h(k)\right]. \tag{4.1.23}$$

Proof. By definition of fractional integral and integration by parts

$$\mathcal{I}_{k+}^{\nu+n,\psi} h_{\psi}^{[n]}(l) = \int_k^l \frac{(\psi(l)-\psi(s))^{\nu+n-1}}{\Gamma(\nu+n)} \frac{d}{ds} h_{\psi}^{[n-1]}(s) ds$$
$$= -\frac{(\psi(l)-\psi(k))^{\nu+n-1}}{\Gamma(\nu+n)} h_{\psi}^{[n-1]}(k) + \mathcal{I}_{k+}^{\nu+n-1,\psi} h_{\psi}^{[n-1]}(l).$$

A repeated application of above process leads to

$$\mathcal{I}_{k+}^{\nu+n,\psi} h_{\psi}^{[n]}(l) = -\sum_{i=1}^{n} \frac{(\psi(l)-\psi(k))^{\nu+n-i}}{\Gamma(\nu+n+1-i)} h_{\psi}^{[n-i]}(k) + \mathcal{I}_{k+}^{\nu,\psi} h(l). \tag{4.1.24}$$

In a similar way, one can show that

$$\mathcal{I}_{l-}^{\nu+n,\psi} h_{\psi}^{[n]}(k) = \sum_{i=1}^{n} (-1)^{i+1} \frac{(\psi(l)-\psi(k))^{\nu+n-i}}{\Gamma(\nu+n+1-i)} h_{\psi}^{[n-i]}(l) + (-1)^n \mathcal{I}_{l-}^{\nu,\psi} h(k). \tag{4.1.25}$$

Subtracting (4.1.25) from (4.1.24), we get (4.1.23). □

Theorem 4.1.9. *Assume that $h : [k, l] \to \mathbb{R}$ is n-times ψ-differentiable and $h_{\psi}^{[n]} \circ \psi^{-1}$ is convex. Then*

$$\sum_{i=1}^{n} \frac{(\psi(l)-\psi(k))^{\nu+n-i}}{\Gamma(\nu+n-i+1)} \left[(-1)^{i+1} h_{\psi}^{[n-i]}(l) + h_{\psi}^{[n-i]}(k)\right] \leq \left[\mathcal{I}_{k+}^{\nu,\psi} h(l) + (-1)^{n+1} \mathcal{I}_{l-}^{\nu,\psi} h(k)\right]$$
$$+ \frac{(\psi(l)-\psi(k))^{\nu+n}}{\Gamma(\nu+n+1)} \left(1 - \frac{1}{2^{\nu+n-1}}\right) (|h_{\psi}^{[n]}(k)| + |h_{\psi}^{[n]}(l)|). \tag{4.1.26}$$

Proof. Let $z = \frac{\psi(s)-\psi(l)}{\psi(k)-\psi(l)}$ which implies $\psi(s) = z\psi(k) + (1-z)\psi(l)$. Therefore

$$\left|\mathcal{I}_{l-}^{\nu+n,\psi} h_{\psi}^{[n]}(k) - \mathcal{I}_{k+}^{\nu+n,\psi} h_{\psi}^{[n]}(l)\right| \leq \frac{1}{\Gamma(\nu+n)} \int_k^l (\psi(s)-\psi(k))^{\nu+n-1} - (\psi(l)-\psi(s))^{\nu+n-1} \psi'(s) h_{\psi}^{[n]}(s) ds$$

$$= \frac{(\psi(l)-\psi(k))^{\nu+n}}{\Gamma(\nu+n)} \int_0^1 \left|(1-z)^{\nu+n-1} - z^{\nu+n-1}\right| h_{\psi}^{[n]} \circ \psi^{-1}(z\psi(k) + (1-z)\psi(l)) dz$$
$$\leq \frac{(\psi(l)-\psi(k))^{\nu+n}}{\Gamma(\nu+n)} \int_0^1 \left|(1-z)^{\nu+n-1} - z^{\nu+n-1}\right| (z|h_{\psi}^{[n]}(k)| + (1-z)|h_{\psi}^{[n]}(l)|) dz.$$

By similar calculations as in proof of Theorem 3 [40], we have

$$\int_0^1 \left|(1-z)^{\nu+n-1} - z^{\nu+n-1}\right| (z|h_{\psi}^{[n]}(k)| + (1-z)|h_{\psi}^{[n]}(l)|) dz$$
$$= \frac{2}{\nu+n} \left(1 - \frac{1}{2^{\nu+n-1}}\right) (|h_{\psi}^{[n]}(k)| + |h_{\psi}^{[n]}(l)|).$$

Consequently, we have

$$\left|\mathcal{I}_{l-}^{\nu+n,\psi} h_{\psi}^{[n]}(k) - \mathcal{I}_{k+}^{\nu+n,\psi} h_{\psi}^{[n]}(l)\right| \leq \frac{(\psi(l)-\psi(k))^{\nu+n}}{(\nu+n)\Gamma(\nu+n)} \left(1 - \frac{1}{2^{\nu+n-1}}\right) (|h_{\psi}^{[n]}(k)| + |h_{\psi}^{[n]}(l)|). \tag{4.1.27}$$

Combining inequality (4.1.27) with Equation (4.1.23) we get the desired result in (4.1.26).

□

Remark 4.1.10. Inequality (4.1.26) generalizes the inequality (3.5) in [40]. However it is still an open problem whether we can derive a similar result for generalized Hadamard-type fractional integrals with respect to a function. In future work, we plan to generalize various integral inequalities for the new generalized context of Hadamard-type operators with respect to functions.

Chapter 5

Summary

Chapter 1 begins by recalling some basic functions (gamma function, beta function, Mittag-Leffler function) and their properties. Furthermore we discuses properties of differential and integral operators and some function spaces. We study the different properties of Riemann-Liouville fractional integral and derivative, Caputo derivative, Hadamard fractional integral and derivative and Hadamard-type fractional integral with respect to a function ϕ. Chapter 1 gives a broad view on the different properties of fractional integrals and derivatives including composition property of integral over derivative and vice versa. we also define generalized fractional integral with respect to another function, Hadamard-type fractional integral, Tempered fractional integral and Katugampola fractional integral.

In Chapter 2 we present an introduction of the classical Hermite-Hadamard inequality and some of its applications. In Chapter 3, Hermite-Hadamard inequality, Hermite-Hadamard type inequality and Fejér inequality for Riemann-Liouville fractional integral are discussed. The major part of Chapter 3 consists of survey on Hermite-Hadamard and Hermite-Hadamard type inequalities for different fractional integrals.

In Chapter 4, we established Hermite-Hadamard inequality for Hadamard-type fractional integral with respect to a function. The main significance of the inequality is that it contains Hermite-Hadamard inequalities for many fractional integrals as special cases. Generalized result of Hermite-Hadamard type inequality for fractional integral with respect to another function is also established.

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