Entanglement of Indistinguishable Particles: Second Quantization Approach

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Dedicated to my Family especially, to My Beloved Parents.

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Abstract

Quantum entanglement is a key resource for numerous quantum information tasks, quantum computing, quantum sensing and related quantum technologies. However, the generation, detection and quantification of entanglement may be quite complicated depending on the nature and properties of the pertaining system. In this thesis, we study various methods for detection and quantification as resource theory of entanglement for a system of particles in the context of indistinguishability transition of particles' identity.

We start our discussion with the system of distinguishable particles and discuss different methods of entanglement detection and quantification such as Schmidt decomposition and von Neumann Entropy, with all their merits and demerits. The notion of entanglement is extended for the quantum systems composed of indistinguishable particles and, its detection and quantification is analyzed. It is found that the conventional Schmidt decomposition deeds to be modified, in the form of Slater-Schmidt decomposition, when the particles' identity becomes indistinguishable. In order to build a comprehensive analysis, we review various techniques to make a global entanglement detection scheme, such as, so-called no label approach which becomes complicated in the case of indistinguishable particles. Finally, we discuss the second quantization approach which helps to reduce the complications of symmetrization postulate for indistinguishable particles and suggest a way to develop a unified approach to all quantum systems.

Contents

List of Abbreviations

IP	Identical Particles
LFC	Lo Franco and Compagno's method
LHVM	Local Hidden Variable Model
LOCC	Local Operations and Classical Communication
LT	Local Trace
NIP	Non-identical Particles
SD	Schmidt Decomposition
SPDC	Spontaneous Parametric Down Conversion
SQ	Second Quantization

List of Figures

Chapter 1

Introduction

Entanglement is a unique feature of a composite quantum system that basically originates from the principle of superposition. It questions our sense of locality as first discussed by Einstein, Podolsky, and Rosen [1]. They questioned the completeness of quantum mechanics on the basis of "locality" which disagrees with the concept of remote influence and "realism" which emphasizes the intrinsic values of physical properties before observation. Schrodinger [2] recognized its importance and stated,

> "I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought"

It was a revolutionary resource for quantum communication that defeat the classical devices with its high computing power and time-saving characteristics. These features and their vast application motivated scientists to perform experiments and put forward theoretical proposals in order to generate entanglement. In 1964 Bell [3] describe the EPR concept of incompleteness in terms of the Local Hidden Variable Model (LHVM). He proved that by applying the assumptions of realism, locality, and free will we can impose restrictions on correlations that involve bipartite systems in the form of Bell inequalities. He showed that some states violate the bell inequality on measurement which was termed entangled states. So it was clear that this unique feature cannot be simulated in the classical world. The transition from thought experiments to laboratory

started in the mid-60s but the convincing violation of Bell inequalities was reported by Aspect, J.Dalibard, and G.Roger in 1982 [4]. After that many interesting experimental tests were performed against LHVM in the laboratories that confirms the prediction of quantum description. In the early 90s, this concept evolves in terms of entropic inequalities on the basis of von Neumann entropy. The violation of these inequalities is the indicator of entanglement in quantum states but the physical meaning of this was unclear. In 1997 Cerf and Adami [5] tried to explain it in term of conditional entropy and considered it a fundamental quantity responsible for the capabilities of transmission of quantum information. All of this was theoretical until a direct violation of the entropic inequalities was experimentally demonstrated and confirmed the breaking of classical statistical order in compound quantum systems in 2005 by Bovino [6]. Later this entanglement theory plays an important role in the discoveries of quantum cryptography [7], quantum dense coding [8] and quantum teleportation [9, 10]. Due to enormous application in the modern world, nowadays many experiments aim at the generation of entanglement. In any of these experiments, typical questions arise: How can one be sure that entanglement was indeed produced? How can one detect the presence of entanglement? Can we quantify the entanglement in the experiment? These questions are difficult to answer and many possible ways to tackle this problem have been proposed. Scientist put a lot of effort to study entanglement in distinguishable particle quantum systems that can be evaluated individually. Initially spin $\frac{1}{2}$ particles were used to study entanglement. After that photonic entanglement discovered in interferometry, e.g, parametric down-conversion, and we link the with qubit state in order to devise a simplified explanation.

Entanglement is a property that is exhibited by a system that is composed up of two or more particle/sub-systems. The state of a single particle can be exact or in superposition, e.g, $(|\psi\rangle = a\,|\uparrow\rangle + b\,|\downarrow\rangle)$ that falls to a specific state on observation but it does not affect the state of any other particle in the world while in entanglement we have more than one particle in a system whose states are somehow correlated, e.g, $(|\psi\rangle = a\,|\uparrow,\downarrow\rangle_{1,2} + b\,|\downarrow,\uparrow\rangle_{1,2})$ with each other. This correlation is entanglement, when both the particles in a specific system attain a certain state when one of the

particle's state is observed. So, before entanglement, we will take a look at different systems and various methods that are involved in their state representation [11]. Here we are going to make use of the Stern-Gerlach apparatus in a decaying process of a spin-0 particle into two spin-1/2 particles that travel in opposite directions in order to conserve momentum obviously will have an opposite spin that satisfies the conservation of spin angular momentum. Now the question arises which particle has $+1/2$ spin and which one is with -1/2 spin? For this purpose consider two observers with Stern-Gerlach apparatus who can observe the spin of one of the two particles that is in his vicinity. We can write the state of this system as, $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle_{1,2} - |\downarrow,\uparrow\rangle_{1,2})$ [12] which shows if we know the spin of any of these two particles we can tell the spin of other particle with 100% accuracy. All the systems are not that state forward, some methods are devised to detect entanglement in distinguishable quantum system like separability and Schmidt decomposition [13] of the states. These methods only give information about the presence or absence of entanglement in the system which is not enough. For its use we need to quantify the entanglement that will be done with the help of the Schmidt-coefficient that comes from Schmidt decomposition. von Neumann entropy and Concurrence are considered good quantifier of the entanglement that make use of Schmidt-coefficient. These methods are only suitable for distinguishable systems and are not applicable to indistinguishable particle systems.

Classically, distinguishable and indistinguishable particles are treated identically because we can mark them with labels or we can trace the trajectory they follow but this is not the case in quantum mechanics. The properties of indistinguishable particles differ significantly from distinguishable because we can not apply physical labels or trace their trajectory as in classical systems. Initially, the systems of indistinguishable particles were described with the help of unphysical(mock) labels [13] that make them distinguishable and symmetrization postulates were introduced in the system. These labels make the characterization of quantum states and correlation complicated and add exchange-correlation in the state of the system. Now the question arises that is this correlation is entanglement or not? To counter with exchange-correlation and characterization of entanglement a method called Slater rank was developed [14, 15]

which was analogous to Schmidt decomposition in an indistinguishable system. According to it, the state which can be written in a single Slater determinant or convex combination of them are separable [14, 16]. The quantification of entanglement was the same in both the system through von Neumann entropy but we need to add a shift equals lnN to filter entanglement in indistinguishable systems, where N is the total number of particles in the system. This method suggest different treatment for bosons and fermions. Moreover, labels make particles distinguishable which was practically impossible. So a method with no label approach was introduced which was extended to Schmidt decomposition for indistinguishable particles [17, 18, 19]. Symmetrization was achieved by the permutation of the particles that expel the need for labels. In the end, in order to make a unified approach to all the quantum systems, we introduce second quantization [46] and its authentication will be verified by comparing it with results.

Throughout all this discussion we will chose some quantum systems like Spontaneous Parametric Down Conversion (SPDC) and apply all these methods on them and will compare the results with the help of an entanglement graph that will simplify the effect of localization and non-localization approach on entanglement.

1.1 Thesis Outline

This work is organized as follows: In chapter 2 we will review the methods that are used to represent the state of any system using product Hilbert space. After that, we will discuss the density operator and its properties for the pure and mixed states. In next step will talk about the methods of entanglement detection (separability and Schmidt decomposition). The methods to separate out particles through partial trace will come under discussion. We will apply some of these methods to Spontaneous Parametric Down Conversion (SPDC) photons. Entanglement quantifier like von Neumann entropy will come in the end of this chapter. In chapter 3 we will move from distinguishable to indistinguishable particles. In the first step, we will talk about the state formalism of these particles. After that, we will look at different methods to quantify entanglement in these particles in which we will try to convert indistinguishable particles to distinguishable by freezing some of their freedoms. After that, we will apply the mock label technique and symmetrization postulates which induce entanglement in the system. To solve this theoretical entanglement Slater Schmidt decomposition will be studied. To quantify entanglement we will talk about von Neumann entropy and its behavior in both cases. At the end of this chapter, we discuss no label approach that makes use of the permutation technique instead of symmetrization postulates. Then extend this concept to build Schmidt decomposition for identical particles. In chapter 4, we will apply the second quantization approach to show the behavior of particles when they move from indistinguishable to the distinguishable domain, local and non-local approach will be compared for all quantum systems and our work will be concluded in chapter 5.

Chapter 2

Entanglement in Quantum Systems

A composite system is one that incorporates more than one quantum object (for instance, a molecule comprises protons and electrons). Our task is to assemble a formalism in which the state of the whole system can be represented in terms of its constituent states.

In general, we deal with composite systems which are not isolated but somehow correlated with each other in the quantum mechanical scenarios. In many cases, we deal with multiparticle states. To understand the composite system we build a large Hilbert space H that is composed up of sub Hilbert spaces that are independent of each other and separately represent a complete system. This mechanism is termed as *Kronecker* or *tensor product.*

lets consider two Hilbert spaces H_α and H_β of dimension M_1 and M_2 respectively. To represent it in a composite system we combine these Hilbert spaces and build a large Hilbert space H using a tensor product mechanism which is written as,

$$H = H_\alpha \otimes H_\beta. \tag{2.0.1}$$

The product of dimensions of H_α and H_β represent the dimension of combined Hilbert space.

$$\dim(H) = M_1 M_2 \tag{2.0.2}$$

2.1 Quantum Description of Composite System

A state vector in composite Hilbert space H, which is a combination of H_α and H_β,is written as the tensor product in Dirac notation. Let $|\psi\rangle \in H_\alpha$ and $|\phi\rangle \in H_B$ then the vector space in Hilbert space H is represented as,

$$|\chi\rangle = |\delta\rangle \otimes |\phi\rangle \,, \tag{2.1.1}$$

where $|\chi\rangle$ belongs to composite Hilbert space H. The tensor product follows the linearity property,

$$\begin{aligned} |\delta\rangle \otimes [|\phi_1\rangle + |\phi_2\rangle] &= |\delta\rangle \otimes |\phi_1\rangle + |\delta\rangle \otimes |\phi_2\rangle \\ [|\delta_1\rangle + |\delta_2\rangle] \otimes |\phi\rangle &= |\delta_1\rangle \otimes |\phi\rangle + |\delta_2\rangle \otimes |\phi\rangle \,. \end{aligned} \tag{2.1.2}$$

It also shows linear behavior when multiplied by a scalar,

$$|\delta\rangle \otimes (B\,|\phi\rangle) = B\,|\delta\rangle \otimes |\phi\rangle \,. \tag{2.1.3}$$

Now the basis of large Hilbert space can be written in the form of tensor products of basis vectors for the spaces H_α and H_β. Lets define the basis of H_α by $|X_i\rangle$ and H_β by $|Y_i\rangle$ then the basis $|Z_i\rangle$ for state of composite system is,

$$|Z_i\rangle = |X_i\rangle \otimes |Y_i\rangle \,. \tag{2.1.4}$$

The tensor product is independent of the order; i.e,

$$|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle \,. \tag{2.1.5}$$

The tensor product can also be represented as,

$$|\psi\rangle \otimes |\phi\rangle = |\psi\rangle\,|\phi\rangle = |\psi\phi\rangle \,. \tag{2.1.6}$$

The procedure to calculate the scalar product of two vectors, that belong to the same Hilbert space, is quite simple. Suppose we have two state vectors,

$$\begin{aligned} |Z_A\rangle &= |X_A\rangle \otimes |Y_A\rangle \,, \\ |Z_B\rangle &= |X_B\rangle \otimes |Y_B\rangle \,. \end{aligned} \tag{2.1.7}$$

then, the inner product of above state vectors defined as,

$$\begin{aligned} \langle Z_A|Z_B\rangle &= (\langle X_A| \otimes \langle Y_A|)(|X_B\rangle \otimes |Y_B\rangle), \\ &= \langle X_A|X_B\rangle \langle Y_A|Y_B\rangle . \end{aligned} \tag{2.1.8}$$

We can build a basis of $\mathbb{C}^4$ out of $\mathbb{C}^2$ with the help of inner product in many cases.

2.1.1 Pure State

We will initially understand pure state with the example of a bipartite system and then generalize it to the multi-partite systems. For a two-particle system prepared in a pure state $|\psi_i\ (i = 1, 2)\rangle$; then the separable state of composite system $|\Psi_S\rangle$ can be written as direct product form,

$$|\Psi_S\rangle = |\psi_1\rangle \otimes |\psi_2\rangle . \tag{2.1.9}$$

The pure state can be a combination of independent states or superimposed states. The above state in this form looks like a non-super imposed state that shows only local measurements can be performed on the system, which means that the observer had access to only one system at a time. When we perform local measurement on the composite system e.g, $(\alpha \otimes \mathbb{I})$, where α operates on the first subsystem and $\mathbb{I}$ operates on the second subsystem. The states of the first subsystem will be projected to the eigenstates of the operator α and the second subsystem will remain unchanged. After some time if the observer performs the measurement on the second subsystem it will not affect the first subsystem and will not be affected by the earlier operation which was performed on the first subsystem. This means that both the subsystem are independent of each other.

When a pure state superimpose, then the resultant state is also a pure state that can be written as,

$$|\Psi_E\rangle = \frac{1}{\sqrt{2}}(|\psi_A\rangle \otimes |\psi_B\rangle + |\phi_A\rangle \otimes |\phi_B\rangle), \tag{2.1.10}$$

where $|\psi_i\rangle \neq |\phi_i\rangle$ and (i=A,B). Now the operation of a local operator $\alpha \otimes \mathbb{I}$ on this

system gives the result as,

$$\begin{aligned}\langle\alpha\rangle &= \langle\Psi_E|\,\alpha\otimes\mathbb{I}\,|\Psi_E\rangle\,,\\ &= tr(\alpha\otimes\mathbb{I}\,|\Psi_E\rangle\,\langle\Psi_E|),\\ &= tr_A(\alpha\; tr_B\,|\Psi_E\rangle\,\langle\Psi_E|),\\ &= tr_A(\alpha\rho_A),\end{aligned} \tag{2.1.11}$$

where $tr_{A,B}$ is the partial trace over one of the subsystems (A or B) and $\rho_A = tr_B\,|\Psi_E\rangle\,\langle\Psi_E|$, is the reduced density matrix that belongs to subsystem A. From these arguments, we can conclude that the reduced density matrix of the sub-subsystems can be calculated from the density matrix of the system as,

$$\begin{aligned}\rho &= |\Psi_E\rangle\,\langle\Psi_E|\,,\\ \rho_A &= tr_B\,|\Psi_E\rangle\,\langle\Psi_E|\,,\\ \rho_B &= tr_A\,|\Psi_E\rangle\,\langle\Psi_E|\,.\end{aligned} \tag{2.1.12}$$

Above equations shows that the partial trace method is useful to separate one subsystem from the other.(add generalized states)

For a multipartite system of N particles the pure state can be represented in the tensor product form of N independent state vectors,

$$|\Psi\rangle = |\psi_1\rangle\otimes|\psi_2\rangle\otimes...\otimes|\psi_N\rangle\,. \tag{2.1.13}$$

2.1.2 Mixed State

Generally, the most frequently encountered state in real experiments is a mixed state [20]. It is practically not feasible to isolate every system from the surrounding, so the state which left to be dealt with is mixed. To get the mixed state of the system we simply took the partial trace over the surroundings and then the left reduce state is mixed. e.g for the bipartite system, of pure state, the mixed state is represented as,

$$\rho = \rho^1\otimes\rho^2, \tag{2.1.14}$$

where there is no correlation between ρ^1 and ρ^2 subsystems. In the form of convex sum, we can write it as,

$$\rho = \Sigma_i p_i \rho_i^1 \otimes \rho_i^2, \tag{2.1.15}$$

where p_i is the probability which satisfies the conditions $p_i > 0$ and $\Sigma_i p_i = 1$. For N number of particles mixed state can be written as,

$$\rho = \Sigma_i p_i \rho_i^1 \otimes \rho_i^2 \otimes \ ... \otimes \rho_i^N. \tag{2.1.16}$$

2.2 Density Operator and its Properties

Single quantum systems are easy to handle but in general, we encounter with the collection of systems called *ensemble* and its members can be found in different quantum states with some probability which may be the same or different for every state. Let us consider an example for further understanding [21, 22].

Suppose we have two dimensional Hilbert space whose basis vectors are defined as, $|\psi\rangle$ *and* $|\phi\rangle$. By using defined basis vectors $|\psi\rangle$ *and* $|\phi\rangle$, we can prepare composite system comprised of N number of particles in which each can be in one of the given states, i.e,

$$\begin{aligned} |X\rangle &= \alpha_1 |\psi\rangle + \alpha_2 |\phi\rangle , \\ |Y\rangle &= \beta_1 |\psi\rangle + \beta_2 |\phi\rangle . \end{aligned} \tag{2.2.1}$$

As the given states are normalized to unity so,

$$\begin{aligned} |\alpha_1|^2 + |\alpha_2|^2 &= 1, \\ |\beta_1|^2 + |\beta_2|^2 &= 1. \end{aligned} \tag{2.2.2}$$

By considering N number of particles out of which n_x are prepared in $|X\rangle$ state and n_y are in $|Y\rangle$ state so we can write,

$$N = n_x + n_y. \tag{2.2.3}$$

According to this relation the probability of finding any random member of this sample/ensemble in state $|X\rangle$ can be represented by p $= n_x/N$. As we knew the sum of all

the probabilities is one so the probability of particle to be in state $|Y\rangle$ is 1 - p, which satisfies the law of probabilities.

So, from the above discussion, we can conclude that system probability behaves classically at the ensemble level, which shows that the information we are getting is incomplete; that is we have a simple statistical mixture at ensemble level. This statistical mixture works for classical properties but not in the case of quantum states. The calculated quantities need to be weighted by the probability of every state which can be done by density operators application [21]. In the next section we will use this density method approach for pure and mixed states.

2.2.1 Pure State

A system in some known state $|\Psi\rangle$ given as,

$$|\Psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle + ... + c_N |\psi_N\rangle . \tag{2.2.4}$$

Probabilities of any state can be found with the help of this definite state but here we describe a different way that can be generalized to the statistical mixture as in classical situations. Here we use an operator to describe this statistical mixture which is called density operator(ρ). In a pure state it is given as,

$$\rho = |\Psi\rangle \langle\Psi| . \tag{2.2.5}$$

Density operator can be used to determine the expected value of any operator easily. For example, the expectation value of operator X in the given basis of the pure state is defined as,

$$\begin{aligned} \langle \mathbf{X} \rangle &= \langle \Psi | \mathbf{X} | \Psi \rangle , \\ \langle \mathbf{X} \rangle &= (c_1^* \langle \psi_1| + c_2^* \langle \psi_2| + ... + c_N^* \langle \psi_N|) \mathbf{X} (c_1 |\psi_1\rangle + c_2 |\psi_2\rangle + ... + c_N |\psi_N\rangle), \\ &= \sum_{i,j=1}^{N} c_i^* c_j \langle \psi_i | \mathbf{X} | \psi_j \rangle . \end{aligned} \tag{2.2.6}$$

In above equation the coefficient of expansion can be defined as,

$$\begin{aligned} c_j &= \langle\psi_j|\Psi\rangle\,, \\ c_i^* c_j &= \langle\Psi|\psi_i\rangle\,\langle\psi_j|\Psi\rangle\,, \\ &= \langle\psi_j|\Psi\rangle\,\langle\Psi|\psi_i\rangle\,, \\ &= \langle\psi_j|\rho|\psi_i\rangle\,. \end{aligned} \tag{2.2.7}$$

So the expectation value becomes,

$$\begin{aligned} \langle\mathbf{X}\rangle &= \sum_{i,j=1}^{N} \langle\psi_j|\rho|\psi_i\rangle\,\langle\psi_i|\mathbf{X}|\psi_j\rangle\,, \\ \langle\mathbf{X}\rangle &= \sum_{j=1}^{N} \langle\psi_j|\,\rho\left(\sum_{i=1}^{N}|\psi_i\rangle\,\langle\psi_i|\right)\mathbf{X}\,|\psi_j\rangle\,, \\ \langle\mathbf{X}\rangle &= \sum_{j=1}^{N} \langle\psi_j|\,\rho\mathbf{X}\,|\psi_j\rangle\,, \\ \langle\mathbf{X}\rangle &= Tr(\rho\mathbf{X}). \end{aligned} \tag{2.2.8}$$

If we look at the trace of density operator we get,

$$\begin{aligned} Tr(\rho) &= \Sigma_i\,\langle\psi_i|\Psi\rangle\,\langle\Psi|\psi_i\rangle\,, \\ &= \Sigma_i c_i^* c_i, \\ &= \Sigma_i |c_i|^2, \\ &= 1. \end{aligned} \tag{2.2.9}$$

This equation shows the conservation of probabilities. The density operator is Hermitian which means that,

$$\rho = \rho^\dagger. \tag{2.2.10}$$

In case of pure state,

$$\begin{aligned} \rho^2 &= |\Psi\rangle\,(\langle\Psi|\Psi\rangle)\,\langle\Psi|\,, \\ &= |\Psi\rangle\,\langle\Psi|\,, \\ \rho^2 &= \rho. \end{aligned} \tag{2.2.11}$$

2.2.2 Mixed State

In case of ensemble for mixed state, the statistical mixture or the density matrix can be built by,

- Build density operator for each state in the ensemble.
- Weight it by the probability in the ensemble.
- Sum up the probabilities.

Let us consider an example of an ensemble in which each member can be found in one of the two given states,

$$\begin{aligned} |\psi\rangle &= \mathrm{a}\,|x\rangle + \mathrm{b}\,|y\rangle\,, \\ |\phi\rangle &= \mathrm{c}\,|x\rangle + \mathrm{d}\,|y\rangle\,. \end{aligned} \tag{2.2.12}$$

The density operator of these states can be defined as,

$$\begin{aligned} \rho_\psi &= |\psi\rangle\langle\psi|\,, \\ \rho_\phi &= |\phi\rangle\langle\phi|\,. \end{aligned} \tag{2.2.13}$$

In above example, we have only two states and if we take the probability of state $|\psi\rangle$ be p then the probability of any member of the ensemble for being in the state $|\phi\rangle$ will be $1-p$. So, the density operator can be written in terms of probabilities as,

$$\rho = p\rho_\psi + (1-p)\rho_\phi. \tag{2.2.14}$$

This is the simplest two state system of an ensemble which can be generalize to n state system. For example, $|\Psi_i\rangle$ where $i = 1, ..., n$ and density operator for every single state is given as $\rho_i = |\Psi_i\rangle\langle\Psi_i|$. If the probability of each state is given by p_i then the overall density operator is given as,

$$\begin{aligned} \rho &= \sum_{i=1}^{n} p_i\rho_i \\ &= \sum_{i=1}^{n} p_i\,|\Psi_i\rangle\langle\Psi_i|\,. \end{aligned} \tag{2.2.15}$$

Properties of density operator:

The density operator has the following three key properties ,

1. It is hermitian, which means $\rho = \rho^{\dagger}$.
2. tr $\rho = 1$.
3. ρ is positive operator, $\langle \Psi | \rho | \Psi \rangle \geq 0$.

Characteristics of mixed states:

Different components of a state or different states interfere with each other when they are coherent. In a statistical mixture (mixed state) there is no coherence present while it is the property of the pure state of superposition states. So we can differentiate between pure and mixed states on the basis of coherence which is indicated by the density matrix. If off-diagonal elements in this matrix are non-zero then coherence is present in it. In short,

- A mixed state has all the off-diagonal elements zero.
- A pure state has non-zero off-diagonal elements.

There are other criteria that can be used to differentiate between pure and mixed states which is related to the trace of the square of the density matrix. As we knew that for the pure states $\rho^2 = \rho$ and the $tr(\rho^2) = 1$, but this is not true for the case mixed states. So, the distinction criteria for pure and mixed state is,

- $1 > Tr(\rho^2)$ for mixed state.
- $1 = Tr(\rho^2)$ for pure state.

2.3 Reduced Density Operator by Partial Trace

The density operator is very useful in dealing with the subsystems. We can consider here a composite bipartite system which consists of two subsystems one is under the observation of observer A and the other is with B and they travel in the opposite direction. The whole system is described by a complete state which contains information of the system, but each of them have no information about other half unless they adopt any way of communication. We can build the density operator ρ for the whole system and filter it down into a reduced density operator that just shows what B or A alone see. We calculate it with the help of partial trace and compute separate density operators for A and B. This operation is called *reduced density operation* and the operator we get is reduced density operator [22].

Let's assume that the described system is in one of the Bell's states,

$$|\chi_{10}\rangle = \frac{|1_A\rangle |0_B\rangle + |0_A\rangle |1_B\rangle}{\sqrt{2}}. \tag{2.3.1}$$

The density operator for this system can be written as,

$$\begin{aligned}
\rho &= |\chi_{10}\rangle \langle\chi_{10}|, \\
&= \left(\frac{|1_A\rangle |0_B\rangle + |0_A\rangle |1_B\rangle}{\sqrt{2}}\right)\left(\frac{\langle 1_A| \langle 0_B| + \langle 0_A| \langle 1_B|}{\sqrt{2}}\right), \\
&= \frac{|1_A\rangle |0_B\rangle \langle 1_A| \langle 0_B| + |1_A\rangle |0_B\rangle \langle 0_A| \langle 1_B| + |0_A\rangle |1_B\rangle \langle 1_A| \langle 0_B| + |0_A\rangle |1_B\rangle \langle 0_A| \langle 1_B|}{2}.
\end{aligned} \tag{2.3.2}$$

This density operator contains all the information about the system as according to the situation each of the observer has access to only one subsystem so we need a mathematical tool to identify what Alice see and what Bob will find upon measurement. The tool we use here is the partial trace, this trace is computed over the diagonal states of a single subsystem. Let's assume we are at B's place then we need to trace over A's basis,

$$\begin{aligned}
\rho_B &= Tr_A(\rho), \\
&= Tr_A(|\chi_{10}\rangle \langle\chi_{10}|), \\
&= \langle 0_A|(|\chi_{10}\rangle \langle\chi_{10}|)|0_A\rangle + \langle 1_A|(|\chi_{10}\rangle \langle\chi_{10}|)|1_A\rangle .
\end{aligned} \tag{2.3.3}$$

After computing the above equations the results we get,

$$\begin{aligned} \langle 0_A|(|\chi_{10}\rangle\langle\chi_{10}|)|0_A\rangle &= \frac{|1_B\rangle\langle 1_B|}{2}, \\ \langle 1_A|(|\chi_{10}\rangle\langle\chi_{10}|)|1_A\rangle &= \frac{|0_B\rangle\langle 0_B|}{2}. \end{aligned} \tag{2.3.4}$$

So B's density operator is,

$$\begin{aligned} \rho_B &= Tr_A(\rho), \\ &= Tr_A(|\chi_{10}\rangle\langle\chi_{10}|), \\ &= \frac{|0_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B|}{2}. \end{aligned} \tag{2.3.5}$$

This is the density operator for B. We can evaluate $\mathrm{Tr}\rho_B = 1$ and check that $\mathrm{Tr}\rho_B^2 = \frac{1}{4}$ which shows that B has a completely mixed state.

Partial Transpose

It is the transpose that is taken with respect to the one part of the system (bipartite). The other part remains unchanged. e.g, if we have a density matrix $\rho = \rho^{XY}$ in a Hilbert space $\mathbb{H} = \mathbb{H}_X \otimes \mathbb{H}_Y$ and we want to take transpose over subsystem X then it can be viewed as a map $\mathrm{T}(\rho) \otimes \mathrm{I}$. it can be defined on a composite system as,
P^{T_X} :

$$|1\psi\rangle\langle 0\phi| \implies |0\psi\rangle\langle 1\phi|, |0\psi\rangle\langle 1\phi| \implies |1\psi\rangle\langle 0\phi|.$$

Similarly,
P^{T_Y} :

$$|\psi 1\rangle\langle\phi 0| \implies |\psi 0\rangle\langle\phi 1|, |\psi 0\rangle\langle\phi 1| \implies |\psi 1\rangle\langle\phi 0|,$$

where ψ, $\phi \in$ 1,0.

2.4 Entanglement Detection in States

In quantum mechanics, we can understand those phenomenons that are unknown to classical mechanics e.g, the quantum states superposition, interference, or tunneling. These effects are shown by the systems that consist of single particles and are explained on the basis of quantum systems. This edge of quantum mechanics is not restricted to single-particle systems but extended to multipartite systems as well. The connections between subsystems bring about an extra qualification from classical frameworks. Classically the correlations are dealt with the probability theory but this theory is unable to perform in terms of quantum systems. The very first problem in this scenario was proposed by Einstein, Podolsky and Rosen [1]. According to their views in quantum mechanics, there is remote action and they gave the concept of locality and realism.

States that show such non-classical correlations are referred to as *entangled states*. Now we will look at different tools which can be used to differentiate between classical and quantum correlations.

2.4.1 Separability and Entanglement

Separability is defined via the existence of a decomposition of a state into tensor product states, in the case of pure states, or into a convex sum of tensor products for mixed states. To prove that whether the given state is separable or entangled, one need to look for a representation in which that state can be written in decomposition; that is in tensor product form. Which shows that each tensor is restricted to only one subsystem of the composite system which reveals that the state is separable otherwise the state may be entangled or may be separable but decomposition is non identifiable [23]. In next section we will define entanglement for pure and mixed states.

Pure States

Initially, we consider a bipartite system it is separable if it can be written in the state vector of this system in terms of tensor product of two states which corresponds to each subsystem,

$$|\psi_s\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \,. \tag{2.4.1}$$

If the state cannot be decomposed in this form then it is entangled. e.g,

$$|\psi_e\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle \otimes |\psi_2\rangle + |\phi_1\rangle \otimes |\phi_2\rangle), \tag{2.4.2}$$

where $|\psi_1\rangle \neq |\phi_1\rangle$.

Mixed States

It is represented by the tensor product of the density matrices of subsystems of the composite system [6]. In the case of the bipartite system, it is written as,

$$\begin{aligned} \varrho &= \rho^{(1)} \otimes \rho^{(2)}, \\ \varrho &= \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}, \end{aligned} \tag{2.4.3}$$

where $\rho^{(1)}$ belongs to the first subsystem and $\rho^{(2)}$ to the second.
If this type of decomposition doesn't exist then the state is entangled, meaning $\rho^{(1)}$ does not belong specifically to the first subsystem and vice versa,

$$\varrho \neq \rho^{(1)} \otimes \rho^{(2)}. \tag{2.4.4}$$

Entangled states describe the quantum correlation measurement in different subsystems of a single composite system which can not be described by the probability theory as in the classical cases.

Limitations

The previous definition of separable and entangled states provides information only about the presence or absence (detection) of entanglement for any system. We can not guess anything about the percentage or quantity of entanglement in the system. In next section we will discuss different criterion for entanglement detection and quantification.

2.4.2 Schmidt Decomposition

For a pure bipartite system composed of system 'A' and 'B', there exists a state ($|\Psi\rangle \in \mathcal{H}$) that is the tensor product of two Hilbert spaces belongs to subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$. For this type of system there exists at least one orthonormal basis for which the state can be decomposed as,

$$|\Psi\rangle = \sum_i \alpha_i |\psi_i\rangle \otimes |\phi_i\rangle , \tag{2.4.5}$$

where $|\psi_i(i = 1, 2, ...)\rangle$ are the orthonormal basis belongs to $\mathcal{H}_A$ and $|\phi_i(i = 1, 2, ...)\rangle$ represents the $\mathcal{H}_B$. These basis are called Schmidt basis and the coefficient α_i is termed as Schmidt coefficient that satisfies certain conditions $\alpha_i \geq 0$ and $\sum_i \alpha_i^2 = 1$. This expansion is known as Schmidt decomposition [25].
Entanglement depends upon the number of non-zero Schmidt coefficients.

- If the number of Schmidt coefficients is 1 the state is separable, Sch-no=1.
- If the number of Schmidt coefficients is greater than 1, the state is entangled, Sch-no > 1.

To compute the Schmidt coefficient, we use the reduced density matrix approach as,

$$\begin{aligned} & Tr_{\text{A or B}}(|\Psi\rangle \langle\Psi|), \\ \rho_{\text{A}} &= \sum_i \alpha_i^2 |\psi_i\rangle \langle\psi_i| , \\ \rho_{\text{B}} &= \sum_i \alpha_i^2 |\phi_i\rangle \langle\phi_i| , \end{aligned} \tag{2.4.6}$$

the eigenvalues of these matrices are same and termed as Schmidt coefficients.
Nevertheless, this approach is relatively simple and covers a broad range of systems but it has certain limitations.

Limitations

This method is applicable to the bipartite systems only and useful for entanglement detection but provides no information about the extent of entanglement in the system. In the next step, we will apply this approach to a system and try to evaluate whether the system is entangled or not.

2.5 Entanglement Source

2.5.1 Spontaneous Parametric Down Conversion

This is a technique that is used to create a pure bipartite system composed of two photons. Generally, in Spontaneous Parametric Down Conversion (SPDC), a photon wave which is referred to as a pump wave passes through a birefringent crystal (collinear or noncollinear) that transforms some of the photons into two photons of larger wavelength that are labeled signal(s) and idler(i). All the photons did not split into two which restricts them from interference with each other and chances of parametric amplification reduce during their travel through a crystal. The frequencies of signal (ν_1) and idler (ν_2) obey the conservation law ($\nu_1+\nu_2 = \nu_p$). The angle between wave vectors ($\vec{k}_{1,2}$) of resultant photons and wave vector of pump ($\vec{k}_p$) gives information about their propagation and conservation of momentum.It is the energy and momentum of the pup photon that initiate this conversion and termed as phase-matching when it transform. The polarization of photons is a trait that can be controlled by different factors like frequency of pump photon, angle of incident, crystal type but it can be horizontal(H) or vertical(V). On the basis of polarization of signal(s) and idler(i) photon there are

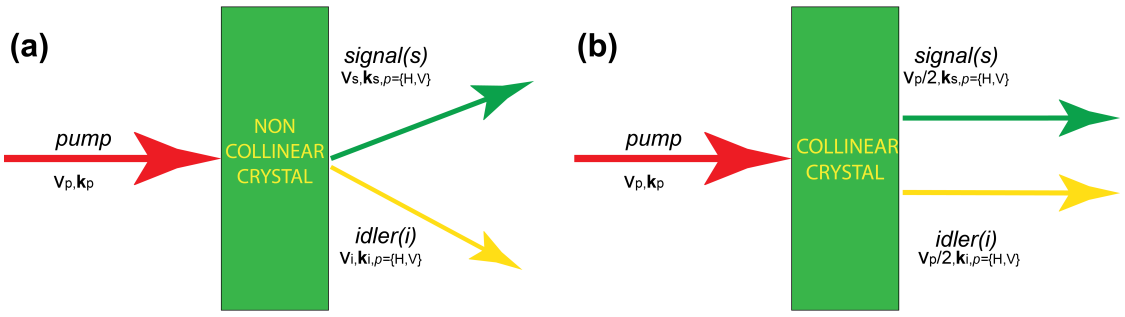

Figure 2.5.1: Spontaneous Parametric Down-Conversion: $\nu_{s,i}$ and $\mathbf{k}_{s,i}$ are frequencies and wave vectors of signal and idler; (a)non-collinear and non-degenerate case, and (b) collinear and degenerate case.

two types of results that can be achieved type-I and type-II. In type-I both s and i have same polarization (HH or VV) while in Type-II they posses different polarization (HV or VH).

2.5.2 Schmidt Decomposition of SPDC

To apply Schmidt Decomposition (SD) on SPDC, we consider the simplest collinear degenerate case in which both signal and idler photons have the same frequency $\nu_p/2$ and they travel in the same direction the only property that can be different between them is polarization which can be our entangled trait. As polarization can be represented in the form of a qubit so we will use this notation to label the state of this system. We can write our system in Schmidt basis as,

$$\Phi(1,2) = \sum_{n=1}^{2}\sum_{m=1}^{2} \alpha_{n,m}\nu_n^{(1)} \otimes \nu_m^{(2)}, \tag{2.5.1}$$

where ν_n and ν_m separately form complete set of basis as $\sum_{n=1}^{2} \nu_n \otimes \nu_n^{\dagger} = 1$.
To specify Schmidt basis and Schmidt coefficient we will follow the standard procedure with certain restrictions of this system. By considering distinguishable particles we have in total four possible terms that are independent of each other and can be written as,

$$\Phi(1,2) = \alpha_{0,0}\left|0,0\right\rangle + \alpha_{0,1}\left|0,1\right\rangle + \alpha_{1,0}\left|1,0\right\rangle + \alpha_{1,1}\left|1,1\right\rangle, \tag{2.5.2}$$

where $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$ with reference to polarization.
As our system is in a pure state so we will build a density matrix for it which is,

$$\begin{aligned}\rho_{1,2} = {} & |\alpha_{0,0}|^2 \left|0,0\right\rangle\left\langle 0,0\right| + \alpha_{0,0}\alpha_{0,1}^* \left|0,0\right\rangle\left\langle 0,1\right| + \alpha_{0,0}\alpha_{1,0}^* \left|0,0\right\rangle\left\langle 1,0\right| + \alpha_{0,0}\alpha_{1,1}^* \left|0,0\right\rangle\left\langle 1,1\right| \\ & + \alpha_{0,1}\alpha_{0,0}^* \left|0,1\right\rangle\left\langle 0,0\right| + |\alpha_{0,1}|^2 \left|0,1\right\rangle\left\langle 0,1\right| + \alpha_{0,1}\alpha_{1,0}^* \left|0,1\right\rangle\left\langle 1,0\right| + \alpha_{0,1}\alpha_{1,1}^* \left|0,1\right\rangle\left\langle 1,1\right| \\ & + \alpha_{1,0}\alpha_{0,0}^* \left|1,0\right\rangle\left\langle 0,0\right| + \alpha_{1,0}\alpha_{0,1}^* \left|1,0\right\rangle\left\langle 0,1\right| + |\alpha_{1,0}|^2 \left|1,0\right\rangle\left\langle 1,0\right| + \alpha_{1,0}\alpha_{1,1}^* \left|1,0\right\rangle\left\langle 1,1\right| \\ & + \alpha_{1,1}\alpha_{0,0}^* \left|1,1\right\rangle\left\langle 0,0\right| + \alpha_{1,1}\alpha_{0,1}^* \left|1,1\right\rangle\left\langle 0,1\right| + \alpha_{1,1}\alpha_{1,0}^* \left|1,1\right\rangle\left\langle 1,0\right| + |\alpha_{1,1}|^2 \left|1,1\right\rangle\left\langle 1,1\right|. \end{aligned} \tag{2.5.3}$$

In matrix form,

$$\rho_{1,2} = \begin{pmatrix} |\alpha_{0,0}|^2 & \alpha_{0,0}\alpha_{0,1}^* & \alpha_{0,0}\alpha_{1,0}^* & \alpha_{0,0}\alpha_{1,1}^* \\ \alpha_{0,1}\alpha_{0,0}^* & |\alpha_{0,1}|^2 & \alpha_{0,1}\alpha_{1,0}^* & \alpha_{0,1}\alpha_{1,1}^* \\ \alpha_{1,0}\alpha_{0,0}^* & \alpha_{1,0}\alpha_{0,1}^* & |\alpha_{1,0}|^2 & \alpha_{1,0}\alpha_{1,1}^* \\ \alpha_{1,1}\alpha_{0,0}^* & \alpha_{1,1}\alpha_{0,1}^* & \alpha_{1,1}\alpha_{1,0}^* & |\alpha_{1,1}|^2 \end{pmatrix}. \tag{2.5.4}$$

Tracing out subsystem 2, the reduced density matrix we will get is,

$$\rho_1 = \begin{pmatrix} |\alpha_{0,0}|^2 + |\alpha_{0,1}|^2 & \alpha_{0,0}\alpha_{1,0}^* + \alpha_{0,1}\alpha_{1,1}^* \\ \alpha_{1,0}\alpha_{0,0}^* + \alpha_{1,1}\alpha_{0,1}^* & |\alpha_{1,0}|^2 + |\alpha_{1,1}|^2 \end{pmatrix}. \tag{2.5.5}$$

This matrix have two eigenvalues that are,

$$\lambda_{\pm} = \frac{1}{2}\left[1 \pm \sqrt{1 - 4\left(|\alpha_{1,1}\alpha_{0,0} - \alpha_{1,0}\alpha_{0,1}|^2 + |\alpha_{1,1}|^2|\alpha_{0,0}|^2 + |\alpha_{1,0}|^2|\alpha_{0,1}|^2\right)}\right]. \quad (2.5.6)$$

Till now we had not chosen our system type, whether it is Type-I or II. If we consider our system to be Type-I in which the polarization of both the photons should be same then the reduced density matrix will become,

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.5.7)$$

This matrix has two eigenvalues $\lambda_1 = \frac{1}{2} = \lambda_2$ which shows that it is an entangled system. Here we reached the conclusion that the states of the system under consideration are entangled but we are still unaware of the amount of entanglement in this system.

2.6 Entanglement Quantification

All the above-discussed techniques only detect the entanglement in the system but we have no information about its quantity. To measure the extent of entanglement in a composite system we devise a function, entanglement quantifier,that have certain requirements [11].

Requirements and qualities for entanglement measures

A good entanglement measure 'E' should fulfill the following axioms [26] but the commonly used entanglement measuring tools may not fulfill all of them.

1. *Separability*: If density operator ρ of a composite system is separable then, E(ρ)=0.

2. *Normalization*: For a maximally entangled two d-dimensional system E is given as,

$$\mathrm{E}(P_+^d) = \log(d). \quad (2.6.1)$$

3. *No increasing effect under LOCC*: We cannot increase entanglement by applying local operations and classical communication (LOCC) to ρ,

$$\mathrm{E}(\Lambda_{LOCC}(\rho)) \leq \mathrm{E}(\rho). \tag{2.6.2}$$

4. *Continuity*: If the distance between two density matrices approaches zero then the difference between their entanglement should be zero, i.e.

$$\mathrm{E}(\rho) - \mathrm{E}(\eta) \to 0 \quad \text{for} \quad \|\rho - \eta\| \to 0. \tag{2.6.3}$$

5. *Additivity*: Identical copies of a system should contain 'm' times entanglement of a single system

$$\mathrm{E}(\rho^{\otimes m}) = m\mathrm{E}(\rho). \tag{2.6.4}$$

6. *Subadditivity*: Entanglement of a state which is in turn is a tensor product of two states can never be greater than the sum of entanglement of each constituent state,

$$\mathrm{E}(\rho \otimes \eta) \leq \mathrm{E}(\rho) + \mathrm{E}(\eta). \tag{2.6.5}$$

7. *Convexity*: The entanglement measure should be a convex function

$$\mathrm{E}(\lambda\rho + (1-\lambda)\eta) \leq \lambda\mathrm{E}(\rho) + (1-\lambda)\mathrm{E}(\eta), \tag{2.6.6}$$

for $0 < \lambda < 1$.

Now we will discuss some entanglement measures in next section.

2.6.1 Schmidt Measures

Schmidt number is used to get an estimated measurement of entanglement which is known as a Schmidt measure (Hartley strength) and is defined as,

$$\mathrm{E}_S(|\Psi\rangle) = \log_2(\lambda), \tag{2.6.7}$$

where λ=Schmidt no of state $|\Psi\rangle$.
The unit of entanglement in this measurement is e-bits (entangled bits) which is defined by the entanglement of Bell states as the Schmidt no of bell state is 2 and consequently, $\mathrm{E}_S = 1$ e-bit. So the unit in this measure is the entanglement present in a Bell state.

von Neumann entropy

Shannon entropy is the measurement of ignorance in the classical information similarly von Neumann entropy [27] is define as the ignorance about a quantum system and is defined as,

$$\mathrm{E} = S(\rho) = -Tr(\rho \log \rho). \tag{2.6.8}$$

In term of eigen values it can be written as

$$\mathrm{E} = S(\rho) = -\sum_j \lambda_j \log \lambda_j, \tag{2.6.9}$$

where λ_j are the eigen values of ρ. Actually, $\mathrm{S}(\rho)$ is the uncertainty in the quantum state before measurement.

Different properties of von Neumann entropy

- For a d-dimensional Hilbert space the von Neumann entropy is,

  $$0 \leq S(\rho) \leq \log_2 d, \tag{2.6.10}$$

  where $\mathrm{S}(\rho) = 0$ for separable state.

- $S(\rho) = \log_2 d$ for completely mixed state.

- For a bipartite system

  $$S(\rho^{XY}) \leq S(\rho^X) + S(\rho^Y), \tag{2.6.11}$$

  the entropy of a bipartite system is always less than or equal to the sum of individual entropy of the subsystems.

- For a separable bipartite system, the above equation turns into equality only [28].

  $$S(\rho^X \otimes \rho^Y) = S(\rho^X) + S(\rho^Y). \tag{2.6.12}$$

To quantify entanglement with von Neumann entropy we consider here an example of an entangled state written as,

$$|\Psi(\alpha)\rangle_{XY} = \sin\alpha\,|01\rangle_{XY} + \cos\alpha\,|10\rangle_{XY}\,. \tag{2.6.13}$$

To extract subsystem A we use the reduced density matrix approach,

$$\begin{aligned}\rho^X &= Tr_Y(\rho^{XY})\\ &= Tr_Y(\sin^2\alpha\,|01\rangle\langle 01| + \cos\alpha\sin\alpha\,|01\rangle\langle 10| + \cos\alpha\sin\alpha\,|10\rangle\langle 01| + \cos^2\alpha\,|10\rangle\langle 10|)\\ &= \cos^2\alpha\,|1\rangle\langle 1| + \sin^2\alpha\,|0\rangle\langle 0|\,.\end{aligned} \tag{2.6.14}$$

In matrix notation,

$$\rho^X = \begin{pmatrix}\sin^2\alpha & 0\\ 0 & \cos^2\alpha\end{pmatrix}. \tag{2.6.15}$$

Therefore,

$$\begin{aligned}\mathrm{E}(\alpha) &= -Tr(\rho^X\log_2\rho^X)\\ &= -Tr\left(\begin{pmatrix}\sin^2\alpha & 0\\ 0 & \cos^2\alpha\end{pmatrix} log_2 \begin{pmatrix}\sin^2\alpha & 0\\ 0 & \cos^2\alpha\end{pmatrix}\right)\\ &= -Tr\left(\begin{pmatrix}\sin^2\alpha & 0\\ 0 & \cos^2\alpha\end{pmatrix}\begin{pmatrix}\log_2\sin^2\alpha & 0\\ 0 & \log_2\cos^2\alpha\end{pmatrix}\right)\\ &= -2(\sin^2\alpha\log_2\cos\alpha + \cos^2\alpha\log_2\sin\alpha).\end{aligned} \tag{2.6.16}$$

Here we can compute the eigenvalues of ρ^X which is $\sin^2\alpha$ and $\cos^2\alpha$. This eigenvalue is used to obtain von Neumann entropy. From here we can obtain maxima and minima of $\mathrm{E}(\alpha)$ and see how it measure entanglement

$$\begin{aligned}\frac{d\mathrm{E}}{d\alpha} &= -2\frac{d}{d\alpha}(\sin^2\alpha\log_2\sin\alpha + \cos^2\alpha\log_2\sin\alpha),\\ &= 2\sin 2\alpha\log_2\cot\alpha,\end{aligned} \tag{2.6.17}$$

where $\dfrac{d\mathrm{E}}{d\alpha}$ will give zero at sin2α=0 or α=$\dfrac{m\pi}{2}$. E is zero at these points which means that this state is separable at these points,

$$\begin{aligned}|\Psi(\alpha)\rangle_{XY} &= |10\rangle_{XY}\,, \quad At\ \ \alpha = 0\,|\Psi(\alpha)\rangle_{XY}\,,\\ &= |01\rangle_{XY}\,, \quad At\ \ \alpha = \frac{\pi}{2}.\end{aligned} \tag{2.6.18}$$

The other set of zeroes of $\dfrac{d\mathrm{E}}{d\alpha}$ correspond to $\log_2\cot\alpha = 0$ or $\cot\alpha = 1$ or $\alpha = \dfrac{\pi}{4}$ $\pm$ 2mπ. At these points $\mathrm{E}(\alpha)$=1, which corresponds to the maxima of von Neumann entropy.

von Neumann entropy in SPDC

The system discussed in section 2.5.1 results into reduced density matrix 2.5.7,

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.6.19}$$

The eigenvalues of this matrix are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and von Neumann entropy was defined as,

$$\mathrm{E} = S(\rho) = -\sum_j \lambda_j \log \lambda_j, \tag{2.6.20}$$

The von Neumann entropy with these eigenvalues is 1 which means the states are completely entangled.

Negativity

It is one of the popularly used measures of entanglement defined as,

$$\begin{aligned} N(\rho) &= \frac{1}{2}(\|\rho^{Tx}\|_1 - 1), \\ &= |\Sigma_i \lambda_i|, \\ &= \frac{1}{2}\Sigma_j(|\lambda_j| - 1), \end{aligned} \tag{2.6.21}$$

where ρ^{Tx} is the partial transpose of the density operator of a bipartite system,'i' runs over the negative eigenvalues of the density operator ρ^{Tx} and j runs over all the eigenvalues of ρ^{Tx}. If all the eigenvalues are positive then negativity vanishes and the state is un-entangled while the non-vanishing negativity indicates entanglement. Negativity is unable to characterize all the entangled states.

2.6.2 Concurrence

For a bipartite system concurrence [13] is a quantitative measure of the amount of entanglement which is defined as,

$$\mathrm{C}(|\Psi\rangle) = |\langle\Psi|\tilde{\Psi}\rangle|, \tag{2.6.22}$$

where $|\tilde{\Psi}\rangle = \mathbf{Y} \otimes \mathbf{Y} |\Psi\rangle^*$, $\mathbf{Y}$ is pauli matrix and $|\Psi\rangle^*$ is complex conjugate of the sate $|\Psi\rangle$. Concurrence can be calculated from the density operator ρ of the state as,

$$\begin{aligned} \mathrm{C}(\rho) &= \max(0, \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4); \\ Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \end{aligned} \tag{2.6.23}$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$ are eigenvalue of the matrix R=$\sqrt{\rho^{1/2}\tilde{\rho}\rho^{1/2}}$, where $\tilde{\rho} = \mathbf{Y}\otimes\mathbf{Y}\rho^*\mathbf{Y}\otimes\mathbf{Y}$. when the concurrence is zero, state is separable otherwise it is entangled. For maximally entangled state concurrence is 1 [29].

2.6.3 Entanglement of Formation

The entanglement of formation E(ρ) for an entangled state(mathematically) is the number of resources needed to create that particular entangled state [27]. On the basis of concurrence, it is defined as,

$$\mathrm{E}(\rho) = H_2 \left(\frac{1 + \sqrt{1 - (\mathrm{C}(\rho))^2}}{2} \right), \tag{2.6.24}$$

where H_2 is binary entropy function given as,

$$H_2(a) = -a \log_2(a) - (1 - a) \log_2(1 - a). \tag{2.6.25}$$

From here we can say that entanglement of formation is the same as the von Neumann entropy of the subsystems for a pure state.

In this chapter, our main focus was on distinguishable particles. We studied different entanglement detection and quantification parameters but what happen if we have a system composed of indistinguishable particles. In next chapter we will look at the complications arise due to indistinguishability and there possible solutions.

Chapter 3

Indistinguishability Transition

In classical physics, it is irrelevant that the system consists of distinguishable or indistinguishable particles. They can always be identified with the help of trajectory and other classical properties while in quantum mechanics, we cannot trace a particle's trajectory and all the quantum properties are the same, it is impossible to identify a certain indistinguishable particle among others. This behavior creates certain complications in indistinguishable systems that cannot be summarized with a distinguishable particle approach. In this chapter, we will discuss all these complications and their possible solutions.

3.1 Distinguishable Particles

Particles can be considered distinguishable either on the basis of their intrinsic properties (mass, electric charge, etc) but generally, we deal with the systems that are composed of particles with the same intrinsic properties. In this case, we can track the trajectory of each particle that makes them distinguishable from others. But this approach is not applicable in quantum systems because the position of particles is not definite. In quantum mechanics, it is defined by wave functions that give a probability of finding particles at a specific position. As the particle move their wave functions overlap and they become indistinguishable. So, we can say that most of the identical particle systems in quantum mechanics are indistinguishable.

Here we consider a case in which a composite system consists of more than two subsystems and all of them are completely distinguishable. There can be different types of correlations possible in this system. In an N-particle system, there may be a particle's group that can be entangled with another group and non-entangled to remaining particles with respect to one property but the same may not be true with respect to any other trait of these particles. We can identify the disentangled groups by repeated analysis of each particle in the system and group them into separate subsystems that are completely non-entangled to each other which is a quite complicated procedure. To deal with this complication we follow the previously discussed procedures of separability and Schmidt decomposition that we use for a bipartite system [30]. By following these procedures we will look at different situations of the system in which the simplest is the completely non-entangled system.

3.1.1 Completely Non-entangled state

The components of a composite system may or may not be entangled and there may be a case in which some of the constituents are entangled among themselves but non-entangled to the other. Here we will see the representation of those systems that are completely non-entangled.

Pure state

The pure state of a composite system consists of 'N' distinguishable particles $|\Psi_{1,...,N}\rangle \in \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_N$ is non-entangled if there exist N one-dimensional projection operator where P^i belongs to $\mathcal{H}_i$ where $1 \leq i \leq N$.

$$Tr^{1+...+N}[P^i \left|\Psi_{1,...,N}\right\rangle \left\langle\Psi_{1,...,N}\right|] = 1. \tag{3.1.1}$$

In terms of generalization to the bipartite system, the pure state is completely non-entangled if we can factorize it completely. i.e, there are N sub-states each belongs to a single particle only. i.e,

$$\left|\Psi_{1,...,N}\right\rangle = \left|\psi_1\right\rangle \otimes ... \otimes \left|\psi_N\right\rangle, \tag{3.1.2}$$

where $|\psi_i\rangle \in \mathbb{H}_i$ where $\forall i = 1, ..., N$

For such a system there are N reduced density matrices which can be calculated from density matrix,

$$\rho^i = Tr^{\forall j \neq k}[|\Psi_{1,..,j,..,k,..,N}\rangle \langle \Psi_{1,..,j,..,k,..,N}|]. \tag{3.1.3}$$

Sometimes a group of particles in a system is entangled with another group in the same system ut non-entangled with others.

3.2 Entanglement between Subsystems

In this section, we'll consider the situation in which different subsystems are entangled with some constituent and non-entangled to the others. On this basis, the system can be divided into different groups which are entangled themselves but are non-entangled to other constituents.

If we consider a many-particle quantum system represented by $|\Phi_{1...N}\rangle$ belongs to the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_N$. This state can be divide into two different groups which contain the particles completely non-entangled. There are three conditions that needs to be fulfilled in order to divide the system into sub-groups ($S_{1,...,m}$ and $S_{m+1,...,N}$) that are not correlated to each other.

1. The state vectors of the system are factorizable. i.e,

$$|\Psi_{1,...,N}\rangle = |\phi_{1,...,m}\rangle \otimes |\chi_{m+1,...,N}\rangle , \tag{3.2.1}$$

   where

   $|\phi_{1,...,m}\rangle \in \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_m$,

   $|\chi_{1+m,...,N}\rangle \in \mathcal{H}_{m+1} \otimes ... \otimes \mathcal{H}_N$.

2. There exists a projection operator $P^{1,...,m}$ belongs to many folds of $\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_m$ that satisfies the following equation.

$$Tr^{(1+...+N)}[P^{(1,...,m)} \otimes \mathbb{I}^{(m+1,...,N)} \rho^{(1,...,N)}] = 1. \tag{3.2.2}$$

3. The reduced density operator should be recovered by using the following relation,

$$\rho^{(1,\ldots,m)} = Tr^{(m+1,\ldots,N)}[\rho^{(1,\ldots,N)}]. \tag{3.2.3}$$

The separable state of a composite system that consists of N particles can be represented in the form of a tensor product as,

$$|\Psi_{1,\ldots,N}\rangle = |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle\,, \tag{3.2.4}$$

where $|\psi_i\rangle$ belongs to i^{th} subsystem only.
If a system is not separable then the entanglement in this composite system can be detected with the strict correlation between appropriate observables of the components of the system. If a subsystem ($S_{1,\ldots,m}$) of this system consists of 'm' particles is non-entangled with subsystem ($S_{m+1,\ldots,N}$) consists of all other particles in that system then the action of the observable, $\mathbf{X}$(1, .. ,m) belong to the Hilbert space $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m$ and $\mathbf{Y}$(m+1, ... ,N) belongs to $\mathcal{H}_{m+1} \otimes \ldots \otimes \mathcal{H}_N$, on the pure state $|\Phi_{1\ldots N}\rangle$ of the system is written as,

$$\langle\Psi_{1\ldots N}|X \otimes Y|\Psi_{1\ldots N}\rangle = \langle\psi_{1,\ldots,m}|X|\psi_{1,\ldots,m}\rangle \langle\psi_{1+m,\ldots,N}|Y|\psi_{1+m,\ldots,N}\rangle\,, \tag{3.2.5}$$

which shows that the joint probabilities of two non-entangled subsystems factorize independent of the measurement process. There are different criteria for identical particles that need to be followed.

3.3 Formalism for Identical Particles

3.3.1 First Quantization Approach

The standard approach to represent quantum identical particles follows the process of labeling different particle states (with numbers or symbols) in first quantization. The usual tools of entanglement detection and quantification e.g, separability criterion, Schmidt decomposition, and von Neumann entropy cannot be used directly in identical particles (IP) cases. So, we will look for a method that can be used to characterize entanglement in IP. In the end, we will test these methods by using the first quantization approach [31].

3.3.2 Exchange Degeneracy and Symmetrization Postulate

In classical mechanics, it is irrelevant for the description of a system to be a combination of IP(identical particles) or NIP(Non-identical particles) because they follow different trajectories that can be traced independently of the identity of the particles. We can label them without affecting their properties. All the label schemes come to the same dynamic conduct of the particles that shows their label independence [31].

In quantum mechanics, the IP consider truly elementary when the system is in the ground state and the exchange of energy between environments is much less than the excited state energy of the particles of the system. The trajectory of the particles can never be found without disturbing the system in quantum physics. So to identify different particles with non-zero probability in a specific region that are assigned with un-physical labels, this approach of identification is called Standard Approach(SA) [15]. The labeling of IP modifies their properties. e.g, consider two IP in a certain region of space. According to SA if we label them as 1 and 2 associated with states $|A\rangle$ and $|B\rangle$ then different quantum mechanical descriptions of the whole system can be given as [35],

$$|\Phi\rangle = |A\rangle_1 \otimes |B\rangle_2 . \tag{3.3.1}$$

This state shows that the particle which is labeled as 1 is in state $|A\rangle$ and particle 2 is in state $|B\rangle$. As these quantum particles are identical so they can be in opposite states which is another possible state of this system,

$$|\Phi\rangle = |A\rangle_2 \otimes |B\rangle_1 , \tag{3.3.2}$$

The overall state of the system with certain probabilities is,

$$|\Phi\rangle = \alpha|A\rangle_1 \otimes |B\rangle_2 + \beta|A\rangle_2 \otimes |B\rangle_1, \tag{3.3.3}$$

This state shows that the probability of any particle is not a hundred percent at any place before measurement. Where $|\alpha|^2 + |\beta|^2 = 1$, which is known as exchange degeneracy. Both the particles are identical but the nonphysical labels 1 and 2 made them

mathematical distinguishable. These label associate different kets to the same sets of eigenvalues and predicts these particles physically. The exchange degeneracy does not allow us to write the state associated with a specific particle uniquely. We cannot know the exact form of the state $|\Phi\rangle$ without actual measurement. Different mathematical possibilities arise analogous to non-identical particles due to this labeling approach [12]. The non-identical behavior in identical particles is counter by the introduction of symmetrization postulates [31] that also remove the exchange degeneracy. According to it the wave function can be symmetric or anti-symmetric depending upon the nature of particles i.e, fermions (half-integral spin $\frac{\hbar}{2}, \frac{3\hbar}{2}, \frac{5\hbar}{2}$,...) or boson (integral spin $0, \hbar, 2\hbar, 3\hbar$,...). For N particles the wave function can be given as,

$$|\Phi(x_1, x_2, ..., x_i, x_j, ..., x_N)| = \pm|\Phi(x_1, x_2, ..., x_j, x_i, ..., x_N)|. \tag{3.3.4}$$

where the negative sign is for fermions and positive sign is for boson on exchange of i^{th} particle with j^{th}.

The identical particles are mainly of two types they can either be bosons or fermions and their wave function is symmetric and anti-symmetric respectively. For the simplest case of two particles, the wave function can be written as,

$$\begin{aligned} \Phi_s(x_1, x_2) &= \frac{1}{\sqrt{2}}[\phi(x_1, x_2) + \phi(x_2, x_1)], \\ \Phi_a(x_1, x_2) &= \frac{1}{\sqrt{2}}[\phi(x_1, x_2) - \phi(x_2, x_1)]. \end{aligned} \tag{3.3.5}$$

For two non-interacting identical particle system the wave function is defined as,

$$\begin{aligned} \Phi_s(x_1, x_2) &= \frac{1}{\sqrt{2}}[\phi_1(x_1)\phi_2(x_2) + \phi_1(x_2)\phi_2(x_1)], \\ \Phi_a(x_1, x_2) &= \frac{1}{\sqrt{2}}[\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)]. \end{aligned} \tag{3.3.6}$$

In general it can be written as,

$$\begin{aligned} \Phi_s(x_1, x_2) &= \frac{1}{\sqrt{2!}}\sum_P (1)^P[\phi_1(x_1)\phi_2(x_2)], \\ \Phi_a(x_1, x_2) &= \frac{1}{\sqrt{2!}}\sum_P (-1)^P[\phi_1(x_1)\phi_2(x_2)], \end{aligned} \tag{3.3.7}$$

where P is the permutation and the determinant form of above anti-symmetric equation is,

$$\Phi_a(x_1, x_2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) \\ \phi_2(x_1) & \phi_2(x_2) \end{vmatrix}. \tag{3.3.8}$$

For three spin-$\frac{1}{2}$ particles it is,

$$\Phi_a(x_1, x_2, x_3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) & \phi_1(x_3) \\ \phi_2(x_1) & \phi_2(x_2) & \phi_2(x_3) \\ \phi_3(x_1) & \phi_3(x_2) & \phi_3(x_3) \end{vmatrix}. \tag{3.3.9}$$

The generalization to 'N' anti-symmetric state it is,

$$\Phi_a(x_1, x_2..., x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) & \cdots & \phi_1(x_N) \\ \phi_2(x_1) & \phi_2(x_2) & \cdots & \phi_2(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_3(x_1) & \phi_3(x_2) & \cdots & \phi_3(x_N) \end{vmatrix}. \tag{3.3.10}$$

This N×N determinant is called **Slater determinant** and it involves only a single-particle state because of the Pauli exclusion principle which states that no two identical fermions can occupy the same state. This determinant is used to determine the Slater-Schmidt number which is analogous to Schmidt number and it is used for entanglement detection [36].

Experimental demonstration of indistinguishability

To illustrate the effects of indistinguishability on the state of the system that look like distinguishable system, we take a system of two electrons located in a double-well potential. We can consider two qubits as a manifestation of electron spin with spin degree of freedom $|\uparrow\rangle, |\downarrow\rangle$ and label spatial wave-functions with $|\psi\rangle$ and $|\phi\rangle$ for left and right potential well respectively. Under all these considerations the Hilbert space of this system is four dimensional, $\{|\psi\downarrow\rangle, |\phi\uparrow\rangle, |\psi\uparrow\rangle, |\phi\downarrow\rangle\}$.

Firstly, we consider two electrons in completely separated square wells as shown in figure 3.3.1. The state of this system is given as,

$$|\Psi_{init}\rangle_{AB} = \frac{1}{\sqrt{2}}[|\psi\downarrow\rangle_A \otimes |\phi\uparrow\rangle_B + |\phi\downarrow\rangle_A \otimes |\psi\uparrow\rangle_B] \tag{3.3.11}$$

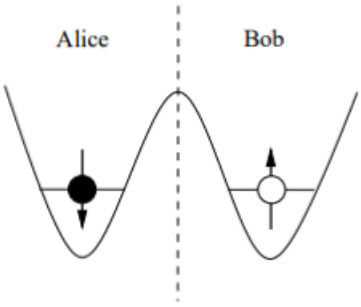

Figure 3.3.1: The initial state of electrons localized in left and right square well observed by Alice and Bob.

After performing measurements or fixing the spatial wave function with Alice and Bob the state will become

$$|\Psi_{init}\rangle_{AB} = |\psi\downarrow\rangle_A \otimes |\phi\uparrow\rangle_B\,, \tag{3.3.12}$$

which represent a distinguishable system.

Now we will disturb square wells potential. In a situation when the energy barrier is removed or both the wells move closer to each other the probability of finding the electrons in each of two wells will gain some value. As both are electrons so they obey fermionic statistics and their wave function will become,

$$\begin{aligned} |\Psi(t_1)\rangle &= \frac{1}{\sqrt{2}}[|\psi\downarrow\rangle_A \otimes |\phi\uparrow\rangle_B - |\phi\uparrow\rangle_A \otimes |\psi\downarrow\rangle_B] \\ &= \frac{1}{\sqrt{2}}\begin{vmatrix} |\psi\downarrow\rangle_A & |\psi\downarrow\rangle_B \\ |\phi\uparrow\rangle_A & |\phi\uparrow\rangle_B \end{vmatrix}. \end{aligned} \tag{3.3.13}$$

The labeling of states is completely un-physical but it introduces the anti-symmetrization

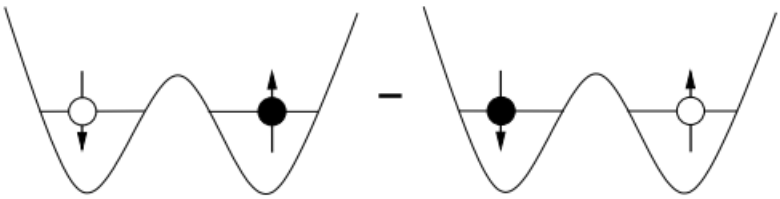

Figure 3.3.2: The de-localization of wave function after lowering the potential tunneling barrier.

in states $|\Psi(t_1)\rangle$ which matches with an entangled state in distinguishable systems but this entanglement is unreal and is due to the indistinguishability of the particles that cannot be used as a resource. To characterize the useful entanglement in indistinguishable particles the term *quantum correlations* is used [33]. The quantum correlation in a system of indistinguishable particles is related to the number of Slater determinants involve in the representation of states. If more than one Slater determinant is involved in writing the simplest state (anti-symmetric combination orthogonal single-particle states) of fermions then it is considered entangled this is analogous to quantum entanglement in the distinguishable systems. To elaborate this point we consider the above system with independent spin coupling J(t) that follow fermionic principles. Then the state of the system will become,

$$\begin{aligned}|\Psi(t_2)\rangle &= \frac{1}{\sqrt{2}}[|\psi\downarrow\rangle_1 \otimes |\phi\uparrow\rangle_2 - |\phi\uparrow\rangle_1 \otimes |\psi\downarrow\rangle_2] + \frac{1}{\sqrt{2}}[|\psi\uparrow\rangle_1 \otimes |\phi\downarrow\rangle_2 - |\phi\downarrow\rangle_1 \otimes |\psi\uparrow\rangle_2], \\ &= \frac{1}{\sqrt{2}}\begin{vmatrix}|\psi\downarrow\rangle_1 & |\psi\downarrow\rangle_2 \\ |\phi\uparrow\rangle_1 & |\phi\uparrow\rangle_2\end{vmatrix} + \frac{1}{\sqrt{2}}\begin{vmatrix}|\psi\uparrow\rangle_1 & |\psi\uparrow\rangle_2 \\ |\phi\downarrow\rangle_1 & |\phi\downarrow\rangle_2\end{vmatrix}.\end{aligned} \tag{3.3.14}$$

That can be seen in figure 3.3.3 . The above equation is in single particle basis and contain two elementary Slater determinants and there is no possible basis that can convert it to single determinant so this state is entangled and contain useful correlations and can be seen by localizing the particles again by raising tunneling barrier. By this the basis separate between Alice and Bob, Alice Hilbert space is $(|\psi\uparrow\rangle, |\psi\downarrow\rangle)$ and Bobs $(|\phi\uparrow\rangle, |\phi\downarrow\rangle)$. Now we can consider both the particles as distinguishable because both the fermions are spatially separated and their final state can be written as [34],

$$|\Psi_{\text{final}}\rangle = \frac{1}{\sqrt{2}}[|\psi\downarrow\rangle_A \otimes |\phi\uparrow\rangle_B + [|\psi\uparrow\rangle_A \otimes |\phi\downarrow\rangle_B]. \tag{3.3.15}$$

That is illustrated in figure 3.3.4. As it is clear from the analysis of this system that labels are creating ambiguity between entanglement and indistinguishability of the particles so we will exercise some other methods that can give better understanding of the system.

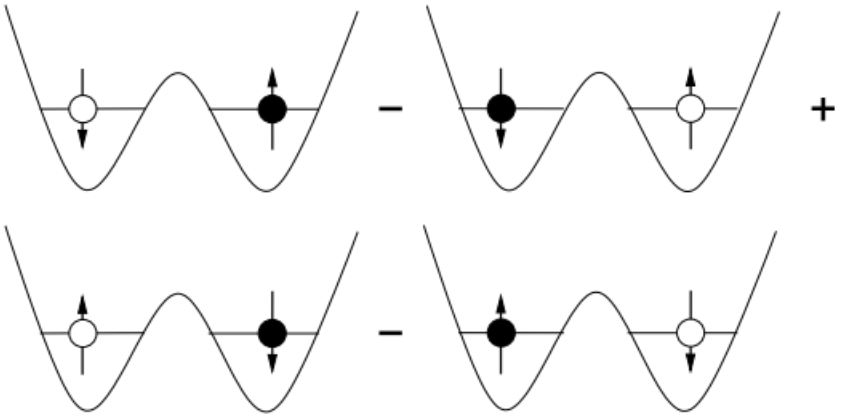

Figure 3.3.3: The de-localization of wave function after lowering the potential tunneling barrier and controlled spin coupling J(t).

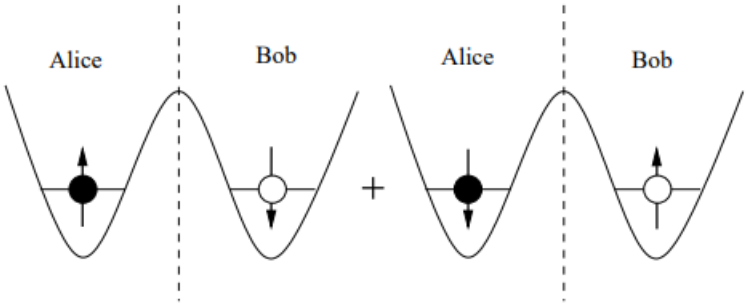

Figure 3.3.4: The final state $|\Psi_{final}\rangle_{AB}$. Localization of wave functions by raising barrier.

3.3.3 Usual Tools and Identical Particle Entanglement

The basic tools we use in entanglement detection and quantification are separability and Schmidt decomposition of the state. A system that consists of two distinguishable particles is separable if it can be represented by a state that is simply a tensor product of two sub-states as,

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle \,, \tag{3.3.16}$$

where $|\Psi\rangle \in \mathcal{H}_s = \mathcal{H}_1 \otimes \mathcal{H}_2$, $|\psi\rangle \in \mathcal{H}_1, |\phi\rangle \in \mathcal{H}_2$. In this type of system, each subsystem is completely independent of others any operation performed on one sub-system does

not affect the state of other system. If they disturb each other state by any mean then they will be considered entangled and their entanglement will be detected by Schmidt decomposition [37],

$$|\Psi\rangle = \sum_{i=1}^{n} \sqrt{\lambda_i}\, |\psi_i\rangle \otimes |\phi_i\rangle\,, \tag{3.3.17}$$

where λ_i is Schmidt coefficient.

If the number of non-zero Schmidt coefficients is more than one then the state is entangled otherwise not. The quantification of entanglement is done with the help of von Neumann entropy. Which is given as,

$$S(\rho_A) = S(\rho_A) = -\sum_{i=1} \lambda_i \log_2 \lambda_i. \tag{3.3.18}$$

In the case of identical particles, these criteria of separation and Schmidt decomposition does not hold and two identical particles non-entangled state is given as,

$$|\Psi\rangle_{\pm} = \frac{1}{\sqrt{2}}(|\psi\rangle_1 \otimes |\phi\rangle_2 \pm |\psi\rangle_2 \otimes |\phi\rangle_1), \tag{3.3.19}$$

where the +ve sign is for bosonic particle state and -ve is for fermionic.

- **Separability:** According to the criteria defined earlier, the states of identical particles are separable until the symmetrization postulate does not apply. After it, each and every state of IP become entangled which is an unrealistic interpretation of symmetrization postulates as it is not possible for two independently prepared identical particles to be entangled regardless of how far they are.[13]

- **Schmidt number:** As the labeling of particles made them distinguishable so one can think that we can apply a non-identical particle approach to detect and quantify entanglement but all these tools give un-physical correlation and most of the times entanglement is only the reflection of the symmetrization.

- **von Neumann entropy:** We can apply the concept of partial trace on IP but it also depends upon mock labels that diminish the concept of indistinguishability. So a careful analysis is required for IP.

- There is a link proposed between von Neumann entropy and IP entanglement which is discussed under different approaches of first quantization (which require a separate approach for bosons and fermions) [38, 37, 39] and second quantization in which the concept of locality is used [40, 39].

A greater part of physicists agrees that two identical particles which are spatially separated and never interacted before are intrinsically uncorrelated and this remains true until they interact with each other. We can consider them distinguishable on the basis of their spatial state. But when both IP has a certain probability of occupying the same spatial state then the particles can not be considered uncorrelated and their entanglement is detected with the help of mock labels. But there are some unrealistic complications that arise due to this approach.

3.3.4 Slater-Schmidt Decomposition

As we had discussed earlier that Schmidt decomposition is used to detect entanglement in the system composed of distinguishable particles and later we use the Schmidt number to quantify by using von-Neuman entropy. Similarly here we introduce the procedure of Slater-Schmidt decomposition which gives an alternative mechanism to deal with indistinguishable particles. In the limit of finite-dimensional single-particle Hilbert space, we deal separately with fermionic and bosonic cases because they show different behavior.

3.3.5 Fermionic Entanglement

Theorem: For any anti-symmetric (N × N) complex matrix X [i.e X $\in \mathcal{M}$(N,$\mathbb{C}$) and $X^T = -X$] there exist a unitary transformation matrix U such that $X = UZU^T$, where Z is the block diagonal matrix of the form,

$$Z = diag[Z_0, Z_1, ..., Z_M], \qquad Z_0 = 0, \qquad Z_i = \begin{pmatrix} 0 & z_i \\ -z_i & 0 \end{pmatrix}, \tag{3.3.20}$$

where Z_0 is the (N - 2M) × (N - 2M) null matrix and z_i are complex numbers. Equivalently, Z is the direct sum of the (N - 2M) × (N - 2M) null matrix and the M (2 × 2)

complex antisymmetric matrices Z_i.

Any state vector representing two fermions of spin 's' can be written as,

$$|\Phi(1,2)\rangle = \sum_{j=1}^{(2s+1)/2} \alpha_j \frac{1}{\sqrt{2}}[|2j-1\rangle_1 \otimes |2j\rangle_2 - |2j\rangle_1 \otimes |2j-1\rangle_2], \tag{3.3.21}$$

where $\{|j-1\rangle, |2j\rangle\}$ are orthonormal basis, α_j is complex coefficient that satisfies the normalization condition $\sum_j |\alpha_j|^2 = 1$. The number of non-zero coefficient α_j is called the Slater rank. If it is greater than 1 then the state is entangled otherwise it is un-entangled. From here we can easily prove that the symmetrization state are uncorrelated e.g, the above equation when $\alpha_j = 1$ can be written as,

$$|\Phi(1,2)\rangle = \frac{1}{\sqrt{2}}[|1\rangle_1 \otimes |2\rangle_2 - |2\rangle_1 \otimes |1\rangle_2], \tag{3.3.22}$$

which is the simple anti-symmetrization of the orthogonal state $\{|1\rangle, |2\rangle\}$ [14, 16, 15]. To quantify entanglement we will use the previously discussed concept of von Neumann Entropy for which we need a reduced density matrix which can be obtained by using,

$$\rho^{(1)} = Tr^{(2)}[|\Phi(1,2)\rangle \langle\Phi(1,2)|]. \tag{3.3.23}$$

By using this reduced density matrix the entropy can be defined as,

$$S(\rho^{(1)}) = -Tr^{(1)}[\rho^{(1)} \log_2 \rho^{(1)}] = 1 - \sum_j |\alpha_j|^2 \log_2 |\alpha_j|^2. \tag{3.3.24}$$

- S_{min}=1 correspond to Slater Number=1 while for distinguishable particles entropy value is zero and the state is un-entangled.
- A state vector describing two identical fermionic states is entangled when the Slater number is greater than 1 and the corresponding von Neumann entropy also exceeds 1.

3.3.6 Bosonic Entanglement

Theorem: For any symmetric (N x N) matrix Y [i.e, Y $\in \mathcal{M}$(N,$\mathbb{C}$) and $Y^T = Y$] there exist a unitary transformation which is given a Y=UβU^T where β is non-negative real

diagonal matrix given as $\beta = \text{diag}[y_1, y_2, ..., y_N]$. The columns in matrix U are an orthonormal set of eigenvectors of $YY^\dagger$ and the diagonal entries of β are the non-negative square roots of the corresponding eigenvalues [45].

A state vector representing two identical bosons which are many fold symmetric $\mathcal{S}(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})$ is written as,

$$|\Phi(1,2)\rangle = \sum_{j=1}^{2s+1} a_j \, |j\rangle \otimes |j\rangle \,, \tag{3.3.25}$$

where state $|j\rangle$ is orthonormal within the range. a_j satisfies the normalization condition $\sum_j |a_j|^2 = 1$.

In contrast to the distinguishable particles and fermions here we cannot conclude that when the number of Schmidt coefficients is 1 the system is non-correlated and for greater than 1 it is entangled rather Bosons shows different criteria for entanglement in the system.

Schmidt number = 1: The state vector in this case is written as

$$|\Phi(1,2)\rangle = |j\rangle \otimes |j\rangle \,. \tag{3.3.26}$$

which shows that both the bosons are in a single state and all the properties are known for each particle. One can claim that we do not have any information exactly about the identity of the particles. This property is useless regarding the useful entanglement. So in this case the system is non-entangled.

Schmidt number = 2: The state vector in this scenario has the form

$$|\Phi(1,2)\rangle = \alpha_1 \, |1\rangle_1 \otimes |1\rangle_2 + \alpha_2 \, |2\rangle_1 \otimes |2\rangle_2 \,. \tag{3.3.27}$$

where $\{|\alpha_1|^2 + |\alpha_2|^2 = 1\}$ which is a normalization condition and in this situation reduced density matrix and von Neumann entropy is,

$$\begin{aligned} \rho^{(1or2)} &= \alpha_1^2 \, |1\rangle \langle 1| + \alpha_2^2 \, |2\rangle \langle 2| \,, \\ S(\rho^{(1or2)}) &= -\alpha_1^2 \log_2 \alpha_1^2 - \alpha_2^2 \log_2 \alpha_2^2. \end{aligned} \tag{3.3.28}$$

Now from here depending upon the values of coefficients, we have two conditions. when $\alpha_1 = \alpha_2$, the state can be written by the symmetrization of the states. To prove this claim consider $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}$. Now the state vector of the system will be,

$$|\Phi(1,2)\rangle = \frac{1}{\sqrt{2}} |1\rangle_1 \otimes |1\rangle_2 + |2\rangle_1 \otimes |2\rangle_2 . \tag{3.3.29}$$

If we define the orthonormal states as $|1\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle + |\chi\rangle)$ *and* $|2\rangle = \frac{i}{\sqrt{2}}(|\psi\rangle - |\chi\rangle)$ then we can write the above state vector as,

$$|\Phi(1,2)\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle_1 \otimes |\chi\rangle_2 + |\chi\rangle_1 \otimes |\psi\rangle_2). \tag{3.3.30}$$

This is the simplest application of symmetrization postulates and this state is non-entangled and von Neumann entropy for this state will come out to be 1 which is solely due to the ignorance of the association of bosons with a particular state vector.
When $\alpha_1 \neq \alpha_2$ then the state can be written by the symmetrization of non-orthagonal states.

$$|\Phi(1,2)\rangle = \frac{1}{\sqrt{2(1 + |\langle\chi|\psi\rangle|^2)}}[|\psi\rangle_1 \otimes |\chi\rangle_2 + |\chi\rangle_1 \otimes |\psi\rangle_2], \quad \langle\chi|\psi\rangle \neq 0. \tag{3.3.31}$$

This state can also be represented in another form if we define a unique normalized state $|\psi_\perp\rangle$ perpendicular to $|\psi\rangle$ and defined in two dimensional manifold $|\psi\rangle$ *and* $|\chi\rangle$. Then the state vector can be written as,

$$|\Psi(1,2)\rangle = a\,|\psi\rangle_1 \otimes |\psi\rangle_2 + \frac{b}{\sqrt{2}}[|\phi\rangle_1 \otimes |\psi_\perp\rangle_2 + |\psi_\perp\rangle_1 \otimes |\psi\rangle_2], \tag{3.3.32}$$

where $(a, b \neq 0)$ are complex coefficients which satisfy the normalization condition and depend upon the modulus of inner product as, $|b| = \frac{\sqrt{(1 - |\langle\chi|\psi\rangle|^2)}}{\sqrt{(1 + |\langle\chi|\psi\rangle|^2)}}$. Now by applying our criteria of entanglement we can conclude that this state is entangled. After applying Schmidt Decomposition on this state we can get,

$$|\Phi(1,2)\rangle = \sqrt{\frac{1 + \sqrt{1 - |b|^4}}{2}}\,|1\rangle_1 \otimes |1\rangle_2 + \sqrt{\frac{1 - \sqrt{1 - |b|^4}}{2}}\,|2\rangle_1 \otimes |2\rangle_2 \ , \ |b| \in (0,1), \tag{3.3.33}$$

and we can quantify the entanglement from here with the help of von-Neuman entropy using Schmidt coefficients.

Schmidt number $\geq$ 3: When it is greater than two the state is genuinely entangled because we cannot obtain our state vector just by factorization of two orthagonal states.

Summary

- Schmidt number of $|\Phi(1,2)\rangle$=1 and $S(\rho^{(1or2)}) = 0$ then the state is **non-entangled** (Tensor product case).
- Schmidt number of $|\Phi(1,2)\rangle$=2 and $S(\rho^{(1or2)}) \in (0,1)$ then the state is **entangled** (Symmetrization of non-orthagonal states).
- Schmidt number of $|\Phi(1,2)\rangle$=2 and $S(\rho^{(1or2)}) = 1$ then the state is **non-entangled** (Symmetrization of orthagonal states).
- Schmidt number of $|\Phi(1,2)\rangle > 2$ and then the state is **entangled**.

3.3.7 No Label Approach to Identical Particles

In the previous section, we studied identical particles with a mock label attached to them which induces some ambiguities and later we introduce symmetrization to rectify that misleading. Now we will look at a method in which no particle will be associated with any label and generalize the concept of Schmidt decomposition to identical particles.

Here in no label approach we incorporate the symmetric and anti-symmetric behavior of bosons and fermions in their state representation [42]. If two particles are in state ϕ and χ then their combine state is given as $|\phi,\chi\rangle$. These two particles are not completely independent of each other and their overall state is whole and can not be written as tensor product i.e, $|\phi,\chi\rangle \neq |\phi\rangle \otimes |\chi\rangle$. However non-separable external symmetric product(wedge product) can be written as $|\phi,\chi\rangle := |\phi\rangle \times |\chi\rangle$. The probability amplitude of finding these particles in some other state i.e, $|\alpha,\beta\rangle$ is given as,

$$\langle\alpha,\beta|\phi,\chi\rangle = \langle\alpha|\phi\rangle\langle\beta|\chi\rangle + \eta\langle\alpha|\chi\rangle\langle\beta|\phi\rangle, \tag{3.3.34}$$

where η is $+1$ for bosons and -1 for fermions.This probability density shows the symmetrization and anti-symmetrization of boson and fermions. Inner product between different dimensions are given as,

$$\langle \Psi_i | \phi, \chi \rangle = \langle \Psi | \phi \rangle | \chi \rangle + \eta \langle \Psi | \chi \rangle | \phi \rangle . \tag{3.3.35}$$

The state $|\phi, \chi\rangle$ spans over two particle Hilbert space $\mathcal{H}^{(2)}$. The two IP state can also be expressed as $|\Psi^{(2)}\rangle = \sum_{i,j} \alpha_{ij} |i, j\rangle$. The density matrix for this state is given as,

$$\rho = |\Psi^{(2)}\rangle \langle \Psi^{(2)}| , \tag{3.3.36}$$

and on the basis of all this discussion reduced density matrix can be given as,

$$\rho^{(1)} = \frac{1}{2} \sum_j \langle j | \Psi^{(2)} \rangle \langle \Psi^{(2)} | j \rangle = \frac{1}{2} Tr^{(1)} \rho. \tag{3.3.37}$$

On the basis of this concept, we will generalize Schmidt decomposition to identical particles in next step.

3.3.8 Schmidt Decomposition for Identical Particles

A pure state of two degenerate IP $|\Psi\rangle$ can be written in the Schmidt decomposition (SD) within a symmetric two-particle Hilbert space $\mathcal{H}^{(2)}$,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \sum_k \sqrt{\lambda_k} |k, \tilde{k}\rangle \quad (\lambda_k > 0, \ \sum_k \lambda_k = 1), \tag{3.3.38}$$

where λ_k are Schmidt coefficients which are the square roots of the eigenvalues of the reduced density matrix.$|k.\tilde{k}\rangle$ are Schmidt basis and $|\tilde{k}\rangle \in \{|k\rangle\}$.

Proof: The state of two identical particles $|\Psi\rangle$ in $|k, l\rangle$ basis can be given as

$$|\Psi\rangle = \frac{1}{2} \sum_{k,l} |k, l\rangle \langle k, l | \Psi \rangle , \tag{3.3.39}$$

where $\frac{1}{2} \sum_{k,l} |k, l\rangle \langle k, l| = \mathbb{I}_2$ is symmetric two particles identity matrix. By considering $|\tilde{k}\rangle = \sum_l \langle k, l | \Psi \rangle |l\rangle$,

$$|\Psi\rangle = \frac{1}{2} \sum_k |k, \bar{k}\rangle . \tag{3.3.40}$$

The states $\{|\bar{k}\rangle\}$ are generally not orthonormal. Nevertheless the basis $\{k\}$ are orthogonal which means $\langle\bar{k}'|\bar{k}\rangle \propto \delta_{kk'}$. The inner product can be written as,

$$\begin{aligned}\langle\bar{k}'|\bar{k}\rangle &= \sum_{l,l'} \langle l'| \langle\Psi|k', l'\rangle \langle k, l|\Psi\rangle |l\rangle , \\ &= \sum_{l} \langle k, l|\Psi\rangle \langle\Psi|k', l\rangle , \\ &= \sum_{l} \langle k| \times \langle l|\Psi\rangle \langle\Psi|l\rangle \times |k'\rangle ,\end{aligned} \tag{3.3.41}$$

where the reduced density matrix of symmetric two particle system is given as, $\rho^{(1)} = \frac{1}{2}\sum_l \langle l|\Psi\rangle \langle\Psi|l\rangle$. So the relation become,

$$\langle\bar{k}'|\bar{k}\rangle = 2\sum_{k,k'} \langle k|\rho^{(1)}|k'\rangle . \tag{3.3.42}$$

The states $\{|k\rangle\}$ are eigenstates of $\rho^{(1)}$, i.e, $\rho^{(1)}|k\rangle = \lambda_k |k\rangle$. As the states $\{|\bar{k}\rangle\}$ are orthagonal so $\langle\bar{k}|\bar{k}'\rangle = 2\lambda_k\delta_{kk'}$. From here we can use orthonormal bases $\{|\tilde{k}\rangle\}$ that are associated with orthagonal basis $\{|\bar{k}\rangle\}$ as,

$$|\tilde{k}\rangle = \frac{1}{\sqrt{2\lambda_k}}|\bar{k}\rangle = \frac{1}{\sqrt{2\lambda_k}}\sum_l \langle k, l|\Psi\rangle |l\rangle , \tag{3.3.43}$$

both $\{|k\rangle\}$ and $\{\tilde{k}\}$ are the eigenstates of the reduced density matrix $\rho^{(1)}$ with same eigenvalue λ_k. Thus, given a set of eigenstates $\{|k\rangle\}$, each $|\tilde{k}\rangle$ state belongs to the same set. Now we have the relation $|\bar{k}\rangle = \sqrt{2\lambda_k}|\tilde{k}\rangle$. By substituting this we can demonstrate Schmidt decomposition. When the state of individual particles in the system is given by the combination of two properties e.g $|k\rangle \equiv |ab\rangle$, which may be a spatial and spin state in one case, then SD is applied by fixing one of them and taking reduced density matrix with respect to other property [43]. Similar to distinguishable particles SD is only the detection of entanglement and Schmidt number is the indicator of it. Whenever the number of non-zero Schmidt coefficients is greater than one the particles are entangled. This entanglement is quantified with the help of von Neumann entropy, $S(\rho^1) = -Tr^{(1)}(\rho^{(1)}\log_2\rho^{(1)}) = -\sum_k \lambda_k \log_2\lambda_k$.

In the light of above discussion SD of any two identical particle system can be obtain by following these simple steps,

- Perform trace of the ρ and find reduced density matrix $\rho^{(1)}$. if the particle is expressed in multiple sub-states then we need to fix all of them except the required one and perform the operation accordingly,
- Calculate eigenvalues and eigenstates of $\rho^{(1)}$, and
- Construct the state $|\tilde{k}\rangle$ and express the state in Schmidt basis $\{|k,\tilde{k}\rangle\}$.

3.3.9 SPDC Qutrits as Identical Particles

As we have discussed in the section 2.5.1 a biphotonic system in which the polarization in 2D is represented in the form of qubits as,

$$\Phi(1,2) = \sum_{n=1}^{2}\sum_{m=1}^{2} \alpha_{n,m}\nu_n^{(1)} \otimes \nu_m^{(2)}. \tag{3.3.44}$$

When the particles are indistinguishable they follow the symmetrization postulates and the number of constants decreases as $(\alpha_{0,1} = \alpha_{1,0})$. So the number of states left behind are three that are independent of each other. Such states are called qutrits. The polarization of photons is restricted to horizontal and vertical only. On basis of this information, the qutrit states can be written as,

$$\begin{aligned} |\phi\rangle_{HH} &= |1\rangle^1 \otimes |1\rangle^2 \ , \ |\phi\rangle_{VV} = |0\rangle^1 \otimes |0\rangle^2 , \\ |\phi\rangle_{HV} &= \frac{1}{\sqrt{2}}\left(|1\rangle^1 \otimes |0\rangle^2 + |0\rangle^1 \otimes |1\rangle^2\right), \end{aligned} \tag{3.3.45}$$

where ϕ_{HH} and ϕ_{VV} represents the photons with similar while ϕ_{HV} state represent particles with different polarization. This qutrit state can be defined by the superposition of basic states,

$$|\Phi\rangle_{qtr}(1,2) = \alpha_1 |\phi\rangle_{HH} + \alpha_2 |\phi\rangle_{HV} + \alpha_3 |\phi\rangle_{VV}, \tag{3.3.46}$$

which obeys the normalization condition $\{|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1\}$. The polarization function of these waves can be measured by taking an inner product with polarization

vectors,

$$\begin{aligned}
\langle\phi\rangle_{HH}(p_1,p_2) &= \langle p_1,p_2|0,0\rangle = \delta_{p_1,H}\delta_{p_2,H},\\
\langle\phi\rangle_{VV}(p_1,p_2) &= \langle p_1,p_2|1,1\rangle = \delta_{p_1,V}\delta_{p_2,V},\\
\langle\phi\rangle_{HV}(p_1,p_2) &= \langle p_1,p_2|0,1\rangle,\\
&= \frac{\delta_{p_1,H}\delta_{p_2,V}+\delta_{p_1,V}\delta_{p_2,H}}{\sqrt{2}}.
\end{aligned} \tag{3.3.47}$$

The density matrix of this system can be written by following the standard procedure with these states which is,

$$\rho_{1,2} = \begin{pmatrix}
|\alpha_{0,0}|^2 & \frac{1}{\sqrt{2}}\alpha_{0,0}\alpha^*_{0,1} & \frac{1}{\sqrt{2}}\alpha_{0,0}\alpha^*_{0,1} & \alpha_{0,0}\alpha^*_{1,1}\\
\frac{1}{\sqrt{2}}\alpha_{0,1}\alpha^*_{0,0} & \frac{1}{2}|\alpha_{0,1}|^2 & \frac{1}{2}|\alpha_{0,1}|^2 & \frac{1}{\sqrt{2}}\alpha_{0,1}\alpha^*_{1,1}\\
\frac{1}{\sqrt{2}}\alpha_{0,1}\alpha^*_{0,0} & \frac{1}{2}|\alpha_{0,1}|^2 & \frac{1}{2}|\alpha_{0,1}|^2 & \frac{1}{\sqrt{2}}\alpha_{0,1}\alpha^*_{1,1}\\
\alpha_{1,1}\alpha^*_{0,0} & \frac{1}{\sqrt{2}}\alpha_{1,1}\alpha^*_{0,1} & \frac{1}{\sqrt{2}}\alpha_{1,1}\alpha^*_{0,1} & |\alpha_{1,1}|^2
\end{pmatrix}. \tag{3.3.48}$$

By tracing over one particle the matrix will reduce to,

$$\rho_1 = \begin{pmatrix}
|\alpha_{0,0}|^2 + \frac{|\alpha_{0,1}|^2}{2} & \frac{\alpha_{0,0}\alpha^*_{1,0}+\alpha_{1,0}\alpha^*_{1,1}}{\sqrt{2}}\\
\frac{\alpha_{0,1}\alpha^*_{0,0}+\alpha_{1,1}\alpha^*_{0,1}}{\sqrt{2}} & \frac{|\alpha_{0,1}|^2}{2} + |\alpha_{1,1}|^2
\end{pmatrix}. \tag{3.3.49}$$

The eigenvalue of this matrix is,

$$\lambda_\pm = \frac{1}{2}(1 \pm (1 - |2\alpha_{0,0}\alpha_{1,1} - \alpha^2_{0,1}|^2)^{1/2}). \tag{3.3.50}$$

We can relate this eigenvalue with concurrence as,

$$C = |2\alpha_{0,0}\alpha_{1,1} - \alpha^2_{0,1}| = 2(\lambda_+\lambda_-)^{1/2}. \tag{3.3.51}$$

To quantify entanglement we use the concept of von Neumann entropy,

$$S = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_-. \tag{3.3.52}$$

All the above discussion revolves around two particle systems only. Now we will try to generalize it to N-particle system and discuss methods that can be implemented to quantify entanglement in it.

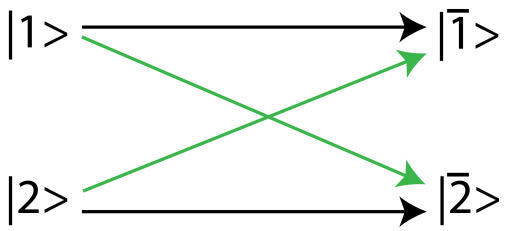

Figure 3.3.5: Probability of 2-particle system changing their state.

3.3.10 N-Particle Formalism

In previous sections, we considered only two-particle systems when we were calculating their probability amplitude in another two-particle state (symmetrized product) and it was,

$$\langle \bar{1}, \bar{2}|1, 2\rangle_{\eta} := \langle \bar{1}|1\rangle \langle \bar{2}|2\rangle + \eta \langle \bar{1}|2\rangle \langle \bar{2}|1\rangle , \tag{3.3.53}$$

where the η is for symmetrization which is $+1$ for bosons and -1 for fermions. The probability of finding both the particles in a single state is,

$$\langle \bar{1}, \bar{1}|1, 2\rangle_{\eta} = (1 + \eta) \langle \bar{1}|1\rangle \langle \bar{1}|2\rangle , \tag{3.3.54}$$

where it is zero for fermions which satisfies the Pauli exclusion principle and for bosons probability is maximum. Generalization of this method to N-identical particles give us,

$$\langle \bar{1}, \bar{2}, ..., \bar{N}|1, 2, ..., N\rangle_{\eta} ; = \sum_{P} \eta^{P} \langle \bar{1}|P_1\rangle \langle \bar{2}|P_2\rangle , ..., \langle \bar{N}|P_N\rangle , \tag{3.3.55}$$

where P is the parity of permutation, for bosons η^P is 1 while in fermionic case it is 1 for even and -1 for odd permutations. In a system of N-identical particles when ever two IP exchange their states with each other the resultant state is become $|1, 2, ..., k, ..., l, ..., N\rangle = \eta\, |1, 2, ..., l, ..., k, ..., N\rangle$, where k,l $\in$ 1,2,...,N and l$\neq$k.

The tensor product state of the N-particle system $\{|1\rangle \otimes |2\rangle \otimes, ..., \otimes |N\rangle\}$ is not possible when two or more IP states are overlapping and the state is not normalized. So the state of N-identical particles in normalized form is,

$$|\Psi^{(N)}\rangle := \frac{1}{\mathcal{N}} |\psi^{(N)}\rangle := \frac{1}{\mathcal{N}} |1, 2, ..., N\rangle , \tag{3.3.56}$$

where $\mathcal{N} = \sqrt{\langle 1,2,...,N|1,2,...,N\rangle}$. Now we will look at the action of a single particle operator on a multiparticle state. On N-identical particle state, the action of an operator $\hat{O}^{(1)}$ is

$$\hat{O}^{(1)}\left|1,2\right\rangle = |\hat{O}^{(1)}1,2\rangle + |1,\hat{O}^{(1)}2\rangle\,, \tag{3.3.57}$$

As all the particles are identical so the operator can act on all of them but the action is not simultaneous rather one by one. We can generalize this idea to N-identical particle state as,

$$\hat{O}^{(1)}\left|1,2,...,N\right\rangle := \sum_{k} |1,...,\hat{O}^{(1)}k,...,N\rangle\,. \tag{3.3.58}$$

3.3.11 Partial Trace and von Neumann Entropy

The inner product of a single particle bra with multi-particle ket can be defined with the action of a 1-particle operator $\hat{O}^{(1)} = |i'\rangle\langle j'|$ on N-particle state which is given as,

$$\begin{aligned}\hat{O}^{(1)}\left|1,2,...,N\right\rangle :&= \sum_{j} |i'\rangle\,, |1,2,...,\langle j'|j\rangle\,,...,N\rangle\,,\\ &= \sum_{j} \eta^{j-1}\langle j'|j\rangle\,|i',1,2,...,\not{j},...,N\rangle\,.\end{aligned} \tag{3.3.59}$$

The state $|i'\rangle$ is displaced from the j^{th} site and placed at the first place due to the individuality of the state and the initial state is canceled. The simple inner product between 1-particle bra and N-particle symmetric Hilbert space $\mathcal{H}^{(N)}$ ket is defined as,

$$\langle j'|.|1,2,...,N\rangle := \sum_{j=1}^{N} \eta^{j-1}\langle j'|j\rangle\,|1,...,\not{j},...,N\rangle\,. \tag{3.3.60}$$

This is un-normalized (N-1)-particle state whose normalized state $|\Psi^{(N-1)}_{j'}\rangle$ is,

$$|\Psi^{(N-1)}_{j'}\rangle = \frac{\langle j'|.|\Psi^{(N)}\rangle}{\sqrt{\langle\Pi^{(1)}_{j'}\rangle_{\Psi^{(N)}}}}, \tag{3.3.61}$$

where $\Pi^{(1)}_{j'} = |j'\rangle\langle j'|$ is projection operator. The reduced density matrix of N-particle state after performing one particle trace is,

$$\begin{aligned}\rho^{(N-1)} &:= \frac{1}{\langle\mathbb{I}^{(1)}\rangle_{\Psi^{(N)}}} Tr^{(1)}\,|\Psi^{(N)}\rangle\langle\Psi^{(N)}|\,,\\ &= \sum_{j'} p_{j'}\,|\Psi^{(N-1)}_{j'}\rangle\langle\Psi^{(N-1)}_{j'}|\,,\end{aligned} \tag{3.3.62}$$

where 1-particle identity operator is defined as $\mathbb{I}^{(1)} = \sum_{j'} \Pi_{j'}^{(1)}$ and its action on N-particle state is $\mathbb{I}^{(1)} |1, 2, ..., N\rangle = N |1, 2, ..., N\rangle$, therefore $p_{j'} = \frac{\langle \Pi_{j'}^{(1)} \rangle_{\Psi^{(N)}}}{\langle \mathbb{I}^{(1)} \rangle_{\Psi^{(N)}}}$. This reduced density matrix is same for tracing out any particle from the system.
For 2-identical particles inner product is defined as,

$$\begin{aligned} \langle i', j'|.|1, 2, ..., N\rangle = & \sum_{k=1}^{N} \eta^{k-1} \langle i'|k\rangle \\ & \left[\sum_{j<k} \eta^{j-1} \langle j'|j\rangle |1, ..., \not{j}, ..., \not{k}, ..., N\rangle + \sum_{j>k} \eta^{j-2} \langle j'|j\rangle |1, ..., \not{k}, ..., \not{j}, ..., N\rangle \right]. \end{aligned} \tag{3.3.63}$$

This results replicates the two IP system when N=2. The identity operator of 2 particles and their projection operator are defined as,

$$\begin{aligned} \mathbb{I}^{(2)} &= (\frac{1}{2!}) \sum_{k'} \Pi_{k'}^{(2)}, \\ \Pi_{k'}^{(2)} &= \frac{1}{\mathcal{N}_{k'}^2} |k_1', k_2'\rangle \langle k_1', k_2'| . \end{aligned} \tag{3.3.64}$$

So, normalized N-2 state is,

$$|\Psi_{k'}^{N-1}\rangle = \frac{\langle k_1', k_2'|.|\Psi^{(N)}\rangle}{\sqrt{\langle \Pi_{k'}^{(2)} \rangle_{\Psi^{(N)}}}}. \tag{3.3.65}$$

The reduced density matrix of (N-2) particle state by using previous result is,

$$\begin{aligned} \rho^{(N-2)} &:= \frac{1}{2! \langle \mathbb{I}^{(2)} \rangle_{\Psi^{(N)}}} Tr^{(2)} |\Psi^{(N)}\rangle \langle \Psi^{(N)}| \\ &= \sum_{k'} p_{k'} |\Psi_{k'}^{(N-2)}\rangle \langle \Psi_{k'}^{(N-2)}| , \end{aligned} \tag{3.3.66}$$

where $p_{k'} = \frac{1}{2!} \frac{\langle \Pi_{k'}^{(2)} \rangle_{\Psi^{(N)}}}{\langle \mathbb{I}^{(2)}|_{\Psi^{(N)}}}$. We can generalize this procedure for any arbitrary number M which ranges between 1 and N to find its reduced density matrix which will be

$$\begin{aligned} \rho^{(N-M)} &:= \frac{1}{M! \langle \mathbb{I}^{(2)} \rangle_{\Psi^{(N)}}} Tr^{(M)} |\Psi^{(N)}\rangle \langle \Psi^{(N)}| , \\ &= \sum_{k'} p_{k'} |\Psi_{k'}^{(N-M)}\rangle \langle \Psi_{k'}^{(N-M)}| . \end{aligned} \tag{3.3.67}$$

We can use this density matrix to calculate von Neumann entropy between any two IP or group of identical particles [44],

$$S(\rho^{(N-M)}) = -Tr(\rho^{(N-M)} \log_2 \rho^{(N-M)}). \tag{3.3.68}$$

This approach is termed as LO FRANCO AND COMPAGNO'S METHOD (LFC). All the discussed methods are complicated and suggest different approaches for bosons and fermions. Moreover symmetrization add more to its complications. So, we will discuss quantum systems in second quantization that is relatively a simple approach.

Chapter 4

Second Quantization Approach

The second quantization was first introduced in quantum field theory where wave functions were treated as field operators and can e expanded in terms of single particle creation and annihilation operators. We are going to introduce a similar approach in entanglement detection and quantification. But the question arises here, why do we use this approach? Our ultimate goal is to introduce a simpler mechanism than the traditional one that we have discussed. The difficulties that arise in first quantization was due to the Pauli exclusion principle that tackled by using symmetrization postulates. The Slater determinants that we had introduced in the previous approach was quite complicated. In this method, the difficulties that arise from symmetrization postulates resolve automatically and at its cost, we have the normal ordering of operators which is relatively a simpler task. Other features of this approach is that it is independent of number of particles that help in understanding behavior of each particle individually and it play an important role in finite basis approximation.

4.1 Representation of States

To represent a system in second quantization we make use of the orthonormal vacuum state ($|0\rangle$), creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operator. By using this state and operators a single particle quantum state can be defined as,

$$|\alpha\rangle = \hat{a}^\dagger_\alpha |0\rangle . \tag{4.1.1}$$

The two particle state in first quantization is represented in determinant form which follows the symmetrization postulates,

$$\Phi_a(x_1, x_2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) \\ \phi_2(x_1) & \phi_2(x_2) \end{vmatrix}, \tag{4.1.2}$$

while in second quantization it is,

$$|\alpha, \beta\rangle = \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta |0\rangle , \tag{4.1.3}$$

which shows that a particle is in state α and other is in state β created in vacuum state using creation operators. Depending upon the bosonic and fermionic nature of particle their creation and annihilation operators follow commutation and anti-commutation relations that is,

$$\begin{aligned} [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] &= \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta - \hat{a}^\dagger_\beta \hat{a}^\dagger_\alpha = 0 \ \text{(for bosons)}, \\ \{\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\} &= \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta + \hat{a}^\dagger_\beta \hat{a}^\dagger_\alpha = 0 \ \text{(for fermions)}, \end{aligned} \tag{4.1.4}$$

which can be computed by using the symmetrization property of the particles,

$$\begin{aligned} |\Phi\rangle &= |\alpha_1, \alpha_2, \alpha_3, ..., \alpha_i, \alpha_j, ..., \alpha_n\rangle , \\ |\Phi\rangle &= \pm |\alpha_1, \alpha_2, \alpha_3, ..., \alpha_j, \alpha_i, ..., \alpha_n\rangle , \end{aligned} \tag{4.1.5}$$

where the +ve sign is for bosons and -ve sign is for fermions. We can derive commutator and anti-commutator relation between creation and annihilation operator by applying this property.

$$\begin{aligned} [\hat{a}_\alpha, \hat{a}^\dagger_\beta] &= 0 \ \text{(for bosons)}, \\ \{\hat{a}_\alpha, \hat{a}^\dagger_\beta\} &= 0 \ \text{(for fermions)}, \\ [\hat{a}_\alpha, \hat{a}^\dagger_\alpha] &= 1 \ \text{(for bosons)}, \\ \{\hat{a}_\alpha, \hat{a}^\dagger_\alpha\} &= 1 \ \text{(for fermions)}. \end{aligned} \tag{4.1.6}$$

N-particles separable state in second quantization is,

$$|\Psi\rangle = |\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\rangle = \hat{a}^\dagger_1 \hat{a}^\dagger_2 \hat{a}^\dagger_3 ... \hat{a}^\dagger_n |0\rangle . \tag{4.1.7}$$

To establish second quantization form of operators, the basic principle in this approach is that both representation should have same expectation value. In case of

indistinguishable particle the fermionic non-entangle state is given in the form of convex combination of Slater determinant,

$$\rho_f = \sum_j p_j {a_1^j}^\dagger ... {a_N^j}^\dagger \left|0\right\rangle \left\langle 0\right| a_N^j ... a_1^j, \tag{4.1.8}$$

where ${a_k^j}^\dagger$ and a_1^j are creation and annihilation operators of fermions.

Single particle fermionic state can be evaluated with the help of trace and is given as,

$$\rho_{f1} \equiv \frac{1}{N} \sum_{j=1}^{N} a_j^\dagger \left|0\right\rangle \left\langle 0\right| a_j, \tag{4.1.9}$$

where the factor $\frac{1}{N}$ arise due to action of single operator on all the possible states. The von Neumann entropy and entanglement in second quantization are shifted and written as,

$$E(\left|\Psi\right\rangle \left\langle\Psi\right|) = S(\rho_f) - lnN. \tag{4.1.10}$$

Bosonic non-entangled pure state is represented as,

$$\left|\Psi\right\rangle = \frac{1}{\sqrt{N!}} (a^\dagger)^N \left|0\right\rangle. \tag{4.1.11}$$

We can compute density matrix and then follow the standard procedure to calculate entropy. One particle reduced density matrix is,

$$\begin{aligned} \rho_b &= \frac{1}{N} Tr(a_i^\dagger a_j \left|\Psi\right\rangle \left\langle\Psi\right|), \\ \rho_b &= \frac{1}{N} \sum_{i=1}^{N} n_i a_i^\dagger \left|0\right\rangle \left\langle 0\right| a_i \qquad (i = j). \end{aligned} \tag{4.1.12}$$

and the entropy in this case will come,

$$S(\rho_b) = -\sum_{i=1}^{N} \left(\frac{n_i}{N}\right) \ln \left(\frac{n_i}{N}\right), \tag{4.1.13}$$

which is the quantitative measure of entanglement. To test the authenticity of this approach we will compare it with LFC's method.

4.1.1 LFC's Method and Second Quantization

In this section, we will compare the LFC's method to second quantization (SQ) in order to prove that both techniques agree with each other. The state of two indistinguishable particles in LFC's method is defined as,

$$|\psi\rangle = \sum_{A,B,i,j} \alpha_{Bj}^{Ai} |Ai, Bj\rangle \,, \tag{4.1.14}$$

where {A, B} represents the spatial degree of freedom and {i, j} is for another trait like spin or polarization as discussed. The density matrix for this state is,

$$\rho = |\psi\rangle \langle\psi| = \sum_{A,B,i,j} \sum_{C,D,k,l} \alpha_{Bj}^{Ai} \alpha_{Dl}^{Ck*} |Ai, Bj\rangle \langle Ck, Dl| \,, \tag{4.1.15}$$

where indices represent finite orthogonal states and are generalized as,
A,B,C,D=$\{E_1, E_2, ..., E_s\}$ and i,j,k,l=$\{x_1, x_2, ..., x_t\}$.
Localized partial trace in region E_1 in non-normalized form is,

$$\begin{aligned}
\tilde{\rho}_N^{(1)} = Tr_N(\rho) &= \sum_{n=x_1}^{x_t} \langle E_1 n|\rho|E_1 n\rangle \,, \\
&= \sum_{n=x_1}^{x_t} \sum_{A,B,i,j} \sum_{C,D,k,l} \alpha_{Bj}^{Ai} \alpha_{Dl}^{Ck*} \begin{pmatrix} \langle E_1 n|Ai\rangle \langle Dl|E_1 n\rangle |Bj\rangle \langle Ck| \\ + \eta \langle E_1 n|Ai\rangle \langle Ck|E_1 n\rangle |Bj\rangle \langle Dl| \\ + \eta \langle E_1 n|Bj\rangle \langle Dl|E_1 n\rangle |Ai\rangle \langle Ck| \\ + \langle E_1 n|Bj\rangle \langle Ck|E_1 n\rangle |Ai\rangle \langle Dl| \end{pmatrix}, \\
&= \sum_{n=x_1}^{x_t} \begin{pmatrix} \sum_{B,j} \sum_{C,k} \alpha_{Bj}^{E_1 n} \alpha_{Ck}^{E_1 n*} |Bj\rangle \langle Ck| + \eta \sum_{B,j} \sum_{D,l} \alpha_{Bj}^{E_1 n} \alpha_{E_1 n}^{Dl*} |Bj\rangle \langle Dl| \\ + \eta \sum_{A,i} \sum_{C,k} \alpha_{E_1 n}^{Ai} \alpha_{Ck}^{E_1 n*} |Ai\rangle \langle Ck| + \sum_{A,i} \sum_{D,l} \alpha_{E_1 n}^{Ai} \alpha_{E_1 n}^{Dl*} |Ai\rangle \langle Dl| \end{pmatrix}.
\end{aligned} \tag{4.1.16}$$

This is the reduced density operator that we obtain by using LFC's method.
Now we will derive reduced density matrix using second quantization. For this purpose we will write density matrix in this formalism as,

$$\rho = |\psi\rangle \langle\psi| = \sum_{A,B,i,j} \sum_{C,D,k,l} \alpha_{Bj}^{Ai} \alpha_{Dl}^{Ck*} a_{Ai}^{\dagger} a_{Bj}^{\dagger} |0\rangle \langle 0| a_{Ck} a_{Dl}. \tag{4.1.17}$$

The non-normalized reduced density matrix for single particle by using commutation relations in this approach is,

$$\begin{aligned}
\tilde{\rho}^{(1)} &= \sum_{n=x_1}^{x_t} \langle 0| a_{E_1 n} \rho a^{\dagger}_{E_1 n} |0\rangle , \\
&= \sum_{n=x_1}^{x_t} \sum_{A,B,i,j} \sum_{C,D,k,l} \alpha^{Ai}_{Bj} \alpha^{Ck*}_{Dl} \begin{pmatrix} (\delta_{E_1 A}\delta_{ni} a^{\dagger}_{Bj} + \eta \delta_{E_1 B}\delta_{nj} a^{\dagger}_{Aj}) |0\rangle \\ \langle 0| (\delta_{E_1 D}\delta_{nl} a_{Ck} + \eta \delta_{E_1 C}\delta_{nk} a_{Dl}) \end{pmatrix} , \\
&= \sum_{n=x_1}^{x_t} \begin{pmatrix} \sum_{B,j}\sum_{C,k} \alpha^{E_1 n}_{Bj} \alpha^{E_1 n*}_{Ck} a^{\dagger}_{Bj} |0\rangle \langle 0| a_{Ck} + \eta \sum_{B,j}\sum_{D,l} \alpha^{E_1 n}_{0} \alpha^{Dl*}_{E_1 n} a^{\dagger}_{Bj} |0\rangle \langle 0| a_{Dl} \\ + \eta \sum_{A,i}\sum_{C,k} \alpha^{Ai}_{E_1 n} \alpha^{E_1 n*}_{Ck} a^{\dagger}_{Aj} |0\rangle \langle 0| a_{Ck} + \sum_{A,i}\sum_{D,l} \alpha^{Ai}_{E_1 n} \alpha^{Dl*}_{E_1 n} a^{\dagger}_{Aj} |0\rangle \langle 0| a_{Dl} \end{pmatrix} .
\end{aligned} \tag{4.1.18}$$

As both the reduced density matrices 4.1.16 and 4.1.18 so we can quantify entanglement by computing von Neumann entropy of normalized reduced density matrix. Now we will apply second quantization to detect and quantify entanglement in a quantum system.

4.1.2 Application to SPDC

To apply second quantization here we will consider the type-II regime of SPDC in which both the photons signal and idler have different polarization which will be treated as an entangled trait. If the spatial degree of freedom is $\{|A\rangle, |B\rangle\}$ and polarization is $\{|H\rangle, |V\rangle\}$ then the output state of this system is represented as,

$$|\Psi\rangle = a |AH, BV\rangle + b e^{i\theta} |AV, BH\rangle , \tag{4.1.19}$$

where $|A\rangle = \kappa |U\rangle + \sqrt{1-\kappa^2} |L\rangle$, $|B\rangle = \kappa |L\rangle + \sqrt{1-\kappa^2} |U\rangle , \kappa = \langle U|L\rangle , b = \sqrt{1-a^2}$ ($|U\rangle$ and $|L\rangle$ represents the spatial region of upper and lower region). In type-I both the photons have either vertical polarization (V) or Horizontal (H) but in this setup polarization of both photons can never be same.

The state of the system after plugging $\{|A\rangle, |B\rangle\}$ is,

$$\begin{aligned}
|\Psi\rangle =& ((a + \eta b e^{i\theta})\kappa\sqrt{1-\kappa^2}) |UH, UV\rangle + (a\kappa^2 + \eta b e^{i\theta}(1-\kappa^2)) |UH, LV\rangle , \\
&+ (a(1-\kappa^2) + \eta b e^{i\theta}\kappa^2) |LH, UV\rangle + ((a + \eta b e^{i\theta})\kappa\sqrt{1-\kappa^2}) |LH, LV\rangle .
\end{aligned} \tag{4.1.20}$$

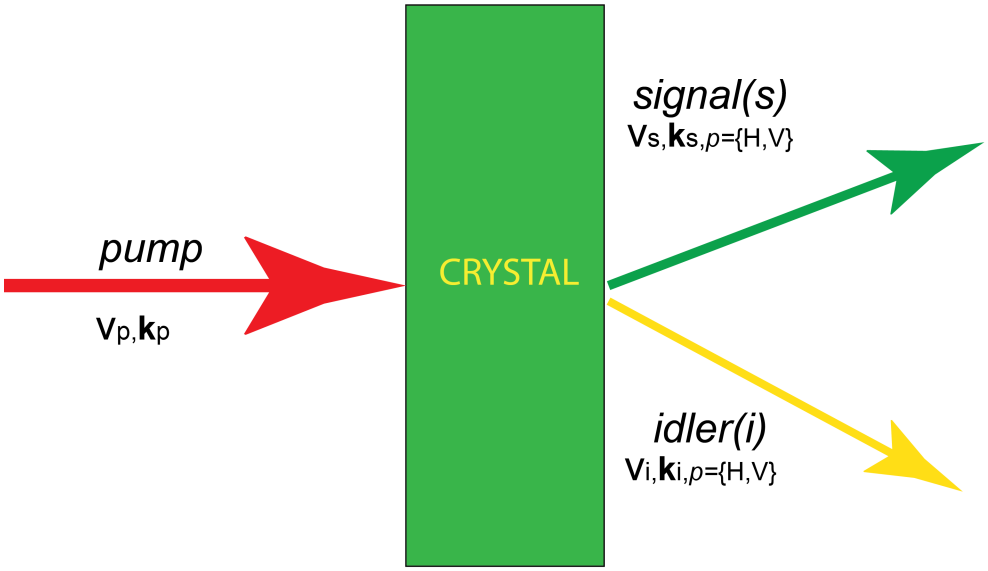

Figure 4.1.1: Type-II Spontaneous Parametric Down-Conversion:$\nu_{s,i}$ and $\mathbf{k}_{s,i}$ are frequencies and wave vectors of signal and idler.

Now second quantization form of this state is,

$$\begin{aligned}|\Psi\rangle =&((a+\eta be^{i\theta})\kappa\sqrt{1-\kappa^2})a^\dagger_{UH}a^\dagger_{UV}\left|0\right\rangle + (a\kappa^2+\eta be^{i\theta}(1-\kappa^2))a^\dagger_{UH}a^\dagger_{LV}\left|0\right\rangle \\ &+(a(1-\kappa^2)+\eta be^{i\theta}\kappa^2)a^\dagger_{LH}a^\dagger_{UV}\left|0\right\rangle + ((a+\eta be^{i\theta})\kappa\sqrt{1-\kappa^2})a^\dagger_{LH}a^\dagger_{LV}\left|0\right\rangle .\end{aligned} \tag{4.1.21}$$

From here we can easily calculate density matrix which is $\rho = |\Psi\rangle\langle\Psi|$. Now we will compute entanglement both by localized partial trace method and trace over entire degree of freedom of one particle separately. Firstly, the localized trace over $\{a^\dagger_{LH}\left|0\right\rangle, a^\dagger_{LV}\left|0\right\rangle\}$ will give reduced density matrix which in normalized form is,

$$\rho_1^{LT} = \frac{1}{N}\begin{pmatrix}\alpha_1 & 0 & \alpha_4 & 0\\ 0 & \alpha_1 & 0 & \alpha_5\\ \alpha_4^* & 0 & \alpha_2 & 0\\ 0 & \alpha_5^* & 0 & \alpha_3\end{pmatrix}, \tag{4.1.22}$$

where

$$\begin{aligned}
\alpha_1 &= (1+2\eta ab\cos\theta)\kappa(1-\kappa),\\
\alpha_2 &= b^2(1-\kappa)+a^2\kappa+2\eta ab\cos\theta(1-\kappa)\kappa,\\
\alpha_3 &= a^2(1-\kappa)+b^2\kappa+2\eta ab\cos\theta(1-\kappa)\kappa,\\
\alpha_4 &= (1+2\eta ab\cos\theta)\kappa(1-\kappa),\\
\alpha_5 &= a(a+\eta be^{i\theta}+2ib\sin\theta)\sqrt{(1-\kappa)\kappa},\\
N &= 2\alpha_1+\alpha_2+\alpha_3 = 1+(2+8\eta ab\cos\theta)(1-\kappa)\kappa.
\end{aligned} \tag{4.1.23}$$

The eigenvalues of this reduced density matrix are,

$$\begin{aligned}
\lambda_1 &= \frac{a^2+(b^2+4\eta ab\cos\theta)(1-\kappa)\kappa}{1+(2+8\eta ab\cos\theta)(1-\kappa)\kappa},\\
\lambda_2 &= 1-\lambda_1,\\
\lambda_3 &= \lambda_4 = 0.
\end{aligned} \tag{4.1.24}$$

Now we can compute entanglement by von Neumann entropy of this reduced density matrix which is,

$$E^{LT}(|\Psi\rangle) = -\sum_{j=1}^{4}\lambda_j\ln\lambda_j. \tag{4.1.25}$$

The above calculations were with local trace (LT) but entanglement behave differently when we trace over the entire degree of freedom (non-local),$\{a^\dagger_{LH}|0\rangle, a^\dagger_{LV}|0\rangle, a^\dagger_{UH}|0\rangle, a^\dagger_{UV}|$ of a single particle. The reduced density matrix in this scenario is,

$$\rho_1^{SQ} = \frac{1}{2(N-\alpha_1)}\begin{pmatrix} \alpha_1+\alpha_3 & 0 & \alpha_4 & 0\\ 0 & \alpha_1+\alpha_2 & 0 & \alpha_5\\ \alpha_4^* & 0 & \alpha_2 & 0\\ 0 & \alpha_5^* & 0 & \alpha_3 \end{pmatrix}. \tag{4.1.26}$$

The eigenvalues of this matrix are,

$$\begin{aligned}
\lambda_1 = \lambda_2 &= \frac{1}{4}\left\{1+\sqrt{1-\frac{8a^2(1-a^2)(1-\kappa)^2\kappa^2}{\left[4\sqrt{a^2(1-a^2)}(1-\kappa)\kappa\cos\theta+\eta\right]^2}}\right\},\\
\lambda_3 = \lambda_4 &= \frac{1}{4}\left\{1-\sqrt{1-\frac{8a^2(1-a^2)(1-\kappa)^2\kappa^2}{\left[4\sqrt{a^2(1-a^2)}(1-\kappa)\kappa\cos\theta+\eta\right]^2}}\right\}.
\end{aligned} \tag{4.1.27}$$

By using these eigenvalue it is straight forward to compute entanglement through von Neumann entropy which is,

$$E^{SQ}(|\Psi\rangle) = -\sum_{j=1}^{4} \lambda_j \ln \lambda_j - \ln 2. \tag{4.1.28}$$

The last term ln2 discount the entanglement that arise because of exchange correlation of the particles. As single Slater determinant of separable state is equals to ln2.

4.2 Comparison Between Local and Non-local Approach

To study the effect of local and non-local approach on distinguishable and indistinguishable particles we compare the entanglements that we obtain in bosonic, fermionic and distinguishabble particle systems by using local and non local trace as shown in graph. The overlapping parameter between the entangled states is 0.3 in case of indistinguishable particles and 0 when particles are distinguishable. In graph blue dotted lines show

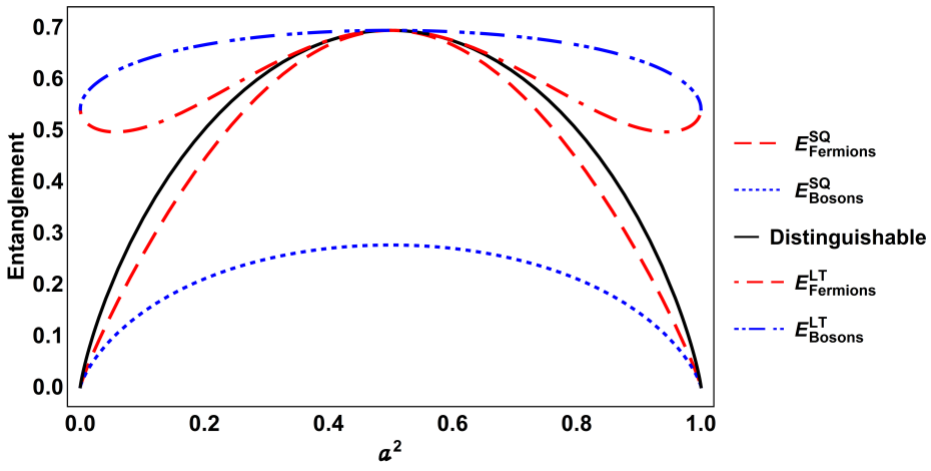

Figure 4.2.1: Entanglement as a function of a^2 with $\theta = 0$ for bosons, fermion and distinguishable particles case. Comparison between entanglement calculated by local trace and trace over entire system with overlap parameter $\kappa = 0.3$ for indistinguishable particles and $\kappa = 0$ for distinguishable particles [46]

.

the entanglement of bosons, red dotted lines are for fermions and the black line is for

distinguishable particles. As the graph clearly depicts the deflection in entanglement when we move from distinguishable to the indistinguishable world. In distinguishable particle systems, the nature of the particles has a minimalist effect on the entanglement while indistinguishable case there is a huge difference recorded in entanglement behavior under similar circumstances but different nature of the particles (fermions and bosons) in the system.

On the basis of approaches that we use to quantify entanglement, distinguishable particle results agree both in local and non-local approaches but indistinguishable particles give different results. As it is ridiculous to say that a system has two different values for a single quantity under the same conditions so there must be some factor that restricts the results to be the same. This disagreement arise due to the contribution of exchange correlation that is taken into account in case of non-local approach ($E^{SQ}(\Psi)$) but in local approach it was ignored.

Chapter 5

Discussion and Conclusion

In this thesis, we have studied various approaches for the detection and quantification of entanglement in indistinguishable particles. Entanglement is a basic resource that is useful in quantum information, quantum communication, quantum computation, quantum cryptography, superdense coding, quantum teleportation, and many other advanced applications. However, generation, detection, and quantification are much involved and challenging processes and may be complicated in some systems depending on their nature and properties. In general, there are two types of quantum systems, distinguishable and indistinguishable. Different techniques to detect the entanglement in distinguishable systems are separability and Schmidt decomposition. Usually, Schmidt decomposition is used to detect entanglement in distinguishable particles however it doesn't give any information about its quantification in this contest different methods of quantification are used such as von Neumann entropy, concurrence, negativity, and entanglement of formation. The von Neumann entropy is most comprehensive among them which is linked with Schmidt coefficients.

Entanglement detection and quantification become complicated in the systems of indistinguishable particle. Here we try to present various tools to analyze the entanglement in indistinguishable domains but certain complications arise. To tackle them firstly, we introduced symmetrization postulates and develop Slater Schmidt decomposition which was similar to Schmidt decomposition but depends upon the nature of particles in the system. To develop a unified approach, we try to generalize Schmidt

decomposition for identical particles by using, so called no-labeling approach. All these approaches were complicated largely due to symmetrization postulates. So, to circumvent this complication, we introduced a second quantization approach, which is based on the commutation relations. This approach is less complicated and can be applied universally on both distinguishable and indistinguishable particles. Finally, it is concluded that both of these approaches agree in the case of distinguishable particles but deviate in the case of indistinguishable particles. This deviation is because of exchange-correlation that contributes to non-local measurements. It is found the Second quantization approach is relatively simple, equally effective to all kinds of quantum systems and it circumvents the complication of symmetrization postulates.

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