Some Topological Properties in the Category of Interval Space

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Dedication

This work is dedicated to my beloved parents who supported me continuously. They are passionate to motivate me for goal achievement. May Allah Almighty bless them always for their endless love, support and encouragement.

Acknowledgement

In the name of **Allah Almighty** who is most beneficent and merciful. I would like to express my special thanks of gratitude to my supervisor, **Dr. Muhammad Qasim** for his mentorship and encouragement towards me.I deeply indebted to him for his continuous support. May **Allah Almighty** bless him always. I am nothing without support of **Allah Almighty** and prayers of my parents. Their countless blessing on me and confidence they had given to me enhance my abilities. I would like to thank my all research mates and class fellows for their support throughout the research work. I would like to thank my sweet brothers **Afraz Jehan Khan** and **Alwaz Mumtaz** they always behind me in hard times. Special thanks to my very close friends **Shoaib Assar** and **Shahid Ali** for their supportive attiude towards me. I would like to thank my research mate **Ch M Afaq Aslam**, **Sana khadim** and **Naveed Ahmad Malik** for their help and support.

Abstract

In this thesis, we considered the category of interval space and show that it is normalized topological category. Further, we examined the local T_0 and local T_1 objects and notion of closedness in this category. Moreover, we extended the point free T_0, $\mathbf{T_0}$ and T_1 interval space and examined their mutual relationship. Finally, we characterized the zero dimensionality and D-connectedness in interval space and compared all the results.

Contents

Introduction

Category theory is a field of pure mathematics introduced by Saunders Mac Lane and Eilenberg [5] in 1945 that gains reputation from algebraic topology to define the different structural ideas from various mathematical fields. It has vast applications in computer science, algebra and geometry. Category theory is applied in engineering field including electrical and biomedical engineering. It becomes more vital tool in programming language and domain theory where it has previously recognized itself as a regular linguistic of dissertation [4]. A category contains the collection of objects that are associated with each other. Objects cannot exist individually because they are linked with mapping known as morphism. Vector space and sets are example for indication of object where linear maps and functions are example of what mapping means. The study of category theory helps us to independently explain the arguments by connecting it with structure preserving characteristics which differentiate between diverse class possess mathematical structure. In 1971, Horst Herrlich [6] developed a subbranch of mathematics known as categorical topology. It is an area that plays the role of bridge between general topology and category theory. Categorical thoughts are applicable to topological ideas uniquely. Concept of separation axiom indicate some restriction on several topological structure.

In 1992, Baran [9] extended the local T_0 and T_1 of general topology to category topology and the purpose was to introduce notion of closedness, several important concepts of general topology such as connectedness [16], compactness [17], hausdorffness and closure operators [18] can be studied in category theory.

Baran [9] also introduced the point free (generic) T_0 and T_1 objects in categorical topology which are also valid for topos theory. The main purpose for these notions is to extend T_2 [19], T_3 [2], $T_{3\frac{1}{2}}$, T_4 [2], normalized regular objects [1] in category theory. Connectedness and zero dimensionality are the basic concepts of general topology. In 1997 and 2006, Stine[12] and Baran[10] extended these notions of topology to categorical topology using initial lift and discrete objects.

Convexity is a fundamental feature in many fields of mathematics. However, it is not the best environment for understanding the basic features of convex sets in some actual mathematical settings, such as vector spaces. To avoid this flaw, abstract convex structures (convex structures are defined similarly to topological structures) are used. Convex structures appear every where in mathematics such as in lattice [20], graphs [21] and in topology [22]. A unique type of closure operator known as algebraic closure operators is responsible for the entire convex structure. In fact, algebraic closure operators behaved as convex space hull operators. Interval operators, being a generalization of intervals, give a natural and common means of expressing or constructing convex structures, aside from algebraic closure operators. Convex structures and interval operators have a close relationship as well.

In the first chapter, some fundamental concepts of topology including continuity, separation axioms, initial topology and its application, connectedness and zero dimensionality are revised. Furthermore, some basic concepts of category theory such as categories, functors, some types of functors, and their properties and some relevant examples are provided. Finally, definition of categorical topology, relevant examples and normalized topological functor are defined.

In the second chapter, interval space, convex space and its basic properties are narrated. Later on, interval preserving mapping and relation between interval space and convex space are examined. Finally, it is shown that category of interval space and interval preserving mapping is a normalized topological category and denoted by **IS** and its initial lifts, discrete and indiscrete objects are explained.

In third chapter, local T_0 and local T_1 interval spaces are explicitly characterized and it is shown that all objects of interval spaces are T_0 at p (resp T_1 at p).

In fourth chapter, generic T_0 and T_1 and $\mathbf{T_0}$ are studied are interval spaces and their mutual relationship are examined and examples are provided.

In fifth chapter, notion of closedness and D-connectedness are characterized in interval spaces. Further, zero dimensionality are examined in the category **IS**. Finally, the relationship among D-connectedness and zero dimensionality are examined.

In last chapter, some concluding remarks about T_0, T_1, $\mathbf{T_0}$, closedness, D-connectedness and zero dimensionality are made and their relationship is studied.

Chapter 1

Basic Definitions

In this chapter we will discuss some fundamental concepts which will be used in later chapters.

1.1 Topological Spaces

The topological space introduced by Hausdorff in 1914 under the name of "Hausdorff Space" and the current, definition of topology is given by Kuratowski in 1922. The description of definitions are given in [7].

Definition 1.1.1. *Let $Y \neq \emptyset$ and τ is any subset of $P(Y)$. Then τ is said to be a topology if the following axioms hold:*

1. *Empty set $\emptyset$ and Y belongs to τ;*

2. *Arbitrary union of elements of τ belongs to τ;*

3. *Finite intersection of elements of τ also belongs to τ.*

Example 1.1.2. *Suppose $Y = \mathbb{R}$; where (set of real numbers), form the usual topology on $\mathbb{R}$ represented by, $\tau = \{\bigcup_{i \in I} G_i \mid G_i = (x_i, y_i); \forall\ x_i, y_i \in \mathbb{R}\}$.*

Example 1.1.3. *Suppose Y is nonempty set and $P(Y)$ establish a topology on Y it is said to be a* ***discrete topology****. The set having only $\emptyset$ and Y is known as trivial topology and also called* ***indiscrete topology****.*

Definition 1.1.4. *Suppose (Y,τ) represents a topological space.Then every set G belongs to τ is known as an **open set**.*

Definition 1.1.5. *Let (Y, τ) be a topological space, a set G belongs to τ is said to be closed if G^c belongs to τ*

Example 1.1.6. *Let $Y = \{1, 2, 3\}$ and $\tau = \{\emptyset, Y, \{3\}\}$ represents the topology on Y. Then the complement Y-$\{3\} = \{1, 2\}$ denotes the closed set in (Y, τ).*

Definition 1.1.7. *Let Y be a nonempty set and set $\mathcal{B}$ be the sub collection for elements of power set. Then $\mathcal{B}$ is known as basis for Y if it holds the following axioms:*

1. $\forall\ r \in Y,\ \exists\ C \in \mathcal{B}$ *such that* $r \in C$.
2. $\forall\ C_1, C_2 \in \mathcal{B}$, *if* $r \in C_1 \cap C_2$, *then there is a basis element* C_3 *containing* r *such that* $C_3 \subset C_1 \cap C_2$.

Example 1.1.8. *Let $Y \neq \emptyset$, and $\tau = P(Y)$. Then its basis are $\mathcal{B} = \{\ \{\ y\ \}\ |\ y \in Y\}$.*

Definition 1.1.9. *Let (Y, τ) be a topological space, the closure of E which is subset of Y is defined as*

$$\bar{E} = \bigcap_{(closed\ \ B)\subseteq E} B.$$

Example 1.1.10. *Let $Y = \{1, 2, 9, 8\}$ with topology $\tau = \{\emptyset.\{1\}, \{1, 2, 9\}, \{2, 9\}, Y\}$ and $E = \{2, 8\}$ be a subset of Y.*
Open sets: $\emptyset,\ Y, \{2, 9\}, \{1, 2, 9\}, \{1\}$.
closed sets: $\emptyset,\ \{2, 9, 8\}, \{1, 8\}, \{8\}, Y$.
Closed sets containing E : Y, $\{2, 9, 8\}$.
$\bar{E} = Y \cap \{2, 9, 8\} = \{2, 9, 8\}$.

Definition 1.1.11. *Let (L, σ) and (W, τ) be two topological spaces and a mapping $g : (L, \sigma) \longrightarrow (W, \tau)$ is continuous iff $\forall\ \ V \in \tau$, $g^{-1}(V) \in \sigma$.*

Theorem 1.1.12. *Every function $f : (X, \tau) \rightarrow (Y, \sigma)$ with $\tau = P(X)$ is always continuous.*

Example 1.1.13. *$f : (\mathbb{R}, P(\mathbb{R}) \to (\mathbb{R}, \tau^l)$ represented as $f(x) = x^2$, f is continuous but $f : (\mathbb{R}, \tau^l) \to (\mathbb{R}, P(\mathbb{R}))$ is not continuous.*

Definition 1.1.14. *Let (Y, σ) be topological space. If $B \subseteq Y$, then the subspace topology on B is defined by*

$$\sigma_B = \{B \cap U \mid U \in \sigma\}.$$

Example 1.1.15. *Let $Y = \mathbb{R}$ endowed with standard topology and let $\mathbb{N} \subset \mathbb{R}$. Then $\tau_{\mathbb{N}} = \{U \cap \mathbb{N} : U \in \tau_{st}\} = P(\mathbb{N}.)$*

Example 1.1.16. *Suppose $A = \{1, 2, 3\} \subset \mathbb{R}$, then $\tau_A = \{U \cap \{1, 2, 3\} \mid U \in \tau_{st}\}$. We get $\tau_A = P(A)$*

Definition 1.1.17. *Suppose $(Y_j, \tau_j)_{j \in J}$ is the collection of topological spaces and $Y \neq \emptyset$, define $g_j : (Y, \tau_*) \to (Y_j, \tau_j)$, then*

$$\tau_* = \bigcup_{j \in J} \bigcap_{l=1}^{n} \{g_{jl}^{-1}\ (V_{jl})\ ;\ V_{jl}\ \in\ \tau_j\}.$$

is called initial Topology.

Remark 1.1.18. *The subspace topology is the initial topology on the subspace with respect to the inclusion map.*

1.2 Separation Axioms

Definition 1.2.1. *Let (Y, τ) be Topological Space and $p \in Y$ if $\forall\ r \in Y$ with $r \neq p$ $\exists$ $G_1 \in \tau$ with $r \in G_1$, $p \notin G_1$ or $\exists$ $G_2 \in \tau$ with $p \in G_2$, $r \notin G_2$. Then (Y, τ) is called local T_0 or T_0 at p.*

Example 1.2.2. *If $Y = \{l, m, n\}$ and $\tau = \{\emptyset, \{l\}, \{m, n\}, Y\}$ is T_0 at l but not T_0 at m.*

Definition 1.2.3. *A topological space (Y, τ) is called T_0 if $\forall$ s,$t \in Y$ with $s \neq t$, $\exists$ G_1 $\in \tau$ such that $s \in G_1$, $t \notin G_1$ or $\exists$ $G_2 \in \tau$ such that $t \in G_2$, $s \notin G_2$.*

Example 1.2.4. *If* $Y = \{l, m\}$ *with topology* $\tau = \{\emptyset, \{l\}, \{l, m\}\}$ *on* Y *is a* T_0 *Space.*

Theorem 1.2.5. *([8])* (Y, τ) *is* T_0 *iff* (Y, τ) *is* T_0 *at* p, $\forall\ p \in Y$.

Theorem 1.2.6. *A topological space* (Y, τ) *is said to be* $T_0 \iff \forall\ s,t \in Y$ *with* $s \neq t$, $\overline{\{s\}} \neq \overline{\{t\}}$, *where* $\{s\}$ *and* $\{t\}$ *stand for closure of* $\{s\}$ *and* $\{t\}$ *respectively.*

Theorem 1.2.7. *Every subspace of* T_0 *space is again* T_0.

Definition 1.2.8. *Let* (Y, τ) *be a topological space and* $p \in Y$ *if for all* $r \in Y$ *with* $r \neq p$, $\exists\ G \in \tau$ *with* $r \in G$, $p \notin G$ *and* $\exists\ H \in \tau$ *with* $p \in H$, $r \notin H$*.Then* (Y, τ) *is called local* T_1 *or* T_1 *at* p.

Example 1.2.9. *If* $Y = \{6, 7, 8\}$ *and* $\tau = \{\emptyset, \{6\}, \{7, 8\}, Y\}$ *is* T_1 *at 6 but not* T_1 *at 7.*

Definition 1.2.10. *A topological space* Y *is said to be a* T_1 *for each* $r,s \in Y$ *with* $r \neq s$, *then their exists two open sets* G_1 *and* G_2 *such that* $r \notin G_1$ *and* $s \in G_1$, *and* $s \notin G_2$ *and* $r \in G_2$.

Example 1.2.11. $Y = \{6, 7, 8\}$ *with topology* $\tau = \{\emptyset, Y, \{7\}, \{7, 6\}, \{6\}\}$ *is not* T_1 *However,* $\tau = P(Y)$ *is* T_1.

Theorem 1.2.12. *([8])* (Y, τ) *is* T_1 *iff* (Y, τ) *is* T_1 *at* p, $\forall\ p \in Y$.

Theorem 1.2.13. *Every subspace of* T_1 *space is again* T_1.

Theorem 1.2.14. *A space* Y *is said to be* T_1 *iff every singleton is closed, i.e;* $\overline{\{y\}} = \{y\}$.

Lemma 1.2.15. *Every* T_1 *space is a* T_0 *space but converse may or may not be true.*

1.3 Connectedness and Zero Dimensonality

Definition 1.3.1. *([7, 16]) A topological space* (Y, τ) *is connected iff the following equivalent conditions hold:*

(i) The only clopen subsets of Y *are* $\emptyset$ *and* Y.

(ii) The continuous mapping $f : (Y, \tau) \to \{\{0, 1\}, \tau_{dis}\}$ *is constant.*

(iii) If Y *cannot be expressed as union of two disjoint open sets.*

Example 1.3.2. *Every indiscrete topological space is connected.*

Example 1.3.3. *Let* $Y = \{6, 7, 8, 9, 10\}$ *and* $\tau = \{\emptyset, Y, \{6\}, \{10, 7\}, \{6, 7, 10\}, \{7, 8, 9, 10\}\}$ Y *is disconnected. Since* E=$\{6\}$ *and* C=$\{7, 8, 9, 10\}$ *are clopen sets for* τ.

Definition 1.3.4. *A topological space* (Y, τ) *is called zero dimensional if* Y *contains the basis of clopen sets.*

Example 1.3.5. *Every discrete topological space is zero dimensional.*

Theorem 1.3.6. *([12]) Let* (Y, τ) *be topological space. Then,* (Y, τ) *is zero-dimensional iff* (Y, τ) *is an initial topology induced by* $f_j : (Y, \tau) \to (Y_j, \tau_{j_{dis}})$*, where* $\tau_{j_{dis}}$ *is the discrete topological space.*

Theorem 1.3.7. *([7, 16]) Every disconnected (not connected) topological space is zero dimensional.*

1.4 Category Theory

Definition 1.4.1. *[3] A* ***Category*** *defined on quadruple* $\mathcal{G} = (\mathcal{T}, \boldsymbol{Hom}, id, \circ)$. *Each member of the class* $\mathcal{T}$ *are said to be* $\mathcal{G}$***-objects*** *such that following condition hold*

1. *Each pair (**L**,**S**) such that* L,$S \in \mathcal{T}$, *then there exists* ***Hom****(L,S) having* $\mathcal{G}$*-morphisms from* ***L*** *to* ***S***.

2. *Every Y from $\mathcal{G}$- objects have morphism $id_Y : Y \to Y$ is **$\mathcal{G}$-identity** on Y.*

3. *Composition law corresponding to each $\mathcal{G}$-morphisms $g : E \longrightarrow F$ and each $\mathcal{G}$-morphisms $k : F \longrightarrow H$, a $\mathcal{G}$-morphisms $g \circ k : E \longrightarrow H$ is called composition of g and k.*

   a) *Every morphisms $g : E \to F$, $k : F \to M$ and $f : M \to L$, the following equation satisfies*
$$f \circ (g \circ k) = (f \circ g) \circ k.$$

   b) *For each $\mathcal{G}$-identity proceed as identity in correspondence with composition i.e., for every $\mathcal{G}$-morphisms $h : G \longrightarrow K$ we have*
$$h \circ id_X = h \text{ and } id_Y \circ h = h.$$

Example 1.4.2. *$\mathcal{G}$ = **Set**: indicates the objects **Hom**(C,D) represent the set element which are map from C to D, $\circ$ is the function of composition and id_Y form the identity function on $\mathcal{G}$ = **Set**.*

Example 1.4.3. *$\mathcal{G}$ = **Top**, where topological spaces represents objects and continuous maps are morphism, respectively. $id_{(Y,\tau)} : (Y, \tau) \to (Y, \tau)$ form the identity morphism on $\mathcal{G}$ = **Top**.*

Example 1.4.4. *$\mathcal{G}$ = **Vec**, where vector spaces indicates the objects and linear transformation are morphisms between objects having vectors.*

Example 1.4.5. *$\mathcal{G}$ = **Grp**, where group represents objects and group homorphism are morphisms between groups.*

Definition 1.4.6. *([3])*

(i) *Subcategory $\mathcal{K}$ of the category $\mathcal{G}$ is defined if following holds :*

   (a) *$Obj(\mathcal{K}) \subseteq Obj(\mathcal{G})$,*

   (b) *for every C_1, $C_2 \in Obj(\mathcal{K})$, $\boldsymbol{Hom}_{\mathcal{K}}(C_1, C_2) \subseteq \boldsymbol{Hom}_{\mathcal{G}}(C_1, C_2)$,* *(c) for every* $C_1 \in \mathcal{K}$*-object, a* $\mathcal{G}$*-identity on* C_1 *is the* $\mathcal{K}$*-identity on* C_1*,*

(d) Law of composition in $\mathcal{K}$ *is restriction of composition law in* $\mathcal{G}$ *to the morphisms of* $\mathcal{K}$*.*

(ii) A full subcategory of $\mathcal{K}$ *of* $\mathcal{G}$ *if condition (a)-(d) hold for every* $C_1, C_2 \in Obj(\mathcal{K})$, $\boldsymbol{Hom}_{\mathcal{K}}(C_1, C_2) = \boldsymbol{Hom}_{\mathcal{G}}(C_1, C_2)$.

Definition 1.4.7. *([3]) Let* $\mathbf{K}$ *and* $\mathcal{F}$ *be two categories.* $\mathcal{U} : \mathbf{K} \longrightarrow \mathcal{F}$ *is said to be functor if*

1. $\forall\ Z \in Obj(\mathbf{K}) \ \Rightarrow \mathcal{U}(Z) \in Obj(\mathcal{F})$*,*

2. *For every morphism* $g : L \longrightarrow M$ *in* $\mathbf{K}$ *lead towards morphism* $\mathcal{U}(g) : \mathcal{U}(L) \longrightarrow \mathcal{U}(M)$ *in* $\mathcal{F}$*,*

3. $\mathcal{U}$ *have identity morphism; i.e.,*

$$\mathcal{U}(1_L) = 1_{\mathcal{U}(L)},$$

4. *Composition preserves under* $\mathcal{U}$ *for each morphism in* $\mathbf{K}$*. If* $L \xrightarrow{f} M \xrightarrow{g} N$ *in* $\mathbf{K}$ *then,*

$$\mathcal{U}(g) \circ \mathcal{U}(f) = \mathcal{U}(g \circ f).$$

Example 1.4.8. *A mapping* $\mathcal{U} : \boldsymbol{Top} \longrightarrow \boldsymbol{Set}$ *defined as* $\mathcal{U}((X, \tau)) = X$ *for some set* X *and* $\mathcal{U}(g) = g$ *for a continuous map* $g : (Z, \tau) \longrightarrow (L, \Psi)$*. Then* $\mathcal{U}$ *is a functor.*

Definition 1.4.9. *([3]) A functor* $\mathcal{U} : \mathbf{K} \longrightarrow \mathcal{F}$ *from a category* $\mathcal{G}$ *to* $\mathcal{H}$ *is called* ***full functor****, if for each pair* $X, Y \in Obj(\mathbf{K})$ *functor* $\mathcal{U}$ *there exists a morphism such that*

$$f_{X,Y} : \boldsymbol{Hom}_{\mathbf{K}}(X, Y) \longrightarrow \boldsymbol{Hom}_{\mathcal{H}}(\mathcal{U}(X), \mathcal{U}(Y)) \text{ is surjective.}$$

Example 1.4.10. *A functor defined by* $\mathcal{U} : \boldsymbol{Set} \rightarrow \boldsymbol{Top}$ *is a full functor.*

Definition 1.4.11. *([3]) A functor* $\mathcal{U} : \mathbf{K} \longrightarrow \mathcal{F}$ *from a category* $\mathbf{K}$ *to* $\mathcal{F}$ *is called* ***faithful functor****, if for each pair* $X, Y \in Obj(\mathbf{K})$ *functor* $\mathcal{F}$ *there exists a morphism such that*

$$f_{X,Y} : \boldsymbol{Hom}_{\mathbf{K}}(X, Y) \longrightarrow \boldsymbol{Hom}_{\mathcal{F}}(\mathcal{U}(X), \mathcal{U}(Y)) \text{ is injective.}$$

Definition 1.4.12. *([3]) A functor $\mathcal{U} : \mathbf{K} \longrightarrow \mathcal{F}$ from a category $\mathbf{K}$ to $\mathcal{F}$ is **amnestic** if any $\mathbf{K}$-morphisms f is an identity whenever $\mathcal{U}(f)$ is an identity $\mathcal{F}$-morphism, i.e., $\mathcal{U}$ reflects identity morphism.*

Definition 1.4.13. *([3]) A functor $\mathcal{U} : \mathbf{K} \longrightarrow \mathcal{F}$ is known as **concrete functor** iff $\mathcal{U}$ is faithful and amnestic.*

Example 1.4.14. *A functor $\mathcal{U}$: **Grp** $\longrightarrow$ **Set** known as amnestic and faithful in other words it is a concrete functor.*

1.5 Categorical Topology

Definition 1.5.1. *([3]) Let $\mathcal{U} : \mathbf{K} \longrightarrow \mathcal{F}$ be a functor, the set H is defined by*

$$H = \{Z \in obj(\mathbf{K}) \; ; \; \mathcal{U}(Z) = L\}.$$

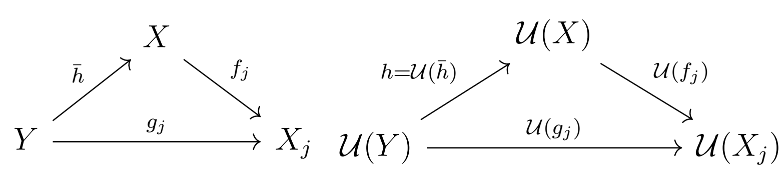
*is known as **fiber** of $L \in \mathcal{F}$.*

Definition 1.5.2. *([3]) A functor $\mathcal{U} : \mathbf{K} \longrightarrow \mathcal{F}$ is said to be **topological functor** if $\mathcal{U}$ the following holds,*

(i) $\mathcal{U}$ have small fibers.

(ii) $\mathcal{U}$ is concrete functor.

(iii) A source $\boldsymbol{Z} = f_j : X \to X_j$, $(\forall\ j \in J)$ is called $\mathcal{U}$- initial provided that for each source $\tau = g_j : Y \to X_j$ and $(\forall\ j \in J)$ in $\mathbf{K}$ with the same co-domain as $\boldsymbol{Z}$ and each $\mathcal{F}$-morphism $h : \mathcal{U}(Y) \to \mathcal{U}(X)$ with $\mathcal{U}(\tau) = \mathcal{U}(H \circ h)$, $\exists$ a unique $\mathbf{K}$-morphism $\bar{h} : Y \to X$ with $\tau = \boldsymbol{Z} \circ \bar{h}$ and h= $\mathcal{U}(\bar{h})$.



Example 1.5.3. *A functor* $\mathcal{U}$*:* $\boldsymbol{Top} \to \boldsymbol{Set}$ *is a topological functor and initial lift corresponds to initial topology.*

Example 1.5.4. *The functor* $\mathcal{U}$*:* $\boldsymbol{Grp} \to \boldsymbol{Set}$ *has no initial lift exist in groups. In other words, the subset of a group may or may not hold the axiom of subgroup.*

Definition 1.5.5. *([3]) A topological functor* $\mathcal{U} : \mathbf{K} \to \mathcal{F}$ *is said to be normalized topological functor if the unique structures exists on* $\mathbf{K}$ *whenever,* $Y = \{a\}$ *or* $Y = \emptyset$*, where* $Y \in obj(\mathbf{K})$

Example 1.5.6. *The functor* $\mathcal{U} : \boldsymbol{Top} \to \boldsymbol{Set}$ *is a normalized topological functor since a unique structure of topology exists on singleton set.*

Definition 1.5.7. *([3]) Let* $\boldsymbol{K}$ *and* $\mathcal{F}$ *be two categories. A left adjoint* $D : \boldsymbol{K} \longrightarrow \mathcal{F}$ *of the topological functor* $\mathcal{U} : \boldsymbol{K} \longrightarrow \mathcal{F}$ *is called discrete functor.*

Definition 1.5.8. *([3]) Let* $\boldsymbol{K}$ *and* $\mathcal{F}$ *be two categories. A right adjoint* $D : \boldsymbol{K} \longrightarrow \mathcal{F}$ *of the topological functor* $\mathcal{U} : \boldsymbol{K} \longrightarrow \mathcal{F}$ *is called indiscrete functor.*

Chapter 2

Interval Space and Convex Spaces

2.1 Interval Space

Definition 2.1.1. *([13]) Let $Y \neq \emptyset$ and 2^Y be a power set of Y. $\{B_k\}_{k\in K}$ represents the directed subset of 2^Y such that $\forall\ L, E \in \{B_k\}_{k\in K}$, there exists $F \in \{B_k\}_{k\in K}$ such that $L \subseteq F$ and $E \subseteq F$.*

Example 2.1.2. *Let $D = \{1, 2\}$, $E = \{3\}$ where $F = \{1, 2, 3\}$ such that $D \subseteq F$ and $E \subseteq F$ hold. Hence D and E are directed subset of F respectively.*

Definition 2.1.3. *([13]) Let $g : K \to L$ be a mapping where K and L are two nonempty sets then $g^{\to}(K) = \{g(e) \mid e \in K\}$ and $g^{\leftarrow}(L) = \{e \mid g(e) \in L\}$ represents forward and backward mapping, respectively.*

Definition 2.1.4. *([13]) A mapping $J : Y \times Y \to 2^Y$ is called interval operator if it satisfies the following:*

1. $x, y \in J(x, y)$
2. $J(x, y) = J(y, x)$

Then, (Y, J) represents the interval space for interval operator J on Y.

Example 2.1.5. *Suppose $\mathbb{R}$ represents set of real numbers. A map $J : \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is defined by*

$$\forall s,t \in \mathbb{R},\ J_{\mathbb{R}}(s,t) = [min\{s,t\}, max\{s,t\}]$$

where $J_{\mathbb{R}}$ indicates the interval operator on $\mathbb{R}$.

Example 2.1.6. *Let d be a metric on Y. A map $J_d : Y \times Y \to 2^Y$ is defined by*

$$k,l \in Y,\ J_d(k,l) = \{m \in Y \mid d(k,l) = d(k,m) + d(m,l)\},$$

where J_d indicates the interval operator on Y.

Example 2.1.7. *If $Y = \{s,t\}$ and $2^Y = \{\emptyset\ , \{s\}, \{t\}, \{s,t\}\}$, $\forall s,t \in Y$ then a map $J : Y \times Y \to 2^Y$ gives $J(s,s) = \{s\}$ or $\{s,t\}$ and $J(s,t) = \{s,t\}$ Similarly, $J(t,s) = J(s,t)$ and $J(t,t) = \{t\}$ or $\{s,t\}$. Clearly, (Y,J) is an interval space.*

Definition 2.1.8. *([13]) Let $\mathfrak{C}$ is a convex structure on Y and $\mathfrak{C} \subset P(Y)$ which hold:*

1. *$\emptyset, Y \in \mathfrak{C}$;*
2. *$\{B_k\}_{k\in K} \subseteq \mathfrak{C}$ implies $\bigcap\limits_{k\in K} B_k \in \mathfrak{C}$, where $\{B_k\}_{k\in K}$ is subset of $P(Y)$;*
3. *$\{B_k\}_{k\in K} \subseteq \mathfrak{C}$ implies $\bigcup\limits_{k\in K} B_k \in \mathfrak{C}$, where$\{B_k\}_{k\in K}$ is directed subset of $P(Y)$.*

A pair $(X, \mathfrak{C})$ represents the convex space for the convex structure $\mathfrak{C}$ on X.

Example 2.1.9. *Let $Y = \{3,4\}$ and $P(Y) = \{\emptyset, \{3\}, \{4\}, \{3,4\}\}$. Then by forming convex structure $\mathfrak{C}$ on Y satisfy the above condition. Hence, $\mathfrak{C}$ is convex space on Y.*

Definition 2.1.10. *A mapping $g : (X, \mathfrak{C}_X) \to (Y, \mathfrak{C}_Y)$ is called convexity preserving map provided that $B \in \mathfrak{C}_Y$ implies $g^{\leftarrow}(B) \in \mathfrak{C}_X$.*

Example 2.1.11. *Let $g : (E, \mathfrak{C}_E) \to (E, \mathfrak{C}_E)$ be a map defined as $g(s) = s$, $\forall s \in E$ is clearly an convexity preserving map.*

Remark 2.1.12. CS *denotes the category of convex space $(X, \mathfrak{C})$ and convexity preserving map.*

Definition 2.1.13. *([13]) A closure operator on Y is a map $C : 2^Y \to 2^Y$ that satisfies the followings:*

1. $C(\emptyset) = \emptyset$;

2. $E \subseteq C(E)$;

3. $E \subseteq F \Rightarrow C(E) \subseteq C(F)$;

4. $C(C(F)) = C(F)$

5. $C(E) = \cup\{C(F) \mid F$ *is finite subset of* $E\}$

Then closure space on Y *is represented by pair* (Y, C) *and it is said to be algebric closure operator if 5 holds.*

Definition 2.1.14. *([13]) A mapping* $g : (E, C_E) \rightarrow (F, C_F)$ *between two closure spaces is called algebric closure preserving map such that* $g^{\rightarrow}(C_E(D) \subseteq C_F(g^{\rightarrow}(D), \forall D \in 2^E$.

Remark 2.1.15. *(i)* ***ACLS*** *denotes the category of algebric closure operator and algebric closure preserving mapping.*

(ii) Note that ***ACLS*** $\cong$ ***CS***. *([13])*

Definition 2.1.16. *([13]) A hull operator on* Y *is a mapping* $co : 2^Y \rightarrow 2^Y$ *which satisfies:*

1. $co(\emptyset) = \emptyset$, $co(Y) = Y$;

2. $E \subseteq co(E)$;

3. $E \subset F \Rightarrow co(E) \subset co(F)$;

4. $co(co(E)) = co(E)$;

5. $co \bigcup\limits_{j \in J}^{dir} E_j = \bigcup\limits_{j \in J} co(E_j)$.

For a hull operator co *on* Y*, the pair* (Y, co) *is called a hull space. Actually, a hull operator on* Y *is a closure operator on* Y *which satisfies 5.*

Definition 2.1.17. *([13]) A map $g : (E, co_E) \to (F, co_F)$ is said to be hull preserving map $g^{\to}(co_E(D)) \subseteq co_F(g^{\to}(D))$, $\forall D \in 2^Y$.*

Theorem 2.1.18. *([13]) Let $(Y, \mathfrak{C})$ be a convex space and define $co^{\mathfrak{c}} : 2^Y \to 2^Y$*

$\forall E \in 2^Y$, $co^{\mathfrak{c}}(E) = \bigcap_{E \subseteq F \in C} F$. *Then $co^{\mathfrak{c}}$ is a hull operator on Y.*

Proposition 2.1.19. *[13] Let (E, co) be a hull space and define $\mathfrak{C}^{co} = \{D \in 2^E \mid D = co(D)\}$. Then $\mathfrak{C}^{co}$ is convex structure on E.*

Remark 2.1.20. *(i) **HS** denotes the category of hull space and hull preserving maps.*

(ii) ***HS*** $\cong$ ***CS*** [13].

We discuss the relation between interval space and convex space. For this purpose we have full subcategory of convex spaces known as arity 2 convex spaces. Furthermore, arity 2 convex spaces have relation with interval space.

Definition 2.1.21. *([13]) A convex space $(Y, \mathfrak{C})$ is known as arity 2 convex space if $\forall B \in 2^Y$, where $s, t \in B$ gives $co^{\mathfrak{c}}\{s, t\} \subseteq B$ implies $B \in \mathfrak{C}$*

Proposition 2.1.22. *([13]) Suppose $(Y, \mathfrak{C})$ is convex space and $J^{\mathfrak{C}} : Y \times Y \to 2^Y$ be a map defined by*

$$\forall s, t \in Y, J^{\mathfrak{C}}(s, t) = co^{c}(s, t) = \bigcap_{s, t \in B \in C} B$$

Then $J^{\mathfrak{C}}$ represents the interval operator on Y.

Proposition 2.1.23. *([13]) Suppose (Y, J) is interval space and define $\mathfrak{C}^{\boldsymbol{J}}$ as*

$$\mathfrak{C}^{\boldsymbol{J}} = \{B \in 2^Y \mid \forall s, t \in B, J(s, t) \subseteq B\}.$$

Then $(Y, \mathfrak{C}^{\boldsymbol{J}})$ is an aritry 2 convex space.

2.2 Interval Preserving Mapping

Definition 2.2.1. *([13]) A map $g : (E, J_E) \to (F, J_F)$ hold interval preserving map*

$$\forall s, t \in E,\ g^{\to}(J_E(s,t)) \subseteq J_F((g(s), g(t))$$

Example 2.2.2. *Let $E = \{s, t\}$ and $F = \{u, v\}$ be two interval spaces and a map $g : (E, J_E) \to (F, J_F)$ is defined by $g(s) = u$ and $g(t) = v$. Now for $s, t \in E$ we have $J_E(s,t) = \{s, t\}$ and $J_F(g(s), g(t)) = J_F(u, v) = \{u, v\}$. Therefore $g^{\to}(J_E(s,t)) = g^{\to}\{s, t\} = \{u, v\}$. Above map g clearly satisfies*

$$\forall s, t \in E,\ g^{\to}(J_E(s,t)) \subseteq J_F((g(s), g(t))$$

Hence, f is interval preserving map.

Proposition 2.2.3. *Every identity map is interval preserving map.*

Proof. Suppose (E, J) and (F, J) are two interval spaces and a map $g : (E, J) \to (F, J)$ defined by $g(s) = s\ \forall s \in E$. Then $J(s,s) = g^{\leftarrow}(J(g(s), g(s))) = g^{\leftarrow}(J(s,s)) = g^{\leftarrow}\{s\} = \{s\}$. Let g is an interval preserving map then following hold

$$\forall s \in E,\ g^{\to}(J_E(s,s)) \subseteq J_F((g(s), g(s))$$

Clearly g is an interval preserving map. □

Proposition 2.2.4. *Every constant map is an interval preserving map.*

Proof. Suppose (E, J) and (F, J) are two interval spaces and a map $g : (E, J) \to (F, J)$ defined by $g(s) = c,\ \forall s \in E$. Then $J(s,s) = g^{\leftarrow}(J(g(s), g(s))) = g^{\leftarrow}(J(c,c)) = g^{\leftarrow}\{c\} = \{c\}$. Let g is an interval preserving map then following hold

$$\forall s \in E,\ g^{\to}(J_E(s,s)) \subseteq J_F((g(s), g(s))$$

Clearly g is an interval preserving map. □

Proposition 2.2.5. *([13]) Let (F, J_F), (G, J_G) and (H, J_H) be interval spaces. If $l : F \to G$ and $k : G \to H$ are interval preserving, then $k \circ l : (F, J_F) \to (H, J_H)$ is also interval preserving map.*

Proposition 2.2.6. *([13]) A map $g : (E, \mathfrak{C}_E) \to (F, \mathfrak{C}_F)$ is a convexity preserving map, then $g : (E, J_E^{\mathfrak{C}}) \to (F, J_F^{\mathfrak{C}})$ is an interval preserving map.*

Proposition 2.2.7. *([13]) If $g : (X, J_X) \to (Y, J_Y)$ have interval preserving map, then $g : (E, \mathfrak{C}_E^J) \to (F, \mathfrak{C}_F^J)$ have convexity preserving map.*

Remark 2.2.8. *([13])*

1. ***IS** denotes the category of interval space as objects and interval preserving map as morphism.*
2. ***IS** is full subcategory of **CS**.*
3. *The category **CS(2)** can be embedded in the category of **IS** as reflexive subcategory.*

2.3 IS as Normalized Topological Category

Theorem 2.3.1. *([13]) The functor U : **IS** $\to$ **Set** is given by for each (Y, J) interval space, $U(Y, J) = Y$, a set and for each $g : (Y, J) \to (Y', J')$ interval preserving mapping $U(g) = g : Y \to Y'$ function is a topological functor.*

Proposition 2.3.2. *([13]) Let (Y_i, J_i) be the collection of interval space and Y be a nonempty set and $(f_i : Y \to (Y_i, J_i))_{i \in I}$ be a source*

$$\forall x, y \in Y,\ J(x, y) = \bigcap_{i \in I} f_i^{\leftarrow}(J_i(f_i(x), f_i(y))$$

Definition 2.3.3. *([13]) Let Y is a nonempty set and (Y, J) be an interval space*

1. *The discrete interval space on Y such that $J : Y \times Y \to 2^Y$ defined by $J'(x, y) = \{x, y\}, \forall x, y \in Y$ or $x \in Y$, $J'(x, x) = \{x\}$. Then J' is said to be discrete interval operator on Y.*
2. *The indiscrete interval structure on Y is given by $J_{\prime}(x, y) = Y$. Here $J_{\prime}$ is indiscrete interval operator on Y.*

Proposition 2.3.4. *([13]) The functor U : $\boldsymbol{IS} \to \boldsymbol{Set}$ is a normalized topological functor.*

Proof. Since for $Y = \{s\}$, it carries only one structure that $J(s, s) = \{s\}$. Thus U : $\mathbf{IS} \to \mathbf{Set}$ is a normalized functor. $\square$

Chapter 3

Local T_0 and Local T_1 Interval Space

3.1 Local T_0 Interval Space

Definition 3.1.1. *([8]) Let* (Z, τ) *be topological space and* $p \in Z$ *if for all* $s \in Z$ *with* $s \neq p$ $\exists$ $G \in \tau$ *with* $s \in G$, $p \notin G$ *or* $\exists$ $H \in \tau$ *with* $p \in H$, $s \notin H$*. Then* (Z, τ) *is called Local* T_0 *or* T_0 *at* p*(in classical sense).*

Example 3.1.2. *If* $Z = \{l, m, n\}$ *and* $\tau = \{\emptyset, \{l\}, \{m, n\}, Z\}$ *is* T_0 *at* l *but not* T_0 *at* m*.*

Theorem 3.1.3. *([8])* (Z, τ) *is* T_0 *iff* (Z, τ) *is* T_0 *at* p*,* $\forall$ $p \in Z$*.*

In 1991, Baran ([9]) introduced T_0 at p in classical sense of topology in form of Categorical Topology in the terms of initial and discrete structure.

Definition 3.1.4. *([9])Suppose* Z *is a nonempty set and* p *is a point in* Z*.Let* $Z \bigvee_p Z$ *be a wedge product of* Z *with itself and* $Z^2 = Z \times Z$ *represents the cartesian product of* Z*. A point* z *in* $Z \bigvee_p Z$ *is* z_1*(resp.* z_2*) if it is in first component (resp. 2nd component).*

Definition 3.1.5. *([9]) A map* $A_p : Z \bigvee_p Z \to Z^2$ *is said to be principal axis map defined by*

$$A_p(z_i) = \begin{cases} (z, p) & , \ i = 1 \\ (p, z) & , \ i = 2 \end{cases}$$

Definition 3.1.6. *([9]) A map $S_p : Z \bigvee_p Z \to Z^2$ is said to be skewed principal axis map defined by*

$$S_p(z_i) = \begin{cases} (z,z) & , \ i=1 \\ (p,z) & , \ i=2 \end{cases}$$

Definition 3.1.7. *([9]) A map $\nabla_p : Z \bigvee_p Z \to Z$ is said to be fold map $\nabla_p(z_i) = z$ for $i = 1, 2$.*

Definition 3.1.8. *([9]) Suppose (Z, τ) is a top space. (Z, τ) is called local T_0 iff the initial structure on $Z \bigvee_p Z$ induced by $A_p : Z \vee_p Z \to Z^2$ and $\nabla_p : Z \vee_p Z \to Z$ is discrete top space.*

Theorem 3.1.9. *([8]) Suppose (Z, τ) is a top space. (Z, τ) is local T_0 in (classical sense) iff (Z, τ) is local T_0.*

Now categorically we have following definition ([9])

Definition 3.1.10. *Let $\boldsymbol{U} : F \to Set$ be a topological functor and $Z \in Obj(F)$ provided that $\boldsymbol{U}(Z) = E$. Z is said to be T_0 at p in the case that initial lift on $Z \vee_p Z$ defined by $A_p : Z \vee_p Z \to Z^2$ and $\nabla_p : Z \vee_p Z \to Z$ is discrete.*

Theorem 3.1.11. *All the objects in interval space are T_0 at p.*

Proof. Let (Z, J) be an interval space. We have to show that (Z, J) is T_0 at p. Let $\bar{J}$ be an initial structure on $Z \bigvee_p Z$ induced by

$$A_p : (Z \bigvee_p Z, \bar{J}) \to (Z^2, J^2) \text{ and } \nabla_p : (Z \bigvee_p Z, \bar{J}) \to (Z, J_{dis})$$

where, J^2 and J_{dis} are product interval structure and discrete interval structure on Z^2 and Z respectively.
Let $u, v \in Z \vee_p Z$.
Case-I
If $u = v$ then $\nabla_p u = \nabla_p v$ also $\pi_k A_p u = \pi_k A_p v$, $k = 1, 2$ where π_k is projection map

$\pi_k : Z^2 \to Z$ for $k = 1, 2$.
On other hand

$$\nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v)) = \nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v)) = \nabla_P^{\leftarrow}(\{\nabla_p u\}) = \{u\}$$

and

$$\pi_k A_p^{\leftarrow}(J(\pi_k A_p u, \pi_k A_p v)) = \pi_k A_p^{\leftarrow}(J(\pi_k A_p u, \pi_k A_p u)), k = 1, 2$$

It follows that

$$u \in \pi_k A_p^{\leftarrow}(J(\pi_k A_p u, \pi_k A_p u)), k = 1, 2$$

By Proposition 2.3.2,

$$\bar{J}(u, u) = \pi_k A_p^{\leftarrow}(J(\pi_k A_p u, \pi_k A_p u)) \cap \nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p u)), k = 1, 2$$

$$\bar{J}(u, u) = \pi_k A_p^{\leftarrow}(J(\pi_k A_p u, \pi_k A_p u)) \cap \{u\}$$

Since

$$\pi_k A_p u \in (J(\pi_k A_p u, \pi_k A_p u)) \Rightarrow u \in \pi_k A_p^{\leftarrow}(J(\pi_k A_p u, \pi_k A_p u))$$

$$\bar{J}(u, u) = \{u\}$$

indicates the discrete structure by Definition 2.3.3.
Case-II Let $u \neq v$ and $\nabla_p u = \nabla_p v$. If $\nabla_p u = p = \nabla_p v$ implies $u = p = v$, a contradiction Since $u \neq v$. Suppose $\nabla_p u = x = \nabla_p v \Rightarrow u = x_i, v = x_j$, $i, j = 1, 2$ and $i \neq j$. Since $u \neq v$. If $u = x_1$ and $v = x_2$
Note that

$$(J_{dis}(\nabla_p u, \nabla_p v)) = (J_{dis}(\nabla_p x_1, \nabla_p x_2)) = (J_{dis}(x, x)) = \{x\}$$

and

$$\nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v)) = \nabla_p^{\leftarrow}\{x\} = \{x_1, x_2\}$$

$$\pi_1 A_p^{\leftarrow}(J(\pi_1 A_p u, \pi_1 A_p v) = \pi_1 A_p^{\leftarrow}(J(\pi_1 A_p x_1, \pi_1 A_p x_2) = \pi_1 A_p^{\leftarrow}(J(x, p))$$

$$\pi_2 A_p^{\leftarrow}(J(\pi_2 A_p u, \pi_2 A_p v) = \pi_2 A_p^{\leftarrow}(J(\pi_2 A_p x_1, \pi_2 A_p x_2) = \pi_2 A_p^{\leftarrow}(J(p, x))$$

Since

$$x, p \in J(x,p) \Rightarrow x = \pi_1 A_p x_1 \in J(x,p) \Rightarrow x_1 \in \pi_1 A_p^{\leftarrow}(J(x,p)$$

and

$$p = \pi_1 A_p x_2 \in J(x,p) \Rightarrow x_2 \in \pi_1 A_p^{\leftarrow}(J(x,p)$$

and it follows that

$$\{x_1, x_2\} \in \pi_1 A_p^{\leftarrow}(J(x,p) \text{ and } \{x_1, x_2\} \in \pi_2 A_p^{\leftarrow}(J(p,x)$$

$$\bar{J}(u,v) = \pi_1 A_p^{\leftarrow}(J(\pi_1 A_p u, \pi_1 A_p v) \cap \pi_2 A_p^{\leftarrow}(J(\pi_2 A_p u, \pi_2 A_p v) \cap \nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v))$$

$$\bar{J}(u,v) = \pi_1 A_p^{\leftarrow}(J(x,p) \cap \pi_1 A_p^{\leftarrow}(J(x,p) \cap \{x_1, x_2\}$$

$$\bar{J}(u,v) = \{x_1, x_2\} = \{u, v\}$$

indicates the discrete structure by Definition 2.3.3 Similarly If $u = x_2, v = x_1$ then alike Case-II gives us

$$\bar{J}(u,v) = \{x_1, x_2\} = \{u, v\}$$

By Definition 3.1.10, we have $(X, \bar{J})$ is T_0 at p.

□

3.2 Local T_1 Interval Space

Definition 3.2.1. *([8]) Let (Z, τ) be topological space and $p \in Z$ if for all $r \in Z$ with $r \neq p$ $\exists$ $G \in \tau$ with $r \in G$, $p \notin G$ and $\exists$ $H \in \tau$ with $p \in H$, $r \notin H$. Then (Z, τ) is called Local T_1 or T_1 at p.*

Example 3.2.2. *If $Z = \{6, 7, 8\}$ and $\tau = \{\emptyset, \{6\}, \{7, 8\}, Z\}$ is T_1 at 6 but not T_1 at 7.*

Theorem 3.2.3. *([8]) (Z, τ) is T_1 iff (Z, τ) is T_1 at p, $\forall$ $p \in Z$.*

In 1991, Baran ([9]) introduced T_1 at p in classical sense of topology in form of Categorical Topology in the terms of initial and discrete structure.

Definition 3.2.4. *([9]) Suppose (Z, τ) is a top space. (Z, τ) is called local T_1 iff the initial structure on $Z \bigvee_p Z$ induced by $A_p : Z \vee_p Z \to Z^2$ and $\nabla_p : Z \vee_p Z \to (Z, P(Z))$ is discrete top space.*

Theorem 3.2.5. *([8]) Suppose (Z, τ) is a top space. (Z, τ) is local T_1 in (classical sense) iff (Z, τ) is local T_1.*

Now let categorically, we have following definition ([9])

Definition 3.2.6. *Let $\boldsymbol{U} : F \to Set$ is said to be a topological functor and $Z \in Obj(F)$ provided that $\boldsymbol{U}(Z) = E$. Z is said to be T_1 at p in the case that initial lift on $Z \vee_p Z$ defined by $S_p : Z \vee_p Z \to Z^2$ and $\nabla_p : Z \vee_p Z \to Z$ is discrete.*

Theorem 3.2.7. *All the objects in interval space are T_1 at p.*

Proof. Let (Z, J) be an interval space. We have to show that (Z, J) is T_1 at p.
Let $\bar{J}$ be an initial structure on $Z \vee_p Z$ induced by
$S_p : (Z \vee_p Z, \bar{J}) \to (Z^2, J^2)$
$\nabla_p : (Z \vee_p Z, \bar{J}) \to (Z, J_{dis})$, where J^2 and J_{dis} are product interval structure and discrete interval structure on Z^2 and Z respectively.
Let $u, v \in Z \vee_p Z$.
Case-I: If $u = v$ then $\nabla_p u = \nabla_p v$ also $\pi_k A_p u = \pi_k A_p v$, k=1,2
where π_k is projection map $\pi_k : Z^2 \to Z$ for k=1,2
On other hand

$$\nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v)) = \nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v))$$

and it follows that

$$\nabla_P^{\leftarrow}(\{\nabla_p u\}) = \{u\}$$

and

$$\pi_k S_p^{\leftarrow}(J(\pi_k S_p u, \pi_k S_p v)) = \pi_k S_p^{\leftarrow}(J(\pi_k S_p u, \pi_k S_p u), k = 1, 2$$

It follows that

$$u \in \pi_k S_p^{\leftarrow}(J(\pi_k S_p u, \pi_k S_p u)), k = 1, 2$$

By Proposition 2.3.2

$$\bar{J}(u,u) = \pi_k S_p^{\leftarrow}(J(\pi_k S_p u, \pi_k S_p u)) \cap \nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p u)), k = 1, 2$$

$$\bar{J}(u,u) = \pi_k S_p^{\leftarrow}(J(\pi_k S_p u, \pi_k S_p u)) \cap \{u\}$$

Since

$$\pi_k S_p u \in (J(\pi_k S_p u, \pi_k S_p u)), k = 1, 2$$

$$u \in \pi_k S_p^{\leftarrow}(J(\pi_k S_p u, \pi_k S_p u)), k = 1, 2$$

$$\bar{J}(u,u) = \{u\}$$

Case-II: Let $u \neq v$ and $\nabla_p u = \nabla_p v$
If $\nabla_p u = p = \nabla_p v \Rightarrow u = p = v$, a contradiction. Since $u \neq v$. Suppose $\nabla_p u = x = \nabla_p v$ $\Rightarrow u = x_i$, $v = x_j$, $i, j = 1, 2$ and $i \neq j$. Since $u \neq v$. If $u = x_1$ and $v = x_2$.
Note that

$$(J_{dis}(\nabla_p u, \nabla_p v)) = (J_{dis}(\nabla_p x_1, \nabla_p x_2)) = (J_{dis}(x, x)) = \{x\}$$

and

$$\nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v)) = \nabla_p^{\leftarrow}\{x\} = \{x_1, x_2\}$$

$$\pi_1 S_p^{\leftarrow}(J(\pi_1 S_p u, \pi_1 S_p v) = \pi_1 S_p^{\leftarrow}(J(\pi_1 S_p x_1, \pi_1 S_p x_2) = \pi_1 S_p^{\leftarrow}(J(x, p))$$

$$\pi_2 S_p^{\leftarrow}(J(\pi_2 S_p u, \pi_2 S_p v) = \pi_2 S_p^{\leftarrow}(J(\pi_2 S_p x_1, \pi_2 S_p x_2) = \pi_2 S_p^{\leftarrow}(J(x, x))$$

Since

$$x, p \in J(x, p)$$

$$x = \pi_1 S_p x_1 \in J(x, p)$$

$$x_1 \in \pi_1 S_p^{\leftarrow}(J(x, p)$$

and

$$p = \pi_1 S_p x_2 \in J(x, p)$$

$$x_2 \in \pi_1 S_p^{\leftarrow}(J(x, p)$$

and it follows that

$$\{x_1, x_2\} \in \pi_1 S_p^{\leftarrow}(J(x,p)$$

and

$$\{x_1, x_2\} \in \pi_2 S_p^{\leftarrow}(J(x,x)$$

$$\bar{J}(u,v) = \pi_1 S_p^{\leftarrow}(J(\pi_1 S_p u, \pi_1 S_p v) \cap \pi_2 S_p^{\leftarrow}(J(\pi_2 A_p u, \pi_2 S_p v) \cap \nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v))$$

$$\bar{J}(u,v) = \pi_1 S_p^{\leftarrow}(J(x,p) \cap \pi_2 S_p^{\leftarrow}(J(x,x) \cap \{x_1, x_2\}$$

$$\bar{J}(u,v) = \{x_1, x_2\} = \{u, v\}$$

Similarly If $u = x_2$, $v = x_1$ then

$$J_{dis}(\nabla_p u, \nabla_p v) = J_{dis}(\nabla_p x_2, \nabla_p x_1) = J_{dis}(x,x) = \{x\}$$

and

$$\nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v)) = \nabla_p^{\leftarrow}\{x\} = \{x_1, x_2\}$$

$$\pi_1 S_p^{\leftarrow}(J(\pi_1 S_p u, \pi_1 A_p v) = \pi_1 S_p^{\leftarrow}(J(\pi_1 S_p x_2, \pi_1 S_p x_1)) = \pi_1 S_p^{\leftarrow}(J(p,x))$$

$$\pi_2 S_p^{\leftarrow}(J(\pi_2 S_p u, \pi_2 S_p v) = \pi_2 S_p^{\leftarrow}(J(\pi_2 S_p x_2, \pi_2 S_p x_1)) = \pi_2 S_p^{\leftarrow}(J(x,x))$$

Since

$$x, p \in J(x,p)$$

$$x = \pi_1 S_p x_1 \in J(x,p)$$

$$x_1 \in \pi_1 S_p^{\leftarrow}(J(x,p)$$

and

$$p = \pi_1 S_p x_2 \in J(x,p)$$

$$x_2 \in \pi_1 S_p^{\leftarrow}(J(x,p)$$

and it follows that

$$\{x_1, x_2\} \in \pi_1 S_p^{\leftarrow}(J(x,p)$$

and

$$\{x_1, x_2\} \in \pi_2 S_p^{\leftarrow}(J(x,x)$$

$$\bar{J}(u,v) = \pi_1 S_p^{\leftarrow}(J(\pi_1 S_p u, \pi_1 S_p v) \cap \pi_2 S_p^{\leftarrow}(J(\pi_2 S_p u, \pi_2 S_p v) \cap \nabla_p^{\leftarrow}(J_{dis}(\nabla_p u, \nabla_p v))$$

$$\bar{J}(u,v) = \pi_1 S_p^{\leftarrow}(J(x,p) \cap \pi_1 S_p^{\leftarrow}(J(x,x) \cap \{x_1, x_2\}$$

$$\bar{J}(u,v) = \{x_2, x_1\} = \{v, u\}$$

Hence $(Z, \bar{J})$ is T_1 at p.

□

Chapter 4

T_0 and T_1 Interval Spaces

4.1 T_0 Interval Space

Definition 4.1.1. *([9])A topological space (Z, τ) is called T_0 if $\forall$ $r,s \in Z$ with $r \neq s$, $\exists$ $G \in \tau$ such that $r \in G$, $s \notin G$ or $\exists$ $H \in \tau$ such that $r \notin H$, $s \in H$.*

Example 4.1.2. *If $Z = \{l, m\}$ with topology $\tau = \{\emptyset, \{l\}, \{l, m\}\}$ on Z is a T_0 Space.*

Theorem 4.1.3. *([8]) (Z, τ) is T_0 iff (Z, τ) is T_0 at p, $\forall$ $p \in Z$.*

In 1991, Baran ([9]) introduced T_0 in classical sense of topology in form of Categorical Topology in the terms of initial and discrete Structure.

Definition 4.1.4. *([9]) Let $Z^2 = Z \times Z$ be a cartesian product and $Z^2 \vee_\Delta Z^2$ two any disjoint copies of Z^2 Here Δ denotes the diagonal. It means that two disjoint copies of Z^2 intersect the wedge at the diagonal and its image lies in three dimensional space. Here (c, d) be any arbitrary point in $Z^2 \vee_\Delta Z^2$ where by distinction corresponding to first and second component it is denoted by $(c, d)_1$ and $(c, d)_2$ accordingly.*

Definition 4.1.5. *([9]) A map $A : Z^2 \vee_\Delta Z^2 \to Z^3$ is said to be principal axis map defined by*

$$A(c,d)_j = \begin{cases} (c,d,c) & , \ j = 1 \\ (c,c,d) & , \ j = 2 \end{cases}$$

Definition 4.1.6. *([9]) A map $S : Z^2 \vee_\Delta Z^2 \to Z^3$ is said to be skewed axis map defined by*

$$S(c,d)_j = \begin{cases} (c,d,d) & , \ j=1 \\ (c,c,d) & , \ j=2 \end{cases}$$

Definition 4.1.7. *([9]) A map $\nabla : Z^2 \vee_\Delta Z^2 \to Z^2$ is said to be fold map defined by $\nabla(c,d)_j = (c,d)$ for $j = 1,2$.*

Definition 4.1.8. *([9]) Suppose (Z, τ) is a top space (Z,τ) is called T_0 iff the initial structure on $Z^2 \vee_\Delta Z^2$ induced by $A : Z^2 \vee_\Delta Z^2 \to Z^3$ and $\nabla : Z^2 \vee_\Delta Z^2 \to Z^2$ is discrete top space.*

Theorem 4.1.9. *([8]) Suppose (Z, τ) is a top space. (Z, τ) is T_0 in (classical sense) iff (Z, τ) is T_0.*

Now let categorically, we have following definition ([9])

Definition 4.1.10. *Let $\boldsymbol{U} : E \to Set$ be a topological functor and $Z \in Obj(E)$ provided that $\boldsymbol{U}(Z) = F$ then Z is said to be T_0 in the case that initial structure on $Z^2 \vee_\Delta Z^2$ defined by $A : Z^2 \vee_\Delta Z^2 \to Z^3$ and $\nabla_: Z^2 \vee_\Delta Z^2 \to (Z^2, P(Z^2))$ is discrete.*

Theorem 4.1.11. *All the objects in interval space are T_0.*

Proof. Let (Z, J) be an interval space. We have to show that (Z, J) is T_0. Let $\bar{J}$ be an initial structure on $Z^2 \bigvee_\Delta Z^2$ induced by
A :$Z^2 \bigvee_\Delta Z^2 \to (Z^3, J^3)$ and $\nabla : Z^2 \bigvee_\Delta Z^2 \to (Z^2, J^2_{dis})$
where, J^3 and J^2 are cubic product and discrete interval structure on Z^3 and Z^2 respectively.
Let $u, v \in Z^2 \bigvee_\Delta Z^2$ where $u = (c,d)_1$ and $v = (c,d)_2$
Case-I
If

$$u = v \Rightarrow \nabla u = \nabla v$$

$$\nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla v) = \nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla u)) = \nabla^{\leftarrow}(J_{dis}(\nabla(c,d)_1, \nabla(c,d)_1))$$

and it follows that

$$\nabla^{\leftarrow}(\nabla(c,d)_1) = \{(c,d)_1\} = \{u\}$$

where π_k is projection map $\pi_k : Z^3 \to Z$ for k=1,2

$$\pi_k A^{\leftarrow}(J(\pi_k Au, \pi_k Av)) = \pi_k A^{\leftarrow}(J(\pi_k Au, \pi_k Au)$$

$$\pi_k A^{\leftarrow}(J(\pi_k Au, \pi_k Au) = \pi_k A^{\leftarrow}(J(\pi_k A(c,d)_1, \pi_k A(c,d)_1)$$

It follows that

$$\pi_k A(c,d)_1 \in J(\pi_k A(c,d)_1, \pi_k A(c,d)_1) \Rightarrow (c,d)_1 \in \pi_k A^{\leftarrow}(J(\pi_k A(c,d)_1, \pi_k A(c,d)_1)$$

By Preposition 2.3.2,

$$\bar{J}(u,u) = \pi_k A^{\leftarrow}(J(\pi_k Au, \pi_k Au) \cap \nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla u)) = \pi_k A^{\leftarrow}(J(\pi_k A(c,d)_1, \pi_k A(c,d)_1) \cap \{(c,d)_1\}$$

$$\bar{J}(u,u) = \{u\} = \{(c,d)_1\}$$

Case-II

Let $u \neq v$ and $\nabla u = \nabla v$. Consider $u = (c,d)_1$, $v = (c,d)_2$. If $\nabla u = (c,d)_1 = \nabla v$, a contradiction. Since $u \neq v$

Suppose $\nabla u = (c,d)_2 = \nabla v$. Since $u \neq v$

Note that

$$J_{dis}(\nabla u, \nabla v) = J_{dis}(\nabla(c,d)_1, \nabla(c,d)_2)$$

$$\nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla v)) = \nabla^{\leftarrow}(\nabla(c,d)_1, \nabla(c,d)_2) = \{(c,d)_1, (c,d)_2\}$$

$$\pi_1 A^{\leftarrow}(J(\pi_1 Au, \pi_1 Av)) = \pi_1 A^{\leftarrow}(J(\pi_1 A(c,d)_1, \pi_1 A(c,d)_2) = \pi_1 A^{\leftarrow}(J(c,c))$$

$$\pi_2 A^{\leftarrow}(J(\pi_2 Au, \pi_2 Av)) = \pi_2 A^{\leftarrow}(J(\pi_2 A(c,d)_1, \pi_2 A(c,d)_2) = \pi_2 A^{\leftarrow}(J(d,c))$$

$$\pi_3 A^{\leftarrow}(J(\pi_3 Au, \pi_2 Av)) = \pi_3 A^{\leftarrow}(J(\pi_3 A(c,d)_1, \pi_3 A(c,d)_2) = \pi_3 A^{\leftarrow}(J(c,d))$$

Since

$$(c,d)_1 = \pi_1 A(c,d)_1 \in J(\pi_1 A(c,d)_1, \pi_1 A(c,d)_2) \Rightarrow (c,d)_1 \in \pi_1 A^{\leftarrow}(J(\pi_1 A(c,d)_1, \pi_1 A(c,d)_2)$$

$$(c,d)_2 = \pi_1 A(c,d)_2 \in J(\pi_1 A(c,d)_1, \pi_1 A(c,d)_2) \Rightarrow (c,d)_2 \in \pi_1 A^{\leftarrow}(J(\pi_1 A(c,d)_1, \pi_1 A(c,d)_2)$$

$$\{(c,d)_1,(c,d)_2\} = \pi_1 A^{\leftarrow}(J(\pi_1 A(c,d)_1, \pi_1 A(c,d)_2)$$

and

$$\{(c,d)_1,(c,d)_2\} = \pi_2 A^{\leftarrow}(J(\pi_2 A(c,d)_1, \pi_2 A(c,d)_2)$$

and

$$\{(c,d)_1,(c,d)_2\} = \pi_3 A^{\leftarrow}(J(\pi_3 A(c,d)_1, \pi_3 A(c,d)_2)$$

By Prepoistion 2.3.2,

$$\bar{J}(u,v) = \pi_1 A^{\leftarrow}(J(\pi_1 A(c,d)_1, \pi_1 A(c,d)_2) \cap \pi_2 A^{\leftarrow}(J(\pi_2 A(c,d)_1, \pi_2 A(c,d)_2) \cap \pi_3 A^{\leftarrow}(J(\pi_3 A(c,d)_1, \pi_3 A(c$$

$$\bar{J}(u,v) = \{u,v\} = \{(c,d)_1,(c,d)_2\}$$

a discrete structure by Definition 4.1.10

Case-III

Now If $u = (c,d)_2$ and $v = (c,d)_1$ is similarly done as in Case-2 which gives

$$\bar{J}(u,v) = \{u,v\} = \{(c,d)_2,(c,d)_1\}$$

Then by Definition 4.1.10 we have $(Z,\bar{J})$ is T_0 in Interval space. □

Theorem 4.1.12. *([8]) Suppose (Z,τ) is a top space. (Z,τ) is $\mathbf{T_0}$ in (classical sense) if Z cannot contain an indiscrete subspace with atleast two points.*

Definition 4.1.13. *([14]) Let $\boldsymbol{U}: E \to Set$ be a topological functor and $Z \in Obj(E)$ provided that $\boldsymbol{U}(Z) = F$ then Z is said to be $\mathbf{T_0}$ if Z cannot contain an indiscrete subspace with atleast two points.*

Theorem 4.1.14. *([8]) Suppose (Z,τ) is a top space. (Z,τ) is $\mathbf{T_0}$ in (classical sense) iff (Z,τ) is $\mathbf{T_0}$.*

Theorem 4.1.15. *Let (Z,J) be an interval space. (Z,J) is $\mathbf{T_0}$ iff $\forall x,y \in Z$ with $x \neq y$, $J(x,y) \neq Z$.*

Proof. Let (Z,J) be $\mathbf{T_0}$, for distinct $x,y \in Z$, and $M = \{x,y\} \subset Z$ and J_M be the initial interval structure induced by $i : M \to (Z,J)$. For all $x,y \in Z$ with $x \neq y$

$$J_M(x,y) = i^{\leftarrow}(J(i(x),i(y)) = i^{\leftarrow}(J(x,y)) = J(x,y)$$

It follows that $J(x,y) \neq Z$ otherwise $J_M(x,y) = J(x,y) = Z$ and Z contains an indiscrete subspace with atleast two elements. Conversely, suppose $J(x,y) \neq Z$ for all $x, y \in Z$ with $x \neq y$. Let M be an indiscrete subspace of Z with atleast two elements $x, y \in M$ with $x \neq y$. Let J_M be the initial interval structure, induced by $i : M \to Z$, the inclusion map. It follows that

$$Z = J_M(x,y) = i^{\leftarrow}(J(i(x), i(y)) = J(x,y)$$

a contradiction. Thus by definition 4.1.11, (Z, J) is $\mathbf{T_0}$. □

4.2 T_1 Interval Space

Definition 4.2.1. *([8]) A topological space Z is said to be a T_1 for each s,t $\in Z$ with $s \neq t$, then there exists two open sets G_1,G_2 such that $s \notin G_1$ and $t \in G_1$, $t \notin G_2$ and $s \in G_2$*

Example 4.2.2. *If $Z = \{6,7,8\}$ with topology $\tau = \{\emptyset, Z, \{7\}, \{7,6\}, \{6\}\}$ is not T_1 However, $\tau = P(Z)$ is T_1.*

Theorem 4.2.3. *([8]) (Z, τ) is T_1 iff (Z, τ) is T_1 at p, $\forall$ $p \in Z$.*

In 1991, Baran ([9]) introduced T_1 in classical sense of topology in form of Categorical Topology in the terms of initial and discrete structure.

Definition 4.2.4. *([9]) Suppose (Z, τ) is a top space. (Z, τ) is called T_1 iff the initial topology on $Z^2 \vee_\Delta Z^2$ induced by $S : Z^2 \vee_\Delta Z^2 \to Z^3$ and $\nabla : Z^2 \vee_\Delta Z^2 \to (Z^2, P(Z^2))$ is discrete top space.*

Theorem 4.2.5. *([8]) Suppose (Z, τ) is a top space. (Z, τ) is T_1 in (classical sense) iff (Z, τ) is T_1.*

Now let categorically, we have following Definition ([9])

Definition 4.2.6. *Let $\boldsymbol{U} : K \to Set$ be a topological functor and $Z \in Obj(K)$ provided that $\boldsymbol{U}(Z) = F$ then Z is said to be T_1 in the case that initial structure on $Z^2 \vee_\Delta Z^2$ defined by $S : Z^2 \vee_\Delta Z^2 \to Z^3$ and $\nabla_: Z^2 \vee_\Delta Z^2 \to (Z^2, P(Z^2))$ is discrete.*

Theorem 4.2.7. *All the objects in interval space are T_1.*

Proof. Let (Z, J) be an interval space. We have to show that (Z, J) is T_1. Let $\bar{J}$ be an initial structure on $Z^2 \bigvee_\Delta Z^2$ induced by
S :$Z^2 \bigvee_\Delta Z^2 \to (Z^3, J^3)$ and $\nabla : Z^2 \bigvee_\Delta Z^2 \to (Z^2, J^2_{dis})$
J^3 and J^2 are cubic product and discrete interval structure on Z^3 and Z^2 respectively.
Let $u, v \in Z^2 \vee_\Delta Z^2$ where $u = (c, d)_1$ and $v = (c, d)_2$
Case-I
If

$$u = v \Rightarrow \nabla u = \nabla v$$

$$\nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla v) = \nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla u)) = \nabla^{\leftarrow}(J_{dis}(\nabla(c, d)_1, \nabla(c, d)_1))$$

and it follows that

$$\nabla^{\leftarrow}(\nabla(c, d)_1) = \{(c, d)_1\} = \{u\}$$

where π_k is projection map $\pi_k : Z^3 \to Z$ for k=1,2

$$\pi_k S^{\leftarrow}(J(\pi_k Su, \pi_k Sv)) = \pi_k S^{\leftarrow}(J(\pi_k Su, \pi_k Su)$$

$$\pi_k S^{\leftarrow}(J(\pi_k Su, \pi_k Su) = \pi_k S^{\leftarrow}(J(\pi_k S(c, d)_1, \pi_k S(c, d)_1)$$

It follows that

$$\pi_k S(c, d)_1 \in J(\pi_k S(c, d)_1, \pi_k S(c, d)_2) \Rightarrow (c, d)_1 \in \pi_k S^{\leftarrow}(J(\pi_k S(c, d)_1, \pi_k S(c, d)_1)$$

By Proposition 2.3.2,

$$\bar{J}(u, u) = \pi_k S^{\leftarrow}(J(\pi_k Su, \pi_k Su) \cap \nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla u)) = \pi_k S^{\leftarrow}(J(\pi_k S(c, d)_1, \pi_k S(c, d)_1) \cap \{(c, d)_1\}$$

$$\bar{J}(u, u) = \{u\} = \{(c, d)_1\}$$

Case-II
Let $u \neq v$ and $\nabla u = \nabla v$. Consider $u = (c, d)_1, v = (c, d)_2$. If $\nabla u = (c, d)_1 = \nabla v$, a contradiction. Since $u \neq v$

Suppose $\nabla u = (c,d)_2 = \nabla v$. Since $u \neq v$

Note that

$$J_{dis}(\nabla u, \nabla v) = J_{dis}(\nabla (c,d)_1, \nabla (c,d)_2)$$

$$\nabla^{\leftarrow}(J_{dis}(\nabla u, \nabla v)) = \nabla^{\leftarrow}(\nabla (c,d)_1, \nabla (c,d)_2) = \{(c,d)_1, (c,d)_2\}$$

$$\pi_1 S^{\leftarrow}(J(\pi_1 Su, \pi_1 Sv)) = \pi_1 S^{\leftarrow}(J(\pi_1 S(c,d)_1, \pi_1 S(c,d)_2) = \pi_1 S^{\leftarrow}(J(c,c))$$

$$\pi_2 S^{\leftarrow}(J(\pi_2 Su, \pi_2 Sv)) = \pi_2 S^{\leftarrow}(J(\pi_2 S(c,d)_1, \pi_2 S(c,d)_2) = \pi_2 S^{\leftarrow}(J(d,c))$$

$$\pi_3 S^{\leftarrow}(J(\pi_3 Su, \pi_2 Sv)) = \pi_3 S^{\leftarrow}(J(\pi_3 S(c,d)_1, \pi_3 S(c,d)_2) = \pi_3 S^{\leftarrow}(J(d,d))$$

Since

$$(c,d)_1 = \pi_1 S(c,d)_1 \in J(\pi_1 S(c,d)_1, \pi_1 S(c,d)_2) \Rightarrow (c,d)_1 \in \pi_1 S^{\leftarrow}(J(\pi_1 S(c,d)_1, \pi_1 S(c,d)_2)$$

$$(c,d)_2 = \pi_1 S(c,d)_2 \in J(\pi_1 S(c,d)_1, \pi_1 S(c,d)_2) \Rightarrow (x,y)_2 \in \pi_1 S^{\leftarrow}(J(\pi_1 S(c,d)_1, \pi_1 S(c,d)_2)$$

$$\{(c,d)_1, (c,d)_2\} = \pi_1 S^{\leftarrow}(J(\pi_1 S(c,d)_1, \pi_1 S(c,d)_2)$$

and

$$\{(c,d)_1, (c,d)_2\} = \pi_2 S^{\leftarrow}(J(\pi_2 S(c,d)_1, \pi_2 S(c,d)_2)$$

and

$$\{(c,d)_1, (c,d)_2\} = \pi_3 S^{\leftarrow}(J(\pi_3 S(c,d)_1, \pi_3 S(c,d)_2)$$

By Proposition 2.3.2,

$$\bar{J}(u,v) = \pi_1 S^{\leftarrow}(J(\pi_1 S(c,d)_1, \pi_1 S(c,d)_2) \cap \pi_2 S^{\leftarrow}(J(\pi_2 S(c,d)_1, \pi_2 S(c,d)_2) \cap \pi_3 S^{\leftarrow}(J(\pi_3 S(c,d)_1, \pi_3 S(c,d$$

$$\bar{J}(u,v) = \{u,v\} = \{(c,d)_1, (c,d)_2\}$$

a discrete structure by Definition 4.2.6

Case-III

Now If $u = (c,d)_2$ and $v = (c,d)_1$ is similarly done as in Case-II which gives

$$\bar{J}(u,v) = \{u,v\} = \{(c,d)_2, (c,d)_1\}$$

Then by Definition 4.2.6 we have $(Z, \bar{J})$ is T_1 in interval space. □

Corollary 4.2.8. *By Definition 4.1.10, 4.1.11, 4.2.6 we conclude that*

$$\mathbf{T_0} \Rightarrow T_0 = T_1$$

but converse is not true in general.

Chapter 5

Zero Dimensionality, Notion of Closedness and D-Connectedness in Interval Space

5.1 Zero Dimensionality in Interval Spaces

Definition 5.1.1. *([7]) A topological space (Z, τ) is said to be zero dimensional if Z contains a basis of clopen sets.*

In 1997, Stine ([12]), analyzed the zero dimensionality of topological spaces.

Definition 5.1.2. *Let (Z, τ) be a topological space and $f_i : (Z, \tau) \to (Z_i, \tau_{idis})$ be a initial topology of $f_i : Z \to Z_i$ where (Z_i, τ_{idis}) is the family of discrete topological spaces, (Z, τ) is called zero dimensional topological space.*

Definition 5.1.3. *([15]) Let $U : F \to E$ be a topological functor and $G : E \to F$ be a discrete functor. Any object $Z \in Obj(F)$ is zero dimensional object if $\forall i \in I$ there exist $B_i \in Obj(E)$. $f_i : U(Z) \to B_i$ such that $\bar{f}_i : Z \to G(B_i)_{i\in I}$ is initial topology of $f_i : U(Z) \to U(G(B_i)) = (B_i)_{i\in I}$.*

Theorem 5.1.4. *Every interval space (Z, J) with $Card(Z) = 1$ or 2 is zero dimensional.*

Proof. Let (Z, J) be a interval space and $Z = \{x\}$
Then $J_{dis} = J_{ind} = J$

By remark and Theorem 4.3.4 and Theorem 5.3.1 of ([12])
It is zero dimensional. □

Theorem 5.1.5. *Let (Z, J) be an interval space with $Card(Z) \geq 3$ and $(Z_i, J_i dis)$ be discrete interval space $\forall i \in I$. (Z, J) is zero dimensional iff (Z, J) is discrete interval space.*

Proof. Suppose (Z, J) is zero dimensional. By Definition 5.1.3, then there exist nonempty discrete interval space $(Z_i, J_i)_{i \in I}$ and family of function.
$f_i : Z \to Z_i$ such that $f_i : (Z, J) \to (Z_i, J_i)$ is initial lift of $f_i : Z \to Z_i$
Note that $\forall x, y \in Z$
By Proposition 2.3.2

$$J(x, y) = \bigcap_{i \in I} f_i^{\leftarrow}(J_i dis(f_i(x), f_i(y))$$

$$J(x, y) = \bigcap_{i \in I} f_i^{\leftarrow}(f_i(x), f_i(y))$$

$$J(x, y) = \bigcap_{i \in I} f_i^{\leftarrow}(f_i(x), f_i(y))$$

$$J(x, y) = \bigcap_{i \in I} \{t \mid f_i(t) \in \{f_i(x), f_i(y)\}$$

$$J(x, y) = \bigcap_{i \in I} \{x, y\} = \{x, y\}$$

Conversely, suppose that (Z, J) is discrete. We show that (Z, J) is zero dimensional, i.e by Definition 5.1.3, we have $f_i : (Z, J) \to (Z_i, J_i dis)$ is initial lift of $f_i : Z \to Z_i$. First we show that $\forall i \in I$. $f_i : (Z, J) \to (Z_i, J_i dis)$ is interval preserving map. Indeed, $\forall x, y \in Z$
Since f_i is interval preserving map

$$f_i^{\rightarrow}(J(x, y) = f_i^{\rightarrow}\{x, y\} = \{f_i(x), f_i(y)\}$$

On other hand

$$J_i dis(f_i(x), f_i(y)) = \{f_i(x), f_i(y)\}$$

and clearly

$$f_i^{\rightarrow}(J(x,y) \subseteq J_i dis(f_i(x), f_i(y))$$

Suppose $g : (Y, J') \rightarrow (Z, J)$ is a mapping. We show that g is interval preserving mapping iff $f_i \circ g$ is interval preserving. Now suppose that g is an interval-preserving map, it is obvious that $f_i \circ g$ is an interval preserving map. Let $f_i \circ g$ is an interval preserving map for each $i \in I$. It follows that for $x, y \in Y$.

$$f_i \circ g^{\rightarrow}(J'(x,y)) \subseteq J_i dis(f_i \circ g(x), f_i \circ g(y))$$

$$\{f_i \circ g(t) \mid t \in J'(x,y)\} \subseteq \{f_i \circ g(x), f_i \circ g(y)\}$$

it follows that t could only be x and y which shows

$$J'(x,y) = \{x, y\}$$

represents the discrete structure by Definition 2.3.3

$$Thus, g^{\rightarrow}(J'(x,y)) = \{g(t) \mid t \in J'(x,y)\}$$

$$g^{\rightarrow}(J'(x,y)) = \{g(x), g(y)\} \subseteq J(g(x), g(y))$$

Thus g is an interval preserving map and consequently (Z, J) is zero dimensional. □

5.2 Notion of Closedness in Interval Space

Theorem 5.2.1. *([7]) Let (Z, τ) be a topological space, $p \in Z$. $\{p\}$ is closed iff $\{p\}$ is closed in usual sense i.e, $\{p\}^c \in \tau$.*

Definition 5.2.2. *([9]) Let $Z^\infty = Z \times Z \times Z \times ......$ be the countable cartesian product of Z.*

1. *Suppose Z is any set and p is in Z. Then infinite wedge product $\vee_p^\infty Z$ can be established in forming many different copies of Z and intersect these copies of Z at point p. A point z in $\vee_p^\infty Z$ could be symbolized as z_i if it is ith component.*

2. *A infinite principal axis map at p,* $A_p^\infty : V_p^\infty Z \to Z^\infty$ *is stated as*

$$A_p^\infty(z_i) = (p, p, ...p, z, p....)$$

Here z_i *denotes the ith position.*

3. *A infinite fold map at p,* $\nabla_p^\infty : \vee_p^\infty Z \to Z$ *is stated as*

$$\nabla_p^\infty(z_i) = z$$

for all $i \in I$

Definition 5.2.3. *([9, 10]) Suppose* $B : F \to Set$ *is a topological functor,* $Z \in obj(F)$ *with* $B(Z) = Z$ *for* $p \in Z$.

1. $\{p\}$ *holds closedness iff the initial lift formed by* $A_p^\infty : \vee_p^\infty Z \to Z^\infty$ *and* $\nabla_p^\infty : \vee_p^\infty Z \to Z$ *is discrete.*

2. *Z is said to be D-connected iff any morphism from Z to any discrete structure remains constant.*

Theorem 5.2.4. *Every singleton set* $\{p\}$ *is closed in interval space.*

Proof. Let (Z, J) be an interval space. We have to show that $(Z, \bar{J})$ hold closedness at p. Let $\bar{J}$ be an initial structure on $\vee_p^\infty Z$ induced by

$$A_p^\infty : \vee_p^\infty Z \to (Z^\infty, J^\infty)$$

and

$$\nabla_p^\infty : \vee_p^\infty Z \to (Z, J_{dis})$$

where, J^∞ and J_{dis} are infinite product interval structure and discrete interval structure on Z^∞ and Z respectively. Let $u, v \in \vee_p^\infty Z$. If $u = v$ then $\nabla_p^\infty u = \nabla_p^\infty v$ also $\pi_k A_p^\infty u = \pi_k A_p^\infty v$, where $k \in I$. Here π_k are projection map $\pi_k : X^\infty \to X$ where $k \in I$
On other hand

$$\nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^\infty u, \nabla_p^\infty v)) = \nabla_p^{\leftarrow}(J_{dis}(\nabla_p^\infty u, \nabla_p^\infty v))$$

and it follows that

$$\nabla_P^{\infty\leftarrow}(\{\nabla_p^{\infty}u\}) = \{u\}$$

and

$$\pi_k A_p^{\infty\leftarrow}(J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}v)) = \pi_k A_p^{\infty\leftarrow}(J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}u)), k \in I$$

It follows that

$$u \in \pi_k A_p^{\infty\leftarrow}(J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}u)), k \in I$$

By Proposition 2.3.2

$$\bar{J}(u,u) = \pi_k A_p^{\infty\leftarrow}(J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}u)) \cap \nabla_p^{\leftarrow}(J_{dis}(\nabla_p^{\infty}u, \nabla_p^{\infty}u)), k \in I$$

$$\bar{J}(u,u) = \pi_k A_p^{\infty\leftarrow}(J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}u)) \cap \{u\}$$

Since

$$\pi_k A_p^{\infty}u \in (J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}u)), k \in I$$

$$u \in \pi_k A_p^{\infty\leftarrow}(J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}u)), k \in I$$

$$\bar{J}(u,u) = \{u\}$$

Let $u \neq v$ and $\nabla_p^{\infty}u = \nabla_p^{\infty}v$. If $\nabla_p^{\infty}u = p = \nabla_p^{\infty}v \Rightarrow u = (p,p,p,...p...) = v$, a contradiction. Since $u \neq v$. Suppose $\nabla_p^{\infty}u = x = \nabla_p^{\infty}v \Rightarrow u = x_i, v = x_j$, $i, j \in I$ and $i \neq j$. Since $u \neq v$. If $u = x_i$ and $v = x_j$.
Note that

$$(J_{dis}(\nabla_p^{\infty}u, \nabla_p^{\infty}v)) = (J_{dis}(\nabla_p^{\infty}x_1, \nabla_p^{\infty}x_2)) = (J_{dis}(x,x)) = \{x\}$$

and

$$\nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^{\infty}u, \nabla_p^{\infty}v)) = \nabla_p^{\infty\leftarrow}\{x\} = \{x_i, x_j\} = \{u, v\}$$

$$\bar{J}(u,v) = \pi_K A_p^{\infty\leftarrow}(J(\pi_K A_p^{\infty}u, \pi_K A_p^{\infty}v) \cap \nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^{\infty}u, \nabla_p^{\infty}v))$$

We have

$$J(\pi_k A_p^{\infty}u, \pi_k A_p^{\infty}v) = \begin{cases} J(z,p) & for\ i = k \\ J(p,z) & for\ k = j \\ J(p,p) & for\ k \notin i,j \end{cases} \tag{5.1}$$

Since $z, p \in J(z,p)$

$$z = \pi_K A_p^{\infty} u \in J(z,p) \Rightarrow u \in \pi_K A_p^{\infty\leftarrow}(J(z,p)$$

and

$$p = \pi_K A_p^{\infty} v \in J(z,p) \Rightarrow v \in \pi_k A_p^{\infty\leftarrow}(J(z,p)$$

and it follows that

$$\{u,v\} \in \pi_k A_p^{\infty\leftarrow}(J(z,p)$$

and

$$\{u,v\} \in \pi_k A_p^{\infty\leftarrow}(J(p,z)$$

and

$$\{u,v\} \in \pi_k A_p^{\infty\leftarrow}(J(p,p)$$

$$\bar{J}(u,v) = \pi_k A_p^{\infty\leftarrow}(J(\pi_k A_p^{\infty} u, \pi_k A_p^{\infty} v) \cap \nabla_p^{\leftarrow\infty}(J_{dis}(\nabla_p^{\infty} u, \nabla_p^{\infty} v))$$

$$\bar{J}(u,v) = \pi_k A_p^{\infty\leftarrow}(J(z,p) \cap \pi_k A_p^{\infty\leftarrow}(J(p,z) \cap \pi_k A_p^{\infty\leftarrow}(J(p,p) \cap \{u,v\}$$

$$\bar{J}(u,v) = \{u,v\}$$

Similarly if $u = x_j$ and $v = x_i$ then by repeating above steps we get

$$\bar{J}(u,v) = \{u,v\}$$

Hence $(Z, \bar{J})$ is closed at p.

□

Corollary 5.2.5. *Let (Z,J) be an interval space then the followings are equivalent at $p \in Z$*

1. *$\{p\}$ is closed*
2. *(Z,J) is T_0 at p*
3. *(Z,J) is T_1 at p*

Proof. It follows the Theorem 5.2.4, Theorem 3.2.7 and Theorem 3.1.11. Hence all of above conditions are equivalent at p. □

5.3 D-Connectedness in Interval Space

Theorem 5.3.1. *Let (Z, J) be an interval space. (Z, J) is D-connected iff there exists a subset M of Z such that $\{x, y\} \subset J(x, y)$ for some $x \in M$ and $y \in M^c$.*

Proof. Let (Z, J) be D-connected and there exists a nonempty subset M of Z. $J(x, y) = \{x, y\}$ for all $x \in M$ and $y \in M^c$. Let (Y, J_{dis}) be discrete interval space. Defined as $f : (Z, J) \to (Y, J_{dis})$

$$f(x) = \begin{cases} a, & x \in M \\ b, & x \notin M \end{cases}$$

Let $x, y \in Z$

Case-I: If $x, y \in M$ then

$$f^{\to}(J(x, y)) = f^{\to}(\{x, y\}) = \{f(x), f(y)\} = \{a\}$$

and

$$J_{dis}(f(x), f(y)) = \{a\}$$

and consequently,

$$f^{\to}(J(x, y)) \subseteq J_{dis}(f(x), f(y)).$$

Thus f is an interval preserving map. Similarly if $x, y \in M^c$, then f is an interval preserving map.

Case-II: Let $x \in M$ and $y \in M^c$ (respectively $y \in M$ and $x \in M^c$)

$$f^{\to}(J(x, y)) = \{f(t) \mid t \in J(x, y) = \{x, y\}\} = \{f(x), f(y)\} = \{a, b\}.$$

and

$$J_{dis}((f(x), f(y)) = \{a, b\}$$

Thus,

$$f^{\to}((J(x, y)) \subseteq J_{dis}(f(x), f(y)).$$

Hence, f is an interval preserving map but not constant, a contradiction. Conversely, suppose the condition holds. Let (Y, J_{dis}) be a discrete interval space and $f : (Z, J) \to (Y, J_{dis})$ be an interval preserving map.

If $CardY = 1$, then f is constant. Suppose $CardY > 1$, and f is not constant. Then there exists $x, y \in X$ with $x \neq y$ such that $f(x) = f(y)$ and let $M = f^{\leftarrow}\{f(x)\}$. Note that M is a proper subset of Z. By our assumption $\{x, y\} \subset J(x, y)$ for some $x \in M$ and $y \notin M$

$$\{f(x), f(y)\} \subset f^{\rightarrow}((J(x, y)) \subset J_{dis}\{(f(x), f(y)\} = \{f(x), f(y)\}.$$

It follows that f is not an interval-preserving map, a contradiction. Thus f must be constant and by Definition 5.2.3, (Z, J) is D-connected. □

Corollary 5.3.2. *Every D-disconnected (not D-connected) interval space with cardinality greater than 2 is zero dimensional.*

Proof. Let (Z, J) be an D-disconnected interval space with cardinality greater than 2, by Definition 2.3.3, $\forall x, y \in X,\ J(x, y) = \{x, y\}$ with $x \neq y$. It follows that $J(x, y)$ is discrete and by Theorem 5.3.1, (Z, J) is zero dimensional. □

Chapter 6

Conclusions

We discuss the category of interval space in this dissertation. Firstly we describe the interval space as topological category. Further we characterize the Local T_0, T_1 and closedness at p in the category of interval space by using initial lift. After that we characterize the T_0, T_1 and $\mathbf{T_0}$ and discuss its relation. Lastly we explain zero dimensionality and D-connectedness in category of interval space. We reach to following conclusion

(i) All the objects in category of interval space are T_0, T_1 and hold closedness at p.

(ii) We characterize T_0, T_1,$\mathbf{T_0}$, and

$$\mathbf{T_0} \Rightarrow T_0 = T_1$$

but converse is not true in general.

(iii) Every interval space (Z, J) with $Card(Z) = 1$ or 2 is zero dimensional.

(iv) Let (Z, J) be an interval space with $Card(Z) \geq 3$ and $(Z_i, J_i dis)$ be discrete interval space $\forall i \in I$. (Z, J) is zero dimensional iff (Z, J) is discrete interval space.

(v) Let (Z, J) be an interval space. (Z, J) is D-connected iff there exists a subset M of Z such that $\{x, y\} \subset J(x, y)$ for some $x \in M$ and $y \in M^c$

(vi) Every D-disconnected (not D-connected) interval space with cardinality greater than 2 is zero dimensional.

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