The Noether Symmetries and Invariants of Some Partial Differential Equations

by

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Dedicated To

My Mother

Abstract

A connection is obtained between isometries and Noether symmetries for the area-minimizing Lagrangian. It is shown that the Lie algebra of Noether symmetries for the Lagrangian minimizing an $(n-1)$-area enclosing a constant n-volume in a Euclidean space is $so(n) \oplus_s \mathbb{R}^n$ and in a space of constant curvature the Lie algebra is $so(n)$. Here for the non-compact space this has to be taken in the sense of being cut at a fixed boundary that respects the symmetry of the space and is not a volume enclosing hypersurface otherwise. Further if the space has one section of constant curvature of dimension n_1, another of n_2, etc. to n_k and one of zero curvature of dimension m, with $n \geq \sum_{j=1}^{k} n_j + m$ (as some of the sections may have no symmetry), then the Lie algebra of Noether symmetries is $\oplus_{j=1}^{k} so(n_j+1) \oplus (so(m) \oplus_s \mathbb{R}^m)$.

For a subclass of the general class of linear hyperbolic systems, obtainable from complex base hyperbolic equation, semi-invariant and joint invariants are investigate by complex and real symmetry analysis. A comparison of all the invariants derived by complex and real methods is presented here which shows that the complex procedure provides a few invariants different from those extracted by real symmetry analysis for a linear hyperbolic system.

The equations for the classification of symmetries of the scalar linear elliptic equation are obtained in terms of Cotton's invariants. New joint differential invariants of the scalar linear elliptic equations in two independent variables are derived, in terms of Cotton's invariants by application of the infinitesimal method. Joint differential invariants of the scalar linear elliptic equation are also derived from the bases of the joint differential invariants of the scalar linear hyperbolic equation under the application of the complex linear transformation. We also find a basis of joint differential invariants for such equations by utilization of the operators of invariant differentiation. The other invariants are functions of the bases elements and their invariant derivatives.

Cotton-type invariants for a subclass of a system of two linear elliptic equations, obtainable from a complex base linear elliptic equation, are derived both by splitting the corresponding complex

Cotton invariants of the base complex equation and from the Laplace-type invariants of the system of linear hyperbolic equations equivalent to the system of linear elliptic equations via linear complex transformations of the independent variables. It is shown that Cotton-type invariants derived from these two approaches are identical. Furthermore, Cotton-type and joint invariants for a general system of two linear elliptic equations are also obtained from the Laplace-type and joint invariants for a system of two linear hyperbolic equations equivalent to the system of linear elliptic equations by complex changes of the independent variables.

Contents

List of Publications

- A. Aslam, A. Qadir, Noether Symmetries of the Area-Minimizing Lagrangian, *Math. Comp. Appl.* **16** (4) (2011) 923-34.
- A. Aslam, F. M. Mahomed, Cotton-type and joint invariants for linear elliptic systems, *The Scientific World Journal*, Article ID 540705, (2013).
- F.M. Mahomed, A. G. Johnpillai and A. Aslam, Symmetry classification and joint differential invariants for the scalar linear (1+1) elliptic equation, submitted in *Communications in Nonlinear Science and Numerical Simulation.*
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Chapter 1

Introduction

The history of differential equations (DEs) goes back to Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716) who independently developed the foundation of calculus in the seventeenth century. Newton formulated his principles in the form of DEs. Leibniz mentioned the term *differential equation* for the first time in his letter to Newton in 1676 and then in 1684 used it in his publication [58]. Since then, differential equations are frequently used as mathematical models in science, technology, engineering, physics, economics, epidemiology, cosmology and many other disciplines. Einstein formulated his famous field equations, to explain the evolution of the universe, in the form of DEs.

DEs involve independent variables and dependent variables together with the derivatives of the dependent variables. An nth-order DE is one in which the highest order of the derivative of the dependent variable is n. If the unknown variable depends on a single independent variable, the equations are named as *ordinary differential equations* (ODEs). While, if a dependent variable is a function of several independent variables, so that the given equation involves the independent variables, dependent variable and partial derivatives of the dependent variables of two or more independent variables, then it is a *partial differential equation* (PDE). In the course of development of the theory of DEs, the need arose to determine the functional dependencies between the variables involved. In other words, the problem of solving DEs was born.

In 1691, Leibniz developed the method for the integration of DE, known as the *separation of variables*. He also formulated the techniques for the integration of the first-order homogeneous and nonhomogeneous linear DEs. John and James Bernoulli, who played a fundamental role in the theory of DEs and the expression "separation of variables" was first used by John Bernoulli [11]. Euler [28] developed many important ideas in integration theory of DEs such as integrating factors, power series solutions and method of variation etc. Taylor pointed out a particular solution of the DEs, known as the *singular solutions.* The classical theorems on the existence and uniqueness of the DEs are presented by the renowned mathematician Cauchy in his lectures (1820-1830). Before Cauchy mathematicians were working only on the formulation and solution of the DEs. Later, in 1876, Cauchy's theorems were embellished by Lipschitz and are known as Cauchy-Lipschitz. In 1893, this theory was taken further by Picard [87]. There are many other prominent mathematician, who contributed in the theory of the DEs, such as Liouville [66], Birkhoff, Gronwall, Lyapunov and many more.

There are several integration methods to find the solutions of the DEs, but these methods are applicable only to restricted classes of ODEs as well as PDEs. In the nineteenth century, a Norwegian mathematician, Marius Sophus Lie, introduced an outstanding method to find the solutions of DEs that virtually unify most of the integrating techniques known at that time. Lie's method for integrating the DEs is based on the groups of continuous transformations, known as *Lie groups.* Lie was inspired by the *Galois theory* developed, by Evariste Galois [95], during the investigation of the general algebraic equation and finding its solutions by radicals. In the history of mathematics Galois is one of the most romantic mathematicians because despite his death in a duel at age 21, his ill-fated political activism which often landed him in jail and the mystery surrounding his death, he managed to solve the major outstanding mathematical problem of his time by developing a new branch of mathematics, group theory. To generalize the *Galois theory* for DEs, Lie had to introduce continuously infinite groups instead of finite groups. Lie was not the only person who had the idea of generalizing GaloisŠ results to differential equations. Almost at the same time, there were other attempts to generalize the *Galois theory* like *differential Galois theory* and *Picard-Vessiot-theory.* There are still some questions, which need to be addressed, such as, the relationship between these theories. But Lie theory is the more powerful tool for finding the solutions of nonlinear DEs.

A common aspect of several solution techniques is to find invertible transformations to reduce nonlinear DEs to linear form. This is called the linearization problem, which is a special case of the equivalence problem. The class of all the DEs that can be mapped to one another under an invertible transformation form an equivalence class and the problem of finding all such equivalence classes is known as the equivalence problem [41]. Lie [61] showed that the necessary and sufficient conditions for a scalar second-order ODE to be linearizable by means of invertible point transformations to the simplest linear second order ODE is that the ODE be at most cubic in the first derivative and that its coefficients satisfy an overdetermined system of PDEs for two auxiliary functions.

Lie obtained, by invertible change of variables, both algebraic and practical criteria for a scalar second order ODE to be solvable. Linearization criteria for scalar third order ODEs were derived by Chern [24, 25] using point transformations and by Grebot [34, 35] using contact transformations. Some further improvements in the linearization of third order ODEs were made by Ibragimov and Meleshko [47] following the Lie approach and by Neut and Petitot [83] using Cartan procedure. The linearization problem for fourth order ODEs was addressed by Ibragimov et al. [48] using point and by Suksern et al. [96] using contact transformations. All first order ODEs are linearizable by point transformations while there is only one linearizable class for second order ODEs, i.e., the ODEs having 8 Lie symmetries. For nth-order ODEs ($n \geq 3$), it was proved by Mahomed and Leach [72], there are three linearizable classes, i.e., the ODEs containing $n+1$, $n+2$ or $n+4$ Lie symmetries can be transformed to linear form by the application of the point transformations. By differentiating the linearizable second order ODE and substituting the original equation in the third order ODE, known as conditional linearizability, linearization of third-order ODEs was discussed by Mahomed and Qadir [76, 78]. This procedure of linearization was extended to fourth order scalar ODEs by Mahomed and Qadir [77] and to system of third order ODEs by Mahomed et al. [73]. In addition to point transformations, contact transformations have been found to be of great use. These transformations involve independent variables, a dependent variable and its partial derivatives. In [63], Lie found the complete classification of all finite dimensional Lie groups of contact transformations acting on a space of one independent and one dependent complex variable. This include the groups of point transformations as well. Lie showed that the theory of PDEs of first-order reduces to the theory of groups of contact transformations. Contact transformations can be used to transform a first-order PDE to another PDE such that solving one PDE is tantamount to solving the other. This is not true for systems of PDEs because, in the case of many dependent variables, contact transformations reduce to prolonged Lie point transformations. Similarly these transformations are not sufficient for dealing with higher order DEs. Therefore, Lie raised the problem of the existence of higher order transformations, which was solved by B*ä*cklund [9] and are called the Lie-B*ä*cklund transformations.

A connection was established between the symmetries of the differential equations (geodesic equations) and the isometries of the manifold by Aminova and Aminov [6,7]. Independently, the same idea was further developed by Feroze et al. [31] for maximally symmetric spaces and a conjecture was stated for all spaces. This leads to the linearization criteria for a system of second-order quadratically semi-linear ODEs that have no terms of lower degree and lower order in them by Mahomed and

Qadir [74]. The conjecture stated in [31] was proved as a theorem and linearization criteria for second order ODEs were reviewed by Qadir [88]. A procedure was introduced for the projection of a system of n second order quadratically semi-linear ODE to $(n-1)$ second order cubically semi-linear ODE, following the projection procedure of Aminova and Aminov and by utilization of the above connection, linearization criteria for a system of two cubically semi-linear second order ODEs to the simplest system had been deduced by Mahomed and Qadir [75]. A consequence of this approach is the success in obtaining the Lie linearization conditions for scalar second-order ODEs. Algebraic linearization criteria via invertible change of variables for a system of second order ODEs had been obtained by Wafo Soh and Mahomed [103], Ayub et al. [8] and Bagderina [10]. The symmetry group classification for scalar ODEs was investigated by Mahomed [69] and Mahomed et al. [70] and for system of two second order ODEs by Gorringe and Leach [33] and Wafo Soh and Mahomed [104]. Linearization criteria had been developed further followed from the geometric developments and group classification of Lie by Chandrasekar [18–23]. The approach of complex symmetry analysis (CSA), was utilized by Ali [1] and Ali et al. [2]. This method provides a connection between an n-dimensional system of complex ODEs/PDEs and a $2n$-dimensional real system of ODEs/PDEs by a complex split of the base complex equation into real and imaginary parts. By the application of the CSA linearization criteria had been derived for a system of two ODEs by Ali et al. [4, 5] and by Safdar et al. [92] and for a system of four ODEs by Safdar et al. [91].

So far, some development relating the Lie symmetries of DE have been mentioned. Nothing has been said about the differential invariants of the group of equivalence transformation of DE. Differential invariants play a vital role in the transformation of the differential equation to integrable form. Tressé [97] derived two invariants of the equivalence group of point transformations for a scalar second order ODE and proved that their vanishing provides the necessary and sufficient conditions for its linearization. These conditions had been proved to be equivalent to the Lie linearization conditions by Mahomed and Leach [71]. They were derived by Ibragimov and Magi [46] using geometric arguments and for the Cartan equivalence method by Grissom et al. [36]. For linear ODEs semi-invariants were extensively derived, following the definition of the invariants directly by Laguerre [56], Cockle [26], Forsyth [32], Halphen [37], Harley [38] and Malet [81]. Lie showed that all variational problems and invariant DEs can be written in terms of differential invariants [60, 64, 65]. He also pointed out that the theory of differential invariants is based on the infinitesimal methods. Later, Ovsiannikov [85] and Ibragimov [41, 42] systematically developed the infinitesimal methods to calculate the invariants of the algebraic and differential equations, known as Lie infinitesimal methods.

In the course of the calculation of invariants, equivalence transformations play a fundamental role. A transformation that leaves the DEs form invariant is known as an equivalence transformation. The set of all the equivalence transformations form a continuous group. In 1770, two semi-invariants had been derived by Euler in his integral calculus [28] and then, in 1773, by Laplace [57] in his fundamental memoir on the integration of linear PDEs, known as the Laplace invariants, for the linear hyperbolic PDEs. Euler also proved that the solution of hyperbolic PDEs can be obtain by solving two first order ODEs if and only if one of the Laplace invariants is zero. Since the Laplace invariants are invariants only under a subgroup of equivalence transformation corresponding to the dependent variable, therefore these quantities are called *semi-invariants*. In 1900, for the linear elliptic PDEs Cotton [27] constructed the semi-invariants, known as the Cotton invariants. Laplace and Cotton invariants remain conserved under the linear changes of the dependent variables which respectively map the linear hyperbolic and elliptic equations into themselves. Linear hyperbolic and elliptic equations can be transformed into each other by the application of linear complex transformations [29, 55] of the independent variables, as do Laplace and Cotton invariants. Differential invariants can be used in the group classification of DEs. Ovsiannikov [86] used the Laplace invariants in the group classification of the hyperbolic equation by writing the determining equations for the symmetries of hyperbolic equation in terms of these invariants. The solution of the equivalence problem for scalar linear hyperbolic equations in two independent variables and some new invariants were given by Johnpillai and Mahomed [53] and Ibragimov [43]. Laplace-type and joint invariants for a system of two linear hyperbolic equations were derived by Tsaousi and Sophocleous [99] and Laplace-type invariants for a subclass of a system of two linear hyperbolic equations obtained from a complex linear hyperbolic equation were presented by Mahomed et al. [79]. Johnpillai et al. [54] deduced a complete basis of joint invariants for scalar linear hyperbolic PDE. Tsaousi and Sophocleous had given an extension of differential invariants for higher dimensional hyperbolic PDEs [101] and derived a general form of hyperbolic equations [100] that can be linearized by the application of the invariants. Laplace-type semi-invariants of the linear parabolic equation via a transformation of only the dependent variable had been derived by Ibragimov [44], which are the analogue of the Laplace invariants for hyperbolic equations. Semi-invariants of such equations that arise by transforming only the dependent variables were given by Ibragimov et al. [50] and the semi-invariants of the parabolic equations only for independent variables were constructed by Johnpillai and Mahomed [52] . Involving a change of both the dependent and independent variables reveals joint invariants of the parabolic PDEs (e.g., see [49, 68]). Laplace-type invariants of the linear parabolic equation had been extended to Ibragimov-type semi-invariants for a system of two parabolic-type PDEs using CSA by Mahomed et al. [80]. Semi-invariant and joint invariants for the Parabolic PDEs having three independent variables had been derived [102].

The plan of the thesis is as follows. In the remaining part of this chapter some basic definitions and preliminaries are given. Some crucial concepts are also reviewed here.

In the second chapter a relation between the isometries and Noether symmetries for the area-minimizing Lagrangian is established. Some connections between Noether symmetries and isometries have been found in the context of general relativity [13–15, 39, 40]. Recently, the relation of both the Lie and Noether symmetries of the geodesic for a general Riemannian manifold has been given [98]. The geodesic equations are the Euler-Lagrange (EL) equations for the arc-length-minimizing action. Here, an extension of the connection between the isometries and Neother symmetries to PDEs is found using the area-minimizing Lagrangian. It is shown that the Lie algebra of Noether symmetries for the Lagrangian minimizing an $(n-1)$-area enclosing a constant n-volume in a Euclidean space is $so(n) \oplus_s \mathbb{R}^n$ and in a space of non-zero curvature the Lie algebra is $so(n)$. Further if the space has one section of constant curvature of dimension n_1, another of n_2, etc. to n_k and one of zero curvature of dimension m, with $n \geq \sum_{j=1}^{k} n_j + m$ (as some of the sections may have no symmetry), then the Lie algebra of Noether symmetries is $\oplus_{j=1}^{k} so(n_j+1) \oplus (so(m) \oplus_s \mathbb{R}^m)$.

In the third chapter, a subclass of general system of linear hyperbolic PDEs is investigated for associated invariants by complex as well as real methods. The invariants of a general linear system of two hyperbolic equations have been derived under the transformations of dependent and independent variables by the real infinitesimal method. The complex procedure relies on the correspondence of the system and associated invariants with the base complex equation and related complex invariants, respectively. A comparison of all the invariant quantities obtained by complex and real methods is presented which shows that the complex procedure provides a few invariants different from those extracted by real symmetry analysis.

In the fourth chapter, the equations for the classification of symmetries of the scalar linear elliptic PDE in two independent variables are obtained in terms of Cotton's invariants. This is the analogue of the work of Ovsiannikov [86], who used the Laplace invariants in the group classification of the hyperbolic PDEs by writing the determining equations for the symmetries of hyperbolic equations in terms of Laplace invariants. New joint differential invariants of the scalar linear elliptic PDE in two independent variables are derived in terms of Cotton's invariants by application of the infinitesimal method. A set of the maximum number of independent invariants is said to be a basis of invariants if all other invariants can be obtained from this set of invariants using invariant differentiations. Joint differential invariants of the scalar linear elliptic equations are derived from the basis of the joint differential invariants of the scalar linear hyperbolic equations under the application of the complex linear transformations. Basis of joint differential invariants are also found for such type of equations by utilization of the operators of invariant differentiation. The other invariants are functions of the basis elements and their invariant derivatives.

In the fifth chapter, Cotton-type invariants for a subclass of a systems of two linear elliptic equations, obtainable from a complex linear base elliptic equation, are derived both by split of the corresponding complex Cotton invariants of the base complex equation and from the Laplace-type invariants of the system of linear hyperbolic equations equivalent to the system of linear elliptic equations via linear complex transformations of the independent variables. It is shown that Cotton-type invariants derived from these two approaches are identical. Furthermore, Cotton-type and joint invariants for a general system of two linear elliptic equations are also obtained from the Laplace-type and joint invariants for a system of two linear hyperbolic equations equivalent to the system of linear elliptic equations by complex changes of the independent variables.

In the last chapter, conclusions and discussions are given.

1.1 Manifolds, Lie Derivatives and Isometries

To understand the properties of the geometry, some basic concepts are discussed in this section. Manifolds, that may correspond to a complicated curved space but locally looks like $\mathbb{R}^n$, are one of the fundamental concepts of physics and mathematics.

1.1.1 Manifold

A manifold $\mathcal{M}_n$ [89] of dimension n is a separable, connected, Hausdorff space with a homeomorphism from each element of its open cover into $\mathbb{R}^n$.

A space is said to be *separable* if it has a countable dense subset. A set with a one-to-one correspondence with the set of natural numbers is said to be *countable*. A subset is said to be *dense* if its closure is the original set. The *closure* of a set is the smallest closed set containing it, e.g. $[0,1]$ is the closure of $(0,1)$. This statement is written as $[0,1] = \overline{(0,1)}$. The set of rational numbers, $\mathbb{Q}$, is a dense subset of $\mathbb{R}$, i.e., $\overline{\mathbb{Q}} = \mathbb{R}$. Since $\mathbb{Q}$ is countable, $\mathbb{R}$ is separable but $\mathbb{Q}$ is not. This condition ensures that the space is at least a continuum and there are no accumulation points.

A space is said to be *disconnected* if there exist two sets A and B, whose union is the whole space but which are disjoint, i.e., $A \cap B = \phi$, such that the closure of either is disjoint with the other, i.e., $\overline{A} \cap B = A \cap \overline{B} = \phi$. It is not necessary that $\overline{A} \cap \overline{B} = \phi$. A space is said to be *Hausdorff* if two distinct points possess disjoint neighborhoods. A *neighborhood* of a point is a set containing an open set containing the point. A *homeomorphism* is a one to one invertible, continuous, mapping. It will be used to assign coordinates to points on the manifold.

A differentiable and bijective function g_i such that, $g_i : \mathcal{V}_i \longrightarrow \mathbb{R}^n$, is called *coordinatization*, where $\{\mathcal{V}_i\}_{i\in I}$ is open cover of the manifold $\mathcal{M}_n$.

A collection of open sets $\{\mathcal{V}_i\}_{i\in I}$ is called an open cover of the manifold $\mathcal{M}_n$ if $\bigcup_{i\in I} \mathcal{V}_i = \mathcal{M}_n$. For every V_i, called *coordinate patch*, there exist some V_j such that $V_i \bigcap V_j \neq \phi$. Here (V_i, g_i) and $\mathbb{R}^n$ are called *coordinate chart* and *coordinate system* respectively.

Example 1: An n-dimensional Euclidean space $\mathbb{R}^n$ is an n-dimensional manifold . The single set $V = \mathbb{R}^n$ serve as the *coordinate patch* and the identity function $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as the *coordinatization.* Any open subset $U \subseteq \mathbb{R}^n$ is an example of n-dimensional manifold.

Example 2: The sphere $\mathbb{S}^2$

$$\mathbb{S}^2 = \{(x_1, x_2, x_3), x_1^2 + x_2^2 + x_3^2 = 1\},$$

is a nontrivial example of a 2-dimensional manifold and the two open sets

$$U = \mathbb{S}^2 \setminus \{(0, 0, -1)\}, V = \mathbb{S}^2 \setminus \{(0, 0, 1)\},$$

form an open cover for $\mathbb{S}^2$, got by removing the south and north poles respectively. The stereographic projections $f : U \longrightarrow \mathbb{R}^2$ and $g : V \longrightarrow \mathbb{R}^2$ from the respective pole are defined by

$$f(x_1, x_2, x_3) = (\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}),$$

$$g(x_1, x_2, x_3) = (\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}).$$

In this case, stereographic projections f and g are the *coordinatizations* for the respective *coordinate patches.*

1.1.2 Lie Derivative

The Lie derivative is a derivation along a curve, which transforms a tensor into another tensor of the same valence *without* the effect of coordinatizations. Geometrically, the Lie derivative of a tensor changes its components by transforming it from one point to another neighboring point in the direction of the tangent vector at that point on a curve.

Consider a derivation **T** on the manifold. The Lie derivative of **T** in the direction of the tangent vector **S** is defined by [82, 89]

$$\mathfrak{L}_{\mathbf{S}}(\mathbf{T}) = [\mathbf{S}, \mathbf{T}],$$

this can be written, in index notation notation, as

$$\begin{aligned}(\mathfrak{L}_{\mathbf{S}}\mathbf{T})^a &= \left(\mathrm{S}^{\underline{b}} \nabla_{\underline{b}} \mathrm{T}^a - \mathrm{T}^{\underline{b}} \nabla_{\underline{b}} \mathrm{S}^a\right) \\ &= \mathrm{S}^b \mathrm{T}^a{}_{,b} + \mathrm{S}^b \Gamma^a_{bc} \mathrm{T}^c - \mathrm{T}^b \mathrm{S}^a{}_{,b} - \Gamma^a_{bc} \mathrm{T}^b \mathrm{S}^c.\end{aligned}$$

Interchanging b and c in last term and using the symmetry property of the Christoffel symbol, i.e., $\Gamma^a_{bc} = \Gamma^a_{cb}$, last and second term cancel against each other and we have

$$(\mathfrak{L}_{\mathbf{S}}\mathbf{T})^a = \mathrm{S}^b \mathrm{T}^a{}_{,b} - \mathrm{T}^b \mathrm{S}^a{}_{,b}.$$

To find the Lie derivative of a general tensor, one need to calculate the Lie derivative of a covariant vector A, which turns out to be

$$(\mathfrak{L}_{\mathbf{S}}\mathbf{A})_a = \mathrm{S}^b \mathrm{A}_{a,b} + \mathrm{A}_b \mathrm{S}^b{}_{,a}.$$

For a general tensor **X** of valence $\begin{bmatrix} m \\ n \end{bmatrix}$ on the manifold the Lie derivative is

$$\begin{aligned}(\mathfrak{L}_{\mathbf{S}}\mathbf{X})^{a\ldots c}{}_{d\ldots f} =& \mathrm{S}^q \mathrm{X}^{a\ldots c}{}_{d\ldots f,q} - \mathrm{X}^{q\ldots c}{}_{d\ldots f} \mathrm{S}^a_{,q} - \cdots - \mathrm{X}^{a\ldots q}{}_{d\ldots f} \mathrm{S}^c_{,q} \\ &+ \mathrm{X}^{a\ldots c}{}_{q\ldots f} \mathrm{S}^q_{,d} + \cdots + \mathrm{X}^{a\ldots c}{}_{d\ldots q} \mathrm{S}^q_{,f}.\end{aligned} \tag{1.1}$$

1.1.3 Isometries

Isometries are the directions, $\mathbf{k} = k^a \frac{\partial}{\partial x^a}$, along which the Lie derivative of the metric tensor, **g**, is zero, i.e.

$$\mathfrak{L}_k \mathbf{g} = 0.$$

This equation can be written in component form, using the definition of the Lie derivative (1.1), as

$$g_{ab,c} k^c + g_{bc} k^c_{,a} + g_{ac} k^c_{,b} = 0, \tag{1.2}$$

where ",'' denotes derivative with respect to $x^a (a = 1, 2, ..., n)$. This equation forms a set of $n(n+1)/2$ linear first-order partial differential equations for n functions of n variables in general, called the

Killing equations.

Example: The metric for a two dimensional Euclidean space using Cartesian coordinates is

$$ds^2 = dx^2 + dy^2.$$

The condition for isometries (1.2) results the following system of DEs,

$$k^x_{,x} = 0, \quad k^y_{,y} = 0, \quad k^x_{,y} + k^y_{,x} = 0.$$

The solution of these equations gives the following three isometries

$$\mathbf{k}_1 = \frac{\partial}{\partial x}, \quad \mathbf{k}_2 = \frac{\partial}{\partial y}, \quad \mathbf{k}_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \tag{1.3}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ correspond to the translations along x-axis and y-axis, while $\mathbf{k}_3$ represents a rotation in the xy-plane.

In the next two sections, we present the geodesic equation and use the Kuhn-Tucker theorem [51, 90] to find the $(n-1)$-area minimizing Lagrangian keeping a constant n-volume.

1.1.4 Geodesic Equation

Free particles follow the shortest path to move from one point to another point. In a flat space straight lines are the shortest path from one point to another point but in curved space there are no straight lines. To find the shortest path between two point p and q in any arbitrary space, we use the Euler-Lagrange (EL) equations. The arc length between the points p and q is given by

$$\begin{aligned} s_{pq} &= \int_p^q ds = \int_p^q 1 ds, \\ &= \int_p^q g_{ab}\dot{x}^a\dot{x}^b ds = \int_p^q \mathcal{L}[x^a, \dot{x}^a] ds, \end{aligned} \tag{1.4}$$

where $\dot{x}^a = dx^a/ds$. The corresponding EL equation is given by

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^c}\right) - \frac{\partial \mathcal{L}}{\partial x^c} = 0, \tag{1.5}$$

substituting the value of Lagrangian $\mathcal{L}[x^a, \dot{x}^a] = g_{ab}\dot{x}^a\dot{x}^b$ in (1.5), we have the geodesic equation

$$\ddot{x}^d + \Gamma^d_{ab}\dot{x}^a\dot{x}^b = 0, \tag{1.6}$$

where

$$\Gamma^d_{ab} = \frac{1}{2}g^{df}\left(g_{bf,a} + g_{af,b} - g_{ab,f}\right). \tag{1.7}$$

1.1.5 Area Minimizing Action

In the n-dimensional space we minimize the $(n-1)$-area $A(S)$ of a hypersurface S given by $x^n = x^n(x^\alpha)$, $\alpha = 1, 2, ..., n-1$, keeping the n-volume $V(S)$ fixed. We define $y_\alpha = \partial x^n/\partial x^\alpha$. The area minimizing action is given as [16, 17]

$$I = A(S) + \lambda V(S) = \int_s n^p d^{n-1} s_p + \lambda \int_{\mathbf{V}} d^n V, \quad p = 1, 2, ..., n-1, \tag{1.8}$$

where λ is the Lagrange multiplier for the volume constraint yielding the Lagrangian

$$L = g_{n-1} + \lambda g_n$$

where g_n is the determinant of the n-metric of the volume. The resultant EL-equation is

$$\left(\frac{\partial}{\partial x^n} - D_\alpha \frac{\partial}{\partial y_\alpha}\right) L = 0.$$

As g_{n-1} and g_n do not depend on x^n and y_α respectively, so we have

$$D_\alpha \left(\ln \sqrt{\mid g_{n-1} \mid}\right)_{,}^{\alpha} - \lambda \left(\ln \sqrt{\mid g_n \mid}\right)_{,n} = 0,$$

where "$_,^{\ \alpha}$ "represents $\partial/\partial y_\alpha$.

In the next section, some basic concepts relating the symmetries of the DEs are presented. In particular, Lie symmetries of ODEs are defined and systematic methods are explained to calculate the symmetries of ODEs.

1.2 Lie Symmetry Analysis of ODEs

A symmetry group of a system of DEs is the largest group of transformations acting on the space of dependent and independent variables that maps a solution of the system of DEs into another solution. In other words, the solution manifold of the system of DEs remains invariants under a symmetry transformation of that system of DEs.

1.2.1 Point Transformations and Symmetry Generators

Let x and u be independent and dependent variables respectively. A point transformation

$$\overline{x} = \overline{x}(x, u), \quad \overline{u} = \overline{u}(x, u), \tag{1.9}$$

can be used to simplify a DE

$$E(x, u, u', u'', \cdots, u^{(n)}) = 0. \tag{1.10}$$

A set of invertible transformations that depends on an arbitrary parameter ϵ,

$$\overline{x} = \overline{x}(x, u; \epsilon), \quad \overline{u} = \overline{u}(x, u; \epsilon), \tag{1.11}$$

such that it contains the identity, i.e., for $\epsilon = 0$, $\overline{x}(x, u; 0) = x$, $\overline{u}(x, u; 0) = u$, and composition also belongs to the same set, i.e., $\overline{\overline{x}}(\overline{x}, \overline{u}; \overline{\epsilon}) = \overline{\overline{x}}(x, u; \overline{\overline{\epsilon}})$, for some $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}(\overline{\epsilon}, \epsilon)$ then the set of transformations (1.11) forms a group known as the *one-parameter group of point transformation.*

If the group of transformations (1.11) is such that ϵ is a continuous parameter, transformations are infinitely differentiable with respect to the independent and dependent variables and $\overline{\overline{\epsilon}}(\overline{\epsilon}, \epsilon)$ is an analytic function of $\overline{\epsilon}$ and ϵ then it form a *one-parameter Lie group of continuous transformations.* The one-parameter transformations (1.11) map one point (x, y) to another point $(\overline{x}, \overline{y})$ in the xy-plane and when the parameter ϵ changes from some initial value, say $\epsilon_{\circ}$ to some other value then the point $(\overline{x}, \overline{y})$ moves along some curve. For different initial points, different curves are obtained which can be mapped into one another under the action of the group (1.11). The set of these curves, called the orbits of the groups, can be complectly described by the field of its tangent vectors $\mathbf{U}$ and vice versa.

A concise explanation of the idea can be given by considering the Taylor expansion of the transformations (1.11) about $\epsilon = 0$,

$$\overline{x}(x, u; \epsilon) = x + \epsilon \frac{\partial \overline{x}}{\partial \epsilon}|_{\epsilon=0} + O(\epsilon^2) = x + \epsilon \mathbf{U} x + O(\epsilon^2), \tag{1.12}$$

$$\overline{u}(x, u; \epsilon) = u + \epsilon \frac{\partial \overline{u}}{\partial \epsilon}|_{\epsilon=0} + O(\epsilon^2) = x + \epsilon \mathbf{U} u + O(\epsilon^2). \tag{1.13}$$

Let

$$\xi(x, u) = \frac{\partial \overline{x}}{\partial \epsilon}|_{\epsilon=0}, \quad \eta(x, u) = \frac{\partial \overline{u}}{\partial \epsilon}|_{\epsilon=0}.$$

The transformations $x + \epsilon\xi$ and $u + \epsilon\eta$ are called the *infinitesimal transformations* of the one-parameter Lie group of point transformations and the operator given by

$$\mathbf{U} = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}, \tag{1.14}$$

is known as the *infinitesimal generator.*

Example 1: A one-parameter group of rotations is

$$\overline{x} \approx u \sin\epsilon + x \cos\epsilon, \quad \overline{u} \approx u \cos\epsilon - x \sin\epsilon, \tag{1.15}$$

and the corresponding infinitesimal transformations are

$$\overline{x} = x + \epsilon u, \quad \overline{u} = u - \epsilon x.$$

The infinitesimal generator of the group of rotations in the space spanned by the independent and dependent variables (1.15) is

$$\mathbf{U} = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}.$$

1.2.2 Prolongations of Point Transformations and their Symmetry Generators

To apply the one-parameter of point transformation (1.11) and the infinitesimal generator (1.14) to the DE (1.10). These transformations and generators have to be prolonged to the derivatives. The transformations can be extended as follows

$$d\overline{u} = \left(\frac{\partial \overline{u}}{\partial u}\right) du + \left(\frac{\partial \overline{u}}{\partial u}\right) dx,$$
$$d\overline{x} = \left(\frac{\partial \overline{x}}{\partial u}\right) du + \left(\frac{\partial \overline{x}}{\partial u}\right) dx,$$

$$\begin{aligned}
\overline{u}' =& \frac{d\overline{u}(x,u;\epsilon)}{d\overline{x}(x,u;\epsilon)}, \\
=& \frac{\left(\frac{\partial \overline{u}}{\partial u}\right) u' + \left(\frac{\partial \overline{u}}{\partial x}\right)}{\left(\frac{\partial \overline{x}}{\partial u}\right) u' + \left(\frac{\partial \overline{x}}{\partial x}\right)} = \overline{u}'(x,u,u';\epsilon), \\
\overline{u}'' =& \frac{\left(\frac{\partial \overline{u}'}{\partial u'}\right) u'' + \left(\frac{\partial \overline{u}'}{\partial u}\right) u' + \left(\frac{\partial \overline{u}'}{\partial x}\right)}{\left(\frac{\partial \overline{x}}{\partial u}\right) u' + \left(\frac{\partial \overline{x}}{\partial x}\right)} = \overline{u}'(x,u,u',u'';\epsilon), \\
\vdots \quad & \quad \vdots \\
\overline{u}^{(n)} =& \frac{\left(\frac{\partial \overline{u}^{(n-1)}}{\partial u^{(n-1)}}\right) u^{(n)} + \cdots + \left(\frac{\partial \overline{u}^{(n-1)}}{\partial u}\right) u' + \left(\frac{\partial \overline{u}^{(n-1)}}{\partial x}\right)}{\left(\frac{\partial \overline{x}}{\partial u}\right) u' + \left(\frac{\partial \overline{x}}{\partial x}\right)} = \overline{u}'(x,u,u',u'',\cdots,u^{(n)};\epsilon).
\end{aligned} \tag{1.16}$$

Now, the nth order prolongation of the infinitesimal generator is given by

$$\begin{aligned}
\overline{x} =& x + \epsilon\frac{\partial \overline{x}}{\partial \epsilon}|_{\epsilon=0} + \cdots = x + \epsilon\xi(x,u) + \cdots \\
\overline{u} =& u + \epsilon\frac{\partial \overline{u}}{\partial \epsilon}|_{\epsilon=0} + \cdots = u + \epsilon\eta(x,u) + \cdots \\
\overline{u}' =& u' + \epsilon\frac{\partial \overline{u}'}{\partial \epsilon}|_{\epsilon=0} + \cdots = u' + \epsilon\eta^{(1)}(x,u,u') + \cdots \\
\vdots \quad & \quad \vdots \\
\overline{u}^{(n)} =& u^{(n)} + \epsilon\frac{\partial \overline{u^{(n)}}}{\partial \epsilon}|_{\epsilon=0} + \cdots = u^{(n)} + \epsilon\eta^{(n)}(x,u,u',...,u^{(n)}) + \cdots
\end{aligned} \tag{1.17}$$

Using the expressions (1.17) in (1.16), we have

$$\begin{aligned}
\overline{u}' =& \frac{d\overline{u}}{d\overline{x}} = \frac{du + \epsilon d\eta + \cdots}{dx + \epsilon d\xi + \cdots}, \\
&= \left[u' + \epsilon\left(d\eta/dx\right) + \cdots\right]\left[1 + \epsilon\left(d\xi/dx\right) + \cdots\right]^{-1}, \\
u' + \epsilon\eta^{(1)}(x,u,u') + \cdots =& u' + \epsilon\left[\left(d\eta/dx\right) - u'\left(d\xi/dx\right)\right] + \cdots.
\end{aligned} \tag{1.18}$$

Similarly, for the $\eta^{(n)}(x,u,u',...,u^{(n)})$, we have

$$
\begin{aligned}
\overline{u}^{(n)} = \frac{d\overline{u}^{(n-1)}}{d\overline{x}} &= \left[u^{(n)} + \epsilon\left(\frac{d\eta^{(n-1)}}{dx}\right) + \cdots\right]\left[1 + \epsilon\left(\frac{d\xi}{dx}\right) + \cdots\right]^{-1}, \\
u^{(n)} + \epsilon\eta^{(n)}(x,u,u',...,u^{(n)}) + \cdots &= u^{(n)} + \epsilon\left[\left(\frac{d\eta^{(n-1)}}{dx}\right) - u^{(n)}\left(\frac{d\xi}{dx}\right)\right] + \cdots .
\end{aligned} \tag{1.19}
$$

Here $\eta^{(n)}(x,u,u',...,u^{(n)})$ is the nth prolongation of $\eta(x,u)$.

We summarize the results in the following theorem.

Theorem 1.2.1. *A one-parameter Lie group of point transformations* (1.11) *acting on the* (x,u)*-space can be extended to the* $(x,u,u',...,u^{(n)})$*-space by* (1.17) *and the corresponding infinitesimals are extended as follows*

$$
\begin{aligned}
\eta^{(1)} &= \frac{d\eta}{dx} - u'\frac{d\xi}{dx}, \\
\eta^{(n)} &= \frac{d\eta^{(n-1)}}{dx} - u^{(n)}\frac{d\xi}{dx}.
\end{aligned} \tag{1.20}
$$

The n*th order extended infinitesimal generator is given by*

$$
\mathbf{U}^{(n)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta^{(1)}\frac{\partial}{\partial u'} + \cdots + \eta^{(n)}\frac{\partial}{\partial u^{(n)}}. \tag{1.21}
$$

Definition 1.2.2. (**Symmetry of Differential Equation**) A one-parameter Lie group of point transformations

$$
\overline{x} = \overline{x}(x,u;\epsilon), \quad \overline{u} = \overline{u}(x,u;\epsilon), \tag{1.22}
$$

is said to be a point symmetry of the DE

$$
E(x,u,u',u'',\cdots,u^{(n)}) = 0. \tag{1.23}
$$

if and only if the DE (1.23) remains invariant under the nth extension of transformations (1.22), i.e., equation (1.23) and (1.22) imply

$$
E(\overline{x},\overline{u},\overline{u}',\overline{u}'',\cdots,\overline{u}^{(n)}) = 0. \tag{1.24}
$$

In other words, the transformations (1.22) map any solution of (1.23) into another solution of the same equation.

Theorem 1.2.3. *A one-parameter Lie group of point transformations* (1.22) *with the* n*th order extended generator* (1.21) *is said to be point symmetry of the DE* (1.23) *if and only if*

$$
\mathbf{U}^{(n)}E(x,u,u',u'',\cdots,u^{(n)}) = 0, \tag{1.25}
$$

whenever

$$E(x, u, u', u'', \cdots, u^{(n)}) = 0,$$

and the infinitesimal generator $\mathbf{U}$ *is called the symmetry generator.*

1.2.3 Multiple-Parameter Lie Groups of Transformations and Lie Algebras

A one-parameter Lie group of point transformations gives only one Lie point symmetry, but a DE may have more than one Lie point symmetries. To find all the Lie point symmetries, we have to consider the multiple-parameter Lie group of point transformations.

Definition 1.2.4. (**Multiple-Parameter Lie Groups of Transformations**) A transformation

$$\overline{x} = \overline{x}(x, u; \epsilon_m), \quad \overline{u} = \overline{u}(x, u; \epsilon_m), \quad m = 1, 2, \cdots, r, \tag{1.26}$$

is called an r-parameter Lie group of transformations if ϵ_m do not dependent on each other and satisfy all the properties for one-parameter Lie group of transformations. For an r-parameter Lie group of transformations, an infinitesimal generator depending on r independent parameters

$$\mathbf{U} = \sum_{m=1}^{r} \epsilon_m \mathbf{U}_m, \tag{1.27}$$

exist and an infinitesimal generator is associated to each parameter ϵ_m, as

$$\mathbf{U}_m = \xi_m \frac{\partial}{\partial x} + \eta_m \frac{\partial}{\partial x}. \tag{1.28}$$

Theorem 1.2.5. *For an r-parameter Lie group of point transformations, the commutators of any two infinitesimal generators is also an infinitesimal generator*

$$[\mathbf{U}_a, \mathbf{U}_b] = \mathbf{U}_a\mathbf{U}_b - \mathbf{U}_b\mathbf{U}_a = C^c_{ab}\mathbf{U}_c, \qquad a, b, c = 1, 2, \cdots, r, \tag{1.29}$$

where the constant coefficients C^c_{ab} *are known as the* structure constants *and these constants satisfy the following relations*

$$C^c_{ab} = C^c_{ba},$$
$$C^d_{ab}C^e_{dc} + C^d_{bc}C^e_{da} + C^d_{ca}C^e_{db} = 0.$$

The relation (1.29) *is known as the commutation relation.*

Definition 1.2.6. (**Lie Algebra**) A set of r-infinitesimal generators corresponding to an r-parameter Lie group of point transformations form an r-dimensional Lie algebra, $\mathcal{L}^r$, over the field of real numbers $\mathbb{R}$, under a product (1.29) (commutation relation), if for any $\mathbf{U}_a, \mathbf{U}_b, \mathbf{U}_c \in \mathcal{L}^r$ and $\alpha, \beta \in \mathbb{R}$:

(i). $\alpha\mathbf{U}_a + \beta\mathbf{U}_b \in \mathcal{L}^r$,

(ii). $\mathbf{U}_a + \mathbf{U}_b = \mathbf{U}_b + \mathbf{U}_a$,

(iii). $\mathbf{U}_a + (\mathbf{U}_b + \mathbf{U}_c) = (\mathbf{U}_a + \mathbf{U}_b) + \mathbf{U}_c$,

(iv). $[\mathbf{U}_a, \mathbf{U}_b] \in \mathcal{L}^r$,

(v). $[\mathbf{U}_a, \mathbf{U}_b] = -[\mathbf{U}_b, \mathbf{U}_a]$,

(vi). $[\mathbf{U}_a, [\mathbf{U}_b, \mathbf{U}_c]] + [\mathbf{U}_b, [\mathbf{U}_c, \mathbf{U}_a]] + [\mathbf{U}_c, [\mathbf{U}_a, \mathbf{U}_b]] = 0$,

(vii). $[\alpha\mathbf{U}_a + \beta\mathbf{U}_b, \mathbf{U}_c] = \alpha\,[\mathbf{U}_a, \mathbf{U}_c] + \beta\,[\mathbf{U}_b, \mathbf{U}_c]$.

An r-dimensional Lie algebra $\mathcal{L}^r$ is a vector space over the field of real numbers $\mathbb{R}$.

Definition 1.2.7. (**Subalgebra**) A subspace $\mathcal{S}$ of the r-dimensional Lie algebra, $\mathcal{L}^r$, is called the subalgebra of $\mathcal{L}^r$ if for $\mathbf{U}_a, \mathbf{U}_b \in \mathcal{S}$, $[\mathbf{U}_a, \mathbf{U}_b] \in \mathcal{S}$.

1.3 Lie Symmetry Analysis of PDEs

Let $\mathbf{x} = (x^i)$ and $\mathbf{u} = (u^\alpha)$ be n independent and m dependent variables respectively. The derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$ are denoted by $\partial\mathbf{u} = u_i^\alpha = D_i(u^\alpha)$, $\partial^2\mathbf{u} = u_{ij}^\alpha = D_iD_j(u^\alpha)$, $\ldots, \partial^k\mathbf{u} = u^\alpha_{i_1 i_2 \ldots i_k} = D_{i_1}D_{i_2}\cdots D_{i_k}(u^\alpha)$, where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \tag{1.30}$$

is the total derivative operator. Then a system $\mathbf{S}$ of kth order N PDEs can be written as

$$S^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, \cdots, \partial^k\mathbf{u}) = 0, \qquad \sigma = 1, 2, ..., N, \tag{1.31}$$

In order to deal with the symmetries of the system of PDEs (1.31), some definitions and terminologies are given in the remaining part of this section.

Definition 1.3.1. (**Point Transformations**) For m dependent $\mathbf{u} = (u^\alpha)$ and n independent $\mathbf{x} = (x^i)$ variables, a one-parameter Lie group of point transformations is of the form

$$\begin{aligned} \overline{x}^i =& f^i(x^i, u^\alpha; \epsilon), \\ \overline{u}^\alpha =& g^\alpha(x^i, u^\alpha; \epsilon), \end{aligned} \tag{1.32}$$

with the infinitesimal transformations

$$\begin{aligned} \overline{x}^i &= x^i + \epsilon\xi^i(x^i, u^\alpha) + O(\epsilon^2), \qquad i = 1, 2, ..., n, \\ \overline{u}^\alpha &= u^\alpha + \epsilon\eta^\alpha(x^i, u^\alpha) + O(\epsilon^2), \qquad \alpha = 1, 2, ..., m, \end{aligned} \tag{1.33}$$

where

$$\xi^i = \frac{\partial x^i}{\partial \epsilon}|_{\epsilon=0}, \qquad \eta^\alpha = \frac{\partial u^\alpha}{\partial \epsilon}|_{\epsilon=0}.$$

The infinitesimal transformations are generated by the infinitesimal generator

$$\mathbf{U} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{1.34}$$

Theorem 1.3.2. *For a one-parameter Lie group of point transformations* (1.32) *the kth extension of the corresponding infinitesimal generator* (1.34) *is given by*

$$\mathbf{U}^{(k)} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_i^{(1)\alpha} \frac{\partial}{\partial u_i^\alpha} + \cdots + \eta_{i_1,i_2,\ldots,i_k}^{(k)\alpha} \frac{\partial}{\partial u_{i_1,i_2,\ldots,i_k}^{(\alpha)}}, \tag{1.35}$$

where

$$\begin{aligned} \eta_i^{(1)\alpha} &= D_i \eta^\alpha - u_j^\alpha D_i \xi^j, \\ \eta_{i_1,i_2,\ldots,i_k}^{(k)\alpha} &= D_{i_k} \eta_{i_1,i_2,\ldots,i_{k-1}}^{(k-1)\alpha} - u_{i_1,i_2,\ldots,i_{k-1}j}^\alpha D_{i_k} \xi^j. \end{aligned} \tag{1.36}$$

Definition 1.3.3. (**Lie Point Symmetry of System of PDEs**) A one-parameter Lie group of point transformations (1.32) is said to be a Lie point symmetry of the system of PDEs (1.31) if and only if the system of PDEs remains invariant under one-parameter Lie group of transformations (1.32), i.e., by applying the transformations (1.32) on the system of PDEs (1.31), we have

$$S^\sigma(\overline{\mathbf{x}}, \overline{\mathbf{u}}, \partial\overline{\mathbf{u}}, \partial^2\overline{\mathbf{u}}, \cdots, \partial^k\overline{\mathbf{u}}) = 0, \qquad \sigma = 1, 2, ..., N. \tag{1.37}$$

In simple words, the solution manifold of the system (1.31) remains invariant under the transformations (1.32).

Theorem 1.3.4. *A one-parameter Lie group of point transformations* (1.32) *with the kth order extended generator* (1.35) *is said to be point symmetry of the DE* (1.31) *if and only if*

$$\mathbf{U}^{(k)} S^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, \cdots, \partial^k\mathbf{u}) = 0, \qquad \sigma = 1, 2, ..., N, \tag{1.38}$$

whenever

$$S^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, \cdots, \partial^k\mathbf{u}) = 0, \qquad \sigma = 1, 2, ..., N,$$

and the infinitesimal generator $\mathbf{U}$ *is called the symmetry generator.*

1.4 Euler-Lagrange Equation and Noether's Theorem

Emmy Noether, in her celebrated paper [84], established a correspondence between variational symmetries and conservation laws for a system of DEs admitting a variational principle. She proved that for any variational symmetry (group of point transformations under which action integral remains invariant) there is a conservation law and for any conservation law of a system of DEs, a variational symmetry exist for the corresponding action integral. If a variational principle is admitted by a system of DEs then the Euler-Lagrange (EL) equations (system of DEs) are obtained from the extremals of its action integral (variational principle).

1.4.1 Euler-Lagrange Equation

Theorem 1.4.1. *Let* $\mathbf{u}(\mathbf{x})$ *be a smooth function and* $A[\mathbf{u}]$ *be an action integral defined on some domain* $\mathfrak{D}$ *as*

$$A[\mathbf{u}] = \int_{\mathfrak{D}} \mathcal{L}[\mathbf{u}] d\mathbf{x}, \tag{1.39}$$

where $L[\mathbf{u}] = L(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, ..., \partial^k\mathbf{u})$ *is the Lagrangian. If* $\mathbf{u}(\mathbf{x})$ *is an extremum of* (1.39), *then* $\mathbf{u}(\mathbf{x})$ *satisfies*

$$E_\alpha(L[\mathbf{u}]) = \frac{\partial L}{\partial u^\alpha} - D_i \frac{\partial L}{\partial u_i^\alpha} + \cdots + (-1)^k D_{i_1} \cdots D_{i_k} \frac{\partial L}{\partial u^\alpha_{i_1 \ldots i_k}}. \tag{1.40}$$

This system of N *DE* (1.40) *is called the EL equations.*

Proof. Consider a Lagrangian $L[\mathbf{u}] = L(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, ..., \partial^k\mathbf{u})$ of order k, depending on m dependent variables $\mathbf{u} = (u^\alpha)$ and n independent variables $\mathbf{x} = (x^i)$ and the derivative of the dependent variables up to kth order defined on some domain $\mathfrak{D}$, then the action functional (action integral) is given by (1.39). Now for the extremum $\mathbf{u}(\mathbf{x})$, the infinitesimal variation of $\mathbf{u} : \mathbf{u}(\mathbf{x}) \longrightarrow \mathbf{u}(\mathbf{x}) + \epsilon\mathbf{w}(\mathbf{x})$ is such that the function $\mathbf{w}(\mathbf{x})$ and all its derivative up to order k becomes zero on the boundary $\partial\mathfrak{D}$. The variation of the Lagrangian $L[\mathbf{u}]$ corresponding to the infinitesimal variation of $\mathbf{u}$ is given by

$$\begin{aligned} \delta L[U] &= L(\mathbf{x}, \mathbf{u} + \epsilon\mathbf{w}(\mathbf{x}), \partial\mathbf{u} + \epsilon\partial\mathbf{w}(\mathbf{x}), ..., \partial^k\mathbf{u} + \epsilon\partial^k\mathbf{w}(\mathbf{x})) - L(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, ..., \partial^k\mathbf{u}) \\ &= \epsilon \left[\frac{\partial L}{\partial u^\alpha} w^\alpha + \frac{\partial L}{\partial u_i^\alpha} w_i^\alpha + \cdots + \frac{\partial L}{\partial u^\alpha_{i_1 i_2 \cdots i_k}} w^\alpha_{i_1 i_2 \cdots i_k} \right] + O(\epsilon^2), \end{aligned} \tag{1.41}$$

Now, applying the integration by parts repeatedly, we have

$$\delta L[U] = \epsilon \left[E_\alpha(L[\mathbf{u}]) w^\alpha + D_i W^i[u, w] \right] + O(\epsilon^2), \tag{1.42}$$

where $E_\alpha(L[\mathbf{u}])$ is given by (1.40) and $W^i[u,w]$ is

$$\begin{aligned} W^i[\mathbf{u},w] =& w^\alpha\left[\frac{\partial L}{\partial u_i^\alpha} - D_j\frac{\partial L}{\partial u_{ij}^\alpha} + \cdots + (-1)^{k-1}D_{i_1}\cdots D_{i_{(k-1)}}\frac{\partial L}{\partial u^\alpha_{ii_1\ldots i_{k-1}}}\right] \\ & + (D_{i_1}w^\alpha)\left[\frac{\partial L}{\partial u^\alpha_{i_1 i}} + D_{i_2}\frac{\partial L}{\partial u^\alpha_{i_1 i i_2}} + \cdots + (-1)^{k-2}D_{i_2}\cdots D_{i_{(k-1)}}\frac{\partial L}{\partial u^\alpha_{i_1 i i_2\ldots i_{k-1}}}\right] \\ & + \cdots + (D_{i_1}D_{i_2}\cdots D_{i_{k-1}}w^\alpha)\frac{\partial L}{\partial u^\alpha_{i_1 i_2\ldots i_{k-1} i}}. \end{aligned} \tag{1.43}$$

Next, we calculate the variation in the action $A[\mathbf{u}]$ (1.39) corresponding to the infinitesimal variation in the function $\mathbf{u}(\mathbf{x})$, which is the extremum of the $A[\mathbf{u}]$, as follows

$$\begin{aligned} \delta A[\mathbf{u}] &= A[\mathbf{u}+\epsilon\mathbf{w}] - A[\mathbf{u}] \\ \int_{\mathfrak{D}} \delta L[U]d\mathbf{x} &= \epsilon\int_{\mathfrak{D}} \left[E_\alpha(L[\mathbf{u}])w^\alpha + D_iW^i[u,w]\right]d\mathbf{x} + O(\epsilon^2). \end{aligned} \tag{1.44}$$

Applying the divergence theorem to the second term on the right hand, we have

$$\int_{\mathfrak{D}} \delta L[U]d\mathbf{x} = \epsilon\left[\int_{\mathfrak{D}} E_\alpha(L[\mathbf{u}])w^\alpha d\mathbf{x} + \int_{\partial\mathfrak{D}} W^i[u,w]p_i d\mathbf{S}\right] + O(\epsilon^2), \tag{1.45}$$

where $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ is an outward unit vector perpendicular to $\partial\mathfrak{D}$ and $\int_{\partial\mathfrak{D}}$ is the surface integral over the boundary of $\mathfrak{D}$. Now for the function $\mathbf{u}(\mathbf{x})$ to be extremum of the action integral $A[\mathbf{u}]$, the coefficient of the $O(\epsilon)$ must be zero in the variation of the action integral $\delta A[\mathbf{u}]$. In the infinitesimal variation of $\mathbf{u} : \mathbf{u}(\mathbf{x}) \longrightarrow \mathbf{u}(\mathbf{x}) + \epsilon\mathbf{w}(\mathbf{x})$ the function $\mathbf{w}(\mathbf{x})$ and all its derivative vanish on the boundary $\partial\mathfrak{D}$. Since $\mathrm{W}^i[u,w]$ is linear in w and its derivatives, so its surface integral also vanishes. Finally we are left with

$$\int_{\mathfrak{D}} E_\alpha(L[\mathbf{u}])w^\alpha d\mathbf{x} = 0, \tag{1.46}$$

with arbitrary $\mathbf{w}(\mathbf{x})$ within $\mathfrak{D}$, then for the extremum $\mathbf{u}$, we have

$$E_\alpha(L[\mathbf{u}]) = 0, \tag{1.47}$$

which is the system of PDEs (1.40), known as the EL-equations.

Theorem 1.4.2. *The EL-equations are the same for two lagrangians* L_1 *and* L_2 *if*

$$\mathrm{L}_1 - \mathrm{L}_2 = div\mathbf{F},$$

where $\mathbf{F}(\mathbf{x},\mathbf{u},\partial\mathbf{u},...,\partial^k\mathbf{u}) = (F^1, F^2, \cdots, F^n)$ *is a vector.*

1.4.2 Noether's Theorem

Theorem 1.4.3. *Suppose a system of EL equations* (1.40) *corresponding to the variational principle* (1.39) *and the solution* $\mathbf{u}(\mathbf{x})$ *of the EL equations are the extrema of the action functional* $A[\mathbf{u}]$. *Suppose the action integral (functional) is invariant under a one-parameter Lie group of point transformations and* $\mathrm{W}^i[\mathbf{u}, w]$ *is defined* (1.43) *for* $\mathbf{u}$ *and* w. *Then the conservation law*

$$D_i\left(\xi^i(\mathbf{x},\mathbf{u})L[\mathbf{u}] + W^i[\mathbf{u}, \hat{\eta}[\mathbf{u}]]\right) = 0, \tag{1.48}$$

holds for any solution $\mathbf{u}(\mathbf{x})$ *of the system of the EL equations.*

Proof. A one-parameter Lie group of point transformations is

$$\begin{aligned} \overline{x}^i &= x^i + \epsilon\xi^i(\mathbf{x},\mathbf{u}) + O(\epsilon^2), \qquad i = 1, 2, ..., n, \\ \overline{u}^\alpha &= u^i + \epsilon\eta^\alpha(\mathbf{x},\mathbf{u}) + O(\epsilon^2), \qquad \alpha = 1, 2, ..., m, \end{aligned} \tag{1.49}$$

and the associated infinitesimal generator is

$$\mathbf{U} = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}. \tag{1.50}$$

Under the transformations (1.49) the domain $\mathfrak{D}$ is transformed to $\overline{\mathfrak{D}}$. The action integral remains invariant under the point transformations (1.49) if and if

$$\int_{\overline{\mathfrak{D}}} L[\overline{\mathbf{u}}]d\overline{\mathbf{x}} = \int_{\mathfrak{D}} L[\mathbf{u}]d\mathbf{x}. \tag{1.51}$$

The variables $\mathbf{x}$ and $\overline{\mathbf{x}}$ can be changed as $d\overline{\mathbf{x}} = \mathrm{J}d\mathbf{x}$, where J is the Jacobian given by

$$\mathrm{J} = \det|D_i(\overline{x}^j)| = 1 + \epsilon D_i\xi^i(\mathbf{x},\mathbf{u}) + O(\epsilon^2). \tag{1.52}$$

The Lagrangian $L[\overline{\mathbf{u}}]$ is obtained by from the Lagrangian $L[\mathbf{u}]$ using the one-parameter Lie group of point transformations (1.49) with the symmetry generator $\mathbf{U}$, so we can write

$$L[\overline{\mathbf{u}}] = \exp(\epsilon\mathbf{U}^{(k)})L[\mathbf{u}],$$

where $\mathbf{U}^{(k)}$) is the kth extension of $\mathbf{U}$. Then the invariance condition can be written as

$$\begin{aligned} \int_{\mathfrak{D}}\left[\mathrm{J}\exp(\epsilon\mathbf{U}^{(k)})L[\mathbf{u}] - L[\mathbf{u}]\right]d\mathbf{x} &= 0, \\ \int_{\mathfrak{D}}\left[\left(1 + \epsilon D_i\xi^i(\mathbf{x},\mathbf{u}) + \cdots\right)\left(1 + \epsilon\mathbf{U}^{(k)} + \cdots\right)L[\mathbf{u}] - L[\mathbf{u}]\right]d\mathbf{x} &= 0. \end{aligned}$$

This is equal to zero if coefficients of all the powers of ϵ are zero. As it holds for all $\mathbf{u}(\mathbf{x})$, so we have

$$L[\mathbf{u}]D_i\xi^i + \mathbf{U}^{(k)}L[\mathbf{u}] = 0. \tag{1.53}$$

The one-parameter group of transformation (1.49) is equivalent to the one-parameter group of transformation

$$\begin{aligned} \overline{x}^i &= x^i, & i &= 1,2,...,n, \\ \overline{u}^\alpha &= u^i + \epsilon\left[\eta^\alpha(\mathbf{x},\mathbf{u}) + u_i^\alpha \xi^i(\mathbf{x},\mathbf{u})\right] + O(\epsilon^2), & \alpha &= 1,2,...,m, \end{aligned} \tag{1.54}$$

with the infinitesimal generator $\hat{\mathbf{U}}$ given by

$$\hat{\mathbf{U}} = \hat{\eta}^\alpha \frac{\partial}{\partial u^\alpha} = \left[\eta^\alpha(\mathbf{x},\mathbf{u}) + u_i^\alpha \xi^i(\mathbf{x},\mathbf{u})\right] \frac{\partial}{\partial u^\alpha}. \tag{1.55}$$

The infinitesimal variation in $\mathbf{u}(\mathbf{x})$ given by $\mathbf{u}(\mathbf{x}) \longrightarrow \mathbf{u}(\mathbf{x}) + \epsilon\mathbf{w}(\mathbf{x})$ is such that $w^\alpha = \eta^\alpha(\mathbf{x},\mathbf{u}) + u_i^\alpha \xi^i(\mathbf{x},\mathbf{u})$. Moreover, the variation in the Lagrangian is

$$\begin{aligned} \delta L &= L[\overline{\mathbf{u}}] - L[\mathbf{u}], \\ &= \exp\left(\epsilon[\hat{\mathbf{U}}^{(k)}]\right) L[\mathbf{u}] - L[\mathbf{u}], \\ \int \delta L dx &= \epsilon \int \hat{\mathbf{U}}^{(k)} L[\mathbf{u}] dx + O(\epsilon^2). \end{aligned} \tag{1.56}$$

Now comparing this with the expression (1.44) and utilizing $w^\alpha = \hat{\eta}^\alpha$, we have

$$\hat{\mathbf{U}}^{(\mathbf{k})} = E_\alpha(L[\mathbf{u}])\hat{\eta}^\alpha + D_i W^i[u,\hat{\eta}]. \tag{1.57}$$

For the kth extended equivalent infinitesimal generators $\mathbf{U}^{(\mathbf{k})}$ and $\hat{\mathbf{U}}^{(\mathbf{k})}$, following identity trivially holds

$$\mathbf{U}^{(\mathbf{k})} L[\mathbf{u}] + L[\mathbf{u}] D_i \xi^i(\mathbf{x},\mathbf{u}) = \hat{\mathbf{U}}^{(\mathbf{k})} L[\mathbf{u}] + D_i\left(L[\mathbf{u}]\xi^i(\mathbf{x},\mathbf{u})\right). \tag{1.58}$$

The left hand side of (1.58) is zero by (1.53), so we have

$$\hat{\mathbf{U}}^{(\mathbf{k})} L[\mathbf{u}] + D_i\left(L[\mathbf{u}]\xi^i(\mathbf{x},\mathbf{u})\right) = 0. \tag{1.59}$$

Now substituting the value of $\hat{\mathbf{U}}^{(\mathbf{k})}$ from (1.57) and for $\mathbf{u}(\mathbf{x})$ to be solution of the EL equations, i.e., $E_\alpha(L[\mathbf{u}]) = 0$, we finally have the conservation law

$$D_i\left(L[\mathbf{u}]\xi^i(\mathbf{x},\mathbf{u}) + W^i[u,\hat{\eta}]\right) = 0. \tag{1.60}$$

Definition 1.4.4. (**Noether Symmetry**) A vector field

$$\mathbf{U} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \tag{1.61}$$

is said to be a *Noether symmetry*, if there exists a vector valued gauge function $\mathbf{A} = (A^1, A^2, ...A^n),\ A^i \in \mathcal{A}$, where $\mathcal{A}$ is the space of differential functions, such that

$$\mathbf{U}(L) + L D_i(\xi^i) = D_i(A^i). \tag{1.62}$$

Theorem 1.4.5. *If the infinitesimal generator* $\mathbf{U}$ *is the Noether symmetry for the action integral* (1.39)*, then the conservation law becomes*

$$D_i\left(L[\mathbf{u}]\xi^i(\mathbf{x},\mathbf{u}) + W^i[u,\hat{\eta}] - A^i\right) = 0. \tag{1.63}$$

Now in the remaining sections of this chapter general theory and few applications of the invariants of the equivalence group of transformations for DEs is presented. Many topics related to the invariants of the DEs like semi-invariants, joint invariants, invariant differentiation, bases of invariants, invariant equations etc. are also defined and illustrated with examples in these sections.

1.5 Differential Invariants of Differential Equations

Invariants and semi-invariants of the group of equivalence transformations are extremely useful tools for transforming the differential equations into integrable forms and simplified forms. The transformation which does not change the differential structure of the DE but may change the constitutive functions and/or parameters of the DE is called the equivalence transformation and the set of all the equivalence transformations form a group. If a group of equivalence transformations is admitted by a DE then the DE can be written in terms of the corresponding differential invariants. The problem of determining the DE which admits a given group of transformations can be solved by utilizing the differential invariants of the group of transformations. If a scalar DE or a system of DEs involves classifying parameters and/or functions, then during the classification of DEs by the classifying parameters and/or functions it is useful to consider the differential invariants. Moreover, using the differential invariants the given DEs are classified for the constitutive parameters and/or functions of DEs which cannot be transformed into each under equivalence transformation. In particular, equivalence transformations can be used to transform DEs into canonical form. Furthermore in the group classification of DEs, differential invariants can be utilized by writing the determining equations in the form of these invariants.

In the course of the calculation of invariants, equivalence transformations play a crucial part. Differential invariants are actually the invariants of the groups of equivalence transformations of the scalar or system of DEs, so first step is to calculate all the equivalence transformations. There are two main methods to calculate the set of all the equivalence transformations. The first method uses directly the definition of the equivalence transformations called the *direct method.* Theoretically, one can calculate the most general group of equivalence transformations but usually this method leads to huge computational difficulties especially when dealing with non-linear DEs. Lie pointed that the theory of differential invariants is based on the the infinitesimal method. Later, Ovsiannikov and Ibragimov systematically develop the infinitesimal methods to calculate the invariants of the algebraic and differential equations, known as *Lie's infinitesimal methods.*

Consider a system $\mathbf{E}$ of kth order N PDEs, written as

$$E^{\sigma}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, \cdots, \partial^k\mathbf{u}, \mathbf{P}) = 0, \qquad \sigma = 1, 2, ..., N, \tag{1.64}$$

involving l constitutive parameters and/or functions $\mathbf{P} = (P^1, P^2, ..., P^l)$, which may depend on independent variables, dependent variables and derivatives of the dependent variables.

Definition 1.5.1. (**Finite Equivalence Group of Point Transformations**) A finite equivalence transformation of the system of PDEs (1.64) is a transformation

$$\begin{aligned} \overline{x}^i &= \overline{x}^i(x^i, u^{\alpha}), \\ \overline{u}^{\alpha} &= \overline{u}^{\alpha}(x^i, u^{\alpha}), \end{aligned} \tag{1.65}$$

such that it maps the PDEs system (1.64) to another PDEs system belonging to the same family but the constitutive parameters and/or functions may change as $\overline{P}^l = \overline{P}^l(x^i, u^{\alpha}, P^l)$ depending on x^i, u^{α} and P^l .

Definition 1.5.2. (**Infinitesimal Equivalence Group of Point Transformations**) A one parameter Lie group of point transformation

$$\begin{aligned} \overline{x}^i &= \overline{x}^i(x^i, u^{\alpha}, \epsilon), \\ \overline{u}^{\alpha} &= \overline{u}^{\alpha}(x^i, u^{\alpha}, \epsilon), \end{aligned} \tag{1.66}$$

is called an infinitesimal equivalence transformation if it leave the system of PDEs (1.64) to the same family of PDEs, in general, with new constitutive parameters and/or functions $\overline{P}^l = \overline{P}^l(x^i, u^{\alpha}, P^l)$.

Now we discuss both the methods for calculating the equivalence transformations for the general second order ODE

$$u''(x) = E(x, u). \tag{1.67}$$

1.5.1 Direct Method

An equivalence transformation for the family of equations (1.69) is a transformation of the dependent and independent variables

$$\overline{x} = f(x, u), \qquad \overline{u} = g(x, u), \tag{1.68}$$

that transforms the ODE (1.69) to an equivalent ODE

$$\overline{u}''(x) = \overline{E}(\overline{x}, \overline{u}) \tag{1.69}$$

with, in general, new function $\overline{E}$ different from E. The most general equivalence transformation can be found by changing the dependent and independent variables under the transformation (1.68) and substituting them back in the ODE to find the restriction on the transformation. The variables are changed as follows

$$\overline{u}' = \frac{g_x + g_u u'}{f_x + f_u u'}, \tag{1.70}$$

and

$$\overline{u}'' = \frac{(f_x + f_u u')(g_{xx} + 2g_{xu}u' + g_u u'') - (g_x + g_u u')(f_{xx} + 2f_{xu}u' + f_u u'')}{(f_x + f_u u')^3}. \tag{1.71}$$

By substituting these in equation (1.69), we have

$$\frac{(f_x + f_u u')(g_{xx} + 2g_{xu}u' + g_u E(x,u)) - (g_x + g_u u')(f_{xx} + 2f_{xu}u' + f_u E(x,u))}{(f_x + f_u u')^3} = \overline{E}(\overline{x}, \overline{u}). \tag{1.72}$$

Now as $E(x,u)$ and $\overline{E}(\overline{x}, \overline{u})$ do not depend on u'. Comparing the coefficient of u', u'^2 and u'^3, we obtain

$$2f_x g_{xu} - g_u f_{xx} = 0, \quad f_x g_{uu} = 0, \quad f_u = 0. \tag{1.73}$$

The solutions of these equations give the equivalence transformations

$$\overline{x} = f(x), \qquad \overline{u} = c\sqrt{f'}u + h(x), \tag{1.74}$$

where c is an arbitrary constant. The terms without u' gives the restrictions on the function $\overline{E}(\overline{x}, \overline{u})$, given by

$$\overline{E}(\overline{x}, \overline{u}) = \frac{c}{(f')^{\frac{3}{2}}}E + c\left[\frac{f'''}{2(f')^{\frac{5}{2}}} - \frac{3(f'')^2}{4(f')^{\frac{7}{2}}}\right]u + \frac{h''}{(f')^2} - \frac{h'f''}{(f')^3}. \tag{1.75}$$

Here the function $\overline{E}(\overline{x}, \overline{u})$ can be calculated from the equivalence transformations (1.74) and the function $E(x,u)$.

Next we calculate the continuous group of equivalence transformation by infinitesimal method.

1.5.2 Lie Infinitesimal Method

Since under an equivalence transformation the function $E(x,u)$ may change so we consider E as a new variable and the infinitesimal generator in the extended (x, u, E)-space is given as

$$\mathbf{U} = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \alpha(x,u,E)\frac{\partial}{\partial E}. \tag{1.76}$$

The second-order prolongation of (1.76) is

$$\mathbf{U}^{(2)} = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \eta^{(1)}\frac{\partial}{\partial u'} + \eta^{(2)}\frac{\partial}{\partial u''} + \alpha(x,u,E)\frac{\partial}{\partial E}, \tag{1.77}$$

where

$$\begin{aligned} \eta^{(1)} &= \eta_x + (\eta_u - \xi_x)u' - \xi_u u'^2, \\ \eta^{(2)} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u' + (\eta_{uu} - 2\xi_{xu})u'^2 - \xi_{uu}u'^3 + (\eta_u - 2\xi_x - 3\xi_u u')u''. \end{aligned} \tag{1.78}$$

The infinitesimal invariance condition is given as

$$\mathbf{U}^{(2)}\left(u''(x) - E(x,u)\right)|_{(u''-E=0)} = 0. \tag{1.79}$$

Substituting $\mathbf{U}^{(2)}$, we have

$$\alpha = \eta_{xx} + (2\eta_{xu} - \xi_{xx})u' + (\eta_{uu} - 2\xi_{xu})u'^2 - \xi_{uu}u'^3 + (\eta_u - 2\xi_x - 3\xi_u u')E. \tag{1.80}$$

This equation satisfied identically for all variables x, u, u' and E. So by separating the coefficients of the different powers of u', we obtain the following equations

$$\begin{aligned} \alpha - (\eta_u - 2\xi_x)E - \eta_{xx} &= 0, \\ 2\eta_{xu} - \xi_{xx} - 3\xi_u E &= 0, \\ \eta_{uu} - 2\xi_{xu} &= 0, \\ \xi_{uu} &= 0. \end{aligned} \tag{1.81}$$

The solutions of these equations give the following three infinite generators

$$\begin{aligned} &\mathbf{U}_1 = h(x)\frac{\partial}{\partial u} + h''(x)\frac{\partial}{\partial E}, \\ &\mathbf{U}_2 = u\frac{\partial}{\partial u} + E\frac{\partial}{\partial E}, \\ &\mathbf{U}_3 = \xi(x)\frac{\partial}{\partial x} + \frac{u}{2}\xi'(x)\frac{\partial}{\partial u} + \frac{1}{2}\left[u\xi'''(x) - 3\xi'(x)E\right]\frac{\partial}{\partial E}. \end{aligned} \tag{1.82}$$

These infinitesimal generators can generate the finite equivalence transformation and vice versa.

1.5.3 Invariants and Semi-Invariants

Before deriving the Invariants of some PDEs. We give proper definitions of differential invariants, invariants and semi-invariants in this subsection.

Definition 1.5.3. (**Differential Invariants**) A function

$$\mathrm{J}(\mathbf{P}, \partial\mathbf{P}, \partial^2\mathbf{P}, \cdots, \partial^s\mathbf{P}), \tag{1.83}$$

where s is the maximal order derivative of $\mathbf{P}$, is called a differential invariant of order s of a system of PDEs (1.64) if it remains invariant under the most general group of equivalence transformation of the system of PDEs (1.64).

Definition 1.5.4. (**Invariants**) A function is called invariant if it is a differential invariant of order zero, i.e., the invariant (1.83) contains no derivative of $\mathbf{P}$.

Definition 1.5.5. (**Semi-invariants**) The differential invariant (1.83) is called semi-invariant if it is left invariant only under a subgroup of the most general group of equivalence transformations.

1.5.4 Laplace Invariants of Hyperbolic Equations

The scalar linear second order hyperbolic PDEs in two independent variables t and z in canonical form is

$$u_{tz} + A(t,z)u_t + B(t,z)u_z + C(t,z)u = 0, \tag{1.84}$$

where A, B and C are given twice differentiable functions of t and z. An invertible transformation of the dependent and independent variables

$$\bar{t} = \phi(t,z,u), \qquad \bar{z} = \psi(t,z,u), \qquad \bar{u} = \omega(t,z,u), \tag{1.85}$$

is called an equivalence transformation if the equation (1.84) remains invariant under the transformations (1.85). Semi-invariants corresponding to only the dependent variables are called the Laplace invariants [44, 57]. To apply the Lie infinitesimal method to calculate the Laplace invariants, we consider the infinitesimal generator

$$\mathbf{U} = \eta\frac{\partial}{\partial u} + \alpha\frac{\partial}{\partial A} + \beta\frac{\partial}{\partial B} + \gamma\frac{\partial}{\partial C}, \tag{1.86}$$

where $\eta = \eta(t,z,u)$, $\alpha = \alpha(t,z,u,A,B,C)$, $\beta = \beta(t,z,u,A,B,C)$ and $\gamma = \gamma(t,z,u,A,B,C)$. The Lie invariance condition is given by

$$\mathbf{U}^{(2)}\left(u_{tz} + A(t,z)u_t + B(t,z)u_z + C(t,z)u\right)|_{(1.84)} = 0. \tag{1.87}$$

Applying the second order prolonged generator on (1.84) and replacing u_{tz} by $-(A(t,z)u_t + B(t,z)u_z + C(t,z)u)$, we obtain

$$\eta_{tz} + u_t\eta_{zu} + u_z\eta_{tu} + A\eta_t + u_tu_z\eta_{uu} + B\eta_z + C\eta - Cu\eta_u + \alpha u_t + \beta u_z + \gamma u = 0. \tag{1.88}$$

Now the coefficients of $u_t u_z, u_t, u_z$ and the remaining terms gives the following system of equations

$$\begin{aligned} \eta_{uu} &= 0, \\ \eta_{zu} + \alpha &= 0, \\ \eta_{tu} + \beta &= 0, \\ \eta_{tz} + A\eta_t + B\eta_z + C\eta - Cu\eta_u + \gamma u &= 0. \end{aligned} \tag{1.89}$$

The solution of the these equations is

$$\eta = \eta(t,z)u, \quad \alpha = -\eta_z, \quad \beta = -\eta_t, \quad \gamma = -(\eta_{tz} + A\eta_t + B\eta_z). \tag{1.90}$$

So the corresponding generator for the infinitesimal transformations in A, B and C is

$$\mathbf{U} = \eta_z \frac{\partial}{\partial A} + \eta_t \frac{\partial}{\partial B} + (\eta_{tz} + A\eta_t + B\eta_z)\frac{\partial}{\partial C}. \tag{1.91}$$

For the invariant $\mathrm{J}(A,B,C)$ the infinitesimal test $\mathbf{U}\mathrm{J}(A,B,C) = 0$, gives

$$\mathrm{J}_A = 0, \qquad \mathrm{J}_B = 0, \qquad \mathrm{J}_C = 0, \tag{1.92}$$

so the zeroth order invariant is $\mathrm{J} = \mathit{const}.$ which is of no interest. Now we find the differential invariants of first order, i.e., $\mathrm{J}(A,B,C,A_t,B_t,C_t,A_z,B_z,C_z)$ using the once extended generator

$$\begin{aligned} \mathbf{U}^{(1)} = {} & \eta_z \frac{\partial}{\partial A} + \eta_t \frac{\partial}{\partial B} + (\eta_{tz} + A\eta_t + B\eta_z)\frac{\partial}{\partial C} + \eta_{tz}\frac{\partial}{\partial A_t} + \eta_{zz}\frac{\partial}{\partial A_z} + \eta_{tt}\frac{\partial}{\partial B_t} + \eta_{tz}\frac{\partial}{\partial B_z} \\ & + (\eta_{ttz} + A_t\eta_t + A\eta_{tt} + B_t\eta_z + B\eta_{tz})\frac{\partial}{\partial C_t} + (\eta_{tzz} + A_z\eta_t + A\eta_{tz} + B_z\eta_z + B\eta_{zz})\frac{\partial}{\partial C_z}. \end{aligned} \tag{1.93}$$

The invariant condition $\mathbf{U}^{(1)}\mathrm{J}(A,B,C,A_t,B_t,C_t,A_z,B_z,C_z)$, after equating to zero the coefficients of $\eta_t, \eta_z, \eta_{tt}, \eta_{tz}, \eta_{zz}, \eta_{ttz}$ and η_{tzz} gives

$$\begin{aligned} \frac{\partial \mathrm{J}}{\partial C_t} = 0, \quad \frac{\partial \mathrm{J}}{\partial C_z} = 0, \quad \frac{\partial \mathrm{J}}{\partial A_z} = 0, \quad \frac{\partial \mathrm{J}}{\partial B_t} &= 0, \\ \frac{\partial \mathrm{J}}{\partial A} + B\frac{\partial \mathrm{J}}{\partial C} = 0, \quad \frac{\partial \mathrm{J}}{\partial B} + A\frac{\partial \mathrm{J}}{\partial C} &= 0, \\ \frac{\partial \mathrm{J}}{\partial C} + \frac{\partial \mathrm{J}}{\partial A_t} + \frac{\partial \mathrm{J}}{\partial B_z} &= 0. \end{aligned} \tag{1.94}$$

The solution of the equations (1.94) deduced two semi-invariants

$$\begin{aligned} h &= A_t + AB - C, \\ k &= B_z + AB - C, \end{aligned} \tag{1.95}$$

for equation (1.84), known as the Laplace invariants. These semi-invariants were derived by Laplace in his fundamental memoir in 1773.

Now we present a few applications of the Laplace invariants to illustrate their role in the integration of PDEs.

1.5.5 Applications of the Laplace Invariants

1. If both the Laplace invariants are zero, i.e., $h = 0$ and $k = 0$, then the hyperbolic PDE (1.84) can be transformed to the simplest form $u_{tz} = 0$ by means of equivalence transformations (1.85) of the dependent and independent variables.

2. The hyperbolic PDE (1.84) can be factorized, viz. the second order operator $L = D_t D_z + A(t,z)D_t + B(t,z)D_z + C(t,z)$ can be written as

$$L = [D_t + \alpha(t,z)][D_z + \beta(t,z)] \quad \text{iff} \quad h = 0, \tag{1.96}$$

   and

$$L = [D_z + \beta(t,z)][D_t + \alpha(t,z)] \quad \text{iff} \quad k = 0. \tag{1.97}$$

   So in either case the general solution of the hyperbolic PDE (1.84) can be found by the successive integration of two ODEs.

3. If the Laplace invariants are equal, i.e., $h = k$ then the by the application of the equivalence transformations the hyperbolic PDE (1.84) can be reduced to the simple form $u_{tz} + C(t,z)u = 0$.

4. If the Laplace invariants are equal and separable, i.e., $h = k = f(t)g(z)$ then the hyperbolic PDE (1.84) can be further simplified as $u_{tz} + Cu = 0$, here C is a constant.

1.5.6 Cotton Invariants of Elliptic PDEs

The scalar linear second order elliptic equation in two independent variables in canonical form is

$$u_{xx} + u_{yy} + au_x + bu_y + cu = 0, \tag{1.98}$$

where a, b and c are given twice differentiable functions of x and y. The Cotton invariants are the semi-invariants corresponding to only the equivalence transformations of the dependent variables under which the the elliptic PDE (1.98) remains invariant. To apply the Lie infinitesimal method to calculate the Cotton invariants, we consider the infinitesimal generator

$$\mathbf{U} = \eta\frac{\partial}{\partial u} + \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial b} + \gamma\frac{\partial}{\partial c}, \tag{1.99}$$

where $\eta = \eta(x,y,u)$, $\alpha = \alpha(x,y,u,a,b,c)$, $\beta = \beta(x,y,u,a,b,c)$ and $\gamma = \gamma(x,y,u,a,b,c)$. The Lie invariance condition is given by

$$\mathbf{U}^{(2)}\left(u_{xx} + u_{yy} + au_x + bu_y + cu\right)|_{(1.98)} = 0. \tag{1.100}$$

This invariance condition gives the following generator for the infinitesimal transformations in a, b and c as

$$\mathbf{U} = 2\eta_x \frac{\partial}{\partial a} + 2\eta_y \frac{\partial}{\partial b} + (\eta_{xx} + \eta_{yy} + a\eta_x + b\eta_y)\frac{\partial}{\partial c}. \qquad (1.101)$$

For this generator the infinitesimal test $\mathbf{U}\mathrm{J}(a, b, c) = 0$, gives no zeroth order invariant. To find the first order semi-invariants, i.e., $\mathrm{J}(a, b, c, a_x, b_x, c_x, a_y, b_y, c_y)$, the invariant condition

$$\mathbf{U}^{(1)}\mathrm{J}(a, b, c, a_x, b_x, c_x, a_y, b_y, c_y) = 0,$$

yields

$$\begin{gathered} \frac{\partial \mathrm{J}}{\partial c_x} = 0, \quad \frac{\partial \mathrm{J}}{\partial C_y} = 0, \quad \frac{\partial \mathrm{J}}{\partial c} + 2\frac{\partial \mathrm{J}}{\partial a_x} = 0, \quad \frac{\partial \mathrm{J}}{\partial c} + 2\frac{\partial \mathrm{J}}{\partial b_y} = 0, \\ 2\frac{\partial \mathrm{J}}{\partial a} + a\frac{\partial \mathrm{J}}{\partial c} = 0, \quad 2\frac{\partial \mathrm{J}}{\partial b} + b\frac{\partial \mathrm{J}}{\partial c} = 0, \quad \frac{\partial \mathrm{J}}{\partial b_x} + \frac{\partial \mathrm{J}}{\partial a_y} = 0. \end{gathered} \qquad (1.102)$$

The solution of the equations (1.102) yields two semi-invariants

$$\begin{aligned} \mu &= a_y - b_x, \\ H &= a_x + b_y + \frac{1}{2}(a^2 + b^2) - 2c. \end{aligned} \qquad (1.103)$$

for equation (1.98), known as the Cotton invariants.

It is well-known that by means of the linear complex transformations,

$$x = \frac{1}{2}(t + z), \quad y = \frac{-i}{2}(t - z), \qquad (1.104)$$

the elliptic equation (1.98) can be mapped to the linear hyperbolic equation (1.84), with

$$A = \frac{1}{4}(a + ib), \quad B = \frac{1}{4}(a - ib), \quad C = \frac{1}{4}c. \qquad (1.105)$$

These Laplace invariants (1.95) can be transformed, by use of the inverse of the transformations (1.104) as well as after the substitution of (1.105) into (1.95) and then splitting the real and imaginary parts, to arrive at the Cotton invariants.

1.5.7 Ibragimov Invariants of Parabolic PDEs

The scalar linear second order parabolic equation in two independent variables in canonical form is

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u, \qquad (1.106)$$

where a, b and c are given twice differentiable functions of t and x. Following the Lie infinitesimal methods Ibragimov semi-invariant under the transformation of the dependent variables for the the parabolic equation (1.106) can be derived as

$$K = (a_t - aa_{xx} + a_x^2)b - \frac{1}{2}b^2 a_x + (ab - aa_x)b_x - ab_t + a^2 b_{xx} - 2a^2 c_x. \tag{1.107}$$

This semi-invariant for the parabolic PDE is analogue of the Laplace and the Cotton invariants for the hyperbolic and elliptic PDEs respectively.

1.6 Joint Invariants and Invariant Equations

In dealing with invariants of the DEs joint invariants and invariant equations are more important as these belong to the most general group of equivalence transformations. In this section, we present the definitions of joint invariants and invariant equations. Joint invariants for the hyperbolic PDEs are also presented.

Definition 1.6.1. (**Joint Invariants**) Let $(h_1, h_2, \cdots)$ be a set of semi-invariants under the transformations of the dependent variables only. A function of the semi-invariants $(h_1, h_2, \cdots)$ and their derivatives is called a joint invariant of a system of PDEs (1.64) if it remains invariant under the most general group of equivalence transformation of the system of PDEs (1.64).

Definition 1.6.2. (**Invariant Equations**) A system of equations $E_i(\mathbf{x}, \mathbf{u}, h_j, \partial h_j, \cdots) = 0$ is said to be an invariant system if for the infinitesimal generator $\mathbf{U}$ it satisfies the following condition

$$\mathbf{U} E_i(\mathbf{x}, \mathbf{u}, h_j, \partial h_j, \cdots)|_{E_i=0} = 0.$$

1.6.1 Joint Invariants for Hyperbolic Equations

Now, we present the joint differential invariants for the hyperbolic PDE (1.84). The Laplace invariants for hyperbolic equation have already been given in the last section. To derive the joint invariants we first obtain the infinitesimal generator of equivalence transformations for independent variables and then transform this infinitesimal generator in terms of the Laplace invariants. This resulting generator gives the joint invariants of the hyperbolic PDE in terms of the Laplace invariants and their derivatives.

First, consider an infinitesimal generator of the form

$$\mathbf{U} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial z} + \alpha \frac{\partial}{\partial A} + \beta \frac{\partial}{\partial B} + \gamma \frac{\partial}{\partial C}, \tag{1.108}$$

where ξ^1 and ξ^2 depend on t, z, u and α, β and γ are functions of t, z, u, A, B and C. The generator corresponds to the equivalence transformations of the independent variables

$$\bar{t} = \phi(t, z), \qquad \bar{z} = \psi(t, z). \tag{1.109}$$

In order to calculate the functions ξ^1, ξ^2, α, β and γ, we invoke the infinitesimal criteria

$$\mathbf{U}^{(2)} \left(u_{tz} + A(t, z)u_t + B(t, z)u_z + C(t, z)u \right) |_{(1.84)} = 0. \tag{1.110}$$

The second prolongation of the generator $\mathbf{U}$ (1.108) is given by

$$\mathbf{U}^{(2)} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial z} + \alpha \frac{\partial}{\partial A} + \beta \frac{\partial}{\partial B} + \gamma \frac{\partial}{\partial C} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_z \frac{\partial}{\partial u_z} + \zeta_{tz} \frac{\partial}{\partial u_{tz}}, \tag{1.111}$$

where

$$\begin{aligned}
\zeta_t &= -u_t \xi^1_t - u_t^2 \xi^1_u - u_z \xi^2_t - u_t u_z \xi^2_u, \\
\zeta_z &= -u_t \xi^1_z - u_t u_z \xi^1_u - u_z \xi^2_z - u_z^2 \xi^2_u, \\
\zeta_{tz} &= -u_t \xi^1_{tz} - u_t u_z \xi^1_{tu} - u_z \xi^2_{tz} - u_z^2 \xi^2_{tu} - u_t^2 \xi^1_{zu} - u_t^2 u_z \xi^1_{uu} - u_t u_z \xi^2_{zu} - u_1 u_z^2 \xi^2_{uu} - u_{tt} \xi^1_z \\
&\quad - u_{tt} u_z \xi^1_u - 2u_{tz} u_t \xi^1_u - u_{tz} \xi^2_z - 2u_{tz} u_z \xi^2_u - u_{tz} \xi^1_t - u_{zz} \xi^2_t - u_{zz} u_t \xi^2_u.
\end{aligned} \tag{1.112}$$

The infinitesimal criteria give

$$\xi^1 = \phi(t), \quad \xi^2 = \psi(z), \quad \alpha = -A\psi_z, \quad \beta = -B\phi_t, \quad \gamma = -(C\phi_t + C\psi_z), \tag{1.113}$$

where $\phi(t)$ and $\psi(z)$ are arbitrary functions of their arguments.

As the Laplace invariants are $h = A_t + AB - C,\ k = B_Z + AB - C$. So to write the generator in the bases of these invariants we seek a generator as

$$\mathbf{U} = \phi(t) \frac{\partial}{\partial t} + \psi(z) \frac{\partial}{\partial z} - A\psi_z \frac{\partial}{\partial A} - B\psi_t \frac{\partial}{\partial B} - (C\phi_t + C\psi_z) \frac{\partial}{\partial C} + \alpha_t \frac{\partial}{\partial A_t} + \beta_z \frac{\partial}{\partial B_z}. \tag{1.114}$$

Here, α_t and β_z can be calculated as

$$\begin{aligned}
\alpha_t &= D_t(\alpha) - A_t D_t \xi^1(t) - A_z D_t \xi^2(z), \\
\beta_z &= D_z(\beta) - B_t D_z \xi^1(t) - B_z D_z \xi^2(z),
\end{aligned} \tag{1.115}$$

where D_t and D_z are the total differentiation operator given by

$$\begin{aligned}
D_t &= \partial_t + A_t \partial_A + A_{tt} \partial_{A_t} + A_{tz} \partial_{A_z} + \cdots + B_t \partial_B \\
&\quad + B_{tt} \partial_{B_t} + B_{tz} \partial_{B_z} + \cdots + C_t \partial_C + C_{tt} \partial_{C_t} + C_{tz} \partial_{C_z} + \cdots, \\
D_z &= \partial_z + A_z \partial_A + A_{tz} \partial_{A_t} + A_{zz} \partial_{A_z} + \cdots + B_z \partial_B \\
&\quad + B_{tz} \partial_{B_t} + B_{zz} \partial_{B_z} + \cdots + C_z \partial_C + C_{tz} \partial_{C_t} + C_z \partial_{C_z} + \cdots.
\end{aligned} \tag{1.116}$$

By utilizing (1.116) in (1.115), we have

$$\begin{aligned} \alpha_t &= -A_t(\phi_t + \psi_z), \\ \beta_z &= -B_z(\phi_t + \psi_z). \end{aligned} \tag{1.117}$$

The generator can be transformed (1.114) in the space of the h and k by using

$$\mathbf{U} = (\mathbf{U}h)\,\frac{\partial}{\partial h} + (\mathbf{U}k)\,\frac{\partial}{\partial k}, \tag{1.118}$$

where the action of the generator $\mathbf{U}$ on h and k is

$$\mathbf{U}h = -h(\phi_t + \psi_z), \quad \mathbf{U}k = -k(\phi_t + \psi_z). \tag{1.119}$$

Therefore, the generator (1.118) becomes

$$\mathbf{U} = -h(\phi_t + \psi_z)\frac{\partial}{\partial h} - k(\phi_t + \psi_z)\frac{\partial}{\partial k}. \tag{1.120}$$

Now, we calculate the joint invariants using the infinitesimal generator (1.120). To find the invariants $\mathrm{J}(h, k)$, the infinitesimal test $\mathbf{U}\mathrm{J} = 0$ gives

$$h\frac{\partial \mathrm{J}}{\partial h} + k\frac{\partial \mathrm{J}}{\partial k} = 0. \tag{1.121}$$

The solution J of this equation gives the joint invariant $p = k/h$. The Laplace invariants, derived by Euler and then by Laplace, are invariants only under the equivalence transformations of the dependent variables but the invariant p was derived by Ovsiannikov, using a different approach, which is invariant under the equivalence transformation of both the dependent and independent variables.

In order to calculate the joint invariant of second order $\mathrm{J}(h, k, h_t, k_t, h_z, k_z)$, the infinitesimal generator (1.120) has to be extended up to first order, which is given as

$$\mathbf{U} = -h(\phi_t + \psi_z)\frac{\partial}{\partial h} - k(\phi_t + \psi_z)\frac{\partial}{\partial k} + \mu_t\frac{\partial}{\partial h_t} + \nu_t\frac{\partial}{\partial k_t} + \mu_z\frac{\partial}{\partial h_z} + \nu_z\frac{\partial}{\partial k_z}. \tag{1.122}$$

Here, μ_t, ν_t, μ_z and ν_z can be calculated as

$$\begin{aligned} \mu_t &= D_t(-h(\alpha_t + \beta_z)) - h_t D_t\phi - h_z D_t\psi, \\ \nu_t &= D_t(-k(\alpha_t + \beta_z)) - k_t D_t\phi - k_z D_t\psi, \\ \mu_z &= D_z(-h(\alpha_t + \beta_z)) - h_t D_z\phi - h_z D_z\psi, \\ \nu_z &= D_z(-k(\alpha_t + \beta_z)) - k_t D_z\phi - k_z D_z\psi, \end{aligned} \tag{1.123}$$

where the total derivative operators are

$$D_t = \partial_t + h_t\partial_h + h_{tt}\partial_{h_t} + h_{tz}\partial_{h_z} + \cdots + k_t\partial_k + k_{tt}\partial_{k_t} + k_{tz}\partial_{k_z} + \cdots ,$$
$$D_z = \partial_z + h_z\partial_h + h_{tz}\partial_{h_t} + h_{zz}\partial_{h_z} + \cdots + k_z\partial_k + k_{tz}\partial_{k_t} + k_{zz}\partial_{k_z} + \cdots . \tag{1.124}$$

The first order generator (1.122) becomes

$$\begin{aligned}\mathbf{U} =& - h(\phi_t + \psi_z)\frac{\partial}{\partial h} - k(\phi_t + \psi_z)\frac{\partial}{\partial k} - (h\phi_{tt} + 2h_t\phi_t + h_t\psi_z)\frac{\partial}{\partial h_t} \\ & - (k\phi_{tt} + 2k_t\phi_t + k_t\psi_z)\frac{\partial}{\partial k_t} - (h_x\phi_t + h\psi_{zz} + 2h_z\psi_z)\frac{\partial}{\partial h_z} \\ & - (k_x\phi_t + k\psi_{zz} + 2h_x\psi_z)\frac{\partial}{\partial k_z}.\end{aligned} \tag{1.125}$$

The infinitesimal criteria $\mathbf{U}\mathrm{J}(h, k, h_t, k_t, h_z, k_z) = 0$, gives the following system of DEs

$$\begin{aligned}& k\frac{\partial \mathrm{J}}{\partial k_z} + h\frac{\partial \mathrm{J}}{\partial h_z} = 0, \quad k\frac{\partial \mathrm{J}}{\partial k_t} + h\frac{\partial \mathrm{J}}{\partial h_t} = 0, \\ & 2k_z\frac{\partial \mathrm{J}}{\partial k_z} + k_t\frac{\partial \mathrm{J}}{\partial k_t} + 2h_z\frac{\partial \mathrm{J}}{\partial h_z} + h_t\frac{\partial \mathrm{J}}{\partial h_t} + k\frac{\partial \mathrm{J}}{\partial k} + h\frac{\partial \mathrm{J}}{\partial h} = 0, \\ & k_z\frac{\partial \mathrm{J}}{\partial k_z} + 2k_t\frac{\partial \mathrm{J}}{\partial k_t} + h_z\frac{\partial \mathrm{J}}{\partial h_z} + 2h_t\frac{\partial \mathrm{J}}{\partial h_t} + k\frac{\partial \mathrm{J}}{\partial k} + h\frac{\partial \mathrm{J}}{\partial h} = 0.\end{aligned} \tag{1.126}$$

The solution of the above system of PDEs can be obtained, using the theory of linear homogeneous PDEs, as

$$\mathrm{J} = \Psi(p, J_2^1),$$

where Ψ is an arbitrary function, p is Ovsiannikov invariant and J_2^1 is a joint invariant of second order given as

$$J_2^1 = \frac{(k_zh - h_zk)(k_th - h_tk)}{h^5} = \frac{1}{h}p_zp_t. \tag{1.127}$$

Similarly, we can derive the joint invariant of third order $\mathrm{J}(h, k, h_t, k_t, h_z, k_z, h_{tt}, k_{tt}, h_{tz}, k_{tz}, h_{zz}, k_{zz})$, using the infinitesimal generator of second order given as

$$\begin{aligned}\mathbf{U} =& - h(\phi_t + \psi_z)\frac{\partial}{\partial h} - k(\phi_t + \psi_z)\frac{\partial}{\partial k} + \mu_t\frac{\partial}{\partial h_t} + \nu_t\frac{\partial}{\partial k_t} + \mu_z\frac{\partial}{\partial h_z} + \nu_z\frac{\partial}{\partial k_z} \\ & + \mu_{tt}\frac{\partial}{\partial h_{tt}} + \nu_{tt}\frac{\partial}{\partial k_{tt}} + \mu_{tz}\frac{\partial}{\partial h_{tz}} + \nu_{tz}\frac{\partial}{\partial k_{tz}} + \mu_{zz}\frac{\partial}{\partial h_{zz}} + \nu_{zz}\frac{\partial}{\partial k_{zz}},\end{aligned} \tag{1.128}$$

where

$$\begin{aligned}\mu_{tt} &= -(h_{tt}\psi_z + 3h_{tt}\phi_t + 3h_t\phi_{tt} + h\phi_{ttt}), \\ \nu_{tt} &= -(k_{tt}\psi_z + 3k_{tt}\phi_t + 3k_t\phi_{tt} + k\phi_{ttt}), \\ \mu_{tz} &= -(2h_{tz}\psi_z + h_t\psi_{zz} + 2h_{tz}\phi_t + h_z\phi_{tt}), \\ \nu_{tz} &= -(2k_{tz}\psi_z + k_t\psi_{zz} + 2k_{tz}\phi_t + k_z\phi_{tt}), \\ \mu_{zz} &= -(3h_{zz}\psi_z + 3h_z\psi_{zz} + h\psi_{zzz} + h_{zz}\phi_t), \\ \nu_{zz} &= -(3k_{zz}\psi_z + 3k_z\psi_{zz} + k\psi_{zzz} + k_{zz}\phi_t).\end{aligned} \tag{1.129}$$

Now, the infinitesimal criteria $\mathbf{U}\mathrm{J}(h,k,h_t,k_t,h_z,k_z,h_{tt},k_{tt},h_{tz},k_{tz},h_{zz},k_{zz}) = 0$, after separating the coefficients of the derivatives ϕ and ψ, yield the following system of linear homogeneous PDEs

$$\begin{aligned}
&h\frac{\partial \mathrm{J}}{\partial h_{tt}} + k\frac{\partial \mathrm{J}}{\partial k_{tt}} = 0, \quad h\frac{\partial \mathrm{J}}{\partial h_{zz}} + k\frac{\partial \mathrm{J}}{\partial k_{zz}} = 0,\\
&h\frac{\partial \mathrm{J}}{\partial h_z} + k\frac{\partial \mathrm{J}}{\partial k_z} + h_t\frac{\partial \mathrm{J}}{\partial h_{tz}} + 3h_x\frac{\partial \mathrm{J}}{\partial h_{zz}} + k_t\frac{\partial \mathrm{J}}{\partial k_{tz}} + 3k_z\frac{\partial \mathrm{J}}{\partial k_{zz}} = 0,\\
&h\frac{\partial \mathrm{J}}{\partial h_t} + k\frac{\partial \mathrm{J}}{\partial k_t} + 3h_t\frac{\partial \mathrm{J}}{\partial h_{tt}} + h_z\frac{\partial \mathrm{J}}{\partial h_{tz}} + 3k_t\frac{\partial \mathrm{J}}{\partial k_{tt}} + k_z\frac{\partial \mathrm{J}}{\partial k_{tz}} = 0,\\
&h\frac{\partial \mathrm{J}}{\partial h} + k\frac{\partial \mathrm{J}}{\partial k} + 2h_t\frac{\partial \mathrm{J}}{\partial h_t} + h_x\frac{\partial \mathrm{J}}{\partial h_z} + 2k_t\frac{\partial \mathrm{J}}{\partial k_t} + k_x\frac{\partial \mathrm{J}}{\partial k_z} + 3h_{tt}\frac{\partial \mathrm{J}}{\partial h_{tt}}\\
&+ 2h_{tz}\frac{\partial \mathrm{J}}{\partial h_{tz}} + h_{zz}\frac{\partial \mathrm{J}}{\partial h_{zz}} + 3k_{tt}\frac{\partial \mathrm{J}}{\partial k_{tt}} + 2k_{tz}\frac{\partial \mathrm{J}}{\partial k_{tz}} + k_{zz}\frac{\partial \mathrm{J}}{\partial k_{zz}} = 0,
\end{aligned} \tag{1.130}$$

$$\begin{aligned}
&h\frac{\partial \mathrm{J}}{\partial h} + k\frac{\partial \mathrm{J}}{\partial k} + h_t\frac{\partial \mathrm{J}}{\partial h_t} + 2h_z\frac{\partial \mathrm{J}}{\partial h_z} + k_t\frac{\partial \mathrm{J}}{\partial k_t} + 2k_z\frac{\partial \mathrm{J}}{\partial k_z} + h_{tt}\frac{\partial \mathrm{J}}{\partial h_{tt}}\\
&+ 2h_{tz}\frac{\partial \mathrm{J}}{\partial h_{tz}} + 3h_{zz}\frac{\partial \mathrm{J}}{\partial h_{zz}} + k_{tt}\frac{\partial \mathrm{J}}{\partial k_{tt}} + 2k_{tz}\frac{\partial \mathrm{J}}{\partial k_{tz}} + 3k_{zz}\frac{\partial \mathrm{J}}{\partial k_{zz}} = 0.
\end{aligned} \tag{1.131}$$

By application of the theory of linear homogeneous PDEs, solutions of linear PDEs (1.130), using an arbitrary function Ψ, can be written as

$$\mathrm{J} = \Psi(p, J_2^1, J_3^1, J_3^2, J_3^3, J_3^4), \tag{1.132}$$

where

$$\begin{aligned}
J_3^1 &= \frac{1}{h^3}(k_{tz}h + h_{tz}k - k_th_x - k_zh_t),\\
J_3^2 &= \frac{1}{h^9}(k_zh - h_zk)^2(3k_th_th - 3h_t^2k - k_{tt}h^2 + h_{tt}kh),\\
J_3^3 &= \frac{1}{h^9}(k_th - h_tk)^2(3k_zh_zh - 3h_z^2k - k_{zz}h^2 + h_{zz}kh),\\
J_3^4 &= \frac{k}{h^4}(h_{tz}h - h_zh_t),
\end{aligned} \tag{1.133}$$

and p and J_2^1 are already given. Here h and k are not equal to zero. If h or k is zero then the hyperbolic PDEs (1.84) can be factorized. The joint invariant J_3^4 can be written as the product of the Ovsiannikov invariants $p = k/h$ and $q = (\partial_t\partial_z \ln h)/h$ as

$$J_3^4 = pq = \frac{k}{h}\frac{(\partial_t\partial_z \ln h)}{h}.$$

1.7 Invariant Differentiation and Bases of Invariants

In this section, we discuss invariant differentiation and bases of invariants. The operator of invariant differentiation gives differential invariants of higher order from the lower order differential invariants by the application of the invariant differentiation. The bases of invariants give the maximum number of independent invariants, such that all other invariant can be obtained from these invariants using invariant differentiation. Bases of invariants enable one to completely classify the family of DEs.

Definition 1.7.1. (**Invariant Differentiation**) The operator $\widetilde{\mathrm{U}}$ which transforms a differential invariant J of a group $\widetilde{\mathcal{G}}$ into another differential invariant $\widetilde{\mathrm{U}}$J is called the operator of invariant differentiation of the group $\widetilde{\mathcal{G}}$.

Theorem 1.7.2. *The set of all operators of invariant differentiation of the group $\widetilde{\mathcal{G}}$ is a Lie algebra over the field of invariants of this group.*

For the hyperbolic PDE (1.84), the operators of invariant differentiation [86] are to be found in the form

$$\mathcal{D} = \lambda_1 D_t + \lambda_2 D_z, \tag{1.134}$$

where D_t and D_z are the total differentiation operators and λ_1 and λ_2 are functions of the Laplace invariants and their derivatives. For infinitesimal generator of first order (1.125), the formula for the operator of invariant differentiation, $\widetilde{\mathbf{U}} = \mathbf{U} + \mathcal{D}(\phi\partial_{\lambda_1} + \psi\partial_{\lambda_2})$

$$\begin{aligned}\widetilde{\mathbf{U}} =& - h(\phi_t + \psi_z)\frac{\partial}{\partial h} - k(\phi_t + \psi_z)\frac{\partial}{\partial k} - (h\phi_{tt} + 2h_t\phi_t + h_t\psi_z)\frac{\partial}{\partial h_t} \\ & - (k\phi_{tt} + 2k_t\phi_t + k_t\psi_z)\frac{\partial}{\partial k_t} - (h_x\phi_t + h\psi_{zz} + 2h_z\psi_z)\frac{\partial}{\partial h_z} \\ & - (k_x\phi_t + k\psi_{zz} + 2h_x\psi_z)\frac{\partial}{\partial k_z} + \lambda_1\phi_t\frac{\partial}{\partial \lambda_1} + \lambda_2\psi_z\frac{\partial}{\partial \lambda_2}.\end{aligned} \tag{1.135}$$

To calculate λ_1 and λ_2, consider the infinitesimal criteria $\widetilde{\mathbf{U}}\mathrm{J}(h, k, h_t, k_t, h_z, k_z; \lambda_1, \lambda_2) = 0$. Since ϕ and ψ are arbitrary functions so by equating the coefficients of the derivatives of the ϕ and ψ, infinitesimal criteria gives the following system of homogeneous linear PDEs,

$$\begin{aligned}& k\frac{\partial \mathrm{J}}{\partial k_z} + h\frac{\partial \mathrm{J}}{\partial h_z} = 0, \quad k\frac{\partial \mathrm{J}}{\partial k_t} + h\frac{\partial \mathrm{J}}{\partial h_t} = 0, \\ & k_z\frac{\partial \mathrm{J}}{\partial k_z} + 2k_t\frac{\partial \mathrm{J}}{\partial k_t} + h_z\frac{\partial \mathrm{J}}{\partial h_z} + 2h_t\frac{\partial \mathrm{J}}{\partial h_t} + k\frac{\partial \mathrm{J}}{\partial k} + h\frac{\partial \mathrm{J}}{\partial h} - \lambda_1\frac{\partial \mathrm{J}}{\partial \lambda_1} = 0, \\ & 2k_z\frac{\partial \mathrm{J}}{\partial k_z} + k_t\frac{\partial \mathrm{J}}{\partial k_t} + 2h_z\frac{\partial \mathrm{J}}{\partial h_z} + h_t\frac{\partial \mathrm{J}}{\partial h_t} + k\frac{\partial \mathrm{J}}{\partial k} + h\frac{\partial \mathrm{J}}{\partial h} - \lambda_2\frac{\partial \mathrm{J}}{\partial \lambda_2} = 0.\end{aligned} \tag{1.136}$$

Using the theory of linear homogeneous PDEs, solutions of the above system can be given as

$$\mathrm{J} = \Psi(p, J_2^1, c_1, c_2), \tag{1.137}$$

where c_1 and c_2 are

$$c_1 = \lambda_1\lambda_2 h, \qquad c_2 = \frac{\lambda_2}{h^2}(k_z h - h_z k). \tag{1.138}$$

This yields the values of the functions λ_1 and λ_2 as

$$\lambda_1 = \frac{k_z h - h_z k}{h^3} c_3, \qquad \lambda_2 = \frac{h^2}{k_z h - h_z k} c_2 \tag{1.139}$$

for some constant c_3. We can obtain two independent operators

$$\widetilde{\mathbf{U}}_1 = \frac{k_z h - h_z k}{h^3} D_t, \tag{1.140}$$

for $c_3 = 0$, $c_2 = 1$ and

$$\widetilde{\mathbf{U}}_2 = \frac{h^2}{k_z h - h_z k} D_z, \tag{1.141}$$

for $c_2 = 0$, $c_3 = 1$. One can calculate the joint differential invariant J_2^1 by applying the invariant differential operator $\widetilde{\mathbf{U}}_1$ on p, i.e., $\widetilde{\mathbf{U}}_1 p = J_2^1$ and similarly $\widetilde{\mathbf{U}}_2 p = 1$ means no new invariant is obtained and J_2^1 can found from p by the operator $\widetilde{\mathbf{U}}_1$. Hence, one has a basis of joint invariants

$$\{q, p, J_3^1, J_3^2, J_3^3\}. \tag{1.142}$$

Theorem 1.7.3. *The set of joint differential invariants* (1.142) *form bases of joint invariants for the linear hyperbolic PDE* (1.84)*. It is a complete set of joint invariants and any other joint invariant can be constructed from these joint invariants and their invariant derivatives* [54].

Chapter 2

The Noether Symmetries of Area-Minimizing Lagrangian

Sophus Lie developed infinitesimal methods to find the Lie point symmetries and introduced methods to use these symmetries to reduce the order of the DEs or the number of variables in the case of PDEs and for the linearization of non-linear DEs [12, 41, 94]. In the theory of integration of DEs, Lie symmetries play a decisive role to determine whether the given DEs are integrable. Lie derived linearization criteria for a scalar second order ODE and showed that all linearizable second order ODEs have eight Lie point symmetries. There is only one equivalence class of linearizable equations for second order ODEs but for ODEs of order $n \geq 3$, it has been proved that there are three linearizable classes, i.e., the ODEs having $n+1$, $n+2$ or $n+4$ Lie point symmetries can be linearized [72]. However, for a systems of n second order ODEs it was shown that there are $(n+3)$ linearizable classes [103].

In 1918, Emmy Noether proved that if a variational principle is admitted by a system of DEs then the extremal of the action gives a system of EL-equations. The symmetry of the variational principle is called a Noether symmetry. The Noether symmetry is also a Lie symmetry of the EL-equation. Indeed, the Lie algebra of the Noether symmetries is a subalgebra of the Lie algebra of the Lie symmetries of the EL-equations. Noether symmetries are more important than the Lie symmetries as these symmetries give double reduction of the DEs and provide conserved quantities [3, 30].

As the differential equations "live" on manifolds, it is natural to search for the connection between symmetries of differential equations and those of geometry. The first such attempt looked for the connection through the system of geodesic equations [6, 31], some connections between Noether symmetries and isometries have been found in the context of general relativity [13–15]. There are some errors in [14] and [15] is incomplete (as regards all Noether symmetries under discussion the corrected list was given in [39, 40]). Recently the relation of both the Lie and Noether symmetries of the geodesic for a general Riemannian manifold has been given [98] . The geodesic equations are the EL-equations for the arc-length minimizing action. Their symmetries and the corresponding geodesic equations are known for maximally and non-maximally symmetric spaces. A connection was obtained between isometries (the symmetries of the geometry) and Lie symmetries of the geodesic equations of the underlying space [31], which leads to the geometric linearization for ordinary differential equations (ODEs) [46, 74, 88]. An additional benefit of this approach is that one can obtain the solution of the linearized equations by the transformation to the metric tensor coordinates given by the geodesic equations from Cartesian coordinates. In searching for an extension of the geometric methods to PDEs, a relation between isometries and Noether symmetries for the area minimizing Lagrangian has been found. Here, it is proved that the Lie algebra of the symmetries for the area minimizing Lagrangian in an n-dimensional Euclidean space is $so(n) \oplus_s \mathbb{R}^n$ (where $\oplus_s$ denotes the semi-direct sum) and in a space of constant curvature is $so(n)$. It is also derived that if the space has one section of constant curvature of dimension n_1, another of n_2, etc. to n_k, and one of zero curvature of dimension m and $n \geq \sum_{j=1}^{k} n_j + m$, so that some of the sections have no symmetry, then the Lie algebra of Noether symmetries is $A = \oplus_{j=1}^{k} so(n_j + 1) \oplus (so(m) \oplus_s \mathbb{R}^m)$. Here for the non-compact space this has to be taken in the sense of being cut at a fixed boundary that respects the symmetry of the space and is not a volume enclosing hypersurface otherwise.

In the subsequent sections of this chapter we present the symmetries of the area-minimizing Lagrangian for maximally and non-maximally symmetric spaces. In the last section, the symmetries for the less symmetric spaces are given.

2.1 Symmetries for Flat Spaces

Two Area Minimization: The flat space metric in spherical coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Let the enclosing surface be $r = r(\theta, \phi)$. The 2-area is then given by [93]

$$A(S) = \int (r^4 \sin^2\theta + r^2 \sin^2\theta r_{,\theta}^2 + r^2 r_{,\phi}^2)^{\frac{1}{2}} d\theta d\phi,$$

and the variational principle for the action (6.1) becomes

$$\delta\int\left[\Sigma+\lambda\frac{r^3\sin\theta}{3}\right]d\theta d\phi=0,$$

where

$$\Sigma=(r^4\sin^2\theta+r^2\sin^2\theta r_{,\theta}^2+r^2r_{,\phi}^2)^{\frac{1}{2}}.$$

Thus the Lagrangian is

$$L=(r^4\sin^2\theta+r^2\sin^2\theta r_{,\theta}^2+r^2r_{,\phi}^2)^{\frac{1}{2}}+\lambda\frac{r^3\sin\theta}{3}. \tag{2.1}$$

The Noether symmetry condition (1.62) results the following system of linear PDEs.

$$\begin{aligned}
&r\xi^\theta\cos\theta+2\eta\sin\theta+r\sin\theta(\xi^\theta_{,\theta}+\xi^\phi_{,\phi})=0, \qquad \eta_{,\theta}+r^2\xi^\theta_{,r}=0,\\
&r\xi^\theta\cos\theta+\eta\sin\theta+r(\eta_{,r}+\xi^\phi_{,\phi})\sin\theta=0, \qquad \eta_{,\phi}+r^2\sin^2\theta\xi^\phi_{,r}=0,\\
&A^\theta_{,\theta}+A^\phi_{,\phi}-\lambda\left[\frac{\xi^\theta r^3}{3}\cos\theta+\eta r\sin\theta+\frac{r^3}{3}\sin\theta(\xi^\theta_{,\theta}+\xi^\phi_{,\phi})\right]=0,\\
&\eta+r(\eta_{,r}+\xi^\theta_{,\theta})=0, \qquad \sin^2\theta\xi^\phi_{,\theta}+\xi^\theta_{,\phi}=0,\\
&A^\theta_{,r}-\frac{\lambda r^3}{3}\xi^\theta_{,r}=0, \qquad A^\phi_{,r}-\frac{\lambda r^3}{3}\xi^\phi_{,r}=0.
\end{aligned}$$

This system gives us the following six symmetries

$$\begin{aligned}
&\mathbf{X_1}=\sin\phi\frac{\partial}{\partial\theta}+\cos\phi\cot\theta\frac{\partial}{\partial\phi}, \quad \mathbf{X_2}=\cos\phi\frac{\partial}{\partial\theta}-\sin\phi\cot\theta\frac{\partial}{\partial\phi},\\
&\mathbf{X_3}=\frac{\partial}{\partial\phi}, \ \mathbf{X_4}=\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}+\sin\theta\sin\phi\frac{\partial}{\partial r},\\
&\mathbf{X_5}=\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}+\sin\theta\cos\phi\frac{\partial}{\partial r}, \ \mathbf{X_6}=\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}-\cos\theta\frac{\partial}{\partial r}.
\end{aligned}$$

The corresponding Lie algebra of the Noether symmetries is $so(3)\oplus_s\mathbb{R}^3$, where $\oplus_s$ is the semi-direct sum, $so(3)=\langle\mathbf{X_1},\mathbf{X_2},\mathbf{X_3}\rangle$, $\mathbb{R}^3=\langle\mathbf{X_4},\mathbf{X_5},\mathbf{X_6}\rangle$, and

$$\begin{aligned}
&A_1=\frac{\lambda r^2}{6}\left[-\cos\theta\sin\theta\sin\phi,-\cos\phi\right], \ A_2=\frac{\lambda r^2}{6}\left[-\sin\theta\cos\theta\cos\phi,\sin\phi\right],\\
&A_3=-\frac{\lambda r^2}{6}\left[\sin^2\theta,0\right],
\end{aligned}$$

are the non-zero vector gauge functions corresponding to the translations ($\mathbb{R}^3$).

Three Area Minimization: Following the same procedure for a 3-area enclosing a constant 4-volume in hyperspherical coordinates, the Lagrangian is

$$L=(r^6\sin\chi^4\sin^2\theta+r^4r_{,\chi}^2\sin\chi^4\sin\theta^2+r^4r_{,\theta}^2\sin\chi^2\sin\theta^2+r^4r_{,\phi}^2\sin\chi^2)^{\frac{1}{2}}+\lambda\frac{r^4\sin\chi^2\sin\theta}{4}.$$

The Lie algebra of the Noether symmetries of this Lagrangian is $so(4)\oplus_s\mathbb{R}^4$ and

$$\begin{aligned}
A_1 &= \frac{\lambda r^3}{12}\left[\sin^2\chi\cos\chi\sin^2\theta\cos\phi, \sin\chi\sin\theta\cos\theta\cos\phi, -\sin\chi\sin\phi\right],\\
A_2 &= -\frac{\lambda r^3}{12}\left[\sin^2\chi\cos\chi\sin^2\theta\sin\phi, \sin\chi\sin\theta\cos\theta\sin\phi, \sin\chi\cos\phi\right],\\
A_3 &= \frac{\lambda r^3}{12}\left[-\sin^2\chi\cos\chi\sin\theta\cos\theta, \sin\chi\sin^2\theta, 0\right],\ A_4 = \frac{\lambda r^3}{12}\left[-\sin^3\chi\sin\theta, 0, 0\right],
\end{aligned}$$

are the non-zero vector gauge functions corresponding to the translations ($\mathbb{R}^4$).

Four Area Minimization: Extending to the 4-area enclosing a constant 5-volume the Lagrangian is

$$\begin{aligned}
L =&(r^8\sin^6\psi\sin^4\chi\sin^2\theta + r^6r_{,\psi}^2\sin^6\psi\sin^4\chi\sin^2\theta + r^6r_{,\chi}^2\sin^4\psi\sin^4\chi\sin^2\theta +\\
&r^6r_{,\theta}^2\sin\psi^4\sin^2\chi\sin^2\theta + r^6r_{,\phi}\sin^4\psi\sin^2\chi)^{\frac{1}{2}} + \lambda\frac{1}{5}r^5\sin^3\psi\sin^2\chi\sin\theta.
\end{aligned}$$

The Lie algebra of the Noether symmetries for this Lagrangian is $so(5)\oplus_s\mathbb{R}^5$ and

$$\begin{aligned}
A_1 =& \frac{\lambda r^4}{20}\left[\sin^3\psi\cos\psi\sin^3\chi\sin^2\theta\cos\phi,\ \cos\chi\cos\phi\sin^2\chi\sin^2\psi\sin^2\theta,\right.\\
&\left.\sin^2\psi\sin\chi\sin\theta\cos\theta\cos\phi,\ -\sin\chi\sin\phi\sin^2\psi\right],\\
A_2 =& -\frac{\lambda r^4}{20}\left[\sin^3\chi\sin^3\psi\cos\psi\sin\phi\sin^2\theta,\ \sin\phi\cos\chi\sin^2\psi\sin^2\chi\sin^2\theta,\right.\\
&\left.\sin^2\psi\sin\chi\sin\theta\cos\theta\sin\phi,\ \sin\chi\cos\phi\sin^2\psi\right],\\
A_3 =& \frac{\lambda r^4}{20}\left[-\sin^3\chi\sin^3\psi\cos\theta\sin\theta\cos\psi,\ -\cos\theta\sin\theta\cos\chi\sin^2\chi\sin^2\psi,\ \sin^2\psi\sin\chi\sin^2\theta,\ 0\right],\\
A_4 =& \frac{\lambda r^4}{20}\left[-\sin^3\psi\cos\psi\cos\chi\sin^2\chi\sin\theta,\ \sin^3\chi\sin\theta\sin^2\psi,\ 0,\ 0\right],\\
A_5 =& \frac{\lambda r^4}{20}\left[-\sin^4\psi\sin^2\chi\sin\theta,\ 0,\ 0,\ 0\right],
\end{aligned}$$

are the non-zero vector gauge functions corresponding to the translations ($\mathbb{R}^5$).

We can now prove the results generalized to $(m-1)$-area minimization for a constant m-volume in a flat space by using a method of reduction and induction as done earlier for the connection between geometry and Lie symmetries [31].

Theorem 2.1.1. *The Lagrangian for minimizing the $(m-1)$-area enclosing a constant m-volume in a Euclidian space, has a Lie algebra of Noether symmetries identical with the Lie algebra of isometries of the Euclidean space, $so(m)\oplus_s\mathbb{R}^{(m)}$, with the non-zero vector gauge functions corresponding to the translations.*

Proof. The Lie algebra of Noether symmetries of the Lagrangian that minimizes the 2-area enclosing a 3-volume in Euclidean space; 3-area enclosing a 4-volume; and the 4-area enclosing a 5-volume in Euclidean space are $so(3)\oplus_s\mathbb{R}^3$; $so(4)\oplus_s\mathbb{R}^4$; and $so(5)\oplus_s\mathbb{R}^5$ respectively. Now, suppose that the Lie algebra of Noether symmetries of the Lagrangian that minimizes an $(n-1)$-area enclosing a constant n-volume in Euclidean space is $so(n)\oplus_s\mathbb{R}^n$. The Lagrangian for minimizing the n-area enclosing a constant $(n+1)$-volume in Euclidean space contains a subset of Noether symmetries identical to the isometries of S^n, i.e. $so(n+1)$. In the Euclidean space, S^n minimizes the n-area enclosing a constant $(n+1)$-volume. For the full set of Lie algebra, first reduce the n-area to an $(n-1)$-area and $(n+1)$-volume to an n-volume. The Lagrangian minimizing the n-area enclosing a constant $(n+1)$-volume reduces to the Lagrangian which minimizes the $(n-1)$-area enclosing a constant n-volume in the Euclidean space. The corresponding Lie algebra is $so(n)\oplus_s\mathbb{R}^n$ (the Lagrangian which minimizes the 4-area enclosing 5-volume can be transformed to the Lagrangian which minimizes 3-area enclosing 4-volume). Now working in reverse, from the Lagrangian minimizing an $(n-1)$-area enclosing a constant n-volume to the Lagrangian minimizing the n-area enclosing a constant $(n+1)$-volume it takes n more generators of rotation and one generator of translation from the previous one, i.e. $(n+1)$ more generators. Thus the Lie algebra of the Noether symmetries of the Lagrangian which minimizes $(m-1)$-area enclosing a constant m-volume in the Euclidean space is identical to the Lie algebra of isometries of the Euclidean space, i.e. $so(m)\oplus_s\mathbb{R}^{(m)}$. □

2.2 Symmetries for Curved Spaces

Two Area Minimization: The metric for a three dimensional curved space is

$$ds^2 = d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi\sin^2\theta d\phi^2. \tag{2.2}$$

Using the variational principle (6.1) for minimizing two area, we obtain the Lagrangian

$$L = \Sigma_1 + \lambda\frac{1}{2}(\sinh\chi\cosh\chi - \chi)\sin\theta, \tag{2.3}$$

where

$$\Sigma_1 = (\sinh^4\chi\sin^2\theta + \chi_{,\theta}^2\sinh^2\chi\sin^2\theta + \chi_{,\phi}^2\sinh^2\chi)^{\frac{1}{2}}.$$

The Lie algebra of the Noether symmetries of the Lagrangian is $so(3)$. Notice that there is no translational symmetry arising here.

Three Area Minimization: The metric for four dimensional curved space is

$$ds^2 = d\psi^2 + \sinh^2 \psi d\chi^2 + \sinh^2 \psi \sin^2 \chi d\theta^2 + \sinh^2 \psi \sin^2 \chi \sin^2 \theta d\phi^2. \tag{2.4}$$

The Lagrangian for minimizing three area is

$$L = \Sigma_2 + \lambda \frac{1}{3}(\sinh^2 \psi \cosh \psi - 2 \cosh \psi) \sin^2 \chi \sin \theta, \tag{2.5}$$

where

$$\Sigma_2 = (\sinh^6 \psi \sin^4 \chi \sin^2 \theta + \sinh^4 \psi \psi_{,\chi}^2 \sin^4 \chi \sin^2 \theta + \sinh^4 \psi \psi_{,\theta}^2 \sin^2 \chi \sin^2 \theta + \sinh^4 \psi \psi_{,\phi}^2 \sin^2 \chi)^{\frac{1}{2}}.$$

The Lie algebra of the Noether symmetries of the Lagrangian is $so(4)$, there is no translation in this case.

For spaces of constant nonzero curvature we present the following theorem.

Theorem 2.2.1. *The Lie algebra of Noether symmetries for the Lagrangian for minimizing the $(m-1)$-area keeping a constant m-volume in a space of non-zero constant curvature is $so(m)$.*

Proof. The proof of this theorem can be provided by arguments similar to those in theorem (1). □

These two theorems provide the Noether symmetries of the area minimizing Lagrangian for maximally symmetric spaces (constant curvature and zero curvature). Notice that when we go to spaces of constant curvature from spaces of zero curvature we lose m symmetries of the area minimizing Lagrangian. In the case of zero curvature m symmetries (translational symmetries) come out only with particular non-zero vector gauge functions, while the remaining symmetries (rotational symmetries) have a zero gauge function. In the case of non-zero curvature there is no translational symmetry and we have only rotational symmetries corresponding to a zero gauge function.

2.3 Symmetries for Spaces having Flat Section

Three Area Minimization in one Dimensional Flat and Three-Dimensional Curved Space: The metric for a four dimensional space having one dimensional flat section is

$$ds^2 = d\psi^2 + d\chi^2 + \sinh^2 \chi d\theta^2 + \sinh^2 \chi \sin^2 \theta d\phi^2. \tag{2.6}$$

Following the same procedure the three area minimizing Lagrangian is

$$L = \Sigma_3 + \lambda\psi \sinh^2\chi \sin\theta, \tag{2.7}$$

where

$$\Sigma_3 = (\sinh^4\chi \sin^2\theta + \psi_{,\chi}^2 \sinh^4\chi \sin^2\theta + \psi_{,\theta}^2 \sinh^2\chi \sin^2\theta + \psi_{,\phi}^2 \sinh^2\chi)^{\frac{1}{2}}.$$

The Lie algebra of the Noether symmetries of this Lagrangian is $so(4) \oplus \mathbb{R}^1$ and $\mathbb{R}^1$ corresponds to the vector gauge function, $A = \lambda(0, 0, \phi \sinh^2\chi \sin\theta)$.

Four Area Minimization in Two Dimensional Flat and Three-Dimensional Curved Space: The metric for a five-dimensional flat space having two-dimensional flat section is

$$ds^2 = dr^2 + r^2 d\chi^2 + d\psi^2 + \sinh^2\psi d\theta^2 + \sinh^2\psi \sin^2\theta d\phi^2. \tag{2.8}$$

Thus the Lagrangian for four area minimization is

$$\begin{aligned} L =&(r^2 \sinh^4\psi \sin^2\theta + r_{,\chi}^2 \sinh^4\psi \sin^2\theta + r^2 r_{,\psi}^2 \sinh^4\psi \sin^2\theta \\ &+ r^2 r_{,\theta}^2 \sinh^2\psi \sin^2\theta + r^2 r_{,\phi}^2 \sinh^2\psi)^{\frac{1}{2}} + \lambda\frac{1}{2} r^2 \sinh^2\psi \sin\theta. \end{aligned}$$

The Lie algebra of Noether symmetries for this Lagrangian is $so(4) \oplus (so(2) \oplus_s \mathbb{R}^2)$, where $\mathbb{R}^2$ corresponds to the vector gauge function

$$\begin{aligned} A_1 &= \frac{\lambda r}{2}\left[-\sin\theta \cos\chi \sinh^2\psi, 0, 0, 0\right], \\ A_2 &= \frac{\lambda r}{2}\left[-\sin\theta \sin\chi \sinh^2\psi, 0, 0, 0\right]. \end{aligned}$$

2.4 Symmetries for the Less Symmetric Spaces

The metric for spheroid which is a less symmetric surface of positive curvature is

$$ds^2 = (a^2\cos^2\theta + b^2\sin^2\theta)d\theta^2 + a^2\sin^2\theta d\phi^2, \tag{2.9}$$

and the Lagrangian is

$$L = \frac{(a^2 - a^2\theta_{,\phi}^2 + b^2\theta_{,\phi}^2)\sin\theta\cos\theta}{(a^2\cos^2\theta\theta_{,\phi}^2 + b^2\sin^2\theta\theta_{,\phi}^2 + a^2\sin^2\theta)^{\frac{1}{2}}} + \lambda a \sin\theta(a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}}. \tag{2.10}$$

This Lagrangian has only one symmetry, i.e. $\frac{\partial}{\partial\phi}$.

The metric for the ellipsoid is

$$\begin{aligned} ds^2 =&(a^2\cos^2\theta\cos^2\phi + b^2\cos^2\theta\sin^2\phi + c^2\sin^2\theta)d\theta^2 + 2(b^2-a^2)\sin\theta\cos\theta\sin\phi\cos\phi d\theta d\phi \\ &+ (a^2\sin^2\theta\sin^2\phi + b^2\sin^2\theta\cos^2\phi)d\phi^2, \end{aligned} \tag{2.11}$$

and the Lagrangian is

$$\begin{aligned} L =&\frac{1}{\Sigma_4}[\sin\theta\cos\theta(c^2 - a^2\cos^2\phi - b^2\sin^2\phi)\theta_{,\phi}^2 + \sin\phi\cos\phi(\cos^2\theta - \sin^2\theta) \\ &(b^2-a^2)\theta_{,\phi} + \sin\theta\cos\theta(b^2\cos^2\phi + a^2\sin^2\phi)] + \lambda(a^2b^2\sin^2\theta\cos^2\theta + \\ &a^2c^2\sin^4\theta\sin^2\phi + b^2c^2\sin^4\theta\cos^2\phi)^{\frac{1}{2}}, \end{aligned} \tag{2.12}$$

where

$$\Sigma_4 = \left(a^2\left(\cos\theta\cos\phi\theta_{,\phi} - \sin\theta\sin\phi\right)^2 + b^2\left(\cos\theta\sin\phi\theta_{,\phi} + \sin\theta\cos\phi\right)^2 + c^2\theta_{,\phi}^2\sin^2\theta\right)^{\frac{1}{2}}.$$

This Lagrangian admits no symmetry.

We can now generalize the results for spaces having sections of different constant curvatures using the geometric method as done in [88].

Theorem 2.4.1. *The Lie algebra of the Noether symmetries for the Lagrangian which minimizes an* $(n-1)$*-area enclosing a constant* n*-volume, in a space which has one section of constant curvature of dimension* n_1*, another of* n_2*, etc. up to* n_k *and a flat section of dimension* m *and* $n \geq \sum_{j=1}^{k} n_j + m$ *(as some of the sections may have no symmetry), is* $\oplus_{j=1}^{k} so(n_j+1) \oplus (so(m) \oplus_s \mathbb{R}^m)$.

Proof. First consider a manifold N of dimension n containing a maximal m-dimensional flat section M such that $N = M \oplus M^{\perp}$. Now the orthogonal subspace $M^{\perp}$, has no flat section, but can be further broken into sections of constant curvature of dimension n_1, n_2, ... up to n_k and possibly a remnant section with no symmetry. For this manifold we have an n-volume in the space having an m-dimensional flat section and one section of constant curvature of dimension n_1, another of n_2, etc. up to n_k. We minimize the $(n-1)$-area in a subspace having an $(m-1)$-dimensional flat section and the sections of constant curvature remaining unchanged. The Lie algebra of Noether symmetries of the $(m-1)$-area minimizing Lagrangian in flat space is $so(m)\oplus_s\mathbb{R}^m$. In the manifold $M^{\perp}$ each section retains its Lie algebra of Noether symmetries. Thus the full Lie algebra of Noether symmetries, in this case, is the direct sum of all these Lie algebras, i.e. $A = \oplus_{j=1}^{k} so(n_j+1) \oplus (so(m) \oplus_s \mathbb{R}^m)$.

If, instead there is no reduction of dimension of the flat section, but one constant curvature section reduces by one dimension, say $n_j \to n_j - 1$, the Lie algebra of the other sections remains unchanged while that of the reduced section now becomes $so(n_j)$. Now consider the case that there is only a one dimensional flat section, i.e. $m = 1$, and $(n-1)$-dimensional section of constant curvature. We have an n volume in a space having a one dimensional flat section and an $(n-1)$-dimensional section of constant curvature. We minimize the $(n-1)$-area in a subspace of constant curvature keeping a constant n-volume. Then the Lie algebra of Noether symmetries is $so(n) \oplus (so(1) \oplus_s \mathbb{R}^1)$, i.e. $so(n) \oplus \mathbb{R}^1$ (as $so(1)$ is the identity).

By increasing the dimension of the flat section by one, as in theorem 1, the algebra for it becomes $so(m+1) \oplus_s \mathbb{R}^{m+1}$. Similarly, increasing the dimension of one constant curvature section by 1, $n_j \rightarrow n_{j+1}$, the Lie algebra of that section becomes $so(n_j + 2)$ while the other sections retain their Lie algebras.

Thus reduction and induction show that the formula continues to hold. This completes the proof. □

In this chapter we have dealt with the Noether symmetries of the $(n-1)$-area minimizing Lagrangian keeping a constant n-volume for the maximally and non-maximally symmetric spaces. For spaces of maximal symmetry, the Lie algebra of the Noether symmetries is $so(n) \oplus_s \mathbb{R}^n$ in an n-dimensional flat space and $so(n)$ in an n-dimensional space of constant curvature. For an n-dimensional space of constant curvature, the area minimizing Lagrangian has $\frac{n(n-1)}{2}$ rotational symmetries with a zero gauge function and for the zero curvature there are n translational symmetries with specific non-zero vector gauge functions, along with $\frac{n(n-1)}{2}$ rotational symmetries with zero gauge function.

The third theorem provides the Neother symmetries for the area minimizing Lagrangian in non-maximally symmetric spaces. In this case we have a space consisting of sections of different constant curvatures, one section of zero curvature and possibly a section with no symmetry. The Lie algebra of Noether symmetries is then the direct sum of the Lie algebras of Noether symmetries of each section. If the space has a flat section of only one-dimension the Lie algebra becomes $\oplus_{j=1}^{k} so(n_j + 1) \oplus \mathbb{R}^1$.

Chapter 3

Invariants for Systems of Linear Hyperbolic Equations by Complex Methods

Differential invariants are extremely useful tools for transforming differential equations into integrable forms. In his fundamental memoir dedicated to the integration of the linear PDEs Laplace [57] derived two semi-invariants, known as the Laplace invariants, for linear hyperbolic PDEs. Laplace invariants remain conserved under the linear change of the dependent variables which maps the linear hyperbolic PDEs into themselves. Euler also proved that the solution of hyperbolic PDEs can be obtain by solving two first order ODEs if and only if one of the Laplace invariants is zero. Later, in 1773 Laplace extended Euler's method for cases when both the Laplace invariant are different from zero, which is known as *cascade method*. In 1911, Louise Petrén extended the Laplace methods and invariants for higher order DEs. Laplace-type and joint invariants for a system of two linear hyperbolic equations were derived in [99]. The approach of complex symmetry analysis (CSA), was utilized to derive Laplace-type invariants for a subclass of a system of two linear hyperbolic equations obtained from a complex linear hyperbolic equation [79].

In this chapter we first present semi-invariants and joint invariants for a subsystem of the general system of linear hyperbolic PDEs that is obtainable from a base complex hyperbolic PDE by real Lie infinitesimal approach. Then for such systems, semi-invariants and joint invariants are derived by splitting the complex semi-invariants and joint invariants of the base complex equation into real and imaginary parts. A comparison of all the invariant quantities derived by complex and real methods is presented here which shows that the complex procedure provides a few invariants different from those extracted by real symmetry analysis for a system of two linear hyperbolic PDEs.

3.1 Invariants of a system of two hyperbolic equations by real procedure

In this section, semi-invariants for both dependent and independent variables are first presented for a subsystem of two hyperbolic PDEs. Then we shall calculate joint invariants for such a system of equations. The scalar complex hyperbolic equation

$$w_{tx} + \alpha(t,x)w_t + \beta(t,x)w_x + \gamma(t,x)w = 0, \tag{3.1}$$

gives the following two hyperbolic equations

$$\begin{aligned} u_{tx} + \alpha_1(t,x)u_t - \alpha_2(t,x)v_t + \beta_1(t,x)u_x - \beta_2(t,x)v_x + \gamma_1(t,x)u - \gamma_2(t,x)v = 0,\\ v_{tx} + \alpha_2(t,x)u_t + \alpha_1(t,x)v_t + \beta_2(t,x)u_x + \beta_1(t,x)v_x + \gamma_2(t,x)u + \gamma_1(t,x)v = 0, \end{aligned} \tag{3.2}$$

for $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$ and $w = u + iv$. The above system of hyperbolic PDEs is a subclass of the general system of hyperbolic PDEs

$$\begin{aligned} u_{tx} + a_1(t,x)u_t + a_2(t,x)v_t + b_1(t,x)u_x + b_2(t,x)v_x + c_1(t,x)u + c_2(t,x)v = 0,\\ v_{tx} + a_3(t,x)u_t + a_4(t,x)v_t + b_3(t,x)u_x + b_4(t,x)v_x + c_3(t,x)u + c_4(t,x)v = 0. \end{aligned} \tag{3.3}$$

Invariants of a system of two linear hyperbolic PDEs (3.3) had already been determined by using the infinitesimal method [99]. The derivation of these invariants starts with the determination of the most general group of the equivalence transformations that maps the system of two linear hyperbolic equations to itself with, in general, different coefficients. It requires the application of the generator

$$\begin{aligned} &\mathbf{Z} = \xi^1\partial_t + \xi^2\partial_x + \eta^1\partial_u + \eta^2\partial_v + \eta^1_t\partial_{u_t} + \eta^1_x\partial_{u_x} + \eta^2_t\partial_{v_t} + \eta^2_x\partial_{v_x} + \eta^1_{tx}\partial_{u_{tx}} + \eta^2_{tx}\partial_{v_{tx}}\\ &+\mu^{11}\partial_{a_1} + \mu^{12}\partial_{a_2} + \mu^{13}\partial_{a_3} + \mu^{14}\partial_{a_4} + \mu^{21}\partial_{b_1} + \mu^{22}\partial_{b_2} + \mu^{23}\partial_{b_3} + \mu^{24}\partial_{b_4}\\ &+\mu^{31}\partial_{c_1} + \mu^{32}\partial_{c_2} + \mu^{33}\partial_{c_3} + \mu^{34}\partial_{c_4}, \end{aligned} \tag{3.4}$$

on both the equations of the system (3.3), here ξ^κ, η^κ are functions of (t, x, u, v), where $\kappa = 1, 2$, and $\mu^{1\lambda}, \mu^{2\lambda}, \mu^{3\lambda}$ for $\lambda = 1, 2, 3, 4$, are functions of $(t, x, u, v, a_\lambda, b_\lambda, c_\lambda)$. This procedure yields the most general group of equivalence transformations of the dependent and independent variables. Using the generators associated with these infinitesimal transformations the invariants of (3.3) had been derived [99].

The invariants associated with the subclass (3.2) of the system of hyperbolic equations (3.3) are determined in the remaining part of this section. The system of two hyperbolic PDEs (3.2) is obtainable from a hyperbolic PDE with two independent variables, when the dependent variable of equation (3.1) is considered complex. Such systems have a CR-structure due to this correspondence. Thus they are said to be *CR-structured systems.* The group of equivalence transformations associated with (3.2) is obtained when the following generator

$$\begin{aligned}\mathbf{Z} = \xi^1\partial_t + \xi^2\partial_x + \eta^1\partial_u + \eta^2\partial_v + \eta^1_t\partial_{u_t} + \eta^1_x\partial_{u_x} + \eta^2_t\partial_{v_t} + \eta^2_x\partial_{v_x} + \eta^1_{tx}\partial_{u_{tx}} + \eta^2_{tx}\partial_{v_{tx}} \\ +\mu^{11}\partial_{\alpha_1} + \mu^{12}\partial_{\alpha_2} + \mu^{21}\partial_{\beta_1} + \mu^{22}\partial_{\beta_2} + \mu^{31}\partial_{\gamma_1} + \mu^{32}\partial_{\gamma_2},\end{aligned} \tag{3.5}$$

where ξ^κ, η^κ are functions of (t, x, u, v) and $\mu^{1\kappa}, \mu^{2\kappa}$ and $\mu^{3\kappa}$ are functions of $(t, x, u, v, \alpha_\kappa, \beta_\kappa, \gamma_\kappa)$, acts on both the equations of the system (3.2). The solution of the system of linear PDEs obtained when (3.5) operates on the system (3.2), is [99]

$$\begin{aligned}&\xi^1 = F_1(t), \quad \xi^2 = F_2(x),\\ &\eta^1 = F_3u + F_4v, \quad \eta^2 = F_3v - F_4u,\\ &\mu^{11} = -F_{3,x} - \alpha_1F_{2,x}, \quad \mu^{12} = F_{4,x} - \alpha_2F_{2,x},\\ &\mu^{21} = -F_{3,t} - \beta_1F_{1,t}, \quad \mu^{22} = F_{4,t} - \beta_2F_{1,t},\\ &\mu^{31} = -F_{3,tx} - \alpha_1F_{3,t} - \alpha_2F_{4,t} - \beta_1F_{3,x} - \beta_2F_{4,x} - \gamma_1(F_{1,t} + F_{2,x}),\\ &\mu^{32} = F_{4,tx} + \alpha_1F_{4,t} - \alpha_2F_{3,t} + \beta_1F_{4,x} - \beta_2F_{3,x} - \gamma_2(F_{1,t} + F_{2,x}),\end{aligned} \tag{3.6}$$

where F_3 and F_4 depends on (t, x). Inserting the above in (3.5) leads to a generator that corresponds to changes of both the dependent and independent variables in the system (3.2). Hence, the resulting transformations of the coefficients are characterized by (3.5) with the above insertions.

In order to find the semi-invariants for only the dependent variables, we consider a change only of dependent variables and hence exclude eta^1 and eta^2. The corresponding generator is

$$\begin{aligned}\mathbf{X} = -F_{3,x}\partial_{\alpha_1} + F_{4,x}\partial_{\alpha_2} - F_{3,t}\partial_{\beta_1} + F_{4,t}\partial_{\beta_2} - (F_{3,tx} + \alpha_1F_{3,t} + \alpha_2F_{4,t} \\ +\beta_1F_{3,x} + \beta_2F_{4,x})\partial_{\gamma_1} + (F_{4,tx} + \alpha_1F_{4,t} - \alpha_2F_{3,t} + \beta_1F_{4,x} - \beta_2F_{3,x})\partial_{\gamma_2}.\end{aligned} \tag{3.7}$$

By adopting the procedure described in the second section, applying the once extended generator on $J(\alpha_\kappa, \beta_\kappa, \gamma_\kappa, \alpha_{\kappa,t}, \beta_{\kappa,t}, \gamma_{\kappa,t}, \alpha_{\kappa,x}, \beta_{\kappa,x}, \gamma_{\kappa,x})$ and solving the resulting linear system of PDEs, we find the first order semi-invariants [99]

$$\begin{aligned}h^r_1 &= \alpha_{1,t} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1,\\ h^r_2 &= \alpha_{2,t} + \alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_2,\\ k^r_1 &= \beta_{1,x} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1,\\ k^r_2 &= \beta_{2,x} + \alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_2.\end{aligned} \tag{3.8}$$

Now considering only a change of the independent variables leads to an infinitesimal generator

$$\begin{aligned}\mathbf{Z}_I = & F_1(t)\partial_t + F_2(x)\partial_x - \alpha_1 F_{2,x}\partial_{\alpha_1} - \alpha_2 F_{2,x}\partial_{\alpha_2} - \beta_1 F_{1,t}\partial_{\beta_1} - \beta_2 F_{1,t}\partial_{\beta_2}\\ & -\gamma_1(F_{1,t}+F_{2,x})\partial_{\gamma_1} - \gamma_2(F_{1,t}+F_{2,x})\partial_{\gamma_2}.\end{aligned} \tag{3.9}$$

Applying it on $J(\alpha_\kappa, \beta_\kappa, \gamma_\kappa)$ yields the following zeroth order invariants

$$I_1^r = \frac{\alpha_2}{\alpha_1}, \quad I_2^r = \frac{\beta_2}{\beta_1}, \quad I_3^r = \frac{\gamma_1}{\alpha_1\beta_1}, \quad I_4^r = \frac{\gamma_2}{\alpha_1\beta_1}. \tag{3.10}$$

Further, the first order invariants are obtained when the once extended generator (3.9) acts on $J(\alpha_\kappa, \beta_\kappa, \gamma_\kappa, \alpha_{\kappa,t}, \beta_{\kappa,t}, \gamma_{\kappa,t}, \alpha_{\kappa,x}, \beta_{\kappa,x}, \gamma_{\kappa,x})$, this leads to a system of PDEs which gives the following quantities

$$\begin{aligned}& I_5^r = \frac{\alpha_{1,t}}{\alpha_1\beta_1}, \quad I_6^r = \frac{\alpha_{2,t}}{\alpha_1\beta_1}, \quad I_7^r = \frac{\beta_{1,x}}{\alpha_1\beta_1}, \quad I_8^r = \frac{\beta_{2,x}}{\alpha_1\beta_1},\\ & I_9^r = \frac{\beta_1\beta_{2,t} - \beta_2\beta_{1,t}}{b_1^3}, \quad I_{10}^r = \frac{\beta_1\gamma_{1,t} - \gamma_1\beta_{1,t}}{\alpha_1\beta_1^3}, \quad I_{11}^r = \frac{\beta_1\gamma_{2,t} - \gamma_2\beta_{1,t}}{\alpha_1\beta_1^3},\\ & I_{12}^r = \frac{\alpha_1\alpha_{2,x} - \alpha_2\alpha_{1,x}}{\alpha_1^3}, \quad I_{13}^r = \frac{\alpha_1\gamma_{1,x} - \gamma_1\alpha_{1,x}}{\alpha_1^3\beta_1}, \quad I_{14}^r = \frac{\alpha_1\gamma_{2,x} - \gamma_2\alpha_{1,x}}{\alpha_1^3\beta_1},\end{aligned} \tag{3.11}$$

including the four zeroth order invariants (3.10).

The joint invariants of the system (3.2)

$$J_1^r = \frac{h_2^r}{h_1^r}, \quad J_2^r = \frac{k_1^r}{h_1^r}, \quad J_3^r = \frac{k_2^r}{h_1^r}, \tag{3.12}$$

are found when the following PDE

$$h_1^r\frac{\partial J}{\partial h_1^r} + h_2^r\frac{\partial J}{\partial h_2^r} + k_1^r\frac{\partial J}{\partial k_1^r} + k_2^r\frac{\partial J}{\partial k_2^r} = 0, \tag{3.13}$$

is solved. This equation appears due to action of the infinitesimal generator (3.9) that is associated with the change of the independent variables to the space of invariants h_κ^r, k_κ^r.

3.2 Invariants of a system of two hyperbolic equations by complex procedure

Semi-invariants associated with a system of two hyperbolic equations (3.2) that is obtained from a scalar linear hyperbolic equation (3.1), are derived in this section by complex methods. Let the generator of the form (1.91) associated with the equation (3.1) be complex by incorporating the complex dependent variable, therefore, generator (1.91) splits into two operators

$$\begin{aligned}\mathbf{X}_1 = & \eta_{1,z_2}\partial_{\alpha_1} + \eta_{2,z_2}\partial_{\alpha_2} + \eta_{1,z_1}\partial_{\beta_1} + \eta_{2,z_1}\partial_{\beta_2} + (\eta_{1,z_1z_2} + \alpha_1\eta_{1,z_1} - \alpha_2\eta_{2,z_1} + \beta_1\eta_{1,z_2}\\ & -\beta_2\eta_{2,z_2})\partial_{\gamma_1} + (\eta_{2,z_1z_2} + \alpha_2\eta_{1,z_1} + \alpha_1\eta_{2,z_1} + \beta_2\eta_{1,z_2} + \beta_1\eta_{2,z_2})\partial_{\gamma_2},\end{aligned} \tag{3.14}$$

$$\mathbf{X}_2 = \eta_{2,z_2}\partial_{\alpha_1} - \eta_{1,z_2}\partial_{\alpha_2} + \eta_{2,z_1}\partial_{\beta_1} - \eta_{1,z_1}\partial_{\beta_2} + (\eta_{2,z_1z_2} + \alpha_2\eta_{1,z_1} + \alpha_1\eta_{2,z_1} + \beta_2\eta_{1,z_2}$$
$$+\beta_1\eta_{2,z_2})\partial_{\gamma_1} - (\eta_{1,z_1z_2} + \alpha_1\eta_{1,z_1} - \alpha_2\eta_{2,z_1} + \beta_1\eta_{1,z_2} - \beta_2\eta_{2,z_2})\partial_{\gamma_2}. \tag{3.15}$$

There are four first order semi-invariants

$$\begin{aligned} h_1 &= \alpha_{1,z_1} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1, \\ h_2 &= \alpha_{2,z_1} + \alpha_2\beta_1 + \alpha_1\beta_2 - \gamma_2, \\ k_1 &= \beta_{1,z_2} + \alpha_1\beta_1 - \alpha_2\beta_2 - \gamma_1, \\ k_2 &= \beta_{2,z_2} + \alpha_2\beta_1 + \alpha_1\beta_2 - \gamma_2, \end{aligned} \tag{3.16}$$

associated with the system (3.2) on employing the pair of operators (3.14)-(3.15) and solving the emerging system of PDEs. These are exactly the same as represented by $\overset{r}{h_\kappa}$, $\overset{r}{k_\kappa}$ in (3.8). Therefore, in this case the real and complex procedures lead to the same semi-invariants of the system (3.2). Notice that all the four semi-invariants (3.16) are readable from the first order semi-invariants associated with the complex hyperbolic linear equation (3.1) and satisfy

$$\mathbf{X}_1^{[1]}h_1\mid_{h_1=0} = \mathbf{X}_2^{[1]}h_2\mid_{h_2=0} = \mathbf{X}_1^{[1]}k_1\mid_{k_1=0} = \mathbf{X}_2^{[1]}k_2\mid_{k_2=0} = 0. \tag{3.17}$$

The linear combination $\mathbf{X}_3$ of both the operators $\mathbf{X}_1$ and $\mathbf{X}_2$ results in the following relations

$$\mathbf{X}_3^{[1]}h_1\mid_{h_1=0} = \mathbf{X}_3^{[1]}h_2\mid_{h_2=0} = \mathbf{X}_3^{[1]}k_1\mid_{k_1=0} = \mathbf{X}_3^{[1]}k_2\mid_{k_2=0} = 0. \tag{3.18}$$

The semi-invariants of the system of two hyperbolic PDEs (3.2) under a transformation of the independent variables are

$$\overset{c}{I_1} = \frac{(\alpha_1\gamma_1 + \alpha_2\gamma_2)\beta_1 + (\alpha_1\gamma_2 - \alpha_2\gamma_1)\beta_2}{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)},$$
$$\overset{c}{I_2} = \frac{(\alpha_1\gamma_2 - \alpha_2\gamma_1)\beta_1 - (\alpha_1\gamma_1 + \alpha_2\gamma_2)\beta_2}{(\alpha_1^2 + \alpha_2^2)(\beta_1^2 + \beta_2^2)},$$

$$\overset{c}{I_3} = \frac{(\alpha_1\beta_1 - \alpha_2\beta_2)\alpha_{1,t} + (\alpha_2\beta_1 + \alpha_1\beta_2)\alpha_{2,t}}{\alpha_{1,t}^2 + \alpha_{2,t}^2},$$
$$\overset{c}{I_4} = \frac{(\alpha_2\beta_1 + \alpha_1\beta_2)\alpha_{1,t} - (\alpha_1\beta_1 - \alpha_2\beta_2)\alpha_{2,t}}{\alpha_{1,t}^2 + \alpha_{2,t}^2},$$

$$\overset{c}{I_5} = \frac{\alpha_{1,t}\beta_{1,x} + \alpha_{2,t}\beta_{2,x}}{\alpha_{1,t}^2 + \alpha_{2,t}^2}, \quad \overset{c}{I_6} = \frac{\alpha_{1,t}\beta_{2,x} - \alpha_{2,t}\beta_{1,x}}{\alpha_{1,t}^2 + \alpha_{2,t}^2},$$

$$\overset{c}{I_7} = \frac{\alpha_{1,t}\gamma_1 + \alpha_{2,t}\gamma_2}{\alpha_{1,t}^2 + \alpha_{2,t}^2}, \quad \overset{c}{I_8} = \frac{\alpha_{1,t}\gamma_2 - \alpha_{2,t}\gamma_1}{\alpha_{1,t}^2 + \alpha_{2,t}^2},$$

$$I_9^c = \frac{(\alpha_{1,t}^2-\alpha_{2,t}^2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)}[\alpha_1\beta_1(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t})-\alpha_2\beta_1(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}$$
$$-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})+\alpha_2\beta_2(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t})+\alpha_1\beta_2(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}$$
$$-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})]+\frac{2\alpha_{1,t}\alpha_{2,t}}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)}[\alpha_2\beta_1(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t}) \tag{3.19}$$
$$+\alpha_1\beta_1(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})-\alpha_1\beta_2(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t})$$
$$+\alpha_2\beta_2(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})],$$

$$I_{10}^c = \frac{(\alpha_{1,t}^2-\alpha_{2,t}^2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)}[\alpha_2\beta_1(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t})+\alpha_1\beta_1(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}$$
$$-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})-\alpha_1\beta_2(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t})+\alpha_2\beta_2(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}$$
$$-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})]-\frac{2\alpha_{1,t}\alpha_{2,t}}{(\alpha_{1,t}^2+\alpha_{2,t}^2)^2(\beta_1^2+\beta_2^2)}[\alpha_1\beta_1(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t}) \tag{3.20}$$
$$-\alpha_2\beta_1(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})+\alpha_2\beta_2(\beta_1\gamma_{1,t}-\beta_2\gamma_{2,t}-\gamma_1\beta_{1,t}+\gamma_2\beta_{2,t})$$
$$+\alpha_1\beta_2(\beta_2\gamma_{1,t}+\beta_1\gamma_{2,t}-\gamma_2\beta_{1,t}-\gamma_1\beta_{2,t})],$$

$$I_{11}^c = \frac{(\alpha_1^2-\alpha_2^2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2}[(\alpha_1\gamma_{1,x}-\alpha_2\gamma_{2,x}-\gamma_1\alpha_{1,x}+\gamma_2\alpha_{2,x})\alpha_{1,t}+(\alpha_2\gamma_{1,x}+\alpha_1\gamma_{2,x}$$
$$-\gamma_2\alpha_{1,x}-\gamma_1\alpha_{2,x})\alpha_{2,t}]+\frac{2\alpha_1\alpha_2}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2}[(\alpha_2\gamma_{1,x}+\alpha_1\gamma_{2,x}-\gamma_2\alpha_{1,x}-\gamma_1\alpha_{2,x})\alpha_{1,t} \tag{3.21}$$
$$-(\alpha_1\gamma_{1,x}-\alpha_1\gamma_{2,x}-\gamma_1\alpha_{1,x}+\gamma_2\alpha_{2,x})\alpha_{2,t}],$$

$$I_{12}^c = \frac{(\alpha_1^2-\alpha_2^2)}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2}[(\alpha_2\gamma_{1,x}+\alpha_1\gamma_{2,x}-\gamma_2\alpha_{1,x}-\gamma_1\alpha_{2,x})\alpha_{1,t}-(\alpha_1\gamma_{1,x}-\alpha_2\gamma_{2,x}$$
$$-\gamma_1\alpha_{1,x}+\gamma_2\alpha_{2,x})\alpha_{2,t}]-\frac{2\alpha_1\alpha_2}{(\alpha_{1,t}^2+\alpha_{2,t}^2)(\alpha_1^2+\alpha_2^2)^2}[(\alpha_1\gamma_{1,x}-\alpha_2\gamma_{2,x}-\gamma_1\alpha_{1,x}+\gamma_2\alpha_{2,x})\alpha_{1,t} \tag{3.22}$$
$$+(\alpha_2\gamma_{1,x}+\alpha_1\gamma_{2,x}-\gamma_2\alpha_{1,x}-\gamma_1\alpha_{2,x})\alpha_{2,t}].$$

The correspondence of these semi-invariants with the system of the hyperbolic equations is established due to the following operators

$$\mathbf{X}_1 = 2\xi_1\partial_t+2\xi_2\partial_x-\alpha_1\xi_{2,x}\partial_{\alpha_1}-\alpha_2\xi_{2,x}\partial_{\alpha_2}-\beta_1\xi_{1,t}\partial_{\beta_1}-\beta_2\xi_{1,t}\partial_{\beta_2}-\gamma_1(\xi_{1,t}$$
$$+\xi_{2,x})\partial_{\gamma_1}-\gamma_2(\xi_{1,t}+\xi_{2,x})\partial_{\gamma_2}, \tag{3.23}$$

$$\mathbf{X}_2 = -\alpha_2\xi_{2,x}\partial_{\alpha_1}+\alpha_1\xi_{2,x}\partial_{\alpha_2}-\beta_2\xi_{1,t}\partial_{\beta_1}+\beta_1\xi_{1,t}\partial_{\beta_2}-\gamma_2(\xi_{1,t}+\xi_{2,x})\partial_{\gamma_1}$$
$$+\gamma_1(\xi_{1,t}+\xi_{2,x})\partial_{\gamma_2}, \tag{3.24}$$

that are the real and imaginary parts of the complex generator of the form (1.114). Using these operators it is observed that

$$\mathbf{X}_1^{[1]}I_1^c\mid_{I_1^c=0}=\mathbf{X}_2^{[1]}I_2^c\mid_{I_2^c=0}=\mathbf{X}_1^{[1]}I_3^c\mid_{I_3^c=0}=\mathbf{X}_2^{[1]}I_4^c\mid_{I_3^c=I_4^c=0}=0,$$
$$\mathbf{X}_1^{[1]}I_5^c\mid_{I_5^c=0}=\mathbf{X}_2^{[1]}I_6^c\mid_{I_5^c=I_6^c=0}=\mathbf{X}_1^{[1]}I_7^c\mid_{I_7^c=0}=\mathbf{X}_2^{[1]}I_8^c\mid_{I_7^c=I_8^c=0}=0, \tag{3.25}$$
$$\mathbf{X}_1^{[1]}I_9^c\mid_{I_9^c=0}=\mathbf{X}_2^{[1]}I_{10}^c\mid_{I_9^c=I_{10}^c=0}=\mathbf{X}_1^{[1]}I_{11}^c\mid_{I_{11}^c=0}=\mathbf{X}_2^{[1]}I_{12}^c\mid_{I_{11}^c=I_{12}^c=0}=0.$$

It is seen that the above invariants are complex splits of their real analogues. Similarly, the linear combination of both $\mathbf{X}_1$ and $\mathbf{X}_2$, if denoted by $\mathbf{X}_3$, satisfies the relations

$$\begin{aligned}
&\mathbf{X}_3^{[1]}I_1^c\mid_{I_1^c=0}=\mathbf{X}_3^{[1]}I_2^c\mid_{I_2^c=0}=\mathbf{X}_3^{[1]}I_3^c\mid_{I_3^c=0}=\mathbf{X}_3^{[1]}I_4^c\mid_{I_3^c=I_4^c=0}=0,\\
&\mathbf{X}_3^{[1]}I_5^c\mid_{I_5^c=0}=\mathbf{X}_3^{[1]}I_6^c\mid_{I_5^c=I_6^c=0}=\mathbf{X}_3^{[1]}I_7^c\mid_{I_7^c=I_8^c=0}=\mathbf{X}_3^{[1]}I_8^c\mid_{I_7^c=I_8^c=0}=0,\\
&\mathbf{X}_3^{[1]}I_9^c\mid_{I_9^c=I_{10}^c=0}=\mathbf{X}_3^{[1]}I_{10}^c\mid_{I_9^c=I_{10}^c=0}=\mathbf{X}_3^{[1]}I_{11}^c\mid_{I_{11}^c=I_{12}^c=0}=\mathbf{X}_3^{[1]}I_{12}^c\mid_{I_{11}^c=I_{12}^c=0}=0.
\end{aligned} \tag{3.26}$$

To work out the joint invariants of the coupled system of two hyperbolic equations (3.2), the operators (3.23) and (3.24) need to be transformed to the space of invariants $h_\kappa,\ k_\kappa$. The same procedure was adopted in [54] before using the generator (1.114) in determining the joint invariants of the scalar linear hyperbolic equation. The complex generator was transformed to h and k, i.e. to the space of the semi-invariants associated with the hyperbolic equation under a change of the dependent variables. The procedure to transform (3.23) and (3.24) to $(h_\kappa, k_\kappa) - space$ starts with splitting (1.118) when $\mathbf{Z}(h)$ and $\mathbf{Z}(k)$ are taken as complex, i.e. $\mathbf{Z}(h) = \mathbf{Z}(h)_1 + i\mathbf{Z}(h)_2$ and $\mathbf{Z}(k) = \mathbf{Z}(k)_1 + i\mathbf{Z}(k)_2$. The real and imaginary parts of (1.118) are

$$\begin{aligned}
\mathbf{X}_1 &= \frac{1}{2}[\mathbf{Z}(h)_1\partial_{h_1} + \mathbf{Z}(h)_2\partial_{h_2} + \mathbf{Z}(k)_1\partial_{k_1} + \mathbf{Z}(k)_2\partial_{k_2}],\\
\mathbf{X}_2 &= \frac{1}{2}[\mathbf{Z}(h)_2\partial_{h_1} - \mathbf{Z}(h)_1\partial_{h_2} + \mathbf{Z}(k)_2\partial_{k_1} - \mathbf{Z}(k)_1\partial_{k_2}],
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
\mathbf{Z}(h)_1 &= \mathbf{X}_1h_1 - \mathbf{X}_2h_2 = -(\xi_{1,t}+\xi_{2,x})h_1,\\
\mathbf{Z}(h)_2 &= \mathbf{X}_2h_1 + \mathbf{X}_1h_2 = -(\xi_{1,t}+\xi_{2,x})h_2,\\
\mathbf{Z}(k)_1 &= \mathbf{X}_1k_1 - \mathbf{X}_2k_2 = -(\xi_{1,t}+\xi_{2,x})k_1,\\
\mathbf{Z}(k)_2 &= \mathbf{X}_2k_1 + \mathbf{X}_1k_2 = -(\xi_{1,t}+\xi_{2,x})k_2.
\end{aligned} \tag{3.28}$$

Using (3.28) in (3.27), the following two operators

$$\begin{aligned}
\mathbf{X}_1 &= -\frac{(\xi_{1,t}+\xi_{2,x})}{2}[h_1\partial_{h_1} + h_2\partial_{h_2} + k_1\partial_{k_1} + k_2\partial_{k_2}],\\
\mathbf{X}_2 &= -\frac{(\xi_{1,t}+\xi_{2,x})}{2}[h_2\partial_{h_1} - h_1\partial_{h_2} + k_2\partial_{k_1} - k_1\partial_{k_2}],
\end{aligned} \tag{3.29}$$

are obtained which are the real and imaginary parts of the complex generator of the form (1.122). These operators are used to arrive at the joint invariants for the system of two linear hyperbolic equations (3.2). We have the following joint invariants

$$\begin{aligned}
J_{11} &= \frac{h_1k_1 + h_2k_2}{k_1^2 + k_2^2},\\
J_{12} &= \frac{h_2k_1 - h_1k_2}{k_1^2 + k_2^2},
\end{aligned}$$

$$J_{13}=\frac{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)(h_1k_{1,t}-h_2k_{2,t}-k_1h_{1,t}+k_2h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_1k_{1,x}-h_2k_{2,x}-k_1h_{1,x}+k_2h_{2,x})$$
$$+\frac{(5h_1^4h_2-10h_1^2h_2^3+h_2^5)(h_2k_{1,t}+h_1k_{2,t}-k_2h_{1,t}-k_1h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_1k_{1,x}-h_2k_{2,x}-k_1h_{1,x}+k_2h_{2,x})$$
$$+\frac{(5h_1^4h_2-10h_1^2h_2^3+h_2^5)(h_1k_{1,t}-h_2k_{2,t}-k_1h_{1,t}+k_2h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_2k_{1,x}+h_1k_{2,x}-k_2h_{1,x}-k_1h_{2,x})$$
$$+\frac{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)(h_2k_{1,t}+h_1k_{2,t}-k_1h_{1,t}-k_1h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_2k_{1,x}+h_1k_{2,x}-k_2h_{1,x}-k_1h_{2,x}),$$

$$J_{14}=\frac{-(5h_1^4h_2-10h_1^2h_2^3+h_2^5)(h_1k_{1,t}-h_2k_{2,t}-k_1h_{1,t}+k_2h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_1k_{1,x}-h_2k_{2,x}-k_1h_{1,x}+k_2h_{2,x})$$
$$+\frac{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)(h_2k_{1,t}+h_1k_{2,t}-k_2h_{1,t}-k_1h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_1k_{1,x}-h_2k_{2,x}-k_1h_{1,x}+k_2h_{2,x})$$
$$+\frac{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)(h_1k_{1,t}-h_2k_{2,t}-k_1h_{1,t}+k_2h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_2k_{1,x}+h_1k_{2,x}-k_2h_{1,x}-k_1h_{2,x})$$
$$+\frac{(5h_1^4h_2-10h_1^2h_2^3+h_2^5)(h_2k_{1,t}+h_1k_{2,t}-k_1h_{1,t}-k_1h_{2,t})}{(h_1^5-10h_1^3h_2^2+5h_1h_2^4)^2+(5h_1^4h_2-10h_1^2h_2^3+h_2^5)^2}(h_2k_{1,x}+h_1k_{2,x}-k_2h_{1,x}-k_1h_{2,x}),$$

$$J_{15}=\frac{(h_1^3-3h_1h_2^2)(k_1h_{1,tx}-k_2h_{2,tx}+h_1k_{1,tx}-h_2k_{2,tx}-h_{1,t}k_{1,x}+h_{2,t}k_{2,x}-h_{1,x}k_{1,t}+h_{2,x}k_{2,t})}{(h_1^3-3h_1h_2^2)^2+(3h_1^2h_2-h_2^3)^2}$$
$$+\frac{(3h_1^2h_2-h_2^3)(k_2h_{1,tx}+k_1h_{2,tx}+h_2k_{1,tx}+h_1k_{2,tx}-h_{2,t}k_{1,x}-h_{1,t}k_{2,x}-h_{2,x}k_{1,t}-h_{1,x}k_{2,t})}{(h_1^3-3h_1h_2^2)^2+(3h_1^2h_2-h_2^3)^2},$$

$$J_{16}=\frac{(3h_1^2h_2-h_2^3)(k_1h_{1,tx}-k_2h_{2,tx}+h_1k_{1,tx}-h_2k_{2,tx}-h_{1,t}k_{1,x}+h_{2,t}k_{2,x}-h_{1,x}k_{1,t}+h_{2,x}k_{2,t})}{(h_1^3-3h_1h_2^2)^2+(3h_1^2h_2-h_2^3)^2}$$
$$+\frac{(h_1^3-3h_1h_2^2)(k_2h_{1,tx}+k_1h_{2,tx}+h_2k_{1,tx}+h_1k_{2,tx}-h_{2,t}k_{1,x}-h_{1,t}k_{2,x}-h_{2,x}k_{1,t}-h_{1,x}k_{2,t})}{(h_1^3-3h_1h_2^2)^2+(3h_1^2h_2-h_2^3)^2},$$

$$J_{17}=\frac{k_1(-6h_1^2h_2^2+h_1^4+h_2^4)+k_2(4h_1^3h_2-4h_1h_2^3)}{(-6h_1^2h_2^2+h_1^4+h_2^4)^2+(4h_1^3h_2-4h_1h_2^3)^2}(h_1h_{1,tx}-h_2h_{2,tx}-h_{1,t}h_{1,x}+h_{2,t}h_{2,x})$$
$$-\frac{k_2(-6h_1^2h_2^2+h_1^4+h_2^4)-k_1(4h_1^3h_2-4h_1h_2^3)}{(-6h_1^2h_2^2+h_1^4+h_2^4)^2+(4h_1^3h_2-4h_1h_2^3)^2}(h_2h_{1,tx}+h_1h_{2,tx}-h_{2,t}h_{1,x}-h_{1,t}h_{2,x}),$$

$$J_{18}=\frac{k_2(-6h_1^2h_2^2+h_1^4+h_2^4)-k_1(4h_1^3h_2-4h_1h_2^3)}{(-6h_1^2h_2^2+h_1^4+h_2^4)^2+(4h_1^3h_2-4h_1h_2^3)^2}(h_1h_{1,tx}-h_2h_{2,tx}-h_{1,t}h_{1,x}+h_{2,t}h_{2,x})$$
$$+\frac{k_1(-6h_1^2h_2^2+h_1^4+h_2^4)+k_2(4h_1^3h_2-4h_1h_2^3)}{(-6h_1^2h_2^2+h_1^4+h_2^4)^2+(4h_1^3h_2-4h_1h_2^3)^2}(h_2h_{1,tx}+h_1h_{2,tx}-h_{2,t}h_{1,x}-h_{1,t}h_{2,x}),$$

$$J_{19}=\frac{\mu_1\nu_1+\mu_2\nu_2}{\mu_1^2+\mu_2^2}\omega_1+\frac{\mu_2\nu_1-\mu_1\nu_2}{\mu_1^2+\mu_2^2}\omega_2,$$
$$J_{20}=\frac{\mu_1\nu_2-\mu_2\nu_1}{\mu_1^2+\mu_2^2}\omega_1+\frac{\mu_1\nu_1+\mu_2\nu_2}{\mu_1^2+\mu_2^2}\omega_2,$$
$$J_{21}=\frac{\mu_1\nu_3+\mu_2\nu_4}{\mu_1^2+\mu_2^2}\omega_3+\frac{\mu_2\nu_3-\mu_1\nu_4}{\mu_1^2+\mu_2^2}\omega_4,$$
$$J_{22}=\frac{\mu_1\nu_4-\mu_2\nu_3}{\mu_1^2+\mu_2^2}\omega_3+\frac{\mu_1\nu_3+\mu_2\nu_4}{\mu_1^2+\mu_2^2}\omega_4,$$

where

$$\mu_1 = h_1^9 - 36h_1^7h_2^2 + 126h_1^5h_2^4 - 84h_1^3h_2^6 + 9h_1h_2^8,$$

$$\mu_2 = 9h_1^8h_2 - 84h_1^6h_2^3 + 126h_1^4h_2^5 - 36h_1^2h_2^7 + h_2^9,$$

$$\begin{aligned}
\nu_1 =& k_2^2h_{2,x}^2 + 2h_1k_{2,x}k_1h_{2,x} - 2h_2k_{2,x}k_2h_{2,x} + 2h_2k_{1,x}k_1h_{2,x} - 4k_1h_{1,x}k_2h_{2,x} - k_1^2h_{2,x}^2 + h_2^2k_{2,x}^2 \\
&+ 2h_2k_{1,x}k_2h_{1,x} - 2h_1k_{1,x}k_1h_{1,x} + 2h_2k_{2,x}k_1h_{1,x} + k_1^2h_{1,x}^2 + h_1^2k_{1,x}^2 + 2h_1k_{2,x}k_2h_{1,x} - h_2^2k_{1,x}^2 \\
&- 4h_1k_{1,x}h_2k_{2,x} - h_1^2k_{2,x}^2 - k_2^2h_{1,x}^2 + 2h_1k_{1,x}k_2h_{2,x}, \\
\nu_2 =& - 2k_2h_{2,x}^2k_1 - 2h_1k_{1,x}k_1h_{2,x} - 2k_1h_{1,x}h_1k_{2,x} + 2h_2k_{2,x}k_2h_{1,x} - 2k_2^2h_{2,x}h_{1,x} + 2h_2k_{2,x}k_1h_{2,x} \\
&+ 2k_2h_{2,x}h_2k_{1,x} - 2h_2^2k_{2,x}k_{1,x} + 2k_2h_{2,x}h_1k_{2,x} + 2h_1k_{1,x}^2h_2 - 2h_1k_{1,x}k_2h_{1,x} - 2h_2k_{2,x}^2h_1 \\
&+ 2h_1^2k_{1,x}k_{2,x} + 2k_1h_{1,x}^2k_2 - 2k_1h_{1,x}h_2k_{1,x} + 2k_1^2h_{1,x}h_{2,x}, \\
\nu_3 =& - 2h_2k_{2,t}k_2h_{2,t} - k_1^2h_{2,t}^2 + 2h_2k_{1,t}k_2h_{1,t} - h_2^2k_{1,t}^2 + 2h_2k_{2,t}k_1h_{1,t} + h_1^2k_{1,t}^2 - 2h_1k_{1,t}k_1h_{1,t} \\
&+ k_1^2h_{1,t}^2 + k_2^2h_{2,t}^2 - 4h_1k_{2,t}h_2k_{1,t} - k_2^2h_{1,t}^2 + 2h_1k_{1,t}k_2h_{2,t} + 2h_1k_{2,t}k_1h_{2,t} - h_1^2k_{2,t}^2 \\
&- 4k_1h_{1,t}k_2h_{2,t} + 2h_2k_{1,t}k_1h_{2,t} + 2h_1k_{2,t}k_2h_{1,t} + h_2^2k_{2,t}^2, \\
\nu_4 =& 2h_2k_{2,t}k_2h_{1,t} + 2h_1k_{1,t}^2h_2 + 2h_1k_{2,t}k_2h_{2,t} + 2k_1h_{1,t}^2k_2 - 2h_1k_{2,t}k_1h_{1,t} + 2h_1^2k_{1,t}k_{2,t} \\
&- 2k_1h_{2,t}^2k_2 - 2h_2^2k_{1,t}k_{2,t} + 2h_2k_{2,t}k_1h_{2,t} + 2h_2k_{1,t}k_2h_{2,t} + 2k_1^2h_{1,t}h_{2,t} - 2k_2^2h_{1,t}h_{2,t} \\
&- 2h_2k_{1,t}k_1h_{1,t} - 2h_1k_{1,t}k_1h_{2,t} - 2h_1k_{1,t}k_2h_{1,t} - 2h_1k_{2,t}^2h_2,
\end{aligned}$$

and

$$\begin{aligned}
\omega_1 =& (h_1k_1 - h_2k_2)h_{1,tt} - (h_2k_1 + h_1k_2)h_{2,tt} + (-h_1^2 + h_2^2)k_{1,tt} + 2h_1h_2k_{2,tt} - 3k_1(h_{1,t}^2 - h_{2,t}^2) \\
&+ 6k_2h_{1,t}h_{2,t} + (3h_1h_{1,t} - 3h_2h_{2,t})k_{1,t} - (3h_2h_{1,t} + 3h_1h_{2,t})k_{2,t}, \\
\omega_2 =& (h_2k_1 + h_1k_2)h_{1,tt} + (h_1k_1 - h_2k_2)h_{2,tt} + (-h_1^2 + h_2^2)k_{2,tt} - 2h_1h_2k_{1,tt} - 3k_2(h_{1,t}^2 - h_{2,t}^2) \\
&- 6k_1h_{1,t}h_{2,t} + (3h_2h_{1,t} + 3h_1h_{2,t})k_{1,t} + (3h_1h_{1,t} - 3h_2h_{2,t})k_{2,t}, \\
\omega_3 =& (h_1k_1 - h_2k_2)h_{1,xx} - (h_2k_1 + h_1k_2)h_{2,xx} + (-h_1^2 + h_2^2)k_{1,xx} + 2h_1h_2k_{2,xx} - 3k_1(h_{1,x}^2 - h_{2,x}^2) \\
&+ 6k_2h_{1,x}h_{2,x} + (3h_1h_{1,x} - 3h_2h_{2,x})k_{1,x} - (3h_2h_{1,x} + 3h_1h_{2,x})k_{2,x}, \\
\omega_4 =& (h_2k_1 + h_1k_2)h_{1,xx} + (h_1k_1 - h_2k_2)h_{2,xx} + (-h_1^2 + h_2^2)k_{2,xx} - 2h_1h_2k_{1,xx} - 3k_2(h_{1,x}^2 - h_{2,x}^2) \\
&- 6k_1h_{1,x}h_{2,x} + (3h_2h_{1,x} + 3h_1h_{2,x})k_{1,x} + (3h_1h_{1,x} - 3h_2h_{2,x})k_{2,x},
\end{aligned}$$

which are found to be associated with the system of two linear hyperbolic PDEs (3.2). These also can be observed to be the complex split of the joint invariants (1.133).

3.3 Applications

In this section a few examples of systems of hyperbolic equations are provided to illustrate the invariance criteria developed.

1. A system of two hyperbolic PDEs

$$\begin{aligned}u_{tx}+&\left(a_1-\frac{1}{x}\right)u_t-a_2v_t+\left(b_1+\frac{2}{t}\right)u_x-b_2v_x+\left(c_1-\frac{b_1}{x}+2\frac{a_1}{t}-\frac{2}{tx}\right)u\\&-\left(c_2-\frac{b_2}{x}+2\frac{a_2}{t}\right)v=0,\\v_{tx}+&a_2u_t+\left(a_1-\frac{1}{x}\right)v_t+b_2u_x+\left(b_1+\frac{2}{t}\right)v_x+\left(c_2-\frac{b_2}{x}+2\frac{a_2}{t}\right)u\\&+\left(c_1-\frac{b_1}{x}+2\frac{a_1}{t}-\frac{2}{tx}\right)v=0,\end{aligned}\tag{3.30}$$

corresponds to a complex hyperbolic equation in two independent variables

$$w_{tx}+\left(a-\frac{1}{x}\right)w_t+\left(b+\frac{2}{t}\right)w_x+\left(c-\frac{b}{x}+2\frac{a}{t}-\frac{2}{tx}\right)w=0,\tag{3.31}$$

where $a=a_1+ia_2$ is a complex constant. The following complex transformation of the dependent variable $w=(x/t^2)\overline{w}$ maps the complex equation (3.31) to

$$\overline{w}_{tx}+a\overline{w}_t+b\overline{w}_x+c\overline{w}=0.\tag{3.32}$$

Both the complex hyperbolic equations (3.31) and (3.32) are transformable to each other because they have the same semi-invariants

$$h=ab-c=k.\tag{3.33}$$

The system of hyperbolic equations (3.30) is transformable to the system

$$\begin{aligned}\overline{u}_{tx}+a_1\overline{u}_t-a_2\overline{v}_t+b_1\overline{u}_x-b_2\overline{v}_x+c_1\overline{u}-c_2\overline{v}=0,\\\overline{v}_{tx}+a_2\overline{u}_t+a_1\overline{v}_t+b_2\overline{u}_x+b_1\overline{v}_x+c_2\overline{u}+c_1\overline{v}=0,\end{aligned}\tag{3.34}$$

with constant coefficient. The systems of hyperbolic equations (3.30) and (3.34) are transformable into each other as these systems have the same semi-invariants

$$\begin{aligned}h_1=a_1b_1-a_2b_2-c_1=k_1,\\h_2=a_1b_2+a_2b_1-c_2=k_2.\end{aligned}\tag{3.35}$$

The real transformations of the dependent variables

$$u=(x/t^2)\overline{u},\quad v=(x/t^2)\overline{v},\tag{3.36}$$

are obtained by splitting the complex transformation of the dependent variable, used to map the complex equations (3.31) and (3.32) into each other.

2. An uncoupled system of PDEs

$$\begin{aligned} u_{z_1z_2} + 2az_1^2u_{z_1} + 2bz_1u_{z_2} + 4cz_1u = 0, \\ v_{z_1z_2} + 2az_1^2v_{z_1} + 2bz_1v_{z_2} + 4cz_1v = 0, \end{aligned} \tag{3.37}$$

is transformable to

$$\begin{aligned} u_{tx} + atu_t + bu_x + cu = 0, \\ v_{tx} + atv_t + bv_x + cv = 0, \end{aligned} \tag{3.38}$$

via invertible transformations of the independent variables

$$z_1 = \sqrt{t}, \quad z_2 = \frac{1}{2}(x-1). \tag{3.39}$$

These are the invertible maps which can also reduce a complex hyperbolic equation of the form

$$w_{z_1z_2} + 2az_1^2w_{z_1} + 2bz_1w_{z_2} + 4cz_1w = 0, \tag{3.40}$$

with the semi-invariants

$$I_1 = \frac{c}{abz_1^2}, \quad I_2 = bz_1^2, \quad I_3 = 0, \quad I_4 = \frac{c}{a}, \quad I_5 = 0 = I_6, \tag{3.41}$$

to a simple linear form

$$w_{tx} + atw_t + bw_x + cw = 0, \tag{3.42}$$

with the following semi-invariants

$$I_1 = \frac{c}{abt}, \quad I_2 = bt, \quad I_3 = 0, \quad I_4 = \frac{c}{a}, \quad I_5 = 0 = I_6. \tag{3.43}$$

Notice that the semi-invariants (3.41) and (3.43) are the same by means of the transformations of the independent variables (3.39). The complex hyperbolic equation (3.40) does not only yield an uncoupled system of the hyperbolic equations (3.37). In fact it gives a coupled system

$$\begin{aligned} u_{z_1z_2} + 2a_1z_1^2u_{z_1} - 2a_2z_1^2v_{z_1} + 2b_1z_1u_{z_2} - 2b_2z_1v_{z_2} + 4c_1z_1u - 4c_2z_1v = 0, \\ v_{z_1z_2} + 2a_2z_1^2u_{z_1} + 2a_1z_1^2v_{z_1} + 2b_2z_1u_{z_2} + 2b_1z_1v_{z_2} + 4c_2z_1u + 4c_1z_1v = 0. \end{aligned} \tag{3.44}$$

This system of two hyperbolic equations can be mapped to

$$
\begin{aligned}
&u_{tx} + a_1 t u_t - a_2 t v_t + b_1 u_x - b_2 v_x + c_1 u - c_2 v = 0,\\
&v_{tx} + a_2 t u_t + a_1 t v_t + b_2 u_x + b_1 v_x + c_2 u + c_1 v = 0,
\end{aligned} \tag{3.45}
$$

under the transformations (3.39) these are already used to map the base complex equation to its canonical form.

3. A coupled system of two hyperbolic equations of the form

$$
\begin{aligned}
&u_{z_1,z_2} + 2a_1 z_2 \ln z_1 u_{z_1} - 2a_2 z_2 \ln z_1 v_{z_1} + \frac{b_1}{z_1} u_{z_2} - \frac{b_2}{z_1} v_{z_2} + \frac{2c_1 z_2}{z_1} u - \frac{2c_2 z_2}{z_1} v = 0,\\
&v_{z_1,z_2} + 2a_2 z_2 \ln z_1 u_{z_1} + 2a_1 z_2 \ln z_1 v_{z_1} + \frac{b_2}{z_1} u_{z_2} + \frac{b_1}{z_1} v_{z_2} + \frac{2c_2 z_2}{z_1} u + \frac{2c_1 z_2}{z_1} v = 0,
\end{aligned} \tag{3.46}
$$

transforms to

$$
\begin{aligned}
&u_{tx} + a_1 t u_t - a_2 t v_t + b_1 u_x - b_2 v_x + c_1 u - c_2 v = 0,\\
&v_{tx} + a_2 t u_t + a_1 t v_t + b_2 u_x + b_1 v_x + c_2 u + c_1 v = 0,
\end{aligned} \tag{3.47}
$$

by the applications of the following change of the independent variables

$$
z_1 = e^t, \quad z_2 = \sqrt{x}. \tag{3.48}
$$

These systems of hyperbolic equations are transformable into each other as these systems have the same semi-invariants. The transformation of these systems under the invertible change of the independent variables follows from the base complex hyperbolic equation

$$
w_{z_1 z_2} + 2a \ln z_1 w_{z_1} + \frac{b}{z_1} w_{z_2} + \frac{2c z_2}{z_1} w = 0. \tag{3.49}
$$

It can be transformed to another linear form

$$
w_{tx} + a t w_t + b w_x + c w = 0, \tag{3.50}
$$

under the invertible transformations (3.48). Similarly, the invertible transformations of the independent variables (3.48) map the following system of PDEs

$$
\begin{aligned}
&u_{z_1,z_2} + 2a_1 z_2 u_{z_1} - 2a_2 z_2 v_{z_1} + \frac{b_1}{z_1} u_{z_2} - \frac{b_2}{z_1} v_{z_2} + \frac{2c_1 z_2}{z_1} u - \frac{2c_2 z_2}{z_1} v = 0,\\
&v_{z_1,z_2} + 2a_2 z_2 u_{z_1} + 2a_1 z_2 v_{z_1} + \frac{b_2}{z_1} u_{z_2} + \frac{b_1}{z_1} v_{z_2} + \frac{2c_2 z_2}{z_1} u + \frac{2c_1 z_2}{z_1} v = 0,
\end{aligned} \tag{3.51}
$$

to

$$u_{tx} + a_1 u_t - a_2 v_t + b_1 u_x - b_2 v_x + c_1 u - c_2 v = 0,$$
$$v_{tx} + a_2 u_t + a_1 v_t + b_2 u_x + b_1 v_x + c_2 u + c_1 v = 0. \tag{3.52}$$

4. Consider an uncoupled system of two hyperbolic type PDEs

$$g_{1,tx} + \frac{\lambda}{2}(g_{1,t} + g_{1,x}) = 0,$$
$$g_{2,tx} + \frac{\lambda}{2}(g_{2,t} + g_{2,x}) = 0, \tag{3.53}$$

for which $h_1 = k_1 = \frac{\lambda^2}{4}$, and $h_2 = k_2 = 0$. This implies that

$$J_1 = 1, \quad J_2 = \cdots = J_{12} = 0. \tag{3.54}$$

The system (3.53) is transformable to another system with the same invariants as given in (3.54) where $h_1 = k_1 = -1$, $h_2 = k_2 = 0$. The transformed system reads as

$$f_{1,z_1z_2} + f_1 = 0,$$
$$f_{2,z_1z_2} + f_2 = 0. \tag{3.55}$$

The correspondence between the systems (3.53) and (3.55) is established by

$$z_1 = \frac{\lambda}{2}t, \quad z_2 = -\frac{\lambda}{2}x, \quad f_1 = g_1 \exp(\frac{\lambda t + \lambda x}{2}), \quad f_2 = g_2 \exp(\frac{\lambda t + \lambda x}{2}). \tag{3.56}$$

These transformations are obtainable from

$$z_1 = \frac{\lambda}{2}t, \quad z_2 = -\frac{\lambda}{2}x, \quad w = u \exp(\frac{\lambda t + \lambda x}{2}), \tag{3.57}$$

by $w = f_1 + if_2$ and $u = g_1 + ig_2$. The complex transformations map the complex scalar PDE

$$w_{,z_1z_2} + \frac{\lambda}{2}(w_{,z_1} + w_{,z_2}) = 0, \tag{3.58}$$

with $h = k = \frac{\lambda^2}{4}$ and $p = 1$, to an equation

$$u_{,tx} + u = 0, \tag{3.59}$$

for which $h = k = -1$ and $p = 1$. Notice that the substitution $\lambda = \lambda_1 + i\lambda_2$, in the equation (3.58) results in a coupled system of two hyperbolic PDEs but it can not be transformed by the complex method. The reason is the complex transformations (3.57) where the two independent variables split into four. Therefore, the complex procedure fails for that case.

5. The complex transformations of the form

$$z_1 = \frac{1}{t}, \quad z_2 = 2x, \quad w = \frac{u}{x}, \tag{3.60}$$

map the following Lie canonical form

$$w_{,z_1 z_2} + \alpha z_2^2 w_{,z_2} + 2w = 0, \tag{3.61}$$

to

$$u_{,tx} - \frac{1}{x}u_{,t} - \frac{\alpha x^2}{t^2}u_{,x} + \frac{1}{t^2}(\alpha x - 2)u = 0. \tag{3.62}$$

The invariant quantities associated with both the scalar Lie canonical form and the hyperbolic equations are $h = -1,\ k = 2\alpha x - 1,\ p = 2(1 - \alpha x)$ and $h = 2/t^2,\ k = \frac{2(1-\alpha x)}{t^2},\ p = 1 - \alpha x$, respectively. Inserting $u = g_1 + ig_2$ in the equation (3.62) while keeping α *a real constant* yields an uncoupled system of two PDEs

$$\begin{aligned} g_{1,tx} - \frac{1}{x}g_{1,t} - \frac{\alpha x^2}{t^2}g_{1,x} + \frac{\alpha x - 2}{t^2}g_1 = 0, \\ g_{2,tx} - \frac{1}{x}g_{2,t} - \frac{\alpha x^2}{t^2}g_{2,x} + \frac{\alpha x - 2}{t^2}g_2 = 0. \end{aligned} \tag{3.63}$$

The system (3.63) is transformable to another system of the form

$$\begin{aligned} f_{1,z_1 z_2} + \alpha x^2 f_{1,z_2} + 2f_1 = 0, \\ f_{2,z_1 z_2} + \alpha x^2 f_{2,z_2} + 2f_2 = 0, \end{aligned} \tag{3.64}$$

under a change of the dependent and independent variables

$$z_1 = \frac{1}{t}, \quad z_2 = 2x, \quad f_1 = \frac{g_1}{x}, \quad f_2 = \frac{g_2}{x}. \tag{3.65}$$

These transformations are the real and imaginary parts of the complex transformations (3.60) and the transformed system is obtained by splitting the Lie canonical form (3.61) into the real and imaginary parts. The invariance criteria that ensure such a transformation of the system are satisfied. These quantities for both the systems (3.63) and (3.64) are

$$h_1 = \frac{2}{t^2}, \quad k_1 = \frac{2(1 - \alpha x)}{t^2}, \quad h_2 = 0 = k_2, \quad p = \frac{-1}{\alpha x - 1}, \tag{3.66}$$

and

$$h_1 = -2, \quad k_1 = 2(\alpha x - 1), \quad h_2 = 0 = k_2, \quad p = \frac{1}{1 - \alpha x}, \tag{3.67}$$

respectively.

A coupled system

$$g_{1,tx} - \frac{1}{x}g_{1,t} - \frac{\alpha_1 x^2}{t^2}g_{1,x} + \frac{\alpha_2 x^2}{t^2}g_{2,x} + \frac{\alpha_1 x - 2}{t^2}g_1 - \frac{\alpha_2 x}{t^2}g_2 = 0,$$
$$g_{2,tx} - \frac{1}{x}g_{2,t} - \frac{\alpha_2 x^2}{t^2}g_{1,x} - \frac{\alpha_1 x^2}{t^2}g_{2,x} + \frac{\alpha_2 x}{t^2}g_1 + \frac{\alpha_1 x - 2}{t^2}g_2 = 0, \quad (3.68)$$

with the invariants

$$h_1 = \frac{2}{t^2}, \quad k_1 = \frac{2(1-\alpha_1 x)}{t^2}, \quad h_2 = 0, \quad k_2 = \frac{-2\alpha_2 x}{t^2},$$
$$J_1 = \frac{1-\alpha_1 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2}, \quad J_2 = \frac{\alpha_2 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2}, \quad (3.69)$$

is obtainable from the complex scalar PDE (3.62) when α is also complex, i.e., $\alpha = \alpha_1 + i\alpha_2$. Employing the transformations (3.65) on (3.68) one arrives at a coupled system

$$f_{1,z_1z_2} + \alpha_1 x^2 f_{1,z_2} - \alpha_2 x^2 f_{2,z_2} + 2f_1 = 0,$$
$$f_{2,z_1z_2} + \alpha_2 x^2 f_{1,z_2} + \alpha_1 x^2 f_{2,z_2} + 2f_2 = 0, \quad (3.70)$$

which is the real analogue of the complex transformed equation (3.61) and satisfies the invariance criteria, where

$$h_1 = -2, \quad k_1 = 2(\alpha_1 x - 1), \quad h_2 = 0, \quad k_2 = 2\alpha_2 x,$$
$$J_1 = \frac{1-\alpha_1 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2}, \quad J_2 = \frac{\alpha_2 x}{(1-\alpha_1 x)^2 + \alpha_2^2 x^2}. \quad (3.71)$$

Semi-invariants of a special class of systems of two hyperbolic PDEs were derived here using real and complex methods developed for such systems of equations. Both the procedures are adopted to find the semi-invariants of the system of two hyperbolic equations that is obtainable from a complex hyperbolic PDE. Semi-invariants associated with the invertible change of the dependent and of the independent variables are deduced by both the real and complex methods. It is shown that same invariant quantities for the system of hyperbolic PDEs appear due to complex and real procedures, in the case of transformations of only the dependent variables. However, the semi-invariants of this system obtained by real symmetry analysis are different from those provided by the complex procedure.

Chapter 4

Symmetry Classification and Joint Differential Invariants for Scalar Linear Elliptic PDEs

Lie showed the successful use of symmetries in the study of integration of DEs and gave a complete classification of second order ODEs. For Lie the central problem in the theory of transformation groups was the classification problem, the problem of determining, up to similarity, all transformation groups of both point and contact transformations in n dimensions. However Lie's success in dealing with the group classification problem was not as great as he had initially hoped. As for the problem for arbitrary n, he expressed the view [62] that it would probably never be resolved. He succeeded in completely classifying all Lie groups in one and two dimensions [59]. Further, in his third volume of his treatise on transformation groups [62], Lie claimed to have completed the three dimensional classification. Later, in 1881, he further discussed in detail the symmetry structure of general scalar linear second order PDEs of the form

$$aw_{xx} + 2bw_{xy} + cw_{yy} + dw_x + ew_y + fw = g, \tag{4.1}$$

where a, b, c, d, e, f and g are given C^2 functions of x and y. He obtained seven canonical forms according to their point symmetries and types of equations. Of these, four belongs to the hyperbolic class and three to the parabolic class. In 1773, two semi-invariants had been derived by Laplace [57] in his fundamental memoir on the integration of linear PDEs, known as the Laplace invariants, for the linear hyperbolic PDEs. In 1900, for the linear elliptic PDEs Cotton [27] constructed the semi-invariants, named after him. The Laplace and the Cotton invariants remain conserved under the linear changes of the dependent variables which respectively map the linear hyperbolic and elliptic equations into themselves. Linear hyperbolic and elliptic PDEs and Laplace and Cotton invariants can be transformed into each other by the application of linear complex transformations of the independent variables [29, 55]. In 1990, Ovsiannikov [86] used the Laplace invariants in the group classification of the hyperbolic equation by writing the determining equations for symmetries of the hyperbolic equation in terms of these invariants.

It is well known that the set of all equivalence transformations of (1.98) is an infinite group which comprises of the linear transformations of the dependent variable

$$\bar{u} = \sigma(t, x)u, \quad \sigma(t, x) \neq 0, \tag{4.2}$$

and invertible changes of the independent variables of the form

$$\bar{t} = \phi(t, x), \quad \bar{x} = \psi(t, x), \quad \phi_t = \psi_x, \quad \phi_x = -\psi_t, \tag{4.3}$$

where $\phi(t, x), \psi(t, x)$ and $\sigma(t, x)$ are arbitrary nonzero functions and $\bar{u}, \bar{t}$ and $\bar{x}$ are new dependent and independent variables respectively.

In the first section of this chapter The equations for the classification of symmetries of the scalar linear elliptic PDE in two independent variables are obtained by in terms of Cotton's invariants. New joint differential invariants of the scalar linear elliptic PDEs in two independent variables are derived in terms of Cotton's invariants by application of the infinitesimal method. Here joint differential invariants of the scalar linear elliptic equation are derived from the basis of the joint differential invariants of the scalar linear hyperbolic equation under the application of the complex linear transformation. We also find a basis of joint differential invariants for such equations by utilization of the operators of invariant differentiation. The other invariants are functions of the basis elements and their invariant derivatives. Examples are given to illustrate our method.

4.1 Symmetry classification

In this section, we obtain the symmetry classification of second order scalar linear elliptic PDEs of two independent variables, E, via Cotton's invariants.

Let

$$\mathbf{X} = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{4.4}$$

be the symmetry operator admitted by the equation (1.25) with ξ^1, ξ^2 and η are unknown functions to be found. Then the symmetry condition for (1.25) is

$$\mathbf{X}^{[2]}(u_{tt} + u_{xx} + au_t + bu_x + cu)|_{(E)} = 0, \tag{4.5}$$

where $\mathbf{X}^{[2]}$ denotes the prolongation of the operator (4.4) to the second-order derivatives:

$$\mathbf{X}^{[2]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}}.$$

From (4.5), we have the determining equation

$$\begin{aligned} &\zeta_{tt} + \zeta_{xx} + a\zeta_t + (a_t\xi^1 + a_x\xi^2)u_t + b\zeta_x + (b_t\xi^1 + b_x\xi^2)u_x \\ &+c\eta + (c_t\xi^1 + c_x\xi^2)u = 0, \end{aligned} \tag{4.6}$$

evaluated on the elliptic equation.

The coefficient functions $\zeta_t, \zeta_x, \zeta_{tt}$ and ζ_{xx} are well-known and given in expanded form by

$$\begin{aligned} \zeta_t &= \eta_t + \eta_u u_t - (\xi^1_t + u_t\xi^1_t)u_t - (\xi^2_t + u_t\xi^2_u)u_x, \\ \zeta_x &= \eta_x + \eta_u u_x - (\xi^1_x + u_x\xi^1_u)u_t - (\xi^2_x + u_x\xi^2_u)u_x, \\ \zeta_{tt} &= \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + u_t^2\eta_{uu} - 2u_{tt}\xi^1_t - u_t\xi^1_{tt} - 2u_t^2\xi^1_{tu} \\ &\quad -3u_tu_{tt}\xi^1_u - u_t^3\xi^1_{uu} - 2u_{tx}\xi^2_t - u_x\xi^2_{tt} - 2u_tu_x\xi^2_{tu} \\ &\quad -(u_xu_{tt} + 2u_tu_{tx})\xi^2_u - u_t^2u_x\xi^2_{uu}, \\ \zeta_{xx} &= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi^2_x - u_x\xi^2_{xx} - 2u_x^2\xi^2_{xu} \\ &\quad -3u_xu_{xx}\xi^2_u - u_x^3\xi^2_{uu} - 2u_{tx}\xi^1_x - u_t\xi^1_{xx} - 2u_tu_x\xi^1_{xu} \\ &\quad -(u_tu_{xx} + 2u_xu_{tx})\xi^1_u - u_tu_x^2\xi^1_{uu}. \end{aligned} \tag{4.7}$$

The insertion of equations (4.7) into (4.6), the replacement of u_{xx} by $-(u_{tt}+au_t+bu_x+cu)$ and the separation of terms with $u_{tx}, u_{tt}, u_t^2, u_t, u_x$ and the remaining terms, result in the following equations:

$$\xi^1 = \xi^1(t,x), \quad \xi^2 = \xi^2(t,x),$$

$$\xi^1_x = -\xi^2_t, \quad \xi^1_t = \xi^2_x, \quad \eta = \alpha(t,x)u + \beta(t,x), \tag{4.8}$$

$$\frac{\partial}{\partial t}(2\alpha + a\xi^1 + b\xi^2) = -H\xi^2, \quad \frac{\partial}{\partial x}(2\alpha + a\xi^1 + b\xi^2) = H\xi^1, \tag{4.9}$$

$$\frac{\partial}{\partial t}(H\xi^1) + \frac{\partial}{\partial x}(H\xi^2) = 0, \quad \frac{\partial}{\partial t}(K\xi^1) + \frac{\partial}{\partial x}(K\xi^2) = 0, \tag{4.10}$$

where α is a function of t and x, and β satisfy the elliptic equation and H and K are the Laplace invariants.

The above first set of equations reveals that ξ^1 and ξ^2 depend on t, x and equations (4.8) show the relationships between them. Equations (4.9) define the function α once ξ^1, ξ^2 are found. The compatibility condition of these equations follows from (4.10). The function α is uniquely defined by (4.9) up to a constant term. As a consequence of equations (4.10), the general solution of the determining equations and the results of group classification follow.

4.2 Joint differential invariants

In this section, we obtain joint differential invariants of the elliptic equation in terms of Cotton's invariants via the infinitesimal method.

We firstly write the operator $\mathbf{X}$ in the form

$$\begin{aligned}\mathbf{X} &= \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} \\ &\quad + \mu\frac{\partial}{\partial a} + \nu\frac{\partial}{\partial b} + \omega\frac{\partial}{\partial c},\end{aligned}$$

where $\xi^1 = \xi^1(t, x, u)$, $\xi^2 = \xi^2(t, x, u)$ and μ, ν and ω are functions of t, x, a, b, c. This enables us to determine the infinitesimals in t, x, a, b and c. If one follows the same way as in Section 4.1, one arrives at the equations

$$\xi^1 = \xi^1(t, x) = p(t, x), \quad \xi^2 = \xi^2(t, x) = q(t, x),$$

$$p_t = q_x, \quad p_x = -q_t, \quad \mu = bp_x - ap_t, \quad \nu = -(ap_x + bp_t), \quad \omega = -2cp_t, \tag{4.11}$$

where $p(t, x)$ and $q(t, x)$ are related by the first two equations of (4.11).

Our aim in the next step is to find the generator in the space of the Cotton invariants. To this end, we next look for a projected generator of the form

$$\begin{aligned}\mathbf{X} &= p(t, x)\frac{\partial}{\partial t} + q(t, x)\frac{\partial}{\partial x} + (bp_x - ap_t)\frac{\partial}{\partial a} - (ap_x + bp_t)\frac{\partial}{\partial b} - 2cp_t\frac{\partial}{\partial c} \\ &\quad + \mu_t\frac{\partial}{\partial a_t} + \mu_x\frac{\partial}{\partial a_x} + \nu_t\frac{\partial}{\partial b_t} + \nu_x\frac{\partial}{\partial b_x},\end{aligned}$$

where μ_t, μ_x, ν_t and ν_x are found by using the total differentiations with respect to t and x

$$\begin{aligned}D_t &= \frac{\partial}{\partial t} + a_t\frac{\partial}{\partial a} + a_{tt}\frac{\partial}{\partial a_t} + a_{tx}\frac{\partial}{\partial a_x} + \cdots + b_t\frac{\partial}{\partial b} + b_{tt}\frac{\partial}{\partial b_t} + b_{tx}\frac{\partial}{\partial b_x} + \cdots \\ &\quad + c_t\frac{\partial}{\partial c} + c_{tt}\frac{\partial}{\partial c_t} + c_{tx}\frac{\partial}{\partial c_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + a_x\frac{\partial}{\partial a} + a_{xx}\frac{\partial}{\partial a_x} + a_{tx}\frac{\partial}{\partial a_t} + \cdots + b_x\frac{\partial}{\partial b} + b_{xx}\frac{\partial}{\partial b_x} + b_{tx}\frac{\partial}{\partial b_t} + \cdots \\ &\quad + c_x\frac{\partial}{\partial c} + c_{xx}\frac{\partial}{\partial c_x} + c_{tx}\frac{\partial}{\partial c_t} + \cdots,\end{aligned} \tag{4.12}$$

and equations (4.11). That is,

$$\begin{aligned}\mu_t &= D_t(\mu) - a_t D_t(\xi^1) - a_x D_t(\xi^2),\\ &= p_t(-2a_t) + p_x(a_x + b_t) + p_{tt}(-a) + p_{tx}(b).\end{aligned}$$

In a similar fashion, one can find

$$\begin{aligned}\mu_x &= p_t(-2a_x) + p_x(b_x - a_t) + p_{tx}(-a) + p_{xx}(b),\\ \nu_t &= p_t(-2b_t) + p_x(b_x - a_t) + p_{tt}(-b) + p_{tx}(-a),\\ \nu_x &= p_t(-2b_x) + p_x(-a_x - b_t) + p_{tx}(-b) + p_{xx}(-a).\end{aligned}$$

Then, we find the action of $\mathbf{X}$ on H and K (assuming that both H and K are nonzero) and seek an infinitesimal generator in the space of the Cotton invariants (1.23) K and H. This gives us the generator

$$\mathbf{X} = -2Hp_t\frac{\partial}{\partial H} - 2Kp_t\frac{\partial}{\partial K}. \tag{4.13}$$

The invariance test $\mathbf{X}J = 0$ for the invariants $J(H, K)$ is

$$H\frac{\partial J}{\partial H} + K\frac{\partial J}{\partial K} = 0 \tag{4.14}$$

from the solution of which one easily finds the first-order joint differential invariant

$$J_1^1 = \frac{K}{H}. \tag{4.15}$$

We obtain the second-order differential invariants, i.e., the invariants of the form $J(H, H_t, H_x; K, K_t, K_x)$, by prolongation of the generator (4.13) to first-order:

$$\mathbf{X} = \mu\frac{\partial}{\partial H} + \nu\frac{\partial}{\partial K} + \mu_t\frac{\partial}{\partial H_t} + \mu_x\frac{\partial}{\partial H_x} + \nu_t\frac{\partial}{\partial K_t} + \nu_x\frac{\partial}{\partial K_x}, \tag{4.16}$$

where $\mu = -2Hp_t$ and $\nu = -2Kp_t$. Also

$$\begin{aligned}\mu_t &= D_t(-2Hp_t) - H_t D_t(p) - H_x D_t(q),\\ &= p_t(-3H_t) + p_x(H_x) + p_{tt}(-2H)\end{aligned}$$

are calculated by utilization of the total differentiations with respect to t and x, viz.

$$\begin{aligned}D_t &= \frac{\partial}{\partial t} + H_t\frac{\partial}{\partial H} + H_{tt}\frac{\partial}{\partial H_t} + H_{tx}\frac{\partial}{\partial H_x} + \cdots + K_t\frac{\partial}{\partial K} + K_{tt}\frac{\partial}{\partial K_t}\\ &\quad + K_{tx}\frac{\partial}{\partial K_x} + \cdots,\\ D_x &= \frac{\partial}{\partial x} + H_x\frac{\partial}{\partial H} + H_{tx}\frac{\partial}{\partial H_t} + H_{xx}\frac{\partial}{\partial H_x} + \cdots + K_x\frac{\partial}{\partial K} + K_{tx}\frac{\partial}{\partial K_t}\\ &\quad + K_{xx}\frac{\partial}{\partial K_x} + \cdots.\end{aligned} \tag{4.17}$$

The following are calculated in a similar manner. We have

$$
\begin{aligned}
\mu_x &= p_t(-3H_x) + p_x(-H_t) + p_{tx}(-2H), \\
\nu_t &= p_t(-3K_t) + p_x(K_x) + p_{tt}(-2K), \\
\nu_x &= p_t(-3K_x) + p_x(-K_t) + p_{tx}(-2K).
\end{aligned}
$$

Thus, the once-extended generator of (4.13) is

$$
\begin{aligned}
\mathbf{X} &= p_t(-2H)\frac{\partial}{\partial H} + p_t(-2K)\frac{\partial}{\partial K} + [p_t(-3H_t) + p_x(H_x) + p_{tt}(-2H)]\frac{\partial}{\partial H_t} \\
&\quad + [p_t(-3H_x) + p_x(-H_t) + p_{tx}(-2H)]\frac{\partial}{\partial H_x} \\
&\quad + [p_t(-3K_t) + p_x(K_x) + p_{tt}(-2K)]\frac{\partial}{\partial K_t} \\
&\quad + [p_t(-3K_x) + p_x(-K_t) + p_{tx}(-2K)]\frac{\partial}{\partial K_x}.
\end{aligned}
$$

The equation $\mathbf{X}J(H, H_t, H_x; K, K_t, K_x) = 0$, upon equating to zero the coefficients of the terms p_{tt}, p_{tx}, p_x and p_t, provides the following system of four equations:

$$
\begin{gathered}
H\frac{\partial J}{\partial H_t} + K\frac{\partial J}{\partial K_t} = 0, \quad H\frac{\partial J}{\partial H_x} + K\frac{\partial J}{\partial K_x} = 0, \\
H_x\frac{\partial J}{\partial H_t} - H_t\frac{\partial J}{\partial H_x} + K_x\frac{\partial J}{\partial K_t} - K_t\frac{\partial J}{\partial K_x} = 0, \\
2H\frac{\partial J}{\partial H} + 2K\frac{\partial J}{\partial K} + 3H_t\frac{\partial J}{\partial H_t} + 3H_x\frac{\partial J}{\partial H_x} + 3K_t\frac{\partial J}{\partial K_t} + 3K_x\frac{\partial J}{\partial K_x} = 0.
\end{gathered} \tag{4.18}
$$

The solution of the system (4.18) provides two functionally independent solutions, viz. J_1^1 and the second-order joint differential invariant

$$
J_2^1 = \frac{1}{H^5}[(HK_t - KH_t)^2 + (HK_x - KH_x)^2]. \tag{4.19}
$$

That is $J = J(J_1^1, J_2^1)$.

We shall now consider the third-order joint differential invariants of the form

$$
J = J(H, H_t, H_x, H_{tt}, H_{tx}, H_{xx}; K, K_t, K_x, K_{tt}, K_{tx}, K_{xx}),
$$

for the twice-extended generator of (4.13)

$$
\begin{aligned}
\mathbf{X} &= \mu\frac{\partial}{\partial H} + \nu\frac{\partial}{\partial K} + \mu_t\frac{\partial}{\partial H_t} + \mu_x\frac{\partial}{\partial H_x} + \mu_{tt}\frac{\partial}{\partial H_{tt}} + \mu_{tx}\frac{\partial}{\partial H_{tx}} + \mu_{xx}\frac{\partial}{\partial H_{xx}} \\
&\quad + \nu_t\frac{\partial}{\partial K_t} + \nu_x\frac{\partial}{\partial K_x} + \nu_{tt}\frac{\partial}{\partial K_{tt}} + \nu_{tx}\frac{\partial}{\partial K_{tx}} + \nu_{xx}\frac{\partial}{\partial K_{xx}},
\end{aligned}
$$

where

$$\begin{aligned}
\mu_{tt} &= p_t(-4H_{tt}) + p_x(2H_{tx}) + p_{tt}(-5H_t) + p_{tx}(H_x) + p_{ttt}(-2H),\\
\mu_{tx} &= p_t(-4H_{tx}) + p_x(H_{xx} - H_{tt}) + p_{tt}(-3H_x) + p_{tx}(-3H_t) + p_{ttx}(-2H),\\
\mu_{xx} &= p_t(-4H_{xx}) + p_x(-2H_{tx}) + p_{tt}(H_t) + p_{tx}(-5H_x) + p_{ttt}(2H),\\
\nu_{tt} &= p_t(-4K_{tt}) + p_x(2K_{tx}) + p_{tt}(-5K_t) + p_{tx}(K_x) + p_{ttt}(-2K),\\
\nu_{tx} &= p_t(-4K_{tx}) + p_x(K_{xx} - K_{tt}) + p_{tt}(-3K_x) + p_{tx}(-3K_t) + p_{ttx}(-2k),\\
\nu_{xx} &= p_t(-4K_{xx}) + p_x(-2K_{tx}) + p_{tt}(K_t) + p_{tx}(-5K_x) + p_{ttt}(2K),
\end{aligned}$$

are calculated in a similar fashion as before.

Upon equating to zero the coefficients of the terms $p_{ttx}, p_{ttt}, p_{tt}, p_{tx}, p_x$ and p_t of the equation $\mathbf{X}J(H, H_t, H_x, H_{tt}, H_{tx}, H_{xx}; K, K_t, K_x, K_{tt}, K_{tx}, K_{xx}) = 0$ yields the system of linear PDEs

$$\begin{aligned}
&H\frac{\partial J}{\partial H_{tx}} + K\frac{\partial J}{\partial K_{tx}} = 0,\\
&H\frac{\partial J}{\partial H_{tt}} - H\frac{\partial J}{\partial H_{xx}} + K\frac{\partial J}{\partial K_{tt}} - K\frac{\partial J}{\partial K_{xx}} = 0,\\
&2H\frac{\partial J}{\partial H_t} + 2K\frac{\partial J}{\partial K_t} + 5H_t\frac{\partial J}{\partial H_{tt}} + 3H_x\frac{\partial J}{\partial H_{tx}} - H_t\frac{\partial J}{\partial H_{xx}} + 5K_t\frac{\partial J}{\partial K_{tt}}\\
&+ 3K_x\frac{\partial J}{\partial K_{tx}} - K_t\frac{\partial J}{\partial K_{xx}} = 0,\\
&2H\frac{\partial J}{\partial H_x} + 2K\frac{\partial J}{\partial K_x} - H_x\frac{\partial J}{\partial H_{tt}} + 3H_t\frac{\partial J}{\partial H_{tx}} + 5H_x\frac{\partial J}{\partial H_{xx}} - K_x\frac{\partial J}{\partial K_{tt}}\\
&+ 3K_t\frac{\partial J}{\partial K_{tx}} + 5K_x\frac{\partial J}{\partial K_{xx}} = 0,\\
&H_x\frac{\partial J}{\partial H_t} - H_t\frac{\partial J}{\partial H_x} + K_x\frac{\partial J}{\partial K_t} - K_t\frac{\partial J}{\partial K_x} + 2H_{tx}\frac{\partial J}{\partial H_{tt}} + (H_{xx} - H_{tt})\frac{\partial J}{\partial H_{tx}}\\
&- 2H_{tx}\frac{\partial J}{\partial H_{xx}} + 2K_{tx}\frac{\partial J}{\partial K_{tt}} + (K_{xx} - K_{tt})\frac{\partial J}{\partial K_{tx}} - 2K_{tx}\frac{\partial J}{\partial K_{xx}} = 0,\\
&2H\frac{\partial J}{\partial H} + 2K\frac{\partial J}{\partial K} + 3H_t\frac{\partial J}{\partial H_t} + 3H_x\frac{\partial J}{\partial H_x} + 3K_t\frac{\partial J}{\partial K_t} + 3K_x\frac{\partial J}{\partial K_x}\\
&+ 4H_{tt}\frac{\partial J}{\partial H_{tt}} + 4H_{tx}\frac{\partial J}{\partial H_{tx}} + 4H_{xx}\frac{\partial J}{\partial H_{xx}} + 4K_{tt}\frac{\partial J}{\partial K_{tt}} + 4K_{tx}\frac{\partial J}{\partial K_{tx}}\\
&+ 4K_{xx}\frac{\partial J}{\partial K_{xx}} = 0.
\end{aligned} \tag{4.20}$$

The solution of (4.20) gives rise to six functionally independent quantities, i.e. we have $J = J(J_1^1, J_2^1, J_3^1, J_3^2, J_3^3, J_3^4)$, where

$$\begin{aligned}
J_3^1 &= H^{-8}[\{2H(HK_{tx} - KH_{tx}) - 3H_t(HK_x - KH_x) - 3H_x(HK_t - KH_t)\}^2\\
&\quad -\{2H(HK_{tt} - KH_{tt}) - 5H_t(HK_t - KH_t) + H_x(HK_x - KH_x\}\\
&\quad \times\{2H(HK_{xx} - KH_{xx}) - 5H_x(HK_x - KH_x) + H_t(HK_t - KH_t\}],
\end{aligned}$$

$$
\begin{aligned}
J_3^2 &= H^9(C_1^2+D_1^2)^{-2}[KD_5(C_1^2-D_1^2)-2KC_1D_1(KD_3-H^2D_2+K^2D_4 \\
&\quad -(C_1^2+D_1^2))]^2[K^2D_5^2+(KD_3-H^2D_2+K^2D_4-(C_1^2+D_1^2))^2]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
J_3^3 &= \frac{1}{K^2}(K_{tt}+K_{xx})-\frac{1}{K^3}(K_t^2+K_x^2), \\
J_3^4 &= \frac{1}{H^2}(H_{tt}+H_{xx})-\frac{1}{H^3}(H_t^2+H_x^2). \qquad (4.21)
\end{aligned}
$$

and

$$
\begin{aligned}
C_1 &= HK_t-KH_t, \\
D_1 &= HK_x-KH_x, \\
D_2 &= K(K_{tt}+K_{xx})-K_t^2-K_x^2, \\
D_3 &= 2H(HK_{tt}-KH_{tt})-5H_t(HK_t-KH_t)+H_x(HK_x-KH_x), \\
D_4 &= H(H_{tt}+H_{xx})-H_t^2-H_x^2, \\
D_5 &= 2H(HK_{tx}-KH_{tx})-3H_t(HK_x-KH_x)-3H_x(HK_t-KH_t).
\end{aligned}
$$

What happens if one or both of H and K are zero? The simplest situation is when $H=K=0$. In this case the elliptic PDE reduces to the Laplace equation via just the linear change of dependent variable (examples are given in Section 4.5). If $H=0$ and K nonzero, then (4.20) reduces to a system with K terms only, i.e. one merely sets the Hs zero in the coefficients of the relevant partial derivatives. The solution of the resultant system in K is then J_3^3. Thus in this case we have $H=0$ and J_3^3. Likewise, if $K=0$ and H nonzero, one has J_3^4. Examples of these are also given in Section 4.5.

We now use finite equivalence transformations of the elliptic PDE to find the transformation formulas and to also verify that the quantities $J_1^1, J_2^1, J_3^1, J_3^2, J_3^3, J_3^4$ are indeed invariants.

By use of the rules of total derivatives we obtain

$$
\begin{aligned}
D_t &= \phi_t\bar{D}_{\bar{t}}+\psi_t\bar{D}_{\bar{x}} = \phi_t\bar{D}_{\bar{t}}-\phi_x\bar{D}_{\bar{x}}, \\
D_x &= \phi_x\bar{D}_{\bar{t}}+\psi_x\bar{D}_{\bar{x}} = \phi_x\bar{D}_{\bar{t}}+\phi_t\bar{D}_{\bar{x}}. \qquad (4.22)
\end{aligned}
$$

From (4.22), we get

$$
\bar{D}_{\bar{t}} = \frac{1}{\phi_t^2+\phi_x^2}(\phi_tD_t+\phi_xD_x), \quad \bar{D}_{\bar{x}} = \frac{1}{\phi_t^2+\phi_x^2}(\phi_tD_x-\phi_xD_t). \qquad (4.23)
$$

The actions of (4.23) on (4.2) give

$$\begin{aligned} \bar{u}_{\bar{t}} &= \frac{1}{\phi_t^2+\phi_x^2}[\phi_t(\sigma_t u+\sigma u_t)+\phi_x(\sigma_x u+\sigma u_x)], \\ \bar{u}_{\bar{x}} &= \frac{1}{\phi_t^2+\phi_x^2}[\phi_t(\sigma_x u+\sigma u_x)-\phi_x(\sigma_t u+\sigma u_t)]. \end{aligned} \tag{4.24}$$

In a similar manner, we can find formulae for $\bar{u}_{\bar{t}\bar{t}}$ and $\bar{u}_{\bar{x}\bar{x}}$. Insertion of (4.24) and $\bar{u}_{\bar{t}\bar{t}}$, $\bar{u}_{\bar{x}\bar{x}}$ into the equation

$$\bar{u}_{\bar{t}\bar{t}}+\bar{u}_{\bar{x}\bar{x}}+\bar{a}\bar{u}_{\bar{t}}+\bar{b}\bar{u}_{\bar{x}}+\bar{c}\bar{u}=0$$

results

$$\begin{aligned} & u_{tt}+u_{xx}+\Big(\bar{a}\phi_t-\bar{b}\phi_x+2\frac{\sigma_t}{\sigma}\Big)u_t+\Big(\bar{a}\phi_x+\bar{b}\phi_t+2\frac{\sigma_x}{\sigma}\Big)u_x \\ & +\Big(\frac{\sigma_{tt}}{\sigma}+\frac{\sigma_{xx}}{\sigma}+\bar{c}(\phi_t^2+\phi_x^2)+(\bar{a}\phi_t-\bar{b}\phi_x)\frac{\sigma_t}{\sigma}+(\bar{a}\phi_x+\bar{b}\phi_t)\frac{\sigma_x}{\sigma}\Big)u=0. \end{aligned} \tag{4.25}$$

From (4.25), we get that

$$\begin{aligned} \bar{a} &= \frac{1}{\phi_t^2+\phi_x^2}\Big[(a-2\frac{\sigma_t}{\sigma})\phi_t+(b-2\frac{\sigma_x}{\sigma})\phi_x\Big], \\ \bar{b} &= \frac{1}{\phi_t^2+\phi_x^2}\Big[(b-2\frac{\sigma_x}{\sigma})\phi_t-(a-2\frac{\sigma_t}{\sigma})\phi_x\Big], \\ \bar{c} &= \frac{1}{\phi_t^2+\phi_x^2}\Big[c-\frac{\sigma_{tt}}{\sigma}-\frac{\sigma_{xx}}{\sigma}+2\frac{\sigma_t^2}{\sigma^2}+2\frac{\sigma_x^2}{\sigma^2}-a\frac{\sigma_t}{\sigma}-b\frac{\sigma_x}{\sigma}\Big]. \end{aligned} \tag{4.26}$$

One can show by routine computations that

$$(\phi_t^2+\phi_x^2)\bar{H}=H, \quad (\phi_t^2+\phi_x^2)\bar{K}=K. \tag{4.27}$$

by use of the equations (4.23) and (4.26). Hence, from equations (4.27) and its consequences, we deduce that

$$\bar{J}_1^1=J_1^1, \quad \bar{J}_2^1=J_2^1, \quad \bar{J}_3^1=J_3^1, \quad \bar{J}_3^2=J_3^2, \quad \bar{J}_3^3=J_3^3, \quad \bar{J}_3^4=J_3^4, \tag{4.28}$$

which verifies that $J_1^1, J_2^1, J_3^1, J_3^2, J_3^3, J_3^4$ are indeed invariants as obtained using the infinitesmial approach. The argument here, though tedious, is the same as that for the hyperbolic equations given in Johnpillai and Mahomed [53] and Johnpillai et al. [54].

4.3 Joint differential invariants obtained from invariants of the hyperbolic equation

In this section the joint differential invariants for the scalar linear elliptic equation are derived from the joint differential invariants of the hyperbolic equation.

For the linear hyperbolic equation the basis of the joint differential invariants are given in [53,54] as follows,

$$
\begin{aligned}
p &= \frac{k}{h}, \\
q &= \frac{(\partial_t \partial_x \ln h)}{h}, \\
J_3^1 &= \frac{1}{h^3}\left(kh_{\bar{t}\bar{x}} + hk_{\bar{t}\bar{x}} - h_{\bar{t}}k_x - h_{\bar{x}}k_{\bar{t}}\right), \\
J_3^2 &= \frac{1}{h^9}\left(hk_{\bar{x}} - kh_{\bar{x}}\right)^2\left(hkh_{\bar{t}\bar{t}} - h^2k_{\bar{t}\bar{t}} - 3kh_{\bar{t}}^2 + 3hh_{\bar{t}}k_{\bar{t}}\right), \\
J_3^3 &= \frac{1}{h^9}\left(hk_{\bar{t}} - kh_{\bar{t}}\right)^2\left(hkh_{\bar{x}\bar{x}} - h^2k_{\bar{x}\bar{x}} - 3kh_{\bar{x}}^2 + 3hh_{\bar{x}}k_{\bar{x}}\right).
\end{aligned} \tag{4.29}
$$

By the application of the inverse the transformations (1.104), the joint differential invariants (4.29) are to transformed to the joint differential invariants of the scalar linear elliptic equations as follows,

$$
\begin{aligned}
H_1 &= \frac{K^2 - H^2}{(K^2 + H^2)}, \\
H_2 &= \frac{2HK}{(K^2 + H^2)}, \\
H_3 &= \frac{-8}{(K^2 + H^2)^3}\Big[\left(H^4 - K^4\right)K_{tx} + \left(-2H^3K - 2HK^3\right)H_{tx} + \Big\{(3H^3K - K^3)H_x \\
&\quad - HK_x(H^2 - 3K^2)\Big\}H_t - K_t\Big\{(H^3 - 3K^2H)H_x + (3H^2K - K^3)K_x\Big\}\Big], \\
H_4 &= \frac{8}{(K^2 + H^2)^3}\Big[\left(H^4 - K^4\right)H_{tx} + \left(2H^3K + 2HK^3\right)K_{tx} + \Big\{\left(-H^3 + 3K^2H\right)H_x \\
&\quad + \left(-3H^2K + K^3\right)K_x\Big\}H_t + K_t\Big\{\left(-3H^2K + K^3\right)H_x + HK_x\left(H^2 - 3K^2\right)\Big\}\Big], \\
H_5 &= \frac{16}{(H^2 + K^2)^3}K\left(K^2 - 3H^2\right)\left(KK_{tx} + HH_{tx} - K_tK_x - H_tH_x\right), \\
H_6 &= \frac{16}{(H^2 + K^2)^3}K\left(3K^2 - H^2\right)\left(KK_{tx} + HH_{tx} - K_tK_x - H_tH_x\right), \\
H_7 &= \frac{512\left(HK_x - KH_x\right)^2\left(KH_t - HK_t\right)}{(K^2 + H^2)^9}\Big[8\left(K^{10}H - 6K^8H^3 + 6K^4H^7 - H^9K^2\right)H_{tt} \\
&\quad + 8\Big(6K^7H^4 - K^9H^2 + KH^{10} - 6K^3H^8\Big)K_{tt} + 3\Big\{\Big(-9KH^8 + 84K^3H^6 - 126K^5H^4 \\
&\quad + 36K^7H^2 - K^9\Big)H_t + HK_t\left(-3K^2 + H^2\right)\left(-3K^6 + 27K^4H^2 - 33K^2H^4 + H^6\right)\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
H_8 &= \frac{512\,(HK_x-KH_x)^2\,(KH_t-HK_t)}{(K^2+H^2)^9}\Big[\Big(-KH^{10}-42K^7H^4-42K^5H^6+27K^9H^2\\
&\quad +27K^3H^8-K^{11}\Big)H_{tt}+\Big(42K^4H^7-27H^9K^2-27K^8H^3+K^{10}H+H^{11}\\
&\quad +42K^6H^5\Big)K_{tt}-3\Big\{\Big(9K^8H+126K^4H^5-36K^2H^7+H^9-84K^6H^3\Big)H_t\\
&\quad +KK_t(3H^2-K^2)\Big(33K^4H^2-K^6-27K^2H^4+3H^6\Big)\Big\}\Big],\\
H_9 &= \frac{512\Big(HK_t-KH_t\Big)^2\Big(KH_x-HK_x\Big)}{\Big(H^2+K^2\Big)}\Big[8\Big(6K^4H^7-6K^8H^3-H^9K^2+K^{10}H\Big)H_{xx}\\
&\quad +8\Big(KH^{10}+6K^7H^4-K^9H^2-6K^3H^8\Big)K_{xx}+3\Big\{\Big(84K^3H^6-9KH^8-126K^5H^4\\
&\quad +36K^7H^2-K^9\Big)H_x+HK_x\Big(H^2-3K^2\Big)\Big(27K^4H^2-3K^6-33K^2H^4+H^6\Big)\Big\}\Big]\\
H_{10} &= \frac{512\Big(HK_t-KH_t\Big)^2\Big(KH_x-HK_x\Big)}{\Big(H^2+K^2\Big)}\Big[\Big(27K^9H^2-42K^7H^4-42K^5H^6+27K^3H^8\\
&\quad -KH^{10}-K^{11}\Big)H_{xx}+\Big(42K^4H^7+42K^6H^5-27H^9K^2+K^{10}H-27K^8H^3\\
&\quad +H^{11}\Big)K_{xx}-3\Big\{\Big(9K^8H-84K^6H^3+126K^4H^5-36K^2H^5-36K^2H^7\\
&\quad +H^9\Big)H_x+KK_x\Big(3H^2-K^2\Big)\Big(33K^4H^2-K^6-27K^2H^4+3H^6\Big)\Big\}\Big] \qquad (4.30)
\end{aligned}
$$

These are the joint differential invariants for the elliptic equation and are obtained from the basis of joint differential invariants of the hyperbolic equations. These joint differential invariants do not form a basis of the elliptic equations.

4.4 Invariant differentiation

In this section, we derive the operators of invariant differentiation to find a basis of invariants and the invariants of higher orders for the scalar linear elliptic PDEs.

We define the operator D by

$$D = \lambda D_t + \kappa D_x,$$

where λ, κ are differential functions of H, K and their derivatives and D_t, D_x are given by (4.17).

From the first prolongation of (4.13), viz. (4.16), and the invariant differentiation operator

$\tilde{\mathbf{X}} = \mathbf{X} + D(\xi^1\partial_\lambda + \xi^2\partial_\kappa)$ (see Ovsiannikov [85]), we find that

$$\begin{aligned}\tilde{\mathbf{X}} &= p_t(-2H)\frac{\partial}{\partial H} + p_t(-2K)\frac{\partial}{\partial K} + [p_t(-3H_t) + p_x(H_x) + p_{tt}(-2H)]\frac{\partial}{\partial H_t}\\ &+[p_t(-3H_x) + p_x(-H_t) + p_{tx}(-2H)]\frac{\partial}{\partial H_x}\\ &+[p_t(-3K_t) + p_x(K_x) + p_{tt}(-2K)]\frac{\partial}{\partial K_t}\\ &+[p_t(-3K_x) + p_x(-K_t) + p_{tx}(-2K)]\frac{\partial}{\partial K_x}\\ &+(\lambda p_t + \kappa p_x)\frac{\partial}{\partial \lambda} + (\kappa p_t - \lambda p_x)\frac{\partial}{\partial \kappa}.\end{aligned}$$

Since the function $p(t, x)$ is arbitrary, there is no relation between its derivatives. Upon equating to zero of the coefficients of the terms p_{tt}, p_{xx}, p_x and p_t, in the equation $\tilde{\mathbf{X}}J(H, H_t, H_x; K, K_t, K_x; \lambda, \kappa) = 0$, the following system of four PDEs results

$$\begin{aligned}&H\frac{\partial J}{\partial H_t} + K\frac{\partial J}{\partial K_t} = 0, \quad H\frac{\partial J}{\partial H_x} + K\frac{\partial J}{\partial K_x} = 0,\\ &H_x\frac{\partial J}{\partial H_t} - H_t\frac{\partial J}{\partial H_x} + K_x\frac{\partial J}{\partial K_t} - K_t\frac{\partial J}{\partial K_x} + \kappa\frac{\partial J}{\partial \lambda} - \lambda\frac{\partial J}{\partial \kappa} = 0,\\ &2H\frac{\partial J}{\partial H} + 2K\frac{\partial J}{\partial K} + 3H_t\frac{\partial J}{\partial H_t} + 3H_x\frac{\partial J}{\partial H_x} + 3K_t\frac{\partial J}{\partial K_t} + 3K_x\frac{\partial J}{\partial K_x}\\ &-\lambda\frac{\partial J}{\partial \lambda} - \kappa\frac{\partial J}{\partial \kappa} = 0.\end{aligned} \tag{4.31}$$

The solution of the system of PDEs (4.31) gives $J = J(J_1^1, J_2^1, c_1, c_2)$, where J_1^1, J_2^1 and

$$\begin{aligned}c_1 &= \frac{1}{H^2}[\kappa(HK_x - KH_x) + \lambda(HK_t - KH_t)],\\ c_2 &= \frac{1}{H^2}[\kappa(HK_t - KH_t) - \lambda(HK_x - KH_x)]\end{aligned} \tag{4.32}$$

are solutions of (4.31). From (4.32), we get

$$\begin{aligned}\lambda &= \frac{H^2[c_1(HK_t - KH_t) - c_2(HK_x - KH_x)]}{[(HK_t - KH_t)^2 + (HK_x - KH_x)^2]},\\ \kappa &= \frac{H^2[c_1(HK_x - KH_x) + c_2(HK_t - KH_t)]}{[(HK_t - KH_t)^2 + (HK_x - KH_x)^2]}.\end{aligned} \tag{4.33}$$

Then the choices of (c_1, c_2) as $(1, 0)$ or $(0, 1)$ give two independent operators of invariant differentiation

$$\begin{aligned}\tilde{\mathbf{X}}_1 &= \frac{H^2[(HK_x - KH_x)D_x + (HK_t - KH_t)D_t]}{[(HK_t - KH_t)^2 + (HK_x - KH_x)^2]},\\ \tilde{\mathbf{X}}_2 &= \frac{H^2[(HK_t - KH_t)D_x - (HK_x - KH_x)D_t]}{[(HK_t - KH_t)^2 + (HK_x - KH_x)^2]}.\end{aligned} \tag{4.34}$$

If we utilize the first operator of invariant differentiation $\tilde{\mathbf{X}}_1$ on J_1^1, we obtain $\tilde{\mathbf{X}}_1(J_1^1) = 1$. Likewise, $\tilde{\mathbf{X}}_2(J_1^1) = 0$. Hence, a basis of joint differential invariants of (1.24) is

$$\{J_1^1, J_2^1, J_3^1, J_3^2, J_3^3, J_3^4\}. \tag{4.35}$$

As a consequence of the preceding results, we can state the following theorems. These theorems exclude the case when both H and K are zero. In this case the elliptic PDE is reducible to Laplace's equation via a linear change of the dependent variable.

Theorem 4.4.1. *The joint differential invariants* (4.35) *of* (1.98) *defined by* (4.15), (4.19) *and* (4.21) *provide a complete set of joint invariants of* (1.98) *if H and K are nonzero. The other joint invariants are functions of the basis of invariants* (4.35) *and their invariant derivatives. If $H = 0$ ($K = 0$) and K (H) is nonzero, we have one basis element given by J_3^3 (J_3^4).*

Theorem 4.4.2. *Two elliptic PDEs of the form* (4.2) *are locally equivalent via the invertible transformations* (4.2) *and* (4.3) *if and only if their joint invariants and invariant equations where applicable remain unchanged under the said transformations.*

4.5 Examples

The first few examples are about the use of Cotton's semi-invariants. Note that two linear elliptic PDEs are locally equivalent to each other under linear homogeneous transformations of the dependent variable only if and only if their Cotton semi-invariants H and K remain invariant under the said transformations. A transformation $\bar{u} = \sigma(t,x)u$ which maps the respective elliptic PDEs to each other is constructed from $\sigma_t/\sigma = (a - \bar{a})/2$, $\sigma_x/\sigma = (b - \bar{b})/2$.

1. The constant coefficient elliptic PDE

$$u_{tt} + u_{xx} + au_t + bu_x + \frac{1}{4}(a^2 + b^2)u = 0,$$

has $H = K = 0$ and is reducible to the Laplace PDE

$$\bar{u}_{tt} + \bar{u}_{xx} = 0, \tag{4.36}$$

by means of the linear transformation $\bar{u} = u\exp(at/2 + bx/2)$. Equation (4.36) has the same values $\bar{H} = \bar{K} = 0$ of the semi-invariants H and K.

2. The variable coefficient elliptic equation

$$u_{tt} + u_{xx} + xu_t + tu_x + \frac{1}{4}(t^2 + x^2)u = 0,$$

possesses $H = K = 0$ and is reducible to the Laplace PDE (4.36) via $\bar{u} = u\exp(tx/2)$.

3. The PDE

$$u_{tt} + u_{xx} + xu_t + x^2u_x + (\frac{1}{4}x^4 + x)u = 0,$$

has $H = 1$ and $K = x^2/2$ and it transforms, via $\bar{u} = u\exp(x^3/6)$, to the simpler equation $\bar{u}_{tt} + \bar{u}_{xx} + x\bar{u}_t = 0$ which has $\bar{H} = 1$ and $\bar{K} = x^2/2$.

We now illustrate local equivalence by joint invariants.

4. The elliptic equation

$$u_{tt} + u_{xx} + c(t, x)u = 0, \tag{4.37}$$

can be mapped to

$$u_{\bar{t}\bar{t}} + u_{\bar{x}\bar{x}} + \bar{c}(\bar{t}, \bar{x})u = 0, \tag{4.38}$$

where $\bar{c} = c/(\phi_t^2 + \phi_x^2)$, under a change of independent variables (4.3) provided $H = \bar{H} = 0$ and $J_3^3 = \bar{J}_3^3$. As a concrete example, the PDE (4.37) with $c = t^2 + x^2$ reduces to the constant coefficient equation (4.38) with $\bar{c} = 1$ under $\bar{t} = t^2/2 - x^2/2$, $\bar{x} = tx$. The invariants are $H = \bar{H} = 0$ and $J_3^3 = \bar{J}_3^3 = 0$

5. The constant coefficient PDE

$$u_{tt} + u_{xx} + au_t + bu_x + cu = 0,$$

where $c \neq (a^2 + b^2)/4$, possesses $H = 0$ and $K = \text{const} \neq 0$ and maps to

$$\bar{u}_{\bar{t}\bar{t}} + \bar{u}_{\bar{x}\bar{x}} + \epsilon\bar{u} = 0, \quad \epsilon = \pm 1,$$

under

$$\bar{t} = \sqrt{\frac{-K}{2\epsilon}}x, \quad \bar{x} = -\sqrt{\frac{-K}{2\epsilon}}t, \quad \bar{u} = u\exp\frac{1}{2}(at + bx)u, \quad K \neq 0,$$

where K and ϵ have opposite signs. Here the invariants are $H = \bar{H} = 0$ and $J_3^3 = \bar{J}_3^3 = 0$.

6. Finally consider

$$u_{tt} + u_{xx} - tu_t - 2tu_x + (tx - \frac{1}{2}x^2 - \frac{1}{2} + \frac{3}{4}t^2)u = 0.$$

This PDE has $H = 2$, $K = (x - t)^2$ and can be reduced to the simpler form

$$\bar{u}_{\bar{t}\bar{t}} + \bar{u}_{\bar{x}\bar{x}} + \bar{x}\bar{u}_{\bar{t}} = 0,$$

via

$$\bar{t} = t + x, \quad \bar{x} = x - t, \quad \bar{u} = u\exp(-\frac{1}{2}tx - \frac{1}{4}x^2).$$

Indeed, the six invariants agree, i.e. $J_1^1 = \bar{J}_1^1 = (x - t)^2/2$, $J_2^1 = \bar{J}_2^1 = (x - t)^2$, $J_3^1 = \bar{J}_3^1 = 0$, $J_3^2 = \bar{J}_3^2 = 0$, $J_3^3 = \bar{J}_3^3 = -4/K^2$ and $J_3^4 = \bar{J}_3^4 = 0$.

Here we have provided the symmetry classification for the elliptic PDE (1.98) in terms of the Cotton invariants. The application of the infinitesimal method have resulted in new joint differential invariants which are in terms of Cotton's invariants. It has been shown via the operators of invariant differentiation that there are six elements of joint invariants which constitute a basis of joint differential invariants for the elliptic PDEs.

An alternate approach to find joint differential invariants for the elliptic equation is by using joint invariants of the hyperbolic PDE. Then one can obtain joint invariants for the elliptic PDEs with the aid of application of the rules of derivatives on the complex transformations and the joint invariants of the hyperbolic PDEs. From the transformed hyperbolic PDE, one can obtain the Laplace invariants in complex form by use of the rules of derivatives. The real and imaginary parts will give Cotton's invariants for the elliptic PDEs. This is easy enough. However, if one proceeds in a similar manner for the five basic invariants of the hyperbolic PDEs one will, in general, get five joint differential invariants in complex form. That is, in general, ten invariants from the real and imaginary parts. It becomes difficult to identify which are elements of the basis of joint differential invariants for the elliptic PDEs. The infinitesimal approach used here avoids this difficulty.

Chapter 5

Cotton-type and Joint Invariants for Linear Elliptic Systems

A complex scalar ODE/PDE provides a system of two real ODEs/PDEs by splitting the complex base equation into real and imaginary parts using CSA. Similarly a system of two elliptic PDEs is obtained from a scalar complex linear elliptic PDE. The system of elliptic PDEs obtained from complex elliptic PDE is a subsystem of the general system of two elliptic PDEs as the former has fewer arbitrary coefficients. For the scalar linear hyperbolic and elliptic PDEs the semi-invariant under the linear change of the dependent variables are known as the Laplace' invariants and Cotton' invariants. Similar invariants for the system of hyperbolic and elliptic PDEs are called the Laplace-type and Cotton-type invariants. Here Cotton-type invariants for a system of two linear elliptic equations, obtainable from a complex base linear elliptic equation, are derived by split of the corresponding complex Cotton invariants of the base complex equation and from the Laplace-type invariants of the system of linear hyperbolic equations equivalent to the system of linear elliptic equations via linear complex transformations of the independent variables. It is shown that Cotton-type invariants derived from these two approaches are identical. Furthermore, Cotton-type and joint invariants for a general system of two linear elliptic equations are also obtained from the Laplace-type and joint invariants for a system of two linear hyperbolic equations equivalent to the system of linear elliptic equations by complex changes of the independent variables.

5.1 Cotton-type invariants for a subclass

In this section, Cotton-type invariants for a subsystem of two linear elliptic equations are first obtained from a complex scalar linear elliptic equation by splitting the complex Cotton invariants of the base complex equation into real and imaginary parts. Then for such a system, we determine invariants from the Laplace-type invariants for the equivalent system of two linear hyperbolic equations. This is achieved by performing complex splits of the Laplace-type invariants. It is concluded, as a proposition, that the Cotton-type invariants are the same for both the approaches.

The subsystem of two elliptic equations

$$\begin{aligned} u_{xx}+u_{yy}+\alpha_1u_x-\alpha_2v_x+\beta_1u_y-\beta_2v_y+\gamma_1u-\gamma_2v &= 0,\\ v_{xx}+v_{yy}+\alpha_2u_x+\alpha_1v_x+\beta_2u_y+\beta_1v_y+\gamma_2u+\gamma_1v &= 0, \end{aligned} \tag{5.1}$$

is obtained by spliting the complex linear elliptic equation

$$w_{xx}+w_{yy}+aw_x+bw_y+cw=0, \tag{5.2}$$

where

$$a=\alpha_1+i\alpha_2, b=\beta_1+i\beta_2,\ c=\gamma_1+i\gamma_2, w=u+iv. \tag{5.3}$$

The Cotton invariants, corresponding to the complex elliptic equation (5.2), are (1.103) which split into the four invariants

$$\begin{aligned} \mu_1 &= \alpha_{1y}-\beta_{1x},\\ \mu_2 &= \alpha_{2y}-\beta_{2x},\\ H_1 &= \alpha_{1x}+\beta_{1y}+\frac{1}{2}(\alpha_1^2+\beta_1^2)-\frac{1}{2}(\alpha_2^2+\beta_2^2)-2\gamma_1,\\ H_2 &= \alpha_{2x}+\beta_{2y}+\alpha_1\alpha_2+\beta_1\beta_2-2\gamma_2. \end{aligned} \tag{5.4}$$

These are precisely the Cotton-type invariants for the linear elliptic system (5.1). The simplest case is when the semi-invariants (5.4) are zero. In this case the elliptic PDE system (5.1) reduces to the Laplace system by linear transformation of the dependent variables. This is similar to the scalar linear elliptic PDE case.

Now for the system of elliptic equations (5.1), we derive the Cotton-type invariants by transforming the system of equations to the corresponding linear hyperbolic equations and then using the inverse transformations of the independent variables to convert the Laplace-type invariants to the

Cotton-type invariants. By means of the transformations (1.104), the system of elliptic equations (5.1) can be mapped to the system of two linear hyperbolic type equations

$$\begin{aligned} u_{tz} + A_1 u_t - A_2 v_t + B_1 u_z - B_2 v_z + C_1 u - C_2 v &= 0, \\ v_{tz} + A_2 u_t + A_1 v_t + B_2 u_z + B_1 v_z + C_2 u + C_1 v &= 0, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} A_1 &= \frac{1}{4}(\alpha_1 + i\beta_1),\ B_1 = \frac{1}{4}(\alpha_1 - i\beta_1),\ C_1 = \frac{1}{4}\gamma_1 \\ A_2 &= \frac{1}{4}(\alpha_2 + i\beta_2),\ B_2 = \frac{1}{4}(\alpha_2 - i\beta_2),\ C_2 = \frac{1}{4}\gamma_2 \end{aligned} \tag{5.6}$$

The system of hyperbolic equations (5.5) has four Laplace-type invariants [79]

$$\begin{aligned} h_1 &= A_{1t} + A_1 B_1 - A_2 A_2 - C_1, \\ h_2 &= A_{2t} + A_1 B_2 + A_2 B_1 - C_2, \\ k_1 &= B_{1z} + A_1 B_1 - A_2 B_2 - C_1, \\ k_2 &= B_{2z} + A_1 B_2 + A_2 B_1 - C_2. \end{aligned} \tag{5.7}$$

Now by the application of the transformations (1.104) and complex splits, the Laplace-type invariants (5.7) become the Cotton-type invariants (5.4). We therefore conclude with the following result.

Proposition 1.

For a class of a system of two linear elliptic equations (5.1) obtained from a complex base linear elliptic equation (5.2) or equivalent to a subsystem of two linear hyperbolic equations (5.5) by complex linear transformations of the independent variables (1.104), Cotton-type invariants either constructed by splitting of the complex Cotton invariants (1.103) of the complex base elliptic equation into real and imaginary parts or those computed by splitting the Laplace-type invariants (5.7) of the system of linear hyperbolic equations are identical to (5.4).

5.2 Cotton-type and joint invariants in general

In this section, Cotton-type and joint invariants for a general system of two linear elliptic equations are obtained. A general system of two linear elliptic equations is

$$\begin{aligned} u_{xx} + u_{yy} + a_1 u_x + a_2 v_x + b_1 u_y + b_2 v_y + c_1 u + c_2 v &= 0, \\ v_{xx} + v_{yy} + a_3 u_x + a_4 v_x + b_3 u_y + b_4 v_y + c_3 u + c_4 v &= 0, \end{aligned} \tag{5.8}$$

By means of the complex transformations of the independent variables (1.104), this system (5.8) is transformed into the system of two linear hyperbolic equations

$$\begin{aligned} u_{tz} + A_1 u_t + A_2 v_t + B_1 u_z + B_2 v_z + C_1 u + C_2 v &= 0, \\ v_{tz} + A_3 u_t + A_4 v_t + B_3 u_z + B_4 v_z + C_3 u + C_4 v &= 0, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} A_1 &= \frac{1}{4}(a_1 + ib_1),\ B_1 = \frac{1}{4}(a_1 - ib_1),\ C_1 = \frac{1}{4}c_1, \\ A_2 &= \frac{1}{4}(a_2 + ib_2),\ B_2 = \frac{1}{4}(a_2 - ib_2),\ C_2 = \frac{1}{4}c_2, \\ A_3 &= \frac{1}{4}(a_3 + ia_3),\ B_3 = \frac{1}{4}(a_3 - ib_3),\ C_3 = \frac{1}{4}c_3, \\ A_4 &= \frac{1}{4}(a_4 + ib_4),\ B_4 = \frac{1}{4}(a_4 - ib_4),\ C_4 = \frac{1}{4}c_4. \end{aligned} \tag{5.10}$$

This system of linear hyperbolic equations (5.9) has five semi-invariants [99] under the linear change of dependent variables. They are [99]

$$\begin{gathered} I_1 = k_1 + k_4,\ I_2 = k_5 + k_8, \\ I_3 = k_1 k_4 - k_2 k_3,\ I_4 = k_5 k_8 - k_6 k_7, \\ I_5 = k_1 k_5 + k_2 k_7 + k_3 k_6 + k_4 k_8. \end{gathered} \tag{5.11}$$

where

$$\begin{aligned} k_1 &= A_1 B_1 + A_3 B_2 + A_{1t} - C_1, \\ k_2 &= A_1 B_3 + A_3 B_4 + A_{3t} - C_3, \\ k_3 &= A_2 B_1 + A_4 B_2 + A_{2t} - C_2, \\ k_4 &= A_2 B_3 + A_4 B_4 + A_{4t} - C_4, \\ k_5 &= A_1 B_1 + A_2 B_3 + B_{1z} - C_1, \\ k_6 &= A_3 B_1 + A_4 B_3 + B_{3z} - C_3, \\ k_7 &= A_1 B_2 + A_2 B_4 + B_{2z} - C_2, \\ k_8 &= A_3 B_2 + A_4 B_4 + B_{4z} - C_4. \end{aligned} \tag{5.12}$$

The system of linear hyperbolic equations (5.9) also has the four joint invariants [99]

$$\begin{aligned} J_1 &= \tfrac{I_2}{I_1},\ J_2 = \tfrac{I_3}{I_1^2}, \\ J_3 &= \tfrac{I_4}{I_1^2},\ J_4 = \tfrac{I_5}{I_1^2}. \end{aligned} \tag{5.13}$$

We utilize the same approach as in the previous section. Indeed via the transformations (1.104), the Laplace-type invariants (5.11) transform to the five Cotton-type invariants

$$\begin{aligned} H_1 &= \mathrm{Im}(K_1 + K_4) = \mathrm{Im}(K_5 + K_8), \\ H_2 &= \mathrm{Re}(K_1 + K_4) = \mathrm{Re}(K_5 + K_8), \\ H_3 &= \mathrm{Im}(K_1K_4 - K_2K_3) = \mathrm{Im}(K_5K_8 - K_6K_7), \\ H_4 &= \mathrm{Re}(K_1K_4 - K_2K_3) = \mathrm{Re}(K_5K_8 - K_6K_7), \\ H_5 &= \mathrm{Re}(K_1K_5 + K_2K_7 + K_3K_6 + K_4K_8). \end{aligned} \tag{5.14}$$

and the invariant equation

$$\mathrm{Im}(K_1K_5 + K_2K_7 + K_3K_6 + K_4K_8) = 0, \tag{5.15}$$

where

$$\begin{aligned} K_1 &= \frac{1}{4^2}(a_1^2 + b_1^2 + a_2a_3 + b_2b_3 + 2a_{1x} + 2b_{1y} - 4c_1) + \frac{i}{4^2}(a_2b_3 - a_3b_2 - 2a_{1y} + 2b_{1x}), \\ K_2 &= \frac{1}{4^2}(a_1a_3 + b_1b_3 + a_3a_4 + b_3b_4 + 2a_{3x} + 2b_{3y} - 4c_3) + \frac{i}{4^2}(a_3b_1 - a_1b_3 + a_4b_3 \\ &\quad -a_3b_4 - 2a_{3y} + 2b_{3x}), \\ K_3 &= \frac{1}{4^2}(a_1a_2 + b_1b_2 + a_2a_4 + b_2b_4 + 2a_{2x} + 2b_{2y} - 4c_2) + \frac{i}{4^2}(a_1b_2 - a_2b_1 + a_2b_4 \\ &\quad -a_4b_2 - 2a_{2y} + 2b_{2x}), \\ K_4 &= \frac{1}{4^2}(a_2a_3 + b_2b_3 + a_4^2 + b_4^2 + 2a_{4x} + 2b_{4y} - 4c_4) + \frac{i}{4^2}(a_3b_2 - a_2b_3 - 2a_{4y} + 2b_{4x}), \\ K_5 &= \frac{1}{4^2}(a_1^2 + b_1^2 + a_2a_3 + b_2b_3 + 2a_{1x} + 2b_{1y} - 4c_1) + \frac{i}{4^2}(a_3b_2 - a_2b_3 + 2a_{1y} - 2b_{1x}), \\ K_6 &= \frac{1}{4^2}(a_1a_3 + b_1b_3 + a_3a_4 + b_3b_4 + 2a_{3x} + 2b_{3y} - 4c_3) + \frac{i}{4^2}(a_1b_3 - a_3b_1 + a_3b_4 \\ &\quad -a_4b_3 + 2a_{3y} - 2b_{3x}), \\ K_7 &= \frac{1}{4^2}(a_1a_2 + b_1b_2 + a_2a_4 + b_2b_4 + 2a_{2x} + 2b_{2y} - 4c_2) + \frac{i}{4^2}(a_2b_1 - a_1b_2 + a_4b_2 \\ &\quad -a_2b_4 + 2a_{2y} - 2b_{2x}), \\ K_8 &= \frac{1}{4^2}(a_2a_3 + b_2b_3 + a_4^2 + b_4^2 + 2a_{4x} + 2b_{4y} - 4c_4) + \frac{i}{4^2}(a_2b_3 - a_3b_2 + 2a_{4y} - 2b_{4x}). \end{aligned} \tag{5.16}$$

Note that we have an invariant equation here. This differs from the invariants of the split elliptic system of section 2. We therefore have the following result.

Proposition 2.

A general system of two linear elliptic equations (5.8) has the five Cotton-type invariants (5.14) and its coefficients satisfy the invariant condition (5.15).

Now the four joint invariants (5.13) reduce to the four invariants of the elliptic equations (5.8) and are

$$\begin{aligned}
\mu_1 &= \frac{(H_1^2 - H_2^2)H_3 + 2H_1H_2H_4}{(H_1^2 - H_2^2)^2 + 4H_1^2H_2^2}, \\
\mu_2 &= \frac{(H_1^2 - H_2^2)H_4 - 2H_1H_2H_3}{(H_1^2 - H_2^2)^2 + 4H_1^2H_2^2}, \\
\mu_3 &= \frac{(H_1^2 - H_2^2)H_5}{(H_1^2 - H_2^2)^2 + 4H_1^2H_2^2}, \\
\mu_4 &= \frac{-2H_1H_2H_5}{(H_1^2 - H_2^2)^2 + 4H_1^2H_2^2},
\end{aligned} \tag{5.17}$$

where the semi-invariants H_1^2 and H_2^2 are both not zero. The situation when both are zero are for the Laplace system discussed earlier. We have thus obtained the Cotton-type and joint invariants for a general linear elliptic system of two equations (5.8) by using the Laplace-type and joint invariants of the general system of linear hyperbolic equations (5.9) by utilizing the known semi- and joint invariants of [99]. We thus state the proposition:

Proposition 3.

A general system of two linear elliptic equations (5.8) has the four joint invariants (5.17).

5.3 Applications

Here we present some examples for illustration. We have $u, v, \overline{u}, \overline{v}$ as dependent variables and x, y, s, t as independent variables.

Example 1.

Consider the system of two linear elliptic equations

$$\begin{aligned}
u_{xx} + u_{yy} + \frac{2}{x}u_x + \frac{4}{y}u_y + \frac{2}{y^2}u &= 0, \\
v_{xx} + v_{yy} - \frac{4}{x}v_x - \frac{2}{y}v_y + 2(\frac{1}{y^2} + \frac{3}{x^2})v &= 0.
\end{aligned} \tag{5.18}$$

This system transforms to the simplest elliptic equations

$$\begin{aligned}
\overline{u}_{xx} + \overline{u}_{yy} &= 0, \\
\overline{v}_{xx} + \overline{v}_{yy} &= 0,
\end{aligned} \tag{5.19}$$

under the transformation

$$\overline{u} = xy^2u, \quad \overline{v} = \frac{v}{x^2y}. \tag{5.20}$$

The systems of elliptic equations (5.18) and (5.19) are transformable into each other as these systems have the same Cotton-type semi-invariants

$$H_1 = H_2 = H_3 = H_4 = H_5 = 0. \tag{5.21}$$

Example 2.

The system of elliptic equations

$$\begin{aligned} u_{xx} + u_{yy} + (1-2y)u_x + (1-2x)u_y + (x^2+y^2-x-y)u &= 0, \\ v_{xx} + v_{yy} + (1-2y)v_x + (1-2x)v_y + (x^2+y^2-x-y)v &= 0, \end{aligned} \tag{5.22}$$

with the Cotton-type invariants

$$H_1 = 0, \quad H_2 = \frac{1}{4}, \quad H_3 = 0, H_4 = \frac{1}{64}, \quad H_5 = \frac{1}{32}, \tag{5.23}$$

reduces to the simple system of elliptic equations

$$\begin{aligned} \overline{u}_{xx} + \overline{u}_{yy} + \overline{u}_x + \overline{u}_y &= 0, \\ \overline{v}_{xx} + \overline{v}_{yy} + \overline{v}_x + \overline{v}_y &= 0, \end{aligned} \tag{5.24}$$

by the application of the transformation

$$\overline{u} = \exp(-xy)u, \qquad \overline{v} = \exp(-xy)v. \tag{5.25}$$

The system (5.24) also has the Cotton-type invariants (5.23).

Example 3.

The uncoupled system two of elliptic equations

$$\begin{aligned} u_{xx} + u_{yy} + (\frac{2}{x}+1)u_x + (1-\frac{4}{y})u_y + (\frac{1}{x}+\frac{6}{y^2}-\frac{2}{y})u &= 0, \\ v_{xx} + v_{yy} + (1-\frac{2}{x})v_x + (1+\frac{4}{y})v_y + (\frac{2}{y^2}+\frac{2}{y}-\frac{1}{x}+\frac{2}{x^2})v &= 0, \end{aligned} \tag{5.26}$$

has the Cotton-type invariants

$$H_1 = 0, \quad H_2 = \frac{1}{4}, \quad H_3 = 0, H_4 = \frac{1}{64}, \quad H_5 = \frac{1}{32}. \tag{5.27}$$

Therefore it is reducible to the simple system

$$\begin{aligned} \overline{u}_{xx} + \overline{u}_{yy} + \overline{u}_x + \overline{u}_y &= 0, \\ \overline{v}_{xx} + \overline{v}_{yy} + \overline{v}_x + \overline{v}_y &= 0, \end{aligned} \tag{5.28}$$

by means of the transformation

$$\overline{u} = \frac{x}{y^2}u, \quad \overline{v} = \frac{y^2}{x}v. \tag{5.29}$$

Example 4.

Consider now the linear system of elliptic equations

$$\begin{aligned} u_{xx} + u_{yy} + (\frac{2}{x} + \frac{1}{2})u_x + (\frac{1}{2} - \frac{2}{y})u_y + \frac{1}{2}(\frac{4}{y^2} - \frac{1}{y} + \frac{1}{x})u &= 0, \\ v_{xx} + v_{yy} + (\frac{2}{x} + \frac{1}{2})v_x + (\frac{1}{2} - \frac{2}{y})v_y + \frac{1}{2}(\frac{4}{y^2} - \frac{1}{y} + \frac{1}{x})v &= 0, \end{aligned} \tag{5.30}$$

which has the joint invariants

$$\mu_1 = 0, \quad \mu_2 = -\frac{1}{4}, \quad \mu_3 = -\frac{1}{2}, \quad \mu_4 = 0. \tag{5.31}$$

By using the transformation

$$s = \frac{x}{2}, \quad t = \frac{y}{2}, \quad ,\overline{u} = \frac{x}{y}u, \quad \overline{v} = \frac{x}{y}v. \tag{5.32}$$

the above system reduces to the simple system

$$\begin{aligned} \overline{u}_{ss} + \overline{u}_{tt} + \overline{u}_s + \overline{u}_t &= 0, \\ \overline{v}_{ss} + \overline{v}_{tt} + \overline{v}_s + \overline{v}_t &= 0, \end{aligned} \tag{5.33}$$

because this system has the joint invariant identical to the system (5.30).

Example 5.

Finally, the coupled system of elliptic equations

$$\begin{aligned} u_{xx} + u_{yy} + (2 + \frac{1}{x})u_x + 2y^3x^{-\frac{3}{2}}v_x - \frac{2}{y}u_y + (\frac{1}{x} - \frac{1}{4x^2} + \frac{2}{y^2})u - 2y^3x^{-\frac{5}{2}}v &= 0, \\ v_{xx} + v_{yy} + \frac{2x^{\frac{3}{2}}}{y^3}u_x + 2(1 - \frac{1}{x})v_x + \frac{4}{y}v_y + \frac{x^{\frac{1}{2}}}{y^3}u + 2(\frac{1}{y^2} - \frac{1}{x} + \frac{1}{x^2})v &= 0, \end{aligned} \tag{5.34}$$

with the joint invariants

$$\mu_1 = 0, \quad \mu_2 = 0, \quad \mu_3 = -1, \quad \mu_4 = 0, \tag{5.35}$$

simplifies to the system

$$\begin{aligned} \overline{u}_{ss} + \overline{u}_{tt} + \overline{u}_s + \overline{u}_t + \overline{v}_s + \overline{v}_t &= 0, \\ \overline{v}_{ss} + \overline{v}_{tt} + \overline{u}_s + \overline{u}_t + \overline{v}_s + \overline{v}_t &= 0, \end{aligned} \tag{5.36}$$

which has the same joint invariants as the system (5.34). The transformation that does this reduction is

$$s = x + y, \quad t = x - y, \quad ,\overline{u} = \frac{\sqrt{x}}{y}u, \quad \overline{v} = \frac{y^2}{x}v. \tag{5.37}$$

In this chapter, a complex scalar linear elliptic equation has been transformed into a system of two linear elliptic equations by splitting the complex equation, which is a subclass of the general system of two linear elliptic equations. Cotton-type semi-invariants for this system of elliptic equations are obtained by two approaches. One is by split of the complex Cotton invariants that correspond to the complex base scalar linear elliptic equation into real and imaginary parts, and the second by transformation of the subsystem of the linear elliptic equations into linear hyperbolic equations and application of the linear inverse transformations on the Laplace-type semi-invariants of the hyperbolic equations to deduce the Cotton-type invariants for the required subsystem of linear elliptic equations. It is found that the Cotton-type invariants by both the approaches are the same. For a general system of linear elliptic equations, the Cotton-type and joint invariants have been constructed by transformation of the system of two linear elliptic equations into a system of two linear hyperbolic equations and thereafter applying the linear inverse transformations on the Laplace-type and joint invariants of [99] to deduce the Cotton-type and joint invariants for the linear system of elliptic equations.

Chapter 6

Summary and Conclusions

Sophus Lie developed infinitesimal methods to find the Lie groups of continuous transformations. Lie group theory play a fundamental role in finding the solutions of the differential equations. Lie also pointed out that the theory of differential invariants of the equivalence group of point transformation of the differential equations is also based on the infinitesimal methods. Knowledge of differential invariants is extremely useful in the integration of the differential equations. In this thesis, a relation between the isometries and the Noether symmetries is presented. In the third chapter of this thesis, differential invariants for the system of hyperbolic PDEs are discussed using the complex symmetry analysis. Cotton's invariants and joint invariants for the scalar and system of elliptic PDEs are also derived in of this thesis.

6.1 The Noether Symmetries of the Area-Minimizing Lagrangian

The importance of Lie symmetries in the theory of integration of DEs provides mathematician an incentive to find the symmetries of the DEs. As the differential equations "live" on manifolds, it is natural to search for the connection between symmetries of differential equations and those of geometry. The first such attempt looked for the connection through the system of geodesic equations [6, 31], some connections between Noether symmetries and isometries have been found in the context of general relativity [13–15]. The geodesic equations are the EL-equations for the arc-length minimizing action. Their symmetries and the corresponding geodesic equations are known for maximally and non-maximally symmetric spaces. A connection was obtained between isometries (the symmetries of the geometry) and Lie symmetries of the geodesic equations of the underlying space [31], which leads to the geometric linearization for ODEs [46, 74, 88]. An additional benefit of this approach is that one can obtain the solution of the linearized equations by the transformation to the metric tensor coordinates given by the geodesic equations from Cartesian coordinates. There are three questions that needs to be addressed. First, can one extend geometric approach to higher order ODEs? Second, is there exist any connection between the symmetries in geometry and higher order symmetries of the corresponding equations? Third and most important of all, how to extend the geometric methods for PDEs. In the second chapter of this thesis, we discuss this last question. For this, first we formulate the $(n-1)$-area minimizing Lagrangian keeping constant n-volume using the Kuhn-Tucker theorem [51, 90], as [16, 17]

$$I = A(S) + \lambda V(S) = \int_s n^p d^{n-1} s_p + \lambda \int_{\mathbf{V}} d^n V, \quad p = 1, 2, ..., n-1, \tag{6.1}$$

here λ is the Lagrange multiplier. Here for the non-compact space this has to be taken in the sense of being cut at a fixed boundary that respects the symmetry of the space and is not a volume enclosing hypersurface otherwise. Then a relation between isometries and Noether symmetries for the area minimizing Lagrangian has been found and presented in the form of following theorems.

Theorem 6.1.1. *The Lagrangian for minimizing the $(m-1)$-area enclosing a constant m-volume in a Euclidian space, has a Lie algebra of Noether symmetries identical with the Lie algebra of isometries of the Euclidean space, $so(m) \oplus_s \mathbb{R}^{(m)}$, with the vector gauge functions corresponding to the translations.*

Theorem 6.1.2. *The Lie algebra of Noether symmetries for the Lagrangian for minimizing the $(m-1)$-area keeping a constant m-volume in a space of non-zero constant curvature is $so(m)$.*

Theorem 6.1.3. *The Lie algebra of the Noether symmetries for the Lagrangian which minimizes an $(n-1)$-area enclosing a constant n-volume, in a space which has one section of constant curvature of dimension n_1, another of n_2, etc. up to n_k and a flat section of dimension m and $n \geq \sum_{j=1}^{k} n_j + m$ (as some of the sections may have no symmetry), is $\oplus_{j=1}^{k} so(n_j + 1) \oplus (so(m) \oplus_s \mathbb{R}^m)$.*

6.2 Invariants of the Group of Equivalence Transformations for Hyperbolic and Elliptic PDEs

Third, fourth and fifth chapters of this thesis deals with the differential invariants of the group of equivalence transformations of hyperbolic and elliptic PDEs. Differential invariants are extremely useful tools in the integration of DEs. There are two main methods to calculate the set of all the equivalence transformation. The first method uses directly the definition of the equivalence transformation called the *direct method*. Theoretically, one can calculate the most general group of equivalence transformations using direct method but usually this method leads to huge computational difficulties especially when dealing with non-linear DEs. Indeed, when Roger Liouville [66,67] calculated the differential invariants using the direct methods for the second-order cubically nonlinear ODE, introduced by Lie, the calculations were done on seventy pages [45].

The approach of complex symmetry analysis (CSA), was utilized in [2,4]. This method provides a connection between a complex scalar ODE/PDE and a system of real ODEs/PDEs by a complex split of the base complex equation into real and imaginary parts. Applying the CSA on the scalar complex hyperbolic equation

$$w_{tx} + \alpha(t,x)w_t + \beta(t,x)w_x + \gamma(t,x)w = 0, \tag{6.2}$$

gives the following two hyperbolic equations

$$\begin{aligned} u_{tx} + \alpha_1(t,x)u_t - \alpha_2(t,x)v_t + \beta_1(t,x)u_x - \beta_2(t,x)v_x + \gamma_1(t,x)u - \gamma_2(t,x)v = 0, \\ v_{tx} + \alpha_2(t,x)u_t + \alpha_1(t,x)v_t + \beta_2(t,x)u_x + \beta_1(t,x)v_x + \gamma_2(t,x)u + \gamma_1(t,x)v = 0. \end{aligned} \tag{6.3}$$

if $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$ and $w = u + iv$. The system of hyperbolic PDEs is a subclass of the general system of hyperbolic PDEs

$$\begin{aligned} u_{tx} + a_1(t,x)u_t + a_2(t,x)v_t + b_1(t,x)u_x + b_2(t,x)v_x + c_1(t,x)u + c_2(t,x)v = 0, \\ v_{tx} + a_3(t,x)u_t + a_4(t,x)v_t + b_3(t,x)u_x + b_4(t,x)v_x + c_3(t,x)u + c_4(t,x)v = 0. \end{aligned} \tag{6.4}$$

The scalar linear hyperbolic PDE (6.2) has two semi-invariants under the linear change of the dependent variables, six semi-invariants under the change of the independent variables and six joint invariants of which five form a bases of joint invariants [43,53,54]. The system of two linear hyperbolic PDEs (6.4) has four joint invariant and five semi-invariants under the linear change of the dependent variables [99] and four semi-invariants associated with the change of only the dependent variables for a subclass of a system of two linear hyperbolic equations (6.3) obtained from a complex linear hyperbolic equation [79]. Here, for a subclass of the system (6.3) semi-invariants associated with the invertible change of the dependent as well as independent variables are first derived using the Lie's infinitesimal methods. The equivalence transformations of the system of equations (6.3) is an infinite group of the dependent variable

$$\overline{u} = \sigma_1(t,x)u + \sigma_2(t,x)v, \quad \overline{v} = \sigma_1(t,x)v - \sigma_2(t,x)u, \tag{6.5}$$

and the invertible change of the independent variables

$$\bar{t} = \phi(t), \quad \bar{x} = \psi(x). \tag{6.6}$$

The system (6.3) has four and six semi-invariant under change of the dependent and independent variables, respectively. It is found that this system has six joint invariants. Semi-invariants and joint invariants of the system are also obtained using the complex procedure. This procedure reveals the correspondence of such systems and associated invariants with the base complex hyperbolic equation and related complex invariants, respectively. Complex procedure give four and twelve semi-invariants under the change of the dependent variables, respectively. This is achieved by performing complex splits of the semi-invariants. It is also found that the corresponding system has twelve joint invariants by the splitting the corresponding six joint invariants for the complex scalar equation. It is shown that same invariant quantities for the system of hyperbolic PDEs appear due to complex and real procedures, in the case of transformations of only the dependent variables. However, the semi-invariants of this system associated with only independent variables obtained by real symmetry analysis are different from those provided by the complex procedure. Furthermore, the joint invariants of this system of hyperbolic equations obtained by both the methods are also found to be different.

In the fourth chapter of this thesis, we consider a scalar linear second order elliptic equation in two independent variables in canonical form

$$u_{xx} + u_{yy} + au_x + bu_y + cu = 0. \tag{6.7}$$

It is well-known that by means of the linear complex transformations [29, 55],

$$x = \frac{1}{2}(t+z), \quad y = \frac{-i}{2}(t-z), \tag{6.8}$$

the elliptic equation (6.7) can be mapped to the linear hyperbolic equation

$$u_{tz} + Au_t + Bu_z + Cu = 0. \tag{6.9}$$

Because under the transformation (6.8), $u_{xx} = u_{tt} + 2u_{tz} + u_{zz}$ and $u_{yy} = -u_{tt} + 2u_{tz} - u_{zz}$, so u_{tt} and u_{zz} are canceled with each other and the only remaining second order term is u_{tz}. As the hyperbolic and elliptic equations can be transformed into each other, so do their corresponding Laplace and Cotton invariants. The Laplace invariants

$$\begin{aligned} h &= A_t + AB - C, \\ k &= B_z + AB - C, \end{aligned} \tag{6.10}$$

for equation (6.9) can be transformed, by use of the inverse of the transformations (6.8) as well as after the substitution of the values of A, B and C into (6.10) and then splitting the real and imaginary parts, to arrive at the Cotton invariants

$$\begin{aligned}\mu &= a_y - b_x,\\ H &= a_x + b_y + \frac{1}{2}(a^2+b^2) - 2c.\end{aligned} \tag{6.11}$$

Here, an important question arises, can one find the bases of the joint invariants of the elliptic equations from the bases of the joint invariants of the hyperbolic equations. The application of the Lie's infinitesimal method have resulted in new joint differential invariants which are in terms of Cotton's invariants. It has been shown via the operators of invariant differentiation that there are six elements of joint invariants which constitute a bases of joint differential invariants for the elliptic PDE (6.7).

It is thus worthwhile to apply an alternate approach to find joint differential invariants for the elliptic equation (6.7) via knowledge of joint invariants of the hyperbolic PDE (6.9). Then one can obtain joint invariants for (6.7) with the aid of application of the rules of derivatives on the complex transformations and the joint invariants of the hyperbolic equation (6.9). From the transformed hyperbolic PDE, one can obtain the Laplace invariants in complex form by use of the rules of derivatives. The real and imaginary parts will give Cotton's invariants for (6.7). This is easy enough. However, if we proceed in a similar manner for the five basic invariants of the hyperbolic PDE, one will, in general, get five joint differential invariants in complex form. That is, in general, ten invariants from the real and imaginary parts. It becomes difficult to identify which are elements of the bases of joint differential invariants for the elliptic PDE. The infinitesimal approach used here averts this difficulty. Differential invariants can be used in the group classification of DEs. Ovsiannikov [86] used the Laplace invariants in the group classification of the hyperbolic equation by writing the determining equations for the symmetries of hyperbolic equation in terms of these invariants. We have also extended this by providing the symmetry classification for the elliptic equations (6.7) in terms of the Cotton invariants.

In the last chapter we consider a subsystem of two elliptic equations

$$\begin{aligned}u_{xx} + u_{yy} + \alpha_1 u_x - \alpha_2 v_x + \beta_1 u_y - \beta_2 v_y + \gamma_1 u - \gamma_2 v &= 0,\\ v_{xx} + v_{yy} + \alpha_2 u_x + \alpha_1 v_x + \beta_2 u_y + \beta_1 v_y + \gamma_2 u + \gamma_1 v &= 0,\end{aligned} \tag{6.12}$$

which can be obtained by splitting the complex linear elliptic equation

$$w_{xx} + w_{yy} + aw_x + bw_y + cw = 0, \tag{6.13}$$

where

$$a = \alpha_1 + i\alpha_2, b = \beta_1 + i\beta_2,\ c = \gamma_1 + i\gamma_2, w = u + iv. \tag{6.14}$$

By the application of the complex transformations (6.8), the system of elliptic equations (6.12) can be mapped to the system of two linear hyperbolic equations

$$\begin{aligned} u_{tz} + A_1u_t - A_2v_t + B_1u_z - B_2v_z + C_1u - C_2v &= 0, \\ v_{tz} + A_2u_t + A_1v_t + B_2u_z + B_1v_z + C_2u + C_1v &= 0, \end{aligned} \tag{6.15}$$

Cotton-type semi-invariants for this system of elliptic equations are obtained by two approaches. One is by split of the complex Cotton invariants that correspond to the complex base scalar linear elliptic equation into real and imaginary parts, and the second by transformation of the subsystem of the linear elliptic equations into linear hyperbolic equations and application of the linear inverse transformations on the Laplace-type semi-invariants of the hyperbolic equations to deduce the Cotton-type invariants for the required subsystem of linear elliptic equations. It is shown that for a class of a system of two linear elliptic equations (6.12) obtained from a complex base linear elliptic equation (6.13) or equivalent to a system of two linear hyperbolic equations (6.15) by complex linear transformations of the independent variables (6.8), Cotton-type invariants either constructed by splitting of the complex Cotton invariants of the complex elliptic equation into real and imaginary parts or those computed by the Laplace-type invariants of the system of the system of linear hyperbolic equations are identical.

For a general system of linear elliptic equations, the Cotton-type and joint invariants have been constructed by transformation of the system of two linear elliptic equations into a system of two linear hyperbolic equations and thereafter applying the linear inverse transformations on the Laplace-type and joint invariants of hyperbolic equations to deduce the Cotton-type and joint invariants for the linear system of elliptic equations. It is also shown that the general system of two linear elliptic equations has five Cotton-type invariants and four joint invariants.

Chapter 7

Appendix

7.1 Appendix A-1

For 2-area

The EL-equation corresponding to the Lagrangian (2.1) is

$$\begin{aligned}&\left(2r^5+3r^3r_\theta^2-r^4r_{\theta\theta}\right)\sin^3\theta-r^2r_\theta\cos\theta\left(r^2+{r_\theta}^2\right)\sin^2\theta\\&-r^2\left(r_{\theta\theta}{r_\phi}^2-2r_{\theta\phi}r_\theta r_\phi-3r{r_\phi}^2+r_{\phi\phi}{r_\theta}^2+r^2r_{\phi\phi}\right)\sin\theta\\&-2r^2r_\theta\cos\theta{r_\phi}^2+\lambda\left(r^4\sin^2\theta+r^2\sin\theta^2{r_\theta}^2+r^2{r_\phi}^2\right)^{3/2}=0,\end{aligned}\tag{7.1}$$

and the conserved quantities for the Noether symmetries $\mathbf{X_1},\mathbf{X_2},\mathbf{X_3},\mathbf{X_4},\mathbf{X_5},\mathbf{X_6}$ are

$$\begin{aligned}\mathbf{I_1}=&\frac{r^2}{\Sigma}\Big[\frac{1}{3}r\lambda\sin\phi\sin\theta\Sigma+(r^2\sin\phi-r_\theta r_\phi\cot\theta\cos\theta)\sin^2\theta+r_\phi^2\sin\phi,\\&\frac{1}{3}r\lambda\cos\theta\cos\phi\Sigma+\cos\phi(r^2+r_\theta^2)\cos\theta\sin\theta-r_\theta r_\phi\sin\phi\Big],\\\mathbf{I_2}=&\frac{r^2}{\Sigma}\Big[\frac{1}{3}r\lambda\cos\phi\sin\theta\Sigma+(r^2\cos\phi+r_\theta r_\phi\cot\theta\sin\phi)\sin^2\theta+r_\phi^2\cos\phi,\\&-\frac{1}{3}r\lambda\cos\theta\sin\phi\Sigma-\sin\phi(r^2+r_\theta^2)\cos\theta\sin\theta-r_\theta r_\phi\cos\phi\Big],\\\mathbf{I_3}=&\frac{r^2}{\Sigma}\Big[-r_\theta r_\phi\sin^2\theta,\frac{1}{3}r\lambda\sin\theta\Sigma+(r^2+r_\theta^2)\sin^2\theta\Big],\\\mathbf{I_4}=&\frac{r}{\Sigma}\Big[\frac{1}{2}\lambda r\cos\theta\sin\theta\sin\phi\Sigma+rr_\theta\sin^3\theta\sin\phi\cos\theta\sin\phi(2r_\theta^2+r^2)\sin^2\theta-r_\theta r_\phi\sin\theta\cos\phi\\&+r_\phi^2\cos\theta\sin\phi,\ \frac{1}{2}r\lambda\cos\phi\Sigma+((r^2+r_\theta^2)\cos\phi+r_\theta r_\phi\sin\phi)\sin\theta-r_\theta r_\phi\cos\theta\sin\phi\Big],\end{aligned}\tag{7.2}$$

$$\begin{aligned}
\mathbf{I_5} &= \frac{r}{\Sigma}\Big[r^2\cos\theta\cos\phi\sin^2\theta + r_\phi^2\cos\theta\cos\phi + \frac{1}{2}r\lambda\cos\theta\sin\theta\cos\phi\Sigma + rr_\theta\sin^3\theta\cos\phi \\
&\quad + r_\theta r_\phi\sin\theta\sin\phi, -\frac{1}{2}r\lambda\sin\phi\Sigma - ((r^2+r_\theta^2)\sin\phi - rr_\phi\cos\phi)\sin\theta - r_\theta r_\phi\cos\theta\cos\phi\Big], \\
\mathbf{I_6} &= \frac{r}{\Sigma}\Big[r^2\sin^3\theta + r_\phi^2\sin\theta + \frac{1}{2}r\lambda\sin^2\theta\Sigma - rr_\theta\sin^2\theta\cos\theta, \\
&\quad - rr_\phi\cos\theta - r_\theta r_\phi\sin\theta\Big],
\end{aligned} \tag{7.3}$$

where

$$\Sigma = (r^4\sin^2\theta + r^2r_\theta^2\sin^2\theta + r^2r_\phi^2)^{\frac{1}{2}}.$$

For 3-area

The metric for a 4-dimensional flat space in hyperspherical coordinates is

$$ds^2 = dr^2 + r^2d\chi^2 + r^2\sin^2\chi d\theta^2 + r^2\sin^2\chi\sin^2\theta d\phi^2.$$

Let the enclosing surface be $r = r(\chi, \theta, \phi)$. The 3-area is

$$A(S) = \int (r^6\sin^4\chi\sin^2\theta + r^4r_{,\chi}^2\sin^4\chi\sin^2\theta + r^4r_{,\theta}^2\sin^2\chi\sin^2\theta + r^4r_{,\phi}^2\sin^2\chi)^{\frac{1}{2}}d\chi d\theta d\phi.$$

Then the variational principle (6.1) becomes

$$\delta\int\Big[\Sigma + \lambda\frac{1}{4}r^4\sin^2\chi\sin\theta\Big]d\chi d\theta d\phi = 0,$$

where

$$\Sigma = (r^6\sin^4\chi\sin^2\theta + r^4r_{,\chi}^2\sin^4\chi\sin^2\theta + r^4r_{,\theta}^2\sin^2\chi\sin^2\theta + r^4r_{,\phi}^2\sin^2\chi)^{\frac{1}{2}}.$$

Thus the Lagrangian is

$$L = \Sigma + \lambda\frac{1}{4}r^4\sin^2\chi\sin\theta.$$

For 4-area

The metric for a 5-dimensional flat space in hyperspherical coordinates is

$$ds^2 = dr^2 + r^2d\psi^2 + r^2\sin^2\psi d\chi^2 + r^2\sin^2\psi\sin^2\chi d\theta^2 + r^2\sin^2\psi\sin^2\chi\sin^2\theta d\phi^2.$$

Let the enclosing surface be $r = r(\psi, \chi, \theta, \phi)$. The 4-area is

$$\begin{aligned}
A(S) = \int (r^8\sin^6\psi\sin^4\chi\sin^2\theta + r^6r_{,\psi}^2\sin^6\psi\sin^4\chi\sin^2\theta + r^6r_{,\chi}^2\sin^4\psi\sin^4\chi\sin^2\theta \\
+ r^6r_{,\theta}^2\sin^4\psi\sin^2\chi\sin^2\theta + r^6r_{,\phi}^2\sin^4\psi\sin^2\chi)^{\frac{1}{2}}d\psi d\chi d\theta d\phi.
\end{aligned}$$

Then the variational principle (6.1) becomes

$$\delta\int\Big[\Sigma + \lambda\frac{1}{5}r^5\sin^3\psi\sin^2\chi\sin\theta\Big]d\psi d\chi d\theta d\phi = 0,$$

where

$$\Sigma =(r^8 \sin^6 \psi \sin^4 \chi \sin^2 \theta + r^6 r_{,\psi}^2 \sin^6 \psi \sin^4 \chi \sin^2 \theta \\ + r^6 r_{,\chi}^2 \sin^4 \psi \sin^4 \chi \sin^2 \theta + r^6 r_{,\theta}^2 \sin \psi^4 \sin^2 \chi \sin^2 \theta + r^6 r_{,\phi} \sin^4 \psi \sin^2 \chi)^{\frac{1}{2}}.$$

Thus the Lagrangian is

$$L = \Sigma + \lambda \frac{1}{5} r^5 \sin^3 \psi \sin^2 \chi \sin \theta.$$

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