Stanley Depth and Depth of the Quotient Rings of Edge Ideals Corresponding to Some ϱ-Fold Bristled Graphs

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MS THESIS WORK

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I dedicate this thesis to my loving parents, venerable supervisor and siblings for their limitless support and encouragement.

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Abstract

In this thesis, we discuss some fundamental concept of Abstract Algebra like rings and modules. Furthermore, we discuss the Algebraic invariants Stanley depth and depth. We also discuss some known results related to these invariants. Afterwards, we compute the exact value of Stanley depth and depth of the quotient rings of the edge ideals associated with ϱ-fold bristled graph of ladder graph, circular ladder graph, and strong product of two graphs when both graphs are paths or when one of them is a cycle and other is a path. We also proved that both these invariants have the same values for all the classes, we considered.

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Introduction

Richard P. Stanley is known for his work to develop a relationship in Algebra and Geometry. In 1982, Stanley proposed a conjecture [25]. According to Stanley conjecture, Stanley depth of a module is atleast the depth of a module. This concept garbed the attention of algebraist but also Stanley decomposition plays an important role in applied mathematics. In [24] Sturmfels et al. shows that Stanley decomposition can be used to describe finitely generated graded algebras.

Herzog and Popescu [15, 21] gave some remarkable results related to this conjecture. A while later, many articles have been published in which this conjecture was proved for various special cases. In 2016, Dual et al. [10] disproved it by using result of Herzog et al. [15] they constructed explicit counter example for which the conjecture was not satisfied.

In this thesis we calculate the exact values of Stanley depth and depth for the quotient module of edge ideal associated with some ϱ-fold bristled graphs.

This thesis has four chapters. Chapter 1 is devoted for preliminaries. This chapter is divided into three parts. In the first part we recall some concepts related to polynomial ring and monomial ideals. In second part we covers the exact sequence, graded ring together with other fundamentals of Module Theory. The third part of this chapter give a precise overview of Graph Theory. Chapter 2 covers introduction to depth, Stanley decomposition, Stanley depth and Stanley conjecture. At the end some known results and bounds for Stanley depth are given.

In Chapter 3, the edge ideal associated with ϱ-fold bristled graph of ladder graph and strong product of two paths are considered. The exact values for Stanley depth and depth of the quotient ring associated to these edge ideals are computed.

In Chapter 4, the edge ideal associated with ϱ-fold bristled graph of circular ladder graph and strong product of cycle and path are considered. The exact values for Stanley depth and depth of the quotient ring associated to these edge ideals are computed.

Chapter 1

Preliminaries

1.1 Introduction

In 1914, Fraenkel gave the definition of ring [13]. Ring Theory have two main classifications commutative and non commutative. Basic theories of each came from different sources. Problems and theories from Algebraic Number Theory and Algebraic Geometry plays a central role in the origin of Commutative Ring Theory. The origin of non commutative ring came from the approaches to extend complex number in hypercomplex number system. Noether and Artin play an important role to made the abstract ring concepts focal in Algebra, they presented and gave importance to the algebraic concepts as module, ideal and both ascending and descending chain conditions. See [27, 6].

In 18th century, Euler solved the Konigsberg's bridge problem which lead to new branch of mathematics called Graph Theory. Graph Theory is considered as a field of modern mathematics. In 1991 Anderson et al. [2] gave the idea of associating a graph to a commutative ring , which is widely used these days in research.

1.2 Ring Theory

In this section we will discuss some basics of Ring Theory. For these definitions we refer to [11].

Definition 1.2.1. A ring $(\omega, +, \times)$ is a set with two binary operations, addition and multiplication denoted by $u + v$ and uv, respectively. Such that for $u, v, s \in \omega$ satisfy the following axioms:

- ω is an abelian group under addition.
- Multiplication is associative
$$(uv)s = u(vs).$$
- Multilplication is distributive over addition
$$u(v + s) = uv + us, \ (v + s)u = vu + su.$$

Definition 1.2.2. Let $(\omega, +, \times)$ be a ring. If multiplication is commutative in ω, that is $uv = vu$, for all $u, v \in \omega$, then ω is a commutative ring.

If there is an element e $\in \omega$ such that $ue = u = eu$, for all $u \in \omega$, we say ω is a ring with multiplicative identity (or a ring with unity). Multiplicative identity or unity of ω is denoted by symbol 1.
In this thesis we will consider only commutative ring with unity.

Example 1.2.3.
1. Integers, real and complex number sets are examples of commutative rings having unity 1.
2. $\mathbb{Z}/n\mathbb{Z}$ with multiplicative identity 1 under multiplication and addition of residue classes, forms a commutative ring.
3. The set $\mathcal{M}_{2\times 2}(\mathbb{Z}) = \left\{ \begin{bmatrix} s & o \\ u & \nu \end{bmatrix} : s, o, u, \nu \in \mathbb{Z} \right\}$ is a non-commutative ring with unity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with the standard matrix addition and multiplication.

Definition 1.2.4. An element ν in a commutative ring ω with unity is said to be invertible if there exists $x \in \omega$ such that $\nu x = e$.
Unity and unit are different concepts and should not be confused with each other.

Definition 1.2.5. Let $(\omega, +, \cdot)$ be a ring. If elements in $\omega^* = \omega\backslash\{0\}$ have multiplicative inverse, then $(\omega, +, \cdot)$ is called a field.

Example 1.2.6. $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$ are fields. But $\mathbb{Z}$ is not a field since $\mathbb{Z}$ does not have multiplicatve inverse of its elements except $\{1, -1\}$.

Definition 1.2.7. Let ω be a ring and $0 \neq s \in \omega$ and if $\exists\, t \in \omega$ suct that $st = 0$. Then s and t are called zero divisors.

Example 1.2.8. Let $\mathbb{Z}_8$ be a ring, then $\bar{2}, \bar{4}, \bar{6}$ are zero divisors because $\bar{4}.\bar{2} = \bar{8} = \bar{0}$ and $\bar{6}.\bar{4} = \bar{2}\bar{4} = \bar{0}$.

Definition 1.2.9. Let ω be a ring then it is called a non-zero divisor ring if and only if $\forall s, \varkappa \in \omega$ if $s \cdot \varkappa = 0$, then $s = 0$ or $\varkappa = 0$.

Definition 1.2.10. A commutative ring with unity is called an integral domain if it has no zero divisors.

Example 1.2.11.

1. $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{Z}_7$ are all integral domains.
2. Let $n \geq 2$, then $n\mathbb{Z}$ is not an integral domain as $n\mathbb{Z}$ does not have unity.

1.2.1 Polynomial ring

Definition 1.2.12. Let T be a commutative ring with unity. For $n \geq 0$ and $\mu_i \in \mathsf{T}$.

$$\mathcal{P}(\varkappa) = \mu_n \varkappa^n + \mu_{n-1}\varkappa^{n-1} + \cdots + \mu_1 \varkappa + \mu_0$$

is termed polynomial in indeterminate $\varkappa$. For $0 \leq i \leq n$, μ_i is called co-efficient of $\mathcal{P}(\varkappa)$ and $\mu_i \varkappa^i$ are terms of polynomial $\mathcal{P}(\varkappa)$. The polynomial is of degree n if $\mu_n \neq 0$. If $\mathcal{P}_1(\varkappa)$ and $\mathcal{P}_2(\varkappa)$ be any two polynomials, then

1. $\deg(\mathcal{P}_1(\varkappa)) + \deg(\mathcal{P}_2(\varkappa)) = \max\{\deg(\mathcal{P}_1(\varkappa), \deg(\mathcal{P}_2(\varkappa)\}$.
2. $\deg(\mathcal{P}_1(\varkappa) \cdot \mathcal{P}_2(\varkappa)) = \deg(\mathcal{P}_1(\varkappa)) + \deg(\mathcal{P}_2(\varkappa))$.

Definition 1.2.13. Let T be a commutative ring. The set of formal symbols

$$\omega = \mathsf{T}[\varkappa] = \{\mu_n \varkappa^n + \mu_{n-1}\varkappa^{n-1} + \cdots + \mu_1 \varkappa + \mu_0 : n \geq o, \mu_i \in \mathsf{T}\}$$

is called the ring of polynomials over T in the variable $\varkappa$. The zero of the polynomial ring is $\mathfrak{f}(\varkappa) = 0$, and unity is $\mathfrak{g}(\varkappa) = 1$. The polynomial ring for n variables $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$ with co-efficients in T is defined as

$$\mathsf{T}[\varkappa_1, \varkappa_2, \ldots, \varkappa_n] = \mathsf{T}[\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1}][\varkappa_n].$$

Example 1.2.14. $\mathbb{R}[\varkappa], \mathbb{Q}[\varkappa]$ and $\mathbb{Z}_p[\varkappa]$ are all polynomial rings.

Proposition 1.2.15. *Let* $\mathsf{T}[\varkappa]$ *be commutative ring with unity. Then*

1. *The units in* $\mathsf{T}[\varkappa]$ *are the units of* T.

2. *If* ω *is an integral domain then so is* $\mathsf{T}[\varkappa]$.

Definition 1.2.16. Let ω be a ring and $S \subset \omega$. Then S is a subring of ω if it is a ring under the same operations as ω.

Definition 1.2.17. A map $\Gamma : \omega_1 \longrightarrow \omega_2$ that preserves both operations of ω_1 that is

1. $\Gamma(\varkappa + \nu) = \Gamma(\varkappa) + \Gamma(\nu), \quad \forall\ \varkappa, \nu \in \omega_1$ and

2. $\Gamma(\varkappa\nu) = \Gamma(\varkappa)\Gamma(\nu), \quad \forall\ \varkappa, \nu \in \omega_1$

is called a ring homomorphism. The kernal of the map Γ is termed as,

$$\operatorname{Ker}\Gamma = \{\varkappa \in \omega_1 \mid \Gamma(\varkappa) = 0_{\omega_2}\}.$$

The image of map Γ is termed as $\operatorname{Im}\Gamma = \{\Gamma(\varkappa) \mid \varkappa \in \omega_1\}$. A bijective ring homomorphism is a ring isomorphism and injective ring homomorphism is monomorphism.

Theorem 1.2.18. *Let* $\Gamma : \omega_1 \longrightarrow \omega_2$ *be a ring homomorphism. Then* $\operatorname{Ker}\Gamma = \{0\}$ *iff* Γ *is a monomorphism.*

Definition 1.2.19. A subring $\mathfrak{O}$ of a ring ω is called an ideal if $u\varkappa \in \mathfrak{O}$ for all $u \in \omega$, $\varkappa \in \mathfrak{O}$.

Definition 1.2.20. Consider $\mathfrak{J}$ and $\mathfrak{O}$ be two ideals of ω, then

1. $\mathfrak{J} + \mathfrak{O} = \{\varkappa + \nu : \varkappa \in \mathfrak{J}, \nu \in \mathfrak{O}\}$.

2. $\mathfrak{J}\mathfrak{O} = \{\varkappa_1\nu_1 + \varkappa_2\nu_2 + \cdots + \varkappa_r\nu_r : \varkappa_1, \ldots \varkappa_r \in \mathfrak{J}, \quad \nu_1 \ldots \nu_r \in \mathfrak{O} \text{ and } r \in \mathbb{Z}^+\}$.

Definition 1.2.21. Two ideals $\mathfrak{J}$ and $\mathfrak{O}$ of ω, are comaximal if $\mathfrak{J} + \mathfrak{O} = \omega$.

Proposition 1.2.22. *If $\mathfrak{J}$ and $\mathfrak{O}$ comaximal then, $\mathfrak{J}\mathfrak{O} = \mathfrak{J} \cap \mathfrak{O}$.*

Remark 1.2.23. *The condition $\mathfrak{J}+\mathfrak{O} = \omega$ is not absolutely necessary for $\mathfrak{J}\mathfrak{O} = \mathfrak{J}\cap\mathfrak{O}$. For example in ring $\mathbb{Z}_5$, $\mathfrak{J} = \mathfrak{O} = (3)$ then $\mathfrak{J}\mathfrak{O} = (3) = \mathfrak{J} \cap \mathfrak{O}$, even though $\mathfrak{J} + \mathfrak{O} \neq \omega$*

Definition 1.2.24. Let $\mathfrak{O}$ be an ideal of ω, then the radical of $\mathfrak{O}$, is denoted by $\sqrt{\mathfrak{O}}$ and given as $\sqrt{\mathfrak{O}} = \{\varkappa \in \omega \mid \varkappa^t \in \mathfrak{O}, \quad \text{for some } t > 0\}$. $\sqrt{\mathfrak{O}}$ is an ideal containing $\mathfrak{O}$.

Example 1.2.25. Following are some examples of the radical of an ideal.

1. Let $\mathfrak{J} = (2^2 3^2 5^4)$ be an ideal in $\mathbb{Z}$, then $\sqrt{\mathfrak{J}} = (2 \cdot 3 \cdot 5) = (30)$.

2. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4]$ be a polynomial ring and $\mathfrak{J} = (\varkappa_1^2\varkappa_2^5, \varkappa_1^3\varkappa_4, \varkappa_3^2\varkappa_4^3)$, then $\sqrt{\mathfrak{J}} = (\varkappa_1\varkappa_2, \varkappa_1\varkappa_4, \varkappa_3\varkappa_4)$.

Definition 1.2.26. An ideal $\mathfrak{O}$ of ω is radical ideal if $\sqrt{\mathfrak{O}} = \mathfrak{O}$.

Definition 1.2.27. Let ω be a ring and $\mathfrak{O}$ be an ideal of ω. Then $\omega/\mathfrak{O} = \{\mathfrak{O}+\varkappa \mid \varkappa \in \omega\}$ is also a ring called factor ring. For any $\varkappa, \nu \in \omega$, the multiplication and addition are defined as,

$$(\mathfrak{O} + \varkappa) + (\mathfrak{O} + \nu) = \mathfrak{O} + (\varkappa + \nu).$$

$$(\mathfrak{O} + \varkappa)(\mathfrak{O} + \nu) = \mathfrak{O} + \varkappa\nu.$$

Example 1.2.28. Let $\mathbb{Z}$ be a ring and $3\mathbb{Z}$ be its subring. Also $3\mathbb{Z}$ be an ideal.

$$\begin{aligned} \mathbb{Z}/3\mathbb{Z} &= \{3\mathbb{Z} + u | u \in 3\mathbb{Z}\} \\ &= \{3\mathbb{Z} + 0, 3\mathbb{Z} + 1, 3\mathbb{Z} + 2\} \\ &\approx \mathbb{Z}_3. \end{aligned}$$

Definition 1.2.29. Let $\mathfrak{J}$ be an ideal in ω. If for a single element $\varkappa \in \omega$, $\mathfrak{J}$ can be written as $\mathfrak{J} = (\varkappa) = \{\varkappa\mu \mid \mu \in \omega\}$ then it is a principal ideal.

Example 1.2.30. The principal ideal in polynomial ring $\mathbb{Z}[\varkappa]$ is

$$\langle \mathfrak{g}(\varkappa) \rangle = \langle 2 \rangle = \{2 \cdot \gamma(\varkappa) \mid \gamma(\varkappa) \in \mathbb{Z}\}$$

set of all polynomial in $\mathbb{Z}[\varkappa]$ with even co-efficients.

Proposition 1.2.31. *Let ω be a field, then every ideal of the polynomial ring $\omega[\varkappa]$ is a principal ideal.*

Definition 1.2.32. A proper ideal $\mathfrak{J}$ is a prime ideal in a commutative ring ω if $u, v \in \omega$ and $uv \in \mathfrak{J}$, then $u \in \mathfrak{J}$ or $v \in \mathfrak{J}$.

Example 1.2.33.

1. $13\mathbb{Z}$ is a prime ideal in ring $\mathbb{Z}$.
2. The ideal $\langle \varkappa^3 \rangle = \{\Gamma(\varkappa)\varkappa^3 : \Gamma(\varkappa) \in \mathbb{Z}[\varkappa]\}$ is not prime ideal in ring $\mathbb{Z}[\varkappa]$, as $\varkappa^2\varkappa = \varkappa^3 \in \langle \varkappa^3 \rangle$ but $\varkappa^2 \notin \langle \varkappa^3 \rangle$ and $\varkappa \notin \langle \varkappa^3 \rangle$.

Theorem 1.2.34. *Let ω be a commutative ring with unity and $\mathfrak{J}$ be an ideal of ω, then $\mathfrak{J}$ is prime iff $\omega/\mathfrak{J}$ is an integral domain.*

Definition 1.2.35. A proper ideal $\mathfrak{O}$ of a ring ω is said to be a maximal ideal if $\mathfrak{I}$ is an ideal of ω with $\mathfrak{O} \subseteq \mathfrak{I} \subseteq \omega$ then either $\mathfrak{O} = \mathfrak{I}$ or $\omega = \mathfrak{I}$.

Example 1.2.36. $2\mathbb{Z}$ and $5\mathbb{Z}$ are maximal ideals in $\mathbb{Z}$. But $8\mathbb{Z}$ is not a maximal as $8\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$.

Theorem 1.2.37. *Let ω be a commutative ring with unity and $\mathfrak{J}$ be an ideal. Then, $\omega/\mathfrak{J}$ is a field iff $\mathfrak{J}$ is a maximal ideal.*

Definition 1.2.38. If ring ω has a unique maximal ideal, then ω is called a local ring.

Definition 1.2.39. If ω is ring and ω have only finite number of maximal ideals then, it is called semi-local ring.

Example 1.2.40. $\mathbb{Z}_{10}$ have only two maximal ideals $\{0, 2, 4, 6, 8\}$ and $\{0, 5\}$, so $\mathbb{Z}_{10}$ is semi local ring.

Definition 1.2.41. Let ω be a ring, then intersection of all maximal ideals of ω is called Jacobson radical of ω. It is denoted by $J(\omega)$.

Definition 1.2.42. Let $\mathcal{A} = \{\omega_i : \ i \in J\}$ be the collection of rings, where J is countable set. The direct product is defined as

$$\prod_{i \in J} \omega_i = \omega_1 \times \omega_2 \cdots = \{(\varkappa_1, \varkappa_2, \cdots, \varkappa_n, \cdots) \ \ \varkappa_i \in \omega_i\}.$$

This direct product satisfy all axioms of ring under the following binary operations of addition and multiplication for $\varkappa_i, \nu_i \in \omega_i$

$$(\varkappa_1, \varkappa_2, \varkappa_3, \cdots) + (\nu_1, \nu_2, \nu_3, \cdots) = (\varkappa_1 + \nu_1, \varkappa_2 + \nu_2, \varkappa_3 + \nu_3, \cdots),$$
$$(\varkappa_1, \varkappa_2, \varkappa_3, \cdots)(\nu_1, \nu_2, \nu_3, \cdots) = (\varkappa_1\nu_1, \varkappa_2\nu_2, \varkappa_3\nu_3, \cdots).$$

The direct sum of collection of $\mathcal{A}$ is defined as follows

$$\bigoplus_{i \in I} \omega_i = \{(\varkappa_1, \varkappa_2, \cdots) \in \prod_{i \in I} \omega_i : \ \ \varkappa_i \text{ is zero for all but finitely many } i\}.$$

1.2.2 Monomial ideals

Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \ldots, \varkappa_m]$ be a ring of polynomials with m variables, where T is a field. A monomial is any product of $\varkappa_1^{c_1}, \varkappa_2^{c_2}, \ldots, \varkappa_m^{c_n}$ with $c_i \in \mathbb{Z}_+$. Consider the monomial $\mu = \varkappa_1^{c_1}\varkappa_2^{c_2} \cdots \varkappa_m^{c_n}$ then we write it as $\mu = \varkappa^c$, where $c = (c_1, c_2, \cdots, c_n) \in \mathbb{Z}_+^n$. If $\mathscr{A}$ is the set of all monomials of ω then $\mathscr{P}$ form a T-basis of ω. Therefore any polynomial in ω can be written as a linear combination of monomials with coefficients from T. So polynomial $\mathcal{L} \in \omega$ can be uniquely written as

$$\mathcal{L} = \sum_{\nu \in \mathscr{P}} c_\mu \mu, \quad with \ \ c_\nu \in \mathsf{T}.$$

The support of $\mathcal{L}$ is defined as $\operatorname{supp}(\mathcal{L}) = \{\mu \in \mathscr{A} : c_\mu \neq 0\}$, and the support of monomial μ is $\operatorname{supp}(\mu) = \{\varkappa_j : \varkappa_j \mid \mu\}$. A monomial $\mu = \varkappa_1^{c_1}\varkappa_2^{c_2} \cdots \varkappa_m^{c_n}$ is called square free, if c_i's are 0 or 1.

Definition 1.2.43. Let the monomial ideal $\mathfrak{J} \subset \omega$. If generating set of $\mathfrak{J}$ consist of square free monomials, then it is called a square free monomial ideal.

Example 1.2.44. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4]$ be a polynomial ring. Then, $\mathfrak{J} = (\varkappa_3\varkappa_4, \varkappa_3\varkappa_2, \varkappa_1\varkappa_2\varkappa_4)$ is a square free monomial ideal.

Definition 1.2.45. Let $\mu = \varkappa_1^{\Upsilon_1}\varkappa_2^{\Upsilon_2}\cdots\varkappa_m^{\Upsilon_n}$ and $\nu = \varkappa_1^{a_1}\varkappa_2^{a_2}\cdots\varkappa_m^{a_n}$ be two monomials, then

1. $\mu \mid \nu$ if $\Upsilon_i \leq a_i$ for all i.
2. $\gcd(\mu, \nu) = \varkappa_1^{\min\{\Upsilon_1, a_1\}}\varkappa_2^{\min\{\Upsilon_2, a_2\}}\cdots\varkappa_n^{\min\{\Upsilon_n, a_n\}}$.
3. $\operatorname{lcm}(\mu, \nu) = \varkappa_1^{\max\{\Upsilon_1, a_1\}}\varkappa_2^{\max\{\Upsilon_2, a_2\}}\cdots\varkappa_n^{\max\{\Upsilon_n, a_n\}}$.

Proposition 1.2.46. *Every monomial ideal $\mathfrak{J} \subset \omega$ has a unique minimal monomial set of generators, denoted by $\mathcal{G}(\mathfrak{J})$.*

Example 1.2.47. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4]$ be a polynomial ring. Then, $\mathcal{G}(\mathfrak{J})$ of monomial ideal $\mathfrak{J} = (\varkappa_3^2\varkappa_4^3, \varkappa_1^2\varkappa_2^4\varkappa_4, \varkappa_3^2\varkappa_2, \varkappa_3^2\varkappa_4, \varkappa_1)$ is $(\varkappa_3^2\varkappa_2, \varkappa_3^2\varkappa_4, \varkappa_1)$.

Definition 1.2.48. Let $\mathcal{G}(\mathfrak{J})$ and $\mathcal{G}(\mathfrak{O})$ be minimal set of monomial generators of $\mathfrak{J}$ and $\mathfrak{O}$ respectively, then

1. $\mathcal{G}(\mathfrak{J} + \mathfrak{O}) \subseteq \mathcal{G}(\mathfrak{J}) \cup \mathcal{G}(\mathfrak{O})$.
2. $\mathcal{G}(\mathfrak{J}\mathfrak{O}) \subseteq \mathcal{G}(\mathfrak{J})\mathcal{G}(\mathfrak{O})$, where $\mathcal{G}(\mathfrak{J}\mathfrak{O}) = \{\mu\nu : \mu \in \mathfrak{J}, \nu \in \mathfrak{O}\}$.

Example 1.2.49. Let $\mathfrak{O} = (\varkappa_1\varkappa_2, \varkappa_2\varkappa_3^2)$ and $\mathfrak{J} = (\varkappa_1^2\varkappa_2, \varkappa_2\varkappa_3)$ be two ideals, then

1. $\mathcal{G}(\mathfrak{J} + \mathfrak{O}) = \{\varkappa_1\varkappa_2, \varkappa_2\varkappa_3^2\} \subseteq \mathcal{G}(\mathfrak{J}) \cup \mathcal{G}(\mathfrak{O}) = \{\varkappa_1\varkappa_2, \varkappa_2\varkappa_3^2, \varkappa_1^2\varkappa_2, \varkappa_2\varkappa_3\}$.
2. $\mathcal{G}(\mathfrak{J}\mathfrak{O}) = \{\varkappa_1^3\varkappa_2^2, \varkappa_1\varkappa_2^2\varkappa_3, \varkappa_2^2\varkappa_3^3\} \subseteq \mathcal{G}(\mathfrak{J})\mathcal{G}(\mathfrak{O}) = \{\varkappa_1^2\varkappa_2^2\varkappa_3^2, \varkappa_1^3\varkappa_2^2, \varkappa_1\varkappa_2^2\varkappa_3, \varkappa_2^2\varkappa_3^3\}$.

Proposition 1.2.50. *Let $\mathfrak{O}$ and $\mathfrak{J}$ be two monomial ideals with $\mathcal{G}(\mathfrak{O}) = \{\mu_1, \mu_2, \ldots, \mu_s\}$ and $\mathcal{G}(\mathfrak{J}) = \{\varkappa_1, \varkappa_2, \ldots, \varkappa_t\}$, then $\mathfrak{J} \cap \mathfrak{O} = (\{\operatorname{lcm}(\mu_i, \varkappa_k) : i = 1, \cdots, s,\ k = 1, \cdots, t\})$.*

Example 1.2.51. Let $\mathfrak{O} = (\varkappa_1\varkappa_2, \varkappa_2\varkappa_3^2)$ and $\mathfrak{J} = (\varkappa_1^2\varkappa_2, \varkappa_2\varkappa_3)$ be two ideals, then

$$\mathfrak{J} \cap \mathfrak{O} = (\varkappa_1^2\varkappa_2, \varkappa_1\varkappa_2\varkappa_3, \varkappa_1^2\varkappa_2\varkappa_3^2, \varkappa_2\varkappa_3^2) = (\varkappa_1^2\varkappa_2, \varkappa_1\varkappa_2\varkappa_3, \varkappa_2\varkappa_3^2).$$

Proposition 1.2.52. *Let $\mathcal{G}(\mathfrak{J})$ and $\mathcal{G}(\mathfrak{O})$ be monomial ideals with $\mathcal{G}(\mathfrak{O}) = \{\mu_2, \ldots, \mu_m\}$ and $\mathcal{G}(\mathfrak{J}) = \{\varkappa_1, \varkappa_2, \ldots, \varkappa_n\}$, then*

$$\mathfrak{O} : \mathfrak{J} = \bigcap_{j=1}^{n} (\mathfrak{J} : \varkappa_j),$$

and $\mathfrak{J} : (\varkappa_j) = (\mu_i / \gcd(\mu_i, \varkappa_j) : i = 1, \cdots, m)$.

Example 1.2.53. Let $\mathfrak{O} = (\varkappa_1\varkappa_2^2, \varkappa_2\varkappa_3^2)$ and $\mathfrak{J} = (\varkappa_1\varkappa_2\varkappa_3, \varkappa_2^2)$ be two ideals, then

$$\begin{aligned}
\mathfrak{O} : \mathfrak{J} &= ((\varkappa_1\varkappa_2^2, \varkappa_2\varkappa_3^2) : (\varkappa_1\varkappa_2\varkappa_3)) \cap ((\varkappa_1\varkappa_2^2, \varkappa_2\varkappa_3^2) : (\varkappa_2^2)) \\
&= (\frac{\varkappa_1\varkappa_2^2}{\gcd(\varkappa_1\varkappa_2^2, \varkappa_1\varkappa_2\varkappa_3)}, \frac{\varkappa_2\varkappa_3^2}{\gcd(\varkappa_2\varkappa_3^2, \varkappa_1\varkappa_2\varkappa_3)}) \cap (\frac{\varkappa_1\varkappa_2^2}{\gcd(\varkappa_1\varkappa_2^2, \varkappa_2^2)}, \frac{\varkappa_2\varkappa_3^2}{\gcd(\varkappa_2\varkappa_3^2, \varkappa_2^2)}) \\
&= (\frac{\varkappa_1\varkappa_2^2}{\varkappa_1\varkappa_2}, \frac{\varkappa_2\varkappa_3^2}{\varkappa_2\varkappa_3}) \cap (\frac{\varkappa_1\varkappa_2^2}{\varkappa_2^2}, \frac{\varkappa_2\varkappa_3^2}{\varkappa_2}) \\
&= (\varkappa_2, \varkappa_3) \cap (\varkappa_1, \varkappa_3^2) \\
&= (\varkappa_1\varkappa_2, \varkappa_2\varkappa_3^2, \varkappa_1\varkappa_3, \varkappa_3^2) \\
&= (\varkappa_1\varkappa_2, \varkappa_1\varkappa_3, \varkappa_3^2).
\end{aligned}$$

Definition 1.2.54. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \ldots, \varkappa_m]$ be a polynomial ring, then prime monomial ideal is the ideal generated by subsets of the variables of type $\varkappa_{j_1}, \varkappa_{j_2}, \ldots, \varkappa_{j_n}$ where $\{j_1, j_2, \ldots, j_n\} \subseteq \{1, 2, \ldots, m\}$.

Example 1.2.55. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4]$ be a polynomial ring then, $\mathfrak{O} = (\varkappa_1, \varkappa_2, \varkappa_3)$ is prime ideal but $\mathfrak{J} = (\varkappa_1\varkappa_3, \varkappa_2, \varkappa_4)$ is not a prime ideal.

Corollary 1.2.56. *Let $\mathfrak{O}$ be a squrefree monomial ideal, then $\mathfrak{O}$ is a finite intersection of monomial prime ideals.*

Example 1.2.57. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4]$ be polynomial ring then, the ideal $\mathfrak{J} = (\varkappa_2\varkappa_3, \varkappa_1, \varkappa_4) = (\varkappa_3, \varkappa_1, \varkappa_4) \cap (\varkappa_2, \varkappa_1, \varkappa_4)$, where $(\varkappa_2, \varkappa_1, \varkappa_4)$ and $(\varkappa_3, \varkappa_1, \varkappa_4)$ are prime ideals.

Definition 1.2.58. Let ω be a ring, then ideal $\mathfrak{J} \neq \omega$ is primary if for $\mu, \nu \in \omega$, $\mu\nu \in \mathfrak{J}$ then either $\mu \in \mathfrak{J}$ or $\nu^m \in \mathfrak{J}$ for some $m \geq 1$.

Example 1.2.59.

1. The primary ideals of integers ring are 0 and $p^m\mathbb{Z}$, where p is a prime number $m \geq 1$.
2. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2]$ be a polynomial ring, then $\mathfrak{J} = (\varkappa_2^2, \varkappa_1\varkappa_2)$ is not primary as $\varkappa_1, \varkappa_2 \in \omega$ and $\varkappa_1\varkappa_2 \in \mathfrak{J}$ but neither $\varkappa_2 \in \mathfrak{J}$ nor some m power of $\varkappa_1$ is in $\mathfrak{J}$.

Definition 1.2.60. Let $\mathfrak{J}$ be an ideal then the presentation $\mathfrak{J} = \bigcap_{i=1}^{m} \mathcal{M}_i$ is irredundant if none of the ideals $\mathcal{M}_i$ can be omitted in this presentation.

Example 1.2.61.

1. Let $\mathfrak{J} = (\varkappa^4 z, y^4 z)$ be an ideal, then presentation $\mathfrak{J} = (\varkappa, y) \cap (\varkappa^4, y^4) \cap (z)$ is not irredundant presentation of $\mathfrak{J}$ as $(\varkappa, y)$ can be omitted in this presentation.
2. Let $\mathfrak{J} = (\varkappa y, z)$ be an ideal, then presentation $\mathfrak{J} = (\varkappa, z) \cap (y, z)$ is irredundant presentation of $\mathfrak{J}$.

Definition 1.2.62. Let $\mathfrak{J}$ be an ideal then it is irreducible if it cannot be written as an intersection of two other monomial ideals containing $\mathfrak{J}$. If $\mathfrak{J}$ is not irreducible, then it is called reducible.

Example 1.2.63. $\mathfrak{O} = (\varkappa^4, y^4)$ is irreducible, where as $\mathfrak{J} = (\varkappa y, z)$ is reducible as it can be written as proper intersection of two ideals $(\varkappa, z)$ and (y, z).

Definition 1.2.64. A presentation of an ideal $\mathfrak{J}$ as $\mathfrak{J} = \mathcal{M}_1 \cap \mathcal{M}_2 \cdots \cap \mathcal{M}_r$, where $\mathcal{M}_i$ is a primary monomial ideal for all i is called a primary decomposition of $\mathfrak{J}$ and if none of the $\mathcal{M}_i$ can be omitted in this intersection and $\sqrt{\mathcal{M}_i} \neq \sqrt{\mathcal{M}_j}$ if $i \neq j$, then it is called irredundant primary decomposition.

Example 1.2.65. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4]$ and $\mathfrak{J} = (\varkappa_1\varkappa_4, \varkappa_3^2\varkappa_2, \varkappa_3\varkappa_4)$, then the irredundant primary decomposition of $\mathfrak{O}$ is

$$\begin{aligned}\mathfrak{J} &= (\varkappa_1\varkappa_4, \varkappa_3^2\varkappa_2, \varkappa_3\varkappa_4, \varkappa_3^2) \cap (\varkappa_1\varkappa_4, \varkappa_3^2\varkappa_2, \varkappa_3\varkappa_4 : \varkappa_3^2)\\ &= (\varkappa_1\varkappa_4, \varkappa_3\varkappa_4, \varkappa_3^2) \cap (\varkappa_1\varkappa_4, \varkappa_2, \varkappa_4)\\ &= (\varkappa_1\varkappa_4, \varkappa_3\varkappa_4, \varkappa_3^2, \varkappa_4) \cap (\varkappa_1\varkappa_4, \varkappa_3\varkappa_4, \varkappa_3^2 : \varkappa_4) \cap (\varkappa_2, \varkappa_4)\\ &= (\varkappa_3^2, \varkappa_4) \cap (\varkappa_2, \varkappa_4) \cap (\varkappa_1, \varkappa_3, \varkappa_3^2)\\ &= (\varkappa_3^2, \varkappa_4) \cap (\varkappa_2, \varkappa_4) \cap (\varkappa_1, \varkappa_3).\end{aligned}$$

Definition 1.2.66. A P-graded ring is such type of a ring ω having a decomposition

$$\omega = \bigoplus_{p\in\mathsf{P}} \omega_p,$$

such that $\omega_p\omega_q \subset \omega_{p+q}\ \forall\ p, q \in \mathsf{P}$.
Then for $r \in \omega$, we can write a unique expression

$$r = \sum_{p\in\mathsf{P}} r_p,$$

where $r_p \in \omega_p$ and almost all $r_p = 0$. The element r_p is called the *pth* homogeneous component and if $r = r_p$, then r is homogeneous of degree p.

1.3 Module Theory

In this section we will discuss some basics of Module Theory.

Definition 1.3.1. Let ω be a ring. The ω -module $\daleth$ is a abelian group and action of ω on $\daleth$ is a map

$$\cdot : \omega \times \daleth \to \daleth$$

defined as $\cdot((\varkappa, \tau)) = \gamma\tau$, satisfying these axioms

1. $\varkappa(\tau_1 + \tau_2) = \varkappa\tau_1 + \varkappa\tau_2$,

2. $(\varkappa_1 + \varkappa_2)\tau_1 = \varkappa_1\tau_1 + \varkappa_2\tau_1$,

3. $(\varkappa_1\varkappa_2)\tau = \varkappa_1(\varkappa_2\tau_1)$,

4. $1\tau_1 = \tau_1$,

$$\forall\ \varkappa_1, \varkappa_2 \in \omega \ and\ \tau_1, \tau_2 \in \daleth.$$

Modules satisfying axim 4 are called unital modules.

Example 1. 1. $(\mathbb{Z}(\sqrt{a}), +)$ is a $\mathbb{Z}$-module, where a is any integer.

2. All abelian groups are examples of $\mathbb{Z}$-modules.

Definition 1.3.2. Let ω be a ring and $\daleth$ be an ω-module, then $\mathsf{N} \subseteq \daleth$ is submodule of $\daleth$ if it meets the following axioms

1. $\mathsf{N} \neq \varnothing$ and

2. $\tau_1 + \varkappa\tau_2 \in \mathsf{N}$, where $\varkappa \in \omega$ and $\tau_1, \tau_2 \in \mathsf{N}$.

Definition 1.3.3. Let $\daleth_1$ and $\daleth_2$ be ω-modules, then there sum is defined as

$$\daleth_1 + \daleth_2 = \{\tau_1 + \tau_2 \quad \tau_1 \in \daleth_1, \tau_2 \in \daleth_2\}.$$

Definition 1.3.4. Let $\daleth$ and N be ω-modules. A map $\Gamma : \daleth \to \mathsf{N}$ is known as ω-module homomorphism if it satisfies

- $\Gamma(\tau_1 + \tau_2) = \Gamma(\tau_1) + \Gamma(\tau_2)$, for all $\tau_1, \tau_2 \in \daleth$.
- $\Gamma(\varkappa\tau) = \varkappa\Gamma(\tau)$, for all $\varkappa \in \omega$, $\tau \in \daleth$.

Remark 1.3.5. An ω-module homomorphism also satisfies the axioms of additive group homomorphism but converse is not always true. If ω is a ring and ω is ω-module, then ω-module homomorphim need not be ring homomorphim.

Example 2. Let $\mathbb{Z}$ be the ring and map $\Gamma : \mathbb{Z} \to \mathbb{Z}$ is defined as $\Gamma(\varkappa) = n\varkappa$, where n is any positive interger. Γ is ω-module homomorphim but not ring homomorphim.

Definition 1.3.6. Let $\daleth$ be ω-module and $\mathsf{A} \subset \daleth$, then

$$\omega\mathsf{A} = \{\varkappa_1\alpha_1 + \varkappa_2\alpha_2 + \cdots + \varkappa_m\alpha_m :\ \varkappa_1, \varkappa_2, \cdots, \varkappa_m \in \omega,\ \alpha_1, \alpha_2, \cdots, \alpha_m \in \mathsf{A} \quad \text{and } m \in \mathbb{Z}^+\},$$

is called submodule of $\daleth$ generated by A. For any submodule N of $\daleth$ if $\mathsf{N} = \omega\mathsf{A}$ then A is generating set of N and if A is finit set then N is finitely generated submodule of $\daleth$. If $\alpha \in \daleth$ and $\mathsf{N} = \omega\alpha = \{\varkappa\alpha : \quad \varkappa \in \omega\}$ then N is called cyclic submodule of $\daleth$.

Example 3. 1. Let $\mathbb{Z}_3 \times \mathbb{Z}_4$ be $\mathbb{Z}$-module then $\mathsf{A} = \{(1,0),(0,1)\}$ will be generating set of $\mathbb{Z}_3 \times \mathbb{Z}_4$.

2. Let $\mathbb{Z}$-module $\mathbb{Z}$, then $\mathbb{Z}$ is cyclic and generated by $\mathsf{A} = \{1\}$

Definition 1.3.7. Consider $\{\daleth_i\}_{i\in I}$ be the collection of ω-modules, then direct product $\prod_{i\in I} \daleth_i$ is Cartesian product of $\{\daleth_i\}_{i\in I}$ whose elements are of the from $(\tau_i)_{i\in I}$ and $\tau_i \in \daleth_i$ and operations of addition and scalar multiplication is defined as

$$(\tau_i)_{i\in I} + (y_i)_{i\in I} = (\tau_i + y_i)_{i\in I}$$

$$\gamma(\tau_i)_{i\in I} = (\gamma\tau_i)_{i\in I}.$$

The external direct sum of $\{\daleth_i\}_{i\in I}$ are defined as

$$\bigoplus_{i\in I} \daleth_i = \{(\tau_i)_{i\in I} \in \prod_{i\in I} \daleth_i : \text{ only finitely many } \tau_i \neq 0\}.$$

Remark 1.3.8. *If I in above definition is finite, then*

$$\bigoplus_{i\in I} \daleth_i = \prod_{i\in I} \daleth_i.$$

Proposition 1.3.9. *Consider $\mathsf{I}_1, \mathsf{I}_2, \cdots, \mathsf{I}_n$ be submodules of ω-module $\daleth$ then the following are equivalent.*

1. *The function $\Gamma : \mathsf{I}_1 \oplus \mathsf{I}_2 \oplus \cdots \oplus \mathsf{I}_n \to \mathsf{I}_1 + \mathsf{I}_2 + \cdots + \mathsf{I}_n$ defined by $\Gamma(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)$ is an isomorphism that is $\mathsf{I}_1 \oplus \mathsf{I}_2 \oplus \cdots \oplus \mathsf{I}_n \cong \mathsf{I}_1 + \mathsf{I}_2 + \cdots + \mathsf{I}_n$,*

2. *$\mathsf{I}_j \cap (\mathsf{I}_1 + \mathsf{I}_2 + \cdots + \mathsf{I}_{j-1} + \mathsf{I}_{j+1} + \cdots + \mathsf{I}_n) = \{0\}$, for all $j \in \{1, 2, \cdots, n\}$.*

Example 4. Let $\daleth = \omega^3 = \{(\alpha_1, \alpha_2, \alpha_3) : \ \alpha_i \in \omega\}$ be an ω-module

1. Let $\mathsf{I}_1 = \{(\alpha_1, 0, 0) : \ \alpha_1 \in \omega\}$ and $\mathsf{I}_2 = \{(0, \alpha_2, \alpha_3) : \ \alpha_2, \alpha_3 \in \omega\}$ be submodules. As we can see $\daleth = \mathsf{I}_1 + \mathsf{I}_2$ also $\mathsf{I}_1 \cap \mathsf{I}_2 = \{(0,0,0)\}$. Which shows that $\mathsf{I}_1 \oplus \mathsf{I}_2 \cong \mathsf{I}_1 + \mathsf{I}_2$.

2. Let $\mathsf{I}_1 = \{(0, \Upsilon_2, \Upsilon_3) : \ \Upsilon_2, \Upsilon_3 \in \omega\}$ and $\mathsf{I}_2 = \{(\Upsilon_1, 0, \Upsilon_3) : \ \Upsilon_1, \Upsilon_3 \in \omega\}$ be submodules. As we can see $\daleth = \mathsf{I}_1 + \mathsf{I}_2$ and $\mathsf{I}_1 \cap \mathsf{I}_2 \neq \emptyset$. Which shows that $\mathsf{I}_1 \oplus \mathsf{I}_2 \ncong \mathsf{I}_1 + \mathsf{I}_2$.

1.3.1 Free modules

Definition 1.3.10. If $\daleth$ is ω-module and $\mathsf{A} \subset \daleth$, then A is linearly independent if for $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathsf{A}$ and $\varkappa_1, \varkappa_2, \cdots, \varkappa_n \in \omega$.

$$\alpha_1\varkappa_1 + \alpha_2\varkappa_2 + \cdots + \alpha_n\varkappa_n = 0,$$

then $\varkappa_i = 0$ for all i. If for all $m \in \daleth$, we have

$$m = \alpha_1\varkappa_1 + \alpha_2\varkappa_2 + \cdots + \alpha_n, \varkappa_n,$$

then we say A spans $\daleth$.

Definition 1.3.11. Let $\daleth$ is ω-module. $\mathsf{A} \subset \daleth$ are bases of $\daleth$ if A is linearly independent and A spans $\daleth$, then $\daleth$ is called a free ω-module with basis A. $\mid \mathsf{A} \mid$ is called rank of $\daleth$.

Remark 1.3.12. *For $\daleth$ to be free ω-module on subset A it is necessary that every $\tau \in \daleth$ it have unique representation such that $\tau = \tau_1\varkappa_1 + \tau_2\varkappa_2 + \cdots + \tau_n\varkappa_n$. Where $\tau_1, \tau_2, \cdots, \tau_n \in \mathsf{A}$ and $\varkappa_1, \varkappa_2, \cdots, \varkappa_n \in \omega$, for some $n \in \mathbb{Z}^+$.*

Example 5. let $\mathbb{Z}$-module $\mathbb{Z}_4$ and $\mathsf{A} = \{\bar{1}\}$ be generating set. But $\mathbb{Z}_4$ is not free module as $3 \in \mathbb{Z}_4$ and it have more then one representations $3 = 3.\bar{1}$ and $3 = 7.\bar{1}$.

Lemma 1.3.13. *Let $\daleth$ be a finitely generated ω-module of ring and $\mathfrak{J}$ is an ideal of ω such that $\mathfrak{J}$ is subset of Jacobson radical $J(\omega)$. If $\mathfrak{J}\daleth = \daleth$, then $\daleth = 0$.*

Definition 1.3.14. Let ω be a ring. A ω-module $\daleth$ is called Noetherian if every ascending chain of ω-submodule of $\daleth$ is stationary.

Definition 1.3.15. Let ω be a ring. A ω-module $\daleth$ is called Artinian if every descending chain of ω-submodule of $\daleth$ is stationary.

Example 6.
1. Every finite abelian group is both Noetherian and Artinian.
2. The ring $\mathbb{Z}$ (as $\mathbb{Z}$-module) does not satisfy the descending chain condition so it is not Artinian.

Corollary 1.3.16. *If* $\daleth_1, \daleth_2, \cdots, \daleth_n$ *are Noetherian* ω*-module then,* $\bigoplus_{i=1}^{n} \mathsf{M}_\mathsf{i}$ *is also Noetherian.*

Definition 1.3.17. Let $\daleth$ be an ω-module. The annihilator of $\daleth$ is defined in this way

$$\mathrm{Ann}(\daleth) = \{\varkappa \in \omega : \varkappa\daleth = 0\}.$$

1.3.2 Exact sequences

A sequence of ω-module and homomorphisms

$$\cdots \longrightarrow \Gamma_{j-1} \xrightarrow{h_j} \Gamma_j \xrightarrow{h_{j+1}} \Gamma_{j+1} \xrightarrow{h_{i+2}} \Gamma_{j+2} \cdots$$

is said to be exact at Γ_j, if $\mathrm{Im}\,(h_j) = \mathrm{Ker}\,(h_{j+1})$. If the sequence is exact at each Γ_j, then it is called exact sequence.

Proposition 1.3.18. *Let* ω *be a ring and* Γ_a, Γ *and* Γ_b *be* ω*-modules, then*

1. *The sequence* $0 \to \Gamma_a \xrightarrow{h} \Gamma$ *is said exact at* Γ_a *iff* h *is one to one.*

2. *The sequence* $\Gamma \xrightarrow{g} \Gamma_b \to 0$ *is said to be exact at* Γ_b *iff* g *is onto .*

Remark 1.3.19. *The sequence* $0 \to \Gamma_a \xrightarrow{h} \Gamma \xrightarrow{g} \Gamma_b \to 0$ *is short exact iff* h *is one to one,* g *is onto and* $\mathrm{Im}\,(h) = \mathrm{Ker}\,(g)$.

Example 7. Consider sequence $0 \to \mathbb{Z} \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z}_6 \xrightarrow{g} \mathbb{Z}_6 \to 0$, where $h(\gamma) = (\gamma, 0)$ and $g((\alpha, \gamma)) = \gamma$. Then clearly h is one to one and g is onto, so this sequence is a short exact sequence.

Definition 1.3.20. For a P-graded ring ω and ω-module $\daleth$

$$\daleth = \bigoplus_{p \in \mathsf{P}} \daleth_p,$$

with $\omega_p \daleth_q \subset \daleth_{p+q}$ for all $p, q \in \mathsf{P}$, then $\daleth$ is said to be a P-graded module. A non zero element of $\daleth_p$ is called a homogeneous element of degree p.

Definition 1.3.21. For a polynomial ring ω defined over the field K, suppose $\mathbf{b} \in \mathbb{Z}^n$, then $h \in \omega$ is said to be homogeneous of degree $\mathbf{b}$ when h has the form $\beta x^{\mathbf{b}}$, where $\beta \in K$. Also ω is $\mathbb{Z}^n$-graded with graded components:

$$\omega_{\mathbf{b}} = \begin{cases} Kx^{\mathbf{b}}, & \text{if } \mathbf{b} \in \mathbb{Z}^n_+; \\ 0\,, & \text{otherwise.} \end{cases}$$

An ω-module $\daleth$ is $\mathbb{Z}^n$-graded if $\daleth = \bigoplus_{\mathbf{b} \in \mathbb{Z}^n} \daleth_{\mathbf{b}}$ and $\omega_{\mathbf{b_1}} \daleth_{\mathbf{b_2}} \subset \daleth_{\mathbf{b_1}+\mathbf{b_2}}$ for all $\mathbf{b_1}, \mathbf{b_2} \in \mathbb{Z}^n$.

1.4 Graph Theory

In this section, we discuss some fundamentals of Graph Theory. We also discuss different types of graph which we will use in next chapters.

Definition 1.4.1. A graph $\mathcal{W}$ is an ordered pair $(\mathsf{V}(\mathcal{W}), \mathsf{E}(\mathcal{W}))$, where $\mathsf{V}(\mathcal{W})$ can be referred as vertex and $\mathsf{E}(\mathcal{W})$ can be referred as edge set. Each edge consists of two vertices which are its endpoints. If e_1 is an edge whose end points are same then e_1 is a loop. If e_2 and e_3 are the edges with exactly the same set of endpoints then e_2 and e_3 are multiple edges. If edges e_2 and e_3 have a common endpoint then they are adjacent edges. Two vertices joined by an edge is known as adjacent vertices.

Definition 1.4.2. A simple graph is a graph which has no loops and multiple edges.

Definition 1.4.3. The degree of $v \in \mathsf{V}(\mathcal{W})$ in graph $\mathcal{W}$ is the number of edges incident to v. Which can be represented by $\deg(V)$. Each loop at v counts twice. The maximum degree in $\mathcal{W}$ is represented by $\Delta(\mathcal{W})$ and minimum degree in $\mathcal{W}$ is represented by $\delta(\mathcal{W})$.

Definition 1.4.4. A graph with $\Delta(\mathcal{W}) = \delta(\mathcal{W})$ is known as regular graph. If $\Delta(\mathcal{W}) = \delta(\mathcal{W}) = t$, then graph $\mathcal{W}$ is known as $t-$regular graph.

Definition 1.4.5. The totat number of vertices in graph $\mathcal{W}$ is the order of graph represented by $n(\mathcal{W})$ and totat number of edges is size of graph $\mathcal{W}$ represented by $e(\mathcal{W})$.

Definition 1.4.6. The union of graphs $\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_n$ is the graph with vertix set $\cup_{i=1}^n \mathsf{V}(\mathcal{W}_i)$ and edge set $\cup_{i=1}^n \mathsf{E}(\mathcal{W}_i)$.

Definition 1.4.7. A simple graph represented by P_n is a path if its vertices can be ordered in such a way that two vertices have an edge between them iff they are consecutive in the list.

Figure 1.1: P_5

Definition 1.4.8. A simple graph represented by C_n is a cycle if its $\mid \mathsf{V}(\mathcal{W}) \mid=\mid \mathsf{E}(\mathcal{W}_i) \mid$ and vertices can be placed around the circle in such a way that two vertices have an edge between them iff they are consecutively in the circle.

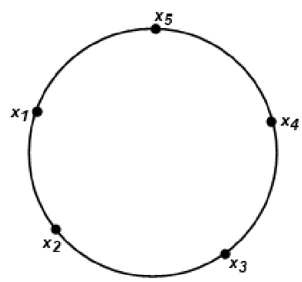


Figure 1.2: C_5

Definition 1.4.9. If in a graph $\mathcal{W}$ there is a path between any two vertices $\nu, \varkappa \in \mathsf{V}(\mathcal{W})$ then $\mathcal{W}$ is termed as connected graph.

Definition 1.4.10. Let $\mathcal{W} = (\mathsf{V}(\mathcal{W}), \mathsf{E}(\mathcal{W}))$, then complement of $\mathcal{W}$, denoted by $\bar{\mathcal{W}}$ is defined by $\mathsf{V}(\mathcal{W}) = \mathsf{V}(\bar{\mathcal{W}})$ howerver, the edge $\nu\varkappa \in \mathsf{E}(\bar{\mathcal{W}})$ iff $\nu\varkappa \notin \mathsf{E}(\mathcal{W})$.

Definition 1.4.11. Let D be a graph and $\mathsf{V}(\mathsf{D}) \subseteq \mathsf{V}(\mathcal{W})$ also $\mathsf{E}(\mathsf{D}) \subseteq \mathsf{E}(\mathcal{W})$ then, D is subgraph of $\mathcal{W}$.

Definition 1.4.12. A decomposition of a graph is a collection of its subgraphs in such a way that each edge apears in exactly one subgraph of the collection.

Definition 1.4.13. The component of a graph $\mathcal{W}$ is its maximal connected subgraph.

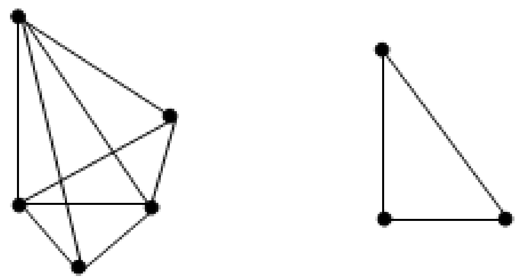


Figure 1.3: Graph on left with its subgraph on right.

Definition 1.4.14. If a vertex $\varkappa \in \mathsf{V}(\mathcal{W})$ having degree zero, then $\varkappa$ is known as isolated vertex. If $\varkappa$ have degree one then it is known as pendent and all the vertices $\nu \in \mathsf{V}(\mathcal{W})$ with $\deg(\nu) \geq 2$ are known as internal vertices.

Definition 1.4.15. A graph with no cycle is known as acyclic graph and such acyclic graphs are called forest.

Definition 1.4.16. A connected acyclic graph is known as tree.

Definition 1.4.17. The radius of a graph is the least of all eccentricities of its vertices.

Definition 1.4.18. Let U and D be two graphs with vertex set $\mathsf{V}(\mathsf{U}) = \{\alpha_1, \alpha_2, \cdots, \alpha_\upsilon\}$ and $\mathsf{V}(\mathsf{D}) = \{\varkappa_1, \varkappa_2, \cdots, \varkappa_\upsilon\}$. The Cartesian product $\mathsf{U}\square\mathsf{D}$ is a graph with $V(\mathsf{U}\square\mathsf{D}) = V(\mathsf{U}) \times V(\mathsf{D})$, and for $\{(\alpha_1, \varkappa_1), (\alpha_2, \varkappa_2)\} \in V(\mathsf{U}\square\mathsf{D})$, whenever

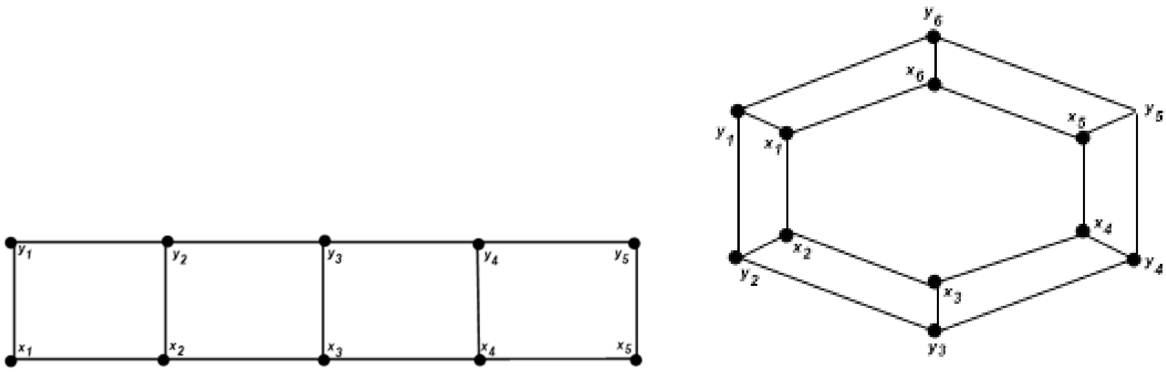
1. $\{\alpha_1, \varkappa_2\} \in E(\mathsf{U})$ and $\varkappa_1 = \varkappa_2$ or

2. $\alpha_1 = \alpha_2$ and $\{\varkappa_1, \varkappa_2\} \in E(\mathsf{D})$.

The ladder graph is Cartesian product of P_2 and P_υ, where $\upsilon \geq 2$. The circular ladder graph is the Cartesian product of P_2 and C_υ, where $\upsilon \geq 3$.

Definition 1.4.19. Let $\omega = \mathsf{T}\{\varkappa_1, \varkappa_2, \cdots, \varkappa_n\}$ and $\mathcal{W}$ be a graph with vertex set $\mathsf{V}(\mathcal{W}) = \{\varkappa_1, \varkappa_2, \cdots, \varkappa_n\}$ and edge set $\mathsf{E}(\mathcal{W})$. A square free monomial ideal generated by the elements related to the edges of the graph $\mathcal{W}$ is called edge ideal.

$$\mathsf{I}(\mathcal{W}) = (\varkappa_j \varkappa_{j+1} \mid \{\varkappa_j, \varkappa_{j+1}\} \in \mathsf{E}(\mathcal{W})) \subset \omega.$$



(a) $P_2 \square P_5$ (b) $P_2 \square C_6$

Figure 1.4: Example of Cartesian product of two graphs.

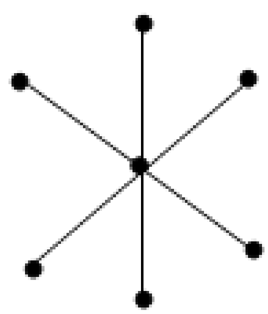


Figure 1.5: $\mathcal{S}_7$

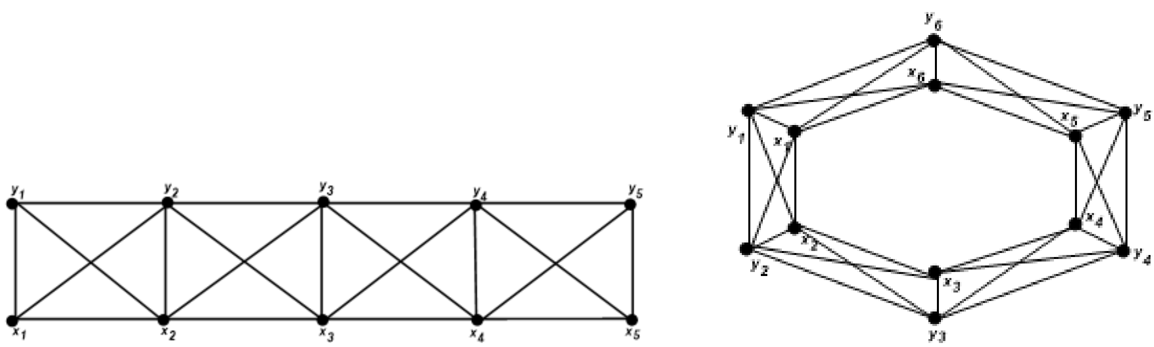
Definition 1.4.20. The $\varrho-$star is a tree on ϱ vertices with one internal vertex and $\varrho - 1$ pendents that are attached to internal vertex. It is denote by $\mathcal{S}_\varrho$.

Definition 1.4.21. Let U and D be two graphs with vertex set $\mathsf{V}(\mathsf{U}) = \{\alpha_1, \alpha_2, \cdots, \alpha_\upsilon\}$ and $\mathsf{V}(\mathsf{D}) = \{\varkappa_1, \varkappa_2, \cdots, \varkappa_\upsilon\}$. The strong product $\mathsf{U} \boxtimes \mathsf{D}$ is a graph with $V(\mathsf{U} \boxtimes \mathsf{D}) = V(\mathsf{U}) \times V(\mathsf{D})$, and for $\{(\alpha_1, \varkappa_1), (\alpha_2, \varkappa_2)\} \in V(\mathsf{U} \boxtimes \mathsf{D})$, whenever

1. $\{\alpha_1, \varkappa_2\} \in E(\mathsf{U})$ and $\varkappa_1 = \varkappa_2$ or

2. $\alpha_1 = \alpha_2$ and $\{\varkappa_1, \varkappa_2\} \in E(\mathsf{D})$ or

3. $\{\alpha_1, \alpha_2\} \in E(\mathsf{U})$ and $\{\varkappa_1, \varkappa_2\} \in E(\mathsf{D})$.

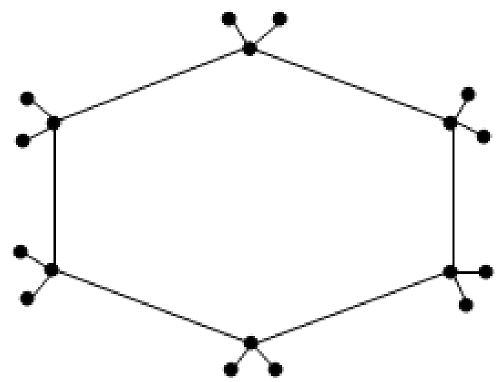


(a) $\mathsf{P}_5 \boxtimes \mathsf{P}_2$ (b) $\mathsf{C}_6 \boxtimes \mathsf{P}_2$

Figure 1.6: Example of strong product of two graphs.

Definition 1.4.22. For any given graph G, the $\varrho-$fold bristled graph $Brs_\varrho(G)$ is obtained by attaching ϱ pendants to each vertex of G.

(a) $Brs_2(\mathsf{P}_5)$



(b) $Brs_2(\mathsf{C}_6)$

Figure 1.7: Example of 2-fold bristled graph.

Chapter 2

Depth and Stanley depth

Richard P. Stanley is known for his work to develop a relationship in Algebra and Geometry. In 1982, Stanley proposed a conjecture [25], which relates the algebraic invariants called Stanley depth and depth. Herzog and Popescu [15, 21] gave some remarkable results related to this conjecture. A while later, many articles have been published in which this conjecture was proved for various special cases. In 2016, Dual et al. [10] disproved it by using result of Herzog et al. [15] they constructed explicit counter example for which the conjecture was not satisfied. In this chapter we will discuss about Stanley's conjecture and some results related to Stanley depth obtained in recent years.

2.1 Depth

Definition 2.1.1. Let ω be a ring and $\daleth$ be a ω-module, then for any subset A of ω, the set

$$\mathrm{Ann}_{\daleth}\,\mathsf{A} = \{\alpha \in \daleth \mid \alpha\mathsf{A} = 0\},$$

is called annihilator of α in $\daleth$.

Definition 2.1.2. Let $\daleth$ be a ω-module. An element $0 \neq \varkappa \in \omega$ is called a regular element on $\daleth$ if whenever $\tau \in \daleth$ and $\varkappa\tau = 0$, then $\tau = 0$.

Definition 2.1.3. Let ω be a ring and $\daleth$ be a ω-module. An element $0 \neq \varkappa \in \omega$ is called zero divisor on module $\daleth$ if there exists $\mathsf{A} \neq 0$ in $\daleth$ such that $\mathbf{A}\varkappa = 0$.

Example 8. Let $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3]$ and $\mathcal{J} = (\varkappa_1^2, \varkappa_1\varkappa_2)$ and consider $\daleth = \omega/\mathcal{J} = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3]/(\varkappa_1^2, \varkappa_1\varkappa_2)$, $\varkappa_1 \in \omega$ also $\varkappa_1 + (\varkappa_1^2, \varkappa_1\varkappa_2) \in \daleth$.

$$\varkappa_1(\varkappa_1 + (\varkappa_1^2, \varkappa_1\varkappa_2)) = \varkappa_1^2 + (\varkappa_1^2, \varkappa_1\varkappa_2)$$

$\varkappa_1$ is a zero divisor on $\daleth$. Now, let $\alpha \neq 0$ in $\daleth$ such that $\alpha + (\varkappa_1^2, \varkappa_1\varkappa_2) \neq (\varkappa_1^2, \varkappa_1\varkappa_2)$. Then clearly $\varkappa_3(\alpha + (\varkappa_1^2, \varkappa_1\varkappa_2)) = \varkappa_3\alpha + (\varkappa_1^2, \varkappa_1\varkappa_2) \neq (\varkappa_1^2, \varkappa_1\varkappa_2)$. Which shows that $\varkappa_3$ is regular on $\daleth$.

Definition 2.1.4. A sequence $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ of elements of ring ω is called an $\daleth$-regular sequence if it satisfies the following conditions:

1. $\varkappa_j$ is regular on $\daleth/(\varkappa_1, \varkappa_2, \dots, \varkappa_{j-1})\daleth$ for any j.
2. $\daleth \neq (\varkappa_1, \varkappa_2, \dots, \varkappa_n)\daleth$.

Definition 2.1.5. Let ω be a local Noetherian ring with unique maximal ideal $\mathfrak{J}$ and $\daleth$ be a finitely generated ω-module. The common length of all maximal $\mathfrak{J}$-sequences in $\mathfrak{J}$ is called the depth of $\daleth$ and is denoted by depth($\daleth$).

2.2 Stanley depth

Let $\omega := \mathsf{T}[\varkappa_1, \varkappa_2, \dots, \varkappa_n]$ be a polynomial ring and $\daleth$ be a $\mathbb{Z}^n$-graded ω-module. Let $\Upsilon_i \in \daleth$ be a homogeneous element and $H_i \in \{\varkappa_1, \varkappa_2, \dots, \varkappa_n\}$, then $\Upsilon_i\mathsf{T}[H_i]$ is T-subspace of $\daleth$ whose generating set is all elements of $\Upsilon_i\mu$, here μ is a monomial in $\mathsf{T}[H_i]$. The space $\Upsilon_i\mathsf{T}[H_i]$ is called a Stanley space of dimension $\mid H_i \mid$, if it is a free $\Upsilon_i\mathsf{T}[H_i]$-module and $\mid H_i \mid$ represents the number of indeterminates of H_i. A Stanley decomposition is finite direct sum of Stanley spaces defined as

$$\mathcal{P} : \daleth = \bigoplus_{i=1}^{s} \Upsilon_i\mathsf{T}[H_i].$$

And

$$\operatorname{sdepth} \mathcal{P} = \min\{\mid H_i \mid, i = 1, \dots, s\}.$$

The Stanley depth of $\daleth$ is

$$\text{sdepth}\,(\daleth) = \max\{\mathcal{P} : \ \mathcal{P} \text{ is a Stanley decomposition of } \daleth\}.$$

In 1982 Richard P. Stanley [25] presented a conjecture given as

$$\text{sdepth}\,(\daleth) \leq \text{depth}\,(\daleth).$$

In [4, 3, 21] this conjecture was proved for $\omega/\mathfrak{J}$, where the polynomial ring ω over field T in atmost three, four and five variables respectively and $\mathfrak{J}$ is an ideal of ω. In 2016, Duval et al. [10] proved that this conjecture is not generally true with the help of a counter example.

2.2.1 The method to comute Stanley depth of a monomial ideal

In this section, we will discuss the method to determine the Stanley depth of $\mathcal{I}/\mathcal{J}$ $\mathbb{Z}^n$-graded module by the method proposed by Herzog et al in [14]. Let $\varkappa^{\alpha_1}, \varkappa^{\alpha_2}, \dots, \varkappa^{\alpha_t}$ be the monomial set of generated of $\mathcal{I}$ and $\mu_1^{\Upsilon_1}, \mu_2^{\Upsilon_2}, \dots, \mu_t^{\Upsilon_t}$ be the monomial set of generated of $\mathcal{J}$. Here, monomial $\varkappa_1^{\alpha(1)}, \varkappa_2^{\alpha(2)}, \dots, \varkappa_n^{\alpha(t)}$ is denoted by $\varkappa^{\alpha}$. Now, choosing $\sigma \in \mathbb{N}^p$ with the condition that $\alpha_i \leq \sigma$ and $\Upsilon_i \leq \sigma$. We define subpost $A^{\sigma}_{\mathcal{I}/\mathcal{J}}$ of $\mathbb{N}^p$ given as

$$A^{\sigma}_{\mathcal{I}/\mathcal{J}} = \{\alpha \in \mathbb{N}^n : \varkappa_\alpha \in \mathcal{I}/\mathcal{J}, \alpha \leq \sigma\}.$$

For square free monomial ideal $\mathcal{I}$ we consider $\mathcal{J} = 0$ and $\sigma = (1, \dots, 1)$. For any $s, m \in A^{\sigma}_{\mathcal{I}}$, where $s \subseteq m$, we define

$$[s, m] = \{\gamma \in A^{(1,\dots,1)}_{\mathcal{I}} : s \subseteq \gamma \subseteq m\}.$$

Partition of $A^{(1,\dots,1)}_{\mathcal{I}}$ is disjoint union of intervals

$$\mathcal{E} : A^{(1,\dots,1)}_{\mathcal{I}} = \cup_{r=1}^{q} [s_r, m_r].$$

The Stanley decomposition corresponding to each partition of $A^{(1,\dots,1)}_{\mathcal{I}}$ is

$$\mathcal{D}(\mathcal{E}) : \mathcal{I} = \bigoplus_{r=1}^{h} \varkappa^{s_r} \mathsf{T}[\{\varkappa_t \mid t \in m_r\}].$$

Clearly, $\operatorname{sdepth}\mathcal{D}(\mathcal{E}) = \min\{|m_1|, \ldots, |m_r|\}$ and

$$\operatorname{sdepth}(\mathfrak{I}) = \max\{\operatorname{sdepth}\mathcal{D}(\mathcal{E}) \mid \mathcal{E} \textit{ is a partition of } A_{\mathfrak{I}}^{(1,\ldots,1)}\}.$$

Example 9. Consider $\omega = \mathsf{T}[\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5]$, $\mathfrak{I} = (\varkappa_1\varkappa_4, \varkappa_2\varkappa_3, \varkappa_1\varkappa_2, \varkappa_1\varkappa_3\varkappa_5)$ be a square-free monomial ideal and $\mathfrak{J} = 0$. Set $\alpha_1 = (1,0,0,1,0)$, $\alpha_2 = (0,1,1,0,0)$, $\alpha_3 = (1,1,0,0,0)$, $\alpha_4 = (1,0,1,0,1)$. Thus $\mathfrak{I}$ is generated by $\varkappa^{\alpha_1}$, $\varkappa^{\alpha_2}$, $\varkappa^{\alpha_3}$ and $\varkappa^{\alpha_4}$. We choose $\xi = (1,1,1,1,1)$. The poset $A = A^{\xi}_{\mathfrak{I}/\mathfrak{J}}$ is

$$\begin{aligned} A = \{&(1,0,0,1,0),(0,1,1,0,0),(1,1,0,0,0),(1,1,0,1,0),(1,0,1,1,0),(1,0,0,1,1),\\ &(1,0,1,0,1),(1,1,1,0,0),(0,1,1,1,0),(0,1,1,0,1),(1,1,0,0,1),(1,1,1,1,0),\\ &(1,1,1,0,1),(1,1,0,1,1),(0,1,1,1,1),(1,0,1,1,1),(1,1,1,1,1)\}. \end{aligned}$$

Partitions of A are given by

$$\begin{aligned} A_1 : &[(1,1,0,0,0),(1,1,1,1,1)]\bigcup[(0,1,1,0,0),(0,1,1,1,1)]\bigcup\\ &[(1,0,0,1,0),(1,0,0,1,1)]\bigcup[(1,0,1,1,0),(1,0,1,1,1)]\bigcup\\ &[(1,0,1,0,1),(1,0,1,0,1)], \end{aligned}$$

$$\begin{aligned} A_2 : &[(1,0,0,1,0),(1,0,1,1,1)]\bigcup[(0,1,1,0,0),(1,1,1,0,1)]\bigcup\\ &[(1,1,0,0,0),(1,1,0,1,1)]\bigcup[(0,1,1,1,0),(1,1,1,1,0)]\bigcup\\ &[(0,1,1,0,1),(1,1,1,1,1)]. \end{aligned}$$

Then, the corresponding Stanley decomposition is

$$\begin{aligned} \mathcal{D}(A_1) := {}&\varkappa_1\varkappa_2\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_2\varkappa_3\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_1\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_4,\varkappa_5]\oplus\\ &\varkappa_1\varkappa_3\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_1\varkappa_3\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_3,\varkappa_5], \end{aligned}$$

$$\begin{aligned} \mathcal{D}(A_2) := {}&\varkappa_1\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_2\varkappa_3\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_5] \oplus \varkappa_1\varkappa_2\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5]\\ &\oplus \varkappa_2\varkappa_3\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4] \oplus \varkappa_2\varkappa_3\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_5]. \end{aligned}$$

Then, it follows that

$$\operatorname{sdepth}(\mathfrak{I}) \geq \max\{\operatorname{sdepth}(\mathcal{D}(A_1)), \operatorname{sdepth}\mathcal{D}(A_2)\}$$

$$\geq \max\{3, 4\}$$
$$= 4.$$

Now for $\omega/\mathcal{I}$, the poset $\mathcal{P} = \mathcal{P}^{\xi}_{\omega/\mathcal{I}}$ is

$$\begin{aligned}\mathcal{P} = \{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),\\&(1,0,1,0,0),(1,0,0,0,1),(0,1,0,1,0),(0,1,0,0,1),(0,0,1,1,0),(0,0,1,0,1),\\&(0,0,0,1,1),(0,1,0,1,1),(0,0,1,1,1)\}.\end{aligned}$$

Partitions of $\mathcal{P}$ are given by

$$\begin{aligned}\mathcal{P}_1 : &[(0,0,0,0,0),(0,0,1,1,1)] \bigcup [(0,1,0,0,0),(0,1,0,1,1)] \bigcup\\&[(1,0,0,0,0),(1,0,0,0,1)] \bigcup [(1,0,1,0,0),(1,0,1,0,0)],\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2 : &[(0,0,0,0,0),(0,1,0,1,1)] \bigcup [(1,0,0,0,0),(1,0,0,0,1)] \bigcup\\&[(0,0,1,0,0),(0,0,1,1,1)] \bigcup [(1,0,1,0,0),(1,0,1,0,0)].\end{aligned}$$

The corresponding Stanley decomposition is

$$\mathcal{D}(\mathcal{P}_1) := \mathsf{T}[\varkappa_2,\varkappa_4,\varkappa_5] \oplus \varkappa_2\mathsf{T}[\varkappa_2,\varkappa_4,\varkappa_5] \oplus \varkappa_1\mathsf{T}[\varkappa_1,\varkappa_5] \oplus \varkappa_1\varkappa_3\mathsf{T}[\varkappa_1,\varkappa_3],$$

$$\mathcal{D}(\mathcal{P}_2) := \mathsf{T}[\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_1\mathsf{T}[\varkappa_1,\varkappa_5] \oplus \varkappa_3\mathsf{T}[\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_1\varkappa_3\mathsf{T}[\varkappa_1,\varkappa_3].$$

$$\begin{aligned}\operatorname{sdepth}(\omega/\mathcal{I}) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\}\\&\geq \max\{2, 2\}\\&\geq 2.\end{aligned}$$

Example 10. Consider $\omega = \mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5]$, $\mathcal{I} = (\varkappa_1\varkappa_3, \varkappa_1\varkappa_5, \varkappa_2\varkappa_4, \varkappa_2\varkappa_5)$ be a square-free monomial ideal and and $\mathcal{J} = 0$. Set $\alpha_1 = (1,0,1,0,0)$, $\alpha_2 = (1,0,0,0,1)$,

$\alpha_3 = (0,1,0,1,0)$, $\alpha_4 = (0,1,0,0,1)$. Thus $\mathcal{I}$ is generated by $\varkappa^{\alpha_1}$, $\varkappa^{\alpha_2}$, $\varkappa^{\alpha_3}$ and $\varkappa^{\alpha_4}$. We choose $\xi = (1,1,1,1,1)$. The poset $A = A^{\xi}_{\mathcal{I}/\mathcal{J}}$ is

$$
\begin{aligned}
A = \{&(1,0,1,0,0),(1,0,0,0,1),(0,1,0,1,0),(0,1,0,0,1),(1,1,1,0,0),(1,0,1,1,0),\\
&(1,0,1,0,1),(1,1,0,0,1),(1,0,0,1,1),(1,1,0,1,0),(0,1,0,1,1),(1,1,1,1,0),\\
&(1,1,1,0,1),(1,1,0,1,1),(0,1,1,1,1),(1,1,1,1,1)\}.
\end{aligned}
$$

Partitions of A are given by

$$
\begin{aligned}
A_1 : &[(0,1,0,1,0),(0,1,0,1,1)] \bigcup [(1,1,1,0,0),(1,1,1,1,1)] \bigcup\\
&[(1,1,0,1,0),(1,1,0,1,1)] \bigcup [(1,0,1,0,0),(1,0,1,0,1)] \bigcup\\
&[(0,1,0,0,1),(0,1,1,1,1)] \bigcup [(1,0,0,0,1),(1,1,0,0,1)] \bigcup\\
&[(1,0,1,1,0),(1,0,1,1,0)] \bigcup [(1,0,0,1,1),(1,0,0,1,1)],
\end{aligned}
$$

$$
\begin{aligned}
A_2 : &[(1,0,1,0,0),(1,1,1,1,1)] \bigcup [(1,0,0,0,1),(1,1,0,1,1)] \bigcup\\
&[(1,0,1,0,1),(0,1,1,1,1)] \bigcup [(0,1,0,0,1),(0,1,0,0,1)].
\end{aligned}
$$

The corresponding Stanley decomposition is

$$
\begin{aligned}
\mathcal{D}(A_1) := {}&\varkappa_2\varkappa_4\mathsf{T}[\varkappa_2,\varkappa_4,\varkappa_5] \oplus \varkappa_1\varkappa_2\varkappa_3\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4] \oplus \varkappa_1\varkappa_2\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5]\oplus\\
&\varkappa_1\varkappa_3\mathsf{T}[\varkappa_1,\varkappa_3,\varkappa_5] \oplus \varkappa_2\varkappa_5\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_1\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_5]\oplus\\
&\varkappa_1\varkappa_3\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_3,\varkappa_4] \oplus \varkappa_1\varkappa_4\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_4,\varkappa_5],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}(A_2) := {}&\varkappa_1\varkappa_3\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_1\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5] \oplus \varkappa_2\varkappa_4\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5]\oplus\\
&\varkappa_2\varkappa_5\mathsf{T}[\varkappa_2,\varkappa_5].
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\operatorname{sdepth}(\mathcal{I}) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(A_1)), \operatorname{sdepth}\mathcal{D}(A_2)\}\\
&\geq \max\{3,2\}\\
&\geq 3.
\end{aligned}
$$

Now for $\omega/\mathcal{I}$, the poset $\mathcal{P} = \mathcal{P}^{\xi}_{\omega/\mathcal{I}}$ is

$$\begin{aligned}\mathcal{P} = \{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),\\ &(1,1,0,0,0),(1,0,0,1,0),(0,1,1,0,0),(0,0,1,1,0),(0,0,1,0,1),(0,0,0,1,1),\\ &(0,1,1,1,0),(0,1,1,0,1),(0,0,1,1,1)\}.\end{aligned}$$

Partitions of $\mathcal{P}$ are given by

$$\begin{aligned}\mathcal{P}_1 : \ &[(0,0,0,0,0),(0,0,1,1,1)]\bigcup[(1,0,0,0,0),(1,1,0,0,0)]\bigcup\\ &[(0,1,0,0,0),(0,1,1,1,0)]\bigcup[(0,1,1,0,0),(0,1,1,0,1)]\bigcup\\ &[(1,0,0,1,0),(1,0,0,1,0)],\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2 : \ &[(0,0,0,0,0),(0,1,1,1,0)]\bigcup[(1,0,0,0,0),(1,0,0,1,0)]\bigcup\\ &[(0,0,0,0,1),(0,0,1,1,1)]\bigcup[(1,1,0,0,0),(1,1,0,0,0)]\bigcup\\ &[(0,1,1,0,1),(0,1,1,0,1)].\end{aligned}$$

The corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) := \ &\mathsf{T}[\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_1\mathsf{T}[\varkappa_1,\varkappa_2] \oplus \varkappa_2\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_4] \oplus \varkappa_2\varkappa_3\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_5]\oplus\\ &\varkappa_1\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_4],\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) := \ &\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_4] \oplus \varkappa_1\mathsf{T}[\varkappa_1,\varkappa_4] \oplus \varkappa_5\mathsf{T}[\varkappa_3,\varkappa_4,\varkappa_5]\varkappa_1\varkappa_2\mathsf{T}[\varkappa_1,\varkappa_2]\oplus\\ &\varkappa_2\varkappa_3\varkappa_5\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_5].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(\omega/\mathcal{I}) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\}\\ &\geq \max\{2,2\}\\ &\geq 2.\end{aligned}$$

Example 11. Consider $\omega = \mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5]$, $\mathcal{I} = (\varkappa_3\varkappa_4, \varkappa_3\varkappa_5, \varkappa_1\varkappa_2, \varkappa_2\varkappa_4)$ be a square-free monomial ideal and and $\mathcal{J} = 0$. Set $\alpha_1 = (0,0,1,1,0)$, $\alpha_2 = (0,0,1,0,1)$, $\alpha_3 = (0,1,0,1,0)$, $\alpha_4 = (1,1,0,0,0)$. Thus, $\mathcal{I}$ is generated by $\varkappa^{\alpha_1}$, $\varkappa^{\alpha_2}$, $\varkappa^{\alpha_3}$ and $\varkappa^{\alpha_4}$.

We choose $\xi = (1, 1, 1, 1, 1)$. The poset $A = A^{\xi}_{\mathcal{J}/\mathcal{J}}$ is

$$A = \{(0,0,1,1,0),(0,0,1,0,1),(0,1,0,1,0),(1,1,0,0,0),(0,1,1,1,0),(0,0,1,1,1),$$
$$(0,1,1,0,1),(1,1,1,0,0),(1,1,0,1,0),(1,1,0,0,1),(0,1,0,1,1),(1,0,1,0,1),$$
$$(1,0,1,1,0),(1,1,1,1,0),(0,1,1,1,1),(1,0,1,1,1),(1,1,1,0,1),(1,1,0,1,1),$$
$$(1,1,1,1,1)\}.$$

Partitions of A are given by

$$A_1 : [(0,1,0,1,0),(1,1,1,1,1)] \bigcup [(0,0,1,1,0),(0,0,1,1,1)] \bigcup$$
$$[(0,0,1,0,1),(1,0,1,1,1)] \bigcup [(1,1,0,0,0),(1,1,1,0,1)] \bigcup$$
$$[(0,1,1,0,1),(0,1,1,0,1)] \bigcup [(1,0,1,1,0),(1,0,1,1,0)],$$

$$A_2 : [(1,1,0,0,0),(1,1,0,1,1)] \bigcup [(0,0,1,1,0),(1,0,1,1,1)] \bigcup$$
$$[(0,0,1,0,1),(1,1,1,0,1)] \bigcup [(0,1,0,1,0),(0,1,1,1,1)] \bigcup$$
$$[(0,0,1,1,1),(1,1,1,1,1)] \bigcup [(1,1,1,0,0),(1,1,1,1,0)].$$

The corresponding Stanley decomposition is

$$\mathcal{D}(A_1) := \varkappa_2\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_3\varkappa_4\mathsf{T}[\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_3\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_3,\varkappa_4,\varkappa_5] \oplus$$
$$\varkappa_1\varkappa_2\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_5] \oplus \varkappa_2\varkappa_3\varkappa_5\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_5,] \oplus \varkappa_1\varkappa_3\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_3,\varkappa_4],$$

$$\mathcal{D}(A_2) := \varkappa_1\varkappa_2\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_4,\varkappa_5] \oplus \varkappa_3\varkappa_4\mathsf{T}[\varkappa_1,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_3\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_5] \oplus$$
$$\varkappa_2\varkappa_4\mathsf{T}[\varkappa_2,\varkappa_3,\varkappa_4,\varkappa_5] \oplus \varkappa_3\varkappa_4\varkappa_5\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4] \oplus \varkappa_1\varkappa_2\varkappa_3\mathsf{T}[\varkappa_1,\varkappa_2,\varkappa_3,\varkappa_4].$$

Then

$$\operatorname{sdepth}(\mathcal{I}) \geq \max\{\operatorname{sdepth}(\mathcal{D}(A_1)), \operatorname{sdepth}\mathcal{D}(A_2)\}$$
$$\geq \max\{3,4\}$$
$$= 4.$$

Now for $\omega/\mathcal{I}$, the poset $\mathcal{P} = \mathcal{P}^{\xi}_{\omega/\mathcal{I}}$ is

$$\mathcal{P} = \{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),(1,0,1,0,0),\\ (1,0,0,1,0),(1,0,0,0,1),(0,1,1,0,0),(0,1,0,0,1),(0,0,0,1,1),(1,0,0,1,1),\\ (0,0,0,0,0)\}.$$

Partitions of $\mathcal{P}$ are given by

$$\mathcal{P}_1 : [(1,0,0,0,0),(1,0,0,1,1)]\bigcup[(0,0,0,0,0),(0,0,0,1,1)]\bigcup\\ [(0,1,0,0,0),(0,1,1,0,0)]\bigcup[(0,0,1,0,0),(1,0,1,0,0)]\bigcup\\ [(0,1,0,0,1),(0,1,0,0,1)],$$

$$\mathcal{P}_2 : [(0,0,0,0,0),(1,0,0,1,1)]\bigcup[(0,1,0,0,0),(0,1,0,0,1)]\bigcup\\ [(0,0,1,0,0),(1,0,1,0,0)]\bigcup[(0,1,1,0,0),(0,1,1,0,0)].$$

The corresponding Stanley decomposition is

$$\mathcal{D}(\mathcal{P}_1) := \varkappa_1\mathsf{T}[\varkappa_1,\varkappa_4,\varkappa_5] \oplus \mathsf{T}[\varkappa_4,\varkappa_5] \oplus \varkappa_2\mathsf{T}[\varkappa_2,\varkappa_3] \oplus \varkappa_3\mathsf{T}[\varkappa_1,\varkappa_3] \oplus \varkappa_2\varkappa_5\mathsf{T}[\varkappa_2,\varkappa_5],$$

$$\mathcal{D}(\mathcal{P}_2) := \mathsf{T}[\varkappa_1,\varkappa_4,\varkappa_5] \oplus \varkappa_2\mathsf{T}[\varkappa_2,\varkappa_5] \oplus \varkappa_3\mathsf{T}[\varkappa_1,\varkappa_3] \oplus \varkappa_2\varkappa_3\mathsf{T}[\varkappa_2,\varkappa_3].$$

Then

$$\begin{aligned}\operatorname{sdepth}(\omega/\mathcal{I}) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1)),\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\}\\ &\geq \max\{2,2\}\\ &\geq 2.\end{aligned}$$

2.3 Some results on Stanley depth and depth

Lemma 2.3.1 ([7]). *If $0 \to \daleth_1 \to \daleth_2 \to \daleth_3 \to 0$ is a short exact sequence of modules over a local ring ω, or a Noetherian graded ring with ω_0 local, then*

1. $\operatorname{depth}(\daleth_2) \geq \min\{\operatorname{depth}(\daleth_3), \operatorname{depth}(\daleth_1)\}$.

2. $\operatorname{depth}(\daleth_1) \geq \min\{\operatorname{depth}(\daleth_2), \operatorname{depth}(\daleth_3) + 1\}$.

3. $\operatorname{depth}(\daleth_3) \geq \min\{\operatorname{depth}(\daleth_1) - 1, \operatorname{depth}(\daleth_2)\}$.

Lemma 2.3.2 ([23, Lemma 2.2]). *Let the sequence* $0 \to \daleth_1 \to \daleth_2 \to \daleth_3 \to 0$ *be short exact sequence of* $\mathbb{Z}^n$*-graded* $\daleth$*-module. Then,*

$$\operatorname{sdepth} \daleth_2) \geq \min\{\operatorname{sdepth} \daleth_1), \operatorname{sdepth} (\daleth_3)\}.$$

Lemma 2.3.3 ([14, lemma 3.6]). *Let* $\mathfrak{J} \subset \omega$ *be an ideal generated by monomials. If* $\omega^* = \omega \otimes_{\mathsf{T}} \mathsf{T}[\varkappa_{p+1}] \cong \omega[\varkappa_{p+1}]$ *and* $\mathfrak{J}^* \subset \omega^*$*. Then,*

$$\operatorname{depth}(\omega^*/\mathfrak{J}^*\omega^*) = \operatorname{depth}(\omega/\mathfrak{J}) + 1.$$

And

$$\operatorname{sdepth}(\omega^*/\mathfrak{J}^*\omega^*) = \operatorname{sdepth}(\omega/\mathfrak{J}) + 1.$$

Lemma 2.3.4 ([1]). *Let* $\mathfrak{J} \subset \omega$ *and* $\mathfrak{J} = I(\mathcal{S}_n) \subseteq \omega$ *is a edge ideal of p-star. Then,*

$$\operatorname{depth}(\omega/\mathfrak{J}) = \operatorname{sdepth}(\omega/\mathfrak{J}) = 1.$$

Lemma 2.3.5 ([12, Lemma 2.12]). *Let* $\mathfrak{J}^* \subset \omega^* = \mathsf{T}[\varkappa_1, \ldots, \varkappa_r]$*,* $\mathfrak{J}^{**} \subset \omega{**} = \mathsf{T}[\varkappa_{r+1}, \ldots, x_p]$ *be the ideal generated by monomials, where* $1 \leq r < p$*. Then,*

$$\operatorname{depth}(\omega^*/\mathfrak{J}^* \otimes_{\mathsf{T}} \omega^{**}/\mathfrak{J}^{**}) = \operatorname{depth}_{\omega^*}(\omega^*/\mathfrak{J}^*) + \operatorname{depth}_{\omega^{**}}(\omega^{**}/\mathfrak{J}^{**}).$$

Corollary 2.3.6 ([23, Corollary 1.3]). *Let* $\mathfrak{J} \subset \omega$ *be the ideal generated by monomials. Then*

$$\operatorname{depth}(\omega/(\mathfrak{J} : \curlyvee)) \geq \operatorname{depth}(\omega/\mathfrak{J}),$$

for all monomials $\curlyvee \notin \mathfrak{J}$*.*

Lemma 2.3.7 ([12, Lemma 2.13]). *Let* $\mathfrak{J}^* \subset \omega^* = \mathsf{T}[\varkappa_1, \ldots, \varkappa_r]$*,* $\mathfrak{J}^{**} \subset \omega^{**} = \mathsf{T}[\varkappa_{r+1}, \ldots, \varkappa_p]$ *be monomial ideals and* $\omega = \mathsf{T}[\varkappa_1, \ldots, \varkappa_r, \varkappa_{r+1}, \ldots, \varkappa_p]$*. Then,*

$$\operatorname{sdepth}_{\omega^*}(\omega^*/\mathfrak{J}^*) + \operatorname{sdepth}_{\omega^{**}}(\omega^{**}/\mathfrak{J}^{**}) \leq \operatorname{sdepth}(\omega^{**}/\mathfrak{J}^* \otimes_{\mathsf{T}} \omega^{**}/\mathfrak{J}^{**}).$$

Proposition 2.3.8 ([9, Proposition 2.7]). *Let $\mathfrak{J} \subset \omega$ be the ideal generated by monomials. Then, for all monomial $\Upsilon \notin \mathfrak{J}$*

$$\operatorname{sdepth}(\omega/\mathfrak{J}) \leq \operatorname{sdepth}(\omega/(\mathfrak{J} : \Upsilon)).$$

Lemma 2.3.9 ([17, Lemma 3.3]). *Let $\mathfrak{J} \subset \omega$ be a monomial ideal and ideal $\mathfrak{J}$ be square-free with $\operatorname{supp}(\omega) = \{\varkappa_1, \varkappa_2, \dots, \varkappa_p\}$, let $\mu := \varkappa_{i_1}\varkappa_{i_2}\dots\varkappa_{i_p} \in \omega/\mathfrak{J}$, such that $\varkappa_n\mu \in \mathfrak{J}$, for all $n \in \{\varkappa_1, \varkappa_2, \dots, \varkappa_p\} \setminus \operatorname{supp}(\mu)$. Then $\operatorname{sdepth}(\omega/\mathfrak{J}) \leq p$.*

Lemma 2.3.10 ([20, Lemma 2.8]). *Let $I = I(P_s)$ be an edge ideal of a path graph on s vertices and $s \geq 2$. Then,*

$$\operatorname{depth}(\omega/I) = \lceil \frac{s}{3} \rceil.$$

Proposition 2.3.11 ([8, Proposition 1.3]). *Let $I = I(C_s)$ be an edge ideal of a cycle on s vertices and $s \geq 3$. Then,*

$$\operatorname{depth}(\omega/I) = \lceil \frac{s-1}{3} \rceil.$$

Proposition 2.3.12 ([14, Proposition 3.4]). *Let $\mathfrak{J} \subset \omega$ be the ideal minimally generated by r elements. Then,*

$$\operatorname{sdepth}(\mathfrak{J}) \geq \max\{n - r + 1, 1\}.$$

Lemma 2.3.13. *Let $I = I(\mathcal{C}_{3,m})$ be an edge ideal of m-fold bristled graph of cycle C_3. Then,*

$$\operatorname{depth}(\omega/I(\mathcal{C}_{3,m})) = \operatorname{sdepth}(\omega/I(\mathcal{C}_{3,m})) = 2m + 1.$$

Proposition 2.3.14 ([26, Proposition 4.2]). *Let square free monomial ideal $\mathfrak{J} \subset \omega = \mathsf{T}[\varkappa_1, \varkappa_2, \dots, \varkappa_s]$ be minimally generated by 4 elements. Then,*

$$\operatorname{sdepth}(\mathfrak{J}) \geq s - 2.$$

Theorem 2.3.15 ([5, Theorem 2.2]). *Let maximal ideal $\mathfrak{J}$ generated by $(\varkappa_1, \dots, \varkappa_s) \subseteq \omega$. Then,*

$$\operatorname{sdepth}(\mathfrak{J}) = \lceil \frac{s}{2} \rceil.$$

Lemma 2.3.16 ([22, Lemma 3.1]). *Let* T *be a tree graph and* $I = I(\mathsf{T})$ *be the edge ideal of it. Let* s *be the total number of leaves and* $r \geq 1$ *be the diameter of graph. Then,*

$$\text{sdepth}\,(\omega/I) \geq \lceil \frac{s+r-1}{3} \rceil.$$

Theorem 2.3.17 ([19, Theorem 5.1]). *Let* $I = I(P_{s,t})$ *be the edge ideal of strong product of two paths on* s *and* t *vertices. Then for* $s \geq 2$,

$$\text{depth}\,(\omega_{s,t}/I(P_{s,t})), \text{sdepth}\,(\omega_{s,t}/I(P_{s,t})) \geq \lceil \frac{s}{3} \rceil \lceil \frac{t}{3} \rceil.$$

Theorem 2.3.18 ([19, Theorem 5.3]). *Let* $I = I(C_{s,t})$ *be the edge ideal of strong product of path and cycle on* s *and* t *vertices respectively. Then for* $s \geq 3$ *and* $m \geq 1$,

$$\text{depth}\,(\omega_{s,t})/I(C_{s,t}) \leq \begin{cases} \lceil \frac{s-1}{3} \rceil + (\lceil \frac{r}{3} \rceil - 1) \lceil \frac{s}{3} \rceil, & \textit{if } t \equiv 1, 2 (mod 3); \\ \lceil \frac{s}{3} \rceil \lceil \frac{r}{3} \rceil, & \textit{if } t \equiv 0 (mod 3). \end{cases}$$

Lemma 2.3.19 ([18, Lemma 3.2]). *Let* H_s *be the edge ideal of the line graph of ladder graph. Then for* $s \geq 2$,

$$\lceil \frac{s}{2} \rceil \leq \text{depth}\,(\omega_s/H_s), \text{sdepth}\,(\omega_s/H_s) \leq s-1.$$

Lemma 2.3.20 ([18, Lemma 3.6]). *Let* A_s *be the edge ideal of the line graph of circular ladder graph. Then for* $s \geq 3$.

$$\lceil \frac{s}{2} \rceil \leq \text{depth}\,(\omega_s/A_s) \leq s-1.$$

And

$$\lceil \frac{s}{2} \rceil \leq \text{depth}\,(\omega_s/A_s) \leq s.$$

Theorem 2.3.21 ([16, Theorem 2.1]). *Let* $\mathfrak{J}$ *and* $\mathfrak{O}$ *be ideals in* ω *generated by monomials such that* $\mathfrak{J} \subset \mathfrak{O}$. *Then,*

$$\text{sdepth}\,(\mathfrak{O}/\mathfrak{J}) \leq \text{sdepth}\,(\sqrt{\mathfrak{O}}/\sqrt{\mathfrak{J}}).$$

Chapter 3

Stanley depth and depth of the quotient rings of the edge ideals corresponding to ladder graph and a class of strong product of two paths

In this chapter, we compute Stanley depth and depth of the quotient modules of edge ideal associated with the $\varrho-$fold bristled graph of ladder graph and L_υ graph, where $L_\upsilon = \mathsf{P}_2 \boxtimes \mathsf{P}_\upsilon$ and $\upsilon \geq 1$. We prove for these graphs, the depth and Stanley depth values are equal.

Througout this chapter, we set $\digamma_{\upsilon,\varrho} := \mathsf{T}[\cup_{i=1}^{\upsilon}\{x_i, y_i\}, \cup_{j=1}^{\varrho}\{x_{1j}, x_{2j}, \dots, x_{\upsilon j}, y_{1j}, y_{2j}, \dots, y_{\upsilon j}\}]$, here υ is the total of vertices of path P_υ and ϱ is the total number of pendant vertices attached at each x_i.

Definition 3.0.1. The ladder graph is Cartesian product of P_υ and P_2, where $\upsilon \geq 2$. We denote this graph by $D_\upsilon = \mathsf{P}_2 \square \mathsf{P}_\upsilon$. The $\varrho-$fold bristled graph $Brs_\varrho(D_\upsilon)$ is obtained by attaching ϱ pendants to each vertex of D_υ. Figure 3.1 shows $Brs_2(D_4)$.

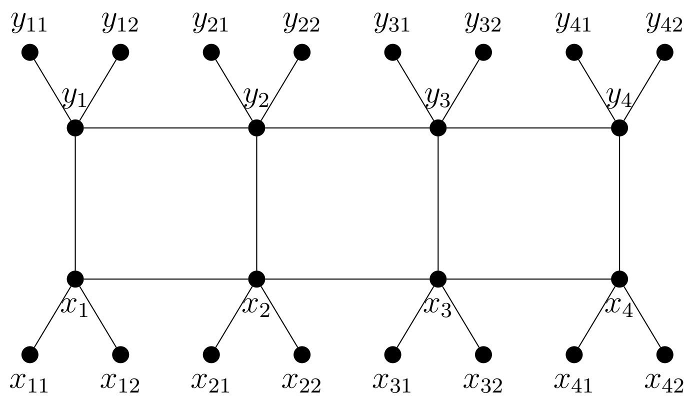


Figure 3.1: $Brs_2(D_4)$

Definition 3.0.2. Let L_υ denotes the strong product of two paths P_2 and P_υ by $L_\upsilon = P_2 \boxtimes \mathsf{P}_\upsilon$, where $\upsilon \geq 1$. The $\varrho-$fold bristled graph $Brs_\varrho(L_\upsilon)$ is obtained by attaching ϱ pendants to each vertex of L_υ. Figure 3.2 shows $Brs_2(L_4)$.

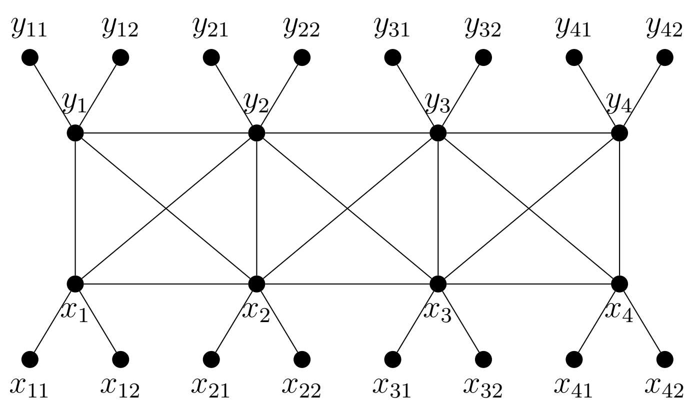


Figure 3.2: $Brs_2(L_4)$

Definition 3.0.3. Let $\varrho \geq 1$ and $\upsilon \geq 2$. If edge ideal $I_{\upsilon,\varrho} = I(Brs_\varrho(D_\upsilon))$, then its minimal generating set is stated as

$$\mathcal{G}(I_{\upsilon,\varrho}) := \bigcup_{i=1}^{\upsilon-1}\{x_i x_{i+1}, y_i y_{i+1}\} \bigcup_{i=1}^{\upsilon}\{x_i y_i\} \bigcup_{j=1}^{\varrho}\{y_1 y_{1j}, \dots, y_\upsilon y_{\upsilon j}, x_1 x_{1j}, \dots, x_\upsilon x_{\upsilon j}\}.$$

We also define a modified graph of $Brs_{\varrho}(D_{\upsilon})$ denoted by $D'_{\upsilon,\varrho}$, with the set of vertices $V(D'_{\upsilon,\varrho}) := V(Brs_{\varrho}(D_{\upsilon})) \bigcup \{y_{\upsilon+1}\} \bigcup_{j=1}^{\varrho} y_{(\upsilon+1)j}$ and edge set $E(D'_{\upsilon,\varrho}) := E(I_{\upsilon,\varrho}) \bigcup \{y_{\upsilon}y_{\upsilon+1}\} \bigcup_{j=1}^{\varrho} \{y_{\upsilon}y_{(\upsilon+1)j}\}$. Figure 3.3 shows $D'_{4,2}$ graph. We assume $\digamma^{*}_{\upsilon,\varrho} := \digamma_{\upsilon,\varrho}[y_{\upsilon+1}, \bigcup_{j=1}^{\varrho} y_{(\upsilon+1)j}]$ and edge ideal $I^{*}_{\upsilon,\varrho} = I(D'_{\upsilon,\varrho})$, then its minimal generating set is stated as

$$\mathcal{G}(I^{*}_{\upsilon,\varrho}) := \mathcal{G}(I_{\upsilon,\varrho}) \bigcup \{y_{\upsilon}y_{\upsilon+1}\} \bigcup_{j=1}^{\varrho} \{y_{\upsilon}y_{(\upsilon+1)j}\}.$$

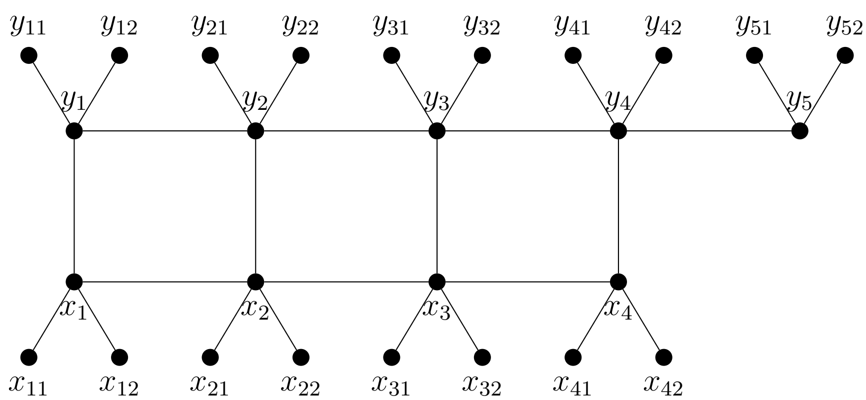


Figure 3.3: $D'_{4,2}$

Definition 3.0.4. Let $\upsilon = 1$, then minimal generating set of the edge ideal of $\varrho-$fold bristled graph of L_1 is stated as

$$\mathcal{G}(L_{1,\varrho}) := \{x_1y_1\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, y_1y_{1j}\}.$$

When $\upsilon \geq 2$ it is given as:

$$\mathcal{G}(L_{\upsilon,\varrho}) := \cup_{i=1}^{\upsilon-1}\{x_ix_{i+1}, y_iy_{i+1}\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, \ldots, x_{\upsilon}x_{\upsilon j}, y_1y_{1j}, \ldots, y_{\upsilon}y_{\upsilon j}\} \cup_{i=1}^{\upsilon} \{x_iy_i\}$$
$$\cup\{y_1x_2, y_{\upsilon}x_{\upsilon-1}\} \cup_{j=2}^{\upsilon-1} \{y_ix_{i-1}, y_ix_{i+1}\}.$$

To prove our major results in our coming section, first we will prove this Lemma.

Remark 3.0.1. The $\varrho-$fold bristled graph of the path P_{υ} is denote by $P_{\upsilon,\varrho}$ Let $\digamma'_{\upsilon,\varrho} = [\bigcup_{i=1}^{\upsilon}\{x_i\}, \bigcup_{j=1}^{\varrho}\{x_{1j}, x_{2j}, \ldots, x_{\upsilon j}\}]$ and $\mathcal{G}(P_{\upsilon,\varrho}) := \bigcup_{i=1}^{\upsilon-1}\{x_ix_{i+1}\} \bigcup_{j=1}^{\varrho}\{x_1x_{1j}, \ldots, x_{\upsilon}x_{\upsilon j}\}$.

Lemma 3.0.2. *Let $\varrho \geq 1$ and $n = 2, 3$, then*

$$\operatorname{depth}(\digamma'_{\upsilon,\varrho}/P_{\upsilon,\varrho}) = \operatorname{sdepth}(\digamma'_{\upsilon,\varrho}/P_{\upsilon,\varrho}) = \begin{cases} \varrho+1, & \text{if } \upsilon = 2; \\ \varrho+2, & \text{if } \upsilon = 3. \end{cases}$$

Proof. We will prove this for each value of υ separately. Consider the following exact sequence.

$$0 \longrightarrow \digamma'_{\upsilon,\varrho}/(P_{\upsilon,\varrho} : x_\upsilon) \xrightarrow{\cdot y_\upsilon} \digamma'_{\upsilon,\varrho}/P_{\upsilon,\varrho} \longrightarrow \digamma'_{\upsilon,\varrho}/(P_{\upsilon,\varrho}, x_\upsilon) \longrightarrow 0.$$

Case 1. For $\upsilon = 2$, $\varrho \geq 1$ we have $\digamma'_{2,\varrho}/(P_{2,\varrho} : x_2) \cong \mathsf{T}[x_2, \cup_{i=1}^{\upsilon}\{x_{1j}\}]$. Thus by Lemma 2.3.3 $\operatorname{depth}(\digamma'_{2,\varrho}/(P_{2,\varrho} : x_2)) = \varrho + 1$. Also it is clear that $\digamma'_{2,\varrho}/(P_{2,\varrho}, x_2) \cong \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{i=1}^{\upsilon}\{x_{2j}\}]$. By Lemma 2.3.4 and Lemma 2.3.3,
$\operatorname{depth}(\digamma'_{2,\varrho}/(P_{2,\varrho}, x_2)) = \varrho + 1$. Therefore by Depth Lemma $\operatorname{depth}(\digamma'_{2,\varrho}/P_{2,\varrho}) = \varrho + 1$.
Case 2. For $\upsilon = 3$, $\varrho \geq 1$ as we can see $\digamma'_{3,\varrho}/(P_{3,\varrho} : x_3) \cong \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[x_3, \cup_{i=1}^{\upsilon}\{x_{2j}\}]$, by Lemma 2.3.4 and Lemma 2.3.3 $\operatorname{depth}(\digamma'_{3,\varrho}/(P_{3,\varrho} : x_3)) = 1+\varrho+1 = \varrho+2$. Now $\digamma'_{3,\varrho}/(P_{3,\varrho}, x_3) \cong \digamma'_{2,\varrho}/(P_{2,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{i=1}^{\upsilon}\{x_{3j}]$, using previous case and Lemma 2.3.3 we get $\operatorname{depth}(\digamma'_{3,\varrho}/(P_{3,\varrho}, x_3)) = \varrho + 1 + \varrho = 2\varrho + 1$. As $\operatorname{depth}(\digamma'_{3,\varrho}/(P_{3,\varrho} : x_3)) \leq \operatorname{depth}(\digamma'_{3,\varrho}/(P_{3,\varrho}, x_3))$, by Depth Lemma $\operatorname{depth}(\digamma'_{3,\varrho}/P_{3,\varrho}) = \varrho + 2$.

For the case of Stanley depth we get the same result by following Lemma 2.3.2 instead of Depth Lemma, for upper bound in case 2 as $\operatorname{depth}(\digamma'_{3,\varrho}/(P_{3,\varrho} : x_3)) = \varrho+2$, using Proposition 2.3.8 $\operatorname{depth}(\digamma'_{3,\varrho}/P_{3,\varrho}) \geq \varrho + 2$. □

3.1 Stanley depth and depth of the quotient rings of the edge ideals corresponding to $\varrho-$fold bristled graph of ladder graph and strong product of two paths.

In this section, we determine the Stanley depth and depth of the quotient modules of the edge ideal of $\varrho-$fold bristled graph of ladder graph and L_υ graph. We prove for these graphs, the depth and Stanley depth values are equal.
To determine these invariants, we shall first determine these values for the quotient modules associated with edge ideal of $D'_{\upsilon,\varrho}$ graph.

Lemma 3.1.1. *Let $\varrho \geq 1$ and $2 \leq \upsilon \leq 3$. Then*

$$\operatorname{depth}(\mathsf{F}^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) = \operatorname{sdepth}(\mathsf{F}^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) = (\varrho+1)\upsilon + 1.$$

Proof. We will prove this for each value of υ separately. Consider the following exact sequence.

$$0 \longrightarrow \mathsf{F}^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho} : y_\upsilon) \xrightarrow{\cdot y_\upsilon} \mathsf{F}^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho} \longrightarrow \mathsf{F}^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho}, y_\upsilon) \longrightarrow 0.$$

Let $\upsilon = 2$,

$$\begin{aligned}\mathcal{G}(I^*_{2,\varrho} : y_2) &= \{x_1x_2, x_1y_1, x_2, y_1, y_3\} \cup_{j=1}^{\varrho} \{y_1y_{1j}, y_3y_{3j}\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, x_2x_{2j}\} \cup_{j=1}^{\varrho} \{y_{2j}\}.\\ &= \{y_1, y_3, x_2\} \cup_{j=1}^{\varrho} \{x_1x_{1j}\} \cup_{j=1}^{\varrho} \{y_{2j}\}.\end{aligned}$$

We have $(\mathsf{F}^*_{2,\varrho}/(I^*_{2,\varrho} : y_2)) \cong \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_2, \cup_{j=1}^{\varrho}\{y_{1j}, y_{3j}, x_{3j}\}]$. Using Lemma 2.3.3

$$\operatorname{depth}(\mathsf{F}^*_{2,\varrho}/(I^*_{2,\varrho} : y_2)) = \operatorname{depth}(\mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1})) + 3\varrho + 1.$$

Thus by Lemma 2.3.4 we get

$$\operatorname{depth}(\mathsf{F}^*_{2,\varrho}/(I^*_{2,\varrho} : y_2)) = 1 + 3\varrho + 1 = 3\varrho + 2.$$

Here

$$\mathcal{G}(I^*_{2,\varrho}, y_2) = \{y_2\} \cup_{j=1}^{\varrho} \{x_1x_2, x_1y_1, x_1x_{1j}, x_2x_{2j}, y_1y_{1j}\} \cup_{j=1}^{\varrho} \{y_3y_{3j}\}.$$

As we can see that $(\mathsf{F}^*_{2,\varrho}/(I^*_{2,\varrho}, y_2)) \cong \mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} y_{2j}]$. Therefore, by Lemma 2.3.5 and Lemma 2.3.3

$$\operatorname{depth}(\mathsf{F}^*_{2,\varrho}/(I^*_{2,\varrho}, y_2)) = \operatorname{depth}(\mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho})) + \operatorname{depth}(\mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1})) + \varrho.$$

By Lemma 2.3.4 and Lemma 3.0.1, we have

$$\operatorname{depth}(\mathsf{F}^*_{2,\varrho}/(I^*_{2,\varrho}, y_2)) = (\varrho + 2) + 1 + \varrho = 2\varrho + 3.$$

Now by Depth Lemma

$$\operatorname{depth}(\mathsf{F}^*_{2,\varrho}/I^*_{2,\varrho}) \geq \min\{\operatorname{depth}(\mathsf{F}^*_{2,\varrho}/(I^*_{2,\varrho} : y_2)), \operatorname{depth}(\mathsf{F}^*_{2,\varrho}/(I_{2,\varrho}, y_2))\}.$$

This implies that

$$\operatorname{depth}(F^*_{2,\varrho}/I^*_{2,\varrho}) \geq 2\varrho + 3.$$

For upper bound as we have $\mathcal{G}(I^*_{2,\varrho} : y_3) = \{y_2\} \cup \{x_1x_2, x_1y_1\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, x_2x_{2j}, y_1y_{1j}\} \cup_{j=1}^{\varrho} \{y_{3j}\}$, and $(F^*_{2,\varrho}/(I^*_{2,\varrho} : y_3)) \cong \mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_3, \cup_{j=1}^{\varrho} y_{2j}]$. Thus by Lemma 2.3.3 and Lemma 3.0.1.

$$\begin{aligned}\operatorname{depth}(F^*_{2,\varrho}/(I^*_{2,\varrho} : y_3)) &= \operatorname{depth}(\mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho})) + 1 + \varrho \\ &= \varrho + 2 + 1 + \varrho = 2\varrho + 3,\end{aligned}$$

as $y_3 \notin I^*_{2,\varrho}$ so by Corollary 2.3.6.

$$\operatorname{depth}(F^*_{2,\varrho}/I^*_{2,\varrho}) \leq \operatorname{depth}(F^*_{2,\varrho}/(I^*_{2,\varrho} : y_3)) = 2\varrho + 3.$$

This prove the result for $\upsilon = 2$.
Now let $\upsilon = 3$, we have

$$\mathcal{G}(I^*_{3,\varrho} : y_3) = \{y_4, x_3, y_2\} \cup_{j=1}^{\varrho} \{x_1x_2, x_1y_1, x_1x_{1j}, x_2x_{2j}, y_1y_{1j}\} \cup_{j=1}^{\varrho} \{y_{3j}\}.$$

Therefore $(F^*_{3,\varrho}/(I^*_{3,\varrho} : y_3)) \cong \mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_3, \cup_{j=1}^{\varrho}\{y_{2j}, y_{4j}, x_{3j}\}]$. By Lemma 3.0.1 and Lemma 2.3.3

$$\operatorname{depth}(F^*_{3,\varrho}/(I^*_{3,\varrho} : y_3)) = \operatorname{depth}(\mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho})) + 3\varrho + 1 = \varrho + 2 + 3\varrho + 1 = 4\varrho + 3.$$

Clearly

$$\begin{aligned}\mathcal{G}(I^*_{3,\varrho}, y_3) = \{y_3\} \cup_{j=1}^{\varrho} \{x_1x_2, x_1x_3, y_1y_2, x_2y_2, x_1y_1, x_1x_{1j}, x_2x_{2j}, x_3x_{3j}, y_1y_{1j}, y_2y_{2j}\} \\ \cup_{j=1}^{\varrho} \{y_4y_{4i}\},\end{aligned}$$

and $(F^*_{3,\varrho}/(I^*_{3,\varrho}, y_3)) \cong F^*_{2,\varrho}/I^*_{2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} y_{3j}]$. By previous case and Lemma 2.3.4.

$$\begin{aligned}\operatorname{depth}(F^*_{3,\varrho}/(I^*_{3,\varrho}, y_3)) &= \operatorname{depth}(F^*_{2,\varrho}/I^*_{2,\varrho}) + \operatorname{depth}(\mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1})) + \varrho \\ &= (\varrho + 1)2 + 1 + 1 + \varrho = 3\varrho + 4.\end{aligned}$$

So by Depth Lemma

$$\operatorname{depth}(\digamma^*_{3,\varrho}/I^*_{3,\varrho}) \geq 3\varrho + 4.$$

For upper bound as $(\digamma^*_{3,\varrho}/(I^*_{3,\varrho} : y_4)) \cong (\digamma^*_{2,\varrho}/I^*_{2,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_4, \cup_{j=1}^{\varrho} y_{3j}]$. Thus by Lemma 2.3.3 $\operatorname{depth}(\digamma^*_{3,\varrho}/(I^*_{3,\varrho} : y_4)) = \operatorname{depth}(\digamma^*_{2,\varrho}/I^*_{2,\varrho}) + \varrho + 1$. By previous case

$$\operatorname{depth}(\digamma^*_{3,\varrho}/(I^*_{3,\varrho} : y_4)) = 2\varrho + 3 + \varrho + 1 = 3\varrho + 4.$$

As, $y_4 \notin I^*_{3,\varrho}$ by Corollary 2.3.6 $\operatorname{depth}(\digamma^*_{3,\varrho}/I^*_{3,\varrho}) \leq 3\varrho + 4$, and it follows that

For Stanley depth when $2 \leq \upsilon \leq 3$, we follow the same procedure on the short exact sequence as for depth using Lemma 2.3.2. We use Lemma 2.3.7 instead of Lemma 2.3.5 and Proposition 2.3.8 instead of Corollary 2.3.6. We get

$$\operatorname{sdepth}(\digamma^*_{3,\varrho}/I^*_{3,\varrho}) = (\varrho + 1)\upsilon + 1.$$

□

Theorem 3.1.2. *Let $\upsilon \geq 2$ and $\varrho \geq 1$. Then*

$$\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) = \operatorname{sdepth}(\digamma^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) = (\varrho + 1)\upsilon + 1.$$

Proof. When $\upsilon = 2, 3$, the result hold by Lemma 3.1.1. Let $\upsilon \geq 4$. Consider the following exact sequence.

$$0 \longrightarrow \digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho} : y_\upsilon) \xrightarrow{\cdot y_\upsilon} \digamma^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho} \longrightarrow \digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho}, y_\upsilon) \longrightarrow 0.$$

Here

$$\mathcal{G}(I^*_{\upsilon,\varrho} : y_\upsilon) = \{y_{\upsilon+1}, x_\upsilon, y_{\upsilon-1}\} \cup_{i=1}^{\upsilon-1} \{x_i x_{i+1}, x_i y_i\} \cup_{i=1}^{\upsilon-2} \{y_i y_{i+1}\} \cup_{j=1}^{\varrho} \{y_1 y_{1j}, y_2 y_{2j}, \ldots,$$
$$y_{\upsilon-1} y_{(\upsilon-1)j}, y_{\upsilon+1} y_{(\upsilon+1)j}, x_1 x_{1j}, x_2 x_{2j}, \ldots, x_\upsilon x_{\upsilon j}\} \cup_{j=1}^{\varrho} \{y_{\upsilon j}\}.$$

Clearly, $(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho} : y_\upsilon)) \cong \digamma^*_{\upsilon-2,\varrho}/I^*_{\upsilon-2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[y_\upsilon, \cup_{j=1}^{\varrho}\{x_{\upsilon j}, y_{(\upsilon-1)j}, y_{(\upsilon+1)j}\}]$. By Lemma 2.3.3 and induction, $\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho} : y_\upsilon)) = 1 + (\varrho+1)(\upsilon-2) + 3\varrho + 1 = (\varrho+1)\upsilon + \varrho$. Now

$$\mathcal{G}(I^*_{\upsilon,\varrho}, y_\upsilon) = \{y_\upsilon\} \cup_{i=1}^{\upsilon-1} \{x_i x_{i+1}\} \cup_{i=1}^{\upsilon-2} \{y_i y_{i+1}\} \cup_{i=1}^{\upsilon-1} \{x_i y_i\} \cup_{j=1}^{\varrho} \{y_1 y_{1j}, y_2 y_{2j}, \ldots,$$
$$y_{\upsilon-1} y_{(\upsilon-1)j}, x_1 x_{1j}, x_2 x_{2j}, \ldots, x_\upsilon x_{\upsilon j}\} \cup_{j=1}^{\varrho} \{y_{\upsilon+1} y_{(\upsilon+1)j}\}.$$

Here $(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho}, y_\upsilon)) \cong (\digamma^*_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} y_{\upsilon j}]$. Therefore using Lemma 2.3.3 and Lemma 2.3.5 we get

$$\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho}, y_\upsilon)) = \operatorname{depth}(\mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1})) + \operatorname{depth}(\digamma^*_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho}) + \varrho.$$

Using Lemma 2.3.4 also induction on υ

$$\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho}, y_\upsilon)) = (\varrho+1)(\upsilon-1)+1+1+\varrho = 1+(\varrho+1)\upsilon.$$

By Depth Lemma

$$\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho})) \geq (\varrho+1)\upsilon+1,$$

and for upper bound as

$$\mathcal{G}(I^*_{\upsilon,\varrho} : y_{\upsilon+1}) = \{y_\upsilon\} \cup_{j=1}^{\varrho} y_{(\upsilon+1)j} \cup_{i=1}^{\upsilon-1} \{x_i x_{i+1}\} \cup_{i=1}^{\upsilon-2} \{y_i y_{i+1}\} \cup_{j=1}^{\varrho} \{y_1 y_{1j}, y_2 y_{2j}, \dots, \\ y_{\upsilon-1} y_{(\upsilon-1)j}, x_1 x_{1j}, \dots, x_\upsilon x_{\upsilon j}\} \cup_{i=1}^{\upsilon-1} \{x_i y_i\}.$$

Clearly we have $(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho} : y_{\upsilon+1})) \cong \digamma^*_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[y_{\upsilon+1}, \cup_{j=1}^{\varrho}\{y_{\upsilon j}\}]$, using Lemma 2.3.3, we have $\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho} : y_{\upsilon+1})) = \operatorname{depth}(\digamma^*_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho}) + \varrho + 1$. By induction we get

$$\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/(I^*_{\upsilon,\varrho} : y_{\upsilon+1})) = (\varrho+1)(\upsilon-1)+1+\varrho+1 = (\varrho+1)\upsilon+1.$$

As $y_{\upsilon+1} \notin I^*_{\upsilon,\varrho}$, from Corollary 2.3.6. $\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) \leq (\varrho+1)\upsilon+1$. We get

$$\operatorname{depth}(\digamma^*_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) = (\varrho+1)\upsilon+1.$$

For Stanley depth we get the same result using Lemma 2.3.2 instead of Depth Lemma, Lemma 2.3.7 instead of Lemma 2.3.5 and Proposition 2.3.8 instead of Corollary 2.3.6. □

Lemma 3.1.3. *Let $\varrho \geq 1$ and $\upsilon = 2, 3$. Then,*

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) = \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) = (\varrho+1)\upsilon.$$

Proof. We will prove for each υ separately. We have a short exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho} : x_\upsilon) \xrightarrow{\cdot x_\upsilon} \digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho} \longrightarrow \digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho}, x_\upsilon) \longrightarrow 0.$$

Let $\upsilon = 2$, we have

$$\mathcal{G}(I_{2,\varrho} : x_2) = \{x_1, y_2\} \cup_{j=1}^{\varrho} \{y_1 y_{1j}\} \cup_{j=1}^{\varrho} \{x_{2j}\}.$$

Clearly, $(\digamma_{2,\varrho}/(I_{2,\varrho} : x_2)) \cong \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[x_2, \cup_{j=1}^{\varrho}\{x_{1j}, y_{2j}\}]$. Using Lemma 2.3.3 $\operatorname{depth}(\digamma_{2,\varrho}/(I_{2,\varrho} : x_2)) = \operatorname{depth}(\mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1})) + 2\varrho + 1$ by Lemma 2.3.4

$$\operatorname{depth}(\digamma_{2,\varrho}/(I_{2,\varrho} : x_2)) = 1 + 2\varrho + 1 = 2(\varrho + 1).$$

Here,

$$\mathcal{G}(I_{2,\varrho}, x_2) = \{x_2\} \cup_{j=1}^{\varrho} \{x_1 y_1, y_1 y_2, y_1 y_{1j}, y_2 y_{2j}, x_1 x_{1j}\}.$$

As $(\digamma_{2,\varrho}/(I_{2,\varrho}, x_2)) \cong \mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} x_{2j}]$. Therefore using Lemma 2.3.3 we get $\operatorname{depth}(\digamma_{2,\varrho}/(I_{2,\varrho}, x_2)) = \operatorname{depth}(\mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho})) + \varrho$. By Lemma 3.0.1

$$\operatorname{depth}(\digamma_{2,\varrho}/(I_{2,\varrho}, x_2)) = (\varrho + 2) + \varrho = 2(\varrho + 1).$$

So by Depth Lemma, this prove the result for $\upsilon = 2$.
Now let $\upsilon = 3$, Clearly

$$\mathcal{G}(I_{3,\varrho} : x_3) = \{x_2, y_3\} \cup_{j=1}^{\varrho} \{x_1 y_1, y_1 y_2, x_1 x_{1j}, y_1 y_{1j}, y_2 y_{2j}\} \cup_{j=1}^{\varrho} \{x_{3j}\}.$$

Since $(\digamma_{3,\varrho}/(I_{3,\varrho} : x_3)) \cong \mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[x_3, \cup_{j=1}^{\varrho}\{y_{3j}, x_{2j}\}]$, therefore by Lemma 2.3.3 $\operatorname{depth}(\digamma_{3,\varrho}/(I_{3,\varrho} : x_3)) = \operatorname{depth}(\mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho})) + 2\varrho + 1$. By Lemma 3.0.1.

$$\operatorname{depth}(\digamma_{3,\varrho}/(I_{3,\varrho} : x_3)) = (\varrho + 2) + 2\varrho + 1 = 3(\varrho + 1).$$

Now

$$\begin{aligned}\mathcal{G}(I_{3,\varrho}, x_3) &= \{x_3, x_1 x_2, x_1 y_1, y_1 y_2, y_2 y_3\} \cup_{j=1}^{\varrho} \{x_1 x_{1j}, x_2 x_{2j}, y_1 y_{1j}, y_2 y_{2j}, y_3 y_{3j}\} \\ &= \{\mathcal{G}(I_{2,\varrho}^*), x_3\}.\end{aligned}$$

We have $(\digamma_{3,\varrho}/(I_{3,\varrho},x_3)) \cong \digamma^*_{2,\varrho}/I^*_{2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho}\{x_{3j}\}]$. By Lemma 2.3.3

$$\operatorname{depth}(\digamma_{3,\varrho}/(I_{3,\varrho},x_3)) = \operatorname{depth}(\digamma^*_{2,\varrho}/I^*_{2,\varrho}) + \varrho.$$

Using Lemma 3.1.1 we get, $\operatorname{depth}(\digamma_{3,\varrho}/(I_{3,\varrho},x_3)) = 2(\varrho+1)+1+\varrho = 3(\varrho+1)$. So, by depth Lemma, $\operatorname{depth}(\digamma_{3,\varrho}/I_{3,\varrho}) = 3(\varrho+1)$.

For Stanley depth we get the same result using Lemma 2.3.2 instead of Depth Lemma. We have $\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) \geq \upsilon(\varrho+1)$. For upper bound as $x_\upsilon \notin I_{\upsilon,\varrho}$ by Proposition 2.3.8.

$$\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) \leq \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho}:x_\upsilon)) = (\varrho+1)\upsilon.$$

Which completes the proof for Stanley depth. □

Theorem 3.1.4. *Let $\varrho \geq 1$ and $\upsilon \geq 2$. Then*

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) = \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) = (\varrho+1)\upsilon.$$

Proof. We will show this by induction on υ, when $\upsilon = 2,3$, it is already proved in previous Lemma 3.1.3. Let $\upsilon \geq 4$, consider the following exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho}:x_\upsilon) \xrightarrow{\cdot x_\upsilon} \digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho} \longrightarrow \digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho},x_\upsilon) \longrightarrow 0.$$

Note that

$$\mathcal{G}(I_{\upsilon,\varrho}:x_\upsilon) = \{x_{\upsilon-1},y_\upsilon\} \cup_{i=1}^{\upsilon-3}\{x_ix_{i+1}\} \cup_{i=1}^{\upsilon-2}\{y_iy_{i+1}\} \cup_{i=1}^{\upsilon-2}\{x_iy_i\} \cup_{j=1}^{\varrho}\{y_1y_{1j},\dots, y_{\upsilon-1}y_{(\upsilon-1)j}, x_1x_{1j}, x_2x_{2j},\dots, x_{\upsilon-2}x_{(\upsilon-2)j}\} \cup_{j=1}^{\varrho}\{x_{\upsilon j}\}.$$

$$\mathcal{G}(I_{\upsilon,\varrho}:x_\upsilon) = \{\mathcal{G}(I^*_{\upsilon-2,\varrho}), x_{\upsilon-1}, y_\upsilon\} \cup_{j=1}^{\varrho}\{x_{\upsilon j}\}.$$

Where $(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho}:x_\upsilon)) \cong (\digamma^*_{\upsilon-2,\varrho}/I^*_{\upsilon-2,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[x_\upsilon, \cup_{j=1}^{\varrho}\{y_{\upsilon j}, x_{(\upsilon-1)j}\}]$. By Lemma 2.3.3 .

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho}:x_\upsilon)) = \operatorname{depth}(\digamma^*_{\upsilon-2,\varrho}/I^*_{\upsilon-2,\varrho}) + 2\varrho + 1.$$

By Theorem 3.1.2

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho}:x_\upsilon)) = (\varrho+1)(\upsilon-2)+1+2\varrho+1 = (\varrho+1)\upsilon.$$

Now

$$\mathcal{G}(I_{\upsilon,\varrho},x_\upsilon) = \{x_\upsilon\}\cup_{i=1}^{\upsilon-2}\{x_ix_{i+1}\}\cup_{i=1}^{\upsilon-1}\{y_iy_{i+1}\}\cup_{i=1}^{\upsilon-1}\{x_iy_i\}\cup_{j=1}^{\varrho}\{y_1y_{1j},\dots,y_\upsilon y_{\upsilon j},x_1x_{1j},\dots,x_{\upsilon-1}x_{(\upsilon-1)j}\}.$$

Clearly $\mathcal{G}(I_{\upsilon,\varrho},x_\upsilon) = \{\mathcal{G}(I^*_{\upsilon-1,\varrho}),x_\upsilon\}$. Also $(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho},x_\upsilon)) \cong (\digamma^*_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho})\bigotimes_{\mathsf{T}}\mathsf{T}[\cup_{j=1}^{\varrho}x_{\upsilon j}]$, using Lemma 2.3.3 $\operatorname{depth}(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho},x_\upsilon)) = \operatorname{depth}(\digamma^*_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho})+\varrho$. By Theorem 3.1.2.

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho},x_\upsilon)) = (\varrho+1)(\upsilon-1)+1+\varrho = (\varrho+1)\upsilon.$$

Thus by Depth Lemma, $\operatorname{depth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) = (\varrho+1)\upsilon$.

For Stanley depth the get the same result using Lemma 2.3.2 instead of Depth Lemma. We have $\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) \geq \upsilon(\varrho+1)$. For upper bound as $x_\upsilon \notin I_{\upsilon,\varrho}$ by Proposition 2.3.8.

$$\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/I_{\upsilon,\varrho}) \leq \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/(I_{\upsilon,\varrho}:x_\upsilon)) = (\varrho+1)\upsilon.$$

Which completes the proof for Stanley depth. □

Lemma 3.1.5. *Let* $1\leq\upsilon\leq3$ *and* $\varrho\geq1$. *Then*

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) = \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) = \lfloor\frac{3\upsilon}{2}\rfloor\varrho+\lceil\frac{\upsilon}{2}\rceil.$$

Proof. We will prove for each υ separately. We have the following short exact sequence.

$$0\longrightarrow\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho}:x_\upsilon)\xrightarrow{\cdot x_\upsilon}\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}\longrightarrow\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho},x_\upsilon)\longrightarrow0.$$

When $\upsilon=1$ as $L_{1,\varrho}\cong P_{2,\varrho}$, it is clear from Lemma 3.0.1 that, $\operatorname{depth}(\digamma_{1,\varrho}/L_{1,\varrho}) = \operatorname{sdepth}(\digamma_{1,\varrho}/L_{1,\varrho}) = \varrho+1$.
For $\upsilon=2$,

$$\begin{aligned}\mathcal{G}(L_{2,\varrho}:x_2) &= \{x_1y_1,y_1y_2,x_1y_2,x_1,y_2,y_1\}\cup_{j=1}^{\varrho}\{x_1x_{1j},y_1y_{1j},y_2y_{2j}\}\cup_{j=1}^{\varrho}\{x_{2j}\}\\ &= \{x_1,y_2,,y_1\}\cup_{j=1}^{\varrho}\{x_{2j}\}.\end{aligned}$$

So $(\digamma_{2,\varrho}/(L_{2,\varrho}:x_2)) \cong \mathsf{T}[x_2, \cup_{j=1}^{\varrho}\{x_{1j}, y_{1j}, y_{2j}\}]$. Therefore by Lemma 2.3.3

$$\operatorname{depth}(\digamma_{2,\varrho}/(L_{2,\varrho}:x_2)) = 3\varrho + 1.$$

Here

$$\mathcal{G}(L_{2,\varrho}, x_2) = \{x_2\} \cup_{j=1}^{\varrho} \{x_1y_1, y_1y_2, x_1y_2, x_1x_{1j}, y_1y_{1j}, y_2y_{2j}\}.$$

We have $(\digamma_{2,\varrho}/(L_{2,\varrho}, x_2)) \cong \mathsf{T}[V(\mathcal{C}_{3,\varrho})]/I(\mathcal{C}_{3,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} x_{2j}]$. Hence by Lemma 2.3.3 and Lemma 2.3.13

$$\operatorname{depth}(\digamma_{2,\varrho}/(L_{2,\varrho}, x_2)) = \operatorname{depth}(\mathsf{T}[V(\mathcal{C}_{3,\varrho})]/I(\mathcal{C}_{3,\varrho})) + \varrho = 2\varrho + 1 + \varrho = 3\varrho + 1.$$

So, by Depth Lemma

$$\operatorname{depth}(\digamma_{2,\varrho}/(L_{2,\varrho}) = 3\varrho + 1.$$

Now let $\upsilon = 3$,

$$\mathcal{G}(L_{3,\varrho}:x_3) = \{x_1x_2, y_1y_2, y_2y_3, x_1y_2, x_2y_1, x_2y_3, x_1y_1, x_2y_2, , y_2, y_3, x_2\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, y_1y_{1j}, y_2y_{2j}, y_3y_{3j}\} \cup_{j=1}^{\varrho} \{x_{3j}\}$$

$$= \{y_2, y_3, x_2\} \cup_{j=1}^{\varrho} \{x_1y_1, x_1x_{1j}, y_1y_{1j}\} \cup_{j=1}^{\varrho} \{x_{3j}\}.$$

Since $(\digamma_{3,\varrho}/(L_{3,\varrho}:x_3)) \cong \mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[x_3, \cup_{j=1}^{\varrho}\{y_{3j}, x_{2j}, y_{2j}\}]$. Using Lemma 2.3.3 and Lemma 3.0.1,

$$\begin{aligned}\operatorname{depth}(\digamma_{3,\varrho}/(L_{3,\varrho}:x_3)) &= \operatorname{depth}(\mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho})) + 3\varrho + 1\\ &= \varrho + 1 + 3\varrho + 1 = 4\varrho + 2.\end{aligned}$$

And

$$\mathcal{G}(L_{3,\varrho}, x_3) = \{x_1x_2, y_1y_2, y_2y_3, x_1y_2, x_2y_1, x_2y_3, x_1y_1, x_2y_2, x_3\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, x_2x_{2j}, y_1y_{1j}, y_2y_{2j}, y_3y_{3j}\}.$$

Now let $J := \mathcal{G}(L_{3,\varrho}, x_3)$. Consider the following exact sequence.

$$0 \longrightarrow \digamma_{3,\varrho}/(J : y_3) \xrightarrow{\cdot y_3} \digamma_{3,\varrho}/J \longrightarrow \digamma_{3,\varrho}/(J, y_3) \longrightarrow 0.$$

$$\begin{aligned}\mathcal{G}(J : y_3) &= \{x_1x_2, y_1y_2, x_1y_2, x_2y_1, x_1y_1, x_2y_2, , y_2, x_2, x_3\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, y_1y_{1j}\} \cup_{j=1}^{\varrho} \{y_{3j}\} \\ &= \{y_2, x_2, x_3\} \cup_{j=1}^{\varrho} \{x_1y_1, x_1x_{1j}, y_1y_{1j}\} \cup_{j=1}^{\varrho} \{y_{3j}\}.\end{aligned}$$

As we can see $(\digamma_{3,\varrho}/(J : y_3)) \cong \mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_3, \cup_{j=1}^{\varrho}\{x_{2j}, y_{2j}, x_{3j}\}]$, by Lemma 2.3.3 and Lemma 3.0.1

$$\begin{aligned}\operatorname{depth}(\digamma_{3,\varrho}/(J : y_3)) &= \operatorname{depth}(\mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho})) + 3\varrho + 1 \\ &= \varrho + 1 + 3\varrho + 1 = 4\varrho + 2.\end{aligned}$$

Now $\mathcal{G}(J, y_3) = \{x_1x_2, y_1y_2, x_1y_2, x_2y_1, x_1y_1, x_2y_2, y_3, x_3\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, x_2x_{2j}, y_1y_{1j}, y_2y_{2j}\}$

$$= \{\mathcal{G}(L_{2,\varrho}), x_3, y_3\}.$$

We have $(\digamma_{3,\varrho}/(J, y_3)) \cong \digamma_{2,\varrho}/L_{2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho}\{x_{3j}, y_{3j}\}]$. Therefore by Lemma 2.3.3 and previous case, we obtain

$$\operatorname{depth}(\digamma_{3,\varrho}/(J, y_3)) = \operatorname{depth}(\digamma_{2,\varrho}/L_{2,\varrho}) + 2\varrho = 3\varrho + 1 + 2\varrho = 5\varrho + 1,$$

by Depth Lemma, $\operatorname{depth}(\digamma_{3,\varrho}/J) = \operatorname{depth}(\digamma_{3,\varrho}/(L_{3,\varrho}, x_3)) \geq 4\varrho + 2$. As by depth Lemma, $\operatorname{depth}(\digamma_{3,\varrho}/L_{3,\varrho}) \geq \min\{\operatorname{depth}(\digamma_{3,\varrho}/(L_{3,\varrho} : x_3)), \operatorname{depth}(\digamma_{3,\varrho}/(L_{3,\varrho}, x_3))\}$.

$$\operatorname{depth}(\digamma_{3,\varrho}/L_{3,\varrho}) \geq 4\varrho + 2.$$

For upper bound as $\operatorname{depth}(\digamma_{3,\varrho}/(L_{3,\varrho} : y_3)) = \operatorname{depth}(\mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho})) + 3\varrho + 1 = \varrho + 1 + 3\varrho + 1 = 4\varrho + 2$ as $y_3 \notin \digamma_{3,\varrho}$, using Corollary 2.3.6 we get $\operatorname{depth} \digamma_{3,\varrho}/(L_{3,\varrho}) \leq \operatorname{depth}(\digamma_{3,\varrho}/(L_{3,\varrho} : y_3)) = 4\varrho + 2$. By depth Lemma

$$\operatorname{depth}(\digamma_{3,\varrho}/L_{3,\varrho}) = 4\varrho + 2.$$

For Stanley depth we get the same result using Lemma 2.3.2 instead of Depth Lemma and Proposition 2.3.8 instead of Corollary 2.3.6. □

Theorem 3.1.6. *Let $\upsilon \geq 1$ and $\varrho \geq 1$. Then*

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) = \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) = \lfloor \frac{3\upsilon}{2} \rfloor \varrho + \lceil \frac{\upsilon}{2} \rceil.$$

Proof. We will show this result with help of induction on υ. For $1 \leq \upsilon \leq 3$, then we have the desired result by Lemma 3.1.5. we have the following exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho} : x_\upsilon) \xrightarrow{\cdot x_\upsilon} \digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho} \longrightarrow \digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho}, x_\upsilon) \longrightarrow 0.$$

By Depth Lemma

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) \geq \min\{\operatorname{depth}(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho} : x_\upsilon)), \operatorname{depth}(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho}, x_\upsilon))\},$$

and

$$\mathcal{G}(L_{\upsilon,\varrho} : x_\upsilon) = \{y_1x_2, y_{\upsilon-2}x_{\upsilon-3}, x_{\upsilon-1}, y_\upsilon, y_{\upsilon-1}\} \cup_{i=1}^{\upsilon-3} \{x_ix_{i+1}, y_iy_{i+1}\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, \ldots,$$
$$x_{\upsilon-2}x_{(\upsilon-2)j}, y_1y_{1j}, \ldots, y_{\upsilon-2}y_{(\upsilon-2)j}\} \cup_{i=1}^{\upsilon-2} x_iy_i \cup_{j=2}^{\upsilon-3} \{y_jx_{j-1}, y_jx_{j+1}\} \cup_{j=1}^{\varrho} \{x_{\upsilon j}\}.$$

$$\mathcal{G}(L_{\upsilon,\varrho} : x_\upsilon) = \{\mathcal{G}(L_{\upsilon-2,\varrho}), x_{\upsilon-1}, y_\upsilon, y_{\upsilon-1}\} \cup_{j=1}^{\varrho} \{x_{\upsilon j}\}.$$

Here $(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho} : x_\upsilon)) \cong (\digamma_{\upsilon-2,\varrho}/(L_{\upsilon-2,\varrho}) \bigotimes_K \mathsf{T}[x_\upsilon, \cup_{j=1}^{\varrho}\{y_{\upsilon j}, x_{(\upsilon-1)j}, y_{(\upsilon-1)j}\}]$. Using Lemma 2.3.3 and induction on υ, clearly

$$\begin{aligned}\operatorname{depth}(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho} : x_\upsilon)) &= \operatorname{depth}(\digamma_{\upsilon-2,\varrho}/L_{\upsilon-2,\varrho}) + 3\varrho + 1\\ &= \lfloor \frac{3(\upsilon-2)}{2} \rfloor \varrho + \lceil \frac{\upsilon-2}{2} \rceil + 3\varrho + 1\\ &= \lfloor \frac{3\upsilon}{2} \rfloor \varrho + \lceil \frac{\upsilon}{2} \rceil.\end{aligned}$$

Now let $J := \mathcal{G}(L_{\upsilon,\varrho}, x_\upsilon) = \{\mathcal{G}(L_{\upsilon-1,\varrho}), y_\upsilon x_{\upsilon-1}, y_\upsilon y_{\upsilon-1}, x_\upsilon\} \cup_{j=1}^{\varrho} \{y_\upsilon y_{\upsilon j}\}$. Consider the following exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(J : y_\upsilon) \xrightarrow{\cdot y_\upsilon} \digamma_{\upsilon,\varrho}/J \longrightarrow \digamma_{\upsilon,\varrho}/(J, y_\upsilon) \longrightarrow 0. \tag{3.1}$$

Here

$$\mathcal{G}(J : y_\upsilon) = \{\mathcal{G}(L_{\upsilon-2,\varrho}), x_{\upsilon-1}, x_\upsilon, y_{\upsilon-1}\} \cup_{j=1}^{\varrho} \{y_{\upsilon j}\},$$

and $(\digamma_{\upsilon,\varrho}/(J:y_\upsilon) \cong (\digamma_{\upsilon-2,\varrho}/L_{\upsilon-2,\varrho}) \bigotimes_K \mathsf{T}[y_\upsilon, \cup_{j=1}^{\varrho}\{y_{(\upsilon-1)j}, x_{(\upsilon-1)j}, x_{\upsilon j}\}]$. Using Lemma 2.3.3 and induction on υ,

$$\begin{aligned}\operatorname{depth}(\digamma_{\upsilon,\varrho}/(J:y_\upsilon) &= \operatorname{depth}(\digamma_{\upsilon-2,\varrho}/L_{\upsilon-2,\varrho}) + 3\varrho + 1\\ &= \lfloor\frac{3(\upsilon-2)}{2}\rfloor\varrho + \lceil\frac{\upsilon-2}{2}\rceil + 3\varrho + 1\\ &= \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil.\end{aligned}$$

It is easy to check that $\mathcal{G}(J, y_\upsilon) = \{\mathcal{G}(L_{\upsilon-1,\varrho}), x_\upsilon, y_\upsilon\}$ and $(\digamma_{\upsilon,\varrho}/(J, y_\upsilon)) \cong \digamma_{\upsilon-1,\varrho}/L_{\upsilon-1,\varrho} \bigotimes_K \mathsf{T}[\cup_{j=1}^{\varrho}\{y_{\upsilon j}, x_{\upsilon j}\}]$. By Lemma 2.3.3 and induction on υ, we get

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/(J, y_\upsilon)) = \operatorname{depth}(\digamma_{\upsilon-1,\varrho}/L_{\upsilon-1,\varrho}) + 2\varrho = \lfloor\frac{3(\upsilon-1)}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil + 2\varrho$$

$$= \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil.$$

As we can see $\lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil = \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil$. When υ is odd then $\lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil = \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil$ for $\varrho = 1$ and $\lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil \geq \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil$ for $\varrho \geq 2$. So by Depth Lemma $\operatorname{depth}(\digamma_{\upsilon,\varrho}/J) \geq \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil$. Again by Depth Lemma $\operatorname{depth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) \geq \min\{\operatorname{depth}(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho}:x_\upsilon)), \operatorname{depth}(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho}, x_\upsilon))\}$.

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) \geq \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil.$$

For upper bound as $y_\upsilon \notin \digamma_{\upsilon,\varrho}$ and

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho}:y_\upsilon)) = \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil.$$

By Corollary 2.3.6, $\operatorname{depth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) \leq \operatorname{depth}(\digamma_{\upsilon,\varrho}/(L_{\upsilon,\varrho}:y_\upsilon)) = \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil$. We get

$$\operatorname{depth}(\digamma_{\upsilon,\varrho}/L_{\upsilon,\varrho}) = \lfloor\frac{3\upsilon}{2}\rfloor\varrho + \lceil\frac{\upsilon}{2}\rceil.$$

For Stanley depth we get the same result using Lemma 2.3.2 instead of Depth Lemma and Proposition 2.3.8 instead of Corollary 2.3.6. □

Chapter 4

Stanley depth and depth of the quotient rings of edge ideals corresponding to $\varrho-$fold bristled graph of circular ladder graph and a class of strong product of a path and a cycle

In this chapter, we determine the Stanley depth and depth of the quotient rings of edge ideals of $\varrho-$fold bristled graph of circular ladder graph and T_υ graph, where $T_\upsilon = P_2 \boxtimes \mathcal{C}_\upsilon$, where $\upsilon \geq 3$.

Througout this chapter, we assume $\digamma_{\upsilon,\varrho} := \mathsf{T}[\cup_{i=1}^{\upsilon}\{x_i, y_i\}, \cup_{j=1}^{\varrho}\{x_{1j}, x_{2j}, \dots, x_{\upsilon j}, y_{1j}, y_{2j}, \dots, y_{\upsilon j}\}]$, here υ represents the total vertices of cycle C_υ and ϱ shows the total pendant vertices attached at each x_i.

Definition 4.0.1. The circular ladder graph is the cartesian product of P_2 and C_υ, where $\upsilon \geq 3$. We denote this graph by H_υ. The $\varrho-$fold bristled graph $Brs_\varrho(H_\upsilon)$ is obtained by attaching ϱ pendants to each vertex of H_υ. Figure 4.1 shows $Brs_2(H_6)$.

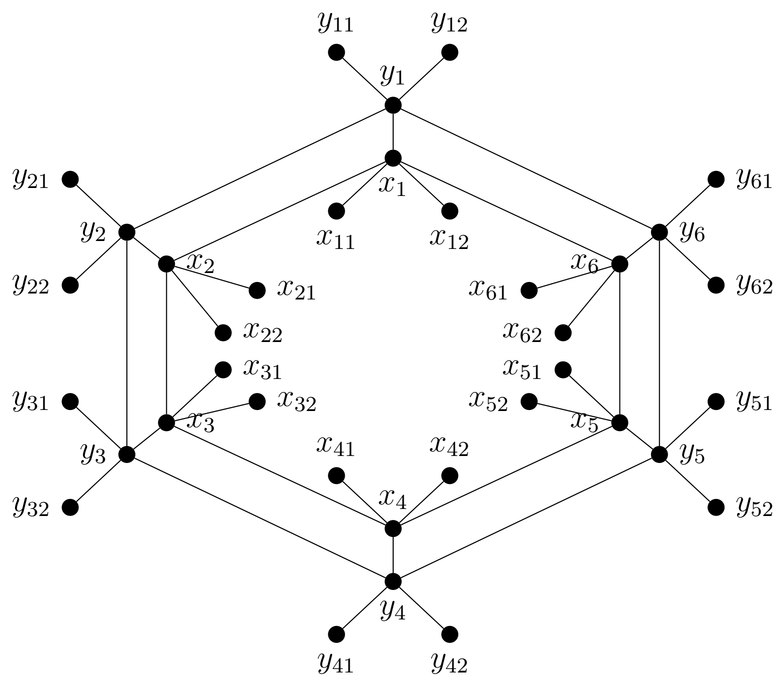


Figure 4.1: $Brs_6(H_2)$

Definition 4.0.2. Let T_υ denotes the strong product of path P_2 and cycle C_υ by $T_\upsilon = P_2 \boxtimes \mathsf{C}_\upsilon$, where $\upsilon \geq 3$. The ϱ-fold bristled graph $Brs_\varrho(T_\upsilon)$ is obtained by attaching ϱ pendants to each vertex of T_υ. Figure 4.2 shows $Brs_2(T_6)$.

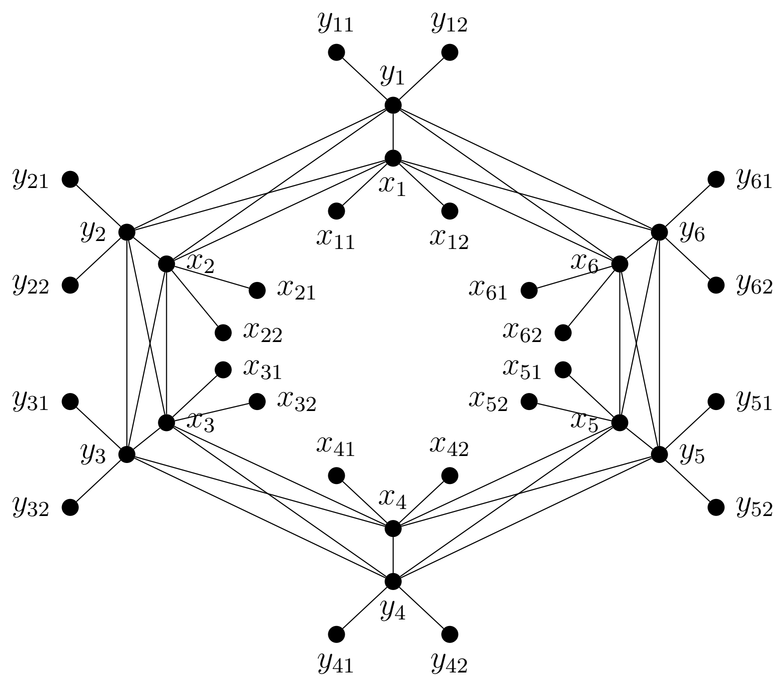


Figure 4.2: $Brs_2(T_6)$

Definition 4.0.3. Let $\varrho \geq 1$ and $\upsilon \geq 3$, then the generating set of edge ideal $\mathfrak{C}_{\upsilon,\varrho} = I(Brs_\varrho(H_\upsilon))$ is

$$\mathcal{G}(\mathfrak{C}_{\upsilon,\varrho}) := \mathcal{G}(I_{\upsilon,\varrho}) \cup \{x_1x_\upsilon, y_1y_\upsilon\}.$$

We also define a new graph $D''_{\upsilon,\varrho}$ with the vertix set $V(D''_{\upsilon,\varrho}) := \bigcup_{i=1}^{\upsilon}\{x_i\} \bigcup_{i=1}^{\upsilon+2}\{y_i\}$ $\bigcup_{j=1}^{\varrho}\{x_{1j}, \dots, x_{\upsilon\varrho}, y_{1\upsilon}, \dots, y_{(\upsilon+2)\varrho}\}$ and edge set of graph is $E(D''_{\upsilon,\varrho}) := \bigcup_{i=1}^{\upsilon-1}\{x_ix_{i+1}\}$ $\bigcup_{i=1}^{\upsilon+1}\{y_iy_{i+1}\} \bigcup_{i=1}^{\upsilon}\{x_iy_{i+1}\} \bigcup_{j=1}^{\varrho}\{y_1y_{1j}, \dots, y_{\upsilon+2}y_{(\upsilon+2)j}, x_1x_{1j}, \dots, x_\upsilon x_{\upsilon j}\}$. Figure 4.3 shows $E_{3,2}$ graph. We set

$$\digamma^{**}_{\upsilon,\varrho} := \digamma_{\upsilon,\varrho}[y_{\upsilon+1}, y_{\upsilon+2}, \bigcup_{j=1}^{\varrho} y_{(\upsilon+1)j}, \bigcup_{j=1}^{\varrho} y_{(\upsilon+2)j}].$$

Then minimally generating set of associated edge ideal for $\upsilon = 1$ is $\mathcal{G}(E_{1,\varrho}) := \{x_1y_2, y_1y_2, y_2y_3\} \bigcup_{j=1}^{\varrho}\{x_1x_{1j}, y_1y_{1j}, y_2y_{2j}, y_3y_{3j}\}$, for $\upsilon \geq 2$ and edge ideal $E_{\upsilon,\varrho} := I(D''_{\upsilon,\varrho})$, it is stated as:

$$\mathcal{G}(E_{v,\varrho}) := \bigcup_{i=1}^{v-1}\{x_ix_{i+1}\} \cup_{i=1}^{v+1} \{y_iy_{i+1}\} \bigcup_{i=1}^{v}\{x_iy_{i+1}\} \bigcup_{j=1}^{\varrho}\{y_1y_{1j}, \dots , y_{v+2}y_{(v+2)j}, x_1x_{1j}, \dots , x_vx_{vj}\}.$$

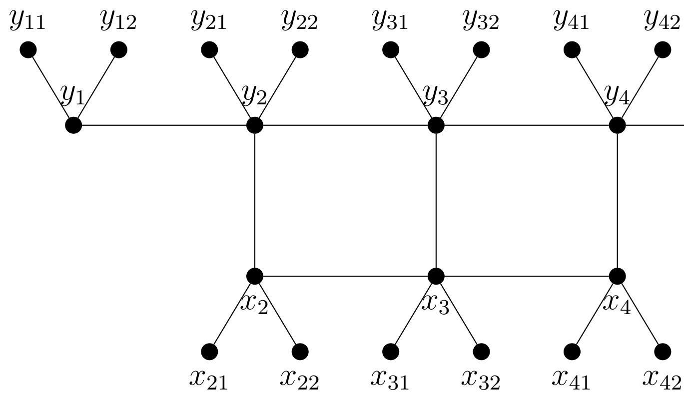


Figure 4.3: $E_{3,2}$

4.1 Results for $\varrho-$fold bristled graph of circular ladder graph and a strong product related graph

In this section, we determine the Stanley depth and depth and of the quotient rings corresponding with edge ideals of $\varrho-$fold bristled graph of circular ladder graph and T_v graph. We prove for these graphs, the depth and Stanley depth values are equal.

To determine these invariants of $\varrho-$fold bristled graph of circular ladder graph, we shall first determine these values for the quotient module associated with edge ideal of $D''_{v,\varrho}$ graph.

Lemma 4.1.1. *Let $\varrho \geq 1$. Then,*

$$\operatorname{sdepth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho}) = \operatorname{depth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho}) = \varrho + 3.$$

Also

$$\operatorname{sdepth}(\digamma^{**}_{2,\varrho}/E_{2,\varrho}) = \operatorname{depth}(\digamma^{**}_{2,\varrho}/E_{2,\varrho}) = 3\varrho + 3.$$

Proof. We have a short exact sequence.

$$0 \longrightarrow \digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho} : y_{\upsilon+2}) \xrightarrow{\cdot y_{\upsilon+2}} \digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho} \longrightarrow \digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho}, y_{\upsilon+2}) \longrightarrow 0.$$

Let $\upsilon = 1$, then $\digamma^{**}_{1,\varrho}/(E_{1,\varrho} : y_3) \cong \bigotimes_{i=1}^{2} \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_3, \cup_{j=1}^{\varrho} y_{2j}]$, by Lemma 2.3.3, Lemma 2.3.5 and Lemma 2.3.4, we have

$$\operatorname{depth}(\digamma^{**}_{1,\varrho}/(E_{1,\varrho} : y_3)) = 2\operatorname{depth}(\mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1})) + \varrho + 1 = 2 + \varrho + 1 = \varrho + 3.$$

Since $\digamma^{**}_{1,\varrho}/(E_{1,\varrho}, y_3) \cong \mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} y_{3j}]$, by Lemma 2.3.3 and Theorem 3.0.1 $\operatorname{depth} \digamma^{**}_{1,\varrho}/(E_{1,\varrho}, y_3) = \operatorname{depth}(\mathsf{T}[V(P_{3,\varrho})]/I(P_{3,\varrho})) + \varrho = \varrho + 2 + \varrho = 2\varrho + 2$. Now by Depth Lemma if

$$\operatorname{depth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho} : y_3) \leq \operatorname{depth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho}, y_3).$$

Then,

$$\operatorname{depth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho}) = \operatorname{depth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho} : y_3) = \varrho + 3.$$

This prove the result for $\upsilon = 1$.

Now let $\upsilon = 2$, we have $\digamma^{**}_{2,\varrho}/(E_{2,\varrho} : y_4) \cong \mathsf{T}[V(P_{4,\varrho})]/I(P_{4,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_4, \cup_{j=1}^{\varrho} y_{3j}]$. Therefore by Lemma 2.3.3 and Lemma 3.0.1,

$$\begin{aligned}\operatorname{depth}(\digamma^{**}_{2,\varrho}/(E_{2,\varrho} : y_4)) &= \operatorname{depth}(\mathsf{T}[V(P_{4,\varrho})]/I(P_{4,\varrho})) + \varrho + 1 \\ &= 2(\varrho + 1) + \varrho + 1 = 3\varrho + 3.\end{aligned}$$

Now $\digamma^{**}_{2,\varrho}/(E_{2,\varrho}, y_4) \cong \digamma^{*}_{2,\varrho}/I^{*}_{2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} y_{4j}]$. Using Lemma 2.3.3 and Lemma 3.1.1, we get $\operatorname{depth}(\digamma^{**}_{2,\varrho}/(E_{2,\varrho}, y_4)) = \operatorname{depth}(\digamma^{*}_{2,\varrho}/I^{*}_{2,\varrho}) + \varrho = 2(\varrho+1)+1+\varrho = 2\varrho+2+1+\varrho = 3\varrho + 3$. So by Depth Lemma

$$\operatorname{depth}(\digamma^{**}_{2,\varrho}/E_{2,\varrho})) = 3\varrho + 3.$$

For Stanley depth when $\upsilon = 1$, by applying Lemma 2.3.2 instead of Depth Lemma, Lemma 2.3.7 instead of Lemma 2.3.5 on the exact sequence. We have

$$\operatorname{sdepth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho}) \geq \varrho + 3.$$

For upper bound, consider $\mu = y_{21}\dots y_{2\varrho}y_1y_3x_1 \in \digamma^{**}_{1,\varrho}/E_{1,\varrho}$, clearly $x_t\mu \in E_{1,\varrho}$, for all $t \in [4\varrho+4] \setminus \operatorname{supp}(\mu)$, therefore by Lemma 2.3.9 $\operatorname{sdepth}(\digamma^{**}_{1,\varrho}/E_{1,\varrho}) \leq \varrho+3$.
For $\upsilon = 2$, by applying Lemma 2.3.2 instead of Depth Lemma we get $\operatorname{sdepth}(\digamma^{**}_{2,\varrho}/E_{2,\varrho}))$ $\geq 3\varrho+3$. For upper bound, as $y_4 \notin E_{2,\varrho}$ by Proposition 2.3.8 $\operatorname{sdepth}(\digamma^{**}_{2,\varrho}/E_{2,\varrho})) \leq$ $\operatorname{sdepth}(\digamma^{**}_{2,\varrho}/E_{2,\varrho} : y_4)) = 3\varrho+3$. □

Theorem 4.1.2. *Let $\varrho \geq 1$ and $\upsilon \geq 1$, then*

$$\operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) = \operatorname{sdepth}(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) = \begin{cases} (\upsilon+1)(\varrho+1), & \text{if } \upsilon \text{ is even;} \\ \upsilon(\varrho+1)+2, & \text{if } \upsilon \text{ is odd.} \end{cases}$$

Proof. For $\upsilon = 1, 2$, it is already proved in Lemma 4.1.1. Now we will prove for $\upsilon \geq 3$, we will prove this result with the help of induction on υ. Consider the following exact sequence.

$$0 \longrightarrow \digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho} : y_{\upsilon+2}) \xrightarrow{\cdot y_{\upsilon+2}} \digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho} \longrightarrow \digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho}, y_{\upsilon+2}) \longrightarrow 0.$$

By Depth Lemma

$$\operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) \geq \min\{\operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho} : y_{\upsilon+2})), \operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho}, y_{\upsilon+2}))\}.$$

Clearly $\mathcal{G}(E_{\upsilon,\varrho} : y_{\upsilon+2}) = \{\mathcal{G}(E_{\upsilon-2,\varrho}), x_{\upsilon-2}x_{\upsilon-1}, x_{\upsilon-1}y_\upsilon, x_{\upsilon-1}x_\upsilon, y_{\upsilon+1}\} \cup_{j=1}^{\varrho} \{y_{(\upsilon+2)j}\}$.
Let $J^* := (E_{\upsilon,\varrho} : y_{\upsilon+2})$. Now consider following exact sequence.

$$0 \longrightarrow \digamma^{**}_{\upsilon,\varrho}/(J^* : x_\upsilon) \xrightarrow{\cdot x_\upsilon} \digamma^{**}_{\upsilon,\varrho}/J^* \longrightarrow \digamma^{**}_{\upsilon,\varrho}/(J^*, x_\upsilon) \longrightarrow 0.$$

Since $\digamma^{**}_{\upsilon,\varrho}/(J^* : x_\upsilon) \cong \digamma^{**}_{\upsilon,\varrho}/E_{\upsilon-2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[x_\upsilon, y_{\upsilon+2}, \cup_{j=1}^{\varrho} y_{(\upsilon+1)j}, \cup_{j=1}^{\varrho} x_{(\upsilon-1)j}]$, and $\digamma^{**}_{\upsilon,\varrho}/(J^*, x_\upsilon) \cong \digamma^{*}_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[y_{\upsilon+2}, \cup_{j=1}^{\varrho} x_{\upsilon j}, \cup_{j=1}^{\varrho} y_{(\upsilon+1)j}]$. Thus by Lemma 2.3.3 $\operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/(J^* : x_\upsilon)) = \operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon-2,\varrho}) + 2\varrho+2$, also

$$\operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/(J^*, x_\upsilon)) = \operatorname{depth}(\digamma^{*}_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho}) + 2\varrho+1.$$

Case 1.
If υ is even then by induction on υ, $\operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/(J^* : x_\upsilon)) = \operatorname{depth}(\digamma^{**}_{\upsilon-2,\varrho}/E_{\upsilon-2,\varrho}) + 2\varrho+2 = (\upsilon-2+1)(\varrho+1)+2\varrho+2 = \upsilon(\varrho+1)-\varrho-1+2\varrho+2 = (\upsilon+1)(\varrho+1)$. Similarly, Lemma 3.1.1 gives us $\operatorname{depth}(\digamma^{**}_{\upsilon,\varrho}/(J^*, x_\upsilon)) = \operatorname{depth}(\digamma^{*}_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho}) +$

$2\varrho + 1 = (\upsilon - 1)(\varrho + 1) + 1 + 2\varrho + 1 = \upsilon(\varrho + 1) - \varrho - 1 + 1 + 2\varrho + 1 = \upsilon(\varrho + 1) + \varrho + 1$. Applying Depth Lemma we get depth $(\digamma^{**}_{\upsilon,\varrho}/J^*) = \upsilon(\varrho + 1) + \varrho + 1$. Now $\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho}, y_{\upsilon+2}) \cong \digamma^{*}_{\upsilon,\varrho}/I^*_{\upsilon,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} y_{(\upsilon+2)j}]$. Using Lemma 2.3.3 and Lemma 3.1.1 we get depth $(\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho}, y_{\upsilon+2})) =$ depth $(\digamma^{*}_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) + \varrho = \upsilon(\varrho + 1) + 1 + \varrho$. By Depth Lemma

$$\text{depth}\,(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) = \upsilon(\varrho + 1) + \varrho + 1.$$

Case 2.

If υ is odd, then by induction on υ, depth $(\digamma^{**}_{\upsilon,\varrho}/(J^* : x_\upsilon)) =$ depth $(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon-2,\varrho}) + 2\varrho + 2 = (\upsilon - 2)(\varrho + 1) + 2 + 2\varrho + 2 = \upsilon(\varrho + 1) - 2\varrho - 2 + 2 + 2\varrho + 2 = \upsilon(\varrho + 1) + 2$. Also using Lemma 3.1.1 depth $(\digamma^{**}_{\upsilon,\varrho}/(J^*, x_\upsilon)) =$ depth $(\digamma^{*}_{\upsilon-1,\varrho}/I^*_{\upsilon-1,\varrho}) + 2\varrho + 1 = (\upsilon - 1)(\varrho + 1) + 1 + 2\varrho + 1 = \upsilon(\varrho + 1) - \varrho - 1 + 1 + 2\varrho + 1 = \upsilon(\varrho + 1) + \varrho + 1$. By Depth Lemma depth $(\digamma^{**}_{\upsilon,\varrho}/J^*) \geq \upsilon(\varrho + 1) + 2$. Here $\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho}, y_{\upsilon+2}) \cong \digamma^{*}_{\upsilon,\varrho}/I^*_{\upsilon,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} y_{(\upsilon+2)j}]$. Using Lemma 2.3.3 and Lemma 3.1.1 we get
depth $(\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho}, y_{\upsilon+2})) =$ depth $(\digamma^{*}_{\upsilon,\varrho}/I^*_{\upsilon,\varrho}) + \varrho = \upsilon(\varrho + 1) + 1 + \varrho$. Therefore by Depth Lemma depth $(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) \geq \upsilon(\varrho + 1) + 2$. For upper bound as $x_\upsilon \notin E_{\upsilon,\varrho}$, and $\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho} : x_\upsilon) \cong \digamma^{**}_{\upsilon,\varrho}/E_{\upsilon-2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1}) \bigotimes_{\mathsf{T}} \mathsf{T}[x_\upsilon, \cup_{j=1}^{\varrho} y_{(\upsilon+1)j}, \cup_{j=1}^{\varrho} x_{(\upsilon-1)j}]$. Thus, by Lemma 2.3.3, Lemma 2.3.4 and principal of induction on υ.

$$\text{depth}\,(\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho} : x_\upsilon)) = \text{depth}\,(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon-2,\varrho}) + \text{depth}\,(\mathsf{T}[V(\mathcal{S}_{\varrho+1})]/I(\mathcal{S}_{\varrho+1})) + 2\varrho + 1.$$

$$\begin{aligned}
\text{depth}\,(\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho} : x_\upsilon)) &= (\upsilon - 2)(\varrho + 1) + 2 + 1 + 2\varrho + 1 \\
&= \upsilon(\varrho + 1) - 2\varrho - 2 + 2 + 2\varrho + 2 \\
&= \upsilon(\varrho + 1) + 2.
\end{aligned}$$

Using Corollary 2.3.6

$$\text{depth}\,(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) \leq \text{depth}\,(\digamma^{**}_{\upsilon,\varrho}/(E_{\upsilon,\varrho} : x_\upsilon)) = \upsilon(\varrho + 1) + 2.$$

Using Depth Lemma

$$\text{depth}\,(\digamma^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) = \upsilon(\varrho + 1) + 2.$$

For the case of Stanley depth this value come after applying Lemma 2.3.2 instead of Depth Lemma, Lemma 2.3.7 instead of Lemma 2.3.5 and Proposition 2.3.8 instead of

Corollary 2.3.6. When υ is even we get, sdepth $(F^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) \geq (\upsilon+1)(\varrho+1)$. For upper bound consider $\mu = y_{11}\dots y_{1\varrho}\dots y_{(\upsilon-1)1}\dots y_{(\upsilon-1)\varrho}y_{(\upsilon+1)1}\dots y_{(\upsilon+1)\varrho}x_{11}\dots x_{1\varrho}\dots x_{(\upsilon-3)1}$ $\dots x_{(\upsilon-3)\varrho}x_{(\upsilon-1)1}\dots x_{(\upsilon-1)\varrho}y_2y_4\dots y_\upsilon y_{\upsilon+2}x_2x_4\dots x_{\upsilon-2}x_\upsilon \in F^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}$, clearly $x_t\mu \in E_{\upsilon,\varrho}$, for all $t \in [2(\upsilon+1)(\varrho+1)] \setminus \mathrm{supp}(\mu)$, therefore using Lemma 2.3.9 sdepth $(F^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) \leq (\upsilon+1)\varrho+\upsilon+1 = (\upsilon+1)(\varrho+1)$. Hence

$$\mathrm{sdepth}\,(F^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) = (\upsilon+1)(\varrho+1).$$

When υ is odd we get, sdepth $(F^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) \geq \upsilon(\varrho+1)+2$. For upper bound consider $\mu = y_{21}\dots y_{2\varrho}\dots y_{(\upsilon-1)1}\dots y_{(\upsilon-1)\varrho}y_{(\upsilon+1)1}\dots y_{(\upsilon+1)\varrho}x_{21}\dots x_{2\varrho}\dots x_{(\upsilon-3)1}\dots x_{(\upsilon-3)\varrho}x_{(\upsilon-1)1}\dots$ $x_{(\upsilon-1)\varrho}y_1y_3\dots y_\upsilon y_{\upsilon+2}x_1x_3\dots x_{\upsilon-2}x_\upsilon \in F^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}$, clearly $x_t\mu \in E_{\upsilon,\varrho}$, for all $t \in [2(\upsilon+1)(\varrho+1)] \setminus \mathrm{supp}(\mu)$, therefore using Lemma 2.3.9 sdepth $(F^{**}_{\upsilon,\varrho}/E_{\upsilon,\varrho}) \leq \upsilon\varrho+\upsilon+2 = \upsilon(\varrho+1)+2$. □

Theorem 4.1.3. *Let $\varrho \geq 1$ and $\upsilon \geq 3$, then*

$$\mathrm{depth}\,(F_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) = \mathrm{sdepth}\,(F_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) = \begin{cases} \upsilon(\varrho+1), & \text{if } \upsilon \text{ is even;} \\ \upsilon(\varrho+1)+\varrho-1, & \text{if } \upsilon \text{ is odd.} \end{cases}$$

Proof. First we will prove the result for depth. Consider the following exact sequence.

$$0 \longrightarrow F_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho} : x_\upsilon) \xrightarrow{\cdot x_\upsilon} F_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho} \longrightarrow F_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho}, x_\upsilon) \longrightarrow 0. \tag{4.1}$$

Let $\upsilon = 3$,
Here $F_{3,\varrho}/(\mathfrak{C}_{3,\varrho} : x_3) \cong \mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[x_3, \cup_{j=1}^{\varrho} y_{3j}, \cup_{j=1}^{\varrho} x_{1j}, \cup_{j=1}^{\varrho} x_{2j}]$. Using Lemma 2.3.3 and Lemma 3.0.1, we have

$$\begin{aligned}\mathrm{depth}\,(F_{3,\varrho}/(\mathfrak{C}_{3,\varrho} : x_3)) &= \mathrm{depth}\,(\mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho})) + 3\varrho + 1 \\ &= (\varrho+1) + 3\varrho + 1 = 4\varrho + 2.\end{aligned}$$

Now let $A := (\mathfrak{C}_{3,\varrho}, x_3)$ and $\mathcal{G}(A) = \{\mathcal{G}(I_{2,\varrho}), y_1y_3, y_2y_3, x_3\} \cup_{j=1}^{\varrho} \{y_3y_{3j}\}$. Consider the following exact sequence.

$$0 \longrightarrow F_{3,\varrho}/(A : y_3) \xrightarrow{\cdot y_3} F_{3,\varrho}/A \longrightarrow F_{3,\varrho}/(A, y_3) \longrightarrow 0.$$

Since $F_{3,\varrho}/(A : y_3) \cong \mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho}) \bigotimes_{\mathsf{T}} \mathsf{T}[y_3, \cup_{j=1}^{\varrho} x_{3j}, \cup_{j=1}^{\varrho} y_{1j}, \cup_{j=1}^{\varrho} y_{2j}]$. By Lemma 2.3.3 and Lemma 3.0.1, depth $(F_{3,\varrho}/(A : y_3)) = \mathrm{depth}\,(\mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho})) + 3\varrho + 1 =$

$(\varrho+1)+3\varrho+1 = 4\varrho+2$. Similarly $\digamma_{3,\varrho}/(A,y_3) \cong \digamma_{2,\varrho}/I_{2,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} x_{3j}, \cup_{j=1}^{\varrho} y_{3j}]$. Applying Lemma 2.3.3 and Lemma 3.1.3,

$$\operatorname{depth}(\digamma_{3,\varrho}/(A,y_3)) = \operatorname{depth}(\digamma_{2,\varrho}/I_{2,\varrho}) + 2\varrho = 2(\varrho+1)+2\varrho = 4\varrho+2.$$

By Depth Lemma, $\operatorname{depth}(\digamma_{3,\varrho}/A) = 4\varrho+2$. By Depth Lemma

$$\operatorname{depth}(\digamma_{3,\varrho}/\mathfrak{C}_{3,\varrho}) = 4\varrho+2.$$

Now let $\upsilon \geq 4$, here $\digamma_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho} : x_\upsilon) \cong \digamma^{**}_{\upsilon-3,\varrho}/E_{\upsilon-3,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[x_\upsilon, \cup_{j=1}^{\varrho} x_{1j}, \cup_{j=1}^{\varrho} x_{(\upsilon-1)j}, \cup_{j=1}^{\varrho} y_{\upsilon j}]$. By Lemma 2.3.3 $\operatorname{depth}(\digamma_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho} : x_\upsilon)) = \operatorname{depth}(\digamma^{**}_{\upsilon-3,\varrho}/E_{\upsilon-3,\varrho}) + 3\varrho+1$. Let $A^* := (\mathfrak{C}_{\upsilon,\varrho}, x_\upsilon)$ and $\mathcal{G}(A^*) = \{\mathcal{G}(I_{\upsilon-1,\varrho}), y_1y_\upsilon, y_\upsilon y_{\upsilon-1}, x_\upsilon\} \cup_{j=1}^{\varrho} \{y_\upsilon y_{\upsilon j}\}$. Consider the following exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(A^* : y_\upsilon) \xrightarrow{y_\upsilon} \digamma_{\upsilon,\varrho}/A^* \longrightarrow \digamma_{\upsilon,\varrho}/(A^*, y_\upsilon) \longrightarrow 0.$$

where

$\digamma_{\upsilon,\varrho}/(A^* : y_\upsilon) \cong \digamma^{**}_{\upsilon-3,\varrho}/E_{\upsilon-3,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[y_\upsilon, \cup_{j=1}^{\varrho} x_{\upsilon j}, \cup_{j=1}^{\varrho} y_{1j}, \cup_{j=1}^{\varrho} y_{(\upsilon-1)j}]$, also it very clear $\digamma_{\upsilon,\varrho}/(A^*, y_\upsilon) \cong \digamma_{\upsilon-1,\varrho}/I_{\upsilon-1,\varrho} \bigotimes_{\mathsf{T}} \mathsf{T}[\cup_{j=1}^{\varrho} x_{\upsilon j}, \cup_{j=1}^{\varrho} y_{\upsilon j}]$.

Case 1.

When υ is even, using Lemma 2.3.3 $\operatorname{depth}(\digamma_{\upsilon,\varrho}/(A^* : y_\upsilon)) = \operatorname{depth}(\digamma^{**}_{\upsilon-3,\varrho}/E_{\upsilon-3,\varrho}) + 3\varrho+1$. As υ is even so $\upsilon-3$ will be an odd number so by Theorem 4.1.2 we have $\operatorname{depth}(\digamma_{\upsilon,\varrho}/(A^* : y_\upsilon)) = (\upsilon-3)(\varrho+1)+2+3\varrho+1 = \upsilon(\varrho+1)-3\varrho-3+3\varrho+3 = \upsilon(\varrho+1)$. Now by Lemma 2.3.3 and Theorem 3.1.4 $\operatorname{depth}(\digamma_{\upsilon,\varrho}/(A^*, y_\upsilon)) = \operatorname{depth}(\digamma_{\upsilon-1,\varrho}/I_{\upsilon-1,\varrho}) + 2\varrho = (\upsilon-1)(\varrho+1)+2\varrho = \upsilon(\varrho+1)-\varrho-1+2\varrho = \upsilon(\varrho+1)+\varrho-1$. By Depth Lemma $\operatorname{depth}(\digamma_{\upsilon,\varrho}/A^*) \geq \upsilon(\varrho+1)$.

Similarly by Theorem 4.1.2 $\operatorname{depth}(\digamma_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho} : x_\upsilon)) = \operatorname{depth}(\digamma^{**}_{\upsilon-3,\varrho}/E_{\upsilon-3,\varrho}) + 3\varrho+1 = (\upsilon-3)(\varrho+1)+2+3\varrho+1 = \upsilon(\varrho+1)-3\varrho-3+3\varrho+3 = \upsilon(\varrho+1)$. Applying Depth Lemma we get $\operatorname{depth}(\digamma_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) \geq \upsilon(\varrho+1)$. For upper bound as $x_\upsilon \notin \mathfrak{C}_{\upsilon,\varrho}$ by Corollary 2.3.6 $\operatorname{depth}(\digamma_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) \leq \operatorname{depth}(\digamma_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho} : x_\upsilon)) = \upsilon(\varrho+1)$. This completes the proof when υ is even.

Case 2.

If υ is odd, using Lemma 2.3.3 $\operatorname{depth}(\digamma_{\upsilon,\varrho}/(A^* : y_\upsilon)) = \operatorname{depth}(\digamma^{**}_{\upsilon-3,\varrho}/E_{\upsilon-3,\varrho}) + 3\varrho +$

1. As υ is odd so $\upsilon - 3$ will be an even number so by Theorem 4.1.2 we have depth $(\digamma_{\upsilon,\varrho}/(A^* : y_\upsilon)) = (\upsilon - 3 + 1)(\varrho + 1) + 3\varrho + 1 = \upsilon(\varrho + 1) - 2\varrho - 2 + 3\varrho + 1 = \upsilon(\varrho + 1) + \varrho - 1$.

Now by Lemma 2.3.3 and Theorem 3.1.4 depth $(\digamma_{\upsilon,\varrho}/(A^*, y_\upsilon)) =$ depth $(\digamma_{\upsilon-1,\varrho}/I_{\upsilon-1,\varrho}) + 2\varrho = (\upsilon - 1)(\varrho + 1) + 2\varrho = \upsilon(\varrho + 1) - \varrho - 1 + 2\varrho = \upsilon(\varrho + 1) + \varrho - 1$. By Depth Lemma depth $(\digamma_{\upsilon,\varrho}/A^*) = \upsilon(\varrho + 1) + \varrho - 1$.

By Theorem 4.1.2 depth $(\digamma_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho} : x_\upsilon)) =$ depth $(\digamma^{**}_{\upsilon-3,\varrho}/E_{\upsilon-3,\varrho}) + 3\varrho + 1 = (\upsilon - 3 + 1)(\varrho + 1) + 3\varrho + 1 = \upsilon(\varrho + 1) - 2\varrho - 2 + 3\varrho + 1 = \upsilon(\varrho + 1) + \varrho - 1$. By Depth Lemma we get depth $(\digamma_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) = \upsilon(\varrho + 1) + \varrho - 1$.

For Stanley depth the we get the same result using Lemma 2.3.2 instead of Depth Lemma, Lemma 2.3.7 instead of Lemma 2.3.5 and Proposition 2.3.8 instead of Corollary 2.3.6.

When υ is even we have, sdepth $(\digamma_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) = \upsilon(\varrho + 1)$. Similarly when υ is odd we get sdepth $(\digamma_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) \geq \upsilon(\varrho + 1) + \varrho - 1$. For upper bound as $x_\upsilon \notin \mathfrak{C}_{\upsilon,\varrho}$ by Proposition 2.3.8 sdepth $(\digamma_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) \leq$ sdepth $(\digamma_{\upsilon,\varrho}/(\mathfrak{C}_{\upsilon,\varrho} : x_\upsilon)) = \upsilon(\varrho + 1) + \varrho - 1$. Hence

$$\text{sdepth}\,(\digamma_{\upsilon,\varrho}/\mathfrak{C}_{\upsilon,\varrho}) = \upsilon(\varrho + 1) + \varrho - 1.$$

□

Lemma 4.1.4. *Let $\upsilon = 3, 4,$ and $\varrho \geq 1$. Then*

$$\text{depth}\,(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) = \text{sdepth}\,(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) = \lfloor\frac{3\upsilon + 1}{2}\rfloor\varrho + \lceil\frac{\upsilon - 1}{2}\rceil.$$

Proof. Consider the following exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho} : x_\upsilon) \xrightarrow{\cdot x_\upsilon} \digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho} \longrightarrow \digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho}, x_\upsilon) \longrightarrow 0. \qquad (4.2)$$

Let $\upsilon = 3$,

$$\mathcal{G}(C_{3,\varrho} : x_3) = \{y_1y_2, y_2y_3, x_2y_2, x_1y_2, x_2y_1, x_2y_3, , y_1y_3, x_1y_3, x_2, y_3, y_2, x_1, y_1\} \cup_{j=1}^{\varrho} \{ x_1x_{1i}, x_2x_{2j}, y_1y_{1j}, y_2y_{2j}, y_3y_{3j}\} \cup_{j=1}^{\varrho} \{x_{3j}\}.$$

$$\mathcal{G}(C_{3,\varrho} : x_3) = \{x_2, y_3, y_2, x_1, y_1\} \cup_{j=1}^{\varrho} \{x_{3j}\}.$$

Clearly $(\digamma_{3,\varrho}/(C_{3,\varrho}:x_3)) \cong \mathsf{T}[\cup_{j=1}^{\varrho}\{x_{1j},x_{2j},y_{1j},y_{2j},y_{3j}\}]$. Thus by Lemma 2.3.3 depth $(\digamma_{3,\varrho}/(C_{3,\varrho}:x_3)) = 5\varrho+1$. Also we have

$$\mathcal{G}(C_{3,\varrho},x_3) = \{x_1x_2,y_1y_2,y_2y_3,x_1y_1,x_2y_2,x_1y_2,x_2y_1,x_2y_3,y_1y_3,x_1y_3,x_3\} \cup_{j=1}^{\varrho} \{x_1x_{1j}, x_2x_{2j},y_1y_{1j},y_2y_{2j},y_3y_{3j}\}.$$

Let $J' := (C_{3,\varrho},x_3)$. Consider the following exact sequence.

$$0 \longrightarrow \digamma_{3,\varrho}/(J':y_3) \xrightarrow{\cdot y_3} \digamma_{3,\varrho}/J' \longrightarrow \digamma_{3,\varrho}/(J',y_3) \longrightarrow 0.$$

$$\mathcal{G}(J':y_3) = \{x_1x_2,y_1y_2,x_1y_1,x_2y_2,x_1y_2,x_2y_1,y_1,x_1,y_2,x_2,x_3\}\cup_{j=1}^{\varrho}\{x_1x_{1j},x_2x_{2j},y_1y_{1j} ,y_2y_{2j}\} \cup_{j=1}^{\varrho} \{y_{3j}\}.$$

$$\mathcal{G}(J':y_3) = \{y_1,x_1,y_2,x_2,x_3\} \cup_{j=1}^{\varrho} \{y_{3j}\}.$$

As $(\digamma_{3,\varrho}/(J':y_3)) \cong \mathsf{T}[y_3,\cup_{j=1}^{\varrho}\{x_{1j},x_{2j},x_{3j},y_{1j},y_{2j}\}]$. Thus by Lemma 2.3.3 depth $(\digamma_{3,\varrho}/(J:y_3)) = 5\varrho+1$. Now

$$\mathcal{G}(J',y_3) = \{y_3,x_3\} \cup \{x_1x_2,y_1y_2,x_1y_1,x_2y_2,x_1y_2,x_2y_1\} \cup_{j=1}^{\varrho} \{x_1x_{1j},x_2x_{2j},y_1y_{1j},y_2y_{2j}\}.$$

Where $(\digamma_{3,\varrho}/(J',y_3)) \cong \digamma_{2,\varrho}/L_{2,\varrho} \bigotimes_K \mathsf{T}[\cup_{j=1}^{\varrho}\{y_{3j},x_{3j}\}]$. Using Lemma 2.3.3 and Lemma 3.1.5 we get depth $(\digamma_{3,\varrho}/(J',y_3)) =$ depth $(\digamma_{2,\varrho}/L_{2,\varrho}) + 2\varrho = 3\varrho+1+2\varrho = 5\varrho+1$. By Depth Lemma results depth $(\digamma_{3,\varrho}/J') =$ depth $(\digamma_{3,\varrho}/(C_{3,\varrho},x_3)) = 5\varrho+1$. Again Depth Lemma results gives us 4.2 depth $(\digamma_{3,\varrho}/C_{3,\varrho}) = 5\varrho+1$. This confirm the result for $\upsilon = 3$.
Now we have $\upsilon = 4$, we have

$$\mathcal{G}(C_{4,\varrho}:x_4) = \{y_1y_4,x_1y_4,x_3,y_4,y_3,x_1,y_1,\}\cup_{j=2}^{2}\{y_ix_{i+1}\}\cup_{j=2}^{3}\{y_ix_{i-1}\}\cup_{i=1}^{3}\{x_iy_i,y_1x_2, y_4x_3\} \cup_{j=1}^{\varrho} \{x_1x_{1j},\dots,x_3x_{3j},y_1y_{1j},\dots,y_4y_{4j}\} \cup_{i=1}^{3} \{y_iy_{i+1}\} \cup_{i=1}^{2} \{x_ix_{i+1}\} \cup_{j=1}^{\varrho} \{x_{4j}\}$$

$$= \{x_3,y_4,y_3,x_1,y_1\} \cup_{j=1}^{\varrho} \{x_2y_2,x_2x_{2j},y_2y_{2j}\} \cup_{j=1}^{\varrho} \{x_{4j}\}.$$

As we can see that $(\digamma_{4,\varrho}/(C_{4,\varrho} : x_4)) \cong \mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho}) \bigotimes_K \mathsf{T}[x_4, \cup_{j=1}^{\varrho}\{x_{1j}, x_{3j}, y_{1j}, y_{3j}, y_{4j}\}]$. By Lemma 2.3.3 and Lemma 3.0.1.

$$\operatorname{depth}(\digamma_{4,\varrho}/(C_{4,\varrho} : x_4)) = \operatorname{depth}(\mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho})) + 5\varrho + 1 = \varrho + 1 + 5\varrho + 1 = 6\varrho + 2.$$

Now let $B := (C_{4,\varrho}, x_4)$, where $\mathcal{G}(B) = \mathcal{G}(L_{3,\varrho}) \cup \{x_3y_4, y_3y_4, y_4x_1, y_4y_1, x_4\}$. Consider the following exact sequence.

$$0 \longrightarrow \digamma_{4,\varrho}/(B : y_4) \xrightarrow{\cdot y_4} \digamma_{4,\varrho}/B \longrightarrow \digamma_{4,\varrho}/(B, y_4) \longrightarrow 0.$$

Clearly,

$$\mathcal{G}(B : y_4) = \{y_1, x_1, x_3, y_3, x_4\} \cup_{j=1}^{\varrho} \{x_2y_2, x_2x_{2j}, y_2y_{2j}\} \cup_{j=1}^{\varrho} \{y_{4j}\}.$$

Since $(\digamma_{4,\varrho}/(B : y_4)) \cong \mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho}) \bigotimes_K \mathsf{T}[y_3, \cup_{j=1}^{\varrho}\{y_{3j}, y_{1j}, x_{1j}, x_{3j}, x_{4j}\}]$. By Lemma 2.3.3 and Lemma 3.0.1

$$\begin{aligned}\operatorname{depth}(\digamma_{\upsilon,\varrho}/(B : y_4)) &= \operatorname{depth}(\mathsf{T}[V(P_{2,\varrho})]/I(P_{2,\varrho})) + 5\varrho + 1 \\ &= \varrho + 1 + 5\varrho + 1 = 6\varrho + 2.\end{aligned}$$

Note that $\mathcal{G}(B, y_4) = (\mathcal{G}(L_{3,\varrho}), x_4, y_4)$ and $(\digamma_{4,\varrho}/(B, y_4)) \cong \digamma_{3,\varrho}/L_{3,\varrho} \bigotimes_K \mathsf{T}[\cup_{j=1}^{\varrho}\{y_{4j}, x_{4j}\}]$, by Lemma 2.3.3 and Lemma 3.1.5

$$\operatorname{depth}(\digamma_{4,\varrho}/(B, y_4)) = \operatorname{depth}(\digamma_{3,\varrho}/L_{3,\varrho}) + 2\varrho = 4\varrho + 2 + 2\varrho = 6\varrho + 2,$$

by Depth Lemma

$$\operatorname{depth}(\digamma_{4,\varrho}/B) = \operatorname{depth}(\digamma_{4,\varrho}/(C_{4,\varrho}, x_4)) = 6\varrho + 2,$$

and consequently, Depth Lemma, we get $\operatorname{depth}(\digamma_{4,\varrho}/C_{4,\varrho}) = 6\varrho + 2$.

For Stanley depth when $\upsilon = 3, 4$, we get this result by applying Lemma 2.3.2 instead of Depth Lemma and Proposition 2.3.8 instead of Corollary 2.3.6. We get $\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) \geq \lfloor \frac{3\upsilon+1}{2} \rfloor \varrho + \lceil \frac{\upsilon-1}{2} \rceil$. For upper bound as $x_\upsilon \notin C_{\upsilon,\varrho}$ by Proposition 2.3.8.

$$\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) \leq \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho} : x_\upsilon) = \lfloor \frac{3\upsilon + 1}{2} \rfloor \varrho + \lceil \frac{\upsilon - 1}{2} \rceil.$$

So,

$$\text{sdepth}\,(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) = \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil.$$

□

Theorem 4.1.5. *Let $\upsilon \geq 3$ and $\varrho \geq 1$. Then*

$$\text{depth}\,(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) = \text{sdepth}\,(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) = \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil.$$

Proof. For $\upsilon = 3, 4$, Lemma 4.1.4 shows that result hold Now we will prove this for $\upsilon \geq 5$. Consider the following exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho} : x_\upsilon) \xrightarrow{\cdot x_\upsilon} \digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho} \longrightarrow \digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho}, x_\upsilon) \longrightarrow 0. \tag{4.3}$$

$$\mathcal{G}(C_{\upsilon,\varrho} : x_\upsilon) = \{x_{\upsilon-1}, y_\upsilon, y_{\upsilon-1}, x_1, y_1\} \cup_{j=2}^{\upsilon-3} \{y_i x_{i+1}\} \cup_{j=3}^{\upsilon-2} y_j x_{j-1}\} \cup_{i=2}^{\upsilon-2} \{x_i y_i\} \cup_{j=1}^{\varrho} \{x_2 x_{2j}, \dots, x_{\upsilon-2} x_{(\upsilon-2)j}, y_2 y_{2j}, \dots, y_{\upsilon-2} y_{(\upsilon-2)j}\} \cup_{i=2}^{\upsilon-3} \{x_i x_{i+1}, y_i y_{i+1}\} \cup_{j=1}^{\varrho} \{x_{\upsilon j}\}.$$

Now $(\digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho} : x_\upsilon)) \cong \digamma_{\upsilon-3,\varrho}/L_{\upsilon-3,\varrho} \bigotimes_K \mathsf{T}[x_\upsilon, \cup_{j=1}^{\varrho}\{x_{1j}, x_{(\upsilon-1)j}, y_{1j}, y_{\upsilon j}, y_{(\upsilon-1)j}\}]$. Using Lemma 2.3.3 and Theorem 3.1.6.

$$\begin{aligned}\text{depth}\,(\digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho} : x_\upsilon)) &= \text{depth}\,(\digamma_{\upsilon-3,\varrho}/L_{\upsilon-3,\varrho}) + 5\varrho + 1\\ &= \lfloor\frac{3(\upsilon-3)}{2}\rfloor\varrho + \lceil\frac{\upsilon-3}{2}\rceil + 5\varrho + 1\\ &= \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil.\end{aligned}$$

Let $J' := (C_{\upsilon,\varrho}, x_\upsilon)$, where $\mathcal{G}(J') = (\mathcal{G}(I_{\upsilon-1}), x_{\upsilon-1}y_\upsilon, y_{\upsilon-1}y_\upsilon, y_\upsilon y_1, y_\upsilon x_1, x_\upsilon) \cup_{j=1}^{\varrho} \{y_\upsilon y_{\upsilon j}\}$. Consider the short exact sequence.

$$0 \longrightarrow \digamma_{\upsilon,\varrho}/(J' : y_\upsilon) \xrightarrow{\cdot y_\upsilon} \digamma_{\upsilon,\varrho}/J' \longrightarrow \digamma_{\upsilon,\varrho}/(J', y_\upsilon) \longrightarrow 0.$$

$$\mathcal{G}(J' : y_\upsilon) = \{x_1, y_1, x_\upsilon, x_{\upsilon-1}, y_{\upsilon-1}\} \cup_{i=2}^{\upsilon-3} \{x_i x_{i+1}, y_i y_{i+1}\} \cup_{j=1}^{\varrho} \{x_2 x_{2j}, \dots, x_{\upsilon-2} x_{(\upsilon-2)j}, y_2 y_{2j}, \dots, y_{\upsilon-2} y_{(\upsilon-2)}\} \cup_{i=2}^{\upsilon-2} \{x_i y_i, y_{\upsilon-2} x_{\upsilon-3}\} \cup \{y_2 x_3\} \cup_{j=3}^{\upsilon-3} \{y_i x_{i-1}, y_i x_{i+1}\} \cup_{j=1}^{\varrho} \{y_{\upsilon j}\}.$$

Since $(\digamma_{\upsilon,\varrho}/(J' : y_\upsilon)) \cong \digamma_{\upsilon-3,\varrho}/L_{\upsilon-3,\varrho} \bigotimes_K \mathsf{T}[y_\upsilon, \cup_{j=1}^{\varrho}\{y_{1j}, x_{1j}, x_{\upsilon j}, x_{(\upsilon-1)j}, y_{(\upsilon-1)j}\}]$, using Lemma 2.3.3 and Theorem 3.1.6

$$\begin{aligned}
\operatorname{depth}(\digamma_{\upsilon,\varrho}/(J' : y_\upsilon)) &= \operatorname{depth}(\digamma_{\upsilon-3,\varrho}/L_{\upsilon-3,\varrho}) + 5\varrho + 1 \\
&= \lfloor\frac{3(\upsilon-3)}{2}\rfloor\varrho + \lceil\frac{\upsilon-3}{2}\rceil + 5\varrho + 1 \\
&= \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil.
\end{aligned}$$

Now $\mathcal{G}(J', y_\upsilon) = \{\mathcal{G}(L_{\upsilon-1,\varrho}), y_\upsilon, x_\upsilon\}$ and $(\digamma_{\upsilon,\varrho}/(J', y_\upsilon)) \cong \digamma_{\upsilon-1,\varrho}/L_{\upsilon-1,\varrho} \bigotimes_K \mathsf{T}[\cup_{j=1}^{\varrho}$ $\{x_{\upsilon j}, y_{\upsilon j}\}]$. Using Lemma 2.3.3 and Theorem 3.1.6 we have

$$\begin{aligned}
\operatorname{depth}(\digamma_{\upsilon,\varrho}/(J', y_\upsilon)) &= \operatorname{depth}(\digamma_{\upsilon-1,\varrho}/L_{\upsilon-1,\varrho}) + 2\varrho \\
&= \lfloor\frac{3(\upsilon-1)}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil + 2\varrho \\
&= \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil.
\end{aligned}$$

By Depth Lemma $\operatorname{depth}(\digamma_{\upsilon,\varrho}/J') = \operatorname{depth}(\digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho}, x_\upsilon)) = \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil$. Applying Depth Lemma $\operatorname{depth}(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) = \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil$.

For the case of Stanley depth we get this value by applying Lemma 2.3.2 instead of Depth Lemma. We get $\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) \geq \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil$. For upper bound as $x_\upsilon \notin C_{\upsilon,\varrho}$ by Proposition 2.3.8. $\operatorname{sdepth}(\digamma_{\upsilon,\varrho}/C_{\upsilon,\varrho}) \leq \operatorname{sdepth}(\digamma_{\upsilon,\varrho}/(C_{\upsilon,\varrho} : x_\upsilon)) = \lfloor\frac{3\upsilon+1}{2}\rfloor\varrho + \lceil\frac{\upsilon-1}{2}\rceil$. □

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