Generalized Taylor's Method for ψ-Fractional Differential Equations

by

Momna Tariq

A thesis submitted in partial fulfillment of the requirements for the degree of MS in Mathematics

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
Islamabad, Pakistan

August, 2022

Generalized Taylor's Method for ψ-Fractional Differential Equations

Momna Tariq
Reg No. 330935

A thesis submitted in partial fulfillment of the requirements for the degree of MS in Mathematics

Supervised by: Dr. Mujeeb ur Rehman

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
Islamabad, Pakistan

August, 2022

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: **Momna Tariq**, Regn No. **00000330935** Titled: **"Generalized Taylor's Method for ψ-Fractional Differential Equations"** accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. MERAJ MUSTAFA HASHMI — Signature:

2. Name: DR. MUHAMMAD ASIF FAROOQ — Signature:

Supervisor's Name: DR. MUJEEB UR REHMAN — Signature:

Head of Department

30/08/2022
Date

COUNTERSINGED

Date: 31.08.2022

Dean/Principal

Dedication

This work is dedicated to my beloved Parents, Tariq Masood and Faria Tariq. The reason of what i become today. Thanks for your endless love, support and encouragement.

Acknowledgements

By the Name of Almighty Allah, The Most beneficent and The Most Gracious, He Who showed upon me His blessings throughout the thick and thin of my life, all the glories belong to Him. First and foremost, I thank Allah, the Most Merciful, for the completion of my MS thesis. He gave me so much courage, strength and patience to deal with all hardships during this time.

I would like to thank those who made this thesis possible. I would like to express my sincere gratitude to Dr Mujeeb Ur Rehman, who has always been helpful throughout my research work. I consider it my privilege to have accomplished this thesis under his guidance.

Words are not enough to explain gratefulness of my parents who always stand on my side no matter what and always supported me. Dear parents, thank you for all the love and support. Having parents like them is the biggest of all blessings. Thank you for all that you've done for me.

Abstract

In this work, we present the Generalization of fractional Taylor's theorem with respect to a function. Also, we discuss the general form of fractional power series, convergence of power series, series solutions by using fractional Taylor's formula and fractional power series. Moreover, we present an operational matrix method for the numerical solution of ψ-Caputo fractional ordinary and partial differential equations. For this purpose, a fractional version of the Taylor's theorem is presented in the framework of ψ-fractional calculus. The method converts the underlying ordinary or partial differential equations to systems of algebraic equations. The method is accompanied by examples in which ψ-fractional differential equations are solved, to verify the applicability and effectiveness. Further, estimates of upper bounds of error for the approximations have been derived.

Contents

List of Figures

List of Tables

Introduction

Fractional calculus is a field of mathematics that deals with the derivatives and integrals of non-integer order. It tends to be set apart as old as classical calculus that deals with derivatives and integrals of integer order. Derivatives and integrals of fractional order are considered as non-local operators since both these operators involve integration which is a non-local operator (as it is defined on an interval). This property makes these operators a proficient and incredible asset to characterize long term memory effects, asymptotic scaling and hereditary properties of various physical phenomena.

Working with fractional differential equations is relatively difficult as compared with their ordinary counterparts. In most cases, we cannot find the exact solutions to fractional differential equations, so we use numerical approaches to find approximate solutions for fractional differential equations. As the classical Taylor's theorem has been widely studied, the Taylor series are widely used to approximate solutions to complicated problems. Recently, the numerical study of fractional differential equations with variable coefficients by the Taylor basis function has attracted some attention [10]. There are several books, articles and research papers [5, 11] which exhibits the wide assortment of applications of fractional calculus. Comprehensive history of fractional calculus was first time discovered in [21]. A brief history of fractional differential equations was first introduced in [19]. Fractional differential equations and its applications were discussed in [28]. Analysis of fractional differential equations was discussed in [4] in detail. Recently, the numerical study of fractional differential equations with variable coefficients by Taylor basis function has attracted much attention [10]. In [12], the author used numerical approach to solve fractional relaxation-oscillation equations. In [5], series solutions of linear and nonlinear fractional differential equations are produced using the fractional power series technique. Motivated by the works cited above, in this paper, we are concerned with the numerical solution of the ψ-fractional differential equations. The principal findings of this work are: Generalized Taylor's theorem for the differential operator $D^{n,\psi} = (\frac{1}{\psi(x)}\frac{d}{dx})^n$ where n is nonnegative integer. An alternate proof of the Generalized Taylor's theorem for fractional derivative with respect to a function. Approximations of a function by Generalized Taylor's formula with respect to another function. ψ-Fractional power series is introduced for demonstrating the general form of generalized Taylor's theorem. Discussion on convergence and divergence of ψ-fractional power series and remainder theorem. Development of a method to find the series solution by using ψ-fractional Taylor series and ψ-fractional power series. Fractional integration matrix is developed for generalized Taylor polynomials. Development of a numerical method for the solution of ψ-fractional ordinary and partial differential equations. The method is similar to the general operational matrix method commonly used in literature. Estimate for the error in approximation by the $\psi-$Taylor polynomial is presented.

The operational matrix method reduces the ψ-fractional partial differential equation to a system of algebraic equations. For partial differential equations, this algebraic system forms the Sylvester equation. MATLAB programs are developed for the numerical computation of entries of the fractional Taylor operational matrices. The applicability of the method is tested with several examples.
This thesis consists of four chapters. It is organized as follows: Chapter 1 is devoted to fundamental definitions and preliminary concepts of fractional calculus, properties, applications and important results. In Chapter 2, important concepts of generalized Taylor's theorem with respect to a function and ψ-fractional power series are introduced. Chapter 3 focuses on development of Taylor series method, through which ψ-fractional differential equations are solved. Numerical examples for ψ-fractional ordinary and partial differential equations are presented to show the applicability and effectiveness of the proposed method. Chapter 3.4.1 is the summary of the thesis.

Chapter 1

Basic Concepts

We start by reviewing some classical facts of calculus. In this chapter, we discuss fundamental concepts of special functions of fractional calculus that can be used in other chapters. We provide definitions of gamma function, beta function, Mittag-Leffler function, Norm, Leibniz rule and function spaces in detail. Also, we discuss definitions of Reimann-Liouville fractional integral and derivatives, Caputo's fractional derivative and their fundamental properties. Moreover, we present basic idea of ψ-fractional calculus. Some basic definitions and important properties will be discussed.

1.1 Historical background

The development of calculus was started in $17th$ century. Isaac Newton and Gottfried Leibniz autonomously discovered the idea of differential calculus (1642-1727). Fractional calculus was first invented when letters were exchanged between mathematicians Marquis de L'Hopital and Leibniz. Leibniz developed the notation $\frac{d^n y}{dx^n}$ for nth order derivative and he assume that $n \in N$. L'Hopital raised a question in one of his letter to Leibniz that "what is the derivative of non-integer number". Leibniz replied on 30th September 1695, wrote that one day we will draw useful consequences for this".
Fractional derivatives were first introduced by S. F. de Lacroix in his published text in 1819. Tremendous contributions were made by many great mathematicians to fractional calculus throughout 19th and 20th century. Few of the great mathematicians who worked for fractional calculus are are J.P.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832), B. Riemann (1847), W. Center (1850), H. R. Greer (1859), Z. Wastchenxo (1861), A.K. Grunwald (1867), A.V. Letnikov (1868), A. Cayley (1880), G. Oltramare (1893), R. E. Mortiz (1902), H. Weyl (1917), H. T. Davis (1936), A. Erdlyi (1939), H. Kober (1940), A. S. Peters (1961), K. B. Oldham (1972), S. G. Samko, A. A. Kilbas, O. I. Marichev (1993).
Fractional calculus has long and rich history, yet because of absence of reasonable physical and geometrical interpretations, it stayed new to researchers working in applied mathematics up to ongoing years and was considered as numerical curiosities, not helpful for tackling issues emerging from applied sciences. A few endeavours have been made to give physical and geometrical interpretations to fractional operators. In 2002, I. Podlubny [28] developed the physical and geometric interpretation for the first time in detail.

First definition of fractional derivative was introduced by Lacroix in 1819. In 1823 Abel was the first who solved the tautochrone problem by using arbitrary order derivative. In 1834 J. Liouville worked on complementary functions, gave a reasonable definition of a fractional derivative. He has great contributions in fractional calculus.
In recent years, several definitions of fractional derivatives and integrals were developed. Some of mathematicians developed their own definitions which include the Hadamard, the Riemann-Liouville, the Erdelyi-Kober, the Weyl, the Marchaud, the Granwald-Letnikov and the Caputo fractional derivatives and integrals. Mostly we use definitions of Riemann-Liouville. But the situations in which this definition is not applicable we prefer to use Caputo's approach, which was introduced in 1967 by M. Caputo.

1.2 Special functions

Special functions play an important role in theory of fractional calculus. We will present the basic implications and characteristics of fractional calculus in this section. To continue further in this work, we give necessary information about gamma functions, beta functions, Mittag-Leffler function and Leibniz rule in detail. Also, we discuss function spaces in which will be used in further development.

1.2.1 Gamma function

The gamma function is one of the basic function and plays an important role in fractional calculus. Swiss mathematician Euler discovered the gamma function in order to convert the factorial into non-integer case. It was studied by other mathematicians due of it's great importance.

Definition 1.2.1. [4] The function is defined by

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt, \;\; y > 0$$

is known as Euler's Gamma function. $\Gamma(y)$ converges for all $y > 0$.

Properties: Some important properties of Gamma function are

1. $\Gamma(y+1) = y\Gamma(y)$.
2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
3. $\Gamma(p+1) = p!$, where p is an integer.

Proof. 1. By the definition of gamma function:

$$\Gamma(y+1) = \int_0^\infty t^y e^{-t} dt.$$

Using integration by parts

$$\begin{aligned}\Gamma(y+1) =& t^y e^{-t}|_0^\infty + \int_0^\infty e^{-t} y t^{y-1} dt = 0 + \int_0^\infty y t^{y-1} e^{-t} dt \\ =& y \int_0^\infty t^{y-1} e^{-t} dt = y\Gamma(y).\end{aligned}$$

2. By substituting, $y = \frac{1}{2}$ in definition of gamma function and then integrating, we get

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt = \int_0^\infty t^{\frac{-1}{2}} e^{-t} dt.$$

We use substitution method to evaluate this integral. Let $t = u^2$, then

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-u^2}}{\sqrt{u}} 2udu = 2\int_o^\infty e^{-u^2} du = 2\int_0^\infty e^{-v^2} dv.$$

Now,

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dudv.$$

Now, we use the transformation of rectangular coordinates to polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, $0 < \theta \le \frac{\pi}{2}$, $dudv = Jdrd\theta$ where J is called the Jacobi matrix. Here in this case $J = r$. Thus

$$\begin{aligned}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 =& 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta \\ =& 4\left(\frac{\pi}{2} - 0\right) \int_0^\infty e^{-r^2} r dr.\end{aligned}$$

Let $y = r^2$, $dy = 2rdr$, then

$$\Gamma\left(\frac{1}{2}\right)^2 = 2\frac{\pi}{2}\int_0^\infty e^{-y} dy = \pi(e^{-\infty} + e^0) = \pi.$$

Taking square root on both sides, we get the desired result. □

3. Since $\Gamma(1) = 1$, by part (1). Thus the relation is true for $p = 1$. Assume that for $p = j$,

$$\Gamma(j+1) = l!.$$

For $p = j + 1$, we have

$$\Gamma(j+1+1) = (j+1)\Gamma(j+1) = (j+1)l! = (j+1)!.$$

Thus $\Gamma(p+1) = p!$ is true for all integers p.

Extension of domain of gamma function

The functional equation

$$\Gamma(w+1) = w\Gamma(w), \tag{1.2.1}$$

can be used to extend definition from $w > 0$ to all real number except $w \neq 0, -1, -2, \cdots$. From (1.2.1) we have,

$$\Gamma(w+1) = w\Gamma(w) \ or \ \Gamma(w) = \frac{\Gamma(w+1)}{w}.$$

Right hand side is defined for $w+1 > 0$, $w > -1$, $w \neq 0$ where $\Gamma(w)$ is defined for $w > 0$. Now from (1.2.1)

$$\begin{aligned}\Gamma(w+2) =& (w+1)(w)\Gamma(w)\\ \Gamma(w) =& \frac{\Gamma(w+2)}{w(w+1)}, \ w \neq 0, -1, -2.\end{aligned}$$

By repeating the above procedure l-times, we get

$$\Gamma(w) = \frac{\Gamma(w+l)}{w(w+1)(w+2)\cdots(w+l-1)}; \ w \neq 0, -1, -2, \cdots .$$

Thus domain of $\Gamma(w)$ is extended for all real numbers except $w \neq 0, -1, -2, \cdots$.
In mathematics, Euler integral of first kind is also called beta function, which is closely related to the gamma function and to the binomial co-efficient.

1.2.2 Beta function

Definition 1.2.2. [4] We define beta function as

$$B(\alpha, \beta) = \int_0^1 y^{(\alpha-1)}(1-y)^{(\beta-1)} dy, \quad \alpha, \ \beta > 0.$$

Incomplete beta function is defined as

$$B(y; \alpha, \beta) = \int_0^y s^{\alpha-1}(1-s)^{\beta-1} ds, \quad y \in [0, 1]. \tag{1.2.2}$$

Relation between gamma and beta functions

Gamma and beta functions are related as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

where $B(\alpha, \beta)$ is two variable function and $\Gamma(y)$ is function of one independent variable.

1.2.3 Mittag-Leffler function

Mittag-Leffler function is a generalized form of exponential function. We define Mittag-Leffler function for two variables as follows.

Definition 1.2.3. [4] Let $\alpha, \beta > 0$, then Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(y) = \sum_{m=0}^{\infty} \frac{y^m}{\Gamma(\beta + \alpha m)}, \ y \in \mathbb{R}.$$

1.2.4 Function Spaces

Before we proceed to our principle work, we present function spaces which will be used in forthcoming chapters.

Definition 1.2.4. [4] The space $L_r[c, d]$ $(1 \leq r \leq \infty)$ is the set of Lebesgue measurable functions on $A = [c, d]$ such that $||\phi||_r < \infty$, where

$$||\phi||_r = \left(\int_c^d |\phi(s)|^r ds \right)^{\frac{1}{r}}, \ 1 \leq r < \infty$$

and

$$||\phi||_\infty = ess \sup_{c \leq y \leq d} |\phi(y)|.$$

Definition 1.2.5. [4] The space $A^m[c, d]$ is the set of functions ϕ for which there exists a function $g \in L_1[c, d]$ almost everywhere such that

$$\phi^{n-1}(y) = \phi^{n-1}(c) + \int_c^y g(t)dt$$

defines functions with absolutely continuous $(n - 1)$ derivatives.

Theorem 1.2.6. *If $\phi(y_1, y_2)$ is continuous on $R = [l_1, l_2] \times [m_1, m_2]$ then,*

$$\int_R\int \phi(y_1, y_2)dA = \int_{l_1}^{l_2}\int_{m_1}^{m_2} \phi(y_1, y_2)dy_2dy_1 = \int_{m_1}^{m_2}\int_{l_1}^{l_2} \phi(y_1, y_2)dy_1dy_2. \tag{1.2.3}$$

The integrals are called iterated integral.

Leibniz rule

Lemma 1.2.7. *Suppose $\phi(w, u)$ is continuous and $\frac{\partial}{\partial w}\phi$ be continuous in the domain of the yu−plane that contains the rectangular region R:= $a \leq w \leq b, u_0 \leq u \leq u_1$ and limit of integration $\alpha(w)$ and $\beta(w)$ are functions having continuous derivatives on $a \leq w \leq b$. Then*

$$\frac{d}{dw}\int_{\alpha(w)}^{\beta(w)} \phi(w, u)du = \phi(w, \beta(w))\frac{d\beta}{dw} - \phi(w, \alpha(w))\frac{d\alpha}{dw} + \int_{\alpha(w)}^{\beta(w)} \frac{\partial}{\partial w}\phi(w, u)du.$$

1.3 Differential and integral operators

Derivatives and integrals play an important role in mathematics. Integrals help us either to obtain the area under the graph or to find the function whose derivative is integrated. Fractional integral can be determined by repeated integration.

1.3.1 Properties of classical differential and integral operators

Lemma 1.3.1. *[4] Let $\phi : [k, l] \to \mathbb{R}$ is a continuous function, and $\Phi : [k, l] \to \mathbb{R}$ be defined as*

$$\Phi(t) := \int_k^t \phi(s)ds.$$

Then, Φ is differentiable and $\Phi'(t) = \phi(t)$.

Theorem 1.3.2. *If $m \geq n$, then for $t \in [a, b]$*

$$D^m I_a^n \phi(t) = D^{m-n}\phi(t),$$

and if $m \leq n$, then

$$D^m I_a^n \phi(t) = I_a^{n-m}\phi(t).$$

Proof. Lemma (1.3.1) can also be read as

$$DI_a\phi(t) = \phi(t), \text{ where } D = \frac{d}{dt} \text{ and } I_a\phi(t) = \int_a^t \phi(s)ds. \tag{1.3.1}$$

Now we state the composition properties of differential and integral operators as follows. Repeated application of (1.3.1) gives

$$D^2 I_a^2 \phi(t) = D(DI_a(I_a\phi(t))) = DI_a\phi(t) = \phi(t). \tag{1.3.2}$$

By using Eq (1.3.2) we can deduce that

$$D^3 I_a^2 \phi(t) = D(D^2 I_a^2 \phi(t)) = D\phi(t),$$
$$D^2 I_a^3 \phi(t) = D^2 I_a^2 (I_a\phi(t)) = I_a\phi(t).$$

Similarly we have

$$D^3 I_a^3 \phi(t) = D^2(DI_a(I_a^2\phi(t))) = D^2 I_a^2 \phi(t) = D(DI_a(I_a\phi(t))) = DI_a\phi(t) = \phi(t). \tag{1.3.3}$$

By using Eq (1.3.3)

$$D^4 I_a^3 \phi(t) = D(D^3 I_a^3 \phi(t)) = D\phi(t),$$
$$D^3 I_a^4 \phi(t) = D^3 I_a^3 (I_a\phi(t)) = I_a\phi(t).$$

In general

$$D^m I_a^n \phi(t) = D^{m-n}\phi(t), \ m \geq n$$
$$D^m I_a^n \phi(t) = I_a^{n-m}\phi(t), \ m \leq n.$$

□

1.3.2 Riemann-Liouville fractional integral and derivative

Riemann-Liouville fractional integral is obtained from Cauchy iterated formula.

Lemma 1.3.3. *Let ϕ be Riemann integrable on $[r, u]$. Then, for $r \leq y \leq u$ and $\alpha \in \mathbb{N}$ we have*

$$I_r^\alpha \phi(y) = \frac{1}{(\alpha-1)!} \int_r^y (y-s)^{\alpha-1} \phi(s) ds.$$

Proof. Let us start from the simple integral

$$I_r \phi(y) = \int_r^y \phi(s) ds. \tag{1.3.4}$$

Iterating integral (1.3.4)

$$\begin{aligned} I_r^2 \phi(y) = I_r(I_r\phi)y &= \int_r^y \int_r^{t_1} \phi(s) dt_1 dt_2 \\ &= \int_r^y \int_{t_1}^y \phi(t_1) dt_2 dt_1 \\ &= \int_r^y \phi(t_1)(y-t_1) dt_1 \\ &= \int_r^y \phi(t)(y-t) dt. \end{aligned}$$

The third iterate gives

$$I_r^3 \phi(y) = I_r(I_r(I_r\phi(y))) = \int_r^y \int_r^{t_1} \int_r^{t_2} \phi(t) dt dt_2 dt_1.$$

By using Theorem (3.4.11)

$$I_r^3 \phi(y) = \int_r^y \int_r^{t_1} \int_t^{t_1} \phi(t) dt_2 dt dt_1.$$

Now this becomes

$$I_r^3 \phi(y) = \int_r^y \phi(t) \frac{(y-t)^2}{2!} dt.$$

Repeating the above process upto α-times we have

$$I_r^\alpha \phi(y) = \frac{1}{(\alpha-1)!} \int_r^y \phi(t)(y-t)^{\alpha-1} dt. \tag{1.3.5}$$

The last integral is called Cauchy iterated integral formula. □

Using relation between gamma function and factorial function, we can define fractional integral. Replacing integer n with real $\alpha > 0$ in Eq (1.3.5). The integral (1.3.5) becomes fractional integral.

Definition 1.3.4. [4, 28] Let $\alpha \in \mathbb{R}^+$, $\phi \in L_1[a, b]$ we define Riemann-Liouville fractional integral as

$$I_a^\alpha \phi(y) = \frac{1}{\Gamma(\alpha)} \int_a^y (y-s)^{\alpha-1} \phi(s) ds. \tag{1.3.6}$$

Example 1.3.5. [4, 28] For $\phi(y) = y^\beta$ we have

$$I_0^\alpha \phi(y) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} y^{\alpha+\beta}. \tag{1.3.7}$$

By definition (1.3.4)

$$I_0^\alpha \phi(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-s)^{\alpha-1} s^\beta ds$$

$$I_0^\alpha y^\beta = \frac{1}{\Gamma(\alpha)} \int_0^y y^{\alpha-1} \left(1 - \frac{s}{y}\right)^{\alpha-1} s^\beta ds.$$

We evaluate the integral by substituting $v = \frac{s}{y}$.

$$\begin{aligned} I_0^\alpha y^\beta =& \frac{1}{\Gamma(\alpha)} \int_0^1 y^{\alpha-1}(1-v)^{\alpha-1} y^\beta v^\beta dv \\ =& \frac{y^{\alpha+\beta+1-1}}{\Gamma(\alpha)} \int_0^1 (1-v)^{\alpha-1} v^\beta dv. \end{aligned}$$

Since

$$\int_0^1 (1-v)^{\alpha-1} v^\beta dv = B(\alpha, \beta+1).$$

Therefore, we have

$$I_0^\alpha y^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} y^{\beta+\alpha}.$$

Fractional integral of a function, which can be expressed in the form of Maclaurin series can be computed by formula 1.3.5. As an example, here we will find the fractional integral of $\sin y$ for $y \in \mathbb{R}$.

Example 1.3.6. [4, 28] To find the fractional integral of $\sin y$. For this purpose, we expand $\sin y$ into its Maclaurin series

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} \cdots, \tag{1.3.8}$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p y^{2p+1}}{(2p+1)!}, \tag{1.3.9}$$

where p is non-negative integer. Using Eq (1.3.7) and property of gamma function we get

$$I_0^\alpha \sin y = \sum_{q=0}^{\infty} \frac{(-1)^q y^{2q+1+\alpha}}{\Gamma(2q+2+\alpha)}.$$

We can write in terms of Mittag-Leffler function as

$$E_{2,2+\alpha}(-y^2) = \sum_{q=0}^{\infty} \frac{(-y^2)^q}{\Gamma(2+\alpha+2q)}.$$

Therefore

$$\begin{aligned} I_0^\alpha \sin y &= y^{1+\alpha}\sum_{q=0}^{\infty}\frac{(-y^2)^q}{\Gamma(2+\alpha+2q)} \\ &= y^{1+\alpha}E_{2,2+\alpha}(-y^2). \end{aligned}$$

Now, we present the semi-group property of Reimann-Liouville fractional integral.

Theorem 1.3.7. *[28] Let $\alpha,\beta \geq 0$ and $\phi \in L_1[p,q]$. Then*

$$I_p^\alpha I_p^\beta \phi(y) = I_p^{\alpha+\beta}\phi(y).$$

If $\phi \in C[p,q]$ or $\alpha+\beta \geq q$, then the identity holds everywhere on $[p,q]$.

Proof.

$$I_p^\alpha I_p^\beta \phi(y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_p^{y_1}\int_p^{t_1}(y_1-t_1)^{\alpha-1}(t_1-s)^{\beta-1}\phi(s)dsdt_1.$$

Now, we interchange the integration order

$$I_p^\alpha I_p^\beta \phi(y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_p^{y_1}\int_s^{y_1}(y_1-t_1)^{\alpha-1}(t_1-s)^{\beta-1}\phi(s)dt_1ds.$$

By substituting $t_1 = s + y(y_1-s)$, we have

$$I_p^\alpha I_p^\beta \phi(y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_p^{y_1}\int_0^{1}(y_1-s-y(y_1-s))^{\alpha-1}(s+y(y_1-s)-s)^{\beta-1}(y_1-s)\phi(s)dyds$$

By rearranging the above integral, we get

$$I_p^\alpha I_p^\beta \phi(y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_p^{y_1}(y_1-s)^{\alpha+\beta-1}\int_0^{1}(1-y)^{\alpha-1}y^{\beta-1}\phi(s)dyds,$$

where $B(\alpha,\beta) = \int_0^1(1-y)^{\alpha-1}y^{\beta-1}\phi(s)dy$. Therefore, by using relation between gamma and beta function, we have

$$I_p^\alpha I_p^\beta \phi(y) = \frac{1}{\Gamma(\alpha+\beta)}\int_p^{y}(y-s)^{\alpha+\beta-1}\phi(s)ds = I_p^{\alpha+\beta}\phi(y).$$

□

We have discussed all the fundamental properties of Riemann-Liouville fractional integral. Now, we introduce the notation D_a^α, which will represent the Riemann-Liouville fractional derivative.

Definition 1.3.8. [4, 28] Let $\alpha \in \mathbb{R}^+$, $\phi \in L_1[r,u]$ and $m = \lceil \alpha \rceil$, we define Riemann-Liouville fractional derivative as

$$D_r^\alpha \phi(y) = D^m I_r^{m-\alpha}\phi(y) = \left(\frac{d}{dy}\right)^m\left(\frac{1}{\Gamma(m-\alpha)}\int_r^y (y-s)^{m-\alpha-1}\phi(s)ds\right). \tag{1.3.10}$$

Example 1.3.9. [4, 28] We will find the fractional derivative of $\phi(y) = (y-e)^{\beta}$, $\beta \geq -1$ and $\alpha \geq 0$. We use the definition and evaluate the resulting integral. By definition (1.3.8)

$$D_e^{\alpha}\phi(y) = D^p I_e^{p-\alpha}\phi(y) = D^p I_e^{p-\alpha}(y-e)^{\beta}.$$

From Eq (1.3.7) we have

$$\begin{aligned} D_e^{\alpha}\phi(y) =& D^p\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+p-\alpha+1)}(y-e)^{p-\alpha+\beta}\right] \\ =& \frac{\Gamma(\beta+1)}{\Gamma(\beta+p-\alpha+1)}D^p(y-e)^{p-\alpha+\beta}. \end{aligned}$$

Case 1: If $\alpha - \beta \in \mathbb{N}$, then we have

$$D^p(y-e)^{\alpha-\beta} = 0,$$

for all $\alpha \geq 0$, $p \in 1, 2, \cdots, [\alpha]$.
Case 2: If $\alpha - \beta \neq \mathbb{N}$, here we generalize the integer-order derivative of a power function

$$\begin{aligned} D(y-e)^v &= v(y-e)^{v-1} \\ D^2(y-e)^v &= v(v-1)(y-e)^{v-2} \\ D^3(y-e)^v &= v(v-1)(v-2)(y-e)^{v-3}. \end{aligned}$$

In general

$$D^p(y-e)^v = \frac{\Gamma(v+1)}{\Gamma(v-p+1)}(y-e)^{v-p}. \tag{1.3.11}$$

Equation (1.3.9) becomes

$$D_e^{\alpha}(y-e)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+p-\alpha+1)}\frac{\Gamma(\beta+p-\alpha+1)}{\Gamma(p+1-\alpha+\beta-p)}(y-e)^{p-\alpha+\beta-p}$$

$$D_e^{\alpha}(y-e)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(y-e)^{\beta-\alpha}. \tag{1.3.12}$$

Example 1.3.10. Now, we will find the fractional derivative of $\sin y$. For this purpose, we expand $\sin y$ into its Maclaurin series:

$$\begin{aligned} \sin y &= y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!}\cdots, \\ &= \sum_{q=0}^{\infty}\frac{(-1)^q y^{2q+1}}{(2q+1)!}, \end{aligned}$$

where q is non-negative integer. Using Eq (1.3.9) and property of gamma function we get

$$\begin{aligned} D_0^{\alpha}\sin y &= \sum_{q=0}^{\infty}\frac{(-1)^q D_0^{\alpha} y^{2q+1}}{(2q+1)!} \\ &= \sum_{q=0}^{\infty}\frac{(-1)^q D^q I_0^{q-\alpha} y^{2q+1}}{(2q+1)!} \\ &= \sum_{q=0}^{\infty}\frac{(-1)^q\Gamma((2q+1)+1)D^q y^{q-(\alpha-2q-1)}}{(2q+1)!\Gamma(2q+2-\alpha+q)}. \end{aligned}$$

Case 1: If $\alpha - 2q - 1 \in \mathbb{N}$ then

$$D_0^\alpha \sin y = 0.$$

Case 2: If $\alpha - 2q - 1 \notin \mathbb{N}$ then using Eq (1.3.11)

$$D_0^\alpha \sin y = \sum_{q=0}^{\infty} \frac{(-1)^q y^{2q+1-\alpha}}{\Gamma(2q+2-\alpha)}.$$

As,

$$E_{2,2-\alpha}(-y^2) = \sum_{q=0}^{\infty} \frac{(-y^2)^q}{\Gamma(2-\alpha+2q)}.$$

Therefore

$$\begin{aligned} D_0^\alpha \sin y &= y^{1+\alpha} \sum_{q=0}^{\infty} \frac{(-y^2)^q}{\Gamma(2+2q-\alpha)}, \\ &= y^{1+\alpha} E_{2,2-\alpha}(-y^2). \end{aligned}$$

1.3.3 Properties of Reimann-Liouville fractional integral and derivative

Next we come to show the relationship between Riemann-Liouville fractional integral with derivative and vice versa. Moreover, we discuss basic properties, which will be helpful for further chapters.

Theorem 1.3.11. *[28] Let $\alpha > 0$. Then for every $\phi \in L_1[r, u]$*

$$D_r^\alpha I_r^\alpha \phi(y) = \phi(y)$$

holds almost everywhere.

Proof. By using the definition (1.3.8), theorem (1.3.7) and Eq (1.3.1), we get

$$\begin{aligned} D_r^\alpha I_r^\alpha \phi(y) =& D^m I_r^{m-\alpha} I_r^\alpha \phi(y) = D^m I_r^{m-\alpha+\alpha} \phi(y) \\ =& D^m I_r^m \phi(y) = \phi(y). \end{aligned}$$

□

Theorem 1.3.12. *[28] Assume that $\alpha \geq 0$, $k = \lceil \alpha \rceil$ and $\phi \in A^k[r, u]$. Then*

$$I_r^\alpha D_r^\alpha \phi(y) = \phi(y) - \sum_{j=1}^{k} D_r^{\alpha-j} \phi(y)|_{y=r} \frac{(y-r)^{\alpha-j}}{\Gamma(\alpha-j+1)}.$$

Proof. By definition of Riemann-Liouville fractional integral

$$I_r^\alpha D_r^\alpha \phi(y) = \frac{1}{\Gamma(\alpha)} \int_r^y (y-s)^{\alpha-1} D_r^\alpha \phi(s) ds. \tag{1.3.13}$$

By using Leibniz's Rule, we have

$$\frac{d}{dy}\left(\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha D_r^\alpha \phi(s)ds\right) = \frac{1}{\Gamma(\alpha)}\int_r^y (y-s)^{\alpha-1} D_r^\alpha \phi(s)ds. \tag{1.3.14}$$

Let us consider the left hand side of (1.3.14)

$$\begin{aligned}
&\frac{d}{dy}\left(\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha D_r^\alpha \phi(s)ds\right)\\
=&\frac{1}{\Gamma(\alpha+1)}(y-s)^\alpha D_r^\alpha \phi(s)|_{s=y}\frac{d}{dy}(y)\\
&-\frac{1}{\Gamma(\alpha+1)}(y-s)^\alpha D_r^\alpha \phi(s)|_{s=r}\frac{d}{dy}(r)+\frac{1}{\Gamma(\alpha+1)}\int_r^y \frac{\partial}{\partial y}(y-s)^\alpha D_a^\alpha \phi(s)ds\\
=&\frac{\alpha}{\Gamma(\alpha+1)}\int_r^y (y-s)^{\alpha-1} D_r^\alpha \phi(s)ds\\
=&\frac{1}{\Gamma(\alpha)}\int_r^y (y-s)^{\alpha-1} D_r^\alpha \phi(s)ds.
\end{aligned}$$

On the other hand, repeatedly integrating and by Theorem (1.3.7) we have

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha D_r^\alpha \phi(s)ds =&\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha D^k I_r^{k-\alpha}\phi(s)ds\\
=&\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha \frac{d^k}{dy^k} I_r^{k-\alpha}\phi(s)ds.
\end{aligned}$$

Now we evaluate this by using iterative method. Let $k=1$

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha \frac{d}{ds} I_r^{1-\alpha}\phi(s)ds\\
&\qquad=\frac{1}{\Gamma(\alpha+1)}(y-s)^\alpha I_r^{1-\alpha}\phi(s)|_r^y+\frac{\alpha}{\Gamma(\alpha+1)}\int_r^y (y-s)^{\alpha-1} I_r^{1-\alpha}\phi(s)ds\\
&\qquad=-\frac{1}{\Gamma(\alpha+1)}(y-s)^\alpha I_r^{1-\alpha}\phi(s)|_r+\frac{1}{\Gamma(\alpha)}\int_r^y (y-s)^{\alpha-1} I_r^{1-\alpha}\phi(s)ds.
\end{aligned}$$

For $k=2$

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha \frac{d^2}{ds^2} I_r^{2-\alpha}\phi(s)ds\\
&\quad=\left[\frac{1}{\Gamma(\alpha+1)}(y-s)^\alpha \frac{d}{ds} I_r^{2-\alpha}\phi(s)\right]_r^y + \frac{\alpha}{\Gamma(\alpha+1)}\int_r^y (y-s)^{\alpha-1}\frac{d}{ds} I_r^{2-\alpha}\phi(s)ds.\\
&\quad=-\frac{1}{\Gamma(\alpha+1)}(y-r)^\alpha \frac{d}{ds} I_r^{2-\alpha}\phi(s)|_r + \frac{1}{\Gamma(\alpha)}\int_r^y (y-s)^{\alpha-1}\frac{d}{ds} I_r^{1-\alpha}\phi(s)ds\\
&\quad=-\frac{1}{\Gamma(\alpha+1)}(y-r)^\alpha \frac{d}{ds} I_r^{2-\alpha}\phi(s)|_r + \big[\frac{1}{\Gamma(\alpha)}(y-s)^{\alpha-1} I_r^{2-\alpha}\phi(s)\big]_r^y\\
&\qquad+\frac{\alpha-1}{\Gamma(\alpha)}\int_r^y (y-s)^{\alpha-2} I_r^{2-\alpha}\phi(s)ds\\
&\quad=-\frac{1}{\Gamma(\alpha+1)}(y-r)^\alpha \frac{d}{ds} I_r^{2-\alpha}\phi(s)|_r - \frac{1}{\Gamma(\alpha)}(y-r)^{\alpha-1} I_r^{2-\alpha}\phi(s)|_r\\
&\qquad+\frac{1}{\Gamma(\alpha-1)}\int_r^y (y-s)^{\alpha-2} I_r^{2-\alpha}\phi(s)ds.\\
&\quad=\frac{1}{\Gamma(\alpha-1)}\int_r^y (y-s)^{\alpha-2} I_r^{2-\alpha}\phi(s)ds - \sum_{j=1}^{2}\frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|_r \frac{(y-r)^{\alpha-j+1}}{\Gamma(\alpha-j+2)}.
\end{aligned}
$$

In general

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha \frac{d^k}{dy^k} I_r^{k-\alpha}\phi(s)ds =& \frac{1}{\Gamma(\alpha-k+1)}\int_r^y (y-s)^{\alpha-k} I_r^{k-\alpha}\phi(s)ds\\
&-\sum_{j=1}^{n}\frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|r \frac{(y-r)^{\alpha-j+1}}{\Gamma(\alpha-j+2)}\\
=& I_r^{\alpha-k+1}[I_r^{k-\alpha}\phi(s)] - \sum_{j=1}^{n}\frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|_r \frac{(y-r)^{\alpha-j+1}}{\Gamma(\alpha-j+2)}.
\end{aligned}
$$

$$
\frac{1}{\Gamma(\alpha+1)}\int_r^y (y-s)^\alpha \frac{d^k}{dy^k} I_r^{k-\alpha}\phi(s)ds = I_r^1\phi(s) - \sum_{j=1}^{n}\frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|_r \frac{(y-r)^{\alpha-j+1}}{\Gamma(\alpha-j+2)}. \tag{1.3.15}
$$

Combing (1.3.15) and (1.3.14) we obtain

$$
\frac{1}{\Gamma(\alpha)}\int_r^y (y-s)^{\alpha-1} D_r^\alpha\phi(s)ds = \frac{d}{dy} I_r^1\phi(s) - \sum_{j=1}^{n}\frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|_r \frac{(y-r)^{\alpha-j}}{\Gamma(\alpha-j+1)}.
$$

By Lemma (1.3.1)

$$
\begin{aligned}
I_r^\alpha D_r^\alpha\phi(y) =& \phi(s) - \sum_{j=1}^{n}\frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|_r \frac{(y-r)^{\alpha-j}}{\Gamma(\alpha-j+1)}.\\
=& \phi(s) - \sum_{j=1}^{n}\frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|_r \frac{(y-r)^{\alpha-j}}{\Gamma(\alpha-j+1)}.
\end{aligned}
$$

□

An important particular case of Theorem (1.3.12) is for $0 < \alpha < 1$,

$$I_r^\alpha D_r^\alpha \phi(y) = \phi(y)[-D_r^{\alpha-1}\phi(s)]_r \frac{(y-r)^{\alpha-1}}{\Gamma(\alpha)}. \tag{1.3.16}$$

Having established the relation between Riemann-Liouville fractional integral with derivative. Now, we proceed to define the Caputo fractional derivative.

1.3.4 Caputo's fractional derivative

The Riemann-Liouville has specific disadvantages while attempting to show this present reality peculiarities with fractional differential equations. Function need not to be continuous while managing them. And also, the derivative of the constant term is not equal to zero. We will now discuss the modified concept, which is known as Caputo derivative. It has more advantages than the first one.

Definition 1.3.13. [4, 28] Let $\alpha \in \mathbb{R}^+$, $\phi \in L_1[r,u]$ and $l = \lceil \alpha \rceil$. Then we define the Caputo derivative as

$$_cD_r^\alpha \phi(y) = I_r^{l-\alpha} D^l \phi(y). \tag{1.3.17}$$

Example 1.3.14. [28] We find the Caputo derivative of $\phi(y) = y^\beta$ as under.
By definition (1.3.13)

$$_cD_0^\alpha \phi(y) = I_0^{l-\alpha} D^l \phi(y) = \frac{1}{\Gamma(l-\alpha)} \int_0^y (y-s)^{l-\alpha-1} D^l s^\beta ds. \tag{1.3.18}$$

Case 1: If $\beta < l$, then $D^l y^\beta = 0$.
Case 2: If $\beta \in \mathbb{N}$ and $l \leq \beta$. Then, for integer case, we have

$$\begin{aligned} Dy^l &= ly^{l-1} \\ D^2y^l &= l(l-1)y^{l-2} \\ D^3y^l &= l(l-1)(l-2)y^{l-3}. \end{aligned}$$

In general

$$\begin{aligned} D^ky^l =& l(l-1)(l-2)\cdots(l-k-1)y^{l-k} \\ =& \frac{l(l-1)(l-2)\cdots(l-k-1)(l-k)!}{(l-k)!}y^{l-k} \\ =& \frac{l!}{(l-k)!}y^{l-k} \\ D^ky^l =& \frac{\Gamma(l+1)}{\Gamma(l-k+1)}y^{l-k}. \end{aligned}$$

Eq (1.3.18) reduce to

$$
\begin{aligned}
{}_cD_0^{\alpha}\phi(y) =& \frac{1}{\Gamma(l-\alpha)}\int_0^y (y-s)^{l-\alpha-1} D^l s^{\beta} ds. \\
=& \frac{\Gamma(\beta+1)}{\Gamma(\beta-m+1)\Gamma(l-\alpha)}\int_0^y (y-s)^{l-\alpha-1} s^{\beta-l} ds. \\
=& \frac{\Gamma(\beta+1)}{\Gamma(\beta-l+1)\Gamma(l-\alpha)}\int_0^y \left(1-\frac{s}{y}\right)^{l-\alpha-1} y^{l-\alpha-1} s^{\beta-l} ds.
\end{aligned}
$$

Let $z = \frac{s}{y}$

$$
{}_cD_0^{\alpha}\phi(y) = \frac{\Gamma(\beta+1)y^{l-\alpha}}{\Gamma(\beta-l+1)\Gamma(l-\alpha)}\int_0^1 (1-z)^{l-\alpha-1} z^{\beta-l} dz.
$$

Since $\int_0^1 (1-z)^{l-\alpha-1} z^{\beta-l} dz = B(l-\alpha, \beta-l+1) = \frac{\Gamma(l-\alpha)\Gamma(\beta-l+1)}{\Gamma(l-\alpha+\beta-l+1)}$.
Thus

$$
{}_cD_0^{\alpha} y^{\beta} = \frac{\Gamma(\beta+1)y^{l-\alpha}}{\Gamma(\beta-\alpha+1)}.
$$

Example 1.3.15. Now, we find the Caputo derivative of $\sin y$. For this purpose, we proceed as follows, we expand $\sin y$ into its Maclaurin series:

$$
\begin{aligned}
\sin y =& y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!}\cdots, \\
=& \sum_{p=0}^{\infty} \frac{(-1)^p y^{2p+1}}{(2p+1)!},
\end{aligned}
$$

where p is non-negative integers. Using Eq (1.3.11) and property of gamma function we get

$$
\begin{aligned}
{}_cD_0^{\alpha} \sin y =& I_0^{m-\alpha} \sum_{p=0}^{\infty} \frac{(-1)^p D^m y^{2p+1}}{(2p+1)!} \\
=& \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma((2p+1)+1) I_0^{m-\alpha} y^{2p+1-m}}{(2p+1)!\Gamma(2p+2-m)} \\
=& \sum_{p=0}^{\infty} \frac{(-1)^p I_0^{m-\alpha} y^{2p+1-m}}{\Gamma(2p+2-m)}
\end{aligned}
$$

$$
{}_cD_0^{\alpha} \sin y == \sum_{p=0}^{\infty} \frac{(-1)^p y^{2p+1-\alpha}}{\Gamma(2p+2-\alpha)}.
$$

As,

$$
E_{2,2-\alpha}(-y^2) = \sum_{p=0}^{\infty} \frac{(-y^2)^p}{\Gamma(2-\alpha+2p)}.
$$

Thus

$$
\begin{aligned}
{}_cD_0^\alpha \sin y &= y^{1+\alpha} \sum_{p=0}^{\infty} \frac{(-y^2)^p}{\Gamma(2+2p-\alpha)}, \\
&= y^{1+\alpha} E_{2,2-\alpha}(-y^2).
\end{aligned}
$$

1.3.5 Properties of Caputo fractional derivative

In the following we discuss the composition of Caputo derivative with Riemann-Liouville fractional integral and vice versa.

Theorem 1.3.16. *[4, 28] If ϕ is continuous and $\alpha \geq 0$, then*

$$
{}_cD_a^\alpha I_a^\alpha \phi(y) = \phi(y). \tag{1.3.19}
$$

Proof. Let $\Phi = I_a^\alpha \phi(y)$, where $D^k\Phi(a) = 0$, then we have

$$
{}_cD_a^\alpha I_a^\alpha \phi(y) ={}_c D_a^\alpha \phi(y) = D_a^\alpha \phi(y) = D^m I_a^{m-\alpha} \phi(y) = \phi(y).
$$

□

Theorem 1.3.17. *[4] Assume that, if $\phi \in A^m(r,u)$, $\alpha \geq 0$ and $m = [\alpha]$, then*

$$
I_r^\alpha {}_cD_r^\alpha \phi(y) = \phi(y) - \sum_{k=0}^{m-1} D^k \frac{\phi(r)}{k!}(y-r)^k.
$$

Proof. By using fundamental theorem of calculus and iterative method we obtain

$$
\begin{aligned}
I_r D\phi(y) &= \int_r^y \frac{d}{dy}\phi(y)dy = \phi(y) - \phi(r) \\
I_r^2 D^2\phi(y) &= I_r(I_r D(D\phi(y))) \\
&= I_r\left(\int_r^y \frac{d}{dy} D\phi(y)dy\right) \\
&= I_r D\phi(y) - I_r D\phi(r) \\
&= \int_r^y \frac{d}{dy}\phi(y)dy - D\phi(r)\int_r^y dy \\
&= \phi(y) - \phi(r) - D\phi(r)(y-r)
\end{aligned}
$$

$$
\begin{aligned}
I_r^3 D^3\phi(y) &= I_r(I_r(I_r D(D(D\phi(y))))) \\
&= I_r\left(\int_r^y \frac{d}{dy} D^2\phi(y)dy\right) \\
&= I_r^2 D^2\phi(y) - I_r^2 D^2\phi(r) \\
&= I_r D\phi(y) - I_r D\phi(r) - D^2\phi(r)\frac{(y-r)^2}{2} \\
&= \phi(y) - \phi(r) - D\phi(r)(y-r) - D^2\phi(r)\frac{(y-r)^2}{2}.
\end{aligned}
$$

In general

$$I_r^m D^m \phi(y) = \phi(y) - \sum_{k=0}^{m-1} Dy^k \phi(r) \frac{(y-r)^k}{k!}. \tag{1.3.20}$$

By definition (1.3.13)

$$_cD^n \phi(y) = I_r^{m-n} D^m \phi(y). \tag{1.3.21}$$

Apply I_a^n on both sides of Eq (1.3.21), and using semi-group property, we achieve

$$I_r^n {}_cD^n \phi(y) = I_r^n I_r^{m-n} D^m \phi(y) = I_r^m D^m \phi(y). \tag{1.3.22}$$

Use Eq (1.3.22) into (1.3.20) and replacing n by α we can get the desired result.

□

In particular, if $0 < \alpha \leq 1$ and $\phi(y) \in C[r,u]$ then $I_r^\alpha {}_cD^\alpha \phi(y) = \phi(y) - \phi(r)$.

1.4 ψ-Fractional calculus

Fractional differentiation and integration of a function with respect to another function is ψ-fractional calculus. The initial genesis idea of this work was presented in [22, 7], it was additionally evolved in standard books [6]. We provide the necessary information about ψ-fractional integral and derivatives and their important properties, where ψ: $(a,\infty) \to [a,\infty)$, is a one-to-one and increasing function such that $\psi'(0) \neq 0$.

1.4.1 ψ-Riemann-Liouville fractional integral and derivatives

ψ-Riemann-Liouville fractional integral can be obtained from the Cauchy iterated formula.

Lemma 1.4.1. *Let ϕ be Riemann integrable on $[r,u]$. Then, for $r \leq y \leq u$ and $n \in \mathbb{N}$ we have*

$$I_r^{n,\psi} \phi(y) = \frac{1}{\Gamma(\alpha)} \int_r^y (\psi(y) - \psi(s))^{n-1} \psi'(s) \phi(s) ds. \tag{1.4.1}$$

Proof. We will start from integral

$$I_r^{1,\psi} \phi(y) = \int_r^y \phi(s) \psi'(s) ds. \tag{1.4.2}$$

By iterating the integral in (1.4.2), we have

$$\begin{aligned} I_r^{2,\psi} \phi(y) = I_r^{1,\psi}(I_r^{1,\psi}\phi)y &= \int_r^y \int_r^{t_1} \phi(s)\psi'(s) dt_1 dt_2 \\ &= \int_r^y \int_{t_1}^y \phi(t_1)\psi'(t_1)\psi'(t_2) dt_2 dt_1 \\ &= \int_r^y \phi(t_1)\psi'(t_1)(\psi(y) - \psi(t_1)) dt_1 \\ &= \int_r^y \phi(t)\psi'(t)(\psi(y) - \psi(t)) dt. \end{aligned}$$

The third iteration gives

$$\begin{aligned}
I_r^{3,\psi}\phi(y) =& I_r^{1,\psi}(I_r^{1,\psi}(I_r^{1,\psi}\phi(y))) = \int_r^y \int_r^{t_1} \int_r^{t_2} \phi(t)\psi'(t)dtdt_2dt_1. \\
=& \int_r^y \int_r^{t_1} \phi(t)\psi'(t)\psi'(t_1)(\psi(t_1)-\psi(t))dtdt_1, \\
=& \int_r^y \phi(t)\frac{(\psi(y)-\psi(t))^2}{2!}dt.
\end{aligned}$$

If we repeat the process for α-times, we will get the desired result. □

Definition 1.4.2. [1] Let ϕ be Riemann integrable on $[r, u]$. Then, for $r \leq y \leq u$ and $\alpha > 0$ we have

$$I_r^{\alpha,\psi}\phi(y) = \frac{1}{\Gamma(\alpha)} \int_r^y (\psi(y)-\psi(s))^{\alpha-1}\psi'(s)\phi(s)ds. \tag{1.4.3}$$

Example 1.4.3. For $\phi(y) = \psi(y)^\beta$ we will show that

$$I_0^{\alpha,\psi}\phi(y) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(\psi(y))^{\alpha+\beta}. \tag{1.4.4}$$

Using the definition (1.4.2), we achieve

$$I_0^{\alpha,\psi}\phi(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (\psi(y)-\psi(t))^{\alpha-1}\psi'(t)(\psi(t))^\beta dt,$$

$$I_0^{\alpha,\psi}(\psi(y))^\beta = \frac{1}{\Gamma(\alpha)} \int_0^y (\psi(y))^{\alpha-1}\left(1-\frac{\psi(t)}{\psi(y)}\right)^{\alpha-1}\psi'(t)(\psi(t))^\beta dt.$$

We will evaluate the above integral by substituting $u = \frac{\psi(t)}{\psi(y)}$ and $du = \frac{\psi'(t)}{\psi(y)}dt$

$$\begin{aligned}
I_0^{\alpha}(\psi(y))^\beta =& \frac{1}{\Gamma(\alpha)} \int_0^1 (\psi(y))^{\alpha-1}(1-u)^{\alpha-1}(\psi(y))^\beta u^\beta du \\
=& \frac{(\psi(y))^{\alpha+\beta+1-1}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1}u^\beta du.
\end{aligned}$$

Since,

$$\int_0^1 (1-u)^{\alpha-1}u^\beta dv = B(\alpha, \beta+1).$$

Therefore, we have

$$I_0^{\alpha}(\psi(y))^\beta = \frac{\Gamma(\beta+1)(\psi(y))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

Definition 1.4.4. [1] Let $\beta > 0$ and $p = \lceil\beta\rceil$, the ψ-Riemann-Liouville fractional derivative is defined by

$$D_r^{\beta,\psi}\phi(y) = D^{p,\psi}I_r^{p-\beta,\psi}\phi(y) = \frac{1}{\Gamma(p-\beta)}\left(\frac{d}{dy}\right)^p \int_r^y (\psi(y)-\psi(s))^{p-\beta-1}\psi'(s)\phi(s)ds.$$

where $D^{p,\psi} = \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^p$.

Example 1.4.5. We find the ψ-Riemann-Liouville fractional derivative of $y \in [r, u]$, such that $\phi(y) = (\psi(y))^{\beta}$, $\beta \geq -1$ and $\alpha \geq 0$. We use the definition and evaluate the resulting integral.

$$D_0^{\alpha,\psi}\phi(y) = D^{p,\psi} I_0^{p-\alpha,\psi}\phi(y) = D^{p,\psi} I_0^{p-\alpha,\psi}(\psi(y))^{\beta}.$$

From Eq (1.3.7) we have

$$D_0^{\alpha,\psi}\phi(y) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+p-\alpha+1)} D^{p,\psi}(\psi(y))^{p-\alpha+\beta}. \tag{1.4.5}$$

Case 1: If $\alpha - \beta \in \mathbb{N}$, the right hand side is the pth derivative of a classical polynomial of degree $p - (\alpha - \beta)$ and so the expression vanishes that is

$$D^{p,\psi}(\psi(y))^{\alpha-\beta} = 0,$$

for all $\alpha \geq 0$, $p \in 1, 2, \cdots, [\alpha]$.
Case 2: If $\alpha - \beta \neq \mathbb{N}$, here we generalize the integer-order derivative of a power function

$$\begin{aligned}
D^{1,\psi}(\psi(y))^s = D^{p,\psi} &= \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^p s(\psi(y))^{s-1}\psi'(y)\\
D^{2,\psi}(\psi(y))^s = D^{p,\psi} &= \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^p s(s-1)(\psi(y))^{s-2}\psi'(y)\\
D^{3,\psi}(\psi(y))^s = D^{p,\psi} &= \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^p s(s-1)(s-2)(\psi(y))^{s-3}\psi'(y).
\end{aligned}$$

In general

$$D^{p,\psi}(\psi(y))^s = \frac{(s)!}{(s-p)!}(\psi(y))^{s-p}.$$

Now, we have

$$D^{p,\psi}(\psi(y))^s = D^{p,\psi} = \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)^p \frac{\Gamma(s+1)}{\Gamma(s-p+1)}(\psi(y))^{s-p}\psi'(y). \tag{1.4.6}$$

Eq (1.4.5) becomes

$$D_0^{\alpha}(\psi(y))^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+p-\alpha+1)}\frac{\Gamma(\beta+p-\alpha+1)}{\Gamma(p+1-\alpha+\beta-p)}(\psi(y))^{p-\alpha+\beta-p}$$

$$D_0^{\alpha,\psi}(\psi(y))^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\psi(y))^{\beta-\alpha}. \tag{1.4.7}$$

1.4.2 Properties of ψ-Reimann-Lioville fractional integral and derivatives

Now, we present the relationship between ψ-Riemann-Liouville fractional integral with derivative and vice versa. First, we discuss the property for classical fractional integral and derivative.

Theorem 1.4.6. *[20] If $m \geq n$, then*

$$D^{m,\psi} I_r^{n,\psi} \phi(t) = D^{m-n,\psi} \phi(t).$$

And if $m \leq n$, then

$$D^{m,\psi} I_r^{n,\psi} \phi(t) = I_r^{n-m,\psi} \phi(t).$$

Proof. From Lemma (1.3.1), we can deduce that

$$D^{1,\psi} I_r^{1,\psi} \phi(t) = \phi(t), \text{ where } D^{1,\psi} = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right), \text{ and } I_r^{1,\psi} \phi(t) = \int_r^t \phi(s)\psi'(s)ds. \quad (1.4.8)$$

Now, we state the composition properties of differential and integral operators as follows. Repeated application of (1.4.8) gives

$$D^{2,\psi} I_r^{2,\psi} \phi(t) = D^{1,\psi}(D^{1,\psi} I_r^{1,\psi}(I_r^{1,\psi} \phi(t))) = D^{1,\psi} I_r^{1,\psi} \phi(t) = \phi(t). \quad (1.4.9)$$

By using Eq (1.4.9), we get

$$D^{3,\psi} I_r^{2,\psi} \phi(t) = D^{1,\psi}(D^{2,\psi} I_r^{2,\psi} \phi(t)) = D^{1,\psi} \phi(t),$$
$$D^{2,\psi} I_r^{3,\psi} \phi(t) = D^{2,\psi} I_r^{2,\psi}(I_r^{1,\psi} \phi(t)) = I_r^{1,\psi} \phi(t).$$

Similarly we have

$$\begin{aligned} D^{3,\psi} I_r^{3,\psi} \phi(t) =& D^{2,\psi}(D^{1,\psi} I_r^{1,\psi}(I_r^{2,\psi} \phi(t))) = D^{2,\psi} I_r^{2,\psi} \phi(t) \\ =& D^{1,\psi}(D^{1,\psi} I_r^{1,\psi}(I_r^{1,\psi} \phi(t))) = D^{1,\psi} I_r^{1,\psi} \phi(t) = \phi(t). \end{aligned} \quad (1.4.10)$$

Using Eq (1.4.10)

$$D^{4,\psi} I_r^{3,\psi} \phi(t) = D^{1,\psi}(D^{3,\psi} I_r^{3,\psi} \phi(t)) = D^{1,\psi} \phi(t),$$
$$D^{3,\psi} I_r^{4,\psi} \phi(t) = D^{3,\psi} I_r^{3,\psi}(I_r^{1,\psi} \phi(t)) = I_r^{1,\psi} \phi(t).$$

In general

$$\begin{aligned} D^{m,\psi} I_r^{n,\psi} \phi(t) &= D^{m-n,\psi} \phi(t), \ m \geq n \\ D^{m,\psi} I_r^{n,\psi} \phi(t) &= I_r^{n-m,\psi} \phi(t), \ m \leq n. \end{aligned} \quad (1.4.11)$$

□

Now, we discuss properties for ψ-Reimann-Liouville fractional integral and derivatives.

Theorem 1.4.7. *[28] Let $\alpha > 0$. Then for every $\phi \in L_1[r, u]$*

$$D_r^{\alpha,\psi} I_r^{\alpha,\psi} \phi(y) = \phi(y)$$

holds almost everywhere.

Proof. By the definition 1.4.2, and by semi-group property of Reimann-Liouville fractional integral, we have

$$\begin{aligned} D_a^{\alpha,\psi} I_r^{\alpha,\psi}\phi(y) =& D^{m,\psi} I_r^{m-\alpha,\psi} I_r^{\alpha,\psi}\phi(y) = D^{m,\psi} I_r^{m-\alpha+\alpha,\psi}\phi(y) \\ =& D^{m,\psi} I_r^{m,\psi}\phi(y) = \phi(y). \end{aligned}$$

□

Theorem 1.4.8. *[28] Assume that, if $\phi \in A^k[r,u]$, $\alpha \geq 0$ and $k = m = \lceil\alpha\rceil$. Then*

$$I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) = \phi(y) - \sum_{j=1}^{k} D_r^{\alpha-j,\psi}\phi(y)|_{y=r} \frac{(\psi(y)-\psi(r))^{\alpha-j}}{\Gamma(\alpha-j+1)}.$$

Proof. By definition of Riemann-Liouville fractional integral

$$I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) = \frac{1}{\Gamma(\alpha)}\int_r^y (\psi(y)-\psi(s))^{\alpha-1}\psi'(s) D_r^{\alpha,\psi}\phi(s)ds. \tag{1.4.12}$$

By using Leibniz's Rule, we have

$$\begin{aligned} &\frac{d}{dy}\left(\frac{1}{\Gamma(\alpha+1)}\int_r^y (\psi(y)-\psi(s))^{\alpha}\psi'(s) D_r^{\alpha,\psi}\phi(s)ds\right) \\ =&\frac{1}{\Gamma(\alpha)}\int_r^y (\psi(y)-\psi(s))^{\alpha-1}\psi'(s) D_r^{\alpha,\psi}\phi(s)ds. \end{aligned} \tag{1.4.13}$$

Let us consider the left hand side of (1.4.13)

$$\begin{aligned} &\frac{d}{dy}\left(\frac{1}{\Gamma(\alpha+1)}\int_r^y (\psi(y)-\psi(s))^{\alpha}\psi'(s) D_r^{\alpha,\psi}\phi(s)ds\right) \\ =&\frac{1}{\Gamma(\alpha)}\int_r^y ((\psi(y)-\psi(s))^{\alpha-1} D_r^{\alpha,\psi}\phi(s)ds. \end{aligned}$$

On the other hand, by definition of Riemann-Liouville fractional derivative, repeatedly integrating and by Theorem (1.3.7) we have

$$\frac{1}{\Gamma(\alpha+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha} D_r^{\alpha,\psi}\psi'(s)\phi(s)ds = \frac{1}{\Gamma(\alpha+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha} D^{k,\psi} I_r^{k-\alpha,\psi}\phi(s)ds.$$

Now we evaluate this by using iterative method. Let $k = 1$

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha}\frac{d}{ds} I_r^{1-\alpha,\psi}\psi'(s)\phi(s)ds \\ =&-\frac{1}{\Gamma(\alpha+1)}((\psi(y)-\psi(s))^{\alpha} I_r^{1-\alpha,\psi}\phi(s)|_r + \frac{1}{\Gamma(\alpha)}\int_r^y ((\psi(y)-\psi(s))^{\alpha-1} I_r^{1-\alpha,\psi}\psi'(s)\phi(s)ds. \end{aligned}$$

For $k = 2$

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha}\frac{d^2}{ds^2} I_r^{2-\alpha,\psi}\phi(s)ds \\ =&\frac{1}{\Gamma(\alpha+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha}\frac{d}{ds}[\frac{d}{ds} I_r^{2-\alpha,\psi}\phi(s)ds]. \end{aligned}$$

By doing all the calculations in the similar way, we get

$$= \frac{1}{\Gamma(\alpha-1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha-2} I_r^{2-\alpha,\psi}\phi(s)dt - \sum_{j=1}^{2} \frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha,\psi}\phi(s)|r \frac{((\psi(y)-\psi(r))^{\alpha-j+1}}{\Gamma(\alpha-j+2)}.$$

In general

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha} \frac{d^k}{dy^k} I_r^{k-\alpha,\psi}\phi(s)ds \\
&\quad= \frac{1}{\Gamma(\alpha-k+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha-k} I_r^{k-\alpha,\psi}\phi(s)ds \\
&\quad - \sum_{j=1}^{n} \frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha,\psi}\phi(s)|r \frac{((\psi(y)-\psi(s))^{\alpha-j+1}}{\Gamma(\alpha-j+2)} \\
&\quad= I_r^{\alpha-k+1}[I_r^{k-\alpha,\psi}\phi(s)] - \sum_{j=1}^{n} \frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha,\psi}\phi(s)|r \frac{((\psi(y)-\psi(r))^{\alpha-j+1}}{\Gamma(\alpha-j+2)}.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_r^y ((\psi(y)-\psi(s))^{\alpha} \frac{d^k}{dy^k} I_r^{k-\alpha,\psi}\phi(s)ds \\
&\qquad\qquad = I_r^1\phi(s) - \sum_{j=1}^{n} \frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha,\psi}\phi(s)|r \frac{((\psi(y)-\psi(r))^{\alpha-j+1}}{\Gamma(\alpha-j+2)}.
\end{aligned} \tag{1.4.14}$$

We have

$$\begin{aligned}
I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) &= \phi(s) - \sum_{j=1}^{n} \frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha,\psi}\phi(s)|r \frac{((\psi(y)-\psi(r))^{\alpha-j}}{\Gamma(\alpha-j+1)}. \\
&= \phi(s) - \sum_{j=1}^{n} \frac{d^{k-j}}{ds^{k-j}} I_r^{k-\alpha}\phi(s)|r \frac{((\psi(y)-\psi(r))^{\alpha-j}}{\Gamma(\alpha-j+1)}.
\end{aligned}$$

By solving the above expression, we get the desired result. Now, we discuss the ψ-Caputo's fractional derivative and its application. □

1.4.3 ψ-Caputo's fractional derivative

ψ-Caputo's fractional derivative was first studied in [1]. ψ-Caputo's fractional derivative is more convinent to deal with the ψ-fractional differential equations.

Definition 1.4.9. [2] Let $\beta > 0$, $y \in [r, u]$ and $m = \lceil \beta \rceil$, the ψ-Caputo fractional derivative is defined by

$$D_r^{\beta,\psi}\phi(y) = I_r^{m-\beta,\psi} D^{m,\psi}\phi(y) = \frac{1}{\Gamma(m-\beta)}\int_r^y (\psi(y)-\psi(s))^{m-\beta-1}\psi'(s) D^{m,\psi}\phi(s)ds. \tag{1.4.15}$$

Example 1.4.10. We find the Caputo derivative of $\phi(y) = \psi(y)^\beta$ as under

$$ {}_cD_0^{\alpha,\psi}(\psi(y))^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(\psi(y))^{\alpha+\beta}. \tag{1.4.16}$$

By definition, we have

$$ {}_cD_0^{\alpha,\psi}\phi(y) = I_0^{k-\alpha,\psi}D^k\phi(y) = \frac{1}{\Gamma(k-\alpha)}\int_0^y (\psi(y)-\psi(t))^{k-\alpha-1}\psi'(t)D^{k,\psi}(\psi(t))^\beta dt. \tag{1.4.17}$$

Case 1: If $\beta < k$, then $D^{k,\psi}(\psi(y))^\beta = 0$.
Case 2: If $\beta \in \mathbb{N}$ and $k \leq \beta$. Then we generalize the integer-order derivative of a power function

$$\begin{aligned}
D^{1,\psi}(\psi(y))^k &= \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)k(\psi(y))^{k-1}\psi'(y)\\
D^{2,\psi}(\psi(y))^k &= \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)k(k-1)(\psi(y))^{k-2}\psi'(y)\\
D^{3,\psi}(\psi(y))^k &= \left(\frac{1}{\psi'(y)}\frac{d}{dy}\right)k(k-1)(k-2)(\psi(y))^{k-3}\psi'(y).
\end{aligned}$$

In general

$$D^{j,\psi}(\psi(y))^k = \frac{\Gamma(k+1)}{\Gamma(k-j+1)}(\psi(y))^{k-j}.$$

Eq (1.4.17) reduce to

$$\begin{aligned}
{}_cD_0^{\alpha,\psi}\phi(y) =& \frac{1}{\Gamma(k-\alpha)}\int_0^y ((\psi(y)-\psi(t))^{k-\alpha-1}D^{k,\psi}(\psi(t))^\beta dt.\\
=& \frac{\Gamma(\beta+1)}{\Gamma(\beta-k+1)\Gamma(k-\alpha)}\int_0^y ((\psi(y)-\psi(t))^{k-\alpha-1}(\psi(t))^{\beta-k}dt.\\
=& \frac{\Gamma(\beta+1)}{\Gamma(\beta-k+1)\Gamma(k-\alpha)}\int_0^y \left(1-\frac{\psi(t)}{\psi(y)}\right)^{k-\alpha-1}(\psi(y))^{k-\alpha-1}(\psi(t))^{\beta-k}dt.
\end{aligned}$$

Let $w = \frac{\psi(t)}{\psi(y)}$

$$ {}_cD_0^{\alpha}\phi(y) = \frac{\Gamma(\beta+1)(\psi(y))^{k-\alpha}}{\Gamma(\beta-k+1)\Gamma(k-\alpha)}\int_0^1 (1-w)^{k-\alpha-1}w^{\beta-k}dw.$$

Since $\int_0^1 (1-w)^{k-\alpha-1}w^{\beta-k}dw = B(k-\alpha, \beta-m+1) = \frac{\Gamma(k-\alpha)\Gamma(\beta-k+1)}{\Gamma(k-\alpha+\beta-k+1)}$.
Thus

$$ {}_cD_0^{\alpha}(\psi(y))^\beta = \frac{\Gamma(\beta+1)(\psi(y))^{k-\alpha}}{\Gamma(\beta-\alpha+1)}.$$

1.4.4 Properties of ψ-Caputo fractional derivative

In this section, we discuss the composition of Caputo derivative with Riemann-Liouville fractional integral and vice versa.

Theorem 1.4.11. *[4] If ϕ is continuous and $\alpha \geq 0$, then*

$$ {}_cD_r^{\alpha,\psi} I_r^{\alpha,\psi}\phi(y) = \phi(y). \tag{1.4.18}$$

Proof. Let $\Phi = I_r^{\alpha,\psi}\phi(y)$, where $D^{k,\psi}\Phi(r) = 0$, then we have

$$ {}_cD_r^{\alpha,\psi} I_r^{\alpha,\psi}\phi(y) ={}_c D_r^{\alpha,\psi}\phi(y) = D_r^{\alpha,\psi}\phi(y) = D^{m,\psi} I_r^{m-\alpha,\psi}\phi(y) = \phi(y).$$

□

Theorem 1.4.12. *[4] Assume that $\alpha \geq 0$, $m = m = \lceil \alpha \rceil$ and $\phi \in A^m(r,u)$, then*

$$ I_r^{\alpha,\psi}{}_cD_r^{\alpha,\psi}\phi(y) = \phi(y) - \sum_{k=0}^{m-1} D^{k,\psi}\frac{\phi(r)}{k!}(\psi(y)-\psi(r))^k.$$

Proof. By using fundamental theorem of calculus and iterative method we obtain

$$\begin{aligned}
I_r^{1,\psi}D^{1,\psi}\phi(y) =& \int_r^y \frac{1}{\psi'(y)}\frac{d}{dy}\phi(y)\psi'(y)dy = \phi(y) - \phi(r)\\
I_r^{2,\psi}D^2\phi(y) =& I_r^{1,\psi}(I_r^{1,\psi}D^{1,\psi}(D^{1,\psi}\phi(y)))\\
=& I_r^{1,\psi}\left(\int_r^y \frac{1}{\psi'(y)}\frac{d}{dy}D\phi(y)\psi'(y)dy\right)\\
=& I_r^{1,\psi}D^{1,\psi}\phi(y) - I_r^{1,\psi}D^{1,\psi}\phi(r)\\
=& \int_r^y \frac{1}{\psi'(y)}\frac{d}{dy}\phi(y)\psi'(y)dy - D^{1,\psi}\phi(r)\int_r^y \psi'(y)dy\\
=& \phi(y) - \phi(r) - D^{1,\psi}\phi(r)(\psi(y)-\psi(r))
\end{aligned}$$

$$\begin{aligned}
I_r^{3,\psi}D^{3,\psi}\phi(y) =& I_r^{1,\psi}(I_r^{1,\psi}(I_r^{1,\psi}D^{1,\psi}(D^{1,\psi}(D^{1,\psi}\phi(y)))))\\
=& I_r^{1,\psi}\left(\int_r^y \frac{1}{\psi'(y)}\frac{d}{dy}D^{2,\psi}\psi'(y)\phi(y)dy\right)\\
=& \phi(y) - \phi(r) - D^{1,\psi}\phi(r)(\psi(y)-\psi(r)) - D^{2,\psi}\phi(r)\frac{(\psi(y)-\psi(r))^2}{2}.
\end{aligned}$$

In general

$$ I_r^{m,\psi}D^{m,\psi}\phi(y) = \phi(y) - \sum_{k=0}^{m-1} D^{1,\psi}y^k\phi(r)\frac{(\psi(y)-\psi(r))^k}{k!}. \tag{1.4.19}$$

By thw definition of ψ-Caputo derivative

$$ {}_cD^{n,\psi}\phi(y) = I_r^{m-n,\psi}D^{m,\psi}\phi(y). \tag{1.4.20}$$

Apply $I_r^{n,\psi}$ on both sides of Eq (1.4.20), we have

$$ I_r^{n,\psi}{}_cD^{n,\psi}\phi(y) = I_r^{n,\psi}I_r^{m-n,\psi}D^{m,\psi}\phi(y) = I_r^{m,\psi}D^{m,\psi}\phi(y). \tag{1.4.21}$$

Use Eq (1.4.21) into (1.4.19) we obtain

$$I_r^{n,\psi}{}_cD^{n,\psi}\phi(y) = \phi(y) - \sum_{k=0}^{m-1} D^{k,\psi}\phi(r)\frac{(\psi(y)-\psi(r))^k}{k!}.$$

Replacing n with real α we have

$$I_r^{\alpha,\psi}{}_cD^{\alpha,\psi}\phi(y) = \phi(y) - \sum_{k=0}^{m-1} D^{k,\psi}\phi(r)\frac{(\psi(y)-\psi(r))^k}{k!}.$$

$\square$

In particular, if $0 < \alpha \leq 1$ and $\phi(y) \in C[r,u]$ then $I_r^{\alpha,\psi}{}_cD^{\alpha,\psi}\phi(y) = \phi(y) - \phi(r)$.

Chapter 2

Generalized Taylor's formula with respect to a function

In this chapter, we discuss a generalization of Taylor's theorem. To begin with the generalization, we first discuss the properties of the ψ-Reimann-Liouville fractional integral. We prove the Mean Value theorem for this purpose. Then, we prove the Taylor's theorem for integer case and then for the fractional case. Moreover, we discuss approximations of a function by using generalized Taylor's formula with respect to a function. Furthermore, we present the convergence and remainder theorem for Taylor's theorem. We also present the Taylor series solution and fractional power series solution for ψ-fractional differential equations.

2.1 Introduction

Some properties of fractional integral were proved in [20]. We will generalize those properties of fractional integral with respect to another function. First, we find fractional integral of power of ψ.

Example 2.1.1. The fractional integral $I_r^{\alpha,\psi}$ of $\phi(w) = (\psi(w))^\beta$ is,

$$I_r^{\alpha,\psi}(\psi(w))^\beta = \frac{(\psi(w))^{\beta+\alpha}}{\Gamma(\alpha)} B\left(\frac{\psi(w)-\psi(r)}{\psi(w)}; \alpha, \beta+1\right). \tag{2.1.1}$$

This can be computed as:
From the defination of fractional integral, we have

$$\begin{aligned} I_r^{\alpha,\psi}\phi(w) =& \frac{1}{\Gamma(\alpha)} \int_r^w (\psi(w)-\psi(s))^{\alpha-1}\psi'(s)(\psi(s))^\beta ds \\ =& \frac{\psi(w)^{\alpha-1}}{\Gamma(\alpha)} \int_r^w \left(1-\frac{\psi(w)}{\psi(s)}\right)^{\alpha-1} \psi'(s)(\psi(s))^\beta ds. \end{aligned}$$

Let $y = \left(1 - \frac{\psi(s)}{\psi(w)}\right)$ and $dy = \frac{\psi'(s)}{\psi(w)}ds$, substituting this in integral we get,

$$\begin{aligned} I_r^{\alpha,\psi}\phi(w) =& - \frac{(\psi(w))^{\alpha-1}}{\Gamma(\alpha)} \int_{\frac{\psi(w)-\psi(r)}{\psi(w)}}^{0} y^{\alpha-1}(1-y)^{\beta}(\psi(w))^{\beta}\psi(w)dy \\ =& \frac{(\psi(w))^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^{\frac{\psi(w)-\psi(r)}{\psi(w)}} y^{\alpha-1}(1-y)^{\beta}dy. \end{aligned}$$

From the defination of incomplete beta function (1.2.2), we get

$$I_r^{\alpha,\psi}(\psi(w))^{\beta} = \frac{(\psi(w))^{\beta+\alpha}}{\Gamma(\alpha)} B\left(\frac{\psi(y)-\psi(r)}{\psi(y)}; \alpha, \beta+1\right). \tag{2.1.2}$$

Example 2.1.2. Let us consider $\phi(w) = e^{c\psi(w)}$, then show that

$$I_r^{\alpha,\psi} e^{c\psi(w)} = e^{ac}(\psi(w) - \psi(r)^{\alpha}) \sum_{k=0}^{\infty} \frac{(c(\psi(w)-\psi(r)))^k}{\Gamma(\alpha+k+1)}. \tag{2.1.3}$$

From defination

$$\begin{aligned} I_r^{\alpha,\psi} e^{c\psi(w)} =& \frac{1}{\Gamma(\alpha)} \int_r^w (\psi(w)-\psi(s))^{\alpha-1}\psi'(s)e^{c\psi(s)}ds \\ =& \frac{1}{\Gamma(\alpha)} \int_r^w (\psi(w)+\psi(r)-\psi(r)-\psi(s))^{\alpha-1}\psi'(s)e^{c\psi(s)}ds \\ =& \frac{1}{\Gamma(\alpha)} \int_r^w (\psi(w)-\psi(r))^{\alpha-1}\left(1 - \frac{(\psi(s)-\psi(r))}{(\psi(w)-\psi(r))}\right)^{\alpha-1} \psi'(s)e^{c\psi(s)}ds. \end{aligned}$$

Let $z = \frac{(\psi(s)-\psi(r))}{(\psi(w)-\psi(r))}$ and $dz = \frac{\psi'(s)ds}{(\psi(w)-\psi(r))}$ Putting these values in above equation, we get

$$\begin{aligned} I_r^{\alpha,\psi} e^{c\psi(w)} =& \frac{e^{c\psi(r)}}{\Gamma(\alpha)}(\psi(w)-\psi(r))^{\alpha} \int_0^1 (1-z)^{\alpha-1} e^{cz(\psi(w)-\psi(r))}dz, \\ =& \frac{e^{c\psi(r)}}{\Gamma(\alpha)}(\psi(w)-\psi(r))^{\alpha} \int_0^1 (1-z)^{\alpha-1} \sum_{k=0}^{\infty} \frac{(cz(\psi(w)-\psi(r)))^k}{\Gamma(k+1)}, \\ =& \frac{e^{c\psi(r)}}{\Gamma(\alpha)}(\psi(w)-\psi(r))^{\alpha} \sum_{k=0}^{\infty} \frac{(c(\psi(w)-\psi(r)))^k}{\Gamma(k+1)} \int_0^1 (1-z)^{\alpha-1} z^k dz. \end{aligned}$$

Using defination of Beta function (1.2.2) we get,

$$I_r^{\alpha,\psi} e^{c\psi(w)} = e^{ac}(\psi(w)-\psi(r))^{\alpha} \sum_{k=0}^{\infty} \frac{(c(\psi(w)-\psi(r)))^k}{\Gamma(\alpha+k+1)}.$$

2.2 Generalized Taylor's Theorem

In this section, we introduce an alternate proof of generalization of Taylor's Theorem in the framework of fractional differential equations involving fractional derivative of functions with respect to functions. First, we discuss integer order case, and then we generalize for ψ-fractional operators.

Theorem 2.2.1. *Assume that, if $\phi \in C^n[a,b]$. Then*

$$\phi(x) = \sum_{j=0}^{n} \frac{(\psi(x)-\psi(a))^j}{\Gamma(j+1)} D^{j,\psi}\phi(a) + \frac{D^{(n+1),\psi}\phi(\eta)(\psi(x)-\psi(a))^{n+1}}{\Gamma(n+2)} \tag{2.2.1}$$

with $a \leq \eta \leq x$ and $x \in (a,b]$, where here $D^{j,\psi} = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^j$.

Proof. We introduce a polynomial in $\psi(x)$ as:

$$P_n^{\psi}(x) = \sum_{k=0}^{n} c_k(\psi(x)-\psi(a))^k. \tag{2.2.2}$$

Since $\phi \in C^m[a,b]$, the polynomial $P_n^{\psi}(x)$ of order n in terms of ψ at a is the unique polynomial of degree atmost n such that:

$$D^{j,\psi}\phi(a) = D^{j,\psi}P_n^{\psi}(a), \qquad 0 \leq j \leq n \tag{2.2.3}$$

Now applying D^j on Eq (2.2.2), we get

$$D^{j,\psi}P_n^{\psi}(x) = \sum_{k=j}^{n} c_k \frac{\Gamma(j+1)}{\Gamma(j+1-k)}(\psi(x)-\psi(a))^{k-j}. \tag{2.2.4}$$

Using Eq (2.2.4) in Eq (2.2.3), we get

$$D^{j,\psi}\phi(a) = c_j\frac{\Gamma(j+1)}{\Gamma(1)} + \sum_{k=j+1}^{n} \frac{\Gamma(j+1)}{\Gamma(j+1-k)}(\psi(x)-\psi(a))^{k-j}. \tag{2.2.5}$$

Now for $x = a$, we have, $D^{j,\psi}\phi(a) = c_j\Gamma(j+1)$, which gives, $c_j = \frac{D^{j,\psi}\phi(a)}{\Gamma(j+1)}$.
Thus ψ-Taylor's polynomial with respect to a function is:

$$P_n^{\psi}(x) = \sum_{j=0}^{n} \frac{(\psi(x)-\psi(a))^j}{\Gamma(j+1)} D^{j,\psi}\phi(a)$$

Let $R_n^{\psi}(x) = \phi(x) - P_n^{\psi}(x)$. Then applying $D^{(n+1),\psi}$, we achieve

$$D^{(n+1),\psi}R_n^{\psi}(x) = D^{(n+1),\psi}\phi(x) - D^{(n+1),\psi}P_n^{\psi}(x). \tag{2.2.6}$$

Since P_n is polynomial of degree n, thus

$$D^{(n+1),\psi}P_n^{\psi}(x) = 0.$$

Therefore Eq (2.2.6) becomes

$$D^{(n+1),\psi}R_n^{\psi}(x) = D^{(n+1),\psi}\phi(x).$$

Now applying integral $I_a^{(n+1),\psi}$ on both sides , we get

$$I_a^{(n+1),\psi}D^{(n+1),\psi}R_n^{\psi}(x) = I_a^{(n+1),\psi}D^{(n+1),\psi}\phi(x). \tag{2.2.7}$$

Using Theorem 1.3.4, we get

$$R_n^{\psi}(x) = I_a^{(n+1),\psi} D^{(n+1),\psi}\phi(x). \tag{2.2.8}$$

By the defination of fractional integral, Eq (2.2.8) can be written as,

$$R_n^{\psi}(x) = \frac{1}{\Gamma((n+1))}\int_a^x (\psi(x)-\psi(s))^n \psi'(s) D^{(n+1),\psi}\phi(s)ds. \tag{2.2.9}$$

Using mean value theorem of integral calculus and then evaluating the integral in Eq: (2.2.9), we get

$$R_n^{\psi}(x) = \frac{D^{(n+1),\psi}\phi(\eta)(\psi(x)-\psi(a))^{n+1}}{\Gamma(n+2)}.$$

So, ψ-Taylor's theorem with respect to a function is:

$$\phi(x) = \sum_{j=0}^{n} \frac{(\psi(x)-\psi(a))^j}{\Gamma(j+1)} D^{j,\psi}\phi(a) + \frac{D^{(n+1),\psi}\phi(\eta)(\psi(x)-\psi(a))^{n+1}}{\Gamma(n+2)}.$$

□

2.3 Generalized fractional Taylor's Theorem

Polynomials are often used in approximation theory to approximate functions. The intention here is to approximate unknown functions which are solutions of fractional differential equations with suitably generalized polynomial type functions which are easier to work with, at the price of some reasonably small difference between the solution and the approximation functions. Taylor series is often used to developed various numerical methods for approximating solutions of differential and integral equations. Here, we generalize for fractional case.
Method 1: Here, we generalized the Taylor's theorem with respect to a function. We followed the method in [20].

Theorem 2.3.1. *For $0 < \alpha \leq 1$, suppose that $\phi \in C[r,u]$ and $D_r^{\alpha}\phi \in C[r,u]$, then*

$$\phi(y) = \phi(r) + \frac{1}{\Gamma(\alpha)} D_r^{\alpha,\psi} f(\eta)(\psi(y)-\psi(r))^{\alpha} \tag{2.3.1}$$

with $r \leq \eta \leq y$ for all $y \in [r,u]$.

Proof. From the defination of fractional integral

$$I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) = \frac{1}{\Gamma(\alpha)}\int_r^y (\psi(y)-\psi(s))^{\alpha-1}\psi'(s) D^{\alpha,\psi}\phi(s)ds.$$

Using Mean Value Theorem of integral calculus,

$$I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) = \frac{D^{\alpha,\psi}\phi(\eta)}{\Gamma(\alpha)}\int_r^y (\psi(y)-\psi(s))^{\alpha-1}\psi'(s)ds. \tag{2.3.2}$$

Now by evaluating the integral in (2.3.2) we get,

$$I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) = \frac{D^{\alpha,\psi}\phi(\eta)}{\Gamma(\alpha+1)}(\psi(y)-\psi(s))^{\alpha}. \tag{2.3.3}$$

We know that,

$$I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) = \phi(y)-\phi(r). \tag{2.3.4}$$

From (2.3.3) and (2.3.4) we get,

$$\phi(y) = \phi(r) + \frac{1}{\Gamma(\alpha)} D_r^{\alpha,\psi} f(\eta)(\psi(y)-\psi(r))^{\alpha}.$$

□

Theorem 2.3.2. *For $0 < \alpha \leq 1$, suppose that $D_a^{(n+1)\alpha}\phi(x) \in C[a,b]$ then,*

$$I_r^{m\alpha,\psi} D_r^{m\alpha,\psi}\phi(y) - I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) = \frac{(\psi(y)-\psi(r))^{m\alpha}}{\Gamma(m\alpha+1)} D_r^{m\alpha,\psi}\psi(r). \tag{2.3.5}$$

Proof. By linearity of fractional integral, we have

$$\begin{aligned} I_r^{m\alpha,\psi} D_r^{m\alpha,\psi}\phi(y) - I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) =& I_r^{m\alpha,\psi}[D_r^{m\alpha,\psi}\phi(y) - I_r^{\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y)], \\ =& I_r^{m\alpha,\psi}[D_r^{m\alpha,\psi}\phi(y) - I_r^{\alpha,\psi} D_r^{\alpha,\psi} D_r^{m\alpha}\phi(y)]. \end{aligned}$$

From Theorem (1.3.4), we have

$$I_r^{\alpha,\psi} D_r^{\alpha,\psi}\phi(y) = \phi(y)-\phi(r).$$

So, we get

$$\begin{aligned} I_r^{m\alpha,\psi} D_r^{m\alpha,\psi}\phi(y) -& I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) \\ =& I_r^{m\alpha,\psi}[D_r^{m\alpha,\psi}\phi(y) - D_r^{m\alpha,\psi}\phi(y) + D_r^{m\alpha,\psi}\phi(r)] \\ =& I_r^{m\alpha,\psi}[D_r^{m\alpha,\psi}\phi(r)] \\ =& \frac{1}{\Gamma(m\alpha)}\int_r^y (\psi(y)-\psi(s))^{m\alpha-1}\psi'(s) D^{m\alpha,\psi}\phi(r)ds. \end{aligned}$$

By evaluating the integral on the right hand side,

$$I_r^{m\alpha,\psi} D_r^{m\alpha,\psi}\phi(y) - I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) = \frac{(\psi(y)-\psi(r))^{\alpha}}{\Gamma(m\alpha+1)} D_r^{m\alpha,\psi}\psi(r).$$

□

In the following we generalize the Theorem (2.2.1) for ψ-Caputo fractional derivative.

Theorem 2.3.3. *For $0 < \alpha \leq 1$, suppose that $D_a^{\alpha}\phi \in C[r,u]$, then*

$$\phi(y) = \sum_{k=0}^{m} \frac{(\psi(y)-\psi(r))^{k\alpha}}{\Gamma(k\alpha+1)} D_r^{k\alpha,\psi}\phi(r) + \frac{D_r^{(m+1)\alpha,\psi}\phi(\eta)(\psi(y)-\psi(r))^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)} \tag{2.3.6}$$

with $r \leq \eta \leq y$ and $y \in (r,u]$.

Proof. By theorem (2.3.2), we know that

$$\sum_{i=0}^{m} I_r^{i\alpha,\psi} D_r^{i\alpha,\psi}\phi(y) - I_r^{(i+1)\alpha,\psi} D_r^{(i+1)\alpha,\psi}\phi(y) = \sum_{i=0}^{m} \frac{(\psi(y)-\psi(r))^{\alpha}}{\Gamma(i\alpha+1)} D_r^{i\alpha,\psi}\psi(r) \tag{2.3.7}$$

By using the result of Theorem (1.4.8), we get

$$\phi(y) - I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) = \sum_{i=0}^{m} \frac{(\psi(y)-\psi(r))^{\alpha}}{\Gamma(i\alpha+1)} D_r^{i\alpha,\psi}\psi(r). \tag{2.3.8}$$

Consider,

$$I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) = \frac{1}{\Gamma((m+1)\alpha)} \int_r^y (\psi(y)-\psi(s))^{(m+1)\alpha+1}\psi'(s) D^{(m+1)\alpha,\psi}\phi(s)ds.$$

Using mean value theorem,

$$I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) = \frac{D_r^{(m+1)\alpha,\psi}\phi(\eta)}{\Gamma((m+1)\alpha)} \int_r^y (\psi(y)-\psi(s))^{(m+1)\alpha-1}\psi'(s)ds.$$

Evaluating the integral, we get,

$$I_r^{(m+1)\alpha,\psi} D_r^{(m+1)\alpha,\psi}\phi(y) = \frac{D_r^{(m+1)\alpha,\psi}\phi(\eta)}{\Gamma((m+1)\alpha+1)} (\psi(y)-\psi(s))^{(m+1)\alpha}. \tag{2.3.9}$$

Substituting Eq (2.3.9) in Eq (2.3.8), we get the desired result.

□

Method 2: Here we present an alternate method for the proof of fractional Taylor's theorem.

Lemma 2.3.4. *Assume $\psi \in C[a,b]$ is non-negative increasing function with $\psi(x) \neq 0$, $f \in C^n[a,b]$. Then there exists unique polynomial P_n^{ψ}, in $y = (\psi(x)-\psi(a))^{\alpha}$, $0 < \alpha \leq 1$, of degree at most n for $f(x)$ such that $D_a^{\alpha k,\psi} P_n^{\psi}(a) = D_a^{\alpha k,\psi} f(a)$ for $k = 0,1,2,\ldots,n$.*

Proof. Consider the polynomial of degree at most n in $y = (\psi(x)-\psi(a))^{\alpha}$ of the form $P_n^{\psi}(x) = \sum_{i=0}^{n} a_n(\psi(x)-\psi(a))^{i\alpha}$. Obviously, at $x = a$, we have $a_0 = P_n^{\psi}(a) = f(a)$. Applying $D^{\alpha,\psi}$, we have

$$D_a^{\alpha,\psi} P_n^{\psi}(x) = a_1\Gamma(\alpha+1) + \sum_{i=2}^{n} a_i \frac{\Gamma(i\alpha+1)}{\Gamma(i\alpha-\alpha+1)} (\psi(x)-\psi(a))^{i\alpha-\alpha}. \tag{2.3.10}$$

Substituting $x = a$ in (2.3.10) we have $D_a^{\alpha,\psi} P_n^{\psi}(a) = D_a^{\alpha,\psi} f(a) = \Gamma(\alpha+1)a_1$. Therefor $a_1 = \frac{D_a^{\alpha,\psi} f(a)}{\Gamma(\alpha+1)}$. Now applying D_a^{α} on Eq (2.3.10), we achieve

$$D_a^{2\alpha,\psi} P_n^{\psi}(x) = a_2\Gamma(2\alpha+1) + \sum_{i=3}^{n} a_i \frac{\Gamma(i\alpha+1)}{\Gamma(i\alpha-2\alpha+1)} (\psi(x)-\psi(a))^{i\alpha-2\alpha}. \tag{2.3.11}$$

Repeating the above process k-times, we have

$$D_a^{k\alpha,\psi} P_n^{\psi}(x) = a_k \Gamma(k\alpha+1) + \sum_{i=k+1}^{n} a_i \frac{\Gamma(i\alpha+1)}{\Gamma(i\alpha - k\alpha + 1)} (\psi(x)-\psi(a))^{i\alpha - k\alpha}, \tag{2.3.12}$$

where $D_a^{k\alpha} f(x) = D_a^{\alpha} D_a^{\alpha} D_a^{\alpha} \cdots D_a^{\alpha} f(x)$. Substituting $x = a$ in (2.3.12) we have $D_a^{k\alpha,\psi} P_n^{\psi}(a) = D_a^{k\alpha,\psi} f(a) = \Gamma(k\alpha+1) a_k$. Therefore $a_k = \frac{D_a^{k\alpha,\psi} f(a)}{\Gamma(k\alpha+1)}$.

Hence the unique Taylor's polynomial, in $y = (\psi(x) - \psi(a))$, of degree at most n is

$$P_n^{\psi}(x) = \sum_{i=0}^{n} \frac{D_a^{i\alpha,\psi} f(a)}{\Gamma(i\alpha+1)} (\psi(x)-\psi(a))^{i\alpha}. \tag{2.3.13}$$

□

The applicability of $P_n^{\psi}(x)$ for function approximation indeed depends on error estimate. Just like classical Taylor polynomials, in this case, the estimate is reasonably good in sufficiently small neighbourhood of a provided ψ-derivatives of f are bounded. Following theorem is the generalized case of Theorem 2.2.1 for the fractional case.

Theorem 2.3.5. *For $0 < \alpha \leq 1$, suppose that $D_a^{(n+1)\alpha} f \in C([a,b])$. Then, for $x \in [a,b]$, there exists $\xi \in (a,x)$ such that*

$$f(x) = \sum_{i=0}^{n} \frac{D_a^{i\alpha,\psi} f(a)}{\Gamma(i\alpha+1)} (\psi(x)-\psi(a))^{i\alpha} + D_a^{(n+1)\alpha,\psi} f(\xi) \frac{(\psi(x)-\psi(a))^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}. \tag{2.3.14}$$

where $D_a^{n\alpha} f(x) = D_a^{\alpha} D_a^{\alpha} D_a^{\alpha} \cdots D^{n} (n - times)$.

Proof. Since $R_n^{\psi}(x) = f(x) - P_n^{\psi}(x)$. Note that, by Lemma 2.3.4, we have

$$D_a^{k\alpha,\psi} R_n(a) = D_a^{k\alpha,\psi} f(a) - D_a^{k\alpha,\psi} P_n^{\psi}(a) = 0.$$

Since P_n^{ψ} is a polynomial of degree at most n in $y = (\psi(x) - \psi(a))^{\alpha}$. Therefore $D_a^{(n+1)\alpha,\psi} P_n^{\psi}(x) = 0$ for all $x \in [a,b]$. Hence

$$D_a^{(n+1)\alpha,\psi} R_n(x) = D_a^{(n+1)\alpha,\psi} f(x) - D_a^{(n+1)\alpha,\psi} P_n^{\psi}(x) = D_a^{(n+1)\alpha,\psi} f(x). \tag{2.3.15}$$

Applying $I_a^{(n+1)\alpha,\psi}$ on both sides of Equation (2.3.15)

$$I_a^{(n+1)\alpha,\psi} D_a^{(n+1)\alpha,\psi} R_n(x) = I_a^{(n+1)\alpha,\psi} D_a^{(n+1)\alpha,\psi} f(x) \tag{2.3.16}$$

Now using Theorem 1.3.4 on both sides, we have

$$R_n^{\psi}(x) - \sum_{i=0}^{n} \frac{(\psi(x)-\psi(a))^{i}}{\Gamma(i+1)} D_a^{i,\psi} R_n(x) = \int_a^x \frac{(\psi(x)-\psi(s))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} D_a^{(n+1)\alpha,\psi} f(s) \psi'(s) ds. \tag{2.3.17}$$

Since $D_a^{i,\psi} R_n(x) = 0$ for $i = 1, 2, 3, \ldots, n$ and using mean value theorem for integral calculus, we have

$$R_n^{\psi}(x) = D_a^{(n+1)\alpha,\psi} f(\xi) \int_a^x \frac{(\psi(x)-\psi(s))^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} \psi'(s) ds \tag{2.3.18}$$

$$= D_a^{(n+1)\alpha,\psi} f(\xi) \frac{(\psi(x)-\psi(a))^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}. \tag{2.3.19}$$

□

Using Eq (2.3.13) and Eq (2.3.18) in $R_n^{\psi}(x) = f(x) - P_n^{\psi}(x)$, we get the required result.

Remark 2.3.6. Theorem 2.3.5 is a generalization of the fractional Taylor's theorem for ψ-Caputo fractional derivative introduced in [1]. For $\psi(x) = x$ it reduces to the fractional Taylor's theorem for Caputo fractional derivative. Furthermore, the strategy adopted here to prove the theorem is different form the Zaid M.Odibat et.[20] al approach.

2.4 Approximation of a function

In this section, we approximate the functions by using generalized Taylor's formula with respect to a function (2.3.6). We describe the method of approximation as follows:

Theorem 2.4.1. *[20] For $0 < \alpha \leq 1$, suppose that $D_a^{\alpha,\psi}\phi(y) \in C[a,b]$. If $y \in [a,b]$, then*

$$\phi(y) \cong P_n^{\alpha,\psi}(y) = \sum_{i=0}^{n} \frac{(\psi(y) - \psi(a))^{i\alpha}}{\Gamma(i\alpha + 1)} D_a^{i\alpha,\psi}\phi(a). \tag{2.4.1}$$

Moreover, if there exists ζ with $a \leq \zeta \leq y$, then the error term $R_n^{\alpha}(\psi(y))$ is

$$R_n^{\alpha,\psi}(y) = \frac{D^{(n+1)\alpha,\psi}\phi(\zeta)}{\Gamma((n+1)\alpha + 1)}(\psi(y) - \psi(s))^{(n+1)\alpha}. \tag{2.4.2}$$

If n is very large then accuracy of $P_n^{\alpha}(y)$ increases and if we moves away the value of x from center then it decreases an accuracy. Hence, for the error to not exceed the specific bound, n should be large.

Example 2.4.2. [20] We define Mittag-Leffler function with $\alpha > 0$ as

$$E_\alpha(\psi(y)^\alpha) \cong P_n^{\alpha,\psi}(y) = \sum_{i=0}^{n} \frac{(\psi(y))^{i\alpha}}{\Gamma(i\alpha + 1)}. \tag{2.4.3}$$

And the error term is

$$R_n^{\alpha,\psi}(y) = \frac{E_\alpha(\zeta)}{\Gamma((n+1)\alpha + 1)}(\psi(y) - \psi(s))^{(n+1)\alpha}, \quad a \leq \zeta \leq y. \tag{2.4.4}$$

Table 2.1 shows approximate values of $E_\alpha(\psi(y)^\alpha)$ for different values of y and α, where $N = 10$ and $\psi(y) = \log(y^3 + 1)$.

Example 2.4.3. [20] The wright function with α and β is defined as

$$W(\psi(y);\alpha,\beta) = \sum_{k=0}^{\infty} \frac{(\psi(y))^k}{k!\Gamma(\alpha k + \beta)}. \tag{2.4.5}$$

And the error term is

$$R_n^{\alpha}(\psi(y)) = \frac{(-1)^n W(-\zeta^\alpha, -\alpha, 1)}{\Gamma((n+1)\alpha + 1)}(\psi(y) - \psi(s))^{(n+1)\alpha}, a \leq \zeta \leq y. \tag{2.4.6}$$

y	$\alpha=0.2$	$\alpha=0.4$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=1.0$
0.0	1	1	1	1	1
0.5	2.9634	1.7751	1.3948	1.2189	1.1250
1.0	7.3820	4.2731	3.0152	2.3811	1.9999
1.5	14.2812	10.0606	7.0381	5.3813	4.3749
2.0	21.9231	19.7650	14.6433	11.1718	8.9998
2.5	29.3865	32.9640	26.6548	20.6581	16.6221
3.0	36.4042	48.9563	43.4934	34.6517	27.9806

Table 2.1: Approximations for different values of y and α.

2.5 Fractional power series expansion

The work in this section is generalization of work presented in [20], we introduce a new definition of fractional power series with respect to a function which will be very helpful in demonstrating the general form of generalized Taylor's theorem which contains ψ-Caputo's definition of fractional derivative. Also we discuss some important theorems and results. We will present the results for convergent of the series and the remainder theorem of error bound.

Definition 2.5.1. A ψ-fractional power series expansion about $\psi(y)=\psi(y_0)$ is defined as

$$\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y)-\psi(y_0))^{q+k\alpha}, \quad 0\le p-1<\alpha\le p, y\ge y_0. \tag{2.5.1}$$

where $\psi(y)$ is a function and c_{kq}'s are constants. For a special case, when $\psi(y_0)=0$, then the expansion $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y)-\psi(y_0))^{q+k\alpha}$ is called fractional Maclaurin series. If we write a term corresponding to $k=0$ and $q=0$ in Eq (2.5.1) we have $(\psi(y)-\psi(y_0))^0=1$ even when $\psi(y)=\psi(y_0)$. Now we discuss the convergence and divergence of ψ-fractional power series. The proof of following theorem is similar to the proof in [20].

Theorem 2.5.2. *For the ψ-fractional power series $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}, 0\le p-1<\alpha\le p,\ y\ge 0$, we have*

(1) *if $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$, for $y=a>0$ converges, then it converges for $0\le y<a$,*

(2) *if $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$, for $y=e>0$ diverges, then it diverges for $y>e$.*

Proof. We can write ψ-fractional power series as $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$ as $\sum_{k=0}^{\infty}\delta_n(y)(\psi(y))^{k\alpha},\ y\ge 0$ where $\delta_n(y)=\sum_{q=0}^{p-1} c_{kq}(\psi(y))^q$.
To prove the first part, we assume that $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(a))^{q+k\alpha}$ converges for $a>0$ and $\psi(a)>0$. Then we can say that $\lim_{n\to\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(a))^{q+k\alpha}=\lim_{n\to\infty}\delta_n(a)(\psi(a))^{k\alpha}=0$. According to the defination of limit of function with $\epsilon=M_1/M_2$, where $M_1=min|\ \delta_n(y)\ |: 0\le y\le a$ and $M_2=max|\ \delta_n(y)\ |: 0\le y\le a$, there is a positive integer

N such that $\mid \delta_n(a)(\psi(a))^{k\alpha} \mid< \epsilon$, we have
$|\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}| = |\delta_n(y)(\psi(y))^{k\alpha}| \leq M_2|\psi^{k\alpha}| = |\frac{(\psi(a))^{k\alpha}\delta_n(a)(\psi(y))^{k\alpha}}{(\psi(a))^{k\alpha}\delta_n(a)}| < |\frac{\psi(y)}{\psi(a)}|^{k\alpha}\frac{M_1}{|\delta_n(a)|} \leq |(\frac{\psi(y)}{\psi(a)})^{\alpha}|^k$.
Again, if $0 \leq y < a$, then $|\psi(y)/\psi(a)|^{\alpha} < 1$ and $\sum_{k=N}^{\infty}|\psi(y)/\psi(a)|^{\alpha}$ is called as geometric series and also convergent. So the series $\sum_{k=N}^{\infty}|\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}|$ is convergent by the comparison test. So series converges absolutely.
To prove the second part, we consider that $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(e))^{q+k\alpha}$ is divergent. Now, if we have any function $\psi(y)$ such that $y > e > 0$, then $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$ cannot converge by part (1), the convergence of $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$ depends on the convergence of $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(e))^{q+k\alpha}$. Therefore, $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$ diverges whenever $y > e$. □

Theorem 2.5.3. *For the ψ-fractional power series $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}, 0 \leq p-1 < \alpha \leq p, y \geq 0$, then we have*

(1) *If $y = 0$, only then it is convergent*

(2) *For all $y \geq 0$, it is also convergent series*

(3) *If there exists a positive number R then the series is convergent for $0 \leq y < R$ and divergent if $y > R$, where R is radius of convergence*

Proof. For the proof of first part, consider a series $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$, if $y = 0$ then $\psi(0) = 0$ and $R = 0$, so series converges for $y = 0$.
For the proof of second part, consider a series $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$, $R = \lim_{n\to\infty}|\frac{a_n}{a_{n+1}}| = \frac{c_{kq}}{c_{(k+1)q}}$ as $k \to \infty$, $R \to \infty$. So series converges for $y \geq 0$.
Now, we prove third part as follows: assume that case (1) and (2) are not true. Then there exist numbers a and e which are non-zero such that the series $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$ is convergent for $y = a$ and divergent for $y = e$. Therefore, the set S=$\{$y$|\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$ is convergent$\}$ is non-empty. By Theorem (2.5.2), the series is divergent if $y > e$, so $0 \leq y \leq e$ for all $y \in S$. We can say that e is an upper bound for S. Thus by completeness axiom , the series $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$ converges. □

Theorem 2.5.4. *Suppose that ψ-fractional power series $\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$, $0 \leq p-1 < \alpha \leq p$ has a radius of convergence $R > 0$. If a function ϕ is defined as $\phi(y) = \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y))^{q+k\alpha}$, $0 \leq p-1 < \alpha \leq p$, $0 \leq y < R$, then the following hold:*

$$\begin{aligned}
D_0^{\alpha,\psi}\phi(y) &= c_{10}\Gamma(1+\alpha) + c_{11}\Gamma(2+\alpha)\psi(y) + \cdots + c_{1(p-1)}\frac{\Gamma(p+\alpha)}{\Gamma(p)}(\psi(y))^{p-1} \\
&+c_{20}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(\psi(y))^{\alpha} + c_{21}\frac{\Gamma(2+2\alpha)}{\Gamma(2+\alpha)}(\psi(y))^{1+\alpha} + c_{2(p-1)}\frac{\Gamma(p+2\alpha)}{\Gamma(p+\alpha)}(\psi(y))^{p-1+\alpha} \\
&+\cdots = \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{\Gamma(q+k\alpha+1)}{\Gamma(q+(k-1)\alpha+1)}(\psi(y))^{q+(k-1)\alpha},
\end{aligned} \tag{2.5.2}$$

$$
\begin{aligned}
D_0^{\psi} D_0^{\alpha,\psi}\phi(y) &= c_{11}\Gamma(2+\alpha) + c_{12}\Gamma(3+\alpha)\psi(y) + \cdots + c_{1(p-1)}\frac{\Gamma(p+\alpha)}{\Gamma(p-1)}(\psi(y))^{p-2} + \\
&\quad c_{20}\frac{\Gamma(1+2\alpha)}{\Gamma(\alpha)}(\psi(y))^{\alpha-1} + c_{21}\frac{\Gamma(2+2\alpha)}{\Gamma(1+\alpha)}(\psi(y))^{\alpha} + c_{2(p-1)}\frac{\Gamma(p+2\alpha)}{\Gamma(p+\alpha-1)}(\psi(y))^{p-2+\alpha} + \cdots \\
&= \sum_{q=1}^{p-1} c_{1j}\frac{\Gamma(q+\alpha+1)}{q}(\psi(y))^{q-1} + \sum_{k=2}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{\Gamma(q+k\alpha+1)}{\Gamma(q+(k-1)\alpha+1)}(\psi(y))^{q-1+(k-1)\alpha}.
\end{aligned} \tag{2.5.3}
$$

Proof. From the defination (1.4.9), we know that

$$
\begin{aligned}
{}_cD_a^{\alpha,\psi}\phi(y) =& \frac{1}{\Gamma(p-\alpha)}\int_0^y (\psi(y)-\psi(s))^{p-\alpha-1}\psi'(s) D^{p,\psi}\phi(s)ds \\
=& \frac{1}{\Gamma(p-\alpha)}\int_0^y (\psi(y)-\psi(s))^{p-\alpha-1}\psi'(s)\left(\frac{1}{\psi'(s)}\frac{d^p}{ds^p}\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(s))^{q+k\alpha}\right)ds \\
=& \frac{1}{\Gamma(p-\alpha)}\int_0^y (\psi(y)-\psi(s))^{p-\alpha-1}\psi'(s)\left(\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{1}{\psi'(s)}\frac{d^p}{ds^p}(\psi(s))^{q+k\alpha}\right)ds \\
=& \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{1}{\Gamma(p-\alpha)}\int_0^y (\psi(y)-\psi(s))^{p-\alpha-1}\psi'(s)\left(\frac{1}{\psi'(s)}\frac{d^p}{ds^p}(\psi(s))^{q+k\alpha}\right)ds.
\end{aligned}
$$

By using the substitution $w = \frac{\psi(s)}{\psi(y)}$ and $dw = \frac{\psi'(s)}{\psi(y)}ds$, and solving the above integral, we get,

$$
D_0^{\alpha,\psi}\phi(y) = \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{\Gamma(q+k\alpha+1)}{\Gamma(q+(k-1)\alpha+1)}(\psi(y))^{q+(k-1)\alpha}.
$$

To proceed the proof, we use definition of ordinary derivative

$$
\begin{aligned}
D^{1,\psi} D_0^{\alpha,\psi}\phi(y) =& \frac{1}{\psi'(y)}\frac{d}{dy}\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{\Gamma(q+k\alpha+1)}{\Gamma(q+(k-1)\alpha+1)}(\psi(y))^{q+(k-1)\alpha}, \\
=& \frac{1}{\psi'(y)}\frac{d}{dy}\left(\sum_{q=0}^{p-1} c_{1q}\frac{\Gamma(q+\alpha+1)}{\Gamma(q+1)}(\psi(y))^{q} + \sum_{k=2}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{\Gamma(q+k\alpha+1)}{\Gamma(q+(k-1)\alpha+1)}(\psi(y))^{q+(k-1)\alpha}\right), \\
=& \frac{1}{\psi'(y)}\left(\sum_{q=0}^{p-1} c_{1q}\frac{\Gamma(q+\alpha+1)}{\Gamma(q+1)}\frac{d}{dy}((\psi(y))^{q} + \sum_{k=2}^{\infty}\sum_{q=0}^{p-1} c_{kq}\frac{\Gamma(q+k\alpha+1)}{\Gamma(q+(k-1)\alpha+1)}\frac{d}{dy}(\psi(y))^{q+(k-1)\alpha}\right), \\
=& \sum_{q=1}^{p-1} c_{1q}\frac{\Gamma(q+\alpha+1)}{\Gamma(q)}(\psi(y))^{q-1} + \sum_{k=2}^{\infty}\sum_{q=0}^{k-1} c_{kq}\frac{\Gamma(q+k\alpha+1)}{\Gamma(q+(k-1)\alpha)}(\psi(y))^{q-1+(k-1)\alpha}.
\end{aligned}
$$

So, we get the desired result. □

Theorem 2.5.5. *Let ϕ has a ψ-fractional power series representation at $\psi(y) = \psi(y_0)$ of the form*

$$
\phi(y) = \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} c_{kq}(\psi(y)-\psi(y_0))^{q+k\alpha},\ 0 \le p-1 < \alpha \le p,\ y_0 \le y < y_0 + R. \tag{2.5.4}
$$

If $\phi(y) \in C[y_0, y_0 + R)$, $D^{k\alpha}_{y_0}\phi(y) \in C(y_0, y_0 + R)$, and we can differentiate $D^{k\alpha}_{y_0}\phi(y)$, $(p-1)$-times on $(y_0, y_0 + R)$ for $k = 0, 1, 2, 3...$ where $0 \leq p - 1 < \alpha \leq p$. Then the coefficients c_{kq} are given by formula

$$c_{kq} = \frac{D^{q,\psi}_{y_0} D^{k\alpha,\psi}_{y_0}\phi(y_0)}{\Gamma(q + k\alpha + 1)}, k = 0, 1, 2, \cdots, q = 0, 1, 2, \cdots, p - 1$$

Proof. Assume that ϕ is a function denoted by ψ-fractional power series (2.5.1). We notice that, if we put $\psi(y) = \psi(y_0)$ in Eq (2.5.4), then all terms after the first are vanishing, and we get $\phi(y_0) = c_{00}$. From the result of Eq. (2.5.2), we can write

$$\begin{aligned} D^{\alpha,\psi}_0\phi(y) =& c_{10}\Gamma(1+\alpha) + c_{11}\Gamma(2+\alpha)(\psi(y) - \psi(y_0)) + \cdots + c_{1(p-1)}\frac{\Gamma(p+\alpha)}{\Gamma(p)}(\psi(y) - \psi(y_0))^{p-1} \\ &+ c_{20}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(\psi(y) - \psi(y_0))^{\alpha} + c_{21}\frac{\Gamma(2+2\alpha)}{\Gamma(2+\alpha)}(\psi(y) - \psi(y_0))^{1+\alpha} \\ &+ c_{2(p-1)}\frac{\Gamma(p+2\alpha)}{\Gamma(p+\alpha)}(\psi(y) - \psi(y_0))^{p-1+\alpha} + \cdots . \end{aligned} \tag{2.5.5}$$

The substitution $\psi(y) = \psi(y_0)$ in Eq (2.5.5) gives $c_{10} = \frac{D^{\alpha,\psi}_{y_0}\phi(y_0)}{\Gamma(1+\alpha)}$. Again, from the result of Eq (2.5.3), we can write

$$\begin{aligned} D^{1,\psi} D^{\alpha,\psi}_0\phi(y) = & c_{11}\Gamma(2+\alpha) + c_{12}\Gamma(3+\alpha)(\psi(y) - \psi(y_0)) + \cdots \\ &+ c_{1(p-1)}\frac{\Gamma(p+\alpha)}{\Gamma(p-1)}(\psi(y) - \psi(y_0))^{p-2} + c_{20}\frac{\Gamma(1+2\alpha)}{\Gamma(\alpha)}(\psi(y) - \psi(y_0))^{\alpha-1} \\ &+ c_{21}\frac{\Gamma(2+2\alpha)}{\Gamma(1+\alpha)}(\psi(y) - \psi(y_0))^{\alpha} + c_{2(p-1)}\frac{\Gamma(p+2\alpha)}{\Gamma(p+\alpha-1)}(\psi(y) - \psi(y_0))^{p-2+\alpha} + \cdots . \end{aligned} \tag{2.5.6}$$

Again substitution $\psi(y) = \psi(y_0)$ in Eq. (2.5.6) gives $c_{11} = \frac{D^{\psi}_{y_0} D^{\alpha,\psi}_{y_0} f(y_0)}{\Gamma(2+\alpha)}$. Now applying operator $D^{\psi}_{y_0}$ one more time on Eq (2.5.6), we obtain

$$\begin{aligned} D^{2,\psi}_0 D^{\alpha,\psi}_0\phi(y) =& c_{12}\Gamma(3+\alpha) + \cdots + c_{1(p-1)}\frac{\Gamma(p+\alpha)}{\Gamma(p-2)}(\psi(y) - \psi(y_0))^{p-3} \\ &+ c_{20}\frac{\Gamma(1+2\alpha)}{\Gamma(\alpha-1)}(\psi(y) - \psi(y_0))^{\alpha-2} + c_{21}\frac{\Gamma(2+2\alpha)}{\Gamma(\alpha)}(\psi(y) - \psi(y_0))^{\alpha-1} + \cdots \\ &+ c_{2(p-1)}\frac{\Gamma(p+2\alpha)}{\Gamma(p+\alpha-2)}(\psi(y) - \psi(y_0))^{p-3+\alpha} + \cdots . \end{aligned} \tag{2.5.7}$$

Again using substitution $\psi(y) = \psi(y_0)$ in Eq. (2.5.7) , we get $c_{12} = \frac{D^{2,\psi}_{y_0} D^{\alpha,\psi}_{y_0}\phi(y_0)}{\Gamma(3+\alpha)}$. As we can see that successive pattern is developed. If we pursue to operate D_{y_0}, qth times with $q \leq p-1$ and then use the substitution $\psi(y) = \psi(y_0)$. We can get the first qth coefficient as $c_{1q} = \frac{D^{q,\psi}_{y_0} D^{\alpha,\psi}_{y_0}\phi(y_0)}{\Gamma(q+\alpha+1)}$, $q = 0, 1, 2, \cdots, p-1$. Now, we develop the successive pattern form for the second qth coefficient c_{2q}. For this purpose, we apply the operator $D_{y_0}^{2\alpha,\psi}$, we get the following result

$$\begin{aligned} D_0^{2\alpha,\psi}\phi(y) =& c_{20}\Gamma(1+2\alpha) + c_{21}\Gamma(2+2\alpha)(\psi(y)-\psi(y_0)) + \cdots \\ &+ c_{2(p-1)}\frac{\Gamma(p+2\alpha)}{\Gamma(p)}(\psi(y)-\psi(y_0))^{p-1+\alpha} + \cdots . \end{aligned} \tag{2.5.8}$$

By substituting $\psi(y) = \psi(y_0)$ in Eq. (2.5.8) , we get $c_{20} = \frac{D_{y_0}^{2\alpha,\psi}\phi(y_0)}{\Gamma(1+2\alpha)}$. Again applying the operator D_{y_0} on Eq (2.5.8) and then substituting $\psi(y) = \psi(y_0)$ in resulting formula ,we will obtain $c_{21} = \frac{D_{y_0}^{2\alpha,\psi}\phi(y_0)}{\Gamma(2+2\alpha)}$. Anyway, if we pursue to operate D_{y_0} qth-times with $q \le p-1$ and then substitute $\psi(y) = \psi(y_0)$, we can have second qth coefficient as $c_{2q} = \frac{D_{y_0}^{q,\psi}D_{y_0}^{2\alpha,\psi}\phi(y_0)}{\Gamma(q+2\alpha+1)}, q = 0,1,2,\cdots,p-1$ As a result, we develop general successive pattern form completely. If we apply $D^{q,\psi}D_{y_0}^{k\alpha}$ Eq (2.5.4) we can find the qth coefficient of c_{kq} which is given by the form of $c_{kq} = \frac{D_{y_0}^{q,\psi}D_{y_0}^{k\alpha,\psi}\phi(y_0)}{\Gamma(q+k\alpha+1)}, k = 0,1,2,\cdots q = 0,1,2,\cdots,p-1$. Substituting in series representation Eq. (2.5.4) then we can see that ϕ has ψ-fractional power series expansion at $\psi(y) = \psi(y_0)$, then it's discretized form will be: $\phi(y) = \sum_{k=0}^{\infty}\sum_{q=0}^{p-1}\frac{D_{y_0}^{q,\psi}D_{y_0}^{k\alpha,\psi}\phi(y_0)}{\Gamma(q+k\alpha+1)}(\psi(y)-\psi(y_0))^{q+k\alpha}, \quad 0 \le p-1 < \alpha \le p, \quad y_0 \le y < y_0 + R.$ □

Lemma 2.5.6. *Suppose that $\phi(y) \in C[y_0, y_0+R)$, $D_{y_0}^{k\alpha}\phi(y) \in C(y_0, y_0+R)$, and we can differentiate $D_{y_0}^{k\alpha}\phi(y)$ for $(p-1)$-times on (y_0, y_0+R) for $k = 0,1,2,3,\cdots,m+1$ where $0 \le p-1 < \alpha \le p$. Then*
$I_{y_0}^{(m+1)\alpha,\psi}D_{y_0}^{(m+1)\alpha,\psi}\phi(y) = \phi(y) - \sum_{k=0}^{m}\sum_{q=0}^{p-1}\frac{D_{y_0}^{q,\psi}D_{y_0}^{k\alpha,\psi}\phi(y_0)}{\Gamma(q+k\alpha+1)}(\psi(y)-\psi(y_0))^{q+k\alpha}, \quad 0 \le p-1 < \alpha \le p, \quad y_0 \le y < y_0 + R.$

Proof. By using the properties of operator, we have

$$\begin{aligned} I_{y_0}^{(m+1)\alpha,\psi}D_{y_0}^{(m+1)\alpha,\psi}\phi(y) &= I_{y_0}^{m\alpha,\psi}\left(\left(I_{y_0}^{\alpha,\psi}D_{y_0}^{\alpha,\psi}\right)D_{y_0}^{m\alpha,\psi}\phi(y)\right) = I_{y_0}^{m\alpha,\psi}\left(\left(I_{y_0}^{p,\psi}D_{y_0}^{p,\psi}\right)D_{y_0}^{m\alpha,\psi}\phi(y)\right) \\ &= I_{y_0}^{m\alpha,\psi}\left(D_{y_0}^{m\alpha,\psi}\phi(y) - \sum_{q=0}^{p-1}\frac{D^{j,\psi}D_{y_0}^{m\alpha,\psi}\phi(y_0)}{q!}(\psi(y)-\psi(y_0))^q\right) \\ &= I_{y_0}^{m\alpha,\psi}D_{y_0}^{m\alpha,\psi}\phi(y) - I_{y_0}^{m\alpha,\psi}\left(\sum_{q=0}^{p-1}\frac{D^{q,\psi}D_{y_0}^{m\alpha,\psi}\phi(y_0)}{q!}(\psi(y)-\psi(y_0))^q\right) \\ &= I_{y_0}^{(m-1)\alpha,\psi}\left(\left(I_{y_0}^{p,\psi}D_{y_0}^{p,\psi}\right)D_{y_0}^{(m-1)\alpha,\psi}\phi(y)\right) - \left(\sum_{q=0}^{p-1}\frac{D^{q,\psi}D_{y_0}^{m\alpha,\psi}\phi(y_0)}{\Gamma(q+m\alpha+1)}(\psi(y)-\psi(y_0))^{q+m\alpha}\right) \\ &= I_{y_0}^{(m-1)\alpha,\psi}\left(D_{y_0}^{(m-1)\alpha,\psi}\phi(y) - \sum_{q=0}^{p-1}\frac{D^{q,\psi}D_{y_0}^{(m-1)\alpha,\psi}\phi(y_0)}{q!}(\psi(y)-\psi(y_0))^q\right) \\ &- \left(\sum_{q=0}^{p-1}\frac{D^{q,\psi}D_{y_0}^{m\alpha,\psi}\phi(y_0)}{\Gamma(q+m\alpha+1)}(\psi(y)-\psi(y_0))^{q+m\alpha}\right). \end{aligned}$$

By repeating this process for m-times, we can get the desired result. □

Theorem 2.5.7. *If* $\mid D_{y_0}^{(n+1)\alpha}\phi(y) \mid\leq M$ *on* $y_0 \leq y \leq e$, *where* $n-1<\alpha\leq n$, *then the remainder* $R_n(y)$ *satisfies the inequality of the general form of generalized Taylor's series with respect to a function*

$$\mid R_n(y) \mid\leq \frac{M}{\Gamma((n+1)\alpha+1)}(\psi(y)-\psi(y_0))^{(n+1)\alpha}, \quad y_0 \leq y \leq e$$

Proof. From theorem (2.3.3), we know that

$$R_n(y) = \frac{D_a^{(n+1)\alpha,\psi}\phi(y)(\psi(y)-\psi(a))^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}$$

And given that $\mid D_{y_0}^{(n+1)\alpha}\phi(y) \mid\leq M$, So from the defination of upper bound we can say that

$$\mid R_n(y) \mid\leq \frac{M}{\Gamma((n+1)\alpha+1)}(\psi(y)-\psi(y_0))^{(n+1)\alpha}, \quad y_0 \leq y \leq e.$$

□

2.6 Series solutions by fractional Taylor series

In this section, we use generalized Taylor's formula with respect to a function to find series solution. This is a very useful method and can be used to find series solutions of many ψ-fractional differential equations with non-constant coefficient.

Example 2.6.1. Consider an initial value problem

$$D_0^{\alpha,\psi}\phi(y) = \lambda\phi(y), \quad \phi(0)=\phi_0, \quad 0<\alpha\leq 1, \quad \lambda\in\mathbb{R}. \tag{2.6.1}$$

Let the solution is of the form

$$\phi(y) = \sum_{k=0}^{\infty} c_k \frac{(\psi(y))^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{2.6.2}$$

By the definition (1.4.9), we achieve

$$D_0^{\alpha,\psi}\phi(y) = \sum_{k=1}^{\infty} c_k \frac{(\psi(y))^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)}. \tag{2.6.3}$$

Substituting (2.6.2) and (2.6.3) into (2.6.1) yields

$$\sum_{k=0}^{\infty} c_{k+1}\frac{(\psi(y))^{k\alpha}}{\Gamma(k\alpha+1)} - \lambda\sum_{k=0}^{\infty} c_k\frac{(\psi(y))^{k\alpha}}{\Gamma(k\alpha+1)} = 0.$$

Equating the coefficients of $(\psi(y))^{k\alpha}$ to zero, we get

$$c_{k+1} = \lambda c_k, \quad (c_0 = \phi_0),$$

this yields:

$$c_k = \lambda^k \phi_0.$$

Substituting value of c_k in Eq (2.6.2), we obtain the solution:

$$\begin{aligned}\phi(y) &= \phi_0 \sum_{k=0}^{\infty} \lambda^k \frac{(\psi(y))^{k\alpha}}{\Gamma(k\alpha+1)},\\ &= \phi_0 E_\alpha(\lambda \psi(y)^\alpha),\end{aligned}$$

where $E_\alpha(\psi(y))$ is a Mittag-Leffler function.

Example 2.6.2. Consider an initial value problem

$$D_0^{2\alpha,\psi}\phi(y) + D_0^{\alpha,\psi}\phi(y) - 2\phi(y) = 0. \tag{2.6.4}$$

Let the solution is of the form

$$\phi(y) = \sum_{k=0}^{\infty} c_k \frac{(\psi(y))^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{2.6.5}$$

By the definition (1.4.9), we obtain

$$D_0^{\alpha,\psi}\phi(y) = \sum_{k=1}^{\infty} c_k \frac{(\psi(y))^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)}. \tag{2.6.6}$$

and

$$D_0^{2\alpha,\psi}\phi(y) = \sum_{k=2}^{\infty} c_k \frac{(\psi(y))^{(k-2)\alpha}}{\Gamma((k-2)\alpha+1)}. \tag{2.6.7}$$

Substituting all these values in (2.6.4) yields

$$\sum_{k=0}^{\infty} (c_{k+2} + c_{k+1} - c_k) \frac{(\psi(y))^{k\alpha}}{\Gamma(k\alpha+1)} = 0.$$

Equating coefficients of $\psi(y)^{k\alpha}$ to zero and identifying coefficients , we obtain

$$c_{k+2} = 2c_k - c_{k+1}. \tag{2.6.8}$$

Therefore, we obtain following solutions

$$\begin{aligned}\phi_1(y) =& c_0 \Bigg(1 + \frac{2}{\Gamma(2\alpha+1)}(\psi(y))^{2\alpha} - \frac{2}{\Gamma(3\alpha+1)}(\psi(y))^{3\alpha} + \frac{6}{\Gamma(4\alpha+1)}(\psi(y))^{4\alpha}\\ &- \frac{10}{\Gamma(5\alpha+1)}(\psi(y))^{5\alpha} + \cdots\Bigg),\\ \phi_2(y) =& c_1 \Bigg(\frac{1}{\Gamma(\alpha+1)}(\psi(y))^{\alpha} - \frac{1}{\Gamma(2\alpha+1)}(\psi(y))^{2\alpha} + \frac{3}{\Gamma(3\alpha+1)}(\psi(y))^{3\alpha}\\ &- \frac{5}{\Gamma(4\alpha+1)}(\psi(y))^{4\alpha} + \cdots\Bigg).\end{aligned}$$

A family of solutions of 2.6.4 is given by

$$\phi(y) = c_0\phi_1(y) + c_1\phi_2(y), \;\; c_0, c_1 \in R.$$

2.7 Series solutions by fractional power series.

In this section, we use the ψ-fractional power series to solve two linear fractional differential equations with nonhomogeneous initial conditions at ordinary derivatives.

Example 2.7.1. Consider the following homogeneous FDE:

$$D_{y_0}^{\alpha}\phi(y) = \lambda\phi(y), \quad p-1 < \alpha \leq p, \quad y \geq y_0 \tag{2.7.1}$$

subject to the nonhomogeneous initial equations

$$\phi^{(j)}(y_0) = \delta_j, \quad j = 0, 1, 2, 3, \cdots, p-1. \tag{2.7.2}$$

where δ_j and λ are real constants.

Assume that solution has a form,

$$\phi(y) = \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} \frac{c_{kq}}{\Gamma(q+k\alpha+1)}(\psi(y)-\psi(y_0))^{q+k\alpha}, \ 0 \leq p-1 < \alpha \leq p, \ y_0 \leq y < y_0 + R. \tag{2.7.3}$$

where $c_{kq} = D_{y_0}^{q} D_{y_0}^{k\alpha}\phi(y_0), k = 0, 1, 2, 3, \cdots, q = 0, 1, 2, 3, \cdots, p-1$. From Eq. (2.5.2), one can obtain

$$\begin{aligned} D_{y_0}^{\alpha}\phi(y) &= \sum_{k=1}^{\infty}\sum_{q=0}^{p-1} \frac{c_{kq}}{\Gamma(q+(k-1)\alpha+1)}(\psi(y)-\psi(y_0))^{q+(k-1)\alpha} \\ &= \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} \frac{c_{(k+1)q}}{\Gamma(q+k\alpha+1)}(\psi(y)-\psi(y_0))^{q+k\alpha}. \end{aligned} \tag{2.7.4}$$

By using the expansion formulas of Eq. (2.7.3) and (2.7.4) in both sides of Eq(2.7.1), we have

$$\sum_{k=0}^{\infty}\sum_{q=0}^{p-1} \frac{c_{(k+1)q}}{\Gamma(q+k\alpha+1)}(\psi(y)-\psi(y_0))^{q+k\alpha} = \lambda \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} \frac{c_{kq}}{\Gamma(q+k\alpha+1)}(\psi(y)-\psi(y_0))^{q+k\alpha}.$$

Equating the coefficients of $(\psi(y)-\psi(y_0))^{q+k\alpha}$ in above equation will lead us to:

$$c_{(k+1)q} = \lambda c_{(kq)}, k = 0, 1, 2, 3, \cdots, q = 0, 1, 2, 3, \cdots, p-1.$$

Using the initial conditions of Eq (2.7.2), we can get $c_{0q} = \delta_q, q = 0, 1, 2, 3, ..., p-1$. Thus, the recurrence relation can be written as

$$c_{kq} = \lambda^k \delta_q, \qquad k = 0, 1, 2, \cdots \qquad q = 0, 1, 2, \cdots, p-1.$$

Now, put value of c_{kq} in Eq (2.7.3) to formulate the solution of Eq (2.7.1) and (2.7.2) in the form

$$\phi(y) = \sum_{k=0}^{\infty}\sum_{q=0}^{p-1} \frac{\lambda^k \delta_q}{\Gamma(q+k\alpha+1)}(\psi(y)-\psi(y_0))^{q+k\alpha}$$

Remark 2.7.2. If we consider a special case $0 < \alpha \leq 1$, then series solution of Eq (2.7.1) and (2.7.2) will be

$$\phi(y) = \delta_0 \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\alpha+1)} (\psi(y) - \psi(y_0))^{k\alpha}, \ y \geq y_0.$$

In terms of Mittag-Leffler function

$$\phi(y) = \delta_0 E_\alpha(\lambda(\psi(y) - \psi(y_0))^\alpha).$$

Example 2.7.3. Consider the following homogeneous fractional differential equation:

$$D_0^\alpha \phi(y) = t\phi'(y), \qquad 1 < \alpha \leq 2, \qquad y \geq 0, \tag{2.7.5}$$

with nonhomogeneous initial conditions

$$\phi(0) = \delta_0, \qquad \phi'(0) = \delta_1, \tag{2.7.6}$$

where δ_0 and δ_1 are real finite constants. We assume the solutions of Eq (2.7.5) and (2.7.6) is of the form

$$\phi(y) = \sum_{n=0}^{\infty} \sum_{j=0}^{1} \frac{c_{nj}}{\Gamma(j+n\alpha+1)} (\psi(y))^{j+n\alpha}. \tag{2.7.7}$$

And the ordinary and fracional derivatives can be obtained as follows,

$$\phi'(t) = c_{01} + \sum_{k=1}^{\infty} \sum_{q=0}^{1} \frac{c_{kq}}{\Gamma(q+k\alpha)} (\psi(y))^{q-1+k\alpha}, \tag{2.7.8}$$

$$D_{y_0}^\alpha \phi(y) = \sum_{k=1}^{\infty} \sum_{q=0}^{1} \frac{c_{kq}}{\Gamma(q+(k-1)\alpha+1)} (\psi(y))^{q+(k-1)\alpha} = \sum_{k=0}^{\infty} \sum_{q=0}^{1} \frac{c_{(k+1)q}}{\Gamma(q+k\alpha+1)} (\psi(y))^{q+k\alpha}. \tag{2.7.9}$$

Substituting above values in Eq. (2.7.5), yields that

$$\begin{aligned} (a_1 + b_1 y) + \sum_{k=1}^{\infty} \left(\frac{a_{k+1}}{\Gamma(1+k\alpha)} (\psi(y))^{k\alpha} + \frac{b_{k+1}}{\Gamma(2+k\alpha)} (\psi(y))^{1+k\alpha} \right) \\ = b_0 y + \sum_{k=1}^{\infty} \left(\frac{a_k}{\Gamma(k\alpha)} (\psi(y))^{k\alpha} + \frac{b_k}{\Gamma(1+k\alpha)} (\psi(y))^{1+k\alpha} \right) \end{aligned} \tag{2.7.10}$$

where $a_k = c_{k0}$ and $a_k = c_{k1}$. By equating coefficients of $(\psi(y))^{k\alpha}$ and $(\psi(y))^{1+k\alpha}$ in both sides of above equation, we get recursively the following results: $a_0 = \phi(0) = \delta_0$, $b_0 = \phi'(0) = \delta_1.a_1 = 0$, $b_1 = \delta_1$,and $a_{k+1} = (k\alpha)a_k$, $b_{k+1} = (1+k\alpha)b_k, k = 1, 2, \cdots$. That is, $a_0 = \delta_0$, $b_0 = \delta_1$, $a_k = 0, k = 1, 2, \cdots$, and $b_k = \delta_1 \prod_{q=1}^{k}(1+(q-1)\alpha)b_k, \quad k = 1, 2, \cdots$. If we substitute these values back in Eq (2.7.7), the we have

$$\phi(y) = \delta_0 + \delta_1 t + \delta_1 \sum_{k=0}^{\infty} \left(\prod_{q=1}^{k} (1+(q-1)\alpha) \right) \frac{1}{\Gamma(2+k\alpha)} (\psi(y))^{1+k\alpha}.$$

Chapter 3

Taylor-operational matrices method

In this chapter, Taylor series method is developed for approximations. In this method, the solution is approximated by ψ-Taylor basis vector. Moreover, Taylor operational matrix of integration for ψ-fractional differential equation is provided. We use Taylor basis operational matrix of ψ-fractional integration for solving ψ-fractional ordinary and partial differential equations which reduces the ψ-fractional partial differential equation to a special type of algebraic equation called as Sylvester equation.

3.1 Generalization of Taylor series method for ψ-fractional differential equations

Since the motivation is to find numerical solutions. For this purpose, we use Taylor's method. The ψ-Taylor basis vector is given by

$$T_n^{\psi}(t) = [1, \psi(t), (\psi(t))^2, \cdots, (\psi(t))^n]^T,$$

where n is a positive integer. The ψ-fractional integration $I_a^{\alpha,\psi}$ of a Taylor vector is

$$\begin{aligned} &I_0^{\alpha,\psi} T_n^{\psi}(t) \\ &= \left[\frac{\Gamma(1)}{\Gamma(\alpha+1)}(\psi(t))^{\alpha}, \frac{\Gamma(2)}{\Gamma(\alpha+2)}(\psi(t))^{1+\alpha}, \frac{\Gamma(3)}{\Gamma(\alpha+3)}(\psi(t))^{2+\alpha}, \cdots, \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}(\psi(t))^{n+\alpha}\right]^T. \end{aligned} \tag{3.1.1}$$

We can write Eq (3.1.1) as

$$I_0^{\alpha,\psi} T_n^{\psi}(t) = [a_0(\psi(t))^{\alpha}, a_1(\psi(t))^{1+\alpha}, a_2(\psi(t))^{2+\alpha}, ..., a_n(\psi(t))^{n+\alpha}]^T, \tag{3.1.2}$$

where $a_n := \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}(\psi(t))^{n+\alpha}, n = 0, 1, 2, \cdots, n$. We can write Eq (3.1.2) in the matrix form:

$$I_0^{\alpha,\psi} T_n^{\psi}(t) = \begin{bmatrix} a_0(\psi(t))^{\alpha} \\ a_1(\psi(t))^{1+\alpha} \\ a_2(\psi(t))^{2+\alpha} \\ \vdots \\ a_n(\psi(t))^{n+\alpha} \end{bmatrix} = \begin{bmatrix} a_0 l_0(t) \\ a_1 l_1(t) \\ a_2 l_2(t) \\ \vdots \\ a_n l_n(t) \end{bmatrix}, \tag{3.1.3}$$

where $l_n(t) := (\psi(t))^{n+\alpha}$. Now, rewrite matrix in Eq (3.1.3) as

$$I_0^{\alpha,\psi} T_n^{\psi}(t) = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} l_0(t) \\ l_1(t) \\ l_2(t) \\ \vdots \\ l_n(t) \end{bmatrix}. \tag{3.1.4}$$

We can write Eq (3.1.4) as

$$I_0^{\alpha,\psi} T_n^{\psi}(t) = AL(t), \tag{3.1.5}$$

where $A = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$ and $L(t) = \begin{bmatrix} l_0(t) \\ l_1(t) \\ l_2(t) \\ \vdots \\ l_n(t) \end{bmatrix}$.

Now, we have to calculate $L(t)$. For this purpose, we approximate $l_n(t)$ as:

$$l_n(t) = \sum_{j=0}^{m} w_{nj} T_j^{\psi}(t),$$

where $T_j^{\psi}(t) = \psi(t)^j$. Now,

$$\begin{bmatrix} l_0(t) \\ l_1(t) \\ l_2(t) \\ \vdots \\ l_n(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{n} w_{0j} T_j^{\psi}(t) \\ \sum_{j=0}^{n} w_{1j} T_j^{\psi}(t) \\ \sum_{j=0}^{n} w_{2j} T_j^{\psi}(t) \\ \vdots \\ \sum_{j=0}^{n} w_{mj} T_j^{\psi}(t) \end{bmatrix} = \begin{bmatrix} w_{00} T_0^{\psi} + w_{01} T_1^{\psi} + \cdots + w_{0n} T_n^{\psi} \\ w_{10} T_0^{\psi} + w_{11} T_1^{\psi} + \cdots + w_{1n} T_n^{\psi} \\ w_{20} T_0^{\psi} + w_{21} T_1^{\psi} + \cdots + w_{2n} T_n^{\psi} \\ \vdots \\ w_{n0} T_0^{\psi} + w_{n1} T_1^{\psi} + \cdots + w_{nn} T_{mn}^{\psi} \end{bmatrix}.$$

Rewrite the above matrix, we have

$$\begin{bmatrix} l_0(t) \\ l_1(t) \\ l_2(t) \\ \vdots \\ l_n(t) \end{bmatrix} = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ w_{20} & w_{21} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n0} & w_{n1} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} T_0^{\psi}(t) \\ T_1^{\psi}(t) \\ T_2^{\psi}(t) \\ \vdots \\ T_n^{\psi}(t) \end{bmatrix} \tag{3.1.6}$$

We write this as

$$L(t) = WT^{\psi}(t), \tag{3.1.7}$$

where $L(t) = \begin{bmatrix} l_0(t) \\ l_1(t) \\ l_2(t) \\ \vdots \\ l_n(t) \end{bmatrix}$, $W = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0n} \\ w_{10} & w_{11} & \cdots & w_{1n} \\ w_{20} & w_{21} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n0} & w_{n1} & \cdots & w_{nn} \end{bmatrix}$ and $T^{\psi}(t) = \begin{bmatrix} T_0^{\psi}(t) \\ T_1^{\psi}(t) \\ T_2^{\psi}(t) \\ \vdots \\ T_n^{\psi}(t) \end{bmatrix}$, where W is called a vandermonde matrix. Now using Eq (3.1.7) in Eq (3.1.5), we get

$$I_0^{\alpha,\psi} T_n^{\psi}(t) = AWT^{\psi}(t) = M^{\alpha,\psi} T^{\psi}(t). \tag{3.1.8}$$

The matrix $M^{\alpha,\psi}$ is called a ψ-Taylor basis operational matrix of fractional integration.

3.1.1 Development of the method

The generalized ψ-Taylor series method is developed to find the approximations of ψ-fractional order differential equations. In this section, we develop the main idea of using ψ-Taylor basis matrix to solve ψ-fractional differential equations. For this purpose, we consider

$$D_0^{\alpha,\psi}\phi(t) + \lambda_1 D_0^{\beta,\psi}\phi(t) + \lambda_2\phi(t) = f(t) \tag{3.1.9}$$

with initial conditions $\phi(0) = \phi_0$ and $D^{1,\psi}\phi(0) = \phi_1$. Approximating the highest order derivative term by ψ-Taylor polynomial as

$$D_0^{\alpha,\psi}\phi(t) \cong C^T T^{\psi}(t). \tag{3.1.10}$$

Apply $I_0^{\alpha,\psi}$ on (3.1.10) and using (1.4.8), we have

$$\phi(t) - \phi(0) - D^{1,\psi}\phi(0)(\psi(t) - \psi(0)) = C^T I_0^{\alpha,\psi} T^{\psi}(t). \tag{3.1.11}$$

Using initial conditions

$$\phi(t) \cong C^T M^{\alpha,\psi} T^{\psi}(t) + \phi_0 + \phi_1(\psi(t) - \psi(0)), \tag{3.1.12}$$

where $M^{\alpha,\psi}$ is an integration matrix. Now applying $D_0^{\beta,\psi}$ on Eq (3.1.12)

$$D_0^{\beta,\psi}\phi(t) = C^T D_0^{\beta,\psi} M^{\alpha,\psi} T^{\psi}(t) + \phi_1 D_0^{\beta,\psi}(\psi(t) - \psi(0)). \tag{3.1.13}$$

Note that

$$\begin{aligned} D_0^{\beta,\psi} M^{\alpha,\psi} T^{\psi}(t) =& D_0^{\beta,\psi} I_0^{\alpha,\psi} T^{\psi}(t) \\ =& D_0^{\beta,\psi} I_0^{\beta,\psi} (I_0^{\alpha-\beta,\psi}) T^{\psi}(t). \end{aligned}$$

Thus

$$D_0^{\beta,\psi} M^{\alpha,\psi} T^{\psi}(t) = I_0^{\alpha-\beta,\psi} T^{\psi}(t). \tag{3.1.14}$$

Also

$$D_0^{\beta,\psi}(\psi(t) - \psi(0)) = \frac{1}{\Gamma(2-\beta)}(\psi(t) - \psi(0))^{1-\beta}. \tag{3.1.15}$$

Now using Eq (3.1.14) and (3.1.15) in Eq (3.1.13), we get

$$D_0^{\beta,\psi}\phi(t) = C^T I_0^{\alpha-\beta,\psi} T^\psi(t) + \frac{\phi_1}{\Gamma(2-\beta)}(\psi(t)-\psi(0))^{1-\beta}. \tag{3.1.16}$$

Substituting equations (3.1.10), (3.1.12), (3.1.16) in (3.1.9) we get

$$\begin{aligned} C^T T^\psi(t) &+ \lambda_1 \left(C^T I_0^{\alpha-\beta,\psi} T^\psi(t) + \frac{\phi_1}{\Gamma(2-\beta)}(\psi(t)-\psi(0))^{1-\beta} \right) \\ &+ \lambda_2 (C^T M^{\alpha,\psi} T^\psi(t) + \phi_0 + \phi_1(\psi(t)-\psi(0)) = f(t) \end{aligned}$$

$$\begin{aligned} C^T T^\psi(t) &+ \lambda_1 C^T I_0^{\alpha-\beta,\psi} T^\psi(t) + \lambda_2 C^T I_0^{\alpha,\psi} T^\psi(t) \\ &= f(t) - \lambda_1 \frac{\phi_1}{\Gamma(2-\beta)}(\psi(t)-\psi(0))^{1-\beta} - \lambda_2(\phi_0 + \phi_1(\psi(t)) - \psi(0)). \end{aligned}$$

We can write above equation as

$$C^T(I + \lambda_1 C^T I_0^{\alpha-\beta,\psi} + \lambda_2 C^T I^{\alpha,\psi}) T^\psi(t) = G(t), \tag{3.1.17}$$

where

$$G(t) = f(t) - \lambda_1 \frac{\phi_1}{\Gamma(2-\beta)}(\psi(t) - \psi(0)^{1-\beta}) - \lambda_2(\phi_0 + \phi_1(\psi(t)) - \psi(0)).$$

Thus, in matrix notation, we have

$$C^T Q = G, \tag{3.1.18}$$

where $Q = I + \lambda_1 C^T I^{\alpha-\beta,\psi} + \lambda_2 C^T I^{\alpha,\psi}$. and $G(t) = f(t) - \lambda_1 \frac{\phi_1}{\Gamma(2-\beta)}(\psi(t)-\psi(0))^{1-\beta} - \lambda_2(\phi_0 + \phi_1(\psi(t)) - \psi(0))$. Solving Eq (3.1.18) for $C^T$and using it in Eq (3.1.12) gives a numerical solution of Eq (3.1.9).

3.2 Error Analysis

We present error analysis and convergence analysis for the proposed numerical method for solving ψ-fractional differential equations. For this purpose, we state the following results.

Theorem 3.2.1. *Let $\phi_0(t)$ be the best approximation of ϕ and $\phi \in C^{m+1}[0,1]$ then*

$$\|\phi - \phi_0\| \le \frac{M(\psi(1))^{2m+3}}{(m+1)!} \sqrt{\frac{1}{2m+3}}, \tag{3.2.1}$$

where $M = \sup_{t\in[0,1]} \left\| D_0^{(m+1),\psi} \phi \right\|$.

Proof. Since $\phi(t) \in C^{m+1}[0,1]$. By Taylor's Theorem 2.2.1,

$$\phi(t) = \phi_0(t) + R_m(t).$$

where $\phi_0(t) = \sum_{m=0}^{n} \frac{(\psi(t))^m}{\Gamma(m+1)}$ and $R_m(t) = \frac{D_a^{(m+1),\psi}\phi(\eta)(\psi(t))^{m+1}}{\Gamma(m+2)}$.
Since $M = \sup\limits_{t\in[0,1]} \left\| D^{(m+1),\psi}\phi \right\|$, we have

$$|\phi(t) - \phi_0(t)| \leq \frac{M}{(m+1)!}(\psi(t))^{m+1}.$$

By using the defination of norm, we have

$$\|\phi(t) - \phi_0(t)\|^2 \leq \frac{M^2}{(m+1)!^2}\frac{(\psi(1))^{2m+3}}{2m+3}. \tag{3.2.2}$$

Taking square root on both sides, we get the desired result.

□

3.3 Numerical Illustrations

In this segment, we present the implementation of newly developed numerical method based on the operational matrix of fractional integration for ψ-Taylor series method. In order to show the effectiveness of the method, four numerical illustrations are considered. In the first example, we present the numerical approximation of a fractional integral of a function.

Example 3.3.1. (Fractional Integration) Consider the function

$$\phi(t) = (\psi(t))^{\beta+1} - (\psi(t))^{\beta}, \tag{3.3.1}$$

where $t \in [0,1]$ and $\psi(t) = \sqrt{t}$. The exact integral of Eq (3.3.1) is

$$I_0^{\alpha,\psi}\phi(t) = \frac{\Gamma(\beta+2)}{\Gamma(\beta+\alpha+2)}(\psi(t))^{\beta+\alpha+1} - \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(\psi(t))^{\beta+\alpha}. \tag{3.3.2}$$

For the numerical evaluation of fractional integral, we approximate ϕ as

$$\phi(t) \cong \sum_{i=0}^{m} c_i T^{\psi}(t). \tag{3.3.3}$$

Applying fractional integration on Eq (3.3.3), we get

$$I_0^{\alpha,\psi}\phi(t) \cong C^T I^{\alpha,\psi} T^{\psi}(t). \tag{3.3.4}$$

From Eq (3.3.4)

$$I_0^{\alpha,\psi}\phi(t) \cong C^T M^{\alpha,\psi} T^{\psi}(t),$$

where $M^{\alpha,\psi}$ is an integration matrix.
Eq (3.3.4) in vector notation can be written as $\Phi(t) = C^T T^{\psi}(t)$. Thus the numerical approximation of fractional integral is

$$I_0^{\alpha,\psi}\phi(t) \cong \Phi(T^{\psi}(t))^{-1} M^{\alpha,\psi} T^{\psi}(t). \tag{3.3.5}$$

m	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 2.0$
4	0	0.0168	0.0157	0.0120
6	0	7.8142e-04	5.0177e-04	5.5597e-16
8	0	3.1123e-04	1.2698e-04	6.8820e-14
10	0	1.6122e-04	5.8177e-05	1.6517e-12

Table 3.1: Absolute errors.

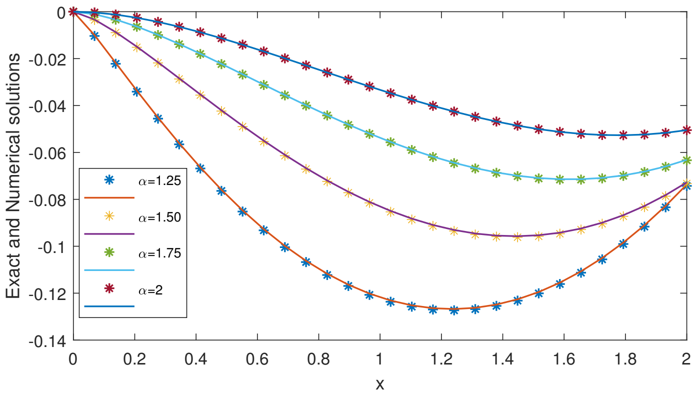


Figure 3.1: Comparison between exact and numerical values of fractional integral.

Absolute errors for different values of α and m are shown in Table 3.1. We observe that error decreases as we decrease the value of m. Exact and numerical values of fractional integral for function given in (3.3.1) are shown in Figure 3.1.

In the above example, we compared the exact and numerical values of fractional integral by using a Taylor series method. Now, we consider the fractional differential equations to solve by Taylor series method and also compare it with other methods available in the literature.

Example 3.3.2. Consider the fractional differential equation

$$D_0^{\alpha,\psi}\phi(t) - \phi(t) = f(t), \quad 1 < \alpha < 2, \quad t \in [0,2] \tag{3.3.6}$$

with $\phi(0) = 0$ and $D^{1,\psi}\phi(0) = 0$.
The exact solution of Eq (3.3.6) is $\phi(t) = ((\psi(t))^{\beta+1} - 1)(\psi(t) - 2)(\psi(t) - 3)$, where $\psi(t) = \sqrt{t} + t$. Applying fractional integral on Eq (3.3.6), we get

$$\phi(t) = I_0^{\alpha,\psi}\phi(t) + I_0^{\alpha,\psi}f(t). \tag{3.3.7}$$

For numerical evaluation, let approximate ϕ is

$$\phi(t) \cong \sum_{i=0}^{m} c_i T^{\psi}(t). \tag{3.3.8}$$

We can write this in vector form as:

$$\phi(t) \cong C^T T^{\psi}(t). \tag{3.3.9}$$

Now, using Eq (3.3.9) in Eq (3.3.7), we get

$$C^T T^\psi(t) = I_0^{\alpha,\psi} C^T T(t) + I_0^{\alpha,\psi} f(t) \tag{3.3.10}$$

$$C^T T^\psi(t) = C^T M^{\alpha,\psi} T^\psi(t) + G(t),$$

where $M^{\alpha,\psi}$ is an integration matrix and $G(t) := I_0^{\alpha,\psi} f(t)$.
In order to determine C, we can write above equation as

$$C^T(I - M^{\alpha,\psi})T^\psi = G$$

which can be written as

$$C^T Q = G, \tag{3.3.11}$$

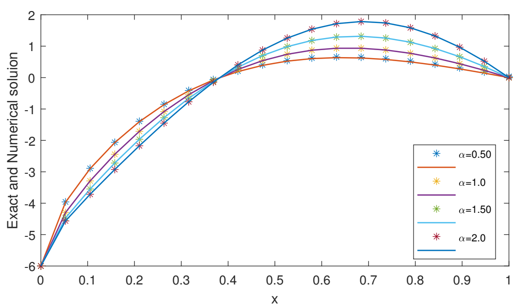
where $Q := (I - M^{\alpha,\psi})T^\psi$. Substituting value of C from Eq (3.3.11) in Eq (3.3.9), gives the solution

$$\phi(t) \cong G Q^{-1} T^\psi(t).$$

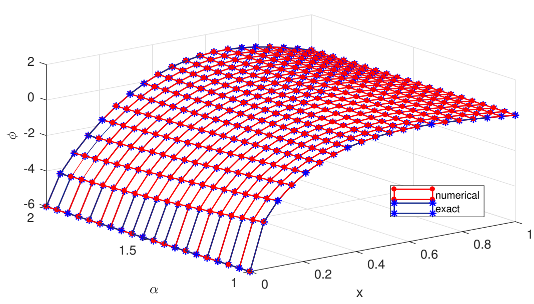
The numerical implementation for $m = 5, 6, 7, 8$ are shown by their absolute errors in Table 3.2. As we can see that error decreases by increases m. Since the exact solution in this case is known, we compare solutions for different choices of α which are depicted in Figure 3.2.

m	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 2.0$
5	0	2.2204e-14	0.1405	0.1436
6	0	3.0198e-14	0.0187	9.0594e-14
7	0	1.7319e-13	0.0048	3.5837e-13
8	0	2.3110e-12	7.0266e-04	1.7625e-12

Table 3.2: : Errors for different values of α and m



(a) Solution for different α.



(b) Solution for $1 < \alpha \le 2$.

Figure 3.2: Comparison of exact and numerical solution.

Example 3.3.3. Consider the fractional differential equation

$$D_0^{\alpha,\psi}\phi(t) + \lambda^2 D_0^{\beta,\psi}\phi(t) = \psi(t), \quad 1 < \alpha < 2, \quad 0 < \beta < 1, \quad t \in [0,2]. \tag{3.3.12}$$

with initial conditions $\phi(0) = 0$ and $D^{1,\psi}\phi(0) = 1$.
When $\alpha = 2$ and $\beta = 0$, the exact solution of Eq (3.3.12) is

$$\phi(t) = \left(\frac{1}{\lambda} - \frac{1}{\lambda^3}\right)\sin(\lambda\psi(t)) + \frac{1}{\lambda^2}\psi(t), \quad \lambda > 0. \tag{3.3.13}$$

When $\alpha = 2$ and $\beta = 1$, the exact solution of Eq (3.3.12) is

$$\phi(t) = \frac{1}{2\lambda^6}\left(-2\lambda^2\psi(t) - 2e^{-\lambda^2\psi(t)} + \lambda^4(-2e^{-\lambda^2\psi(t)} + (\psi(t))^2 + 2) + 2\right), \quad \lambda > 0 \tag{3.3.14}$$

where $\psi(t) = \log(x^2 + x + 1)$. Applying fractional integral on Eq (3.3.12), we get

$$\phi(t) + \lambda^2 I_0^{\alpha-\beta,\psi}\phi(t) = I_0^{\alpha,\psi}\psi(t). \tag{3.3.15}$$

For numerical evaluation, let approximate ϕ as

$$\phi(t) \cong \sum_{i=0}^{m} c_i T^{\psi}(t). \tag{3.3.16}$$

We can write this in vector form as:

$$\phi(t) \cong C^T T^{\psi}(t). \tag{3.3.17}$$

Now, using Eq (3.3.17) in Eq (3.3.15), we get

$$C^T T^{\psi}(t) + \lambda^2 I_0^{\alpha-\beta,\psi} C^T T(t) = I_0^{\alpha,\psi}\psi(t), \tag{3.3.18}$$

$$C^T T^{\psi}(t) + C^T M^{\alpha,\psi} T^{\psi}(t) = G(t).$$

where $M^{\alpha,\psi}$ is an integration matrix and $G(t) := I_0^{\alpha,\psi}\psi(t) + .$
In order to determine C, we can write above equation as

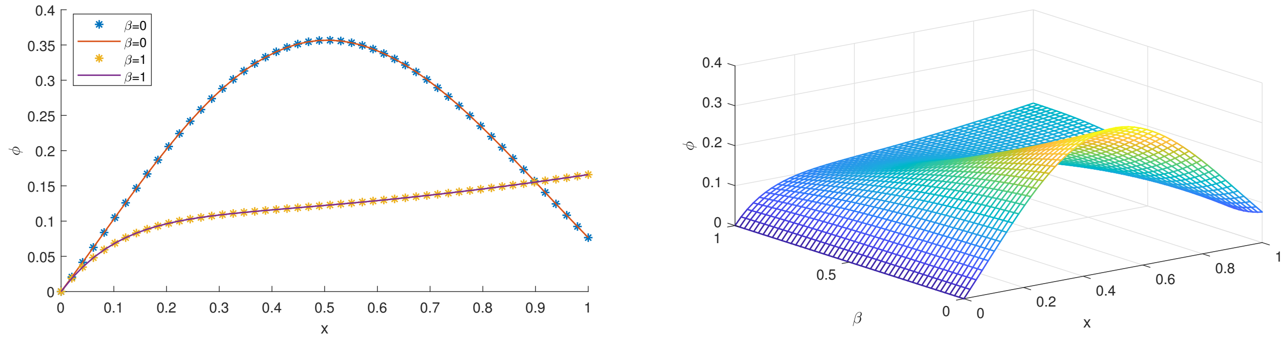
$$C^T(I + M^{\alpha,\psi})T^{\psi} = G$$

which can be written as

$$C^T Q = G, \tag{3.3.19}$$

where $Q = (I + M^{\alpha,\psi})T^{\psi}$. Using value of C from Eq (3.3.19) in Eq (3.3.17) gives us the numerical solution.

As we have ψ-fractional differential equation (3.3.12) whose exact solution is known only in integer case. In Figure 3.3a, we have ploted the exact and numerical soltion for $\alpha = 2$, $\beta = 0$ and $\beta = 1$. Figure 3.3b shows the exact and numerical solution for $0 \le \beta \le 1$ and $1 \le \alpha \le 2$. We observe that in the integer case($\alpha = 2$, $\beta = 0$ and $\beta = 1$), the exact and numerical solutions are close to each other. Although the exact solution is not availabe for fractional case, but from the behaviour of solution for integer case we predict that method works well for fractional case.



(a) Exact and numerical solution for $\beta = 0$ and $\beta = 1$. (b) Numerical solution for $1 < \alpha \leq 2$ and $0 < \beta \leq 1$.

Figure 3.3: Comparison of exact and numerical solution.

Example 3.3.4. Consider the fractional differential equation

$$D_0^{\alpha,\psi}\phi(t) + \lambda_1 D_0^{\beta,\psi}\phi(t) + \lambda_2\phi(t) = 5(\psi(t))^4 - 7(\psi(t))^2,$$

where $\phi(0) = 0$ and $D^{1,\psi}\phi(0) = 1$. And $\lambda_1 = \lambda_2 = 1$.
When $\alpha = 2$ and $\beta = 0$, the exact solution of Eq (3.1.9) is

$$\phi(t) = \frac{1}{2}\left(5(\psi(t))^4 - 37(\psi(t))^2 + \sqrt{2}\sin(\sqrt{2}\psi(t)) - 37\cos(\sqrt{2}\psi(t)) + 37\right). \qquad (3.3.20)$$

When $\alpha = 2$ and $\beta = 1$, the exact solution of Eq (3.3.12) is

$$\begin{aligned}\phi(t) =& 5(\psi(t))^4 - 20(\psi(t))^3 - 7(\psi(t))^2 + 134\psi(t) - 146e^{\frac{-\psi(t)}{2}}\frac{\sin(\frac{\sqrt{3}}{2}\psi(t))}{\sqrt{3}} \\ &+120e^{\frac{-\psi(t)}{2}}\cos(\frac{\sqrt{3}}{2}\psi(t)) - 120,\end{aligned} \qquad (3.3.21)$$

where $\psi(t) = t(t+1)$. First we approximate ϕ as:

$$D_0^{\alpha}\phi(t) \cong C^T T^{\psi}(t). \qquad (3.3.22)$$

Applying the result of (1.4.8) on (3.3.22) with initial conditions, we have

$$\phi(t) \cong C^T I_0^{\alpha,\psi} T^{\psi}(t) + I_0^{\alpha,\psi} f(t). \qquad (3.3.23)$$

Now $D^{\beta,\psi}\phi(t)$ can be evaluated as:

$$D_0^{\beta,\psi}\phi(t) \cong C^T I_0^{\alpha-\beta,\psi} T^{\psi}(t) + D_0^{\beta,\psi} t. \qquad (3.3.24)$$

Substituting equations (3.3.22), (3.3.23), (3.3.24) in (3.1.9), we get

$$C^T T^{\psi}(t) + \lambda_1(C^T I_0^{\alpha-\beta,\psi} T^{\psi}(t) + D_0^{\beta,\psi} t) + \lambda_2(C^T I_0^{\alpha,\psi} T^{\psi}(t) + I_0^{\alpha,\psi} f(t)) = f(t) \qquad (3.3.25)$$

$$C^T T^{\psi}(t) + \lambda_1 C^T I_0^{\alpha-\beta,\psi} T^{\psi}(t) + \lambda_2 C^T I_0^{\alpha,\psi} T^{\psi}(t) = f(t) - \lambda_1 D_0^{\beta,\psi} t - \lambda_2 I_0^{\alpha,\psi} f(t). \qquad (3.3.26)$$

In order to determine C^T, we can write above equation as

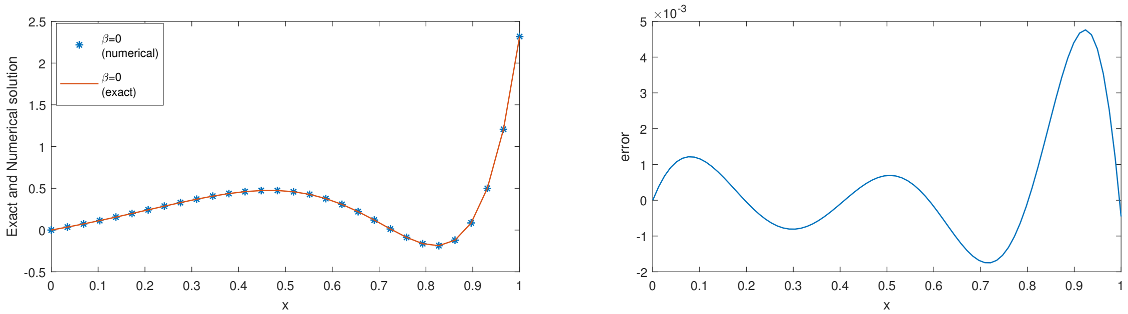
$$C^T(I + \lambda_1 C^T I_0^{\alpha-\beta,\psi} + \lambda_2 C^T I_0^{\alpha,\psi})T^{\psi}(t) = G(t), \tag{3.3.27}$$

where

$$G(t) = f(t) - \lambda_1 D_0^{\beta,\psi} t - \lambda_2 I_0^{\alpha,\psi} f(t).$$

$$C^T Q = G \tag{3.3.28}$$

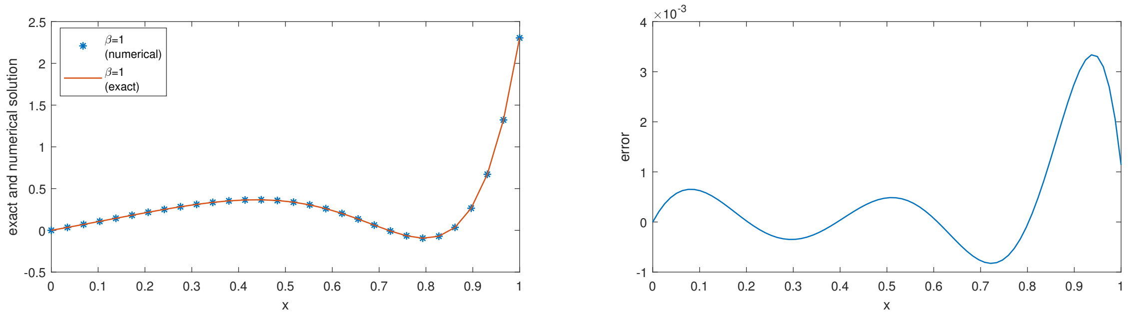
where $Q = I + \lambda_1 C^T I_0^{\alpha-\beta,\psi} + \lambda_2 C^T I_0^{\alpha,\psi}$. and $G(t) = f(t) - \lambda_1 D_0^{\beta,\psi} t - \lambda_2 I_0^{\alpha,\psi} f(t)$. Using Eq (3.3.28) in Eq (3.3.23) gives numerical solution.
In this case, exact solution is not available for the fractional case, so we first find the solution in the integer case for $\alpha = 2, \beta = 0, \beta = 1$. In Figure 3.4a, the graph shows the exact and numerical solution for $\beta = 0$ and Figure 3.4b, shows an absolute error. In Figure 3.5a, the graph shows the exact and numerical solution for $\beta = 1$ and Figure 3.5b shows an absolute error. Figures 3.6 predicts the numerical solution for fractional case.



(a) Exact and numerical solution for $\beta = 0$.

(b) Absolute error for $\beta = 0$.

Figure 3.4: Comparison of solutions, and absolute error for $\beta = 0$ and $n = 15$.



(a) Exact and numerical solution for $\beta = 1$.

(b) Absolute error for $\beta = 1$.

Figure 3.5: Comparison of solutions, and absolute error for $\beta = 1$ and $n = 15$

Example 3.3.5. Consider the linear fractional differential equation [24]

$$D_0^{\alpha,\psi}\phi(t) = (\psi(t))^2 + \frac{2}{\Gamma(3-\alpha)}(\psi(t))^{2-\alpha} - \phi(t), \quad 0 < \alpha \leq 1. \tag{3.3.29}$$

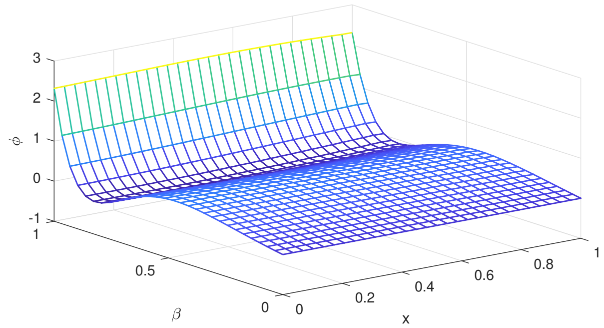


Figure 3.6: Numerical solution for $0 \leq x \leq 1$ and $0 \leq \beta \leq 1$.

with initial conditions $\phi(0) = 0$. The exact solution of Eq (3.3.29) is $\phi(t) = (\psi(t))^2$. First, we apply fractional integral $I_0^{\alpha,\psi}$ on Eq (3.3.29), we have

$$\phi(t) + I_0^{\alpha,\psi}\phi(t) = (\psi(t))^2 + \frac{2}{\Gamma(3+\alpha)}(\psi(t))^2. \tag{3.3.30}$$

Let approximate ϕ as

$$\phi(t) \cong C^T T^{\psi}(t). \tag{3.3.31}$$

Now, using Eq (3.3.31) in Eq (3.3.29), we achieve

$$C^T T^{\psi}(t) + I_0^{\alpha,\psi} C^T T^{\psi}(t) = (\psi(t))^2 + \frac{2}{\Gamma(3+\alpha)}(\psi(t))^2 \tag{3.3.32}$$

$$C^T T^{\psi}(t) + C^T M^{\alpha,\psi} T^{\psi}(t) = G(t), \tag{3.3.33}$$

where $M^{\alpha,\psi}$ is an integration matrix and $G(t) := (\psi(t))^2 + \frac{2}{\Gamma(3+\alpha)}(\psi(t))^2$. To determine C, we write Eq (3.3.33) as

$$C^T(I + M^{\alpha,\psi})T^{\psi}(t) = G(t),$$

which can be written as

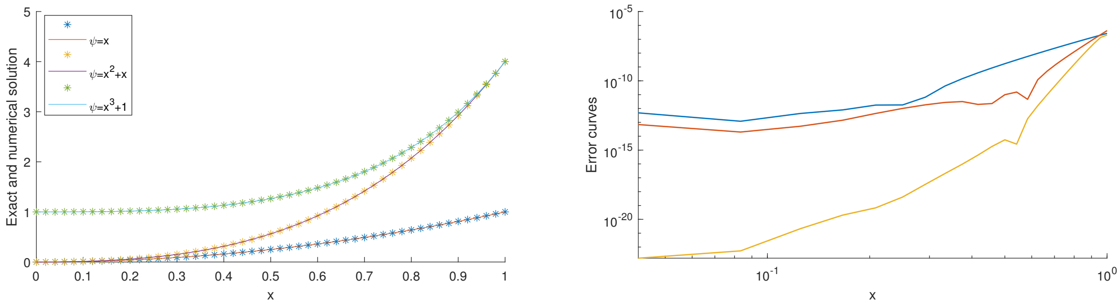
$$C^T Q = G, \tag{3.3.34}$$

where $Q := (I + M^{\alpha,\psi})T^{\psi}(t)$. Substituting value of C from Eq (3.3.34) in Eq (3.3.31), gives the solution.

We have compared the numerical solution obtained by our method with the method in [24]. We show the accuracy of the method by comparing maximum absolute errors in Table 3.3. We observe that absolute error decreases by increases m. Moreover, it is observed that our method gives the better approximation. In Figure 3.7a, we can see that the numerical solution has as good accuracy for $\psi(x) = x$ as method in [24]. Also the graphs for $\psi(x) = x^2 + x$ and $\psi(x) = x^3 + 1$ are shown. Figure 3.7b shows the error curves for $\psi(x) = x$, $\psi(x) = x^2 + x$ and $\psi(x) = x^3 + 1$.

m	Method in [24]	Presented method
6	6.4141e-4	2.7311e-12
8	5.6212e-4	1.5831e-12
10	4.4967e-4	1.1628e-12
12	4.1567e-4	2.9199e-13
14	4.0986e-4	2.6686e-13

Table 3.3: Maximum absolute errors for different choices of m.



(a) Exact and numerical solution for different ψ.

(b) Error curves for different ψ.

Figure 3.7: Exact and numerical solution, and error curves.

3.4 Taylor series method for ψ- fractional partial differntial equations

In this section, our aim is to approximate the numerical solutions by using the ψ-Taylor series method. Main objective is to approximate numerically and compare it with exact solution. And also discuss an absolute error. We use ψ-Taylor basis operational matrix of ψ-fractional integration derived in section (3.1).

3.4.1 Development of a method

The implementation of the method for ψ-fractional partial differential equations is the same as ψ-fractional ordinary differential equations. The method reduces the ψ-fractional partial differential equation to a special type of system of algebraic equation called as Sylvester equation.

Numerical Illustrations

We apply the newly developed Taylor series method to some ψ-fractional partial differential equations with given initial conditions. To outline the fundamental thought of this technique, we solve some examples for ψ-fractional partial differential equations. These examples are discussed to test their validity and relevance.

Example 3.4.1. Consider fractional partial differential equation

$$D_y^{\mu_1,\psi}\phi(y,t) + D_t^{\mu_2,\psi}\phi(y,t) = f(y,t), \quad 0 < \mu_1,\ \mu_2 < 1 \tag{3.4.1}$$

with initial conditions $\phi(y,0) = (\psi(y))^{\mu_1+1} - (\psi(y))^{\mu_1}$ and $\phi(0,t) = (\psi(t))^{\mu_1+1} - (\psi(t))^{\mu_1}$, and $f(y,t) = \Gamma(\mu_1+2)\psi(y) - \Gamma(\mu_1+1) + \Gamma(\mu_2+2)\psi(t) - \Gamma(\mu_2+1)$. The exact solution is $\phi(y,t) = (\psi(y))^{\mu_1+1} - (\psi(y))^{\mu_1} + (\psi(t))^{\mu_2+1} - (\psi(t))^{\mu_2}$, where $\psi(y) = y^3 + y$ and $\psi(t) = t^3 + t$. For the numerical evaluation, we approximate $D_t^{\mu_2,\psi}\phi(y,t)$ as

$$D_t^{\mu_2,\psi}\phi(y,t) \cong \sum_{i=0}^{m}\sum_{j=0}^{m} c_{ij} T_i^{\psi}(y) * T_j^{\psi}(t).$$

This can be written as

$$D_t^{\mu_2,\psi}\phi(y,t) \cong T^{T,\psi}(y) C T^{\psi}(t). \tag{3.4.2}$$

Applying fractional integral $I_t^{\mu_2,\psi}$ on both sides of Eq (3.4.2), we obtain

$$I_t^{\mu_2,\psi} D_t^{\mu_2,\psi}\phi(y,t) \cong T^{T,\psi}(y) C I_t^{\mu_2,\psi} T^{\psi}(t).$$

Using the result of Eq: (1.4.8),

$$\phi(y,t) - \phi(y,0) \cong T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t).$$

Using initial conditions, we have

$$\phi(y,t) \cong T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t) + (\psi(y))^{\mu_1+1} - (\psi(y))^{\mu_1}, \tag{3.4.3}$$

where $M^{\mu_2,\psi}$ is an integration matrix. Now using Eq (3.4.2) in Eq (3.4.1),

$$D_y^{\mu_1,\psi}\phi(y,t) + T^{T,\psi}(y) C T^{\psi}(t) = \Gamma(\mu_1+2)\psi(y) - \Gamma(\mu_1+1) + \Gamma(\mu_2+2)\psi(t) - \Gamma(\mu_2+1). \tag{3.4.4}$$

Applying the fractional integral $I_y^{\mu_1,\psi}$ on both sides of Eq (3.4.4)

$$\begin{aligned} I_y^{\mu_1,\psi} D_y^{\mu_1,\psi}\phi(y,t) + &I_y^{\mu_1,\psi} T^{T,\psi}(y) C T^{\psi}(t) \\ &= I_y^{\mu_1,\psi}(\Gamma(\mu_1+2)\psi(y) - \Gamma(\mu_1+1) + \Gamma(\mu_2+2)\psi(t) - \Gamma(\mu_2+1)). \end{aligned} \tag{3.4.5}$$

Using the result of Eq (1.4.8)

$$\begin{aligned} \phi(y,t) - &\phi(0,t) + T^{T,\psi}(y) M^{T\mu_1,\psi} C T^{\psi}(t) \\ &= \frac{\Gamma(\mu_2+2)}{\Gamma(\mu_1+1)}\psi(t)(\psi(y))^{\mu_1} - \frac{\Gamma(\mu_2+1)}{\Gamma(\mu_1+1)}(\psi(y))^{\mu_1} + (\psi(y))^{\mu_2+1} - (\psi(y))^{\mu_2}. \end{aligned} \tag{3.4.6}$$

Now, using Eq (3.4.3) in Eq (3.4.6)

$$\begin{aligned} T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t) + &T^{T,\psi}(y) M^{T\mu_1,\psi} C T^{\psi}(t) \\ &= \frac{\Gamma(\mu_2+2)}{\Gamma(\mu_1+1)}\psi(t)(\psi(y))^{\mu_1} - \frac{\Gamma(\mu_2+1)}{\Gamma(\mu_1+1)}(\psi(y))^{\mu_1} + (\psi(t))^{\mu_2+1} - (\psi(t))^{\mu_2}, \end{aligned}$$

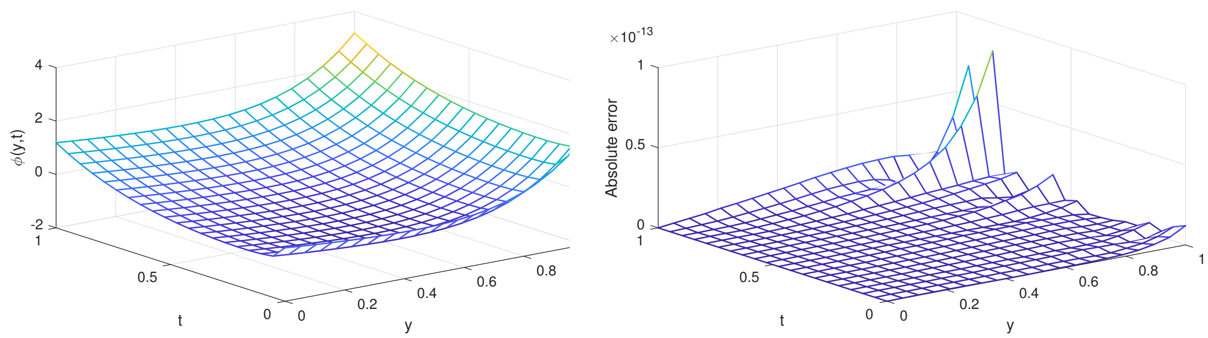
where $M^{T\mu_1,\psi}$ is an integration matrix.

$$T^{T,\psi}(y)CM^{\mu_2,\psi}T^{\psi}(t) + T^{T,\psi}(y)M^{T\mu_1,\psi}CT^{\psi}(t) = G(y,t), \tag{3.4.7}$$

where $G(y,t) = \frac{\Gamma(\mu_2+2)}{\Gamma(\mu_1+1)}\psi(t)(\psi(y))^{\mu_1} - \frac{\Gamma(\mu_2+1)}{\Gamma(\mu_1+1)}(\psi(y))^{\mu_1} + (\psi(t))^{\mu_2+1} - (\psi(t))^{\mu_2}$.
We obtain

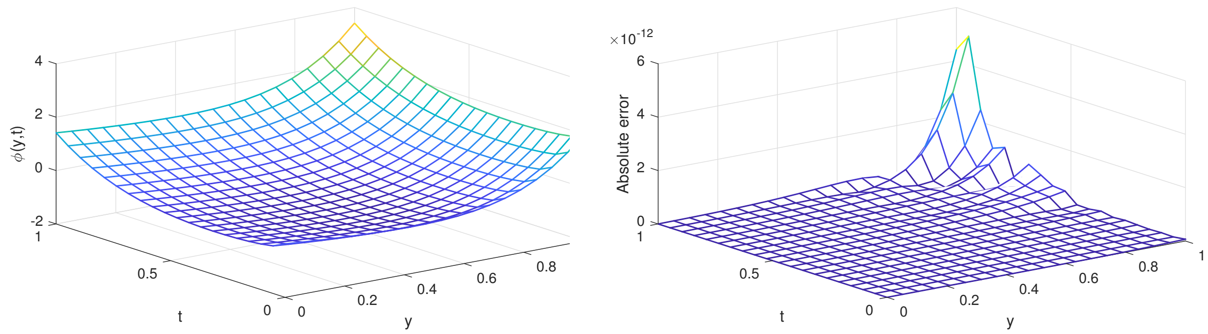
$$CM^{\mu_2,\psi} + M^{T\mu_1,\psi}C = (T^{T,\psi}(y))^{-1}G(y,t)(T^{\psi}(t))^{-1}. \tag{3.4.8}$$

When discretized, Eq (3.4.8) is a well known Sylvester equation with unknown vector C, which can be solved using MATLAB's built-in program.
In this case, the exact solution is known. So, Figure 3.8a shows the exact and numerical solutions for $m = 5$, $\mu_1 = 0.5$ and $\mu_2 = 1$. Figure 3.8b shows an absolute error. Figure 3.9a shows the numerical results for $m = 8$, $\mu_1 = 0.5$ and $\mu_2 = 1$. Figure 3.9b shows an absolute error. We observed that by increasing m, absolute error decreases. Fact is, we can reduce the absolute error by increasing number of points m.



(a) Solutions for $\mu_1 = 0.5$ and $\mu_2 = 1$. (b) Absolute error for $\mu_1 = 0.5$ and $\mu_2 = 1$.

Figure 3.8: Exact and numerical solutions for $m = 5$, $\mu_1 = 0.5$ and $\mu_2 = 1$



(a) Solutions for $\mu_1 = 0.25$ and $\mu_2 = 1$. (b) Absolute error for $\mu_1 = 0.25$ and $\mu_2 = 1$..

Figure 3.9: Exact and numerical solution for $m = 8$, $\mu_1 = 0.25$ and $\mu_2 = 1$

Example 3.4.2. Consider fractional partial differential equation

$$D_y^{\mu_1,\psi}\phi(y,t) + D_t^{\mu_2,\psi}\phi(y,t) + \phi(y,t) = f(y,t), \quad 0 < \mu_1, \mu_2 < 1, \tag{3.4.9}$$

with initial conditions $u(y,0) = (\psi(y))^{\mu_1+2}$ and $u(0,t) = (\psi(t))^{\mu_1+1}$, and $f(y,t) = \frac{\Gamma(\mu_1+3)}{2}(\psi(y))^2 + \Gamma(\mu_2+2)(\psi(t)) + (\psi(y))^{\mu_1+2} + (\psi(t))^{\mu_2+1}$. The exact solution is $\phi(y,t) = (\psi(y))^{\mu_1+2} + (\psi(t))^{\mu_2+1}$, where $\psi(y) = y^2 + y$ and $\psi(t) = t^2 + t$. For the numerical evaluation, we approximate the the term $D_t^{\mu_2,\psi}\phi(y,t)$ as

$$D_t^{\mu_2,\psi}\phi(y,t) \cong \sum_{i=0}^{m}\sum_{j=0}^{m} c_{ij} T_i^{\psi}(y) * T_j^{\psi}(t).$$

This can be written as

$$D_t^{\mu_2,\psi}\phi(y,t) \cong T^{T,\psi}(y) C T^{\psi}(t). \tag{3.4.10}$$

Applying fractional integral $I_t^{\mu_2,\psi}$ on both sides of Eq (3.4.10), we obtain

$$I_t^{\mu_2,\psi} D_t^{\mu_2,\psi}\phi(y,t) \cong T^{T,\psi}(y) C I_t^{\mu_2,\psi} T^{\psi}(t).$$

Using the result of Eq (1.4.8),

$$\phi(y,t) - \phi(y,0) \cong T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t).$$

Applying the initial conditions, we get

$$\phi(y,t) \cong T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t) + (\psi(y))^{\mu_1+2}, \tag{3.4.11}$$

where $M^{\mu_2,\psi}$ is an integration matrix. Now using Eq (3.4.10) and Eq (3.4.11) in Eq (3.4.9)

$$\begin{aligned} D_y^{\mu_1,\psi}\phi(y,t) + T^{T,\psi}(y) C T^{\psi}(t) &+ T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t) + (\psi(y))^{\mu_1+2} \\ &= \frac{\Gamma(\mu_1+3)}{2}(\psi(y))^2 + \Gamma(\mu_2+2)\psi(t) + (\psi(y))^{\mu_1+2} + (\psi(t))^{\mu_2+1}. \end{aligned} \tag{3.4.12}$$

Applying the fractional integral $I_y^{\mu_1,\psi}$ on both sides of Eq (3.4.12)

$$\begin{aligned} I_y^{\mu_1,\psi} D_y^{\mu_1,\psi}\phi(y,t) &+ I_y^{\mu_1,\psi} T^{T,\psi}(y) C T^{\psi}(t) + I_y^{\mu_1,\psi}(T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t) + (\psi(y))^{\mu_1+2}) \\ &= I_y^{\mu_1,\psi}\left(\frac{\Gamma(\mu_1+3)}{2}(\psi(y))^2 + \Gamma(\mu_2+2)\psi(t) + (\psi(y))^{\mu_1+2} + (\psi(t))^{\mu_2+1}\right). \end{aligned} \tag{3.4.13}$$

Using the result of Eq (1.4.8)

$$\begin{aligned} \phi(y,t) - \phi(0,t) &+ T^{T,\psi}(y) M^{T\mu_1,\psi} C T^{\psi}(t) + T^{T,\psi}(y) M_y^{T\mu_1,\psi} C M^{\mu_2,\psi} T^{\psi}(t) + I_y^{\mu_1,\psi}(\psi(y))^{\mu_1+2} \\ &= \frac{\Gamma(\mu_2+2)}{\Gamma(\mu_1+1)}\psi(t)(\psi(y))^{\mu_1} - \frac{\Gamma(\mu_2+1)}{\Gamma(\mu_1+1)}(\psi(y))^{\mu_1} + (\psi(y))^{\mu_2+1} - (\psi(y))^{\mu_2}, \end{aligned}$$

$$\begin{aligned} T^{T,\psi}(y) C M^{\mu_2,\psi} T^{\psi}(t) &+ T^{T,\psi}(y) M^{T\mu_1,\psi} C T^{\psi}(t) + T^{T,\psi}(y) M^{T\mu_1,\psi} C M^{\mu_2,\psi} T^{\psi}(t) \\ &= \frac{\Gamma(\mu_2+2)}{\Gamma(\mu_1+1)}\psi(t)(\psi(y))^{\mu_1} + \frac{1}{\Gamma(\mu_1+1)}(\psi(y))^{\mu_1} + (\psi(y))^{\mu_2+1} + (\psi(y))^{\mu_2+1}, \end{aligned} \tag{3.4.14}$$

where $M^{T\mu_1,\psi}$ is an integration matrix. Now, we get

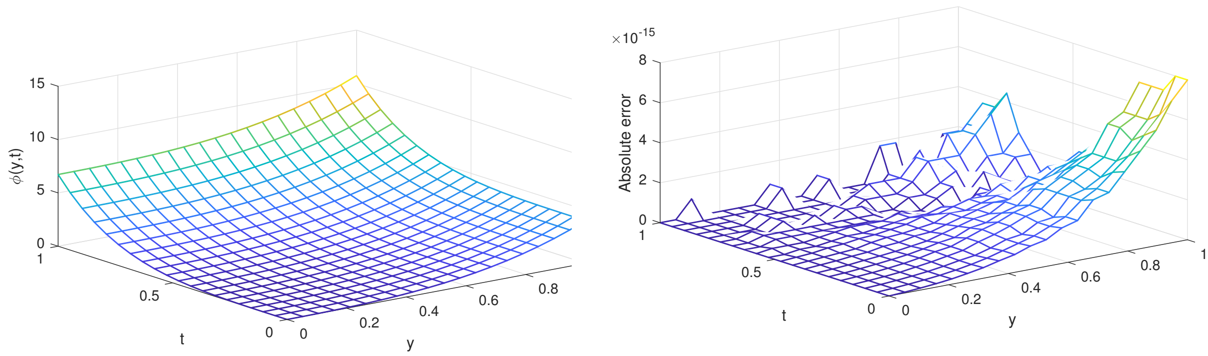
$$T^{T,\psi}(y)CM^{\mu_2,\psi}T^{\psi}(t) + T^{T,\psi}(y)M^{T\mu_1,\psi}CT^{\psi}(t) + T^{T,\psi}(y)M^{T\mu_1,\psi}CM^{\mu_2,\psi}T^{\psi}(t) = G(y,t), \tag{3.4.15}$$

where $G(y,t) = \frac{\Gamma(\mu_2+2)}{\Gamma(\mu_1+1)}\psi(t)(\psi(y))^{\mu_1} + \frac{1}{\Gamma(\mu_1+1)}(\psi(y))^{\mu_1} + (\psi(t))^{\mu_2+1} + (\psi(t))^{\mu_2+1}$.

We obtain,

$$\begin{aligned} CM^{\mu_2,\psi}+&(T^{T,\psi}(y) + T^{T,\psi}(y)M^{T\mu_1,\psi})^{-1}T^{T,\psi}(y)M^{T\mu_1,\psi}C \\ =&(T^{T,\psi}(y) + T^{T,\psi}(y)M^{T\mu_1,\psi})^{-1}G(y,t)(T^{\psi}(t))^{-1}. \end{aligned} \tag{3.4.16}$$

When we discretize Eq (3.4.16), it becomes an eminent Sylvester equation with unknown vector C. Eq (3.4.16) can be solve for C.
Numerical results for different values of μ_1 and μ_2 are shown in Figure 3.10 and Figure 3.11. Figure 3.10a shows the numerical results for exact and numerical solutions for $m = 4$, $\mu_1 = 0.75$ and $\mu_2 = 1$. Figure 3.10b shows an absolute error. Figure 3.11a shows the numerical results for $m = 4$, $\mu_1 = 1$ and $\mu_2 = 1$. Figure 3.11b shows an absolute error. We observe that error decreases while increasing m.



(a) Solution for $m = 4$, $\mu_1 = 0.75$ and $\mu_2 = 1$.

(b) Absolute error for $m = 4$, $\mu_1 = 0.75$ and $\mu_2 = 1$.

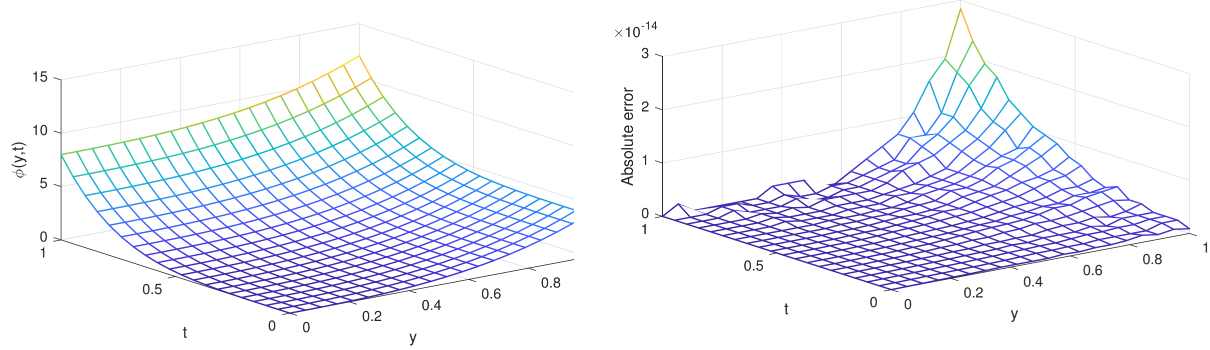
Figure 3.10: Exact and numerical solution.

In example (3.4.1) and (3.4.2), we presented the numerical solutions of fractional partial differential equations and compared it with the exact solutions, with the help of graphs of exact and numerical solutions with their absolute errors. In the following example, we compare the numerical solution obtained by our method and method presented in [25].

Example 3.4.3. Consider the linear fractional partial differential equation [25]:

$$D_y^{1/4,\psi}\phi(y,t) + D_t^{1/5,\psi}\phi(y,t) = f(y,t), \tag{3.4.17}$$

with initial conditions $\phi(y,0) = (\psi(y))^2$, $\phi(0,t) = \psi(t)$,where $f(y,t) = \frac{\Gamma(3)}{\Gamma(11/4)}(\psi(y))^{7/4} + \frac{\Gamma(2)}{\Gamma(9/5)}(\psi(t))^{4/5}$. The exact solution is $\phi(y,t) = (\psi(y))^2 + \psi(t)$, where $\psi(y) = 3y$ and



(a) Solutions for $m = 4$, $\mu_1 = 1$ and $\mu_2 = 1$.

(b) Absolute error for $m = 4$, $\mu_1 = 1$ and $\mu_2 = 1$.

Figure 3.11: Exact and numerical solutions.

$\psi(t) = 3t$. For the numerical evaluation, we approximate the term $D_t^{1/5,\psi}\phi(y,t)$ as

$$D_t^{1/5,\psi}\phi(y,t) \cong \sum_{i=0}^{m}\sum_{j=0}^{m} c_{ij} T_i^{\psi}(y) * T_j^{\psi}(t).$$

This can be written as

$$D_t^{1/5,\psi}\phi(y,t) \cong T^{T,\psi}(y) C T^{\psi}(t). \tag{3.4.18}$$

Applying fractional integral $I_t^{1/5,\psi}$ on both sides of Eq (3.4.18), we obtain

$$I_t^{1/5,\psi} D_t^{1/5,\psi}\phi(y,t) \cong T^{T,\psi}(y) C I_t^{1/5,\psi} T^{\psi}(t).$$

Using the result of Eq: (1.4.8),

$$\phi(y,t) - \phi(y,0) \cong T^{T,\psi}(y) C M^{1/5,\psi} T^{\psi}(t),$$

$$\phi(y,t) \cong T^{T,\psi}(y) C M^{1/5,\psi} T^{\psi}(t) + (\psi(y))^2, \tag{3.4.19}$$

where $M^{1/5,\psi}$ is an integration matrix. Now using Eq: (3.4.18) in Eq: (3.4.17)

$$D_y^{1/4,\psi}\phi(y,t) + T^{T,\psi}(y) C T^{\psi}(t) = f(y,t). \tag{3.4.20}$$

Applying the fractional integral $I_y^{1/4,\psi}$ on both sides of Eq: (3.4.20)

$$I_y^{1/4,\psi} D_y^{1/4,\psi}\phi(y,t) + I_y^{1/4,\psi} T^{T,\psi}(y) C T^{\psi}(t) = I_y^{1/4,\psi}\left(\frac{\Gamma(3)}{\Gamma(11/4)}(\psi(y))^{7/4} + \frac{\Gamma(2)}{\Gamma(9/5)}(\psi(t))^{4/5}\right). \tag{3.4.21}$$

Using the result of Eq: (1.4.8),

$$\phi(y,t) - \phi(0,t) + T^{T,\psi}(y) M_y^{1/4,\psi} C T^{\psi}(t) = I_y^{1/4,\psi}\left(\frac{\Gamma(3)}{\Gamma(11/4)}(\psi(y))^{7/4} + \frac{\Gamma(2)}{\Gamma(9/5)}(\psi(y))^{4/5}\right).$$

$$T^{T,\psi}(y) C M^{1/5,\psi} T^{\psi}(t) + T^{T,\psi}(y) M_y^{1/4,\psi} C T^{\psi}(t) = \frac{1}{\Gamma(9/5)\Gamma(5/4)}(\psi(y))^{1/4}(\psi(t))^{4/5} + \psi(t). \tag{3.4.22}$$

where $M^{T1/4,\psi}$ is an integration matrix. Now, we get

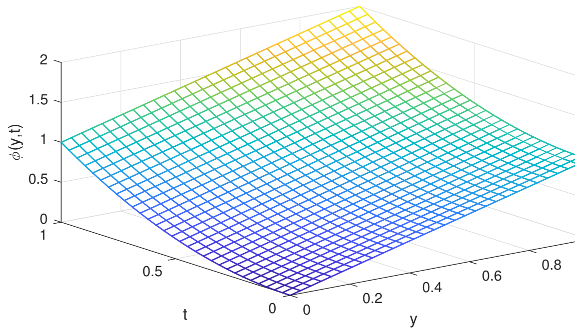
$$T^{T,\psi}(y)CM^{1/5,\psi}T^{\psi}(t) + T^{T,\psi}(y)M^{1/4,\psi}CT^{\psi}(t) = G(y,t), \tag{3.4.23}$$

where $G(y,t) = \frac{1}{\Gamma(9/5)\Gamma(5/4)}(\psi(y))^{1/4}(\psi(t))^{4/5} + \psi(t)$.

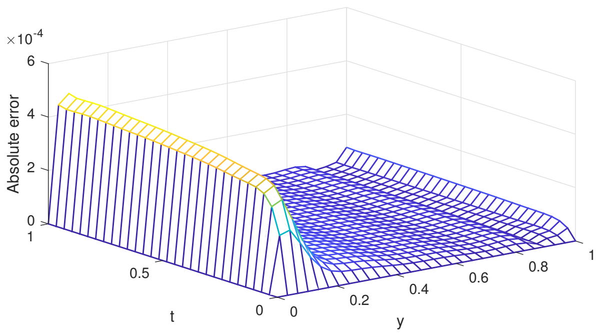
We obtain,

$$CM^{1/5,\psi} + M^{T1/4,\psi}C = (T^{T,\psi}(y))^{-1}G(y,t)(T^{\psi}(t))^{-1}. \tag{3.4.24}$$

When we discretize Eq (3.4.24), it reduces to Sylvester equation with vector C which is unknown, and can be calculated by using built-in function in MATLAB.
Figure 3.13a and Figure 3.13b depicts the comparison of solutions for $m = 10$ and an absolute error of (3.4.17) respectively. As we can see that upper bound of an absolute error is same as in [25].



(a) Solution for $m = 10$, $\mu_1 = 1/4$ and $\mu_2 = 1/5$.



(b) Absolute error for $m = 10$, $\mu_1 = 1/4$ and $\mu_2 = 1/5$.

Figure 3.12: Exact and numerical solution.

Summary

The aim of this paper is to develop the new concept for generalization of Taylor's theorem in the framework of differntial equations of integer order and fractional order involving fractional derivative of functions with respect to functions, we introduce the new concept of ψ-fractional power series which is helpful in the general form of generalized Taylor's theorem, some important results for the convergence and divergence, remainder theorem of error bound and series solutions are discussed.
Moreover, we inroduced a new method of ψ-Taylor series approximation which helps us to find numerical solutions of ψ-fractional differential equations. We provide an operational matrix of fraction integration for ψ-Taylor basis vector. Development of the method was discussed in detail. Furthermore, error analysis and convergence analysis for ψ-Taylor basis approximations are also discussed.
Numerical illustrations are also discussed to show the feasibility of the proposed method by calculating absolute errors. In section 3.3 of Chapter 3, we discuss one example for the approximation of fractional integration, one example for the ψ-differential equations in which exact solution was known, two examples for the ψ-differential equations in which exact solution was not known, and one example for comparison. In section 3.4.1 of Chapter 3, we present three examples for numerical approximations of ψ-fractional partial differential equations. The present method is expected to be further employed to solve other similar problems.

Bibliography

[1] Almeida R. A Caputo fractional derivative of a function with respect to another function. Communications in Nonlinear Science and Numerical Simulation. 2017 Mar 1;44:460-81.

[2] Almeida R. Functional differential equations involving the ψ-Caputo fractional derivative. Fractal and Fractional. 2020 Jun 23;4(2):29.

[3] Das S, Pan I. Fractional order signal processing: introductory concepts and applications. Springer Science and Business Media; 2011 Sep 15.

[4] Diethelm K. A fractional calculus based model for the simulation of an outbreak of dengue fever. Nonlinear Dynamics. 2013 Mar;71(4):613-9.

[5] El-Ajou A, Arqub OA, Zhour ZA, Momani S. New results on fractional power series: theories and applications. Entropy. 2013 Dec 2;15(12):5305-23.

[6] El-Nabulsi RA, Torres DF. Fractional actionlike variational problems. Journal of Mathematical Physics. 2008 May 20;49(5):053521.

[7] Erdelyi A. An integral equation involving Legendre functions. J Soc Ind Appl Math. 1964;12(1):15-30.

[8] Ferreira C, Lpez J, Prez Sinusa E. Analysis of singular one-dimensional linear boundary value problems using two-point Taylor expansions. Electronic Journal of Qualitative Theory of Differential Equations. 2020;2020(22):1-21.

[9] Freihet A, Hasan S, Al-Smadi M, Gaith M, Momani S. Construction of fractional power series solutions to fractional stiff system using residual functions algorithm. Advances in Difference Equations. 2019 Dec;2019(1):1-5.

[10] Garrappa R. Numerical solution of fractional differential equations: A survey and a software tutorial. Mathematics. 2018 Jan 23;6(2):16.

[11] Gorenflo R. Fractional calculus: some numerical methods. Courses and lectures-international centre for mechanical sciences. 1997 Jan 1:277-90.

[12] Gulsu M, Ozturk Y, Anapali A. Numerical approach for solving fractional relaxationoscillation equation. Applied Mathematical Modelling. 2013 Apr 15;37(8):5927-37.

[13] Hardy GH. Riemann's form of Taylor's series. Journal of the London Mathematical Society. 1945 Jan 1;1(1):48-57.

[14] Heydari MH, Avazzadeh Z, Cattani C. Taylors series expansion method for nonlinear variable-order fractional 2D optimal control problems. Alexandria Engineering Journal. 2020 Dec 1;59(6):4737-43.

[15] Hilfer R, editor. Applications of fractional calculus in physics. World scientific; 2000 Mar 2.

[16] Katsikadelis JT. The BEM for numerical solution of partial fractional differential equations. Computers and Mathematics with Applications. 2011 Aug 1;62(3):891-901.

[17] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. elsevier; 2006 Mar 2.

[18] Lopez JL, Sinusa EP, Temme NM. A three-point Taylor algorithm for three-point boundary value problems. Journal of Differential Equations. 2011 Jul 1;251(1):26-44.

[19] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. Wiley; 1993.

[20] Odibat ZM, Shawagfeh NT. Generalized Taylors formula. Applied Mathematics and Computation. 2007 Mar 1;186(1):286-93.

[21] Oldham K, Spanier J. The fractional calculus theory and applications of differentiation and integration to arbitrary order. Elsevier; 1974 Sep 5.

[22] Osler TJ. Leibniz rule for fractional derivatives generalized and an application to infinite series. J SIAM Appl Math. 1970;18(3):658-674.

[23] Postavaru O, Toma A. Numerical solution of two-dimensional fractional-order partial differential equations using hybrid functions. Partial Differential Equations in Applied Mathematics. 2021 Dec 1;4:100099.

[24] Qu H, Liu X. A numerical method for solving fractional differential equations by using neural network. Advances in Mathematical Physics. 2015 Jan 1;2015.

[25] Singh H, Singh CS. Stable numerical solutions of fractional partial differential equations using Legendre scaling functions operational matrix. Ain Shams Engineering Journal. 2018 Dec 1;9(4):717-25.

[26] Singh H, Singh CS. Stable numerical solutions of fractional partial differential equations using Legendre scaling functions operational matrix. Ain Shams Engineering Journal. 2018 Dec 1;9(4):717-25.

[27] Wantanable Y. Notes on the generalized derivatives of Riemann-Liouville and its application to Leibntz's formula. Tohoku Math. J. 1931;24:8-41.

[28] Ye H, Gao J, Ding Y. A generalized Gronwall inequality and its application to a fractional differential equation. Journal of Mathematical Analysis and Applications. 2007 Apr 15;328(2):1075-81.