The Mei Symmetries for the Lagrangian Corresponding to the Kerr Metric

Nimra Sher Asghar
Regn.#00000330917

A thesis submitted in partial
fulfillment of the requirements for the degree of
Master of Science
in
Mathematics

Supervised by: Prof. Tooba Feroze

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan

2022

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: **Nimra Sher Asghar**, Regn No. **00000330917** Titled: **"The Mei Symmetries for the Lagrangian Corresponding to the Kerr Metric"** accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: PROF. AZAD A. SIDDIQUI Signature:

2. Name: DR. IBRAR HUSSAIN Signature:

Supervisor's Name: PROF. TOOBA FEROZE Signature:

Head of Department

14/07/2022
Date

COUNTERSINGED

Date: 18.07.2022

Dean/Principal

Dedication

I would like to dedicate this thesis to my **parents**, who sacrificed all they had to guarantee that I could get education. Their sacrifices and sufferings have given me the key to unlocking the mysteries of our life and beyond.

Acknowledgements

First and foremost, I thank **Allah Almighty** for blessing me with health and the ability to complete this maze-like work.

I would like to express my heartfelt appreciation to my supervisor, **Dr. Tooba Feroze**, for allowing me to do research and for offering vital advice during this process. Her energy, vision, enthusiasm, and motivation have all left an indelible impression on me. It was a great honor and privilege to gain knowledge under her supervision.

I would like to extend my thanks to my GEC members, **Prof. Azad Akhter Siddiqui** and **Dr. Ibrar Hussain**, for their insightful discussions and advice.

My deepest love and affection go out to my **parents**, who have always had huge dreams for me and whose prayers continue to pave the road for me. I value their expert guidance and empathetic ear. In addition, I would want to express my thanks to my brothers, **Muhammad Hassan** and **Shehroz**, for serving as stepping stones in this regard.

My heartfelt gratitude goes to my most steadfast supporter, **Syed Abdul Rehman**, whose concern for me has no boundaries and whose constant encouragement is the most precious commodity I own.

Last but not Least, I extend my warmest gratefulness to my closest friend **Rubab Zehra**, for all the comfort she has given me and the time she has spent making sure I am always on the right track.

Nimra Sher Asghar

Abstract

In this thesis, the Mei symmetries for the Lagrangian corresponding to an axially symmetric metric are examined. For this purpose, the Kerr metric is considered. Using the Mei symmetries criteria, we determined four Mei symmetries for the Lagrangian of Kerr metric. Furthermore, the Lie point symmetries and Noether symmetries for the same Lagrangian are reviewed. The results reveal that, in the case of the Kerr metric, the Noether symmetry set is a subset of the Mei symmetry set and that Mei symmetries are same as that of Lie point symmetries. In the end, the obtained Mei symmetries are also verified.

Contents

1 Introduction **1**

1.1 A Brief History 1

1.2 Lie Point Transformations and Infinitesimal Generators 2

1.2.1 1-Parameter Groups of Lie Point Transformations 3

1.2.2 Infinitesimal Generator 4

1.2.3 N-Parameter Group of Lie Point Transformations 5

1.2.4 Prolonged Lie Point Transformations and Generators 6

1.3 Lie Point Symmetries of ODEs 9

1.3.1 General Procedure of Finding Lie Point Symmetry 11

1.4 Lie Algebras and Lie Brackets 16

1.5 Lagrangian-Based Systems 18

1.5.1 The Noether Symmetries 20

1.5.2 The Relationship between Lie and Noether symmetries 20

1.5.3 The Mei Symmetries 23

1.5.4 The Relationship of Noether and Mei Symmetries 25

2 Mei Symmetries for the Lagrangian of Kerr Metric **27**

2.1 The Kerr Metric 27

2.2 Review of the Noether and Lie Point Symmetries for the Lagrangian of Kerr Metric 28
2.3 The Kerr metric's Mei Symmetry 29
2.4 Mei Symmetries' Verification: 51

3 Summary **54**

Bibliography **57**

List of Tables

3.1 Lie point symmetries, Noether symmetries and Mei symmetries of the Kerr metric. 56

Chapter 1

Introduction

The background of DEs is reviewed in this chapter. A brief history of symmetries of ODEs and PDEs along with some examples is also discussed. Fundamental concepts, definitions, and notations of the Lie point symmetries, Noether symmetries, and Mei symmetries are given here. The procedure for finding these symmetries is discussed in detail and some examples are also given.

1.1 A Brief History

It is difficult to trace back the origin of DEs but after much deliberation on the topic, historians have marked 1693 as the origin year of DEs. Sir Isaac Newton's first-ever work on calculus "the method of fluxions and infinite series" [1] was not published until 1671. He was not officially credited for his commendable work until 1693 when Gottfried Leibniz solved the first DE. This is why, 1693 holds a special place in the history of mathematics, and Newton and Leibniz were declared the founding titans of DEs. Expanding on the basis provided by these two geniuses, Jacob Bernoulli and Johann Bernoulli created Bernoulli's equation in 1695, a new form of an ODE..

In 10 years, another well-known scientist came to the surface with his extraordinary work in the field of mathematics, Leonhard Euler. He reached a pedestal for his work in infinitesimal calculus, trigonometry, algebra, geometry, and number theory. He paved way for thousands of other mathematics to reach newer horizons in this field. In solving DEs, it is seen that in most of his work, he has extensively used the Euler's identity and formula. Last but not the least, he developed the calculus of variations.

With time, scientists kept digging deeper into this vast and at the same time, astonishing field. Scientists such as Joseph Louise Lagrange, Pierre Simon Laplace, Adrien-Marie Legendre, and Joseph Fourier came forth with concepts of the Lagrangian multiplier, Legendre transformations, Laplace's equations, Legendre polynomials, and Fourier series.

Taking the work of Daniel Bernoulli, a step further, Friedrich Bessel generalized Bessel functions. Meanwhile, Augustin Louis Cauchy brought forward the concept of solutions. The list of these great names does not end here. It will be remiss to not mention all the great mathematicians such as Rudolf Lipchitz, Bernhard Riemann, Carl Friedrich Gauss, Emmy Noether and George David Birkhoff. All these names have a solid reputation is taking DEs to a new level [2].

Evariste Galois marked his place in history by using group theory to solve polynomial equations. Whereas, a Norwegian mathematician, named Marius Sophus Lie used groups to solve DEs. He suggested that to obtain solutions, one can use groups of symmetries in standard techniques. Comprehending symmetries require that one must explore transformations and particular generators.

1.2 Lie Point Transformations and Infinitesimal Generators

A Lie point transformation is a transformation that transforms a point (x, y) into a new point (x^*, y^*)

$$x^* = x^*(x, y), \quad y^* = y^*(x, y), \tag{1.1}$$

where x is independent variable and y is dependent variable. Point transformations that are dependent on at least one parameter must be considered in the context of symmetries.

1.2.1 1-Parameter Groups of Lie Point Transformations

1-parameter group of Lie point transformations are the transformations that depends on at least 1-parameter $\varepsilon \in \mathbb{R}$,

$$x^* = x^*(x, y, \varepsilon), \quad y^* = y^*(x, y, \varepsilon), \tag{1.2}$$

with the group properties of closure, inverse and identity being satisfied. Setting $\varepsilon = 0$ yields the identity transformation,

$$x^*(x, y, 0) = x, \quad y^*(x, y, 0) = y. \tag{1.3}$$

Consider the translations

$$x^* = x, \quad y^* = y + \varepsilon. \tag{1.4}$$

After deciding on a value ε for the parameter, a second transformation of the group, corresponding to the value ε_1, is

$$x_1^* = x^*, \quad y_1^* = y^* + \varepsilon_1. \tag{1.5}$$

The outcome of the two's sequential performances is

$$x_1^* = x, \quad y_1^* = y + \varepsilon + \varepsilon_1, \tag{1.6}$$

which is another translation of the set, with $\varepsilon + \varepsilon_1$ as the parameter value. At $\varepsilon = 0$, translation group gets the identity, and at $\varepsilon = -\varepsilon$, the inverse transformation. As a result, all translations of type (1.4) form a group.

Another example, the rotations

$$x^* = xcos\varepsilon - ysin\varepsilon, \quad y^* = xsin\varepsilon + ycos\varepsilon, \tag{1.7}$$

represent a 1-parameter group of Lie point transformations, since they depend on only 1-parameter and satisfy all Lie group axioms.

Scaling is also included in the 1-parameter group, e.g.

$$x^* = e^{\varepsilon} x, \quad x^* = e^{\varepsilon} y. \tag{1.8}$$

In contrast to this, the reflection

$$x^* = -x, \quad y^* = -y, \tag{1.9}$$

does not belong to a 1-parameter group of Lie point transformations, but it is still a point transformation [3].

It is also a symmetry transformation if you translate one solution of a DE to another while preserving its structure. If we take the ODE

$$U = (x, y, y', ..., y^{(n)}) = 0, \tag{1.10}$$

and apply a Lie point transform equation (1.1) to it, we get

$$U = (x^*, y^*, y^{*'}, ..., y^{*(n)}) = 0. \tag{1.11}$$

This Lie point transformation is symmetric since it does not change the equation.

1.2.2 Infinitesimal Generator

Applying the Taylor series on equation (1.2) about $\varepsilon = 0$ gives an infinitesimal representation of Lie point transformation [3]

$$x^* = x + \varepsilon \left(\frac{\partial x^*}{\partial \varepsilon} \right) \bigg|_{\varepsilon=0} + O(\varepsilon^2), \qquad \hat{y} = y + \varepsilon \left(\frac{\partial y^*}{\partial \varepsilon} \right) \bigg|_{\varepsilon=0} + O(\varepsilon^2), \tag{1.12}$$

where the coefficients of infinitesimal transformations are set to be the functions

$$\frac{\partial x^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \xi(x, y), \qquad \frac{\partial y^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \eta(x, y). \tag{1.13}$$

As a result, an infinitesimal generator of transformation is established as

$$\boldsymbol{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{1.14}$$

Consider the following examples to demonstrate transformations and their generators. For a translational group,

$$x^* = x, \qquad y^* = y + \varepsilon, \tag{1.15}$$

we have,

$$\xi(x,y) = \left.\frac{\partial x^*}{\partial \varepsilon}\right|_{\varepsilon=0} = 0, \qquad \eta(x,y) = \left.\frac{\partial y^*}{\partial \varepsilon}\right|_{\varepsilon=0} = 1, \tag{1.16}$$

and the generator is

$$\boldsymbol{X} = \frac{\partial}{\partial y}. \tag{1.17}$$

An infinitesimal generator can be tranformed into new coordinates by

$$\boldsymbol{X} = (\boldsymbol{X}m^{\mu})\frac{\partial}{\partial m^{\mu}} = (\boldsymbol{X}m^{\mu'})\frac{\partial}{\partial m^{\mu'}}, \qquad \mu = 1, ..., N, \tag{1.18}$$

where $m^{\mu'}$ represents the new coordinates and $m^{\mu'} = \frac{\partial z^{\mu'}}{\partial z^{\mu}} m^{\mu}$ represents the new components of tangent vector $\boldsymbol{X}$.

1.2.3 N-Parameter Group of Lie Point Transformations

Lie point transformations may be influenced by multiple parameters. This means that, in contrast to equation (1.2), the following can be written:

$$x^* = x^*(x, y, \varepsilon_{\mathrm{N}}), \qquad y^* = y^*(x, y, \varepsilon_{\mathrm{N}}), \qquad \mathrm{N} = 1, ..., n. \tag{1.19}$$

If all these parameters satisfy all axioms of the group, and if they do not depend on each other, then these Lie point transformations form a N-parameter group (G_{N}) [5]. For each parameter ε_{N} of the N-parameter Lie point transformation group, an infinitesimal generator can be constructed as

$$\boldsymbol{X}_{\mathrm{N}} = \xi_{\mathrm{N}}\frac{\partial}{\partial x} + \eta_{\mathrm{N}}\frac{\partial}{\partial y}, \tag{1.20}$$

where the infinitesimals are

$$\xi_{\mathrm{N}}(x, y) = \left.\frac{\partial x^*}{\partial \varepsilon_{\mathrm{N}}}\right|_{\varepsilon_{\mathrm{N}}=0}, \tag{1.21}$$

$$\eta_{\mathrm{N}}(x, y) = \left.\frac{\partial y^*}{\partial \varepsilon_{\mathrm{N}}}\right|_{\varepsilon_{\mathrm{N}}=0}. \tag{1.22}$$

Example

Consider a 3-parameter group of Lie point transformations [6],

$$x^* = xcos\varepsilon_1 - ysin\varepsilon_1 + \varepsilon_2, \tag{1.23}$$

$$y^* = xsin\varepsilon_1 + ycos\varepsilon_1 + \varepsilon_3. \tag{1.24}$$

Application of equation (1.21) on equation (1.23) for each parameter ε_{N} ($\mathrm{N} = 1, 2, 3.$) provides

$$\xi_1 = -y, \quad \xi_2 = 1, \quad \xi_3 = 0. \tag{1.25}$$

Similarly, application of equation (1.22) on equation (1.24) for each parameter ε_{N} ($\mathrm{N} = 1, 2, 3.$) provides

$$\eta_1 = x, \quad \eta_2 = 0, \quad \eta_3 = 1. \tag{1.26}$$

Hence, the infinitesimal generators can be listed as

$$\boldsymbol{X}_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{1.27}$$

$$\boldsymbol{X}_2 = \frac{\partial}{\partial x}, \tag{1.28}$$

$$\boldsymbol{X}_3 = \frac{\partial}{\partial y}. \tag{1.29}$$

1.2.4 Prolonged Lie Point Transformations and Generators

Consider the DE (1.10). If we want to apply a Lie point transformation (1.2) to equaton (1.10), we have to prolong this transformation to its derivatives $y^{(m)}, \quad m = 1, 2, ...n.$

Recursively compute $y^{*(m)}$ [3]

$$y^{*(m)} = \frac{d\hat{y}}{dx^{*(m)}} = \frac{dx^{*(m-1)}}{dx^{*}} = \frac{D_x(y^{*(m-1)})}{D_x(x^{*})}, \tag{1.30}$$

here D_x is the total derivative

$$D_x = \frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + ... \tag{1.31}$$

As suggested by equation (1.12)

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,y) + O(\varepsilon^2), \\ y^* &= y + \varepsilon\eta(x,y) + O(\varepsilon^2), \\ y^{*'} &= y' + \varepsilon\eta^{(1)}(x,y,y') + O(\varepsilon^2), \\ &\vdots \\ y^{*(n)} &= y^n + \varepsilon\eta^{(n)}(x,y,y',......., y^{(n)}) + O(\varepsilon^2), \end{aligned} \tag{1.32}$$

where

$$\eta^{(1)} = \left.\frac{\partial y^{*'}}{\partial \varepsilon}\right|_{\varepsilon=0}, \eta^{(2)} = \left.\frac{\partial y^{*''}}{\partial \varepsilon}\right|_{\varepsilon=0}, ..., \eta^{(n)} = \left.\frac{\partial y^{*(n)}}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{1.33}$$

By putting equation (1.32) in equation (1.10), we get

$$\begin{aligned} y^{*'} &= \frac{D_x(y^*)}{D_x(x^*)} = \frac{dy + \varepsilon d\eta + ...}{dx + \varepsilon d\xi + ...}, \\ &= y' + \varepsilon((d\eta/dx) - y'(d\xi/dx)) + ... \end{aligned} \tag{1.34}$$

On comparing $y^{*'}$ from equation (1.34) to $y^{*'}$ from equation (1.32) , we get

$$\eta^{(1)} = D_x\eta - y'D_x\xi, \tag{1.35}$$

where D_x is defined by equation (1.31).

Likewise, for $y^{*(n)}$, we get

$$y^{*(n)} = y^{(n)} + \varepsilon\left(\frac{d\eta^{(n-1)}}{dx} - y^n\frac{d\xi}{dx}\right) + O(\varepsilon^2). \tag{1.36}$$

In addition, when we compare $y^{*(n)}$ from equation (1.36) to $y^{*(n)}$ from equation (1.32), we get

$$\eta^{(n)} = D_x \eta^{(n-1)} - y^{(n)} D_x \xi. \tag{1.37}$$

As a result, we can generalize it to

$$\eta^{(k)} = D_x \eta^{(k-1)} - y^{(k)} D_x \xi, \quad k = 1, ..., n \tag{1.38}$$

where $\eta^{(k)}$ is the kth-prolongation and D_x is defined by equation (1.31). We can also compute the prolongation of η by substituting equation (1.31) in equation (1.38) as the first two prolongations are

$$\eta^{(1)} = \eta_x + y'(\eta_y - \xi_x) - y'^2 \eta_y, \tag{1.39}$$

$$\eta^{(2)} = \eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3 \xi_{yy} + y''(\eta_y - 2\xi_x - 3y'\xi_y). \tag{1.40}$$

The infinitesimal transformations are expressed as

$$\begin{aligned} x^* &= x + \varepsilon \boldsymbol{X} x + O(\varepsilon^2), \\ y^* &= y + \varepsilon \boldsymbol{X} y + O(\varepsilon^2), \\ y^{*'} &= y' + \varepsilon \boldsymbol{X} y' + O(\varepsilon^2), \\ &\vdots \\ y^{*(n)} &= y^n + \varepsilon \boldsymbol{X} y^{(n)} + O(\varepsilon^2), \end{aligned} \tag{1.41}$$

and the prolongation of an infinitesimal generator is as follows

$$\boldsymbol{X}^{[n]} = \boldsymbol{X} + \eta^{(1)} \frac{\partial}{\partial y'} + ... + \eta^{(n)} \frac{\partial}{\partial y^{(n)}}. \tag{1.42}$$

Example

Consider following generator

$$\boldsymbol{X} = -x \frac{\partial}{\partial x} - 3y \frac{\partial}{\partial y}. \tag{1.43}$$

Now we will find its second order extended generator, i.e. $\eta^{(1)}$ and $\eta^{(2)}$ are required to find. We can deduce from equation (1.43) that

$$\xi = -x, \quad \eta = -3y. \tag{1.44}$$

Using equation (1.38), we can calculate prolongation co-efficients as follows

$$\eta^{(1)} = -2y'. \tag{1.45}$$

$$\eta^{(2)} = -y''. \tag{1.46}$$

As a result, the prolonged generator is obtained as

$$\boldsymbol{X}^{[2]} = -x\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y} - 2y'\frac{\partial}{\partial y'} - y''\frac{\partial}{\partial y''}. \tag{1.47}$$

1.3 Lie Point Symmetries of ODEs

Consider a set of symmetric transformations that are dependent on at least one parameter, then this symmetry is called **Lie point symmetry**, named after the Norwegian mathematician Sophus Lie [3].

Consider the following ordinary differential equation (ODE) (1.10), which admits a group of symmetries with generator $\boldsymbol{X}$ if and only if

$$\boldsymbol{X}^{[n]}U|_U = 0, \tag{1.48}$$

holds.

Where $\boldsymbol{X}^{[n]}$ is the nth prolongation of an infinitesimal generator given by equation (1.42).

Example

Assume we have a DE

$$y' = \frac{y+1}{x} + \frac{y^2}{x^3}, \tag{1.49}$$

admitting a generator

$$\boldsymbol{X} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{1.50}$$

From equation (1.50), we obtain

$$\xi = x^2, \qquad \eta = xy. \tag{1.51}$$

Using definition (1.38), we now find the prolongation co-efficient as follows:

$$\eta^{(1)} = y - xy'. \tag{1.52}$$

So the first prolonged generator is obtained as

$$\boldsymbol{X}^{[1]} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'}. \tag{1.53}$$

This implies that

$$\boldsymbol{X}^{[1]}U\big|_{\mathrm{U=0}} = \left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'}\right)\left(y' - \frac{y+1}{x} - \frac{y^2}{x^3}\right) = 0. \tag{1.54}$$

So the given ODE (1.49) admits the symmetry.

Example

Suppose we have a DE

$$y'' - x^{-5}y^2 = 0, \tag{1.55}$$

admitting a generator

$$\boldsymbol{X} = -x^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}. \tag{1.56}$$

From equation (1.56), we obtain

$$\xi = -x^2, \qquad \eta = -xy. \tag{1.57}$$

Using definition (1.38), we now find the prolongation co-efficients

$$\eta^{(1)} = xy' - y, \qquad \eta^{(2)} = 3xy''. \tag{1.58}$$

So the second prolonged generator is obtained as

$$\boldsymbol{X}^{[2]} = -x^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} + (xy' - y)\frac{\partial}{\partial y'} + 3xy''\frac{\partial}{\partial y''}. \tag{1.59}$$

This implies that

$$\boldsymbol{X}^{[2]}U\big|_{U=0} = (-x^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} + (xy' - y)\frac{\partial}{\partial y'} + 3xy''\frac{\partial}{\partial y''})(y'' - x^{-5}y^2) = 0. \tag{1.60}$$

So the given ODE (1.55) admits the symmetry.

1.3.1 General Procedure of Finding Lie Point Symmetry

Finding the Lie point symmetries of the DE (1.10) to get ξ and η only. To ensure the regularity of DE (1.10), and because many DEs naturally originate as linear equations with the highest derivative, we choose to begin with the $y^{(n)} = \omega(x, y, y', y''..., y^{(n-1)})$ form of DE. Since the symmetry condition [3] is

$$\boldsymbol{X}^{[n]}\omega = \left(\boldsymbol{X} + \eta'\frac{\partial}{\partial y'} + \eta''\frac{\partial}{\partial y''} + ... + \eta^{(n-1)}\frac{\partial}{\partial y^{(n-1)}}\right)\omega = \eta^{(n)}, \tag{1.61}$$

where $\boldsymbol{X}$ is given by an equation (1.14), $\eta^{(k)}$ is defined by an equation (1.38) and $y^{(n)}$ in $\eta^{(n)}$ must be replaced by ω.

Case1: 1^{st} Order ODE

Consider a 1^{st} order ODE

$$y' = \omega(x, y). \tag{1.62}$$

The associated PDE is

$$\begin{aligned} A &= \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}, \\ &= \frac{\partial}{\partial x} + \omega\frac{\partial}{\partial y}. \end{aligned} \tag{1.63}$$

The symmetry condition (1.61) is as follows

$$\boldsymbol{X}\omega = \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\omega = \eta^{(1)}, \tag{1.64}$$

using the definition of $\eta^{(1)}$ from equation (1.39) in equation (1.64) yields [3]

$$\xi\omega_x + \xi_x\omega + \xi_y\omega^2 = \eta_x + \eta_y\omega - \eta\omega_y. \tag{1.65}$$

The function $\omega(x, y)$ is given, this PDE (1.65) always have non zero solutions $\xi(x, y)$ and $\eta(x, y)$. In fact, one may prescribe η (or ξ) and then deduce ξ (or η) from equation (1.65). In this scenario, a 1^{st} order DE has an infinite number of symmetries, but there is no symmetric way to find them.

Example

Consider 1^{st} order ODE

$$y' = x^2y, \tag{1.66}$$

here

$$\omega = x^2y. \tag{1.67}$$

Equation (1.65) contains two unknowns, ξ and η. To find ξ, we assume $\eta = 0$ and vice versa. So, the equation (1.65) becomes

$$\xi\omega_x + \xi_x\omega + \xi_y\omega^2 = 0. \tag{1.68}$$

This implies that

$$\xi(2xy) + \frac{\partial\xi}{\partial x}(x^2y) + \frac{\partial\xi}{\partial y}(x^4y^2) = 0. \tag{1.69}$$

The characterstic equation of equation (1.69) is

$$\frac{dx}{x^2y} = \frac{dy}{(x^4y^2)} = \frac{d\xi}{\xi(2xy)}. \tag{1.70}$$

Solving equation (1.70) gives

$$\xi = \frac{1}{x^2}. \tag{1.71}$$

Hence, the symmetry generator is

$$\boldsymbol{X} = \frac{1}{x^2}\frac{\partial}{\partial x}. \tag{1.72}$$

Using the condition $\boldsymbol{X}U|_{U=0} = 0$, we can say $\boldsymbol{X}$ is symmetric.

Case2: 2^{nd} Order ODE

In the following example, we show how this procedure is used to find symmetries for 2^{nd} order DE.

Example

Consider a 2^{nd} order DE [7]

$$y'' = 4\frac{y'^2}{y}. \tag{1.73}$$

Now we use the condition (1.61) to find the symmetries of 2^{nd} order DE (1.73)

$$\eta^{(2)} = \boldsymbol{X}^{[2]}\omega. \tag{1.74}$$

Using equation (1.40) in equation (1.74), we have

$$\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} + y''(\eta_y - 2\xi_x - 3y'\xi_y) = -4\eta\frac{y'^2}{y^2} + 8\eta^{(1)}\frac{y'}{y}. \tag{1.75}$$

Putthing $y'' = 4\frac{y'^2}{y}$ from equation (1.73) and $\eta^{(1)}$ fron equation (1.39), we get

$$\begin{aligned}&\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} + 4\frac{y'^2}{y}(\eta_y - 2\xi_x - 3y'\xi_y) + 4\eta\frac{y'^2}{y^2} \\ &- 8\frac{y'}{y}(\eta_x + y'(\eta_y - \xi_x) - y'^2\eta_y) = 0.\end{aligned} \tag{1.76}$$

Comparing the co-efficients of y' and solving it further yields

$$\xi(x, y) = 3c_1x^2 + c_2x + c_3, \tag{1.77}$$

$$\eta(x, y) = -c_1xy + c_4y + c_5y^4 + c_6xy^4. \tag{1.78}$$

In this case, $c_l,\ l = 1, 2, .., 6$ are arbitrary constants. As a result of equation (1.77) and equation (1.78), the infinitesimal generator of 1-parameter Lie point symmetries of equation (1.73) established as

$$\boldsymbol{X} = \left(3c_1x^2 + c_2x + c_3\right)\frac{\partial}{\partial x} + \left(-c_1xy + c_4y + c_5y^4 + c_6xy^4\right)\frac{\partial}{\partial y}. \tag{1.79}$$

and for each $c_i = 1, c_j = 0$, we get following six symmetries

$$\begin{aligned} \boldsymbol{X_1} &= 3x^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}, & \boldsymbol{X_2} &= x\frac{\partial}{\partial x}, \\ \boldsymbol{X_3} &= \frac{\partial}{\partial x}, & \boldsymbol{X_4} &= y\frac{\partial}{\partial y}, \\ \boldsymbol{X_5} &= y^4\frac{\partial}{\partial y}, & \boldsymbol{X_6} &= xy^4\frac{\partial}{\partial y}. \end{aligned} \tag{1.80}$$

A 2^{nd} order DE can accept a maximum of eight symmetries. It is important to mention that every 2^{nd} order linear homogeneous DE is transformed into $y'' = 0$, and hence admits eight symmetries. To understand this consider a 2^{nd} order linear homogeneous DE [3]

$$y'' + p(x)y' + q(x)y = 0. \tag{1.81}$$

The general solution of equation (1.81) is

$$y = c_1 y_1(x) + c_2 y_2(x), \tag{1.82}$$

where y_1 and y_2 are linearly indepedent solution of equation (1.81). Now dividing y_2 on both sides of equation (1.82)

$$\frac{y}{y_2} = c_1\frac{y_1}{y_2} + c_2. \tag{1.83}$$

Let

$$y^* = \frac{y}{y_2}, \quad x^* = \frac{y_1}{y_2}, \tag{1.84}$$

then transformed equation is

$$y^* = c_1 x^* + c_2, \tag{1.85}$$

this implies that

$$y^{*''} = 0. \tag{1.86}$$

So, there are 8 symmetries.

Case3: Higher Order DE

Consider a 3^{rd} order DE [6]

$$y''' = -xy. \tag{1.87}$$

Now we use the condition (1.61) to find the symmetries of 3^{rd} order DE (1.87).

$$\eta^{(3)} = \boldsymbol{X}^{[3]}\omega. \tag{1.88}$$

Solving equation (1.88), we have

$$\eta^{(3)} + \xi y + \eta x = 0. \tag{1.89}$$

Putthing value of $\eta^{(3)}$ in equation (1.89) , we get

$$\begin{aligned}
&\eta_{xxx} + y'(3\eta_{xxy} - \xi_{xxx}) + 3y'^2(\eta_{xyy} - \xi_{xxy}) - y'^3(\eta_{yyy} - 3\xi_{xyy}) - y'^4\xi_{yyy} + \\
&3y''[\eta_{xy} - \xi_{xx} + y'(\eta_{yy} - 3\xi_{xy}) - 2y'^2\xi_{yy}] - 3y''^2\xi_y - xy[\eta_y - 3\xi_x - 4y'\xi_y] \\
&+ \xi y + \eta x = 0.
\end{aligned} \tag{1.90}$$

Comparing coefficients provides PDE

$$\begin{aligned}
(constant):&\quad \eta_{xxx} - xy(\eta_y - 3\xi_x) + \xi y + \eta x = 0,\\
(y'):&\quad 3\eta_{xxy} - \xi_{xxx} + 4xy\xi_y = 0,\\
(y'^2):&\quad 3(\eta_{xyy} - \xi_{xxy}) = 0,\\
(y'^3):&\quad \eta_{yyy} - 3\xi_{xyy} = 0,\\
(y'^4):&\quad \xi_{yyy} = 0,\\
(y''):&\quad 3(\eta_{xy} - \xi_{xx}) = 0,\\
(y'y''):&\quad 3(\eta_{yy} - 3\xi_{xy}) = 0,\\
(y'^2y''):&\quad -2\xi_{yy} = 0,\\
(y''^2):&\quad \xi_y = 0.
\end{aligned} \tag{1.91}$$

Solving the system (1.91), yields

$$\xi(x,y) = c_3x^2 + c_4x + c_5, \tag{1.92}$$

$$\eta(x,y) = y(c_1x+c_2)+c_3(\frac{5}{48}x^4y-\frac{x^4}{24})-c_4\frac{x^4y}{6}-c_2\frac{x^4y}{24}+c_5\frac{x^3y}{6}+c_6\frac{x^2}{2}+c_7x+c_8. \tag{1.93}$$

In this case, $c_l,\ l = 1,2,..,8$ are constants. Thus, the infinitesimal generator of 1-parameter Lie point symmetries of DE (1.87) is established as

$$\begin{aligned}\boldsymbol{X} =& \big(c_3x^2 + c_4x + c_5\big)\frac{\partial}{\partial x} + \big(y(c_1x + c_2) + c_3(\frac{5}{48}x^4y - \frac{x^4}{24}) - c_4\frac{x^4y}{6} - c_2\frac{x^4y}{24} \\ &+ c_5\frac{x^3y}{6} + c_6\frac{x^2}{2} + c_7x + c_8\big)\frac{\partial}{\partial y},\end{aligned} \tag{1.94}$$

and for each $c_i = 1, c_j = 0$, we get following eight symmetries

$$\begin{aligned} &\boldsymbol{X_1} = xy\frac{\partial}{\partial y}, && \boldsymbol{X_2} = \left(y - \frac{x^4y}{24}\right)\frac{\partial}{\partial y}, \\ &\boldsymbol{X_3} = x^2\frac{\partial}{\partial x} + \left(\frac{5}{48}x^4y - \frac{x^4}{24}\right)\frac{\partial}{\partial y}, && \boldsymbol{X_4} = x\frac{\partial}{\partial x} - \frac{x^4y}{6}\frac{\partial}{\partial y}, \\ &\boldsymbol{X_5} = \frac{\partial}{\partial x} + \frac{x^3y}{6}\frac{\partial}{\partial y}, && \boldsymbol{X_6} = \frac{x^2}{2}\frac{\partial}{\partial y}, \\ &\boldsymbol{X_7} = x\frac{\partial}{\partial y}, && \boldsymbol{X_8} = \frac{\partial}{\partial y}. \end{aligned} \tag{1.95}$$

A DE of order $n \geq 3$ admits at most $(n+4)$-parameter group of Lie point symmetries. The Lie point symmetry method is used to solve a variety of DE systems [8–10].

1.4 Lie Algebras and Lie Brackets

Before delving into the details of Lie algebra, we must first define Lie group [5].

Lie Group

A Lie group is a group and a finite-dimensional real smooth manifold in which the group operations of multiplication and inversion are smooth maps. Lie groups were introduced by a Norwegian mathematician Sophus Lie who formulated the theory of continuous transformation groups in order to model the continuous symmetries.

A Lie algebra, which entirely determines the local structure of the Lie group, can be associated with any Lie group. The definition of Lie algebra is

Lie Algebra

The Lie algebra L is a vector space defined on a field $\mathbb{R}$ together with an operation known as the Lie bracket that fulfils the properties [5]

1. Bilinearity : $[\boldsymbol{X}, f\boldsymbol{Y} + g\boldsymbol{Z}] = [\boldsymbol{X}, f\boldsymbol{Y}] + [\boldsymbol{X}, +g\boldsymbol{Z}],\ \forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in L\ \ and\ \ f, g \in \mathbb{R}$.
2. Skew symmetry : $[\boldsymbol{X}, \boldsymbol{Y}] = -[\boldsymbol{Y}, \boldsymbol{X}],\ \forall \boldsymbol{X}, \boldsymbol{Y} \in L$. .
3. Jaccobi identity : $[[\boldsymbol{X}, \boldsymbol{Y}], \boldsymbol{Z}] + [[\boldsymbol{Z}, \boldsymbol{X}], \boldsymbol{Y}] + [[\boldsymbol{Z}, \boldsymbol{Y}], \boldsymbol{X}] = 0,\ \forall \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in L$.

We can deduce $[\boldsymbol{X}, \boldsymbol{X}] = 0$ from a skew-symmetry property whereas the Lie algebra is called abelian when $[\boldsymbol{X}, \boldsymbol{Y}] = 0$. The commutator relation of $\boldsymbol{X}_K, \boldsymbol{X}_L \in L$ is

$$[\boldsymbol{X}_K, \boldsymbol{X}_L] = C^M_{KL}\boldsymbol{X}_M, \tag{1.96}$$

where C^M_{KL} is the structure constant.

Example

The generators of the Lie algebra with 3-parameters ε_N are

$$\boldsymbol{X}_1 = \frac{\partial}{\partial x}, \quad \boldsymbol{X}_2 = \frac{\partial}{\partial y}, \quad \boldsymbol{X}_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}. \tag{1.97}$$

and the corresponding Lie algebra is

$$\begin{aligned}
&[\boldsymbol{X}_1, \boldsymbol{X}_2] = 0,\\
&[\boldsymbol{X}_1, \boldsymbol{X}_3] = \frac{\partial}{\partial x} = \boldsymbol{X}_1 = C^1_{13}\boldsymbol{X}_1,\\
&[\boldsymbol{X}_2, \boldsymbol{X}_3] = \frac{\partial}{\partial y} = \boldsymbol{X}_2 = C^2_{23}\boldsymbol{X}_2.
\end{aligned} \tag{1.98}$$

The structure constants are $C^1_{13} = 1$ and $C^2_{23} = 1$. Due to skew-symmetric property of structure constants $C^1_{13} = -1$, $C^2_{23} = -1$.

Example

The generators of the Lie algebra with 6-parameters ε_N are

$$\begin{aligned} \boldsymbol{X}_1 &= \frac{\partial}{\partial x}, \quad \boldsymbol{X}_2 = \frac{\partial}{\partial y}, \quad \boldsymbol{X}_3 = \frac{\partial}{\partial v}, \quad \boldsymbol{X}_4 = y\frac{\partial}{\partial v}, \\ \boldsymbol{X}_5 &= x\frac{\partial}{\partial x} + 3v\frac{\partial}{\partial v}, \quad \boldsymbol{X}_6 = y\frac{\partial}{\partial y} - 2v\frac{\partial}{\partial v}. \end{aligned} \tag{1.99}$$

and the corresponding Lie algebra is

$$\begin{aligned} &[\boldsymbol{X}_1, \boldsymbol{X}_2] = 0, \quad [\boldsymbol{X}_1, \boldsymbol{X}_3] = 0, \quad [\boldsymbol{X}_1, \boldsymbol{X}_4] = 0, \quad [\boldsymbol{X}_1, \boldsymbol{X}_5] = \frac{\partial}{\partial x} = C^1_{15}\boldsymbol{X}_1, \\ &[\boldsymbol{X}_1, \boldsymbol{X}_6] = 0, \quad [\boldsymbol{X}_2, \boldsymbol{X}_3] = 0, \quad [\boldsymbol{X}_2, \boldsymbol{X}_4] = 0, \quad [\boldsymbol{X}_2, \boldsymbol{X}_6] = \frac{\partial}{\partial y} = C^2_{26}\boldsymbol{X}_2, \\ &[\boldsymbol{X}_3, \boldsymbol{X}_4] = 0, \quad [\boldsymbol{X}_3, \boldsymbol{X}_5] = 3\frac{\partial}{\partial u} = C^3_{35}\boldsymbol{X}_3, \quad [\boldsymbol{X}_3, \boldsymbol{X}_6] = -2\frac{\partial}{\partial u} = C^3_{36}\boldsymbol{X}_3, \\ &[\boldsymbol{X}_4, \boldsymbol{X}_5] = 3y\frac{\partial}{\partial u} = C^4_{45}\boldsymbol{X}_4, \quad [\boldsymbol{X}_4, \boldsymbol{X}_6] = -2y\frac{\partial}{\partial u} = C^4_{46}\boldsymbol{X}_4, \quad [\boldsymbol{X}_5, \boldsymbol{X}_6] = 0. \end{aligned} \tag{1.100}$$

The structure constants are $C^1_{15} = 1$, $C^2_{26} = 1$, $C^3_{35} = 3$, $C^3_{36} = -2$, $C^4_{45} = 3$ and $C^4_{46} = -2$. Due to skew-symmetric property of structure constants $C^1_{15} = -1$, $C^2_{26} = -1$, $C^3_{35} = -3$, $C^3_{36} = 2$, $C^4_{45} = -3$ and $C^4_{46} = 2$.
For detailed discussion one may refer to [11–13]

1.5 Lagrangian-Based Systems

Classical mechanics is primarily made up of systems of second order DEs. In classical mechanics, the concept $\dot{q}^i = \frac{dq^i}{dt}$ is frequently used, where time t is independent variable and generalised coordinates q^i are dependent variables. The system of 2^{nd} order DEs can be written as [3]

$$\ddot{q}^i = \omega^i(t, q^j, \dot{q}^j), \quad i, j = 1, ..., N \tag{1.101}$$

which corresponds to the linear PDE

$$\boldsymbol{A}f = \left(\frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \omega^i(t, q^j, \dot{q}^j)\frac{\partial}{\partial \dot{q}^i}\right)f = 0. \tag{1.102}$$

The solutions ϕ^i of equation (1.102) translate into the $2N$ first integrals of equation (1.101), just as they do in case of ODEs. Taking infinitesimal transformation of time t and generalised coordinates into account

$$t^* = t + \varepsilon\xi(t, q, q^*), \qquad q^{*i} = q^i + \varepsilon\eta^i(t, q, q^*). \tag{1.103}$$

In this coordinate system, the generator and its prolongation can be written as

$$\begin{aligned} \boldsymbol{X} &= \xi(t, q^j)\frac{\partial}{\partial t} + \eta^i(t, q^j)\frac{\partial}{\partial q^i}, \\ \boldsymbol{X}^{[1]} &= \boldsymbol{X} + \dot{\eta}^i(t, q^j, \dot{q}^j)\frac{\partial}{\partial \dot{q}^i}, \end{aligned} \tag{1.104}$$

where $\dot{\eta}^i$ is denoted by

$$\dot{\eta}^i = \frac{d\eta^i}{dt} - \dot{q}^i\frac{d\xi}{dt}. \tag{1.105}$$

By recursion, successive prolongation $\boldsymbol{X}^{[n]}$ of $\boldsymbol{X}$ can be obtained (if convenient, write $\boldsymbol{X}$ for the prolongations as well) and system symmetries can be deduced if

$$[\boldsymbol{X}, \boldsymbol{A}] = \lambda\boldsymbol{A} \tag{1.106}$$

holds.

Once the symmetries are known, the first integral corresponding to each symmetry can be found using the Lagrangian of the system. Lagrangian, on the other hand, is just the kinetic energy T minus potential energy V

$$L(t, q^j, \dot{q}^j) = T - V. \tag{1.107}$$

This correspondence between symmetries and first integrals cannot be established for symmetries less than 2N, but it is possible if the system can be deduced from an action

$$N = \int_{t_a}^{t_b} L(t, q^i, \dot{q}^i)dt. \tag{1.108}$$

The expression N leads to Lagrange (geodesic) equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0. \tag{1.109}$$

1.5.1 The Noether Symmetries

Noether symmetries are those infinitesimal symmetry generators that leave a Lagrangian $L(t, q^j, \dot{q}^j)$ invariant. A generator of infinitesimal transformations $\boldsymbol{X}$ is said to be Noether symmetry if it satisfies

$$\boldsymbol{X}^{[1]}L + L\boldsymbol{A}\xi = \boldsymbol{A}V(t, q^i), \tag{1.110}$$

where $V(t, q^i)$ is a gauge function, $\boldsymbol{A}$ is an operator defined by

$$\boldsymbol{A} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} \tag{1.111}$$

and $\boldsymbol{X}^{[1]}$ is the first prolonged generator defined by equation (1.144). When $V(t, q^i) = 0$, the Noether symmetries lead to variational symmetries [14].

1.5.2 The Relationship between Lie and Noether symmetries

To illustrate the relationship between Noether symmetries and first integrals corresponding to each Noether symmetry, a conserved quantity [15]

$$\phi = \xi[\dot{q}^\alpha L_{\dot{q}^\alpha} - L] - \eta^\alpha L_{\dot{q}^\alpha} + V(t, q^\beta), \tag{1.112}$$

satisfying $\boldsymbol{X}\phi = 0$ may be found. Noether symmetries are always form a subalgebra of Lie point symmetries.

For a more in-depth discussion, see [16] and [17].

Example

Consider the Lagrangian

$$L = \dot{x}^2 + \dot{y}^2. \tag{1.113}$$

From Lagrange equation (1.109), we get

$$\ddot{x} = 0, \quad \ddot{y} = 0. \tag{1.114}$$

By inserting values, equation (1.110) becomes

$$2\dot{x}[\eta^1_t + \dot{x}(\eta^1_x - \xi_t) + \dot{y}\eta^1_y - \dot{x}^2\xi_x - \dot{x}\dot{y}\xi_y] + 2\dot{y}[\eta^2_t + \dot{x}\eta^2_x + \dot{y}(\eta^2_y - \xi_t) - \dot{x}\dot{y}\xi_x - \dot{y}^2\xi_y]$$
$$+ [\dot{x}^2 + \dot{y}^2(\xi_t + \dot{x}\xi_x + \dot{y}^2\xi_y)] = V_t + \dot{x}V_x + \dot{y}V_y. \tag{1.115}$$

Through the comparison of coefficients

$$\begin{aligned}
(\dot{x}) &: 2\eta^1_t = V_x,\\
(\dot{x}^2) &: 2\eta^1_x - \xi_{xt} = 0,\\
(\dot{x}\dot{y}) &: 2\eta^1_y + 2\eta^1_x = 0,\\
(\dot{x}^3) &: \xi_x = 0,\\
(\dot{x}^2\dot{y}) &: \xi_y = 0,\\
(\dot{y}) &: 2\eta^2_t = V_y,\\
(\dot{y}^2) &: 2\eta^2_y - \xi_t = 0,\\
\text{constant} &: V_t = 0.
\end{aligned} \tag{1.116}$$

Now from the equations $(\dot{x}^3)$ and $(\dot{x}\dot{y})$ we get

$$\xi = \xi(t). \tag{1.117}$$

From the equation $V_t = 0$ yields

$$V = (x, y). \tag{1.118}$$

Differentiating equation $(\dot{x})$ w.r.t. t, we get

$$\eta^1 = ta_1(x, y) + a_2(x, y). \tag{1.119}$$

Differentiating equation $(\dot{x}^2)$ w.r.t. x, yields

$$\begin{aligned}
a_1 &= xb_1(y) + b_2(y),\\
a_2 &= xb_3(y) + b_4(y).
\end{aligned} \tag{1.120}$$

Equation (1.119) imples that

$$\eta^1 = t(xb_1 + b_2) + xb_3 + b_4. \tag{1.121}$$

Differentiating equation ($\dot{y}$) w.r.t. t, we get

$$\eta^2 = ta_3(x,y) + a_4(x,y). \tag{1.122}$$

Differentiating equation ($\dot{y}^2$) w.r.t. y, we get

$$\begin{aligned} a_3 &= yd_1(x) + d_2(x), \\ a_4 &= yd_3(x) + d_4(x). \end{aligned} \tag{1.123}$$

Equation (1.122) imples that

$$\eta^2 = t(yd_1 + d_2) + yd_3 + d_4. \tag{1.124}$$

From the equation ($\dot{x}\dot{y}$), we get

$$\begin{aligned} &b_1 = c_1, \quad b_2 = c_5 y + c_6, \quad b_3 = c_2, b_4 = c_8 y + c_9, \\ &d_1 = c_3, \quad d_2 = -c_5 x + c_7, \quad d_3 = c_4, d_4 = -c_8 x + c_{10}. \end{aligned} \tag{1.125}$$

This implies that equation (1.121) and equation (1.124) becomes

$$\eta^1 = t(c_1 x + c_5 y + c_6) + c_2 x + c_8 y + c_9, \tag{1.126}$$

$$\eta^2 = t(c_3 y - c_5 x + c_7) + c_4 y - c_8 x + c_{10}. \tag{1.127}$$

Now from equation ($\dot{x}^2$), we get

$$\xi = c_1 t^2 + c_2 t + c_3, \tag{1.128}$$

and from equation ($\dot{x}$), we get

$$V = c_1 x^2 + 2c_5 cy + 2c_6 x. \tag{1.129}$$

As a result, the symmetry generator looks like this:

$$\begin{aligned} \boldsymbol{X} &= (c_1 t^2 + c_2 t + c_3)\frac{\partial}{\partial t} + (t(c_1 x + c_5 y + c_6) + c_2 x + c_8 y + c_9)\frac{\partial}{\partial x} \\ &+ t(c_3 y - c_5 x + c_7) + c_4 y - c_8 x + c_{10}\frac{\partial}{\partial y}, \end{aligned} \tag{1.130}$$

and for each $c_k = 1, c_l = 0$, we get the following Noether symmetries

$$\begin{aligned}
\boldsymbol{X}_1 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}, & \boldsymbol{X}_2 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\\
\boldsymbol{X}_3 &= \frac{\partial}{\partial t} + ty\frac{\partial}{\partial y}, & \boldsymbol{X}_4 &= y\frac{\partial}{\partial y},\\
\boldsymbol{X}_5 &= ty\frac{\partial}{\partial x} - tx\frac{\partial}{\partial y}, & \boldsymbol{X}_6 &= t\frac{\partial}{\partial x},\\
\boldsymbol{X}_7 &= t\frac{\partial}{\partial y}, & \boldsymbol{X}_8 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\
\boldsymbol{X}_9 &= \frac{\partial}{\partial x}, & \boldsymbol{X}_{10} &= \frac{\partial}{\partial y}.
\end{aligned} \tag{1.131}$$

Mei symmetry is another intriguing symmetry.

1.5.3 The Mei Symmetries

In the year 2000, Mei proposed a new symmetry called form invariance. Form invariance, also known as Mei symmetry. It states that the dynamical functions (such as Lagrangian etc.) appearing in the mechanical system's dynamical equations still fulfil the original equations after the infinitesimal transformation. Mei symmetry, like Noether symmetry, admits first integrals known as Mei conserved quantities.

To be able to find Mei symmetries, we must first define and implement a method for finding them [18].

Assume we have a Lagrangian

$$L = L(t, q^i, \dot{q}^i), \tag{1.132}$$

Consider the infinitesimal transfirmation group with a 1-parameter.

$$\begin{aligned}
t^* &= t + \varepsilon\xi(t, q^j),\\
q^{*i} &= q^i + \varepsilon\eta^i(t, q^j),
\end{aligned} \tag{1.133}$$

where $i, j = 1, ..., n$ and $\varepsilon \in \mathbb{R}$. The associated infinitesimal generator is

$$\boldsymbol{X} = \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i}. \tag{1.134}$$

As a result of the transformation (1.133), the Lagrangian (1.132) becomes

$$\begin{aligned} L^* &= L^*(t^*, q^{*i}, \dot{q}^{*i}), \\ &= \hat{L}\left(t + \varepsilon\xi, q^i + \varepsilon\eta^i, \frac{\dot{q}^i + \varepsilon\dot{\eta}^i}{1 + \varepsilon\dot{\xi}}\right). \end{aligned} \tag{1.135}$$

The Taylor series expansion of equation (1.135) about $\varepsilon = 0$ yields

$$L^* = L(t, q^i, \dot{q}^i) + \varepsilon \boldsymbol{X}^{[1]} L + O(\varepsilon^2), \tag{1.136}$$

where

$$\boldsymbol{X}^{[1]} = \boldsymbol{X} + (\dot{\eta}^i - \dot{\xi}\dot{q}^i)\frac{\partial}{\partial \dot{q}^i}, \tag{1.137}$$

is the first prolongation of the infinitesimal generator $\boldsymbol{X}$.

The Euler-Lagrange equation is written as

$$E_i(L) = 0, \tag{1.138}$$

where E_i denotes the Euler operator

$$E_i = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i}. \tag{1.139}$$

If equation (1.138) remains unchanged when the new Lagrangian $\hat{L}$ from equation (1.136) is substituted in place of the Lagrangian, i.e.

$$E_i(L^*) = 0, \tag{1.140}$$

this invariance is known as the Mei symmetries corresponding to the Lagrangian. As a result, we can present the method for determining Mei symmetries [19–22].

Method for Determining Mei Symmetries

If the infinitesimals ξ and η satisfy

$$E_i[\boldsymbol{X}^{[1]} L] = 0, \qquad i = 1, ..., n. \tag{1.141}$$

then the corresponding invariance is the Mei symmetry for the Lagrangian.
Before using this method to find Mei symmetries, we should investigate the relationship between Mei symmetries and Noether symmetries, as it is crucial in determining Mei conserved quantities and Noether conserved quantities.

1.5.4 The Relationship of Noether and Mei Symmetries

To begin, we will present an important theorem [23].

Theorem

If the Mei symmetry of the system (1.132) and the infinitesimals ξ and η^i of the gauge function $g(t, q^i, \dot{q}^i)$ admit

$$\boldsymbol{X}^{[1]}L\dot{\xi} + \boldsymbol{X}^{[1]}(\boldsymbol{X}^{[1]}L) + z(t)\frac{\partial(\boldsymbol{X}^{[1]}L)}{\partial q^i}\dot{q}^i\dot{\xi} + \dot{g} = 0, \tag{1.142}$$

then the Mei symmetry can lead to new conserved quantity

$$\phi_1 = \frac{\partial(\boldsymbol{X}^{[1]}L)}{\partial \dot{q}}\eta^i + (\boldsymbol{X}^{[1]}L - \frac{\partial(\boldsymbol{X}^{[1]}L)}{\partial \dot{q}}\dot{q} - z(t)\frac{\partial(\boldsymbol{X}^{[1]}L)}{\partial t})\xi + g. \tag{1.143}$$

This theorem aids in the construction of a relationship between the Noether and Mei symmetries.
Consider the integral function

$$S(q) = \int_{t_1}^{t_2} \boldsymbol{X}^{[1]}L(L(t, q^i(t), \dot{q}^i(t)))dt, \tag{1.144}$$

with boundary conditions $q^i(t)|_{t=a} = q^i(a)$ and $q^i(t)|_{t=b} = q^i(b)$ where $i = 1, .., n$.
The same form as equation (1.141) can be deduced from Lagrange equations of equation (1.144). Furthermore, we know that Noether symmetry refers to action invariance, so if

$$S^*(q^*) = S(q) \tag{1.145}$$

remains true under infinitesimal transformations, the invariance is known as Noether Symmetry. There exists a boundary function $g(t, q^i, \dot{q}^i)$ for ξ and η, such that

$$\frac{\partial(\boldsymbol{X}^{[1]}L)}{\partial t}\xi + \frac{\partial(\boldsymbol{X}^{[1]}L)}{\partial q^i}\eta^i + \frac{\partial(\boldsymbol{X}^{[1]}L)}{\partial \dot{q}^i}(\dot{\eta}^i - \dot{q}^i\dot{\xi}) + \boldsymbol{X}^{[1]}L\dot{\xi} = -\dot{g}. \tag{1.146}$$

We get the same equation as in equation (1.142), and this is known as the Noether identity for the problem (1.144). We can deduce the Noether first integral or Noether conserved quantity from this, which is the same as equation (1.143). From equations (1.141) and (1.146), it is clear that Mei symmetry differs from Noether symmetry in general.

The Lie point symmetry method and the Noether symmetry method have evolved significantly over time and are now used to solve various problems. On the contrary, much work and research on Mei symmetries remains unfinished, and they are still on their way to being applied to a variety of problems. Our primary goal is to find Mei symmetries for a specific Lagrangian presented in Section 2. More detailed discussion is given in [24].

Chapter 2

Mei Symmetries for the Lagrangian of Kerr Metric

Before finding the Mei symmetries for the Lagrangian of Kerr metric, a brief introduction of the Kerr metric is presented.

2.1 The Kerr Metric

The Kerr metric, discovered by Roy Kerr, describes the geometry of an empty spacetime in the vicinity of a spinning uncharged axisymmetric black hole. The Kerr metric is one of the well-known solutions to Einstein's field equations. The nonlinearity of these equations makes precise solutions extremely difficult to obtain..
In Boyer-Lindquist coordinates, the metric [25] is given by

$$\begin{aligned} ds^2 &= -c^2 d\tau^2, \\ &= \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[adt - (r^2+a^2)d\phi]^2 - \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2, \end{aligned} \tag{2.1}$$

where

$$\rho^2 = r^2 + a^2 cos^2\theta, \qquad \Delta = r^2 + a^2 - 2mr, \tag{2.2}$$

m is the mass of the rotational object, a is the spin parameter or specific angular momentum and is related to the angular momentum J by $a = \frac{J}{m}$. This spacetime admits two isometries or Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$. Thus, the only conserved quantities in this spacetime are energy and angular momentum.

2.2 Review of the Noether and Lie Point Symmetries for the Lagrangian of Kerr Metric

In this section, we review the approximate Noether symmetries of the geodesic equations for the charged-Kerr spacetime and rescaling of energy. **Ibrar Hussain, Fazal M. Mahomed and Asghar Qadir** [26]:

To begin, we write the Lagrangian of Kerr metric as

$$L = -\left(1 - \frac{2mr}{\rho^2}\right)\dot{t}^2 + \frac{\rho^2}{\Delta}\dot{r}^2 + \rho^2\dot{\theta}^2 + \frac{\sin^2\theta}{\rho^2}\Sigma\dot{\phi}^2 - \frac{4mar\sin^2\theta}{\rho^2}\dot{t}\dot{\phi}, \tag{2.3}$$

where

$$\Sigma = \left[(r^2 + a^2)^2 - a^2\sin^2\theta\Delta\right] \tag{2.4}$$

According to [26], equation (1.110) can be solved to get Noether symmetries for the Lagrangian of Kerr metric given by equation (2.3). The obtained Noether symmetries are

$$\boldsymbol{X}_1 = \frac{\partial}{\partial s}, \quad \boldsymbol{X}_2 = \frac{\partial}{\partial t}, \quad \boldsymbol{X}_3 = \frac{\partial}{\partial \phi}. \tag{2.5}$$

We can infer from this that the isometries are a subalgebra of the Noether symmetries. Furthermore, the geodesic equations (1.139) for Lagrangian given by equation (2.3) are

$$\begin{aligned}\ddot{t} = &-\frac{2m(r^2+a^2)\Omega}{\rho^4\Delta}\dot{t}\dot{r} + \frac{4ma^2 r\sin\theta\cos\theta}{\rho^4}\dot{t}\dot{\theta} - \frac{4ma^3 r\sin^3\theta\cos\theta}{\rho^4}\dot{\theta}\dot{\phi}\\ &+ \frac{2ma\sin^2\theta[(r^2+a^2)\Omega + 2r\rho^2]}{\rho^4\Delta}\dot{r}\dot{\phi},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\ddot{r} = &\frac{m\Omega - a^2 r\sin^2\theta}{\rho^2\Delta}\dot{r}^2 + \frac{2a^2\sin\theta\cos\theta}{\rho^2}\dot{r}\dot{\theta} - \frac{m\Delta\Omega}{\rho^6}\dot{t}^2 + \frac{2ma\sin^2\theta\Delta\Omega}{\rho^6}\dot{t}\dot{\phi}\\ &- \frac{[ma^2\sin^4\theta\Delta\Omega - r\sin^2\theta\Delta\rho^4]}{\rho^6}\dot{\phi}^2 + \frac{r\Delta}{\rho^2}\dot{\theta}^2,\end{aligned} \tag{2.7}$$

$$\ddot{\theta} = \frac{a^2 \sin\theta\cos\theta}{\rho^2}\dot{\theta}^2 + \frac{2ma^2 r\sin\theta\cos\theta}{\rho^6}\dot{t}^2 - \frac{4mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\dot{t}\dot{\phi}$$
$$- \frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\dot{r}^2 + \frac{\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^6}\dot{\phi}^2 - \frac{2r}{\rho^2}\dot{r}\dot{\theta}, \tag{2.8}$$
$$\ddot{\phi} = -\frac{2ma\Omega}{\rho^4\Delta}\dot{t}\dot{r} + \frac{4mar\cot\theta}{\rho^4}\dot{t}\dot{\theta} + \frac{2ma^2\sin^2\theta\Omega + 2r(-\rho^2+2mr)\rho^2}{\rho^4\Delta}\dot{r}\dot{\phi}$$
$$- \frac{2\cot\theta\rho^4 + 4ma^2 r\sin\theta\cos\theta}{\rho^4}\dot{\theta}\dot{\phi}, \tag{2.9}$$

where

$$\Omega = (r^2 - a^2\cos^2\theta). \tag{2.10}$$

Applying the condition (1.48) for the Lie point symmetries of ODEs on these equations (2.6)-(2.9) yields the exact or Lie point symmetries corresponding to Kerr metric. The obtained Lie point symmetries are

$$\boldsymbol{X}_1 = \frac{\partial}{\partial s}, \quad \boldsymbol{X}_2 = s\frac{\partial}{\partial s}, \quad \boldsymbol{X}_3 = \frac{\partial}{\partial t}, \quad \boldsymbol{X}_4 = \frac{\partial}{\partial \phi}. \tag{2.11}$$

In addition to the previously described Noether symmetries, one additional Lie point symmetry $s\partial/\partial s$ is obtained. The set of Noether symmetry is thus said to be a subset of the set of Lie point symmetry.

Next, we compute the Mei symmetries for the same Lagrangian as in equation to examine how they compare to the Lie and Noether symmetries.

2.3 The Kerr metric's Mei Symmetry

Considering the method for the Mei symmetries as

$$E_i[\boldsymbol{X}^{[1]}L] = 0. \tag{2.12}$$

Here L is the Lagrangian, whereas $E_i = \frac{d}{ds}\frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i}$ is the Euler operator and $\boldsymbol{X}^{[1]} = \xi\frac{\partial}{\partial s} + \eta^i\frac{\partial}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial}{\partial \dot{q}^i}$ is the first extended infinitesimal generator.

Applying first prolonged generator on the Lagrangian given in (2.3) yields

$$\boldsymbol{X}^{[1]}L = (\dot{\eta}^1 - \dot{t}\dot{\xi})\Big[-2(1-\frac{2mr}{\rho^2})\dot{t} - \frac{4mar\sin^2\theta}{\rho^2}\dot{\phi}\Big] + \eta^2\Big[-\frac{2m\Omega}{\rho^4}\dot{t}^2 - \frac{2m\Omega - a^2 r\sin^2\theta}{\Delta^2}\dot{r}^2 + 2r\dot{\theta}^2 - \frac{2ma^2\sin^4\theta\Omega - 2r\sin^2\theta}{\rho^4}\dot{\phi}^2 + \frac{4ma\sin^2\theta\Omega}{\rho^4}\dot{t}\dot{\phi}\Big] + (\dot{\eta}^2 - \dot{r}\dot{\xi})\Big[\frac{2\rho^2}{\Delta}\dot{r}\Big] + \eta^3\Big[-\frac{2a^2\sin\theta\cos\theta}{\Delta}\dot{r}^2 + \frac{4ma^2 r\sin\theta\cos\theta}{\rho^4}\dot{t}^2 + \frac{8mar\sin\theta\cos\theta(r^2+a^2)}{\rho^4}\dot{t}\dot{\phi} + 2\sin\theta \frac{\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^4}\dot{\phi}^2 - 2a^2\sin\theta\cos\theta\dot{\theta}^2\Big] + \big(\dot{\eta}^3 - \dot{\theta}\dot{\xi}\big)\Big[2\rho^2\dot{\theta}\Big] + (\dot{\eta}^4 - \dot{\phi}\dot{\xi})\Big[\frac{2\sin^2\theta}{\rho^2}\Sigma\dot{\phi} - \frac{4mar\sin^2\theta}{\rho^2}\dot{t}\Big]. \tag{2.13}$$

For $q^1 = t$, equation (2.12) yields

$$\Big[\frac{d}{ds}\frac{\partial}{\partial\dot{t}} - \frac{\partial}{\partial t}\Big]\big[\boldsymbol{X}^{[1]}L\big] = 0. \tag{2.14}$$

Using equation (2.13) in equation (2.14) and substituting equations (2.6) to (2.9). Simplifying it further and then powers of $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ are compared to get system of determining equations as follows:

$$(constant) : (-\rho^2 + 2mr)\eta^1_{ss} - 2mar\sin^2\theta\eta^4_{ss} = 0, \tag{2.15a}$$

$$(\dot{t}) : (-\rho^2 + 2mr)\xi_{ss} - (-\rho^2 + 2mr)\eta^1_{st} + 2mar\sin^2\theta\eta^4_{ss} + \frac{m\Omega}{\rho^2}\eta^2_s - \frac{2ma^2 r\sin\theta\cos\theta}{\rho^2}\eta^3_s = 0, \tag{2.15b}$$

$$(\dot{r}) : (-\rho^2 + 2mr)\eta^1_{sr} - 2mar\sin^2\theta\eta^4_{sr} - \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^4_s - \frac{m\Omega}{\rho^2}\eta^1_s = 0, \tag{2.15c}$$

$$(\dot{\theta}) : (-\rho^2 + 2mr)\eta^1_{s\theta} - 2mar\sin^2\theta\eta^4_{s\theta} + \frac{2ma^2 r\sin\theta\cos\theta}{\rho^2}\eta^1_s - \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^4_s = 0, \tag{2.15d}$$

$$(\dot{\phi}) : (2mar\sin^2\theta)\xi_{ss} + (-\rho^2 + 2mr)\eta^1_{s\phi} - 2mar\sin^2\theta\eta^4_{s\phi}$$

$$+\frac{ma\sin^2\theta\Omega}{\rho^2}\eta^2_s-\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^3_s=0, \tag{2.15e}$$

$$(\dot{t}^2):4(-\rho^2+2mr)\xi_{st}-(-\rho^2+2mr)\eta^1_{tt}+2mar\sin^2\theta\eta^4_{tt}+$$
$$\Delta\frac{m(-\rho^2+2mr)\Omega}{\rho^6}\eta^1_r-\frac{2ma^2r\sin\theta\cos\theta(-\rho^2+2mr)}{\rho^6}\eta^1_\theta$$
$$+\frac{2m\Omega}{\rho^2}\eta^2_t-\frac{4ma^2r\sin\theta\cos\theta}{\rho^2}\eta^3_t-\frac{2m^2ar\sin^2\theta\Delta\Omega}{\rho^6}\eta^4_r$$
$$+\frac{4ma^2a^3r^2\sin^3\theta\cos\theta}{\rho^6}\eta^4_\theta=0, \tag{2.15f}$$

$$(\dot{r}^2):(-\rho^2+2mr)\eta^1_{rr}-2mar\sin^2\theta\eta^4_{rr}+\frac{(-\rho^2+2mr)}{\rho^2\Delta}$$
$$\big[m\Omega-a^2r\sin^2\theta\big]\eta^1_r-\frac{2m\Omega}{\rho^2}\eta^1_r-\frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\big(-\rho^2$$
$$+2mr\big)\eta^1_\theta+\frac{2ma^3r\sin^3\theta\cos\theta}{\rho^2\Delta}\eta^4_\theta+\frac{2ma\sin^2\theta\Omega}{\rho^2}\eta^4_r$$
$$-\frac{2mar\sin^2\theta[m\Omega-a^2r\sin^2\theta]}{\rho^2\Delta}\eta^4_r=0, \tag{2.15g}$$

$$(\dot{\theta}^2):(-\rho^2+2mr)\eta^1_{\theta\theta}+\frac{2a^2\sin\theta\cos\theta(-\rho^2+4mr)}{\rho^2}\eta^1_\theta+\frac{r\Delta}{\rho^2}$$
$$\big(-\rho^2+2mr\big)\eta^1_r-\frac{4mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^4_\theta-2ma^3r$$
$$\big[\frac{\sin^3\theta\cos\theta(-\rho^2+2mr)}{\rho^2}\big]\eta^4_\theta-\frac{2mar^2\sin^2\theta\Delta}{\rho^2}\eta^4_r-2mar$$
$$\sin^2\theta\eta^4_{\theta\theta}=0, \tag{2.15h}$$

$$(\dot{\phi}^2):(8mar\sin^2\theta)\xi_{s\phi}+\frac{2ma\sin^2\theta\Omega}{\rho^2}\eta^2_\phi-2mar\sin^2\theta\eta^4_{\phi\phi}+$$
$$\frac{(-\rho^2+2mr)}{\rho^6}\big[\sin\theta\cos\theta(r^2+a^2)\Sigma-a^2\sin^3\theta\cos\theta\Delta\rho^2\big]\eta^1_\theta$$
$$-\frac{(-\rho^2+2mr)\Delta}{\rho^6}\big[ma^2\sin^4\theta\Omega-r\sin^2\theta\rho^4\big]\eta^1_r-\frac{2mar}{\rho^6}$$
$$\sin^2\theta\big[\sin\theta\cos\theta(r^2+a^2)\Sigma-a^2\sin^3\theta\cos\theta\Delta\rho^2\big]\eta^4_\theta+\Delta$$
$$\frac{(2mar\sin^2\theta)}{\rho^6}\big[ma^2\sin^4\theta\Omega-r\sin^2\theta\rho^4\big]\eta^4_r+\big(-\rho^2+2$$
$$mr\big)\eta^1_{\phi\phi}-\frac{4mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^3_\phi=0, \tag{2.15i}$$

$$(\dot{t}\dot{r}):(-\rho^2+2mr)\xi_{sr}-(-\rho^2+2mr)\eta^1_{tr}+2mar\sin^2\theta\eta^4_{tr}$$

$$+\frac{m\Omega}{\rho^2}\eta^1_t+\frac{(-\rho^2+2mr)}{\rho^4\Delta}\big[m(r^2+a^2)\Omega\big]\eta^1_t+\frac{(-\rho^2+2mr)}{\rho^4\Delta}$$

$$\big[ma\Omega\big]\eta^1_\phi+\frac{m\Omega}{\rho^2}\eta^2_r-\frac{2mr}{\rho^4}\big[-3a^2\cos^2\theta+r^2\big]\eta^2-\frac{2m^2\Omega^2}{\rho^4\Delta}\eta^2$$

$$-2ma^2r\big[\frac{\sin\theta\cos\theta}{\rho^2}\big]\eta^3_r-\big[\frac{2mr(r^2+a^2)}{\rho^2\Delta}+1\big]\big[\frac{ma\sin^2\theta\Omega}{\rho^2}\big]$$

$$\eta^4_t-\frac{2m^2a^2r\sin^2\theta\Omega}{\rho^4\Delta}\eta^4_\phi+\frac{2ma^2\sin\theta\cos\theta(3r^2-a^2\cos^2\theta)}{\rho^4}\eta^3$$

$$=0, \tag{2.15j}$$

$$(\dot{t}\dot{\theta}):2(-\rho^2+2mr)\xi_{s\theta}-(-\rho^2+2mr)\eta^1_{t\theta}+2mar\sin^2\theta\eta^4_{t\theta}+2$$

$$mar\cot\theta\frac{(-\rho^2+2mr)}{\rho^4}\eta^1_\phi-\frac{4m^2a^2r\sin\theta\cos\theta}{\rho^4}\eta^1_t+\frac{m}{\rho^2}\Omega\eta^2_\theta$$

$$-\frac{2ma^2r\sin\theta\cos\theta}{\rho^2}\eta^3_\theta+\frac{2ma^2r\cos^2\theta\big(4mr\rho^2-1\big)}{\rho^2}\eta^3+2$$

$$(\frac{ma^2r\sin^2\theta\Omega}{\rho^4})\eta^3+\frac{2mar\sin\theta cos\theta(r^2+a^2)}{\rho^2}\eta^4_t+\frac{4m^2a^3r^2}{\rho^4}$$

$$(\sin^3\theta\cos\theta)\eta^4_t+\frac{4m^2a^2r^2\sin\theta\cos\theta}{\rho^4}\eta^4_\phi-\frac{2ma^2\sin\theta\cos\theta}{\rho^4}$$

$$\big(3r^2-a^2\cos^2\theta\big)\eta^2=0, \tag{2.15k}$$

$$(\dot{t}\dot{\phi}):(-\rho^2+2mr)\eta^1_{t\phi}+\frac{ma\sin^2\theta\Omega}{\rho^2}\eta^2_t+\frac{(-\rho^2+2mr)}{\rho^6}\big[ma\Delta$$

$$\sin^2\theta\Omega\big]\eta^1_r-\frac{(-\rho^2+2mr)}{\rho^6}\big[2mar\sin\theta\cos\theta(r^2+a^2)\big]\eta^1_\theta$$

$$-\frac{m\Omega}{\rho^2}\eta^2_\phi-\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^3_t+4mar\sin^2\theta\xi_{st}$$

$$+\frac{2ma^2r\sin\theta\cos\theta}{\rho^2}\eta^3_\phi-\frac{2m^2a^2r\sin^4\theta\Delta\Omega}{\rho^6}\eta^4_r-2\big(-\rho^2$$

$$+2mr\big)\xi_{s\phi}+\frac{4m^2a^2r^2\sin^3\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^4_\theta+2mar$$

$$\sin^2\theta\eta^4_{t\phi}=0, \tag{2.15l}$$

$$(\dot{r}\dot{\theta}):(-\rho^2+2mr)\eta^1_{r\theta}-2mar\sin^2\theta\eta^4_{r\theta}-2mar\sin\theta\cos\theta\eta^4_r$$

$$+\frac{a^2\sin\theta\cos\theta(-\rho^2+4mr)}{\rho^2}\eta^1_r-\frac{1}{\rho^2}\big[m\Omega+r\big(-\rho^2+2$$

$$mr\big)\big]\eta^1_\theta+\frac{ma\sin^2\theta\Omega+2mar^2\sin^2\theta}{\rho^2}\eta^4_\theta=0, \tag{2.15m}$$

$$(\dot{r}\dot{\phi}) : +4mar\sin^2\theta\xi_{sr} + (-\rho^2+2mr)\eta^1_{r\phi} - 2mar\sin^2\theta\eta^4_{r\phi}$$
$$+ \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^2_r + \frac{ma\sin^2\theta(-\rho^2+2mr)}{\rho^4\Delta}\big[(r^2+a^2)\Omega$$
$$+ 2r^2\rho^2\big]\eta^1_t + \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^3_r - \frac{m\Omega}{\rho^2}\eta^1_\phi$$
$$+ \frac{(-\rho^2+2mr)}{\rho^4\Delta}\big[ma^2\sin^2\theta\Omega + r(-\rho^2+2mr)\rho^2\big]\eta^1_\phi$$
$$+ \frac{8ma^3\sin\theta\cos\theta\Omega}{\rho^4}\eta^3 - \frac{2m^2a^2r\sin^4\theta}{\rho^4\Delta}\big[(r^2+a^2)\Omega$$
$$+ 2r^2\rho^2\big]\eta^4_t - \frac{(2mar\sin^2\theta)}{\rho^4\Delta}\big[ma^2\sin^2\theta\Omega + r\big(-\rho^2$$
$$+ 2mr\big)\rho^2\big]\eta^4_\phi - \frac{2ma\sin^2\theta\Omega}{\rho^4\Delta}[m\Omega + r\rho^2]\eta^2 - \frac{2mar}{\rho^4}$$
$$\sin^2\theta\big(r^2 - 3a^2\cos^2\theta\big)\eta^2 + \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^4_\phi + \big(r^2+a^2\big)$$
$$\frac{2mar\sin\theta\cos\theta}{\rho^2}\eta^3_r = 0, \tag{2.15n}$$
$$(\dot{\theta}\dot{\phi}) : 4mar\sin^2\theta\xi_{s\theta} + \big(-\rho^2+2mr\big)\eta^1_{\theta\phi} - 2mar\sin^2\theta\eta^4_{t\theta}$$
$$- (\frac{\cot\theta\rho^4 - 2ma^2r\sin\theta\cos\theta}{\rho^4})\big(-\rho^2+2mr\big)\eta^1_\phi + 2$$
$$\frac{ma^2r\sin\theta\cos\theta}{\rho^2}\eta^1_\phi - \frac{2ma^3r\sin^3\theta\cos\theta(-\rho^2+2mr)}{\rho^4}\eta^1_t$$
$$+ \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^2_\theta + \frac{4ma^3r^2\sin^3\theta\cos\theta}{\rho^4}\eta^2 + 2mar\sin^2\theta$$
$$\frac{(r^2-3a^2\cos^2\theta)}{\rho^4}\eta^3 + \frac{8m^2a^3r^2\sin^2\theta\cos^2\theta}{\rho^4}\eta^3 + 2mar$$
$$\frac{\cos^2\theta(r^2+a^2)}{\rho^2}\eta^3 - \frac{2mar\sin\theta cos\theta(r^2+a^2)}{\rho^2}\eta^3_\theta + 4m^2$$
$$\frac{a^4r^2\sin^5\theta\cos\theta}{\rho^4}\eta^4_t + \frac{2ma^3r\sin^3\theta\cos\theta(-\rho^2+2mr)}{\rho^2}\eta^4_\phi$$
$$= 0, \tag{2.15o}$$
$$(\dot{t}^2\dot{r}) : \xi_{tt} - \frac{m\Delta\Omega}{\rho^6}\xi_r + \frac{2ma^2r\sin\theta\cos\theta}{\rho^6}\xi_\theta = 0, \tag{2.15p}$$
$$(\dot{r}^3) : \xi_{rr} + \frac{m\Omega - a^2r\sin^2\theta}{\rho^2\Delta}\xi_r - \frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\xi_\theta = 0, \tag{2.15q}$$

$$(\dot{r}\dot{\theta}^2): \xi_{\theta\theta} + \frac{r\Delta}{\rho^2}\xi_r + \frac{a^2\sin\theta\cos\theta}{\rho^2}\xi_\theta = 0, \tag{2.15r}$$

$$(\dot{r}\dot{\phi}^2): +\xi_{\phi\phi} + \frac{\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^6}\xi_\theta$$
$$- \frac{\Delta[ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^6}\xi_r = 0, \tag{2.15s}$$

$$(\dot{t}\dot{r}^2): \xi_{tr} - \frac{m(r^2+a^2)\Omega}{\rho^4\Delta}\xi_t - \frac{ma\Omega}{\rho^4\Delta}\xi_\phi = 0, \tag{2.15t}$$

$$(\dot{t}\dot{r}\dot{\theta}): \xi_{t\theta} + \frac{2ma^2r\sin\theta\cos\theta}{\rho^4}\xi_t + \frac{2mar\cot\theta}{\rho^4}\xi_\phi = 0, \tag{2.15u}$$

$$(\dot{t}\dot{r}\dot{\phi}): \xi_{t\phi} + \frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\xi_r - \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\xi_\theta = 0, \tag{2.15v}$$

$$(\dot{r}^2\dot{\theta}): \xi_{r\theta} + \frac{a^2\sin\theta\cos\theta}{\rho^2}\xi_r - \frac{r}{\rho^2}\xi_\theta = 0, \tag{2.15w}$$

$$(\dot{r}^2\dot{\phi}): \xi_{r\phi} + \frac{ma\sin^2\theta[(r^2+a^2)\Omega + 2r^2\rho^2]}{\rho^4\Delta}\xi_t$$
$$+ \frac{[ma^2\sin^2\theta\Omega + r(-\rho^2+2mr)\rho^2]}{\rho^4\Delta}\xi_\phi = 0, \tag{2.15x}$$

$$(\dot{r}\dot{\theta}\dot{\phi}): \xi_{\theta\phi} - \frac{2ma^3r\sin^3\theta\cos\theta}{\rho^4}\xi_t - \frac{[\cot\theta\rho^4 - 2ma^2r\sin\theta\cos\theta]}{\rho^4}\xi_\phi$$
$$= 0. \tag{2.15y}$$

When $q^2 = r$ is substituted into equation (2.12), we get

$$\left[\frac{d}{ds}\frac{\partial}{\partial\dot{r}} - \frac{\partial}{\partial r}\right]\left[\boldsymbol{X}^{[1]}L\right] = 0. \tag{2.16}$$

Again, using equation (2.13) in equation (2.16), further simplification after substituting equations (2.6) to (2.9), as well as the coefficients of $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ and their powers' comparison, yields some identical equations as subequations ((2.15p)-(2.15y)) and the remaining ones are listed as

$$(constant): \eta^2_{ss} = 0, \tag{2.17a}$$

$$(\dot{t}): \eta^2_{st} + \frac{m\Delta\Omega}{\rho^6}\eta^1_s + \frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^4_s = 0, \tag{2.17b}$$

$$(\dot{r}): \xi_{ss} - \eta^2_{sr} + \frac{m\Omega - a^2r\sin^2\theta}{\rho^2\Delta}\eta^2_s + \frac{a^2\sin\theta\cos\theta}{\rho^2}\eta^3_s = 0, \tag{2.17c}$$

$$(\dot{\theta}) : \eta^2_{s\theta} - \frac{a^2 \sin\theta\cos\theta}{\rho^2}\eta^2_s - \frac{r\Delta}{\rho^2}\eta^3_s = 0, \tag{2.17d}$$

$$(\dot{\phi}) : \eta^2_{s\phi} + \Delta\frac{ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4}{\rho^6}\eta^4_s - ma\sin^2\theta\frac{\Delta\Omega}{\rho^6}\eta^1_s = 0, \tag{2.17e}$$

$$(\dot{t}^2) : \eta^2_{tt} + \frac{2m\Delta\Omega}{\rho^6}\eta^1_t - \frac{2m\Delta\Omega}{\rho^6}\eta^2_r + \frac{2m\Omega[m\Omega - a^2r\sin^2\theta]}{\rho^8}\eta^2 - 2$$
$$[\frac{m\Delta(r^2 - 3a^2\cos^2\theta)}{\rho^8}]\eta^2 + \frac{4ma^2\sin\theta\cos\theta\Delta(2r^2 - a^2\cos^2\theta)}{\rho^6}\eta^3$$
$$+ \frac{2ma^2r\sin\theta\cos\theta}{\rho^6}\eta^2_\theta - \frac{2ma\sin^2\theta\Delta\Omega}{\rho^2}\eta^4_t = 0, \tag{2.17f}$$

$$(\dot{r}^2) : 4\xi_{sr} - \eta^2_{rr} + \frac{m\Omega - a^2r\sin^2\theta}{\rho^2\Delta}\eta^2_r + \frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\eta^2_\theta + \frac{2a^2}{\rho^2}$$
$$\sin\theta\cos\theta\eta^3_r - \frac{(2r - 2m)[m\Omega - a^2r\sin^2\theta]}{\rho^2\Delta}\eta^2 - \frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}$$
$$\big[2r - 2m\big]\eta^3 + \frac{a^2\sin\theta\Omega + 4ma^2r\cos^2\theta}{\rho^2\Delta}\eta^2 + \frac{2a^2\sin\theta\cos\theta}{\rho^4\Delta}$$
$$\big[m\Omega - a^2r\sin^2\theta\big]\eta^3 = 0, \tag{2.17g}$$

$$(\dot{\theta}^2) : \eta^2_{\theta\theta} + \frac{r\Delta}{\rho^2}\eta^2_r - \frac{a^2\sin\theta\cos\theta}{\rho^2}\eta^2_\theta - \frac{2r\Delta}{\rho^2}\eta^3_\theta - \frac{2a^2r\sin\theta\cos\theta\Delta}{\rho^4}\eta^3$$
$$- \frac{2r[m\Omega - a^2r\sin^2\theta] + \Delta\rho^2}{\rho^4}\eta^2 = 0, \tag{2.17h}$$

$$(\dot{\phi}^2) : \eta^2_{\phi\phi} - \frac{\Delta[ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^6}\eta^2_r + \frac{\sin\theta\cos\theta}{\rho^6}\big[(r^2 + a^2)$$
$$\Sigma - a^2\sin^2\theta\Delta\rho^2\big]\eta^2_\theta - \frac{2ma^2\sin^2\theta\Delta\Omega}{\rho^6}\eta^1_\phi - \frac{4mar\sin\theta\cos\theta}{\rho^2}$$
$$(r^2 + a^2)\eta^3_\phi + \frac{2[m\Omega - a^2r\sin^2\theta]}{\rho^8}\big[ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4\big]\eta^2$$
$$+ \frac{2\Delta[ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^6}\eta^4_\phi + \frac{2\cot\theta\Delta(r^2 + a^2)}{\rho^8}\big[ma^2$$
$$\sin^4\theta\Omega - r\sin^2\theta\rho^4\big]\eta^3 - \frac{2ma^2\sin^3\theta\Delta}{\rho^8}\big[\cos\theta\big[\rho^4 - a^2\sin^2\theta$$
$$\Omega\big]\big]\eta^3 + \frac{2a^2r\sin^2\theta\Sigma}{\rho^8}\eta^3 - 2ma^2r\sin^4\theta\big[\frac{(r^2 - 3a^2\cos^2\theta)}{\rho^8} +$$
$$\frac{\sin^2\theta\Delta}{\rho^2}\big]\eta^2 = 0, \tag{2.17i}$$

$$(\dot{t}\dot{r}) : \xi_{st} - \eta^2_{tr} - \frac{m\Delta\Omega}{\rho^6}\eta^1_r + \frac{m(r^2 + a^2)\Omega}{\rho^4\Delta}\eta^2_t + \frac{m\Omega - a^2r\sin^2\theta}{\rho^2\Delta}\eta^2_t$$

$$+\frac{ma\Omega}{\rho^4\Delta}\eta^2_\phi+\frac{a^2\sin\theta\cos\theta}{\rho^2}\eta^3_t+\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^4_r=0, \tag{2.17j}$$

$$(\dot{t}\dot{\theta}):\eta^2_{t\theta}+\frac{m\Delta\Omega}{\rho^6}\eta^1_\theta+\frac{a^2\sin\theta\cos\theta(-\rho^2+2mr)}{\rho^2}\eta^2_t-\frac{r\Delta}{\rho^2}\eta^3_t$$
$$-\frac{2mar\cot\theta}{\rho^4}\eta^2_\phi-\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^4_\theta=0, \tag{2.17k}$$

$$(\dot{t}\dot{\phi}):\eta^2_{t\phi}-\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^1_t+\frac{m\Delta\Omega}{\rho^6}\eta^1_\phi+\frac{\Delta}{\rho^6}\big[ma^2\sin^4\theta\Omega-r$$
$$\sin^2\theta\rho^4\big]\eta^4_t-\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^2_\theta-\frac{2ma\sin^2\theta\Delta}{\rho^8}$$
$$\big[-r(r^2-3a^2\cos^2\theta)+\Omega\big(m\Omega-a^2r\sin^2\theta\big)\big]\eta^2-8mar$$
$$\sin\theta\Delta\cos\theta\big[\frac{(r^2+a^2)\Omega}{\rho^8}\big]\eta^3-2ma^3r\sin^3\theta\cos^2\theta\Delta\frac{\Omega}{\rho^8}\eta^3$$
$$+\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^2_r-ma\Delta\frac{\sin^2\theta\Omega}{\rho^6}\eta^4_\phi=0, \tag{2.17l}$$

$$(\dot{r}\dot{\theta}):2\xi s\theta-\eta^2_{r\theta}-\frac{r}{\rho^2}\eta^2_\theta+\frac{m\Omega-a^2r\sin^2\theta}{\rho^2}\eta^1_\theta+\frac{a^2\sin\theta\cos\theta}{\rho^2}\eta^3_\theta$$
$$-\frac{a^2\sin\theta\cos\theta(2r-2m)}{\rho^2\Delta}\eta^2-\frac{a\sin^2\theta\Omega-a^2\cos^2\theta\rho^2}{\rho^4}\eta^3-r$$
$$\frac{\Delta}{\rho^2}\eta^3_r=0, \tag{2.17m}$$

$$(\dot{r}\dot{\phi}):2\xi_{s\phi}-\eta^2_{r\phi}+\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^1_r+\frac{m\Omega-a^2r\sin^2\theta}{\rho^2\Delta}\eta^2_\phi+a^2$$
$$\frac{\sin\theta\cos\theta}{\rho^2}\eta^3_\phi-\frac{ma^2\sin^2\theta\Omega+r(-\rho^2+2mr)\rho^2}{\rho^4\Delta}\eta^2_\phi-\frac{\Delta}{\rho^6}$$
$$[ma^2\sin^2\theta\Omega-r\sin^2\theta\rho^4]\eta^4_r-\frac{ma\sin^2\theta}{\rho^4\Delta}\big((r^2+a^2)\Omega+2$$
$$r^2\rho^2\big)\eta^2_t=0, \tag{2.17n}$$

$$(\dot{\theta}\dot{\phi}):\eta^2_{\theta\phi}-\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^1_\theta-\frac{2ma^3r\sin^3\theta\cos\theta}{\rho^4}\eta^2_\phi+\frac{\Delta}{\rho^6}\big[ma^2$$
$$\sin^4\theta\Omega-r\sin^2\theta\rho^4\big]\eta^4_\theta+\frac{\cot\theta\rho^4-2ma^2r\sin\theta\cos\theta-r\Delta\rho^2}{\rho^4}$$
$$\eta^3_\phi=0. \tag{2.17o}$$

When $q^3 = \theta$ is substituted into equation (2.12), it becomes

$$\left[\frac{d}{ds}\frac{\partial}{\partial\dot{\theta}} - \frac{\partial}{\partial\theta}\right]\left[\boldsymbol{X}^{[1]}L\right] = 0. \tag{2.18}$$

Now, by using equation (2.13) in equation (2.18) and substituting equations (2.6) to (2.9), it is further simplified, which provides some similar equations ((2.15p)-(2.15y)) are written above. The determining equations are

$$(constant) : \eta^3_{ss} = 0, \tag{2.19a}$$

$$(\dot{t}) : \eta^3_{st} - \frac{2ma^2 r\sin\theta\cos\theta}{\rho^6}\eta^1_s + \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^4_s = 0, \tag{2.19b}$$

$$(\dot{r}) : \eta^3_{sr} + \frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\eta^2_s + \frac{r}{\rho^2}\eta^3_s = 0, \tag{2.19c}$$

$$(\dot{\theta}) : \xi_{ss} - \eta^3_{s\theta} - \frac{r}{\rho^2}\eta^2_s + \frac{a^2\sin\theta\cos\theta}{\rho^2}\eta^3_s = 0, \tag{2.19d}$$

$$(\dot{\phi}) : \eta^3_{s\phi} + \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^1_s - \frac{\sin\theta\cos\theta}{\rho^6}\big[(r^2+a^2)\Sigma \\ - a^2\sin^2\theta\Delta\rho^2\big]\eta^4_s = 0, \tag{2.19e}$$

$$(\dot{t}^2) : -\frac{4ma^2 r\sin\theta\cos\theta}{\rho^6}\eta^1_t - \frac{m\Delta\Omega}{\rho^6}\eta^3_r + \frac{2ma^2 r}{\rho^6}\sin\theta\cos\theta\eta^3_\theta \\ + \frac{4mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^4_t + \frac{2ma^2\sin\theta\cos\theta}{\rho^8}\big(5r^2 - \\ a^2\cos^2\theta\big)\eta^2 + \frac{2ma^2 r[\sin^2\theta(r^2-5a^2\cos^2\theta) - \cos^2\theta\rho^2]}{\rho^8}\eta^3 \\ + \eta^3_{tt} = 0, \tag{2.19f}$$

$$(\dot{r}^2) : \frac{2a^2\sin\theta\cos\theta}{\rho^2\Delta}\eta^2_r - \frac{a^2\sin\theta\cos\theta[2r\Delta + (2r-2m)\rho^2]}{\rho^4\Delta^2}\eta^2 \\ + \frac{[m\Omega - a^2 r\sin^2\theta] + 2r\Delta}{\rho^2\Delta}\eta^3_r - \frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\eta^3_\theta + \eta^3_{rr} \\ + \frac{a^2\rho^2(\cos^2\theta - \sin^2\theta) + 2a^4\sin^2\theta\cos^2\theta}{\rho^2\Delta}\eta^3 = 0, \tag{2.19g}$$

$$(\dot{\theta}^2) : -\eta^3_{\theta\theta} - \frac{2r}{\rho^2}\eta^2_\theta - \frac{r\Delta}{\rho^2}\eta^3_r + \frac{a^2\sin\theta\cos\theta}{\rho^2}\eta^3_\theta - \frac{2a^2 r\sin\theta\cos\theta}{\rho^4}\eta^2 \\ + \frac{a^2\rho^2(\cos^2\theta - \sin^2\theta) + 2a^4\sin^2\theta\cos^2\theta}{\rho^4}\eta^3 + 4\xi_{s\theta} = 0, \tag{2.19h}$$

$$(\dot{\phi}^2):\ \eta^3_{\phi\phi}-\frac{\Delta[ma^2\sin^4\theta\Omega-r\sin^2\theta\rho^4]}{\rho^6}\eta^3_r+\frac{\sin\theta\cos\theta}{\rho^6}\big[(r^2+a^2)$$

$$\Sigma-a^2\sin^2\theta\Delta\rho^2\big]\eta^3_\theta-\frac{2\sin\theta\cos\theta}{\rho^6}\big[(r^2+a^2)\Sigma-a^2\sin^2\theta\Delta$$

$$\rho^2\big]\eta^4_\phi+\frac{2r\sin\theta\cos\theta[(r^2+a^2)\Sigma-a^2\sin^2\theta\Delta\rho^2]}{\rho^8}\eta^2+2\sin\theta$$

$$\frac{\cos\theta}{\rho^8}\big[-r\rho^6+2ma\sin^2\theta(r^2+a^2)\Omega+ma^3\sin^4\theta\big]\eta^2+\sin\theta$$

$$\frac{(r^2-5a^2\cos^2\theta)}{\rho^6}\big[-a^2\sin^2\theta\Delta\rho^2+(r^2+a^2)\Sigma\big]\eta^3-\frac{\cos^2\theta}{\rho^6}$$

$$\big[\big[(r^2+a^2)\Sigma-a^2\sin^2\theta\Delta\rho^2\big]-4a^2\sin^2\theta\Delta\rho^2\big]\eta^3+4mar\sin\theta$$

$$\frac{\cos\theta(r^2+a^2)}{\rho^4}\eta^1_\phi=0, \tag{2.19i}$$

$$(\dot{t}\dot{r}):\ \eta^3_{tr}-\frac{2ma^2r\sin\theta\cos\theta}{\rho^6}\eta^1_r+\frac{m(r^2+a^2)\Omega-r\rho^2\Delta}{\rho^4\Delta}\eta^3_t+\frac{ma\Omega}{\rho^4\Delta}$$

$$\eta^3_\phi+\frac{2a^2\sin\theta\cos\theta}{\rho^2\Delta}\eta^2_t+\frac{2mar\sin\theta(r^2+a^2)}{\rho^6}\cos\theta\eta^4_r=0, \tag{2.19j}$$

$$(\dot{t}\dot{\theta}):\ 2\xi_{st}-\eta^3_{t\theta}+\frac{2ma^2r\sin\theta\cos\theta}{\rho^6}\eta^1_\theta-\frac{r}{\rho^2}\eta^2_t-\frac{2mar\cot\theta}{\rho^4}\eta^3_\phi$$

$$-\frac{a^2\sin\theta\cos\theta(-\rho^2+2mr)}{\rho^2}\eta^3_t-\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^4_\theta$$

$$=0, \tag{2.19k}$$

$$(\dot{t}\dot{\phi}):\ \eta^3_{t\phi}+\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^1_t-\frac{2ma^2r\sin\theta\cos\theta}{\rho^6}\eta^1_\phi$$

$$-\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^3_\theta+\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^4_\phi$$

$$-\frac{\sin\theta(r^2-5a^2\cos^2\theta)}{\rho^6}\big[(r^2+a^2)\Sigma-a^2\sin^2\theta\Delta\rho^2\big]\eta^4_t-2$$

$$\frac{ma\sin\theta\cos\theta}{\rho^8}\big[2a^2r^2\sin^2\theta+(r^2+a^2)(3r^2-a^2\cos^2\theta\rho^2)\big]\eta^2$$

$$-\frac{2mar}{\rho^8}\big[-\cos^2\theta(r^2+a^2)+\sin^2\theta(r^2+a^2)\big(r^2-5a^2\cos^2\theta$$

$$\big)\big]\eta^3=0, \tag{2.19l}$$

$$(\dot{r}\dot{\theta}):\ 2\xi sr-\eta^3_{r\theta}-\frac{2a^2r\sin\theta\cos\theta}{\rho^4}\eta^3+\frac{\Omega}{\rho^4}\eta^2+\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^3_r$$

$$-\frac{r}{\rho^2}\eta^3_\theta-\frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\eta^2_\theta=0, \tag{2.19m}$$

$$(\dot{r}\dot{\phi}): \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^1_r + \frac{a^2\sin\theta\cos\theta}{\rho^2\Delta}\eta^2_\phi + \frac{r(\rho^2+2mr)}{\Delta}\eta^3_\phi$$
$$+\eta^3_{r\phi} - \frac{\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^6}\eta^4_r + \frac{1}{\rho^4\Delta}\big[ma\sin^2\theta$$
$$[(r^2+a^2)\Omega] + 2r^2\rho^2\big]\eta^3_t = 0, \tag{2.19n}$$
$$(\dot{\theta}\dot{\phi}): +2\xi_{s\phi} - \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^1_\theta - \frac{2ma^3r\sin^3\theta\cos\theta}{\rho^4}\eta^3_t$$
$$-\frac{r}{\rho^2}\eta^2_\phi - \eta^3_{\theta\phi} - \frac{\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^6}\eta^4_t$$
$$+\frac{\cot\theta(r^2+a^2)\rho^2 + 2ma^2r\sin\theta\cos\theta}{\rho^4}\eta^3_\phi = 0. \tag{2.19o}$$

For last variable $q^4 = \phi$, equation (2.12) returns

$$\left[\frac{d}{ds}\frac{\partial}{\partial\dot{\phi}} - \frac{\partial}{\partial\phi}\right]\big[\boldsymbol{X}^{[1]}L\big] = 0. \tag{2.20}$$

It is simplified by using equation (2.13) in equation (2.20) and substituting equations (2.6) to (2.9), then equating to zero the coefficients of $(\dot{t},\dot{r},\dot{\theta},\dot{\phi})$ and their powers produce some similar equations ((2.15p)-(2.15y)). The remaining equations are

$$(constant): 2mar\sin^2\theta\eta^1_{ss} - \sin^2\theta\Sigma\eta^4_{ss} = 0, \tag{2.21a}$$
$$(\dot{t}): 2mar\sin^2\theta\xi_{ss} - 2mar\sin^2\theta\eta^1_{st} + \sin^2\theta\Sigma\eta^4_{ss} + \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^2_s$$
$$-\frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^3_s = 0, \tag{2.21b}$$
$$(\dot{r}): 2mar\sin^2\theta\eta^1_{sr} + \frac{ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4}{\rho^2}\eta^4_s - \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^1_s$$
$$-\sin^2\theta\Sigma\eta^4_{sr} = 0, \tag{2.21c}$$
$$(\dot{\theta}): +2mar\sin^2\theta\eta^1_{s\theta} - \sin^2\theta\Sigma\eta^4_{s\theta} + \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^1_s$$
$$-\frac{\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\rho^2]}{\rho^2}\eta^4_s = 0, \tag{2.21d}$$
$$(\dot{\phi}): \sin^2\theta\Sigma\xi_{ss} - \sin^2\theta\Sigma\eta^4_{s\phi} + \frac{ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4}{\rho^2}\eta^2_s + 2mar$$
$$\sin^2\theta\eta^1_{s\phi} - \frac{\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^2}\eta^3_s = 0, \tag{2.21e}$$

$$(\dot{t}^2):-2mar\sin^2\theta\eta^1_{tt}+8mar\sin^2\theta\xi_{st}+\sin^2\theta\Sigma\eta^4_{tt}+\frac{2ma\sin^2\theta\Omega}{\rho^2}\eta^2_t$$
$$+\frac{2mar\sin^2\theta\Delta(-\rho^2+2mr)\Omega}{\rho^6}\eta^1_r-\frac{4m^2a^3r^2\sin^3\theta\cos\theta}{\rho^6}\eta^1_\theta-4m$$
$$(\frac{ar\sin\theta\cos\theta(r^2+a^2)}{\rho^2})\eta^3_t-\frac{m\sin^2\theta\Delta\Omega\Sigma}{\rho^6}\eta^4_r+\frac{2ma^2r\sin^3\theta\cos\theta}{\rho^6}$$
$$\Sigma\eta^4_\theta=0, \tag{2.21f}$$
$$(\dot{r}^2):-\frac{2ma^3r\sin^3\theta\cos\theta}{\rho^2\Delta}\eta^1_\theta+\frac{2mar\sin^2\theta[m\Omega-a^2r\sin^2\theta]}{\rho^2\Delta}\eta^1_r+2m$$
$$ar\sin^2\theta\eta^1_{rr}-\frac{2ma\sin^2\theta\Omega}{\rho^2}\eta^1_r-\frac{\sin^2\theta}{\rho^2\Delta}\big[\Sigma\big(m\Omega-a^2r\sin^2\theta\big)\big]\eta^4_r$$
$$+\frac{2ma^2\sin^4\theta\Omega-2r\sin^2\theta\rho^4}{\rho^2}\eta^4_r+\frac{a^2\sin^3\theta}{\rho^2\Delta}\big[\cos\theta\Sigma\big]\eta^4_\theta=0, \tag{2.21g}$$
$$(\dot{\theta}^2):2mar\sin^2\theta\eta^1_{\theta\theta}+\frac{2a^2\sin\theta\cos\theta(-\rho^2+2mr+mar\sin^2\theta)}{\rho^2}\eta^1_\theta$$
$$+\frac{2mar^2\sin^2\theta\Delta}{\rho^2}\eta^1_r-\frac{\sin\theta\cos\theta\Sigma[2(r^2+a^2)+a^2\sin^2\theta]}{\rho^2}\eta^4_\theta$$
$$-\frac{r\sin^2\theta\Delta\Sigma}{\rho^2}\eta^4_r-\sin^2\theta\Sigma\eta^4_{\theta\theta}=0, \tag{2.21h}$$
$$(\dot{\phi}^2):4\sin^2\theta\Sigma\xi_{s\phi}+2mar\sin^2\theta\eta^1_{\phi\phi}+\frac{2ma^2\sin^4\theta\Omega-2r\sin^2\theta\rho^4}{\rho^2}\eta^2_\phi$$
$$-\sin^2\theta\Sigma\eta^4_{\phi\phi}-\frac{2mar\sin^2\theta\Delta[ma^2\sin^4\theta\Omega-r\sin^2\theta\rho^4]}{\rho^6}\eta^1_r+2$$
$$(\frac{mar\sin^2\theta\cos\theta}{\rho^6})\big[(r^2+a^2)\Sigma-a^2\sin^2\theta\Delta\rho^2\big]\eta^1_\theta-\frac{2\sin\theta\cos\theta}{\rho^2}$$
$$\big[-a^2\sin^2\theta\Delta\rho^2+(r^2+a^2)\Sigma\big]\eta^3_\phi+\frac{\Delta\sin^2\theta\Sigma}{\rho^6}\big[ma^2\sin^4\theta\Omega$$
$$-r\sin^2\theta\rho^4\big]\eta^4_r-\big[(r^2+a^2)\Sigma-a^2\sin^2\theta\Delta\rho^2\big](\frac{\sin^3\theta\cos\theta\Sigma}{\rho^6})\eta^4_\theta$$
$$=0, \tag{2.21i}$$
$$(\dot{t}\dot{r}):4mar\sin^2\theta\xi_{sr}-\frac{2mar\sin^2\theta(r^2-3a^2\cos^2\theta)}{\rho^4}\eta^2-2mar$$
$$\sin^2\theta\eta^1_{tr}+\frac{ma\sin^2\theta\Omega[\rho^2\Delta+2mr(r^2+a^2)]}{\rho^2}\eta^1_t+\frac{ma\sin^2\theta\Omega}{\rho^2}\eta^2_r$$
$$+\frac{2m^2a^2r\sin^2\theta\Omega}{\rho^4\Delta}\eta^1_\phi-2ma\sin^2\theta\big[\frac{\Omega[m\Omega-\rho^2]}{\rho^4\Delta}\big]\eta^2+2ma^2$$

$$\sin\theta\cos\theta[\frac{3r^2 - a^2\cos^2\theta}{\rho^4}]\eta^3 - \frac{2ma\sin\theta\cos\theta\Omega}{\rho^4}\eta^3 - 2mar$$
$$[\frac{\sin\theta\cos\theta(r^2+a^2)}{\rho^2}]\eta^3_r - (\frac{\Omega - r\sin^2\theta\rho^4}{\rho^2})ma^2\sin^4\theta\eta^4_t - \Omega\Sigma$$
$$\frac{m\sin^2\theta(r^2+a^2)}{\rho^4\Delta}\eta^4_t - \frac{ma\sin^2\theta\Omega\Sigma}{\rho^4\Delta}\eta^4_\phi + \sin^2\theta\Sigma\eta^4_{tr} = 0, \tag{2.21j}$$
$$(\dot{t}\dot{\theta}) : 4mar\sin^2\theta\xi_{s\theta} - 2mar\sin^2\theta\eta^1_{t\theta} + \sin^2\theta\Sigma\eta^4_{t\theta} + \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^2_\theta$$
$$- \frac{2mar\sin\theta\cos\theta[\rho^2(r^2+a^2) + 2ma^2r\sin^2\theta]}{\rho^4}\eta^1_t - (\frac{4m^2a^2r^2}{\rho^4})$$
$$\sin\theta\cos\theta\eta^1_\phi + \frac{2ma\sin\theta\cos\theta(r^2+a^2)(3r^2 - a^2\cos^2\theta)}{\rho^4}\eta^2 + 2$$
$$\frac{mar\sin^2\theta[(r^2+a^2)(r^2 - 3a^2\cos^2\theta) + 4ma^2r\cos^2\theta]}{\rho^4}\eta^3 - 2ma$$
$$\frac{r\sin\theta cos\theta(r^2+a^2)}{\rho^2}\eta^3_\theta + \frac{\sin\theta cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^2}\eta^4_t$$
$$+ \frac{2ma^2r^2\sin^3\theta\cos\theta\Sigma}{\rho^4}\eta^4_t + \frac{2mar\cos^2\theta(r^2+a^2)}{\rho^2}\eta^3 + 2mar$$
$$\sin\theta\cos\theta\Sigma\eta^4_\phi = 0, \tag{2.21k}$$
$$(\dot{t}\dot{\phi}) : 4mar\sin^2\theta\xi_{s\phi} - 2\sin^2\theta\Sigma\xi_{st} + \sin^2\theta\Sigma\eta^4_{t\phi} - 2mar\sin^2\theta\eta^1_{t\phi}$$
$$- \frac{2m^2a^2r\sin^4\theta\Delta\Omega}{\rho^6}\eta^1_r + 4m^2a^2r^2\cos\theta(\frac{\sin^3\theta(r^2+a^2)}{\rho^6})\eta^1_\theta$$
$$- \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^3_\phi - \frac{ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4}{\rho^2}\eta^2_t$$
$$+ \frac{\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^2}\eta^3_t + \frac{ma\sin^4\theta\Delta\Omega\Sigma}{\rho^6}\eta^4_r$$
$$- 2mar[\frac{\sin^3\theta\cos\theta(r^2+a^2)\Sigma}{\rho^6}]\eta^4_\theta + ma\sin^2\theta\frac{\Omega}{\rho^2}\eta^2_\phi = 0, \tag{2.21l}$$
$$(\dot{r}\dot{\theta}) : 2mar\sin^2\theta\eta^1_{r\theta} + \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^2}\eta^1_r - \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^1_\theta$$
$$- \sin^2\theta\Sigma\eta^4_{r\theta} + \frac{2ma^3r\sin^3\theta\cos\theta}{\rho^2}\eta^1_r - \frac{a^2\sin^3\theta\Sigma}{\rho^2}\eta^4_r - \sin\theta\cos\theta$$
$$\frac{[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^2}\eta^4_r + \frac{ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4}{\rho^2}\eta^4_\theta$$
$$+ \frac{r\sin^2\theta\Sigma}{\rho^2}\eta^4_\theta = 0, \tag{2.21m}$$

$$(\dot{r}\dot{\phi}) : 2\sin^2\theta\Sigma\xi_{sr} + 2mar\sin^2\theta\eta^1_{r\phi} - \sin^2\theta\Sigma\eta^4_{tr} - \frac{ma\sin^2\theta\Omega}{\rho^2}\eta^1_\phi + \frac{2mar\sin^2\theta[ma^2\sin^2\theta\Omega + r(-\rho^2+2mr)\rho^2]}{\rho^4\Delta}\eta^1_\phi - \sin\theta\cos\theta \frac{[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^2}\eta^3_r + \frac{2a^2\sin^3\theta\cos\theta}{\rho^4}\big[r\Sigma + m\rho^2 (-\rho^2+2mr)\big]\eta^3 - \frac{ma\sin^4\theta\Sigma}{\rho^4\Delta}[(r^2+a^2)\Omega + 2r^2\rho^2]\eta^4_t + \frac{\Sigma}{\rho^4\Delta} \big[ma^2\sin^4\theta\Omega + r\rho^2\sin^2\theta(-\rho^2+2mr)\big]\eta^4_\phi - \frac{2ma^2\sin^4\theta\Omega}{\rho^4\Delta}\big[r\rho^2 + m\Omega\big]\eta^2 - \frac{2ma^2r\sin^4\theta(r^2-3a^2\cos^2\theta) + \sin^2\theta\rho^6}{\rho^4}\eta^2 + ma^2r \frac{\sin^4\theta}{\rho^2}\big[(r^2+a^2)\Omega + 2r^2\rho^2\big]\eta^2_r + \frac{2m^2a^3r}{\rho^4\Delta}\big[\sin^4\theta(r^2+a^2)\Omega + 2r^2 \sin^4\theta\rho^2\big]\eta^1_t - ma^2\sin^4\theta(\frac{\Omega + r\sin^2\theta\rho^4}{\rho^4\Delta})\eta^4_\phi = 0, \tag{2.21n}$$

$$(\dot{\theta}\dot{\phi}) : 2\sin^2\theta\Sigma\xi_{s\theta} + 2mar\sin^2\theta\eta^1_{\theta\phi} - \sin^2\theta\Sigma\eta^4_{t\theta} + \frac{\sin\theta\cos\theta\Sigma}{\rho^2}\big[\big(r^2 + a^2\big) + (-\rho^2+2mr)a^2\sin^2\theta\big]\eta^4_\phi - \frac{\sin\theta\cos\theta}{\rho^2}\big[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2\big]\eta^4_\phi - \frac{4m^2a^4r^2\sin^5\theta\cos\theta}{\rho^4}\eta^1_t - \frac{2ma^3r\sin^3\theta\cos\theta}{\rho^4} (-\rho^2+2mr)\eta^1_\phi + \frac{ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4}{\rho^2}\eta^2_\theta + \frac{2ma^3r\sin^5\theta}{\rho^4} \cos\theta\Sigma\eta^4_t - \frac{2\sin\theta\cos\theta[ma^2\sin^2\theta(r^2-a^2) + r\rho^4]}{\rho^2}\eta^2 - \sin\theta \frac{\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^2}\eta^3_\theta + \frac{\sin^2\theta(r^2-3a^2\cos^2\theta)}{\rho^4} \big[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2\big]\eta^3 + \frac{2a^2\sin^2\theta\cos^2\theta\Delta}{\rho^4}\big(\rho^4 + 2ma^2 r\sin^2\theta\big)\eta^3 = 0. \tag{2.21o}$$

We now solve the above system of PDEs to determine values of $\xi, \eta^1, \eta^2, \eta^3$ and η^4. Differentiating equation (2.15v) w.r.t. t and differentiating equation (2.15p) w.r.t. ϕ, and when we solve them, we get the equation

$$2mar\sin^2\theta\xi_t - (-\rho^2 + 2mr)\xi_\phi = 0, \tag{2.22}$$

and by differentiating equation (2.15s) w.r.t. t and equation (2.15v) w.r.t. ϕ, we get the equation

$$(r^2 + a^2)\xi_t - a\xi_\phi = 0. \tag{2.23}$$

By solving equations (2.22), (2.23) simultaneously, we get

$$\xi_t = 0, \tag{2.24}$$

and

$$\xi_\phi = 0. \tag{2.25}$$

Differentiate equation (2.15t) w.r.t. t, we get the equation

$$\xi_{ttr} - \frac{m\Omega^2}{\rho^8}\xi_r = 0. \tag{2.26}$$

Using equations (2.24),(2.25), and (2.15t) yields

$$\xi_{tr} = 0. \tag{2.27}$$

Now from equation (2.26), we get the result

$$\xi_r = 0. \tag{2.28}$$

Similarly, by differentiating equation (2.15u) w.r.t. t, we get the equation

$$\xi_{tt\theta} + \frac{4m^2a^2r^2\cos^2\theta}{\rho^8}\xi_\theta = 0. \tag{2.29}$$

Using equations (2.24),(2.25), and (2.15u) yields

$$\xi_{t\theta} = 0. \tag{2.30}$$

Now from equation (2.29), we get the result

$$\xi_\theta = 0. \tag{2.31}$$

From equations (2.24),(2.25),(2.28) and (2.31) we know that ξ is a function of s only i.e.

$$\xi = \xi(s). \tag{2.32}$$

Now by solving equation (2.15a) and equation (2.21a) simultaneously, yields

$$\eta^1_{ss} = 0, \tag{2.33}$$

and

$$\eta^4_{ss} = 0, \tag{2.34}$$

If we differentiate equation (2.15b) w.r.t. s and making use of equations (2.33),(2.17a), (2.19a) and (2.34), we get

$$\xi_{sss} = 0. \tag{2.35}$$

By integrating above equation (2.38), gives

$$\xi = c_1 s^2 + c_2 s + c_3. \tag{2.36}$$

Now from equation (2.33), we can write η^1 as

$$\eta^1 = a_1(t, r, \theta, \phi)s + a_2(t, r, \theta, \phi), \tag{2.37}$$

where a_1, a_2 are arbitrary functions of mentioned arguments.
Differentiating equation (2.15b) w.r.t. ϕ and equation (2.15e) w.r.t. t, results

$$\eta^4_s = 0, \tag{2.38}$$

and differentiating equation (2.21b) w.r.t. ϕ and equation (2.21e) w.r.t. t, results

$$\eta^1_s = 0. \tag{2.39}$$

Using equations (2.38), (2.39) in equations (2.15c), (2.21c) and solving it further yields

$$\eta^1_{sr} = 0, \tag{2.40}$$

and

$$\eta^4_{sr} = 0. \tag{2.41}$$

Similarly, using equations (2.38), (2.39) in equations (2.15d), (2.21d) and solving them yields

$$\eta^1_{s\theta} = 0, \tag{2.42}$$

and

$$\eta^4_{s\theta} = 0. \tag{2.43}$$

From equations (2.40),(2.42), equation (2.37) implies

$$\eta^1 = a_1(t, \phi)s + a_2(t, r, \theta, \phi), \tag{2.44}$$

differentiating equation (2.15b) w.r.t. ϕ and equation (2.21e) w.r.t. t, results

$$\eta^1_s = 0. \tag{2.45}$$

This implies that $a_1(t, \phi)$ must be zero. Therefore,

$$\eta^1 = a_2(t, r, \theta, \phi). \tag{2.46}$$

Solving equation (2.15n) and equation (2.21j), we get

$$\eta^3_s = 0, \tag{2.47}$$

and similarly, by solving equation (2.15o) and equation (2.21k), we get

$$\eta^2_s = 0. \tag{2.48}$$

Since $\eta^1_s = \eta^2_s = \eta^3_s = \eta^4_s = 0$, equation (2.15b) becomes

$$(-\rho^2 + 2mr)\xi_{ss} = 0. \tag{2.49}$$

Either $(-\rho^2+2mr) = 0$ or $\xi_{ss} = 0$. In our case, we considered $\xi_{ss} = 0$ and $(-\rho^2+2mr) \neq 0$. This implies that equation (2.49) becomes

$$\xi_{ss} = 0. \tag{2.50}$$

Therefore,

$$\xi = c_1 s + c_2. \tag{2.51}$$

From equation (2.17a) η^2 can be written as

$$\eta^2 = b_1(t, r, \theta, \phi)s + b_2(t, r, \theta, \phi). \tag{2.52}$$

Since $\eta^2_s = 0$, equation (2.52) becomes

$$\eta^2 = b_2(t, r, \theta, \phi). \tag{2.53}$$

Solving equation (2.19a) gives

$$\eta^3 = d_1(t, r, \theta, \phi)s + d_2(t, r, \theta, \phi). \tag{2.54}$$

Since $\eta^3_s = 0$, equation (2.54) becomes

$$\eta^3 = d_2(t, r, \theta, \phi). \tag{2.55}$$

Equation (2.21a) can be solved to get η^4 as

$$\eta^4 = e_1(t, r, \theta, \phi)s + e_2(t, r, \theta, \phi). \tag{2.56}$$

Since $\eta^4_s = 0$, equation (2.56) becomes

$$\eta^4 = e_2(t, r, \theta, \phi). \tag{2.57}$$

Differentiating equation (2.15i) w.r.t. t and differentiating equation (2.15l) w.r.t. ϕ, we get

$$\begin{aligned}&(-\rho^2 + 2mr)(2mar\sin^2\theta)\eta^1_t - (-\rho^2 + 2mr)^2\eta^1_\phi - (2mar\sin^2\theta)^2\eta^4_t \\ &+ (-\rho^2 + 2mr)(2mar\sin^2\theta)\eta^4_\phi = 0.\end{aligned} \tag{2.58}$$

Similarly, differentiating equation (2.15f) w.r.t. ϕ and equation (2.15l) w.r.t. t, we get.

$$\begin{aligned}&(-\rho^2+2mr)(r^2+a^2)\eta^1_t - a(-\rho^2+2mr)\eta^1_\phi - (2mar\sin^2\theta)(r^2+a^2)\eta^4_t\\&+2ma^2r\sin^2\theta\eta^4_\phi = 0.\end{aligned} \tag{2.59}$$

On the other hand, differentiating equation (2.21i) w.r.t. t and DE (2.21l) w.r.t. ϕ, we get

$$\begin{aligned}&(2mar\sin^2\theta)^2\eta^1_t - (-\rho^2+2mr)(2mar\sin^2\theta)\eta^1_\phi - 2mar\sin^3\theta\\&\Sigma\eta^4_t + \sin^2\theta(-\rho^2+2mr)\Sigma\eta^4_\phi = 0.\end{aligned} \tag{2.60}$$

Similarly, by differentiating equation (2.21f) w.r.t. ϕ and equation (2.21l) w.r.t. t, we get

$$\begin{aligned}&(2mar\sin^2\theta)(r^2+a^2)\eta^1_t - a(2mar\sin^2\theta)\eta^1_\phi - \sin^2\theta(r^2+a^2)\\&\Sigma\eta^4_t + a\sin^2\theta\Sigma\eta^4_\phi = 0.\end{aligned} \tag{2.61}$$

Equation (2.58) and equation (2.60), yields

$$2mar\sin^2\theta\eta^1_t - (-\rho^2+2mr)\eta^1_\phi = 0, \tag{2.62}$$

$$2mar\sin^2\theta\eta^4_t - (-\rho^2+2mr)\eta^4_\phi = 0. \tag{2.63}$$

Similarly, equation (2.59) and equation (2.61), yields

$$(r^2+a^2)\eta^1_t - a\eta^1_\phi = 0, \tag{2.64}$$

$$(r^2+a^2)\eta^4_t - a\eta^4_\phi = 0. \tag{2.65}$$

By solving equation (2.63) and equation (2.64), we get

$$\eta^1_t = 0, \quad \text{and} \quad \eta^1_\phi = 0. \tag{2.66}$$

Similarly, by solving equation (2.62) and equation (2.65), we get

$$\eta^4_t = 0, \quad \text{and} \quad \eta^4_\phi = 0. \tag{2.67}$$

Differentiate equation (2.15j) w.r.t. t and equation (2.21j) w.r.t. t we get

$$\frac{m\Omega^2(-\rho^2+2mr)}{\rho^8}\eta^1_r - \frac{2m^2ar\sin^2\theta\Omega^2}{\rho^8}\eta^4_r = 0, \tag{2.68}$$

$$\frac{2m^2ar\sin^2\theta\Omega^2}{\rho^8}\eta^1_r - \frac{m\sin^2\theta\Omega^2\Sigma}{\rho^8}\eta^4_r = 0. \tag{2.69}$$

Solving equation (2.68) and equation (2.69), yields

$$\eta^1_r = 0, \quad \text{and} \quad \eta^4_r = 0. \tag{2.70}$$

Similarly, by differentiating equation (2.15k) w.r.t. t and equation (2.21k) w.r.t. t we get

$$\frac{4m^2a^2r^2\cos^2\theta(-\rho^2+2mr)}{\rho^8}\eta^1_\theta - \frac{8m^3a^3r^3\cos^2\theta\sin^2\theta}{\rho^8}\eta^4_\theta = 0, \tag{2.71}$$

$$\frac{8m^3a^3r^3\cos^2\theta\sin^2\theta}{\rho^8}\eta^1_r - \frac{4m^2a^2r^2\cos^2\theta\sin^2\theta\Sigma}{\rho^8}\eta^4_r = 0. \tag{2.72}$$

Solving equation (2.71) and equation (2.3), yields

$$\eta^1_\theta = 0, \quad \text{and} \quad \eta^4_\theta = 0. \tag{2.73}$$

Since $\eta^1_t = \eta^1_r = \eta^1_\theta = \eta^1_\phi = 0$ equation (2.46) becomes

$$\eta^1 = c_4, \tag{2.74}$$

where c_4 is an arbitrary constant.

Since $\eta^4_t = \eta^4_r = \eta^4_\theta = \eta^4_\phi = 0$ equation (2.57) becomes

$$\eta^4 = c_5, \tag{2.75}$$

where c_5 is an arbitrary constant.

Using the results from equations (2.66), (2.67) in equations (2.15o), (2.21k) and solving them yields

$$\eta^2 = 0. \tag{2.76}$$

Similarly, when we use the results in equation (2.15n) and equation (2.21j) we get

$$\eta^3 = 0. \tag{2.77}$$

Differentiating equation (2.17l) w.r.t. t and differentiating equation (2.17f) w.r.t. ϕ, we get

$$2mar\sin^2\theta\eta^2_t - (-\rho^2 + 2mr)\eta^2_\phi = 0, \tag{2.78}$$

and by differentiating equation (2.17i) w.r.t. t and differentiating equation (2.17l) w.r.t. ϕ, we get

$$(r^2 + a^2)\eta^2_t - a\eta^2_\phi = 0. \tag{2.79}$$

Solving equation (2.78) and equation (2.79) simultaneously, yields

$$\eta^2_t = 0, \quad \text{and} \quad \eta^2_\phi = 0. \tag{2.80}$$

Similarly, differentiating equation (2.19l) w.r.t. t and differentiating equation (2.19f) w.r.t. ϕ, when we solve them, we get

$$2mar\sin^2\theta\eta^3_t - (-\rho^2 + 2mr)\eta^3_\phi = 0, \tag{2.81}$$

and by differentiating equation (2.19i) w.r.t. t and differentiating equation (2.19l) w.r.t. ϕ, we get

$$(r^2 + a^2)\eta^3_t - a\eta^3_\phi = 0. \tag{2.82}$$

Solving equation (2.81) and equation (2.82) simultaneously, yields

$$\eta^3_t = 0, \quad \text{and} \quad \eta^3_\phi = 0. \tag{2.83}$$

Solving equation (2.17f) and equation (2.17l) by using above results, we get

$$\begin{aligned} &\frac{m\Delta\Omega}{\rho^6}\eta^2_r - \frac{2ma^2r\sin\theta\cos\theta}{\rho^6}\eta^2_\theta = 0, \\ &\frac{ma\sin^2\theta\Delta\Omega}{\rho^6}\eta^2_r - \frac{2mar\sin\theta\cos\theta(r^2+a^2)}{\rho^6}\eta^2_\theta = 0. \end{aligned} \tag{2.84}$$

Solving the system (2.84), we get

$$\eta^2_r = 0, \quad \text{and} \quad \eta^2_\theta = 0. \tag{2.85}$$

Likewise, using the above results in equation (2.19f) and equation (2.19l), we get

$$\eta^3_r = 0, \quad \text{and} \quad \eta^3_\theta = 0. \tag{2.86}$$

So we have found all of the necessary infinitesimals, and if we assume $(c_1, c_2, c_4, c_5) = (C_1, C_2, C_3, C_4)$, we can write

$$\begin{aligned} &\xi = C_1 + sC_2, \quad \eta^1 = C_3, \\ &\eta^2 = 0, \quad \eta^3 = 0, \quad \textit{and} \quad \eta^4 = C_4. \end{aligned} \tag{2.87}$$

Hence, the generator can be written as

$$\boldsymbol{X}^{[1]} = (C_1 + sC_2)\frac{\partial}{\partial s} + C_3\frac{\partial}{\partial t} + C_4\frac{\partial}{\partial \phi}. \tag{2.88}$$

For $C_k = 0$ where $k = 1, 2, 3, 4$, we get four symmetries

$$\begin{aligned} &\boldsymbol{X}_1 = \frac{\partial}{\partial s}, \qquad \boldsymbol{X}_2 = s\frac{\partial}{\partial s}, \\ &\boldsymbol{X}_3 = \frac{\partial}{\partial t}, \qquad \boldsymbol{X}_4 = \frac{\partial}{\partial \phi}. \end{aligned} \tag{2.89}$$

These four symmetries are the required Mei symmetries.

We can see that three of the four Mei symmetries, $\boldsymbol{X}_1, \boldsymbol{X}_3$, and $\boldsymbol{X}_4$, are identical to the Noether symmetries given in equation (2.5), which correspond to the Lagrangian provided by equation (2.3). However, these three symmetries satisfy equation (1.146), and $\boldsymbol{X}_2$ does not satisfy equation (1.146), so it is not a Noether symmetry. Thus, Noether symmetries form a sub-algebra of the Mei symmetries. One can also observe that all the four Mei symmetries are also Lie point symmetries of the system of equations of motion given by equations (2.6) to (2.9).

The obtained Mei symmetries satisfy the Lie algebra

$$\begin{aligned} &[\boldsymbol{X}_1, \boldsymbol{X}_2] = \boldsymbol{X}_1, \qquad [\boldsymbol{X}_1, \boldsymbol{X}_3] = 0, \\ &[\boldsymbol{X}_1, \boldsymbol{X}_4] = 0, \qquad [\boldsymbol{X}_2, \boldsymbol{X}_3] = 0, \\ &[\boldsymbol{X}_2, \boldsymbol{X}_4] = 0, \qquad [\boldsymbol{X}_3, \boldsymbol{X}_4] = 0. \end{aligned} \tag{2.90}$$

2.4 Mei Symmetries' Verification:

One may verify to see whether the resulting symmetries satisfy the Mei symmetries condition given by equation (2.12). Using the acquired values of infinitesimals, we write $\boldsymbol{X}^{[1]}L$ as

$$\boldsymbol{X}^{[1]}L = C_1\left[2(1-\frac{2mr}{\rho^2})\dot{t}^2 + \frac{4mar\sin^2\theta}{\rho^2}\dot{t}\dot{\phi}\right] - C_1\left[\frac{2\rho^2}{\Delta}\dot{r}^2\right] - C_1\left[2\rho^2\dot{\theta}^2\right]$$
$$- C_1\left[\frac{2\sin^2\theta}{\rho^2}\Sigma\dot{\phi}^2 - \frac{4mar\sin^2\theta}{\rho^2}\dot{t}\dot{\phi}\right]. \tag{2.91}$$

As required by the condition stated in equation (2.12), we apply the Euler operator for each dependent variable one by one.

Condition (2.12) for $q^1 = t$ yields

$$\left(\frac{d}{ds}\frac{\partial}{\partial\dot{t}} - \frac{\partial}{\partial t}\right)[\boldsymbol{X}^{[1]}L] = 0. \tag{2.92}$$

Equation (2.91) yields the left-hand side of equation (2.92)

$$\frac{d}{ds}\left(4C_1(1-\frac{2mr}{\rho^2})\dot{t} + C_1\frac{8mar\sin^2\theta}{\rho^2}\dot{\phi}\right) = \frac{8m\Omega}{\rho^4}C_1\dot{t}\dot{r} - \frac{8m\Omega}{\rho^4}C_1\dot{t}\dot{r}$$
$$- \frac{16ma^2r\sin\theta\cos\theta}{\rho^4}C_1\dot{t}\dot{\theta} + \frac{16ma^2r\sin\theta\cos\theta}{\rho^4}C_1\dot{t}\dot{\theta} - \frac{8ma\sin^2\theta\Omega}{\rho^4}$$
$$C_1\dot{r}\dot{\phi} + \frac{8ma\sin^2\theta\Omega}{\rho^4}C_1\dot{r}\dot{\phi} + \frac{16mar\sin\theta\cos\theta(r^2+a^2)}{\rho^4}C_1\dot{\theta}\dot{\phi} - 16m$$
$$\frac{ar\sin\theta\cos\theta(r^2+a^2)}{\rho^4}C_1\dot{\theta}\dot{\phi} = 0. \tag{2.93}$$

This signifies that the condition (2.12) is valid for $q^1 = t$.

Condition (2.12) for $q^2 = r$ gives

$$\left(\frac{d}{ds}\frac{\partial}{\partial\dot{r}} - \frac{\partial}{\partial r}\right)[\boldsymbol{X}^{[1]}L] = 0. \tag{2.94}$$

When we substitute equation (2.91) into equation (2.94), we obtain

$$\frac{d}{ds}\left[-\frac{4\rho^2}{\Delta}C_1\dot{r}\right] - \left[\frac{4m\Omega}{\rho^4}C_1\dot{t}^2 - \frac{8ma\sin^2\theta\Omega}{\rho^4}C_1\dot{t}\dot{\phi} - 4rC_1\dot{\theta}^2\right.$$

$$
+ \frac{4[m\Omega - a^2 r \sin^2\theta]}{\rho^4} C_1 \dot{r}^2 + \frac{4[ma^2 \sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^4} C_1 \dot{\phi}^2 \Bigg]
$$

$$
= \frac{4[m\Omega - a^2 r \sin^2\theta]}{\Delta^2} C_1 \dot{r}^2 + \frac{8a^2 \sin\theta\cos\theta}{\Delta} C_1 \dot{r}\dot{\theta} - \frac{8a^2 \sin\theta\cos\theta}{\Delta}
$$

$$
C_1 \dot{r}\dot{\theta} + \frac{4m\Omega}{\rho^4} C_1 \dot{t}^2 + \frac{4[ma^2 \sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^4} C_1 \dot{\phi}^2 - \frac{8ma\sin^2\theta}{\rho^4}
$$

$$
\Omega C_1 \dot{t}\dot{\phi} - 4rC_1\dot{\theta}^2 - \Bigg[\frac{4m\Omega}{\rho^4} C_1 \dot{t}^2 - \frac{8ma\sin^2\theta\Omega}{\rho^4} C_1 \dot{t}\dot{\phi} - 4rC_1\dot{\theta}^2
$$

$$
+ \frac{4[m\Omega - a^2 r \sin^2\theta]}{\rho^4} C_1 \dot{r}^2 + \frac{4[ma^2 \sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^4} C_1
$$

$$
\dot{\phi}^2 \Bigg] = 0, \tag{2.95}
$$

it goes to zero and therefore condition (2.12) holds for $q^2 = r$.

For $q^3 = \theta$ condition (2.12) becomes

$$
\left(\frac{d}{ds}\frac{\partial}{\partial\dot{\theta}} - \frac{\partial}{\partial\theta}\right)[\boldsymbol{X}^{[1]} L] = 0. \tag{2.96}
$$

Solving left hand side we obtain

$$
\frac{d}{ds}\Bigg[-4\rho^2 C_1 \dot{\theta}\Bigg] - \Bigg[-\frac{8ma^2 r\sin\theta\cos\theta}{\rho^4} C_1 \dot{t}^2 + 4a^2 \sin\theta\cos\theta C_1 \dot{\theta}^2
$$

$$
+ \frac{16mar\sin\theta\cos\theta(r^2+a^2)}{\rho^4} C_1 \dot{t}\dot{\phi} + \frac{4a^2\sin\theta\cos\theta}{\Delta} C_1 \dot{r}^2 - 4\sin\theta
$$

$$
\frac{\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^4} C_1 \dot{\phi}^2 \Bigg]
$$

$$
= 4a^2\sin\theta\cos\theta C_1 \dot{\theta}^2 - \frac{8ma^2 r\sin\theta\cos\theta}{\rho^4} C_1 \dot{t}^2 + \frac{4a^2\sin\theta\cos\theta}{\Delta}
$$

$$
C_1 \dot{r}^2 + \frac{16mar\sin\theta\cos\theta(r^2+a^2)}{\rho^4} C_1 \dot{t}\dot{\phi} - \frac{4\sin\theta\cos\theta}{\rho^4} C_1 \dot{\phi}^2 \big[\big(r^2
$$

$$
+ a^2\big)\Sigma - a^2\sin^2\theta\Delta\rho^2\big] - \Bigg[-\frac{8ma^2 r\sin\theta\cos\theta}{\rho^4} C_1 \dot{t}^2 + 16mar
$$

$$
\frac{\sin\theta\cos\theta(r^2+a^2)}{\rho^4} C_1 \dot{t}\dot{\phi} - \frac{4\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^4}
$$

$$
C_1 \dot{\phi}^2 + \frac{4a^2\sin\theta\cos\theta}{\Delta} C_1 \dot{r}^2 + 4a^2\sin\theta\cos\theta C_1 \dot{\theta}^2 \Bigg] = 0. \tag{2.97}
$$

As a result, condition (2.12) also holds for $q^3 = \theta$.

Similarly for $q^4 = \phi$ condition (2.12) becomes

$$\left(\frac{d}{ds}\frac{\partial}{\partial\dot{\phi}} - \frac{\partial}{\partial\phi}\right)[\boldsymbol{X}^{[1]}L] = 0. \tag{2.98}$$

Left hand side gives

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{8mar\sin^2\theta}{\rho^2}C_1\dot{t} + \frac{4\sin^2\theta}{\rho^2}\Sigma C_1\dot{\phi}\right) = -\frac{8ma\sin^2\theta\Omega}{\rho^4}C_1\dot{t}\dot{r}\\
&+\frac{8ma\sin^2\theta\Omega}{\rho^4}C_1\dot{t}\dot{r} + \frac{8[ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^4}C_1\dot{r}\dot{\phi}\\
&-\frac{8[ma^2\sin^4\theta\Omega - r\sin^2\theta\rho^4]}{\rho^4}C_1\dot{r}\dot{\phi} - \frac{16mar\sin\theta\cos\theta(r^2+a^2)}{\rho^4}\\
&C_1\dot{t}\dot{\theta} + \frac{8\sin\theta\cos\theta[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^4}C_1\dot{\theta}\dot{\phi} - 8\sin\theta\cos\theta\\
&\frac{[(r^2+a^2)\Sigma - a^2\sin^2\theta\Delta\rho^2]}{\rho^4}C_1\dot{\theta}\dot{\phi}\frac{16mar\sin\theta\cos\theta(r^2+a^2)}{\rho^4}C_1\dot{t}\dot{\theta}\\
&= 0,
\end{aligned} \tag{2.99}$$

As a result, condition (2.12) holds true for $q^4 = \phi$ as well. Hence, equation (2.89) presents four Mei symmetries for the Lagrangian corresponding to the Kerr metric.

Chapter 3

Summary

In mathematics and mechanics, the study of symmetry and conserved quantity is extremely significant. Noether symmetries are the modern way of determining a mechanical system's conserved quantities. The Noether symmetry is an invariance of the Lagrange equation under infinitesimal transformations. In the last decade, several significant results in study of the Lie point symmetry [27] and the Noether symmetry [28] have been obtained. Mei introduced a new symmetry known as Mei symmetry or form invariance which varies from the Lie point symmetry or Noether symmetry [17]. Mei symmetry states that the dynamical functions (such as Lagrangian etc.) appearing in the mechanical system's dynamical equations still fulfil the original equations after the infinitesimal transformation.

Recently Mei symmetries for the Lagrangian corresponding to the Schwarzschild metric [29], which is the spherically symmetric, static, homogenous, and isotropic gravitational field has been studied. In this thesis Mei symmetries for the Lagrangian corresponding to the Kerr metric are obtained. Kerr black hole is a more realistic scenario that represents an uncharged revolving black hole and is no longer spherically symmetric. The Kerr metric is one of the well-known solutions to Einstein's field equations. The nonlinearity of these equations makes precise solutions extremely difficult to obtain.

In this thesis, important breakthroughs in DEs over time are briefly addressed from the vast history of DEs. The dynamic research naturally encompasses both ODEs and PDEs. The definition of symmetry groups of point transformations and infinitesimal generators is discussed in detail. The method of Lie point symmetry is analysed and used in several well-known DEs. The Lie algebras and Lie brackets of the basic symmetry generators are evaluated. Following the definition of the Lagrangian, Noether symmetries and Mei symmetries are defined along with their conditions. Relationship between Lie symmetry and Noether symmetry [15], and relationship between Noether symmetry and Mei symmetry [23] are established using historical facts.

The second chapter focuses on Mei symmetries for the Lagrangian of rotating uncharged axially symmetric metric. The Kerr metric is considered in this case. This chapter contains a review of the Noether and Lie point symmetries for the Lagrangian corresponding to the Kerr metric from the research paper [26]. The Lagrangian of Kerr metric is provided. First, the Noether symmetries for the Lagrangian of Kerr metric obtained are presented. Second, the geodesic equations for the four Boyer-Lindquist coordinates (t, r, θ, ϕ) are then compiled one by one. The Lie point symmetries obtained for the Kerr metric are provided using the definition of the Lie point symmetries of ODEs on the Lagrange equations [26]. Following that, the main task of obtaining Mei symmetries corresponding to the Lagrangian of the Kerr metric is executed. Using the Mei symmetries criteria, the infinitesimal generator is extended and the system of determining equations for all dependent variables is achieved. After that, the system is solved to determine the values of the infinitesimals $(\xi, \eta^1, \eta^2, \eta^3, \eta^4)$. Two of them are determined to be zero while the remaining three are dependent on four arbitrary constants for which we obtained four Mei symmetries.

Lie point symmetries, Noether symmetries, and Mei symmetries of the Kerr metric are listed as

Lie point symmetries	$\frac{\partial}{\partial s}, s\frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial \phi}$.
Noether symmetries	$\frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial \phi}$.
Mei symmetries	$\frac{\partial}{\partial s}, s\frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial \phi}$.

Table 3.1: Lie point symmetries, Noether symmetries and Mei symmetries of the Kerr metric.

The results reveal that, in the case of the Kerr metric, the Noether symmetries are subset of the Mei symmetries and that Mei symmetries are same as that of Lie point symmetries. Finally, the obtained Mei symmetries are verified.

Bibliography

[1] Isaac Newton. *The Method of Fluxions and Infinite Series: With its Application to the Geometry of Curve Lines*. Henry Woodfall, (1736).

[2] Tom Archibald, Craig Fraser, and Ivor Grattan-Guinness. *The History of DEs, 1670-1950*. Oberwolfach Reports, **1**(4):2729-2794 (2005).

[3] Hans Stephani. *DEs: Their Solution Using Symmetries*. Cambridge University Press, (1989).

[4] Peter E Hydon and Peter Ellsworth Hydon. *Symmetry Methods for DEs: A Beginner's Guide*. Cambridge University Press, (2000).

[5] Daniel J Arrigo. *Symmetry Analysis of DEs: An Introduction*. John Wiley & Sons, (2015).

[6] George W Bluman and Sukeyuki Kumei. *Symmetries and DEs*. Springer Science and Business Media, (2013).

[7] Kruglikov, Boris. *Symmetries of Second Order ODEs*. Journal of Mathematical Analysis and Applications, **461**(1):591–594, (2018).

[8] Ruth A Steinhour. *The Truth About Lie Symmetries: Solving DEs With Symmetry Methods*. The College of Wooster, (2013).

[9] Jia Man. *Lie Point Symmetry Algebras and Finite Transformation Groups of the General Broer-Kaup System.* Chinese Physics, **16**(6):1534-1544, (2007).

[10] Robert J Gray. *The Lie Point Symmetry Generators Admitted by Systems of Linear DEs.* Proceedings of the Royal Society A: Mathematical , Physical and Engineering Sciences, **470**(2166):20130779, (2014).

[11] Willi-Hans Steeb. *Continuous Symmetries, Lie Algebras, DEs and Computer Algebra.* World Scientific Publishing Company, (2007).

[12] J Fuchs and Christoph Schweigert *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists.* Cambridge University Press, (2003).

[13] Fritz Schwarz. *Symmetries of DEs: from Sophus Lie to Computer Algebra.* Siam Review, **30**(3):450-481, (1988).

[14] Amlan K Halder, Andronikos Paliathanasis, and Peter GL Leach. *Noether's Theorem and Symmetry.* Symmetry, **10**(12):744:1021, (2018).

[15] Muhammad Safdar and Asghar Qadir and Muhammad Umar Farooq. *Comparison of Noether Symmetries and First Integrals of Two-Dimensional Systems of Second Order ODEs by Real and Complex Methods.* Symmetry, **11**(9):1180:1-12, (2019).

[16] PGL Leach. *Lie Symmetries and Noether Symmetries.* Applicable Analysis and Discrete Mathematics, **6**(2):238-246, (2012).

[17] Mei Feng-Xiang. *Form Invariance of Lagrange System.* Journal of Beijing Institute of Technology, **9**(2):120-124, (2000).

[18] Mei Feng-Xiang. *Form Invariance of Appell Equations.* Chinese Physics, **10**(3):177-180, (2001).

[19] Fang Jian-Hui. *Mei Symmetry and Lie Symmetry of the Rotational Relativistic Variable Mass System*. Communications in Theoretical Physics, **40**(3):269-272, (2003).

[20] XH Zhai and Y Zhang. *Mei Symmetry of Time-Scales Euler-Lagrange Equations and Its Relation to Noether Symmetry*. Acta Physica Polonica, A., **136**(3):439-443, (2019).

[21] Yi Zhang. *Mei Symmetry and Conservation Laws for Time-Scale Nonshifted Hamilton Equations*. Advances in Mathematical Physics, **2021**:1-8 (2021).

[22] Shao-Kai, L. *Mei Symmetry, Noether Symmetry and Lie Symmetry of Hamiltonian System [J]*. Acta Physica Sinica, **52**,2941, (2003).

[23] Yi Zhang. *The Relation Between the Mei Symmetry and the Noether Symmetry-Taking the Lagrange System as an Example*. Mechanics in Engineering, **38**(2):169-171, (2016).

[24] Fang Jian-Hui. *Mei Symmetry and Noether Symmetry of the Relativistic Variable Mass System*. Communications in Theoretical Physics, **41**(3):349-352, (2004).

[25] Roy P Kerr. *Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics*. Physical Review Letters, **11**(5):237-238, (1963).

[26] Ibrar Hussain, FM Mahomed, and Asghar Qadir. *Approximate Noether Symmetries of the Geodesic Equations for the Charged-Kerr Spacetime and Rescaling of Energy*. General Relativity and Gravitation, **41**(10):2399-2414, (2009).

[27] Sophus Lie. *Über Die Integration Durch Bestimmte Integrale Von Einer Classe Linearer Partieller Differentialgleichungen*. Cammermeyer, **6**:328-368, (1880).

[28] Emmy Noether. *Invariant Variation Problems.* In Transport theory and Statistical Physics, **1**(3):186–207, (1971).

[29] Kinza Iftikhar. *The Mei Symmetries Corresponding to Lagrangian of the Schwarzschild Metric*, MS Thesis. National University of Sciences and Technology Islamabad, (2021).