Quantum Optical Metrology Using Entangled Photonic States

By

Syed Zakir Hussain

Reg no: 00000330834

Supervisor

Dr. Shahid Iqbal

Department of Physics

School of Natural Sciences (SNS)

National University of Sciences and Technology (NUST)

Islamabad, Pakistan

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MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: **Syed Zakir Hussain**, Regn No. **00000330834** Titled: **"Quantum Optical Metrology using Entangled Photonic States"** accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. FAHAD AZAD Signature:

2. Name: DR. MUZZAMAL IQBAL SHAUKAT Signature:

Supervisor's Name: DR. SHAHID IQBAL Signature:

Head of Department

05/08/2022
Date

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Date: 05.08.2022

Dean/Principal

This thesis is dedicated to *my parents*

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I meekly thank ALLAH ALMIGHTY, the most merciful, who gave me health, thoughts and cooperative staff to enable me to achieve this goal.

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Declarations

I, **Syed Zakir Hussain** declare that this thesis titled *"Quantum Optical Metrology Using Entangled Photonic States"* and the work presented in it is my own and have been generated by me as a result of my original research. I confirm that:

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Syed Zakir Hussain

Reg: no 00000330834

Contents

List of Figures

List of Tables

Abstract

Quantum metrology is one of the most sophisticated disciplines of quantum technologies. The primary purpose of this research domain is to investigate unknown physical quantities by using quantum mechanical resources. Quantum metrology has an extensive range of applications including, in the development of quantum sensors, photonic quantum systems, and various metrological tasks. Quantum resources have shown better improvement in phase estimation as compared to classical measurements which are limited by standard quantum limit. Quantum entanglement has been considered one of the counter-intuitive concepts in quantum physics which has played a vital role in overcoming the standard quantum limit. Herein we have deployed the quantum Fisher information approach and used various quantum mechanical states for phase estimation. Our main focus was using squeezed states in different modes to develop ways for quantum metrology. We have found better results that surpass the shot-noise limit and approach the more fundamental Heisenberg's limit in modified structures with given schemes. Moreover, we have introduced an entangled squeezed vacuum state (ESVS) as the best quantum resource for phase estimation. This research aims to contribute to the area of precision measurements and hence to develop photonic quantum sensing.

Keywords: *Quantum Metrology, Quantum Cramer-Rao bound, Quantum Fisher information, Phase uncertainty*

Chapter 1

Introduction

1.1 Prologue

In science, measurement is one of the best ways to study the nature of an event or process. To better understand the behavior of nature, physicists have drawn their attention to precision measurements and in these endeavors, they developed the theory of quantum mechanics in 1900 intending to achieve insights into nature that were previously unreachable. Three steps participate in the measurement process: the experimental setup (laboratory, instruments, detectors, etc.), the experimental process (doing experiment), and then reading the scales on the experimental setup. Statistical or systematic errors are responsible for reducing the precision in the measurement and may be caused due to insufficient control of the probes or arise due to environmental situations. Whatever the situation is the error can be minimized by repeating the experiment many times and taking averages of the results. Additionally, there are fundamental restrictions on uncertainty for example Heisenberg's uncertainty inequalities that are enforced by the principles of quantum physics. Since quantum theory has been the most victorious theory for explaining events at the atomic and subatomic dimensions throughout history. This implies that the estimation techniques, along with the resultant precision limits, must be conducted in the context of such a paradigm.

Astonishingly, the deployment of quantum mechanical resources to measure the required (unknown) parameters surpass the precision bounds that can be established through classical resources. This is the basic idea behind the fast-growing field of quantum metrology which is the theoretical and experimental study of elevated resolution and high precision processes for the estimation of physical quantities using the aspects of quantum mechanics, particularly by exploiting quantum entanglement. Various quantum states including entangled photonic quantum states and squeezed states can be used for achieving the ultimate limit on the precision of the measurement, some of which are within the scope of this thesis.

1.2 Preliminary Introduction

To study nature fundamentally, physics provides the best platform for this purpose. For example, measurement is the first step toward the research of nature and events occurring around us. Physics has offered a various detailed understanding of the basic workings of nature, by providing and devising highly accurate theories and instruments respectively, which have enlightened the knowledge of humans in different areas including, medicines, archaeology, biology, chemistry, material science, astronomy, etc. Optics enables extremely sensitive ways of measuring ultra-small physical components. The quantum nature of the electromagnetic field bounds the ultimate sensitivity in these investigations, as it does in every optical measurement. The example of the optical interferometer has been thoroughly investigated, and it has also been demonstrated that interferometer-induced vacuum disturbances causes the so-called standard quantum limit or shot noise limit. It has long been acknowledged that the shot-noise limit, does not constitute the fundamental bound in photonic quantum metrology, and non-classical states (quantum states) of light may be helpful to enhance measurements. The development of contemporary quantum optics has been oriented toward improving interferometric measurements using non-classical states of light.

As quoted by Galileo, the motive of experimental science as being to, "Measure what is measurable and make measurable what is not so" [1]. While this idea is great, it raises the question of what would happen if there was a fundamental bound to what we could estimate. Such a deadlock would create a problem in scientific development because it would bound how deeply one can estimate. The precision of measurements is limited by the shot-noise limit or standard quantum limit upon using classical estimation theory [2]. However, these concepts were changed when quantum theory was introduced.

In the advancement of physical theories to study nature at the atomic and subatomic level physicists of the 20^{th} century have developed the theory of quantum mechanics, which has been considered the most successful and predictive theory of all time regardless of its indeterministic nature [3]. Since the precision of measurement is the basic factor behind every technological development so, the theory or experimental setup for ultimate bounds on precision must be raised under the paradigm of quantum mechanics. The laws and concepts of quantum mechanics govern all measuring operations. Quantum mechanics also impose fundamental limits (Heisenberg's limit) on the precision of measurements [4]. However, the measurements which are performed by using the resources of quantum mechanics are overly sensitive and précised and cannot be done through classical resources. Quantum metrology is the study of elevated accuracy measurements and high precision resolutions by the deployment of quantum resources, particularly by using *quantum entanglement*. Furthermore, quantum correlated light may be utilized to circumvent the standard quantum limit, hence this field's primary focus is on developing strategies to improve optical measurement precision. Quantum metrology is the way of measuring physical parameters, which uses quanta (individual packets of energy) for elevated precision research. Typically, four steps are involved in quantum metrological tasks for the measurement of unknown variables: preparation of probe state, parameterization (interaction between probe state and the system being studied), measurement of the system, and final readout or classical estimation process [5]. Particularly, phase estimation is the primary goal of quantum metrology that can be conducted by using an interferometric setup. To realize quantum metrological tasks and other quantum information processes various physical systems can be deployed, photons have been considered the most fundamental because of their different properties [6–9]. In the recent several years, there has been exponential growth in the field of quantum metrology, as documented in numerous review articles on theoretical exposures of quantum measurements and phase estimation dilemmas [10–14], optical metrology [15], multiparameter problems [16–18] and other physical systems used for metrological tasks [5, 14, 19].

The best error estimation through 'm' classical probes (classically correlated states), each interacting with the system under consideration will scale at $\Delta\phi = 1/\sqrt{m}$. This classical bound is the result of the central limit theorem and is called the shot-noise limit or standard quantum limit (SQL) [20]. However, if quantum resources and probes are used the standard quantum limit can be circumvented, and the uncertainty will approach the more fundamental limit $\Delta\phi = 1/m$. This fundamental limit, known as the Heisenberg's limit (HL) improves the precision by a factor of $\sqrt{m}$ over the classical limit [20].

Quantum metrology has widespread applications in addition to the underlying interest in ultimate accuracy bounds. Therefore, numerous scientific disciplines can benefit quantum-enhanced sensitivity, including measurement on biological systems [5, 21], atomic clocks [21, 22], interferometry with atomic and molecular matter waves [5, 23], gravitational waves detection [24, 25], magnetometry [26, 27], plasmonic sensing [16, 28], spectroscopy [29] and frequency measurement [30], microscopy and imaging [5, 31–36], Hamiltonian estimation [37, 38] and other general sensing technologies [39].

A vital role has been played in the estimation theory by quantum entanglement. This phenomenon has been considered the most counter-intuitive in quantum mechanics and is completely juxtaposed with the classical world as recognized by Einstein et al [40]. Quantum entanglement has widely been researched for quantum metrological tasks as discussed in these papers [11, 41–45]. Furthermore, squeezed states of electromagnetic field has also been suggested as a quantum mechanical tool for the problem of quantum metrology.

Non-classical states, such as squeezed states of light, are the most relevant in quantum optics because they have less uncertainty in one quadrature component at the expense of another quadrature component [46]. The non-classicality of squeezed states can be measured through Glauber-Sudarshan P-function [47]. The positivity of the P-function demonstrates that the state is classical for example coherent state ($|\alpha\rangle$) or a mixture of coherent states. However, if the P-function is negative it confirms that the state is quantum mechanical since the negativity of the P-function is an adequate and inevitable condition for a state to be non-classical [47]. One of the dominant characteristics of the squeezed state is that its zero-point fluctuations are no longer distributed randomly as compared to coherent states where the zero-point fluctuations are random [46]. Squeezed states have been widely researched for various applications including quantum optical communication [48–50], improving the sensitivity of laser interferometers [2, 51], gravitational waves detection [52], and quantum key distribution (secure quantum communication) [53, 54]. In this study, the author has used squeezed states, squeezed vacuum states ($|\chi\rangle = S(\chi)|0\rangle$), squeezed coherent states ($|\alpha, \chi\rangle$), and entangled squeezed vacuum states ($|\psi\rangle_{\chi 00 \chi}$) for quantum metrology and hence overcome the shot-noise limit (standard quantum limit). Moreover, the results of these states are compared with each other for further analysis.

1.3 Outline of the Thesis

The paradigm of the thesis is organized as follows:

Chapter 2 provides the basic concepts of quantum optical states. We introduced well known quantum states and their connections with quantum mathematical entities. In short, this section discussed all the tools and background of the study. First, the quantum state is explained, then tools have been discussed that are useful for further descriptions.

Chapter 3 describes basic formalism of quantum estimation theory. We provided all the formalism that we may use for quantum metrological tasks, including Fisher information, Cramer Rao bound classically and then quantum mechanical framework has presented. Moreover, other quantum resources have been investigated like quantum entanglement and non-classicality.

Chapter 4 deals with the applications of various photonic states for the problem of photonic quantum metrology and compares the results of quantum states with each other.

Chapter 5 puts concluding remarks on the work that has been presented in this thesis. Moreover, the quantum states presented in this work are thoroughly discussed and analyzed.

Chapter 2

Quantum Optical States: Fundamentals

2.1 A Quantum State

Quantum mechanics has stimulated the scientific and technological advancements that have impacted our daily lives in many respects ranging from super-fast computing to elevated precision measurements and secure communication [1, 55–57]. Quantum mechanics explains the behavior of matter and light at the atomic and subatomic levels, at these levels, the physical systems behave very differently from what we are experiencing in daily life. The information about these physical systems can be encoded into a proper state called a "quantum state". However, quantum mechanically it is impossible to extract all the information exactly because of the uncertainty principle so, a probabilistic way is introduced to collect information about the physical system by using proper mathematics of quantum mechanics. This implies that quantum states are important objects in quantum physics. The quantum states are usually symbolized by a Greek letter Psi (ψ) in a quantum mechanical notation introduced by Dirac called bra and ket. Therefore, the quantum state $|\psi\rangle$ is a ket and the $\langle\psi|$, is a bra, representing the states of a quantum system. Usually, quantum physicists are always dealing with bra-ket notation for various purposes to know about a quantum system. For example, let us consider two states represented by $|\psi\rangle$, and $|\phi\rangle$ respectively, if we invert the direction of one state and combine it like,

$$\langle\phi|\psi\rangle, \tag{2.1.1}$$

is called the inner product. When we calculate this inner product, it gives a probability amplitude, and its absolute square provides the probability of the system going from state $|\phi\rangle$ to the state $|\psi\rangle$. Similarly, the observables (e.g., momentum) are of immense importance in quantum mechanics which are represented by an operator and act on a quantum state. In literature, an operator acts on a state ket from the left, let the operator be $\hat{A}$, then

$$\hat{A}|\psi\rangle = A|\psi\rangle, \tag{2.1.2}$$

which is another ket. However, if we sandwich the operator $\hat{A}$ in a bra-ket, we will get an important result called the expectation value of a particular observable under consideration

$$\langle\psi|\hat{A}|\psi\rangle = A\langle\psi|\psi\rangle, \tag{2.1.3}$$

where the product $\langle\psi|\psi\rangle$ is called the inner product and is equal to 1. Similarly, if we put quantum states in the following way then it is called an identity operator

$$\sum_{\psi} |\psi\rangle\langle\psi| = 1. \tag{2.1.4}$$

The unity on the right side of the equation (2.1.4) confirms that this is an identity operator and is also called completeness relation or closure. However, the detailed mathematical descriptions for quantum mechanics are available in various outstanding books, some of them are encoded here in references for the ease of readers [58–62].

2.2 Density Operator or Density Matrix

Quantum states can be pure or mixed. Pure states can be easily characterized through simple wave functions, while mixed states are statistical composites with unequal or bizarre information about the given system. The quantum state is mixed if it is considered an ensemble of a microscopic or macroscopic system having no initial phase connection between elements of the mixture [63]. In other words, state vectors can explain pure quantum states, while mixed states cannot be represented by state vectors, which cannot be separated or explained individually. Such types of states can be well understood through density operator or density matrix as given by,

$$\hat{\rho} = \sum_{\lambda} P_{\lambda} |\psi_{\lambda}\rangle\langle\psi_{\lambda}|, \tag{2.2.1}$$

where, P_{λ} are the probabilities of estimating the system in the λ^{th} members of the ensemble (quantum system). For pure states, the probability is equal to one otherwise are,

$$0 \leq P_{\lambda} \leq 1, \sum_{\lambda} P_{\lambda} = 1, and \sum_{\lambda} P_{\lambda}^{2} = 1. \tag{2.2.2}$$

Similarly, if one wants to calculate the average value of a state of the ensemble, $|\psi_{\lambda}\rangle$ for an operator $\hat{O}$ then, expectation value can be written as

$$\langle\hat{O}\rangle = \langle\psi_{\lambda}|\hat{O}|\psi_{\lambda}\rangle. \tag{2.2.3}$$

Equation (2.2.3) can be further defined for mixed states in the following way,

$$\langle\hat{O}\rangle = \sum_{\lambda} P_{\lambda} \langle\psi_{\lambda}|\hat{O}|\psi_{\lambda}\rangle. \tag{2.2.4}$$

In terms of density matrix or density operator, equation (2.2.4) can be easily obtained as,

$$\langle\hat{O}\rangle = Tr(\hat{O}\hat{\rho}). \tag{2.2.5}$$

The density matrix is Hermitian in nature that is $\hat{\rho}^{*} = \hat{\rho}$ and if $Tr(\rho^{2}) = 1$, then the state is pure, however for entangled states $Tr(\rho^{2}) < 1$ [47]. Furthermore, we have investigated the principles of quantum mechanics in quantum optics by exploiting some basic quantum states. Quantum optics is one of the advanced areas of physics where semi-classical physics and quantum mechanics are used to research and deal with how photons of the electromagnetic field interact with matter. The great pace of research in quantum optics can be linked with the quantum states that have no classical analogs. These include single-photon states, squeezed states, EPR states, and other complex scenarios that can be involved [46, 64, 65]. Moreover, we have considered some basic quantum states that can be later used for quantum optical metrology.

2.3 Number State or Fock State

The number state is a quantum mechanical state that belongs to Fock-space and has well-defined photons (or particles). The Number states are named after Soviet theoretical physicist

Vladimir Fock. Fock states have significant importance in quantum mechanics and here, we have presented its applications in quantum metrology [66]. As the quantization of a single-mode field satisfy Maxwell's equation under certain boundary condition that is,

$$E_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}} q_{(t)} Sin(kz). \tag{2.3.1}$$

Since a quantized electromagnetic field can be considered as an infinite pack of uncoupled harmonic oscillators, each can be explained through a Hamiltonian obtained by using Maxwell equations with minor mathematical manipulation

$$H = \frac{1}{2}(p^2 + \omega^2 q^2), \tag{2.3.2}$$

where the magnetic and electric field plays the role of canonical momentum and position respectively. Consequently, using the correspondence rule these variables can be used as operators that are, $q = \hat{q}$ and $p = \hat{p}$ and they must satisfy the following commutation relations

$$[\hat{q}, \hat{p}] = i\hbar\hat{I}, \tag{2.3.3}$$

so, the corresponding Hamiltonian becomes,

$$H = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2). \tag{2.3.4}$$

Now, we will look at the non-Hermitian annihilation and creation operators, which are denoted as,

$$\hat{a} = \frac{1}{2\hbar\omega}(\omega\hat{q} + i\hat{p}), \tag{2.3.5}$$

and,

$$\hat{a}^\dagger = \frac{1}{2\hbar\omega}(\omega\hat{q} - i\hat{p}). \tag{2.3.6}$$

These annihilation and creation operators, along with the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, are used. We get the following result for the Hamiltonian,

$$\hat{H} = \hbar\omega\left[\hat{a}^\dagger\hat{a} + \frac{1}{2}\right]. \tag{2.3.7}$$

The product of annihilation and creation operator $\hat{a}^\dagger\hat{a}$ has particular importance and is called *number operator* denoted by $\hat{n}$. The corresponding Fock or number state is $|n\rangle$. If one applies the Hamiltonian operator on a number state a new eigenvalue equation can be formed as given by

$$\hat{H}|n\rangle = \hbar\omega\left[\hat{a}^\dagger\hat{a} + \frac{1}{2}\right]|n\rangle = E_n|n\rangle, \tag{2.3.8}$$

for ground state $n = 0$ the operation of $\hat{H}$ on the ground, the state gives us zero-point energy and is given by $\hbar\omega/2$. Let us consider a general state $|\psi_n\rangle$ is an eigenstate of the number operator $\hat{n}$ then the following eigenvalue equation is obtained,

$$\hat{n}|\psi_n\rangle = n|\psi_n\rangle, \tag{2.3.9}$$

where, (n) is the eigenvalue. The number operator is Hermitian in nature and therefore its eigenvectors build a complete set of orthogonal states with real eigenvalues as given by,

$$\langle\psi_n|\hat{n}|\psi_n\rangle = \langle\psi_n|\hat{a}^\dagger\hat{a}|\psi_n\rangle = n\langle\psi_n|\psi_n\rangle = n. \tag{2.3.10}$$

Now let us apply the annihilation operator on the number state and using the commutation relation, $[\hat{a}, \hat{a}^\dagger] = 1$ in the following manner we get,

$$\hat{n}\hat{a}|n\rangle = (\hat{a}\hat{a}^\dagger - 1)a|n\rangle = (n-1)a|n\rangle. \tag{2.3.11}$$

This indicate that if $|n\rangle$ is an eigen-state of number operator then $a|n\rangle$ is also the eigen-state of the same operator with eigenvalue $(n-1)$ as a result, we are free to write,

$$\hat{a}|n\rangle = C_n|n-1\rangle. \tag{2.3.12}$$

By using the normalization condition one can find the value of $C_n = \sqrt{n}$ so,

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{2.3.13}$$

and similar calculations can be done for the creation operator and are found as,

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2.3.14}$$

For n^{th} Fock states we can operate the creation operator on the ground state in the following manner

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}|0\rangle, \tag{2.3.15}$$

as eigenvectors of number operators are orthogonal and make a complete set of orthonormal states, $\langle n|m\rangle = \delta_{nm}$ and $\sum_n |n\rangle\langle n| = \hat{I}$ where, $\hat{I}$ is an identity operator in the Hilbert space [47].

2.4 Coherent State

Coherent states provide a critical role in the development of quantum dynamics, especially when the quantum evolution is closed to classical. Erwin Schrödinger was the first one who introduced the coherent states as Gaussian wave packets to explain the transformation of a harmonic oscillator in 1926 [67]. The coherent states have gained more significance with the invention of the laser and to study of quantum electrodynamics with coherent systems. Because the electromagnetic field in free space is a superposition of numerous classical modes, each one is governed by a simple harmonic oscillator equation. In this regard, the coherent states have been considered a remarkable tool to connect quantum and classical optics. These minimum uncertainty states have been studied for a long time however, the discussion became hot after the pioneer works of Glauber and Sudarshan in the 1960s in the context of quantum optics [68, 69]. Coherent states have several applications however, we are interested to study these classical states in quantum metrology to find, which bound can be saturated with coherent states [70]? Coherent states can be generated by three methods as reported in the literature. In the ladder operator approach, where a coherent or Glauber state ($|\alpha\rangle$) is considered as an eigenstate of the annihilation operator ($\hat{a}$), or vacuum state can be displaced with the displacement operator to generate a coherent state and the other way is to consider the minimum uncertainty condition where the Heisenberg's relation equate $\hbar/2$. Here, we used the first approach to develop equation for coherent states

$$\hat{a}|\alpha\rangle = (\alpha)|\alpha\rangle, \tag{2.4.1}$$

and,

$$\langle\alpha|\hat{a}^{\dagger} = (\alpha^{*})\langle\alpha|, \tag{2.4.2}$$

where $\alpha \in C$ and is called displacement. Since a coherent state contains an indefinite number of photons so, we can expand it on the following basis,

$$|\alpha\rangle = \sum_{n} |n\rangle\langle n|\alpha\rangle. \tag{2.4.3}$$

The term $\langle n|\alpha\rangle$ is called the transition amplitude and let us call it C_n. Also, we know the operation of annihilation and creation operator on number state as well as on coherent state that are, $\hat{a}^{\dagger}|n\rangle = (\sqrt{n+1})|n+1\rangle$ and $\langle n|\hat{a} = (\sqrt{n+1})\langle n+1|$ using these relations one can find the following compact form for the coherent state,

$$|\alpha\rangle = C_n \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.4.4}$$

Now, using the normalization condition to get the value for C_n and above equation can be written in compact form for normalized coherent states as,

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.4.5}$$

Therefore, coherent state can be thought as a superposition of number state or coherent state has infinite number of photons and can be expanded in number state basis.

2.5 Squeezed States of Light

An electric field may be decomposed into two quadratures $Sin\omega t$ and $Cos\omega t$ for a monochromatic plane wave. The coherent state has been considered the most classical state having minimum uncertainty divided into the two quadratures given by Heisenberg's uncertainty product. These fluctuations in the quadratures are called zero-point fluctuations and are distributed randomly in phase space. Moreover, this zero-point noise shows the short noise limit (SNL) to the minimization of noise in a wave signal. One can manipulate these vacuum fluctuations in the two quadratures in a way to increase the fluctuations in one quadrature at the expense of other quadratures, this will create fewer or zero fluctuations in one of the quadratures. Such states have been referred to as *quadrature squeezed states* of light or generalized coherent states where quantum noise is no more available haphazardly [46]. On the other hand, we have photon-number-squeezed states where the uncertainty in photon number is less as compared to that of a coherent state ($\Delta n < \sqrt{\langle n \rangle}$). This decrease in uncertainty can only be achieved at the expense of squeezing in phase uncertainty. These states obey sub-Poissonian distribution and are referred by different names like sub-Poisson light, quit light [71]. Squeezed states offer various applications in different areas including optical communication, quantum information, and precision measurements [72]. Before discussing various examples of squeezed states, we need to provide some prerequisites for further explanation.

To generate single-mode squeezed states mathematically, an operator is acted on some general state corresponding to a physical process. The squeezing operator is defined as follows,

$$\hat{S}(\chi) = \exp\left[\frac{1}{2}\left(\chi^* \hat{a}^2 - \chi \hat{a}^{\dagger 2}\right)\right], \tag{2.5.1}$$

the factor χ is a complex number and is defined as $\chi = re^{i\theta}$, where r is called the squeezing parameter and ranges as $0 \leq r < \infty$ and θ is the squeezing angle ($0 \leq \theta \leq 2\pi$). Furthermore, the nature of the squeezing operator is unitary, and one can write $\hat{S}(\chi)\hat{S}(-\chi) = I$, which implies that $\hat{S}(\chi)\hat{S}^\dagger(\chi) = I$. Let us consider an arbitrary state, $|\psi\rangle$ for the moment and apply the squeezing parameter given by the above equation (2.5.1) to see the action of the squeeze parameter,

$$|\psi_s\rangle = \hat{S}(\chi)|\psi\rangle. \tag{2.5.2}$$

The operation of the squeezing operator $\hat{S}(\chi)$ on the general state $|\psi\rangle$ is visible in equation (2.5.2), where the squeezed state is denoted by $|\psi_s\rangle$. Moreover, to find variances and photon number statistics of the quadratures $\hat{X}_1$ and $\hat{X}_2$, defined as $(\hat{a} + \hat{a}^\dagger)/2$ and $(\hat{a} - \hat{a}^\dagger)/2i$ then one must find the following relations, by using the Baker-Hausdorf lemma [47]. For details, and calculations visit A.2

$$\hat{S}^\dagger(\chi)\hat{a}\hat{S}(\chi) = \hat{a}Coshr - \hat{a}^\dagger e^{i\theta} Sinhr, \tag{2.5.3}$$

and,

$$\hat{S}^\dagger(\chi)\hat{a}^\dagger\hat{S}(\chi) = \hat{a}^\dagger Coshr - \hat{a}e^{-i\theta} Sinhr, \tag{2.5.4}$$

if we have two operators let us say $\hat{A}$ and $\hat{B}$ then, the Baker-Hausdorf lemma can be written as [73],

$$exp(\hat{A})\hat{B}exp(-\hat{A}) = \hat{B} + [\hat{A},\hat{B}] + \frac{\left[\hat{A}\left[\hat{A},\hat{B}\right]\right]}{2!} + \frac{\left[\hat{A}\left[\hat{A}\left[\hat{A},\hat{B}\right]\right]\right]}{3!}. \tag{2.5.5}$$

Next, we have considered some particular cases, where the arbitrary state can be replaced with vacuum states and also if one applies both displacement and squeeze operator on vacuum states.

2.5.1 Squeezed Vacuum State

Squeezed vacuum state can be generated mathematically by applying the squeezing operator $\hat{S}(\chi)$ on a vacuum. The vacuum means here zero average amplitude however, the average photons number is not zero. Squeezed vacuum state can be established as,

$$|\chi\rangle = \hat{S}(\chi)|0\rangle. \tag{2.5.6}$$

Experimentally these states can be created or produced from a vacuum state by deploying some non-linear optical interactions [74–76]. For example, a squeezed vacuum with less noise in one quadrature component can be generated by using optical parametric amplifiers with injected input as the vacuum [77].

2.5.2 Squeezed Coherent State

Moreover, from previous definitions of squeezed state, we can conclude that squeezed coherent state can be generated by first squeezing the vacuum and then displacing the vacuum in phase space as given by [76],

$$|\alpha,\chi\rangle = \hat{D}(\alpha)\hat{S}(\chi)|0\rangle, \tag{2.5.7}$$

where we know that coherent and squeezed vacuum states (2.5.6) can be constructed as,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, \tag{2.5.8}$$

where $\hat{D}(\alpha)$ and $\hat{S}(\chi)$ is displacement and squeezing operator (2.5.1) respectively, and is defined as follows,

$$\hat{D}(\alpha) = \exp\left[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}\right]. \tag{2.5.9}$$

Therefore, these states are also known as displaced squeezed vacuum states. Furthermore, squeezed coherent states are also the minimum uncertainty states having less noise in one quadrature component at the expense of another component. This amazing characteristic of displaced squeezed vacuum state has been used in many quanta informational tasks including precision measurements, secure communications, gravitational wave detection, and others [76, 78, 79].

2.6 The Entangled Quantum States

Quantum entanglement has been considered one of the bizarre phenomena where separated quantum states get connected in such a way, that no matter how far they are in space, these states remain connected [80]. In simple words, they share the same quantum state. Therefore, measurements of one state can directly affect the other state regardless of the distance between them. The fundamental notions about quantum entanglement were put forward as physicists investigated the mechanics of the quantum world in the early decades. They found something called a quantum state to properly describe the mechanics of the subatomic world. The strong correlation between quantum states can destroy the individual quantum states and instead share the same or unified state known as a quantum entangled state as recognized by EPR in 1935 [40]. Quantum entanglement is a quantum mechanical resource like other physical entities having unusual non-classical correlations between separated quantum states. Entangled quantum systems can be used for various quantum informational tasks like quantum information, quantum computations, quantum metrology, quantum cryptography, and others [16, 81]. Let us build a quantum mechanical framework for a better understanding of quantum entanglement. Suppose we have two arbitrary quantum systems P and Q having corresponding Hilbert spaces H_P and H_Q respectively. Then we know that, the tensor product of individual Hilbert spaces can be considered as the Hilbert space of a composite system as given by,

$$H = H_p \otimes H_Q. \tag{2.6.1}$$

Moreover, if system P is in a state $|\psi\rangle_P$ and system Q is in state $|\psi\rangle_Q$ then state for the composite system can be constructed as,

$$|\psi\rangle = |\psi\rangle_P \otimes |\psi\rangle_Q. \tag{2.6.2}$$

Quantum states that can be written in the format of equation (2.6.2) are called separable or pure states. Since all states cannot be product states, therefore, we are free to write the most general state as,

$$|\psi\rangle_{PQ} = \sum_{ij} C_{ij}|i\rangle_P \otimes |j\rangle_Q. \tag{2.6.3}$$

The state is product state or separable if there exist vectors $\left[C_i^P\right]$, $\left[C_j^Q\right]$ satisfies $C_{ij} = C_i^P C_j^Q$. Therefore, one can write the following results

$$|\psi\rangle_p = \sum_i C_i^P |i\rangle_P, \tag{2.6.4}$$

and,

$$|\phi\rangle_Q = \sum_j C_j^Q |j\rangle_Q. \tag{2.6.5}$$

However, for entangled or inseparable state $C_i j \neq C_i^P C_j^Q$ at least for one pair of coordinates. For example, if we have basis vectors corresponding to each Hilbert space H_P and H_Q as $\{|0\rangle_P, |1\rangle_P\}$ of H_P and $\{|0\rangle_Q, |1\rangle_Q\}$ of H_Q then the entangled quantum state can be written for such system as,

$$\frac{1}{\sqrt{2}}\left(|0\rangle_P \otimes |1\rangle_Q + |1\rangle_P \otimes |0\rangle_Q\right). \tag{2.6.6}$$

Other particular examples can be also considered as entangled states like GHZ states, Bell states, Cat states (not necessarily), etc [82, 83]. However, other particular examples can be seen in a later chapter and their applications to quantum metrology have also been investigated.

Chapter 3

Quantum Estimation Theory: Basic Formalism

3.1 Estimation Process

Measurement is one of the fundamental entities that have applications in every field of science and technology ranging from astronomical measurements to atomic scale investigations. The purpose of measurement is to relate a particular value to a physical quantity or parameter under measurement, providing an estimate of it. Since no estimation process is perfect, therefore, there will be an associated uncertainty with estimated values. These uncertainties are due to statistical errors that may be technical or fundamental. Systematic or insufficient control of probes or environmental difficulties may be considered technical errors however, on the hand there are fundamental bounds imposed by physical laws [84]. Because quantum theory has been considered the most fundamental so these bounds can be raised and studied under the regime of such theory. Astonishingly, measurements through quantum resources surpass the precision bounds that can be attained by using classical systems. Consequently, the process is an estimation process that can be measuring any unknown parameter encoded in a system physical. The estimation process can be done in the following four phases. (i): initial state preparation, (ii): interaction of probe state with the system under consideration or parameter encoding through unitary evolution, (iii): extracting useful information from the output results, and final (iv): a suitable estimator is used to provide an approximate value of the unknown parameter θ as shown in the figure [5]. An estimator is a function that provides an estimate of an unknown parameter from the obtained results. Various estimators can be used to post-process the available data however we used a maximum likelihood estimator (MLE) that maximizes the likelihood probability [85]. Moreover, we considered an unbiased estimator whose mean value converges to the true value of the unknown parameter θ

$$\langle\theta\rangle = \sum_{\xi} P(\xi|\theta)\theta_{est}(\xi) = \theta, \tag{3.1.1}$$

where $P(\xi|\theta)$ is known is the conditional output probability of getting a series of measurement ξ for a certain value of parameter θ. The Born rule $P(\xi|\theta) = Tr(E_{\xi}\rho_{\theta})$ provides such a probability, which is also known as likelihood.

To investigate the accuracy of an estimation process we need to define the mean square error, which is provided by,

$$MSE(\theta) = \sum_{\xi} (\theta(\xi) - \theta)^2 \, P(\xi|\theta). \tag{3.1.2}$$

Furthermore, we have considered an unbiased estimator therefore, the mean square error is equal to the variance on the estimated value of the unknown parameter θ

$$\Delta\theta^2 \equiv \sum_{\xi} (\theta(\xi) - \langle\theta\rangle)^2 \, P(\xi|\theta). \tag{3.1.3}$$

In the later section, this equation can be connected to the bound of precision of any unbiased estimator.

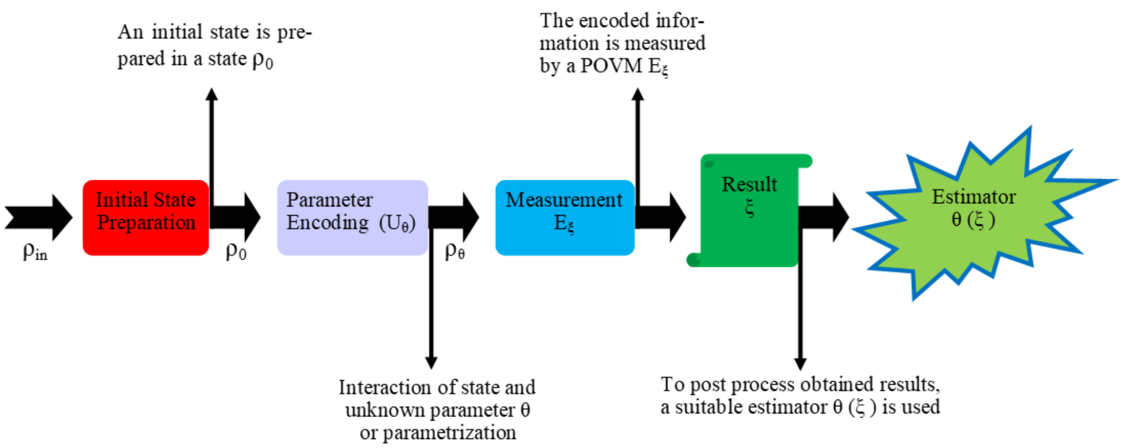

Figure 3.1: Conceptual illustration of parameter estimation process

3.2 Quantum Estimation Theory

Many parameters of relevance in physics cannot be measured directly, either theoretically or experimentally. Because some quantum mechanical properties, such as quantum entanglement and purity, are nonlinear functions of density matrices, they cannot even be used to describe appropriate quantum observables. So, in such a scenario one should consider indirect measurement and estimate the parameter of interest from a set of data obtained from the estimation of various observables. This is nothing but a parameter evaluation problem and can be formally described in the paradigm of quantum estimation theory (QET) [86]. The framework of quantum estimation theory is further classified into two parts. Global quantum estimation theory seeks out the POVM (positive operator-valued measure) with the lowest cost functional, averaging all potential values of the parameter to be evaluated. In contrast, local quantum estimation theory seeks out the POVM that maximizes Fisher information while reducing estimator variance for a given parameter value [87–89]. As a side note, one can predict excellent performances through local QET, because it only works with a single value of a parameter, it requires some sort of back-action technique to reach ultimate limits [90]. Global QET has widely been researched to study lower bounds on accuracy for the measurement of parameters evolved under linear transformation [91, 92]. On the other hand, local quantum estimation theory (LQET) has been used to estimate quantum phase [93], open quantum systems, and other non-unitary transformations [94]. To obtain the required quantity, classical estimation theory employs a suitable estimator, such as the mean square error (MSE) estimation technique, which is lower-limited by the Cramer-Rao bound. In this section, we have presented concepts that are related to the estimation of an unknown parameter, ultimate bounds on the precision (sensitivity). Furthermore, we developed explicit relations for Fisher information, quantum Fisher information, and their relationships with the symmetric logarithmic derivative (SLD) and the Cramer-Rao inequality.

3.2.1 Fisher Information and Cramer-Rao Bound

Researchers in quantum metrology is concerned with the ultimate limits on the resolution or sensitivity of the estimation process, one of the fundamental tools to study and achieve bounds on the precision is *Fisher information (F)*. The information that is encrypted in the resultant probabilities of the measurement process is known as Fisher information or expected information

$$F(\theta) = \sum_{\xi} P(\xi|\theta) \left[\frac{\partial log(P(\xi|\theta))}{\partial \theta} \right]^2 \tag{3.2.1}$$

or,

$$F(\theta) = \sum_{\xi} \frac{1}{P(\xi|\theta)} \left[\frac{\partial (P(\xi|\theta))}{\partial \theta} \right]^2 . \tag{3.2.2}$$

$P(\xi|\theta)$ is known as conditional or provisional probability. When the parameter has the value "θ", $P(\xi|\theta)$ is the conditional probability of getting ξ. The result in equation (3.2.2) can be easily extended for continuous measurement outcomes just by replacing sum with integral, the calculations are presented in A.1. From the above equations, the greater amount of information is proportional to a greater change in the output probabilities which implies that Fisher information is a great concept to measure the sensitivity or uncertainty of a given setup (quantum system). The compact form of this formalism is abridged into a rudimental equation, called the Cramer-Rao bound (CRB), given by equation (3.2.4) that links Fisher information to the ultimately achievable bounds. Furthermore, μ is the number of observations and $F(\theta)$ is the so-called Fisher information.

The main concern about the parameter estimation problem is to calculate the estimator, and various estimators have been used for different situations [5, 95]. Let us consider an example from a set of measurement outcomes $\theta = \theta(\xi_1, \xi_2, \xi_3 ...)$, the optimal estimators that can be realized in classical estimation theory (CET) are those satisfying the Cramer-Rao inequality [96]. Visit A.1 for detailed calculations of the below equations. As Fisher information is proportional to the changes in the output probabilities with respect to the parameter. This allow us to measure the system's sensitivity to fluctuations in θ. This concept was used with fundamental result known as Cramer Rao inequality that connects Fisher information to the optimum attainable limit on the variance of any unbiased estimator. Therefore,

$$\Delta\theta^2 = \sum_{\xi} \left(\theta(\xi) - \langle\theta\rangle\right)^2 P(\xi|\theta) \geq \frac{(\partial\langle\theta\rangle/\partial\theta)^2}{\mu F(\theta)}. \tag{3.2.3}$$

However, in the presence of an unbiased estimator ($\partial\langle\theta\rangle/\partial\theta = 1$), the Cramer Rao bound becomes

$$\Delta\theta^2 \geq \frac{1}{\mu F(\theta)}. \tag{3.2.4}$$

Equation (3.2.4) develops a lower limit on the mean square error (MSE) that is, $MSE = Var(\theta) = E_\theta[(\theta(\xi) - \langle\theta\rangle)^2]$ on any estimator of the parameter θ and for unbiased estimators where the mean value agreed with the true value of the parameter of interest.

3.2.2 Quantum Fisher Information and Quantum Cramer-Rao Bound

Quantum Fisher information is a core concept in quantum metrology that defines a bottom, limit to the variance of an unbiased estimator for the quantity of interest under the estimation process. Now, if we consider a quantum system then the estimation problem is no longer simply because the system is now an ensemble of quantum states,also known as a quantum statistical representation and is defined by ρ_θ. In this case, the parameter θ does not represent any quantum observable so, we need to find its values through the measurement of observables on ρ_θ. Even under ideal conditions, this indirect technique introduces extra uncertainty in the calculated value that cannot be eliminated. Therefore, the primary goal of the QET is nothing but to reduce this extra uncertainty which will lead to higher precision in the measurement. Typically, one knows a quantum system in a state ρ_θ that is a function of the required variable θ to be estimated for quantum parameter estimation. The quantum system can be further manipulated with a suitable measurement procedure and the outcomes of the measurement are processed with an estimator to get a true value for the required parameter.

The Born rule in quantum physics gives the likelihood probability, which is stated as follows:

$$P(\xi|\theta) = Tr(E_\xi \hat{\rho}_\theta), \tag{3.2.5}$$

where, $\hat{\rho}_\theta$ is the density operator and E_ξ is a selected positive operator value measure (POVM). Therefore, quantum Fisher information is the maximization over all POVMs E_ξ. According to [97, 98] the quantum Fisher information F_Q is defined like $F_Q = Tr(\rho L^2)$, where L is a significant quantity and is known as the symmetric logarithmic derivative (SLD). Let θ be the parameter of interest to estimate, then the SLD satisfy the following equation,

$$\partial_\theta \rho_\theta = \frac{(\rho_\theta L + L\rho_\theta)}{2}. \tag{3.2.6}$$

The symmetric logarithmic derivative has been widely researched because of its core role in metrological tasks and its importance can be linked in two ways: first, if the SLD operator is known then quantum Fisher information (QFI) can be directly defined as $F_Q = \langle \Delta^2 L \rangle$, this indicates that the variance of the symmetric logarithmic derivative (SLD) operator represents the quantum Fisher information (QFI) [89, 99]. Equation (3.2.5) can be further written as,

$$\partial_\theta (P(\xi|\theta)) = Re\left(Tr(\rho_\theta E_\xi L_\theta)\right). \tag{3.2.7}$$

This leads to the following relation for Fisher information, which is linked to symmetric logarithmic derivative (SLD) operator as,

$$F(\theta) = \sum_\xi \frac{Re\left(Tr(\rho_\theta E_\xi L_\theta)\right)^2}{Tr(\rho_\theta E_\xi)}. \tag{3.2.8}$$

This puts a classical limit on the precision, which can be obtained by using proper data processing or using a proper estimator like a maximum likelihood estimator. However, if one wants to measure the ultimate bound to precision, it is crucial to optimize the Fisher information over quantum estimation. In this situation, the Fisher information can be expressed as,

$$F(\theta) \leq \left| \frac{Tr(\rho_\theta E_\xi L_\theta)}{\sqrt{Tr(\rho_\theta E_\xi)}} \right|^2, \tag{3.2.9}$$

after a little mathematical manipulations above equation (3.2.9) can be further reduce as,

$$F(\theta) \leq Tr(\rho_\theta L_\theta^2). \tag{3.2.10}$$

It is apparent from the above inequalities that any quantum measurement's Fisher information (F) is bounded by the so-called quantum Fisher information (QFI), let suppose $F_Q(\theta) = Tr(\rho_\theta L_\theta^2)$ then,

$$F(\theta) \leq F_Q(\theta). \tag{3.2.11}$$

A huge amount of information provides better estimation protocols, so we have maximized the classical Fisher information. Therefore, it can be estimated from the above equation (3.2.11) that quantum Fisher information (QFI) puts a fundamental upper limit on classical Fisher information (F). So, the Cramer-Rao bound can be qualified to quantum Cramer Rao bound (QCRB) for unbiased estimator as follows,

$$\Delta\theta^2 \geq \frac{1}{\mu F_Q(\theta)}. \tag{3.2.12}$$

This implies that the quantum Cramer Rao bound depends on the geometrical behavior of the quantum statistical representation and is independent of the measurement [10, 100]. Moreover, we have provided a relation for quantum fisher information that can be used for any general system. Let us consider an N-dimensional system (quantum statistical model) with the density operator ρ_θ (the density matrix is a function of the parameter θ) given by,

$$\hat{\rho}_\theta = \sum_{\lambda=1}^{z} P_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|, \tag{3.2.13}$$

where P_λ, is an eigenvalue, $|\psi_\lambda\rangle$ is the eigen-state and "z" is the dimension of a set. Since we found that the following relation defines quantum fisher information

$$F_Q(\theta) = Tr(\rho_\theta L_\theta^2). \tag{3.2.14}$$

The symmetric logarithmic derivative (SLD) is determined by the Lyapunov matrix relation, which is provided by

$$\partial_\theta \rho_\theta = \frac{(\rho_\theta L + L\rho_\theta)}{2}. \tag{3.2.15}$$

Let us expand equation (3.2.15) in the basis of ρ_θ, we get the following result

$$\langle\psi_\lambda|\partial_\theta\rho_\theta|\psi_k\rangle = \frac{(P_\lambda + P_k)L_{\lambda k}}{2}, \tag{3.2.16}$$

where $L_{\lambda k} = \langle\psi_\lambda|L|\psi_k\rangle$ and $\partial_\theta = \frac{\partial}{\partial_\theta}$. From equation (3.2.16) one can get the idea that $L_{\lambda k}$ can take any value for $\lambda, k > z$, because this equation can be establish for any value of $L_{\lambda k}$. This can be seen in the following equation if one uses equation (3.2.13) and the normalization equation given by equation (3.2.17)

$$\hat{I} = \sum_{k=1}^{N} |\psi_k\rangle\langle\psi_k|. \tag{3.2.17}$$

In equation (3.2.14), one can get the following result

$$F_Q(\theta) = \sum_{\lambda=1}^{z}\sum_{k=1}^{N} P_\lambda L_{k\lambda} L_{\lambda k}. \tag{3.2.18}$$

The identity operator is given by equation (3.2.17) and all $P_\lambda > 0$ which satisfies $\sum_{\lambda=1}^{s} P_\lambda = 1$ so, equation (3.2.16) can be modified as,

$$L_{\lambda k} = 2\frac{\langle\psi_\lambda|\partial_\theta\rho_\theta|\psi_k\rangle}{P_\lambda + P_k}, \tag{3.2.19}$$

or,

$$L_{\lambda k} = 2\frac{(\partial_\theta\rho_\theta)_{\lambda k}}{P_\lambda + P_k}, \tag{3.2.20}$$

and,

$$L_{k\lambda} = 2\frac{(\partial_\theta\rho_\theta)_{k\lambda}}{P_\lambda + P_k}, \tag{3.2.21}$$

where $(\partial_\theta\rho_\theta)_{k\lambda} = \langle\psi_\lambda|\partial_\theta\rho_\theta|\psi_k\rangle$, now using equations (3.2.20) and (3.2.21) in (3.2.18) for quantum Fisher information we get,

$$F_Q(\theta) = \sum_{\lambda=1}^{z}\sum_{k=1}^{N} \frac{4P_\lambda}{(P_\lambda + P_k)^2}\left|(\partial_\theta\rho_\theta)_{\lambda k}\right|^2. \tag{3.2.22}$$

It is apparent from the above relations that quantum Fisher information is dependent on two factors: the eigen vectors $|\psi_\lambda\rangle$, and the eigenvalues ρ_θ. To explicitly write the two contributions we have expanded $(\partial_\theta\rho_\theta)_{\lambda k}$ as follow and used orthogonality property $-\langle\psi_\lambda|\partial_\theta\psi_k\rangle = \langle\partial_\theta\psi_\lambda|\psi_k\rangle$. Therefore, one can write,

$$(\partial_\theta\rho_\theta)_{\lambda k} = \partial_\theta P_\lambda\delta_{\lambda k} + (P_k - P_\lambda)\langle\psi_\lambda|\partial_\theta\psi_k\rangle. \tag{3.2.23}$$

Now substitute equation (3.2.23) in equation (3.2.22) for quantum Fisher information, the following equation can be found

$$F_Q(\theta) = \sum_{\lambda=1}^{z}\frac{1}{P_\lambda}(\partial_\theta P_\lambda)^2 + \sum_{\lambda=1}^{z}\sum_{k=1}^{N}\frac{4P_\lambda(P_\lambda - P_k)^2}{(P_\lambda + P_k)^2}\left|\langle\psi_\lambda|\partial_\theta\psi_k\rangle\right|^2. \tag{3.2.24}$$

Since we know that $\sum_{k=1}^{N} = \sum_{k=1}^{z} + \sum_{k=z+1}^{N}$, this implies that the second part of the equation (3.2.24) can be separated into two parts as F_1 and F_2

$$F_1 = \sum_{\lambda,k=1}^{z}\frac{4P_\lambda(P_\lambda - P_k)^2}{(P_\lambda + P_k)^2}\left|\langle\psi_\lambda|\partial_\theta\psi_k\rangle\right|^2, \tag{3.2.25}$$

and F_2 is

$$F_2 = \sum_{\lambda=1}^{z} + \sum_{k=z+1}^{N} 4P_\lambda \left|\langle\psi_\lambda|\partial_\theta\psi_k\rangle\right|^2 . \tag{3.2.26}$$

Using normalization condition one can find the following relation,

$$\sum_{k=z+1}^{N} |\psi_k\rangle\langle\psi_k| = I - \sum_{k=1}^{z} |\psi_k\rangle\langle\psi_k| . \tag{3.2.27}$$

Therefore, F_2 can be further modified as

$$F_2 = \sum_{\lambda=1}^{z} 4P_\lambda\langle\partial_\theta\psi_\lambda|\partial_\theta\psi_i\rangle - \sum_{\lambda,k=1}^{z} 4P_\lambda \left|\langle\psi_\lambda|\partial_\theta\psi_k\rangle\right|^2 . \tag{3.2.28}$$

Now adding F_1 and F_2 and use it in equation (3.2.24) the result for quantum Fisher information will read as,

$$F_Q(\theta) = \sum_{\lambda=1}^{z} \frac{1}{P_\lambda}(\partial_\theta P_\lambda)^2 + \sum_{\lambda=1}^{z} 4P_\lambda\langle\partial_\theta\psi_\lambda|\partial_\theta\psi_\lambda\rangle - \sum_{\lambda,k=1}^{z} \frac{8P_\lambda P_k}{P_\lambda + P_k} \left|\langle\psi_\lambda|\partial_\theta\psi_k\rangle\right|^2 . \tag{3.2.29}$$

Equation (3.2.29) has two parts: the first part shows the classical Fisher information of the distribution P_λ and the second part is purely quantum mechanical contribution. Therefore, one can predict that there are two contributions to quantum Fisher information, one comes from classical Fisher information and the other is quantum mechanical contribution. These can be written individually as,

$$F_Q(\theta) = F_{Cl}(\theta) + F_{Qt}(\theta), \tag{3.2.30}$$

where $F_{Cl}(\theta)$ and $F_{Qt}(\theta)$ reads as

$$F_{Cl}(\theta) = \sum_{\lambda=1}^{z} \frac{1}{P_\lambda}(\partial_\theta P_\lambda)^2, \tag{3.2.31}$$

and,

$$F_{Qt}(\theta) = \sum_{\lambda=1}^{z} 4P_\lambda\langle\partial_\theta\psi_\lambda|\partial_\theta\psi_\lambda\rangle - \sum_{\lambda,k=1}^{z} \frac{8P_\lambda P_k}{P_\lambda + P_k} \left|\langle\psi_\lambda|\partial_\theta\psi_k\rangle\right|^2 . \tag{3.2.32}$$

The classical component of the quantum Fisher information is a special case of the classical Fisher information, as shown by the above mentioned relations. Also, it is easy to see that classical Fisher information is just the derivative of the eigenvalues, which confirms that this piece of the information is contributed from classical distribution in eigen-space.

Additionally, we have identified a few essential features of quantum Fisher information that can be useful in comprehending the quantum metrology problem.

(i): For pure states, the classical contribution vanishes, and the relation for quantum Fisher information is reduced to the given expression [10, 101, 102]. Furthermore, for unitary parametrization $\exp{(i\theta H)}$, the classical information vanishes, and one can get the following result. However, for non-unitary parametrization, the classical part does influence the precision if one uses it for estimation problems

$$F_{Qt}(\theta) = 4(\langle\psi_\lambda|H^2|\psi_\lambda\rangle - |\langle\psi_\lambda|H|\psi_\lambda\rangle|^2), \tag{3.2.33}$$

and it is nothing but the variance of the Hermitian operator H and can be written as,

$$F_{Qt}(\theta) = 4(\Delta H)^2_{|\psi_i\rangle}. \tag{3.2.34}$$

(ii): For states that are not pure one can get the following inequality for quantum Fisher information [102],

$$F_{Qt}(\theta) < 4(\Delta H)^2. \tag{3.2.35}$$

(iii): Quantum Fisher information is convex in the state as a universal characteristics, for instance, a general mixed state $\rho = \sum_k P_k \rho_k$ with $\sum_k P_k = 1$ and the measurement is considered as fixed, then:

$$F(\theta) \leq \sum_k P_k F^k(\theta). \tag{3.2.36}$$

In equation (3.2.36), $F(\theta)$ is the Fisher information of the state and $F^j(\theta)$ is the Fisher information of the singlet state from a mixture of P_k states [5].

3.3 Quantum Entanglement

The word entanglement means "state of being involved in complicated circumstances". So, in quantum mechanics "quantum entanglement" is a complicated affair between two or more states or particles. The word "quantum entanglement" was first introduced by Erwin Schrödinger [103]. One of the key concepts, which distinguish quantum mechanics from the classical world is the quantum correlation between quantum states, and this interesting property has no classical analog. This concept was examined by Albert Einstein, Nathan Rosen, and Boris Podolsky (APR) in their 1935 article that how correlated states would interact with each other, and they agreed that when two particles are deeply entangled, they destroy their own quantum state and can be regarded as one quantum state [40]. Quantum entanglement is an important and counter-intuitive concept of quantum mechanics, which has deep connections with quantum information science [104]. The quantum correlated states are extensively employed for the tests of quantum mechanical concepts and have been widely researched in the areas of quantum communication [81], quantum computing [105], quantum cryptography [80], and quantum metrology [106]. Here the author has investigated the importance of quantum entanglement in quantum metrology.

3.3.1 Role of Quantum Entanglement in Quantum Metrology

Quantum correlation has been considered a key to improvement in metrology and this concept has proved experimentally. However, the limitations in these experiments are due to noise that destroys the entanglement between particles [106]. Measurement operations performed on a single quantum system in a state ρ_θ may be replicated N times on N separate versions of the quantum system available in the same state ρ_θ. In this situation (where quantum states are prepared independently) the quantum Fisher information works like classical Fisher information and is calculated as $NF_Q(\theta)$ with associated mean square, error decreasing as $1/N$ creating standard quantum limit. However, if one uses quantum entanglement and generates quantum states that are not separable but entangled then the measurement of mean square error scaled as $1/N^2$ [44]. This $\sqrt{N}$ improvement in the precision is due to quantum correlation between quantum states because there is no counterpart of quantum entanglement [11]. However, such enhancement of quantum estimation through entanglement is very much fragile to decoherence or quantum noise. When N is very large then a little amount of noise is enough to destroy the quantum correlation and brings back the precision limit to $1/N$ [107]. Moreover, the concept of quantum entanglement is used in this work to achieve quantum-enhanced measurements. These include *N00N* states, entangled coherent states, and other states.

If the states are separable or product states as denoted by

$$|\psi\rangle = |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes ... \otimes |\psi\rangle_n, \tag{3.3.1}$$

then the variance of any unbiased estimator is bound by standard quantum limit or shot noise limit $\Delta\theta^2 \geq \frac{1}{m}$. Where m is the probe state (classical). For example coherent state $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ or number state $\hat{n}|\psi_n\rangle = n|\psi_n\rangle$. This limit is actually the consequence of central limit theorem [108, 109].

However, for entangled states the scenario is different. A generalized entangled state can be formed as

$$|\psi\rangle_{PQ} = \sum_{ij} C_{ij}|i\rangle_P \otimes |j\rangle_Q. \tag{3.3.2}$$

For entangled or inseparable state $C_ij \neq C_i^P C_j^Q$ at least for one pair of coordinates. For example, if we have basis vectors corresponding to each Hilbert space H_P and H_Q as $\{|0\rangle_P, |1\rangle_P\}$ of H_P and $\{|0\rangle_Q, |1\rangle_Q\}$ of H_Q then the entangled state can be written for such system as,

$$\frac{1}{\sqrt{2}} \left(|0\rangle_P \otimes |1\rangle_Q + |1\rangle_P \otimes |0\rangle_Q\right). \tag{3.3.3}$$

For such states as above the variance is limited by Heisenberg's limit. This fundamental limit is obtained due to usage of quantum mechanical resources (quantum entanglement). Some particular examples have been provided in chapter 4.

Chapter 4

Photonic Quantum States: Applications in Quantum Metrology

Since the primary purpose of quantum metrology is to estimate certain parameters by introducing probe states. Therefore, in this chapter, we have enlisted basic photonic quantum states which are important for the problem of quantum metrology.

4.1 Phase Sensitivity using Number State

Let us first consider the average energy associated with a single-mode system in a number state given by

$$\langle n|\hat{H}|n\rangle = \hbar\omega\left[n + \frac{1}{2}\right]. \tag{4.1.1}$$

This implies that energy is related to each photon, and it can be also examined that even at zero photon state ($n = 0$), there is $\hbar\omega/2$ energy associated with vacuum state and is called ground state energy. To find the photon number fluctuations we used the variance of the number operator $\Delta\hat{n}^2 = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2$ and is calculated as,

$$\Delta\hat{n}^2 = \langle n|\hat{n}^2|n\rangle - \langle n|\hat{n}|n\rangle^2 = 0, \tag{4.1.2}$$

because of the sharp estimate of photon number, the fluctuations in the Fock state vanishes [47].

To find the minimum uncertainty in phase upon using the Fock state or number state, we have considered the Mach-Zehnder interferometer (MZI) as depicted in Figure 4.1. The Mach-Zehnder interferometer we have considered has 50-50 beam splitters, two total reflecting mirrors, phase shifters $\phi - 1$ and $\phi - 2$, and detectors at the output ports.

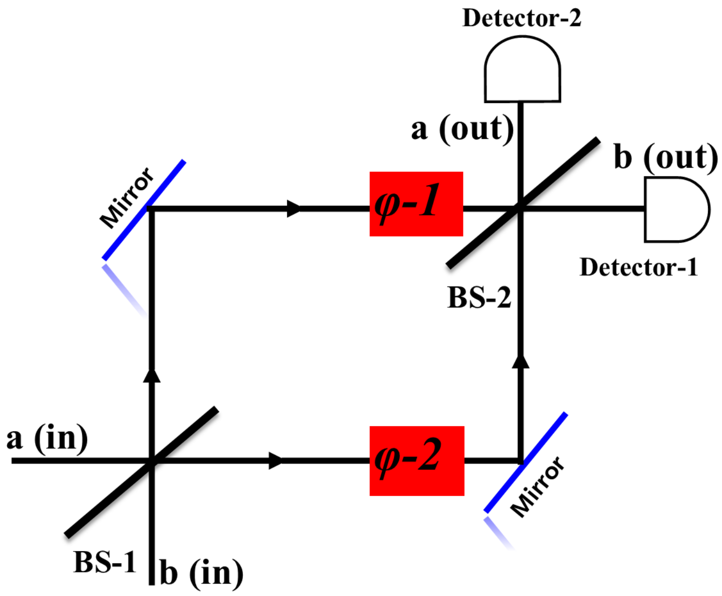

Figure 4.1: The Mach-Zehnder interferometer, with $a(in) = \hat{a}_1$, $b(in) = \hat{b}_1$ and $a(out) = \hat{a}_2$ and $b(out) = \hat{b}_2$are the corresponding annihilation operators at input and output ports respectively. Where $\phi - 1$ and $\phi - 2$ are phase shifters along with detectors at the output ports.

The action of MZI can be easily seen in the following transformation matrices or other words, the input states transform in the following way,

$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix}. \tag{4.1.3}$$

Equation (4.1.3) can be further simplified just by multiplying these matrices and using the corresponding annihilation operators,

$$\begin{pmatrix} \hat{a}_1 \\ \hat{b}_1 \end{pmatrix} = e^{i(\phi_1+\phi_2)/2} \begin{pmatrix} cos(\frac{\phi}{2}) & -sin(\frac{\phi}{2}) \\ sin(\frac{\phi}{2}) & cos(\frac{\phi}{2}) \end{pmatrix} \begin{pmatrix} \hat{a}_2 \\ \hat{b}_2 \end{pmatrix}. \tag{4.1.4}$$

To better understand the action of MZI in terms of angular momentum operators we have deployed the following angular momentum operators that satisfy the algebra of angular momentum as recognized by Yurke et al [110]. Let us define the angular momentum operators are,

$$\hat{j}_x = \frac{1}{2}\left[\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right], \hat{j}_y = \frac{i}{2}\left[\hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}\right], \hat{j}_z = \frac{1}{2}\left[\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\right]. \tag{4.1.5}$$

These operators obey the following commutation relations,

$$[\hat{j}_x, \hat{j}_y] = i\hbar\hat{j}_z, [\hat{j}_y, \hat{j}_z] = i\hbar\hat{j}_x, [\hat{j}_z, \hat{j}_x] = i\hbar\hat{j}_y. \tag{4.1.6}$$

The total angular momentum operator is then given by,

$$\hat{J}^2 = \hat{j}_x^2 + \hat{j}_y^2 + \hat{j}_z^2, \tag{4.1.7}$$

and the corresponding photon number operator is $\hat{n} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$. With these tools in hand now one can manipulate the action of MZI as rotation in spin space. So, it is convenient to use the transformations of angular momentum operators instead of annihilation operators.

$$\begin{pmatrix} \hat{j}_x \\ \hat{j}_y \\ \hat{j}_z \end{pmatrix}_{out} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{j}_x \\ \hat{j}_y \\ \hat{j}_z \end{pmatrix}_{in}, \tag{4.1.8}$$

or we can say that angular momentum operators transform as follow,

$$(\hat{j}_x)_{out} = (\hat{j}_x cos\phi + \hat{j}_z sin\phi)_{in}, (\hat{j}_y)_{out} = (\hat{j}_y)_{in}, and, (\hat{j}_z)_{out} = (\hat{j}_z cos\phi - \hat{j}_x sin\phi)_{in}. \tag{4.1.9}$$

Now if one considers Fock state in one arm of the MZI and vacuum at another input port, the input state may be constructed in the given style,

$$|\psi\rangle_{in} = |n\rangle|0\rangle = |n, 0\rangle. \tag{4.1.10}$$

Furthermore, expectation values can be evaluated as follow by using the above transformations and some in-hand knowledge of finding variances we have,

$$\langle\hat{j}_z\rangle_{in} = \frac{n}{2}, and, \langle\hat{j}_z^2\rangle_{in} = \frac{n^2}{4}. \tag{4.1.11}$$

This directly results in zero variance that is,

$$(\Delta\hat{j}_z^2)_{in} = 0. \tag{4.1.12}$$

From equation (4.1.12) one can predict that the Fock state is the eigenstate of $\hat{j}_z$, that corresponds to the variance with respect to number operator as given in photon number statistics above. and the only contribution to $\hat{j}_z$ is coming from $(\Delta\hat{j}_x^2)_{in}$. The expectation values with respect to other operators are,

$$\langle\hat{j}_x\rangle_{in} = 0, and, \langle\hat{j}_x^2\rangle_{in} = \frac{n^2}{4}, \tag{4.1.13}$$

this implies that,

$$(\Delta\hat{j}_x^2)_{in} = \frac{n^2}{4}. \tag{4.1.14}$$

Similar calculations can be done for variance in the output states and one can easily get the following results,

$$(\Delta\hat{j}_x^2)_{out} = cos^2\phi\frac{n^2}{4}for...\phi = 0, (\Delta\hat{j}_x^2)_{out} = \frac{n^2}{4}, \tag{4.1.15}$$

and,

$$(\Delta\hat{j}_z^2)_{out} = sin^2\phi\frac{n^2}{4}for...\phi = \frac{\pi}{2}, (\Delta\hat{j}_z^2)_{out} = \frac{n^2}{4}. \tag{4.1.16}$$

Moreover, to find the metrological power or to find the minimum uncertainty in the phase we have used the quantum Fisher information approach as defined by equations (3.2.34) and (3.2.12). so, the uncertainty is then related by the following form,

$$\Delta\phi = \frac{1}{\sqrt{n}}. \tag{4.1.17}$$

This is referred to as the short noise limit or standard quantum limit; nevertheless, the phase uncertainty estimate is not affected by phase angle ϕ. The relationship between phase uncertainty and the number of photons is demonstrated in Figure 4.2.

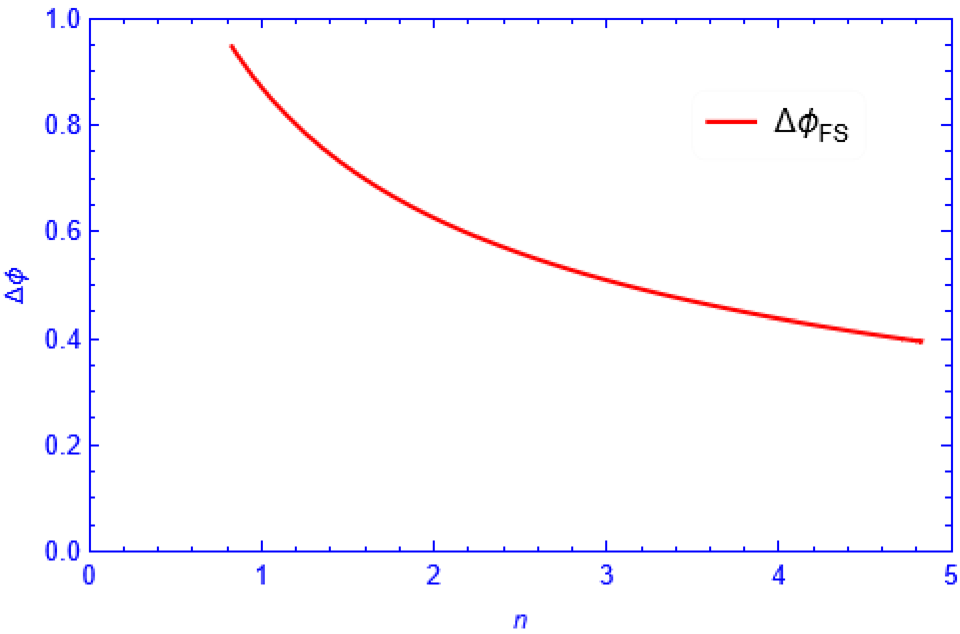

Figure 4.2: Behavior of uncertainty in phase concerning the number of photons.

It is apparent from Figure 4.2 that the high number of photons can lead to better precision as uncertainty will be less. However, this limit is still achievable with classical approaches without considering quantum correlations, this can be seen in the next section.

4.2 Coherent State

In this section we considered coherent state as probe state in MZI for phase estimation problem. Since we know coherent state is an eigenstate of an annihilation operator as given by,

$$\hat{a}|\alpha\rangle = (\alpha)|\alpha\rangle, \langle\alpha|\hat{a}^{\dagger} = (\alpha^{*})\langle\alpha|, \tag{4.2.1}$$

where $\alpha \in C$ and is called displacement.

Photon-number statistics can be calculated using the average value and deviation of the photon number operator $(\hat{n})$,

$$\langle\hat{n}\rangle = \langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle, \tag{4.2.2}$$

and,

$$\Delta\hat{n}^2 = \left[\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2\right] = \langle\alpha|\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a}|\alpha\rangle = |\alpha|^2. \tag{4.2.3}$$

If one applies coherent states for a phase estimation problem so, only a standard quantum limit can be achieved given by, $1/\sqrt{\langle n \rangle} = 1/|\alpha|$. This implies that coherent states have more classical statistics and can be used as a bridge between classical and quantum states. Furthermore, the expectation values with respect with $\hat{j}_z$ operator can be obtained as,

$$\langle \hat{j}_z \rangle = \frac{1}{2}|\alpha|^2, and, \langle \hat{j}_z^2 \rangle = \frac{1}{4}|\alpha|^2 + \frac{1}{4}|\alpha|^4, \tag{4.2.4}$$

so, the variance in $\hat{j}_z$ is calculated as

$$\Delta \hat{j}_z^2 = \frac{1}{4}|\alpha|^2. \tag{4.2.5}$$

Therefore, the corresponding quantum Fisher information is $F_Q = |\alpha|^2$, and hence the metrological power or minimum uncertainty through using coherent state is given by equation (3.2.12), using this relation one can get the following phase sensitivity [111]

$$\Delta\phi = \frac{1}{|\alpha|} = \frac{1}{\sqrt{\langle n \rangle}}. \tag{4.2.6}$$

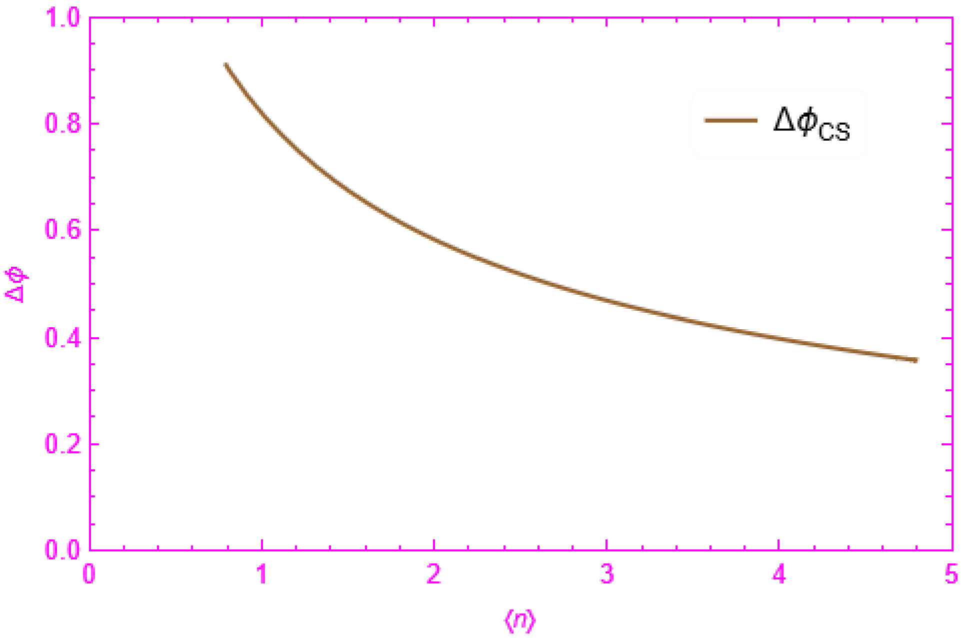

Figure 4.3: Phase uncertainty of coherent state and its relationship with average number of photons

4.3 *N00N* State

N00N states are highly entangled multipartite states and are the most significant class of quantum states. The essential quantum limit known as the Heisenberg limit in quantum metrology has been extensively researched using *N00N* states [112]. Due to its greater variance between the two modes, *N00N* states satisfy the Heisenberg limit for single-phase estimation. It has recently been investigated to generalize the concept of *N00N* states to multimode *N00N* states to find a way for multiparameter estimation. Quantum enhancement through *N00N* states has been demonstrated experimentally beyond the standard quantum limit [113]. However, the generation of multimode *N00N* states is abstract and its experimental demonstration is very much fragile because its entanglement is sensitive to noise [114]. Mathematically,

$$|\psi\rangle_{N00N} = \frac{1}{\sqrt{2}}\left[|N,0\rangle + e^{i\phi}|0,N\rangle\right], \tag{4.3.1}$$

that is, in the first mode, a coherent superposition of N photons with vacuum in the second mode, or vice versa. Where, "ϕ" is the phase between the two modes. To find the achievable sensitivity through N00N states, the variance of the number operator is estimated as $\Delta N^2 = N^2/4$ and then using equations (3.2.34) and (3.2.12), we get

$$\Delta\phi_{N00N} \geq \frac{1}{N}. \tag{4.3.2}$$

That is reached to Heisenberg limit and that's why *N00N* states provide a critical role in quantum estimation (particularly in phase measurement) [115, 116]. The evolution of *N00N* states through an ideal interferometer is given by,

$$\frac{1}{\sqrt{2}}\left[|N,0\rangle + |0,N\rangle\right] \longrightarrow U_\phi \longrightarrow \frac{1}{\sqrt{2}}\left[|N,0\rangle + e^{iN\phi}|0,N\rangle\right]. \tag{4.3.3}$$

The improved precision is due to the faster deviation in phase shift which is related to photons number $N\phi$. The phase sensitivity of quantum interferometry upon using *N00N* state is graphed in figure 4.4

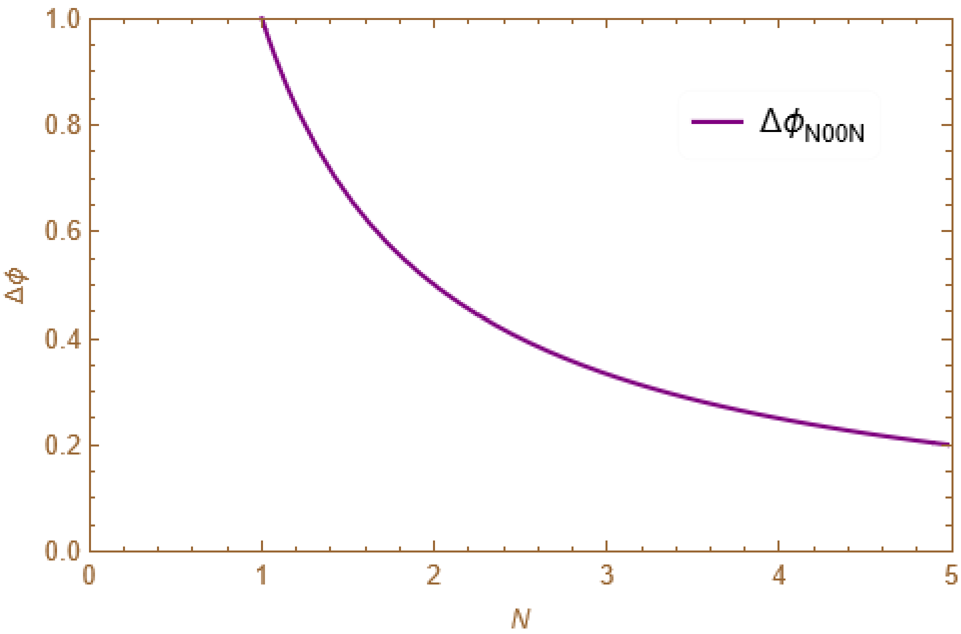

Figure 4.4: The uncertainty becomes very minimum for *N00N* states at a large number of photons.

Moreover, the enhanced sensitivity and resolution through *N00N* states have been researched for biological microscopy, tissue imaging, quantum optical coherence tomography, refractive index sensing, and others [117]. However, with such amazing applications of *N00N* states, certain limitations restrict its uses including the generation of high photon number states and its fragile nature makes these states of less usage [5]. For these reasons, several other states have been researched for the practical implication of quantum metrology.

4.4 The Entangled Coherent State

As discussed in the section on coherent states that these quantum states provide a significant role in connecting the classical dynamics of a system to quantum mechanical. To further study coherent states from quantum perspectives quantum superposition can be invoked. This quantum superposition was studied by various researchers at different times and concluded enhanced results for quantum metrology [67]. These states can be generated by mixing coherent and squeezed vacuum states on a beam splitter [118]. The application of entangled coherent states has shown improved sensitivity for phase measurement under modest photons number as compared to the *N00N* state [119]. For pure states, the entangled coherent states can be regarded as a superposition of *N00N* states with a different number of photons, this leads to better sensitivity with a large number of photons as compared to the average photon number

[42]. Let us prepare the initial state $|\psi^{in}\rangle_{12}$ in modes 1 and mode 2 and apply unitary transformation in mode 2, then entangled coherent state can be stated as following by using the definition of coherent state,

$$|\psi\rangle_{\alpha 00\alpha} = \mathbb{Z}_\alpha[|\alpha, 0\rangle + |0, \alpha\rangle], \tag{4.4.1}$$

where $\mathbb{Z}_\alpha$ is normalization constant and is given by,

$$\mathbb{Z}_\alpha = \frac{1}{\sqrt{(1 + e^{-|\alpha|^2})}}. \tag{4.4.2}$$

The phase optimization is given by quantum Cramer Rao bound (3.2.12) and the quantum Fisher information for the pure states (3.2.34) can be used, the average of number operator is given by,

$$\langle \hat{n} \rangle = \mathbb{Z}_\alpha^2 |\alpha|^2, \tag{4.4.3}$$

and,

$$\langle \hat{n}^2 \rangle = \mathbb{Z}_\alpha^2 |\alpha|^2 + \mathbb{Z}_\alpha^2 |\alpha|^4. \tag{4.4.4}$$

Moreover, using equations (4.4.3) and (4.4.4) for quantum Fisher information we get,

$$F_Q(\phi) = 4\mathbb{Z}_\alpha^2 |\alpha|^2 \left[1 + (1 - \mathbb{Z}_\alpha^2)|\alpha|^2\right]. \tag{4.4.5}$$

Therefore, the sensitivity can be written as follow,

$$\Delta\phi_{ECS} \geq \frac{1}{2\mathbb{Z}_\alpha \alpha \sqrt{1 + (1 - \mathbb{Z}_\alpha^2)|\alpha|^2}}. \tag{4.4.6}$$

From the above equation, one can get the idea if N is very large then the sensitivity of the entangled coherent state becomes equal to that of the *N00N* state, $\Delta\phi_{ECS} = \Delta\phi_{N00N}$. Furthermore, it can be also seen from equation (4.4.6) that the sensitivity is improved by a factor of $2\mathbb{Z}_\alpha \alpha \sqrt{1 + (1 - \mathbb{Z}_\alpha^2)}$. The phase sensitivity is plotted against "α" in Figure 4.5. Moreover, this improvement in sensitivity is welcomed, however, the generation of entangled coherent states is extremely hard due to its sensitive nature towards environmental decoherence, like *N00N* states these states are also fragile and vanish their entanglement in presence of noise [67]. So other states have been explored including squeezed states. Therefore, we devoted next section of this chapter to squeezed states of light.

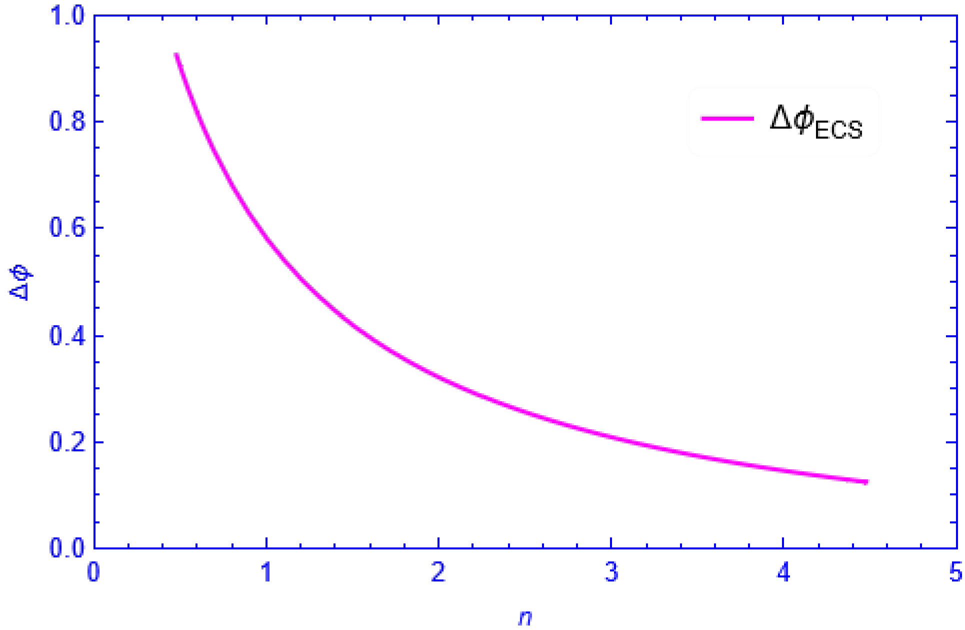

Figure 4.5: Uncertainty in phase is plotted against "n" for the entangled coherent state.

The next section focuses on the applications of squeezed vacuum states ($|\chi\rangle$), squeezed coherent states or displaced squeezed vacuum states ($|\alpha, \chi\rangle$), and entangled squeezed vacuum states ($|\psi\rangle_{\chi 00 \chi}$) in quantum metrology.

4.5 Squeezed Vacuum State

The deployment of a squeezing operator to a vacuum state result in a squeezed vacuum state; in other words, the arbitrary state is now a vacuum state that is,

$$|\chi\rangle = \hat{S}(\chi)|0\rangle, \tag{4.5.1}$$

where $|\chi\rangle$, is the squeezed vacuum state and $\hat{S}(\chi)$ is a single mode squeezing operator given by equation (2.5.1). Furthermore, if one wants to find the quadrature squeezing, the expectation values of $\hat{a}^2$, $\hat{a}$ etc can be obtained by using equations (4.5.7) and (4.5.8). After mathematical manipulations, one can get the following results,

$$\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle = \frac{1}{4}\left[sinh^2 r + cosh^2 r - 2 sinhr coshr cos\theta\right], \tag{4.5.2}$$

and,

$$\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle = \frac{1}{4}\left[sinh^2r + cosh^2r + 2sinhrcoshrcos\theta\right]. \tag{4.5.3}$$

For $\theta = 0$, the above equations are reduced to,

$$\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle = \frac{1}{4}e^{-2r}, \tag{4.5.4}$$

and,

$$\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle = \frac{1}{4}e^{2r}. \tag{4.5.5}$$

From these equations, it is easy to see that squeezing in $\hat{X}_1$ observable is at the expense of other quadrature component and for $\theta = \pi$, there will be squeezing in the $\hat{X}_2$ quadrature.

Such states have been used for enhanced quantum estimation tasks. The application of squeezed vacuum states for phase measurement in Mach–Zehnder interferometer (MZI) has widely been researched. The photon number statistics of squeezed vacuum state are given by,

$$\langle n \rangle = sinh^2r, \tag{4.5.6}$$

and, similarly one can find the following result as well by using the transformation equations given by

$$\hat{S}^\dagger(\chi)\hat{a}\hat{S}(\chi) = \hat{a}Coshr - \hat{a}^\dagger e^{i\theta}Sinhr, \tag{4.5.7}$$

and,

$$\hat{S}^\dagger(\chi)\hat{a}^\dagger\hat{S}(\chi) = \hat{a}^\dagger Coshr - \hat{a}e^{-i\theta}Sinhr. \tag{4.5.8}$$

Therefore,

$$\langle n^2 \rangle = 2cosh^2rsinh^2r + sinh^2r, \tag{4.5.9}$$

hence the variance is,

$$\Delta n^2 = 2\left[\langle n \rangle + \langle n \rangle^2\right]. \tag{4.5.10}$$

Now, using equation (3.2.34) therefore, the quantum Fisher information (QFI) for squeezed vacuum state leads to the following relation,

$$F_Q(\phi) = 8\left[\langle n\rangle + \langle n\rangle^2\right]. \tag{4.5.11}$$

This implies the quantum Cramer Rao bound (3.2.12) in the following structure,

$$\Delta\phi \geq \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{\langle n\rangle + \langle n\rangle^2}}. \tag{4.5.12}$$

Equation (4.5.12) reduces to Heisenberg's limited sensitivity for large $\langle n\rangle$. This result shows an excellent furtherance in phase measurement in contrast to the standard quantum-limited measurements [5, 120]. The phase sensitivity in equation (4.5.12) is shown in the below figure depending on the average number of photons.

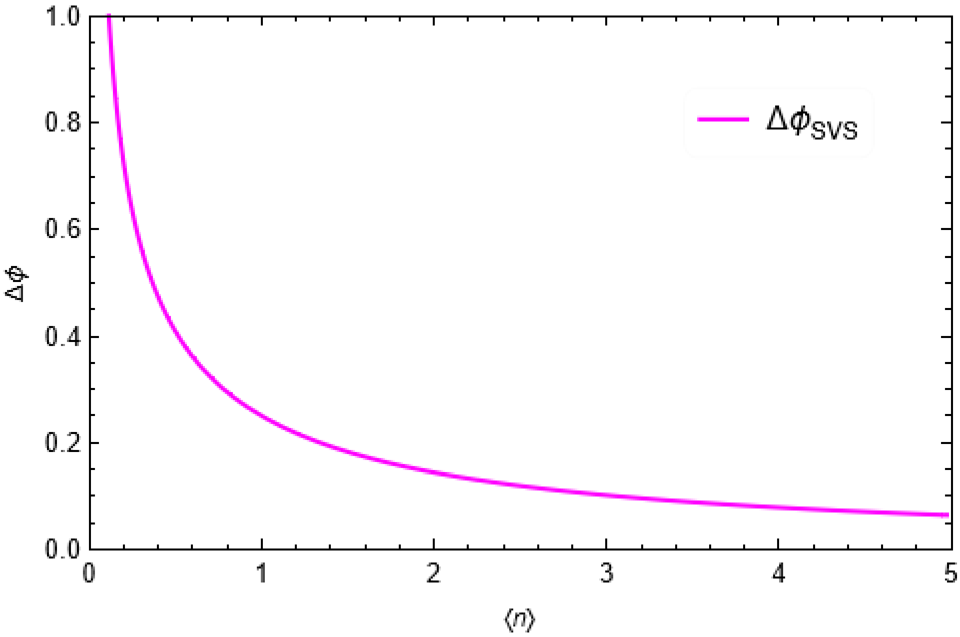

Figure 4.6: Phase sensitivity of interferometric setup using squeezed vacuum state with average number of photons $\langle n\rangle$

4.6 Squeezed Coherent State

Furthermore, another scenario has been considered where the squeezed coherent state is displaced in the phase space [121]. This type of interferometric structure has been implemented in advanced interferometers (LIGO) for the detection of gravitational waves [122]. Squeezed vacuum and coherent state are used as inputs modes in the interferometric setup which makes the input state as,

$$|\psi\rangle_{in} = |\alpha, \chi\rangle = \hat{D}(\alpha)\hat{S}(\chi)|0\rangle. \tag{4.6.1}$$

Since coherent and squeezed vacuum states can be written as,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle, and, |\chi\rangle = \hat{S}(\chi)|0\rangle, \tag{4.6.2}$$

where $\hat{D}(\alpha)$ and $\hat{S}(\chi)$ are displacement and squeezing operators respectively, and are defined as follows,

$$\hat{D}(\alpha) = \exp\left[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}\right], \tag{4.6.3}$$

and,

$$\hat{S}(\chi) = \exp\left[\frac{1}{2}\left(\chi^{*}\hat{a}^{2} - \chi\hat{a}^{\dagger 2}\right)\right]. \tag{4.6.4}$$

To calculate the expectation values of various operators in this section we will need certain types of transformation equations which can be easily obtained by using the Baker-Hausdorf lemma given by equation (2.5.5). The transformation equations are, given by following relations (proofs are given in A.2)

$$\hat{D}(\alpha)\hat{a}\hat{D}(\alpha) = (\hat{a} + \alpha), \tag{4.6.5}$$

and,

$$\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = (\hat{a}^{\dagger} - \alpha^{*}). \tag{4.6.6}$$

The relevant quantities that are important to calculate the uncertainty or estimation precision are given,

$$\langle\hat{n}\rangle = Sinh^{2}r + |\alpha|^{2}. \tag{4.6.7}$$

It is easy to interpret from the above equation that a complex "α" is contributed to the expectation value of the number operator because of the involvement of the coherent state in one of the input ports. Moreover, similar calculations can be done to find the following result. Remember that we have used the above transformation equations (4.6.4) and (4.6.6) in each of the derivations

$$\begin{aligned}\langle\hat{n}^2\rangle = 2Cosh^2rSinh^2r + Sinh^4r + 3|\alpha|^2Sinh^2r + |\alpha|^2Cosh^2r - \\ SinhrCoshr(\alpha^2e^{-i\theta} + \alpha^{*2}e^{i\theta}) + |\alpha|^4.\end{aligned} \tag{4.6.8}$$

Therefore, the variance can be extracted as,

$$\Delta n^2 = 2Cosh^2rSinh^2r + |\alpha|^2Sinh^2r + |\alpha|^2Cosh^2r - SinhrCoshr(\alpha^2e^{-i\theta} + \alpha^{*2}e^{i\theta}), \tag{4.6.9}$$

so, the final result for phase estimation is obtained as,

$$\Delta\phi = \frac{1}{\sqrt{8Cosh^2rSinh^2r + 4|\alpha|^2Sinh^2r + 4|\alpha|^2Cosh^2r - S}}, \tag{4.6.10}$$

where, $S = SinhrCoshr(\alpha^2e^{-i\theta} + \alpha^{*2}e^{i\theta})$. Since the maximum point is for $\theta = \pi/2$ and we have considered the real value of "α". So, the last term in square root goes to zero. Furthermore, to make a better comparison with other states we have supposed that $\langle n\rangle$ is very large ($\langle n\rangle >> 1$). In this situation, the squeezed vacuum should carry $\sqrt{\langle n\rangle}/2$ of photons, therefore the squeezing factor can be approximated as $Sinh^2r = \sqrt{\langle n\rangle}/2$. Using these approximations one can find the following result and then plotted concerning the mean number of photons as shown in Figure 4.7

$$\Delta\phi = \frac{1}{\sqrt{4\sqrt{\langle n\rangle} + 6\langle n\rangle + 4\langle n\rangle^{\frac{3}{2}}}} \approx \frac{1}{2\langle n\rangle^{\frac{3}{4}}} \approx \frac{1}{\langle n\rangle^{\frac{3}{4}}}. \tag{4.6.11}$$

The above result shows better improvement as compared to shot noise-limited interferometry. Moreover, from the above example, it is clear that quantum resources offer the best strategies for overcoming the classical limit of scaling.

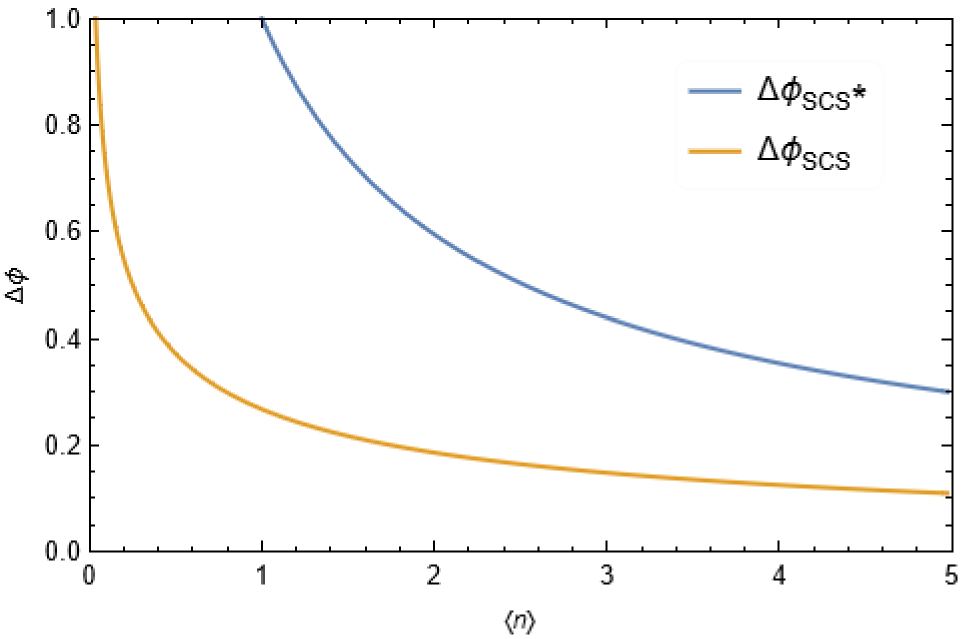

Figure 4.7: Behavior of squeezed coherent state. For large $\langle n \rangle >> 1$, the uncertainty goes like blue plot while if we consider the original form of the equation that increases the denominator by some factor and reaches Heisenberg's limited scaling as shown in the figure by the orange line.

4.7 Entangled Squeezed Vacuum State

Thus far in this chapter, we have looked at squeezed vacuum states and squeezed coherent states and their applications towards phase estimation and interestingly, these states have shown better improvement in the precision of measurement. However, in this section, the author has introduced a new class of entangled states "*the so-called entangled squeezed vacuum states (ESVS)*". We have developed this state as an analogous state to an entangled coherent state, moreover, it can be prepared through various schemes as discussed in [77, 123]. Therefore, one can write the following state in contrast to an entangled coherent state

$$|\psi\rangle_{\chi 00\chi} = Z(\chi)\left[|\chi\rangle|0\rangle + |0\rangle|\chi\rangle\right], \tag{4.7.1}$$

where, $|\chi\rangle$ is the squeezed vacuum state, which can be obtained through the application of a single-mode squeezing operator and $Z(\chi)$ is an appropriate normalization constant

$$|\chi\rangle = \hat{S}(\chi)|0\rangle. \tag{4.7.2}$$

The single-mode squeezing operator is defined by equation (2.5.1). To obtain the exact relation for normalization constant $Z(\chi)$ we need to expand squeezed vacuum state in its Fock basis by invoking some basic formalism of lie algebra we will get the following compact form

$$|\chi\rangle = \hat{S}(\chi)|0\rangle = \frac{1}{\sqrt{Coshr}} \sum_{n=0}^{\infty} e^{in\phi}(tanhr)^n \frac{\sqrt{(2n)!}}{2^n n!}|2n\rangle, \tag{4.7.3}$$

where we have used disentangling theorem to disentangle the squeezing operator. If we have two operators let us say $\hat{A}$ and $\hat{B}$, then the disentangling lemma (proof is available in A.3) is defined as,

$$\exp\left[\hat{A} + \hat{B}\right] = \exp\left[\hat{A}\right] \exp\left[\hat{B}\right] \exp\left[-\frac{1}{2}\left[\hat{A}, \hat{B}\right]\right], \tag{4.7.4}$$

provided that $[\hat{A}, \hat{B}] \neq 0$. Therefore, the normalization constant is obtained as,

$$Z(\chi) = \frac{1}{\sqrt{2\left(1 + \frac{1}{Coshr}\right)}}. \tag{4.7.5}$$

Furthermore, we are interested to use an entangled squeezed vacuum state (ESVS) as a quantum resource in the phase estimation problem. The related expressions that we need to find the sensitivity in phase through quantum Cramer-Rao bound (QCRB) and quantum Fisher information (QFI) are given by,

$$\langle \hat{n} \rangle = Z^2(\chi) Sinh^2 r. \tag{4.7.6}$$

The contribution from entanglement is visible and this will enhance the final estimation process as we have expected. Similarly, one may write,

$$\langle \hat{n}^2 \rangle = 2Z^2(\chi) Sinh^2 r + 3Z^2(\chi) Sinh^4 r, \tag{4.7.7}$$

where we have used the transformation relations given by equations (4.5.7), (4.5.8), and expanded form of squeezed vacuum state in Fock basis as constructed in equation (4.7.3). Therefore, the variance can be extracted as follow,

$$\Delta n^2 = 2\langle n \rangle + \frac{3\langle n \rangle^2}{Z^2(\chi)} - \langle n \rangle^2. \tag{4.7.8}$$

Since we know that the estimated phase sensitivity is related to the inverse of quantum Fisher information $F_Q(\theta) = 4\Delta n^2$ as defined by equations (3.2.12) and (3.2.34). Therefore, the phase uncertainty obtained as,

$$\Delta\phi \geq \frac{1}{\sqrt{8\langle n\rangle + \frac{12\langle n\rangle^2}{Z^2(\chi)} - 4\langle n\rangle^2}}. \tag{4.7.9}$$

To develop optimal results for phase sensitivity or to reduce the phase uncertainty one needs to use a specific value of "r" for which the normalization constant gives a maximum outcome. Therefore, $Z^2(\chi) = 1/4$ for $r = 0$, However, $r = 0$ means no squeezing accordingly, we can at least use value of $r = 1$ therefore, $Z^2(\chi) = 1/3.3$ for better comparison, then the phase uncertainty becomes,

$$\Delta\phi \geq \frac{1}{\sqrt{8\langle n\rangle + 36\langle n\rangle^2}}. \tag{4.7.10}$$

Moreover, if we consider a limiting case where the average number of photons is very high then the expecting result is very much modified Heisenberg's limit, and the phase sensitivity is improved to a high level as compared to other states considered in this study

$$\Delta\phi \geq \frac{1}{\sqrt{36\langle n\rangle^2}}. \tag{4.7.11}$$

The critical analysis of the above relations has been plotted in the below diagram in comparison with squeezed vacuum state. It is obvious from the below figure that using entangled squeezed vacuum states can improve the phase sensitivity up to an extreme level as compared to other states discussed in this study. Moreover, if we consider greater value of squeezing parameter r, then $1/Coshr$ term can be ignored and the obtained result will modify equation (4.7.8) in the following form

$$\Delta n^2 = 2\langle n\rangle + 5\langle n\rangle^2. \tag{4.7.12}$$

Finally, for the case of greater value of r the phase sensitivity can be modified as

$$\Delta\phi \geq \frac{1}{\sqrt{8\langle n\rangle + 20\langle n\rangle^2}}. \tag{4.7.13}$$

However, for greater value of $\langle n\rangle$, the phase sensitivity will reduce to Heisenberg's limited scaling.

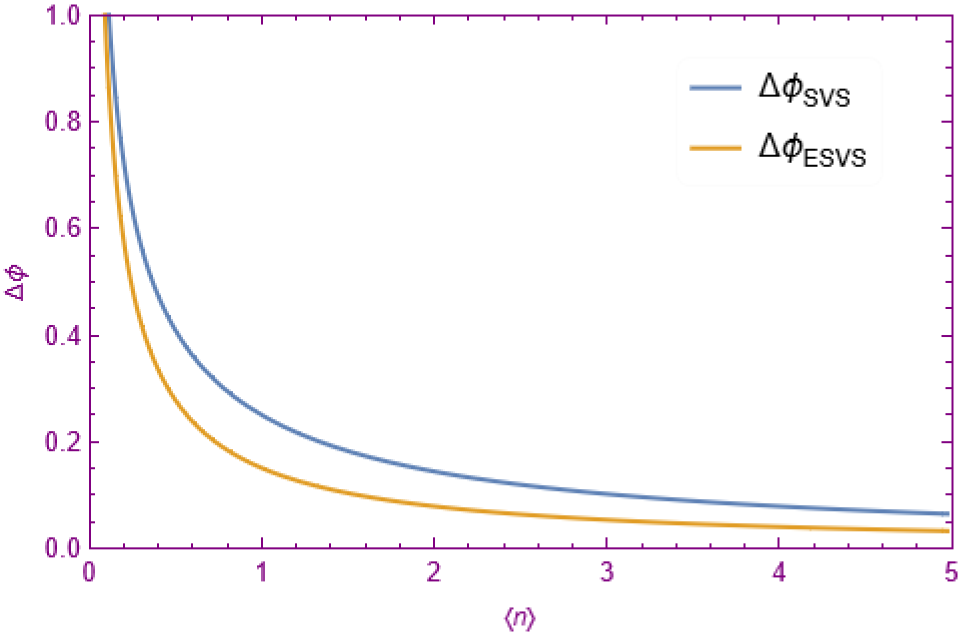

Figure 4.8: Phase sensitivity of squeezed vacuum and entangled squeezed vacuum state (ESVS), blue and orange respectively.

Chapter 5

Results and Discussion

5.1 Discussion

Precision measurements through quantum resources have been thoroughly researched at different times since the development of quantum mechanics. Quantum mechanics has offered various excellent platforms for different technologies including quantum information, quantum computing, quantum key distribution, quantum metrology, etc. In this task, the author has presented a comprehensive investigation of quantum resources for their applications in quantum metrology. The precision of measurements is bounded by certain limits fundamentally, classically these bounds on precision are well studied through classical statistical mechanics, the limit achieved on the estimation of a certain parameter is named as shot noise limit due to the presence of vacuum fluctuations [124]. Because of the limitations of classical physics, the bound on precision can only reach to standard quantum limit or shot noise limit [124]. However, quantum optics offer better strategies for overcoming the shot-noise limit and an ultimate limit (Heisenberg's limit) on precision can be achieved with less effort [125]. Detailed research has been done in this area as discussed in the literature of the study. Moreover, we have investigated various quantum states for quantum phase estimation problems, including Fock state, coherent state, entangled coherent state (ECS), *N00N* state, squeezed vacuum state, squeezed coherent state, and entangled squeezed vacuum state (ESVS). Quantum entanglement and non-classicality have shown better improvement towards the achievement of ultimate limit on precision. All the results have been compiled in Table 5.1. For phase estimation, we have used the quantum Fisher information approach which provides the direct and easiest way for calculations. Although the Fock state is non-classical, its results can only reach shot noise scaling because it is the eigenstate of the number operator and shows zero variance, however, variance in angular momentum operator components have variances in input and output ports for different scenarios. Similarly, a coherent state is considered to be a semi-classical state due to its characteristics and the phase uncertainty achievable is shot noise limited. On the other hand, non-classicality along with quantum entanglement have shown improved interest as given by the *N00N* state and others. The phase uncertainty provided by these states is either Heisenberg's limited or modified Heisenberg's scaling. To get better insights and comparison of these states with each other we have considered an approximation that is $n \approx \langle n \rangle$ because the nature of "n" is same. The complied results are shown in Table 5.1 and the analytics of the results are presented in Figure 5.1.

Moreover, we have considered all the results given in this work for better comparison in the following figure. It is apparent that various quantum states have different analytics for the same number of photons. The best result we generated is by using an entangled squeezed vacuum state (ESVS) as a probe state in quantum interferometry.

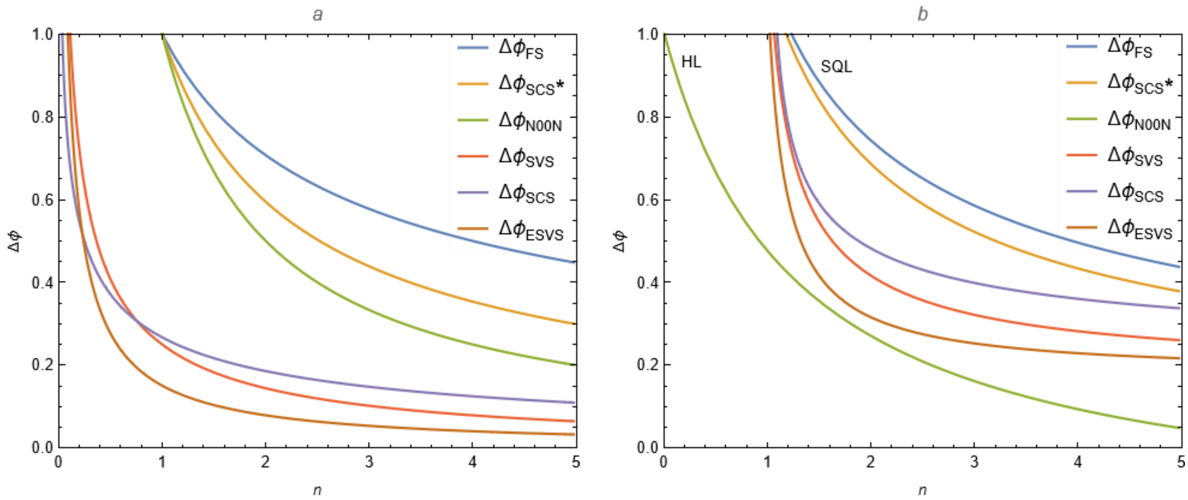

Figure 5.1: Comparison of precision phase uncertainties of various photonic states

Relationship of phase uncertainty and number of photons: Whereas in 5.1 (a), we investigated an ideal example $\langle n \rangle = n$ and produced improved results that circumvented the standard quantum limit as well as Heisenberg's limit. (Blue): it is obtained when input states are Fock state or coherent state and is called shot noise scaling. (Orang *): a result of squeezed coherent state for $\theta = 0$ and $n >> 1$ and $Sinh^2 r = \frac{\sqrt{n}}{2}$. (Green): this result is called Heisenberg's limit and is achieved after using *N00N* states as input probes. (Red): this shows the phase estimation through using squeezed vacuum states as input source and limit obtained as modified

Heisenberg's limit. (Purple): this is modified Standard quantum limit for squeezed coherent state for a smaller number of photons and has shown better sensitivity finally, (Brown) graph shows the result of entangled squeezed vacuum state (ESVS) and has provided Heisenberg's limited sensitivity. In figure 5.1 (b), on the other hand, if we treated Heisenberg's limit as a reference bound that cannot be surpassed as established in the literature on quantum limitations in optical interferometry, we obtained superior results for phase uncertainty when employing entangled squeezed vacuum state and other.

Furthermore, other factors that may influence our precision phase estimation, such as squeezing factor "r" must be considered. Because, as seen in equation, (4.7.6) the average number of photons is associated with the squeezing parameter "r". As a result, by squeezing parameter one can vary the variance. Figure 5.2 depicts the behavior of phase sensitivity in response to the parameter "r".

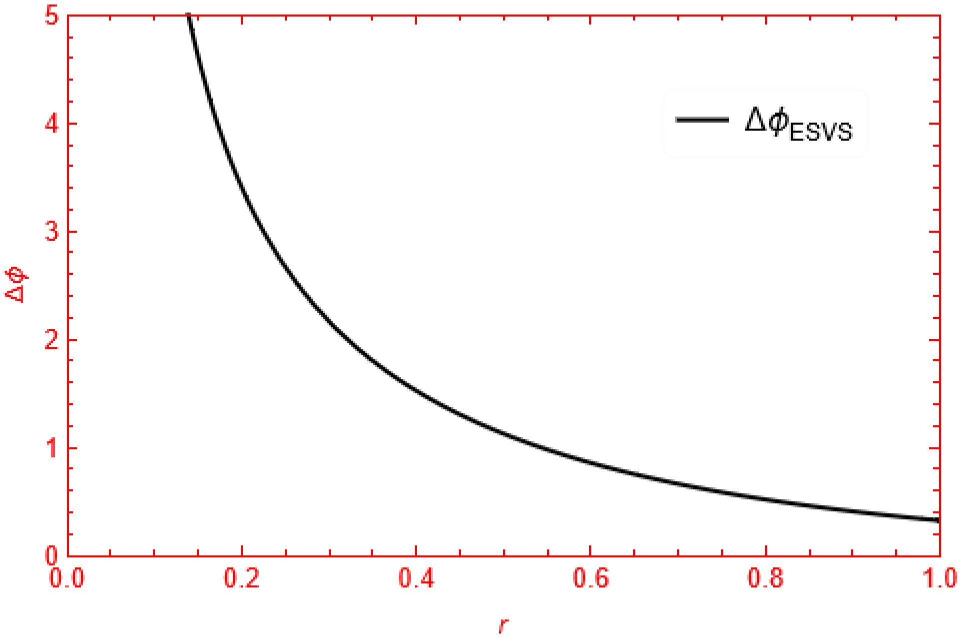

Figure 5.2: Phase uncertainty and its relationship with squeezing parameter "r"

As a result, it is clear that as the squeezing parameter is increased, the phase uncertainty decreases rapidly, as illustrated in the above figure. As a consequence, estimating the phase using an entangled squeezed vacuum state (ESVS) is more accurate than using other states as probes.

Table 5.1: Various quantum states and their metrological power or estimated phase uncertainties.

S.no	**Quantum State**	**Method used**	**Results**	**Type of Limit**
1	Fock State	Algebra of angular momentum and (QFI)	$\Delta\phi = \frac{1}{\sqrt{n}}$	Standard Quantum Limit
2	Coherent State	(QFI) and QCRB	$\Delta\phi = \frac{1}{\lvert\alpha\rvert} = \frac{1}{\sqrt{\langle n\rangle}}$	Standard Quantum Limit
3	Entangled Coherent State	(QFI) and QCRB	$\Delta\phi_{ECS} \geq \frac{1}{2\mathbb{Z}_\alpha \alpha\sqrt{1+(1-\mathbb{Z}_\alpha^2)\lvert\alpha\rvert^2}}$	Modified Standard Quantum Limit
4	N00N State	(QFI) and QCRB	$\Delta\phi_{N00N} \geq \frac{1}{N}$	Heisenberg's Limit
5	Squeezed Vacuum State	(QFI) and QCRB	$\Delta\phi \geq \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{\langle n\rangle + \langle n\rangle^2}}$	Modified Heisenberg's Limit
6	Squeezed Coherent State	(QFI) and QCRB	$\Delta\phi = \frac{1}{\sqrt{4\sqrt{\langle n\rangle}+6\langle n\rangle+4\langle n\rangle^{\frac{3}{2}}}}$	Modified Standard Quantum Limit
7	Entangled Squeezed Vacuum State (ESVS)	(QFI) and QCRB	$\Delta\phi \geq \frac{1}{\sqrt{8\langle n\rangle + \frac{12\langle n\rangle^2}{Z^2(\chi)} - 4\langle n\rangle^2}}$	Modified Heisenberg's Limit

5.2 Conclusion

In the shadow of the above discussions and calculations, we declared that quantum resources offer excellent possibilities for enhancing precision measurements. The resources of quantum mechanics (quantum states and manipulation of these states, non-classicality, quantum entanglement, etc.) have provided the ultimate way of improving metrological tasks. Furthermore, the achievable bounds on precision have been approached through quantum Fisher information (QFI) and quantum Cramer-Rao bound (QCRB). Various quantum states have become an adornment of this work including, coherent state, Fock state, *N00N* state, entangled coherent state, squeezed vacuum state, squeezed coherent state, and entangled squeezed vacuum state (ESVS), and has shown different results as compared in Table 5.1.Coherent and Fock states have provided a result that saturates the shot-noise limit, however, the other states have either circumvented the standard quantum limit or modified the Heisenberg's limit. The phase sensitivity of the *N00N* state is limited by Heisenberg's bound, though it is challenging to generate a high *N00N* state and in a noisy environment, it only maintains its optimality at a smaller number of photons. Moreover, an entangled coherent state beats the standard quantum limit by some factor but these states are also fragile like the *N00N* state. In continuation, we have considered squeezed states in different scenarios that provide better phase sensitivity for a large number of photons but yet their generation and manipulation is a hard task. Furthermore, the experimental manipulation of these states is a challenging task, particular the entangled states are sensitive to environmental noise therefore, one needs a highly sophisticated laboratory to deal with such states.

Therefore, we predict that the discussed and proposed states for quantum metrology may equip us with photonic quantum sensors and other quantum technologies in the future if we become able to generate and control these states for particular circumstances.

Appendix A

Supplementary Data

In this section, we have provided all the supportive materials including proofs and essential lemmas that have been used in this work.

A.1 Proof of Classical Fisher Information

If we have a certain amount of data and we are interested to find an estimated value of a specific parameter then Fisher information tells us about the particular parameter, or in simple words how much information can be extracted about a certain parameter from a sample of data or formally speaking Fisher information is the expected amount of data about an unknown parameter "θ" encoded on "ξ". Let us consider a random variable "ξ" and "θ" to be the parameter of interest. Intuitively, if the occurrence of an event has less probability, then it's happening can bring us much information. Furthermore, we defined a log-likelihood function because it will make our calculations less cumbersome. Also, the log-likelihood function helps us to maximize the likelihood and this maximization only happens if the function is concave downward due to the presence of a logarithmic function. Therefore,

$$l(\xi|\theta) = lnf(\xi|\theta), \tag{A.1.1}$$

or,

$$l^{'}(\xi|\theta) = \frac{\partial}{\partial\theta}lnf(\xi|\theta) = \frac{f^{'}(\xi|\theta)}{f(\xi|\theta)}. \tag{A.1.2}$$

It is obvious that if $l^{'}(\xi|\theta) \sim 0$, then there may not be much information about the unknown variable "θ" in "ξ", so we need to consider the square of the derivative then the Fisher information is related by,

$$F(\theta) = E_\theta\left[\left(l^{'}(\xi|\theta)\right)^2\right] = \int \left(l^{'}(\xi|\theta)\right)^2 f(\xi|\theta)d\xi = \int \left(\frac{\partial}{\partial\theta} lnf(\xi|\theta)\right)^2 f(\xi|\theta)d\xi. \tag{A.1.3}$$

To simplify equation (A.1.3) further we know that,

$$var(z) = E\left[(z - E(z))^2\right] = E[z] - (E[z])^2\,, \tag{A.1.4}$$

in this case $z = l^{'}(\xi|\theta)$, Therefore,

$$F(\theta) = var\left(\left[l^{'}(\xi|\theta)\right]\right) + \left(E_\theta\left(\left[l^{'}(\xi|\theta)\right]\right)\right)^2. \tag{A.1.5}$$

For now, consider the second part of equation (A.1.5) which is

$$E_\theta\left(\left[l^{'}(\xi|\theta)\right]\right) = \int \left(l^{'}(\xi|\theta)\right) f(\xi|\theta)d\xi = \int f^{'}(\xi|\theta)d\xi = \int \frac{\partial}{\partial\theta} f(\xi|\theta)d\xi. \tag{A.1.6}$$

If we are allowed to change the order of differentiation and integration then the above equation (A.1.6) becomes zero and Fisher information can now be written as,

$$F(\theta) = var\left[l^{'}(\xi|\theta)\right] = \left\langle \left(l^{'}(\xi|\theta)\right)^2 \right\rangle - \left\langle \left(l^{'}(\xi|\theta)\right)\right\rangle^2, \tag{A.1.7}$$

or,

$$F(\theta) = \left\langle \left(\frac{\partial}{\partial\theta} lnf(\xi|\theta)\right)^2 \right\rangle = \int \left(\frac{\partial}{\partial\theta} lnf(\xi|\theta)\right)^2 f(\xi|\theta)d\xi, \tag{A.1.8}$$

or, for discrete cases, one can write Fisher information as,

$$F(\theta) = \sum_\xi \frac{1}{f(\xi|\theta)} \left(\frac{\partial}{\partial\theta} f(\xi|\theta)\right)^2. \tag{A.1.9}$$

As we have used the symbol $P(\xi|\theta)$ so, the final equation for Fisher information becomes,

$$F(\theta) = \sum_\xi \frac{1}{P(\xi|\theta)} \left(\frac{\partial}{\partial\theta} P(\xi|\theta)\right)^2, \tag{A.1.10}$$

where, $P(\xi|\theta)$ is the conditional or provisional probability.

A.1.1 Proof of Cramer Rao Bound

Cramer-Rao bound (CRB) puts a lower limit on the precision of measurements of a specific unknown parameter. The estimation process is the measurement of the true value of the required parameter "θ" which is encoded in a physical system or data. To find the bound on the precision we supposed that our estimator is unbiased this implies that its average value is equal to the true value of the unknown parameter "θ"

$$\theta = \langle \theta_{est}(\xi) \rangle , \tag{A.1.11}$$

with the help of this equation, we can build a trivial identity as given by the definition of expectation value.

$$\int d\xi \left[\left(P\left(\xi|\theta\right) \left[\theta_{est}(\xi) - \theta \right] \right) \right] . \tag{A.1.12}$$

Taking the derivative of the part inside square brackets in equation (A.1.12) with respect to "θ",

$$\int d\xi \frac{1}{P(\xi|\theta)} \frac{\partial P(\xi|\theta)}{\partial\theta} \left[\theta_{est}(\xi) - \theta \right] = \int d\xi \frac{\partial lnP(\xi|\theta)}{\partial\theta} \left[\theta_{est}(\xi) - \theta \right] = 1. \tag{A.1.13}$$

Let us consider the following terms from the above equation

$$\frac{\partial lnP(\xi|\theta)}{\partial\theta} = A, and, \left[\theta_{est}(\xi) - \theta \right] = B, \tag{A.1.14}$$

using Cauchy–Schwarz inequality to simplify the equation further,

$$\langle A^2 \rangle \langle B^2 \rangle \geq |\langle AB \rangle|^2, \tag{A.1.15}$$

therefore, one can write,

$$\left\langle \left(\frac{\partial lnP(\xi|\theta)}{\partial\theta} \right)^2 \right\rangle \left\langle \left[\theta_{est}(\xi) - \theta \right]^2 \right\rangle \geq \left\langle \left(\frac{\partial lnP(\xi|\theta)}{\partial\theta} \right)^2 \right\rangle \Delta\theta_{est}^2 \geq 1. \tag{A.1.16}$$

Since $\left\langle \left(\frac{\partial lnP(\xi|\theta)}{\partial\theta} \right)^2 \right\rangle = \int \left(\frac{\partial}{\partial\theta} lnf(\xi|\theta) \right)^2 P(\xi|\theta) d\xi = F(\theta)$, so the Cramer-Rao bound (CRB) can be written in a simple representation

$$\Delta\theta \geq \frac{1}{\sqrt{F(\theta)}}. \tag{A.1.17}$$

A.2 Proof of Transformation Equations

(a):

$$\hat{S}^{\dagger}(\chi)\hat{a}\hat{S}(\chi) = \hat{a}Coshr - \hat{a}^{\dagger}e^{i\theta}Sinhr,$$

as, the squeezing operator is given by equation (2.5.1) and the lemma that is important to get the above result is defined by equation (2.5.5), therefore,

$$\hat{A} = \left[\frac{1}{2}\left(\chi\hat{a}^{\dagger 2} - \chi^{*}\hat{a}^{2}\right)\right], and, \hat{B} = \hat{a}, \tag{A.2.1}$$

so, let us find each term in equation (2.5.5),

$$[\hat{A}, \hat{B}] = \frac{1}{2}\left[\chi\hat{a}^{\dagger 2} - \chi^{*}\hat{a}^{2}, \hat{a}\right]. \tag{A.2.2}$$

Evaluating the above equation and using the commutation relation between $\hat{a}$ and $\hat{a}^{\dagger}$

$$[\hat{A}, \hat{B}] = -\chi\hat{a}^{\dagger}, \tag{A.2.3}$$

similarly,

$$\left[\hat{A}[\hat{A}, \hat{B}]\right] = \frac{1}{2}\left[\chi\hat{a}^{\dagger 2} - \chi^{*}\hat{a}^{2}, -\chi\hat{a}^{\dagger}\right] = |\chi|^{2}\hat{a}, \tag{A.2.4}$$

and,

$$\left[\hat{A}\left[\hat{A}[\hat{A}, \hat{B}]\right]\right] = \frac{1}{2}\left[\chi\hat{a}^{\dagger 2} - \chi^{*}\hat{a}^{2}, |\chi|^{2}\hat{a}\right] = -\chi|\chi|^{2}\hat{a}^{\dagger}, \tag{A.2.5}$$

collecting these equations into equation (2.5.5) and then using a series of $Sinhr$ and $Coshr$, one can get the following result

$$\hat{S}^{\dagger}(\chi)\hat{a}\hat{S}(\chi) = \hat{a}Coshr - \hat{a}^{\dagger}e^{i\theta}Sinhr. \tag{A.2.6}$$

(b):

(b) Furthermore, by the taking the dagger of the above equation, the other transformation equation can be obtained as,

$$\hat{S}^{\dagger}(\chi)\hat{a}^{\dagger}\hat{S}(\chi) = \hat{a}^{\dagger}Coshr - \hat{a}e^{-i\theta}Sinhr, \tag{A.2.7}$$

(c):

$$\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha) = (\hat{a} + \alpha),$$

as the displacement operator is provided by the given equations

$$\hat{D}(\alpha) = \exp\left[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}\right], and, \hat{D}(-\alpha) = \exp\left[\alpha^{*}\hat{a} - \alpha\hat{a}^{\dagger}\right] = \hat{D}^{\dagger}(\alpha). \tag{A.2.8}$$

Once again use equation (2.5.5) and let us find each term in this equation

$$\hat{A} = \alpha^{*}\hat{a} - \alpha\hat{a}^{\dagger}, and, \hat{B} = \hat{a}, \tag{A.2.9}$$

then,

$$[\hat{A}, \hat{B}] = [\alpha^{*}\hat{a} - \alpha\hat{a}^{\dagger}, \hat{a}] = \alpha. \tag{A.2.10}$$

The other terms in the sequence are zero, therefore,

$$\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha) = (\hat{a} + \alpha), \tag{A.2.11}$$

and using the unitary property of the displacement operator the other transformation equation can be read as,

(d):

$$\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = (\hat{a}^{\dagger} - \alpha^{*}). \tag{A.2.12}$$

A.3 Proof of Disentangling Theorem

In the special case where $[\hat{A}, \hat{B}] \neq 0$, the following theorem obeys,

$$\exp\left[\hat{A} + \hat{B}\right] = \exp\left[\hat{A}\right]\exp\left[\hat{B}\right]\exp\left[-\frac{1}{2}\left[\hat{A}, \hat{B}\right]\right], \tag{A.3.1}$$

or one can write in another way as,

$$\exp\left[\hat{A}+\hat{B}\right]=\exp\left[\hat{B}\right]\exp\left[\hat{A}\right]\exp\left[\frac{1}{2}\left[\hat{A},\hat{B}\right]\right]. \tag{A.3.2}$$

We are interested to show these equations, therefore, Let us consider the general function as

$$f(x)=e^{\hat{A}x}e^{\hat{B}x}, \tag{A.3.3}$$

the first-order derivative of the above equation is

$$f^{'}(x)=\hat{A}e^{\hat{A}x}e^{\hat{B}x}+e^{\hat{A}x}\hat{B}e^{\hat{B}x}=\left(\hat{A}+e^{\hat{A}x}\hat{B}e^{-\hat{A}x}\right), \tag{A.3.4}$$

as we know from the properties of commutation relation,

$$\left[\hat{B},\hat{A}^{n}\right]=n\hat{A}^{n}\left[\hat{B},\hat{A}\right]. \tag{A.3.5}$$

Therefore, we are free to write,

$$\left[\hat{B},e^{-\hat{A}x}\right]=\sum_{n}\left[\hat{B},\frac{(-\hat{A}x)^{n}}{n!}\right]=\sum_{n}(-1)^{n}\frac{x^{n}}{n!}\left[\hat{B},\hat{A}^{n}\right], \tag{A.3.6}$$

and after a short mathematical manipulation, one can write,

$$e^{-\hat{A}x}\hat{B}e^{\hat{A}x}=\hat{B}-e^{\hat{A}x}[\hat{B},\hat{A}]x, \tag{A.3.7}$$

using the above equation in equation (A.3.4), the following equation may be obtained

$$f^{'}(x)=\left(\hat{A}+\hat{B}-[\hat{B},\hat{A}]\right)f(x). \tag{A.3.8}$$

The solution of the above (A.3.8) differential equation can be obtain as for $x=1$,

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}. \tag{A.3.9}$$

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