NOETHER SYMMETRIES OF BIANCHI TYPE IV SPACETIME

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MS THESIS WORK

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Dedication

I would like to dedicate this thesis to my Mom, Dad, Nana Abu, Siblings, Mamu's and my beloved Daughter.

Acknowledgement

All praises square measure for Almighty Supreme Being, the foremost beneficent and also the most merciful, WHO created this whole universe. I'm extremely grateful to Almighty Supreme Being for showering multitudinous blessings upon me and giving the power and strength to finish this thesis with success and blessing me quite I merit. I'm deeply grateful to my supervisor Dr. Tooba Feroze, for her continuous support and steering throughout my thesis. My understanding and appreciation of the topic square measure entirely thanks to her efforts and positive response to my queries. I'd additionally wish to convey the complete college of the academic department for his/her kind facilitate throughout my analysis work. My studies at NUST are created a lot of unforgettable and that I learned heaps from returning here. I'd wish to provide special because of my category mates for valuable advices, suggestions, and friendly relationship.

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Abstract

In this thesis, we find Noether symmetries for Bianchi type IV spacetime. Different cases have been studied. It is found that the minimal set contains four symmetries. There are various cases, where we get five Noether symmetries. Maximum number of Noether symmetries that the Bianchi type IV spacetime metric may admit is six.

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Chapter 1

Introduction

1.1 Background

Differential equations (DEs) have a long history dating back to the seventeenth century, when Newton and Leibniz separately formulated the basis of calculus. Newton's ideas were expressed in the form of DEs. First time word differential equation used by Leibniz in a letter to Newton in 1676, since then DEs have been widely employed as mathematical models in science and technology.

DEs are divided in two categories, ordinary differential equations (ODEs) and partial differential equations (PDEs). ODEs are those differential equations, which contain ordinary derivatives of unknown variable that depend on a single independent variable. However, PDEs consist of the partial derivatives of one or more dependent variables with respect to two or more independent variables.

In 1691, Lebniz introduced the technique of seperation of variables for solving both homogeneous and non-homogeneous DEs. Johnn and Jacob Bernoulli are pivotal figures in the theory of DEs, they coined phrase "separation of variables" [1]. Many significant ideas in integration theory of DEs derived by Euler [2], such as method of parameters, power series solutions, and the integrating factors, among others. Taylor drew attention to the singular solutions of the differential equations.

There are numerous integration methods for seeking of DEs solutions, however those

approaches are only applicable to restricted classes of ODEs and PDEs. In nineteenth century, a Norwegien Mathematician, Lie motivated by Galois provided a useful approach for solving the DEs in 1881. This method is known as Lie Symmetries method [3, 4, 5, 6]. In recent years, these approaches are commonly employed to the DEs.
Another remarkable point symmetry that is very important in the field of applied mathematics is known as Noether symmetry(NS), named after German Mathematician Emmy Noether [7]. Noether's famous dissertation on NS theory and the invariance of Hamilton actions under infinitesimal transformation was published in 1918 [8]. This technique reduced the amount of effort involved in solving DEs using Lagrangians.
In the fields of geometry and physics, the groups of 3 dimensional Riemannian manifolds have great significance. The real 3-dimensional Lie algebras G_3, enumerated by Bianchi is classified as G_3A and G_3B. There are nine types, Bianchi type I to Bianchi type IX. Type I, II, VI_0, VII_0, VIII and IX are contain in G_3A and G_3B contains type III, IV, V,VI_h, VII_h, Here for type VII, $h > 0$ and for VII_h, $h < 0$ [9].
For batter understanding of the comparison between NSs and conformal Killing vectors, Bokhari and Kara studied on NSs of Friedmann model [10]. They demonstrated that the flat Friedman model admits additional conservation laws, and these conserved variables cannot be derived from the Killing or conformal Killing vectors. Tsamparlis examined the dynamical system and demonstrated that Lie symmetries of equations of motion are generated by a Lie algebra of projective collineations, whereas NSs are generated by the homothetic algebra [11]. Jamil presented their results in cosmology by using the NS technique to a flat Friedmann-Robertson-Walker metric and obtained the tachyon potential [12]. Shabir worked on multiple Bianchi types. He found conformal vector fields of Bianchi type I space-times in $f(R)$ gravity [13]. Using a direct integration approach, he classified Bianchi type II, VIII and IX space-times based on their teleparallel Killing vector fields in the teleparallel theory of gravity [14, 15]. Shabir and Ali investigated the existance of proper curvature collineations (CCS) in Bianchi type IV spacetimes [16]. Ali, Khan and Hussain applied direct integration to inves-

tigate technique appropriate homothetic symmetry for Bianchi type IV space-times. Using this approach, it is determined that the given space-times admit only one case for proper homothetic symmetry [17]. Hickman and Yazdan investigate NSs of Bianchi type II spacetimes. They proved that,both Killing vectors and homothetic motions contain in NSs [18]. Akhtar and Hussain investigated the NSs of LRS Bianchi type V. They established the conservation laws and Lie algebra for all NS generators. Furthermore, several physical applications of the obtained metrics are described, including the investigation of various energy conditions [19].

1.2 Objective of the Thesis

Objective of the thesis, is to find NSs for Bianchi type IV spacetime given as [16]

$$ds^2 = -dt^2 + e^{-2z}\left[P(t)dx^2 + \left(z^2P(t) + Q(t)\right)dy^2 + 2zP(t)dxdy\right] + R(t)dz^2. \quad (1.1)$$

The process of finding NSs leads to a set of PDEs which are solved by using the appropriate methods of integration and differentiation.

1.3 Scheme of Work

There are four chapters in this thesis. Chapter one contains the review of significant background information related to our thesis. Basic terms used in this study are given in chapter two along with examples that are required to establish a framework for the next chapter. In the third chapter, NSs for Bianchi type IV spacetime are obtained. The NS condition is utilized to get the system of partial differential equations that are solved analytically using methods known to us. The final chapter of this thesis is devoted to a summary of the thesis.

Chapter 2

Preliminaries

2.1 Lie Point Symmetries

2.1.1 Infinitesimal Generator

Consider an invertible point transformation with parameter ζ

$$x^* = x^*(x, y; \zeta), \qquad y^* = y^*(x, y; \zeta),$$

where x and y are the independent and dependent variables respectively. Expanding above transformations at $\zeta = 0$, we get

$$x^*(x, y; \zeta) = x + \zeta \left[\frac{\partial x^*(x, y; \zeta)}{\partial \zeta}|_{\zeta=0} \right] + \cdots = x + \zeta \xi(x, y) + \ldots,$$

$$y^*(x, y; \zeta) = y + \zeta \left[\frac{\partial y^*(x, y; \zeta)}{\partial \zeta}|_{\zeta=0} \right] + \cdots = y + \zeta \eta(x, y) + \ldots.$$

Here ξ and η are infinitesimals and defined by

$$\xi(x, y) = \frac{\partial x^*}{\partial \zeta} \; |_{\zeta=0}, \tag{2.1}$$

$$\eta(x, y) = \frac{\partial y^*}{\partial \zeta} \; |_{\zeta=0} \; . \tag{2.2}$$

Infinitesimal generator is given by,

$$\mathbf{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

Now consider DE of the form

$$H(x, y, y', y'', \dots, y^{(n)}) = 0, \tag{2.3}$$

$y', y'', \dots, y^{(n)}$ are the derivatives of y w.r.t. x. In order to apply point transformation in DE, we should know how to find transformed derivatives of y. To do this let us define

$$y*^{'} = \frac{dy^*(x,y;\zeta)}{dx^*(x,y;\zeta)} = y*^{'}(x, y, y^{'}; \zeta), \tag{2.4}$$

$$y^{*''} = \frac{dy^{*'}}{dx^*} = y*^{''}(x, y, y^{'}; \zeta). \tag{2.5}$$

So above derivatives are derivative of transformed variables. Now we prolong infinitesimal generator [7]

$$\begin{aligned}
x^*(x,y;\zeta) &= x + \zeta\xi(x,y) + \dots = x + \zeta X x + \dots, \\
y^*(x,y;\zeta) &= y + \zeta\eta(x,y) + \dots = y + \zeta X y + \dots, \\
y*^{'}(x,y,y^{'};\zeta) &= y^{'} + \zeta\eta^{'}(x,y,y^{'}) + \dots = y^{'} + \zeta X y^{'} + \dots, \\
&\vdots \\
y*^{(n)}(x,y,y',\dots,y^{(n)};\zeta) &= y^{(n)} + \zeta\eta^{n}(x,y,y^{'},\dots,y^{(n)}) + \dots = y^{(n)} + \zeta\mathbf{X}y^{(n)} + \dots,
\end{aligned} \tag{2.6}$$

hare η, $\eta^{'}, \dots, \eta^{n}$ are given by

$$\eta^{'} = \frac{\partial y*^{'}}{\partial \zeta}\,|_{\zeta=0} = 0, \dots, \eta^{n} = \frac{\partial y*^{(n)}}{\partial \zeta}\,|_{\zeta=0} = 0. \tag{2.7}$$

Using eq(2.6) and eq(2.7), we get

$$\begin{aligned}
y*^{'} &= y^{'} + \zeta\eta^{'} = \frac{dy^*}{dx^*} = \frac{dy + \zeta d\eta + \dots}{dx + \zeta d\xi + \dots} = \frac{y^{'} + \zeta\dfrac{d\eta}{dx} + \dots}{1 + \zeta\dfrac{d\xi}{dx} + \dots} = y^{'} + \zeta\left(\frac{d\eta}{dx} - y^{'}\frac{d\xi}{dx}\right) + \dots, \\
y*^{(n)} &= y^{(n)} + \zeta\eta^{n} = \frac{dy*^{(n-1)}}{dx^*} = y^{(n)} + \zeta\left(\frac{d\eta^{n-1}}{dx} - y^{(n)}\frac{d\xi}{dx}\right) + \dots.
\end{aligned}$$

From above equation, we get

$$\eta^n = \frac{d\eta^{n-1}}{dx} - y^{(n)}\frac{d\xi}{dx}, \tag{2.8}$$

here $\dfrac{d}{dx}$ is known as total derivative, given by

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y^{'}\frac{\partial}{\partial y} + y^{''}\frac{\partial}{\partial y'} + \ldots. \tag{2.9}$$

So prolongation of infinitesimal generators up-to nth-order is

$$\mathbf{X}^{[\mathbf{n}]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{'}\frac{\partial}{\partial y'} + \cdots + \eta^{n}\frac{\partial}{\partial y^{(n)}}. \tag{2.10}$$

Theorem 2.1.1. The nth-order DE of the form [7]

$$H(x, y, y^{'}, y^{''}, \ldots, y^{(n)}) = 0, \tag{2.11}$$

where $y', y'', \ldots, y^{(n)}$ are the derivatives of y w.r.t. x admits a Lie point symmetry with generator eq(2.11) iff

$$\mathbf{X}^{[\mathbf{n}]}H = 0, \qquad (mod \;\; H = 0).$$

Example 1. Consider second order DE

$$y^{''} = 0.$$

To find infinitesimal generators(symmetries), we use symmetry condition for 2nd-order ODE

$$\eta^2 = 0,$$

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y^{'} + (\eta_{yy} - 2\xi_{xy})y'^{2} - \xi_{yy}y'^{3} = 0.$$

Comparing coefficients of y'^3, y'^2, $y^{'}$ and y'^0

$$y'^3 \quad : \qquad \xi_{yy} = 0, \tag{2.12}$$

$$y'^2 \quad : \qquad \eta_{yy} - 2\xi_{xy} = 0, \tag{2.13}$$

$$y^{'} \quad : \qquad 2\eta_{xy} - \xi_{xx} = 0, \tag{2.14}$$

$$y'^0 \quad : \qquad \eta_{xx} = 0. \tag{2.15}$$

From eq(2.12), we get

$$\xi(x,y) = ym(x) + n(x). \tag{2.16}$$

Using eq(2.16) in eq(2.13), we get

$$\eta(x,y) = y^2 m^{'}(x) + yp(x) + q(x). \tag{2.17}$$

Now using eq(2.17)in eq(2.15)

$$y^2 m^{'''}(x) + yp^{''}(x) + q^{''}(x) = 0,$$

$$3ym^{''}(x) + 2p^{'}(x) - n^{''}(x) = 0. \tag{2.18}$$

Comparing coefficients of y^3, y^2, y and y^0, we get

$$m^{''}(x) = 0, \quad p^{''}(x) = 0, \quad q^{''}(x) = 0, \quad n^{''}(x) = 2p^{'}(x),$$

on integrating

$$\begin{aligned} m(x) &= k_1 x + k_2, \\ n(x) &= k_3 x^2 + k_7 x + k_8, \\ p(x) &= k_3 x + k_4, \\ q(x) &= k_5 x + k_6. \end{aligned}$$

Substituting in eq(2.16) and eq(2.17), we get

$$\xi(x,y) = y(k_1 x + k_2) + k_3 x^2 + k_7 x + k_8, \tag{2.19}$$

$$\eta(x,y) = y^2 k_1 + y(k_3 x + k_4) + k_5 x + k_6. \tag{2.20}$$

Where $k_1, k_2, \ldots, k_8$ are arbitrary constants, so we have eight corresponding symmetries,

$$
\begin{aligned}
\mathbf{X_1} &= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \quad & \mathbf{X_2} = y\frac{\partial}{\partial x}, \\
\mathbf{X_3} &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad & \mathbf{X_4} = y\frac{\partial}{\partial y}, \\
\mathbf{X_5} &= x\frac{\partial}{\partial y}, & \mathbf{X_6} = \frac{\partial}{\partial y}, \\
\mathbf{X_7} &= x\frac{\partial}{\partial x}, & \mathbf{X_8} = \frac{\partial}{\partial x}.
\end{aligned}
$$

2.2 Noether Symmetry

Noether symmetry is the form of symmetry that satisfies following condition [7],

$$\mathbf{X}^{[1]}L + (D\xi)L = DV. \tag{2.21}$$

Here $V(s, q^a)$ is a guage function, D is an operator given by

$$D = \frac{\partial}{\partial s} + \dot{q}^a\frac{\partial}{\partial q^a}, \tag{2.22}$$

and L is Lagrangian

$$L = L(s, q^a, \dot{q}^a), \tag{2.23}$$

where dot (.) is derivative w.r.t. s, and

$$\mathbf{X}^{[1]} = \xi(s, q^i)\frac{\partial}{\partial s} + \eta^a(s, q^i)\frac{\partial}{\partial q^a} + \dot{\eta}^a(s, q^i, \dot{q}^i)\frac{\partial}{\partial \dot{q}^a}, \qquad a, i = 1, 2, 3, 4, ..., n. \tag{2.24}$$

Where $\mathbf{X}^{[1]}$ is the 1st-order prolonged generator and to explore this we have to find $\dot{\eta}^a$ and its formula is given by

$$\dot{\eta}^a = \frac{\partial}{\partial s}\eta^a - \dot{q}^a\frac{\partial}{\partial s}\xi. \tag{2.25}$$

Example 2. Let Lagrangian of free particle is

$$L = \frac{1}{2}y'^2, \tag{2.26}$$

we have Euler-Lagrange equation

$$\frac{\partial L}{\partial y}=\frac{d}{dx}\frac{\partial L}{\partial y'}, \tag{2.27}$$

substituting the Lagrangian given by eq(2.26) in Euler-Lagrange equation, we have DE corresponding to eq(2.26) is

$$y''=0. \tag{2.28}$$

Using Lagrangian given by eq(2.26) in NS condition eq(2.21), we have

$$\left(\eta_x+(\eta_y-\xi_x)y'-y'^2\xi_y\right)y'+(\xi_x+y'\xi_y)\frac{1}{2}y'^2=V_x+y'V_y,$$

where ξ, η, V are function of x and y.

Now comparing coefficients of y'^3, y'^2, y' and constant, we have

$$y'^3 \quad : \quad \xi_y=0, \tag{2.29}$$

$$y'^2 \quad : \quad \eta_y-\frac{1}{2}\xi_x=0, \tag{2.30}$$

$$y' \quad : \quad \eta_x=V_y, \tag{2.31}$$

$$Constant \quad : \quad V_x=0. \tag{2.32}$$

Solving eq(2.29), we have

$$\xi=a_1(x), \tag{2.33}$$

using eq(2.33) in eq(2.30) then integrating w.r.t. y we have

$$\eta=\frac{1}{2}a_{1,x}y+a_2(x). \tag{2.34}$$

Using eq(2.34) in eq(2.31), we have

$$\frac{1}{2}a_{1,xx}y+a_{2,x}=V_y,$$

$$V=\frac{1}{4}a_{1,xx}y^2+a_{2,x}y+a_3(x). \tag{2.35}$$

Substituting value of $V(x,y)$ in eq(2.32), we have

$$\frac{1}{4}a_{1,xxx}y^2 + a_{2,xx}y + a_{3,x} = 0. \tag{2.36}$$

Comparing coefficients of y^2, y and y^0, we obtain the following differential equations

$$y^2 \quad : \qquad a_{1,xxx} = 0, \tag{2.37}$$

$$y \quad : \qquad a_{2,xx} = 0, \tag{2.38}$$

$$constant \quad : \qquad a_{3,x} = 0. \tag{2.39}$$

Therefore

$$\begin{aligned} a_1(x) &= \frac{1}{2}C_1x^2 + C_2x + C_3, \\ a_2(x) &= C_4x + C_5, \\ a_3(x) &= C_6. \end{aligned}$$

Substituting values of a_1, a_2 and a_3 in eq(2.33-2.35), we get values of ξ, η and V i.e.

$$\begin{aligned} \xi &= \frac{1}{2}C_1x^2 + C_2x + C_3, \\ \eta &= \frac{1}{2}(C_1x + C_2)y + C_4x + C_5, \\ V &= \frac{1}{4}C_1y^2 + C_4y + C_6. \end{aligned}$$

Here $C_1, \ldots, C_6$ are the arbitrary constants. The NSs and corresponding gauge functions are given by

$$\begin{aligned} \mathbf{X_1} &= \frac{1}{2}x^2\frac{\partial}{\partial x} + \frac{1}{2}xy\frac{\partial}{\partial y}, \qquad && V_1(y) = \frac{1}{4}y^2, \\ \mathbf{X_2} &= x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y}, \\ \mathbf{X_3} &= \frac{\partial}{\partial x}, \\ \mathbf{X_4} &= x\frac{\partial}{\partial y}, && V_4(y) = y, \\ \mathbf{X_5} &= \frac{\partial}{\partial y}. \end{aligned}$$

Chapter 3

Noether Symmetries Associated with Bianchi Type IV Metric

In this chapter we discuss NSs for Bianchi type IV. The Lagrangian of corresponding metric given by eq(1.1) is

$$L = -\dot{t}^2 + e^{-2z}\left[P(t)\dot{x}^2 + \left(z^2P(t) + Q(t)\right)\dot{y}^2 + 2zP(t)\dot{x}\dot{y}\right] + R(t)\dot{z}^2, \tag{3.1}$$

here $P(t)$, $Q(t)$ and $R(t)$ are no where zero functions of t. So substituting eq(3.1) in NS condition eq(2.21), we have

$$\begin{aligned} & \eta^0[e^{-2z}(P'(t)\dot{x}^2 + (z^2P'(t) + Q'(t))\dot{y}^2 + 2zP'(t)\dot{x}\dot{y}) + R'(t)\dot{z}^2] + 2\eta^3 e^{-2z}[-P(t)\dot{x}^2 - \\ & (z^2P(t) + Q(t))\dot{y}^2 - 2zP(t)\dot{x}\dot{y} + zP(t)\dot{y}^2 + P(t)\dot{x}\dot{y}] - 2\dot{t}[\eta^0_s + (\eta^0_t - \xi_s)\dot{t} + \eta^0_x\dot{x} + \\ & \eta^0_y\dot{y} + \eta^0_z\dot{z} - \xi_t\dot{t}^2 - \xi_x\dot{x}\dot{t} - \xi_y\dot{y}\dot{t} - \xi_z\dot{z}\dot{t}] + 2e^{-2z}(P(t)\dot{x} + zP(t)\dot{y})[\eta^1_s + (\eta^1_x - \xi_s)\dot{x} + \\ & \eta^1_t\dot{t} + \eta^1_y\dot{y} + \eta^1_z\dot{z} - \xi_x\dot{x}^2 - \xi_t\dot{x}\dot{t} - \xi_y\dot{x}\dot{y} - \xi_z\dot{x}\dot{z}] + 2e^{-2z}((z^2P(t) + Q(t))\dot{y} + zP(t)\dot{x}) \\ & [\eta^2_s + (\eta^2_y - \xi_s)\dot{y} + \eta^2_t\dot{t} + \eta^2_x\dot{x} + \eta^2_z\dot{z} - \xi_y\dot{y}^2 - \xi_t\dot{t}\dot{y} - \xi_x\dot{x}\dot{y} - \xi_z\dot{y}\dot{z}] + 2R(t)\dot{z}[\eta^3_s + (\eta^3_z - \\ & \xi_s)\dot{z} + \eta^3_t\dot{t} + \eta^3_x\dot{x} + \eta^3_y\dot{y} - \xi_z\dot{z}^2 - \xi_t\dot{t}\dot{z} - \xi_x\dot{x}\dot{z} - \xi_y\dot{y}\dot{z}] + (\xi_s + \dot{t}\xi_t + \dot{x}\xi_x + \dot{y}\xi_y + \dot{z}\xi_z) \\ & [-\dot{t}^2 + e^{-2z}(P(t)\dot{x}^2 + (z^2P(t) + Q(t))\dot{y}^2 + 2zP(t)\dot{x}\dot{y}) + R(t)\dot{z}^2] \\ = \; & V_s + \dot{t}V_t + \dot{x}V_x + \dot{y}V_y + \dot{z}V_z. \end{aligned}$$

Comparing coefficients of $\dot{t}^3$, $\dot{x}^3$, $\dot{y}^3$, $\dot{z}^3$, $\dot{t}^2$, $\dot{x}^2$, $\dot{y}^2$, $\dot{z}^2$, $\dot{t}\dot{x}$, $\dot{t}\dot{y}$, $\dot{t}\dot{z}$, $\dot{x}\dot{y}$, $\dot{x}\dot{z}$, $\dot{y}\dot{z}$, $\dot{t}$, $\dot{x}$, $\dot{y}$, $\dot{z}$ and constant respectively,

$$\dot{t}^3 \quad : \qquad \xi_t = 0, \tag{3.2}$$

$$\dot{x}^3 \quad : \qquad \xi_x = 0, \tag{3.3}$$

$$\dot{y}^3 \quad : \qquad \xi_y = 0, \tag{3.4}$$

$$\dot{z}^3 \quad : \qquad \xi_z = 0, \tag{3.5}$$

$$\dot{t}^2 \quad : \qquad -\eta^0_t + \frac{1}{2}\xi_s = 0, \tag{3.6}$$

$$\dot{x}^2 \quad : \qquad P'(t)\eta^0 + 2P(t)\left(\eta^1_x + z\eta^2_x - \eta^3 - \frac{1}{2}\xi_s\right) = 0, \tag{3.7}$$

$$\begin{aligned} \dot{y}^2 \quad : \qquad & (z^2P'(t) + Q'(t))\eta^0 + \left(2zP(t) - 2(z^2P(t) + Q(t))\right)\eta^3 \\ & +2zP(t)\eta^1_y + 2\left(z^2P(t) + Q(t)\right)\left(\eta^2_y - \frac{1}{2}\xi_s\right) = 0, \end{aligned} \tag{3.8}$$

$$\dot{z}^2 \quad : \qquad R'(t)\eta^0 + 2R(t)(\eta^3_z - \frac{1}{2}\xi_s) = 0, \tag{3.9}$$

$$\dot{t}\dot{x} \quad : \qquad -\eta^0_x + P(t)e^{-2z}(\eta^1_t + z\eta^2_t) = 0, \tag{3.10}$$

$$\dot{t}\dot{y} \quad : \qquad -\eta^0_y + e^{-2z}(zP(t)\eta^1_t + (z^2P(t) + Q(t))\eta^2_t) = 0, \tag{3.11}$$

$$\dot{t}\dot{z} \quad : \qquad -\eta^0_z + R(t)\eta^3_t = 0, \tag{3.12}$$

$$\begin{aligned} \dot{x}\dot{y} \quad : \qquad & zP'(t)\eta^0 + (1-2z)\,P(t)\eta^3 + P(t)\eta^1_y + \left(z^2P(t) + Q(t)\right)\eta^2_x \\ & +zP(t)\left(\eta^1_x + \eta^2_y - \xi_s\right) = 0, \end{aligned} \tag{3.13}$$

$$\dot{x}\dot{z} \quad : \qquad P(t)e^{-2z}(\eta^1_z + z\eta^2_z) + R(t)\eta^3_x = 0, \tag{3.14}$$

$$\dot{y}\dot{z} \quad : \qquad e^{-2z}\left(zP(t)\eta^1_z + (z^2P(t) + Q(t))\eta^2_z\right) + R(t)\eta^3_y = 0, \tag{3.15}$$

$$\dot{t} \quad : \qquad -2\eta^0_s = V_t, \tag{3.16}$$

$$\dot{x} \quad : \qquad 2e^{-2z}P(t)\left(\eta^1_s + z\eta^2_s\right) = V_x, \tag{3.17}$$

$$\dot{y} \quad : \qquad 2e^{-2z}\left(zP(t)\eta^1_s + \left(z^2P(t) + Q(t)\right)\eta^2_s\right) = V_y, \tag{3.18}$$

$$\dot{z} \quad : \qquad 2R(t)\eta^3_s = V_z, \tag{3.19}$$

$$constant \quad : \qquad V_s = 0. \tag{3.20}$$

Eq(3.20) implies

$$V = V(t, x, y, z).$$

taking derivative of the eq(3.16-3.19) w.r.t. s, we get

$$\eta^0_{ss} = 0, \tag{3.21}$$

$$\eta^1_{ss} = 0, \tag{3.22}$$

$$\eta^2_{ss} = 0, \tag{3.23}$$

$$\eta^3_{ss} = 0. \tag{3.24}$$

On integrating twice eq(3.21-3.24) w.r.t. s, we have

$$\eta^0 = a_1(t, x, y, z)s + a_2(t, x, y, z), \tag{3.25}$$

$$\eta^1 = b_1(t, x, y, z)s + b_2(t, x, y, z), \tag{3.26}$$

$$\eta^2 = d_1(t, x, y, z)s + d_2(t, x, y, z), \tag{3.27}$$

$$\eta^3 = e_1(t, x, y, z)s + e_2(t, x, y, z). \tag{3.28}$$

Now taking derivative of eq(3.6) w.r.t. s, we have

$$-\eta^0_{tss} + \frac{1}{2}\xi_{sss} = 0,$$

using eq(3.21) in above, we get

$$\xi = \frac{1}{2}c_1 s^2 + c_2 s + c_3. \tag{3.29}$$

So eq(3.6) gives

$$\eta^0_t = \frac{1}{2}(c_1 s + c_2),$$

$$\eta^0_{tt} = \eta^0_{tx} = \eta^0_{ty} = \eta^0_{tz} = 0. \tag{3.30}$$

Now multiplying eq(3.10) with z then subtract from eq(3.11), we get

$$z\eta^0_x - \eta^0_y + e^{-2z}Q(t)\eta^2_t = 0. \tag{3.31}$$

Multiplying eq(3.14) with z then subtract from eq(3.15), we get

$$e^{-2z}Q(t)\eta_z^2 + R(t)\left(\eta_y^3 - z\eta_x^3\right) = 0. \tag{3.32}$$

Using eq(3.25) and eq(3.30), we have

$$\eta_{tt}^0 = a_{1,tt}s + a_{2,tt} = 0,$$

comparing coefficients of s and s^0, we ave

$$s: \quad a_{1,tt} = 0, \qquad s^0: \quad a_{2,tt} = 0.$$

On integrating twice w.r.t. t, we get

$$a_1(t,x,y,z) = a_3(x,y,z)t + a_4(x,y,z),$$
$$a_2(t,x,y,z) = a_5(x,y,z)t + a_6(x,y,z).$$

Substituting above in eq(3.25) we have

$$\eta^0 = \left(a_3(x,y,z)t + a_4(x,y,z)\right)s + a_5(x,y,z)t + a_6(x,y,z). \tag{3.33}$$

Again from eq(3.30) and eq(3.33), we have

$$a_{3,x} = a_{3,y} = a_{3,z} = 0 \quad \Rightarrow \quad a_3 = c_4,$$
$$a_{5,x} = a_{5,y} = a_{5,z} = 0 \quad \Rightarrow \quad a_5 = c_5.$$

Using above values in eq(3.33), we get

$$\eta^0 = \left(c_4t + a_4(x,y,z)\right)s + c_5t + a_6(x,y,z). \tag{3.34}$$

Using eq(3.34) and eq(3.16), we have

$$-2(c_4t + a_4(x,y,z)) = V_t.$$

from above, we have

$$-2a_{4,z} = V_{tz}. \tag{3.35}$$

Substituting eq(3.35) in eq(3.19), we get

$$R^{'}(t)\eta^3_s + R(t)\eta^3_{st} = -a_{4,z}. \tag{3.36}$$

Now substituting eq(3.6) in eq(3.9), we have

$$R^{'}(t)\eta^0 + 2R(t)\left(\eta^3_z - \eta^0_t\right) = 0. \tag{3.37}$$

From eq(3.37), we get

$$R^{'}(t)\eta^0_x + 2R(t)\eta^3_{xz} = 0, \tag{3.38}$$

$$R^{'}(t)\eta^0_y + 2R(t)\eta^3_{yz} = 0, \tag{3.39}$$

$$R^{'}(t)\eta^0_z + 2R(t)\eta^3_{zz} = 0. \tag{3.40}$$

Differentiating eq(3.12) w.r.t z then using eq(3.34) and eq(3.28), we have

$$a_{4,zz}s + a_{6,zz} = R(t)(e_{1,tz}s + e_{2,tz}).$$

Comparing coefficients of s and s^0, we have

$$s: \quad a_{4,zz} = R(t)e_{1,tz}; \qquad s^0: \quad a_{6,zz} = R(t)e_{2,tz},$$

$$e_{1,z} = a_{4,zz}\int\frac{dt}{R(t)} + e_3(x, y, z), \tag{3.41}$$

and

$$e_{2,z} = a_{6,zz}\int\frac{dt}{R(t)} + e_4(x, y, z). \tag{3.42}$$

From eq(3.28) and eq(3.40), we have

$$R^{'}(t)(a_{4,z}s + a_{6,z}) + 2R(t)(e_{1,zz}s + e_{2,zz}) = 0.$$

Comparing coefficients of s and s^0,

$$s: \quad R^{'}(t)a_{4,z} + 2R(t)e_{1,zz} = 0, \tag{3.43}$$

$$s^0: \quad R^{'}(t)a_{6,z} + 2R(t)e_{2,zz} = 0. \tag{3.44}$$

Using eq(3.41) in eq(3.43),

$$R^{'}(t)a_{4,z} + 2R(t)(a_{4,zzz}\int \frac{dt}{R(t)} + e_{3,z}) = 0.$$

Similarly, using eq(3.42) in eq(3.44), we have

$$R^{'}(t)a_{6,z} + 2R(t)(a_{6,zzz}\int \frac{dt}{R(t)} + e_{4,z}) = 0.$$

We find the NSs for $W\left(R^{'}(t), R(t), R(t)\int \frac{dt}{R(t)}\right) \neq 0$,
where W is the Wronskian. Comparing coefficients of $R^{'}(t)$, we have

$$a_{4,z} = 0 \quad \Rightarrow a_4(x,y,z) = a_7(x,y), \tag{3.45}$$

$$a_{6,z} = 0 \quad \Rightarrow a_6(x,y,z) = a_8(x,y). \tag{3.46}$$

Substituting above in eq(3.34), we have

$$\eta^0 = (c_4t + a_7(x,y))\,s + c_5t + a_8(x,y). \tag{3.47}$$

Using eq(3.47) in eq(3.12), we have

$$\eta^3_t = 0,$$

so from eq(3.28), we get

$$\eta^3 = e_5(x,y,z)s + e_6(x,y,z).$$

Using eq(3.36), we have

$$R^{'}(t)\eta^3_s = 0,$$

since $R^{'}(t) \neq 0$, so

$$\eta^3_s = 0, \quad \Rightarrow \quad \eta^3 = e_6(x,y,z). \tag{3.48}$$

Now using eq(3.47) and eq(3.48) in eq(3.38), we get

$$R^{'}(t)(a_{7,x}s + a_{8,x}) + 2R(t)e_{6,xz} = 0.$$

Comparing coefficients of $R(t)$ and $R^{'}(t)$,

$$R(t): \quad e_{6,xz} = 0 = \eta^3_{xz}, \tag{3.49}$$

$$R^{'}(t): \quad a_{7,x}s + a_{8,x} = 0.$$

Comparing coefficients of s and s^0,

$$s: \quad a_{7,x} = 0; \quad s^0: \quad a_{8,x} = 0,$$

$$a_7(x, y) = a_9(y) \quad and \quad a_8(x, y) = a_{10}(y).$$

So eq(3.47) implies,

$$\eta^0 = (c_4t + a_9(y))\, s + c_5t + a_{10}(y). \tag{3.50}$$

Using eq(3.50) and eq(3.48) in eq(3.39), we get

$$R^{'}(t)(a_{9,y}s + a_{10,y}) + 2R(t)e_{6,yz} = 0.$$

Comparing coefficients of $R(t)$ and $R^{'}(t)$,

$$R(t): \quad e_{6,yz} = 0 = \eta^3_{yz}, \tag{3.51}$$

$$R^{'}: \quad a_{9,y}s + a_{10,y} = 0.$$

Comparing coefficients of s and s^0,

$$s: \quad a_{9,y} = 0; \quad s^0: \quad a_{10,y} = 0,$$

on integrating

$$a_9 = c_6; \quad a_{10} = c_7.$$

Substituting values in eq(3.50),

$$\eta^0 = (c_4t + c_6)\, s + c_5t + c_7. \tag{3.52}$$

Using above equation in eq(3.6), we get

$$\xi = c_4 s^2 + 2c_5 s + c_3.$$

Using eq(3.52) in eq(3.31), we get

$$e^{-2z} Q(t) \eta^2_t = 0,$$

$$e^{-2z} Q(t) \neq 0 \quad \Rightarrow \quad \eta^2_t = 0. \tag{3.53}$$

Now using eq(3.52) and eq(3.53) in eq(3.10), we get

$$e^{-2z} P(t)(\eta^1_t + z\eta^2_t) = 0,$$

$$e^{-2z} P(t) \neq 0 \quad \Rightarrow \quad \eta^1_t = 0. \tag{3.54}$$

Using above in eq(3.26) and eq(3.27), we get

$$\eta^1 = b_1(x, y, z)s + b_2(x, y, z),$$

$$\eta^2 = d_1(x, y, z)s + d_2(x, y, z).$$

Using eq(3.52) in eq(3.38-3.40), we get

$$\eta^3_{xz} = \eta^3_{yz} = \eta^3_{zz} = 0.$$

Using eq(3.48) in above, we get

$$\eta^3 = c_8 z + e_7(x, y).$$

So, we have

$$\begin{aligned} \xi &= c_4 s^2 + 2c_5 s + c_3, \\ \eta^0 &= (c_4 t + c_6)\, s + c_5 t + c_7, \\ \eta^1 &= b_1(x, y, z)s + b_2(x, y, z), \\ \eta^2 &= d_1(x, y, z)s + d_2(x, y, z), \\ \eta^1 &= b_1(x, y, z)s + b_2(x, y, z), \\ \eta^3 &= c_8 z + e_7(x, y), \\ V &= V(t, x, y). \end{aligned} \tag{3.55}$$

Taking derivative of eq(3.14) and eq(3.15) w.r.t. y and x respectively, then subtract, we get

$$zP(t)\eta^1_{xz} - P(t)\eta^1_{yz} + (z^2P(t) + Q(t))\eta^2_{xz} - zP(t)\eta^2_{yz} = 0. \tag{3.56}$$

From eq(3.7), we have

$$\eta^1_{xx} + z\eta^2_{xx} - \eta^3_x = 0, \tag{3.57}$$

$$\eta^1_{xy} + z\eta^2_{xy} - \eta^3_y = 0. \tag{3.58}$$

From eq(3.8), we get

$$\left(zP(t) - (z^2P(t) + Q(t))\right)\eta^3_x + zP(t)\eta^1_{xy} + (z^2P(t) + Q(t))\eta^2_{xy} = 0, \tag{3.59}$$

$$\left(zP(t) - (z^2P(t) + Q(t))\right)\eta^3_y + zP(t)\eta^1_{yy} + (z^2P(t) + Q(t))\eta^2_{yy} = 0. \tag{3.60}$$

Multiplying eq(3.58) with $zP(t)$ then subtract from eq(3.59), we get

$$\left((z - z^2)\eta^3_x + z\eta^3_y\right)P(t) + (\eta^2_{xy} - \eta^3_x)Q(t) = 0. \tag{3.61}$$

Now using eq(3.13), we get

$$\left((1 - 2z)\eta^3_x + \eta^1_{xy} + z^2\eta^2_{xx} + z(\eta^1_{xx} + \eta^2_{xy})\right)P(t) + Q(t)\eta^2_{xx} = 0, \tag{3.62}$$

$$\left((1 - 2z)\eta^3_y + \eta^1_{yy} + z^2\eta^2_{xy} + z(\eta^1_{xy} + \eta^2_{yy})\right)P(t) + Q(t)\eta^2_{xy} = 0. \tag{3.63}$$

From eq(3.59), we have

$$\left((z - z^2)\eta^3_x + z\eta^1_{xy} + z^2\eta^2_{xy}\right)P(t) + (\eta^2_{xy} - \eta^3_x)Q(t) = 0. \tag{3.64}$$

We have the following cases,

(Case a) When $P(t)$ and $Q(t)$ are linearly independent

I: $P(t)$ and $Q(t)$ are linearly independent, but $P(t)$ and $R(t)$ are linearly dependent.

II: $P(t)$ and $Q(t)$ are linearly independent, but $Q(t)$ and $R(t)$ are linearly dependent.

III: $P(t)$, $Q(t)$ and $R(t)$ are linearly independent.

(Case b) When $P(t)$ and $Q(t)$ are linearly dependent

IV: $P(t), Q(t)$ and $R(t)$ are linearly dependent.

V: $P(t)$ and $Q(t)$ are linearly dependent but $R(t)$ is independent.

3.1 (Case a)

Since $P(t)$ and $Q(t)$ are linearly independent, so from eq(3.64) we have

$$(z - z^2)\eta^3_x + z\eta^1_{xy} + z^2\eta^2_{xy} = 0, \tag{3.65}$$

and

$$\eta^2_{xy} - \eta^3_x = 0 \Rightarrow \quad \eta^3_x = \eta^2_{xy}. \tag{3.66}$$

Substituting eq(3.66) in eq(3.65),

$$z(\eta^3_x + \eta^1_{xy}) = 0,$$

$$z \neq 0 \Rightarrow \quad \eta^3_x + \eta^1_{xy} = 0. \tag{3.67}$$

Using eq(3.66) in eq(3.67),

$$\eta^1_{xy} + \eta^2_{xy} = 0. \tag{3.68}$$

Similarly from eq(3.60), we have

$$\left((z - z^2)\eta^3_y + z\eta^1_{yy} + z^2\eta^2_{yy}\right) P(t) + (\eta^2_{yy} - \eta^3_y)Q(t) = 0. \tag{3.69}$$

Since $P(t)$ and $Q(t)$ are linearly independent, so

$$(z - z^2)\eta^3_y + z\eta^1_{yy} + z^2\eta^2_{yy} = 0, \tag{3.70}$$

and

$$\eta^2_{yy} - \eta^3_y \Rightarrow \quad \eta^2_{yy} = \eta^3_y. \tag{3.71}$$

Substituting eq(3.71) in eq(3.70), we get

$$z(\eta^1_{yy} + \eta^3_y) = 0,$$

$$z \neq 0 \Rightarrow \quad \eta^1_{yy} + \eta^3_y = 0. \tag{3.72}$$

Using eq(3.71) in eq(3.72), we have

$$\eta^1_{yy} + \eta^2_{yy} = 0. \tag{3.73}$$

Now using eq(3.66) and eq(3.67) in eq(3.58), we get

$$-\eta^3_x + z\eta^3_x - \eta^3_y = 0.$$

Comparing coefficients of z andz^0

$$z: \quad \eta^3_x = 0, \qquad z^0: \quad -\eta^3_x - \eta^3_y = 0,$$

$$\eta^3_x = \eta^3_y = 0,$$

so eq(3.55) implies,

$$e_7(x,y) = c_9 \Rightarrow \quad \eta^3 = zc_8 + c_9. \tag{3.74}$$

From eq(3.72) and eq(3.67), we have

$$\begin{aligned} \eta^1_{yy} &= 0, \\ \eta^1_{xy} &= 0, \end{aligned} \tag{3.75}$$

using eq(3.71) and eq(3.66), we get

$$\begin{aligned} \eta^2_{yy} &= 0, \\ \eta^2_{xy} &= 0. \end{aligned} \tag{3.76}$$

Now using eq(3.55) in eq(3.76), we get

$$\eta^2_{yy} = d_{1,yy}s + d_{2,yy} = 0. \tag{3.77}$$

Comparing coefficients of s and s^0

$$s: \quad d_{1,yy} = 0; \quad s^0: \quad d_{2,yy} = 0,$$

on integrating

$$\begin{aligned} d_1 &= yd_3(x,z) + d_4(x,z), \\ d_2 &= yd_5(x,z) + d_6(x,z). \end{aligned}$$

Substituting above equations in eq(3.55), we get

$$\eta^2 = (yd_3(x,z) + d_4(x,z))s + yd_5(x,z) + d_6(x,z). \tag{3.78}$$

Again using eq(3.76), we have

$$\eta^2_{xy} = d_{3,x}s + d_{5,x} = 0,$$

comparing coefficients of s and s^0,

$$s: \quad d_{3,x} = 0; \qquad s^0: \quad d_{5,x} = 0,$$

$$d_3(x,z) = d_7(z); \quad d_5(x,z) = d_8(z).$$

Substituting above equations in eq(3.78), we get

$$\eta^2 = (yd_7(z) + d_4(x,z))s + yd_8(z) + d_6(x,z). \tag{3.79}$$

Using eq(3.74) in eq(3.32), we get

$$\eta^2_z = 0,$$

so eq(3.79) implies

$$(yd_{7,z} + d_{4,z}(x,z))s + yd_{8,z}(z) + d_{6,z}(x,z) = 0.$$

Comparing coefficients of s and s^0, we have

$$s: \quad yd_{7,z} + d_{4,z}(x,z) = 0, \qquad s^0: \quad yd_{8,z}(z) + d_{6,z}(x,z) = 0,$$

comparing coefficients of y and y^0, we have

$$\begin{array}{lcll} y & : & d_{7,z} = 0, & d_{8,z}(z) = 0, \\ y^0 & : & d_{4,z} = 0, & d_{6,z}(x,z) = 0, \end{array}$$

on integrating

$$d_7 = c_{10}; \quad d_8 = c_{11}; \quad d_4(x,z) = d_9(x); \quad d_6(x,z) = d_{10}(x).$$

Substituting above equations in eq(3.79), we get

$$\eta^2 = (yc_{10} + d_9(x))s + yc_{11} + d_{10}(x). \tag{3.80}$$

Using eq(3.80) and eq(3.14), we have

$$P(t)e^{-2z}\eta^1_z = 0,$$

$$P(t)e^{-2z} \neq 0 \Rightarrow \eta^1_z = 0.$$

So eq(3.55) become,

$$\eta^1 = b_1(x, y)s + b_2(x, y). \tag{3.81}$$

Now using eq(3.74) in eq(3.57), we have

$$\eta^1_{xx} + z\eta^2_{xx} = 0,$$

comparing coefficients of z andz^0, we have

$$z: \qquad \eta^2_{xx} = 0, \tag{3.82}$$

$$z^0: \qquad \eta^1_{xx} = 0. \tag{3.83}$$

Using eq(3.80) in eq(3.82), we have

$$d_{9,xx}s + d_{10,xx} = 0,$$

comparing coefficients of s and s^0,

$$s: \quad d_{9,xx} = 0; \qquad s^0: \quad d_{10,xx} = 0,$$

on integrating

$$d_9(x) = xc_{12} + c_{13},$$

$$d_{10}(x) = xc_{14} + c_{15}.$$

Substituting above equations in eq(3.80), we get

$$\eta^2 = (yc_{10} + xc_{12} + c_{13})s + yc_{11} + xc_{14} + c_{15}. \tag{3.84}$$

Now using eq(3.81) in eq(3.75), we have

$$\eta^1_{yy} = b_{1,yy}s + b_{2,yy} = 0.$$

Comparing coefficients of s and s^0,

$$s: \quad b_{1,yy} = 0; \qquad s^0: \quad b_{2,yy} = 0.$$

on integrating

$$b_1 = yb_3(x) + b_4(x),$$
$$b_2 = yb_5(x) + b_6(x).$$

So eq(3.81) implies

$$\eta^1 = (yb_3(x) + b_4(x))s + yb_5(x) + b_6(x), \tag{3.85}$$

using eq(3.75), we have

$$\eta^1_{xy} = b_{3,x}s + b_{5,x} = 0.$$

Comparing coefficients of s and s^0,

$$s: \quad b_{3,x} = 0; \qquad s^0: \quad b_{5,x} = 0,$$

on integrating,

$$b_3 = c_{16}; \qquad b_5 = c_{17}.$$

Substituting in eq(3.85),

$$\eta^1 = (yc_{16} + b_4(x))s + yc_{17} + b_6(x). \tag{3.86}$$

Using eq(3.86) in eq(3.83),

$$\eta^1_{xx} = b_{4,xx}s + b_{6,xx} = 0.$$

Comparing coefficients of s and s^0,

$$s: \quad b_{4,xx} = 0; \qquad s^0: \quad b_{6,xx} = 0,$$

on integrating,

$$b_4 = xc_{18} + c_{19},$$

$$b_6 = xc_{20} + c_{21}.$$

Substituting above equations in eq(3.86),

$$\eta^1 = (yc_{16} + xc_{18} + c_{19})s + yc_{17} + xc_{20} + c_{21}. \tag{3.87}$$

Using eq(3.74) in eq(3.19), we get

$$V_z = 0. \tag{3.88}$$

Now from eq(3.17), we have

$$2e^{-2z}P(t)(-2\eta^1_s - 2z\eta^2_s + \eta^2_s) = V_{xz} = 0,$$

$$2e^{-2z}P(t) \neq 0 \Rightarrow \quad -2\eta^1_s - 2z\eta^2_s + \eta^2_s = 0.$$

Comparing coefficients of z and z^0 we get

$$\eta^1_s = 0, \qquad \eta^2_s = 0. \tag{3.89}$$

Using eq(3.89) in eq(3.17) and eq(3.18), we get

$$V_x = 0, \qquad V_y = 0.$$

So using eq(3.55) and eq(3.16), we get

$$V = -c_4t^2 - 2c_6t - c_0. \tag{3.90}$$

Now using (3.89) in (3.84) and (3.87), we have

$$\eta^1 = yc_{17} + xc_{20} + c_{21}, \tag{3.91}$$

and

$$\eta^2 = yc_{11} + xc_{14} + c_{15}.$$

Comparing coefficients of z and z^0 from eq(3.7), we get

$$z: \quad \eta^2_x = 0, \quad \Rightarrow \quad c_{14} = 0,$$

$$\eta^2 = yc_{11} + c_{15}, \tag{3.92}$$

$$z^0: \quad P'(t)\eta^0 + 2P(t)\left(\eta^1_x - \eta^3 - \frac{1}{2}\xi_s\right) = 0. \tag{3.93}$$

On differentiating eq(3.8) thrice w.r.t. z, we get

$$\eta^3_z = 0,$$

so eq(3.74) gives,

$$c_8 = 0, \quad \Rightarrow \quad \eta^3 = c_9. \tag{3.94}$$

Using above in eq(3.9), we get

$$R^{'}(t)\eta^0 - R(t)\xi_s = 0. \tag{3.95}$$

Now comparing coefficients of z^2, z and z^0, of eq(3.8), we have

$$z^2 \quad : \quad P^{'}(t)\eta^0 + 2P(t)(-\eta^3 + \eta^2_y - \frac{1}{2}\xi_s) = 0, \tag{3.96}$$

$$z \quad : \quad P(t)(\eta^3 + \eta^1_y) = 0 \Rightarrow \quad \eta^3 + \eta^1_y = 0, \tag{3.97}$$

$$z^0 \quad : \quad Q^{'}(t)\eta^0 + 2Q(t)(-\eta^3 + \eta^2_y - \frac{1}{2}\xi_s) = 0. \tag{3.98}$$

Using eq(3.94) and eq(3.91) in eq(3.97), we get

$$c_{17} = -c_9 \quad \Rightarrow \quad \eta^1 = -yc_9 + xc_{20} + c_{21}.$$

Subtracting eq(3.93) and eq(3.96), we get

$$\eta^1_x = \eta^2_y \quad \Rightarrow \quad c_{20} = c_{11},$$

$$\eta^1 = -yc_9 + xc_{11} + c_{21}.$$

From eq(3.55), eq(3.90), eq(3.92) and eq(3.94), we have

$$\begin{aligned}
\xi &= c_4 s^2 + 2c_5 s + c_3, \\
\eta^0 &= (c_4 t + c_6)\, s + c_5 t + c_7, \\
\eta^1 &= -yc_9 + xc_{11} + c_{21}, \\
\eta^2 &= yc_{11} + c_{15}, \\
\eta^3 &= c_9, \\
V &= -c_4 t^2 - 2c_6 t - c_0.
\end{aligned} \tag{3.99}$$

3.1.1 Case I:

$P(t)$ and $Q(t)$ are linearly independent, but $P(t)$ and $R(t)$ are linearly dependent. Suppose $R(t) = hP(t)$, using in eq(3.95), we

$$hP^{'}(t)\eta^0 - hP(t)\xi_s = 0,$$

$$P^{'}(t)\eta^0 - P(t)\xi_s = 0. \tag{3.100}$$

Subtracting eq(3.100) from eq(3.96), we get

$$\eta^2_y = \eta^3 \quad \Rightarrow \quad c_{11} = c_9$$

Using above in eq(3.99), we have

$$\begin{aligned}
\xi &= c_4 s^2 + 2c_5 s + c_3, \\
\eta^0 &= (c_4 t + c_6)\, s + c_5 t + c_7, \\
\eta^1 &= (x - y)c_9 + c_{21}, \\
\eta^2 &= yc_9 + c_{15}, \\
\eta^3 &= c_9, \\
V &= -c_4 t^2 - 2c_6 t - c_0.
\end{aligned} \tag{3.101}$$

Multiplying eq(3.96) and eq(3.98) with Q(t) and P(t) respectively, then subtract both equations, we get

$$(P'(t)Q(t) - Q'(t)P(t))\eta^0 = 0,$$

since P(t) and Q(t) are linearly independent, so

$$P'(t)Q(t) - Q'(t)P(t) \neq 0 \quad \Rightarrow \eta^0 = 0.$$

Using eq(3.101), we have

$$c_4 = c_5 = c_6 = c_7 = 0.$$

Substituting in eq(3.101), we have

$$\begin{aligned} \xi &= c_3, \\ \eta^0 &= 0, \\ \eta^1 &= (x-y)c_9 + c_{21}, \\ \eta^2 &= yc_9 + c_{15}, \\ \eta^3 &= c_9, \\ V &= c_0. \end{aligned} \tag{3.102}$$

Here $c_0,\ c_3,\ c_9,\ c_{15}\ \textit{and}\ c_{21}$, are the arbitrary constant. We have following NSs,

$$\begin{aligned} \mathbf{X_0} &= \frac{\partial}{\partial s}, \\ \mathbf{X_1} &= (x-y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \\ \mathbf{X_2} &= \frac{\partial}{\partial x}, \\ \mathbf{X_3} &= \frac{\partial}{\partial y}. \end{aligned} \tag{3.103}$$

Which is the minimal set of NSs for Bianchi type IV.

3.1.2 Case II:

$P(t)$ and $Q(t)$ are linearly independent, but $Q(t)$ and $R(t)$ are linearly dependent. Suppose $R(t) = jQ(t)$, using in eq(3.95), we

$$jQ^{'}(t)\eta^0 - jQ(t)\xi_s = 0,$$

$$Q^{'}(t)\eta^0 - Q(t)\xi_s = 0. \tag{3.104}$$

Subtracting eq(3.104) from eq(3.98), we get

$$\eta_y^2 = \eta^3 \quad \Rightarrow \quad c_{11} = c_9$$

Using above in eq(3.99), we have

$$\begin{aligned} \xi &= c_4 s^2 + 2c_5 s + c_3, \\ \eta^0 &= (c_4 t + c_6)\, s + c_5 t + c_7, \\ \eta^1 &= (x-y)c_9 + c_{21}, \\ \eta^2 &= yc_9 + c_{15}, \\ \eta^3 &= c_9, \\ V &= -c_4 t^2 - 2c_6 t - c_0. \end{aligned} \tag{3.105}$$

Multiplying eq(3.96) and eq(3.98) with Q(t) and P(t) respectively, then subtract both equations, we get

$$(P^{'}(t)Q(t) - Q^{'}(t)P(t))\eta^0 = 0,$$

since P(t) and Q(t) are linearly independent, so

$$P^{'}(t)Q(t) - Q^{'}(t)P(t) \neq 0 \quad \Rightarrow \eta^0 = 0.$$

Using eq(3.105), we have

$$c_4 = c_5 = c_6 = c_7 = 0.$$

Substituting in eq(3.105), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_9 + c_{21}, \\
\eta^2 &= yc_9 + c_{15}, \\
\eta^3 &= c_9, \\
V &= c_0.
\end{aligned}$$

From above, we again get the minimal set of NSs given by eq(3.103).

3.1.3 Case III:

$P(t)$, $Q(t)$ and $R(t)$ are linearly independent. Multiplying eq(3.96) and eq(3.98) with Q(t) and P(t) respectively, then subtract both equations, we get

$$(P^{'}(t)Q(t) - Q^{'}(t)P(t))\eta^0 = 0,$$

since P(t) and Q(t) are linearly independent, so

$$P^{'}(t)Q(t) - Q^{'}(t)P(t) \neq 0 \quad \Rightarrow \eta^0 = 0.$$

Using eq(3.99), we have

$$c_4 = c_5 = c_6 = c_7 = 0,$$

substituting in eq(3.99), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= xc_{11} - yc_9 + c_{21}, \\
\eta^2 &= yc_{11} + c_{15}, \\
\eta^3 &= c_9, \\
V &= c_0.
\end{aligned} \tag{3.106}$$

Using above eq(3.96), we get

$$\eta^2_y = \eta^3 \quad \Rightarrow \quad c_{11} = c_9. \tag{3.107}$$

Substituting above in eq(3.106), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_9 + c_{21}, \\
\eta^2 &= yc_9 + c_{15}, \\
\eta^3 &= c_9, \\
V &= c_0.
\end{aligned}$$

From above, we again get the minimal set of NSs given by eq(3.103).

3.2 (Case b)

$P(t)$ and $Q(t)$ are linearly dependent. Suppose we have $Q(t) = kP(t)$, using eq(3.56),

$$zP(t)\eta^1_{xz} - P(t)\eta^1_{yz} + (z^2P(t) + kP(t))\eta^2_{xz} - zP(t)\eta^2_{yz} = 0,$$

$$P(t) \neq 0 \quad \Rightarrow z\eta^1_{xz} - \eta^1_{yz} + (z^2+k)\eta^2_{xz} - z\eta^2_{yz} = 0. \tag{3.108}$$

Using eq(3.59),

$$\left(zP(t) - (z^2P(t) + kP(t))\right)\eta^3_x + zP(t)\eta^1_{xy} + (z^2P(t) + kP(t))\eta^2_{xy} = 0,$$

since $P(t) \neq 0$, above equation become

$$\left(z - (z^2+k)\right)\eta^3_x + z\eta^1_{xy} + (z^2+k)\eta^2_{xy} = 0. \tag{3.109}$$

Using eq(3.60), we have

$$\left(zP(t) - (z^2P(t) + kP(t))\right)\eta^3_y + zP(t)\eta^1_{yy} + (z^2P(t) + kP(t))\eta^2_{yy} = 0,$$

since $P(t) \neq 0$, above equation become

$$\left(z - (z^2 + k)\right) \eta^3_y + z\eta^1_{yy} + (z^2 + k)\eta^2_{yy} = 0. \tag{3.110}$$

From eq(3.61), we get

$$(z - z^2)\eta^3_x + z\eta^3_y + k(\eta^2_{xy} - \eta^3_x) = 0. \tag{3.111}$$

From eq(3.62) and eq(3.63), we get

$$(1 - 2z)\eta^3_x + \eta^1_{xy} + z^2\eta^2_{xx} + z(\eta^1_{xx} + \eta^2_{xy}) + k\eta^2_{xx} = 0, \tag{3.112}$$

$$(1 - 2z)\eta^3_y + \eta^1_{yy} + z^2\eta^2_{xy} + z(\eta^1_{xy} + \eta^2_{yy}) + k\eta^2_{xy} = 0. \tag{3.113}$$

From eq(3.7), we have

$$\eta^1_{xz} + z\eta^2_{xz} + \eta^2_x - \eta^3_z = 0. \tag{3.114}$$

Now multiplying eq(3.17) with z then subtract from eq(3.18), we get

$$2e^{-2z}Q(t)\eta^2_s = V_y - zV_x,$$

on differentiating twice w.r.t. z, we get

$$\eta^2_{szz} - 4\eta^2_{sz} + 4\eta^2_s = 0.$$

Using eq(3.55) in above equation, we get

$$d_{1,zz}(x, y, z) - 4d_{1,z}(x, y, z) + 4d_1(x, y, z) = 0,$$

solving above DE, we get

$$d_1(x, y, z) = e^{2z}(d_3(x, y) + zd_4(x, y)).$$

Substituting above equation in eq(3.55), we have

$$\eta^2 = e^{2z}(d_3(x, y) + zd_4(x, y))s + d_2(x, y, z). \tag{3.115}$$

Now from eq(3.17), we have

$$\eta^1_{sz} + \eta^2_s - 2\eta^1_s + z(\eta^2_{sz} - 2\eta^2_s) = 0, \tag{3.116}$$

from eq(3.18), we have

$$P(t)\left(z\eta^1_{sz} + (1-2z)\eta^1_s + z^2\eta^2_{sz} + (2z - 2z^2)\eta^2_s\right) + Q(t)(\eta^2_{sz} - 2\eta^2_s) = 0. \tag{3.117}$$

Now multiplying eq(3.116) with $zP(t)$ then subtract from eq(3.117), we get

$$P(t)(z\eta^2_s + \eta^1_s) + Q(t)(\eta^2_{sz} - 2\eta^2_s) = 0,$$

but $Q(t) = kP(t)$, we have

$$z\eta^2_s + \eta^1_s + k(\eta^2_{sz} - 2\eta^2_s) = 0. \tag{3.118}$$

Using eq(3.32), we get

$$-Q(t)e^{-2z}\eta^2_{sz} = 0,$$

$$-Q(t)e^{-2z} \neq 0 \quad \Rightarrow \eta^2_{sz} = 0,$$

so using above equation in eq(3.115),

$$(d_4(x,y) + 2d_3(x,y) + 2zd_4(x,y))e^{2z} = 0,$$

$$e^{2z} \neq 0 \Rightarrow \quad d_4(x,y) + 2d_3(x,y) + 2zd_4(x,y) = 0.$$

Comparing coefficients of z and z^0 of above equation, we get

$$d_4(x,y) = 0, \quad d_3(x,y) = 0, \tag{3.119}$$

therefore

$$\eta^2 = d_2(x,y,z),$$

so above equation implies

$$\eta^2_s = 0.$$

Using above equation in eq(3.118), we get

$$\eta^1_s = 0.$$

Using above result in eq(3.55), we have

$$\begin{aligned} \xi &= c_4 s^2 + 2c_5 s + c_3, \\ \eta^0 &= (c_4 t + c_6)\, s + c_5 t + c_7, \\ \eta^1 &= b_2(x, y, z), \\ \eta^2 &= d_2(x, y, z), \\ \eta^3 &= c_8 z + e_7(x, y) \\ V &= V(t). \end{aligned} \tag{3.120}$$

Now multiplying eq(3.58) with z, then subtract from eq(3.109), we get

$$(z - z^2 - k)\eta^3_x + z\eta^3_y + k\eta^2_{xy} = 0, \tag{3.121}$$

subtracting eq(3.58) and product of z and eq(3.57) from eq(3.112), we get

$$(1 - 2z)\eta^3_x + \eta^3_y + z\eta^3_x + k\eta^2_{xx} = 0,$$

$$(1 - z)\eta^3_x + \eta^3_y + k\eta^2_{xx} = 0. \tag{3.122}$$

Now differentiating eq(3.121) and eq(3.122) w.r.t. x and y respectively, then subtract their result, we get

$$(z - z^2 - k)\eta^3_{xx} + (2z - 1)\eta^3_{xy} - \eta^3_{yy} = 0,$$

comparing coefficients of z^2, z and z^0, we have

$$z^2 : \quad \eta^3_{xx} = 0, \tag{3.123}$$

$$z : \quad \eta^3_{xy} = 0, \tag{3.124}$$

$$z^0 : \quad \eta^3_{yy} = 0. \tag{3.125}$$

Using eq(3.120) in eq(3.124), we have

$$e_7 = xe_9(y) + e_{10}(y), \tag{3.126}$$

using above equation in eq(3.124), we have

$$e_{9,y} = 0 \Rightarrow \quad e_9 = c_{11},$$

using above equation in eq(3.126),

$$e_7 = c_{11}x + e_{10}(y). \tag{3.127}$$

Substituting above in eq(3.125), we have

$$e_{10,yy} = 0 \Rightarrow \quad e_{10}(y) = c_{12}y + c_{13},$$

using above equation and eq(3.126) in eq(3.120), we get

$$\eta^3 = c_{11}x + c_{12}y + c_8 z + c_{13}. \tag{3.128}$$

Now from eq(3.32), we have

$$\eta^2_{xz} = 0 =, \quad \eta^2_{yz} = 0, \tag{3.129}$$

again using eq(3.32), we get

$$\eta^2_{zzz} - 4\eta^2_{zz} + 4\eta^2_z = 0.$$

Using eq(3.120), we have

$$d_{2,zzz} - 4d_{2,zz} + 4d_{2,z} = 0,$$

by solving above equation, we get

$$d_2(x, y, z) = d_5(x, y) + e^{2z}(d_6(x, y) + zd_7(x, y)) \tag{3.130}$$

using above equation in eq(3.129), we have

$$d_{2,xz} = e^{2z}(2d_{6,x} + 2zd_{7,x} + d_{7,x}) = 0,$$

$$e^{2z} \neq 0, \quad 2d_{6,x} + 2zd_{7,x} + d_{7,x} = 0.$$

Comparing coefficients of z and z^0, we have

$$z: \quad d_{7,x}(x,y) = 0 \quad \Rightarrow \quad d_7(x,y) = d_9(y),$$

$$z^0: \quad d_{6,x}(x,y) = 0 \quad \Rightarrow \quad d_6(x,y) = d_8(y).$$

using above in eq(3.130),

$$d_2(x,y,z) = d_5(x,y) + e^{2z}(d_8(y) + zd_9(y)), \tag{3.131}$$

again using above in eq(3.129), we have

$$d_{2,yz} = e^{2z}(d_{9,y} + 2zd_{9,y} + d_{8,y}) = 0,$$

$$e^{2z} \neq 0, \quad d_{9,y} + 2zd_{9,y} + d_{8,y} = 0.$$

Comparing coefficients of z and z^0, we have

$$z: \quad d_{9,y} = 0 \quad \Rightarrow \quad d_9(y) = c_{15},$$

$$z^0: \quad d_{8,y} = 0 \quad \Rightarrow \quad d_8(y) = c_{14}$$

using above equation in eq(3.130),

$$\eta^2 = d_2(x,y,z) = d_5(x,y) + e^{2z}(c_{14} + zc_{15}). \tag{3.132}$$

Now from eq(3.122), we have

$$\eta^2_{xxx} = 0, \quad \Rightarrow \quad \eta^2_{xxy} = 0. \tag{3.133}$$

Using eq(3.132), we have

$$d_{5,xxx}(x,y) = 0,$$

by solving above, we have

$$d_5(x,y) = \frac{1}{2}d_{10}(y)x^2 + d_{11}(y)x + d_{12}(y). \tag{3.134}$$

Again using eq(3.133), we have

$$\eta^2_{xxy} = d_{5,xxy}(x,y) = d_{10,y} = 0 \Rightarrow \quad d_{10} = c_{16},$$

so

$$d_5(x,y) = \frac{1}{2}c_{16}x^2 + d_{11}(y)x + d_{12}(y).$$

Using above equation and eq(3.128) in eq(3.121),

$$(z - z^2 - k)c_{11} + zc_{12} + kd_{11,y}(y) = 0.$$

Comparing coefficients of z^2, z and z^0, we have

$$\begin{aligned} z^2 &: \quad c_{11} = 0, \\ z &: \quad c_{12} = 0, \\ z^0 &: \quad d_{11,y}(y) = 0 \quad \Rightarrow \quad d_{11}(y) = c_{17}. \end{aligned}$$

Substituting above result in eq(3.128) and eq(3.134), we have

$$\begin{aligned} \eta^3 &= c_8 z + c_{13}, \\ d_5(x,y) &= \frac{1}{2}c_{16}x^2 + c_{17}x + d_{12}(y), \end{aligned} \tag{3.135}$$

using above equation eq(3.122), we have

$$\eta^2_{xx} = d_{5,xx} = c_{16} = 0,$$

so using in above equation, which implies

$$d_5(x,y) = c_{17}x + d_{12}(y).$$

Substituting in eq(3.132), we have

$$\eta^2 = c_{17}x + d_{12}(y) + e^{2z}(c_{14} + zc_{15}), \tag{3.136}$$

using eq(3.135) and above equation in eq(3.32), we have

$$\eta^2_z = 0,$$

so using eq(3.136) in above equation,

$$2c_{14} + 2zc_{15} + c_{15} = 0.$$

Comparing coefficients of z and z^0, we get

$$z: \quad c_{15} = 0, \quad z^0: \quad c_{14} = 0,$$

So eq(3.136) implies,

$$\eta^2 = c_{17}x + d_{12}(y). \tag{3.137}$$

using eq(3.136), in eq(3.14), we have

$$\eta^1_z = 0 \quad \Rightarrow \quad \eta^1 = b_2(x, y), \tag{3.138}$$

from eq(3.110), we have

$$z\eta^1_{yy} + (z^2 + k)\eta^2_{yy} = 0.$$

Comparing coefficients of z^2 and z, we have

$$z^2: \quad \eta^2_{yy} = 0, \quad z: \quad \eta^1_{yy} = 0, \tag{3.139}$$

using eq(3.137) in above, we have

$$\eta^2_{yy} = d_{12,yy} = 0,$$

so

$$d_{12}(y) = c_{18}y + c_{19},$$

so eq(3.137) implies,

$$\eta^2 = c_{17}x + c_{18}y + c_{19}. \tag{3.140}$$

Using eq(3.138) in eq(3.139), we have

$$\eta^1_{yy} = b_{2,yy} = 0,$$

$$\eta^1 = b_2(x, y) = yb_3(x) + b_4(x), \tag{3.141}$$

from eq(3.112), we have

$$z\eta^1_{xx} + \eta^1_{xy} = 0.$$

Comparing coefficients of z and z^0, we have

$$z : \quad \eta^1_{xx} = 0 \quad z^0 : \quad \eta^1_{xy} = 0, \tag{3.142}$$

using eq(3.141) in above equation, we have

$$\eta^1_{xy} = b_{3,x}(x) = 0 \quad \Rightarrow \quad b_3(x) = c_{20},$$

so eq(3.141) implies,

$$\eta^1 = b_2(x, y) = yc_{20} + b_4(x). \tag{3.143}$$

Again using eq(3.142), we have

$$\eta^1_{xx} = b_{4,xx} = 0, \quad \Rightarrow \quad b_4(x) = c_{21}x + c_{22},$$

substituting eq(3.143), we have

$$\eta^1 = b_2(x, y) = yc_{20} + c_{21}x + c_{22}. \tag{3.144}$$

Using eq(3.16), we get

$$V_t = -2c_4t - 2c_6.$$

From eq(3.120), we have

$$\begin{aligned}
\xi &= c_4 s^2 + 2c_5 s + c_3, \\
\eta^0 &= (c_4 t + c_6)s + c_5 t + c_7, \\
\eta^1 &= y c_{20} + c_{21} x + c_{22}, \\
\eta^2 &= c_{17} x + c_{18} y + c_{19}, \\
\eta^3 &= c_8 z + c_{13}, \\
V &= -c_4 t^2 - 2c_6 t - c_0.
\end{aligned} \tag{3.145}$$

Differentiating eq(3.8) thrice w.r.t. z, we have

$$P(t)\eta^3_z = 0, \quad P(t) \neq 0 \quad \Rightarrow \quad \eta^3_z = 0.$$

So

$$c_8 = 0,$$

eq(3.145) implies,

$$\eta^3 = c_{13}. \tag{3.146}$$

Comparing coefficient of z and z^0 from eq(3.7), we get

$$z: \quad \eta^2_x = 0 \quad \Rightarrow \quad c_{17} = 0,$$

$$z^0: \quad P^{'}(t)\eta^0 + 2P(t)(\eta^1_x - \eta^3 - \frac{1}{2}\xi_s) = 0. \tag{3.147}$$

Now comparing coefficients of z^2, z and z^0, of eq(3.8), we have

$$z^2 \ : \ P^{'}(t)\eta^0 + 2P(t)(-\eta^3 + \eta^2_y - \frac{1}{2}\xi_s) = 0, \tag{3.148}$$

$$z \ : \ P(t)(\eta^3 + \eta^1_y) = 0 \Rightarrow \quad \eta^3 + \eta^1_y = 0. \tag{3.149}$$

Using eq(3.145), we get

$$c_{20} = -c_{13},$$

$$z^0: \quad Q^{'}(t)\eta^0 + 2Q(t)(-\eta^3 + \eta^2_y - \frac{1}{2}\xi_s) = 0,$$

but Q(t)=kP(t), so above equation gives

$$P^{'}(t)\eta^0 + 2P(t)(-\eta^3 + \eta^2_y - \frac{1}{2}\xi_s) = 0.$$

Subtracting eq(3.147) and eq(3.148), we get

$$\eta^1_x = \eta^2_y, \quad \Rightarrow \quad c_{21} = c_{18}.$$

From eq(3.145), we have

$$\begin{aligned} \xi &= c_4 s^2 + 2c_5 s + c_3, \\ \eta^0 &= (c_4 t + c_6)s + c_5 t + c_7, \\ \eta^1 &= c_{18}x - c_{13}y + c_{22}, \\ \eta^2 &= c_{18}y + c_{19}, \\ \eta^3 &= c_{13}, \\ V &= -c_4 t^2 - 2c_6 t - c_0. \end{aligned} \tag{3.150}$$

3.2.1 Case IV :

$P(t), Q(t)$ and $R(t)$ are linearly dependent. since $R(t) = hP(t)$, so eq(3.9), we have

$$hP^{'}(t)\eta^0 - hP(t)\xi_s = 0, \tag{3.151}$$

$$P^{'}(t)\eta^0 - P(t)\xi_s = 0. \tag{3.152}$$

Subtracting eq(3.152) from eq(3.147), we get

$$\eta^1_x = \eta^3,$$

but

$$\eta^3 = c_{13}.$$

Using eq(3.150) and above equation, we have

$$c_{18} = c_{13},$$

substituting above values in eq(3.150), we have

$$\begin{aligned}
\xi &= c_4 s^2 + 2c_5 s + c_3, \\
\eta^0 &= (c_4 t + c_6)s + c_5 t + c_7, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= c_{13} y + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= -c_4 t^2 - 2c_6 t - c_0.
\end{aligned} \tag{3.153}$$

Now using eq(3.9) and eq(3.6), we have,

$$R^{'}(t)\eta^0 - 2R(t)\eta^0_t = 0, \tag{3.154}$$

from above equation,we get,

$$R^{''}(t)\eta^0 - R^{'}(t)\eta^0_t = 0. \tag{3.155}$$

Since $R(t)$ and $R^{'}(t)$ are linearly independent, which implies $R^{'}(t) \neq 0$. So from eq(3.155), we have following possibilities

1: $\eta^0 = 0$,

2: $R^{''}(t) = 0 = \eta^0_t$,

3: $R^{''}(t)\eta^0 = R^{'}(t)\eta^0_t$.

Case IV-1

$\eta^0 = 0$,
using eq(3.153), we have $c_4 = c_5 = c_6 = c_7 = 0$,
so eq(3.153) implies

from eq(3.153), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

Here $c_0,\ c_3,\ c_{13},\ c_{19}$ and c_{22}, are the arbitrary constant. Corresponding to these constants, we again get the minimal set of NSs given by eq(3.103).

Case IV-2

$R^{''}(t) = 0 = \eta^0_t$,
using above in eq(3.154), we have

$$R^{'}(t)\eta^0 = 0, \tag{3.156}$$

since $R^{'}(t) \neq 0$, so $\eta^0 = 0$, in both **(1)** and **(2)** cases , we get $\eta^0 = 0$, from eq(3.153), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

From above, we again get the minimal set of NSs given by eq(3.103).

Case IV-3

From eq(3.155), we have

$$R^{''}(t)\eta^0 = R^{'}(t)\eta^0_t \quad \Rightarrow R^{'}(t) = c_{23}\eta^0, \tag{3.157}$$

multiplying eq(3.154) and eq(3.155) with $R^{''}(t)$ and $R^{'}(t)$ respectively, then subtract

$$\left(R^{'2}(t) - 2R^{''}(t)R(t)\right)\eta^0_t = 0.$$

If $\eta^0_t = 0$ then, eq(3.154) gives $\eta^0 = 0$, it is already done and we have result in eq(3.157), so we solve for

$$R^{'2}(t) - 2R^{''}(t)R(t) = 0. \tag{3.158}$$

Using eq(3.157) in above equation, we have

$$c_{23}^2\eta^{0^2} - 2\left(c_{23}\eta^0_t\right)\left(c_{23}\int \eta^0 dt\right) = 0,$$

using eq(3.120) in above equation, we have

$$((c_4t + c_6)s + c_5t + c_7)^2 - 2(c_4s + c_5)((\frac{1}{2}c_4^2t^2 + c_6t)s + \frac{1}{2}c_5^2t^2 + c_7t + \frac{c_{24}}{c_{23}^2}).$$

Suppose $\frac{c_{24}}{c_{23}^2} = c_{25}$ then simplifying above equation, we have

$$c_6^2s^2 + (2c_6c_7 - 2c_4c_{25})s + c_7^2 - 2c_5c_{25} = 0.$$

Comparing coefficients of s^2, s and s^0

$$s^2 \ : \ c_6 = 0, \tag{3.159}$$

$$s \ : \ c_6c_7 = c_4c_{25}, \tag{3.160}$$

$$s^0 \ : \ c_7^2 = 2c_5c_{25}. \tag{3.161}$$

Since $c_6 = 0$, therefore $c_4c_{25} = 0$. Hence, either $c_4 = 0$ or $c_{25} = 0$.
If $c_{25} = 0$, then eq(3.161) implies, $c_7 = 0$.
From above, we have three possibilities

(i) $c_6 = 0, \quad c_4 = 0, \quad c_7 = 0,$

(ii) $c_6 = 0, \quad c_4 = 0, \quad c_7 \neq 0,$

(iii) $c_6 = 0, \quad c_4 \neq 0, \quad c_7 = 0.$

Case IV-3-(i)

$c_6 = 0, \quad c_4 = 0, \quad c_7 = 0,$
from eq(3.153), we have

$$\begin{aligned}
\xi &= 2c_5 s + c_3, \\
\eta^0 &= c_5 t, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= c_{13} y + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= -c_0.
\end{aligned}$$

From above, we get one more NS in addition to the minimal set of NSs given by eq(3.103),

$$X_4 = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}. \tag{3.162}$$

Case IV-3-(ii)

$c_6 = 0, \quad c_4 = 0,$
using eq(3.153), we have

$$\begin{aligned}
\xi &= 2c_5 s + c_3, \\
\eta^0 &= c_5 t + c_7, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= c_{13} y + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= -c_0.
\end{aligned}$$

From above, we get two more NSs in addition to the minimal set of NSs given by eq(3.103),

$$\begin{aligned}
\mathbf{X_4} &= 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}, \\
\mathbf{X_5} &= \frac{\partial}{\partial t}.
\end{aligned} \tag{3.163}$$

Case IV-3-(iii)

$c_6 = 0, \quad c_7 = 0,$
using eq(3.153), we have

$$\begin{aligned}
\xi &= c_4 s^2 + 2c_5 s + c_3, \\
\eta^0 &= c_4 ts + c_5 t, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= c_{13} y + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= -c_4 t^2 - c_0.
\end{aligned}$$

From above, we get two more NSs in addition to the minimal set of NSs given by eq(3.103) along with corresponding gauge function,

$$\begin{aligned} \mathbf{X_4} &= 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}, \\ \mathbf{X_5} &= s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t}, \quad V = -t^2. \end{aligned} \tag{3.164}$$

3.2.2 Case V :

$P(t)$ and $Q(t)$ are linearly dependent but $R(t)$ is independent. From eq(3.37), we have

$$R^{'}(t)\eta^0 - 2R(t)\eta^0_t = 0, \tag{3.165}$$

from equation, we get

$$R^{''}(t)\eta^0 - R^{'}(t)\eta^0_t = 0. \tag{3.166}$$

Since $R(t)$ and $R^{'}(t)$ are linearly independent, which implies $R^{'}(t) \neq 0$. So from above equation, we have following possibilities,

1: $\eta^0 = 0,$

2: $R^{''}(t) = 0 = \eta^0_t,$

3: $R^{''}(t)\eta^0 = R^{'}(t)\eta^0_t.$

Case V-1

$\eta^0 = 0,$

using eq(3.148), we have

$$\eta^2_y = \eta^3, \quad \Rightarrow \quad c_{18} = c_{13},$$

using eq(3.150), we have

$c_4 = c_5 = c_6 = c_7 = 0,$

so eq(3.150) implies,

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

Here $c_0,\ c_3,\ c_{13},\ c_{19}$ and ,c_{22}, are the arbitrary constant. Corresponding to these constants, we again get the minimal set of NSs given by eq(3.103).

Case V-2

$R^{''}(t) = 0 = \eta^0_t$,
using above in eq(3.165), we have

$$R^{'}(t)\eta^0 = 0.$$

Since $R^{'}(t) \neq 0$, so $\eta^0 = 0$, in both **(1)** and **(2)** cases , we get $\eta^0 = 0$,
using eq(3.148), we have

$$\eta^2_y = \eta^3, \quad \Rightarrow \quad c_{18} = c_{13},$$

so eq(3.150) implies,

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

From above, we again get the minimal set of NSs given by eq(3.103).

Case V-3

From eq(3.166), we have

$$R^{''}(t)\eta^0 = R^{'}(t)\eta^0_t \quad \Rightarrow R^{'}(t) = c_{27}\eta^0. \tag{3.167}$$

Multiplying eq(3.165) and eq(3.166) with $R^{''}(t)$ and $R^{'}(t)$ respectively, then subtract

$$\left(R^{'^2}(t) - 2R^{''}(t)R(t)\right)\eta^0_t = 0.$$

If $\eta^0_t = 0$ then, eq(3.154) gives $\eta^0 = 0$, it is already done and we have result in eq(3.167). So we solve for

$$R^{'^2}(t) - 2R^{''}(t)R(t) = 0,$$

using eq(3.167) in above equation, we have

$$c_{27}^2\eta^{0^2} - 2\left(c_{27}\eta^0_t\right)\left(c_{27}\int \eta^0 dt\right) = 0.$$

Using eq(3.145) in above equation, we have

$$((c_4t + c_6)s + c_5t + c_7)^2 - 2(c_4s + c_5)((\frac{1}{2}c_4^2t^2 + c_6t)s + \frac{1}{2}c_5^2t^2 + c_7t + \frac{c_{28}}{c_{27}^2}).$$

Suppose $\frac{c_{28}}{c_{27}^2} = c_{29}$ then simplifying above equation, we have

$$c_6^2s^2 + (2c_6c_7 - 2c_4c_{25})s + c_7^2 - 2c_5c_{29} = 0.$$

Comparing coefficients of s^2, s and s^0

$$s^2 \;:\; c_6 = 0, \tag{3.168}$$

$$s \;:\; c_6c_7 = c_4c_{29}, \tag{3.169}$$

$$s^0 \;:\; c_7^2 = 2c_5c_{29}. \tag{3.170}$$

Since $c_6 = 0$, therefore $c_4c_{29} = 0$. Hence, either $c_4 = 0$ or $c_{29} = 0$.
If $c_{29} = 0$, then eq(3.170) implies, $c_7 = 0$.
From above, we have three possibilities

(i) $c_6 = 0, \quad c_4 = 0, \quad c_7 = 0,$

(ii) $c_6 = 0, \quad c_4 = 0, \quad c_7 \neq 0,$

(iii) $c_6 = 0, \quad c_4 \neq 0, \quad c_7 = 0.$

and from eq(3.147), we have

$$P^{'}(t)\eta^0 + 2P(t)(\eta^1_x - \eta^3 - \eta^0_t) = 0. \tag{3.171}$$

Case V-3-(i)

$c_6 = 0, \quad c_4 = 0, \quad c_7 = 0,$
from eq(3.150), we have

$$\begin{aligned} \xi &= 2c_5 s + c_3, \\ \eta^0 &= c_5 t, \\ \eta^1 &= x c_{18} - y c_{13} + c_{22}, \\ \eta^2 &= c_{18} y + c_{19}, \\ \eta^3 &= c_{13}, \\ V &= -c_0. \end{aligned} \tag{3.172}$$

Using above in eq(3.171), we get

$$t c_5 P^{'}(t) + 2P(t)(c_{18} - c_{13} - c_5) = 0. \tag{3.173}$$

From above equation we have two possibilities
$\alpha : \quad tP^{'}(t) \quad and \quad P(t)$ are linearly independent.
$\beta : \quad tP^{'}(t) \quad and \quad P(t)$ are linearly dependent.

Case V-3-(i)-α

$tP^{'}(t)$ *and* $P(t)$ are linearly independent,
using above eq(3.172) and eq(3.173), we get

$$c_5 = 0, \quad and \quad c_{18} = c_{13},$$

using above in eq(3.172), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

From above, we again get the minimal set of NSs given by eq(3.103).

Case V-3-(i)-β

$tP^{'}(t)$ *and* $P(t)$ are linearly dependent.
Suppose $tP^{'}(t) = 2P(t)$,
using above in eq(3.172) and eq(3.173), which gives $c_{18} = c_{13}$.
Substituting above in eq(3.172), we have

$$\begin{aligned}
\xi &= 2c_5 s + c_3, \\
\eta^0 &= c_5 t, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

From above, we get the same NSs as we get in Case IV-3-(i) given by eq(3.162).

Case V-3-(ii)

$c_6 = 0, \quad c_4 = 0,$
from eq(3.150), we have

$$\begin{aligned} \xi &= 2c_5 s + c_3, \\ \eta^0 &= c_5 t + c_7, \\ \eta^1 &= xc_{18} - yc_{13} + c_{22}, \\ \eta^2 &= c_{18} y + c_{19}, \\ \eta^3 &= c_{13}, \\ V &= -c_0. \end{aligned} \tag{3.174}$$

Using above in eq(3.171), we get

$$c_5 tP^{'}(t) + c_7 P^{'}(t) + 2P(t)(c_{18} - c_{13} - c_5) = 0. \tag{3.175}$$

From above equation we have two possibilities
$\alpha :$ $P^{'}(t),\ tP^{'}(t)\ and\ P(t)$ are linearly independent.
$\beta :$ $tP^{'}(t)\ and\ P(t)$ are linearly dependent but $P^{'}(t)\ and\ P(t)$ are linearly independent.
$\gamma :$ $P^{'}(t)\ and\ P(t)$ are linearly dependent but $tP^{'}(t)\ and\ P(t)$ are linearly independent.

Case V-3-(ii)-α

$P^{'}(t),\ tP^{'}(t)\ and\ P(t)$ are linearly independent,
using in eq(3.174) and eq(3.175), we get

$$c_5 = 0, \quad c_{7=0}, \quad c_{18} = c_{13}.$$

using above in eq(3.172), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

From above, we again get the minimal set of NSs given by eq(3.103).

Case V-3-(ii)-β

$tP^{'}(t)$ *and* $P(t)$ are linearly dependent, suppose $tP^{'}(t) = 2P(t)$,
using above in eq(3.175), we have

$$c_7P^{'}(t) + 2P(t)(c_{18} - c_{13} + c_5 - c_5) = 0,$$

$$c_7P^{'}(t) + 2P(t)(c_{18} - c_{13}) = 0.$$

Since $P^{'}(t)$ *and* $P(t)$ are linearly independent, so above equation gives

$$c_{7=0}, \quad c_{18} = c_{13}.$$

Substituting in eq(3.174), we have

$$\begin{aligned}
\xi &= 2c_5s + c_3, \\
\eta^0 &= c_5t, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

From above, we get the same NSs as we get in Case IV-3-(i) given by eq(3.162).

Case V-3-(ii)-γ

$P^{'}(t)$ *and* $P(t)$ are linearly dependent, suppose $P^{'}(t) = 2P(t)$,
using above in eq(3.175), we have

$$c_7 + c_{18} - c_{13} + tc_5 - c_5 = 0,$$

from above we get

$$c_5 = 0, \quad c_{18} = -c_7 + c_{13}.$$

Substitute above in eq(3.174), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= c_7, \\
\eta^1 &= -xc_7 + (x-y)c_{13} + c_{22}, \\
\eta^2 &= -c_7 y + c_{13} y + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= -c_0.
\end{aligned}$$

From above, we get the one more NS in addition to the minimal set of NSs given by eq(3.103),

$$\mathbf{X_4} = \frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{3.176}$$

Case V-3-(iii)

$c_6 = 0, \quad c_7 = 0,$

from eq(3.150), we have

$$\begin{aligned} \xi &= c_4 s^2 + 2c_5 s + c_3, \\ \eta^0 &= c_4 t s + c_5 t, \\ \eta^1 &= x c_{18} - y c_{13} + c_{22}, \\ \eta^2 &= c_{18} y + c_{19}, \\ \eta^3 &= c_{13}, \\ V &= -c_4 t^2 - c_0. \end{aligned} \tag{3.177}$$

Using above in eq(3.171), we get

$$(c_4 s + c_5)P^{'}(t) + 2P(t)(c_{18} - c_{13} - c_4 s - c_5) = 0. \tag{3.178}$$

From above equation we have two possibilities

$\alpha: \quad tP^{'}(t) \quad and \quad P(t)$ are linearly independent.

$\beta: \quad tP^{'}(t) \quad and \quad P(t)$ are linearly dependent.

Case V-3-(iii)-α

$tP^{'}(t)$ *and* $P(t)$ are linearly independent,

using above in eq(3.177) and eq(3.178), we have

$$c_4 s + c_5 = 0, \quad c_{18} - c_{13} - c_4 s - c_5 = 0,$$

from above we get

$$c_4 = 0, \quad c_5 = 0, \quad c_{18} = c_{13}.$$

Substituting above in eq(3.177), we have

$$\begin{aligned}
\xi &= c_3, \\
\eta^0 &= 0, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= yc_{13} + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= c_0.
\end{aligned}$$

From above, we again get the minimal set of NSs given by eq(3.103).

Case V-3-(iii)-β

$tP^{'}(t)$ *and* $P(t)$ are linearly dependent,
suppose $tP^{'}(t) = 2P(t)$,
using above in eq(3.178), we have

$$2P(t)(c_4 s + c_5 + c_{18} - c_{13} - c_4 s - c_5) = 0,$$

$$c_{18} = c_{13}.$$

Substitute above in eq(3.177), we have

$$\begin{aligned}
\xi &= c_4 s^2 + 2c_5 s + c_3, \\
\eta^0 &= c_4 ts + c_5 t, \\
\eta^1 &= (x-y)c_{13} + c_{22}, \\
\eta^2 &= c_{13} y + c_{19}, \\
\eta^3 &= c_{13}, \\
V &= -c_4 t^2 - c_0.
\end{aligned}$$

From above, we get the same set of NSs as we get in Case IV-3-(iii) given by eq(3.164).

Chapter 4

Summary

In this thesis we have investigated Noether symmetries for Bianchi type IV spacetime. To find Noether symmetries we used Lagrangian given by eq(3.1) in the Noether symmetry condition given by eq(2.21) which led to a system of nineteen partial differential equations given by eq(3.2) to eq(3.20). Solving this system of equations for various cases we obtain corresponding symmetries. The summarized results are presented in the following Table 4.1.

Table 4.1: Noether symmetries of Bianchi type IV spacetimes corresponding to different cases.

Sr. No	Case No.	NSs	Ref. Eqn
1	I, II, III, IV-1, IV-2, V-1, V-2, V-3-(i)-α, V-3-(ii)-α, V-3-(iii)-α.	$\mathbf{X_0} = \frac{\partial}{\partial s}$, $\mathbf{X_1} = (x-y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, $\mathbf{X_2} = \frac{\partial}{\partial x}$, $\mathbf{X_3} = \frac{\partial}{\partial y}$. (which is the minimal set of NSs.)	3.103
2	IV-3-(i), V-3-(i)-β. V-3-(ii)-β	$\mathbf{X_4} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}$, and minimal set.	3.162
3	IV-3-(ii).	$\mathbf{X_4} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}$. $\mathbf{X_5} = \frac{\partial}{\partial t}$, and minimal set.	3.163
4	IV-3-(iii). V-3-(iii)-β	$\mathbf{X_4} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}$. $\mathbf{X_5} = s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t}$, $V = -t^2$, and minimal set.	3.164
5	V-3-(ii)-γ.	$\mathbf{X_4} = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, and minimal set.	3.176

Bibliography

[1] Bernoulli, J., Schafheitlin, P. *Die differentialrechnung*. Akademische verlagsgesellschaft mbh, (1924).

[2] Euler, L. *Institvtionvm Calcvli Integrelis Volvmen Privm in Qvo Methodvs Integrandi a primis principiis Vsqve ad Integrationem Aeqvationvm Differentialvm Primi Gradvs Pertractatvr*. Impensis Academiae Imperialis Scientiarum, (1768).

[3] Lie, S. *On integration of a class of linear partial differential equations by means of definite integrals*. CRC handbook of Lie group analysis of differential equations, **2**, 473, (1881).

[4] Lie, S. *Theorie der transformationsgruppen...: abschnitt. I. Die endlichen continuirlichen gruppen der geraden linie und der ebene. II. Die endlichen continuirlichen gruppen des gewöhnlichen raumes. III. Die projectiven gruppen des gewöhnlichen raumes. IV. Untersuchungen über verschiedene arten von gruppen des n-fach ausgedehnten raumes. V. Untersuchungen über die grundlagen der geometrie. VI. Allgemeine betrachtungen über endliche continuirliche gruppen*. BG Teubner, (1893).

[5] Cantwell, B. J. *Introduction to symmetry analysis*. Cambridge University Press, (2002).

[6] Bluman, G., Anco, S. *Symmetry and integration methods for differential equations*. Springer Science Business Media, (2008).

[7] Stephani, H. *Differential equations: their solution using symmetries*. Cambridge University Press, (1989).

[8] Noether, E. *Invariant variation problems*. Transport theory and statistical physics, **1**(3), 186, (1971).

[9] Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E. *Exact solutions of Einstein's field equations*. Cambridge University Press, (2009).

[10] Bokhari, A. H., Kara, A. H. *Noether versus Killing symmetry of conformally flat Friedmann metric*. General Relativity and Gravitation, **39**(12), 2053, (2007).

[11] Tsamparlis, M., Paliathanasis, A. *The geometric nature of Lie and Noether symmetries*. General Relativity and Gravitation, **43**(6), 1861, (2011).

[12] Jamil, M., Mahomed, F. M., Momeni, D. *Noether symmetry approach in f (R)–tachyon model*. Physics Letters B, **702**(5), 315, (2011).

[13] Shabbir, G., Hussain, F., Jamal, S., Ramzan, M. *Existence of conformal vector fields of Bianchi type I space-times in f (R) gravity*. International Journal of Geometric Methods in Modern Physics, **17**(08), 2050113, (2020).

[14] Shabbir, G., Khan, S. *A Note on Killing Vector Fields of Bianchi Type II Spacetimes in Teleparallel Theory of Gravitation*. Modern Physics Letters A, **25**(20), 1733, (2010).

[15] Shabbir, G., Ali, A., Khan, S. *A note on teleparallel Killing vector fields in Bianchi type VIII and IX space—times in teleparallel theory of gravitation*. Chinese Physics B, **20**(7), 070401, (2011).

[16] Shabbir, G., Ali, A. *A note on proper curvature collineations in Bianchi type IV space-times*. arXiv preprint arXiv:1512.07252. (2015).

[17] Ali, A., Khan, J., Hussain, T. *A Note on Proper Homothetic Symmetry in Bianchi Type IV Space-Times.* Advanced Studies in Theoretical Physics, **6**, 187, (2012).

[18] Hickman, M., Yazdan, S. *Noether symmetries of Bianchi type II spacetimes.* General Relativity and Gravitation, **49**(5), 1, (2017).

[19] Akhtar, S. S., Hussain, T. *A study of Noether symmetries in LRS Bianchi type V spacetimes.* Modern Physics Letters A, **35**(06), 2050026, (2020).