Depth and Stanley Depth of Cyclic Modules Associated to q-Fold Bristled Graphs of Multi Triangular Snake and Ouroboros Snake Graphs.

By

Ayesha Saqib

Supervised by

Dr. Muhammad Ishaq

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan
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National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: **Ayesha Saqib**, Regn No. **00000330914** Titled: **"Depth and Stanley Depth of Cyclic Modules Associated to q-Fold Brisrtled Graphs of Multi Triangular Snake and Ouroboros Snake Graphs"** accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: PROF. MATLOOB ANWAR Signature: ____________

2. Name: DR. MUHAMMAD QASIM Signature: ____________

Supervisor's Name: DR. MUHAMMAD ISHAQ Signature: ____________

Head of Department

27/09/2022
Date

COUNTERSINGED

Date: 27.9.2022

Dean/Principal

This thesis is dedicated to my beloved parents, venerable supervisor, fellows and seniors for their boundless encouragement and support.

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Abstract

This dissertation deals with the algebraic and geometric invariants such as depth and Stanley depth, respectively. In this thesis, we compute exact values of depth and Stanley depth of cyclic modules associated to q-fold bristled graphs of triangular and multi triangular snake and ourorboros snake graphs. It is shown that the values of both invariants are same for the classes of graphs we considered.

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Introduction

Monomial ideals are vital in understanding the relationship between combinatorics and commutative algebra. In general, combinatorial problems are translated as monomial ideals and solved using commutative algebra methods and techniques. Stanley depth is an invariant for finitely produced $\mathbb{Z}^n$-graded modules over the commutative ring proposed by Stanley [23] in 1982. He also proposed a relationship between Stanley depth and module depth known as Stanley conjecture. Later on it was proved by Duval et al. [11] in year 2015 that Stanley's conjecture generally does not hold for A/Υ type modules, where A is defined as a ring of polynomials with n variables and Υ is a monomial ideal. Yet, finding classes which still satisfy the Stanley's inequality is a challenging task. In this thesis exact values of Depth and Staley depth for the cyclic modules associated to q-fold bristled graphs of some graphs are computed.

The first chapter provides an overview, definitions and findings for abstract and commutative algebra. This chapter introduces ring and module theory. This chapter also provides a brief overview of fundamental graph theory and some important graph operations. The second chapter discusses the fundamental theory of depth, Stanley depth, and Stanley decomposition of ideals and modules. Furthermore, previously known results are thoroughly explored. In chapter 3, depth and Stanley depth of cyclic modules associated to q-fold bristled graphs of triangular and multi triangular snake are computed. At the end, in chapter 4, depth and Stanley depth of cyclic modules assiciated to q-fold bristled graphs of triangular and multi triangular ouroboros snake are computed by using induction and some known results.

Chapter 1

Preliminaries

The following chapter includes basic definitions and significant results of abstract algebra and commutative algebra in order to provide the reader with a firm background for further ideas, that will be revealed in forthcoming chapters.

1.1 Ring theory

Ring theory deals with the study of algebraic structures, called rings.

Definition 1.1.1. A set $\mathcal{T}$ that is non-empty is a ring with the two defined binary operations; addition and multiplication that satisfy the below axioms:

1. $\mathcal{T}$ is an abelian group under " $+$ ".

2. The law of associativity under " $\times$ " holds in $\mathcal{T}$.

3. Distributive laws (left and right) holds in $\mathcal{T}$, that is, for all $æ_1, æ_2, æ_3 \in \mathcal{T}$

   - $æ_1 \times (æ_2 + æ_3) = (æ_1 \times æ_2) + (æ_1 \times æ_3)$.
   - $(æ_1 + æ_2) \times æ_3 = (æ_1 \times æ_3) + (æ_2 \times æ_3)$.

The ring $\mathcal{T}$ is called a commutative ring if all the elements $æ_1, æ_2 \in \mathcal{T}$ commute w.r.t multiplication. That is,

$$æ_1 \times æ_2 = æ_2 \times æ_1.$$

Throughout this thesis, we are dealing with commutative rings possessing a multiplicative identity 1 which is known as the unity of $\mathcal{T}$.

Example 1.1.2. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{Z}_n$ are examples of commutative rings with unity. The set of even integers is an example of commutative ring without unity.

Definition 1.1.3. Consider a ring $\mathcal{T}$ with unity. If each element of $\mathcal{T}$ that is non zero has a multiplicative inverse then $\mathcal{T}$ is known as a division ring. A field is an example of division ring which is commutative with respect to multiplication.

Example 1. $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$ are fields. But $\mathbb{Z}$ is not a field.

1.1.1 Ring of polynomials

The specific type of ring that is obtained by a set of polynomials is called a polynomial ring. These polynomials are in one or more than one variable where the coefficients belong to ring. Polynomial rings are used in several disciplines of mathematics and studying their characteristics is one of the key motivations for the advancement of commutative algebra and ring theory.

Definition 1.1.4. Consider a commutative ring $\mathcal{T}$ with unity, a polynomial in variable ϑ has the form

$$t_0 + t_1\vartheta + \cdots + t_{n-1}\vartheta^{n-1} + t_n\vartheta^n$$

where, $n \geq 0$ and each $t_i \in \mathcal{T}$. If $t_n \neq 0$, then the polynomial is said to be of degree n, where $t_n z^n$ is known as the leading term. The set of polynomials is denoted by $\mathcal{T}[z]$. Thus

$$\mathcal{T}[\vartheta] = \{t_0 + t_1\vartheta + \cdots + t_{n-1}\vartheta^{n-1} + t_n\vartheta^n : n \geq 0, t_i \in \mathcal{T}\}.$$

$\mathcal{T}[\vartheta]$ is a commutative ring with unity under addition and multiplication of polynomials and the unity of $\mathcal{T}[\vartheta]$ is the unity of $\mathcal{T}$.

Definition 1.1.5. The polynomial ring in the variables $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ and coefficients belonging to $\mathcal{T}$ is defined inductively

$$\mathcal{T}[\vartheta_1, \vartheta_2, \ldots, \vartheta_n] = \mathcal{T}[\vartheta_1, \vartheta_2, \ldots, \vartheta_{n-1}][\vartheta_n].$$

Definition 1.1.6. 1. Consider two rings $\mathcal{T}_1$ and $\mathcal{T}_2$. A ring homomorphism is a map $\xi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ satisfying

- $\xi(\mu_1 + \mu_2) = \xi(\mu_1) + \xi(\mu_2), \qquad \forall\ \mu_1, \mu_2 \in \mathcal{T}$
- $\xi(\mu_1\mu_2) = \xi(\mu_1)\xi(\mu_2), \qquad \forall\ \mu_1, \mu_2 \in \mathcal{T}$

2. The kernal of ξ is the set $ker\,\xi = \{\mu \in \mathcal{T} : \xi(\mu) = 0\}$

3. A biiective ring homomorphism is called a ring isomorphism.

Example 1.1.7. A map $\psi : \mathbb{Z} \longrightarrow \mathbb{Z}_2$ defined by

$$\psi(q) = \begin{cases} 0, & \text{if} \quad q \text{ is even;} \\ 1, & \text{if} \quad q \text{ is odd.} \end{cases}$$

is a ring homomorphism.

Remark 1.1.8. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be rings and let $\psi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be a ring homomorphism. Then ψ is iniective iff $ker\psi$ is trivial.

1.1.2 Properties and operations on ideals

Definition 1.1.1. A subset $\mathcal{L}$ of a ring $\mathcal{T}$ that is non-empty is known as an ideal iff $l_1 - l_2 \in \mathcal{L}$, $lt \in \mathcal{L}$ and $tl \in \mathcal{L}\ \forall\ l_1, l_2, l \in \mathcal{L}$ and $t \in \mathcal{T}$.

Remark 1.1.9. If $\mathcal{L}$ is an ideal of $\mathcal{T}$ then $\mathcal{L}$ is called a proper ideal if $\mathcal{L} \neq \mathcal{T}$. The ideal $\{0\}$ is called the trivial ideal of $\mathcal{T}$.

Definition 1.1.10. Consider a ring $\mathcal{T}$ and a proper ideal $\mathcal{L}$, a quotient ring $\mathcal{T}/\mathcal{L}$ can be formed, that consists of cosets $t + \mathcal{L}$, where $t \in \mathcal{T}$, and the multiplication of cosets is defined as:

$$(t_1 + \mathcal{L})(t_2 + \mathcal{L}) = t_1t_2 + \mathcal{L}.$$

Next there are the isomorphism theorems for rings.

Theorem 1.1.11. *(Isomorphism theorems)*

1. (First isomorphism theorem) For a ring homomorphism $\psi : \mathcal{T}_1 \to \mathcal{T}_2$, image of ψ is isomorphic to $\mathcal{T}_1/\ker(\psi)$, i.e.,

$$\mathcal{T}_1/\ker(\psi) \cong \operatorname{Im}(\psi).$$

2. (Second isomorphism theorem) For an ideal I and a subring $\mathcal{S}$ of the ring $\mathcal{T}_1$

$$(\mathcal{S} + I)/I \cong S/\mathcal{S} \cap I$$

3. (Third isomorphism theorem) Consider the ideals I_1 and I_2 of ring $\mathcal{T}_1$, with $I_1 \subseteq I_2$, then I_2/I_1 is an ideal of $\mathcal{T}_1/I_1$. Also

$$(\mathcal{T}_1/I_1)/(I_2/I_1) \cong \mathcal{T}_1/I_2.$$

Definition 1.1.12. Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ be the ideals of ring $\mathcal{T}$. Product of two ideals, say $\mathcal{L}_1$ and $\mathcal{L}_2$, is a set consisting of all possible finite sums of the form xy, where $x \in \mathcal{L}_1$ and $y \in \mathcal{L}_2$. It is represented by $\mathcal{L}_1\mathcal{L}_2$.

Example 1.1.13. Let $J_1 = 10\mathbb{Z}$ and $J_2 = 15\mathbb{Z}$ in $\mathbb{Z}$. Then $J_1 + J_2$ comprises all integers of the form $10q_1 + 15q_2$ with $q_1, q_2 \in \mathbb{Z}$. Since each such type of integer is divisible by 5, so $10\mathbb{Z} + 15\mathbb{Z} \subseteq 5\mathbb{Z}$. On the other hand, $5 = 10(-1) + 15(1)$ shows that $5\mathbb{Z}$ is contained in $10\mathbb{Z} + 15\mathbb{Z}$, hence $10\mathbb{Z} + 15\mathbb{Z} = 5\mathbb{Z}$. In general, $r_1\mathbb{Z} + r_2\mathbb{Z} = d\mathbb{Z}$, whereas $d = (r_1, r_2)$. The product J_1J_2 comprises all possible finite sums of the components of the form $(10q_1)(15q_2)$ where $q_1, q_2 \in \mathbb{Z}$, which clearly gives the ideal $150\mathbb{Z}$.

Definition 1.1.14. For a ring $\mathcal{T}$, principal ideal is an ideal with a single element in its generating set. A finitely generated ideal is an ideal with finite elements in its generating set.

Definition 1.1.15. Consider an arbitrary ring $\mathcal{T}$, a proper ideal $\mathcal{M}$ is known as maximal ideal if there is no proper ideal in between $\mathcal{M}$ and $\mathcal{T}$. In other words, if an ideal $\mathcal{J}$ contains $\mathcal{M}$, then either $\mathcal{M} = \mathcal{J}$ or $\mathcal{J} = \mathcal{T}$.

Definition 1.1.16. . A ring $\mathcal{T}$ is said to be local if it contains a unique maximal ideal.

Example 1.1.17. Ideal generated by $(2) = \{0, 2\}$ is the maximal ideal in $\mathbb{Z}_4$. (2) is also the unique maximal ideal in $\mathbb{Z}_4$. So $\mathbb{Z}_4$ is a local ring.

Definition 1.1.18. A prime ideal $\mathcal{Q}$ is a proper ideal of a ring $\mathcal{T}$ such that for $b_1, b_2 \in \mathcal{T}$, if $b_1 b_2 \in \mathcal{Q}$, then either $b_1 \in \mathcal{Q}$ or $b_2 \in \mathcal{Q}$.

Definition 1.1.19. For a ring $\mathcal{T}$, let us suppose two ideals $\mathcal{L}_1$ and $\mathcal{L}_2$. Then their ideal quotient is defined as

$$(\mathcal{L}_1 : \mathcal{L}_2) = \{t \in \mathcal{T} \ : \ t\mathcal{L}_2 \subseteq \mathcal{L}_1\}.$$

Definition 1.1.20. Consider a ring $\mathcal{T}$ and its ideal $\mathcal{L}$. Then $(0 : \mathcal{L})$ is an ideal known as the annihilator of $\mathcal{L}$ represented as $Ann(\mathcal{L})$ defined as

$$Ann(\mathcal{L}) = \{t \in \mathcal{R} : t\mathcal{L} = 0\}.$$

Definition 1.1.21. Consider any ideal $\mathcal{Y}$ of $\mathcal{T}$. $\mathcal{Y}$ is said to be primary if $p_1 p_2 \in \mathcal{Y}$, where $p_1, p_2 \in \mathcal{T}$, then either $p_1 \in \mathcal{Y}$ or $p_2^l \in \mathcal{Y}$ for any $l \geq 1$.

1.1.3 Monomial ideals

Let $S = K[\rho_1, \ldots, \rho_n]$ is a ring over field K, monomials forms the natural K-basis for S. Let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n$ where every $d_i \geq 0$. A monomial is any product of the form $\rho_1^{d_1} \ldots \rho_n^{d_n}$ with $b_i \in \mathbb{Z}_+$. If $\nu = \rho_1^{d_1} \ldots \rho_n^{d_n}$ is a monomial, then we write $\nu = \rho^{\mathbf{d}}$ with $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}_+^n$, and

$$\rho^{\mathbf{d_1}} \rho^{\mathbf{d_2}} = \rho^{\mathbf{d_1} + \mathbf{d_2}}.$$

A monomial ideal is an ideal whose generating set only consists of monomials. The set consisting of all the monomials in S is denoted by $\mathrm{Mon}(S)$. The set of all monomials in S form a K-basis of S. For any polynomial $h \in S$ and for $d_\nu \in K$

$$h = \sum_{\nu \in Mon(S)} d_\nu \nu,$$

where support of h is defined as

$$\mathrm{supp}(h) = \{\nu \in Mon(S) \ : \ d_\nu \neq 0\}.$$

Proposition 1.1.22. Consider two monomial ideals Υ_1 and Υ_2. Then $\Upsilon_1 \cap \Upsilon_2$ is a monomial ideal, and $\{lcm(m_1, m_2) : m_1 \in G(\Upsilon_1), m_2 \in G(\Upsilon_2)\}$ is a generating set of $\Upsilon_1 \cap \Upsilon_2$.

Proposition 1.1.23. Consider two monomial ideals Υ_1 and Υ_2. Then $(\Upsilon_1 : \Upsilon_2)$ is a monomial ideal and $(\Upsilon_1 : \Upsilon_2) = \bigcap_{m_2 \in G(\Upsilon_2)} (\Upsilon_1 : m_2)$. Furthermore $\{\frac{m_1}{gcd(m_1,m_2)} : m_1 \in G(\Upsilon_1)\}$ is the set of monomial generators of $\Upsilon_1 : m_2$.

A monomial $\rho^{\mathbf{d}}$ is said to be squarefree if $\mathbf{d}$ has components 0 and 1. An ideal with a generating set containing only squarefree monomials is called squarefree monomial ideal.

1.1.4 Primary decomposition

For an ideal I, primary decomposition is a way of representing I as an intersection $I = \bigcap_{m=1}^{n} \mathcal{P}_m$, whereas each $\mathcal{P}_m$ is a primary ideal. Let $\mathcal{N}_m = \mathrm{Ass}(\mathcal{P}_m)$. If none of the $\mathcal{P}_m$ can be omitted in this intersection and $\mathcal{N}_r \neq \mathcal{N}_s$ for all $r \neq s$ then it is called irredundant primary decomposition.

Example 1.1.24. Let $I = (\rho_1^2\rho_3 , \rho_4^3 , \rho_2^4\rho_4^2 , \rho_1\rho_2\rho_3^3)$, then

$$\begin{aligned}
I &= (\rho_1^2 , \rho_4^3 , \rho_2^4\rho_4^2 , \rho_1\rho_2\rho_3^3) \cap (\rho_3 , \rho_4^3 , \rho_2^4\rho_4^2 , \rho_1\rho_2\rho_3^3) \\
&= (\rho_1^2 , \rho_4^3 , \rho_2^4\rho_4^2 , \rho_1\rho_2\rho_3^3) \cap (\rho_3 , \rho_4^3 , \rho_2^4\rho_4^2) \\
&= (\rho_1^2 , \rho_4^3 , \rho_2^4 , \rho_1\rho_2\rho_3^3) \cap (\rho_1^2 , \rho_4^3 , \rho_4^2 , \rho_1\rho_2\rho_3^3) \cap (\rho_3 , \rho_4^3 , \rho_2^4) \cap (\rho_3 , \rho_4^3 , \rho_4^2) \\
&= (\rho_1^2 , \rho_4^3 , \rho_2^4 , \rho_1\rho_2\rho_3^3) \cap (\rho_1^2 , \rho_4^2 , \rho_1\rho_2\rho_3^3) \cap (\rho_3 , \rho_4^2 , y\rho_2^4) \cap (\rho_3 , \rho_4^2).
\end{aligned}$$

In the above example, the obtained primary decomposition is irredundant as $\mathcal{N}_r \neq \mathcal{N}_s$ for all $r \neq s$ and $1 \leq r, s \leq 4$. But generally it does not happen, as in the next example.

Example 1.1.25. Let $I = (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_2^3\rho_4^3\,,\,\rho_2\rho_3\rho_4^3\,,\,\rho_3^3\rho_4^3\,)$, then

$$\begin{aligned}
I &= (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_2^3\,,\,\rho_2\rho_3\rho_4^3\,,\,\rho_3^3\rho_4^3\,) \cap (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_4^3\,,\,\rho_2\rho_3\rho_4^3\,,\,\rho_3^3\rho_4^3\,)\\
&= (\,\rho_2^3\,,\,\rho_3^4\,,\,\rho_2\rho_3\rho_4^3\,,\,\rho_3^3\rho_4^3\,) \cap (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_4^3\,)\\
&= (\,\rho_2^3\,,\,\rho_3^4\,,\,z_2\,,\,z_3^3z_4^3\,) \cap (\,\rho_2^3\,,\,\rho_3^4\,,\,\rho_3\rho_4^3\,,\,\rho_3^3\rho_4^3\,) \cap (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_4^3\,)\\
&= (\,\rho_2\,,\,\rho_3^4\,,\,\rho_3^3\rho_4^3\,) \cap (\,\rho_2^3\,,\,\rho_3^4\,,\,\rho_3z_4^3\,) \cap (\,\rho_2^4\,,\,\rho_3^3\,,\,\rho_4^3\,)\\
&= (\,\rho_2\,,\,\rho_3^4\,,\,\rho_3^3\,) \cap (\,\rho_2\,,\,\rho_3^4\,,\,\rho_4^3\,) \cap (\,\rho_2^3\,,\,\rho_3^4\,,\,\rho_3\,) \cap (\,\rho_2^3\,,\,\rho_3^4\,,\,\rho_4^3\,) \cap (\,\rho_2^4\,,\,\rho_3^3\,,\,\rho_4^3\,)\\
&= (\,y_2\,,\,z_3^3\,) \cap (\,\rho_2\,,\,\rho_3^4\,,\,\rho_4^3\,) \cap (\,\rho_2^3\,,\,\rho_3\,) \cap (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_4^3\,)\\
&= (\,\rho_2\,,\,\rho_3^3\,) \cap (\,\rho_2^3\,,\,\rho_3\,) \cap (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_4^3\,).
\end{aligned}$$

It is the primary decompostion of I but not irredundant. Here $\mathrm{Ass}((\,\rho_2\,,\,\rho_3^3\,)) = \mathrm{Ass}((\,\rho_2^3\,,\,\rho_3\,)) = \{(\,\rho_2\,,\,\rho_3\,)\}$. Now for irredundant primary decomposition, take an intersection of $(\,\rho_2\,,\,\rho_3^3\,)$ and $(\,\rho_2^3\,,\,\rho_3\,)$, that is

$$(\,\rho_2\,,\,\rho_3^3\,) \cap (\,\rho_2^3\,,\,\rho_3\,) = (\,\rho_2^3\,,\,\rho_2\rho_3\,,\,\rho_3^3\,).$$

Hence

$$I = (\,\rho_2^4\,,\,\rho_3^4\,,\,\rho_4^3\,) \cap (\,\rho_2^3\,,\,\rho_2\rho_3\,,\,\rho_3^3\,).$$

Example 1.1.26. Let $U = (\,\mu_1\mu_3\,,\,\mu_2\mu_4\,,\,\mu_1\mu_3\mu_4\,)$ be an ideal of S, then

$$\begin{aligned}
U &= (\,\mu_1\mu_3\,,\,\mu_2\mu_4\,,\,\mu_1\mu_3\mu_4\,)\\
&= (\mu_1, \mu_2\mu_4, \mu_1\mu_3\mu_4) \cap (\mu_3, \mu_2\mu_4, \mu_1\mu_3\mu_4)\\
&= (\mu_1, \mu_2\mu_4) \cap (\mu_3, \mu_2\mu_4)\\
&= (\mu_1, \mu_2) \cap (\mu_1, \mu_4) \cap (\mu_3, \mu_2) \cap (\mu_3, \mu_4).
\end{aligned}$$

Since U is square free monomial ideal, so it can be seen that (μ_1, μ_2), (μ_1, μ_4), (μ_3, μ_2) and (μ_3, μ_4) are minimal prime ideals of U.

1.2 Module theory

Definition 1.2.1. For a commutative ring $\mathcal{T}$, a $\mathcal{T}$-module Γ is abelian group under addition, along with a scalar multiplication map $\cdot : \mathcal{T} \times \Gamma \to \Gamma$, defined as $\cdot((t, \beta)) = t\beta$, which satisfies the below axioms:

1. $t(\beta_1 + \beta_2) = t\beta_1 + t\beta_2$,

2. $(t_1 + t_2)\beta = t_1\beta + t_2\beta$,

3. $(t_1 t_2)\beta = t_1(t_2\beta)$,

4. $1\beta = \beta$,

$$\forall\ t_1, t_2, t \in \mathcal{T} \textit{ and } \beta_1, \beta_2 \in \mathit{\Gamma}.$$

Examples 1.2.2. 1. Consider a commutative group E, let $\vartheta \in \mathbb{Z}$ and $b \in B$, then define $\cdot : \mathbb{Z} \times B \to B$, such that

$$\cdot(\vartheta, b) = \vartheta b = \begin{cases} (-b) + \cdots + (-b)\,, & \text{if } \ \vartheta < 0; \\ b + b + \cdots + b\,, & \text{if } \ \vartheta > 0; \\ 0\,, & \text{if } \ \vartheta = 0. \end{cases}$$

Then B is a $\mathbb{Z}$-module.

2. The ideals of the ring are also $\mathcal{T}$-modules.

Definition 1.2.1. Consider a $\mathcal{T}$-module $\mathit{\Gamma}$. A subset $\mathcal{S}$ of $\mathit{\Gamma}$ that is non-empty is called a submodule of $\mathit{\Gamma}$, if $\mathcal{S}$ is a subgroup of the additive group $\mathit{\Gamma}$ as well as it satisfies the module axioms using the scalar multiplication on $\mathit{\Gamma}$.

1.2.1 Module homomorphism and quotient module

Definition 1.2.3. For a ring $\mathcal{T}$, let us suppose two $\mathcal{T}$-modules, $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$. A function $\psi : \mathit{\Gamma}_1 \to \mathit{\Gamma}_2$ is known as $\mathcal{T}$-module homomorphism if

- $\psi(\beta_1 + \beta_2) = \psi(\beta_1) + \psi(\beta_2)$, for all $\beta_1, \beta_2 \in \mathit{\Gamma}_1$.

- $\psi(\sigma\beta) = \sigma\psi(\beta)$, for all $\sigma \in \mathcal{T}$, $\beta \in \mathit{\Gamma}_1$.

If φ is both iniective and surjective then it becomes a $\mathcal{T}$-module isomorphism.

Examples 1.2.4. 1. For a ring $\mathcal{T}$, consider $\mathcal{T}$-module $\mathcal{T}$. Then $\mathcal{T}$-module homomorphism (even from $\mathcal{T}$ into itself) need not be a ring homomorphism. Consider $\mathcal{T} = \mathbb{Z}$, then $\mathbb{Z}$-module homomorphism $z \mapsto 2z$ is not a ring homomorphism.

2. When $\mathcal{T} = K[w]$, the ring homomorphism $\psi : g(w) \mapsto g(w^2)$ is not an $K[w]$-module homomorphism.

Definition 1.2.5. For a ring $\mathcal{T}$, let us suppose a submodule $\mathcal{S}$ of $\mathcal{T}$-module $\varGamma$. Then (additive abelian) quotient group $\varGamma/\mathcal{S}$ becomes an $\mathcal{T}$-module by using scalar multiplication defined as

$$t(\beta + \mathcal{S}) = t\beta + \mathcal{S},$$

$\forall\ t \in \mathcal{T}$, $\beta + \mathcal{S} \in \varGamma/\mathcal{S}$.

1.2.2 Generation of modules

Consider any subset N of $\mathcal{T}$-module $\varGamma$, suppose

$$\mathcal{T}N = \{t_1c_1 + \cdots + t_mc_m \,:\, t_1, \ldots, t_m \in \mathcal{T}\,,\, c_1, \ldots, c_m \in N \text{ and } m \in \mathbb{Z}^+\}.$$

If N is a finite set $\{c_1, \ldots, c_m\}$, then $\mathcal{T}N = \mathcal{T}c_1 + \mathcal{T}c_2 + \cdots + \mathcal{T}c_m$. And $\mathcal{T}N$ is called the submodule of $\varGamma$ generated by N. For any submodule $\mathcal{S}$ of $\mathcal{T}$-module $\varGamma$, if there exist an element $\beta \in \varGamma$ such that $\mathcal{S} = \mathcal{T}\beta = \{t\beta : t \in \mathcal{T}\}$ that is $\mathcal{S}$ is generated by single element. Then $\mathcal{S}$ is called cyclic submodule.

Definition 1.2.6. Let F be a $\mathcal{T}$-module then it is called free on the subset N of F if for $0 \neq f \in F$, there are unique non-zero elements $t_1, \ldots, t_i$ of $\mathcal{T}$ and unique $c_1, \ldots, c_i$ in N, such that

$$f = t_1c_1 + \cdots + t_ic_i.$$

1.2.3 Noetherian rings and Noetherian modules

Proposition 1.2.7. For any poset P with respect to $\leq$, the following are equivalent.

1. Any increasing sequence $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_p \leq \ldots$ in P is stationary, that is there exist $p \in \mathbb{N}$ for which $\beta_q = \beta_p$, for all $p \leq q$.

2. Any $\emptyset \neq U \subset P$ possesses a maximal element.

If P be the set of submodules of Γ which is ordered w.r.t the relation $\subseteq$ then statement 1 is known as ascending chain condition and statement 2 is known as the maximal condition.

Definition 1.2.8. Consider a commutative ring $\mathcal{T}$, a $\mathcal{T}$-module Γ is known as Noetherian if each ascending chain of $\mathcal{T}$-submodules of Γ is stationary. A ring $\mathcal{T}$ is Noetherian if $\mathcal{T}$ is Noetherian as a $\mathcal{T}$-module.

Theorem 1.2.9. A $\mathcal{T}$-module Γ is Noetherian iff every submodule of Γ is finitely generated.

Definition 1.2.10. Consider a finitely generated $\mathcal{T}$-module Γ where $\mathcal{T}$ is a Noetherian ring, a prime ideal $\mathcal{Q}$ of the ring $\mathcal{T}$ is said to be associated prime ideal of Γ if there exist an elemet $\beta \in \Gamma$ such that $\mathcal{Q} = \mathrm{Ann}(\beta)$, where $\mathrm{Ann}(\beta) = \{t \in \mathcal{T} : t\beta = 0\}$ is an ideal of $\mathcal{T}$. The set of all associated prime ideals of Γ is represented as $\mathrm{Ass}(\Gamma)$.

1.2.4 Exact sequences

Definition 1.2.11. Let $\mathcal{T}$ be a commutative ring, consider a sequence of $\mathcal{T}$-modules and homomorphisms

$$\ldots \longrightarrow \mathcal{A}_{i-1} \xrightarrow{k_i} \mathcal{A}_i \xrightarrow{k_{i+1}} \mathcal{A}_{i+1} \xrightarrow{k_{i+2}} \ldots$$

it is exact at $\mathcal{A}_i$ if $Im(k_i) = ker(k_{i+1})$. A sequence is known to be exact if it is exact at every $\mathcal{A}_i$. Particularly, $0 \longrightarrow \mathcal{A}' \xrightarrow{k} \mathcal{A}$ is exact at $\mathcal{A}'$ if and only if k is one to one and $\mathcal{A} \xrightarrow{k} \mathcal{A}'' \longrightarrow 0$ is exact at $\mathcal{A}''$ if and only if k is onto.

Proposition 1.2.12. The sequence

$$0 \longrightarrow \mathcal{A}' \xrightarrow{j} \mathcal{A} \xrightarrow{k} \mathcal{A}'' \longrightarrow 0$$

is an exact sequence if and only if j is one to one, k is onto and $\mathrm{Im}(j) = \ker(k)$.

Remark 1.2.13. The sequence in Proposition 1.2.12 is called a short exact sequence.

Corollary 1.2.14. Let $\mathcal{S}$ be a submodule of Γ. Then Γ is Noetherian iff $\mathcal{S}$ and $\Gamma/\mathcal{S}$ are Noetherian.

Corollary 1.2.15. If $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_m$ are Noetherian $\mathcal{T}$-modules then $\bigoplus_{j=1}^{m} \Gamma_j$ is also Noetherian.

1.2.5 Graded rings

Consider a commutative semigroup (w.r.t addition) $\mathcal{H}$. An $\mathcal{H}$-graded ring is such type of a ring $\mathcal{R}$ alongside a decomposition

$$\mathcal{R} = \bigoplus_{e \in \mathcal{H}} \mathcal{R}_e \text{ (as a group)},$$

such that $\mathcal{R}_e \mathcal{R}_f \subset \mathcal{R}_{e+f}$ $\forall$ $e, f \in \mathcal{H}$.
Then for $h \in \mathcal{R}$, we can write a unique expression

$$h = \sum_{e \in \mathcal{H}} h_e,$$

where $h_e \in \mathcal{R}_e$ and almost all $h_e = 0$. The element h_e is called the *eth* homogeneous component and if $h = h_e$, then h is homogeneous of degree e. $\mathcal{R}[\rho_1]$ and $\mathcal{R}[\rho_1, \rho_2]$ are $\mathbb{Z}$-graded rings as

- $\mathcal{R}[\rho_1] = \mathcal{R} \oplus \mathcal{R}\rho_1 \oplus \mathcal{R}\rho_1^2 \oplus \mathcal{R}\rho_1^3 \oplus \mathcal{R}\rho_1^4 \oplus \mathcal{R}\rho_1^5 \oplus \cdots$.
- $\mathcal{R}[\rho_1, \rho_2] = \mathcal{R} \oplus (\mathcal{R}\rho_1 + \mathcal{R}\rho_2) \oplus (\mathcal{R}\rho_1^2 + \mathcal{R}\rho_1\rho_2 + \mathcal{R}\rho_2^2) \oplus (\mathcal{R}\rho_1^3 + \mathcal{R}\rho_1^2\rho_2 + \mathcal{R}\rho_1\rho_2^2 + \mathcal{R}\rho_2^3) \oplus \cdots$.

For a $\mathcal{H}$-graded ring $\mathcal{R}$ and $\mathcal{R}$-module $\mathcal{G}$

$$\mathcal{G} = \bigoplus_{e \in \mathcal{H}} \mathcal{G}_e \text{ (as a group)},$$

with $\mathcal{R}_e \mathcal{G}_f \subset \mathcal{G}_{e+f}$ for all $e, f \in \mathcal{H}$, then $\mathcal{G}$ is said to be a $\mathcal{H}$-graded module. An element of $\mathcal{G}_e$ that is non zero is called a homogeneous element of degree u.

For a polynomial ring $\mathcal{S}$ defined over the field K, suppose $\mathbf{d} \in \mathbb{Z}^n$, then $g \in \mathcal{S}$ is said to be homogeneous of degree $\mathbf{d}$ when g has the form $\eta\rho_1^{\mathbf{d}}$, where $\eta \in K$. Also $\mathcal{S}$ is $\mathbb{Z}^n$-graded with graded components:

$$\mathcal{S}_{\mathbf{d}} = \begin{cases} K\rho_1^{\mathbf{d}}, & \text{if } \mathbf{d} \in \mathbb{Z}_+^n; \\ 0\,, & \text{otherwise.} \end{cases}$$

An $\mathcal{S}$-module $\mathcal{M}$ is $\mathbb{Z}^n$-graded if $\mathcal{M} = \bigoplus_{\mathbf{d} \in \mathbb{Z}^n} \mathcal{M}_{\mathbf{d}}$ and $\mathcal{S}_{\mathbf{d_1}} \mathcal{M}_{\mathbf{d_2}} \subset \mathcal{M}_{\mathbf{d_1+d_2}}$ for all $\mathbf{d_1}, \mathbf{d_2} \in \mathbb{Z}^n$.

1.3 Graph theory

The most basic structures in Mathematics are the finite graphs. For this specific aspect, many graph-theoretic problems remained unsolved, before any systematic study of graph theory itself. Leonhard Euler's 1735 Königsberg bridges Problem [21] is the example of this type of problem and the Four-Color Problem that was initially introduced by Francis Guthrie, as a coloring problem of the map of England's countries in 1852. Such notable experiments comprise research on polyhedra cycles by Thomas Kirkman and William Hamilton [15], the circuit laws by Gustav Kirchhoff [25] and research by Arthur Cayley and James [6] that had ties to theoretical chemistry to the structure of molecules specifically. The name of "Graph" was suggested by Sylvester in 1878. Commutative algebraists have been studying the characteristics of finite simple graphs using monomial ideals over the last 10 years. The pioneers in this field include Simis, Fróberg, Vasconcelos and Villarreal. The departure point for these approaches is to generate a monomial ideal by using the edges of a finite simple graph, which is commonly referred to as the edge ideal, and then investigating the qualities of the monomial ideal using graph properties, and vice versa.

In this chapter primary definition and notion of graph theory are given. This chapter provides a full description of many types of graphs, distinct operations of graphs and outcomes that will be used in the forthcoming chapters.

1.3.1 Fundamentals of graph theory

Graph theory deals with the analysis of graphs, that are the mathematical framework used to establish the relationship between the objects. This section introduces the essential concepts of graph theory.

Definition 1.3.1. A graph K is a collection of points and lines that link some subset of them (possibly empty). The points are most commonly known as graph vertices. Similarly, lines connecting graph vertices are frequently referred to as graph edges.

The vertex and edge sets of a graph K are usually represented by $V(K)$ and $E(K)$ respectively.

Definition 1.3.2. An edge having exactly the same end points is known as loop. The edges having precisely the same set of endpoints are said to be multiple edges. A simple graph is one that has no loops and multiple edges. A simple graph with the labelling is shown below.

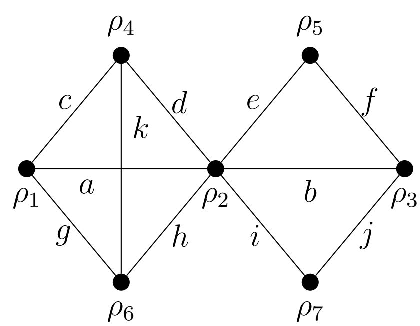

Figure 1.1: Simple Graph

Consider an edge with endpoints ρ_1, ρ_2. Then ρ_1, ρ_2 are said to be adjacent and they are neighbors of each other. The focus is restricted to only basic graphs in several principal applications.

Definition 1.3.3. The number of edges those are incident on vertex w of a graph K is called degree of w, which is generally represented by $d_G(w)$ or $d(w)$.

Definition 1.3.4. The number of vertices in vertex set $V(K)$ is known as order of graph K, written as $n(K)$. And the total number of edges in edge set $E(K)$ indicates the size of graph, represented by $e(K)$.

Definition 1.3.5. Any graph N consisting of the edge and vertex set represented by $E(N)$ and $V(N) = \{w_1, w_2, \ldots, w_n\}$ is said to be a null graph if its edge set is empty.

Figure 1.2: Null graph

Definition 1.3.6. Let $r \geq 2$. An r-star denoted by S_r is a graph on $r + 1$ vertices, having one internal vertex of degree r and all other vertices having degree 1.

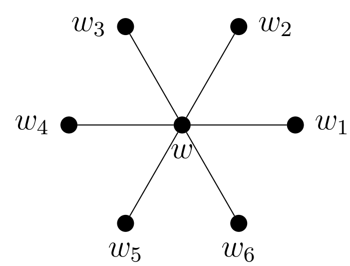

Figure 1.3: A labeled star graph S_6 with 7 vertices and 6 edges.

Definition 1.3.7. A path graph is the sequence of vertices $w_1, w_2, \ldots, w_n$ such as whenever two vertices are consecutive in the sequence, there is an edge between them.

A graph consisting of n vertices ($n \geq 3$) is known as a cycle if we join first and last vertices of path graph by an edge. Deleting one edge from a cycle forms a path. A path and cycle on n vertices are represented by P_n and C_n, respectively.

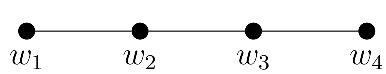

Figure 1.4: P_4

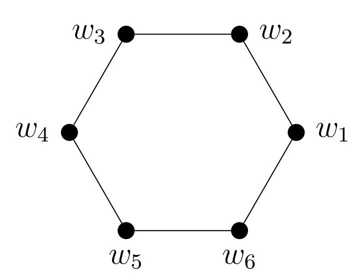

Figure 1.5: C_6

Definition 1.3.8. A graph L is said to be a subgraph of another graph M, written as $L \subseteq M$, if $V(L) \subseteq V(M)$ and $E(L) \subseteq E(M)$ and the endpoints of edges in L are exactly the same as in M.

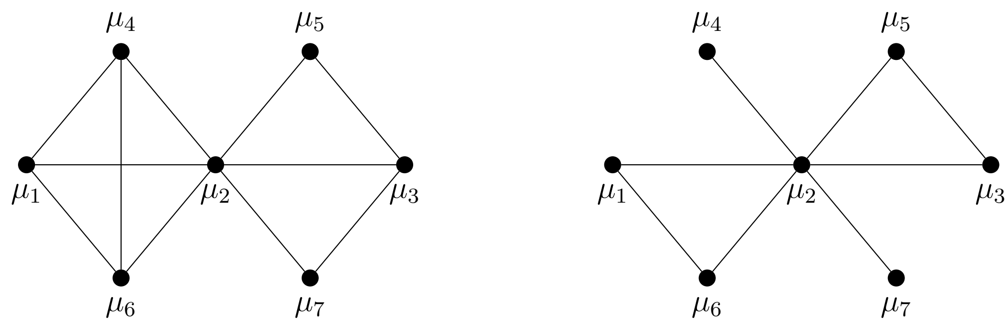

Figure 1.6: Graph and its subgraph

Definition 1.3.9. A graph is considered as a connected graph if it includes a path connecting every two vertices of graph, otherwise the graph is said to be disconnected.

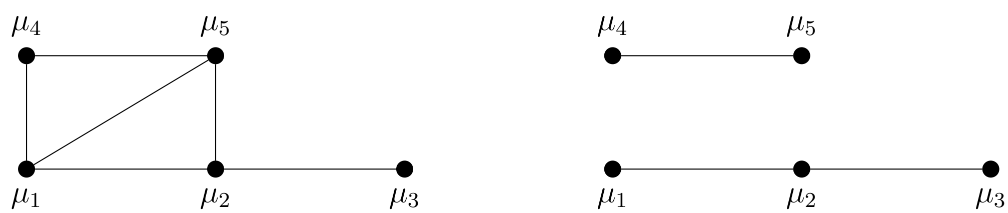

Figure 1.7: Connected and disconnected graph

Proposition 1.3.10. Any graph consisting of w vertices and h edges has at least $w-h$ components.

Proposition 1.3.11 ([26])**.** *(Hand Shaking Lemma)* The sum of the degrees of all the vertices of a graph K is twice the total number of its edges,

$$2E(K) = \sum_{w \in V(K)} deg(w).$$

Definition 1.3.12. Let us have a e, h-path in graph K. The distance from e to h is said to be the minimum length of e, h-path, written as $d(e, h)$. The path with the maximum length in K gives the diameter i.e.,

$$\text{diam}K = \max_{e,h \in V(K)} d(e, h).$$

Definition 1.3.13. A vertex cover of a graph K is a collection of vertices which consists of at least single endpoint of each edge of the graph. The minimal vertex cover of a graph K is one that is not proper subset of any other vertex cover.

Definition 1.3.14. A vertex ρ in a connected graph is a cut vertex whose deletion together with incident edges disconnects the graph. In fig 1.1 the vertex ρ_2 is the cut vertex.

Definition 1.3.15. If there is no cut vertex in a maximal connected subgraph of K, then it is called a block. K is itself a block if it is connected and has no cut vertex.

Definition 1.3.16 ([4])**.** Let K be a connected graph. The block cut vertex graph denoted by $bc(K)$, is a graph in which vertices are the cut vertices and blocks of K. The edges of $bc(K)$ connect cut vertices with those blocks to which they belong.

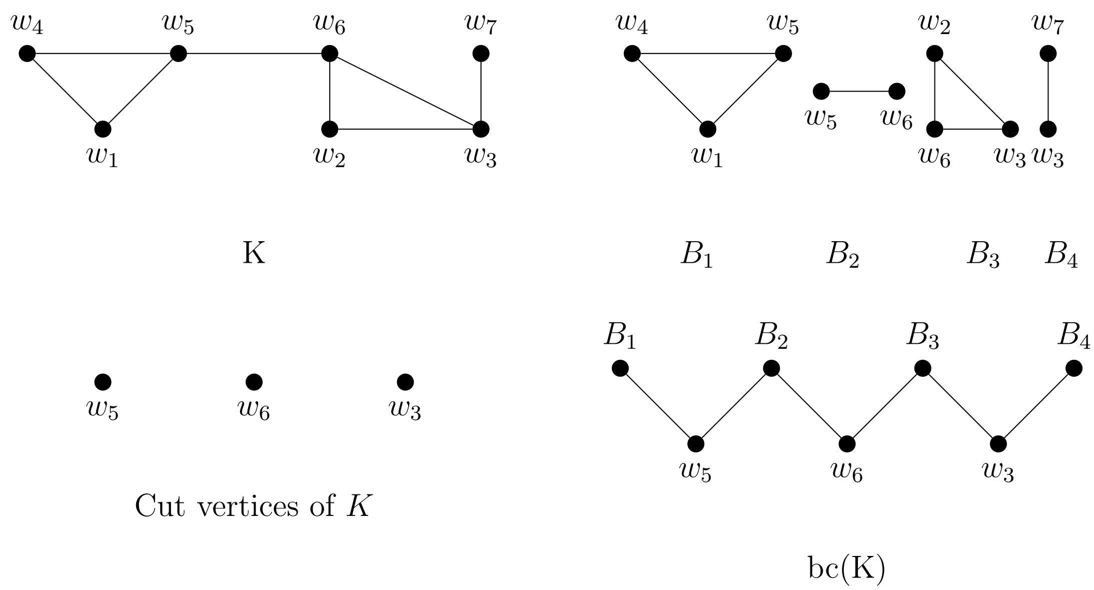

Figure 1.8: A graph K, its blocks, cut vertices and $bc(K)$

Definition 1.3.17 ([20])**.** A connected graph Λ_n, consisting of n blocks is said to be a triangular snake if every block is a triangle and the block cut vertex graph is the path.

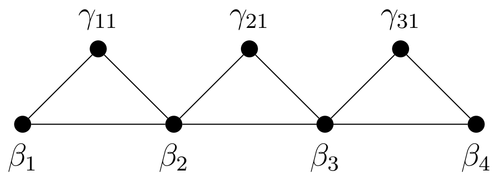

Figure 1.9: Triangular Snake Λ_3

Definition 1.3.18. *(Fusion /Merged/Identified)* The vertices w_1 and w_2 in a graph K is said to be merged, if these two vertices are replaced by one new vertex w such that every edge that was adjacent to either w_1 or w_2 or both, is adjacent to w.

Definition 1.3.19 ([22])**.** Let $n \geq 1$ and $p \geq 1$, A p-triangular snake graph denoted by $\Lambda_{n,p}$ is a triangular snake consisting of n blocks such that each block includes p number of triangles having single same edge.

If we merge vertices x_1 and x_{n+1} in the $\Lambda_{n,p}$ graph, we get a new graph denoted $\mho_{n,p}$, which is called a p-triangular ouraboros snake. In particular, if $p = 1$, then we call $\mho_{n,1}$ a triangular ouroboros snake, and if $p \geq 2$, then we call $\mho_{n,p}$ a multi triangular ouroboros snake.

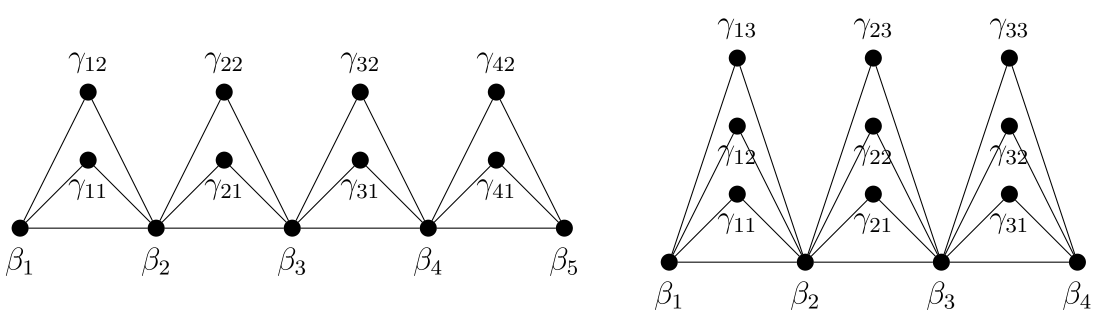

Figure 1.10: $\Lambda_{4,2}$ and $\Lambda_{3,3}$

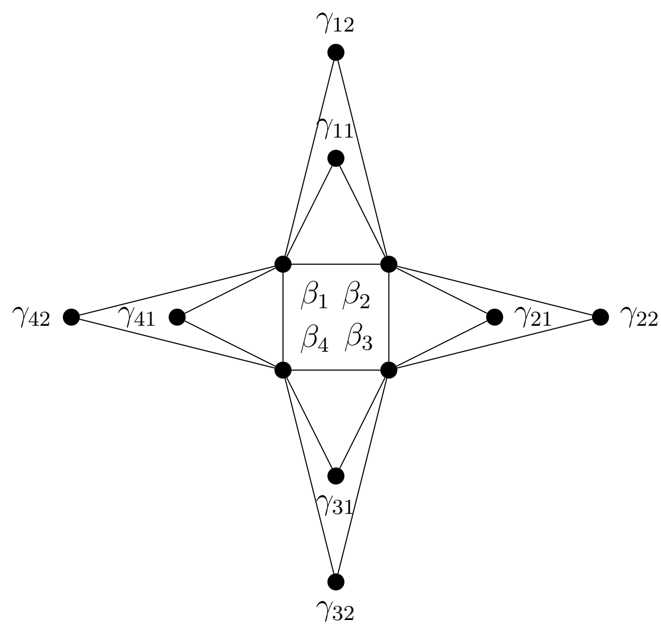

Figure 1.11: 2-triangular ouroboros snake $\mho_{4,2}$

1.3.2 Graph operations

Definition 1.3.20. Let $Y_1 = (V, E)$ and $Y_2 = (\bar{V}, \bar{E})$ are two simple graphs. The union of these two graphs is a simple graph having edge set $E \cup \bar{E}$ and vertex set $V \cup \bar{V}$. The union of Y_1 and Y_2 is denoted by $Y_1 \cup Y_2$.

Definition 1.3.21 ([12])**.** Corona product of two (same or different) graphs Y_1 and Y_2 denoted by $Y_1 \odot Y_2$ is produced by picking one copy of Y_1 and $\mid V(Y_1) \mid$ copies of Y_2 and connecting the jth vertex of Y_1 to each vertex of the jth copy of Y_2, where $1 \leq j \leq \mid V(Y_1) \mid$.

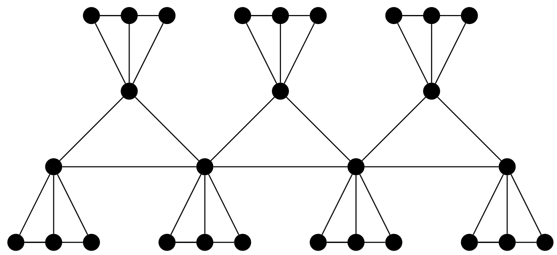

Figure 1.12: Corona product of Λ_4 and P_3 ($\Lambda_4 \odot P_3$)

Definition 1.3.22 ([16])**.** The q-fold bristled graph of a given graph K, denoted by $Brs_q(K)$ is obtained by connecting q vertices of degree 1 to every vertex of K. This graph can also be obtained by taking corona product of K with empty graph consisting of q vertices. The q-fold bristled graph of a given graph K is also called its q-thorny graph.

Here are some examples of q-fold bristled graphs:

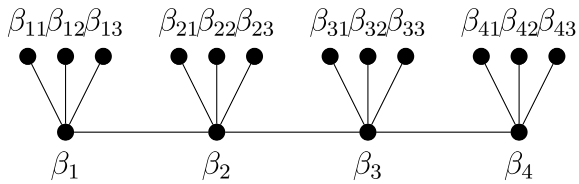

Figure 1.13: $Brs_3(P_4)$
(3-fold bristled graph of P_4)

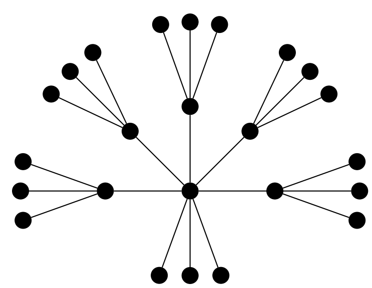

Figure 1.14: $Brs_3(S_5)$
(3-fold bristled graph of S_5)

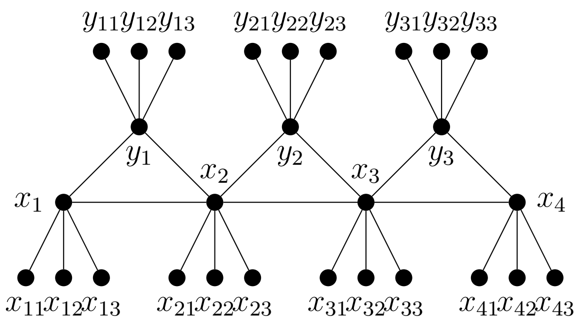

Figure 1.15: $Brs_3(\Lambda_4)$
(3-fold bristled graph of Λ_4)

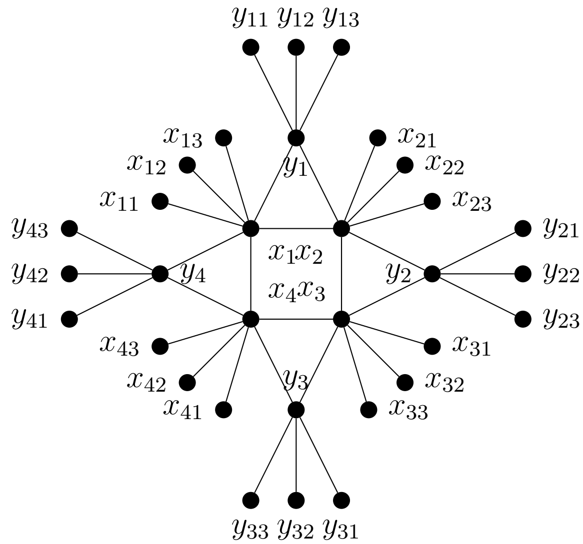

Figure 1.16: $Brs_3(\mho_4)$
(3-fold bristled graph of $\mho_4$)

Chapter 2

Depth and Stanley depth

This chapter concerns the Stanley depth and depth (named after Richard Stanley [23] in 1982) of $\mathbb{Z}^n$-graded modules over a commutative ring, including the Stanley's conjecture. It summarises the known values and bounds of depth and Stanley depth for monomial ideals of polynomial rings and their quotients. Throughout this chapter, ring $\mathcal{T}$ has identity $1 \neq 0$.

2.1 Depth

Definition 2.1.1. Consider a $\mathcal{T}$ module Γ. A zero divisor of a module Γ is an element $0 \neq t \in \mathcal{T}$ such that $t\beta = 0$, where $0 \neq \beta \in \Gamma$.

Definition 2.1.2. Suppose Γ be a $\mathcal{T}$-module. A non-zero element t of $\mathcal{T}$ is Γ-regular if for every $\beta \in \Gamma$, $t\beta = 0$ implies $\beta = 0$.

Definition 2.1.3. A sequence $\beta = \beta_1, \ldots, \beta_n$ of elements of $\mathcal{T}$ is said to be Γ-regular if it satisfies the given axioms:

1. β_m is $\Gamma/(\beta_1, \ldots, \beta_{m-1})\Gamma$ regular for any m;

2. $\Gamma \neq (\beta)\Gamma$.

Example 2.1.4. Consider $R = K[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ as a module over itself. As σ_1 is regular in $R/(0)R$, σ_2 is regular in $R/(\sigma_1)R$, σ_3 is regular in $R/(\sigma_1, \sigma_2)R$, σ_4 is regular in $R/(\sigma_1, \sigma_2, \sigma_3)R$. $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ is the Γ-regular sequence in R.

Definition 2.1.5. Consider a finitely generated $\mathcal{T}$-module Γ and suppose $\mathcal{M}$ be a unique maximal ideal of local Noetherian ring $\mathcal{T}$. Then, depth of Γ is common length of all maximal Γ-sequences in $\mathcal{M}$, represented by depth(Γ).

Lemma 2.1.6 ([13])**.** *(Depth Lemma)* Given a short exact sequence $0 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 0$ of $\mathcal{T}$-modules where $\mathcal{T}$ is a local ring, then

1. $\operatorname{depth}(\Gamma_2) \geq \min\{\operatorname{depth}(\Gamma_3), \operatorname{depth}(\Gamma_1)\}$.
2. $\operatorname{depth}(\Gamma_3) \geq \min\{\operatorname{depth}(\Gamma_2), \operatorname{depth}(\Gamma_1) + 1\}$.
3. $\operatorname{depth}(\Gamma_1) \geq \min\{\operatorname{depth}(\Gamma_3) - 1, \operatorname{depth}(\Gamma_2)\}$.

Lemma 2.1.7 ([18, Lemma 2.2])**.** Given a short exact sequence of $\mathbb{Z}^n$-graded $\mathcal{T}$-modules

$$0 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 0.$$

Then

$$\operatorname{sdepth}(\Gamma_2) \geq \min\{\operatorname{sdepth}(\Gamma_1), \operatorname{sdepth}(\Gamma_3)\}.$$

2.2 Stanley decomposition and Stanley depth

Definition 2.2.1. Let $\mathcal{T} = Z[\beta_1, \ldots, \beta_n]$ be a ring of polynomials and consider $\mathbb{Z}^n$-graded $\mathcal{T}$-module Γ. Suppose $\beta \in \Gamma$ and also consider $W \subset \{\beta_1, \ldots, \beta_n\}$, then $\beta Z[W]$ represents the Z-subspace of Γ, whose generating set comprises of elements (homogeneous in degree) of the form βw, where w is a monomial in $Z[W]$. If $\beta Z[W]$ be a free $Z[W]$-module so it is known as a Stanley space having dimension $|W|$. The Stanley decomposition of Γ is defined as:

$$\mathcal{D} \, : \, \Gamma = \bigoplus_{j=1}^{k} \beta_j Z[W_j],$$

and

$$\operatorname{sdepth} \mathcal{D} = \min\{\, |W_j| \, , \, j = 1, \ldots, k\}.$$

Also,

$$\operatorname{sdepth}_S(\Gamma) = \max\{\, \operatorname{sdepth} \mathcal{D} \, : \, \mathcal{D} \text{ is a Stanley decomposition of } \Gamma\}.$$

2.2.1 Stanley's conjecture

In 1982, Stanley [23] gave a conjecture about an upper bound for the depth of a $\mathbb{Z}^n$-graded S-modules.

$$\operatorname{depth}(\Gamma) \leq \operatorname{sdepth}(\Gamma).$$

It has been extremely significant as it gave a comparison of two very different invariants of modules. For a ring of polynomials $\mathcal{T}$ in n number of variables, consider $J \subset \mathcal{T}$ be the monomial ideal, then for $n \leq 3$, $n = 4$ and $n = 5$ the conjecture for $\mathcal{T}/J$ is proved by Apel [3], Anwar [2] and Popescu [17], respectively. Also, when J is an intersection of three monomial prime ideals, or three monomial primary ideals or four monomial prime ideals of $\mathcal{T}$, the conjecture holds for J. But in 2016, Duval et al. [11] proved that Stanley's coniecture is generally false, by giving a counter example for the module of type $\mathcal{T}/J$ for which the conjecture does not hold.

2.2.2 Method of computing Stanley depth for squarefree monomial ideals

In 2009, Herzog et al. [13] presented a method of computing the lower bound for Stanley depth of monomial ideals in finite number of steps by using posets. Suppose F be a squarefree monomial ideal with $G(F) = \{f_1, \ldots, f_m\}$. The characteristic poset of F w.r.t $h = (1, \ldots, 1)$, written as $\mathcal{O}_F^{(1,\ldots,1)}$ is defined as

$$\mathcal{O}_F^{(1,\ldots,1)} = \{\beta \subset [n] \mid \beta \text{ contains } \operatorname{supp}(f_j) \text{ for some } j\},$$

where $\operatorname{supp}(f_j) = \{i : x_i | f_j\} \subseteq [n] := \{1, \ldots, n\}$. For each $\rho, \sigma \in \mathcal{O}_F^{(1,\ldots,1)}$ where $\rho \subseteq \sigma$, and

$$[\rho\,,\, \sigma] = \{\beta \in \mathcal{O}_F^{(1,\ldots,1)} : \rho \subseteq \beta \subseteq \sigma\}.$$

Let $\mathcal{O} : \mathcal{O}_F^{(1,\ldots,1)} = \cup_{j=1}^{k} [\beta_j\,,\, \eta_j]$ be a partition of $\mathcal{O}_F^{(1,\ldots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\operatorname{supp}(x^{s(j)}) = \beta_j$, then the Stanley decomposition $\mathcal{D}(\mathcal{O})$ of F is given as

$$\mathcal{D}(\mathcal{O}) : F = \bigoplus_{j=1}^{r} x^{s(j)} K[\{x_k \mid k \in \eta_j\}].$$

Clearly, $\operatorname{sdepth} \mathcal{D}(\mathcal{O}) = \min\{|\eta_1|, \ldots, |\eta_r|\}$ and

$$\operatorname{sdepth}(F) = \max\{\operatorname{sdepth} \mathcal{D}(\mathcal{O}) \mid \mathcal{O} \textit{ is a partition of } \mathcal{O}_F^{(1,\ldots,1)}\}.$$

Example 2.2.2. Consider a square-free monomial ideal $I = (\sigma_1\sigma_2, \sigma_1\sigma_4, \sigma_2\sigma_3, \sigma_2\sigma_4) \subset K[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ and $J = 0$. Set $\vartheta_1 = (1,1,0,0)$, $\vartheta_2 = (1,0,0,1)$, $\vartheta_3 = (0,1,1,0)$ and $\vartheta_4 = (0,1,0,1)$. Thus I is generated by $\sigma^{\vartheta_1}, \sigma^{\vartheta_2}, \sigma^{\vartheta_3}, \sigma^{\vartheta_4}$ and choose $h = (1,1,1,1)$. The poset $P = P^h_{I/J}$ is given by

$$P = \{(1,1,0,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1), (1,1,1,1)\}.$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1 : \ & [(1,1,0,0),(1,1,0,0)] \bigcup [(1,0,0,1),(1,0,0,1)] \bigcup [(0,1,1,0),(0,1,1,0)] \bigcup \\ & [(0,1,0,1),(0,1,0,1)] \bigcup [(1,1,1,0),(1,1,1,0)] \bigcup [(1,1,0,1),(1,1,0,1)] \bigcup \\ & [(1,0,1,1),(1,0,1,1)] \bigcup [(0,1,1,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2 : \ & [(1,1,0,0),(1,1,0,1)] \bigcup [(1,0,0,1),(1,0,1,1)] \bigcup [(0,1,1,0),(1,1,1,0)] \bigcup \\ & [(0,1,0,1),(1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_3 : \ & [(1,1,0,0),(1,1,1,0)] \bigcup [(1,0,0,1),(1,1,0,1)] \bigcup [(0,1,1,0),(0,1,1,1)] \bigcup \\ & [(1,0,1,1),(1,1,1,1)] \bigcup [(0,1,0,1),(0,1,0,1)].\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) \ := \ & \sigma_1\sigma_2K[\sigma_1,\sigma_2] \oplus \sigma_1\sigma_4K[\sigma_1,\sigma_4] \oplus \sigma_2\sigma_3K[\sigma_2,\sigma_3] \oplus \sigma_2\sigma_4K[\sigma_2,\sigma_4] \oplus \\ & \sigma_1\sigma_2\sigma_3K[\sigma_1,\sigma_2,\sigma_3] \oplus \sigma_1\sigma_2\sigma_4K[\sigma_1,\sigma_2,\sigma_4] \oplus \sigma_1\sigma_3\sigma_4K[\sigma_1,\sigma_3,\sigma_4] \oplus \\ & \sigma_2\sigma_3\sigma_4K[\sigma_2,\sigma_3,\sigma_4] \oplus \sigma_1\sigma_2\sigma_3\sigma_4K[\sigma_1,\sigma_2,\sigma_3,\sigma_4].\end{aligned}$$

$$\mathcal{D}(\mathcal{P}_2) \ := \ \sigma_1\sigma_2K[\sigma_1,\sigma_2,\sigma_4] \oplus \sigma_1\sigma_4K[\sigma_1,\sigma_3,\sigma_4] \oplus \sigma_2\sigma_3K[\sigma_1,\sigma_2,\sigma_3] \oplus \sigma_2\sigma_4K[\sigma_1,\sigma_2,\sigma_3,\sigma_4].$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_3) \ := \ & \sigma_1\sigma_2K[\sigma_1,\sigma_2,\sigma_3] \oplus \sigma_1\sigma_4K[\sigma_1,\sigma_2,\sigma_4] \oplus \sigma_2\sigma_3K[\sigma_2,\sigma_3,\sigma_4] \oplus \\ & \sigma_1\sigma_3\sigma_4K[\sigma_1,\sigma_2,\sigma_3,\sigma_4] \oplus \sigma_2\sigma_4K[\sigma_2,\sigma_4].\end{aligned}$$

Then

$$\begin{aligned}
\text{sdepth}(I) &\geq \max\{\text{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\, \text{sdepth}(\mathcal{D}(\mathcal{P}_2))\,,\, \text{sdepth}(\mathcal{D}(\mathcal{P}_3))\} \\
&= \max\{2, 3, 2\} \\
&= 3.
\end{aligned}$$

Since I is not principal, so $\text{sdepth}(I) = 3$.

Example 2.2.3. Consider $I = (\sigma_1\sigma_5, \sigma_2\sigma_4, \sigma_1\sigma_3\sigma_4) \subset K[\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5]$ and $J = 0$. Set $\vartheta_1 = (1, 0, 0, 0, 1)$, $\vartheta_2 = (0, 1, 0, 1, 0)$ and $\vartheta_3 = (1, 0, 1, 1, 0)$. Thus I is generated by $\sigma^{\vartheta_1}, \sigma^{\vartheta_2}, \sigma^{\vartheta_3}$ and choose $h = (1, 1, 1, 1, 1)$. The poset $P = P^h_{I/J}$ is given by

$$\begin{aligned}
P = \{&(1,0,0,0,1),(0,1,0,1,0),(1,1,0,0,1),(1,0,1,0,1),(1,0,0,1,1),(1,1,0,1,0), \\
&(0,1,1,1,0),(0,1,0,1,1),(1,0,1,1,0),(1,1,1,1,0),(1,1,0,1,1),(1,1,1,0,1), \\
&(1,0,1,1,1),(0,1,1,1,1),(1,1,1,1,1)\}.
\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}
\mathcal{P}_1 :\; &[(1,0,0,0,1),(1,0,0,0,1)] \bigcup [(0,1,0,1,0),(0,1,0,1,0)] \bigcup \\
&[(1,1,0,0,1),(1,1,0,0,1)] \bigcup [(1,0,1,0,1),(1,0,1,0,1)] \bigcup \\
&[(1,0,0,1,1),(1,0,0,1,1)] \bigcup [(1,1,0,1,0),(1,1,0,1,0)] \bigcup \\
&[(0,1,1,1,0),(0,1,1,1,0)] \bigcup [(0,1,0,1,1),(0,1,0,1,1)] \bigcup \\
&[(1,0,1,1,0),(1,0,1,1,0)] \bigcup [(1,1,1,1,0),(1,1,1,1,0)] \bigcup \\
&[(1,1,0,1,1),(1,1,0,1,1)] \bigcup [(1,1,1,0,1),(1,1,1,0,1)] \bigcup \\
&[(1,0,1,1,1),(1,0,1,1,1)] \bigcup [(0,1,1,1,1),(0,1,1,1,1)] \bigcup \\
&[(1,1,1,1,1),(1,1,1,1,1)].
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_2 :\; &[(1,0,0,0,1),(1,1,0,0,1)] \bigcup [(0,1,0,1,0),(1,1,0,1,1)] \bigcup \\
&[(1,0,1,0,1),(1,1,1,0,1)] \bigcup [(1,0,0,1,1),(1,0,1,1,1)] \bigcup \\
&[(1,0,1,1,0),(1,1,1,1,0)] \bigcup [(0,1,1,1,0),(1,1,1,1,1)].
\end{aligned}$$

$$\mathcal{P}_3: \ [(1,0,0,0,1),(1,1,1,0,1)] \bigcup [(0,1,0,1,0),(1,1,0,1,1)] \bigcup$$
$$[(1,0,0,1,1),(1,0,1,1,1)] \bigcup [(0,1,1,1,0),(0,1,1,1,1)] \bigcup$$
$$[(1,0,1,1,0),(1,1,1,1,1)].$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) \ := \ & \sigma_1\sigma_5K[\sigma_1,\sigma_5] \oplus \sigma_2\sigma_4K[\sigma_2,\sigma_4] \oplus \sigma_1\sigma_2\sigma_5K[\sigma_1,\sigma_2,\sigma_5] \oplus \sigma_1\sigma_3\sigma_5K[\sigma_1,\sigma_3,\sigma_5] \oplus \\ & \sigma_1\sigma_4\sigma_5K[\sigma_1,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_2\sigma_4K[\sigma_1\sigma_2\sigma_4] \oplus \sigma_2\sigma_3\sigma_4K[\sigma_2,\sigma_3,\sigma_4] \oplus \\ & \sigma_2\sigma_4\sigma_5K[\sigma_2,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_3\sigma_4K[\sigma_1,\sigma_3,\sigma_4] \oplus \sigma_1\sigma_2\sigma_3\sigma_4K[\sigma_1,\sigma_2,\sigma_3,\sigma_4] \oplus \\ & \sigma_1\sigma_2\sigma_4\sigma_5K[\sigma_1,\sigma_2,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_2\sigma_3\sigma_5K[\sigma_1,\sigma_2,\sigma_3,\sigma_5] \oplus \sigma_1\sigma_3\sigma_4\sigma_5K[\sigma_1,\sigma_3,\sigma_4,\sigma_5] \oplus \\ & \sigma_2\sigma_3\sigma_4\sigma_5K[\sigma_2,\sigma_3,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5K[\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) \ := \ & \sigma_1\sigma_5K[\sigma_1,\sigma_2,\sigma_5] \oplus \sigma_2\sigma_4K[\sigma_1,\sigma_2,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_3\sigma_5K[\sigma_1,\sigma_2,\sigma_3,\sigma_5] \oplus \\ & \sigma_1\sigma_4\sigma_5K[\sigma_1,\sigma_3,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_3\sigma_4K[\sigma_1,\sigma_2,\sigma_3,\sigma_4] \oplus \sigma_2\sigma_3\sigma_4K[\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_3) \ := \ & \sigma_1\sigma_5K[\sigma_1,\sigma_2,\sigma_3,\sigma_5] \oplus \sigma_2\sigma_4K[\sigma_1,\sigma_2,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_4\sigma_5K[\sigma_1,\sigma_3,\sigma_4,\sigma_5] \oplus \\ & \sigma_2\sigma_3\sigma_4K[\sigma_2,\sigma_3,\sigma_4,\sigma_5] \oplus \sigma_1\sigma_3\sigma_4K[\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(I) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\,,\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_3))\} \\ &= \max\{2,3,4\} \\ &= 4.\end{aligned}$$

Since I is not principal, so $\operatorname{sdepth}(I) = 4$. The next example illustrates the method of finding the Stanley depth of S/I.

Example 2.2.4. For $S = K[\beta_1,\beta_2,\beta_3,\beta_4,\beta_5]$, consider $I = (\beta_1\beta_3, \beta_2\beta_4, \beta_3\beta_4, \beta_2\beta_3\beta_5)$. Then choose $h = (1,1,1,1,1)$ and the poset $P = P^h_{S/I}$ is given by

$$\begin{aligned}P = \{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1), \\ &(1,1,0,0,0),(1,0,0,1,0),(1,0,0,0,1),(0,1,1,0,0),(0,1,0,0,1),(0,0,1,0,1), \\ &(0,0,0,1,1),(1,1,0,0,1),(1,0,0,1,1)\}.\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1 : \ & [(0,0,0,0,0),(1,0,0,0,0)] \bigcup [(0,1,0,0,0),(0,1,0,0,1)] \bigcup \\ & [(0,0,1,0,0),(0,1,1,0,0)] \bigcup [(0,0,0,1,0),(1,0,0,1,0)] \bigcup \\ & [(0,0,0,0,1),(0,0,1,0,1)] \bigcup [(1,0,0,0,1),(1,0,0,1,1)] \bigcup \\ & [(1,1,0,0,0),(1,1,0,0,1)] \bigcup [(0,0,0,1,1),(0,0,0,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2 : \ & [(0,0,0,0,0),(1,0,0,1,1)] \bigcup [(0,1,0,0,0),(1,1,0,0,1)] \bigcup \\ & [(0,0,1,0,0),(0,0,1,0,1)] \bigcup [(0,1,1,0,0),(0,1,1,0,0)].\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) \ &:= \ K[\beta_1] \oplus \beta_2 K[\beta_2,\beta_5] \oplus \beta_3 K[\beta_2,\beta_3] \oplus \beta_4 K[\beta_1,\beta_4] \oplus \beta_5 K[\beta_3,\beta_5] \oplus \\ & \quad \beta_1\beta_5 K[\beta_1,\beta_4,\beta_5] \oplus \beta_1\beta_2 K[\beta_1,\beta_2,\beta_5] \oplus \beta_4\beta_5 K[\beta_4,\beta_5]. \\ \mathcal{D}(\mathcal{P}_2) \ &:= \ K[\beta_1,\beta_4,\beta_5] \oplus \beta_2 K[\beta_1,\beta_2,\beta_5] \oplus \beta_3 K[\beta_3,\beta_5] \oplus \beta_2\beta_3 K[\beta_2,\beta_3].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(S/I) \ &\geq \ \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\, \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\} \\ &= \ \max\{1,2\} \\ &= \ 2.\end{aligned}$$

Example 2.2.5. Let $S = M[\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6]$, consider $U = (\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_2\sigma_6, \sigma_1\sigma_3\sigma_6)$. Then select $g = (1,1,1,1,1,1)$ and the poset $\rho = \rho^g_{S/U}$ is given by

$$\begin{aligned}\mathcal{P} = \{ & (0,0,0,0,0,0),(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),(0,0,0,1,0,0), \\ & (0,0,0,0,1,0),(0,0,0,0,0,1),(1,0,1,0,0,0),(1,0,0,1,0,0),(1,0,0,0,1,0), \\ & (1,0,0,0,0,1),(0,1,0,1,0,0),(0,1,0,0,1,0),(0,0,1,1,0,0),(0,0,1,0,1,0), \\ & (0,0,1,0,0,1),(0,0,0,1,1,0),(0,0,0,1,0,1).(0,0,0,0,1,1),(1,0,1,1,0,0), \\ & (1,0,1,0,1,0),(1,0,0,1,1,0),(1,0,0,1,0,1),(1,0,0,0,1,1),(0,1,0,1,1,0), \\ & (0,0,1,1,1,0),(0,0,1,1,0,1),(0,0,1,0,1,1),(0,0,0,1,1,1),(1,0,1,1,1,0), \\ & (1,0,1,1,0,1),(1,0,1,0,1,1),(1,0,0,1,1,1),(0,1,1,0,1,1),(0,0,1,1,1,1), \\ & (1,0,1,1,1,1)\}.\end{aligned}$$

The partitions of ρ can be written as

$$
\begin{aligned}
\mathcal{P}_1 : \ & [(0,0,0,0,0,0),(1,0,0,0,0,0)] \bigcup [(0,1,0,0,0,0),(0,1,0,0,0,0)] \bigcup \\
& [(0,0,1,0,0,0),(0,0,1,0,0,0)] \bigcup [(0,0,0,1,0,0),(0,0,0,1,0,0)] \bigcup \\
& [(0,0,0,0,1,0),(0,0,0,0,1,0)] \bigcup [(0,0,0,0,0,1),(0,0,0,0,0,1)] \bigcup \\
& [(1,0,1,0,0,0),(1,0,1,0,0,0)] \bigcup [(1,0,0,1,0,0),(1,0,0,1,0,0)] \bigcup \\
& [(1,0,0,0,1,0),(1,0,0,0,1,0)] \bigcup [(1,0,0,0,0,1),(1,0,0,0,0,1)] \bigcup \\
& [(0,1,0,1,0,0),(0,1,0,1,0,0)] \bigcup [(0,1,0,0,1,0),(0,1,0,0,1,0)] \bigcup \\
& [(0,0,1,1,0,0),(0,0,1,1,0,0)] \bigcup [(0,0,1,0,1,0),(0,0,1,0,1,0)] \bigcup \\
& [(0,0,1,0,0,1),(0,0,1,0,0,1)] \bigcup [(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup \\
& [(0,0,0,1,0,1),(0,0,0,1,0,1)] \bigcup [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup \\
& [(1,0,1,1,0,0),(1,0,1,1,0,0)] \bigcup [(1,0,1,0,1,0),(1,0,1,0,1,0)] \bigcup \\
& [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup [(1,0,0,1,0,1),(1,0,0,1,0,1)] \bigcup \\
& [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup [(0,1,0,1,1,0),(0,1,0,1,1,0)] \bigcup \\
& [(0,0,1,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,1,0,1),(0,0,1,1,0,1)] \bigcup \\
& [(0,0,1,0,1,1),(0,0,1,0,1,1)] \bigcup [(0,0,0,1,1,1),(0,0,0,1,1,1)] \bigcup \\
& [(1,0,1,1,1,0),(1,0,1,1,1,0)] \bigcup [(1,0,1,1,0,1),(1,0,1,1,0,1)] \bigcup \\
& [(1,0,1,0,1,1),(1,0,1,0,1,1)] \bigcup [(1,0,0,1,1,1),(1,0,0,1,1,1)] \bigcup \\
& [(0,1,1,0,1,1),(0,1,1,0,1,1)] \bigcup [(0,0,1,1,1,1),(0,0,1,1,1,1)] \bigcup \\
& [(1,0,1,1,1,1),(1,0,1,1,1,1)].
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_2 : \ & [(0,0,0,0,0,0),(0,1,1,0,1,1)] \bigcup [(1,0,0,0,0,0),(1,0,1,1,0,1)] \bigcup \\
& [(0,0,0,1,0,0),(0,0,0,1,1,1)] \bigcup [(1,0,0,0,1,0),(1,0,0,1,1,1)] \bigcup \\
& [(0,1,0,1,0,0),(0,1,0,1,1,0)] \bigcup [(0,0,1,1,0,0),(0,0,1,1,1,1)] \bigcup \\
& [(0,0,1,0,1,0),(1,0,1,1,1,1)].
\end{aligned}
$$

So the corresponding Stanley decomposition is of the partitions will be

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) &:= M[\sigma_1] \oplus \sigma_2 M[\sigma_2] \oplus \sigma_3 M[\sigma_3] \oplus \sigma_4 M[\sigma_4] \oplus \sigma_5 M[\sigma_5] \oplus \sigma_6 M[\sigma_6] \oplus \sigma_1\sigma_3 M[\sigma_1, \sigma_3] \oplus \\ &\quad \sigma_1\sigma_4 M[\sigma_1, \sigma_4] \oplus \sigma_1\sigma_5 M[\sigma_1, \sigma_5] \oplus \sigma_1\sigma_6 M[\sigma_1, \sigma_6] \oplus \sigma_2\sigma_4 M[\sigma_2, \sigma_4] \oplus \\ &\quad \sigma_2\sigma_5 M[\sigma_2, \sigma_5] \oplus \sigma_3\sigma_4 M[\sigma_3, \sigma_4] \oplus \sigma_3\sigma_5 M[\sigma_3, \sigma_5] \oplus \sigma_3\sigma_6 M[\sigma_3, \sigma_6] \oplus \\ &\quad \sigma_4\sigma_5 M[\sigma_4, \sigma_5] \oplus \sigma_5\sigma_6 M[\sigma_5, \sigma_6] \oplus \sigma_1\sigma_3\sigma_4 M[\sigma_1, \sigma_3, \sigma_4] \oplus \sigma_1\sigma_3\sigma_5 M[\sigma_1, \sigma_3, \sigma_5] \oplus \\ &\quad \sigma_1\sigma_4\sigma_5 M[\sigma_1, \sigma_4, \sigma_5] \oplus \sigma_1\sigma_4\sigma_6 M[\sigma_1, \sigma_4, \sigma_6] \oplus \sigma_1\sigma_5\sigma_6 K[\sigma_1, \sigma_5, \sigma_6] \oplus \\ &\quad \sigma_2\sigma_4\sigma_5 M[\sigma_2, \sigma_4, \sigma_5] \oplus \sigma_3\sigma_4\sigma_5 M[\sigma_3, \sigma_4, \sigma_5] \oplus \sigma_3\sigma_4\sigma_6 M[\sigma_3, \sigma_4, \sigma_6] \oplus \\ &\quad \sigma_3\sigma_5\sigma_6 M[\sigma_3, \sigma_5, \sigma_6] \oplus \sigma_4\sigma_5\sigma_6 M[\sigma_4, \sigma_5, \sigma_6] \oplus \sigma_1\sigma_3\sigma_4\sigma_5 M[\sigma_1, \sigma_3, \sigma_4, \sigma_5] \oplus \\ &\quad \sigma_1\sigma_3\sigma_4\sigma_6 M[\sigma_1, \sigma_3, \sigma_4, \sigma_6] \oplus \sigma_1\sigma_3\sigma_5\sigma_6 M[\sigma_1, \sigma_3, \sigma_5, \sigma_6] \oplus \sigma_1\sigma_4\sigma_5\sigma_6 M[\sigma_1, \sigma_4, \sigma_5, \sigma_6] \oplus \\ &\quad \sigma_2\sigma_3\sigma_5\sigma_6 M[\sigma_2, \sigma_3, \sigma_5, \sigma_6] \oplus \sigma_3\sigma_4\sigma_5\sigma_6 M[\sigma_3, \sigma_4, \sigma_5, \sigma_6] \oplus \sigma_1\sigma_3\sigma_4\sigma_5\sigma_6 M[\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6]. \\ \mathcal{D}(\mathcal{P}_2) &:= M[\sigma_2, \sigma_3, \sigma_5, \sigma_6] \oplus \sigma_1 M[\sigma_1, \sigma_3, \sigma_4, \sigma_6] \oplus \sigma_4 M[\sigma_4, \sigma_5, \sigma_6] \oplus \sigma_1\sigma_5 M[\sigma_1, \sigma_4, \sigma_5, \sigma_6] \oplus \\ &\quad \sigma_2\sigma_4 M[\sigma_2, \sigma_4, \sigma_5] \oplus \sigma_3\sigma_4 M[, \sigma_3, \sigma_4, \sigma_5, \sigma_6] \oplus \sigma_3\sigma_5 M[\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_6].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(S/U) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1)),\ \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\} \\ &= \max\{1, 3\} \\ &= 3.\end{aligned}$$

Some fundamental results on Stanley depth and depth of S-modules are given below.

Theorem 2.2.6 ([11, Theorem 1.3])**.** Let $\iota_1, \ldots, \iota_m$ be some positive integers, then

$$\operatorname{sdepth}((w_1^{\iota_1}, \ldots, w_m^{\iota_m})) = \operatorname{sdepth}((w_1, \ldots, w_m)) = \lceil \frac{m}{2} \rceil.$$

In particular, for any $1 \leq n \leq m$

$$\operatorname{sdepth}((w_1^{\iota_1}, \ldots, w_n^{\iota_n})) = m - n + \lceil \frac{n}{2} \rceil.$$

Corollary 2.2.7 ([18, Corollary 1.3])**.** Consider a monomial ideal $\mathcal{L} \subset \mathcal{T}$. Then $\operatorname{depth}(\mathcal{T}/\mathcal{L}) \leq \operatorname{depth}(\mathcal{T}/(\mathcal{L} : l))$ $\forall$ monomials $l \notin \mathcal{L}$.

Proposition 2.2.8 ([8, Proposition 2.7])**.** Let $\mathcal{L} \subset \mathcal{T}$ is a monomial ideal. Then $\operatorname{sdepth}(\mathcal{T}/\mathcal{L}) \leq \operatorname{sdepth}(\mathcal{T}/(\mathcal{L} : l))$ for all monomials $l \notin \mathcal{L}$.

Lemma 2.2.9 ([9, Lemma 2.12])**.** Assume that $\mathcal{L}_1 \subset \Gamma' = K[x_1, \ldots, x_q], \mathcal{L}_2 \subset \Gamma'' = K[x_{q+1}, \ldots, x_m]$ be monomial ideals, with $1 \leq q \leq m$, then

$$\operatorname{depth}(\Gamma'/\mathcal{L}_1 \otimes_K \Gamma''/\mathcal{L}_2) = \operatorname{depth}_{\Gamma}(\Gamma/(\mathcal{L}_1\Gamma + \mathcal{L}_2\Gamma)) = \operatorname{depth}_{\Gamma'}(\Gamma'/\mathcal{L}_1) + \operatorname{depth}_{\Gamma''}(\Gamma''/\mathcal{L}_2).$$

Lemma 2.2.10 ([18, Theorem 3.1])**.** Assume that $\mathcal{L}_1 \subset \Gamma' = K[x_1, \ldots, x_q], \mathcal{L}_2 \subset \Gamma'' = K[x_{q+1}, \ldots, x_m]$ be monomial ideals, with $1 \leq q \leq m$, then

$$\operatorname{sdepth}_{\Gamma}(\Gamma/(\mathcal{L}_1\Gamma + \mathcal{L}_2\Gamma)) \geq \operatorname{sdepth}_{\Gamma'}(\Gamma'/\mathcal{L}_1) + \operatorname{sdepth}_{\Gamma''}(\Gamma''/\mathcal{L}_2).$$

Lemma 2.2.11 ([9, Lemma 2.13])**.** Assume that $\mathcal{L}_1 \subset \Gamma' = K[x_1, \ldots, x_q], \mathcal{L}_2 \subset \Gamma'' = K[x_{q+1}, \ldots, x_m]$ be monomial ideals, with $1 \leq q \leq m$, then

$$\operatorname{sdepth}(\Gamma'/\mathcal{L}_1 \otimes_K \Gamma''/\mathcal{L}_2)) \geq \operatorname{sdepth}_{\Gamma'}(\Gamma'/\mathcal{L}_1) + \operatorname{sdepth}_{\Gamma''}(\Gamma''/\mathcal{L}_2).$$

Lemma 2.2.12 ([13, Lemma 3.6])**.** Consider a monomial ideal $\mathcal{L} \subset \Gamma = K[x_1, \ldots, x_n]$ and $\Gamma' = \Gamma[x_{n+1}, \ldots, x_{n+s}]$ be a ring of polynomials then

$$\operatorname{depth}(\Gamma'/\mathcal{L}\Gamma') = \operatorname{depth}(\Gamma/\mathcal{L}\Gamma) + s \quad \text{and} \quad \operatorname{sdepth}(\Gamma'/\mathcal{L}\Gamma) = \operatorname{sdepth}(\Gamma/\mathcal{L}\Gamma) + s.$$

Theorem 2.2.13 ([1, Theorem 2.6])**.** Let S_q be a q-star. If $\mathcal{I} = \mathcal{I}(S_q)$, then

$$\operatorname{depth}(\mathcal{U}/\mathcal{I}) = \operatorname{sdepth}(\mathcal{U}/\mathcal{I}) = 1,$$

where, $\mathcal{U} = K[V(S_q)]$ and

$$\operatorname{depth}(\mathcal{U}/\mathcal{I}^t), \operatorname{sdepth}(\mathcal{U}/\mathcal{I}^t) \geq 1$$

.

Corollary 2.2.14. Let $q \geq 1$ and $p \geq 2$ and $I = I(S_{p,q})$ where $S_{p,q} = Brs_q(S_p)$ then

$$\operatorname{depth}(K[V(Brs_q(S_p))]/I), \operatorname{sdepth}(K[V(Brs_q(S_p))]/I) = p + q$$

Chapter 3

Depth and Stanley depth of cyclic modules associated with q-fold bristled graphs of triangular and multi triangular snake graphs

Let $n, p, q \geq 1$, a q-fold bristled graph of p-tringular snake denoted by $Brs_q(\Lambda_{n,p})$ is obtained by connecting q vertices of degree 1 to every vertex of p-tringular snake $\Lambda_{n,p}$. In particular, if $p = 1$, then $\Lambda_{n,1} = \Lambda_n$ is a tringular snake and its q-fold bristled graph is denoted by $Brs_q(\Lambda_n)$. Clearly, $|V(Brs_q(\Lambda_{n,p}))|= (1+q)(1+n+np)$. The graph $Brs_q(\Lambda_{n,p})$ has np vertices of degree $q+2$, $n-1$ vertices having degree $2p+q+2$, two vertices having degree $p+q+1$ and $(1+n+np)q$ vertices are of degree 1. So by using Lemma 1.3.11, we have $|E(Brs_q(\Lambda_{n,p}))|= n+nq+q+2np+npq = 2np+n+(1+n+np)q$. For example see figure 1.15 and 3.1. The vertices of the $Brs_q(\mathit{\Lambda}_{n,p})$ graph are labelled by using the following sets of variables $\{\varphi_1, \varphi_2, \dots, \varphi_{n+1}\}$, $\Big\{\{\varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}\}, \{\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}\}, \dots, \{\varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}\}\Big\}$, $\Big\{\{\xi_{11}, \xi_{21}, \dots, \xi_{n1}\}, \{\xi_{12}, \xi_{22}, \dots, \xi_{n2}\}, \dots, \{\xi_{1p}, \xi_{2p}, \dots, \xi_{np}\}\Big\}$, $\Big\{\{\xi_{111}, \xi_{112}, \dots, \xi_{11q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q}, \dots, \xi_{n11}, \xi_{n12}, \dots, \xi_{n1q}\}, \{\xi_{121}, \xi_{122}, \dots, \xi_{12q}, \xi_{221}, \xi_{222}, \dots, \xi_{22q}, \dots, \xi_{n21}, \xi_{n22}, \dots, \xi_{n2q}\}, \dots, \{\xi_{1p1}, \xi_{1p2}, \dots, \xi_{1pq}, \xi_{2p1}, \xi_{2p2}, \dots, \xi_{2pq}, \dots, \xi_{np1}, \xi_{np2}, \dots, \xi_{npq}\}\Big\}$ see figure 3.1. Let $\amalg_{n,p,q} := K[\varphi_1, \varphi_2, \dots, \varphi_{n+1}, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q} \ \dots, \ \varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}, \xi_{11}, \xi_{21}, \dots, \xi_{n1}, \xi_{12}, \xi_{22}, \dots, \xi_{n2} \ \dots, \ \xi_{1p}, \xi_{2p}, \dots, \xi_{np}, \xi_{111}, \xi_{112}, \dots, \xi_{11q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q} \ \dots, \ \xi_{n11}, \xi_{n12}, \dots, \xi_{n1q}, \xi_{121}, \xi_{122}, \dots, \xi_{12q}, \xi_{221}, \xi_{222}, \dots, \xi_{22q} \ \dots, \ \xi_{n21}, \xi_{n22}, \dots,$

$\xi_{n2q}\ \ \ldots,\ \ \xi_{1p1}, \xi_{1p2}, \ldots, \xi_{1pq}, \xi_{2p1}, \xi_{2p2}, \ldots, \xi_{2pq}\ \ \ldots,\ \ \xi_{np1}, \xi_{np2}, \ldots, \xi_{npq}]$ be the ring of polynomials in these variables over the field K. Then $I_{n,p,q}$ is squarefree monomial ideal of $\amalg_{n,p,q}$. Now with the labelling as shown in Figure 3.1, the minimal set of monomial generators of $I_{n,p,q}$ is given as:

$$M(I_{n,p,q}) = \bigcup_{i=1}^{n}\{\varphi_i\varphi_{i+1}\}\bigcup\bigcup_{i=1}^{n}\Big\{\bigcup_{j=1}^{p}\{\varphi_i\xi_{ij}, \varphi_{i+1}\xi_{ij}\}\Big\}\bigcup\bigcup_{i=1}^{n+1}\Big\{\bigcup_{k=1}^{q}\{\varphi_i\varphi_{ik}\}\Big\}\bigcup \bigcup_{i=1}^{n}\Big\{\bigcup_{j=1}^{p}\Big\{\bigcup_{k=1}^{q}\{\xi_{ij}\xi_{ijk}\}\Big\}\Big\}.$$

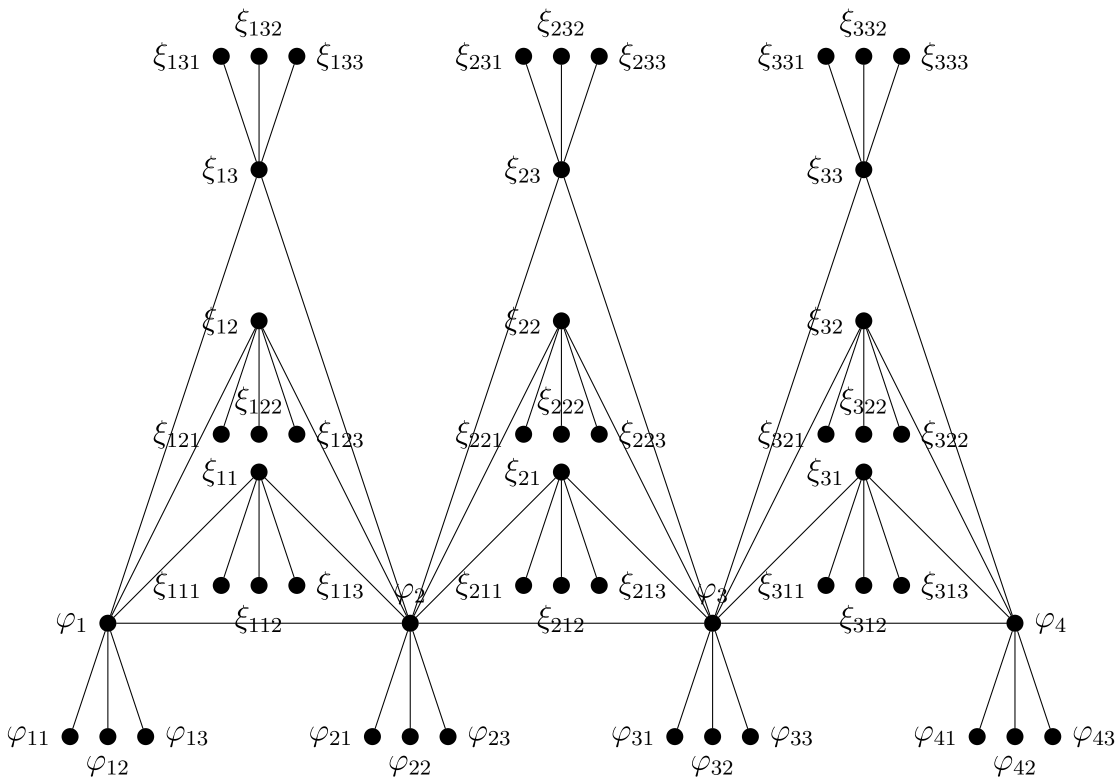

Figure 3.1: $Brs_3(\Lambda_{3,3})$
(3-fold bristled graph of $\Lambda_{3,3}$)

Let us consider a supergraph $Brs_q(\Lambda^{\star}_{n,p})$ of the graph $Brs_q(\Lambda_{n,p})$. The vertex and edge sets of $Brs_q(\Lambda^{\star}_{n,p})$ are $V(Brs_q(\Lambda^{\star}_{n,p}) = V(Brs_q(\Lambda_{n,p})) \bigcup \bigcup\limits_{j=1}^{p} \Big\{ \bigcup\limits_{k=1}^{q} \{\xi_{(n+1)jk}\}, \xi_{(n+1)j} \Big\}$ and

$$E(Brs_q(\Lambda^{\star}_{n,p})) = E(Brs_q(\Lambda_{n,p})) \bigcup \bigcup_{j=1}^{p} \Big\{ \bigcup_{k=1}^{q} \{\xi_{(n+1)j}\xi_{(n+1)jk}\}, \varphi_{n+1}\xi_{(n+1)j} \Big\}.$$

For example of graph $Brs_q(\Lambda^{\star}_{n,p})$, see Figure 3.2. We denote the edge ideal of graph $Brs_q(\Lambda^{\star}_{n,p})$ with $I^{*}_{n,p,q}$, which is the monomial ideal of the polynomial ring $\amalg^{*}_{n,p,q} = \amalg_{n,p,q}[\xi_{(n+1)1}, \xi_{(n+1)2}, \ldots, \xi_{(n+1)p}, \xi_{(n+1)11}, \xi_{(n+1)12}, \ldots, \xi_{(n+1)1q}, \xi_{(n+1)21}, \xi_{(n+1)22}, \ldots,$ $\xi_{(n+1)2q} \ \ \ldots, \ \ \xi_{(n+1)p1}, \xi_{(n+1)p2} \ldots, \xi_{(n+1)pq}]$. The minimal set of monomial generators of $I^{*}_{n,p,q}$ is $M(I^{*}_{n,p,q}) = M(I_{n,p,q}) \bigcup \bigcup\limits_{j=1}^{p} \Big\{ \bigcup\limits_{k=1}^{q} \{\xi_{(n+1)j}\xi_{(n+1)jk}\}, \varphi_{n+1}\xi_{(n+1)j} \Big\}$.

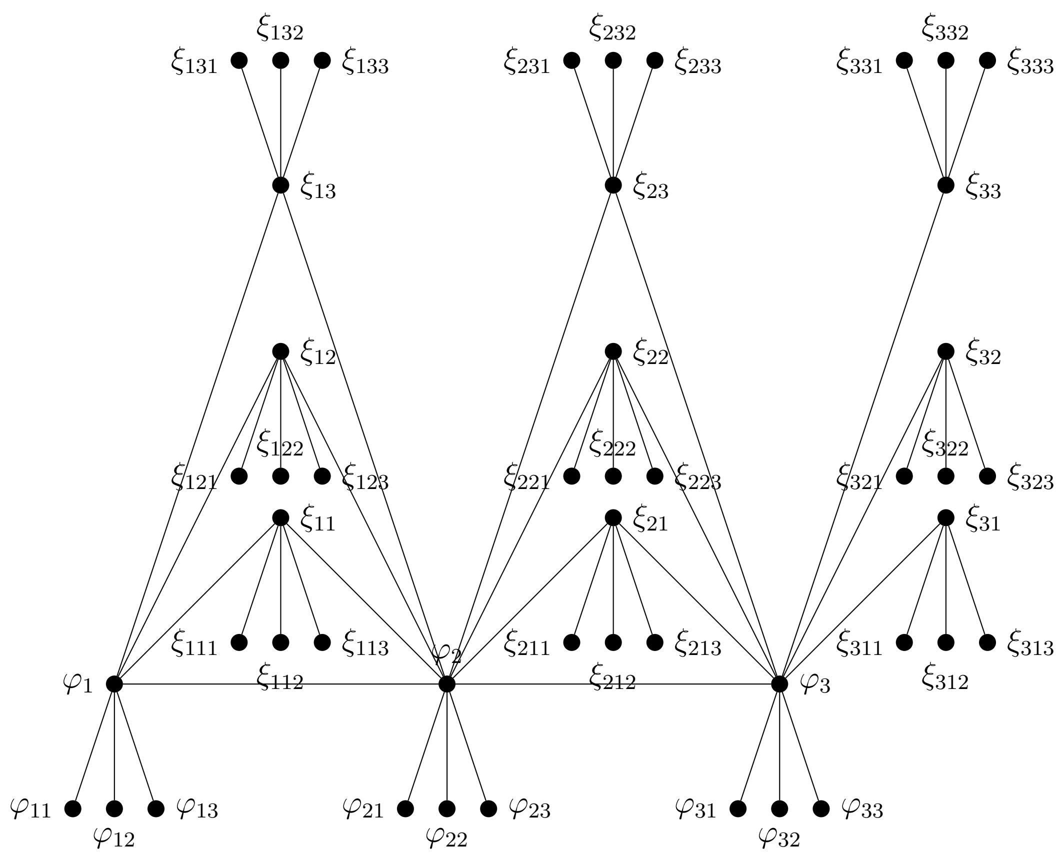

Figure 3.2: $Brs_3(\Lambda^{\star}_{2,3})$

3.0.1 Depth and Stanley depth of cyclic modules associated with q-fold bristled graph of triangular snake graph

If $p = 1$, then we can simply denote the edge ideals by $I_{n,q}$ and $I^*_{n,q}$ and the polynomial rings by $\amalg_{n,q}$ and $\amalg^*_{n,q}$.

Lemma 3.0.1. *Let $n, q \geq 1$ and $p = 1$, then* $\operatorname{depth}(\amalg^\star_{n,q}/I^\star_{n,q}) = \operatorname{sdepth}(\amalg^\star_{n,q}/I^\star_{n,q}) = (q+1)(n+1)$.

Proof. First, we consider depth.

***Case* 1.**

For $q \geq 1$ and $p, n = 1$. Let us consider the exact sequence

$$0 \longrightarrow \amalg^\star_{1,q}/(I^\star_{1,q} : \varphi_2) \xrightarrow{\cdot\varphi_2} \amalg^\star_{1,q}/I^\star_{1,q} \longrightarrow \amalg^\star_{1,q}/(I^\star_{1,q}, \varphi_2) \longrightarrow 0, \tag{3.1}$$

by Depth Lemma

$$\operatorname{depth}(\amalg^\star_{1,q}/I^\star_{1,q}) \geq \min\{\operatorname{depth}(\amalg^\star_{1,q}/(I^\star_{1,q} : \varphi_2), \operatorname{depth}(\amalg^\star_{1,q}/(I^\star_{1,q}, \varphi_2)\}.$$

We have $(I^\star_{1,q} : \varphi_2) = (\varphi_1, \xi_{11}, \xi_{21}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q})$, it follows that

$$\amalg^\star_{1,q}/(I^\star_{1,q} : \varphi_2) \cong K[\varphi_2, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \xi_{111}, \xi_{112}, \dots, \xi_{11q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q}].$$

This implies, $\operatorname{depth}(\amalg^\star_{1,q}/(I^\star_{1,q} : \varphi_2)) = 3q + 1$. Now as

$$(I^\star_{1,q}, \varphi_2) = (I(S_{1,q}), \varphi_2, \varphi_{21}\varphi_{211}, \varphi_{21}\varphi_{212}, \dots, \varphi_{21}\varphi_{21q}) = (I(S_{1,q}), \varphi_2, I(S_q)),$$

so we obtain

$$\amalg^\star_{1,q}/(I^\star_{1,q}, \varphi_2) \cong K[V(S_{1,q})]/I(S_{1,q}) \otimes_K K[V(S_q)]/I(S_q) \otimes_K K[\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}].$$

By using Lemma 2.2.9

$$\operatorname{depth}(\amalg^\star_{1,q}/(I^\star_{1,q}, \varphi_2)) = \operatorname{depth}(K[V(S_{1,q})]/I(S_{1,q})) + \operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth} K[\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}].$$

By Corollary 2.2.14, Theorem 2.2.13 and Lemma 2.2.12,

$$\operatorname{depth}(\text{Ш}^{\star}_{1,q}/(I^{\star}_{1,q},\varphi_2)) = 2q+2.$$

Since $\operatorname{depth}(\text{Ш}^{\star}_{1,q}/(I^{\star}_{1,q}:\varphi_2)) \geq \operatorname{depth}(\text{Ш}^{\star}_{1,q}/(I^{\star}_{1,q},\varphi_2))$, so by Depth Lemma $\operatorname{depth}(\text{Ш}^{\star}_{1,q}/(I^{\star}_{1,q})) \geq \operatorname{depth}(\text{Ш}^{\star}_{1,q}/(I^{\star}_{1,q},\varphi_2))$. That is, $\operatorname{depth}(\text{Ш}^{\star}_{1,q}/I^{\star}_{1,q}) \geq 2q+2$.
Now since $\xi_{11}\xi_{21} \notin I^{\star}_{1,q}$ and $(I^{\star}_{1,q}:\xi_{11}\xi_{21}) = K[\varphi_1,\varphi_2,\xi_{111},\xi_{112},\ldots,\xi_{11q},\xi_{211},\xi_{212},\ldots,\xi_{21q}]$, it follows $\text{Ш}^{\star}_{1,q}/(I^{\star}_{1,q}:\xi_{11}\xi_{21}) \cong K[\xi_{11},\xi_{21},\varphi_{11},\varphi_{12},\ldots,\varphi_{1q},\varphi_{21},\varphi_{22},\ldots,\varphi_{2q}]$. This shows that, $\operatorname{depth}(\text{Ш}^{\star}_{1,q}/(I^{\star}_{1,q}:\xi_{11}\xi_{21})) = 2q+2$. Using Corollary 2.2.7, we get

$$\operatorname{depth}(\text{Ш}_{1,q}/I^{\star}_{1,q}) \leq 2q+2.$$

Hence

$$\operatorname{depth}(\text{Ш}^{\star}_{1,q}/I^{\star}_{1,q}) = 2q+2. \tag{3.2}$$

***Case* 2.**

For $q \geq 1$, $p=1$ and $n=2$. Consider the exact sequence

$$0 \longrightarrow \text{Ш}^{\star}_{2,q}/(I^{\star}_{2,q}:\varphi_3) \xrightarrow{\cdot\varphi_3} \text{Ш}^{\star}_{2,q}/I^{\star}_{2,q} \longrightarrow \text{Ш}^{\star}_{2,q}/(I^{\star}_{2,q},\varphi_3) \longrightarrow 0, \tag{3.3}$$

by Depth Lemma

$$\operatorname{depth}(\text{Ш}^{\star}_{2,q}/I^{\star}_{2,q}) \geq \min\{\operatorname{depth}(\text{Ш}^{\star}_{2,q}/(I^{\star}_{2,q}:\varphi_3),\operatorname{depth}(\text{Ш}^{\star}_{2,q}/(I^{\star}_{2,q},\varphi_3)\}.$$

We have $(I^{\star}_{2,q}:\varphi_3) = (I(S_{1,q}),\varphi_2,\xi_{21},\xi_{31},\varphi_{31},\varphi_{32},\ldots,\varphi_{3q})$. Now

$$\text{Ш}^{\star}_{2,q}\,/(I^{\star}_{2,q}:\varphi_3) \cong K[V(S_{1,q})]/I(S_{1,q}) \otimes_K K[\varphi_3,\varphi_{21},\varphi_{22},\ldots,\varphi_{2q},\xi_{211},\xi_{212},\ldots,\xi_{21q},$$
$$\xi_{311},\xi_{312},\ldots,\xi_{31q}]$$

Applying Lemma 2.2.9, Proposition 2.2.14 and Lemma 2.2.12

$$\operatorname{depth}(\text{Ш}^{\star}_{2,q}/(I^{\star}_{2,q}:\varphi_3)) = 4q+2.$$

Now as $(I^{\star}_{2,q},\varphi_3) = (I^{\star}_{1,q},\varphi_3,I(S_q))$, it follows that

$$\text{Ш}^{\star}_{2,q}/(I^{\star}_{2,q},\varphi_3) \cong \text{Ш}^{\star}_{1,q}/I^{\star}_{1,q} \otimes_K K[V(S_q)]/I(S_q) \otimes_K K[\varphi_{31},\varphi_{32},\ldots,\varphi_{3q}].$$

By using Lemma 2.2.9

$$\operatorname{depth}(\,\amalg^\star_{2,q}\,/(I^\star_{2,q},\varphi_{n+1})) = \operatorname{depth}(\amalg^\star_{1,1,q}/I^\star_{1,1,q}) + \operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth} K[\varphi_{31},\varphi_{32},\ldots,\varphi_{3q}].$$

Using Eq 3.2, Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\amalg^\star_{2,q}/(I^\star_{2,q},\varphi_3)) = 3q+3$. Since $\operatorname{depth}(\amalg^\star_{2,q}/I^\star_{2,q}:\varphi_3) \geq \operatorname{depth}(\amalg^\star_{2,q}/(I^\star_{2,q},\varphi_3))$, so by Depth Lemma $\operatorname{depth}(\amalg^\star_{2,q}/I^\star_{2,q})) \geq 3q+3$. Now since $\xi_{21}\xi_{31} \notin I^\star_{2,q}$ and $(I^\star_{2,q}:\xi_{21}\xi_{31}) = (I(S_{1,q}),\varphi_2,\varphi_3,\xi_{211},\xi_{212},\ldots,\xi_{21q},\xi_{311},\xi_{312},\ldots,\xi_{31q})$, we obtain

$$\amalg^\star_{2,q}/(I^\star_{2,q}:\xi_{21}\xi_{31}) \cong K[V(S_{1,q})]/I(S_{1,q}) \otimes_K K[\xi_{21},\xi_{31},\varphi_{21},\varphi_{22}\ldots,\varphi_{2q},\varphi_{31},\varphi_{32},\ldots,\varphi_{3q}].$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\amalg^\star_{2,q}/(I^\star_{2,q}:\xi_{21}\xi_{31})) = 3q+3$. Using Corollary 2.2.7, $\operatorname{depth}(\amalg^\star_{2,q}/I^\star_{2,q}) \leq 3q+3$. Hence $\operatorname{depth}(\amalg^\star_{2,q}/I^\star_{2,q}) = 3q+3$.

***Case* 3.**

For $q \geq 1$, $p = 1$ and $n \geq 3$, we prove this result inductively. Consider the exact sequence

$$0 \longrightarrow \amalg^\star_{n,q}/(I^\star_{n,q}:\varphi_{n+1}) \xrightarrow{\cdot\varphi_{n+1}} \amalg^\star_{n,q}/I^\star_{n,q} \longrightarrow \amalg^\star_{n,q}/(I^\star_{n,q},\varphi_{n+1}) \longrightarrow 0, \tag{3.4}$$

by Depth Lemma

$$\operatorname{depth}(\amalg^\star_{n,q}/I^\star_{n,q}) \geq \min\{\operatorname{depth}(\amalg^\star_{n,q}/(I^\star_{n,q}:\varphi_{n+1})), \operatorname{depth}(\amalg^\star_{n,q}/(I^\star_{n,q},\varphi_{n+1}))\}.$$

Here $(I^\star_{n,q}:\varphi_{n+1}) = (I^\star_{(n-2),q},\varphi_n,\xi_{n1},\xi_{(n+1)1},\varphi_{(n+1)1},\varphi_{(n+1)2},\ldots,\varphi_{(n+1)q})$, and we have

$$\amalg^\star_{n,q}/(I^\star_{n,q}:\varphi_{n+1}) \cong (\amalg^\star_{(n-2),q}/I^\star_{(n-2),q}) \otimes_K K[\varphi_{n+1},\varphi_{n1},\varphi_{n2},\ldots,\varphi_{nq},\xi_{n11},\xi_{n12},\ldots, \xi_{n1q},\xi_{(n+1)11},\xi_{(n+1)12},\ldots,\xi_{(n+1)1q}].$$

By induction and Lemma 2.2.12, $\operatorname{depth}(\amalg^\star_{n,q}/(I^\star_{n,q}:\varphi_{n+1})) = (1+q)n+2q$.
Now $(I^\star_{n,q},\varphi_{n+1}) = (I^\star_{(n-1),q},\varphi_{n+1},I(S_q))$, it follows

$$\amalg^\star_{n,q}/(I^\star_{n,q},\varphi_{n+1}) \cong (\amalg^\star_{(n-1),q}/I^\star_{(n-1),q}) \otimes_K K[V(S_q)]/(I(S_q) \otimes_K K[\varphi_{(n+1)1},\varphi_{(n+1)2}, \ldots,\varphi_{(n+1)q}].$$

By induction and Theorem 2.2.13 and Lemma 2.2.12,

$$\operatorname{depth}(\text{Ш}^{\star}_{n,q}/(I^{\star}_{n,q}, \varphi_{n+1})) = (n+1)(1+q).$$

Since $\operatorname{depth}(\text{Ш}^{\star}_{n,q}/(I^{\star}_{n,q} : \varphi_{n+1})) \geq \operatorname{depth}(\text{Ш}^{\star}_{n,q}/(I^{\star}_{n,q}, \varphi_{n+1}))$, hence by Depth Lemma $\operatorname{depth}(\text{Ш}^{\star}_{n,q}/I^{\star}_{n,q}) \geq (n+1)(1+q)$. Now since $\xi_{n1}\xi_{(n+1)1} \notin I^{\star}_{n,q}$ and

$$(I^{\star}_{n,q} : \xi_{n1}\xi_{(n+1)1}) = (I^{\star}_{(n-2),q}, \varphi_n, \varphi_{n+1}, \xi_{n11}, \xi_{n12}, \dots, \xi_{n1q}, \xi_{(n+1)11}, \xi_{(n+1)12}, \dots, \xi_{(n+1)1q}).$$

We obtain, $\text{Ш}^{\star}_{n,q}/(I^{\star}_{n,q} : \xi_{n1}\xi_{(n+1)1}) \cong (\text{Ш}^{\star}_{(n-2),q}/I^{\star}_{(n-2),q}) \otimes_K K[\xi_{n1}, \xi_{(n+1)1}, \varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}]$. Again by induction and Lemma 2.2.12

$$\operatorname{depth}(\text{Ш}^{\star}_{n,q}/(I^{\star}_{n,q} : \xi_{n1}\xi_{(n+1)1})) = (n-1)(1+q) + 2q + 2 = (n+1)(1+q).$$

Using Corollary 2.2.7, $\operatorname{depth}(\text{Ш}^{\star}_{n,q}/I^{\star}_{n,q}) \leq (n+1)(1+q)$. Hence

$$\operatorname{depth}(\text{Ш}^{\star}_{n,q}/I^{\star}_{n,q}) = (n+1)(1+q).$$

The consequence for Stanley depth follows by applying Lemma 2.1.7 and Lemma 2.2.11 instead of Depth Lemma and Lemma 2.2.9 on exact sequences 3.1, 3.3 and 3.4, respectively. And we use Proposition 2.2.8 instead of Corollary 2.2.7.

□

Theorem 3.0.2. *For $n, q \geq 1$ and $p = 1$,* $\operatorname{depth}(\text{Ш}_{n,q}/I_{n,q}) = \operatorname{sdepth}(\text{Ш}_{n,q}/I_{n,q}) = (1+q)n + q$.

Proof. First, we consider depth.

***Case* 1.**

For $q \geq 1$ and $n, p = 1$. Consider the exact sequence

$$0 \longrightarrow \text{Ш}_{1,q}/(I_{1,q} : \varphi_2) \xrightarrow{\cdot\varphi_2} \text{Ш}_{1,q}/I_{1,q} \longrightarrow \text{Ш}_{1,q}/(I_{1,q}, \varphi_2) \longrightarrow 0, \tag{3.5}$$

applying Depth Lemma

$$\operatorname{depth}(\text{Ш}_{1,q}/I_{1,q}) \geq \min\{\operatorname{depth}(\text{Ш}_{1,q}/(I_{1,q} : \varphi_2), \operatorname{depth}(\text{Ш}_{1,q}/(I_{1,q}, \varphi_2)\}.$$

Here $(I_{1,q} : \varphi_2) = (\varphi_1, \xi_{11}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2m})$, so we have $\text{Ш}_{1,q}/(I_{1,q} : \varphi_2) \cong K[\varphi_2, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \xi_{111}, \xi_{112}, \dots, \xi_{11q}]$, it implies $\operatorname{depth}(\text{Ш}_{1,q}/(I_{1,q} : \varphi_2)) = 2q + 1$. Also

$$\begin{aligned}(I_{1,q}, \varphi_2) &= (\varphi_1\xi_{11}, \varphi_1\varphi_{11}, \varphi_1\varphi_{12}, \dots, \varphi_1\varphi_{1q}, \xi_{11}\xi_{111}, \xi_{11}\xi_{112}, \dots, \xi_{11}\xi_{11q}, \varphi_2)\\ &= (I(S_{1,q}), \varphi_2),\end{aligned}$$

so we obtain $\text{Ш}_{1,q}/(I_{1,q}, \varphi_2) \cong K[V(S_{1,q})]/I(S_{1,q}) \otimes_K K[\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}]$.
Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\text{Ш}_{1,q}/(I_{1,q}, \varphi_2)) = 2q + 1$. Since $\operatorname{depth}(\text{Ш}_{1,q}/(I_{1,q} : \varphi_2)) = \operatorname{depth}(\text{Ш}_{1,q}/(I_{1,q}, \varphi_2))$, hence by depth lemma

$$\operatorname{depth}(\text{Ш}_{1,q}/I_{1,q}) = 2q + 1.$$

***Case* 2.**

For $q \geq 1$, $p = 1$ and $n = 2$. Consider the exact sequence

$$0 \longrightarrow \text{Ш}_{2,q}/(I_{2,q} : \varphi_3) \xrightarrow{\cdot\varphi_3} \text{Ш}_{2,q}/I_{2,q} \longrightarrow \text{Ш}_{2,q}/(I_{2,q}, \varphi_3) \longrightarrow 0, \tag{3.6}$$

applying Depth Lemma

$$\operatorname{depth}(\text{Ш}_{2,q}/I_{2,q}) \geq \min\{\operatorname{depth}(\text{Ш}_{2,q}/(I_{2,q} : \varphi_3)), \operatorname{depth}(\text{Ш}_{2,q}/(I_{2,q}, \varphi_3))\}.$$

We have $(I_{2,q} : \varphi_3) = (I(S_{1,q}), \varphi_2, \xi_{21}, \varphi_{31}, \varphi_{32}, \dots, \varphi_{3q})$, it follows that

$$\text{Ш}_{2,q}/(I_{2,q} : \varphi_3) \cong K[V(S_{1,q})]/I(S_{1,q}) \otimes_K [\varphi_3, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q}].$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\text{Ш}_{2,q}/(I_{2,q} : \varphi_3)) = 3q + 2$. As

$$(I_{2,q}, \varphi_3) = (I^{\star}_{1,q}, \varphi_3),$$

so we obtain $\text{Ш}_{2,q}/(I_{2,q}, \varphi_3) \cong (S^{\star}_{1,q}/I^{\star}_{1,q}) \otimes_K K[\varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}]$. Using Eq 3.2 and Lemma 2.2.12, $\operatorname{depth}(\text{Ш}_{2,q}/(I_{2,q}, \varphi_3)) = 3q + 2$.
Since $\operatorname{depth}(\text{Ш}_{2,q}/(I_{2,q} : \varphi_3)) = \operatorname{depth}(\text{Ш}_{2,q}/(I_{2,q}, \varphi_3))$, hence by Depth Lemma

$$\operatorname{depth}(\text{Ш}_{2,q}/I_{2,q}) = 3q + 2.$$

***Case* 3.**

For $q \geq 1$, $p = 1$ and $n \geq 3$. We will prove this result by using Lemma 3.0.1. Consider the exact sequence

$$0 \longrightarrow \text{II}_{n,q}/(I_{n,q} : \varphi_{n+1}) \xrightarrow{\cdot \varphi_{n+1}} \text{II}_{n,q}/I_{n,q} \longrightarrow \text{II}_{n,q}/(I_{n,q}, \varphi_{n+1}) \longrightarrow 0, \qquad (3.7)$$

by Depth Lemma

$$\operatorname{depth}(\text{II}_{n,q}/I_{n,q}) \geq \min\{\operatorname{depth}(\text{II}_{n,q}/(I_{n,q} : \varphi_{n+1})), \operatorname{depth}(\text{II}_{n,q}/(I_{n,q}, \varphi_{n+1}))\}.$$

Here $(I_{n,q} : \varphi_{n+1}) = (I^{\star}_{(n-2),q}, \varphi_n, \xi_{n1}, \varphi_{(n+1)1}.\varphi_{(n+1)2}, \dots, \varphi_{(n+1)q})$, and we have

$\text{II}_{n,q}/(I_{n,q} : \varphi_{n+1}) \cong (\text{II}^{\star}_{(n-2),q}/I^{\star}_{(n-2),q}) \otimes_K K[\varphi_{n+1}, \varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}, \xi_{n11}, \xi_{n12}, \dots, \varphi_{n1q}]$.

Using Lemma 3.0.1 and Lemma 2.2.12, $\operatorname{depth}(\text{II}_{n,q}/(I_{n,q}) : \varphi_{n+1})) = (1+q)n + q$. Now

$$(I_{n,q}, \varphi_{n+1}) = (I^{\star}_{(n-1),q}, \varphi_{n+1}),$$

it follows $\text{II}_{n,q}/(I_{n,q}, \varphi_{n+1}) \cong (\text{II}^{\star}_{(n-1),q}/I^{\star}_{(n-1),q}) \otimes_K K[\varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}]$. Using Lemma 3.0.1 and Lemma 2.2.12, $\operatorname{depth}(\text{II}_{n,q}/(I_{n,q}, \varphi_{n+1})) = (1+q)n + q$. Since $\operatorname{depth}(\text{II}_{n,q}/(I_{n,q} : \varphi_{n+1})) = \operatorname{depth}(\text{II}_{n,q}/(I_{n,q}, \varphi_{n+1}))$, hence by Depth Lemma

$$\operatorname{depth}(\text{II}_{n,q}/I_{n,q}) = (1+q)n + q.$$

For Stanley depth, we apply Lemma 2.1.7 and Lemma 2.2.11 rather than Depth Lemma and Lemma 2.2.9 on exact sequences 3.5, 3.6 and 3.7, respectively and have $\operatorname{sdepth}(\text{II}_{n,q}/I_{n,q}) \geq (1+q)n + q$. To compute the upper bound since $\varphi_{(n+1)1}\varphi_{(n+1)2} \notin I_{n,q}$ and $(I_{n,q} : \xi_{(n+1)1}\xi_{(n+1)2}) = (I^{\star}_{(n-1),q}, \varphi_{n+1})$, It follows that
$\operatorname{sdepth}(\text{II}_{n,q}/(I_{n,q} : \varphi_{(n+1)1}\varphi_{(n+1)2})) = (1+q)n + q$. Thus by using Propositon 2.2.8 $\operatorname{sdepth}(\text{II}_{n,q}/I_{n,q}) \leq (1+q)n + q$. Hence $\operatorname{sdepth}(\text{II}_{n,q}/I_{n,q}) = (1+q)n + q$

□

3.0.2 Depth and Stanley depth of cyclic module associated with q-fold bristled graph of multi triangular snake graph

Lemma 3.0.3. *For $n, q \geq 1$ and $p \geq 2$,* $\operatorname{depth}(\text{II}^{\star}_{n,p,q}/I^{\star}_{n,p,q}) = \operatorname{sdepth}(\text{II}^{\star}_{n,p,q}/I^{\star}_{n,p,q}) = (p+q)(n+1)$.

Proof. First, we consider depth.

***Case* 1.**

For $q \geq 1$, $p \geq 2$ and $n = 1$. Consider the exact sequence

$$0 \longrightarrow \amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q} : \varphi_2) \xrightarrow{\cdot\varphi_2} \amalg^{\star}_{1,p,q}/I^{\star}_{1,p,q} \longrightarrow \amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q}, \varphi_2) \longrightarrow 0, \tag{3.8}$$

applying Depth Lemma

$$\operatorname{depth}(\amalg^{\star}_{1,p,q}/I^{\star}_{1,p,q}) \geq \min\{\operatorname{depth}(\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q} : \varphi_2), \operatorname{depth}(\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q}, \varphi_2)\}.$$

Here $(I_{1,p,q} : \varphi_2) = K[\varphi_1, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}, \xi_{11}, \xi_{12}, \ldots, \xi_{1p}, \xi_{21}, \xi_{22}, \ldots, \xi_{2p}]$, we obtain

$$\begin{aligned}\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q} : \varphi_2) \cong K[&\varphi_2, \varphi_{11}, \varphi_{12}, \ldots, \varphi_{1q}, \xi_{111}, \xi_{112}, \ldots, \xi_{11q}, \xi_{121}, \xi_{122}, \ldots, \xi_{12q} \ \ldots, \\ &\xi_{1p1}, \xi_{1p2}, \ldots, \xi_{1pq}, \xi_{211}, \xi_{212}, \ldots, \xi_{21q}, \xi_{221}, \xi_{222}, \ldots, \xi_{22q} \ \ldots, \ \ \xi_{2p1}, y_{2p2}, \ldots, y_{2pq}].\end{aligned}$$

This implies $\operatorname{depth}(\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q} : \varphi_2)) = 2pq + q + 1$. Also $(I^{\star}_{1,p,q}, \varphi_2) = (I(S_{p,q}), \varphi_2, \xi_{21}\xi_{211}, \xi_{21}\xi_{212}, \ldots, \xi_{21}\xi_{21q}, \xi_{22}\xi_{221}, \xi_{22}\xi_{222}, \ldots, \xi_{22}\xi_{22q} \ \ldots,$ $\xi_{2p}\xi_{2p1}, \xi_{2p}\xi_{2p2}, \ldots, \xi_{2p}\xi_{2pq})$. It follows

$$\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q}, \varphi_2) \cong K[V(S_{p,q})]/I(S_{p,q}) \bigotimes_{j=1}^{p}{}_K K[V(S_q)]/I(S_q) \otimes_k K[\varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}].$$

Applying Lemma 2.2.9

$$\begin{aligned}\operatorname{depth}(\ \amalg^{\star}_{1,p,q}\ /(I^{\star}_{1,p,q}, \varphi_2)) &= \operatorname{depth}(K[V(S_{p,q})]/I(S_{p,q})) \\ &\quad + \sum_{j=1}^{p} \operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth} K[\varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}].\end{aligned}$$

Using Corollary 2.2.7, Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q}, \varphi_2)) = 2p + 2q$. Since $\operatorname{depth}(\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q} : \varphi_2)) \geq \operatorname{depth}(\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q}, \varphi_2))$, hence by Depth Lemma $\operatorname{depth}(\amalg^{\star}_{1,p,q}/I^{\star}_{1,p,q}) \geq 2p + 2q$. Now since $\xi_{21}\xi_{22}\ldots\xi_{2p} \notin I^{\star}_{1,p,q}$ and

$$\begin{aligned}(I^{\star}_{1,p,q} : \xi_{21}\xi_{22}\ldots\xi_{2p}) = (I(S_{p,q}), \varphi_2, \xi_{211}, \xi_{212}, \ldots, \xi_{21q}, \xi_{221}, \xi_{222}, \ldots, \xi_{22q} \ \ldots, \ \ \xi_{2p1}, \\ \xi_{2p2}, \ldots, \xi_{2pq}).\end{aligned}$$

We have

$$\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q}:\xi_{21}\xi_{22}\dots\xi_{2p}) \cong K[V(S_{p,q})]/(I(S_{p,q})\otimes_k K[\xi_{21},\xi_{22},\dots,\xi_{2p},\varphi_{21},\varphi_{22},\dots,\varphi_{2q}].$$

By Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\amalg^{\star}_{1,p,q}/(I^{\star}_{1,p,q}:\xi_{21}\xi_{22}\dots\xi_{2p}) = 2p+2q$. Now using Corollary 2.2.7, we obtain $\operatorname{depth}(\amalg_{1,p,q}/I^{\star}_{1,p,q}) \leq 2p+2q$. Hence

$$\operatorname{depth}(\amalg^{\star}_{1,p,q}/I^{\star}_{1,p,q}) = 2p+2q. \tag{3.9}$$

***Case* 2.**

For $q \geq 1$, $p \geq 2$ and $n = 2$. Consider the exact sequence

$$0 \longrightarrow \amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q}:\varphi_3) \xrightarrow{\cdot\varphi_3} \amalg^{\star}_{2,p,q}/I^{\star}_{2,p,q} \longrightarrow \amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q},\varphi_3) \longrightarrow 0, \tag{3.10}$$

by Depth Lemma

$$\operatorname{depth}(\amalg^{\star}_{2,p,q}/I^{\star}_{2,p,q}) \geq \min\{\operatorname{depth}(\amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q}:\varphi_3),\operatorname{depth}(\amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q},\varphi_3)\}.$$

We have $(I_{2,p,q}:\varphi_3) = (I(S_{p,q}),\varphi_2,\varphi_{31},\varphi_{32},\dots,\varphi_{3q},\xi_{21},\xi_{22},\dots,\xi_{2p},\xi_{31},\xi_{32},\dots,\xi_{3p})$, it follows that, $\amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q}:\varphi_3) \cong K[V(S_{p,q})]/I(S_{p,q}) \otimes_K K[\varphi_3,\varphi_{21},\varphi_{22},\dots,\varphi_{2q},\xi_{211},$ $\xi_{212},\dots,\xi_{21q},\xi_{221},\xi_{222},\dots,\xi_{22q}\ \dots,\ \xi_{2p1},\xi_{2p2},\dots,\xi_{2pq},\xi_{311},\xi_{312},\dots,\xi_{31q},\xi_{321},\xi_{322},\dots,$ $\xi_{32q}\ \dots,\ \xi_{3p1},\xi_{3p2},\dots,\xi_{3pq}]$. By Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q}:\varphi_3)) = 2pq+2q+p+1$, also we have

$$(I^{\star}_{2,p,q},\varphi_3) = (I^{\star}_{1,p,q},\varphi_3,\xi_{31}\xi_{311},\xi_{31}\xi_{312},\dots,\xi_{31}\xi_{31q},\xi_{32}\xi_{321},\xi_{32}\xi_{322},\dots,\xi_{32}\xi_{32q}\ \dots,$$
$$\xi_{3p}\xi_{3p1},\xi_{3p}\xi_{3p2},\dots,\xi_{3p}\xi_{3pq}),$$

it shows that, $\amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q},\varphi_3) \cong \amalg^{\star}_{1,p,q}/I^{\star}_{1,p,q} \bigotimes_{j=1}^{p}{}_K K[V(S_q)]/I(S_q)\otimes_K K[\varphi_{31},\varphi_{32},\dots,\varphi_{3q}]$. By using Lemma 2.2.9

$$\operatorname{depth}(\amalg^{\star}_{2,p,q}/(I^{\star}_{2,p,q},\varphi_3)) = \operatorname{depth}(\amalg^{\star}_{1,p,q}/I^{\star}_{1,p,q}) + \sum_{j=1}^{p}\operatorname{depth}\left(K[V(S_q)]/I(S_q)\right) +$$
$$\operatorname{depth} K[\varphi_{31},\varphi_{32},\dots,\varphi_{3q}].$$

Using Eq 3.9, Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(Ш^{\star}_{2,p,q}/(I^{\star}_{2,p,q}, \varphi_3)) = 3p + 3q$. So by Depth Lemma $\operatorname{depth}(Ш^{\star}_{2,p,q}/I^{\star}_{2,p,q}) \geq \operatorname{depth}(Ш^{\star}_{2,p,q}/(I^{\star}_{2,p,q}, \varphi_3))$.
Hence $\operatorname{depth}(Ш^{\star}_{2,p,q}/I^{\star}_{2,p,q})) \geq 3p + 3q$. Now since $\xi_{31}\xi_{32}\ldots\xi_{3p} \notin I^{\star}_{2,p,q}$ and we have

$$(I^{\star}_{2,p,q} : \xi_{31}\xi_{32}\ldots\xi_{3p}) = (I^{\star}_{1,p,q}, \varphi_3, \xi_{311}, \xi_{312}, \ldots, \xi_{31q}, \xi_{321}, \xi_{322}, \ldots, \xi_{32q} \ \ldots, \ \xi_{3p1}, \xi_{3p2}, \ldots, \xi_{3pq}),$$

it follows

$$Ш^{\star}_{2,p,q}/(I^{\star}_{2,p,q} : \xi_{31}\xi_{32}\ldots\xi_{3p}) \cong S^{\star}_{1,p,q}/I^{\star}_{1,p,q} \otimes_K K[\xi_{31}, \xi_{32}, \ldots, \xi_{3p}, \varphi_{31}, \varphi_{32}, \ldots, \varphi_{3q}].$$

Using Eq 3.9 and Theorem 2.2.13 and Lemma 2.2.12, we obtain

$$\operatorname{depth}(Ш^{\star}_{2,p,q}/(I^{\star}_{2,p,q} : \xi_{31}\xi_{32}\ldots\xi_{3p})) = 3p + 3q.$$

By Corollary 2.2.7, $\operatorname{depth}(S_{2,p,q}/I^{\star}_{2,p,q}) \leq 3p + 3q$. Hence $\operatorname{depth}(S^{\star}_{2,p,q}/I^{\star}_{2,p,q}) = 3p + 3q$.

***Case* 3.**

For $q \geq 1$, $p \geq 2$ and $n \geq 3$. We prove this result inductively. Consider the exact sequence

$$0 \longrightarrow Ш^{\star}_{n,p,q}/(I^{\star}_{n,p,q} : \varphi_{n+1}) \xrightarrow{\cdot\varphi_{n+1}} Ш^{\star}_{n,p,q}/I^{\star}_{n,p,q} \longrightarrow Ш^{\star}_{n,p,q}/(I^{\star}_{n,p,q}, \varphi_{n+1}) \longrightarrow 0, \quad (3.11)$$

using Depth Lemma

$$\operatorname{depth}(Ш^{\star}_{n,p,q}/I^{\star}_{n,p,q}) \geq \min\{\operatorname{depth}(Ш^{\star}_{n,p,q}/(I^{\star}_{n,p,q} : \varphi_{n+1})), \operatorname{depth}(Ш^{\star}_{n,p,q}/(I_{n,p,q}, \varphi_{n+1}))\}.$$

Here $(I_{n,p,q} : \varphi_{n+1}) = (I^{\star}_{(n-2),p,q}, \varphi_n, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \ldots, \varphi_{(n+1)q}, \xi_{n1}, \xi_{n2}, \ldots, \xi_{np}, \xi_{(n+1)1}, \xi_{(n+1)2}, \ldots, \xi_{(n+1)p}])$, so we have

$$Ш^{\star}_{n,p,q}/(I^{\star}_{n,p,q} : \varphi_{n+1}) \cong (Ш^{\star}_{(n-2),p,q}/I^{\star}_{(n-2),p,q}) \otimes_K [\varphi_{n+1}, \varphi_{n1}, \varphi_{n2}, \ldots, \varphi_{nq}, \xi_{n11}, \xi_{n12}, \ldots, \xi_{n1q}, \xi_{n21}, \xi_{n22}, \ldots, \xi_{n2q} \ \ldots, \ \xi_{np1}, \xi_{np2}, \ldots, \xi_{npq}, \xi_{(n+1)11}, \xi_{(n+1)12}, \ldots, \xi_{(n+1)1q}, \xi_{(n+1)21}, \xi_{(n+1)22}, \ldots, \xi_{(n+1)2q} \ \ldots, \ \xi_{(n+1)p1}, \xi_{(n+1)p2}, \ldots, \xi_{(n+1)pq}].$$

By induction and Lemma 2.2.12, $\operatorname{depth}(\amalg^{\star}_{n,p,q}/(I^{\star}_{n,p,q} : \varphi_{n+1})) = 2pq + nq + np - p + 1$, and

$$(I_{n,p,q}, \varphi_{n+1}) = (I^{\star}_{(n-1),p,q}, \varphi_{n+1}, \xi_{(n+1)1}\xi_{(n+1)11}, \xi_{(n+1)1}\xi_{(n+1)12}, \ldots, \xi_{(n+1)1}\varphi_{(n+1)1q},$$
$$\xi_{(n+1)2}\varphi_{(n+1)21}, \xi_{(n+1)2}\xi_{(n+1)22}, \ldots, \xi_{(n+1)2}\xi_{(n+1)2q} \quad \ldots, \quad \xi_{(n+1)p}\xi_{(n+1)p1}, \xi_{(n+1)p}\xi_{(n+1)p2},$$
$$\ldots, \xi_{(n+1)p}\xi_{(n+1)pq}),$$

it follows

$$\amalg^{\star}_{n,p,q} / (I^{\star}_{n,p,q}, \varphi_{n+1}) \cong \amalg^{\star}_{(n-1),p,q}/I^{\star}_{(n-1),p,q} \bigotimes_{j=1}^{p}{}_K K[V(S_q)]/(I(S_q)) \otimes_K$$
$$K[\varphi_{(n+1)1}, \varphi_{(n+1)2}, \ldots, \varphi_{(n+1)q}].$$

By using Lemma 2.2.9

$$\operatorname{depth}(\amalg^{\star}_{n,p,q}/(I^{\star}_{n,p,q}, \varphi_{n+1})) = \operatorname{depth}(\amalg^{\star}_{(n-1),p,q}/I^{\star}_{(n-1),p,q}) + \sum_{j=1}^{p} \operatorname{depth}(K[V(S_q)]/I(S_q))$$
$$+ \operatorname{depth} K[\varphi_{(n+1)1}, \varphi_{(n+1)2}, \ldots, \varphi_{(n+1)q}].$$

Again by induction and Theorem 2.2.13 and Lemma 2.2.12,

$$\operatorname{depth}(\amalg^{\star}_{n,p,q}/(I^{\star}_{n,p,q}, \varphi_{n+1})) = (n+1)(p+q).$$

Since $\operatorname{depth}(\amalg^{\star}_{n,p,q}/(I^{\star}_{n,p,q} : \varphi_{n+1})) \geq \operatorname{depth}(\amalg^{\star}_{n,p,q}/(I^{\star}_{n,p,q}, \varphi_{n+1}))$, hence by Depth Lemma $\operatorname{depth}(\amalg^{\star}_{n,p,q}/I^{\star}_{n,p,q}) \geq (n+1)(p+q)$. Now since $\xi_{(n+1)1}\xi_{(n+1)2}\ldots\xi_{(n+1)q} \notin I^{\star}_{n,p,q}$ and

$$(I^{\star}_{n,p,q} : \xi_{(n+1)1}\xi_{(n+1)2}\ldots\xi_{(n+1)q}) = (I^{\star}_{(n-1),p,q}, \varphi_{n+1}, \xi_{(n+1)11}, \xi_{(n+1)12}, \ldots, \xi_{(n+1)1q},$$
$$\xi_{(n+1)21}, \xi_{(n+1)22}, \ldots, \xi_{(n+1)2q} \quad \ldots, \xi_{(n+1)p1}, \xi_{(n+1)p2}, \ldots, \xi_{(n+1)pq}),$$

we obtain

$$\amalg^{\star}_{n,p,q} / (I^{\star}_{n,p,q} : \xi_{(n+1)1}\xi_{(n+1)2}\ldots\xi_{(n+1)q}) \cong \amalg^{\star}_{(n-1),p,q}/I^{\star}_{(n-1),p,q}$$
$$\otimes_K K[\xi_{(n+1)1}, \xi_{(n+1)2}, \ldots, \xi_{(n+1)2}, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \ldots, \varphi_{(n+1)q}].$$

Thus by induction and Lemma 2.2.12

$$\operatorname{depth}(\amalg^{\star}_{n,p,q}/(I^{\star}_{n,p,q} : \xi_{(n+1)1}\xi_{(n+1)2}\ldots\xi_{(n+1)q})) = (n+1)(p+q).$$

By Corollary 2.2.7, $\operatorname{depth}(\amalg^{\star}_{n,p,q}/I^{\star}_{n,p,q}) \leq (n+1)(p+q)$. Hence

$$\operatorname{depth}(\amalg^{\star}_{n,p,q}/I^{\star}_{n,p,q}) = (n+1)(p+q).$$

The consequence for the Stanley depth follows by applying Lemma 2.1.7 and Lemma 2.2.11 rather than Depth Lemma and Lemma 2.2.9 on exact sequences 3.8, 3.10 and 3.11, respectively. And we use Proposition 2.2.8 instead of Corollary 2.2.7.

□

Theorem 3.0.4. *For $n, q \geq 1$ and $p \geq 2$,* $\operatorname{depth}(\amalg_{n,p,q}/I_{n,p,q}) = \operatorname{sdepth}(\amalg_{n,p,q}/I_{n,p,q}) = (p+q)n+q$.

Proof. First, we consider depth.

***Case* 1.**

For $q \geq 1$, $p \geq 2$ and $n = 1$. Consider the exact sequence

$$0 \longrightarrow \amalg_{1,p,q}/(I_{1,p,q} : \varphi_2) \xrightarrow{\cdot\varphi_2} \amalg_{1,p,q}/I_{1,p,q} \longrightarrow \amalg_{1,p,q}/(I_{1,p,q}, \varphi_2) \longrightarrow 0, \tag{3.12}$$

by Depth Lemma

$$\operatorname{depth}(\amalg_{1,p,q}/I_{1,p,q}) \geq \min\{\operatorname{depth}(\amalg_{1,p,q}/(I_{1,p,q} : \varphi_2), \operatorname{depth}(\amalg_{1,p,q}/(I_{1,p,q}, \varphi_2)\}.$$

Here $(I_{1,p,q} : \varphi_2) = (\varphi_1, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}, \xi_{11}, \xi_{12}, \dots, \xi_{1p})$, it follows that

$$\amalg_{1,p,q}/(I_{1,p,q} : \varphi_2) \cong K[\varphi_2, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \xi_{111}, \xi_{112}, \dots, \xi_{11q}, \xi_{121}, \xi_{122}, \dots, \xi_{12q} \ \dots, \\ \xi_{1p1}, \xi_{1p2}, \dots, \xi_{1pq}].$$

This implies $\operatorname{depth}(\amalg_{1,p,q}/(I_{1,p,q} : \varphi_2)) = pq+q+1$, also $(I_{1,p,q}, \varphi_2) = (I(S_{p,q}), \varphi_2)$, so we obtain $\amalg_{1,p,q}/(I_{1,p,q}, \varphi_2) \cong K[V(I(S_{p,q})]/I(S_{p,q}) \otimes_K [\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}]$.
Using Theorem 2.2.14 and Lemma 2.2.12, we find $\operatorname{depth}(\amalg_{1,p,q}/(I_{1,p,q}, \varphi_2)) = 2q+p$. By Depth Lemma $\operatorname{depth}(\amalg_{1,p,q}/I_{1,p,q}) \geq \operatorname{depth}(\amalg_{1,p,q}/(I_{1,p,q}, \varphi_2))$. Hence

$$\operatorname{depth}(\amalg_{1,p,q}/I_{1,p,q})) \geq 2q+p.$$

Now since $\varphi_{21}\varphi_{22} \notin I_{1,p,q}$ and $(I_{1,p,q} : \varphi_{21}\varphi_{22}) = (I(S_{p,q}), \varphi_2)$, it follows

$$\text{Ш}_{1,p,q}/(I_{1,p,q} : \varphi_{21}\varphi_{22}) \cong K[V(I(S_{p,q}))]/I(S_{p,q}) \otimes_K [\varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}].$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\text{Ш}_{1,p,q}/(I_{1,p,q} : \varphi_{21}\varphi_{22})) = 2q + p$. Now by Corollary 2.2.7, $\operatorname{depth}(\text{Ш}_{1,p,q}/I_{1,p,q}) \leq 2q + p$. Hence $\operatorname{depth}(\text{Ш}_{1,p,q}/I_{1,p,q}) = 2q + p$.

***Case* 2.**

For $q \geq 1$, $p \geq 2$ and $n = 2$. Consider the exact sequence

$$0 \longrightarrow \text{Ш}_{2,p,q}/(I_{2,p,q} : \varphi_3) \xrightarrow{\cdot\varphi_3} \text{Ш}_{2,p,q}/I_{2,p,q} \longrightarrow \text{Ш}_{2,p,q}/(I_{2,p,q}, \varphi_3) \longrightarrow 0, \tag{3.13}$$

applying Depth Lemma

$$\operatorname{depth}(\text{Ш}_{2,p,q}/I_{2,p,q}) \geq \min\{\operatorname{depth}(\text{Ш}_{2,p,q}/(I_{2,p,q} : \varphi_3)), \operatorname{depth}(\text{Ш}_{2,p,q}/(I_{2,p,q}, \varphi_3))\}.$$

Here $(I_{2,p,q} : \varphi_3) = (I(S_{p,q}), \varphi_2, \varphi_{31}, \varphi_{32}, \ldots, \varphi_{3p}, \xi_{21}, \xi_{22}, \ldots, \xi_{2p})$, and we have

$\text{Ш}_{2,p,q}/(I_{2,p,q} : \varphi_3) \cong K[V(S_{p,q})]/I(S_{p,q}) \otimes_K K[\varphi_3, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}, \xi_{211}, \xi_{212}, \ldots, \xi_{21q}, \xi_{221}, \xi_{222}, \ldots, \xi_{22q} \ \ \ldots, \ \ \xi_{2p1}, \xi_{2p2}, \ldots, \xi_{2pq}]$. using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\text{Ш}_{2,p,q}/(I_{2,p,q} : \varphi_3)) = pq + 2q + p + 1$, and $(I_{2,p,q}, \varphi_3) = (I^{\star}_{1,p,q}, \varphi_3)$, so we obtain

$$\text{Ш}_{2,p,q}/(I_{2,p,q}, \varphi_3) \cong (\ \text{Ш}^{\star}_{1,p,q}\ /I^{\star}_{1,p,q}) \otimes_K K[\varphi_{31}, \varphi_{32}, \ldots, \varphi_{3q}].$$

Using Eq 3.9 and Lemma 2.2.12, we obtain $\operatorname{depth}(S_{2,p,q}/(I_{2,p,q}, \varphi_3)) = 2p + 3q$. Since $\operatorname{depth}(\text{Ш}_{2,p,q}/(I_{2,p,q} : \varphi_3)) \geq \operatorname{depth}(\text{Ш}_{2,p,q}/(I_{2,p,q}, \varphi_3))$, so by Depth Lemma

$$\operatorname{depth}(\text{Ш}_{2,p,q}/I_{2,p,q}) \geq 2p + 3q.$$

Now since $\varphi_{31}\varphi_{32} \notin I_{2,p,q}$ and $\text{Ш}_{2,p,q}/(I_{2,p,q} : \varphi_{31}\varphi_{32}) \cong (\text{Ш}_{1,p,q}/I^{\star}_{1,p,q}) \otimes_K [\varphi_{31}, \varphi_{32}, \ldots, \varphi_{3q}]$. Using Eq 3.9 and Lemma 2.2.12, $\operatorname{depth}(\text{Ш}_{2,p,q}/(I_{2,p,q} : \varphi_{31}\varphi_{32})) = 2p + 3q$. Thus by Corollary 2.2.7, $\operatorname{depth}(\text{Ш}_{2,p,q}/I_{2,p,q}) \leq 2p + 3q$. Hence $\operatorname{depth}(\text{Ш}_{2,p,q}/I_{2,p,q}) = 2p + 3q$.

***Case* 3.**

For $q \geq 1$, $p \geq 2$ and $n \geq 3$. We will prove this result by using Lemma 3.0.3. Consider the exact sequence

$$0 \longrightarrow \amalg_{n,p,q}/(I_{n,p,q} : \varphi_{n+1}) \xrightarrow{\cdot \varphi_{n+1}} \amalg_{n,p,q}/I_{n,p,q} \longrightarrow \amalg_{n,p,q}/(I_{n,p,q}\varphi_{n+1}) \longrightarrow 0, \qquad (3.14)$$

Using Depth Lemma

$$\operatorname{depth}(\amalg_{n,p,q}/I_{n,p,q}) \geq \min\{\operatorname{depth}(\amalg_{n,p,q}/(I_{n,p,q} : \varphi_{n+1})), \operatorname{depth}(\amalg_{n,p,q}/(I_{n,p,q}, \varphi_{n+1}))\}.$$

Here $(I_{n,p,q} : \varphi_{n+1}) = (I^{\star}_{(n-2),p,q}, \varphi_n, \xi_{n1}, \xi_{n2}, \dots, \xi_{np}, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q})$, so

$$\amalg_{n,p,q}/(I_{n,p,q} : \varphi_{n+1}) \cong (\amalg_{(n-2),p,q}/I^{\star}_{(n-2),p,q}) \otimes_K K[\varphi_{n+1}, \varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}, \xi_{n11}, \xi_{n12}, \dots,$$
$$\xi_{n1q}, \xi_{n21}, \xi_{n22}, \dots, \xi_{n2q} \quad \dots, \quad \xi_{np1}, \xi_{np2}, \dots, \xi_{npq}].$$

By Lemma 3.0.3 and Lemma 2.2.12, $\operatorname{depth}(\amalg_{n,p,q}/(I_{n,p,q} : \varphi_{n+1})) = (p+q)n+pq-p+1$, and $(I_{n,p,q}, \varphi_{n+1}) = (I^{\star}_{(n-1),p,q}, \varphi_{n+1})$, so we obtain

$$\amalg_{n,p,q}/(I_{n,p,q}, \varphi_{n+1}) \cong (\amalg^{\star}_{(n-1),p,q}/I^{\star}_{(n-1),p,q}) \otimes_K K[\varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}].$$

By Lemma 3.0.3 and Lemma 2.2.12, $\operatorname{depth}(\amalg_{n,p,q}/(I_{n,p,q}, \varphi_{n+1})) = n(p+q)+q$. Since

$$\operatorname{depth}(\amalg_{n,p,q}/(I_{n,p,q} : \varphi_{n+1})) \geq \operatorname{depth}(\amalg_{n,p,q}/(I_{n,p,q}, \varphi_{n+1})).$$

Hence by Depth Lemma $\operatorname{depth}(\amalg_{n,p,q}/I_{n,p,q}) \geq (p+q)n+q$. Now since $\varphi_{(n+1)1}\varphi_{(n+1)2} \notin I_{n,p,q}$ and $(I_{n,p,q} : \varphi_{(n+1)1}\varphi_{(n+1)2}) = (I^{\star}_{(n-1),p,q}, \varphi_{n+1})$, it follows

$$\amalg_{n,p,q}/(I_{n,p,q} : \varphi_{(n+1)1}\varphi_{(n+1)2}) \cong (\ \amalg^{\star}_{(n-1),p,q}\ /I^{\star}_{(n-1),p,q}) \otimes_K [\varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}].$$

By Lemma 3.0.3 and Lemma 2.2.12, $\operatorname{depth}(\amalg_{n,p,q}/(I_{n,p,q} : \varphi_{(n+1)1}\varphi_{(n+1)2})) = (p+q)n+q$. Thus by using Corollary 2.2.7, $\operatorname{depth}(\amalg_{n,p,q}/I_{n,p,q}) \leq n(p+q)+q$. Hence

$$\operatorname{depth}(\amalg_{n,p,q}/I_{n,p,q}) = (p+q)n+q.$$

The consequence for Stanley depth follows by applying Lemma 2.1.7 and Lemma 2.2.11 rather than Depth Lemma and Lemma 2.2.9 on the exact sequences 3.12, 3.13 and 3.14, respectively. And we use Proposition 2.2.8 instead of Corollary 2.2.7.

□

Chapter 4

Depth and Stanley depth of cyclic modules associated to q-fold bristled graphs of triangular and multi triangular ouroboros snake graphs

Let $p, q \geq 1$ and $n \geq 3$, a q-fold bristled graph of p-tringular ouroboros snake denoted by $Brs_q(\mho_{n,p})$ is obtained by connecting q vertices of degree 1 to each vertex of p-tringular ouroboros snake $\mho_{n,p}$. In particular, if $p = 1$ then $\mho_{n,1} = \mho_n$ is a tringular ouroboros snake and its q-fold bristled graph is denoted by $Brs_q(\mho_n)$.
Clearly $|V(Brs_q(\mho_{n,p}))| = (q+1)(p+1)n$. The graph $Brs_q(\mho_{n,p})$ has np vertices of degree $q+2$, n vertices of degree $2p+q+2$ and $(1+p)nq$ vertices of degree 1. So by using Lemma 1.3.11, we have $|E(Brs_q(\mho_{n,p}))| = ((p+1)q + 2p + 1)n$. For example see Figure 1.16 and 4.1.

The vertices of the $Brs_q(\mho_{n,p})$ graph are labelled by following sets of variables $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$, $\Big\{\{\varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}\}, \{\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}\}, \dots, \{\varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}\}\Big\}$, $\Big\{\{\xi_{11}, \xi_{21}, \dots, \xi_{n1}\}, \{\xi_{12}, \xi_{22}, \dots, \xi_{n2}\}, \dots, \{\xi_{1p}, \xi_{2p}, \dots, \xi_{np}\}\Big\}$, $\Big\{\{\xi_{111}, \xi_{112}, \dots, \xi_{11q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q} \dots, \xi_{n11}, \xi_{n12}, \dots, \xi_{n1q}\}, \{\xi_{121}, \xi_{122}, \dots, \xi_{12q}, \xi_{221}, \xi_{222}, \dots, \xi_{22q} \dots, \xi_{n21}, \xi_{n22}, \dots, \xi_{n2q}\}, \dots, \{\xi_{1p1}, \xi_{1p2}, \dots, \xi_{1pq}, \xi_{2p1}, \xi_{2p2}, \dots, \xi_{2pq} \dots, \xi_{np1}, \xi_{np2}, \dots, \xi_{npq}\}\Big\}$ see figure 1.16. Let $\mathcal{C}_{n,p,q} := K[\varphi_1, \varphi_2, \dots, \varphi_n, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q} \dots, \varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}, \xi_{11}, \xi_{21}, \dots, \xi_{n1}, \xi_{12}, \xi_{22}, \dots, \xi_{n2} \dots, \xi_{1p}, \xi_{2p}, \dots, \xi_{np}, \xi_{111}, \xi_{112}, \dots, \xi_{11q},$

$\xi_{211}, \xi_{212}, \ldots, \xi_{21q} \quad \ldots, \quad \xi_{n11}, \xi_{n12}, \ldots, \xi_{n1q}, \xi_{121}, \xi_{122}, \ldots, \xi_{12q}, \xi_{221}, \xi_{222}, \ldots, \xi_{22q} \quad \ldots,$
$\xi_{n21}, \xi_{n22}, \ldots, \xi_{n2q} \quad \ldots, \quad \xi_{1p1}, \xi_{1p2}, \ldots, \xi_{1pq}, \xi_{2p1}, \xi_{2p2}, \ldots, \xi_{2pq} \quad \ldots, \quad \xi_{np1}, \xi_{np2}, \ldots, \xi_{npq}]$
be the ring of polynomials in these variables over the field K. We can write

$$\text{II}_{n,p,q} = \mathcal{C}_{n,p,q}[\varphi_{n+1}, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \ldots, \varphi_{(n+1)q}].$$

Then $J_{n,p,q}$ is squarefree monomial ideal of $\mathcal{C}_{n,p,q}$. Now with the labelling as shown in Figure 4.1.

$$\mathcal{N}(J_{n,p,q}) = \bigcup_{i=1}^{n-1}\{\varphi_i\varphi_{i+1}\}\bigcup\{\varphi_n\varphi_1\}\bigcup\bigcup_{i=1}^{n}\Big\{\bigcup_{j=1}^{p}\{\varphi_i\xi_{ij}, \varphi_{i+1}\xi_{ij}\}\Big\}\bigcup\bigcup_{i=1}^{n}\Big\{\bigcup_{k=1}^{q}\{\varphi_i\varphi_{ik}\}\Big\}\bigcup$$
$$\bigcup_{i=1}^{n}\Big\{\bigcup_{j=1}^{p}\Big\{\bigcup_{k=1}^{q}\{\xi_{ij}\xi_{ijk}\}\Big\}\Big\}.$$

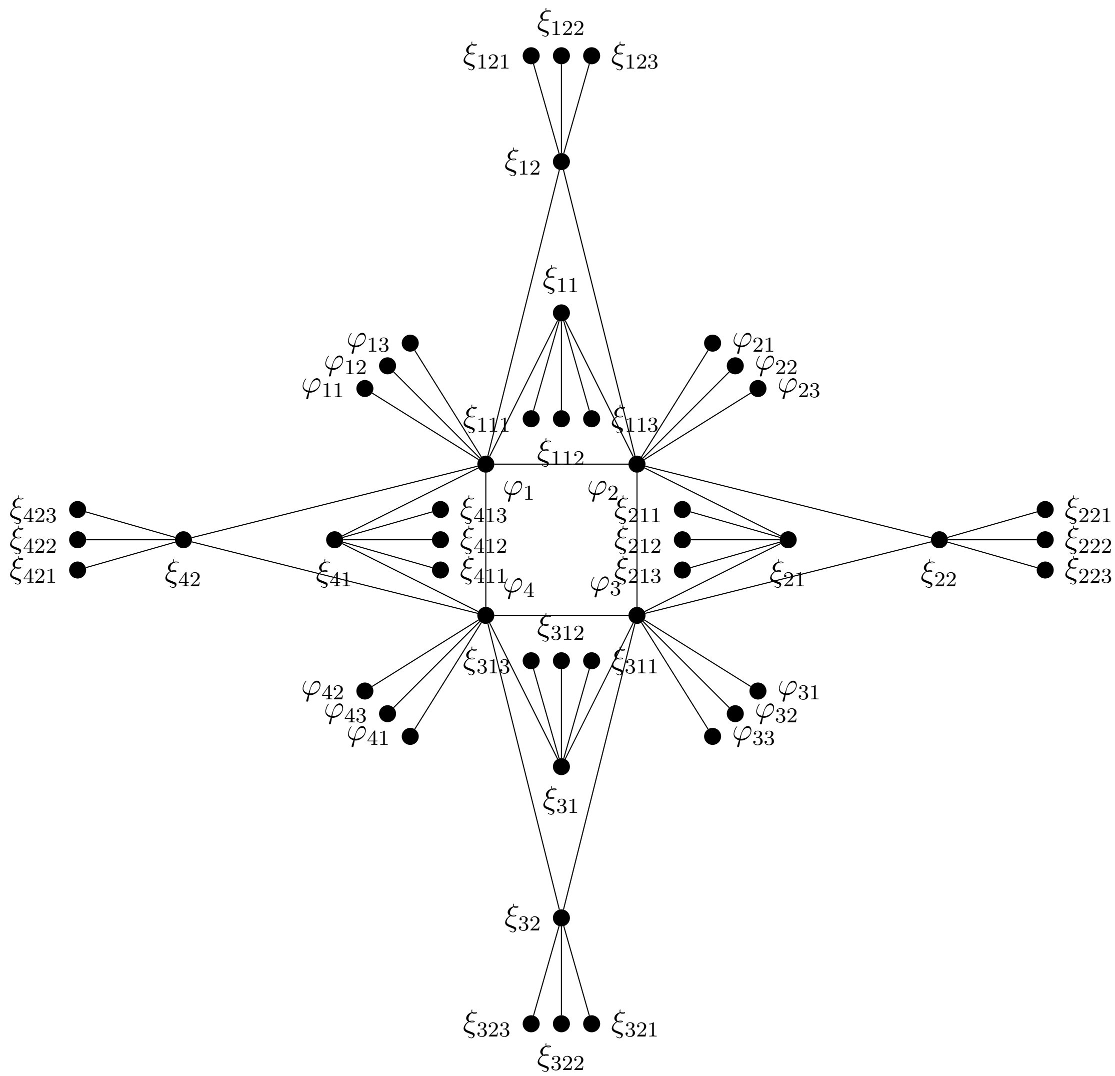

Figure 4.1: $Brs_3(\mho_{4,2})$
(3-fold bristled graph of $\mho_{4,2}$)

Let us consider a supergraph $Brs_q(\Lambda^{\star\star}_{n,p})$ of the graph $Brs_q(\Lambda^{\star}_{n,p})$. The vertex and edge sets of $Brs_q(\Lambda^{\star\star}_{n,p})$ are $V(Brs_q(\Lambda^{\star\star}_{n,p}) = V(Brs_q(\Lambda^{\star}_{n,p}))\bigcup\{\xi_{(n+2)1}, \xi_{(n+2)2}, \ldots, \xi_{(n+2)p}$, $\xi_{(n+2)11}, \xi_{(n+2)12}, \ldots, \xi_{(n+2)1q}, \xi_{(n+2)21}, \xi_{(n+2)22}, \ldots, \xi_{(n+2)2q}, \ldots, \xi_{(n+2)p1}, \xi_{(n+2)p2}, \ldots,$ $\xi_{(n+2)pq}\}$ and $E(Brs_q(\Lambda^{\star\star}_{n,p})) = E(Brs_q(\Lambda^{\star}_{n,p}))\bigcup\bigcup_{j=1}^{p}\Big\{\bigcup_{k=1}^{q}\{\xi_{(n+1)j}\xi_{(n+1)jk}\}, \varphi_{n+1}\xi_{(n+1)j}\Big\}$. For example of graph $Brs_q(\Lambda^{\star\star}_{n,p})$, see Figure 4.2. We denote the edge ideal of graph

$Brs_q(\Lambda^{\star\star}_{n,p})$ with $J^*_{n,p,q}$, where $J^*_{n,p,q}$ is the monomial ideal of the polynomial ring

$$\mathcal{C}^{\star}_{n,p,q} = \text{II}^{\star}_{n,p,q} \bigcup \bigcup_{j=1}^{p} \Big\{ \bigcup_{k=1}^{q} \{\xi_{(n+2)jk}\}, \xi_{(n+2)j} \Big\}.$$

The minimal set of monomial generators of $J^*_{n,p,q}$ is

$$\mathcal{N}(J^*_{n,p,q}) = M(I^*_{n,p,q}) \bigcup \bigcup_{j=1}^{p} \Big\{ \bigcup_{k=1}^{q} \{\xi_{(n+1)j}\xi_{(n+1)jk}\}, \varphi_{n+1}\xi_{(n+1)j} \Big\}$$

.

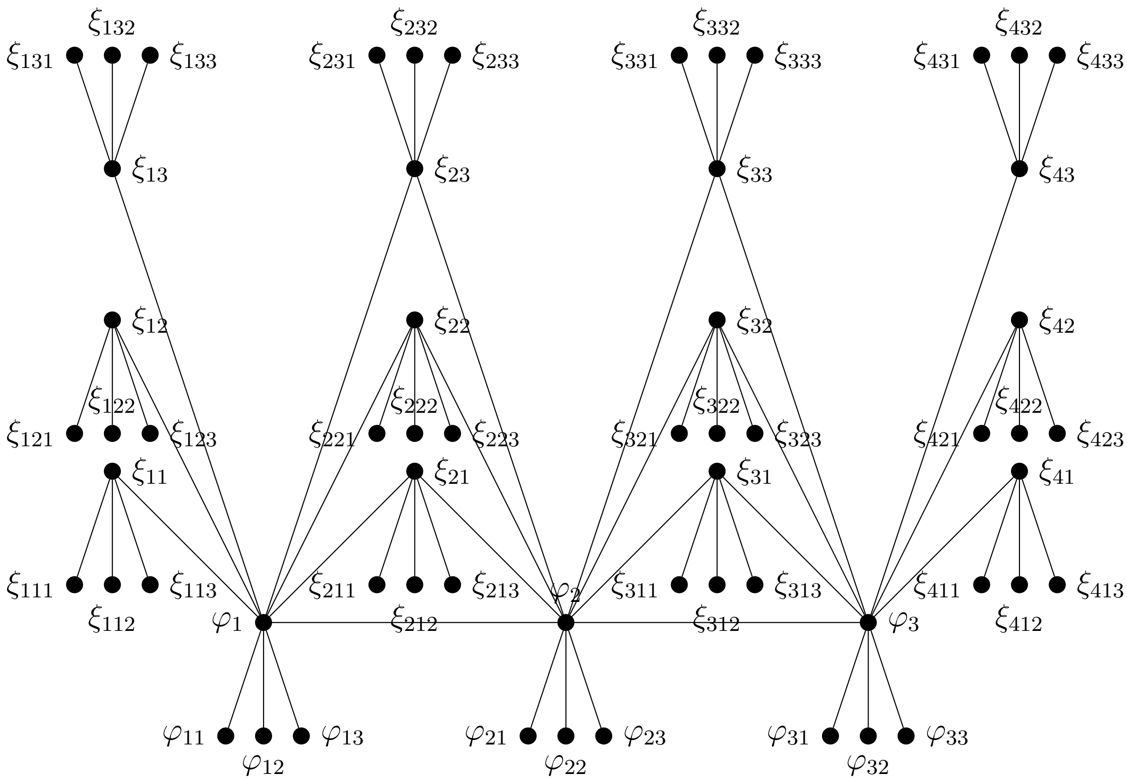

Figure 4.2: $Brs_3(\Lambda^{\star\star}_{2,3})$

4.0.1 Depth and Stanley depht of cyclic module associated with q-fold bristled graph of triangular ouroboros snake graph

If $p = 1$, then we can simply denote the edge ideals by $J_{n,q}$ and $J^*_{n,q}$ and the polynomial rings by $\mathcal{C}_{n,q}$ and $\mathcal{C}^*_{n,q}$.

Lemma 4.0.1. *For $n, q \geq 1$ and $p = 1$,* $\operatorname{depth}(\mathcal{C}^{\star}_{n,q}/J^{\star}_{n,q}) = \operatorname{sdepth}(\mathcal{C}^{\star}_{n,q}/J^{\star}_{n,q}) = (q+1)(n+1)+1$.

Proof. First, we consider depth.

***Case* 1.**

For $q \geq 1$, $n, p = 1$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q} : \varphi_2) \xrightarrow{\cdot\varphi_2} \mathcal{C}^{\star}_{1,q}/J^{\star}_{1,q} \longrightarrow \mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q}, \varphi_2) \longrightarrow 0, \tag{4.1}$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/J^{\star\star}_{1,q}) \geq \min\{\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q} : \varphi_2)), \operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(I^{\star}_{1,q}, \varphi_2))\}.$$

Here $(J^{\star}_{1,q} : \varphi_2) = (\xi_{11}\xi_{111}, \xi_{11}\xi_{112}, \dots, \xi_{11}\xi_{11q}, \varphi_1, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}, \xi_{21}, \xi_{31})$, it follows

$$\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q} : \varphi_2) \cong K[V(S_q)]/I(S_q) \otimes_K K[\varphi_2, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q}, \xi_{311}, \xi_{312}, \dots, \xi_{31q}].$$

Using Theorem 2.2.13 and Lemma 2.2.12, we have $\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q} : \varphi_2) = 3q + 2$. Also $(J^{\star}_{1,q}, \varphi_2) = (I(S_{2,q}), I(S_q), \varphi_2,)$, so we obtain

$$\mathcal{C}^{\star}/(J^{\star}_{1,q}, \varphi_2) \cong K[V(S_{2,q})]/I(S_{2,q}) \otimes_K K[V(S_q)]/I(S_q) \otimes_K [\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}].$$

By using Lemma 2.2.9

$$\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(I^{\star}_{1,q}, \varphi_2)) = \operatorname{depth}(K[V(S_{2,q})]/I(S_{2,q})) + \operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth}(K[\varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}]).$$

Using Corollary 2.2.14 and Theorem 2.2.13 and Lemma 2.2.12

$$\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q}, \varphi_2)) = 2q + 3.$$

As $\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q} : \varphi_2)) \geq \operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q}, \varphi_2))$, so by Depth Lemmadepth$(\mathcal{C}^{\star\star}_{1,q}/J^{\star}_{1,q}) \geq 2q+3$. Now since $\xi_{21}\xi_{31} \notin J^{\star}_{1,q}$ and $(J^{\star}_{1,q} : \xi_{21}\xi_{31}) = (I(S_q), \varphi_1, \varphi_2, \xi_{211}, \xi_{212}, \dots, \xi_{21q}, \xi_{311}, \xi_{312}, \dots, \xi_{31q})$, so we obtain

$$\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q} : \xi_{21}\xi_{31}) \cong K[V(S_q)]/I(S_q) \otimes_K [\xi_{21}, \xi_{31}, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}].$$

By Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/(J^{\star}_{1,q} : \xi_{21}\xi_{31})) = 2q+3$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/J^{\star}_{1,q}) \leq 2q+3$. Hence

$$\operatorname{depth}(\mathcal{C}^{\star}_{1,q}/J^{\star}_{1,q}) = 2q+3. \tag{4.2}$$

***Case* 2.**

For $q \geq 1$, $p=1$ and $n=2$. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q} : \varphi_3) \xrightarrow{\cdot\varphi_3} \mathcal{C}^{\star}_{2,q}/J^{\star}_{2,q} \longrightarrow \mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q}, \varphi_3) \longrightarrow 0, \tag{4.3}$$

applying Depth Lemma

$$\operatorname{depth}(\mathcal{C}^{\star}_{2,q}/J^{\star}_{2,q}) \geq \min\{\operatorname{depth}(\mathcal{C}^{\star}_{2,q}/(J^{\star\star}_{2,q} : \varphi_3), \operatorname{depth}(\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q}, \varphi_3)\}.$$

Here $(J^{\star}_{2,q} : \varphi_3) = (I(S_{2,q}), \varphi_2, \varphi_{31}, \varphi_{32}, \ldots, \varphi_{3q}, \xi_{31}, \xi_{41})$, so we have

$$\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q} : \varphi_3) \cong K[V(S_{2,q})]/I(S_{2,q}) \otimes_K K[\varphi_3, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}, \xi_{311}, \xi_{312}, \ldots, \xi_{31q}, \\ \xi_{411}, \xi_{412} \ldots, \xi_{41q}].$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q} : \varphi_3)) = 4q+3$, and $(J^{\star}_{2,q}, \varphi_3) = (J^{\star}_{1,q}, \varphi_3, \xi_{41}\xi_{411}, \xi_{41}\xi_{412}, \ldots, \xi_{41}\xi_{41q}) = (J^{\star}_{1,q}, I(S_q), \varphi_3)$, it follows that

$$\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q}, \varphi_3) \cong \mathcal{C}^{\star}_{1,q}/J^{\star}_{1,q} \otimes_K K[V(S_q)]/I(S_q) \otimes_K K[\varphi_{31}, \varphi_{32}, \ldots, \varphi_{3q}].$$

By using Lemma 2.2.9

$$\operatorname{depth}(\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q}, \varphi_3)) = \operatorname{depth}(\mathcal{C}^{\star}_{1,q}/J^{\star}_{1,q}) \\ + \operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth}(K[\varphi_{31}, \varphi_{32}, \ldots, \varphi_{3q}]).$$

Using Eq 4.2 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q}, \varphi_3)) = 3q+4$. Since

$$\operatorname{depth}(\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q} : \varphi_3)) \geq \operatorname{depth}(\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q}, \varphi_3)).$$

Therefore by Depth Lemma $\operatorname{depth}(\mathcal{C}^{\star}_{2,q}/J^{\star}_{2,q}) \geq 3q+4$. Now since $\xi_{31}\xi_{41} \notin J^{\star}_{2,q}$ and $(J^{\star}_{2,q} : \xi_{31}\xi_{41}) = (I(S_{2,q}), \varphi_2, \varphi_3, \xi_{311}, \xi_{312}, \ldots, \xi_{31q}, \xi_{411}, \xi_{412}, \ldots, \xi_{41q})$, so we obtain

$$\mathcal{C}^{\star}_{2,q}/(J^{\star}_{2,q} : \xi_{31}\xi_{41}) \cong K[V(S_{2,q})]/I(S_{2,q}) \otimes_K K[\xi_{31}, \xi_{41}, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}, \varphi_{31}, \varphi_{32}, \ldots, \varphi_{3q}].$$

By Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^\star_{2,q}/(J^\star_{2,q} : \xi_{31}\xi_{41})) = 3q+4$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}^\star_{2,q}/J^\star_{2,q}) \leq 3q+4$. Hence

$$\operatorname{depth}(\mathcal{C}^\star_{2,q}/J^\star_{2,q}) = 3q+4. \tag{4.4}$$

***Case* 3.**

For $q \geq 1$, $p = 1$ and $n \geq 3$. We prove this result by using induction on n. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}^\star_{n,q}/(J^\star_{n,q} : \varphi_{n+1}) \xrightarrow{\cdot\varphi_{n+1}} \mathcal{C}^\star_{n,q}/J^\star_{n,q} \longrightarrow \mathcal{C}^\star_{n,q}/(J^\star_{n,q}, \varphi_{n+1}) \longrightarrow 0, \tag{4.5}$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}^\star_{n,q}/J^\star_{n,q}) \geq \min\{\operatorname{depth}(\mathcal{C}^\star_{n,q}/(I^\star_{n,q} : \varphi_{n+1})), \operatorname{depth}(\mathcal{C}^\star_{n,q}/(I^\star_{n,q}, \varphi_{n+1}))\}.$$

Here $(J^\star_{n,q} : \varphi_{n+1}) = (I^\star_{(n-2),q}, \varphi_n, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}, \xi_{(n+1)1}, \xi_{(n+2)1})$, we get

$$\begin{aligned}\mathcal{C}^\star_{n,q}/(J^\star_{n,q} : \varphi_{n+1}) \cong \mathcal{C}^\star_{(n-2),q}/J^\star_{(n-2),q} \otimes_K K[&\varphi_{n+1}, \varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}, \xi_{(n+1)11}, \xi_{(n+1)12},\\ &\dots, \xi_{(n+1)1q}, \xi_{(n+2)11}, \xi_{(n+2)12}, \dots, \xi_{(n+2)1q}].\end{aligned}$$

By induction and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^\star_{n,q}/(J^\star_{n,q}) : \varphi_{n+1})) = (q+1)n + 2q + 1$. Also

$$\begin{aligned}(J^\star_{n,q}, \varphi_{n+1}) &= (J^\star_{(n-1),q}, \varphi_{n+1}, \xi_{(n+2)1}\xi_{(n+2)11}, \xi_{(n+2)1}\xi_{(n+2)12}, \dots, \xi_{(n+2)1}\xi_{(n+2)1q})\\ &= (J^\star_{(n-1),q}, I(S_q), \varphi_{n+1}),\end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{C}^\star_{n,q}/(J^\star_{n,q}, \varphi_{n+1}) \cong \mathcal{C}^\star_{(n-1),q}/(J^\star_{(n-1),q} \otimes_K K[V(S_q)]/I(S_q) \otimes_K&\\ K[\varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}].&\end{aligned}$$

By using Lemma 2.2.9

$$\begin{aligned}\operatorname{depth}(\mathcal{C}^\star_{n,q}/(J^\star_{n,q}, \varphi_{n+1})) = \operatorname{depth}(\mathcal{C}^\star_{(n-1),q}/(J^\star_{(n-1),q}) + \operatorname{depth}(K[V(S_q)]/I(S_q)) +&\\ \operatorname{depth}(K[\varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}]).&\end{aligned}$$

Again by induction and Theorem 2.2.13 and Lemma 2.2.12

$$\operatorname{depth}(\mathcal{C}^{\star}_{n,q}/(J^{\star}_{n,q}, \varphi_{n+1})) = (1+q)(n+1)+1.$$

Since, $\operatorname{depth}(\mathcal{C}^{\star}_{n,q}/(J^{\star}_{n,q} : \varphi_{n+1})) \geq \operatorname{depth}(\mathcal{C}^{\star}_{\varphi_{n+1}}/(J^{\star}_{n,q}, \varphi_{n+1}))$. Hence by Depth Lemma $\operatorname{depth}(\mathcal{C}^{\star}_{n,q}/J^{\star}_{n,q}) \geq (1+q)(n+1)+1$. Now since $\xi_{(n+1)1}\xi_{(n+2)1} \notin J^{\star}_{n,q}$ and

$$(J^{\star}_{n,q} : \xi_{(n+1)1}\xi_{(n+2)1}) = (J^{\star}_{(n-2),q}, \varphi_n, \varphi_{n+1}, \xi_{(n+1)11}, \xi_{(n+1)12}, \dots, \xi_{(n+1)1q}, \xi_{(n+2)11}, \\ \xi_{(n+2)12}, \dots, \xi_{(n+2)1q}).$$

We have

$$\mathcal{C}^{\star}_{n,q}/(J^{\star}_{n,q} : \xi_{(n+1)1}\xi_{(n+2)1}) \cong \mathcal{C}^{\star}_{(n-2),q}/(J^{\star}_{(n-2),q} \\ \otimes_K K[\xi_{(n+1)1}, \xi_{(n+2)1}, \varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \dots, \varphi_{(n+1)q}].$$

By induction and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{n,q}/(J^{\star}_{n,q} : \xi_{(n+1)1}\xi_{(n+2)1})) = (1+q)(n+1)+1$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}^{\star}_{n,q}/J^{\star}_{n,q}) \leq (1+q)(n+1)+1$. Hence

$$\operatorname{depth}(\mathcal{C}^{\star}_{n,q}/J^{\star}_{n,q}) = (1+q)(n+1)+1.$$

For the Stanley depth the consequence follows by applying Lemma 2.1.7 and Lemma 2.2.11 rather than Depth Lemma and Lemma 2.2.9 on the exact sequences 4.1, 4.3 and 4.5, respectively. And we use Proposition 2.2.8 instead of Corollary 2.2.7.

□

Theorem 4.0.2. *For $n, q \geq 1$ and $p = 1$ $\operatorname{depth}(\mathcal{C}_{n,q}/J_{n,q}) = \operatorname{sdepth}(\mathcal{C}_{n,q}/J_{n,q}) = (1+q)n$.*

Proof. First, we consider depth.

***Case* 1.**

For $q \geq$, $p = 1$ and $n = 3$. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}_{3,q}/(J_{3,q} : \varphi_3) \xrightarrow{\cdot\varphi_3} \mathcal{C}_{3,q}/J_{3,q} \longrightarrow \mathcal{C}_{3,q}/(J_{3,q}, \varphi_3) \longrightarrow 0, \tag{4.6}$$

using Depth Lemma

$$\operatorname{depth}(\mathcal{C}_{3,q}/J_{3,q}) \geq \min\{\operatorname{depth}(\mathcal{C}_{3,q}/(J_{3,q}:\varphi_3), \operatorname{depth}(\mathcal{C}_{3,q}/(J_{3,q},\varphi_3)\}.$$

Here $(I_{3,q}:\varphi_3) = (\varphi_1, \varphi_2, \varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}, \xi_{21}, \xi_{31}, \xi_{11}\xi_{111}, \xi_{11}\xi_{112}, \dots, \xi_{11}\xi_{11q})$. We get

$$\mathcal{C}_{3,q}/(J_{3,q}:\varphi_3) \cong K[V(S_q)]/I(S_q) \otimes_K K[\varphi_3, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}, \varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}, \xi_{211}, \\ \xi_{212}, \dots, \xi_{21q}, \xi_{311}, \xi_{312}, \dots, \xi_{31q}].$$

Using Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{3,q}/(J_{3,q}:\varphi_3)) = 4q+2$. Also $(J_{3,q}, \varphi_3) = (J^{\star}_{1,q}, \varphi_3)$, so $\mathcal{C}_{3,q}/(J_{3,q},\varphi_3) \cong \mathcal{C}^{\star}_{1,q}/J^{\star}_{1,q} \otimes_K K[\varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}]$.Using Eq 4.2 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{3,q}/(J_{3,q},\varphi_3)) = 3q+3$. Since, $\operatorname{depth}(\mathcal{C}_{3,q}/(J_{3,q}:\varphi_3)) \geq \operatorname{depth}(\mathcal{C}_{3,q}/(J_{3,q},\varphi_3))$. Hence by Depth Lemma $\operatorname{depth}(\mathcal{C}_{3,q}/J_{3,q}) \geq 3q+3$. Now since $\varphi_{31}\varphi_{32} \notin J_{3,q}$ and $(J_{3,q}:\varphi_{31}\varphi_{32}) = (J^{\star}_{1,q}, \varphi_3)$. So we have $\mathcal{C}_{3,q}/(J_{3,q}:\varphi_{31}\varphi_{32}) \cong \mathcal{C}^{\star}_{1,q}/J^{\star}_{1,q} \otimes_K K[\varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}]$. Using Eq 4.2 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{3,q}/(J_{3,q}:\varphi_{31}\varphi_{32})) = 3q+3$. Thus by Corollary 2.2.7 $\operatorname{depth}(\mathcal{C}_{3,q}/J_{3,q}) \leq 3q+3$. Hence

$$\operatorname{depth}(\mathcal{C}_{3,q}/J_{3,q}) = 3q+3.$$

***Case* 2.**

For $q \geq$, $p = 1$and $n = 4$. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}_{4,q}/(J_{4,q}:\varphi_4) \xrightarrow{\cdot\varphi_4} \mathcal{C}_{4,q}/J_{4,q} \longrightarrow \mathcal{C}_{4,q}/(J_{4,q},\varphi_4) \longrightarrow 0, \tag{4.7}$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}_{4,q}/J_{4,q}) \geq \min\{\operatorname{depth}(\mathcal{C}_{4,q}/(J_{4,q}:\varphi_4)), \operatorname{depth}(\mathcal{C}_{4,q}/(J_{4,q},\varphi_4))\}.$$

Here $(J_{4,q}:\varphi_4) = (I(S_{2,q}), \varphi_1, \varphi_3, \xi_{31}, \xi_{41}, \varphi_{41}, \varphi_{42}, \dots, \varphi_{4q})$. We obtain

$$\mathcal{C}_{4,q}/(J_{4,q}:\varphi_4) \cong K[V(S_{2,q})]/I(S_{2,q}) \otimes_K K[\varphi_4, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}, \xi_{311}, \\ \xi_{312}, \dots, \xi_{31q}, \xi_{411}, \xi_{412}, \dots, \xi_{41q}].$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{4,q}/(J_{4,q}:\varphi_4)) = 5q+3$. Also $(J_{4,q}, x_4) = (J^\star_{2,q}, \varphi_4)$, so we have $\mathcal{C}_{4,q}/(J_{4,q}, \varphi_4) \cong \mathcal{C}^\star_{2,q}/J^\star_{2,q} \otimes_K K[\varphi_{41}, \varphi_{42}, \ldots, \varphi_{4q}]$. Using Eq 4.4 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{4,q}/(J_{4,q}, \varphi_4)) = 4q+4$.
Since $\operatorname{depth}(\mathcal{C}_{4,q}/(J_{4,q}:\varphi_4)) \geq \operatorname{depth}(\mathcal{C}_{4,q}/(J_{4,q}, \varphi_4))$, hence $\operatorname{depth}(\mathcal{C}_{4,q}/J_{4,q}) \geq 4q+4$. Now since $\varphi_{41}\varphi_{42} \notin J_{4,q}$ and $(J_{4,q}:\varphi_{41}\varphi_{42}) = (J^\star_{2,q}, \varphi_4)$. So we have

$$\mathcal{C}_{4,q}/(J_{4,q}:\varphi_{41}\varphi_{42}) \cong \mathcal{C}^\star_{2,q}/J^\star_{2,q} \otimes_K K[\varphi_{41}, \varphi_{42}, \ldots, \varphi_{4q}].$$

Again Using Eq 4.4 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{4,q}/(J_{4,q}:\varphi_{41}\varphi_{42})) = 4q+4$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}_{4,q}/J_{4,q}) \leq 4q+4$. Hence $\operatorname{depth}(\mathcal{C}_{4,q}/J_{4,q}) = 4q+4$.

***Case* 3.**

For $q \geq 1$, $p = 1$ and $n \geq 3$. We will prove this result by using Lemma. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}_{n,q}/(J_{n,q}:\varphi_n) \xrightarrow{\cdot\varphi_n} \mathcal{C}_{n,q}/J_{n,q} \longrightarrow \mathcal{C}_{n,q}/(J_{n,q}, \varphi_n) \longrightarrow 0, \tag{4.8}$$

applying Depth Lemma

$$\operatorname{depth}(\mathcal{C}_{n,q}/J_{n,q}) \geq \min\{\operatorname{depth}(\mathcal{C}_{n,q}/(J_{n,q}:\varphi_n)), \operatorname{depth}(\mathcal{C}_{n,q}/(J_{n,q}, \varphi_n))\}.$$

Here $(J_{n,q}:\varphi_n) = (J^\star_{(n-4),q}, \varphi_1, \varphi_{n-1}, \xi_{n1}, \xi_{(n-1)1}, \varphi_{n1}, \varphi_{n2}, \ldots, \varphi_{nq})$, we have

$$\begin{aligned}\mathcal{C}_{n,q}/(J_{n,q}:\varphi_n) \cong (\mathcal{C}^\star_{(n-4),q}/J^\star_{(n-4),q}) \otimes_K K[&\varphi_n, \varphi_{11}, \varphi_{12}, \ldots, \varphi_{1q}, \varphi_{(n-1)1}, \varphi_{(n-1)2}, \ldots, \\ &\varphi_{(n-1)q}, \xi_{n1}, \xi_{n2}, \ldots, \xi_{nq}, \xi_{(n-1)1}, \xi_{(n-1)2}, \ldots, \xi_{(n-1)q}].\end{aligned}$$

Using Lemma 4.0.1 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{n,q}/(J_{n,q}:\varphi_n)) = (q+1)(n-3)+4q+2$. As $(J_{n,q}, \varphi_n) = (J^\star_{(n-2),q}, \varphi_n)$, so $\mathcal{C}_{n,q}/(J_{n,q}, \varphi_n) \cong \mathcal{C}^\star_{(n-2),q}/J^\star_{(n-2),q} \otimes_K K[\varphi_{n1}, \varphi_{n2}, \ldots, \varphi_{nq}]$. Using Lemma 4.0.1 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{n,q}/(J_{n,q}, \varphi_n)) = (1+q)n$. Since

$$\operatorname{depth}(\mathcal{C}_{n,q}/(J_{n,q}:\varphi_n)) \geq \operatorname{depth}(\mathcal{C}_{n,q}/(J_{n,q}, \varphi_n)),$$

hence by depth lemma $\operatorname{depth}(\mathcal{C}_{n,q}/J_{n,q}) \geq (1+q)n$. Now since $\varphi_{n1}\varphi_{n2} \notin J_{n,q}$ and $(J_{n,q}:\varphi_{n1}\varphi_{n2}) = (J^\star_{(n-2),q}, \varphi_n)$. So we have $\mathcal{C}_{n,q}/(J_{n,q}:\varphi_{n1}\varphi_{n2}) \cong \mathcal{C}^\star_{(n-2),q}/J^\star_{(n-1),q} \otimes_K$

$K[\varphi_{n1}, \varphi_{n2}, \ldots, \varphi_{nq}]$. By induction and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{n,q}/(J_{n,q} : \varphi_{n1}\varphi_{n2})) = (1+q)n$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}_{n,q}/J_{n,q}) \leq (1+q)n$. Hence

$$\operatorname{depth}(\mathcal{C}_{n,q}/J_{n,q}) = (1+q)n.$$

For the Stanley depth the consequence follows by applying Lemma 2.1.7 and Lemma 2.2.11 rather than Depth Lemma and Lemma 2.2.9 on the exact sequences 4.6, 4.7 and 4.8, respectively. And we use Proposition 2.2.8 instead of Corollary 2.2.7.

□

4.0.2 Depth and Stanley depht of cyclic module associated with q-fold bristled graph of multi triangular ouroboros snake graph

Lemma 4.0.3. *For $n, q \geq 1$ and $p \geq 2$,* $\operatorname{depth}(\mathcal{C}^\star_{n,p,q}/J^\star_{n,p,q}) = \operatorname{sdepth}(\mathcal{C}^\star_{n,p,q}/J^\star_{n,p,q}) = (p+q)(n+1)+p$.

Proof. First, we consider depth.

***Case* 1.**

For $q \geq 1$, $q \geq 2$ and $n = 1$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{C}^\star_{1,p,q}/(J^\star_{1,p,q} : \varphi_2) \xrightarrow{\cdot\varphi_2} \mathcal{C}^\star_{1,p,q}/J^\star_{1,p,q} \longrightarrow \mathcal{C}^\star_{1,p,q}/(J^\star_{1,p,q}, \varphi_2) \longrightarrow 0, \tag{4.9}$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}^\star_{1,p,q}/J^\star_{1,p,q}) \geq \min\{\operatorname{depth}(\mathcal{C}^\star_{1,p,q}/(J^\star_{1,p,q} : \varphi_2)), \operatorname{depth}(\mathcal{C}^\star_{1,p,q}/(J^\star_{1,p,q}, \varphi_2))\}.$$

Since $(J^\star_{1,p,q} : \varphi_2) = (\varphi_1, \xi_{21}, \xi_{22}, \ldots, \xi_{2p}, \xi_{31}, \xi_{32}, \ldots, \xi_{3p}, \varphi_{21}, \varphi_{22}, \ldots, \varphi_{2q}, \xi_{11}\xi_{111}, \xi_{11}\xi_{112}, \ldots, \xi_{11}\xi_{11q}, \xi_{12}\xi_{121}, \xi_{12}\xi_{122}, \ldots, \xi_{12}\xi_{12q} \ \ldots, \ \xi_{1p}\xi_{1p1}, \xi_{1p}\xi_{1p2}, \ldots, \xi_{1p}\xi_{1pq})$. We have

$$\mathcal{C}^\star_{1,p,q}/(J^\star_{1,p,q} : \varphi_2) \cong \bigotimes_{j=1}^{p}{}_K K[V(S_q)]/I(S_q) \otimes_K K[\varphi_2, \varphi_{11}, \varphi_{12}, \ldots, \varphi_{1q}, \xi_{211}, \xi_{212}, \ldots, \xi_{21q}, \xi_{221}, \xi_{222}, \ldots, \xi_{22q} \ \ldots, \ \xi_{2p1}, \xi_{2p2}, \ldots, \xi_{2pq}, \xi_{311}, \xi_{312}, \ldots, \xi_{31q}, \xi_{321}, \xi_{322}, \ldots, \xi_{32q} \ \ldots, \xi_{3p1}, \xi_{3p2}, \ldots, \xi_{3pq}].$$

Using Lemma 2.2.9

$$\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q}:\varphi_2)) = \sum_{j=1}^{p}\operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth}(K[\varphi_2,\varphi_{11},\varphi_{12},\ldots,$$
$$\varphi_{1q}, xi_{211},\xi_{212},\ldots,\xi_{21q},\xi_{221},\xi_{222},\ldots,\xi_{22q}\ \ \ldots,\ \ \xi_{2p1},\xi_{2p2},\ldots,\xi_{2pq},\xi_{311},\xi_{312},\ldots,\xi_{31q},\xi_{321},$$
$$\xi_{322},\ldots,\xi_{32q}\ \ \ldots,\ \ \xi_{3p1},\xi_{3p2},\ldots,\xi_{3pq}]).$$

By Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/(I^{\star}_{1,p,q}:\varphi_2)) = 2pq+p+q+1$, and

$$(J^{\star}_{1,p,q},\varphi_2) = (I(S_{2p,q}),\varphi_2,\xi_{31}\xi_{311},\xi_{31}\xi_{312},\ldots,\xi_{31}\xi_{31q},\xi_{32}\xi_{321},\xi_{32}\xi_{322},\ldots,\xi_{32}\xi_{32q}\ \ \ldots,$$
$$\xi_{3p}\xi_{3p1},\xi_{3p}\xi_{3p2},\ldots,\xi_{3p}\xi_{3pq}),$$

so we obtain

$$\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q},\varphi_2) \cong K[V(S_{2p,q})]/I(S_{2p,q}) \bigotimes_{j=1}^{p}{}_{K} K[V(S_q)]/I(S_q) \otimes_K K[\varphi_{21},\varphi_{22},\ldots,\varphi_{2q}].$$

By Lemma 2.2.9

$$\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q},\varphi_2)) = \operatorname{depth}(K[V(S_{2p,q})]/I(S_{2p,q}))$$
$$+\sum_{j=1}^{p}\operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth}(K[\varphi_{21},\varphi_{22},\ldots,\varphi_{2q}]).$$

Using Corollary 2.2.14, Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q},\varphi_2)) = 3p+2q$. Since $\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q}:\varphi_2)) \geq \operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q},\varphi_2))$, hence by Depth Lemma $\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/J^{\star}_{1,p,q}) \geq 3p+2q$. Now since $\xi_{31}\xi_{32}\ldots\xi_{3p} \notin J^{\star}_{1,p,q}$ and

$$(J^{\star}_{1,p,q}:\xi_{31}\xi_{32}\ldots\xi_{3p}) = (I(S_{2p,q}),\varphi_2,\xi_{311},\xi_{312},\ldots,\xi_{31q},\xi_{321},\xi_{322},\ldots,\xi_{32q}\ \ \ldots,\ \ \xi_{3p1},$$
$$\xi_{3p2},\ldots,\xi_{3pq}).$$

It follows that

$$\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q}:\xi_{31}\xi_{32}\ldots\xi_{3p}) \cong K[V(S_{2p,q})]/I(S_{2p,q}) \otimes_K K[\xi_{31},\xi_{32},\ldots,\xi_{3p},\varphi_{21},\varphi_{22},\ldots,\varphi_{2q}].$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/(J^{\star}_{1,p,q}:\xi_{31}\xi_{32}\ldots\xi_{3p})) = 3p+2q$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/J^{\star}{}_{1,p,q}) \leq 3p+2q$. Hence

$$\operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/J^{\star}_{1,p,q}) = 3p+2q. \tag{4.10}$$

***Case* 2.**

For $p, q \geq 1$ and $n = 2$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q} : \varphi_3) \xrightarrow{\cdot\varphi_3} \mathcal{C}^{\star}_{2,p,q}/J^{\star}_{2,p,q} \longrightarrow \mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q}, \varphi_3) \longrightarrow 0, \qquad (4.11)$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/J^{\star}_{2,p,q}) \geq \min\{\operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q} : \varphi_3), \operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q}, \varphi_3)\}.$$

Since $(J^{\star}_{2,p,q} : \varphi_3) = (I(S_{2p,q}), \varphi_2, \xi_{31}, \xi_{32}, \dots, \xi_{3q}, \xi_{41}, \xi_{42}, \dots, \xi_{4q}, \varphi_{31}, \varphi_{32}, \dots, \varphi_{3q})$,
We have

$$\mathcal{C}^{\star}_{2,p,q}/(I^{\star}_{2,p,q} : \varphi_3) \cong K[V(S_{2p,q})]/I(S_{2p,q}) \otimes_K K[\varphi_3, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}, \xi_{311}, \xi_{312}, \dots, \xi_{31q},$$
$$\xi_{321}, \xi_{322}, \dots, \xi_{32q} \ \ \dots, \ \ \xi_{3p1}, \xi_{3p2}, \dots, \xi_{3pq}, \xi_{411}, \xi_{412}, \dots, \xi_{41q}, \xi_{421}, \xi_{422}, \dots, \xi_{42q} \ \ \dots,$$
$$\xi_{4p1}, \xi_{4p2}, \dots, \xi_{4pq}].$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q} : \varphi_3)) = 2pq+2p+2q+1$, and $(J^{\star}_{2,p,q}, \varphi_3) = (J^{\star}_{1,p,q}, \varphi_3, \xi_{41}\xi_{411}, \xi_{41}\xi_{412}, \dots, \xi_{41}\xi_{41q}, \xi_{42}\xi_{421}, \xi_{42}\xi_{422}, \dots, \xi_{42}\xi_{42q} \ \ \dots,$ $\xi_{4p}\xi_{4p1}, \xi_{4p}\xi_{4p2}, \dots, \xi_{4p}\xi_{4pq})$, it follows that

$$\mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q}, \varphi_3) \cong \mathcal{C}^{\star}_{1,p,q}/J^{\star}_{1,p,q} \bigotimes_{j=1}^{p}{}_{K} K[V(S_q)]/I(S_q) \otimes_K K[\varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}].$$

By Lemma 2.2.9

$$\operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q}, \varphi_3)) = \operatorname{depth}(\mathcal{C}^{\star}_{1,p,q}/J^{\star}_{1,p,q}) + \sum_{j=1}^{p} \operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth}(K[\varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}]).$$

Using Eq 4.10, Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q}, \varphi_3)) = 4p + 3q$. By Depth Lemma $\operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/J^{\star}_{2,p,q}) \geq \operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/(J^{\star}_{2,p,q}, \varphi_3))$. Hence $\operatorname{depth}(\mathcal{C}^{\star}_{2,p,q}/J^{\star}_{2,p,q})) \geq 4p+3q$. Now since $\xi_{41}\xi_{42}\dots\xi_{4p} \notin J^{\star}_{2,p,q}$ and $(J^{\star}_{2,p,q} : \xi_{41}\xi_{42}\dots\xi_{4p}) =$ $(J^{\star}_{1,p,q}, \varphi_3, \xi_{411}, \xi_{412}, \dots, \xi_{41q}, \xi_{421}, \xi_{422}, \dots, \xi_{42q} \ \ \dots, \ \ \xi_{4p1}, \xi_{4p2}, \dots, \xi_{4pq})$, we obtain

$$\mathcal{C}^{\star}_{2,p,q}/(I^{\star}_{2,p,q} : \xi_{41}\xi_{42}\dots\xi_{4p}) \cong \mathcal{C}^{\star}_{1,p,q}/J^{\star}_{1,p,q} \otimes_K K[\xi_{41}, \xi_{42}, \dots, \xi_{4p}, \varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}].$$

Using Eq 4.10 and Lemma 2.2.12, we get $\operatorname{depth}(\mathcal{C}^\star_{2,p,q}/(J^\star_{2,p,q} : \xi_{41}\xi_{42}\ldots\xi_{4p})) = 4p+3q$. Thus by Corollary 2.2.7, $\operatorname{depth}(A^\star_{2,p,q}/J^\star_{2,p,q}) \leq 4p+3q$. Hence

$$\operatorname{depth}(\mathcal{C}^\star_{2,p,q}/I^\star_{2,p,q}) = 4p+3q. \tag{4.12}$$

***Case* 3.**

For $p, q \geq 1$ and $n \geq 3$. We will prove this consequence inductively. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}^\star_{n,p,q}/(J^\star_{n,p,q} : \varphi_{n+1}) \xrightarrow{\cdot\varphi_{n+1}} \mathcal{C}^\star_{n,p,q}/J^\star_{n,p,q} \longrightarrow \mathcal{C}^\star_{n,p,q}/(J^\star_{n,p,q}, \varphi_{n+1}) \longrightarrow 0, \tag{4.13}$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}^\star_{n,p,q}/J^\star_{n,p,q}) \geq \min\{\operatorname{depth}(\mathcal{C}^\star_{n,p,q}/(J^\star_{n,p,q} : \varphi_{n+1})), \operatorname{depth}(\mathcal{C}^\star_{n,p,q}/(J_{n,p,q}, \varphi_{n+1}))\}.$$

Since $(J^\star_{n,p,q} : \varphi_{n+1}) = (J^\star_{(n-2),p,q}, \varphi_n, \xi_{(n+1)1}, \xi_{(n+1)2}, \ldots, \xi_{(n+1)p}, \xi_{(n+2)1}, \xi_{(n+2)2}, \ldots, \xi_{(n+2)q}, \varphi_{(n+1)1}, \varphi_{(n+1)2}, \varphi_{(n+1)q})$, so we have

$$\begin{aligned}\mathcal{C}^\star_{n,p,q}/(J^\star_{n,p,q} : \varphi_{n+1}) \cong{}& (\mathcal{C}^\star_{(n-2),p,q}/J^\star_{(n-2),p,q}) \otimes_K [\varphi_{n+1}, \varphi_{n1}, \varphi_{n2}, \ldots, \varphi_{nq}, \xi_{(n+1)11}, \xi_{(n+1)12},\\ &\ldots, \xi_{(n+1)1q}, \xi_{(n+1)21}, \xi_{(n+1)22}, \ldots, \xi_{(n+1)2q} \;\; \ldots, \;\; \xi_{(n+1)p1}, \xi_{(n+1)p2}, \ldots, \xi_{(n+1)pq}, \xi_{(n+2)11},\\ &\xi_{(n+2)12}, \ldots, \xi_{(n+2)1q}, \xi_{(n+2)21}, \xi_{(n+2)22}, \ldots, \xi_{(n+2)2q} \;\; \ldots, \;\; \xi_{(n+2)p1}, \xi_{(n+2)p2}, \ldots, \xi_{(n+2)pq}].\end{aligned}$$

By induction and Lemma 2.2.9 and Lemma 2.2.12,

$$\operatorname{depth}(\mathcal{C}^\star_{n,p,q}/(J^\star_{n,p,q} : \varphi_{n+1})) = (p+q)(n-1) + p + 2pq + q + 1.$$

Also

$$\begin{aligned}(J^\star_{n,p,q}, \varphi_{n+1}) = {}& (J^\star_{(n-1),p,q}, \varphi_{n+1}, \xi_{(n+2)1}\xi_{(n+2)11}, \xi_{(n+2)1}\xi_{(n+2)12}, \ldots, \xi_{(n+2)1}\xi_{(n+2)1q},\\ &\xi_{(n+2)2}\xi_{(n+2)21}, \xi_{(n+2)2}\xi_{(n+2)22}, \ldots, \xi_{(n+2)2}\xi_{(n+2)2q} \;\; \ldots, \;\; \xi_{(n+2)p}\xi_{(n+2)p1}, \xi_{(n+2)p}\xi_{(n+2)p2},\\ &\ldots, \xi_{(n+2)p}\xi_{(n+2)pq}),\end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{C}^\star_{n,p,q}/(J^\star_{n,p,q}, \varphi_{n+1}) \cong{}& (\mathcal{C}^\star_{(n-1),p,q}/J^\star_{(n-1),p,q}) \bigotimes_{j=1}^{p}{}_K K[V(S_q)]/I(S_q) \otimes_K\\ &K[\varphi_{(n+1)1}, \varphi_{(n+1)2}, \ldots, \varphi_{(n+1)q}].\end{aligned}$$

By Lemma 2.2.9

$$\operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/(J^{\star}_{n,p,q},\varphi_{n+1})) = \operatorname{depth}((\mathcal{C}^{\star}_{(n-1),p,q}/J^{\star}_{(n-1),p,q}))+ \sum_{j=1}^{p}\operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth}(K[\varphi_{(n+1)1},\varphi_{(n+1)2},\dots,\varphi_{(n+1)q}]).$$

By induction and Theorem 2.2.13 and Lemma 2.2.12,

$$\operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/(J^{\star}_{n,p,q},\varphi_{n+1})) = (p+q)(n+1)+p.$$

Since $\operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/(J^{\star}_{n,p,q}:\varphi_{n+1})) \geq \operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/(J^{\star}_{n,p,q},\varphi_{n+1}))$, hence by depth lemma $\operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/J^{\star}_{n,p,q}) \geq (p+q)(n+1)+p$. Now as $\xi_{(n+2)1}\xi_{(n+2)2}\dots\xi_{(n+2)p} \notin J^{\star}_{n,p,q}$ and

$$(J^{\star}_{n,p,q}:\xi_{(n+2)1}\xi_{(n+2)2}\dots\xi_{(n+2)p}) = (J^{\star}_{(n-1),p,q},\varphi_{n+1},\xi_{(n+2)11},\xi_{(n+2)12},\dots, \xi_{(n+2)1q},\xi_{(n+2)21},\xi_{(n+2)22},\dots,\xi_{(n+2)2q}\ \ \dots,\ \ \xi_{(n+2)p1},\xi_{(n+2)p2},\dots,\xi_{(n+2)pq}),$$

we obtain

$$\mathcal{C}^{\star}_{n,p,q}/(J^{\star}_{n,p,q}:\xi_{(n+2)1}\xi_{(n+2)2}\dots\xi_{(n+2)p}) \cong (\mathcal{C}^{\star}_{(n-1),p,q}/J^{\star}_{(n-1),p,q}) \otimes_K K[\xi_{(n+2)1},\xi_{(n+2)2},\dots,\xi_{(n+2)p},\varphi_{(n+1)1},\varphi_{(n+1)2},\dots,\varphi_{(n+1)q}].$$

By induction and Lemma 2.2.12

$$\operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/(J^{\star}_{n,p,q}:\xi_{(n+2)1}\xi_{(n+2)2}\dots\xi_{(n+2)p})) = (p+q)(n+1)+p.$$

Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/J^{\star}_{n,p,q}) \leq (p+q)(n+1)+p$. Hence

$$\operatorname{depth}(\mathcal{C}^{\star}_{n,p,q}/J^{\star}_{n,p,q}) = (p+q)(n+1)+p.$$

For the Stanley depth the result follows by applying Lemma 2.1.7 and Lemma 2.2.11 rather than Depth Lemma and Lemma 2.2.9 on the exact sequences 4.9, 4.11 and 4.13, respectively. And we use Proposition 2.2.8 instead of Corollary 2.2.7.

□

Theorem 4.0.4. *For $n,q \geq 1$ and $p \geq 2$,* $\operatorname{depth}(\mathcal{C}_{n,p,q}/J_{n,p,q}) = \operatorname{sdepth}(\mathcal{C}_{n,p,q}/J_{n,p,q}) =$ *$(p+q)n$.*

Proof. First, we consider depth.

***Case* 1.**

For $p, q \geq 1$ and $n = 3$. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{C}_{3,p,q}/(J_{3,p,q} : \varphi_3) \xrightarrow{\cdot\varphi_3} \mathcal{C}_{3,p,q}/J_{3,p,q} \longrightarrow \mathcal{C}_{3,p,q}/(J_{3,p,q}, \varphi_3) \longrightarrow 0, \tag{4.14}$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}_{3,p,q}/J_{3,p,q}) \geq \min\{\operatorname{depth}(\mathcal{C}_{3,p,q}/(J_{3,p,q} : \varphi_3), \operatorname{depth}(\mathcal{C}_{3,p,q}/(J_{3,p,q}, \varphi_3)\}.$$

Since $(J_{3,p,q} : \varphi_3) = (\varphi_1, \varphi_2, \varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}, \xi_{21}, \xi_{22}, \dots, \xi_{2p}\xi_{31}, \xi_{32}, \dots, \xi_{3p}, \xi_{11}\xi_{111}, \xi_{11}\xi_{112}, \dots, \xi_{11}\xi_{11q}, \xi_{12}\xi_{121}, \xi_{12}\xi_{122}, \dots, \xi_{12}\xi_{12q}, \dots, \xi_{1p}\xi_{1p1}, \xi_{1p}\xi_{1p2}, \dots, \xi_{1p}\xi_{1pq})$. We have

$$\mathcal{C}_{3,p,q}/(J_{3,p,q} : \varphi_3) \cong \bigotimes_{j=1}^{p}{}_K K[V(S_q)]/I(S_q) \otimes_K K[\varphi_3, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q}, \xi_{221}, \xi_{222}, \dots, \xi_{22q}, \dots, \xi_{2p1}, \xi_{2p2}, \dots, \xi_{2pq}, \xi_{311}, \xi_{312}, \dots, \xi_{31q}, \xi_{321}, \xi_{322}, \dots, \xi_{32q}, \xi_{3p1}, \xi_{3p2}, \dots, \xi_{3pq}].$$

By Lemma 2.2.9

$$\operatorname{depth}(\mathcal{C}_{3,p,q}/(J_{3,p,q} : \varphi_3)) = \sum_{j=1}^{p} \operatorname{depth}(K[V(S_q)]/I(S_q)) + \operatorname{depth}(K[\varphi_3, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \varphi_{21}, \varphi_{22}, \dots, \varphi_{2q}, \xi_{211}, \xi_{212}, \dots, \xi_{21q}, \xi_{221}, \xi_{222}, \dots, \xi_{22q}, \dots, \xi_{2p1}, \xi_{2p2}, \dots, \xi_{2pq}, \xi_{311}, \xi_{312}, \dots, \xi_{31q}, \xi_{321}, \xi_{322}, \dots, \xi_{32q}, \xi_{3p1}, \xi_{3p2}, \dots, \xi_{3pq}]).$$

Using Theorem 2.2.13 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{3,p,q}/(J_{3,p,q} : \varphi_3)) = 2pq + 2q + p + 1$. As $(J_{3,p,q}, \varphi_3) = (J_{1,p,q}, \varphi_3)$, so $\mathcal{C}_{3,p,q}/(J_{3,p,q}, \varphi_3) \cong \mathcal{C}^{\star}_{1,p,q}/J^{\star}_{1,p,q} \otimes_K K[\varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}]$. Using Eq 4.10 and Lemma 2.2.12, we get $\operatorname{depth}(\mathcal{C}_{3,p,q}/(J_{3,p,q}, \varphi_3)) = 3p + 3q$. Since

$$\operatorname{depth}(\mathcal{C}_{3,p,q}/(J_{3,p,q} : \varphi_3)) \geq \operatorname{depth}(\mathcal{C}_{3,p,q}/(I_{3,p,q}, \varphi_3)).$$

Hence by depth lemma $\operatorname{depth}(\mathcal{C}_{3,p,q}/J_{3,p,q}) \geq 3q + 3q$. Now since $\varphi_{31}\varphi_{32} \notin J_{3,p,q}$ and $(J_{3,p,q} : \varphi_{31}\varphi_{32}) = (J_{1,p,q}, \varphi_3)$, so we obtain

$$\mathcal{C}^{\star}_{3,p,q}/(J^{\star}_{3,p,q} : \varphi_{31}\varphi_{32}) \cong \mathcal{C}^{\star}_{1,p,q}/J^{\star}_{1,p,q} \otimes_K K[\varphi_{31}, \varphi_{32}, \dots, \varphi_{3q}].$$

Again Using Eq 4.10 and Lemma 2.2.12, we have $\operatorname{depth}(\mathcal{C}_{3,2,q}/(J_{3,p,q},\varphi_3)) = 3p+3q$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}_{3,p,q}/J_{3,p,q}) \leq 3p+3q$. Hence

$$\operatorname{depth}(\mathcal{C}_{3,2,q}/J_{3,2,q}) = 3p+3q.$$

***Case* 2.**

For $p,q \geq 1$ and $n=4$. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}_{4,p,q}/(J_{4,p,q}:\varphi_4) \xrightarrow{\cdot\varphi_4} \mathcal{C}_{4,p,q}/J_{4,p,q} \longrightarrow \mathcal{C}_{4,p,q}/(J_{4,p,q},\varphi_4) \longrightarrow 0, \tag{4.15}$$

by Depth Lemma

$$\operatorname{depth}(\mathcal{C}_{4,p,q}/J_{4,p,q}) \geq \min\{\operatorname{depth}(\mathcal{C}_{4,p,q}/(J_{4,p,q}:\varphi_4)), \operatorname{depth}(\mathcal{C}_{4,p,q}/(J_{4,p,q},\varphi_4))\}.$$

Here $(J_{4,p,q}:\varphi_4) = (I(S_{2p,q}),\varphi_1,\varphi_3,\varphi_{41},\varphi_{42},\dots,\varphi_{4q},\xi_{31},\xi_{32},\dots,\xi_{3p},\xi_{41},\xi_{42},\dots,\xi_{4p})$. We have

$$\begin{aligned}\mathcal{C}_{4,p,q}/(J_{4,p,q}:\varphi_4) \cong K[V(S_{2p,q})]/I(S_{2p,q}) \otimes_K [&\varphi_4,\varphi_{11},\varphi_{12},\dots,\varphi_{1q},\varphi_{31},\varphi_{32},\dots,\varphi_{3q},\\ \xi_{311},\xi_{312},\dots,\xi_{31q},\xi_{321},\xi_{322},\dots,\xi_{32q}\ \ \dots,\ \ &\xi_{3p1},\xi_{3p2},\dots,\xi_{3pq},\xi_{411},\xi_{412},\dots,\xi_{41q},\xi_{421},\\ &\xi_{422},\dots,\xi_{42q}\ \ \dots,\ \ \xi_{4p1},\xi_{4p2},\dots,\xi_{4pq}].\end{aligned}$$

Using Corollary 2.2.14 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{4,p,q}/(J_{4,p,q}:\varphi_4)) = 2pq+3q+2p+1$. As $(J_{4,p,q},\varphi_4) = (J^{\star}_{2,p,q},\varphi_4)$, so $\mathcal{C}_{4,p,q}/(J_{4,p,q},\varphi_4) \cong (\mathcal{C}^{\star}_{2,p,q}/J^{\star}_{2,p,q}) \otimes_K [\varphi_{41},\varphi_{42},\dots,\varphi_{4q}]$. Using Eq 4.12 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{4,p,q}/(J_{4,p,q},\varphi_3)) = 4p+4q$. Since

$$\operatorname{depth}(\mathcal{C}_{4,p,q}/(J_{4,p,q}:\varphi_4)) \geq \operatorname{depth}(\mathcal{C}_{4,p,q}/(J_{4,p,q},\varphi_4)).$$

Hence by depth lemma $\operatorname{depth}(\mathcal{C}_{4,p,q}/J_{4,p,q}) \geq 4p+4q$. Now since $\varphi_{41}\varphi_{42} \notin J_{4,p,q}$ and $(J_{4,p,q}:\varphi_{41}\varphi_{42}) = (J^{\star}_{2,p,q},\varphi_4)$, so $\mathcal{C}^{\star}_{4,p,q}/(J^{\star}_{4,p,q}:\varphi_{41}\varphi_{42}) \cong (\mathcal{C}^{\star}_{2,p,q}/J^{\star}_{2,p,q})\otimes_K[\varphi_{41},\varphi_{42},\dots,\varphi_{4q}]$. Again Using Eq 4.12 and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{4,p,q}/(J_{4,p,q}:\varphi_{41}\varphi_{42})) = 4p+4q$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}_{4,p,q}/J_{4,p,q}) \leq 4p+4q$. Hence $\operatorname{depth}(\mathcal{C}_{4,p,q}/J_{4,p,q}) = 4p+4q$.

***Case* 3.**

For $p,q \geq 1$ and $n \geq 5$. We will prove this result by using Lemma. Consider the exact sequence

$$0 \longrightarrow \mathcal{C}_{n,p,q}/(J_{n,p,q} : \varphi_n) \xrightarrow{\cdot\varphi_n} \mathcal{C}_{n,p,q}/J_{n,p,q} \longrightarrow \mathcal{C}_{n,p,q}/(J_{n,p,q}, \varphi_n) \longrightarrow 0, \qquad (4.16)$$

applying Depth Lemma

$$\operatorname{depth}(\mathcal{C}_{n,p,q}/J_{n,p,q}) \geq \min\{\operatorname{depth}(\mathcal{C}_{n,p,q}/(J_{n,p,q} : \varphi_n)), \operatorname{depth}(\mathcal{C}_{n,p,q}/(J_{n,p,q}, \varphi_n))\}.$$

Here $(J_{n,p,q} : \varphi_n) = (J^{\star}_{(n-4),p,q}, \varphi_1, \varphi_{n-1}, \varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}, \xi_{n1}, \xi_{n2}, \dots, \xi_{np}, \xi_{(n-1)1}, \xi_{(n-1)2}, \dots, \xi_{(n-1)p})$, and we have

$$\mathcal{C}_{n,p,q}/(J_{n,p,q} : \varphi_n) \cong \mathcal{C}_{(n-4),p,q}/J^{\star}_{(n-4),p,q} \otimes_K K[\varphi_n, \varphi_{11}, \varphi_{12}, \dots, \varphi_{1q}, \varphi_{(n-1)1}, \varphi_{(n-1)2}, \dots, \varphi_{(n-1)q}, \xi_{n11}, \xi_{n12}, \dots, \xi_{n1q}, \xi_{n21}, \xi_{n22}, \dots, \xi_{n2q} \quad \dots, \quad \xi_{np1}, \xi_{np2}, \dots, \xi_{npq}, \xi_{(n-1)11}, \xi_{(n-1)12}, \dots, \xi_{(n-1)1q}, \xi_{(n-1)21}, \xi_{(n-1)22}, \dots, \xi_{(n-1)2q} \quad \dots, \quad \xi_{(n-1)p1}, \xi_{(n-1)p2}, \dots, \xi_{(n-1)pq}].$$

By induction and Lemma 2.2.12

$$\operatorname{depth}(\mathcal{C}_{n,p,q}/(J_{n,p,q}) : \varphi_n)) = (p+q)(n-3) + p + 2pq + 2q + 1.$$

Since $(J_{n,p,q}, \varphi_n) = (J^{\star}_{(n-2),p,q}, \varphi_n)$, so we obtain

$$\mathcal{C}_{n,2,q}/(J_{n,p,q}, \varphi_n) \cong \mathcal{C}^{\star}_{(n-2),p,q}/J^{\star}_{(n-2),p,q} \otimes_K K[\varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}]).$$

Again by induction and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{n,p,q}/(J_{n,p,q}, \varphi_n)) = (p+q)n$. Since

$$\operatorname{depth}(\mathcal{C}_{n,p,q}/(J_{n,p,q} : \varphi_n)) \geq \operatorname{depth}(\mathcal{C}_{n,p,q}/(J_{n,p,q}\varphi_n)).$$

Hence by depth lemma $\operatorname{depth}(\mathcal{C}_{n,p,q}/J_{n,p,q}) \geq (p+q)n$. Now since $\varphi_{n1}\varphi_{n2} \notin J_{n,p,q}$ and $(J_{n,p,q} : \varphi_{n1}\varphi_{n2}) = (J^{\star}_{(n-2),p,q}, \varphi_n)$. We obtain

$$\mathcal{C}_{n,p,q}/(J_{n,p,q} : \varphi_{n1}\varphi_{n2}) \cong \mathcal{C}^{\star}_{(n-2),p,q}/J^{\star}_{(n-2),p,q} \otimes_K K[\varphi_{n1}, \varphi_{n2}, \dots, \varphi_{nq}].$$

By induction and Lemma 2.2.12, $\operatorname{depth}(\mathcal{C}_{n,p,q}/(J_{n,p,q} : \varphi_{n1}\varphi_{n2})) = (p+q)n$. Thus by Corollary 2.2.7, $\operatorname{depth}(\mathcal{C}_{n,p,q}/J_{n,p,q}) \leq (p+q)n$.Hence $\operatorname{depth}(\mathcal{C}_{n,p,q}/J_{n,p,q}) = (p+q)n$.

For the Stanley depth the consequence follows by applying Lemma 2.1.7 and Lemma 2.2.11 rather than Depth Lemma and Lemma 2.2.9 on the exact sequences 4.14, 4.15 and 4.16, respectively. And we use Proposition 2.2.8 instead of Corollary 2.2.7.

$\square$

Bibliography

[1] Alipour, A., Tehranian, A. (2017). Depth and Stanley Depth of Edge Ideals of Star Graphs. *International Journal of Applied Mathematics and Statistics, 56*(4), 63-69.

[2] Anwar, I., Popescu, D. (2007). Stanley conjecture in small embedding dimension. *Journal of Algebra, 318*(2), 1027-1031.

[3] Apel, J. (2003). On a conjecture of RP Stanley; part II—quotients modulo monomial ideals. *Journal of Algebraic Combinatorics, 17*(1), 57-74.

[4] Barefoot, C. (2002). Block-cutvertex trees and block-cutvertex partitions. *Discrete mathematics, 256*(1-2), 35-54.

[5] Biró, C., Howard, D. M., Keller, M. T., Trotter, W. T., Young, S. J. (2010). Interval partitions and Stanley depth. *Journal of Combinatorial Theory, Series A, 117*(4), 475-482.

[6] Halsted, G. B. (1895). Arthur Cayley. *The American Mathematical Monthly, 2*(4), 102-106.

[7] Cimpoeaş, M. (2008). Stanley depth of complete intersection monomial ideals. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 205-211.

[8] Cimpoeas, M. (2012). Several inequalities regarding Stanley depth. *Romanian Journal of Math. and Computer Science, 2*(1), 28-40.

[9] Din, N. U., Ishaq, M., Sajid, Z. (2021). Values and bounds for depth and Stanley depth of some classes of edge ideals. *AIMS Mathematics, 6*(8), 8544-8566.

[10] Dummit, D. S., Foote, R. M. (2004). *Abstract algebra* (Vol. 3). Hoboken: Wiley.

[11] Duval, A. M., Goeckner, B., Klivans, C. J., Martin, J. L. (2016). A non-partitionable Cohen–Macaulay simplicial complex. *Advances in Mathematics, 299*, 381-395.

[12] Harary, F., Frucht, R. (1970). On the Corona of Two Graphs. *Aequationes mathematicae, 5*, 322-325.

[13] Herzog, J., Vladoiu, M., Zheng, X. (2009). How to compute the Stanley depth of a monomial ideal. *Journal of Algebra, 322*(9), 3151-3169.

[14] Ishaq, M. (2012). Upper bounds for the Stanley depth. *Communications in Algebra, 40*(1), 87-97.

[15] Mckelvie, C. (1975). William Hamilton Maxwell. *Notes and Queries, 22*(10), 450-a.

[16] Pattabiraman, K., Kandan, P. (2014). Weighted PI index of corona product of graphs. *Discrete mathematics, algorithms and applications, 6*(04), 1450055.

[17] Popescu, D. (2009). An inequality between depth and Stanley depth. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 377-382.

[18] Rauf, A. (2010). Depth and Stanley depth of multigraded modules. *Communications in Algebra, 38*(2), 773-784.

[19] Rinaldo, G. (2008). An algorithm to compute the Stanley depth of monomial ideals. *Le Matematiche, 63*(2), 243-256.

[20] Rosa, A. (1967). Cyclic Steiner triple systems and labelings of triangular cacti. *Scientia, 5*, 87-95.

[21] Sachs, H., Stiebitz, M., Wilson, R. J. (1988). An historical note: Euler's Königsberg letters. *Journal of Graph Theory, 12*(1), 133-139.

[22] Selvaraju, P., Balaganesan, P., Vasu, L., Suresh, M. L. (2014). Even sequential harmonious labeling of some cycle related graphs. *International Journal of Pure and Applied Mathematics, 97*(4), 395-407.

[23] Stanley, R. P. (1982). Linear Diophantine equations and local cohomology. *Inventiones mathematicae, 68*(2), 175-193.

[24] Villarreal, R. H. (2001). Monomial Algebras, Monographs and Textbooks in Pure and Applied Mathematics, 238. *New York: Marcel Dekker, Inc.*

[25] Warburg, E.. (1925). Zur Erinnerung an Gustav Kirchhoff. The Science of Nature. 13. 205-212. 10.1007/BF01558883.

[26] West, D. B. (2001). Introduction to graph theory (Vol. 2). Upper Saddle River: Prentice hall.