# The Quotient Reflective Subcategories of Ordered RELative Spaces


By

**Ch Muhammad Afaq Aslam**

Registration No.: 00000331095

Supervisor

**Dr. Muhammad Qasim**

Department of Mathematics

School of Natural Sciences (SNS)

National University of Sciences and Technology (NUST)

Islamabad, Pakistan



June 2022

FORM TH-4

# National University of Sciences & Technology

**MS THESIS WORK**

We hereby recommend that the dissertation prepared under our supervision by: **Ch Muhammad Afaq Aslam**, Regn No. **00000331095** Titled: **"The Quotient Reflective Subcategories of Ordered RELative Spaces"** accepted in partial fulfillment of the requirements for the award of **MS** degree.

**Examination Committee Members**

1. Name: PROF. MATLOOB ANWAR Signature: ________

2. Name: DR. MUHAMMAD ISHAQ Signature: ________

Supervisor's Name: DR. MUHAMMAD QASIM Signature: ________

________ 27/06/2022
Head of Department Date

**COUNTERSINGED**

Date: 27/6/2022 ________
Dean/Principal

This thesis is dedicated to *my beloved parents*

# Acknowledgments

First and foremost, all the praise is to Allah Almighty, the beneficent, the merciful, the creator, the inventor, the ubiquitous, the one who was the first, the one who will be the last, for providing me the courage and will to complete this thesis prosperously with in the designate time. All the esteem for the greatest educator and the benefactor of mankind, Holy Prophet Muhammad (PBUH), who advised all of us to continue getting education from cradle to grave.

I am grateful to my supervisor Dr. Muhammad Qasim for constantly supporting, encouraging and cooperating with me during the completion of this work. Undoubtedly, he provided an admirable guidance and productive help. Also, I am grateful to Prof. Dieter Leseberg for his continuous support and especially providing the final structure of ordered RELative spaces.

Last but not the least I would like to express my gratitude to my parents and my uncle Ghulam Muhammad for their prayers , valuable advice, suggestions, and support. Also, to my siblings for standing by me and encouraging me to accomplish every aim of my life, without them I would not be able to take a single step in my life.

Finally, I would like to give special thanks to my friends and my research fellows for their support, courage and prayers.

# Abstract

In this thesis, we examine the category of ordered RELative spaces. We show that it is a normalized and geometric topological category and find its discrete (resp. indiscrete) structures, and give the characterization of local $\overline{T}_0$, local $T_0^{'}$ and local $T_1$ ordered RELative spaces. Furthermore, we characterize explicitly several notions of $T_0$'s and $T_1$ objects in **O-REL** and study their mutual relationship. Finally, it is shown that the category of $T_0$'s (resp. $T_1$) ordered RELative spaces are quotient reflective subcategories of **O-REL** and $T_0^{'}$**O-REL** is a normalized topological category.

# Contents


**1 Fundamental Concepts** **1**

1.1 Topological Spaces . . . . . . . . . . 1

1.2 Separation Axioms . . . . . . . . . . 5

1.3 Category Theory . . . . . . . . . . 6

1.4 Categorical Topology . . . . . . . . . . 9

**2 RELative Spaces and Ordered RELative Spaces** **12**

2.1 Basic concepts and notations . . . . . . . . . . 12

2.2 **O-REL** as a Normalized and Geometric Topological Category . . . . . . . . . . 15

**3 Local $T_0$ and Local $T_1$ Ordered RELative Spaces** **17**

3.1 Local $T_0$ Ordered-RELspaces . . . . . . . . . . 17

3.2 Local $T_1$ Ordered-RELspaces . . . . . . . . . . 22

**4 Quotient Reflective Subcategories of the category of Ordered-RELative Spaces** **27**

4.1 Generic $T_0$ Ordered-RELspaces . . . . . . . . . . 27

4.2 Generic $T_1$ Ordered-RELspaces . . . . . . . . . . 34

4.3 Quotient-Reflective Subcategories of **O-REL** . . . . . . . . . . 39

**5 Results and Discussions** **41**

5.1 Results and Discussions . . . . . . . . . . 41

**BIBLIOGRAPHY** **43**

# Introduction

Many mathematical concepts were developed to describe certain structures of topology. The concepts of uniform convergences, uniform continuity, cartesian closedness, completeness and total boundedness do not exist in general topology. As a remedy, several approaches have been made to define these concepts in topology by mathematicians. For example, the concepts of uniform convergence in the sense of Kent [25] and Preuss [38], of set-convergence in the sense of Wyler [44], Tozzi [42] (which scrutinize filter convergence to bounded subset and generalizes classical point-convergence and supertopologies), of nearness by Bentely [13] and Herrlich [20] (particularly containing proximities and contiguities), and that of hullness by Čech [15] and Leseberg [33] containing the concepts of b-topologies and closures, respectively. In 2018, Leseberg [34] introduced a global concept which embeds the category of the above mentioned concepts into the category of RELspaces and RELmaps as subcategories. This construct, denoted by **REL**, forms thereby a topological category [34].

Classical separation axioms are very common and important ideas in general topology, and have many applications in all fields of mathematics. With the help of $T_0$ reflection [23] characterization of locally semi-simple morphisms are obtained in algebraic topology. Furthermore, lower separation axioms can be used in digital topology where they describe digital lines, and in image processing and computer graphs to construct cellular complexes [19, 26, 27]. With having the understanding of $T_0$ and $T_1$ separation properties, several mathematicians have extended this idea to arbitrary topological categories [3, 14, 18, 22, 37].

Classical separation axioms at some point were generalized and have been examined in [3], where the generalization motive was to describe the concept of strongly closed and closed sets in any random set based topological categories [4]. Moreover, the notions of compactness in [11], Hausdroffness in [3], regular and normal objects in [10], perfectness in [11], and soberness in [7] have been generalized by using the closed and strongly closed sets in some well defined topological category over sets [11, 30, 31, 40]. Furthermore, the notion of closedness are suitable for the formation of closure operators [16] in several well-known topological categories [12, 17, 41].

This thesis comprises of five chapters.

The first chapter covers the basic definitions of general topology including top. spaces, continuity, initial and final topology and lower separation axioms along with examples. Moreover, some basic concepts of category theory are restated and provided several examples. Furthermore, in third section topological functor, normalized top. functor and geometric top. functor and epireflective, quotient-reflective and bireflective subcategories are defined along with their examples.

In the second chapter, the first section contains some basic concepts and notations of ordered-relative spaces. In second section, the category of ordered-relative spaces can be discussed as normalized and geometric topological category. Furthermore, initial and final structure of ordered-relative spaces are proved.

In the third chapter, we define notions of $T_0$'s and $T_1$ ordered relative spaces at some point $p$ and wedge product of $X$ at p. In first section of this chapter, we define principal p-axis, folding mapping and local $\overline{T}_0$ and $T_0'$ using wedge product of $X$ at $p$. Furthermore, this section contains the characterization of local $T_0'$ and $\overline{T}_0$ objects in **O-REL** and examine their mutual relationship. The second section contains the definitions of skewed p-axis and folding mapping and the characterization of local $T_1$ objects in **O-REL** and examine their mutual relationship.

In fourth chapter, we define generically notions of $T_0$'s and $T_1$ in ordered-relative spaces. Furthermore, we discussed the quotient reflective subcategories of **O-REL** and it is shown that every $\overline{T}_0$**O-REL** (resp. $T_0$**O-REL**, $T_1$**O-REL**) is a quotient-reflective subcategory of **O-REL**. Also it is shown that $T_0'$**O-REL** is a normalized cartesian closed and hereditary topological construct.

The last chapter contains discussion and conclusion of entire study.

Chapter 1

# Fundamental Concepts

In this chapter, we discuss fundamental concepts of topology which are taken from [2].

## 1.1 Topological Spaces

In the year 1906, Fréchet introduced "metric spaces" that is very fruitful concept in many processes in Analysis. But this was not big enough to describe pointwise convergence in function spaces. In order to fix it a structure named "topological spaces" was introduced by Felix Hausdroff in 1914 (that is called as Hausdroff space now-a-days) and by Kuratowski in 1922. The pointwise convergence can be described in topological spaces. Then in 1947, Mac Lane and Eilenberg introduced a theory known as Category Theory. This theory puts processes on equal footing with things (here by "things" we mean "objects" in the category and by "processes" we mean "morphisms" between the objects).

**Definition 1.1.1.** *Let $Z \neq \emptyset$ and $\tau \subseteq \underline{P}Z$ is called topology on set $Z$, if $\tau$ has the following three axioms:*

1. *$Z \in \tau$ and $\emptyset \in \tau$;*
2. *Intersection of any finite subcollection of $\tau$ is in $\tau$ i.e, $U_1, U_2, ...U_n \in \tau \implies \bigcap_{j=1}^{n} U_j \in \tau$;*
3. *Union of the elements of any subcollection of of $\tau$ is in $\tau$ i.e for any index set $I$, $\forall\ U_j \in \tau$ and $\forall\ j \in I$, $\bigcup_{j \in I} U_j \in \tau$.*

*And the pair $(Z, \tau)$ is called top. space.*

**Example 1.1.1.** *Let $Z$ be non-empty set and $\tau = \underline{P}Z$ is a topology on $Z$, called discrete topology. And, if $\tau = \{Z, \emptyset\}$, then $\tau$ is called indiscrete topology on $Z$.*

**Example 1.1.2.** *Let $Z = \{e, f\}$ and $\tau = \{\emptyset, \{e\}, \{e, f\}\}$ be a topology on $Z$, called Sierpinski topology.*

**Example 1.1.3.** *Let $Z = \mathbb{R}$, then the topology $\tau = \{U \subset \mathbb{R} \mid \exists \ r > 0 \ni (a - r, a + r) \subset U\}$ on $\mathbb{R}$ is called standard topology.*

**Example 1.1.4.** *Let $Z = \mathbb{R}$, then the topology $\tau = \{U \subset \mathbb{R} \mid U^c$ is finite $\} \cup \{\emptyset\}$ is called co-finite topology.*

**Definition 1.1.2.** *Let $(Z, \sigma)$ be a top. space and $U \subseteq Z$, then*

1. *If $U \in \tau$ implies $U$ is an open set.*

2. *If $U^c \in \tau$ implies $U$ is closed set.*

**Definition 1.1.3.** *Consider $(Z, \sigma)$ be a top. space and $U \subseteq Z$, then the smallest closed set that contains $U$ is called closure of set $U$ i.e*

$$\overline{U} = \bigcap\{F \subset Z\colon F \text{ is closed and } F \supset U\}.$$

**Example 1.1.5.** *Let $Z = \{1, 2, 3, 4, 5\}$ and $U \subseteq Z = \{2, 3\}$ and $\tau = \{\emptyset, Z, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ be topology on $Z$.*

*Open sets: $\emptyset, Z, \{1\}, \{2, 3\}, \{1, 2, 3\}$*

*Closed sets: $Z, \emptyset, \{2, 3, 4, 5\}, \{1, 4, 5\}, \{4, 5\}$*

*Closed sets containing $U$: $Z, \{2, 3, 4, 5\}$*

*Closure of $U$= $Z \cap \{2, 3, 4, 5\} = \{2, 3, 4, 5\}$.*

**Definition 1.1.4.** *A basis for a topology on set $Z$ is a collection $\mathfrak{A}$ of subsets of $Z$ satisfying the following properties:*

1. *The elements of $\mathfrak{A}$ covers $Z$ i.e $\bigcup_{A \in \mathfrak{A}} A = Z$;*

2. *For any $a \in A_1 \cap A_2$, there exists basis element $A_3$ which contains "a" such that $A_3 \subset A_1 \cap A_2$.*

**Example 1.1.6.** *The basis for usual topology is $\{(a, b) : a, b \in \mathbb{R}\}$.*

**Example 1.1.7.** *Singleton set is the basis for discrete topology.*

**Definition 1.1.5.** *Consider $(Z, \sigma)$ and $(Y, \rho)$ be two top. spaces. A mapping h: $(Z, \sigma) \to (Y, \rho)$ is continuous at a point $a \in Z$ if for all $V \in \rho$ with $h(a) \in U$, $\exists \;\; a \in U \in \sigma$ s.t. $h(U) \subset V$.*

**Example 1.1.8.** *Let $\tau_{st}$ and $\tau^*$ be the usual and upper ray topology on $\mathbb{R}$, respectively. Then the mapping $h : (Z, \tau_{st}) \to (Y, \tau^*)$ defined by $h(a) = a^2$ is continuous.*

**Example 1.1.9.** *Let $\tau_{st}$ and $\tau^l$ be the usual and upper limit topology on $\mathbb{R}$, respectively. Then the mapping $h : (Z, \tau_{st}) \to (Y, \tau^l)$ defined by $h(a) = a^2$ is not continuous.*

**Definition 1.1.6.** *Let $(Z_j, \sigma_j)_{j\in I}$ be the collection of top. spaces and $h_j : Z \to Z_j$ be the mappings, then*

$$\sigma_* = \bigcup_{j\in I} \bigcap_{i=1}^{n} \{h_{ji}^{-1}(U_{ji}) : U_{ji} \in \sigma_j\} \; .$$

*is initial topology on set $Z$.*

**Definition 1.1.7.** *Let $(Z, \sigma)$ be top. space and $A \subseteq Z$, then the subspace topology on $A$ is defined as*

$$\sigma_A = \{A \cap U \mid U \in \sigma\}.$$

**Example 1.1.10.** *Let $A = \{1, 2, 3\}$ and $Z = \mathbb{R}$, then the subspace topology on $A$ is given by $\sigma_A = \{\emptyset, A, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$.*

**Example 1.1.11.** *Suppose $Z = \mathbb{R}$ and $A = \mathbb{N} \subseteq \mathbb{R}$, then subspace topology on $A$ i.e on set of natural numbers is discrete topology.*

**Definition 1.1.8.** *Let $(Z_1, \sigma_1)$ and $(Z_2, \sigma_2)$ be top. spaces, the topology $\sigma$ on $Z_1 \times Z_2$ is called product topology which has the base*

$$\mathfrak{B} = \{U \times V : U \in \sigma_1, V \in \sigma_2\}.$$

**Example 1.1.12.** *Let $Z_1 = \{1, 2\}$ and $\sigma = \{\emptyset, \{1\}, \{1, 2\}\}$. Then the product topology on $Z_1 \times Z_1$ is $\{\emptyset, \{(1, 1)\}, \{(1, 1), (1, 2)\}, \{(1, 1), (2, 1)\}, \{(1, 1), (1, 2), (2, 1), Z_1 \times Z_1\}$.*

**Example 1.1.13.** *Let $Z_1 = Z_2 = \mathbb{R}$ and $(Z_1, \tau_{st})$ and $(Z_2, \tau_{st})$ be top. spaces, where $\tau_{st}$ denotes the standard topology on $\mathbb{R}$. Then the product topology $\sigma$ on $Z_1 \times Z_2$ has the base*

$$\mathfrak{B} = \{(1, 2) \times (3, 4) : 1, 2, 3, 4 \in \mathbb{R}\}.$$

**Example 1.1.14.** *Let $Z_1 = Z_2 = \mathbb{R}$ and $(Z_1, \tau_{st})$ and $(Z_2, \tau^l)$ be top. spaces, where $\tau_{st}$ and $\tau^l$ denotes the standard and upper limit topology, respectively on $\mathbb{R}$. Then the product topology $\rho$ on $Z_1 \times Z_2$ has the base*

$$\mathfrak{B} = \{(1,2) \times (3,4] : 1,2,3,4 \in \mathbb{R}\}.$$

**Definition 1.1.9.** *Let $Z$ be a set and $(Y_j, \tau_j)_{j \in I}$ be top. spaces and $h_j : Y_j \to Z$ be the mappings, then*

$$\sigma_{\mathcal{F}} = \{U \subset Z;\ \forall\ j \in I, h_j^{-1}(U) \in \tau_j\}$$

*is the final topology on $Z$.*

**Example 1.1.15.** *Let $Z = \{1,2,3,4\}$ , $Z_1 = \{a,b,c,d\}$, $Z_2 = \{e,f,g,h\}$ and $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, Z_1\}$ and $\tau_2 = \{\emptyset, \{e\}, \{f\}, \{h\}, \{e,f\}, \{e,h\}, \{f,h\}, \{e,f,h\}, Z_2\}$ be topologies on $Z_1$ and $Z_2$ respectively. The functions $f_1 : Z_1 \to Z$ is defined as*

$$f_1(a) \longrightarrow 3,$$

$$f_1(b) \longrightarrow 2,$$

$$f_1(c) \longrightarrow 3,$$

$$f_1(d) \longrightarrow 4.$$

*and $f_2 : Z_2 \to Z$ is defined as*

$$f_2(e) \longrightarrow 2,$$

$$f_2(f) \longrightarrow 1,$$

$$f_2(g) \longrightarrow 4,$$

$$f_2(h) \longrightarrow 3.$$

*Then $\tau_{\mathcal{F}} = \{\emptyset, \{1\}, \{3\}, \{1,3\}, Z\}$ is the final topology on $Z$.*

**Definition 1.1.16.** *Let $(Z, \tau)$ be a top. space, $R$ be an equivalence relation and define a mapping $q : (Z, \tau) \longrightarrow (Z/R, \tau_{\mathcal{F}})$ by*

$$q(z) = [z],\ \forall\ z \in Z.$$

*Then, the induced topology $\tau_{\mathcal{F}} = \{U \subset Z/R : q^{-1}(U) \in \tau\}$ is called quotient topology on $Z/R$.*

**Example 1.1.17.** *Consider* $\mathbb{Z}_4 = \{[0],[1],[2],[3]\}$ *and* $q : (\mathbb{Z}, \tau_A) \longrightarrow (\mathbb{Z}/mod\ 4\ , \tau_{\mathcal{F}})$ *defined by* $Z \longmapsto q(Z) = [x]$*, where* $\tau_A = \underline{P}\mathbb{Z}$ *is the subspace topology. Then* $\tau_{\mathcal{F}} = \{U \subset \mathbb{Z}_4 : q^{-1}(U) \in \underline{P}\mathbb{Z}\} = \underline{P}\mathbb{Z}_4$ *is the quotient topology.*

## 1.2 Separation Axioms

In this section, we study separation axioms in topology.

**Definition 1.2.1.** *Let* $(Z,\sigma)$ *be top. space on set* $Z$ *and for* $p \in Z$ *if* $\forall\ a \in Z$ *with* $a \neq p$, $\exists$ $U, V \in \sigma$ *with* $a \in U$, $p \notin U$ *or* $p \in V$, $a \notin V$*. Then* $(Z,\sigma)$ *is called local* $\overline{T}_0$ *or* $\overline{T}_0$ *at* $p$*.*

**Example 1.2.1.** *Let* $Z = \{1,2,3\}$ *and* $\sigma = \{\emptyset, \{1,2,3\}, \{1\}, \{3\}, \{1,3\}, \{1,2\}\}$*. Then* $(Z,\sigma)$ *is local* $T_0$*.*

**Definition 1.2.2.** *Let* $(Z,\sigma)$ *be top. space on set* $Z$ *if* $\forall\ a, b \in Z$ *with* $a \neq b$, $\exists\ U, V \in \sigma$ *with* $a \in U$, $b \notin U$ *or* $b \in V$, $a \notin V$*. Then* $(Z,\sigma)$ *is called* $T_0$ *space.*

**Example 1.2.2.** *Let* $Z = \{1,2\}$ *and* $\sigma = \{\emptyset, \{1\}, \{1,2\}\}$*. Then* $(Z,\sigma)$ *is* $T_0$ *space.*

**Example 1.2.3.** *Let* $Z = \mathbb{R}$ *and* $\sigma = \sigma_{st}$ *i.e standard topology on* $Z$*. Then* $(Z,\sigma_{st})$ *is* $T_0$ *space.*

**Example 1.2.4.** *Let* $Z = \mathbb{R}$ *and* $\sigma = \sigma^l$ *i.e upper limit topology on* $Z$*. Then* $(Z,\sigma^l)$ *is* $T_0$ *space.*

**Theorem 1.2.1.** *Let* $(Z,\sigma)$ *be top. space.*

$$(Z,\sigma) \text{ is } T_0 \Longleftrightarrow (Z,\sigma) \text{ is } T_0 \text{ for all } p \in Z.$$

**Example 1.2.5.** *let* $Z = \{1,2,3\}$ *and* $\sigma = \{\emptyset, Z, \{1\}, \{2,3\}\}$*. Then* $(Z,\sigma)$ *is local* $\overline{T}_0$ *(as it is* $\overline{T}_0$ *at at* 1*), but not* $T_0$ *( as it is not local* $T_0$ *for all* $p \in Z$*).*

**Theorem 1.2.2.** *Let* $(Z,\sigma)$ *be top. space and for* $e, f \in Z$

$(Z,\sigma)$ *is* $T_0 \Longleftrightarrow \overline{\{e\}} \neq \overline{\{f\}}$*, where* $\overline{\{e\}}$ *and* $\overline{\{f\}}$ *are closure of* $\{e\}$ *and* $\{f\}$ *with respect to topology* $\sigma$*.*

**Theorem 1.2.3.**

1. *Subspace of* $T_0$ *is again* $T_0$*.*
2. *Infinite product of* $T_0$ *is again* $T_0$ *i.e let* $\{(P_i,\sigma_i) : i \in I\}$ *be top. spaces and* $P = \prod_{i \in I}$ *and* $\sigma^*$ *be topology on* $P$ *, we have*

$$\{(P_i,\sigma_i) : i \in I\} \text{ is } T_0 \Longleftrightarrow (P,\sigma^*) \text{ is } T_0.$$

**Definition 1.2.3.** *Let $(Z, \rho)$ be top. space on set $Z$ and for fixed point $p \in Z$, if $\forall \ a \in Z$ with $a \neq p$, $\exists \ U, V \in \rho$ s.t. $a \in U$, $p \notin U$ and $p \in V$, $a \notin V$. Then $(Z, \rho)$ is called local $T_1$ or $T_1$ at $p$.*

**Example 1.2.6.** *Let $Z = 1, 2, 3$ and $\rho = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Then $(Z, \rho)$ is $T_1$ at 1.*

**Definition 1.2.4.** *Let $(Z, \rho)$ be top. space on set $Z$, if $\forall \ a, b \in Z$ with $a \neq b$, $\exists \ U, V \in \rho$ s.t. $a \in U$, $b \notin U$ and $b \in V$, $a \notin V$. Then the topology $(Z, \rho)$ is called $T_1$ space.*

**Example 1.2.7.** *The co-finte top. space $(Z, \rho_{cof})$, where $\rho_{cof} = \{U \subset \mathbb{R} \mid U^c \text{ is finite } \} \cup \{\emptyset\}$ is a $T_1$ space.*

**Theorem 1.2.4.** *Let $(Z, \sigma)$ be top. space.*

$$(Z, \sigma) \text{ is } T_1 \Longleftrightarrow (Z, \sigma) \text{ is } T_1 \text{ for all } p \in Z.$$

**Theorem 1.2.5.** *Let $(Z, \sigma)$ be top. space and for all $z \in Z$*

*$(Z, \sigma)$ is $T_1 \Longleftrightarrow \overline{\{z\}} = \{z\}$, where $\overline{\{z\}}$ is closure of $\{z\}$ with respect to topology $\sigma$.*

**Theorem 1.2.6.**

1. *Subspace of $T_1$ is again $T_1$.*
2. *Infinite product of $T_1$ is again $T_1$ i.e let $\{(P_i, \rho_i) : i \in I\}$ be top. spaces and $P = \prod_{i \in I}$ and $\rho^*$ be topology on $P$ , we have*

$$\{(P_i, \rho_i) : i \in I\} \text{ is } T_1 \Longleftrightarrow (P, \rho^*) \text{ is } T_1.$$

## 1.3 Category Theory

In 1947, Mac Lane and Eilenberg introduced a theory known as Category Theory. This theory puts processes on equal footing with things (here by "things" we mean "objects" in the category and by "processes" we mean "morphisms" between the objects).

**Definition 1.3.1.** *(cf. [29]) A category $\mathcal{C}$ is a quadruple $\mathcal{C} = (Obj, hom, \circ, id)$ which contains:*

1. *a class of objects denoted by $Obj\mathcal{C}$.*
2. *a set of homomorphisms $Hom_{\mathcal{C}}(C_1, C_2)$ for every pair of objects $C_1, C_2 \in Obj\mathcal{C}$.*

3. *a function called composition;*

$$Hom_{\mathcal{C}}(C_1, C_2) \times Hom_{\mathcal{C}}(C_2, C_3) \longrightarrow Hom_{\mathcal{C}}(C_1, C_3)$$
$$(g_{12}, g_{23}) \longmapsto g_{12} \circ g_{23}.$$

*for each object* $C_1, C_2, C_3 \in Obj\mathcal{C}$ *such that*

(i) *For each* $C_1 \in Obj\,\boldsymbol{C}$*, there is an identity map* $\mathbb{1}_{C_1} \in Hom_{\mathcal{C}}(C_1, C_1)$ *such that for all* $g_{12} \in Hom_{\mathcal{C}}(C_1, C_2)$ *and all* $g_{21} \in Hom_{\mathcal{C}}(C_2, C_1)$*, we have*

$$g_{12} \circ \mathbb{1}_{C_1} = g_{12} \quad \text{and} \quad \mathbb{1}_{C_1} \circ g_{21} = g_{21}.$$

(ii) *For objects* $C_1, C_2, C_3, C_4 \in Obj\mathcal{C}$ *and homomorphisms* $g_{12} \in Hom_{\mathcal{C}}(C_1, C_2)$, $g_{23} \in Hom_{\mathcal{C}}(C_2, C_3)$ *and* $g_{34} \in Hom_{\mathcal{C}}(C_3, C_4)$*, we have*

$$g_{12} \circ (g_{23} \circ g_{34}) = (g_{12} \circ g_{23}) \circ g_{34}.$$

**Examples 1.3.1.** 1. *Category of Rings: We denote it as* ***Ring****, where the objects are rings with identity and the morphisms are ring homomorphisms that preserves identity.*

2. *Category of Abelian Groups: We denote it as* ***Abgrp****, where the objects are abelian groups and the morphism set is the set of group homomorphisms which are closed under composition.*

3. *Category of Preordered Spaces: We denote it as* ***Prord****, where the objects are preordered spaces and the set of morphisms consists of order preserving maps.*

**Definition 1.3.2.** *(cf. [29]) Suppose* $\boldsymbol{H} \subseteq \boldsymbol{C}$ *and* $\boldsymbol{H}$ *is called subcategory if the following conditions holds.*

1. $Obj\,\boldsymbol{H} \subseteq Obj\,\boldsymbol{C}$,

2. $Hom_{\boldsymbol{H}}(H_1, H_2) \subseteq Hom_{\boldsymbol{C}}(C_1, C_2)$,

3. *For all* $H_1 \in Obj\,\boldsymbol{H}$*, the identity map* $\mathbb{1}_{H_1}$ *is same as identity* $\mathbb{1}_{H_1}$ *in* $\boldsymbol{C}$,

4. *Composition law should be preserved.*

**Definition 1.3.3.** *(cf. [29]) A subcategory* $\boldsymbol{H}$ *of* $\boldsymbol{C}$ *is called full subcategory if* $Hom_{\boldsymbol{H}}(H_1, H_2) = Hom_{\boldsymbol{C}}(H_1, H_2)$ *for every pair of* $H_1, H_2 \in Obj\,\boldsymbol{H}$.

**Example 1.3.2.** $\boldsymbol{H} =$ ***Semi-Grp*** *is subcategory of* $\boldsymbol{C} =$ ***Mon****. Also it is not full subcategory.*

**Example 1.3.3.** *$\boldsymbol{H} =$ **Haus** is full subcategory of $\boldsymbol{C} =$ **Top**.*

**Example 1.3.4.** *$\boldsymbol{H} =$ **Grp** is full subcategory of $\boldsymbol{C} =$ **Mon**.*

**Definition 1.3.4.** *(cf. [29]) Let $\mathcal{E}$ and $\mathcal{F}$ be the two categories. $\mathfrak{U} : \mathcal{E} \longrightarrow \mathcal{F}$ is called a functor if*

*(i)* $\forall\ E_1 \in Obj(\mathcal{E}) \Rightarrow \mathfrak{U}(E_1) \in Obj(\mathcal{F})$

*(ii)* $g : E_1 \longrightarrow E_2 \in hom(\mathcal{E}) \Rightarrow \mathfrak{U}(g) : \mathfrak{U}(E_1) \longrightarrow \mathfrak{U}(E_2) \in hom(\mathcal{F})$

*(iii) $\mathfrak{U}$ maintains identity morphism; i.e,*

$$\mathfrak{U}(\mathbb{1}_{E_1}) = \mathbb{1}_{\mathfrak{U}(E_1)}, \quad for\ all\ E_1 \in Obj(\mathcal{E})$$

*(iv) $\mathfrak{U}$ maintains composition; i.e, If $E_1 \xrightarrow{e} E_2 \xrightarrow{f} E \in hom(\mathcal{E})$ then,*

$$\mathfrak{U}(f \circ e) = \mathfrak{U}(f) \circ \mathfrak{U}(e).$$

**Example 1.3.5.** *An operator $\mathfrak{U}$ : **Top** $\longrightarrow$ **Set** given by $\mathfrak{U}(Z, \sigma) = Z$ for some set $Z$ and $\mathfrak{U}(g) = g$ for a continuous map $g : (Z, \sigma) \longrightarrow (Y, \tau)$. Then $\mathfrak{U}$ is a functor.*

**Definition 1.3.5.** *(cf. [29]) Let $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$ be a functor, if for all $E_1, E_2 \in Obj\boldsymbol{E}$ and $g \in Hom(\mathfrak{U}(E_1), \mathfrak{U}(E_2))$ then $\exists\ h : E_1 \longrightarrow E_2$ such that $\mathfrak{U}(h) = g$ then $\mathfrak{U}$ is called full functor.*

**Example 1.3.6.** *An operator $\mathfrak{U}$ : **Set** $\longrightarrow$ **Top** is full functor.*

**Example 1.3.7.** *An operator $\mathfrak{U}$ : **Top** $\longrightarrow$ **Set** is not full functor.*

**Definition 1.3.6.** *(cf. [29]) For a functor $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$, if for all $E_1, E_2 \in Obj\boldsymbol{E}$ and $e, f \in Hom(E_1, E_2)$, we have $\mathfrak{U}(e) = \mathfrak{U}(f) \implies e = f$. Then $\mathfrak{U}$ is called faithful functor.*

**Example 1.3.8.** *An operator $\mathfrak{U}$ : **Top** $\longrightarrow$ **Set** is faithful functor.*

**Definition 1.3.7.** *(cf. [29]) Let $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$ be a functor, and for given $E_1 \in Obj\boldsymbol{E}$ and $g \in Hom(E_1, E_1)$, if $\mathfrak{U}(g) = 1_{E_1} = 1_{\mathfrak{U}(E_1)}$ and $g$ is isomorphism implies $g$ is identity, then $\mathfrak{U}$ is called amnestic functor.*

**Example 1.3.9.** *An operator $\mathfrak{U}$ : **Top** $\longrightarrow$ **Set** is amnestic functor.*

**Example 1.3.10.** *An operator $\mathfrak{U}$ : **Met** $\longrightarrow$ **Top** is not amnestic functor.*

**Definition 1.3.8.** *(cf. [29]) Let $\mathfrak{U} : \mathcal{E} \longrightarrow \mathcal{F}$ be a functor, if $\mathfrak{U}$ is amnestic and faithful then it is called concrete functor.*

**Example 1.3.11.** *An operator $\mathfrak{U} : \boldsymbol{Grp} \longrightarrow \boldsymbol{Set}$ is concrete functor.*

**Example 1.3.12.** *An operator $\mathfrak{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ is concrete functor.*

**Definition 1.3.9.** *(cf. [1]) Consider $\mathcal{E}$ and $\mathcal{F}$ be categories and $\mathcal{H} : \mathcal{E} \to \mathcal{F}$ and $\mathcal{K} : \mathcal{E} \to \mathcal{F}$ be functors. A family of morphisms $\mathfrak{N} : \mathcal{H} \to \mathcal{K}$ is called natural transformation if it satisfies*

1. *$\forall\ a \in Obj(\mathcal{E})$, $\mathfrak{N}$ picks a morphism $\mathfrak{N}_a : \mathcal{H}(a) \to \mathcal{K}(a)$ in $\mathcal{F}$, where $\mathfrak{N}_a$ is component of $\mathfrak{N}$ at $a$.*

2. *for all morphism $g : a \to b$ of $\mathcal{E}$ the following diagram commutes*

$$\begin{array}{ccc} H(a) & \xrightarrow{H(g)} & H(b) \\ \downarrow{\scriptstyle \mathfrak{N}_a} & & \downarrow{\scriptstyle \mathfrak{N}_b} \\ K(a) & \xrightarrow{K(g)} & K(b) \end{array}$$

*i.e $K(g) \circ \mathfrak{N}_a = \mathfrak{N}_b \circ H(g)$.*

**Definition 1.3.10.** *(cf. [1]) Let $\mathcal{E}$ and $\mathcal{F}$ be categories and $\mathcal{H} : \mathcal{E} \to \mathcal{F}$ and $\mathcal{K} : \mathcal{F} \to \mathcal{E}$ be functors. We called $\mathcal{H}$ as left adjoint of $\mathcal{K}$ and $\mathcal{K}$ as right adjoint of $\mathcal{H}$ together with natural transformation $i : id_{\mathcal{E}} \to \mathcal{K} \circ \mathcal{H}$ and $\varepsilon : \mathcal{H} \circ \mathcal{K} \to id_{\mathcal{F}}$ such that below diagrams commutes*

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H} \circ \mathcal{K} \circ \mathcal{H} \\ & \overset{id}{\searrow} & \downarrow{\scriptstyle \varepsilon \cdot \mathcal{H}} \\ & & \mathcal{H} \end{array}$$

*and*

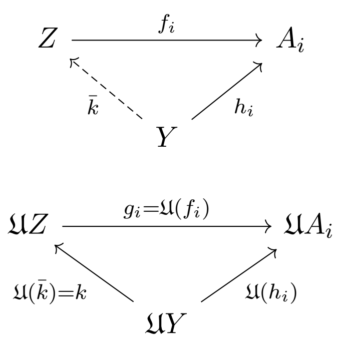
$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{K} \circ \mathcal{H} \circ \mathcal{K} \\ & \overset{id}{\searrow} & \downarrow{\scriptstyle \mathcal{K} \cdot \varepsilon} \\ & & \mathcal{H} \end{array}$$

**Definition 1.3.11.** *A top. functor is said to be indiscrete (respectively discrete) if it has a right (respectively left) adjoint.*

## 1.4 Categorical Topology

In the year 1971, Horst Herrlich [21] presented a novel sub-branch of mathematics termed as "Categorical Topology". It is the field of Mathematics where general topology and category theory overlap. The purpose of introducing this remarkable field was to implement categorical concepts and findings to topological settings, and also to elaborate not only the original topological phenomenon but also similar phenomenon through out topology as well as in other fields.

**Definition 1.4.1** (Initial and Final Lifts)**.** *1. Let $\mathfrak{U} : \boldsymbol{E} \longrightarrow \boldsymbol{F}$ be a functor between two categories. For a $\mathfrak{U}$-source, i.e. family of maps $\mathfrak{U}Z \xrightarrow{g_i} \mathfrak{U}A_i$ in $\boldsymbol{F}$ there is a family $A \xrightarrow{f_i} A_i$ in $\boldsymbol{E}$ such that $\mathfrak{U}(f_i) = g_i$ and if $\mathfrak{U}(h_i) = kg_i$, for every source $Y \xrightarrow{h_i} A_i$ in $\boldsymbol{E}$ along the same domain as in $f_i$. Then there exists a lift $Y \xrightarrow{\bar{k}} A$ of $\mathfrak{U}Y \xrightarrow{k} \mathfrak{U}A$ that is, $\mathfrak{U}(\bar{k}) = k$. In other words, if there exists a morphism $k$ in the codomain then we say that for all $\boldsymbol{F}$-morphism $k$, there is a $\boldsymbol{E}$-morphism $\bar{k}$ (in the domain) so that the diagrams commute.*




*2. Final lift is the dual of initial lift.*

**Definition 1.4.2.** *(cf. [1, 20, 39] Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two categories. A functor $\mathfrak{U} : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is called topological functor or $\boldsymbol{C}$ is topological category over $\boldsymbol{D}$ if the following conditions hold:*

1. *$\mathfrak{U}$ is concrete.*

2. *$\forall D_1 \in \boldsymbol{D}, F^{-1}(D_1) = \{C_1 \in Obj(\boldsymbol{C}) | F(C_1) = D_1\}$ is a set, i.e, $\mathfrak{U}$ contains small fibers.*

3. *Every $\mathcal{U}$-sink has a unique final lift or every $\mathcal{U}$-source has a unique initial lift*

**Example 1.4.1.** *Let $\boldsymbol{C}$=**Set** and $\boldsymbol{D}$=**Top**, then $\mathfrak{U} : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is a topological functor.*

**Definition 1.4.3.** *(cf. [1, 39]) A functor $\mathfrak{U} : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is called a normalized top. functor if constant objects have a unique structure.*

**Example 1.4.2.** *The functor $\mathfrak{U}$ : **Top** $\longrightarrow$ **Set** is normalized functor.*

**Example 1.4.3.** *The functor $\mathfrak{U}$ : **Met** $\longrightarrow$ **Set** is not normalized functor.*

**Definition 1.4.4.** *(cf. [1, 39]) A functor $\mathfrak{U} : \mathcal{E} \longrightarrow \mathcal{F}$ is called geometric functor if the discrete functor preserve finite limits i.e left exact.*

**Example 1.4.4.** *The functor $\mathfrak{U}$ : **Prord** $\longrightarrow$ **Set** is geometric functor, where **Prord** denotes category of preordered spaces and order preserving maps.*

**Definition 1.4.5.** *(cf. [36]) Given a topological functor $\mathfrak{U} : \mathcal{E} \to$ **Set**, and isomorphism-closed full subcategory $\mathcal{H}$ of $\mathcal{C}$, we called $\mathcal{H}$*

*(i) epireflective in $\mathcal{E}$ if and only if $\mathcal{H}$ is closed under the formation of extremal subobjects (i.e., subspaces) and products.*

*(ii) quotient-reflective in $\mathcal{E}$ iff $H \in \mathcal{H}$, $E \in \mathcal{E}$, $\mathfrak{U}(H) = \mathfrak{U}(E)$, and id : $H \to E$ is a $\mathcal{E}$-morphism, then $E \in \mathcal{H}$ (i.e. $\mathcal{H}$ is closed under finer structures and epireflective).*

*(iii) bireflective in $\mathcal{E}$ if and only if $\mathcal{H}$ has the subcategory of all indiscrete objects and is epireflective.*

**Example 1.4.5.** *(i) **$T_0$-TOP** is epireflective in **TOP**.*

*(ii) **$\mathbf{Prord_{0C}}$** is quotient-reflective in **Prord**, where **$\mathbf{Prord_{0C}}$** is a closure operator of **Prord**.*

*(iii) **Dim(Prord)** is bireflective in **Prord**, where **Dim(Prord)** is the category of zero-dimensional preordered space.*

Chapter 2

# RELative Spaces and Ordered RELative Spaces

## 2.1 Basic concepts and notations

For non-empty set $Z$, $\mathcal{R} \subset \underline{P}(Z \times Z)$ is called a relative system for $Z$ and is denoted by **REL**(Z). Moverover, **REL**(Z) can be ordered by setting;

$$\overline{\mathcal{R}} << \mathcal{R} \text{ iff } \forall\, \overline{R} \in \overline{\mathcal{R}},\ \exists\ R \in \mathcal{R} \text{ s.t. } R \subset \overline{R}.$$

Furthermore, we denote **sec** $\mathcal{R} = \{\overline{R} \subset Z \times Z : R \in \mathcal{R}, R \cap \overline{R} \neq \emptyset\}$ and **stack** $\mathcal{R} = \{\overline{R} \subset Z \times Z : \exists\ R \in \mathcal{R}, R \subset \overline{R}\}$.

**Definition 2.1.1.** *(cf. [35]) Let $Z \neq \emptyset$, and $\beta^Z \subset \underline{P}Z$ called boundedness or B-set on $Z$, if $\beta^Z$ holds the following axioms:*

*(i)* $\emptyset \in \beta^Z$,

*(ii)* $B_2 \subset B_1 \in \beta^Z$ *implies* $B_2 \in \beta^Z$,

*(iii)* $a \in Z$ *implies* $\{a\} \in \beta^Z$.

*And for B-sets $\beta^Z$ and $\beta^A$ a function $g : Z \longrightarrow A$ is called bounded if and only if it satisfies;*

$$\{g[\mathcal{B}] : \mathcal{B} \in \beta^Z\} \subset \beta^A.$$

**Example 2.1.2.** *Let $Z = \{1, 2\}$ and $\underline{P}Z = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ be power set on $Z$, then $B_1^Z = \{\emptyset, \{1\}, \{2\}\}$ and $B_2^Z = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ are B-sets on $Z$.*

**Definition 2.1.3.** *(cf. [35]) The triple $(Z, \beta^Z, r)$ is called RELative space (shortly RELspace) if for boundedness $\beta^Z$, a function $r{:}\beta^Z \longrightarrow \underline{P}\mathbf{REL}(\mathbf{Z})$ satisfies the followings:*

*(i)* $B \in \beta^Z$ *and* $\overline{\mathcal{R}} << \mathcal{R} \in r(B)$ *implies* $\overline{\mathcal{R}} \in r(B)$,

*(ii)* $\{\emptyset\} \notin r(B)$ *for* $B \in \beta^Z$,

*(iii)* $\mathcal{R} \in r(\emptyset)$ *iff* $\mathcal{R} = \emptyset$,

*(iv)* $a \in Z$ *implies* $\{\{a\} \times \{a\}\} \in r(\{a\})$.

*The RELspace $(Z, \beta^Z, r)$ is called ordered-RELspace provided that the following axiom hold:*

*(v)* $\emptyset \neq B_1 \subset B \in \beta^Z$ *implies* $r(B_1) \subset r(B)$.

**Example 2.1.4.** *Let $Z = \{1, 2\}$ and $B^Z = \{\emptyset, \{1\}, \{2\}\}$ be the discrete boundendess on $Z$. We define $r{:}\beta^Z \longrightarrow \underline{P}\mathbf{REL}(\mathbf{Z})$ by setting;*

$r(\emptyset) = \{\emptyset\}$,

$r(\{1\}) = \{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(1, 1)\}\}\}$,

$r(\{2\}) = \{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(2, 2)\}\}\}$.

*Then, $(Z, B^Z, r)$ forms a RELspace.*

**Definition 2.1.5.** *(cf. [35]) Let $(Z, \beta^Z, r)$ and $(A, \beta^A, v)$ be two RELspaces and a bounded function $g : Y \longrightarrow A$ is called RELative map (shortly RELmap) iff it satisfies the following condition*

$$B \in \beta^Z \backslash \{\emptyset\} \text{ and } \mathcal{R} \in r(B) \text{ imply } g^Z\mathcal{R} \in v(g[B])$$

*where $g^Z\mathcal{R} = \{(g \times g)[R] : R \in \mathcal{R}\}$ with $(g \times g)[R] = \{(g \times g)(a, c) : (a, c) \in R\} = \{(g(a), g(c)) : (a, c) \in R\}$.*

**Remark 2.1.6.** *Note that,* **REL** *is the category of RELspace and RELmaps, and* ***O-REL*** *is the category of ordered-RELspace, and RELmaps. Also* ***O-REL*** *is a bireflective subcategory of* ***REL*** *[32].*

**Example 2.1.1.** *Let $(Z, T_Z)$ be a preuniform convergence space, then the associated RELspace $(Z, \underline{P}Z, r_{T_Z})$ can be defined as follows:*

$r_{T_Z}(\emptyset) = \{\emptyset\}$ *and for* $B \in \underline{P}Z \backslash \{\emptyset\}$,

$r_{T_Z}(B) = \{\mathcal{R} \in REL(Z) : \exists \ \mathcal{N} \in T_Z, \ \mathcal{N} \subset sec\mathcal{R}\}$.

*Let **PU-REL** be the category, whose objects are* $(Z, \underline{P}Z, r_{T_Z})$ *and morphisms are RELmaps. Note that* ***PUCONV*** $\cong$ ***PU-REL*** *[34], where* ***PUCONV*** *denotes the category of preuniform convergence spaces and uniformly continuous maps as defined in [38].*

**Example 2.1.2.** *Let* $(Z, \beta^Z, t)$ *be a set-convergence space, then the associated RELspace* $(Z, \beta^Z, r_t)$ *can be defined by*

$r_t(\emptyset) = \{\emptyset\}$ *and for* $B \in \beta^Z \backslash \{\emptyset\}$,

$r_t(B) = \{\mathcal{R} \in REL(Z) : \exists \ \mathcal{E} \in FIL(Z)((\mathcal{E}, B) \in t$ *and* $\mathcal{R} \subset sec\,\mathcal{E} \otimes \mathcal{E})\}$, *where* $\mathcal{E} \otimes \mathcal{E} = \{R \subset Y \times Y : \exists E_1, E \in \mathcal{E}$ *such that* $E_1 \times E \subset R\}$ *and FIL(Z) is the collection of all filters defined on* $Z$.

*Let **SET-REL** denotes the category, whose objects are triples* $(Z, \beta^Z, r_t)$ *and morphisms are RELmaps. Note that* ***SETCONV*** $\cong$ ***SET-REL*** *[34], where* ***SETCONV*** *is the category of set-convergence spaces and morphisms are b-continuous maps as defined [44].*

**Example 2.1.3.** *Let* $(Z, \zeta)$ *be prenearness space, then the associated RELspace* $(Z, \underline{P}Z, r_\zeta)$ *can be described as*

$r_\zeta(\emptyset) = \{\emptyset\}$ *and for* $B \in \underline{P}Z \backslash \{\emptyset\}$,

$r_\zeta(B) = \{\mathcal{R} \in REL(Z) : \exists \ \mathcal{Q} \subset \underline{P}Z(\{B\} \cup \mathcal{Q} \in \zeta$ *and* $\mathcal{R} << \mathcal{Q} \times \mathcal{Q})\}$, *where* $\mathcal{Q} \times \mathcal{Q} := \{D \times D : D \in \mathcal{Q}\}$.

*Note that* ***PNEAR*** $\cong$ ***PN-REL*** *[20, 34] , where* ***PNEAR*** *denotes the category of prenearness spaces and nearness preserving maps as defined in [20], and* ***PN-REL*** *is the category of triples* $(Z, \underline{P}Z, r_\zeta)$ *and morphisms are RELmaps.*

**Example 2.1.4.** *For a B-set* $\beta^Z$ *we put* $r_b(\emptyset) := \{\emptyset\}$, *and for* $B \in \beta^Z \backslash \{\emptyset\}$ *we set* $r_b(B) := \{\mathcal{R} \in REL(Z) : \exists \ Z \in B, \mathcal{R} \subset \dot{z} \times \dot{z}\}$, *hence* $(Z, \beta^Z, r_b)$ *defines a RELspace, which is diagonal, meaning that for* $B \in \beta^Z \backslash \{\emptyset\}$ *and* $\mathcal{R} \in s(B)$ *we can find* $Z \in B \ \forall \ R \in \mathcal{R} \ (z, z) \in R$.

*Let* $\Delta$*-**REL** describes the corresponding defined full subcategory of* ***REL****, then* $\Delta$*-**REL*** $\cong$ ***BOUND****.*

**Remark 2.1.1.** *In this context note that* ***BORN****, the full subcategory of* ***BOUND****, whose objects are the bornological spaces, then also has evidently a corresponding counterpart in* ***REL****.*

**Example 2.1.5.** *Let* $(Z, \beta^Z, q)$ *be b-topological space, then the associated RELspace* $(Z, \beta^Z, r_q)$ *is defined by*

$r_q(\emptyset) := \{\emptyset\}$ *and for* $B \in \beta^Z \backslash \{\emptyset\}$, $r_q(B) := \{\mathcal{R} \in REL(Z) : \exists\ \omega \subset \beta^Z,\ \exists\ a \in B(\mathcal{R} << \omega \times \omega$ *and* $a \in \cap\{q(E) : E \in \omega\}\}$.

*Note that* ***b-TOP*** $\cong$ ***bTOP-REL*** *[34], where* ***bTOP-REL*** *denotes the full subcategory of* ***REL****, whose objects are triples* $(Z, \beta^Z, r_q)$*, and* ***b-TOP*** *is the category of b-topological spaces and b-continuous maps as defined in [34].*

## 2.2 O-REL as a Normalized and Geometric Topological Category

Note that the "forgetful functor" $\mathcal{U} : \mathcal{E} \to \mathbf{Set}$, where $\mathcal{E} = \mathbf{REL}$ is a topological in the following sense:

**Lemma 2.2.1.** *Let* $(Z_j, \beta^{Z_j}, r_j)$ *be a collection of RELspaces. A source* $(f_i : (Z, \beta^Z_I, r^Z_I) \longrightarrow (Z_j, \beta^{Z_j}, r_j))_{j\in I}$ *is initial in* ***REL*** *iff*

$$\beta^Z_I \ := \ \{B \subset Z : g_j(B) \in \beta^{Z_j},\ \forall j \in I\}$$

*and for all* $B \in \beta^Z_I$,

$$r^Z_I(B) \ := \ \{\mathcal{R} \in REL(Z) : g^{Z_j}\mathcal{R} \in r_j(g_j[B]),\ \forall j \in I\}$$

*Proof.* It is given in [32]. Consequently, since **O-REL** is a isomorphism-closed full subcategory which is bireflective in **REL** it is topological, too. □

**Lemma 2.2.2.** *Let* $(Z_j, \beta^{Z_j}, r_j)$ *be a collection of ordered-RELspaces. A sink* $(f_i : (Z_j, \beta^{Z_j}, r_j) \longrightarrow (Z, \beta^Z_{fin}, r_{fin}))_{j\in I}$ *is final in* ***O-REL*** *iff*

$$\beta^Z_{fin} \ := \ \{B \subset Z :\ \exists\ j \in I,\ \exists\ B_j \in \beta^Z_j \mid\ B \subset g_j(B_j)\} \cup \mathcal{D}^Z,$$

*where* $\mathcal{D}^Z = \{\emptyset\} \cup \{\{a\} : a \in Z\}$*, and for* $B \in \beta^Z_{fin} \backslash \{\emptyset\}$,

$$\begin{aligned} r_{fin}(B) \ &:= \ \{\mathcal{R} \in REL(Z) : \exists\ j \in I,\ \exists\ B_j \in \beta^{Z_j},\ \exists\ \mathcal{R}_j \in r_j[B_j] \mid \mathcal{R} << g^Z_j \mathcal{R}_j\} \cup \\ &\quad \{\mathcal{R} \in REL(Z) : \exists\ a \in B \mid (a, a) \in \cap\{R : R \in \mathcal{R}\}\} \text{ with } r_{fin}(\emptyset) := \{\emptyset\}. \end{aligned}$$

*Proof.* It is easy to observe that $(Z, \beta^Z_{fin}, r_{fin})$ is an ordered-RELspace and $f_i : (Z_j, \beta^{Z_j}, r_j)_{j\in I} \longrightarrow (Z, \beta^Z_{fin}, r_{fin})$ is a RELmap. Suppose that $g : (Z, \beta^Z_{fin}, r_{fin}) \longrightarrow (Z, \beta^Z, r_Z)$ is a mapping. We show that $g$ is a RELmap iff $g \circ f_j$ is a RELmap. Necessity is obvious since the composition of two RELmaps is RELmap again.

Conversely, let $g \circ f_j : (Z_j, \beta^{Z_j}, r_j) \longrightarrow (A, \beta^A, r_A)$ be a RELmap.

Then, first we show that $g$ is a bounded map. Let $B_i \in \beta_j^Z$, it implies that $g(f_j(B_j)) = g \circ f_j(B_j) \in \beta^Z$. For our own convenience, take $f_j(B_j) = B'$, and since $f_j$ is a RELmap, then $B' \in \beta^Z_{fin}$ and consequently, $g$ is bounded.

Now, let $B_j \in \beta^{Z_j} \backslash \{\emptyset\}$ and $\mathcal{R}_i \in r_j(B_j)$. By the Definition 2.1.5, we have $g(f_j(B_j)) = g \circ f_j(B_j) \in r_A(g(f_j(B_j)))$. On the other hand, $f_j$ is a RELmap, it follows that $f_j(\mathcal{R}_j) \in r_{fin}(f_j(B_j))$. Take $f_j(\mathcal{R}_j) = \mathcal{R}'$. It implies $\mathcal{R}' \in r_{fin}(B')$ and subsequently, $g(\mathcal{R}') \in r_A(g(B'))$ which shows $g$ is a RELmap. □

**Lemma 2.2.3.** *Let $Z \neq \emptyset$, and $(Z, \beta^Z, r)$ be an ordered-RELspace.*

*(i) A RELstructure $(\beta^Z, r)$ is discrete iff $(\beta^Z, r) := (\mathcal{D}^Z, r_{dis})$, where $\mathcal{D}^Z = \{\emptyset\} \cup \{\{a\} : a \in Z\}$ and $r_{dis}(\{a\}) = \{\mathcal{R} \in REL(Z) : (a,a) \in \cap\{R : R \in \mathcal{R}\}\} = \{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a)\}\}\}$ with $r_{dis}(\emptyset) := \{\emptyset\}$.*

*(ii) A RELstructure $(\beta^Z, r)$ is indiscrete iff $(\beta^Z, r) := (\underline{P}Z, r_{id})$, where $r_{id}(B) = \{\mathcal{R} \in REL(Z) : \{\emptyset\} \notin \mathcal{R}\}$ if $\beta^Z \neq \emptyset$ with $r_{id}(\emptyset) := \{\emptyset\}$.*

*Proof.* By applying Lemma 2.2.2 we get the desired result. □

**Theorem 2.2.1.** *The topological functor $\mathcal{U} : \mathcal{C} \to$ **Set**, where $\mathcal{C} =$ **O-REL** is normalized.*

*Proof.* Since an unique RELstructure $\beta^Z = \{\emptyset\}$, and $r(\emptyset) = \{\emptyset\}$ exists whenever $Z = \emptyset$ and an unique RELstructure $\beta^Z = \{\emptyset, \{a\}\}$, $r(\emptyset) = \{\emptyset\}$ and $r(\{a\}) = \{\emptyset, \{(a,a)\}\}$ exists whenever $Z = \{a\}$. □

**Theorem 2.2.1.** *The topological functor $\mathcal{U}$ :* ***O-REL*** *$\to$* ***Set*** *is geometric*

*Proof.* Since the regular sub-object of a discrete RELspace is discrete, and finite product of discrete RELstructure is again discrete. □

Chapter 3

# Local $T_0$ and Local $T_1$ Ordered RELative Spaces

In this section, we define notions for $T_0$ and $T_1$ ordered-RELspaces at some point $p$.

## 3.1 Local $T_0$ Ordered-RELspaces

Let $Z$ be any set and $p \in Z$. We define the *wedge product of* $Z$ *at* $p$ as the two disjoint copies of $Z$ at $p$ and denote it as $Z \bigvee_p Z$. For a point $a \in Z \bigvee_p Z$ we write it as $a_1$ if $Z$ belongs to the first component of the wedge product otherwise we write $a_2$ that is in the second component. Moreover, $Z^2$ is the cartesian product of $Z$.

**Definition 3.1.1.** *Let* $(Z, \sigma)$ *be top. space on set* $Z$ *and for* $p \in Z$ *if* $\forall\ a \in Z$, $a \neq p$, $\exists$ $U, V \in \sigma$ *with* $a \in U$, $p \notin U$ *or* $p \in V$, $a \notin V$. *Then* $(Z, \sigma)$ *is called local* $\overline{T}_0$ *or* $\overline{T}_0$ *at* $p$.

**Remark 3.1.1.** *(i) In* ***TOP***, $\overline{T}_0$ *and* $T'_0$ *at* $p$ *are equivalent to the classical* $T_0$ *at* $p$ *i.e., for each* $a \in Z$ *with* $a \neq p$, *there exists a neighbourhood* $N_a$ *of "*$a$*" not containing "*$p$*" or there exists a neighbourhood* $N_p$ *of "*$p$*" not containing "*$a$*" [9].*

*(ii) A topological space* $Z$ *is* $T_0$ *if and only if* $Z$ *is* $T_0$ *at* $p$ *for each* $p \in Z$ *[9].*

*(iii) Let* $\mathfrak{U} : \mathcal{C} \to$ ***Set*** *be a top. functor,* $Z \in Obj(\mathcal{C})$ *and* $p \in \mathfrak{U}(Z)$ *be a retract of* $Z$. *Then, if* $Z$ *is* $\overline{T}_0$ *at* $p$, *then* $Z$ *is* $T_0^{'}$ *at* $p$ *but not conversely in general [5].*

Now considering the categorical counter part of $T_0$'s we have the following definition.

**Definition 3.1.2.** *(cf. [3])*

*(i) A mapping $A_p : Z\bigvee_p Z \longrightarrow Z^2$ is said to be "**principal $p$-axis mapping**" provided that*

$$A_p(a_j) := \begin{cases} (a,p); & j=1, \\ (p,a); & j=2, \end{cases}$$

*(ii) A mapping $\nabla_p : Z\bigvee_p Z \longrightarrow Z$ is said to be "**fold mapping at $p$**" provided that*

$$\nabla_p(a_j) := a, \qquad j=1,2.$$

**Definition 3.1.3.** *(cf. [3]) Assume that $\mathfrak{U} : \mathcal{C} \longrightarrow$ **Set** is a top. functor, $Z \in Obj(\mathcal{C})$ with $\mathfrak{U}Z = A$ and $p \in A$.*

*(i) $Z$ is $\overline{T}_0$ at $p$ provided that the initial lift of the $\mathfrak{U}$-source $\{A\bigvee_p A \xrightarrow{A_p} \mathfrak{U}(Z^2) = A^2$ and $A\bigvee_p A \xrightarrow{\nabla_p} \mathfrak{U}DA = A\}$ is discrete.*

*(ii) $Z$ is $T_0^{'}$ at $p$ provided that the initial lift of the $\mathfrak{U}$-source $\{A\bigvee_p A \xrightarrow{id} \mathfrak{U}(Z\bigvee_p Z) = A\bigvee_p A$ and $A\bigvee_p A \xrightarrow{\nabla_p} \mathfrak{U}DA = A\}$ is discrete, where $Z\bigvee_p Z$ is the wedge product in $\mathcal{C}$, i.e., the final lift of the $\mathfrak{U}$-sink $\{\mathfrak{U}Z = A \xrightarrow{i_1,\ i_2} A\bigvee_p A\}$, where $i_1,\ i_2$ represent the canonical injections.*

**Corollary 3.1.1.** *(cf. [9])*

*Suppose $\mathfrak{U} : \mathcal{C} \longrightarrow$ **Set** is a top. functor and $\mathcal{C} =$ **TOP**, then the following are equivalent:*

1. *$(Z,\tau)$ is local $\overline{T}_0$ or $\overline{T}_0$ at $p$.*
2. *The initial topology induced by $\left\{Z\bigvee_p Z \xrightarrow{A_p} (Z^2,\tau^2) \text{ and } Z\bigvee_p Z \xrightarrow{\nabla_p} (Z,\tau_{dis})\right\}$ is discrete.*
3. *The initial topology induced by $\left\{Z\bigvee_p Z \xrightarrow{id} (Z\bigvee_p Z,\tau_{\mathcal{F}}) \text{ and } Z\bigvee_p Z \xrightarrow{\nabla_p} (Z,\tau_{dis})\right\}$ is discrete,where $\tau_{\mathcal{F}}$ is the final topology induced by $i_1,i_2 : Z \to Z\bigvee_p Z$ and $i_1,i_2$ are canonical injections.*

**Theorem 3.1.1.** *Suppose $(Z,\beta^Z,r)$ be ordered-RELspace and $p \in Z$. Then, $(Z,\beta^Z,r)$ is $\overline{T}_0$ at $p$ iff for each $a \in Z$ with $a \neq p$, the following holds:*

*(i) $\{a,p\} \notin \beta^Z$.*

*(ii) $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{p\})$.*

*(iii)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{p\})$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{a\})$.

*(iv)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(p,p)\}\}\} \notin r(\{a\})$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,p),(a,a)\}\}\} \notin r(\{p\})$.

*Proof.* Let $(Z, \beta^Z, r)$ be $\overline{T}_0$ at $p$, we show that the conditions $(i)$ to $(iv)$ are holding.

(i) Suppose that $\{a,p\} \in \beta^Z$ for all $a \in Z$ with $a \neq p$. Let $U = \{a_1, a_2\} \in Z\bigvee_p Z$, then since $\nabla_p(U) = \nabla_p(\{a_1,a_2\}) = (\{\nabla_p a_1, \nabla_p a_2\}) = \{a\} \in \mathcal{D}^Z$ and for $j = 1,2$, $\pi_j A_p(U) = \{a,p\} \in \beta^Z$, (by the assumption), where $\pi_j : Z^2 \longrightarrow Z$ for j=1,2 are projection maps. By the Definitions 2.1.1,3.2.3 and Lemma 2.2.1, a contradiction, it follows $\{a,p\} \notin \beta^Z$.

(ii) Assume that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \in r(\{p\})$. Particularly , let $\mathcal{R}_1 = \{\{(a_1,a_2)\}\} \in REL(Z\bigvee_p Z)$ and $B = \{a_1\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, then $\nabla_p \mathcal{R}_1 = \nabla_p\{\{(a_1,a_2)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$. By the assumption, $\pi_1 A_p(\mathcal{R}_1) = \{\{(\pi_1 A_p a_1, \pi_1 A_p a_2)\}\} = \{\{(a,p)\}\} \in r(\{a\})$ and $\pi_2 A_p(\mathcal{R}_1) = \{\{(\pi_2 A_p a_1, \pi_2 A_p a_2)\}\} = \{\{(p,a)\}\} \in r(\{p\})$. Since $(Z, \beta^Z, r)$ is $\overline{T}_0$ at $p$, it follows that $\mathcal{R}_1 \in \overline{r}_{dis}(\{a_1\})$, where $\overline{r}_{dis}$ is the discrete structure on $Z\bigvee_p Z$.

Similarly, for $B = \{a_2\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, we get $\mathcal{R}_1 \in \overline{r}_{dis}(\{a_2\})$, a contradiction to the discreteness of $\overline{r}_{dis}(B)$.

Thus, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{p\})$.

(iii) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \in r(\{p\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \in r(\{a\})$. In particular, let $\mathcal{R}_2 = \{\{(a_2,a_1)\}\} \in REL(Z\bigvee_p Z)$ and $B = \{a_1\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, then $\nabla_p \mathcal{R}_1 = \nabla_p\{\{(a_2,a_1)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$, and by the assumption $\pi_1 A_p(\mathcal{R}_2) = \{\{(p,a)\}\} \in r(\{a\})$ and $\pi_2 A_p(\mathcal{R}_2) = \{\{(a,p)\}\} \in r(\{p\})$. Since $(Z, \beta^Z, r)$ is $\overline{T}_0$ at $p$, we get that $\mathcal{R}_2 \in \overline{r}_{dis}(\{a_1\})$, where $\overline{r}_{dis}$ is the discrete structure on $Z\bigvee_p Z$.

Similarly, for $B = \{a_2\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, we get $\mathcal{R}_2 \in \overline{r}_{dis}(\{a_2\})$, a contradiction.

Therefore, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{p\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{a\})$.

(iv) Assume that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(p,p)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,p),(a,a)\}\}\} \in r(\{p\})$. Let $\mathcal{R}_3 = \{\{(a_1,a_1),(a_2,a_2)\}\} \in REL(Z\bigvee_p Z)$ and

$B = \{a_1\} \in \mathcal{D}^{Z\bigvee_p Z} \backslash \{\emptyset\}$, then $\nabla_p \mathcal{R}_3 = \nabla_p\{\{(a_1,a_1),(a_2,a_2)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$, $\pi_1 A_p(\mathcal{R}_3) = \{\{(a,a),(p,p)\}\}\} \in r(\{a\})$, $\pi_2 A_p(\mathcal{R}_3) = \{\{(p,p),(a,a)\}\}\} \in r(\{p\})$, (by the assumption). Since $(Z, \beta^Z, r)$ is $\overline{T}_0$ at $p$, it follows that $\mathcal{R}_3 \in \overline{r}_{dis}(\{a_1\})$, where $\overline{r}_{dis}$ is the discrete structure on $Z\bigvee_p Z$.

Similarly, for $B = \{a_2\} \in \mathcal{D}^{Z\bigvee_p Z} \backslash \{\emptyset\}$, we get $\mathcal{R}_3 \in \overline{r}_{dis}(\{a_2\})$, a contradiction.

Hence, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(p,p)\}\}\} \notin r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,p),(a,a)\}\}\} \notin r(\{p\})$.

Conversely, suppose $(i)$ to $(iv)$ are holding.

Let $(\beta^{Z\bigvee_p Z}, \overline{r})$ be the initial structure induced by $A_p : Z\bigvee_p Z \to (Z^2, \beta^{Z^2}, r^2)$ and $\nabla_p : Z\bigvee_p Z \to (Z, \mathcal{D}^Z, r_{dis})$, where $(B^{Z^2}, r^2)$ is the product RELstructure on $Z^2$ and $(\mathcal{D}^Z, r_{dis})$ the discrete RELstructure on Z.

We show that $(\beta^{Z\bigvee_p Z}, \overline{r})$ is the discrete REL structure on $Z\bigvee_p Z$, i.e., we show that $\beta^{Z\bigvee_p Z} = \mathcal{D}^{Z\bigvee_p Z} = \{\{\emptyset\} \cup \{a_j\};\ j = 1,2 \text{ and } a_j \in Z\bigvee_p Z\}$ and for $B \in \mathcal{D}^{Z\bigvee_p Z}$, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_j,a_j)\}\}; j = 1,2\}$.

Let $U \in \beta^{Z\bigvee_p Z}$ and $\nabla_p U \in \mathcal{D}^Z$, if $\nabla_p U = \emptyset$ then $U = \emptyset$. Suppose $\nabla_p U \neq \emptyset$. Then, we have $\nabla_p U = \{a\}$ for some $a \in Z$ and if $a = p$ then $U = \{p\}$, let $a \neq p$, then it further implies $U = \{a_1\}$ or $U = \{a_2\}$ or $U = \{a_1, a_2\}$. By the assumption, $\pi_j A_p U = \pi_j A_p\{a_1,a_2\} = \{a,p\} \notin \beta^Z$ (for j=1,2). Thus, $U = \{a_1\}$ and $U = \{a_2\}$, subsequently, $\beta^{Z\bigvee_p Z} = \mathcal{D}^{Z\bigvee_p Z}$.

Now, let $B \in \mathcal{D}^{Z\bigvee_p Z} \backslash \{\emptyset\}$ implies $B = \{a_1\}$ and $B = \{a_2\}$, and by Lemma 2.2.1, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \pi_j A_p(\overline{\mathcal{R}}) \in r(\pi_j A_p(B)) \text{ and } \nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\nabla_p B)),\ where\ j = 1,2\}$

Suppose $B = \{a_1\}$, then

$\overline{r}(\{a_1\}) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \pi_j A_p(\overline{\mathcal{R}}) \in r(\pi_j A_p(\{a_1\}))\ and\ \nabla_p \overline{\mathcal{R}} \in r_{dis}(\nabla_p\{a_1\}),\ where\ j = 1,2\}$, it follows that $\overline{r}(\{a_1\}) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \pi_1 A_p(\overline{\mathcal{R}}) \in r(\{a\})\ and\ \pi_2 A_p(\overline{\mathcal{R}}) \in r(\{p\})\ and\ \nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\{a\})\}$.

Since $\nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\{a\}) = \{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a)\}\}\}$, we have following possibilities of $\mathcal{R}$:

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1,a_1)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2,a_2)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1,a_2)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2,a_1)\}\}\}$, and

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$.

**Case(i)** : Suppose $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1)\}\}\}$. It follows that for all $\overline{R} \in \overline{\mathcal{R}}$ such that $\{(a_1, a_1)\} \subseteq \overline{R}$, and $\pi_1 A_p\{(a_1, a_1)\} \subseteq \pi_1 A_p \overline{R}$, $\pi_1 A_p \overline{\mathcal{R}} << \pi_1 A_p\{\{(a_1, a_1)\}\} = \{\{(\pi_1 A_p a_1, \pi_2 A_p a_1)\}\} = \{\{(a, a)\}\}$. By the Definition 2.1.3, $\pi_1 A_p \overline{\mathcal{R}} << \{\{(a, a)\}\} \in r(\{a\})$. Similarly, $\pi_2 A_p \overline{\mathcal{R}} << \{\{(p, p)\}\} \in r(\{p\})$. Therefore, $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \mathcal{R} << \{\{(a_1, a_1)\}\}\}$ holds.

**Case(ii)** : $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_2)\}\}\}$ holds. The proof is similar to **Case(i)**.

**Case(iii)** : Let $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_2)\}\}\}$. It follows that, for all $\overline{R} \in \overline{\mathcal{R}}$ such that $\{(a_1, a_2)\} \subseteq \overline{R}$ , and $\pi_1 A_p\{(a_1, a_2)\} \subseteq \pi_1 A_p \overline{R}$, $\pi_1 A_p \overline{\mathcal{R}} << \pi_1 A_p\{\{(a_1, a_2)\}\} = \{\{(a, p)\}\}$. By the assumption, we get $\pi_1 A_p \overline{\mathcal{R}} << \{\{(a, p)\}\} \notin r(\{a\})$. Similarly, $\pi_2 A_p \overline{\mathcal{R}} << \{\{(p, a)\}\} \notin r(\{p\})$. Thus, $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_2)\}\}\}$ cannot be possible.

**Case(iv)** : Similar to Case (iii), we conclude $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_1)\}\}\}$ is not possible

**Case(v)** : If $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$. It follows that, for all $\overline{R} \in \overline{\mathcal{R}}$ such that $\{(a_1, a_1), (a_2, a_2)\} \subseteq \overline{R}$, and $\pi_1 A_p\{(a_1, a_1), (a_2, a_2)\} \subseteq \pi_1 A_p \overline{R}$, for all $\overline{R} \in \overline{\mathcal{R}}$ implying $\pi_1 A_p \overline{\mathcal{R}} << \pi_1 A_p\{\{(a_1, a_1), (a_2, a_2)\}\} = \{\{(a, a), (p, p)\}\}$. By the assumption, $\pi_1 A_p \overline{\mathcal{R}} << \{\{(a, a), (p, p)\}\} \notin r(\{a\})$. Similarly, $\pi_2 A_p \overline{\mathcal{R}} << \{\{(p, p), (a, a)\}\} \notin r(\{p\})$. Hence, $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$ is not possible.

Similarly, if $B = \{a_2\}$, only **Case(i)** and **Case(ii)** are holding. By Lemma 2.2.3, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_j, a_j)\}\}; j = 1, 2\}$ is discrete.

Therefore, by the Definition 3.2.3, $(Z, \beta^Z, r)$ is $\overline{T}_0$ at $p$. □

**Theorem 3.1.2.** *All ordered-RELspaces are $T_0'$ at $p$.*

*Proof.* Let $(Z, \beta^Z, r)$ be ordered-RELspace and $p \in Z$. By the Definition 3.2.3. we show that for each $U \in \beta^{Z\bigvee_p Z}, U \subset i_k(V)$, (where k = 1,2) for some $V \in \beta^Z$ and $\nabla_p U \in \mathcal{D}^Z$. $\nabla_p U = \emptyset$ implying $U = \emptyset$. Suppose $\nabla_p U \neq \emptyset$, it implies that $\nabla_p U = \{a\}$ for some $a \in Z$. If $a = p$, then $\nabla_p U = \{p\}$ implying $U = \{p\}$.

Suppose $a \neq p$, it impliess that $U = \{a_1\}$ or $\{a_2\}$ or $\{a_1, a_2\}$. If $U = \{a_1, a_2\}$, then $\{a_1, a_2\} \subset i_1(V)$ for some $V \in \beta^Z$ which shows that $a_2$ should be in the first component of the wedge product $Z \bigvee_p Z$, a contradiction. In similar manner, $\{a_1, a_2\} \not\subset i_2(V)$ for some $V \in \beta^Z$. Hence, $U \neq \{a_1, a_2\}$. Thus, we must have $U = \{a_j\}$ for $j = 1, 2$ only and consequently, $\beta^{Z\bigvee_p Z} =$

$\mathcal{D}^{Z \bigvee_p Z}$, the discrete RELstructure on $Z \bigvee_p Z$.

Now, for $B \in \mathcal{D}^{Z \bigvee_p Z} \backslash \{\emptyset\}$, by Lemma 2.2.1, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << i_1(s)$ for some $s \in r(B)$, $\overline{\mathcal{R}} << i_2(s)$ for some $s \in r(B)$ and $\nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\nabla_p B)\}$. Since $\nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\{B\}) = \{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a_j, a_j)\}\}$ *where* $j = 1, 2\}$ we have the following possibilities of $\mathcal{R}$:

$\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_2)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_2)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_1)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$.

In particular, for $\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_2)\}\}\}$. It follows that, for all $\overline{R} \in \overline{\mathcal{R}}$ such that $\{(a_1, a_2)\} \subset \overline{R}$, and (for k=1,2), $i_k\{(a_1, a_2)\} \subset i_K \overline{R}$ implying $i_k \overline{\mathcal{R}} << i_k\{\{(a_1, a_2)\}\}$. It follows that $a_2$ (respectively $a_1$) in the first (respectively second) component of the wedge product $Z \bigvee_p Z$, a contradiction. Similarly, for $\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_1)\}\}\}$ and $\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$ we get a contradiction.

Therefore, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \mathcal{R} << \{\{(a_j, a_j)\}\}; j = 1, 2\}$. Consequently, by the Definition 3.2.3(i) and Lemma 2.2.1, $(Z, \beta^Z, r)$ is $T_0'$ at $p$.

□

## 3.2 Local $T_1$ Ordered-RELspaces

**Definition 3.2.1.** *Let $(Z, \rho)$ be top. space on set $Z$ and for fixed point $p \in Z$, if for all $a \in Z$ with $a \neq p$, $\exists\ U, V \in \rho$ s.t. $a \in U$, $p \notin U$ and $p \in V$, $a \notin V$. Then $(Z, \rho)$ is called local $\overline{T}_1$ or $\overline{T}_1$ at $p$.*

**Remark 3.2.1.** *(i) In **TOP**, $T_1$ at $p$ is equivalent to the classical $T_1$ at $p$ i.e., for each $a \in Z$ with $a \neq p$, there exists a neighbourhood $N_a$ of "a" not containing "p" or there exists a neighbourhood $N_p$ of "p" not containing "a" [9].*

*(ii) A topological space $Z$ is $T_1$ if and only if $Z$ is $T_1$ at $p$ for each $p \in Z$ [9].*

*(iii) Consider $\mathfrak{U} : \mathcal{C} \to$ **Set** be a top. functor, $Z \in Obj(\mathcal{C})$ and $p \in \mathfrak{U}(Z)$ be a retract of $Z$. Then, if $Z$ is $T_1$ at $p$, then $Z$ is $T_0'$ at $p$ but not conversely in general [5].*

Now considering the categorical counter part of $T_1$ we have the following definition.

**Definition 3.2.2.** *(cf. [3])*

*(i) A mapping $S_p : Z\bigvee_p Z \longrightarrow Z^2$ is called "**skewed $p$-axis mapping**" provided that*

$$S_p(a_j) := \begin{cases} (a,a); & j=1, \\ (p,a); & j=2, \end{cases}$$

*(ii) A mapping $\nabla_p : Z\bigvee_p Z \longrightarrow Z$ is called "**fold mapping at $p$**" provided that*

$$\nabla_p(a_j) := a, \qquad j = 1,2.$$

**Definition 3.2.3.** *(cf. [3])Assume that $\mathfrak{U} : \mathcal{C} \longrightarrow$ **Set** is a topological functor, $Z \in Obj(\mathcal{C})$ with $\mathfrak{U}Z = A$ and $p \in A$.Then, $Z$ is $T_1$ at $p$ provided that the initial lift of the $\mathfrak{U}$-source $\{A\bigvee_p A \xrightarrow{S_p} \mathfrak{U}(Z^2) = A^2$ and $A\bigvee_p A \xrightarrow{\nabla_p} \mathfrak{U}DA = A\}$ is discrete.*

**Theorem 3.2.1.** *Suppose $(Z, \beta^Z, r)$ be an ordered-RELspace and $p \in Z$.*

*$(Z, \beta^Z, r)$ is $T_1$ at $p$ iff for any $a \in Z$ with $a \neq p$, the following holds:*

*(i) $\{a,p\} \notin \beta^Z$,*

*(ii) $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{p\})$.*

*(iii) $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{p\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{a\})$.*

*(iv) $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(p,p)\}\}\} \notin r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,p),(a,a)\}\}\} \notin r(\{p\})$.*

*Proof.* Let $(Z, \beta^Z, r)$ is $T_1$ at $p$, we show that conditions $(i)$ to $(iv)$ hold.

(i) Suppose that $\{a,p\} \in \beta^Z$ for all $a \in Z$ with $a \neq p$. Let $U = \{a_1, a_2\} \in Z\bigvee_p Z$, then since $\nabla_p(U) = \nabla_p(\{a_1,a_2\}) = (\{\nabla_p a_1, \nabla_p a_2\}) = \{a\} \in \mathcal{D}^Z$ and $\pi_1 S_p(U) = \{a,p\} \in \beta^Z$, (by the assumption), where $\pi_1 : Z^2 \longrightarrow Z$ is projection map. By the Definitions 2.1.1,3.2.3 and Lemma 2.2.1, a contradiction, it follows $\{a,p\} \notin \beta^Z$.

(ii) Assume that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \in r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \in r(\{p\})$. Particularly , let $\mathcal{R}_1 = \{\{(a_1,a_2)\}\} \in REL(Z\bigvee_p Z)$ and $B = \{a_1\} \in$

$\mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, then $\nabla_p\mathcal{R}_1 = \nabla_p\{\{(a_1,a_2)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$. By the assumption, $\pi_1 S_p(\mathcal{R}_1) = \{\{(\pi_1 S_p a_1, \pi_1 S_p a_2)\}\} = \{\{(a,p)\}\} \in r(\{a\})$ . Since $(Z,\beta^Z,r)$ is $T_1$ at $p$, it follows that $\mathcal{R}_1 \in \overline{r}_{dis}(\{a_1\})$, where $\overline{r}_{dis}$ is the discrete structure on $Z\bigvee_p Z$.

Similarly, for $B = \{a_2\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, we get $\mathcal{R}_1 \in \overline{r}_{dis}(\{a_2\})$, a contradiction to discreteness of $\overline{r}_{dis}(B)$.

Thus, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{p\})$.

(iii) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \in r(\{p\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \in r(\{a\})$. In particular, let $\mathcal{R}_2 = \{\{(a_2,a_1)\}\} \in REL(Z\bigvee_p Z)$ and $B = \{a_1\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, then $\nabla_p\mathcal{R}_2 = \nabla_p\{\{(a_2,a_1)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$, and by the assumption $\pi_1 S_p(\mathcal{R}_2) = \{\{(p,a)\}\} \in r(\{a\})$. Since $(Z,\beta^Z,r)$ is $T_0$ at $p$, we get that $\mathcal{R}_2 \in \overline{r}_{dis}(\{a_1\})$, where $\overline{r}_{dis}$ is the discrete structure on $Z\bigvee_p Z$.

Similarly, for $B = \{a_2\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, we get $\mathcal{R}_2 \in \overline{r}_{dis}(\{a_2\})$, a contradiction.

Therefore, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,p)\}\}\} \notin r(\{p\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,a)\}\}\} \notin r(\{a\})$.

(iv) Assume that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(p,p)\}\}\} \in r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,p),(a,a)\}\}\} \in r(\{p\})$. Let $\mathcal{R}_3 = \{\{(a_1,a_1),(a_2,a_2)\}\} \in REL(Z\bigvee_p Z)$ and $B = \{a_1\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, then $\nabla_p\mathcal{R}_3 = \nabla_p\{\{(a_1,a_1),(a_2,a_2)\}\} = \{\{(a,a)\}\} \in r_{dis}(\{a\})$, $\pi_1 S_p(\mathcal{R}_3) = \{\{(a,a),(p,p)\}\} \in r(\{a\})$, (by the assumption). Since $(Z,\beta^Z,r)$ is $T_0$ at $p$, it follows that $\mathcal{R}_3 \in \overline{r}_{dis}(\{a_1\})$, where $\overline{r}_{dis}$ is the discrete structure on $Z\bigvee_p Z$.

Similarly, for $B = \{a_2\} \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$, we get $\mathcal{R}_3 \in \overline{r}_{dis}(\{a_2\})$, a contradiction.

Hence, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(p,p)\}\}\} \notin r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(p,p),(a,a)\}\}\} \notin r(\{p\})$.

Conversely, suppose $(i)$ to $(iv)$ holds.

Let $(\beta^{Z\bigvee_p Z},\overline{r})$ be initial structure induced by $S_p : Z\bigvee_p Z \to (Z^2,\beta^{Z^2},r^2)$ and $\nabla_p : Z\bigvee_p Z \to (Z,\mathcal{D}^Z,r_{dis})$, where $(\beta^{Z^2},r^2)$ and $(\mathcal{D}^Z,r_{dis})$ are product REL structure and discrete REL structure respectively on $Z^2$ and $Z$.

We show that $(\beta^{Z\bigvee_p Z},\overline{r})$ is a discrete REL structure on $Z\bigvee_p Z$, i.e., we show that $\beta^{Z\bigvee_p Z} = \mathcal{D}^{Z\bigvee_p Z} = \{\{\emptyset\} \cup \{a_j\};\ j = 1,2 \text{ and } a_j \in Z\bigvee_p Z\}$ and for $B \in \mathcal{D}^{Z\bigvee_p Z}$, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_j,a_j)\}\} : j = 1,2\}$.

Let $U \in \beta^{Z\bigvee_p Z}$ and $\nabla_p U \in \mathcal{D}^Z$, if $\nabla_p U = \emptyset$ then $U = \emptyset$. Suppose $\nabla_p U \neq \emptyset$. Then, we have $\nabla_p U = \{a\}$ for some $a \in Z$ and if $a = p$ then $U = \{p\}$, let $a \neq p$, then it further implies $U = \{a_1\}$ or $U = \{a_2\}$ and $U = \{a_1, a_2\}$. By the assumption, $\pi_1 S_p U = \pi_1 S_p\{a_1, a_2\} = \{a, p\} \notin \beta^Z$. Thus, $U = \{a_1\}$ and $U = \{a_2\}$, subsequently, $\beta^{Z\bigvee_p Z} = \mathcal{D}^{Z\bigvee_p Z}$.

Now, let $B \in \mathcal{D}^{Z\bigvee_p Z}\backslash\{\emptyset\}$ implies $B = \{a_1\}$ and $B = \{a_2\}$ and by Lemma 2.2.1, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \pi_j S_p(\overline{\mathcal{R}}) \in r(\pi_j S_p(B))$ and $\nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\nabla_p B)),\ where\ j = 1, 2\}$

Suppose $B = \{a_1\}$, then

$\overline{r}(\{a_1\}) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \pi_j S_p(\overline{\mathcal{R}}) \in r(\pi_j A_p(\{a_1\}))\ and\ \nabla_p\overline{\mathcal{R}} \in r_{dis}(\nabla_p\{a_1\}),\ where\ j = 1, 2\}$, it implies that $\overline{r}(\{a_1\}) = \{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \pi_1 S_p(\overline{\mathcal{R}}) \in r(\{a\})\ and\ \pi_2 S_p(\overline{\mathcal{R}}) \in r(\{a\})\ and\ \nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\{a\})\}$.

Since $\nabla_p(\overline{\mathcal{R}}) \in r_{dis}(\{a\}) = \{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a, a)\}\}\}$, we have following possibilities of $\mathcal{R}$:

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_2)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_2)\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_1)\}\}\}$, and

$\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$.

**Case(i)** : Suppose $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1)\}\}\}$. It follows that, for all $\overline{R} \in \overline{\mathcal{R}}$ such that $\{(a_1, a_1)\} \subseteq \overline{R}$, and $\pi_1 S_p\{(a_1, a_1)\} \subseteq \pi_1 S_p \overline{R}$ implies $\pi_1 S_p\overline{\mathcal{R}} << \pi_1 S_p\{\{(a_1, a_1)\}\} = \{\{(\pi_1 S_p a_1, \pi_2 S_p a_2)\}\} = \{\{(a, a)\}\}$. By the Definition 2.1.3, $\pi_1 S_p\overline{\mathcal{R}} << \{\{(a, a)\}\} \in r(\{a\})$. Similarly, $\pi_2 S_p\overline{\mathcal{R}} << \{\{(p, p)\}\} \in r(\{a\})$. Therefore, $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \mathcal{R} << \{\{(a_1, a_1)\}\}\}$ holds.

**Case(ii)** : $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_2)\}\}\}$ holds. The proof is similar to **Case(i)**.

**Case(iii)** : Let $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_2)\}\}\}$. It follows that, for all $\overline{R} \in \overline{\mathcal{R}}$ such that $\{(a_1, a_2)\} \subseteq \overline{R}$ , and $\pi_1 S_p\{(a_1, a_2)\} \subseteq \pi_1 S_p\overline{R}$ implies $\pi_1 S_p\overline{\mathcal{R}} << \pi_1 S_p\{\{(a_1, a_2)\}\} = \{\{(a, p)\}\}$. By the assumption, we get $\pi_1 S_p\overline{\mathcal{R}} << \{\{(a, p)\}\} \notin r(\{a\})$. Thus, $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_2)\}\}\}$ cannot be possible.

**Case(iv)** : Similar to Case (iii), we conclude $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_2, a_1)\}\}\}$ is not possible.

**Case(v)** : If $\{\overline{\mathcal{R}} \in REL(Z\bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$. It follows that, for all $\overline{R} \in \overline{\mathcal{R}}$

such that $\{(a_1, a_1), (a_2, a_2)\} \subseteq \overline{R}$, and $\pi_1 S_p\{(a_1, a_1), (a_2, a_2)\} \subseteq \pi_1 S_p\overline{R}$, for all $\overline{R} \in \overline{\mathcal{R}}$ implying $\pi_1 S_p \overline{\mathcal{R}} << \pi_1 S_p\{\{(a_1, a_1), (a_2, a_2)\}\} = \{\{(a, a), (p, p)\}\}$. By the assumption, $\pi_1 S_p \overline{\mathcal{R}} << \{\{(a, a), (p, p)\}\} \notin r(\{a\})$. Hence, $\{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_1, a_1), (a_2, a_2)\}\}\}$ is not possible.

Similarly, if $B = \{a_2\}$, only **Case(i)** and **Case(ii)** holds. By Lemma 2.2.3, $\overline{r}(B) = \{\overline{\mathcal{R}} \in REL(Z \bigvee_p Z) : \overline{\mathcal{R}} << \{\{(a_j, a_j)\}\}; j = 1, 2\}$ is discrete.

Therefore, by the Definition 3.2.3, $(Z, \beta^Z, r)$ is $T_1$ at $p$.

□

Chapter 4

# Quotient Reflective Subcategories of the category of Ordered-RELative Spaces

In this chapter, we define generically notions of $T_0$'s and $T_1$ in Ordered-RELspaces and quotient reflective subcategories of the category of ordered-RELative spaces.

## 4.1 Generic $T_0$ Ordered-RELspaces

In 1991, Baran [3] used generic element method of topos theory defined by Johnstone [24], to define generic separation axioms, due to fact that points doesn't make sense in topos theory but generic does. In general, $Z \bigvee_p Z$ at $p$ can be replaced by $Z^2 \bigvee_\Delta Z^2$ at diagonal $\Delta$. Any element $(a, b) \in Z^2 \bigvee_\triangle Z^2$ is written as $(a, b)_1$ (resp. $(a, b)_2$) if it lies in the first component (resp. second component) of $Z^2 \bigvee_\triangle Z^2$. Clearly, $(a, b)_1 = (a, b)_2$, if and only if $a = b$.

**Definition 4.1.1.** *Let $(Z, \sigma)$ be top. space on set $Z$ if $\forall\ a, b \in Z$, $a \neq b$, $\exists\ U, V \in \sigma$ with $a \in U$, $b \notin U$ or $b \in V$, $a \notin V$. Then $(Z, \sigma)$ is called $T_0$ space.*

**Remark 4.1.1.** *(i) In **TOP**, all $T_0$, $\overline{T}_0$ and $T_0{}'$ are equivalent to the classical $T_0$ i.e., for any $a, b \in Z$ with $a \neq b$, there exists a neighbourhood $N_a$ of "$a$" not containing "$b$" or (respectively and) there exists a neighbourhood $N_b$ of "$b$" not containing "$a$" [3, 37, 43].*

*(ii) Every $\overline{T}_0$ space $Z$ is $T_0'$ but not conversely in general. Also, each of $\overline{T}_0$ and $T_0'$ spaces has no relation with $T_0$ space [8].*

The characterization of $T_0$ objects in categorical topology has been an important ideas in topological universe. Therefore, several attempts has been made such as in 1971 Brümmer [14], in 1973 Marny [37], in 1974 Hoffman [22], in 1977 Harvey [18], and in 1991 Baran [3] to discuss various approaches to generalize classical $T_0$ object and examined relationship between different forms of generalized $T_0$ objects. One of the main purpose of generalization is to define Hausdroff objects in arbitrary topological category. In 1991, Baran [3, 8] also generalizes the classical $T_1$ objects of topology to topological category[3, 8]. In abstract topological category [10], $T_1$ objects are used to define $T_3, T_4$, normal objects, regular and completely regular. To characterize separation axioms Baran's approach was to used initial and final lifts, and discreteness.

**Definition 4.1.2.** *(cf. [3])*

*(i) A mapping $A : Z^2 \bigvee_{\triangle} Z^2 \longrightarrow Z^3$ is called "**principal axis mapping**" provided that*

$$A((a,b)_j) := \begin{cases} (a,b,a); & j = 1, \\ (a,a,b); & j = 2, \end{cases}$$

*(ii) A mapping $\nabla : Z^2 \bigvee_{\triangle} Z^2 \longrightarrow Z^2$ is called "**fold mapping**" provided that*

$$\nabla((a,b)_j) := (a,b), \qquad j = 1,2.$$

any element $(a,b) \in Z^2 \bigvee_{\triangle} Z^2$ is written as $(a,b)_1$ (resp. $(a,b)_1$) if it lies in the first (resp. second) component of $Z^2 \bigvee_{\triangle} Z^2$. Clearly, $(a,b)_1 = (a,b)_2$ if and only if $a = b$.

Now we replace the point $p$ by any generic point $\delta$ and define the following separation axioms.

**Definition 4.1.3.** *Let $\mathfrak{U} : \mathcal{E} \longrightarrow$ **Set** be a top. functor, $Z \in Obj(\mathcal{E})$ with $\mathfrak{U}Z = C$.*

*(i) $Z$ is $\overline{T_0}$ provided that the initial lift of the $\mathfrak{U}$-source $\{C^2 \bigvee_{\Delta} C^2 \xrightarrow{A} \mathfrak{U}(Z^3) = C^3$ and $C^2 \bigvee_{\Delta} C^2 \xrightarrow{\nabla} \mathfrak{U}D(C^2) = C^2\}$ is discrete [3].*

*(ii) $Z$ is $T'_0$ provided that the initial lift of the $\mathfrak{U}$-source $\{C^2 \bigvee_{\Delta} C^2 \xrightarrow{id} \mathfrak{U}(Z^2 \bigvee_{\Delta} Z^2)' = C^2 \bigvee_{\Delta} C^2$ and $C^2 \bigvee_{\Delta} C^2 \xrightarrow{\nabla} \mathfrak{U}D(C^2) = C^2\}$ is discrete, where $(Z^2 \bigvee_{\Delta} Z^2)'$ is the final lift of the $\mathfrak{U}$-sink $\{\mathfrak{U}(Z^2) = C^2 \xrightarrow{i_1,\ i_2} C^2 \bigvee_{\Delta} C^2\}$ [3, 6].*

*(iii) $Z$ is called $T_0$ provided that $Z$ doesn't contain an indiscrete subspace with at least two points [37, 43].*

**Corollary 4.1.1.** *(cf. [9])*

*Suppose $\mathfrak{U} : \mathcal{C} \longrightarrow$ **Set** is a top. functor and $\mathcal{C} =$ **TOP**, then the following are equivalent:*

1. *$(Z,\tau)$ is $T_0$*

2. *The initial topology induced by $\left\{Z^2 \bigvee_\Delta Z^2 \xrightarrow{A} (Z^3,\tau^3) \text{ and } Z^2 \bigvee_\Delta Z^2 \xrightarrow{\nabla} (Z^2,\tau^2_{dis})\right\}$ is discrete.*

3. *The initial topology induced by $\left\{Z^2 \bigvee_\Delta Z^2 \xrightarrow{id} (Z^2 \bigvee_\Delta Z^2,\tau_{\mathcal{F}}) \text{ and } Z^2 \bigvee_\Delta Z^2 \xrightarrow{\nabla} (Z^2,\tau^2_{dis})\right\}$ is discrete,where $\tau_{\mathcal{F}}$ is the final topology induced by $i_1, i_2 : Z^2 \to Z^2 \bigvee_\Delta Z^2$ and $i_1, i_2$ are canonical injections.*

**Theorem 4.1.1.** *Let $(Z,\beta^Z,r)$ be an ordered-RELspace.*

*$(Z,\beta^Z,r)$ is $\overline{T}_0$ iff for any $a,b \in Z$ with $a \neq b$, the following holds:*

*(i)* $\{a,b\} \notin \beta^Z$.

*(ii)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{a\}$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$.

*(iii)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$.

*(iv)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \notin r(\{a\})$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b),(a,a)\}\}\} \notin r(\{b\})$

*Proof.* Suppose $(Z,\beta^Z,r)$ is $\overline{T}_0$, we show that conditions $(i)$ to $(iv)$ are holding.

(i) Suppose that $\{a,b\} \in \beta^Z$ for any $a,b \in Z$ , $a \neq b$. Let $U = \{((a,b)_1,(a,b)_2)\} \in Z^2 \bigvee_\triangle Z^2$. And, $\nabla(U) = \nabla\{((a,b)_1,(a,b)_2)\} = \{(a,b)\} \in \mathcal{D}^{Z^2}$ and $\pi_1 A(U) = \{a\} \in \beta^Z$. By the assumption, $\pi_k A(U) = \pi_k A\{((a,b)_1,(a,b)_2)\} = \{a,b\} \in \beta^Z$, where $\pi_k : Z^3 \longrightarrow Z^2$ (for k=2,3) are projection maps. By the definitions 2.1.1, 3.2.3 and Lemma 2.2.1, contradiction, it implies that $\{a,b\} \notin \beta^Z$.

(ii) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \in r(\{b\})$. Let $\mathcal{R}_1 = \{\{((a,b)_1,(a,b)_2)\}\} \in REL(Z^2 \bigvee_\triangle Z^2)$ and $B = \{(a,b)_1\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2}\backslash\{\emptyset\}$, then $\nabla(\mathcal{R}_1) = \nabla\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(\nabla(a,b)_1,\nabla(a,b)_2)\}\} = \{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$. By the Definition 2.1.3, $\pi_1 A\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(\pi_1 A(a,b)_1,\pi_1 A(a,b)_2)\}\} = \{\{(a,a)\}\} \in r(\{a\})$ and by the assumption, $\pi_2 A\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(b,a)\}\} \in r(\{b\})$ and $\pi_3 A\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(a,b)\}\} \in r(\{a\})$. Since $(Z,\beta^Z,r)$ is $\overline{T}_0$, we conclude $\mathcal{R}_1 \in \overline{r}^2_{dis}(\{(a,b)_1\})$ , where $\overline{r}^2_{dis}$ is the discrete structure on $Z^2 \bigvee_\triangle Z^2$.

Similarly, for $B = \{(a,b)_2\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, we get $\mathcal{R}_1 \in \overline{r}^2_{dis}(\{(a,b)_2\})$, a contradiction.

Therefore, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{a\}$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$.

(iii) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \in r(\{b\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \in r(\{a\})$. In particular, let $\mathcal{R}_2 = \{\{((a,b)_2,(a,b)_1)\}\} \in REL(Z^2 \bigvee_\triangle Z^2)$ and $B = \{(a,b)_1\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, then $\nabla(\mathcal{R}_2) = \nabla\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$. By the Definition 2.1.3, $\pi_1 A\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(a,a)\}\} \in r(\{a\})$ and by the assumption, $\pi_2 A\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(\pi_2 A(a,b)_2, \pi_2 A(a,b)_1)\}\} = \{\{(a,b)\}\} \in r(\{b\})$ and $\pi_3 A\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(\pi_3 A(a,b)_2, \pi_3 A(a,b)_1)\}\} = \{\{(b,a)\}\} \in r(\{a\})$. Since $(Z, \beta^Z, r)$ is $\overline{T}_0$ it follows that $\mathcal{R}_2 \in \overline{r}^2_{dis}(\{(a,b)_2\})$, where $\overline{r}^2_{dis}$ is the discrete structure on $Z^2 \bigvee_\triangle Z^2$.

Similarly, for $B = \{(a,b)_2\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, we get $\mathcal{R}_2 \in \overline{r}^2_{dis}(\{(a,b)_1\})$, a contradiction.

Thus, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$.

(iv) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b),(a,a)\}\}\} \in r(\{b\})$. Let $\mathcal{R}_3 = \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} \in REL(Z^2 \bigvee_\triangle Z^2)$, and $B = \{(a,b)_1\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, then $\nabla\mathcal{R}_3 = \nabla\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\}$ $\{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$, and by the Definition 2.1.3, $\pi_1 A\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} = \{\{(a,a)\}\} \in r(\{a\})$. By the assumption, $\pi_2 A\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} = \{\{(b,b),(a,a)\}\} \in r(\{b\})$ and $\pi_3 A\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} = \{\{(a,a),(b,b)\}\} \in r(\{a\})$. Since $(Z, \beta^Z, r)$ is $\overline{T}_0$, we conclude $\mathcal{R}_3 \in \overline{r}^2_{dis}(\{(a,b)_1\})$, where $\overline{r}^2_{dis}$ is the discrete structure on $Z^2 \bigvee_\triangle Z^2$.

Similarly, for $B = \{(a,b)_2\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, we get $\mathcal{R}_3 \in \overline{r}^2_{dis}(\{(a,b)_2\})$, a contradiction to the discreteness of $\overline{r}^2_{dis}(B)$.

Hence, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \notin r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b),(a,a)\}\}\} \notin r(\{b\})$.

Conversely, suppose $(i)$ to $(iv)$ are holding.

Let $(\beta^{Z^2 \bigvee_\triangle Z^2}, \overline{r}^2)$ be the initial structure induced by $A : Z^2 \bigvee_\triangle Z^2 \longrightarrow (Z^3, \beta^{Z^3}, r^3)$ and $\nabla : Z^2 \bigvee_\triangle Z^2 \longrightarrow (Z^2, \mathcal{D}^{Z^2}, r^2_{dis})$, where $(\beta^{Z^3}, r^3)$ is the product RELstructure on $Z^3$ and $(\mathcal{D}^{Z^2}, r^2_{dis})$ the discrete RELstructure on $Z^2$.

We show that $(\beta^{Z^2 \bigvee_\triangle Z^2}, \overline{r}^2)$ is the discrete RELstructure on $Z^2 \bigvee_\triangle Z^2$ i.e, $\beta^{Z^2 \bigvee_\triangle Z^2} = \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} = \{\{\emptyset\} \cup \{(a,b)_j\} : (a,b)_j \in Z^2 \bigvee_\triangle Z^2 \; for \; j = 1,2\}$ and for $B \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}, \overline{r}^2(B) = \{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_j, (a,b)_j, j = 1,2\}\}$.

Let $U \in \beta^{Z^2 \bigvee_\triangle Z^2}$ and $\nabla U \in B^{Z^2}$. If $\nabla U = \emptyset$ then $U = \emptyset$. Suppose $\nabla U \neq \emptyset$, it implies that $\nabla U = \{(a,b)\}$ for some $(a,b) \in Z^2$. If $a = b$ then $U = \{(b,b)\}$. Next, let $a \neq b$, then we have $U = \{(a,b)_1\}$ or $U = \{(a,b)_2\}$ or $U = \{(a,b)_1, (a,b)_2\}$ and $\pi_1 AU = \pi_1 A\{(a,b)_1, (a,b)_2\} = \{\pi_1 A(a,b)_1, \pi_1 A(a,b)_2\} = \{a,a\}$ and by the assumption, we get $\pi_k A\{(a,b)_1, (a,b)_2\} = \{a,b\} \notin \beta^Z$, (for k=2,3). Thus, $U = \{(a,b)_1\}$ or $U = \{(a,b)_2\}$, and subsequently, $\beta^{Z^2 \bigvee_\triangle Z^2} = \mathcal{D}^{Z^2 \bigvee_\triangle Z^2}$.

Now, let $B \in \beta^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$ implies $B = \{(a,b)_1\}$ and $B = \{(a,b)_2\}$, and by Lemma 2.2.1, $\overline{r}^2(B) = \{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \pi_j A\overline{\mathcal{R}} \in r(\pi_j A(B))$ and $\nabla \overline{\mathcal{R}} \in r^2_{dis}(\nabla B))$, where j=1,2,3 }.

Suppose $B = U = \{(a,b)_1\}$, then

since $\nabla \mathcal{R} \in r^2_{dis}(\{a,b\}) = \{\mathcal{R} \in REL(Z^2) : \mathcal{R} << \{\{((a,b),(a,b))\}\}\}$, we have the following possibilities:

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$ ,

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2, (a,b)_2\}\}\}$ ,

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_2\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2, (a,b)_1\}\}\}$, and

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\}\}$.

**Case(i)** : If $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$. It implies, $\forall \; \overline{R} \in \overline{\mathcal{R}}$ $\{(a,b)_1, (a,b)_1\} \subseteq \overline{R}$ and $\pi_1 A\{(a,b)_1, (a,b)_1\} \subseteq \pi_1 A\overline{R}$, $\pi_1 A\overline{\mathcal{R}} << \pi_1 A\{\{(a,b)_1, (a,b)_1\}\} = \{\{\pi_1 A(a,b)_1, \pi_1 A(a,b)_1\}\} = \{\{(a,a)\}\}$, and by the Definition 2.1.3, we get $\pi_1 A\overline{\mathcal{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

In a similar way, $\pi_2 A\overline{\mathcal{R}} << \{\{(b,b)\}\} \in r(\{b\})$ and $\pi_3 A\overline{\mathcal{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

Thus, $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$ holds.

**Case(ii)** : $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2, (a,b)_2\}\}\}$ holds. The proof is similar to **Case(i)**.

**Case(iii)** : $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_2\}\}\}$. It follows that, for all $\overline{R} \in \overline{\mathcal{R}}$ s.t. $\{(a,b)_1, (a,b)_2\} \subseteq \overline{R}$. And $\pi_1 A\{(a,b)_1, (a,b)_2\} \subseteq \pi_1 A\overline{R}$, $\pi_1 A\overline{\mathcal{R}} << \pi_1 A\{\{(a,b)_1, (a,b)_2\}\} = \{\{(a,a)\}\}$, and by the Definition 2.1.3 $\pi_1 A\overline{\mathcal{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

Similarly, by the assumption $\pi_2 A\overline{\mathcal{R}} << \{\{(b,a)\}\} \notin r(\{b\})$ and $\pi_3 A\overline{\mathcal{R}} << \{\{(a,b)\}\} \notin r(\{a\})$.

Therefore, $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1,(a,b)_2\}\}$ is not possible.

**Case**(**iv**) : Similar to Case (iii), we conclude $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2,(a,b)_1\}\}$ is not possible.

**Case**(**v**) : If $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\}$. It implies, $\forall\ \overline{R} \in \overline{\mathcal{R}}$ such that $\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\} \subseteq \overline{R}$ and $\pi_1 A\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\} \subseteq \pi_1 A\overline{\mathcal{R}}$ implies $\pi_1 A\overline{\mathcal{R}} << \pi_1 A\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} = \{\{(a,a)\}\}$. By the Definition 2.1.3, $\pi_1 A\overline{\mathcal{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

Similarly, by the assumption, $\pi_2 A\overline{\mathcal{R}} << \{\{((b,b),(a,a))\}\} \notin r(\{b\})$ and $\pi_3 A\overline{\mathcal{R}} << \{\{((a,a),(b,b))\}\} \notin r(\{b\})$.

Hence, $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\}$ is not possible.

Similarly, if B= $\{(a,b)_2\}$ only **Case**(**i**) and **Case**(**ii**) are holding. By Lemma 2.2.3, $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{((a,b)_j,(a,b)_j); j=1,2\}\}$ is discrete. Therefore, by the Definition 4.1.3 (i), $(Z,\beta^Z,r)$ is $\overline{T}_0$. □

**Theorem 4.1.2.** *Let $(Z,\beta^Z,r)$ be an ordered-RELspace.*

*$(Z,\beta^Z,r)$ is $T_0$ iff for any $a,b \in Z$ with $a \neq b$, each of the following conditions are satisfied:*

*(i)* $\{a,b\} \notin \beta^Z$.

*(ii)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \notin r(\{a\})$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \notin r(\{b\})$.

*(iii)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\} \notin r(\{a\})$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\} \notin r(\{b\})$.

*(iv)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a)\}\} \notin r(\{b\})$ *or* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b)\}\} \notin r(\{a\})$.

*Proof.* Let $(Z,\beta^Z,r)$ be $T_0$, $\{a,b\} \in \beta^Z$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \in r(\{b\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\} \in r(\{b\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a)\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b)\}\} \in r(\{b\})$.

Let $U = \{a,b\}$. Note that, $(U,\beta^U,r_U)$ is the subspace of $(Z,\beta^Z,r)$, where $(\beta^U,r_U)$ is the initial lift of the ordered-RELsystem induced by the inclusion map $i : S \longrightarrow U$ and for any

$S \subset U$, $S \in \beta^U$, whenever $i(S) = S \in \beta^U$ and for any $\mathcal{R} \in REL(U), \mathcal{R} \in r(S)$, whenever $i(\mathcal{R}) = \mathcal{R} \in r(B)$.

By the assumption, $i(U) = U = \{a, b\} \in \beta^U$ and by the Definition 2.1.1, we get $\beta^U = \underline{P}U$.

Now for any $\mathcal{R} \in REL(U)$ let $\mathcal{R} = \{\{(a,a)\}\} \in REL(U)$. By the Definition 2.1.3, $i(\{\{(a,a)\}\}) = \{\{(a,a)\}\} \in r(\{a\})$. By the assumption, $\mathcal{R} = \{\{(a,a)\}\} \in r(\{b\})$ implying that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a)\}\}\} \in r(\{b\})$.

Similarly, for $\mathcal{R} = \{\{(b,b)\}\} \in REL(U)$, it follows that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b)\}\}\} \in r(\{b\})$.

Now, if $\mathcal{R} = \{\{(a,b)\}\} \in REL(U)$ then by the assumption, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \in r(\{b\})$.

And for $\mathcal{R} = \{\{(b,a)\}\} \in REL(U)$ then by the assumption, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \in r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \in r(\{b\})$.

Therefore, $r_U = \{\mathcal{R} \in REL(U) : \{\emptyset\} \in \mathcal{R}\}$.

and $(\beta^U, r_U) = (P(U), r_{id})$, which is a contradiction by Lemma 2.2.3.Thus $(i) - (iv)$ are holding.

Conversely, sassume that $\forall\ a, b \in Z$ with $a \neq b$, conditions $(i) - (iv)$ are holding. We show that the initial structure $(\beta^U, r_U)$ is not an indiscrete ordered-RELstructure on U. Let $U = \{a, b\} \subset X$. By the assumption, $\{a, b\} \notin \beta^Z$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a)\}\}\} \notin r(\{b\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b)\}\}\} \notin r(\{a\})$. Thus $(U, \beta^U, r)$ is not an indiscrete ordered-RELsubspace of $(Z, \beta^Z, r)$. Hence, by the Definition 4.1.3 (iii), $(Z, \beta^Z, r)$ is $T_0$. □

**Theorem 4.1.3.** *All ordered-RELspaces are $T_0'$.*

*Proof.* Let $(Z, \beta^Z, r)$ be an ordered-RELspace. By the Definition 4.1.3, we show that for any $U \in \beta^{Z^2 \vee_\Delta Z^2}$, $U \subset i_k(V)$ (where k=1,2) for some $V \in \beta^{Z^2}$ and $\nabla U \in \mathcal{D}^{Z^2}$. If $\nabla U = \emptyset$ gives $U = \emptyset$. Suppose $\nabla U \neq \emptyset$, hence $\nabla U = \{(a, b)\}$ for some $(a, b) \in Z^2$.

Suppose $a \neq b$, it implies $U = \{(a,b)_1\}$ *or* $\{(a,b)_2\}$ *or* $\{(a,b)_1, (a,b)_2\}$. If $U = \{(a,b)_1, (a,b)_2\}$ then $\{(a,b)_1, (a,b)_2\} \subset i_1(V)$ for some $V \in \beta^{Z^2}$, which shows that $(a,b)_2$ must be in the first comp. of $Z^2 \vee_\Delta Z^2$, a contradiction. In similar way $\{(a,b)_1, (a,b)_2\} \not\subset i_2(V)$, for $V \in \beta^{Z^2}$. Hence $U = \{\{(a,b)_j\}\}$ for j=1,2. Consequently, $\beta^{Z^2 \vee_\Delta Z^2} = \mathcal{D}^{Z^2 \vee_\Delta Z^2}$, the discrete ordered-RELstructure on $Z^2 \vee_\Delta Z^2$.

Now, for $B \in \mathcal{D}^{Z^2 \vee_\Delta Z^2} \backslash \{\emptyset\}$, and by Lemma 2.2.1, $\overline{r}^2(B) = \{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << i_1(s)$ for some $s \in r(B)$, $\overline{\mathcal{R}} << i_2(s)$ for some $s \in r(B)$ and $\nabla(\overline{\mathcal{R}}) \in r^2_{dis}(\nabla B)\}$. But $\nabla(\overline{\mathcal{R}}) \in r^2_{dis}(\nabla B)$ gives the following possibilities:

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1,(a,b)_1\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2,(a,b)_2\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1,(a,b)_2\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2,(a,b)_1\}\}\}$, and

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\}\}$.

In particular, for $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1,(a,b)_2\}\}\}$.Then, it follows, for all $\overline{R} \in \overline{\mathcal{R}}$ $\{(a,b)_1,(a,b)_2\} \subset \overline{R}$, and consequently, $i_k\{(a,b)_1,(a,b)_2\} \subset \overline{R}$ (for k = 1,2). As a result, $(a,b)_2$ (respectively $(a,b)_1$) is in the first component (respectively second component) of the wedge product $Z^2 \vee_\Delta Z^2$ which leads to a contradiction. Similarly, for $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2,(a,b)_1\}\}\}$ and $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\}\}$ we get a contradiction.

Hence, $\overline{r}^2(B) = \{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{((a,b)_j,(a,b)_j)\};\ j = 1,2\ \}\}$. Thus, by Lemma 2.2.1 and Definition 4.1.3 $(Z, \beta^Z, r)$ is $T_0'$.

□

## 4.2 Generic $T_1$ Ordered-RELspaces

**Definition 4.2.1.** *Let $(Z,\rho)$ be top. space on set $Z$, if for all $a,b \in Z$ with $a \neq b$, $\exists\ U,V \in \rho$ s.t. $a \in U$, $b \notin U$ and $b \in V$, $a \notin V$. Then the topology $(Z,\rho)$ is called $T_1$ space.*

**Remark 4.2.1.** *In **TOP**, $T_1$ is equivalent to the classical $T_1$, i.e., for each $a,b \in Z$ with $a \neq b$, there exists a neighbourhood $N_a$ of "a" not containing "b" or (respectively and) there exists a neighbourhood $N_b$ of "b" not containing "a" [3, 37, 43].*

**Definition 4.2.2.** *(cf. [3])*

*(i) A mapping $S : Z^2 \bigvee_\triangle Z^2 \longrightarrow Z^3$ is called "**skewed axis mapping**" provided that*

$$S((a,b)_j) := \begin{cases} (a,b,b); & j = 1, \\ (a,a,b); & j = 2, \end{cases}$$

*(ii) A mapping* $\nabla : Z^2 \bigvee_{\triangle} Z^2 \longrightarrow Z^2$ *is called "**fold mapping**" provided that*

$$\nabla((a,b)_j) := (a,b), \qquad j = 1,2.$$

any element $(a,b) \in Z^2 \bigvee_{\triangle} Z^2$ is written as $(a,b)_1$ (resp. $(a,b)_1$) if it lies in the first component (resp. second component) of $Z^2 \bigvee_{\triangle} Z^2$. Clearly, $(a,b)_1 = (a,b)_2$ if and only if $a = b$.

Now we replace the point $p$ by any generic point $\delta$ and define the following separation axioms.

**Definition 4.2.3.** *Consider* $\mathfrak{U} : \mathcal{E} \longrightarrow$ ***Set*** *be a top. functor,* $Z \in Obj(\mathcal{E})$ *with* $\mathfrak{U}Z = Z$*. Then,* $Z$ *is* $T_1$ *provided that the initial lift of the* $\mathfrak{U}$*-source* $\{Z^2 \bigvee_{\Delta} Z^2 \xrightarrow{S} \mathfrak{U}(Z^3) = Z^3$ *and* $Z^2 \bigvee_{\Delta} Z^2 \xrightarrow{\nabla} \mathfrak{U}D(Z^2) = Z^2\}$ *is discrete [3].*

**Theorem 4.2.1.** *Let* $(Z, \beta^Z, r)$ *be an ordered-RELspace. Then* $(Z, \beta^Z, r)$ *is* $T_1$ *iff for all* $a, b \in Z$ *with* $a \neq b$*, any of the following holds:*

*(i)* $\{a,b\} \notin \beta^Z$.

*(ii)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{a\})$ *and* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$.

*(iii)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$ *and* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$ .

*(iv)* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \notin r(\{a\})$ *and* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b),(a,a)\}\}\} \notin r(\{b\})$.

*Proof.* Suppose $(Z, \beta^Z, r)$ is $T_1$, we show that conditions $(i)$ to $(iv)$ hold.

(i) Suppose that $\{a,b\} \in \beta^Z$ for all $a,b \in Z$, $a \neq b$. Let $U = \{((a,b)_1,(a,b)_2)\} \in Z^2 \bigvee_{\triangle} Z^2$. Note that, $\nabla(U) = \nabla\{((a,b)_1,(a,b)_2)\} = \{(a,b)\} \in \mathcal{D}^{Z^2}$ and $\pi_1 S(U) == \{\pi_1 S(a,b)_1, \pi_1 S(a,b)_2)\} = \{a\} \in \beta^Z$ and $\pi_3 S(U) == \{\pi_3 S(a,b)_1, \pi_3 S(a,b)_2)\} = \{b\} \in \beta^Z$ . By the assumption, $\pi_2 S(U) = \pi_2 S\{((a,b)_1,(a,b)_2)\} = \{a,b\} \in \beta^Z$, where $\pi_w : Z^3 \longrightarrow Z^2$ is projection map. By the Definitions 2.1.1,3.2.3 and Lemma 2.2.1, it leads to a contradiction, it follows that $\{a,b\} \notin \beta^Z$.

(ii) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \in r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \in r(\{b\})$. Let $\mathcal{R}_1 = \{\{((a,b)_1,(a,b)_2)\}\} \in REL(Z^2 \bigvee_{\triangle} Z^2)$ and $B = \{(a,b)_1\} \in \mathcal{D}^{Z^2 \bigvee_{\triangle} Z^2} \backslash \{\emptyset\}$, then $\nabla(\mathcal{R}_1) = \nabla\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(\nabla(a,b)_1, \nabla(a,b)_2)\}\} = \{\{(a,b)\}\} \in$

$r^2_{dis}(\{(a,b)\})$. By the Definition 2.1.3, $\pi_1 S\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(\pi_1 S(a,b)_1, \pi_1 S(a,b)_2)\}\} = \{\{(a,a)\}\} \in r(\{a\})$ and $\pi_3 S\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(\pi_3 S(a,b)_1, \pi_3 S(a,b)_2)\}\} = \{\{(b,b)\}\} \in r(\{b\})$ and by the assumption, $\pi_2 S\{\{((a,b)_1,(a,b)_2)\}\} = \{\{(a,b)\}\} \in r(\{a\})$. Since $(Z,\beta^Z,r)$ is $T_1$, we conclude $\mathcal{R}_1 \in \overline{r}^2_{dis}(\{(a,b)_1\})$ , where $\overline{r}^2_{dis}$ is discrete structure on $Z^2 \bigvee_\triangle Z^2$.

Similarly, for $B = \{(a,b)_2\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, we get $\mathcal{R}_1 \in \overline{r}^2_{dis}(\{(a,b)_2\})$, a contradiction.

Therefore, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{a\}$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{b\})$.

(iii) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \in r(\{b\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \in r(\{a\})$. In particular, let $\mathcal{R}_2 = \{\{((a,b)_2,(a,b)_1)\}\} \in REL(Z^2 \bigvee_\triangle Z^2)$ and $B = \{(a,b)_1\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, then $\nabla(\mathcal{R}_2) = \nabla\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$. By the assumption, $\pi_2 S\{\{((a,b)_2,(a,b)_1)\}\} = \{\{(b,a)\}\} \in r(\{a\})$. Since $(Z,\beta^Z,r)$ is $T_1$ it follows that $\mathcal{R}_2 \in \overline{r}^2_{dis}(\{(a,b)_2\})$, where $\overline{r}^2_{dis}$ is discrete structure on $Z^2 \bigvee_\triangle Z^2$.

Similarly, for $B = \{(a,b)_2\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, we get $\mathcal{R}_2 \in \overline{r}^2_{dis}(\{(a,b)_1\})$, a contradiction.

Thus, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(\{b\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(\{a\})$.

(iv) Suppose that $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \in r(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b),(a,a)\}\}\} \in r(\{b\})$. Let $\mathcal{R}_3 = \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} \in REL(Z^2 \bigvee_\triangle Z^2)$, and $B = \{(a,b)_1\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, then $\nabla\mathcal{R}_3 = \nabla\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\}$ $\{\{(a,b)\}\} \in r^2_{dis}(\{(a,b)\})$. By the assumption, $\pi_2 S\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} = \{\{(b,b),(a,a)\}\} \in r(\{b\})$. Since $(Z,\beta^Z,r)$ is $T_1$, we conclude $\mathcal{R}_3 \in \overline{r}^2_{dis}(\{(a,b)_1\})$, where $\overline{r}^2_{dis}$ is discrete structure on $Z^2 \bigvee_\triangle Z^2$.

Similarly, for $B = \{(a,b)_2\} \in \mathcal{D}^{Z^2 \bigvee_\triangle Z^2} \backslash \{\emptyset\}$, we get $\mathcal{R}_3 \in \overline{r}^2_{dis}(\{(a,b)_2\})$, a contradiction to discreteness of $\overline{r}^2_{dis}(B)$.

Hence, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \notin r(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b),(a,a)\}\}\} \notin r(\{b\})$.

Conversely, suppose $(i)$ to $(iv)$ holds.

Let $(\beta^{Z^2 \bigvee_\triangle Z^2}, \overline{r}^2)$ be initial structure induced by $S : Z^2 \bigvee_\triangle Z^2 \longrightarrow (Z^3, \beta^{Z^3}, r^3)$ and $\nabla : Z^2 \bigvee_\triangle Z^2 \longrightarrow (Z^2, \mathcal{D}^{Z^2}, r^2_{dis})$, where $(\beta^{Z^3}, r^3)$ and $(\mathcal{D}^{Z^2}, r^2_{dis})$ are product REL structure and discrete REL structure respectively on $Z^3$ and $Z^2$.

We show that $(\beta^{Z^2\bigvee_{\triangle} Z^2}, \overline{r}^2)$ is a discrete RELstructure on $Z^2\bigvee_{\triangle} Z^2$ i.e, $\beta^{Z^2\bigvee_{\triangle} Z^2} = \mathcal{D}^{Z^2\bigvee_{\triangle} Z^2} = \{\{\emptyset\} \cup \{(a,b)_j\} : (a,b)_j \in Z^2\bigvee_{\triangle} Z^2 \ for \ j = 1,2\}$ and for $B \in \mathcal{D}^{Z^2\bigvee_{\triangle} Z^2}\backslash\{\emptyset\}, \overline{r}^2(B) = \{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_j, (a,b)_j, j = 1,2\}\}$.

Let $U \in \beta^{Z^2\bigvee_{\triangle} Z^2}$ and $\nabla U \in B^{Z^2}$. If $\nabla U = \emptyset$ then $U = \emptyset$. Suppose $\nabla U \neq \emptyset$ then it implies that $\nabla U = \{(a,b)\}$ for some $(a,b) \in Z^2$. If $a = b$ then $U = \{(b,b)\}$. Next, let $a \neq b$, then we have $U = \{(a,b)_1\}$ or $U = \{(a,b)_2\}$ or $U = \{(a,b)_1, (a,b)_2\}$ and $\pi_1 SU = \pi_1 S\{(a,b)_1, (a,b)_2\} = \{\pi_1 S(a,b)_1, \pi_1 S(a,b)_2\} = \{a,a\}$ and and $\pi_3 SU = \pi_3 S\{(a,b)_1, (a,b)_2\} = \{\pi_3 S(a,b)_1, \pi_3 S(a,b)_2\} = \{b,b\}$ and by the assumption, we get $\pi_2 S\{(a,b)_1, (a,b)_2\} = \{a,b\} \notin \beta^Z$. Thus, $U = \{(a,b)_1\}$ or $U = \{(a,b)_2\}$, and subsequently, $\beta^{Z^2\bigvee_{\triangle} Z^2} = \mathcal{D}^{Z^2\bigvee_{\triangle} Z^2}$.

Now, let $B \in \beta^{Z^2\bigvee_{\triangle} Z^2}\backslash\{\emptyset\}$ implying $B = \{(a,b)_1\}$ and $B = \{(a,b)_2\}$ and by Lemma 2.2.1, $\overline{r}^2(B) = \{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \pi_j S\overline{\mathcal{R}} \in r(\pi_j S(B))$ and $\nabla\overline{\mathcal{R}} \in r^2_{dis}(\nabla B))$, where j=1,2,3}.

Suppose $B = U = \{(a,b)_1\}$, then

Since $\nabla\mathcal{R} \in r^2_{dis}(\{a,b\}) = \{\mathcal{R} \in REL(Z^2) : \mathcal{R} << \{\{((a,b),(a,b))\}\}\}$, we have that following possibilities:

$\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$ ,

$\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2, (a,b)_2\}\}\}$ ,

$\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_2\}\}\}$,

$\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2, (a,b)_1\}\}\}$, and

$\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1, (a,b)_1), ((a,b)_2, (a,b)_2)\}\}\}$.

**Case(i)** : If $\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$. It implies, $\forall \ \overline{R} \in \ \overline{\mathcal{R}}$ s.t. $\{(a,b)_1, (a,b)_1\} \subseteq \overline{R}$ and $\pi_1 S\{(a,b)_1, (a,b)_1\} \subseteq \pi_1 S\overline{R}$ implies $\pi_1 S\overline{\mathcal{R}} << \pi_1 S\{\{(a,b)_1, (a,b)_1\}\} = \{\{\pi_1 S(a,b)_1, \pi_1 S(a,b)_1\}\} = \{\{(a,a)\}\}$ and by the Definition 2.1.3, we get $\pi_1 S\overline{\mathcal{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

In a similar way, $\pi_2 S\overline{\mathcal{R}} << \{\{(b,b)\}\} \in r(\{b\})$ and $\pi_3 S\overline{\mathcal{R}} << \{\{(b,b)\}\} \in r(\{b\})$.

Thus, $\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_1\}\}\}$ holds.

**Case(ii)** : $\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2, (a,b)_2\}\}\}$ holds. The proof is similar to **Case(i)**.

**Case(iii)** : $\{\overline{\mathcal{R}} \in REL(Z^2\bigvee_{\triangle} Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1, (a,b)_2\}\}\}$. It implies, for all $\overline{R} \in \overline{\mathcal{R}}$ such that $\{(a,b)_1, (a,b)_2\} \subseteq \overline{R}$. And $\pi_1 S\{(a,b)_1, (a,b)_2\} \subseteq \pi_1 S\overline{R}$ implies $\pi_1 S\overline{\mathcal{R}} << \pi_1 S\{\{(a,b)_1, (a,b)_2\}\} = \{\{(a,a)\}\}$ and by the Definition 2.1.3 $\pi_1 S\overline{\mathcal{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

Similarly, by the assumption $\pi_2 S\overline{\mathcal{R}} << \{\{(b,a)\}\} \notin r(\{b\})$.

Therefore, $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_1,(a,b)_2\}\}\}$ is not possible.

**Case(iv)** : Similar to Case (iii), we conclude

$\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(a,b)_2,(a,b)_1\}\}\}$ is not possible.

**Case(v)** : If $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\}\}$. It implies that, $\forall\, \overline{R} \in \overline{\mathcal{R}}$ such that $\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\} \subseteq \overline{R}$ and $\pi_1 S\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\} \subseteq \pi_1 S\overline{\mathcal{R}}$ which further implies,

$\pi_1 S\overline{\mathcal{R}} << \pi_1 S\{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2)\}\} = \{\{(a,a)\}\}$. By the Definition 2.1.3, $\pi_1 S\overline{\mathcal{R}} << \{\{(a,a)\}\} \in r(\{a\})$.

Similarly, by the assumption, $\pi_2 S\overline{\mathcal{R}} << \{\{((b,b),(a,a))\}\} \notin r(\{b\})$.

Hence, $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{(((a,b)_1,(a,b)_1),((a,b)_2,(a,b)_2))\}\}\}$ is not possible.

Similarly, if B$= \{(a,b)_2\}$ only **Case(i)** and **Case(ii)** holds. By Lemma 2.2.3, $\{\overline{\mathcal{R}} \in REL(Z^2 \bigvee_\triangle Z^2) : \overline{\mathcal{R}} << \{\{((a,b)_j,(a,b)_j), j=1,2\}\}\}$ is discrete. Therefore, by the Definition 4.1.3 (i), $(Z,\beta^Z,r)$ is $\overline{T}_0$.

□

**Remark 4.2.2.** *Let $Z$ be an ordered-RELspace.*

*(i) By the theorems 3.1.1 and 4.1.1, $Z$ is $\overline{T}_0$ iff $Z$ is $\overline{T}_0$ at $p$, for each $p \in Z$.*

*(ii) By the theorems 3.2.1 and 4.2.1, $Z$ is $T_1$ iff $Z$ is $T_1$ at $p$, for each $p \in Z$.*

*(iii) By the theorems 3.1.2 and 4.1.3, $Z$ is $T_0'$ iff $Z$ is $T_0'$ at $p$, for each $p \in Z$.*

*(iv) By the theorems 4.1.1-4.1.3, $T_1 \implies \overline{T}_0 \implies T_0 \implies T_0'$ but the converse does not hold in general.*

**Corollary 4.2.1.** *Let $(Z,\underline{P}Z,r)$ be in **PU-REL**. Then, the followings are equivalent.*

*(i) $(Z,\underline{P}Z,r)$ is $\overline{T_0}$.*

*(ii) $(Z,\underline{P}Z,r)$ is $\overline{T_0}$**PUCONV**, where $\overline{T_0}$**PUCONV** denotes the category of $\overline{T_0}$ pre-uniform convergence spaces and uniformly continuous maps.*

*(iii) For each $a,b \in Z$ with $a \neq b$, and for all $B \in \underline{P}Z$, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(B)$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(B)$, and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \notin r(B)$.*

*Proof.* By applying Example 2.1.1 , Theorem 4.1.1, and Theorem 3.1.10 of [28]. □

**Corollary 4.2.2.** *Let $(Z, \underline{P}Z, r)$ be in **PU-REL**. Then, the following are equivalent.*

*(i)* $(Z, \underline{P}Z, r)$ *is* $T_1$.

*(ii)* $(Z, \underline{P}Z, r)$ *is* $T_1$**PUCONV**, *where* $T_1$**PUCONV** *denotes the category of* $T_1$ *pre-uniform convergence spaces and uniformly continuous maps.*

*(iii) For all* $a, b \in Z$ *with* $a \neq b$, *and for all* $B \in \underline{P}Z$, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\}\} \notin r(B)$ *and* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\}\} \notin r(B)$, *and* $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\}\} \notin r(B)$.

*Proof.* This follows from Example 2.1.1, Theorem 4.2.1, and Theorem 3.2.4 of [28]. □

## 4.3 Quotient-Reflective Subcategories of O-REL

**Definition 4.3.1.** *(cf. [36]) Given a topological functor* $\mathfrak{U} : \mathcal{E} \to$ ***Set***, *and an isomorphism-closed full subcategory* $\mathcal{H}$ *of* $\mathcal{E}$, *we say* $\mathcal{H}$ *is*

*(i) epireflective in* $\mathcal{E}$ *and closed if and only if* $\mathcal{H}$ *is closed under the formation of extremal subobjects (i.e., subspaces) and products.*

*(ii) quotient-reflective in* $\mathcal{E}$ *iff* $H \in \mathcal{H}$, $E \in \mathcal{E}$, $\mathfrak{U}(H) = \mathfrak{U}(E)$, *and* $id : H \to E$ *is a* $\mathcal{E}$*-morphism, then* $E \in \mathcal{H}$ *(i.e.* $\mathcal{H}$ *is closed under finer structures and epireflective).*

**Theorem 4.3.1.** *(i) Any* $\overline{T}_0$***O-REL***, $T_0$***O-REL*** *and* $T_1$***O-REL*** *is a quotient-reflective subcategory of* ***O-REL***.

*(ii)* $T_0'$***O-REL*** *is a normalized topological construct.*

*Proof.* (i) Suppose $\mathcal{C} = \overline{T}_0$**O-REL** and $(Z, \beta^Z, r) \in \mathcal{C}$. It can be easily verified that $\mathcal{C}$ is an isomorphism-closed full subcategory of **O-REL** and closed under finer structures. It remains to show that X is closed under extremal subobjects and closed under the formation of products.

Let $A \subset X$ and $(\beta^A, r_A)$ denotes the sub O-REL structure on A, induced by the inclusion map $i : A \longrightarrow X$. We show that $(A, \beta^A, r_A)$ is $\overline{T}_0$O-REL space. Suppose that for all $\{a, b\} \in$ A with $a \neq b$, $\{a, b\} \in \beta^A$, then by the inclusion map $i(\{a, b\}) = \{i(a), i(b)\} = \{a, b\} \in \beta^Z$, a contradiction by Theorem 4.1.1.Thus $\{a, b\} \notin \beta^A$.

Now, suppose $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(a,b)\}\} \in r_A(\{a\})$ and $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(b,a)\}\} \in r_A(\{b\})$. It follows $R \in \mathcal{R}$ implies $\{(a,b)\} \subset R$, and by the inclusion map $i\{(a,b)\} \subset i(R)$ implying $\{(a,b)\} \subset R$. It implies $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \in r_Z(\{a\})$, a contradiction by theorem 4.1.1. Similarly, by the same argument $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(b,a)\}\} \in r_Z(\{b\})$, a contradiction. Therefore, $\{\mathcal{R} \in REL(A)\mathcal{R} << \{\{(a,b)\}\} \notin r_A(\{a\})$ or $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(b,a)\}\} \notin r_A(\{b\})$.

In similar way, $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(a,b)\}\} \notin r_A(\{b\})$ or $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(b,a)\}\} \notin r_A(\{a\})$, and $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(a,a),(b,b)\}\} \notin r_A(\{a\})$ or $\{\mathcal{R} \in REL(A) : \mathcal{R} << \{\{(b,b),(a,a)\}\} \notin r_A(\{b\})$. Hence, $Z$ is closed under the extremal subobjects.

Next, suppose that $Z = \prod_{k\in I} Z_k$, where $(\beta^{Z_k}, r_{Z_k})$ are the $\overline{T}_0$O-REL structures on $Z_k$ induced by projection map $\pi_k : Z_k \to X$ for all $k \in I$, i.e., $(Z_k, \beta^{Z_k}, r_{Z_k}) \in \mathcal{C}$. We show that $(Z, \beta^Z, r_Z)$ is a $\overline{T}_0$O-REL space. Let $\{a,b\} \in \beta^Z$ for any $a,b \in Z$ with $a \neq b$. Then $\pi_k(\{a,b\}) = \{\pi_k(a), \pi_k(b)\} = \{a_k, b_k\} \in \beta^{Z_k}$, a contradiction by Theorem 4.1.1. Thus $\{a,b\} \notin \beta^Z$.

Now, suppose $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \in r_Z(\{a\})$ and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\} \in r_Z(\{b\})$. It follows that, for all $R \in \mathcal{R}$ such that $\{(a,b)\} \subset R$. Then there is $k \in I$ for which $a_k \neq b_k \in Z_k$, and $\pi_k\{(a,b)\} \subset \pi_k R$ implying $\{(\pi_k a, \pi_k b)\} = \{(a_k, b_k)\} \subset \pi_k R$. It follows that $\{\mathcal{R} \in REL(Z) : \pi_k\mathcal{R} << \{\{(a_k,b_k)\}\} \in r_{Z_k}(\{a_k\})$, a contradiction by Theorem 4.1.1. By the same process, $\{\mathcal{R} \in REL(Z) : \pi_k(\mathcal{R}) << \{\{(b_k,a_k)\}\} \in r_{Z_k}(\{b_k\})$, a contradiction. Hence, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \notin r_Z(\{a\}$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\} \notin r_Z(\{b\})$. In similar way, $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,b)\}\} \notin r_Z(\{b\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,a)\}\} \notin r_Z(\{a\})$, and $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(a,a),(b,b)\}\} \notin r_Z(\{a\})$ or $\{\mathcal{R} \in REL(Z) : \mathcal{R} << \{\{(b,b),(a,a)\}\} \notin r_Z(\{b\})$. Hence, $Z$ is closed under the formation of products.

Therefore, the category $\overline{T}_0$**O-REL** is a quotient-reflective subcategory of **O-REL**.

Analogous to the above argument, setting $\mathcal{C} = T_0$**O-REL** or $T_1$**O-REL**, the proof can be easily followed by using Theorem 4.1.2 or Theorem 4.2.1 respectively.

(ii) By the Theorem 4.1.3 and Remark 2.2.1, $T_0'$**O-REL** and **O-REL** are isomorphic categories and thus $T_0'$**O-REL** is normalized.

□

# Chapter 5

# Results and Discussions

## 5.1 Results and Discussions

In this thesis, the category of ordered RELative spaces is taken into consideration. Firstly, initial, final, discrete and indiscrete objects are defined in the category of ordered RELative spaces and it is shown that it is a normalized and geometric topological category. Furthermore, local $\overline{T}_0$, local $T_0'$ and local $T_1$ are characterized in ordered RELative spaces and their mutual relationship were examined. Moreover, point free versions of $\overline{T}_0$, $T_0'$, $T_0$ and $T_1$ objects that makes sense in topos theory are characterized in the category of ordered RELative spaces. Finally, it is shown that the category of $T_0$'s and $T_1$ ordered RELative spaces are quotient reflective subcategories of **O-REL** and $T_0'$**O-REL** is a normalized topological category.

Comparing our results with other topological categories, we have the following conclusions:

Let $(Z, \beta^Z, r)$ be an ordered RELative space.

(i) $Z$ is $\overline{T}_0$ iff $Z$ is $\overline{T}_0$ at $p$, for all $p \in Z$.

(ii) $Z$ is $T_1$ iff $Z$ is $T_1$ at $p$, for all $p \in Z$.

(iii) $Z$ is $T_0'$ iff $Z$ is $T_0'$ at $p$, for all $p \in Z$.

(iv) $T_1 \implies \overline{T}_0 \implies T_0 \implies T_0'$ but the converse does not hold in general.

Considering this study, the following can be examined in **O-REL** as a future research problem.

(i) Can closed and strongly closed be characterized in **O-REL** and what would be their corresponding closure operators in the sense of Dikranjan and Giuli in **O-REL** by using the notion of closedness ?

(ii) What would be the characterization of irreducibility, soberness, connectedness and hyper-connectedness in the category **O-REL** ?

(iii) What would be Pre-Hausdorff, Hausdorff, regular and normal objects in **O-REL** ? How would they be related to classical ones ?

# BIBLIOGRAPHY


[1] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and Concrete Categories. The joy of cats* (Wiley, New York, 1990).

[2] J. R. Munkers, *Topology; A first course. Prentice-Hall* (1975)

[3] M. Baran, *Separation properties*, Indian J. Pure Appl. Math **23** (1992) 333-341.

[4] M. Baran, *The notion of closedness in topological categories. Commentationes Mathematicae Universitatis Carolinae* **34(2)** (1993) 383-395.

[5] M. Baran, *Generalized local separation properties*, Indian J. Pure Appl. Math **25** (1994) 615–620.

[6] M. Baran and H. Altındiş, $T_0$ *objects in topological categories*, J. Univ. Kuwait (Sci.) **22** (1995) 123-127.

[7] M. Baran and H. Abughalwa, *Sober spaces*, Turk J Math **46** (2022) 299-310.

[8] M. Baran and H. Altındiş, $T_2$ *objects in topological categories*, **1996** Acta Math. Hungarica **71(1-2)** (1996) 41-48.

[9] M. Baran, *Separation properties in topological categories*, **1996** Math. Balkanica **10** (1996) 39-48.

[10] M. Baran, *Completely regular objects and normal objects in topological categories*, Acta Mathematica Hungarica **80(3)** (1998) 211-224.

[11] M. Baran, *Compactness, perfectness, separation, minimality and closedness with respect to closure operators*, Applied Categorical Structures **10(4)** (2002) 403-415.

[12] M. Baran, S. Kula, T. M. Baran and M. Qasim, *Closure operators in semiuniform convergence spaces*, Filomat **30(1)** (2016) 131-140.

[13] H. L. Bentely, *Nearness spaces and extension of topological spaces*, In : Studies in topological, Academic Press (1975) 47-66.

[14] G. C. Brümmer, *A categorial study of initiality in uniform topology*, PhD Thesis, University of Cape Town 1971.

[15] E. Čech, *Topological spaces*, revised edition by M. Frolik and M.Katetov, Inter-science, London, 1960.

[16] D. Dikranjan and E. Giuli, *Closure operators I*, Topology and its Applications, **27(2)** 1987 129-143.

[17] A. Erciyes, T. M. Baran, and M. Qasim, *Closure operators in constant filter convergence spaces*, Konuralp Journal of Mathematics **8(1)** 2020 185-191.

[18] J. Harvey, *$T_0$-separation in topological categories*, Quaestiones Mathematicae 1977; 2 (1-3): 1971-1990.

[19] G. T. Herman, *On topology as applied to image analysis*, Computer Vision, Graphics, and Image Processing **52(3)** 1990 409-415.

[20] H. Herrlich, *A concept of nearness*, Gen. Top. Appl. (1974) 191-212.

[21] H. Herrlich, *Topological functors* , General Topology and its Applications 4(2) 1974 125-142.

[22] R. E. Hoffmann, *(E, M)-universally topological functors*, Habilitationsschrift, Universität Düsseldorf, 1974.

[23] G. Janelidze, *Light morphisms for generalized $T_0$-reflections*, Topology and its Applications, **156(12)** 2009 2109-2115.

[24] P. T. Johnstone, *Stone Spaces*, L. M. S. Mathematics Monograph: No. 10. Academic Press, (New York, 1977).

[25] D. C. Kent, *Convergence Functions and Their Related Topologies*, Fund. Math., **54** 1964 125-133.

[26] V. A. Kovalevsky, *Finite topology as applied to image analysis*, Computer vision, graphics, and image processing, **46(2)** 1989 141-161.

[27] V. A. Kovalevsky and R. Kopperman, *Some topology-based Image Processing Algorithms*, Annals of the New York Academy of Sciences, **728(1)** 1994 174–182.

[28] S. Kula, *Closedness and $T_0$, $T_1$ objects in the category of preuniform convergence spaces*, PhD Thesis, Erciyes University, Turkey 2014.

[29] S. Mac Lane, Categories for the working mathematician, Springer Science & Business Media **5** 2013.

[30] M. Kula, T. Maraşli and S. Özkan, *A note on closedness and connectedness in the category of proximity spaces*, Filomat **31(12)** 2014 3837-3846.

[31] M. Kula and S. Özkan, *$T_2$ and $T_3$ at $p$ in the category of proximity spaces*, Mathematica Bohemica **145(2)** 2020 177-190.

[32] D. Leseberg and Z. Vaziry, *Bounded topology*, LAP Lambert Academic Publishing, 2014.

[33] D. Leseberg, *Extension in Bounded topology*, Math. Appl. **3** 2014 97-113.

[34] D. Leseberg, *Relative spaces as a global concept for topology*, EJMS **4(1)** 2018 27-34.

[35] D. Leseberg and Z. Vaziry, *The quasitopos of bounded uniform filter spaces*, Math. Appl. **7** 2018 155-171.

[36] D. Leseberg and Z. Vaziry, *On the completeness of non-symmetrical uniform convergence with some links to approach spaces*, Math. Appl. **8** 2019, 37-57.

[37] T. Marny, *Rechts-Bikategoriestrukturen in topologischen Kategorien: Inaugural-Dissertation*, (Doctoral dissertation, T. Marny), 1973.

[38] G. Preuss, *Semiuniform convergence spaces*, Mathematica Japonicae, **41(3)** 1995 465-491.

[39] G. Preuss, *Foundations of topology: an approach to convenient topology*, Springer Science & Business Media 2014.

[40] M. Qasim and S. Özkan, *The notion of closedness and D-connectedness in quantale-valued approach spaces*, Categories and General Algebraic Structures with Applications **12(1)** 2020 149-173.

[41] M. Qasim, M. Baran and H. Abughalwa, *Closure operators in convergence approach spaces*, Turkish Journal of Mathematics **45(1)** 2021 139-152.

[42] A. Tozzi, *On Categories of supertopological spaces*, Acta Universitatis Carolinae-Mathematica et Physica **28(2)** 1987 137-149.

[43] S. Weck-Schwarz, *$T_0$-objects and separated objects in topological categories*, Quaestiones Mathematicae **14(3)** 1991 315-325.

[44] O. Wyler, *On convergence of filter and ultrafilters to subset*, Lecture Notes in Comp.Sci **393** 1988 340-350.