# Introductory Linear Algebra

## An Applied First Course

EIGHTH EDITION

# INTRODUCTORY LINEAR ALGEBRA

## AN APPLIED FIRST COURSE


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To the memory of Lillie
and to Lisa and Stephen

B. K.

To Suzanne

D. R. H.

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# Preface

## Material Covered

This book presents an introduction to linear algebra and to some of its significant applications. It is designed for a course at the freshman or sophomore level. There is more than enough material for a semester or quarter course. By omitting certain sections, it is possible in a one-semester or quarter course to cover the essentials of linear algebra (including eigenvalues and eigenvectors), to show how the computer is used, and to explore some applications of linear algebra. It is no exaggeration to say that with the many applications of linear algebra in other areas of mathematics, physics, biology, chemistry, engineering, statistics, economics, finance, psychology, and sociology, linear algebra is the undergraduate course that will have the most impact on students' lives. The level and pace of the course can be readily changed by varying the amount of time spent on the theoretical material and on the applications. Calculus is not a prerequisite; examples and exercises using very basic calculus are included and these are labeled "Calculus Required."

The emphasis is on the computational and geometrical aspects of the subject, keeping abstraction to a minimum. Thus we sometimes omit proofs of difficult or less-rewarding theorems while amply illustrating them with examples. The proofs that are included are presented at a level appropriate for the student. We have also devoted our attention to the essential areas of linear algebra; the book does not attempt to cover the subject exhaustively.

## What Is New in the Eighth Edition

We have been very pleased by the widespread acceptance of the first seven editions of this book. The reform movement in linear algebra has resulted in a number of techniques for improving the teaching of linear algebra. The **Linear Algebra Curriculum Study Group** and others have made a number of important recommendations for doing this. In preparing the present edition, we have considered these recommendations as well as suggestions from faculty and students. Although many changes have been made in this edition, our objective has remained the same as in the earlier editions:

> **to develop a textbook that will help the instructor to teach and the student to learn the basic ideas of linear algebra and to see some of its applications.**

To achieve this objective, the following features have been developed in this edition:

- New sections have been added as follows:
  - Section 1.5, *Matrix Transformations*, introduces at a very early stage some geometric applications.
  - Section 2.1, *An Introduction to Coding*, along with supporting material on bit matrices throughout the first six chapters, provides an introduction to the basic ideas of coding theory.
  - Section 7.3, *More on Coding*, develops some simple codes and their basic properties related to linear algebra.
- More geometric material has been added.
- New exercises at all levels have been added. Some of these are more open-ended, allowing for exploration and discovery, as well as writing.
- More illustrations have been added.
- MATLAB M-files have been upgraded to more modern versions.
- Key terms have been added at the end of each section, reflecting the increased emphasis in mathematics on communication skills.
- True/false questions now ask the student to justify his or her answer, providing an additional opportunity for exploration and writing.
- Another 25 true/false questions have been added to the cumulative review at the end of the first ten chapters.
- A glossary, new to this edition, has been added.

## Exercises

The exercises in this book are grouped into three classes. The first class, *Exercises*, contains routine exercises. The second class, *Theoretical Exercises*, includes exercises that fill in gaps in some of the proofs and amplify material in the text. Some of these call for a verbal solution. In this technological age, it is especially important to be able to write with care and precision; therefore, exercises of this type should help to sharpen such skills. These exercises can also be used to raise the level of the course and to challenge the more capable and interested student. The third class consists of exercises developed by David R. Hill and are labeled by the prefix ML (for MATLAB). These exercises are designed to be solved by an appropriate computer software package.

Answers to all odd-numbered numerical and ML exercises appear in the back of the book. At the end of Chapter 10, there is a cumulative review of the introductory linear algebra material presented thus far, consisting of 100 true/false questions (with answers in the back of the book). The **Instructor's Solutions Manual**, containing answers to all even-numbered exercises and solutions to all theoretical exercises, is available (to instructors only) at no cost from the publisher.

## Presentation

We have learned from experience that at the sophomore level, abstract ideas must be introduced quite gradually and must be supported by firm foundations. Thus we begin the study of linear algebra with the treatment of matrices as mere arrays of numbers that arise naturally in the solution of systems of linear equations—a problem already familiar to the student. Much attention has been devoted from one edition to the next to refine and improve the pedagogical aspects of the exposition. The abstract ideas are carefully balanced by the considerable emphasis on the geometrical and computational foundations of the subject.

## Material Covered

Chapter 1 deals with matrices and their properties. Section 1.5, *Matrix Transformations*, new to this edition, provides an early introduction to this important topic. This chapter is comprised of two parts: The first part deals with matrices and linear systems and the second part with solutions of linear systems. Chapter 2 (optional) discusses applications of linear equations and matrices to the areas of coding theory, computer graphics, graph theory, electrical circuits, Markov chains, linear economic models, and wavelets. Section 2.1, *An Introduction to Coding*, new to this edition, develops foundations for introducing some basic material in coding theory. To keep this material at a very elementary level, it is necessary to use lengthier technical discussions. Chapter 3 presents the basic properties of determinants rather quickly. Chapter 4 deals with vectors in $R^n$. In this chapter we also discuss vectors in the plane and give an introduction to linear transformations. Chapter 5 (optional) provides an opportunity to explore some of the many geometric ideas dealing with vectors in $R^2$ and $R^3$; we limit our attention to the areas of cross product in $R^3$ and lines and planes.

In Chapter 6 we come to a more abstract notion, that of a vector space. The abstraction in this chapter is more easily handled after the material covered on vectors in $R^n$. Chapter 7 (optional) presents three applications of real vector spaces: $QR$-factorization, least squares, and Section 7.3, *More on Coding*, new to this edition, introducing some simple codes. Chapter 8, on eigenvalues and eigenvectors, the pinnacle of the course, is now presented in three sections to improve pedagogy. The diagonalization of symmetric matrices is carefully developed.

Chapter 9 (optional) deals with a number of diverse applications of eigenvalues and eigenvectors. These include the Fibonacci sequence, differential equations, dynamical systems, quadratic forms, conic sections, and quadric surfaces. Chapter 10 covers linear transformations and matrices. Section 10.4 (optional), *Introduction to Fractals*, deals with an application of a certain nonlinear transformation. Chapter 11 (optional) discusses linear programming, an important application of linear algebra. Section 11.4 presents the basic ideas of the theory of games. Chapter 12, provides a brief introduction to MATLAB (which stands for MATRIX LABORATORY), a very useful software package for linear algebra computation, described below.

Appendix A covers complex numbers and introduces, in a brief but thorough manner, complex numbers and their use in linear algebra. Appendix B presents two more advanced topics in linear algebra: inner product spaces and composite and invertible linear transformations.

## Applications

Most of the applications are entirely independent; they can be covered either after completing the entire introductory linear algebra material in the course or they can be taken up as soon as the material required for a particular application has been developed. Brief Previews of most applications are given at appropriate places in the book to indicate how to provide an immediate application of the material just studied. The chart at the end of this Preface, giving the prerequisites for each of the applications, and the Brief Previews will be helpful in deciding which applications to cover and when to cover them.

Some of the sections in Chapters 2, 5, 7, 9, and 11 can also be used as independent student projects. Classroom experience with the latter approach has met with favorable student reaction. Thus the instructor can be quite selective both in the choice of material and in the method of study of these applications.

## End of Chapter Material

Every chapter contains a summary of *Key Ideas for Review*, a set of supplementary exercises (answers to all odd-numbered numerical exercises appear in the back of the book), and a chapter test (all answers appear in the back of the book).

## MATLAB Software

Although the ML exercises can be solved using a number of software packages, in our judgment MATLAB is the most suitable package for this purpose. MATLAB is a versatile and powerful software package whose cornerstone is its linear algebra capability. MATLAB incorporates professionally developed quality computer routines for linear algebra computation. The code employed by MATLAB is written in the C language and is upgraded as new versions of MATLAB are released. MATLAB is available from The Math Works, Inc., 24 Prime Park Way, Natick, MA 01760, (508) 653-1415; e-mail: **info@mathworks.com** and is not distributed with this book or the instructional routines developed for solving the ML exercises. The Student Edition of MATLAB also includes a version of *Maple*, thereby providing a symbolic computational capability.

Chapter 12 of this edition consists of a brief introduction to MATLAB's capabilities for solving linear algebra problems. Although programs can be written within MATLAB to implement many mathematical algorithms, *it should be noted that the reader of this book is not asked to write programs. The user is merely asked to use* MATLAB *(or any other comparable software package) to solve specific numerical problems*. Approximately 24 instructional M-files have been developed to be used with the ML exercises in this book and are available from the following Prentice Hall Web site: **www.prenhall.com/kolman**. These M-files are designed to transform many of MATLAB's capabilities into courseware. This is done by providing pedagogy that allows the student to interact with MATLAB, thereby letting the student think through all the steps in the solution of a problem and relegating MATLAB to act as a powerful calculator to relieve the drudgery of a tedious computation. Indeed, this is the ideal role for MATLAB (or any other similar package) in a beginning linear algebra course, for in this course, more than in many others, the tedium of lengthy computations makes it almost impossible to solve a modest-size problem. Thus, by introducing pedagogy and reining in the power of MATLAB, these M-files provide a working partnership between the student and the computer. Moreover, the introduction to a powerful tool such as MATLAB early in the student's college career opens the way for other software support in higher-level courses, especially in science and engineering.

## Supplements

**Student Solutions Manual (0-13-143741-0).** Prepared by Dennis Kletzing, Stetson University, and Nina Edelman and Kathy O'Hara, Temple University, contains solutions to all odd-numbered exercises, both numerical and theoretical. It can be purchased from the publisher.

**Instructor's Solutions Manual (0-13-143742-9).** Contains answers to all even-numbered exercises and solutions to all theoretical exercises—is available (to instructors only) at no cost from the publisher.

**Optional combination packages.** Provide a computer workbook free of charge when packaged with this book.

- *Linear Algebra Labs with MATLAB*, by David R. Hill and David E. Zitarelli, 3rd edition, ISBN 0-13-124092-7 (supplement and text).
- *Visualizing Linear Algebra with Maple*, by Sandra Z. Keith, ISBN 0-13-124095-1 (supplement and text).
- *ATLAST Computer Exercises for Linear Algebra*, by Steven Leon, Eugene Herman, and Richard Faulkenberry, 2nd edition, ISBN 0-13-124094-3 (supplement and text).
- *Understanding Linear Algebra with MATLAB*, by Erwin and Margaret Kleinfeld, ISBN 0-13-124093-5 (supplement and text).

## Prerequisites for Applications

***Prerequisites for Applications***

| | |
|---|---|
| Section 2.1 | Material on bits in Chapter 1 |
| Section 2.2 | Section 1.4 |
| Section 2.3 | Section 1.5 |
| Section 2.4 | Section 1.6 |
| Section 2.5 | Section 1.6 |
| Section 2.6 | Section 1.7 |
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| Section 5.1 | Section 4.1 and Chapter 3 |
| Section 5.2 | Sections 4.1 and 5.1 |
| Section 7.1 | Section 6.8 |
| Section 7.2 | Sections 1.6, 1.7, 4.2, 6.9 |
| Section 7.3 | Section 2.1 |
| Section 9.1 | Section 8.2 |
| Section 9.2 | Section 8.2 |
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| Section 9.5 | Section 9.4 |
| Section 9.6 | Section 9.5 |
| Section 10.4 | Section 8.2 |
| Sections 11.1–11.3 | Section 1.6 |
| Section 11.4 | Sections 11.1–11.3 |

**To Users of Previous Editions:**

During the 29-year life of the previous seven editions of this book, the book was primarily used to teach a sophomore-level linear algebra course. This course covered the essentials of linear algebra and used any available extra time to study selected applications of the subject. *In this new edition we have not changed the structural foundation for teaching the essential linear algebra material. Thus, this material can be taught in exactly the same manner as before*. The placement of the applications in a more cohesive and pedagogically unified manner together with the newly added applications and other material should make it easier to teach a richer and more varied course.

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We thank Dennis Kletzing, Stetson University, and Nina Edelman and

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# TO THE STUDENT

It is very likely that this course is unlike any other mathematics course that you have studied thus far in at least two important ways. First, it may be your initial introduction to abstraction. Second, it is a mathematics course that may well have the greatest impact on your vocation.

Unlike other mathematics courses, this course will not give you a toolkit of isolated computational techniques for solving certain types of problems. Instead, we will develop a core of material called linear algebra by introducing certain definitions and creating procedures for determining properties and proving theorems. Proving a theorem is a skill that takes time to master, so at first we will only expect you to read and understand the proof of a theorem. As you progress in the course, you will be able to tackle some simple proofs. We introduce you to abstraction slowly, keep it to a minimum, and amply illustrate each abstract idea with concrete numerical examples and applications. Although you will be doing a lot of computations, the goal in most problems is not merely to get the "right" answer, but to understand and explain how to get the answer and then interpret the result.

Linear algebra is used in the everyday world to solve problems in other areas of mathematics, physics, biology, chemistry, engineering, statistics, economics, finance, psychology, and sociology. Applications that use linear algebra include the transmission of information, the development of special effects in film and video, recording of sound, Web search engines on the Internet, and economic analyses. Thus, you can see how profoundly linear algebra affects you. A selected number of applications are included in this book, and if there is enough time, some of these may be covered in this course. Additionally, many of the applications can be used as self-study projects.

There are three different types of exercises in this book. First, there are computational exercises. These exercises and the numbers in them have been carefully chosen so that almost all of them can readily be done by hand. When you use linear algebra in real applications, you will find that the problems are much bigger in size and the numbers that occur in them are not always "nice." This is not a problem because you will almost certainly use powerful software to solve them. A taste of this type of software is provided by the third type of exercises. These are exercises designed to be solved by using a computer and MATLAB, a powerful matrix-based application that is widely used in industry. The second type of exercises are theoretical. Some of these may ask you to prove a result or discuss an idea. In today's world, it is not enough to be able to compute an answer; you often have to prepare a report discussing your solution, justifying the steps in your solution, and interpreting your results.

These types of exercises will give you experience in writing mathematics. Mathematics uses words, not just symbols.

## How to Succeed in Linear Algebra

- Read the book slowly with pencil and paper at hand. You might have to read a particular section more than once. Take the time to verify the steps marked "verify" in the text.
- Make sure to do your homework on a timely basis. If you wait until the problems are explained in class, you will miss learning how to solve a problem by yourself. Even if you can't complete a problem, try it anyway, so that when you see it done in class you will understand it more easily. You might find it helpful to work with other students on the material covered in class and on some homework problems.
- Make sure that you ask for help as soon as something is not clear to you. Each abstract idea in this course is based on previously developed ideas—much like laying a foundation and then building a house. If any of the ideas are fuzzy to you or missing, your knowledge of the course will not be sturdy enough for you to grasp succeeding ideas.
- Make use of the pedagogical tools provided in this book. At the end of each section we have a list of key terms; at the end of each chapter we have a list of key ideas for review, supplementary exercises, and a chapter test. At the end of the first ten chapters (completing the core linear algebra material in the course) we have a comprehensive review consisting of 100 true/false questions that ask you to justify your answer. Finally, there is a glossary for linear algebra at the end of the book. Answers to the odd-numbered exercises appear at the end of the book. The *Student Solutions Manual* provides detailed solutions to all odd-numbered exercises, both numerical and theoretical. It can be purchased from the publisher (ISBN 0-13-143742-9).

We assure you that your efforts to learn linear algebra well will be amply rewarded in other courses and in your professional career.

We wish you much success in your study of linear algebra.

Bernard Kolman

David R. Hill

# Introductory Linear Algebra

## an applied first course

CHAPTER 1

# LINEAR EQUATIONS AND MATRICES

## 1.1 LINEAR SYSTEMS

A good many problems in the natural and social sciences as well as in engineering and the physical sciences deal with equations relating two sets of variables. An equation of the type

$$ax = b,$$

expressing the variable $b$ in terms of the variable $x$ and the constant $a$, is called a **linear equation**. The word *linear* is used here because the graph of the equation above is a straight line. Similarly, the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{1}$$

expressing $b$ in terms of the variables $x_1, x_2, \ldots, x_n$ and the known constants $a_1, a_2, \ldots, a_n$, is called a **linear equation**. In many applications we are given $b$ and the constants $a_1, a_2, \ldots, a_n$ and must find numbers $x_1, x_2, \ldots, x_n$, called **unknowns**, satisfying (1).

A **solution** to a linear equation (1) is a sequence of $n$ numbers $s_1, s_2, \ldots, s_n$, which has the property that (1) is satisfied when $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ are substituted in (1).

Thus $x_1 = 2$, $x_2 = 3$, and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13,$$

because

$$6(2) - 3(3) + 4(-4) = -13.$$

This is not the only solution to the given linear equation, since $x_1 = 3$, $x_2 = 1$, and $x_3 = -7$ is another solution.

More generally, a **system of $m$ linear equations in $n$ unknowns $x_1$, $x_2$, $\ldots$, $x_n$**, or simply a **linear system**, is a set of $m$ linear equations each in $n$ unknowns. A linear system can be conveniently denoted by

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{2}$$

The two subscripts $i$ and $j$ are used as follows. The first subscript $i$ indicates that we are dealing with the $i$th equation, while the second subscript $j$ is associated with the $j$th variable $x_j$. Thus the $i$th equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i.$$

In (2) the $a_{ij}$ are known constants. Given values of $b_1, b_2, \ldots, b_m$, we want to find values of $x_1, x_2, \ldots, x_n$ that will satisfy each equation in (2).

A **solution** to a linear system (2) is a sequence of $n$ numbers $s_1, s_2, \ldots, s_n$, which has the property that each equation in (2) is satisfied when $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$ are substituted in (2).

To find solutions to a linear system, we shall use a technique called the **method of elimination**. That is, we eliminate some of the unknowns by adding a multiple of one equation to another equation. Most readers have had some experience with this technique in high school algebra courses. Most likely, the reader has confined his or her earlier work with this method to linear systems in which $m = n$, that is, linear systems having as many equations as unknowns. In this course we shall broaden our outlook by dealing with systems in which we have $m = n$, $m < n$, and $m > n$. Indeed, there are numerous applications in which $m \neq n$. If we deal with two, three, or four unknowns, we shall often write them as $x$, $y$, $z$, and $w$. In this section we use the method of elimination as it was studied in high school. In Section 1.5 we shall look at this method in a much more systematic manner.

**EXAMPLE 1**

The director of a trust fund has \$100,000 to invest. The rules of the trust state that both a certificate of deposit (CD) and a long-term bond must be used. The director's goal is to have the trust yield \$7800 on its investments for the year. The CD chosen returns 5% per annum and the bond 9%. The director determines the amount $x$ to invest in the CD and the amount $y$ to invest in the bond as follows:

Since the total investment is \$100,000, we must have $x + y = 100{,}000$. Since the desired return is \$7800, we obtain the equation $0.05x + 0.09y = 7800$. Thus, we have the linear system

$$\begin{aligned} x + \phantom{0.09}y &= 100{,}000 \\ 0.05x + 0.09y &= 7800. \end{aligned} \tag{3}$$

To eliminate $x$, we add $(-0.05)$ times the first equation to the second, obtaining

$$\begin{aligned} x + \phantom{0.04}y &= 100{,}000 \\ 0.04y &= 2800, \end{aligned}$$

where the second equation has no $x$ term. We have eliminated the unknown $x$. Then solving for $y$ in the second equation, we have

$$y = 70{,}000,$$

and substituting $y$ into the first equation of (3), we obtain

$$x = 30{,}000.$$

To check that $x = 30{,}000$, $y = 70{,}000$ is a solution to (3), we verify that these values of $x$ and $y$ satisfy *each* of the equations in the given linear system. Thus, the director of the trust should invest \$30,000 in the CD and \$70,000 in the long-term bond. ■

**EXAMPLE 2** Consider the linear system

$$\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= \phantom{-}7. \end{aligned} \tag{4}$$

Again, we decide to eliminate $x$. We add $(-2)$ times the first equation to the second one, obtaining

$$\begin{aligned} x - 3y &= -7 \\ 0x + 0y &= 21 \end{aligned}$$

whose second equation makes no sense. This means that the linear system (4) has no solution. We might have come to the same conclusion from observing that in (4) the left side of the second equation is twice the left side of the first equation, but the right side of the second equation is not twice the right side of the first equation. ■

**EXAMPLE 3** Consider the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2. \end{aligned} \tag{5}$$

To eliminate $x$, we add $(-2)$ times the first equation to the second one and $(-3)$ times the first equation to the third one, obtaining

$$\begin{aligned} x + 2y + 3z &= 6 \\ -7y - 4z &= 2 \\ -5y - 10z &= -20. \end{aligned} \tag{6}$$

We next eliminate $y$ from the second equation in (6) as follows. Multiply the third equation of (6) by $\left(-\frac{1}{5}\right)$, obtaining

$$\begin{aligned} x + 2y + 3z &= 6 \\ -7y - 4z &= 2 \\ y + 2z &= 4. \end{aligned}$$

Next we interchange the second and third equations to give

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ -7y - 4z &= 2. \end{aligned} \tag{7}$$

We now add 7 times the second equation to the third one, to obtain

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ 10z &= 30. \end{aligned}$$

Multiplying the third equation by $\frac{1}{10}$, we have

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3. \end{aligned} \tag{8}$$

Substituting $z = 3$ into the second equation of (8), we find $y = -2$. Substituting these values of $z$ and $y$ into the first equation of (8), we have $x = 1$. To check that $x = 1$, $y = -2$, $z = 3$ is a solution to (5), we verify that these values of $x$, $y$, and $z$ satisfy *each* of the equations in (5). Thus, $x = 1$, $y = -2$, $z = 3$ is a solution to the linear system (5). The importance of the procedure lies in the fact that the linear systems (5) and (8) have exactly the same solutions. System (8) has the advantage that it can be solved quite easily, giving the foregoing values for $x$, $y$, and $z$. ■

**EXAMPLE 4** Consider the linear system

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + \phantom{2}y - 3z &= \phantom{-}4. \end{aligned} \tag{9}$$

Eliminating $x$, we add $(-2)$ times the first equation to the second one, to obtain

$$\begin{aligned} x + 2y - 3z &= -4 \\ - 3y + 3z &= \phantom{-}12. \end{aligned} \tag{10}$$

Solving the second equation in (10) for $y$, we obtain

$$y = z - 4,$$

where $z$ can be any real number. Then, from the first equation of (10),

$$\begin{aligned} x &= -4 - 2y + 3z \\ &= -4 - 2(z - 4) + 3z \\ &= z + 4. \end{aligned}$$

Thus a solution to the linear system (9) is

$$\begin{aligned} x &= r + 4 \\ y &= r - 4 \\ z &= r, \end{aligned}$$

where $r$ is any real number. This means that the linear system (9) has infinitely many solutions. Every time we assign a value to $r$, we obtain another solution to (9). Thus, if $r = 1$, then

$$x = 5, \quad y = -3, \quad \text{and} \quad z = 1$$

is a solution, while if $r = -2$, then

$$x = 2, \quad y = -6, \quad \text{and} \quad z = -2$$

is another solution. ■

**EXAMPLE 5** Consider the linear system

$$\begin{aligned} x + 2y &= \phantom{-}10 \\ 2x - 2y &= -4 \\ 3x + 5y &= \phantom{-}26. \end{aligned} \tag{11}$$

Eliminating $x$, we add $(-2)$ times the first equation to the second and $(-3)$ times the first equation to the third one, obtaining

$$\begin{aligned} x + \ 2y &= \ \ 10 \\ -\ 6y &= -24 \\ -y &= \ -4. \end{aligned}$$

Multiplying the second equation by $\left(-\frac{1}{6}\right)$ and the third one by $(-1)$, we have

$$\begin{aligned} x + 2y &= 10 \\ y &= \ 4 \\ y &= \ 4, \end{aligned} \tag{12}$$

which has the same solutions as (11). Substituting $y = 4$ in the first equation of (12), we obtain $x = 2$. Hence $x = 2$, $y = 4$ is a solution to (11). ■

**EXAMPLE 6** Consider the linear system

$$\begin{aligned} x + 2y &= \ 10 \\ 2x - 2y &= -4 \\ 3x + 5y &= \ 20. \end{aligned} \tag{13}$$

To eliminate $x$, we add $(-2)$ times the first equation to the second one and $(-3)$ times the first equation to the third one, to obtain

$$\begin{aligned} x + \ 2y &= \ \ 10 \\ -\ 6y &= -24 \\ -y &= -10. \end{aligned}$$

Multiplying the second equation by $\left(-\frac{1}{6}\right)$ and the third one by $(-1)$, we have the system

$$\begin{aligned} x + 2y &= 10 \\ y &= \ 4 \\ y &= 10, \end{aligned} \tag{14}$$

which has no solution. Since (14) and (13) have the same solutions, we conclude that (13) has no solutions. ■

These examples suggest that a linear system may have one solution (a unique solution), no solution, or infinitely many solutions.

> We have seen that the method of elimination consists of repeatedly performing the following operations:
>
> **1.** Interchange two equations.
> **2.** Multiply an equation by a nonzero constant.
> **3.** Add a multiple of one equation to another.

It is not difficult to show (Exercises T.1 through T.3) that the method of elimination yields another linear system having exactly the same solutions as the given system. The new linear system can then be solved quite readily.

As you have probably already observed, the method of elimination has been described, so far, in general terms. Thus we have not indicated any rules for selecting the unknowns to be eliminated. Before providing a systematic description of the method of elimination, we introduce, in the next section, the notion of a matrix, which will greatly simplify our notation and will enable us to develop tools to solve many important problems.

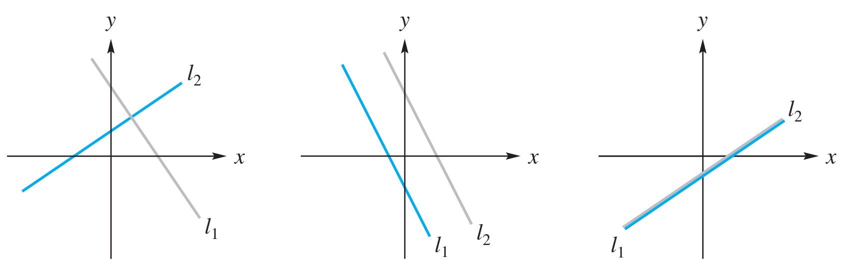
Consider now a linear system of two equations in the unknowns $x$ and $y$:

$$\begin{aligned} a_1x + a_2y &= c_1 \\ b_1x + b_2y &= c_2. \end{aligned} \tag{15}$$

The graph of each of these equations is a straight line, which we denote by $l_1$ and $l_2$, respectively. If $x = s_1$, $y = s_2$ is a solution to the linear system (15), then the point $(s_1, s_2)$ lies on both lines $l_1$ and $l_2$. Conversely, if the point $(s_1, s_2)$ lies on both lines $l_1$ and $l_2$, then $x = s_1$, $y = s_2$ is a solution to the linear system (15). (See Figure 1.1.) Thus we are led geometrically to the same three possibilities mentioned previously.

1. The system has a unique solution; that is, the lines $l_1$ and $l_2$ intersect at exactly one point.
2. The system has no solution; that is, the lines $l_1$ and $l_2$ do not intersect.
3. The system has infinitely many solutions; that is, the lines $l_1$ and $l_2$ coincide.

**Figure 1.1** ►



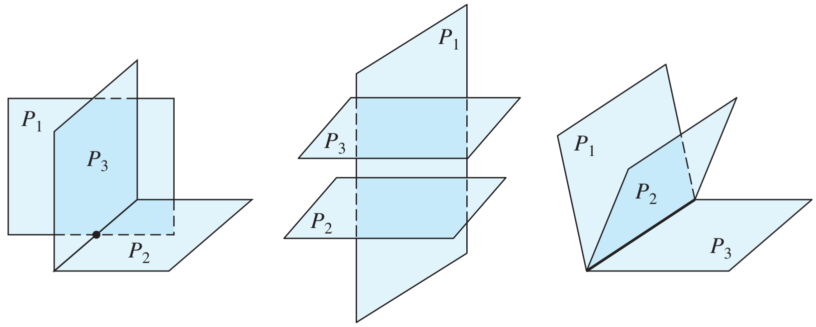

(a) A unique solution (b) No solution (c) Infinitely many solutions

Next, consider a linear system of three equations in the unknowns $x$, $y$, and $z$:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{16}$$

The graph of each of these equations is a plane, denoted by $P_1$, $P_2$, and $P_3$, respectively. As in the case of a linear system of two equations in two unknowns, the linear system in (16) can have a unique solution, no solution, or infinitely many solutions. These situations are illustrated in Figure 1.2. For a more concrete illustration of some of the possible cases, the walls (planes) of a room intersect in a unique point, a corner of the room, so the linear system has a unique solution. Next, think of the planes as pages of a book. Three pages of a book (when held open) intersect in a straight line, the spine. Thus, the linear system has infinitely many solutions. On the other hand, when the book is closed, three pages of a book appear to be parallel and do not intersect, so the linear system has no solution.

**Figure 1.2** ►




(a) A unique solution (b) No solution (c) Infinitely many solutions

**EXAMPLE 7**

**(Production Planning)** A manufacturer makes three different types of chemical products: $A$, $B$, and $C$. Each product must go through two processing machines: $X$ and $Y$. The products require the following times in machines $X$ and $Y$:

1. One ton of $A$ requires 2 hours in machine $X$ and 2 hours in machine $Y$.
2. One ton of $B$ requires 3 hours in machine $X$ and 2 hours in machine $Y$.
3. One ton of $C$ requires 4 hours in machine $X$ and 3 hours in machine $Y$.

Machine $X$ is available 80 hours per week and machine $Y$ is available 60 hours per week. Since management does not want to keep the expensive machines $X$ and $Y$ idle, it would like to know how many tons of each product to make so that the machines are fully utilized. It is assumed that the manufacturer can sell as much of the products as is made.

To solve this problem, we let $x_1$, $x_2$, and $x_3$ denote the number of tons of products $A$, $B$, and $C$, respectively, to be made. The number of hours that machine $X$ will be used is

$$2x_1 + 3x_2 + 4x_3,$$

which must equal 80. Thus we have

$$2x_1 + 3x_2 + 4x_3 = 80.$$

Similarly, the number of hours that machine $Y$ will be used is 60, so we have

$$2x_1 + 2x_2 + 3x_3 = 60.$$

Mathematically, our problem is to find nonnegative values of $x_1$, $x_2$, and $x_3$ so that

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 80 \\ 2x_1 + 2x_2 + 3x_3 &= 60. \end{aligned}$$

This linear system has infinitely many solutions. Following the method of Example 4, we see that all solutions are given by

$$\begin{aligned} x_1 &= \frac{20 - x_3}{2} \\ x_2 &= 20 - x_3 \\ x_3 &= \text{any real number such that } 0 \leq x_3 \leq 20, \end{aligned}$$

since we must have $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. When $x_3 = 10$, we have

$$x_1 = 5, \qquad x_2 = 10, \qquad x_3 = 10$$

while

$$x_1 = \tfrac{13}{2}, \qquad x_2 = 13, \qquad x_3 = 7$$

when $x_3 = 7$. The reader should observe that one solution is just as good as the other. There is no best solution unless additional information or restrictions are given. ■

## Key Terms

Linear equation
Unknowns
Solution to a linear equation
Linear system
Solution to a linear system
Method of elimination
Unique solution
No solution
Infinitely many solutions
Manipulations on a linear system

## 1.1 Exercises

*In Exercises 1 through 14, solve the given linear system by the method of elimination.*

**1.** $\begin{aligned} x + 2y &= 8 \\ 3x - 4y &= 4 \end{aligned}$

**2.** $\begin{aligned} 2x - 3y + 4z &= -12 \\ x - 2y + z &= -5 \\ 3x + y + 2z &= 1 \end{aligned}$

**3.** $\begin{aligned} 3x + 2y + z &= 2 \\ 4x + 2y + 2z &= 8 \\ x - y + z &= 4 \end{aligned}$

**4.** $\begin{aligned} x + y &= 5 \\ 3x + 3y &= 10 \end{aligned}$

**5.** $\begin{aligned} 2x + 4y + 6z &= -12 \\ 2x - 3y - 4z &= 15 \\ 3x + 4y + 5z &= -8 \end{aligned}$

**6.** $\begin{aligned} x + y - 2z &= 5 \\ 2x + 3y + 4z &= 2 \end{aligned}$

**7.** $\begin{aligned} x + 4y - z &= 12 \\ 3x + 8y - 2z &= 4 \end{aligned}$

**8.** $\begin{aligned} 3x + 4y - z &= 8 \\ 6x + 8y - 2z &= 3 \end{aligned}$

**9.** $\begin{aligned} x + y + 3z &= 12 \\ 2x + 2y + 6z &= 6 \end{aligned}$

**10.** $\begin{aligned} x + y &= 1 \\ 2x - y &= 5 \\ 3x + 4y &= 2 \end{aligned}$

**11.** $\begin{aligned} 2x + 3y &= 13 \\ x - 2y &= 3 \\ 5x + 2y &= 27 \end{aligned}$

**12.** $\begin{aligned} x - 5y &= 6 \\ 3x + 2y &= 1 \\ 5x + 2y &= 1 \end{aligned}$

**13.** $\begin{aligned} x + 3y &= -4 \\ 2x + 5y &= -8 \\ x + 3y &= -5 \end{aligned}$

**14.** $\begin{aligned} 2x + 3y - z &= 6 \\ 2x - y + 2z &= -8 \\ 3x - y + z &= -7 \end{aligned}$

**15.** Given the linear system

$$\begin{aligned} 2x - y &= 5 \\ 4x - 2y &= t, \end{aligned}$$

(a) determine a value of $t$ so that the system has a solution.

(b) determine a value of $t$ so that the system has no solution.

(c) how many different values of $t$ can be selected in part (b)?

**16.** Given the linear system

$$\begin{aligned} 2x + 3y - z &= 0 \\ x - 4y + 5z &= 0, \end{aligned}$$

(a) verify that $x_1 = 1$, $y_1 = -1$, $z_1 = -1$ is a solution.

(b) verify that $x_2 = -2$, $y_2 = 2$, $z_2 = 2$ is a solution.

(c) is $x = x_1 + x_2 = -1$, $y = y_1 + y_2 = 1$, and $z = z_1 + z_2 = 1$ a solution to the linear system?

(d) is $3x$, $3y$, $3z$, where $x$, $y$, and $z$ are as in part (c), a solution to the linear system?

**17.** Without using the method of elimination, solve the linear system

$$\begin{aligned} 2x + y - 2z &= -5 \\ 3y + z &= 7 \\ z &= 4. \end{aligned}$$

**18.** Without using the method of elimination, solve the linear system

$$\begin{aligned} 4x \phantom{{}+3y-2z} &= 8 \\ -2x + 3y \phantom{{}-2z} &= -1 \\ 3x + 5y - 2z &= 11. \end{aligned}$$

**19.** Is there a value of $r$ so that $x = 1$, $y = 2$, $z = r$ is a solution to the following linear system? If there is, find it.

$$\begin{aligned} 2x + 3y - z &= 11 \\ x - y + 2z &= -7 \\ 4x + y - 2z &= 12 \end{aligned}$$

**20.** Is there a value of $r$ so that $x = r$, $y = 2$, $z = 1$ is a solution to the following linear system? If there is, find it.

$$\begin{aligned} 3x \qquad - 2z &= 4 \\ x - 4y + z &= -5 \\ -2x + 3y + 2z &= 9 \end{aligned}$$

**21.** Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.2.

**22.** Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.3.

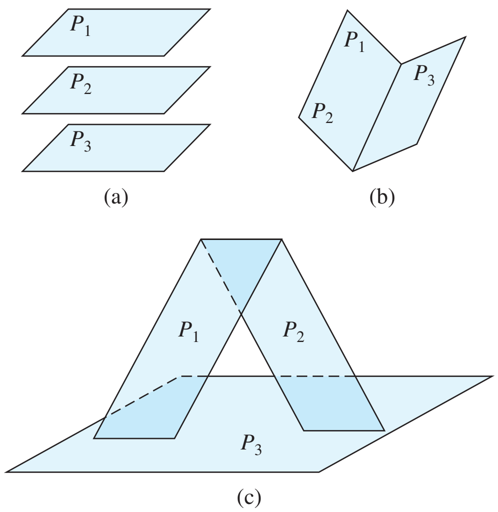



**Figure 1.3** ▲

**23.** An oil refinery produces low-sulfur and high-sulfur fuel. Each ton of low-sulfur fuel requires 5 minutes in the blending plant and 4 minutes in the refining plant; each ton of high-sulfur fuel requires 4 minutes in the blending plant and 2 minutes in the refining plant. If the blending plant is available for 3 hours and the refining plant is available for 2 hours, how many tons of each type of fuel should be manufactured so that the plants are fully utilized?

**24.** A plastics manufacturer makes two types of plastic: regular and special. Each ton of regular plastic requires 2 hours in plant A and 5 hours in plant B; each ton of special plastic requires 2 hours in plant A and 3 hours in plant B. If plant A is available 8 hours per day and plant B is available 15 hours per day, how many tons of each type of plastic can be made daily so that the plants are fully utilized?

**25.** A dietician is preparing a meal consisting of foods A, B, and C. Each ounce of food A contains 2 units of protein, 3 units of fat, and 4 units of carbohydrate. Each ounce of food B contains 3 units of protein, 2 units of fat, and 1 unit of carbohydrate. Each ounce of food C contains 3 units of protein, 3 units of fat, and 2 units of carbohydrate. If the meal must provide exactly 25 units of protein, 24 units of fat, and 21 units of carbohydrate, how many ounces of each type of food should be used?

**26.** A manufacturer makes 2-minute, 6-minute, and 9-minute film developers. Each ton of 2-minute developer requires 6 minutes in plant A and 24 minutes in plant B. Each ton of 6-minute developer requires 12 minutes in plant A and 12 minutes in plant B. Each ton of 9-minute developer requires 12 minutes in plant A and 12 minutes in plant B. If plant A is available 10 hours per day and plant B is available 16 hours per day, how many tons of each type of developer can be produced so that the plants are fully utilized?

**27.** Suppose that the three points $(1, -5)$, $(-1, 1)$, and $(2, 7)$ lie on the parabola $p(x) = ax^2 + bx + c$.

(a) Determine a linear system of three equations in three unknowns that must be solved to find $a$, $b$, and $c$.

(b) Solve the linear system obtained in part (a) for $a$, $b$, and $c$.

**28.** An inheritance of \$24,000 is to be divided among three trusts, with the second trust receiving twice as much as the first trust. The three trusts pay interest at the rates of 9%, 10%, and 6% annually, respectively, and return a total in interest of \$2210 at the end of the first year. How much was invested in each trust?

## Theoretical Exercises

**T.1.** Show that the linear system obtained by interchanging two equations in (2) has exactly the same solutions as (2).

**T.2.** Show that the linear system obtained by replacing an equation in (2) by a nonzero constant multiple of the equation has exactly the same solutions as (2).

**T.3.** Show that the linear system obtained by replacing an equation in (2) by itself plus a multiple of another equation in (2) has exactly the same solutions as (2).

**T.4.** Does the linear system

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

always have a solution for any values of $a$, $b$, $c$, and $d$?

## 1.2 MATRICES

If we examine the method of elimination described in Section 1.1, we make the following observation. Only the numbers in front of the unknowns $x_1$, $x_2$, $\dots$, $x_n$ are being changed as we perform the steps in the method of elimination. Thus we might think of looking for a way of writing a linear system without having to carry along the unknowns. In this section we define an object, a matrix, that enables us to do this—that is, to write linear systems in a compact form that makes it easier to automate the elimination method on a computer in order to obtain a fast and efficient procedure for finding solutions. The use of a matrix is not, however, merely that of a convenient notation. We now develop operations on matrices (plural of matrix) and will work with matrices according to the rules they obey; this will enable us to solve systems of linear equations and solve other computational problems in a fast and efficient manner. Of course, as any good definition should do, the notion of a matrix provides not only a new way of looking at old problems, but also gives rise to a great many new questions, some of which we study in this book.

**DEFINITION**

An $m \times n$ **matrix** $A$ is a rectangular array of $mn$ real (or complex) numbers arranged in $m$ horizontal **rows** and $n$ vertical **columns**:

$$A = \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow i\text{th row} \\ \\ \\ \end{array} \\ \uparrow j\text{th column} & \end{array} \qquad (1)$$

The ***i*th row** of $A$ is

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \qquad (1 \le i \le m);$$

the ***j*th column** of $A$ is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \qquad (1 \le j \le n).$$

We shall say that $A$ is ***m* by *n*** (written as $\boldsymbol{m \times n}$). If $m = n$, we say that $A$ is a **square matrix of order *n*** and that the numbers $a_{11}, a_{22}, \dots, a_{nn}$ form the **main diagonal** of $A$. We refer to the number $a_{ij}$, which is in the $i$th row and $j$th column of $A$, as the ***i*,*j*th element** of $A$, or the **(*i*, *j*) entry** of $A$, and we often write (1) as

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}.$$

For the sake of simplicity, we restrict our attention in this book, except for Appendix A, to matrices all of whose entries are real numbers. However, matrices with complex entries are studied and are important in applications.

**EXAMPLE 1** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}, \qquad E = \begin{bmatrix} 3 \end{bmatrix}, \qquad F = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}.$$

Then $A$ is a $2 \times 3$ matrix with $a_{12} = 2$, $a_{13} = 3$, $a_{22} = 0$, and $a_{23} = 1$; $B$ is a $2 \times 2$ matrix with $b_{11} = 1$, $b_{12} = 4$, $b_{21} = 2$, and $b_{22} = -3$; $C$ is a $3 \times 1$ matrix with $c_{11} = 1$, $c_{21} = -1$, and $c_{31} = 2$; $D$ is a $3 \times 3$ matrix; $E$ is a $1 \times 1$ matrix; and $F$ is a $1 \times 3$ matrix. In $D$, the elements $d_{11} = 1$, $d_{22} = 0$, and $d_{33} = 2$ form the main diagonal. ■

For convenience, we focus much of our attention in the illustrative examples and exercises in Chapters 1–7 on matrices and expressions containing only real numbers. Complex numbers will make a brief appearance in Chapters 8 and 9. An introduction to complex numbers, their properties, and examples and exercises showing how complex numbers are used in linear algebra may be found in Appendix A.

A $1 \times n$ or an $n \times 1$ matrix is also called an ***n*-vector** and will be denoted by lowercase boldface letters. When $n$ is understood, we refer to $n$-vectors merely as **vectors**. In Chapter 4 we discuss vectors at length.

**EXAMPLE 2** $\mathbf{u} = \begin{bmatrix} 1 & 2 & -1 & 0 \end{bmatrix}$ is a 4-vector and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ is a 3-vector. ■

The $n$-vector all of whose entries are zero is denoted by $\mathbf{0}$.

Observe that if $A$ is an $n \times n$ matrix, then the rows of $A$ are $1 \times n$ matrices and the columns of $A$ are $n \times 1$ matrices. The set of all $n$-vectors with real entries is denoted by $R^n$. Similarly, the set of all $n$-vectors with complex entries is denoted by $C^n$. As we have already pointed out, in the first seven chapters of this book we will work almost entirely with vectors in $R^n$.

**EXAMPLE 3** **(Tabular Display of Data)** The following matrix gives the airline distances between the indicated cities (in statute miles).

$$\begin{array}{l} \\ \text{London} \\ \text{Madrid} \\ \text{New York} \\ \text{Tokyo} \end{array} \begin{array}{c} \begin{array}{cccc} \text{London} & \text{Madrid} & \text{New York} & \text{Tokyo} \end{array} \\ \begin{bmatrix} 0 & 785 & 3469 & 5959 \\ 785 & 0 & 3593 & 6706 \\ 3469 & 3593 & 0 & 6757 \\ 5959 & 6706 & 6757 & 0 \end{bmatrix} \end{array}$$ ■

**EXAMPLE 4** **(Production)** Suppose that a manufacturer has four plants each of which makes three products. If we let $a_{ij}$ denote the number of units of product $i$ made by plant $j$ in one week, then the $4 \times 3$ matrix

$$\begin{array}{l} \\ \text{Plant 1} \\ \text{Plant 2} \\ \text{Plant 3} \\ \text{Plant 4} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Product 1} & \text{Product 2} & \text{Product 3} \end{array} \\ \begin{bmatrix} 560 & 340 & 280 \\ 360 & 450 & 270 \\ 380 & 420 & 210 \\ 0 & 80 & 380 \end{bmatrix} \end{array}$$

gives the manufacturer's production for the week. For example, plant 2 makes 270 units of product 3 in one week. ■

**EXAMPLE 5** The wind chill table that follows shows how a combination of air temperature and wind speed makes a body feel colder than the actual temperature. For example, when the temperature is 10°F and the wind is 15 miles per hour, this causes a body heat loss equal to that when the temperature is −18°F with no wind.

| mph | °F 15 | 10 | 5 | 0 | −5 | −10 |
|---|---|---|---|---|---|---|
| 5 | 12 | 7 | 0 | −5 | −10 | −15 |
| 10 | −3 | −9 | −15 | −22 | −27 | −34 |
| 15 | −11 | −18 | −25 | −31 | −38 | −45 |
| 20 | −17 | −24 | −31 | −39 | −46 | −53 |

This table can be represented as the matrix

$$A = \begin{bmatrix} 5 & 12 & 7 & 0 & -5 & -10 & -15 \\ 10 & -3 & -9 & -15 & -22 & -27 & -34 \\ 15 & -11 & -18 & -25 & -31 & -38 & -45 \\ 20 & -17 & -24 & -31 & -39 & -46 & -53 \end{bmatrix}.$$

■

**EXAMPLE 6** With the linear system considered in Example 5 in Section 1.1,

$$\begin{aligned} x + 2y &= 10 \\ 2x - 2y &= -4 \\ 3x + 5y &= 26, \end{aligned}$$

we can associate the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ -4 \\ 26 \end{bmatrix}.$$

In Section 1.3, we shall call $A$ the coefficient matrix of the linear system. ■

**DEFINITION** A square matrix $A = \left[a_{ij}\right]$ for which every term off the main diagonal is zero, that is, $a_{ij} = 0$ for $i \neq j$, is called a **diagonal matrix**.

**EXAMPLE 7**

$$G = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

are diagonal matrices. ■

**DEFINITION**

A diagonal matrix $A = \left[a_{ij}\right]$, for which all terms on the main diagonal are equal, that is, $a_{ij} = c$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$, is called a **scalar matrix**.

**EXAMPLE 8**

The following are scalar matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

■

The search engines available for information searches and retrieval on the Internet use matrices to keep track of the locations of information, the type of information at a location, keywords that appear in the information, and even the way Web sites link to one another. A large measure of the effectiveness of the search engine Google© is the manner in which matrices are used to determine which sites are referenced by other sites. That is, instead of directly keeping track of the information content of an actual Web page or of an individual search topic, Google's matrix structure focuses on finding Web pages that match the search topic and then presents a list of such pages in the order of their "importance."

Suppose that there are $n$ accessible Web pages during a certain month. A simple way to view a matrix that is part of Google's scheme is to imagine an $n \times n$ matrix $A$, called the "connectivity matrix," that initially contains all zeros. To build the connections proceed as follows. When it is detected that Web site $j$ links to Web site $i$, set entry $a_{ij}$ equal to one. Since $n$ is quite large, about 3 billion as of December 2002, most entries of the connectivity matrix $A$ are zero. (Such a matrix is called sparse.) If row $i$ of $A$ contains many ones, then there are many sites linking to site $i$. Sites that are linked to by many other sites are considered more "important" (or to have a higher rank) by the software driving the Google search engine. Such sites would appear near the top of a list returned by a Google search on topics related to the information on site $i$. Since Google updates its connectivity matrix about every month, $n$ increases over time and new links and sites are adjoined to the connectivity matrix.

The fundamental technique used by Google© to rank sites uses linear algebra concepts that are somewhat beyond the scope of this course. Further information can be found in the following sources.

**1.** Berry, Michael W., and Murray Browne. *Understanding Search Engines—Mathematical Modeling and Text Retrieval*. Philadelphia: Siam, 1999.

**2.** www.google.com/technology/index.html

**3.** Moler, Cleve. "The World's Largest Matrix Computation: Google's Page Rank Is an Eigenvector of a Matrix of Order 2.7 Billion," MATLAB *News and Notes*, October 2002, pp. 12–13.

Whenever a new object is introduced in mathematics, we must define when two such objects are equal. For example, in the set of all rational numbers, the numbers $\frac{2}{3}$ and $\frac{4}{6}$ are called equal although they are not represented in the same manner. What we have in mind is the definition that $\frac{a}{b}$ equals $\frac{c}{d}$ when $ad = bc$. Accordingly, we now have the following definition.

**DEFINITION**

Two $m \times n$ matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are said to be **equal** if $a_{ij} = b_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$, that is, if corresponding elements are equal.

**EXAMPLE 9** The matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$$

are equal if $w = -1$, $x = -3$, $y = 0$, and $z = 5$. ■

We shall now define a number of operations that will produce new matrices out of given matrices. These operations are useful in the applications of matrices.

## MATRIX ADDITION

**DEFINITION** If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are $m \times n$ matrices, then the **sum** of $A$ and $B$ is the $m \times n$ matrix $C = \left[c_{ij}\right]$, defined by

$$c_{ij} = a_{ij} + b_{ij} \qquad (1 \le i \le m, 1 \le j \le n).$$

That is, $C$ is obtained by adding corresponding elements of $A$ and $B$.

**EXAMPLE 10** Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 1 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 4+(-4) \\ 2+1 & -1+3 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \end{bmatrix}.$$ ■

It should be noted that the sum of the matrices $A$ and $B$ is defined only when $A$ and $B$ have the same number of rows and the same number of columns, that is, only when $A$ and $B$ are of the same size.

> We shall now establish the convention that when $A + B$ is formed, both $A$ and $B$ are of the same size.

Thus far, addition of matrices has only been defined for two matrices. Our work with matrices will call for adding more than two matrices. Theorem 1.1 in the next section shows that addition of matrices satisfies the associative property: $A + (B + C) = (A + B) + C$. Additional properties of matrix addition are considered in Section 1.4 and are similar to those satisfied by the real numbers.

**EXAMPLE 11** **(Production)** A manufacturer of a certain product makes three models, A, B, and C. Each model is partially made in factory $F_1$ in Taiwan and then finished in factory $F_2$ in the United States. The total cost of each product consists of the manufacturing cost and the shipping cost. Then the costs at each factory (in dollars) can be described by the $3 \times 2$ matrices $F_1$ and $F_2$:

$$\begin{array}{cc} & \begin{array}{cc} \text{Manufacturing cost} & \text{Shipping cost} \end{array} \\ F_1 = & \begin{bmatrix} 32 & 40 \\ 50 & 80 \\ 70 & 20 \end{bmatrix} \begin{array}{l} \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array} \end{array}$$

$$F_2 = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cc} \text{Manufacturing cost} & \text{Shipping cost} \\ \left[\begin{array}{c} 40 \\ 50 \\ 130 \end{array}\right. & \left.\begin{array}{c} 60 \\ 50 \\ 20 \end{array}\right] \end{array}\begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

The matrix $F_1 + F_2$ gives the total manufacturing and shipping costs for each product. Thus the total manufacturing and shipping costs of a model C product are \$200 and \$40, respectively. ■

## SCALAR MULTIPLICATION

**DEFINITION**

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times n$ matrix and $r$ is a real number, then the **scalar multiple** of $A$ by $r$, $rA$, is the $m \times n$ matrix $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, where

$$b_{ij} = ra_{ij} \qquad (1 \le i \le m, 1 \le j \le n).$$

That is, $B$ is obtained by multiplying each element of $A$ by $r$.

If $A$ and $B$ are $m \times n$ matrices, we write $A + (-1)B$ as $A - B$ and call this the **difference** of $A$ and $B$.

**EXAMPLE 12**

Let

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}.$$

Then

$$A - B = \begin{bmatrix} 2-2 & 3+1 & -5-3 \\ 4-3 & 2-5 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -3 & 3 \end{bmatrix}.$$

■

**EXAMPLE 13**

Let $\mathbf{p} = \begin{bmatrix} 18.95 & 14.75 & 8.60 \end{bmatrix}$ be a 3-vector that represents the current prices of three items at a store. Suppose that the store announces a sale so that the price of each item is reduced by 20%.

(a) Determine a 3-vector that gives the price changes for the three items.

(b) Determine a 3-vector that gives the new prices of the items.

**Solution**

(a) Since each item is reduced by 20%, the 3-vector

$$\begin{aligned} 0.20\mathbf{p} &= \begin{bmatrix} (0.20)18.95 & (0.20)14.75 & (0.20)8.60 \end{bmatrix} \\ &= \begin{bmatrix} 3.79 & 2.95 & 1.72 \end{bmatrix} \end{aligned}$$

gives the price reductions for the three items.

(b) The new prices of the items are given by the expression

$$\begin{aligned} \mathbf{p} - 0.20\mathbf{p} &= \begin{bmatrix} 18.95 & 14.75 & 8.60 \end{bmatrix} - \begin{bmatrix} 3.79 & 2.95 & 1.72 \end{bmatrix} \\ &= \begin{bmatrix} 15.16 & 11.80 & 6.88 \end{bmatrix}. \end{aligned}$$

Observe that this expression can also be written as

$$\mathbf{p} - 0.20\mathbf{p} = 0.80\mathbf{p}.$$

■

If $A_1, A_2, \ldots, A_k$ are $m \times n$ matrices and $c_1, c_2, \ldots, c_k$ are real numbers, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k \tag{2}$$

is called a **linear combination** of $A_1, A_2, \ldots, A_k$, and $c_1, c_2, \ldots, c_k$ are called **coefficients**.

**EXAMPLE 14**

(a) If

$$A_1 = \begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix},$$

then $C = 3A_1 - \frac{1}{2}A_2$ is a linear combination of $A_1$ and $A_2$. Using scalar multiplication and matrix addition, we can compute $C$:

$$C = 3\begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{2} & -10 & \frac{27}{2} \\ 3 & 8 & \frac{21}{2} \\ \frac{7}{2} & -5 & -\frac{21}{2} \end{bmatrix}.$$

(b) $2\begin{bmatrix} 3 & -2 \end{bmatrix} - 3\begin{bmatrix} 5 & 0 \end{bmatrix} + 4\begin{bmatrix} -2 & 5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 & -2 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \end{bmatrix}$, and $\begin{bmatrix} -2 & 5 \end{bmatrix}$. It can be computed (verify) as $\begin{bmatrix} -17 & 16 \end{bmatrix}$.

(c) $-0.5\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix} + 0.4\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}$.

It can be computed (verify) as $\begin{bmatrix} -0.46 \\ 0.4 \\ 3.08 \end{bmatrix}$. ■

## THE TRANSPOSE OF A MATRIX

**DEFINITION**

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times n$ matrix, then the $n \times m$ matrix $A^T = \begin{bmatrix} a_{ij}^T \end{bmatrix}$, where

$$a_{ij}^T = a_{ji} \qquad (1 \le i \le n,\ 1 \le j \le m)$$

is called the **transpose** of $A$. Thus, the entries in each row of $A^T$ are the entries in the corresponding column of $A$.

**EXAMPLE 15**

Let

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \\ 0 & 4 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 4 \\ -3 & 2 \\ 2 & -3 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & -5 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}, \qquad B^T = \begin{bmatrix} 6 & 3 & 0 \\ 2 & -1 & 4 \\ -4 & 2 & 3 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 5 & -3 & 2 \\ 4 & 2 & -3 \end{bmatrix}, \quad D^T = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \quad \text{and} \quad E^T = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}.$$

■

## BIT MATRICES (OPTIONAL)

The majority of our work in linear algebra will use matrices and vectors whose entries are real or complex numbers. Hence computations, like linear combinations, are determined using matrix properties and standard arithmetic base 10. However, the continued expansion of computer technology has brought to the forefront the use of binary (base 2) representation of information. In most computer applications like video games, FAX communications, ATM money transfers, satellite communications, DVD videos, or the generation of music CDs, the underlying mathematics is invisible and completely transparent to the viewer or user. Binary coded data is so prevalent and plays such a central role that we will briefly discuss certain features of it in appropriate sections of this book. We begin with an overview of binary addition and multiplication and then introduce a special class of binary matrices that play a prominent role in information and communication theory.

Binary representation of information uses only two symbols 0 and 1. Information is coded in terms of 0 and 1 in a string of **bits**.* For example, the decimal number 5 is represented as the binary string 101, which is interpreted in terms of base 2 as follows:

$$5 = 1(2^2) + 0(2^1) + 1(2^0).$$

The coefficients of the powers of 2 determine the string of bits, 101, which provide the binary representation of 5.

Just as there is arithmetic base 10 when dealing with the real and complex numbers, there is arithmetic using base 2; that is, binary arithmetic. Table 1.1 shows the structure of binary addition and Table 1.2 the structure of binary multiplication.

**Table 1.1**

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

**Table 1.2**

| $\times$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

The properties of binary arithmetic for combining representations of real numbers given in binary form is often studied in beginning computer science courses or finite mathematics courses. We will not digress to review such topics at this time. However, our focus will be on a particular type of matrix and vector that contain entries that are single binary digits. This class of matrices and vectors are important in the study of information theory and the mathematical field of *error-correcting codes* (also called *coding theory*).

*A bit is a binary digit; that is, either a 0 or 1.

**DEFINITION**

An $m \times n$ **bit matrix**† is a matrix all of whose entries are (single) bits. That is, each entry is either 0 or 1.

A bit ***n*-vector** (or **vector**) is a $1 \times n$ or $n \times 1$ matrix all of whose entries are bits.

**EXAMPLE 16**

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is a $3 \times 3$ bit matrix. ■

**EXAMPLE 17**

$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a bit 5-vector and $\mathbf{u} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ is a bit 4-vector. ■

The definitions of matrix addition and scalar multiplication apply to bit matrices provided we use binary (or base 2) arithmetic for all computations and use the only possible scalars 0 and 1.

**EXAMPLE 18**

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Using the definition of matrix addition and Table 1.1, we have

$$A + B = \begin{bmatrix} 1+1 & 0+1 \\ 1+0 & 1+1 \\ 0+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

■

Linear combinations of bit matrices or bit $n$-vectors are quite easy to compute using the fact that the only scalars are 0 and 1 together with Tables 1.1 and 1.2.

**EXAMPLE 19**

Let $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned} c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 &= 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (1+0)+1 \\ (0+0)+1 \end{bmatrix} \\ &= \begin{bmatrix} 1+1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

■

From Table 1.1 we have $0 + 0 = 0$ and $1 + 1 = 0$. Thus the additive inverse of 0 is 0 (as usual) and the additive inverse of 1 is 1. Hence to compute the difference of bit matrices $A$ and $B$ we proceed as follows:

$$A - B = A + (\text{inverse of } 1)\, B = A + 1B = A + B.$$

We see that the difference of bit matrices contributes nothing new to the algebraic relationships among bit matrices.

---

†A bit matrix is also called a **Boolean matrix**.

## Key Terms

Matrix
Rows
Columns
Size of a matrix
Square matrix
Main diagonal of a matrix
Element (or entry) of a matrix
$ij$th element
$(i, j)$ entry
$n$-vector (or vector)
Diagonal matrix
Scalar matrix
**0**, the zero vector
$R^n$, the set of all $n$-vectors
Google©
Equal matrices
Matrix addition
Scalar multiplication
Scalar multiple of a matrix
Difference of matrices
Linear combination of matrices
Transpose of a matrix
Bit
Bit (or Boolean) matrix
Upper triangular matrix
Lower triangular matrix

## 1.2 Exercises

**1.** Let

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 7 & 3 & 2 \\ -4 & 3 & 5 \\ 6 & 1 & -1 \end{bmatrix}.$$

(a) What is $a_{12}$, $a_{22}$, $a_{23}$?

(b) What is $b_{11}$, $b_{31}$?

(c) What is $c_{13}$, $c_{31}$, $c_{33}$?

**2.** If

$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix},$$

find $a$, $b$, $c$, and $d$.

**3.** If

$$\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix},$$

find $a$, $b$, $c$, and $d$.

*In Exercises 4 through 7, let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix},$$

$$\text{and} \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**4.** If possible, compute the indicated linear combination:

(a) $C + E$ and $E + C$ (b) $A + B$

(c) $D - F$ (d) $-3C + 5O$

(e) $2C - 3E$ (f) $2B + F$

**5.** If possible, compute the indicated linear combination:

(a) $3D + 2F$

(b) $3(2A)$ and $6A$

(c) $3A + 2A$ and $5A$

(d) $2(D + F)$ and $2D + 2F$

(e) $(2 + 3)D$ and $2D + 3D$

(f) $3(B + D)$

**6.** If possible, compute:

(a) $A^T$ and $(A^T)^T$

(b) $(C + E)^T$ and $C^T + E^T$

(c) $(2D + 3F)^T$

(d) $D - D^T$

(e) $2A^T + B$

(f) $(3D - 2F)^T$

**7.** If possible, compute:

(a) $(2A)^T$ (b) $(A - B)^T$

(c) $(3B^T - 2A)^T$

(d) $(3A^T - 5B^T)^T$

(e) $(-A)^T$ and $-(A^T)$

(f) $(C + E + F^T)^T$

**8.** Is the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

**9.** Is the matrix $\begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

**10.** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -2 & 3 \\ 5 & 2 & 4 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\lambda$ is a real number, compute $\lambda I_3 - A$.

*Exercises 11 through 15 involve bit matrices.*

**11.** Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Compute each of the following.

(a) $A + B$ (b) $B + C$ (c) $A + B + C$

(d) $A + C^T$ (e) $B - C$

**12.** Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Compute each of the following.

(a) $A + B$ (b) $C + D$ (c) $A + B + (C + D)^T$

(d) $C - B$ (e) $A - B + C - D$

**13.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Find $B$ so that $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) Find $C$ so that $A + C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

**14.** Let $\mathbf{u} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$. Find the bit 4-vector $\mathbf{v}$ so that $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$.

**15.** Let $\mathbf{u} = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$. Find the bit 4-vector $\mathbf{v}$ so that $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$.

## Theoretical Exercises

**T.1.** Show that the sum and difference of two diagonal matrices is a diagonal matrix.

**T.2.** Show that the sum and difference of two scalar matrices is a scalar matrix.

**T.3.** Let

$$A = \begin{bmatrix} a & b & c \\ c & d & e \\ e & e & f \end{bmatrix}.$$

(a) Compute $A - A^T$.

(b) Compute $A + A^T$.

(c) Compute $(A + A^T)^T$.

**T.4.** Let $O$ be the $n \times n$ matrix all of whose entries are zero. Show that if $k$ is a real number and $A$ is an $n \times n$ matrix such that $kA = O$, then $k = 0$ or $A = O$.

**T.5.** A matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is called **upper triangular** if $a_{ij} = 0$ for $i > j$. It is called **lower triangular** if $a_{ij} = 0$ for $i < j$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Upper triangular matrix
(The elements below the main diagonal are zero.)

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Lower triangular matrix
(The elements above the main diagonal are zero.)

(a) Show that the sum and difference of two upper triangular matrices is upper triangular.

(b) Show that the sum and difference of two lower triangular matrices is lower triangular.

(c) Show that if a matrix is both upper and lower triangular, then it is a diagonal matrix.

**T.6.** (a) Show that if $A$ is an upper triangular matrix, then $A^T$ is lower triangular.

(b) Show that if $A$ is a lower triangular matrix, then $A^T$ is upper triangular.

**T.7.** If $A$ is an $n \times n$ matrix, what are the entries on the main diagonal of $A - A^T$? Justify your answer.

**T.8.** If $\mathbf{x}$ is an $n$-vector, show that $\mathbf{x} + \mathbf{0} = \mathbf{x}$.

*Exercises T.9 through T.18 involve bit matrices.*

**T.9.** Make a list of all possible bit 2-vectors. How many are there?

**T.10.** Make a list of all possible bit 3-vectors. How many are there?

**T.11.** Make a list of all possible bit 4-vectors. How many are there?

**T.12.** How many bit 5-vectors are there? How many bit $n$-vectors are there?

**T.13.** Make a list of all possible $2 \times 2$ bit matrices. How many are there?

**T.14.** How many $3 \times 3$ bit matrices are there?

**T.15.** How many $n \times n$ bit matrices are there?

**T.16.** Let 0 represent OFF and 1 represent ON and

$$A = \begin{bmatrix} \text{ON} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{ON} \end{bmatrix}.$$

Find the ON/OFF matrix $B$ so that $A + B$ is a matrix with each entry OFF.

**T.17.** Let 0 represent OFF and 1 represent ON and

$$A = \begin{bmatrix} \text{ON} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{OFF} \\ \text{OFF} & \text{ON} & \text{ON} \end{bmatrix}.$$

Find the ON/OFF matrix $B$ so that $A + B$ is a matrix with each entry ON.

**T.18.** A standard light switch has two positions (or states); either on or off. Let bit matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

represent a bank of light switches where 0 represents OFF and 1 represents ON.

(a) Find a matrix $B$ so that $A + B$ will represent the bank of switches with the state of each switch "reversed."

(b) Let

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Will the matrix $B$ from part (a) also "reverse" that state of the bank of switches represented by $C$? Verify your answer.

(c) If $A$ is any $m \times n$ bit matrix representing a bank of switches, determine an $m \times n$ bit matrix $B$ so that $A + B$ "reverses" all the states of the switches in $A$. Give reasons why $B$ will "reverse" the states in $A$.

## MATLAB Exercises

*In order to use* MATLAB *in this section, you should first read Sections 12.1 and 12.2, which give basic information about* MATLAB *and about matrix operations in* MATLAB. *You are urged to do any examples or illustrations of* MATLAB *commands that appear in Sections 12.1 and 12.2 before trying these exercises.*

**ML.1.** In MATLAB, enter the following matrices.

$$A = \begin{bmatrix} 5 & 1 & 2 \\ -3 & 0 & 1 \\ 2 & 4 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 4 * 2 & 2/3 \\ 1/201 & 5 - 8.2 \\ 0.00001 & (9 + 4)/3 \end{bmatrix}.$$

Using MATLAB commands, display the following.

(a) $a_{23}$, $b_{32}$, $b_{12}$

(b) $\text{row}_1(A)$, $\text{col}_3(A)$, $\text{row}_2(B)$

(c) Type MATLAB command **format long** and display matrix $B$. Compare the elements of $B$ from part (a) with the current display. Note that **format short** displays four decimal places rounded. Reset the format to **format short**.

**ML.2.** In MATLAB, type the command **H = hilb(5);** (Note that the last character is a semicolon, which suppresses the display of the contents of matrix $H$. See Section 12.1.) For more information on the **hilb** command, type **help hilb**. Using MATLAB commands, do the following:

(a) Determine the size of $H$.

(b) Display the contents of $H$.

(c) Display the contents of $H$ as rational numbers.

(d) Extract as a matrix the first three columns.

(e) Extract as a matrix the last two rows.

*Exercises ML.3 through ML.5 use bit matrices and the supplemental instructional commands described in Section 12.9.*

**ML.3.** Use **bingen** to solve Exercises T.10 and T.11.

**ML.4.** Use **bingen** to solve Exercise T.13. (*Hint*: An $n \times n$ matrix contains the same number of entries as an $n^2$-vector.)

**ML.5.** Solve Exercise 11 using **binadd**.

# 1.3 DOT PRODUCT AND MATRIX MULTIPLICATION

In this section we introduce the operation of matrix multiplication. Unlike matrix addition, matrix multiplication has some properties that distinguish it from multiplication of real numbers.

**DEFINITION**

The **dot product** or **inner product** of the $n$-vectors $\mathbf{a}$ and $\mathbf{b}$ is the sum of the products of corresponding entries. Thus, if

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.^{\dagger} \tag{1}$$

Similarly, if $\mathbf{a}$ or $\mathbf{b}$ (or both) are $n$-vectors written as a $1 \times n$ matrix, then the dot product $\mathbf{a} \cdot \mathbf{b}$ is given by (1). The dot product of vectors in $C^n$ is defined in Appendix A.2.

The dot product is an important operation that will be used here and in later sections.

**EXAMPLE 1**

The dot product of

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

is

$$\mathbf{u} \cdot \mathbf{v} = (1)(2) + (-2)(3) + (3)(-2) + (4)(1) = -6. \qquad \blacksquare$$

**EXAMPLE 2**

Let $\mathbf{a} = \begin{bmatrix} x & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = -4$, find $x$.

**Solution**

We have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = 4x + 2 + 6 &= -4 \\ 4x + 8 &= -4 \\ x &= -3. \end{aligned} \qquad \blacksquare$$

**EXAMPLE 3**

**(Application: Computing a Course Average)** Suppose that an instructor uses four grades to determine a student's course average: quizzes, two hourly exams, and a final exam. These are weighted as 10%, 30%, 30%, and 30%, respectively. If a student's scores are 78, 84, 62, and 85, respectively, we can compute the course average by letting

$$\mathbf{w} = \begin{bmatrix} 0.10 \\ 0.30 \\ 0.30 \\ 0.30 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 78 \\ 84 \\ 62 \\ 85 \end{bmatrix}$$

and computing

$$\mathbf{w} \cdot \mathbf{g} = (0.10)(78) + (0.30)(84) + (0.30)(62) + (0.30)(85) = 77.1.$$

Thus, the student's course average is 77.1. $\blacksquare$

---

$^{\dagger}$You may already be familiar with this useful notation, the summation notation. It is discussed in detail at the end of this section.

## MATRIX MULTIPLICATION

**DEFINITION** If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times p$ matrix and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ is a $p \times n$ matrix, then the **product** of $A$ and $B$, denoted $AB$, is the $m \times n$ matrix $C = \begin{bmatrix} c_{ij} \end{bmatrix}$, defined by

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} \\ &= \sum_{k=1}^{p} a_{ik}b_{kj} \quad (1 \le i \le m,\ 1 \le j \le n). \end{aligned} \tag{2}$$

Equation (2) says that the $i, j$th element in the product matrix is the dot product of the $i$th row, $\text{row}_i(A)$, and the $j$th column, $\text{col}_j(B)$, of $B$; this is shown in Figure 1.4.

Figure 1.4 ►

$$\begin{array}{r} \\ \\ \\ \text{row}_i(A) \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{array}{c} \begin{array}{cccccc} & & & \text{col}_j(B) & & \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}.$$

$$\text{row}_i(A) \cdot \text{col}_j(B) = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Observe that the product of $A$ and $B$ is defined only when the number of rows of $B$ is exactly the same as the number of columns of $A$, as is indicated in Figure 1.5.

Figure 1.5 ►

$$\underset{m \times p}{A} \quad \underset{p \times n}{B} \quad = \quad \underset{m \times n}{AB}$$

the same

size of $AB$

**EXAMPLE 4** Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} AB &= \begin{bmatrix} (1)(-2) + (2)(4) + (-1)(2) & (1)(5) + (2)(-3) + (-1)(1) \\ (3)(-2) + (1)(4) + (4)(2) & (3)(5) + (1)(-3) + (4)(1) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}. \end{aligned}$$

■

**EXAMPLE 5** Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}.$$

Compute the (3, 2) entry of $AB$.

**Solution** If $AB = C$, then the (3, 2) entry of $AB$ is $c_{32}$, which is $\text{row}_3(A) \cdot \text{col}_2(B)$. We now have

$$\text{row}_3(A) \cdot \text{col}_2(B) = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = -5.$$

■

**EXAMPLE 6** The linear system

$$\begin{aligned} x + 2y - \phantom{4}z &= 2 \\ 3x \phantom{+ 2y} + 4z &= 5 \end{aligned}$$

can be written (verify) using a matrix product as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

■

**EXAMPLE 7** Let

$$A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}.$$

If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, find $x$ and $y$.

**Solution** We have

$$AB = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix} = \begin{bmatrix} 2 + 4x + 3y \\ 4 - 4 + y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

Then

$$\begin{aligned} 2 + 4x + 3y &= 12 \\ y &= 6, \end{aligned}$$

so $x = -2$ and $y = 6$. ■

The basic properties of matrix multiplication will be considered in the following section. However, multiplication of matrices requires much more care than their addition, since the algebraic properties of matrix multiplication differ from those satisfied by the real numbers. Part of the problem is due to the fact that $AB$ is defined only when the number of columns of $A$ is the same as the number of rows of $B$. Thus, if $A$ is an $m \times p$ matrix and $B$ is a $p \times n$ matrix, then $AB$ is an $m \times n$ matrix. What about $BA$? Four different situations may occur:

1. $BA$ may not be defined; this will take place if $n \neq m$.
2. If $BA$ is defined, which means that $m = n$, then $BA$ is $p \times p$ while $AB$ is $m \times m$; thus, if $m \neq p$, $AB$ and $BA$ are of different sizes.

**3.** If $AB$ and $BA$ are both of the same size, they may be equal.

**4.** If $AB$ and $BA$ are both of the same size, they may be unequal.

**EXAMPLE 8**

If $A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 4$ matrix, then $AB$ is a $2 \times 4$ matrix while $BA$ is undefined. ■

**EXAMPLE 9**

Let $A$ be $2 \times 3$ and let $B$ be $3 \times 2$. Then $AB$ is $2 \times 2$ while $BA$ is $3 \times 3$. ■

**EXAMPLE 10**

Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \quad \text{while} \quad BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}.$$

Thus $AB \neq BA$. ■

One might ask why matrix equality and matrix addition are defined in such a natural way while matrix multiplication appears to be much more complicated. Example 11 provides a motivation for the definition of matrix multiplication.

**EXAMPLE 11**

**(Ecology)** Pesticides are sprayed on plants to eliminate harmful insects. However, some of the pesticide is absorbed by the plant. The pesticides are absorbed by herbivores when they eat the plants that have been sprayed. To determine the amount of pesticide absorbed by a herbivore, we proceed as follows. Suppose that we have three pesticides and four plants. Let $a_{ij}$ denote the amount of pesticide $i$ (in milligrams) that has been absorbed by plant $j$. This information can be represented by the matrix

$$A = \begin{array}{c} \begin{array}{cccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} & \text{Plant 4} \end{array} \\ \begin{bmatrix} 2 & 3 & 4 & 3 \\ 3 & 2 & 2 & 5 \\ 4 & 1 & 6 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Pesticide 1} \\ \text{Pesticide 2} \\ \text{Pesticide 3} \end{array}$$

Now suppose that we have three herbivores, and let $b_{ij}$ denote the number of plants of type $i$ that a herbivore of type $j$ eats per month. This information can be represented by the matrix

$$B = \begin{array}{c} \begin{array}{ccc} \text{Herbivore 1} & \text{Herbivore 2} & \text{Herbivore 3} \end{array} \\ \begin{bmatrix} 20 & 12 & 8 \\ 28 & 15 & 15 \\ 30 & 12 & 10 \\ 40 & 16 & 20 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Plant 1} \\ \text{Plant 2} \\ \text{Plant 3} \\ \text{Plant 4} \end{array}$$

The $(i, j)$ entry in $AB$ gives the amount of pesticide of type $i$ that animal $j$ has absorbed. Thus, if $i = 2$ and $j = 3$, the $(2, 3)$ entry in $AB$ is

$$\begin{aligned} &3(8) + 2(15) + 2(10) + 5(20) \\ &\qquad = 174 \text{ mg of pesticide 2 absorbed by herbivore 3.} \end{aligned}$$

If we now have $p$ carnivores (such as man) who eat the herbivores, we can repeat the analysis to find out how much of each pesticide has been absorbed by each carnivore. ■

It is sometimes useful to be able to find a column in the matrix product $AB$ without having to multiply the two matrices. It can be shown (Exercise T.9) that the $j$th column of the matrix product $AB$ is equal to the matrix product $A\text{col}_j(B)$.

**EXAMPLE 12** Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

Then the second column of $AB$ is

$$A\text{col}_2(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix}.$$

■

**Remark** If **u** and **v** are $n$-vectors, it can be shown (Exercise T.14) that if we view them as $n \times 1$ matrices, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}.$$

This observation will be used in Chapter 3. Similarly, if **u** and **v** are viewed as $1 \times n$ matrices, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T.$$

Finally, if **u** is a $1 \times n$ matrix and **v** is an $n \times 1$ matrix, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}$.

**EXAMPLE 13** Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Then

$$\mathbf{u} \cdot \mathbf{v} = 1(2) + 2(-1) + (-3)(1) = -3.$$

Moreover,

$$\mathbf{u}^T \mathbf{v} = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 1(2) + 2(-1) + (-3)(1) = -3.$$

■

## THE MATRIX-VECTOR PRODUCT WRITTEN IN TERMS OF COLUMNS

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an $m \times n$ matrix and let

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

be an $n$-vector, that is, an $n \times 1$ matrix. Since $A$ is $m \times n$ and $\mathbf{c}$ is $n \times 1$, the matrix product $A\mathbf{c}$ is the $m \times 1$ matrix

$$A\mathbf{c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \mathrm{row}_1(A) \cdot \mathbf{c} \\ \mathrm{row}_2(A) \cdot \mathbf{c} \\ \vdots \\ \mathrm{row}_m(A) \cdot \mathbf{c} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n \\ \vdots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n \end{bmatrix}. \tag{3}$$

The right side of this expression can be written as

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= c_1 \mathrm{col}_1(A) + c_2 \mathrm{col}_2(A) + \cdots + c_n \mathrm{col}_n(A). \tag{4}$$

Thus the product $A\mathbf{c}$ of an $m \times n$ matrix $A$ and an $n \times 1$ matrix $\mathbf{c}$ can be written as a linear combination of the columns of $A$, where the coefficients are the entries in $\mathbf{c}$.

**EXAMPLE 14** Let

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

Then the product $A\mathbf{c}$ written as a linear combination of the columns of $A$ is

$$A\mathbf{c} = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}.$$

■

If $A$ is an $m \times p$ matrix and $B$ is a $p \times n$ matrix, we can then conclude that the $j$th column of the product $AB$ can be written as a linear combination of the columns of matrix $A$, where the coefficients are the entries in the $j$th column of matrix $B$:

$$\mathrm{col}_j(AB) = A\mathrm{col}_j(B) = b_{1j}\mathrm{col}_1(A) + b_{2j}\mathrm{col}_2(A) + \cdots + b_{pj}\mathrm{col}_p(A).$$

**EXAMPLE 15** If $A$ and $B$ are the matrices defined in Example 12, then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 6 \\ 6 & 17 & 16 \\ 17 & 7 & 1 \end{bmatrix}.$$

The columns of $AB$ as linear combinations of the columns of $A$ are given by

$$\mathrm{col}_1(AB) = \begin{bmatrix} 4 \\ 6 \\ 17 \end{bmatrix} = A\mathrm{col}_1(B) = -2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\mathrm{col}_2(AB) = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix} = A\mathrm{col}_2(B) = 3\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\mathrm{col}_3(AB) = \begin{bmatrix} 6 \\ 16 \\ 1 \end{bmatrix} = A\mathrm{col}_3(B) = 4\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}.$$

■

## LINEAR SYSTEMS

We now generalize Example 6. Consider the linear system of $m$ equations in $n$ unknowns,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{5}$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

The entries in the product $A\mathbf{x}$ are merely the left sides of the equations in (5). Hence the linear system (5) can be written in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

The matrix $A$ is called the **coefficient matrix** of the linear system (5), and the matrix

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right],$$

obtained by adjoining column $\mathbf{b}$ to $A$, is called the **augmented matrix** of the linear system (5). The augmented matrix of (5) will be written as $\left[A \,\vdots\, \mathbf{b}\right]$. Conversely, any matrix with more than one column can be thought of as the augmented matrix of a linear system. The coefficient and augmented matrices will play key roles in our method for solving linear systems.

**EXAMPLE 16** Consider the linear system

$$\begin{aligned} -2x \qquad + \ \ z &= 5 \\ 2x + 3y - 4z &= 7 \\ 3x + 2y + 2z &= 3. \end{aligned}$$

Letting

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -4 \\ 3 & 2 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix},$$

we can write the given linear system in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

The coefficient matrix is $A$ and the augmented matrix is

$$\left[\begin{array}{rrr:r} -2 & 0 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right].$$

■

**EXAMPLE 17** The matrix

$$\left[\begin{array}{rrr:r} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$$

is the augmented matrix of the linear system

$$\begin{aligned} 2x - y + 3z &= 4 \\ 3x \qquad + 2z &= 5. \end{aligned}$$

■

It follows from our discussion above that the linear system in (5) can be written as a linear combination of the columns of $A$ as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \tag{6}$$

Conversely, an equation as in (6) always describes a linear system as in (5).

## PARTITIONED MATRICES (OPTIONAL)

If we start out with an $m \times n$ matrix $A = \left[a_{ij}\right]$ and cross out some, but not all, of its rows or columns, we obtain a **submatrix** of $A$.

**EXAMPLE 18** Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 4 & -3 & 5 \\ 3 & 0 & 5 & -3 \end{bmatrix}.$$

If we cross out the second row and third column, we obtain the submatrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -3 \end{bmatrix}.$$

■

A matrix can be partitioned into submatrices by drawing horizontal lines between rows and vertical lines between columns. Of course, the partitioning can be carried out in many different ways.

**EXAMPLE 19** The matrix

$$A = \left[\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right]$$

is partitioned as

$$A = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right].$$

We could also write

$$A = \left[\begin{array}{cc:cc:c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right] = \left[\begin{array}{c:c:c} \widehat{A}_{11} & \widehat{A}_{12} & \widehat{A}_{13} \\ \hdashline \widehat{A}_{21} & \widehat{A}_{22} & \widehat{A}_{23} \end{array}\right], \tag{7}$$

which gives another partitioning of $A$. We thus speak of **partitioned matrices**. ■

**EXAMPLE 20** The augmented matrix of a linear system is a partitioned matrix. Thus, if $A\mathbf{x} = \mathbf{b}$, we can write the augmented matrix of this system as $\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right]$. ■

If $A$ and $B$ are both $m \times n$ matrices that are partitioned in the same way, then $A + B$ is obtained simply by adding the corresponding submatrices of $A$ and $B$. Similarly, if $A$ is a partitioned matrix, then the scalar multiple $cA$ is obtained by forming the scalar multiple of each submatrix.

If $A$ is partitioned as shown in (7) and

$$B = \left[\begin{array}{cc:cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \hdashline b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \\ \hdashline b_{51} & b_{52} & b_{53} & b_{54} \end{array}\right] = \left[\begin{array}{c:c} B_{11} & B_{12} \\ \hdashline B_{21} & B_{22} \\ \hdashline B_{31} & B_{32} \end{array}\right],$$

then by straightforward computations we can show that

$$AB = \left[\begin{array}{c:c} (\widehat{A}_{11}B_{11} + \widehat{A}_{12}B_{21} + \widehat{A}_{13}B_{31}) & (\widehat{A}_{11}B_{12} + \widehat{A}_{12}B_{22} + \widehat{A}_{13}B_{32}) \\ \hdashline (\widehat{A}_{21}B_{11} + \widehat{A}_{22}B_{21} + \widehat{A}_{23}B_{31}) & (\widehat{A}_{21}B_{12} + \widehat{A}_{22}B_{22} + \widehat{A}_{23}B_{32}) \end{array}\right].$$

**EXAMPLE 21** Let

$$A = \left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 \\ \hdashline 2 & 0 & -4 & 0 \\ 0 & 1 & 0 & 3 \end{array}\right] = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right]$$

and let

$$B = \left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ \hdashline 1 & 3 & 0 & 0 & 1 & 0 \\ -3 & -1 & 2 & 1 & 0 & -1 \end{array}\right] = \left[\begin{array}{c:c} B_{11} & B_{12} \\ \hdashline B_{21} & B_{22} \end{array}\right].$$

Then

$$AB = C = \left[\begin{array}{ccc:ccc} 3 & 3 & 0 & 1 & 2 & -1 \\ 6 & 12 & 0 & -3 & 7 & 5 \\ \hdashline 0 & -12 & 0 & 2 & -2 & -2 \\ -9 & -2 & 7 & 2 & 2 & -1 \end{array}\right] = \left[\begin{array}{c:c} C_{11} & C_{12} \\ \hdashline C_{21} & C_{22} \end{array}\right],$$

where $C_{11}$ should be $A_{11}B_{11} + A_{12}B_{21}$. We verify that $C_{11}$ is this expression as follows:

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ -3 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 \\ 6 & 10 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 & 0 \\ 6 & 12 & 0 \end{bmatrix} = C_{11}. \end{aligned}$$

■

This method of multiplying partitioned matrices is also known as **block multiplication**. Partitioned matrices can be used to great advantage in dealing with matrices that exceed the memory capacity of a computer. Thus, in multiplying two partitioned matrices, one can keep the matrices on disk and only bring into memory the submatrices required to form the submatrix products. The latter, of course, can be put out on disk as they are formed. The partitioning must be done so that the products of corresponding submatrices are defined. In contemporary computing technology, parallel-processing computers use partitioned matrices to perform matrix computations more rapidly.

Partitioning of a matrix implies a subdivision of the information into blocks or units. The reverse process is to consider individual matrices as blocks and adjoin them to form a partitioned matrix. The only requirement is that after joining the blocks, all rows have the same number of entries and all columns have the same number of entries.

**EXAMPLE 22**

Let

$$B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 9 & 8 & -4 \\ 6 & 7 & 5 \end{bmatrix}.$$

Then we have

$$\left[\begin{array}{c:c} B & D \end{array}\right] = \left[\begin{array}{c:ccc} 2 & 9 & 8 & -4 \\ 3 & 6 & 7 & 5 \end{array}\right], \quad \left[\begin{array}{c} D \\ \hdashline C \end{array}\right] = \left[\begin{array}{ccc} 9 & 8 & -4 \\ 6 & 7 & 5 \\ \hdashline 1 & -1 & 0 \end{array}\right],$$

and

$$\left[\begin{array}{c:c} \left[\begin{array}{c} D \\ \hdashline C \end{array}\right] & C^T \end{array}\right] = \left[\begin{array}{ccc:c} 9 & 8 & -4 & 1 \\ 6 & 7 & 5 & -1 \\ \hdashline 1 & -1 & 0 & 0 \end{array}\right].$$

■

Adjoining matrix blocks to expand information structures is done regularly in a variety of applications. It is common to keep monthly sales data for a year in a $1 \times 12$ matrix and then adjoin such matrices to build a sales history matrix for a period of years. Similarly, results of new laboratory experiments are adjoined to existing data to update a database in a research facility.

We have already noted in Example 20 that the augmented matrix of the linear system $A\mathbf{x} = \mathbf{b}$ is a partitioned matrix. At times we shall need to solve several linear systems in which the coefficient matrix $A$ is the same but the right sides of the systems are different, say $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$. In these cases we shall find it convenient to consider the partitioned matrix $\left[A \mid \mathbf{b} \mid \mathbf{c} \mid \mathbf{d}\right]$. (See Section 6.7.)

## SUMMATION NOTATION (OPTIONAL)

We shall occasionally use the **summation notation** and we now review this useful and compact notation, which is widely used in mathematics.

By $\sum_{i=1}^{n} a_i$ we mean

$$a_1 + a_2 + \cdots + a_n.$$

The letter $i$ is called the **index of summation**; it is a dummy variable that can be replaced by another letter. Hence we can write

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k.$$

**EXAMPLE 23** If

$$a_1 = 3, \quad a_2 = 4, \quad a_3 = 5, \quad \text{and} \quad a_4 = 8,$$

then

$$\sum_{i=1}^{4} a_i = 3 + 4 + 5 + 8 = 20.$$

■

**EXAMPLE 24** By $\sum_{i=1}^{n} r_i a_i$ we mean

$$r_1 a_1 + r_2 a_2 + \cdots + r_n a_n.$$

It is not difficult to show (Exercise T.11) that the summation notation satisfies the following properties:

(i) $$\sum_{i=1}^{n} (r_i + s_i) a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i.$$

(ii) $$\sum_{i=1}^{n} c(r_i a_i) = c\left(\sum_{i=1}^{n} r_i a_i\right).$$

■

**EXAMPLE 25** If

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

then the dot product $\mathbf{a} \cdot \mathbf{b}$ can be expressed using summation notation as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i . \qquad \blacksquare$$

**EXAMPLE 26** We can write Equation (2), for the $i, j$th element in the product of the matrices $A$ and $B$, in terms of the summation notation as

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad (1 \leq i \leq m, 1 \leq j \leq n). \qquad \blacksquare$$

It is also possible to form double sums. Thus by $\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}$ we mean that we first sum on $i$ and then sum the resulting expression on $j$.

**EXAMPLE 27** If $n = 2$ and $m = 3$, we have

$$\begin{aligned} \sum_{j=1}^{3} \sum_{i=1}^{2} a_{ij} &= \sum_{j=1}^{3} (a_{1j} + a_{2j}) \\ &= (a_{11} + a_{21}) + (a_{12} + a_{22}) + (a_{13} + a_{23}) \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} &= \sum_{i=1}^{2} (a_{i1} + a_{i2} + a_{i3}) \\ &= (a_{11} + a_{12} + a_{13}) + (a_{21} + a_{22} + a_{23}) \\ &= \text{right side of (8).} \end{aligned} \qquad \blacksquare$$

It is not difficult to show, in general (Exercise T.12), that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}. \tag{9}$$

Equation (9) can be interpreted as follows. Let $A$ be the $m \times n$ matrix $\left[a_{ij}\right]$. If we add up the entries in each row of $A$ and then add the resulting numbers, we obtain the same result as when we add up the entries in each column of $A$ and then add the resulting numbers.

## EXAMPLES WITH BIT MATRICES (OPTIONAL)

The dot product and the matrix product of bit matrices are computed in the usual manner, but we must recall that the arithmetic involved uses base 2.

**EXAMPLE 28** Let $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ be bit vectors. Then

$$\mathbf{a} \cdot \mathbf{b} = (1)(1) + (0)(1) + (1)(0) = 1 + 0 + 0 = 1. \qquad \blacksquare$$

**EXAMPLE 29** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ be bit matrices. Then

$$AB = \begin{bmatrix} (1)(0)+(1)(1) & (1)(1)+(1)(1) & (1)(0)+(1)(0) \\ (0)(0)+(1)(1) & (0)(1)+(1)(1) & (0)(0)+(1)(0) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$
■

**EXAMPLE 30** Let $A = \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be bit matrices. If $AB = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $x$ and $y$.

**Solution** We have

$$AB = \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} y+1+x \\ y+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then $y+1+x=1$ and $y+1=1$. Using base 2 arithmetic, it follows that $y=0$ and so then $x=0$. ■

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## Key Terms

Dot product (inner product)
Product of matrices
Coefficient matrix
Augmented matrix
Submatrix
Partitioned matrix
Block multiplication
Summation notation

## 1.3 Exercises

*In Exercises 1 and 2, compute* $\mathbf{a} \cdot \mathbf{b}$.

**1.** (a) $\mathbf{a} = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} -3 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 4 & 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

**2.** (a) $\mathbf{a} = \begin{bmatrix} 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

**3.** Let $\mathbf{a} = \begin{bmatrix} -3 & 2 & x \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = 17$, find $x$.

**4.** Let $\mathbf{w} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$. Compute $\mathbf{w} \cdot \mathbf{w}$.

**5.** Find all values of $x$ so that $\mathbf{v} \cdot \mathbf{v} = 1$, where $\mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ x \end{bmatrix}$.

**6.** Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$. If $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, find $x$ and $y$.

*In Exercises 7 and 8, let*

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -1 & 5 \end{bmatrix},$$
$$C = \begin{bmatrix} 2 & 3 & 1 \\ 3 & -4 & 5 \\ 1 & -1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix},$$
$$E = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 1 & 5 \\ 3 & 4 & 2 \end{bmatrix}, \quad \textit{and} \quad F = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}.$$

**7.** If possible, compute:
(a) $AB$ (b) $BA$ (c) $CB + D$
(d) $AB + DF$ (e) $BA + FD$

**8.** If possible, compute:
(a) $A(BD)$ (b) $(AB)D$ (c) $A(C + E)$
(d) $AC + AE$ (e) $(D + F)A$

**9.** Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.
Compute the following entries of $AB$:
(a) The (1, 2) entry (b) The (2, 3) entry
(c) The (3, 1) entry (d) The (3, 3) entry

**10.** If $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, compute $DI_2$ and $I_2D$.

**11.** Let
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}.$$
Show that $AB \neq BA$.

**12.** If $A$ is the matrix in Example 4 and $O$ is the $3 \times 2$ matrix every one of whose entries is zero, compute $AO$.

*In Exercises 13 and 14, let*
$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$$
*and*
$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 3 & -3 & 4 \\ 4 & 2 & 5 & 1 \end{bmatrix}.$$

**13.** Using the method in Example 12, compute the following columns of $AB$:
(a) The first column (b) The third column

**14.** Using the method in Example 12, compute the following columns of $AB$:
(a) The second column (b) The fourth column

**15.** Let
$$A = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 2 & 3 \\ 5 & -1 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$
Express $A\mathbf{c}$ as a linear combination of the columns of $A$.

**16.** Let
$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}.$$
Express the columns of $AB$ as linear combinations of the columns of $A$.

**17.** Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.
(a) Verify that $AB = 3\mathbf{a}_1 + 5\mathbf{a}_2 + 2\mathbf{a}_3$, where $\mathbf{a}_j$ is the $j$th column of $A$ for $j = 1, 2, 3$.
(b) Verify that $AB = \begin{bmatrix} (\text{row}_1(A))B \\ (\text{row}_2(A))B \end{bmatrix}$.

**18.** Write the linear combination
$$3\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$
as a product of a $2 \times 3$ matrix and a 3-vector.

**19.** Consider the following linear system:
$$\begin{aligned} 2x + \phantom{2y + 3z +} w &= 7 \\ 3x + 2y + 3z \phantom{+ w} &= -2 \\ 2x + 3y - 4z \phantom{+ w} &= 3 \\ x + \phantom{2y +} 3z \phantom{+ w} &= 5. \end{aligned}$$
(a) Find the coefficient matrix.
(b) Write the linear system in matrix form.
(c) Find the augmented matrix.

**20.** Write the linear system with augmented matrix
$$\left[\begin{array}{cccc:c} -2 & -1 & 0 & 4 & 5 \\ -3 & 2 & 7 & 8 & 3 \\ 1 & 0 & 0 & 2 & 4 \\ 3 & 0 & 1 & 3 & 6 \end{array}\right].$$

**21.** Write the linear system with augmented matrix
$$\left[\begin{array}{ccc:c} 2 & 0 & -4 & 3 \\ 0 & 1 & 2 & 5 \\ 1 & 3 & 4 & -1 \end{array}\right].$$

**22.** Consider the following linear system:
$$\begin{aligned} 3x - y + 2z &= 4 \\ 2x + y \phantom{+ 2z} &= 2 \\ y + 3z &= 7 \\ 4x \phantom{- y} - z &= 4. \end{aligned}$$
(a) Find the coefficient matrix.
(b) Write the linear system in matrix form.
(c) Find the augmented matrix.

**23.** How are the linear systems whose augmented matrices are

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 2 & 3 & 6 & 2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 2 & 3 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

related?

**24.** Write each of the following as a linear system in matrix form.

(a) $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix} + z\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $x\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**25.** Write each of the following linear systems as a linear combination of the columns of the coefficient matrix.

(a) $\begin{aligned} x + 2y &= 3 \\ 2x - \phantom{2}y &= 5 \end{aligned}$

(b) $\begin{aligned} 2x - 3y + 5z &= -2 \\ x + 4y - \phantom{5}z &= \phantom{-}3 \end{aligned}$

**26.** Let $A$ be an $m \times n$ matrix and $B$ an $n \times p$ matrix. What if anything can you say about the matrix product $AB$ when:

(a) $A$ has a column consisting entirely of zeros?

(b) $B$ has a row consisting entirely of zeros?

**27.** (a) Find a value of $r$ so that $AB^T = 0$, where

$$A = \begin{bmatrix} r & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}.$$

(b) Give an alternate way to write this product.

**28.** Find a value of $r$ and a value of $s$ so that $AB^T = 0$, where

$$A = \begin{bmatrix} 1 & r & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 2 & s \end{bmatrix}.$$

**29.** Formulate the method for adding partitioned matrices and verify your method by partitioning the matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

in two different ways and finding their sum.

**30.** Let $A$ and $B$ be the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 1 & 2 & 3 & -1 & 4 \\ 2 & 3 & 2 & 1 & 4 \\ 5 & -1 & 3 & 2 & 6 \\ 3 & 1 & 2 & 4 & 6 \\ 2 & -1 & 3 & 5 & 7 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 2 & 1 & 3 & 2 & -1 \\ 1 & 5 & 4 & 2 & 3 \\ 2 & 1 & 3 & 5 & 7 \\ 3 & 2 & 4 & 6 & 1 \end{bmatrix}.$$

Find $AB$ by partitioning $A$ and $B$ in two different ways.

**31.** (***Manufacturing Costs***) A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$A = \begin{array}{c} \begin{array}{cc} \text{Assembly process} & \text{Finishing process} \end{array} \\ \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \end{array} \begin{array}{l} \text{Chair} \\ \text{Table} \end{array}$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$B = \begin{array}{c} \begin{array}{cc} \text{Salt Lake City} & \text{Chicago} \end{array} \\ \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix} \end{array} \begin{array}{l} \text{Assembly process} \\ \text{Finishing process} \end{array}$$

What do the entries in the matrix product $AB$ tell the manufacturer?

**32.** (***Ecology—Pollution***) A manufacturer makes two kinds of products, $P$ and $Q$, at each of two plants, $X$ and $Y$. In making these products, the pollutants sulfur dioxide, nitric oxide, and particulate matter are produced. The amounts of pollutants produced are given (in kilograms) by the matrix

$$A = \begin{array}{c} \begin{array}{ccc} \text{Sulfur dioxide} & \text{Nitric oxide} & \text{Particulate matter} \end{array} \\ \begin{bmatrix} 300 & 100 & 150 \\ 200 & 250 & 400 \end{bmatrix} \end{array} \begin{array}{l} \text{Product } P \\ \text{Product } Q \end{array}$$

State and federal ordinances require that these pollutants be removed. The daily cost of removing each kilogram of pollutant is given (in dollars) by the matrix

$$B = \begin{array}{c} \begin{array}{cc} \text{Plant } X & \text{Plant } Y \end{array} \\ \begin{bmatrix} 8 & 12 \\ 7 & 9 \\ 15 & 10 \end{bmatrix} \end{array} \begin{array}{l} \text{Sulfur dioxide} \\ \text{Nitric oxide} \\ \text{Particulate matter} \end{array}$$

What do the entries in the matrix product $AB$ tell the manufacturer?

**33.** (***Medicine***) A diet research project consists of adults and children of both sexes. The composition of the participants in the project is given by the matrix

$$A = \begin{array}{c} \begin{array}{cc} \text{Adults} & \text{Children} \end{array} \\ \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \end{array} \begin{array}{l} \text{Male} \\ \text{Female} \end{array}$$

The number of daily grams of protein, fat, and carbohydrate consumed by each child and adult is given by the matrix

$$B = \begin{array}{c} \begin{array}{ccc} \text{Protein} & \text{Fat} & \text{Carbo-hydrate} \end{array} \\ \begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix} \end{array} \begin{array}{l} \text{Adult} \\ \text{Child} \end{array}$$

(a) How many grams of protein are consumed daily by the males in the project?

(b) How many grams of fat are consumed daily by the females in the project?

**34.** (***Business***) A photography business has a store in each of the following cities: New York, Denver, and Los Angeles. A particular make of camera is available in automatic, semiautomatic, and nonautomatic models. Moreover, each camera has a matched flash unit and a camera is usually sold together with the corresponding flash unit. The selling prices of the cameras and flash units are given (in dollars) by the matrix

$$A = \begin{array}{c} \begin{array}{ccc} \text{Auto-} & \text{Semi-} & \text{Non-} \\ \text{matic} & \text{automatic} & \text{automatic} \end{array} \\ \begin{bmatrix} 200 & 150 & 120 \\ 50 & 40 & 25 \end{bmatrix} \end{array} \begin{array}{l} \text{Camera} \\ \text{Flash unit} \end{array}$$

The number of sets (camera and flash unit) available at each store is given by the matrix

$$B = \begin{array}{c} \begin{array}{ccc} \text{New York} & \text{Denver} & \text{Los Angeles} \end{array} \\ \begin{bmatrix} 220 & 180 & 100 \\ 300 & 250 & 120 \\ 120 & 320 & 250 \end{bmatrix} \end{array} \begin{array}{l} \text{Automatic} \\ \text{Semiautomatic} \\ \text{Nonautomatic} \end{array}$$

(a) What is the total value of the cameras in New York?

(b) What is the total value of the flash units in Los Angeles?

**35.** Let $\mathbf{s}_1 = \begin{bmatrix} 18.95 & 14.75 & 8.98 \end{bmatrix}$ and $\mathbf{s}_2 = \begin{bmatrix} 17.80 & 13.50 & 10.79 \end{bmatrix}$ be 3-vectors denoting the current prices of three items at stores A and B, respectively.

(a) Obtain a $2 \times 3$ matrix representing the combined information about the prices of the three items at the two stores.

(b) Suppose that each store announces a sale so that the price of each item is reduced by 20%. Obtain a $2 \times 3$ matrix representing the sale prices at the two stores.

*Exercises 36 through 41 involve bit matrices.*

**36.** For bit vectors $\mathbf{a}$ and $\mathbf{b}$ compute $\mathbf{a} \cdot \mathbf{b}$.

(a) $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

**37.** For bit vectors $\mathbf{a}$ and $\mathbf{b}$ compute $\mathbf{a} \cdot \mathbf{b}$.

(a) $\mathbf{a} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**38.** Let $\mathbf{a} = \begin{bmatrix} 1 & x & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} x \\ 1 \\ 1 \end{bmatrix}$ be bit vectors. If $\mathbf{a} \cdot \mathbf{b} = 0$, find all possible values of $x$.

**39.** Let $A = \begin{bmatrix} 1 & 1 & x \\ 0 & y & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be bit matrices. If $AB = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, find $x$ and $y$.

**40.** For bit matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

compute $AB$ and $BA$.

**41.** For bit matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, determine a $2 \times 2$ bit matrix $B$ so that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

## Theoretical Exercises

**T.1.** Let $\mathbf{x}$ be an $n$-vector.

(a) Is it possible for $\mathbf{x} \cdot \mathbf{x}$ to be negative? Explain.

(b) If $\mathbf{x} \cdot \mathbf{x} = 0$, what is $\mathbf{x}$?

**T.2.** Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be $n$-vectors and let $k$ be a real number.

(a) Show that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(b) Show that $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.

(c) Show that $(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$.

**T.3.** (a) Show that if $A$ has a row of zeros, then $AB$ has a row of zeros.

(b) Show that if $B$ has a column of zeros, then $AB$ has a column of zeros.

**T.4.** Show that the product of two diagonal matrices is a diagonal matrix.

**T.5.** Show that the product of two scalar matrices is a scalar matrix.

**T.6.** (a) Show that the product of two upper triangular matrices is upper triangular.

(b) Show that the product of two lower triangular matrices is lower triangular.

**T.7.** Let $A$ and $B$ be $n \times n$ diagonal matrices. Is $AB = BA$? Justify your answer.

**T.8.** (a) Let $\mathbf{a}$ be a $1 \times n$ matrix and $B$ an $n \times p$ matrix. Show that the matrix product $\mathbf{a}B$ can be written as a linear combination of the rows of $B$, where the coefficients are the entries of **a**.

(b) Let $\mathbf{a} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$ and

$$B = \begin{bmatrix} 2 & 1 & -4 \\ -3 & -2 & 3 \\ 4 & 5 & -2 \end{bmatrix}.$$

Write $\mathbf{a}B$ as a linear combination of the rows of $B$.

**T.9.** (a) Show that the $j$th column of the matrix product $AB$ is equal to the matrix product $A\,\mathrm{col}_j(B)$.

(b) Show that the $i$th row of the matrix product $AB$ is equal to the matrix product $\mathrm{row}_i(A)B$.

**T.10.** Let $A$ be an $m \times n$ matrix whose entries are real numbers. Show that if $AA^T = O$ (the $m \times m$ matrix all of whose entries are zero), then $A = O$.

*Exercises T.11 through T.13 depend on material marked optional.*

**T.11.** Show that the summation notation satisfies the following properties:

(a) $\displaystyle\sum_{i=1}^{n} (r_i + s_i)a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i.$

(b) $\displaystyle\sum_{i=1}^{n} c(r_i a_i) = c\left(\sum_{i=1}^{n} r_i a_i\right).$

**T.12.** Show that $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}.$

**T.13.** Answer the following as true or false. If true, prove the result; if false, give a counterexample.

(a) $\displaystyle\sum_{i=1}^{n}(a_i + 1) = \left(\sum_{i=1}^{n} a_i\right) + n$

(b) $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} 1 = mn$

(c) $\displaystyle\sum_{j=1}^{m}\sum_{i=1}^{n} a_i b_j = \left[\sum_{i=1}^{n} a_i\right]\left[\sum_{j=1}^{m} b_j\right]$

**T.14.** Let **u** and **v** be $n$-vectors.

(a) If **u** and **v** are viewed as $n \times 1$ matrices, show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$.

(b) If **u** and **v** are viewed as $1 \times n$ matrices, show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T$.

(c) If **u** is viewed as a $1 \times n$ matrix and **v** as an $n \times 1$ matrix, show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}$.

## MATLAB Exercises

**ML.1.** In MATLAB, type the command **clear**, then enter the following matrices:

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & \frac{5}{4} & \frac{9}{4} \\ 1 & 2 & 3 \end{bmatrix}.$$

Using MATLAB commands, compute each of the following, if possible. Recall that a prime in MATLAB indicates transpose.

(a) $A * C$ (b) $A * B$

(c) $A + C'$ (d) $B * A - C' * A$

(e) $(2 * C - 6 * A') * B'$ (f) $A * C - C * A$

(g) $A * A' + C' * C$

**ML.2.** Enter the coefficient matrix of the system

$$\begin{aligned} 2x + 4y + 6z &= -12 \\ 2x - 3y - 4z &= 15 \\ 3x + 4y + 5z &= -8 \end{aligned}$$

into MATLAB and call it $A$. Enter the right-hand side of the system and call it **b**. Form the augmented matrix associated with this linear system using the MATLAB command **[A b]**. To give the augmented matrix a name, such as **aug**, use the command **aug = [A b]**. (Do not type the period!) Note that no bar appears between the coefficient matrix and the right-hand side in the MATLAB display.

**ML.3.** Repeat the preceding exercise with the following linear system:

$$\begin{aligned} 4x - 3y + 2z - w &= -5 \\ 2x + y - 3z &= 7 \\ -x + 4y + z + 2w &= 8. \end{aligned}$$

**ML.4.** Enter matrices

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 3 & -3 & 4 \\ 4 & 2 & 5 & 1 \end{bmatrix}$$

into MATLAB.

(a) Using MATLAB commands, assign $\mathrm{row}_2(A)$ to **R** and $\mathrm{col}_3(B)$ to **C**. Let $\mathbf{V} = \mathbf{R} * \mathbf{C}$. What is **V** in terms of the entries of the product $\mathbf{A} * \mathbf{B}$?

(b) Using MATLAB commands, assign $\mathrm{col}_2(B)$ to **C**, then compute $\mathbf{V} = \mathbf{A} * \mathbf{C}$. What is **V** in terms of the entries of the product $\mathbf{A} * \mathbf{B}$?

(c) Using MATLAB commands, assign $\mathrm{row}_3(A)$ to **R**, then compute $\mathbf{V} = \mathbf{R} * \mathbf{B}$. What is **V** in terms of the entries of the product $\mathbf{A} * \mathbf{B}$?

**ML.5.** Use the MATLAB command **diag** to form each of the following diagonal matrices. Using **diag** we can form diagonal matrices without typing in all the entries. (To refresh your memory about command **diag**, use MATLAB's help feature.)

(a) The $4 \times 4$ diagonal matrix with main diagonal $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$.

(b) The $5 \times 5$ diagonal matrix with main diagonal $\begin{bmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$.

(c) The $5 \times 5$ scalar matrix with all 5's on the diagonal.

**ML.6.** In MATLAB the dot product of a pair of vectors can be computed using the **dot** command. If the vectors **v** and **w** have been entered into MATLAB as either rows or columns, their dot product is computed from the MATLAB command **dot(v, w)**. If the vectors do not have the same number of elements, an error message is displayed.

(a) Use **dot** to compute the dot product of each of the following vectors.

(i) $\mathbf{v} = \begin{bmatrix} 1 & 4 & -1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 7 & 2 & 0 \end{bmatrix}$

(ii) $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 6 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ -1 \end{bmatrix}$

(b) Let $\mathbf{a} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$. Find a value for $k$ so that the dot product of $\mathbf{a}$ with $\mathbf{b} = \begin{bmatrix} k & 1 & 4 \end{bmatrix}$ is zero. Verify your results in MATLAB.

(c) For each of the following vectors **v**, compute **dot(v,v)** in MATLAB.

(i) $\mathbf{v} = \begin{bmatrix} 4 & 2 & -3 \end{bmatrix}$

(ii) $\mathbf{v} = \begin{bmatrix} -9 & 3 & 1 & 0 & 6 \end{bmatrix}$

(iii) $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -5 \\ -3 \end{bmatrix}$

What sign is each of these dot products? Explain why this is true for almost all vectors **v**. When is it not true?

*Exercises ML.7 through ML.11 use bit matrices and the supplemental instructional commands described in Section 12.9.*

**ML.7.** Use **binprod** to solve Exercise 40.

**ML.8.** Given the bit vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, use **binprod** to compute $\mathbf{a} \cdot \mathbf{b}$.

**ML.9.** (a) Use **bingen** to generate a matrix $B$ whose columns are all possible bit 3-vectors.

(b) Define **A = ones(3)** and compute $AB$ using **binprod**.

(c) Describe why $AB$ contains only columns of all zeros or all ones. (*Hint*: Look for a pattern based on the columns of $B$.)

**ML.10.** Repeat Exercise ML.9 with 4-vectors and **A = ones(4)**.

**ML.11.** Let $B$ be the $n \times n$ matrix of all ones. Compute $BB$ for $n = 2, 3, 4$, and 5. What is $BB$ for $n = k$, where $k$ is any positive integer?

# 1.4 PROPERTIES OF MATRIX OPERATIONS

In this section we consider the algebraic properties of the matrix operations just defined. Many of these properties are similar to familiar properties of the real numbers. However, there will be striking differences between the set of real numbers and the set of matrices in their algebraic behavior under certain operations, for example, under multiplication (as seen in Section 1.3). Most of the properties will be stated as theorems, whose proofs will be left as exercises.

**THEOREM 1.1** **(Properties of Matrix Addition)** *Let $A$, $B$, $C$, and $D$ be $m \times n$ matrices.*

(a) $A + B = B + A$.

(b) $A + (B + C) = (A + B) + C$.

(c) *There is a unique $m \times n$ matrix $O$ such that*

$$A + O = A \tag{1}$$

*for any $m \times n$ matrix $A$. The matrix $O$ is called the $m \times n$* **additive identity** *or* **zero matrix**.

(d) *For each $m \times n$ matrix $A$, there is a unique $m \times n$ matrix $D$ such that*

$$A + D = O. \tag{2}$$

*We shall write $D$ as $(-A)$, so that* (2) *can be written as*

$$A + (-A) = O.$$

*The matrix $(-A)$ is called the* **additive inverse** *or* **negative** *of $A$.*

**Proof** (a) To establish (a), we must prove that the $i, j$th element of $A + B$ equals the $i, j$th element of $B + A$. The $i, j$th element of $A + B$ is $a_{ij} + b_{ij}$; the $i, j$th element of $B + A$ is $b_{ij} + a_{ij}$. Since the elements $a_{ij}$ are real (or complex) numbers,

$$a_{ij} + b_{ij} = b_{ij} + a_{ij} \qquad (1 \le i \le m, 1 \le j \le n),$$

the result follows.

(b) Exercise T.1.

(c) Let $U = \left[u_{ij}\right]$. Then

$$A + U = A$$

if and only if*

$$a_{ij} + u_{ij} = a_{ij},$$

which holds if and only if $u_{ij} = 0$. Thus $U$ is the $m \times n$ matrix all of whose entries are zero; $U$ is denoted by $O$.

(d) Exercise T.1. ■

**EXAMPLE 1** To illustrate (c) of Theorem 1.1, we note that the $2 \times 2$ zero matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix},$$

we have

$$\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4+0 & -1+0 \\ 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}.$$ ■

The $2 \times 3$ zero matrix is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*The connector "if and only if" means that both statements are true or both statements are false. Thus (1) if $A + U = A$, then $a_{ij} + u_{ij} = a_{ij}$ and (2) if $a_{ij} + u_{ij} = a_{ij}$, then $A + U = A$.

**EXAMPLE 2** To illustrate (d) of Theorem 1.1, let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -4 & 5 & -2 \end{bmatrix}.$$

Then

$$-A = \begin{bmatrix} -2 & -3 & -4 \\ 4 & -5 & 2 \end{bmatrix}.$$

We now have $A + (-A) = O$. ■

**EXAMPLE 3** Let

$$A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 & 2 \\ -3 & 4 & 6 \end{bmatrix}.$$

Then

$$A - B = \begin{bmatrix} 3-2 & -2-3 & 5-2 \\ -1+3 & 2-4 & 3-6 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 3 \\ 2 & -2 & -3 \end{bmatrix}.$$ ■

**THEOREM 1.2** **(Properties of Matrix Multiplication)**

(a) *If $A$, $B$, and $C$ are of the appropriate sizes, then*

$$A(BC) = (AB)C.$$

(b) *If $A$, $B$, and $C$ are of the appropriate sizes, then*

$$A(B + C) = AB + AC.$$

(c) *If $A$, $B$, and $C$ are of the appropriate sizes, then*

$$(A + B)C = AC + BC.$$

**Proof** (a) We omit a general proof here. Exercise T.2 asks the reader to prove the result for a specific case.

(b) Exercise T.3.

(c) Exercise T.3. ■

**EXAMPLE 4** Let

$$A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 3 & 0 & -1 & 3 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}.$$

Then

$$A(BC) = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 7 \\ 8 & -4 & 6 \\ 9 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}$$

and

$$(AB)C = \begin{bmatrix} 19 & -1 & 6 & 13 \\ 16 & -8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}.$$ ■

**EXAMPLE 5** Let

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 2 & -2 \end{bmatrix}.$$

Then

$$A(B+C) = \begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 21 & -1 \\ 7 & -2 \end{bmatrix}$$

and

$$AB + AC = \begin{bmatrix} 15 & 1 \\ 7 & -4 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 21 & -1 \\ 7 & -2 \end{bmatrix}.$$

■

**DEFINITION** The $n \times n$ scalar matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

all of whose diagonal entries are 1, is called the **identity matrix of order *n***.

If $A$ is an $m \times n$ matrix, then it is easy to verify (Exercise T.4) that

$$I_m A = A I_n = A.$$

It is also easy to see that every $n \times n$ scalar matrix can be written as $rI_n$ for some $r$.

**EXAMPLE 6** The identity matrix $I_2$ of order 2 is

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 0 & 2 \end{bmatrix},$$

then

$$I_2 A = A.$$

The identity matrix $I_3$ of order 3 is

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$AI_3 = A.$$

■

Suppose that $A$ is a square matrix. If $p$ is a positive integer, then we define the **powers of a matrix** as follows:

$$A^p = \underbrace{A \cdot A \cdots A}_{p \text{ factors}}.$$

If $A$ is $n \times n$, we also define

$$A^0 = I_n.$$

For nonnegative integers $p$ and $q$, some of the familiar laws of exponents for the real numbers can also be proved for matrix multiplication of a square matrix $A$ (Exercise T.5):

$$A^p A^q = A^{p+q}$$

and

$$(A^p)^q = A^{pq}.$$

It should be noted that

$$(AB)^p \neq A^p B^p$$

for square matrices in general. However, if $AB = BA$, then this rule does hold (Exercise T.6).

We now note two other peculiarities of matrix multiplication. If $a$ and $b$ are real numbers, then $ab = 0$ can hold only if $a$ or $b$ is zero. However, this is not true for matrices.

**EXAMPLE 7** If

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix},$$

then neither $A$ nor $B$ is the zero matrix, but

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

■

If $a$, $b$, and $c$ are real numbers for which $ab = ac$ and $a \neq 0$, it then follows that $b = c$. That is, we can cancel $a$ out. However, the cancellation law does not hold for matrices, as the following example shows.

**EXAMPLE 8** If

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix},$$

then

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix},$$

but $B \neq C$. ■

**Remark** In Section 1.7, we investigate a special class of matrices $A$ for which $AB = AC$ does imply that $B = C$.

**EXAMPLE 9** **(Business)** Suppose that only two rival companies, $R$ and $S$, manufacture a certain product. Each year, company $R$ keeps $\frac{1}{4}$ of its customers while $\frac{3}{4}$

switch to $S$. Each year, $S$ keeps $\frac{2}{3}$ of its customers while $\frac{1}{3}$ switch to $R$. This information can be displayed in matrix form as

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} R & S \\ \left[\begin{matrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{matrix}\right. & \left.\right] \end{array}\begin{array}{c} \\ R \\ S \end{array}$$

When manufacture of the product first starts, $R$ has $\frac{3}{5}$ of the market (the market is the total number of customers) while $S$ has $\frac{2}{5}$ of the market. We denote the initial distribution of the market by

$$\mathbf{x}_0 = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}.$$

One year later, the distribution of the market is

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\left(\frac{3}{5}\right) + \frac{1}{3}\left(\frac{2}{5}\right) \\ \frac{3}{4}\left(\frac{3}{5}\right) + \frac{2}{3}\left(\frac{2}{5}\right) \end{bmatrix} = \begin{bmatrix} \frac{17}{60} \\ \frac{43}{60} \end{bmatrix}.$$

This can be readily seen as follows. Suppose that the initial market consists of $k$ people, say $k = 12{,}000$, and no change in this number occurs with time. Then, initially, $R$ has $\frac{3}{5}k$ customers, and $S$ has $\frac{2}{5}k$ customers. At the end of the first year, $R$ keeps $\frac{1}{4}$ of its customers and gains $\frac{1}{3}$ of $S$'s customers. Thus $R$ has

$$\frac{1}{4}\left(\frac{3}{5}k\right) + \frac{1}{3}\left(\frac{2}{5}k\right) = \left[\frac{1}{4}\left(\frac{3}{5}\right) + \frac{1}{3}\left(\frac{2}{5}\right)\right]k = \frac{17}{60}k \text{ customers.}$$

When $k = 12{,}000$, $R$ has $\frac{17}{60}(12{,}000) = 3400$ customers. Similarly, at the end of the first year, $S$ keeps $\frac{2}{3}$ of its customers and gains $\frac{3}{4}$ of $R$'s customers. Thus $S$ has

$$\frac{3}{4}\left(\frac{3}{5}k\right) + \frac{2}{3}\left(\frac{2}{5}k\right) = \left[\frac{3}{4}\left(\frac{3}{5}\right) + \frac{2}{3}\left(\frac{2}{5}\right)\right]k = \frac{43}{60}k \text{ customers.}$$

When $k = 12{,}000$, $S$ has $\frac{43}{60}(12{,}000) = 8600$ customers. Similarly, at the end of 2 years, the distribution of the market will be given by

$$\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0.$$

If

$$\mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix},$$

can we determine $a$ and $b$ so that the distribution will be the same from year to year? When this happens, the distribution of the market is said to be **stable**. We proceed as follows. Since $R$ and $S$ control the entire market, we must have

$$a + b = 1. \tag{3}$$

We also want the distribution after 1 year to be unchanged. Hence

$$A\mathbf{x}_0 = \mathbf{x}_0$$

or

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{3}{4} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then

$$\begin{aligned}\tfrac{1}{4}a + \tfrac{1}{3}b &= a\\ \tfrac{3}{4}a + \tfrac{2}{3}b &= b\end{aligned}$$

or

$$\begin{aligned}-\tfrac{3}{4}a + \tfrac{1}{3}b &= 0\\ \tfrac{3}{4}a - \tfrac{1}{3}b &= 0.\end{aligned} \tag{4}$$

Observe that the two equations in (4) are the same. Using Equation (3) and one of the equations in (4), we find (verify) that

$$a = \tfrac{4}{13} \quad \text{and} \quad b = \tfrac{9}{13}. \qquad \blacksquare$$

The problem described is an example of a **Markov chain**. We shall return to this topic in Section 2.5.

**THEOREM 1.3** **(Properties of Scalar Multiplication)** *If $r$ and $s$ are real numbers and $A$ and $B$ are matrices, then*

(a) $r(sA) = (rs)A$
(b) $(r+s)A = rA + sA$
(c) $r(A+B) = rA + rB$
(d) $A(rB) = r(AB) = (rA)B$

**Proof** Exercise T.12. ■

**EXAMPLE 10** Let $r = -2$,

$$A = \begin{bmatrix} 1 & 2 & 3\\ -2 & 0 & 1\end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & -1\\ 1 & 4\\ 0 & -2\end{bmatrix}.$$

Then

$$A(rB) = \begin{bmatrix} 1 & 2 & 3\\ -2 & 0 & 1\end{bmatrix}\begin{bmatrix} -4 & 2\\ -2 & -8\\ 0 & 4\end{bmatrix} = \begin{bmatrix} -8 & -2\\ 8 & 0\end{bmatrix}$$

and

$$r(AB) = (-2)\begin{bmatrix} 4 & 1\\ -4 & 0\end{bmatrix} = \begin{bmatrix} -8 & -2\\ 8 & 0\end{bmatrix},$$

which illustrates (d) of Theorem 1.3. ■

It is easy to show that $(-1)A = -A$ (Exercise T.13).

**THEOREM 1.4** **(Properties of Transpose)** *If $r$ is a scalar and $A$ and $B$ are matrices, then*

(a) $(A^T)^T = A$
(b) $(A+B)^T = A^T + B^T$
(c) $(AB)^T = B^T A^T$
(d) $(rA)^T = rA^T$

**Proof** We leave the proofs of (a), (b), and (d) as an exercise (Exercise T.14) and prove only (c) here. Thus let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be $m \times p$ and let $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ be $p \times n$. The $i, j$th element of $(AB)^T$ is $c_{ij}^T$. Now

$$\begin{aligned} c_{ij}^T = c_{ji} &= \operatorname{row}_j(A) \cdot \operatorname{col}_i(B) \\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jp}b_{pi} \\ &= a_{1j}^T b_{i1}^T + a_{2j}^T b_{i2}^T + \cdots + a_{pj}^T b_{ip}^T \\ &= b_{i1}^T a_{1j}^T + b_{i2}^T a_{2j}^T + \cdots + b_{ip}^T a_{pj}^T \\ &= \operatorname{row}_i(B^T) \cdot \operatorname{col}_j(A^T), \end{aligned}$$

which is the $i, j$th element of $B^T A^T$. ■

**EXAMPLE 11** Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}.$$

Then

$$(AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}$$

and

$$B^T A^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}.$$

■

**DEFINITION** A matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ with real entries is called **symmetric** if

$$A^T = A.$$

That is, $A$ is symmetric if it is a square matrix for which

$$a_{ij} = a_{ji} \qquad \text{(Exercise T.17)}.$$

If matrix $A$ is symmetric, then the elements of $A$ are symmetric with respect to the main diagonal of $A$.

**EXAMPLE 12** The matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are symmetric. ■

## EXAMPLES WITH BIT MATRICES (OPTIONAL)

All of the matrix operations discussed in this section are valid on bit matrices provided we use arithmetic base 2. Hence the scalars available are only 0 and 1.

**EXAMPLE 13** Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ be a bit matrix. Find the additive inverse of $A$.

**Solution** Let $-A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ (the additive inverse of $A$). Then $A + (-A) = O$. We have

$$\begin{aligned} 1 + a &= 0 \qquad & 0 + b &= 0 \\ 1 + c &= 0 \qquad & 1 + d &= 0 \\ 0 + e &= 0 \qquad & 1 + f &= 0 \end{aligned}$$

so $a = 1$, $b = 0$, $c = 1$, $d = 1$, $e = 0$, and $f = 1$. Hence $-A = A$. (See also Exercise T.38.) ■

**EXAMPLE 14** For the bit matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ determine a $2 \times 2$ bit matrix $B \neq O$ so that $AB = O$.

**Solution** Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

provided $a = b = 0$, $c = 0$ or 1, and $d = 0$ or 1. Thus there are four such matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$ ■

Section 2.2, Graph Theory, which can be covered at this time, uses material from this section.

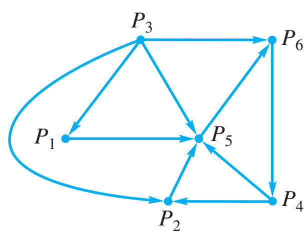
## Preview of an Application

### Graph Theory (Section 2.2)

In recent years, the need to solve problems dealing with communication among individuals, computers, and organizations has grown at an unprecedented rate. As an example, note the explosive growth of the Internet and the promises of using it to interact with all types of media. Graph theory is an area of applied mathematics that deals with problems such as this one:

Consider a local area network consisting of six users denoted by $P_1$, $P_2$, $\ldots$, $P_6$. We say that $P_i$ has "access" to $P_j$ if $P_i$ can directly send a message to $P_j$. On the other hand, $P_i$ may not be able to send a message directly to $P_k$, but can send it to $P_j$, who will then send it to $P_k$. In this way we say that $P_i$ has "2-stage access" to $P_k$. In a similar way, we speak of "$r$-stage access." We may describe the access relation in the network shown in Figure 1.6 by defining the $6 \times 6$ matrix $A = \left[a_{ij}\right]$, where $a_{ij} = 1$ if $P_i$ has access to $P_j$ and 0 otherwise. Thus $A$ may be

Figure 1.6 ▶




$$A = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{array} \begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Using the matrix $A$ and the techniques from graph theory discussed in Section 2.2, we can determine the number of ways that $P_i$ has $r$-stage access to $P_k$, where $r = 1, 2, \ldots$. Many other problems involving communications can be solved using graph theory.

The matrix $A$ above is indeed a bit matrix, but in this situation $A$ is best considered as a matrix in base 10, as will be shown in Section 2.2.

## Key Terms

Properties of matrix addition
Additive identity or zero matrix
Additive inverse or negative of a matrix
Properties of matrix multiplication
Identity matrix
Powers of a matrix
Properties of transpose
Symmetric matrix
Skew symmetric matrix

## 1.4 Exercises

**1.** Verify Theorem 1.1 for
$$A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 5 \end{bmatrix},$$
and
$$C = \begin{bmatrix} -4 & -6 & 1 \\ 2 & 3 & 0 \end{bmatrix}.$$

**2.** Verify (a) of Theorem 1.2 for
$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix},$$
and
$$C = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}.$$

**3.** Verify (b) of Theorem 1.2 for
$$A = \begin{bmatrix} 1 & -3 \\ -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 2 \\ 3 & -1 & -2 \end{bmatrix},$$
and
$$C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -2 \end{bmatrix}.$$

**4.** Verify (a), (b), and (c) of Theorem 1.3 for $r = 6$, $s = -2$, and
$$A = \begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ -4 & 3 \end{bmatrix}.$$

**5.** Verify (d) of Theorem 1.3 for $r = -3$ and
$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix}.$$

**6.** Verify (b) and (d) of Theorem 1.4 for $r = -4$ and
$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & -1 \\ -2 & 1 & 5 \end{bmatrix}.$$

**7.** Verify (c) of Theorem 1.4 for
$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

*In Exercises 8 and 9, let*
$$A = \begin{bmatrix} 2 & 1 & -2 \\ 3 & 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & -2 \end{bmatrix},$$
$$C = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix},$$
$$E = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 2 & -1 \end{bmatrix}, \quad \textit{and} \quad F = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}.$$

**8.** If possible, compute:
(a) $(AB)^T$ (b) $B^TA^T$ (c) $A^TB^T$
(d) $BB^T$ (e) $B^TB$

**9.** If possible, compute:
(a) $(3C - 2E)^TB$ (b) $A^T(D + F)$
(c) $B^TC + A$ (d) $(2E)A^T$
(e) $(B^T + A)C$

**10.** If
$$A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix},$$
show that $AB = O$.

**11.** If
$$A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix},$$
and
$$C = \begin{bmatrix} -4 & -3 \\ 0 & -4 \end{bmatrix},$$
show that $AB = AC$.

**12.** If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, show that $A^2 = I_2$.

**13.** Let $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Find
(a) $A^2 + 3A$
(b) $2A^3 + 3A^2 + 4A + 5I_2$

**14.** Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$. Find
(a) $A^2 - 2A$
(b) $3A^3 - 2A^2 + 5A - 4I_2$

**15.** Determine a scalar $r$ such that $A\mathbf{x} = r\mathbf{x}$, where
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

**16.** Determine a constant $k$ such that $(kA)^T(kA) = 1$, where
$$A = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$
Is there more than one value of $k$ that could be used?

**17.** Let

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 0 \end{bmatrix}$$

and $\mathbf{a}_j = \text{col}_j(A)$, $j = 1, 2, 3$. Verify that

$$A^T A = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \mathbf{a}_1 \cdot \mathbf{a}_3 \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \mathbf{a}_2 \cdot \mathbf{a}_3 \\ \mathbf{a}_3 \cdot \mathbf{a}_1 & \mathbf{a}_3 \cdot \mathbf{a}_2 & \mathbf{a}_3 \cdot \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{a}_1 & \mathbf{a}_1^T \mathbf{a}_2 & \mathbf{a}_1^T \mathbf{a}_3 \\ \mathbf{a}_2^T \mathbf{a}_1 & \mathbf{a}_2^T \mathbf{a}_2 & \mathbf{a}_2^T \mathbf{a}_3 \\ \mathbf{a}_3^T \mathbf{a}_1 & \mathbf{a}_3^T \mathbf{a}_2 & \mathbf{a}_3^T \mathbf{a}_3 \end{bmatrix}.$$

*Exercises 18 through 21 deal with Markov chains, an area that will be studied in greater detail in Section 2.5.*

**18.** Suppose that the matrix $A$ in Example 9 is

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{5} \\ \frac{2}{3} & \frac{3}{5} \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

(a) Find the distribution of the market after 1 year.

(b) Find the stable distribution of the market.

**19.** Consider two quick food companies, $M$ and $N$. Each year, company $M$ keeps $\frac{1}{3}$ of its customers, while $\frac{2}{3}$ switch to $N$. Each year, $N$ keeps $\frac{1}{2}$ of its customers, while $\frac{1}{2}$ switch to $M$. Suppose that the initial distribution of the market is given by

$$\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

(a) Find the distribution of the market after 1 year.

(b) Find the stable distribution of the market.

**20.** Suppose that in Example 9 there were three rival companies $R$, $S$, and $T$ so that the pattern of customer retention and switching is given by the information in the matrix $A$ where

$$\begin{array}{cc} & \begin{array}{ccc} R & S & T \end{array} \\ A = & \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \end{array} \begin{array}{c} R \\ S \\ T \end{array}$$

(a) If the initial market distribution is given by

$$\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix},$$

then determine the market distribution after 1 year; after 2 years.

(b) Show that the stable market distribution is given by

$$\mathbf{x} = \begin{bmatrix} \frac{21}{53} \\ \frac{24}{53} \\ \frac{8}{53} \end{bmatrix}.$$

(c) Which company $R$, $S$, or $T$ will gain the most market share over a long period of time (assuming that the retention and switching patterns remain the same)? Approximately what percent of the market was gained by this company?

**21.** Suppose that in Exercise 20 the matrix $A$ was given by

$$\begin{array}{cc} & \begin{array}{ccc} R & S & T \end{array} \\ A = & \begin{bmatrix} 0.4 & 0 & 0.4 \\ 0 & 0.5 & 0.4 \\ 0.6 & 0.5 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} R \\ S \\ T \end{array}$$

(a) If the initial market distribution is given by

$$\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix},$$

then determine the market distribution after 1 year; after 2 years.

(b) Show that the stable market distribution is given by

$$\mathbf{x} = \begin{bmatrix} \frac{10}{37} \\ \frac{12}{37} \\ \frac{15}{37} \end{bmatrix}.$$

(c) Which company $R$, $S$, or $T$ will gain the most market share over a long period of time (assuming that the retention and switching patterns remain the same)? Approximately what percent of the market was gained by this company?

*Exercises 22 through 25 involve bit matrices.*

**22.** If bit matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, show that $A^2 = O$.

**23.** If bit matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, show that $A^2 = I_2$.

**24.** Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ be a bit matrix. Find

(a) $A^2 - A$ (b) $A^3 + A^2 + A$

**25.** Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ be a bit matrix. Find

(a) $A^2 + A$ (b) $A^4 + A^3 + A^2$

## Theoretical Exercises

**T.1.** Prove properties (b) and (d) of Theorem 1.1.

**T.2.** If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is a $2 \times 3$ matrix, $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ is a $3 \times 4$ matrix, and $C = \begin{bmatrix} c_{ij} \end{bmatrix}$ is a $4 \times 3$ matrix, show that $A(BC) = (AB)C$.

**T.3.** Prove properties (b) and (c) of Theorem 1.2.

**T.4.** If $A$ is an $m \times n$ matrix, show that

$$I_m A = A I_n = A.$$

**T.5.** Let $p$ and $q$ be nonnegative integers and let $A$ be a square matrix. Show that

$$A^p A^q = A^{p+q} \quad \text{and} \quad (A^p)^q = A^{pq}.$$

**T.6.** If $AB = BA$, and $p$ is a nonnegative integer, show that

$$(AB)^p = A^p B^p.$$

**T.7.** Show that if $A$ and $B$ are $n \times n$ diagonal matrices, then $AB = BA$.

**T.8.** Find a $2 \times 2$ matrix $B \neq O$ and $B \neq I_2$ such that $AB = BA$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

How many such matrices $B$ are there?

**T.9.** Find a $2 \times 2$ matrix $B \neq O$ and $B \neq I_2$ such that $AB = BA$, where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

How many such matrices $B$ are there?

**T.10.** Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

(a) Determine a simple expression for $A^2$.

(b) Determine a simple expression for $A^3$.

(c) Conjecture the form of a simple expression for $A^k$, $k$ a positive integer.

(d) Prove or disprove your conjecture in part (c).

**T.11.** If $p$ is a nonnegative integer and $c$ is a scalar, show that

$$(cA)^p = c^p A^p.$$

**T.12.** Prove Theorem 1.3.

**T.13.** Show that $(-1)A = -A$.

**T.14.** Complete the proof of Theorem 1.4.

**T.15.** Show that $(A - B)^T = A^T - B^T$.

**T.16.** (a) Show that $(A^2)^T = (A^T)^2$.

(b) Show that $(A^3)^T = (A^T)^3$.

(c) Prove or disprove that, for $k = 4, 5, \ldots,$ $(A^k)^T = (A^T)^k$.

**T.17.** Show that a square matrix $A$ is symmetric if and only if $a_{ij} = a_{ji}$ for all $i, j$.

**T.18.** Show that if $A$ is symmetric, then $A^T$ is symmetric.

**T.19.** Let $A$ be an $n \times n$ matrix. Show that if $A\mathbf{x} = \mathbf{0}$ for all $n \times 1$ matrices $\mathbf{x}$, then $A = O$.

**T.20.** Let $A$ be an $n \times n$ matrix. Show that if $A\mathbf{x} = \mathbf{x}$ for all $n \times 1$ matrices $\mathbf{x}$, then $A = I_n$.

**T.21.** Show that if $AA^T = O$, then $A = O$.

**T.22.** Show that if $A$ is a symmetric matrix, then $A^k$, $k = 2, 3, \ldots,$ is symmetric.

**T.23.** Let $A$ and $B$ be symmetric matrices.

(a) Show that $A + B$ is symmetric.

(b) Show that $AB$ is symmetric if and only if $AB = BA$.

**T.24.** A matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is called **skew symmetric** if $A^T = -A$. Show that $A$ is skew symmetric if and only if $a_{ij} = -a_{ji}$ for all $i, j$.

**T.25.** Describe all skew symmetric scalar matrices. (See Section 1.2 for the definition of scalar matrix.)

**T.26.** If $A$ is an $n \times n$ matrix, show that $AA^T$ and $A^T A$ are symmetric.

**T.27.** If $A$ is an $n \times n$ matrix, show that

(a) $A + A^T$ is symmetric.

(b) $A - A^T$ is skew symmetric.

**T.28.** Show that if $A$ is an $n \times n$ matrix, then $A$ can be written uniquely as $A = S + K$, where $S$ is symmetric and $K$ is skew symmetric.

**T.29.** Show that if $A$ is an $n \times n$ scalar matrix, then $A = rI_n$ for some real number $r$.

**T.30.** Show that $I_n^T = I_n$.

**T.31.** Let $A$ be an $m \times n$ matrix. Show that if $rA = O$, then $r = 0$ or $A = O$.

**T.32.** Show that if $A\mathbf{x} = \mathbf{b}$ is a linear system that has more than one solution, then it has infinitely many solutions. (*Hint*: If $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions, consider $\mathbf{w} = r\mathbf{u}_1 + s\mathbf{u}_2$, where $r + s = 1$.)

**T.33.** Determine all $2 \times 2$ matrices $A$ such that $AB = BA$ for any $2 \times 2$ matrix $B$.

**T.34.** If $A$ is a skew symmetric matrix, what type of matrix is $A^T$? Justify your answer.

**T.35.** What type of matrix is a linear combination of symmetric matrices? (See Section 1.3.) Justify your answer.

**T.36.** What type of matrix is a linear combination of scalar matrices? (See Section 1.3.) Justify your answer.

**T.37.** Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be the $n \times n$ matrix defined by $a_{ii} = r$ and $a_{ij} = 0$ if $i \neq j$. Show that if $B$ is any $n \times n$ matrix, then $AB = rB$.

**T.38.** If $A$ is any $m \times n$ bit matrix, show that $-A = A$.

**T.39.** Determine all $2 \times 2$ bit matrices $A$ so that $A^2 = O$.

**T.40.** Determine all $2 \times 2$ bit matrices $A$ so that $A^2 = I_2$.

## MATLAB Exercises

*In order to use* MATLAB *in this section, you should first have read Chapter 12 through Section 12.3.*

**ML.1.** Use MATLAB to find the smallest positive integer $k$ in each of the following cases. (See also Exercise 12.)

(a) $A^k = I_3$ for $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $A^k = A$ for $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

**ML.2.** Use MATLAB to display the matrix $A$ in each of the following cases. Find the smallest value of $k$ such that $A^k$ is a zero matrix. Here **tril**, **ones**, **triu**, **fix**, and **rand** are MATLAB commands. (To see a description, use **help**.)

(a) $A = \textbf{tril}(\textbf{ones}(\mathbf{5}), \mathbf{-1})$

(b) $A = \textbf{triu}(\textbf{fix}(\mathbf{10} * \textbf{rand}(\mathbf{7})), \mathbf{2})$

**ML.3.** Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$. Using command **polyvalm** in MATLAB, compute the following matrix polynomials:

(a) $A^4 - A^3 + A^2 + 2I_3$ (b) $A^3 - 3A^2 + 3A$

**ML.4.** Let $A = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$. Using MATLAB, compute each of the following matrix expressions:

(a) $(A^2 - 7A)(A + 3I_3)$.

(b) $(A - I_3)^2 + (A^3 + A)$.

(c) Look at the sequence $A, A^2, A^3, \dots, A^8, \dots$. Does it appear to be converging to a matrix? If so, to what matrix?

**ML.5.** Let $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{3} \end{bmatrix}$. Use MATLAB to compute members of the sequence $A, A^2, A^3, \dots, A^k, \dots$. Write a description of the behavior of this matrix sequence.

**ML.6.** Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & -\frac{1}{5} \end{bmatrix}$. Repeat Exercise ML.5.

**ML.7.** Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Use MATLAB to do the following:

(a) Compute $A^TA$ and $AA^T$. Are they equal?

(b) Compute $B = A + A^T$ and $C = A - A^T$. Show that $B$ is symmetric and $C$ is skew symmetric. (See Exercise T.24.)

(c) Determine a relationship between $B + C$ and $A$.

*Exercises ML.8 through ML.11 use bit matrices and the supplemental instructional commands described in Section 12.9.*

**ML.8.** (a) Use **binrand** to generate a $3 \times 3$ bit matrix $B$.

(b) Use **binadd** to compute $B + B$ and $B + B + B$.

(c) If $B$ were added to itself $n$ times, what would be the result? Explain your answer.

**ML.9.** Let $B = \textbf{triu}(\textbf{ones}(\mathbf{3}))$. Determine $k$ so that $B^k = I_3$.

**ML.10.** Let $B = \textbf{triu}(\textbf{ones}(\mathbf{4}))$. Determine $k$ so that $B^k = I_4$.

**ML.11.** Let $B = \textbf{triu}(\textbf{ones}(\mathbf{5}))$. Determine $k$ so that $B^k = I_5$.

# 1.5 MATRIX TRANSFORMATIONS

In Section 1.2 we introduced the notation $R^n$ for the set of all $n$-vectors with real entries. Thus, $R^2$ denotes the set of all 2-vectors and $R^3$ denotes the set of all 3-vectors. It is convenient to represent the elements of $R^2$ and $R^3$ geometrically as directed line segments in a rectangular coordinate system.[‡] Our approach in this section is intuitive and will enable us to present some interesting geometric applications in the next section (at this early stage of the course). We return in Section 3.1 to a careful and precise study of 2-vectors and 3-vectors.

[‡]You have undoubtedly seen rectangular coordinate systems in your precalculus or calculus courses.