Some representations of the extended Fermi-Dirac and Bose-Einstein functions with applications

By

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Asifa Tassaddiq

Dedicated

To my Father

Abstract

The familiar Fermi-Dirac (FD) and Bose-Einstein (BE) functions are of importance not only for their role in Quantum Statistics, but also for their several interesting mathematical properties in themselves. Here, in my present investigation, I have extended these functions by introducing an extra parameter in a way that gives new insights into these functions and their relationship to the family of zeta functions. This thesis gives applications of their transform and distributional representations. The Weyl and Mellin transform representations are used to derive mathematical properties of these extended functions. The series representations and difference equations presented led to various new results for the FD and BE functions. It is demonstrated that the domain of the real parameter x involved in the definition of the FD and BE functions can be extended to a complex z. These extensions are dual to each other in a sense that is explained in this thesis. Some identities are proved here for each of these general functions and their relationship with the general Hurwitz-Lerch zeta function $\Phi(z, s, a)$ is exploited to derive some new identities. A closely related function to the eFD and eBE functions is also introduced here, which is named as the generalized Riemann zeta (gRZ) function. It approximates the trivial and non-trivial zeros of the zeta function and shows that the original FD and BE functions are related with the Riemann zeta function in the critical strip. Its relation with the Hurwitz zeta functions is used to derive a new series representation for the eBE and the Hurwitz-Lerch zeta functions.

The integrals of the zeta function and its generalizations can be of interest in the proof of the Riemann hypothesis (one of the famous problem in mathematics) as well as in Number Theory. The Fourier transform representation is used to derive various integral formulae involving the eFD, eBE and gRZ functions. These are obtained by using the properties of the Fourier and Mellin transforms. Distributional representation extends some of these formulae to complex variable and yields many new results. In particular, these representations lead to integrals involving the Riemann zeta function and its generalizations. It is also suggested that the Fourier transform and distributional representations of other special functions can be used to evaluate new integrals involving these functions. As an example, I have considered the generalized gamma function. Some of the integrals of products of the gamma function with zeta-related functions can not be expressed in a closed form without defining the eFD, eBE and gRZ functions. It proves the natural occurrence of these generalizations in mathematics. This study led to various new results for the classical FD and BE functions. Integrals of the gamma function and its generalizations are used in engineering mathematics while integrals of the zeta-related functions are essential in Number Theory. Both classes of integrals have been combined first time in this thesis. This in turn gives integrals of product of the modified Bessel functions and zeta-related functions. Further, whereas complex distributions had been defined earlier, and in fact used for different applications, there has been no previous utilization of them for Special Functions in general and for the zeta family in particular. This is provided for the first time in this thesis. An important feature of the approach used is the remarkable simplicity of the proofs by using integral transforms.

Publications from the thesis

1. H. M. Srivastava, M. A. Chaudhry, A. Qadir and A. Tassaddiq, Some extentions of the Fermi-Dirac and Bose-Einstein functions with applications to zeta and related functions, *Russian Math. Phys.* **18**(2011), 107-121.

2. A. Tassaddiq and A. Qadir, Fourier transform representation of the extended Fermi-Dirac and Bose-Einstein functions with applications to the family of the zeta and related functions, *Integral Transforms Spec. Funct.* **22** (2011), 453-466.

3. A. Tassaddiq and A. Qadir, Fourier transform and distributional representation of the generalized gamma function with some applications, *Appl. Math. Comput.* **218** (2011), 1084-1088.

4. M. A. Chaudhry, A. Qadir and A. Tassaddiq, A new generalization of the Riemann zeta function and its difference equation, *Advances in Difference Equations* 2011, 2011:20.

Contents

Chapter 1

Introduction and preliminaries

The history of special functions goes back to the fundamental work of the Greeks, who used trigonometric ratios for measurement. After that John Napier, invented logarithms to make multiplication and division easier. Then special functions appeared as solutions of physical problems. Till the 18th century they were limited to elementary functions: powers; roots; trigonometric functions; and their (algebraic) inverses. Later it was found that solutions for some important physical problems, like the orbital motion of planets, the oscillatory motion of suspended chains and so on, could not always be described in a closed form using only these (elementary) functions. Even in the realm of mathematics, some quantities, such as the circumference of an ellipse could not be discussed in these terms. Solutions to these problems were often expressed as infinite series and integrals.

The solutions of commonly occurring linear second order differential equations of engineering and mathematical physics led to a class of special functions. These arise from the Laplace or wave equations on separation of variables, when they reduce to the Sturm-Liouville problem. For example, the Legendre polynomials and functions, the Bessel functions, the Laguerre and the Hermite polynomials and some others. All of them can be obtained by taking particular parameter values of the hypergeometric or confluent hypergeometric functions [47]. Another collection of special functions does not emerge from differential equations: the gamma function and the family of zeta functions. It is of interest to follow them in more detail.

In the 1720s, Euler defined the gamma function as a generalization of the factorial by extending its domain from integers to the real and thence to the complex numbers. The gamma function is defined over semi-infinite domains. It can be broken into the two incomplete gamma functions [14] by breaking the range of integration at some finite positive number x. It led to another generalization of the gamma function by restricting its domain and proved useful in problems that were formulated over finite domain. Other functions related to incomplete gamma function are the exponential and logarithmic integral functions, sine and cosine integral functions, error functions and Fresnel integrals [47]. Many special functions can be defined in terms of the gamma function, so, it can be regarded as an "elementary function".

In 1737, Euler related the sum over positive integers to the product over primes. In 1859, Bernhard Riemann observed this relation while working on the frequency of prime numbers [18]. He denoted Euler's sum by $\zeta(s)$ and extended its domain from real to complex numbers. He found its integral representation by connecting the sum with gamma function. This sum is now known as Riemann zeta function. It was originally defined for $\Re(s) > 1$ (real part of s greater than 1) and can be analytically continued to the whole complex s-plane with a simple pole at $s = 1$. It has simple zeros at the negative even integers, which are called the trivial zeros. All other zeros of the Riemann zeta function, which are infinitely many [16,18,46], are called the non-trivial zeros. These zeros seem to lie on the critical line $\Re(s) = 1/2$ in the critical strip $0 < \Re(s) < 1$. Despite the attempts of many great mathematicians, who tried to prove it, this statement has remained analytically unproven to date [16,46]. The conjecture that this is true is called the *Riemann Hypothesis*. The Riemann zeta function has played an important role in Number Theory. It has different generalizations in the literature, for example, the polylogarithm, the Hurwitz zeta, the Hurwitz-Lerch zeta functions and some other [21]. Riemann's zeta function and its various extensions, collectively known as the zeta family have many applications in different areas of mathematics and physics.

Fermi-Dirac (FD) and Bose Einstein (BE) functions, arose in the distribution functions for Quantum Statistics [41]. While the Maxwell distribution provides the velocity distribution of molecules of a classical gas, the FD and BE functions come from the velocity distribution of a quantum gas. The crucial difference is that the former are (in principle) distinguishable particles while the latter are, even in principle, indistinguishable. According to the standard theory, all particles either have spins of integer or half integer multiples of Planck's constant h divided by 2π. If the spin is half integer they are called Fermions and their velocity distribution is given by the FD distribution function. If it is integer they are called Bosons and their velocity distribution is given by the BE distribution functions. The FD and BE functions are the respective cumulative probabilities of these distribution functions. Some phenomena were discovered in which the particles behaved as if they were indistinguishable but neither fermions nor bosons. They were called "anyons". The FD and BE functions have close connection with the zeta function. The Riemann zeta function has played an important role in number theory and was not seen to arise from any physical problems. More recently it has (surprisingly) arisen in the theory of Quantum Entanglement. This will be explained in more detail in section (1.3).

Generalization of special functions may prove more useful than the original special functions themselves [58]. In certain situations they did not solve the problem but a generalization, which can be observed in the case of gamma function (as described above). Some other generalizations of the exponential, gamma, zeta, error, beta and hypergeometric functions with applications are discussed in [14, 58] and references therein. These are defined by introducing an extra parameter in the integral representation of original function. All the usual properties of these functions carry over directly for their extensions. For example, the generalized incomplete gamma function $\gamma(s, x; b)$ (and the corresponding complementary function $\Gamma(s, x; b)$) is defined [14, p. 37] as an extension of the incomplete gamma function. It solved many problems [14], which were not solvable by using the original function.

Generalized functions (or distributions) are defined as continuous linear functionals over a space of test functions. As their name implies, a generalized function is a generalization of the concept of a function. The most commonly encountered generalized function is the Dirac delta function. A nice treatment of distributions from a physicist's point of view is given by Vladimirov in [72]. The confluence of the generalized functions with the classical integral transformations extended the theory of the Fourier transformation. This became a remarkably powerful tool in the theory of partial differential equations. A classic and rigorous treatment of the field is given in a multivolume work by Gel'fand and Shilov [26, 27]. The generalized functions have also used to extend various other integral transformations (see [76, 77] and references therein).

One normally thinks of a function as defined by a series or an integral of some variable or in terms of those functions that we regard as "elementary". However, the function needs to be regarded as an entity in itself, which is represented by a series or an integral. Only in this sense can we continue the function beyond its original domain of definition. This is essential for applying the function beyond dealing with the problem it was originally defined to deal with. This consideration becomes especially important when discussing the theory of (special) functions. There is more than one representation for all special functions, for example, the series representation, the asymptotic representation, the integral representation, etc. Further there can be more than one integral or series representation, which help to define the function in different regions. For example the integral representation [47, Equation (1.1.1)] of the gamma function is defined in the positive half of the complex plane. It leads to another representation [47, Equation (1.1.5)], which is defined in the whole complex plane except at negative integers. One can also use a different representation to express the function in terms of simpler functions. It will help to know the behavior of the original function at certain points of its domain. One representation can give simpler proofs of some known properties as compared to other. Here, in my present investigation, I have discussed some representations of the extensions of FD and BE functions.

An important aspect of the analysis of special functions is to find their properties [47]. Parallel to this analysis, the integrals of special functions have attracted the attention of many mathematicians. They developed the tables of integrals of products of various special functions. Among these, especially the tables of W. Gröbner [30], N. Hofreiter [30], A. Erdelyi [20], W. Magnus [20], F. Oberhettinger [20], I. Gradshteyn [25], I. Ryzhik [25], H. Exton [22], H. M. Srivastava [62], A. P. Prudnikov [57], Yu. A. Brychkov [57] and O. I. Marichev [50] are noteworthy. Note that the integral of a product of special functions can produce a new special function, which may prove more useful than the original functions. For example, the gamma function is an integral of the product of the exponential and a simple power. The tables of integrals contain a large number of the integrals of the gamma function, Bessel functions, Legendre polynomials, hypergeometric and related functions. Only a few integrals of the Riemann zeta and related functions have been found in these tables. The integrals of the zeta function and its generalizations are important in the study of Riemann hypothesis and for the analysis of zeta function itself [18, 40]. Integrals involving the zeta function and its generalizations also appear in problems dealing with distributions of $\{nx\}$ for $n \in \mathbb{N}$ and x in not a rational number, where {x} denotes the fractional part of x, see [54]. In this thesis, I have obtained a lot of integrals involving the Riemann zeta function and its generalizations. Integrals of the gamma function and its generalizations are used in engineering mathematics while integrals of the zeta-related functions are essential in Number Theory. Both classes of integrals have been combined first time in this thesis. This in turn gives integrals of product of the modified Bessel functions and zeta-related functions.

I define the *extended Fermi-Dirac* (eFD) and *extended Bose-Einstein* (eBE) functions. On the one hand these extensions give new insights into the study of Fermi-Dirac (FD) and Bose-Einstein (BE) functions, which appear in Quantum Statistics. On the other hand these are related with the zeta function and its generalizations, which appear in Number Theory. A closely related function to the eFD and eBE functions is also introduced here, which is named as the *generalized Riemann zeta* (gRZ) function.

This study is concerned with the mathematical applications of transform and distributional representations of the eFD, eBE and gRZ functions. The Weyl and Mellin transform representations are used to derive mathematical properties of the extended functions. A representation theorem is proved, which is further extended by using the linearity property of the Weyl transform. The Fourier transform representation is used to derive some integral formulae involving these functions. Distributional representation extends some of these formulae to complex variable and yields many new results.

The Mellin transform representation leads to a series representation for the eFD and eBE functions, which also extends the domain of some of the real parameters involved to complex variables. It gives new insights into the study of the FD and BE functions. The extensions presented are dual in a sense that is also explained here. New integral representations for the zeta family are obtained by using the properties of Mellin transform. The relations of the eFD, eBE and gRZ functions with the Riemann, Hurwitz and Lerch zeta functions have proved useful to write explicit representations of the eFD, eBE and gRZ functions in terms of simpler functions, which help to better understand the behaviour of these functions.

The Fourier transform representation of the eFD, eBE and gRZ functions leads to their series representation in terms of Dirac delta functions of complex argument. This series representation is named as "distributional representation" here. It provides a computational technique to evaluate integrals of the products of these functions, which applies to functionals that depend on functions, rather than functions that depend on numbers. I will first check that the results obtained by distributional representation are consistent with the results obtained by classical Fourier transform representation and then obtain some new results. The distributional representation converts the evaluation of the integral of products of special functions into the evaluation of a sum, which is easier to calculate. This in turn proves useful to get analytic extensions of the Fourier transforms of the eFD, eBE, gRZ and other related functions. The distributional representation also points to new formulae that could be obtained by classical means but one would not have thought of doing so. An important aspect of the approach used here is the simplicity of the results by using integral transforms. Further, whereas complex distributions had been defined earlier, and in fact used for different applications, there has been no previous utilization of them for Special Functions in general and for the zeta family in particular. This is provided for the first time in this thesis. As already mentioned, only a few integrals involving the Riemann and Hurwitz zeta functions are given in the "tables of integrals". It is also searched that no integral involving the products of the FD, BE and Hurwitz Lerch zeta functions has been found yet [11, 15, 20, 21, 25, 31, 74]. It is only with the innovations in this thesis that they have become accessible. It is worthwhile to mention that the integrals of the FD, BE and related functions lead to some special and more general cases of the FD and BE distribution functions.

The eFD and eBE functions have also found another use in Physics and had been put forward as possible candidates for the anyon function as they interpolate very naturally between the BE and FD functions [10]. The gRZ function approximates the trivial and non trivial zeros of the Riemann zeta function and has been used in a possible proof of the Riemann hypothesis [9].

This thesis is organized as follows: The rest of the Chapter **1** is devoted to the definitions and fundamental results related to the family of zeta functions, integral transforms and generalized functions, which are in understanding and development of the eFD, eBE, gRZ functions and their representations.

In Chapter **2**, I prove a representation theorem and define the eFD and eBE functions by using the concept of good functions. It is shown that the eFD and eBE functions have the meromorphic continuation to the whole complex s-plane. Some series representations are obtained, which demonstrate that the real parameter x involved in the definition of the FD and BE functions can be extended to a complex z. It led to a mathematical proof of their duality property. Miscellaneous results satisfied by the eFD and eBE functions are given in Section **(2.4)**. It led to new formulae for the classical FD and BE functions. It is shown that the BE function can be expressed as a linear combination of the eBE functions. The relations of the eFD and eBE functions with the Riemann, Hurwitz and Hurwitz-Lerch zeta functions are also discovered here. It is also shown that the FD function is related with Bernoulli polynomials for complex argument. The results of this chapter appeared as [63].

In Chapter **3**, I give further (mathematical) applications of the representation theorem in Section **(3.1)**. By using the analytic continuation of the Hurwitz zeta function, an alternate proof of the series representation for Hurwitz-Lerch zeta function is obtained. Then I define the gRZ function, which is closely related with the eFD and eBE functions. Some properties satisfied by the gRZ function are discussed here. It approximates the zeros of the Riemann zeta function. It also shows that the FD and BE functions are related with the Riemann zeta function in the critical strip. By making use of the representation theorem, a series representation for the gRZ function is obtained. It led to a new series representation for the eBE and Hurwitz-Lerch zeta functions in Section **(3.3)**. It has removable singularities instead of simple poles at $s \in \mathbb{N}$ and generalizes the series representation obtained in section **(3.1)**. The results of this chapter appeared as [12].

In Chapter **4**, after proving some new identities involving the eFD and eBE functions, Fourier transform representation of these functions is given in Section **(4.2)**. Applications of this representation are given in evaluation of some integrals of products of these functions with gamma function in Section **(4.3)** and **(4.4)**. In Section **(4.5)**, I obtain the distributional representation of the extended functions in terms of the Dirac delta function of complex argument. Some applications of this representation are given in Section **(4.6)**. This verifies the exactness of the results obtained by the Fourier transform representation and yields some new results. To get integrals of the Riemann zeta function in the critical strip, Fourier transform and distributional representations of the gRZ function are given in Section **(4.7)**. The results of this chapter appeared as [66].

In Chapter **5**, I discuss about deriving more integrals involving the zeta-related functions by using the Fourier transform and distributional representations of other special functions. For this purpose I have used the Fourier transform and distributional representations of the generalized gamma functions. This in turn provides integrals of the Macdonald function (modified Bessel functions) with zeta-related functions. These integrals cannot be expressed in a closed form without defining the extensions of the Riemann and Hurwitz zeta functions by inserting a regularizer in their original integral representations. This also explores the importance of the generalizations of special functions. Further, the integrals of product of zeta and gamma functions are expressible as special cases of the eFD, eBE and gRZ functions. These new extensions can prove useful to study the integrals of these well studied functions. It is the first step towards their natural occurrence in mathematics. The results of this chapter appeared as [67].

The last Chapter **6** gives the Conclusion and future directions of the research presented in this thesis.

1.1 Gamma and generalized gamma functions

The problem of giving $s!$ a useful meaning when s is any complex number was solved by Euler (1707-1783), who defined what is now called the *gamma function*,

$$\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt \qquad (s = \sigma + i\tau, \sigma = \Re(s) > 0). \tag{1.1.1}$$

This integral can be written as a sum of two integrals

$$\Gamma(s) := \int_0^1 e^{-t}t^{s-1}dt + \int_1^\infty e^{-t}t^{s-1}dt, \tag{1.1.2}$$

where it can be easily shown that the first integral defines a function, which is analytic in the half plane $\Re(s) > 0$, while the second integral defines an entire function. It has no zeros and can be analytically continued to the whole complex plane excluding the points $s = 0, -1, -2, \cdots$, where it has simple poles. Many special functions can be defined in terms of the gamma function, which makes it simplest and most important special function. Therefore, the knowledge of its properties is a first step towards the study of many other special functions. Some of the relations satisfied by the gamma function are given here:

$$\Gamma(s+1) = s\Gamma(s), \tag{1.1.3}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{1.1.4}$$

$$2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2}) = \sqrt{\pi}\Gamma(2s). \tag{1.1.5}$$

Various transformations and calculations involving the gamma function can be simplified by using these relations. The *digamma* or the *psi function*, denoted by $\psi(s)$, is the logarithmic derivative of the gamma function. It is defined as

$$\psi(s) = \frac{d}{ds}\big(\ln\Gamma(s)\big) = \frac{\Gamma'(s)}{\Gamma(s)}. \tag{1.1.6}$$

The *generalized gamma function* is defined by [14, p. 9, Equation (1.66)]

$$\Gamma_b(s) = \int_0^\infty e^{-t-\frac{b}{t}} t^{s-1} dt \qquad (\Re(b) > 0; b = 0, \Re(s) > 0). \tag{1.1.7}$$

Note that this extension is given by inserting the regularizer $e^{-\frac{b}{t}}$ in the original gamma function. This in turn introduces a new parameter b such that for $\Re(b) > 0$, it extends the domain of the gamma function in the whole complex plane so that it is defined at negative integers as well and for $b = 0$, it coincides with (1.1.1). The integral in (1.1.7) can be written in terms of the Macdonald function to give [14, p. 9, Equation (1.67)]

$$\Gamma_b(s) = 2b^{s/2} K_s(2\sqrt{b}) \qquad (\Re(b) > 0). \tag{1.1.8}$$

Macdonald function appears as a special case of the Bessel functions of imaginary argument.

The gamma function (indeed any special function) have infinitely many generalizations, which could be useful in certain types of problems. An important aspect of the above generalization is that it extends the previous results for the function simply. Apart from that, the results obtained for the extension are no less elegant, or more cumbersome, than those for the original function [14,58,67]. Some particularly elegant examples are [14, p. 10, Equation (1.73)]

$$\Gamma_b(s+1) = s\Gamma_b(s) + b\Gamma_b(s-1) \tag{1.1.9}$$

and reflection formula [14, p. 13, Equation (1.88)]

$$b^s\Gamma_b(-s) = \Gamma_b(s) \qquad (\Re(b) > 0). \tag{1.1.10}$$

Similar to the logarithmic derivative of the Euler gamma function, the logarithmic derivative of the generalized gamma function is defined by [14, p. 23, Equation (1.169)]

$$\psi_b(s) = \frac{d}{ds}\big(\ln\Gamma_b(s)\big) = \frac{1}{\Gamma_b(s)}\frac{d}{ds}\big(\Gamma_b(s)\big). \tag{1.1.11}$$

For $b = 0$, it becomes the digamma function. For the extensive applications of the generalized gamma function to a wide variety of problems, I refer to [14, pp. 357-433].

1.2 The family of zeta functions

The *Riemann zeta function* was originally considered by Euler, in 1737, when he proved the following identity

$$\sum_{n=1}^{\infty} n^{-x} = \prod_{p \text{ prime}} \frac{1}{(1-p^{-x})} \quad (x \in \mathbb{R}). \tag{1.2.1}$$

In 1859, while working on his 8-page paper "On the Number of Primes Less than a Given Magnitude", Bernhard Riemann took the above identity as his starting point [18]. He extended it from real x to complex $s = \sigma + i\tau$ and denoted it by $\zeta(s)$. While both representations

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \qquad (\Re(s) > 1) \tag{1.2.2}$$

and

$$\zeta(s) := \prod_{p \text{ prime}} \frac{1}{(1-p^{-s})} \tag{1.2.3}$$

converge only for $\Re(s) > 1$, Riemann showed that it can be analytically continued in the whole complex plane except for $s = 1$, where it has a simple pole. Note that one expression for zeta function (1.2.3) involves *primes* explicitly while the other (1.2.2) does not, which makes it important in the Number Theory.

To obtain the integral representation for the Riemann zeta function, we consider

$$\frac{\Gamma(s)}{k^s} := \int_0^{\infty} e^{-kt} t^{s-1} dt \qquad (\Re(s) > 0). \tag{1.2.4}$$

Summing over both sides of the above equation gives

$$\sum_{k=0}^{\infty} \frac{\Gamma(s)}{k^s} := \sum_{k=0}^{\infty} \int_0^{\infty} e^{-kt} t^{s-1} dt \qquad (\Re(s) > 1), \tag{1.2.5}$$

which leads to

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \quad (\Re(s) > 1). \tag{1.2.6}$$

Here the change of the order of summation and integration is permissible due to the absolute convergence of (1.2.5) for $\Re(s) > 1$. The above equation leads to the following representation [70, p. 19]

$$\zeta(s) := \frac{1}{\Gamma(s)(e^{2\pi i s} - 1)} I(s) \tag{1.2.7}$$

$$= \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} I(s), \tag{1.2.8}$$

where [70, p. 18]

$$I(s) := \int_L \frac{z^{s-1} dz}{e^z - 1} \tag{1.2.9}$$

is the loop integral. The contour L consists of the real axis from ∞ to ρ $(0 < \rho < 2\pi)$, the circle $|z| = \rho$, and the axis from ρ to ∞. The integral $I(s)$ is uniformly convergent in any finite region of the complex plane. The representation (1.2.8) provides an analytic continuation of $\zeta(s)$ in the whole complex s-plane. It follows from (1.2.7) that $\zeta(s)$ has the only simple pole, due to $\Gamma(1-s)$, at $s = 1$ with residue 1. It can be explained by observing that $I(1) = 2\pi i$ and $I(n) = 0 \quad (n = 2, 3, \ldots)$. Therefore, the poles of the gamma function $\Gamma(1-s)$ at $s = 2, 3, 4, \ldots$ are canceled by the zeros of $I(s)$. The zeta function satisfies the *Riemann functional equation* [70, p. 13]

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2} s\pi\right) \Gamma(1-s)\zeta(1-s) \tag{1.2.10}$$

that can also be written as

$$\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-(\frac{1-s}{2})}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.2.11}$$

It relates $\zeta(s)$ with $\zeta(1-s)$ and gives a best way of providing the analytic continuation of the Riemann zeta function. Some other methods for the analytic continuation of the zeta function are discussed in [70].

Next I discuss some other representations of the Riemann zeta function, which will be used in the rest of chapters. Note that (1.2.6) can be written as

$$\Gamma(s)\zeta(s) = \int_0^1 (\frac{1}{e^t-1} - \frac{1}{t})t^{s-1}dt + \frac{1}{s-1} + \int_1^\infty \frac{t^{s-1}}{e^t-1}dt, \tag{1.2.12}$$

which holds for $\Re(s) > 0$. For $0 < \Re(s) < 1$, one can take

$$\frac{1}{s-1} = -\int_1^\infty \frac{t^{s-1}}{t}dt. \tag{1.2.13}$$

These two equations lead to another representation of the Riemann zeta function given by [70, p. 23]

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\frac{1}{e^t-1} - \frac{1}{t})t^{s-1}dt \quad (0 < \Re(s) < 1). \tag{1.2.14}$$

By multiplying (1.2.2) with the factor $1 - 2^{1-s}$, one can get

$$(1-2^{1-s})\zeta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \qquad (\Re(s) > 0), \tag{1.2.15}$$

which has the following integral representation

$$\zeta(s) := \frac{1}{C(s)} \int_0^\infty \frac{t^{s-1}dt}{e^t+1} \qquad (\Re(s) > 0), \tag{1.2.16}$$

where

$$C(s) := \Gamma(s)(1-2^{1-s}). \tag{1.2.17}$$

Here pole of the zeta function at $s = 1$ is canceled by the zero of $1 - 2^{1-s}$.

The *Riemann Hypothesis* states that "All the non trivial zeros of the Riemann zeta function lie in the critical strip $0 < \Re(s) < 1$ on the critical line $\Re(s) = 1/2$". Here, I discuss some facts in support of this statement. It is known that a convergent infinite product of non-zero factors is not zero. This fact along with (1.2.3) implies that the Riemann zeta function has no zeros for $\Re(s) > 1$. However the Riemann functional equation (1.2.10) shows that it has zeros for $s = -2n, (n = 1, 2, \cdots)$. These are called the trivial zeros of the zeta function. These remarks lead to the observation that all other zeros lie between $0 < \Re(s) < 1$. By determining a bound for these zeros it was shown by Hardy [18] that an infinite number of zeros lie on the critical line $\Re(s) = 1/2$ in the critical strip. There are many other facts in favour of this statement but its analytic proof is still unknown [16, 18, 46].

Due to the importance of the Riemann zeta function in *Number Theory* it is also related to some other functions of this field, for example, the Dirichlet $\eta(.)$ and $\Lambda(.)$ functions. The *Dirichlet eta function* is defined by [8, p. 17]

$$\eta(x) := \sum_{n=1}^{\infty}(-1)^{n-1}n^{-x} = \frac{1}{\Gamma(x)}\int_0^{\infty}\frac{t^{x-1}dt}{e^t+1} \qquad (x > 0), \tag{1.2.18}$$

which is also known as the *alternating zeta function*. The *Dirichlet lambda function* is the *Dirichlet L-series* defined by [8, p. 17]

$$\Lambda(x) = \sum_{n=1}^{\infty}\frac{1}{(2n+1)^x} = \frac{1}{\Gamma(x)}\int_0^{\infty}\frac{t^{x-1}dt}{e^t-e^{-t}} \qquad (x > 0). \tag{1.2.19}$$

These are related to zeta function by

$$\eta(x) = (1-2^{1-x})\zeta(x), \quad \Lambda(x) = (1-2^{-x})\zeta(x) \tag{1.2.20}$$

and satisfy the following identity

$$\zeta(x) + \eta(x) = 2\Lambda(x). \tag{1.2.21}$$

The *Dirichlet beta function* (also known as Dirichlet L-function) is defined by [24]

$$\beta(x) := \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^x} \qquad (x > 0), \tag{1.2.22}$$

where $\beta(2) = G$, Catalan's constant. It can be observed from the above equation that, $\beta(x)$, is the alternating version of $\Lambda(x)$, however it can not be directly related to $\zeta(x)$. It is also related to $\eta(x)$ in that the only odd terms are assumed. The Dirichlet beta function satisfies the functional equation

$$\beta(1-s) = (\frac{2}{\pi})^s \sin\left(\frac{1}{2}s\pi\right)\Gamma(s)\beta(s). \tag{1.2.23}$$

It is defined everywhere in the complex plane and has no singularities. Its integral representation is given by [24, p. 56]

$$\beta(x) = \frac{1}{2\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{\cosh t} dt \qquad (x > 0), \tag{1.2.24}$$

which can be rewritten as

$$\beta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}} dt \qquad (x > 0). \tag{1.2.25}$$

It has another representation [24]

$$\beta(x) = \prod_{p \text{ odd prime}} (1 - (-1)^{\frac{p-1}{2}} p^{-x})^{-1}, \tag{1.2.26}$$

which connects it with Number Theory.

There have been several generalizations of the Riemann zeta function, like the *Hurwitz zeta function* defined by [64, Chapter 2]

$$\zeta(s, a) = \sum_{n=0}^\infty \frac{1}{(n+a)^s} \quad (\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-). \tag{1.2.27}$$

It has the following integral representation

$$\zeta(s, a) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(a-1)t} t^{s-1}}{e^t - 1} dt \qquad (\Re(a) > 0; \Re(s) > 1) \tag{1.2.28}$$

and it is related to Riemann zeta function by

$$\zeta(s) = \zeta(s, 1) = \frac{1}{2^s - 1} \zeta(s, \frac{1}{2}) = 1 + \zeta(s, 2). \tag{1.2.29}$$

Hurwitz zeta function has another series representation [64, p. 144, Equation 3.2 (7)]

$$\sum_{n=0}^\infty \frac{(\lambda)_n}{n!} \zeta(\lambda + n, a) t^n = \zeta(\lambda, a - t) \quad (| t |<| a |), \tag{1.2.30}$$

which holds true by the principal of analytic continuation, for all values of $\lambda \neq 1$.

Another important generalization of the Riemann zeta function is the *polylogarithm function*

$$\mathrm{Li}_s(x) := \sum_{n=1}^\infty x^n n^{-s}, \tag{1.2.31}$$

which occurs in the theory of the structure of polymers [71]. The series (1.2.31) defines a function of x analytic in the region $|x| \leq 1-\delta$, for all s and each $0 < \delta < 1$. At the point $x = 1$, it converges for $\Re(s) > 1$ and

$$\phi(1, s) = \zeta(s). \tag{1.2.32}$$

The integral representation for the polylogarithm function is given by

$$\mathrm{Li}_s(x) = \frac{x}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - x} dt. \tag{1.2.33}$$

If x lies anywhere except on the segment of the real axis from 1 to ∞, where a cut is imposed then (1.2.33) defines an analytic function of x provided $\Re(s) > 0$. For $x = 1$ and $\Re(s) > 1$, one obtains the special case

$$\phi(1, s) = \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt. \tag{1.2.34}$$

The polylogarithm function can also be represented by the contour integral

$$\mathrm{Li}_s(x) = -\frac{x\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{(-z)^{s-1}}{e^z - x} dz, \tag{1.2.35}$$

where the contour $(\infty, 0^+)$ encircles the origin once positively and extends to infinity on either side of the cut in the z-plane from 1 to ∞ along the real axis in such a manner as to include no further singularity of the integrand. It has the series representation [71, p. 149 (13)]

$$\mathrm{Li}_s(x) := \Gamma(1-s)(-\log x)^{s-1} + \sum_{n=1}^{\infty} \zeta(s-n) \frac{(\log x)^n}{n!}, \tag{1.2.36}$$

which is also known as Lindelöf representation. The following relation

$$\mathrm{Li}_s(x) + e^{is\pi}\mathrm{Li}_s(1/x) = \frac{(2\pi)^s}{\Gamma(s)} e^{(i\pi s)/2} \zeta\left(1-s, \frac{\log x}{2\pi i}\right) \tag{1.2.37}$$

is useful to simplify various results related to polylogarithm function. Other notations used for the function $\mathrm{Li}_s(x)$ are $F(x, s)$ and $\phi(x, s)$.

A *general Hurwitz-Lerch zeta function* $\Phi(z,s,a)$ defined by [21, p. 27 (1.11.1)]

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$
$$(a \neq 0,-1,-2,\ldots; s \in \mathbb{C} \text{ when } |z|<1; \Re(s)>1 \text{ when } |z|=1) \tag{1.2.38}$$

is related with the polylogarithm function as

$$\mathrm{Li}_s(x) := x\Phi(x,s,1). \tag{1.2.39}$$

It is also related not only with the Riemann and Hurwitz zeta functions

$$\zeta(s) = \Phi(1,s,1), \tag{1.2.40}$$

$$\zeta(s,a) = \Phi(1,s,a), \tag{1.2.41}$$

but with the Lerch zeta function:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\Phi(e^{2\pi i\xi},s,1) \quad (\xi \in \mathbb{R}; \Re(s)>1), \tag{1.2.42}$$

and the Lipschitz-Lerch zeta function defined by [64, p.122]

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi(e^{2\pi i\xi},s,a)$$
$$(a \neq 0,-1,-2,\ldots; \Re(s)>0 \text{ when } \xi \in \mathbb{R}\setminus\mathbb{Z}; \Re(s)>1 \text{ when } \xi \in \mathbb{Z}). \tag{1.2.43}$$

Using the elementary series identity

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q} \left(\sum_{k=0}^{\infty} f(qk+j) \right) \qquad (q \in \mathbb{N}), \tag{1.2.44}$$

it was shown [64, Equation (3.8)] that

$$\Phi(z,s,a) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{a+j-1}{q}\right) z^{j-1}. \tag{1.2.45}$$

For $z=1$, it reduces to the identity [64, Equation (3.10)]

$$\zeta(s,a) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right). \tag{1.2.46}$$

The general Hurwitz-Lerch zeta function has the integral representation [21, p.27, (1.113)]

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-(a-1)t}}{e^t-z}dt$$
$$(\Re(a)>0;\ \text{or } |z|\leq 1, z\neq 1, \Re(s)>0,\ \text{or } z=1, \Re(s)>1). \tag{1.2.47}$$

For further properties of the Riemann zeta and related functions I refer to [1,2,4,13, 18,42,68–71].

Apart from these generalizations of the Riemann zeta function, the extended Riemann and Hurwitz zeta functions are defined in [14, Chapter 7] by introducing a regularizer $e^{-b/t}$ in integral representations (1.2.6), (1.2.16) and (1.2.28).

The *extended Riemann zeta function* is defined by [14, p. 298, Equation (7.78)]

$$\zeta_b(s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{-t-\frac{b}{t}}}{1-e^{-t}}t^{s-1}dt\quad(\Re(b)>0; b=0, \Re(s)>1). \tag{1.2.48}$$

All the properties of the Riemann zeta function can be obtained as a special case of the properties of the extended Riemann zeta function. But it fails to extend these properties in the critical strip. For this purpose, the following extension was defined by [14, p. 305, Equation (7.116)]

$$\zeta_b^*(s)=\frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty\frac{e^{-t-\frac{b}{t}}}{1+e^{-t}}t^{s-1}dt\quad(\Re(b)>0; b=0, \Re(s)>0). \tag{1.2.49}$$

These two extensions are related by

$$\zeta_b^*(s)(1-2^{1-s})=\zeta_b(s)-2^{1-s}\zeta_{2b}(s). \tag{1.2.50}$$

Similar *extensions* for the Hurwitz zeta function are defined by [14, p. 308, Equation (7.140)]

$$\zeta_b(s,\nu)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{-\nu t-b/t}}{1-e^{-t}}t^{s-1}dt\quad(0<\nu\leq 1; \Re(b)>0; b=0, \Re(s)>1). \tag{1.2.51}$$

$$\zeta_b^*(s,\nu)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{-\nu t-b/t}}{1+e^{-t}}t^{s-1}dt\quad(0<\nu\leq 1; \Re(b)>0; b=0, \Re(s)>0). \tag{1.2.52}$$

Note that, to prove several results the parameter ν is restricted between 0 and 1 but the integrals (1.2.51-1.2.52) are convergent for $\nu>1$.

1.3 Fermi-Dirac, Bose-Einstein, Anyon Functions and Quantum Entanglement

Quantum mechanics deals with the particles [23, 59] that are represented by wave functions and satisfy the Schrodinger equation. One shift over to quantum field theory [56, 73] can make the theory relativistic. Klein-Gordon equation, which is achieved by using the Dirac quantization procedure has problems of interpretation because it has a second derivative with respect to the time appearing in it. To overcome these problems a first order equation in space and time variables was obtained by Dirac. The Dirac and the Klein-Gordon equations represent particles with spin and without it respectively. The spin go up in half-integer multiples of the quantity, $\hbar$, defined to be $\frac{h}{2\pi}$, where h is Planck's constant. The wave function is symmetric for identical particles with integer spin and is anti-symmetric for half-integer spin under the exchange of particles. Due to this property, particles with half-integer spin and integer spin can not have the same quantum numbers. The former are said to be subject to the Pauli exclusion principle.

The ensembles of classical particles can be dealt by using the Maxwell distribution:

$$f(x) = e^{\frac{x-\mu}{t}}, \tag{1.3.1}$$

where x denotes the kinetic energy of the given particle and μ for the chemical potential. Thermal energy relates with the temperature denoted by t and f gives the probability of the particle having the given kinetic energy. This distribution can be used as an approximation for a mixture of the two types and did not apply precisely to the quantum particles. It gave a good approximation for the ensemble at high temperatures. While for lower temperatures of systems of particles some other types of distributions were required. For the integer spin particles BE distribution

$$f_{\mathfrak{B}}(x) = \frac{1}{e^{\frac{x-\mu}{t}} - 1} \tag{1.3.2}$$

was proposed by Bose. While for the particles with half-integer spin the FD distribution

$$f_{\mathfrak{F}}(x) = \frac{1}{e^{\frac{x-\mu}{t}} + 1} \tag{1.3.3}$$

was proposed by Fermi. The particles of former type are called bosons and of the latter type fermions. Their respective cumulative probabilities are given by the BE and FD functions. The FD function, $\mathfrak{F}_q(x)$, defined by [17, p. 20, (25)]

$$\mathfrak{F}_q(x) := \frac{1}{\Gamma(q+1)} \int_0^\infty \frac{t^q}{e^{t-x}+1} dt \quad (q > -1; x \geq 0), \tag{1.3.4}$$

and the BE function, $\mathfrak{B}_q(x)$, defined by [17, p. 449, (9)]

$$\mathfrak{B}_q(x) := \frac{1}{\Gamma(q+1)} \int_0^\infty \frac{t^q}{e^{t-x}-1} dt \quad (q > 0; x \geq 0). \tag{1.3.5}$$

These are related with the Riemann zeta function as follows

$$\mathfrak{F}_q(0) = (1 - 2^{-q})\Gamma(q+1)\zeta(q+1) \quad (q > -1), \tag{1.3.6}$$

$$\mathfrak{B}_q(0) = \zeta(q+1) \quad (q > 0). \tag{1.3.7}$$

It appeared that all elementary particles would belong to one of these classes (fermions or bosons). More recently, [3, 45], it was found that under certain conditions, a fermion, can behave as if it is made of more fundamental particles having a fractional spin. These particles are called anyons which interpolate between bosons and fermions. Such particles can only exits in two dimensions. For their non-existence in higher dimensions, two types of reasons (topological, group theoretic) can be given. Since, the wave function of bosons is symmetric under the exchange of two particles and the wave function of the fermions is antisymmetric. Also in three and higher dimensions the symmetry group is the permutation group, which has only two one dimensional representations [10]. The non-trivial representation gives fermions while the trivial representation corresponds to the bosons. The situation is more interesting in two dimensions, where the symmetry group is larger than the permutation group and has a one dimensional representation for every real number ν. It is called a braid group[1] and under the exchange of two particles, a two particle wave function behaves in the following way:

$$\psi(b,a) = \begin{cases} (-1)\psi(a,b), \text{Fermions}, \\ (+1)\psi(a,b), \text{Bosons}, \\ (-1)^{\nu}\psi(a,b), \text{Anyons}. \end{cases}$$

There are functional representations (1.3.2-1.3.5) for bosons and fermions but there is no functional representation for anyons available in the literature. As mentioned earlier, the eFD and eBE functions had been put forward as possible candidates for the anyon function as they interpolate very naturally between the BE and FD functions [10].

In 1935 A. Einstein, B. Podolsky and N. Rosen (EPR) published a famous paper [19] in which they question the completeness of quantum mechanics. They did not question the usefulness of quantum mechanics as a powerful theory but they conclude from their Gedankenexperiment that quantum mechanics is not a complete theory and that there has to be a more fundamental description of the physical reality. EPR use in their analysis a state of two particles I and II that are known at a certain time. It is assumed that particle I and II interact during the time interval $t = [0, T]$. The resulting state created in this interaction is described by the wave packet Ψ. After $t = T$ the two systems do not interact. Starting from this situation EPR make two different reductions of the wave packet Ψ. This means that two different quantities are measured on system I. The two systems (I and II) are so far apart that no interaction

[1] Braid group, B_N, is generated by transpositions $\phi_1, ..., \phi_N$ such that $\phi_i\phi_j = \phi_j\phi_i$ for $|i-j| > 1$ and $\phi_i\phi_{i+1}\phi_i = \phi_{i+1}\phi_i\phi_{i+1}$. The permutation group can be obtained by further imposing the condition $\phi_i^2 = 1$. The possibility of anyons stems from the absence of this last relation in the braid group.

can happen. Nevertheless, the second system (II) is left in states with two different wave functions. EPR conclude therefore that it is possible to assign two different wave functions to the same reality [19]. Subsequently they show that if two operators corresponding to two different physical quantities do not commute, the simultaneous knowledge of both quantities is not possible. From this they conclude that it is indeed possible to assign two different wave functions to the same reality. But since either one or the other, but not both of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way [19]. Finally they conclude that the wave function can not provide a complete description of physical reality. The question of the completeness of quantum mechanics has been of great interests for many decades. It led to the existence of the *quantum mechanical nonlocality* (due to entanglement), which has deep impact on current research topics such as quantum cryptography and quantum computing. These are based on the existence of entanglement.

Quantum cryptography has abstracted strong interests of scientists since it makes it possible to set up unconditional secure key between two remote parties by principles of quantum mechanics. A central problem in cryptography is to establish the existence of one-way function. A one-way function is a function F such that for each x in the domain of F, it is easy to computer $F(x)$; but for essentially all y in the range of F, it is an intractable problem to find an x such that $y = F(x)$. One approach to this problem is to construct candidate one-way functions from seemingly intractable problems in Number Theory. Anshel, and Goldfeld [29] have introduced a new intractable problem arising from the theory of Zeta functions, which leads to a new class of one-way functions based on the arithmetic theory of Zeta functions. For further relevant topics see, [6, 28, 51, 52].

1.4 Bernoulli and Euler polynomials

The classical *Bernoulli polynomials* $B_n(x)$ of degree n are defined by the generating function [61, p. 77, Equation (1.1)]

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\qquad(|t|<2\pi)\tag{1.4.1}$$

and for $x=0$, these polynomials give *Bernoulli numbers*

$$B_n:=B_n(0)=(-1)^nB_n(1)=\frac{1}{2^{1-n}-1}B_n(\frac{1}{2})\quad(n\in\mathbb{N}_0).\tag{1.4.2}$$

The *Euler polynomials* $E_n(x)$ of degree n are defined by [61, p. 77, Equation (1.1)]

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\qquad(|t|<\pi),\tag{1.4.3}$$

which for $x=0$ give *Euler numbers*

$$E_n:=E_n(0)=2^nE_n(\frac{1}{2})\quad(n\in\mathbb{N}_0.\tag{1.4.4}$$

The polynomials in (1.4.1) and (1.4.3) are related by

$$E_n(x)=\frac{2}{n+1}\left[B_{n+1}(x)-2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right]\qquad(n\in\mathbb{N}_0).\tag{1.4.5}$$

A general expression for even and odd Bernoulli numbers is

$$B_{2n}=2(-1)^{n+1}(2n)!\sum_{r=1}^{\infty}(2\pi r)^{-2n},\qquad B_{2n+1}=0.\tag{1.4.6}$$

The Bernoulli and Euler numbers are closely related to the zeta family. For example, [61, (2.6)]

$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1}\qquad(n\in\mathbb{N}_0)\tag{1.4.7}$$

and [61, (2.7)]

$$\zeta(2n)=\frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}\qquad(n\in\mathbb{N}_0).\tag{1.4.8}$$

The Dirichlet beta function can be expressed in terms of Euler numbers

$$\beta(2n+1) = (-1)^n \frac{E_{2n}}{2(2n)!} (\frac{\pi}{2})^{2n+1} \tag{1.4.9}$$

and polylogarithm function is expressed in terms of Bernoulli polynomials

$$B_n(x) = -\frac{n!}{(2\pi i)^n} [(-1)^n \mathrm{Li}_n(e^{-2\pi i x}) + \mathrm{Li}_n(e^{2\pi i x})]. \tag{1.4.10}$$

There are several other relations of the Bernoulli and Euler polynomials and functions of zeta family (for details see, [14, 21, 64]).

1.5 The Mellin, Fourier and Weyl integral transforms

Integral transformations have been used in the study of various problems in applied mathematics, physics and engineering. For example the Fourier transform is used as a basic tool in such problems. Our interest here is to study Mellin, Fourier and Weyl transforms due to their applications in the next chapters.

Following the terminology [48, p. 237] (see also [53, pp. 237-238]), classes of *good functions* $\mathcal{H}(\kappa;\lambda)$ and $\mathcal{H}(\infty;\lambda)$ are defined as follows:

A function $f \in C^\infty(0,\infty)$ is said to be a member of $\mathcal{H}(\kappa;\lambda)$ if:

1. $f(t)$ is integrable on every finite subinterval $[0,T]$ $(0 < T < \infty)$ of $\mathbb{R}_0^+ := [0,\infty)$;
2. $f(t) = O(t^{-\lambda}) \quad (t \to 0^+)$;
3. $f(t) = O(t^{-\kappa})$ $(t \to \infty)$.

Furthermore, if the above relation $f(t) = O(t^{-\kappa})$ $(t \to \infty)$ is satisfied for every exponent $\kappa \in \mathbb{R}_0^+$, then the function $f(t)$ is said to be in the class $\mathcal{H}(\infty;\lambda)$. Clearly,

$$f(t) = e^{-bt} \in \mathcal{H}(\infty;0) \quad (b > 0) \tag{1.5.1}$$

and $\mathcal{H}(\infty;\lambda) \subset \mathcal{H}(\kappa;\lambda), \quad \forall \kappa \in \mathbb{R}_0^+$.

The *Mellin transform* of $f \in \mathcal{H}(\kappa;\lambda)$ is defined as follows (see [20, Vol I, Chapter 6] and [53, p. 79 et seq.])

$$F_{\mathcal{M}}(s) = \mathcal{M}[f(t);s] := \int_0^\infty f(t)t^{s-1}dt \quad (\lambda < \Re(s) < \kappa) \tag{1.5.2}$$

and the *inverse Mellin transform* is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{\mathcal{M}}(s)t^{-s}ds \quad (\lambda < c < k). \tag{1.5.3}$$

The Mellin transform and its inversion formula was first appeared in *Riemann famous Memoir* on prime numbers [18]. For example, a combination of (1.2.6) with (1.5.2) yields

$$\Gamma(s)\zeta(s) = \mathcal{M}\big[\frac{1}{e^t-1};s\big] \tag{1.5.4}$$

and by inverting it we get

$$\frac{1}{e^t-1} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)t^{-s}ds. \tag{1.5.5}$$

Further it is a linear transformation and following properties are useful for the development of theory and applications of the Mellin transform [75]:

1. **Scaling:**

$$\mathcal{M}[f(t/a);s] = a^sF(s) \quad (a > 0), \tag{1.5.6}$$

and

$$\mathcal{M}[f(t^a);s] = \frac{1}{a}F(\frac{s}{a}) \quad (a > 0). \tag{1.5.7}$$

2. **Translation:**

$$\mathcal{M}[t^af(t);s] = F(s+a) \quad (\lambda < \Re(s) + a < \kappa) \tag{1.5.8}$$

and

$$\mathcal{M}[t^{-1}f(t^{-1});s] = F(1-s) \quad (\lambda < 1 - \Re(s) < \kappa). \tag{1.5.9}$$

3. **Differentiation:** If $f \in \mathcal{H}(\infty; \lambda)$ is a differentiable function then

$$\mathcal{M}[f'(t); s] = -(s-1)\mathcal{M}[f(t); s-1] \quad (\lambda < \Re(s) - 1 < \kappa). \tag{1.5.10}$$

Similar but more general results for the derivatives of the Mellin transform also hold.

4. **Integration:**

$$\mathcal{M}\Big[\int_0^x f(t)dt; s\Big] = -\frac{1}{s}\mathcal{M}[f(t); (s+1)] \tag{1.5.11}$$

and

$$\mathcal{M}\Big[\int_x^\infty f(t)dt; s\Big] = -\frac{1}{s}\mathcal{M}[f(t); (s+1)] \tag{1.5.12}$$

The *Fourier integral transform* is defined by [75, Chapter 12]

$$\mathcal{F}[\varphi; \tau] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iy\tau} \varphi(y) dy \quad (\tau \in \mathbb{R}), \tag{1.5.13}$$

whenever the integral exists. It can be obtained by using the transformation $t = e^y$ in the integral representation (1.5.2). It satisfies interesting properties, which can also be obtained by using the properties of Mellin transform. In this thesis, I will use the Parseval's identity and duality property of Fourier transform, which are stated below.

Let f, g be two arbitrary Fourier transformable functions then the *Parseval's identity* [75, p.232] of Fourier transform states that

$$\int_{-\infty}^{+\infty} \mathcal{F}[f(y); \tau]\overline{\mathcal{F}[g(y); \tau]} d\tau = \int_{-\infty}^{+\infty} f(y)\overline{g(y)} dy. \tag{1.5.14}$$

The *duality property* of Fourier transform [15, p. 29] states that

$$\mathcal{F}[\mathcal{F}[\varphi(y); \tau]; \omega] = \varphi(-\omega), \quad \omega \in \mathbb{R}. \tag{1.5.15}$$

For further properties of the Fourier transform I refer to [15, 33, 75–77].

Another integral transformation, which is closely-related to the Mellin transform is known as Weyl transform (or Weyl's fractional integral). The *Weyl transform* of order s of $\omega \in \mathcal{H}(\kappa;0)$ is defined by (see [20, Vol II, Chapter 13], [53, p. 236 et seq.] and [44]),

$$\begin{aligned}\Omega(s;x) &:= \mathcal{W}^{-s}[\omega(t)](x) := \tfrac{1}{\Gamma(s)}\mathcal{M}[\omega(t+x);s] = \tfrac{1}{\Gamma(s)}\int\limits_0^\infty \omega(t+x)t^{s-1}dt \\ &= \tfrac{1}{\Gamma(s)}\int\limits_x^\infty \omega(t)(t-x)^{s-1}dt \quad (0<\Re(s)<\kappa; x\geq 0)\end{aligned} \tag{1.5.16}$$

It is a linear transform and for $\Re(s) \leq 0$, Weyl's fractional derivative of order s of $\omega \in \mathcal{H}(\kappa;0)$ is defined as follows (see [20, Vol II, p. 181] and [53, p. 241, Equation(4.6)]):

$$\Omega(s;x) := \mathcal{W}^{-s}[\omega(t)](x) := (-1)^n \frac{d^n}{dx^n}(\Omega(n+s;x)) \quad (0 \leq n+\Re(s) < k), \tag{1.5.17}$$

where n is the smallest positive integer greater than or equal to $-\Re(s)$ provided that $\omega(0)$ is well defined and that

$$\Omega(0;x) := \omega(x). \tag{1.5.18}$$

We can rewrite Weyl's fractional derivative (1.5.17) alternately as

$$\Omega(s-n;x) := (-1)^n \left[\tfrac{d^n}{dx^n}(\Omega(s;x))\right]_{x=0}, \tag{1.5.19}$$

where n is the smallest positive integer greater than or equal to $\Re(s)$. In particular for $s=n$ $(n=0,1,2,3,...)$, one can observe that

$$\Omega(-n;x) := \mathcal{W}^n[\omega(t)](x) := (-1)^n \frac{d^n}{dx^n}(\Omega(0;x)) = (-1)^n \frac{d^n}{dx^n}(\omega(x)). \tag{1.5.20}$$

Notice that $\{\mathcal{W}^s\}$ $(s \in \mathrm{C})$ is a multiplicative group [53, p. 245] and satisfies

$$\mathcal{W}^{-(\mu+s)}[\omega(t)](x) = \mathcal{W}^{-\mu}[\Omega(s;t)](x) = \Omega(s+\mu;x). \tag{1.5.21}$$

The notations $\Re_s\{f(t);x\}$ and $\mathcal{W}^s_{x+}[f(t)]$ are also used to represent the Weyl transform [20, Vol. II, p. 181] and [63]. Since we have $\mathcal{W}^{-(s+\beta)} = \mathcal{W}^{-s}\mathcal{W}^{-\beta}$ [53, p. 242 (4.10], this shows that

$$\begin{aligned}\Omega(s+\beta;x) &:= \mathcal{W}^{-s}[\Omega(\beta;t)](x) := \frac{1}{\Gamma(s)}\int_0^\infty \Omega(\beta;t+x)t^{s-1}dt \\ &= \frac{1}{\Gamma(s)}\int_x^\infty \Omega(\beta;t)(t-x)^{s-1}dt \quad (\min(\Re(s),\Re(\beta))>0).\end{aligned} \tag{1.5.22}$$

Moreover, it follows from (1.5.20) that

$$\Omega(0; x+t) = \Omega(0; x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Omega(-n; x) t^n \quad (t \geq 0, x \geq 0). \tag{1.5.23}$$

In addition to these, the following interesting properties also hold true for Weyl fractional integral transform.

1. **Translation:**

$$\mathcal{W}^{-s}[T_a \omega(t)](x) = T_a[\mathcal{W}^{-s}\omega(t)](x) \quad (a > 0). \tag{1.5.24}$$

2. **Dilation:**

$$\mathcal{W}^{-s}[D_{\frac{1}{a}} \omega(t)](x) = a^{-s} D_{\frac{1}{a}}[\mathcal{W}^{-s}\omega(t)](x) \quad (a > 0). \tag{1.5.25}$$

3. **Differentiation:**

$$\mathcal{W}^{-s}[\omega^{'}(t)](x) = [\mathcal{W}^{-(s-1)}\omega(t)](x). \tag{1.5.26}$$

4. **Integration:**

$$\mathcal{W}^{-s}\Big[\int_{-\infty}^{t} \omega(u) du\Big] = [\mathcal{W}^{-(s+1)}\omega(t)](x). \tag{1.5.27}$$

For further properties of the fractional derivatives and Weyl's transform, I refer to [44].

1.6 Distributions and test functions

As already mentioned *generalized functions* (or distributions) have played an important role to extend the theory of Fourier and other integral transforms. These are *continuous linear functionals acting on some space of test functions.* In other words the space of distributions is the dual of some space of test functions. In this section, I will discuss some spaces of test functions and distributions, which are necessary in understanding the concepts used in the sequel.

For any function $\phi : \mathbb{R} \to \mathbb{C}$, the set

$$\text{supp } \phi = \overline{\{t \in \mathbb{R} : \phi(t) \neq 0\}}$$

is called the *support* of ϕ.

The space of test functions is the space of all complex valued functions $\phi(t)$, which are C^∞ (infinitely differentiable) and has compact support, for example

$$\phi(t) = \begin{cases} \exp \frac{1}{t^2-1} & (|t| < 1) \\ 0 & (|t| \geq 1) \end{cases}.$$

The space of such test functions is denoted by $\mathcal{D}$, see [76, Chapter 1]. Every complex valued function, $f(t)$, that is continuous for all t and has compact support, can be approximated uniformly by some test function. That is, given an $\epsilon > 0$ there will exist a $\phi(t) \in \mathcal{D}$ such that

$$|f(t) - \phi(t)| \leq \epsilon \quad \forall t. \tag{1.6.1}$$

The space of all continuous linear functionals acting on the space $\mathcal{D}$ is called its dual space denoted by $\mathcal{D}'$. Distributions can be generated by using the following method. Let $f(t)$ be a locally integrable function then corresponding to it, one can define a distribution f through the convergent integral

$$< f, \phi >=< f(t), \phi(t) >:= \int_{-\infty}^{+\infty} f(t)\phi(t)dt \quad \forall \phi \in \mathcal{D}. \tag{1.6.2}$$

It is easy to show that it is a continuous linear functional by using the definition of ϕ and the details are omitted here, [76].

Distributions that can be generated in this way (from locally integrable functions) are called the *regular distributions*. The study of distributions is important because they not only contain representations of locally integrable functions but also include many other entities that are not regular distributions. Therefore many operations like limits, integration and differentiation, which were originally defined for functions, can be extended to these new entities.

All distributions that are not regular are called *singular distributions*, for example, the Dirac delta function is a singular distribution. It is defined by

$$< \delta(t-c), \phi(t) >= \phi(c) \quad (\forall \phi \in \mathcal{D}, c \in \mathbb{R}). \tag{1.6.3}$$

Some other properties of Dirac delta function include

$$\delta(-t) = \delta(t), \tag{1.6.4}$$

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t) \quad (\alpha \neq 0). \tag{1.6.5}$$

The Fourier transform of an arbitrary distribution in $\mathcal{D}'$ is not, in general, a distribution but is instead another kind of continuous linear functional which is defined over a new space of test functions. Such a functional is called an *ultradistribution*, for example delta functional of complex argument is an ultradistribution [76]. Therefore, I first discuss the space of test functions on which such distributions are defined.

The space of test functions denoted by $\mathcal{Z}$ consists of all those entire functions whose Fourier transforms are the elements of $\mathcal{D}$, [76, Section (7.6)]. Since ϕ is an entire function, it cannot be zero on any interval $a < t < b$ except when it is zero everywhere. Thus the spaces $\mathcal{D}$ and $\mathcal{Z}$ do not intersect except in the identically zero testing function. The corresponding dual to the testing function space $\mathcal{Z}$ is $\mathcal{Z}'$ whose elements are the Fourier transforms of the elements of $\mathcal{D}'$. Neither $\mathcal{D}$ nor $\mathcal{D}'$ are subspaces of $\mathcal{Z}$ or $\mathcal{Z}'$ and vice virsa. However, the following inclusion holds

$$\mathcal{Z} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{Z}' \tag{1.6.6}$$

and

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{D}'. \tag{1.6.7}$$

Here $\mathcal{S}$ is the space of testing functions of rapid descent satisfying

$$\lim_{t\to\infty} \mid t \mid^{-N} \phi(t) = 0 \quad \text{for some integer } N \tag{1.6.8}$$

and $\mathcal{S}^{'}$ is the space of distributions of slow growth satisfying [76, Chapter 4]

$$| t |^m \phi^k(t) \leq C_{mk} \quad \text{for each pair of nonnegative integers } m, k; -\infty < t < \infty. \tag{1.6.9}$$

The space $\mathcal{S}^{'}$ is also called the space of *tempered distributions*. From these spaces of test functions and distributions only space $\mathcal{S}$ and its dual are closed under Fourier transform.

An important example of Ultradistributions is [76, p. 204, Equation 7]

$$\mathcal{F}[e^{\alpha t}; \omega] = 2\pi \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \delta^n(\omega) = 2\pi\delta(\omega + i\alpha), \tag{1.6.10}$$

where α is a complex number. Therefore, delta function of complex argument is defined on the testing function space $\mathcal{Z}$.

In contrast to the generalized function spaces which are dual to testing function spaces consisting of smooth complex valued functions of a real variable, the space $\mathcal{G}^{'}$ is the dual of the testing function space $\mathcal{G}$ which consists of entire functions with certain growth conditions. The elements of the space $\mathcal{G}$ are analytic and decrease exponentially. Here some related concepts and results are stated, which will be needed in the sequel. For details and proofs, I refer to [34–39].

The space $\mathcal{G}$ consists of all entire functions $\phi(u)$ of complex variable u such that for every $\alpha > 0$

$$\| \phi \|_\alpha = \sup_{u \in \mathcal{B}_\alpha} e^{\alpha |\Re(u)|} |\phi(u)| < \infty$$

$$\text{where } \mathcal{B}_\alpha = \{x + iy, u \in \mathbb{C} :| y | \leq \alpha\}. \tag{1.6.11}$$

The space $\mathcal{G}$ can be given a Frechét topology with the multi-norm $\{\| \phi \|_\alpha\}_{\alpha \geq 0}$ i.e.,

$$\phi_n \to \phi \text{ in } \mathcal{G} \Leftrightarrow \| \phi_n - \phi \|_\alpha \to 0 \text{ as } n \to \infty \ \forall \ \alpha > 0.$$

The basic analysis on the dual space $\mathcal{G}^{'}$ is given in [36]. For each $\alpha, \beta \in \mathbb{C}$ and $f, g \in \mathcal{G}^{'}$, the linear combination $\alpha f + \beta g$ is the element of $\mathcal{G}^{'}$ defined by

$$< \alpha f + \beta g, \phi > = \alpha < f, \phi > + \beta < g, \phi > . \tag{1.6.12}$$

The weak* topology on $\mathcal{G}'$ will be assumed. Under this topology a sequence f_n converges in $\mathcal{G}'$ to $f \in \mathcal{G}'$, if and only if $< f_n, \phi >$ converges (in the ordinary sense of the convergence of numbers) to $< f, \phi >$ for every $\phi \in \mathcal{G}$.

Theorem 1.6.1 . *For each $\phi \in \mathcal{G}'$ there are finite positive constants C and α such that $| < f, \phi > | \leq C\|\phi\|_\alpha$ for all $\phi \in \mathcal{G}$.*

For each $u \in \mathbb{C}$, $\delta(t-u)$ will denote the corresponding "delta functional" on $\mathcal{G}$ defined by

$$\big\langle \phi(t), \delta(t-u) \big\rangle = \phi(u) \qquad \forall \phi \in \mathcal{G}. \tag{1.6.13}$$

It can be observed that the Fourier transform theory is applicable to all of $\mathcal{G}'$ and the classical theory of the Weierstrass transform, convolution transform with kernel $\frac{\text{sechz}}{2}$ is also extended to this space (for details see [43] and references therein). For further properties of distributions and delta functions, I refer the interested reader to [33, Section IV] and [65].

Chapter 2

The extended Fermi-Dirac and Bose-Einstein functions

In this chapter, I present the extensions of the FD and BE functions by introducing an extra parameter in a way that gives new insights into these functions and their relationship to the family of zeta functions. These functions satisfy interesting mathematical properties, which are also discussed here.

This chapter consists of 4 sections. A general representation theorem for a class of "good functions" is proved in Section (2.1). In the next two sections, the eFD, $\Theta_\nu(s;x)$, and the eBE, $\Psi_\nu(s;x)$, functions are defined. It is shown that these extensions have meromorphic continuation to the whole complex s-plane and the real variable x can be extended to a complex z. This fact leads to a mathematical proof of the duality property of these functions, which is explained in Section (2.4). By using the general representation theorem, a series representation for the eFD function is obtained in Section (2.2). Some properties and functional relationships satisfied by these extended functions are discussed in Section (2.4). It led to various new results for the FD and BE functions. It is shown that the BE function can be expressed as a linear combination of the eBE functions. The relation of the FD function with

Bernoulli polynomials is also given in this chapter.

2.1 A representation theorem for a class of functions

Using the class of "good functions" defined in Section (1.5), I now proceed to establish a representation theorem.

Theorem 2.1.1. *Let* $\omega \in \mathcal{H}(b;0)$ *and* $\Omega(s;x)(x \geq 0)$, *be its Weyl transform. Then*

$$\Omega(s;x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Omega(s-n;0) x^n}{n!} \qquad (0 \leq \Re(s) < b; x \geq 0). \tag{2.1.1}$$

Proof. By the familiar Taylor-Maclaurin expansion, the following equation holds

$$\Omega(s;x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}[\Omega(s;x)]_{x=0} \frac{x^n}{n!}$$

$$(x \geq 0; 0 \leq \Re(s) < b), \tag{2.1.2}$$

which in view of (1.5.19), yields the assertion of theorem. ■

Corollary 2.1.1. *Let* $\omega \in \mathcal{H}(b;0)$. *Then*

$$\omega(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Omega(-n;0) x^n}{n!} \qquad (0 \leq \Re(s) < b; x \geq 0). \tag{2.1.3}$$

Proof. Upon setting $s = 0$ in theorem (2.1.1) and applying the definition (1.5.18) lead to (2.1.3). ■

Example 2.1.1. *Note that* $\omega(t) = e^{-t} \in \mathcal{H}(\infty;0)$ *and* $\Omega(s;0) \equiv 1$, *which leads to the classical series representation*

$$\omega(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \qquad (x \geq 0). \tag{2.1.4}$$

Example 2.1.2. *Equation (2.1.3) can be written as*

$$\Omega(0;x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Omega(-n;0)x^n}{n!} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\Omega(s;0)x^{-s}ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \omega_{\mathcal{M}}(s)x^{-s}ds$$

$$(\omega \in \mathcal{H}(\kappa;0); 0 < c < k; x > 0), \tag{2.1.5}$$

which is the Hardy-Ramanujan master theorem [32, p. 186(B)]. Some special cases of (2.1.5) include

$$\begin{aligned} \Omega(0;x) = \omega(x) &:= (\tfrac{1}{e^x-1} - \tfrac{1}{x}) = \sum_{n=0}^{\infty} \tfrac{(-1)^n \zeta(-n) x^n}{n!} \\ &= \tfrac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s)x^{-s}ds = \tfrac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \omega_{\mathcal{M}}(s)x^{-s}ds \end{aligned} \qquad (0 < c < 1; x > 0) \tag{2.1.6}$$

and

$$\begin{aligned} \mathrm{Z}_a(0;x) = z_a(x) &:= (\tfrac{e^{-ax}}{e^x-1} - \tfrac{1}{x}) = \sum_{n=0}^{\infty} \tfrac{(-1)^n \zeta(-n,a) x^n}{n!} \\ &= \tfrac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s,a)x^{-s}ds = \tfrac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z_{\mathcal{M}}(s)x^{-s}ds \end{aligned} \qquad (0 < c < 1; x > 0), \tag{2.1.7}$$

which shows that $z_a(x) \in \mathcal{H}(1;0) \quad (0 \le a < 1)$. *Similarly, [55, p. 91(3.3.6)] gives*

$$2\cos(2\pi x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(1-s)}{\zeta(s)} x^{-s} ds \quad (0 < c < 1/2; x > 0), \tag{2.1.8}$$

which shows that $\cos(2\pi x) \in \mathcal{H}(1/2;0)$.

2.2 The extended Fermi-Dirac function $\Theta_\nu(s;x)$

I now introduce and investigate the first of the extended functions. An important aspect of this function lies in its transform representation and in its simplicity. It may be regarded as a natural extension of the FD function $\mathfrak{F}_{s-1}$ defined by (1.3.4). In order to introduce the eFD function $\Theta_\nu(s;x)$, consider a function $\vartheta(t;\nu)$, defined by

$$\vartheta(t;\nu) := \frac{e^{-\nu t}}{e^t + 1} \qquad (\Re(\nu) > -1, t \ge 0), \tag{2.2.1}$$

which is obviously integrable on every finite closed interval $[0, T] \quad (0 < T < \infty)$ in $\mathbb{R}_0^+$ and $\vartheta(t; \nu) \in \mathcal{H}(\infty; 0)$. Therefore, the Weyl transform of $\vartheta(t; \nu)$ given by

$$\Theta_\nu(s; x) := \mathcal{W}^{-s}[\vartheta(t; \nu)](x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \vartheta(x + t; \nu) dt$$
$$= \frac{1}{\Gamma(s)} \int_x^\infty (t - x)^{s-1} \vartheta(t; \nu) dt$$
$$(\Re(s) > 0; x \geq 0; \Re(\nu) > -1), \qquad (2.2.2)$$

is well defined. It is called the eFD function. Clearly by comparing the definitions (1.3.4) and (2.2.2), one can get

$$\Theta_0(s; x) = \mathfrak{F}_{s-1}(-x) \quad (\Re(s) > 0; x \geq 0). \qquad (2.2.3)$$

A substitution $x = 0$ in (2.2.2) leads to

$$\Theta_\nu(s; 0) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \vartheta(t; \nu) dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-\nu t} t^{s-1}}{e^t + 1} dt \quad (\Re(s) > 0; \Re(\nu) > -1). \qquad (2.2.4)$$

Since the second integral in (2.2.4) remains absolutely convergent, one can replace the exponential function $e^{-\nu t}$ by its series representation and can invert the order of summation and integration. This leads to

$$\Theta_\nu(s; 0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \vartheta(t; \nu) dt = \sum_{n=0}^\infty \frac{(-1)^n \nu^n \Gamma(s + n)}{n! \Gamma(s)} \left(\frac{1}{\Gamma(s + n)} \int_0^\infty \frac{t^{s+n-1}}{e^t + 1} dt \right)$$
$$= \sum_{n=0}^\infty \frac{(-\nu)^n (s)_n}{n!} \left(\frac{1}{\Gamma(s + n)} \int_0^\infty \frac{t^{s+n-1}}{e^t + 1} dt \right), \qquad (2.2.5)$$

where $(s)_n = \frac{\Gamma(s+n)}{\Gamma(s)}$ denotes the Pochhammer symbol. However, the integral in (2.2.5) can be simplified in terms of zeta function (1.2.16) to give

$$\frac{1}{\Gamma(s + n)} \int_0^\infty \frac{t^{s+n-1}}{e^t + 1} dt = (1 - 2^{1-s-n}) \zeta(s + n) \quad (\Re(s) > 0; n = 0, 1, 2, \ldots). \quad (2.2.6)$$

From (2.2.5) and (2.2.6), one gets

$$\Theta_\nu(s; 0) = \sum_{n=0}^\infty \frac{(s)_n (1 - 2^{1-s-n}) \zeta(s + n) (-\nu)^n}{n!} \quad (\Re(s) > 0; \mid \nu \mid < 1) \qquad (2.2.7)$$

or, equivalently,

$$\Theta_\nu(s;0) = \zeta(s, 1+\nu) - 2^{1-s}\zeta(s, 1+\frac{\nu}{2}) \quad (\Re(s) > 0; \mid \nu \mid < 1), \tag{2.2.8}$$

where the special case $a = 1$ of (1.2.30) is used.

Theorem 2.2.1. *The eFD function can be expressed as an integral of itself as*

$$\Theta_\nu(s+\beta;x) = \frac{1}{\Gamma(s)}\int_0^\infty \Theta_\nu(\beta;t+x)t^{s-1}dt = \frac{1}{\Gamma(\beta)}\int_0^\infty \Theta_\nu(s;t+x)t^{\beta-1}dt$$
$$(\min(\Re(s);\Re(\beta)) > 0; x \geq 0; \Re(\nu) > -1). \tag{2.2.9}$$

Proof. The demonstration of the assertion (2.2.9) is similar to that of the general representation (1.5.22), so the details are being omitted here. ■

Corollary 2.2.1. *The FD function can be expressed as an integral of itself as*

$$\mathfrak{F}_{s+\beta-1}(x) = \frac{1}{\Gamma(s)}\int_0^\infty \mathfrak{F}_{\beta-1}(x+t)t^{s-1}dt = \frac{1}{\Gamma(\beta)}\int_0^\infty \mathfrak{F}_{s-1}(x+t)t^{\beta-1}dt$$
$$(\min(\Re(s);\Re(\beta)) > 0; x \leq 0). \tag{2.2.10}$$

Proof. This follows from (2.2.3) and (2.2.9) when $\nu = 0$. ■

Remark 2.2.1. *In its special case when* $x = 0$*, the eFD function has the series representation* (2.2.7) *or the explicit representation* (2.2.8)*. Hence, for* $\mid \nu \mid < 1$*, it is well defined in terms of the Hurwitz (or generalized) zeta function* $\zeta(s,\nu)$*. In particular, for* $\nu = x = 0$*,* (2.2.8) *implies that*

$$\Theta_0(s;0) = (1-2^{1-s})\zeta(s) \qquad (\Re(s) > 0; s \neq 1). \tag{2.2.11}$$

Furthermore, in view of the relation (2.2.3)*, it is a natural generalization of the FD function* $\mathfrak{F}_{s-1}$ *defined by* (1.3.4)*.*

Upon substituting the value of $\vartheta(t;\nu)$ from (2.2.1) into (2.2.2), one obtains

$$\Theta_\nu(s;x) = \frac{e^{-(\nu+1)x}}{\Gamma(s)}\int_0^\infty \frac{e^{-\nu t}}{e^t + e^{-x}}t^{s-1}dt \quad (\Re(s) > 0; x \geq 0; \Re(\nu) > -1). \tag{2.2.12}$$

Upon substituting ($x = \log\omega$), the eFD function is related with Hurwitz-Lerch zeta function $\Phi(x, s, \nu)$ as follows:

$$\Theta_\nu(s; \log\omega) = \omega^{-(\nu+1)}\Phi(-1/\omega, s, \nu + 1). \tag{2.2.13}$$

Expanding the denominator, inverting the order of summation and integration in (2.2.12) leads to the following series representation for the eFD function

$$\Theta_\nu(s; x) = e^{-(\nu+1)x} \sum_{n=0}^{\infty} \frac{(-1)^n e^{-nx}}{(n + \nu + 1)^s} \quad (\Re(s) > 0; x \geq 0; \Re(\nu) > -1). \tag{2.2.14}$$

This series is convergent for complex numbers ($z = x + iy, \Re(z) > 0$) and therefore extends the domain of the real parameter $x \geq 0$ to a complex z. This representation also extends the domain of parameter $\Re(\nu) > -1$ to $\nu \in \mathbb{C} \setminus \mathbb{Z}^-$. In view of the relation (2.2.3), it leads to a series representation for the FD functions. This in turn extends the domain of the real parameter x involved in the definition of these classical functions to a complex z.

Next, I will obtain a series representation for the eFD function by using representation theorem. For $\nu = 0$, it leads to a series representation for the FD function, $\mathfrak{F}_{s-1}(x)$.

Theorem 2.2.2. *The eFD function has the power series representation*

$$\Theta_\nu(s; x) = \Theta_\nu(s; 0) + \sum_{n=1}^{\infty} \frac{(-1)^n \Theta_\nu(s - n; 0) x^n}{n!} \quad (\Re(s) > 0; x \geq 0; \Re(\nu) > -1). \tag{2.2.15}$$

Proof. It has the Weyl transform representation (2.2.2). Therefore, application of the general result (2.1.1) leads to the proof of (2.2.15). ■

Corollary 2.2.2. *The FD function has the power series representation*

$$\begin{aligned}\mathfrak{F}_{s-1}(x) &= \mathfrak{F}_{s-1}(0) + \sum_{n=1}^{\infty} \frac{\mathfrak{F}_{s-n-1}(0)x^n}{n!} \\ &= (1-2^{1-s})\zeta(s) + \sum_{n=1}^{\infty} \frac{(1-2^{1-s+n})\zeta(s-n)x^n}{n!} \quad (\Re(s)>0; x \geq 0).\end{aligned} \tag{2.2.16}$$

Proof. This follows from theorem (2.2.2) upon setting $x \mapsto -x$ and $\nu = 0$ and using the following reltionship

$$\Theta_0(s;0) = \mathfrak{F}_{s-1}(0) = (1-2^{1-s})\zeta(s) \quad (\Re(s)>0; s \neq 1). \tag{2.2.17}$$

■

2.3 The extended Bose-Einstein function $\Psi_\nu(s;x)$

I now introduce the second extended function, which also has the Weyl and Mellin transform representations. This function $\Psi_\nu(s;x)$ may be regarded as a natural extension of the BE function $\mathfrak{B}_{s-1}$ defined by (1.3.5). In order to introduce the eBE function $\Psi_\nu(s;x)$, consider a function $\psi(t;\nu)$ defined by

$$\psi(t;\nu) := \frac{e^{-\nu t}}{e^t - 1} \qquad (\Re(\nu) > -1; t \geq 0) \tag{2.3.1}$$

and use this function to define $\Psi_\nu(s;x)$

$$\begin{aligned}\Psi_\nu(s;x) &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\psi(t+x;\nu)dt := \mathcal{W}^{-s}[\psi(t;\nu)](x) \\ &= \frac{1}{\Gamma(s)} \int_x^\infty (t-x)^{s-1}\psi(t;\nu)dt \\ &\quad (\Re(s)>1; x \geq 0; \Re(\nu) > -1).\end{aligned} \tag{2.3.2}$$

For $\nu = 0$, it leads to the following relationship with the BE function $\mathfrak{B}_{s-1}(x)$:

$$\Psi_0(s;x) = \mathfrak{B}_{s-1}(-x) \qquad (\Re(s)>1; x \geq 0). \tag{2.3.3}$$

A substitution for the function $\psi(t;\nu)$ from (2.3.1) into (2.3.2), yields

$$\Psi_\nu(s;x) = \frac{e^{-(\nu+1)x}}{\Gamma(s)} \int_0^\infty \frac{e^{-\nu t}}{e^t - e^{-x}} t^{s-1} dt$$
$$(\Re(\nu) > -1; \Re(s) > 1 \text{ when } x = 0; \Re(s) > 0 \text{ when } x > 0). \qquad (2.3.4)$$

The eBE function $\Psi_\nu(s;x)$ is related to Hurwitz-Lerch zeta function $\Phi(x,s,\nu)$ as follows:

$$\Psi_\nu(s;\log\omega) = \omega^{-(\nu+1)}\Phi(1/\omega, s, \nu+1). \qquad (2.3.5)$$

Expanding the denominator, inverting the order of summation and integration in (2.3.4) leads to the following series representation for the eBE function $\Psi_\nu(s;x)$:

$$\Psi_\nu(s;x) = e^{-(\nu+1)x} \sum_{n=0}^{\infty} \frac{e^{-nx}}{(n+\nu+1)^s} \quad (\Re(s) > 1; x \geq 0; \Re(\nu) > -1). \qquad (2.3.6)$$

Again note that this series is convergent for complex numbers $z = x + iy$ such that $\Re(z) > 0$. This representation also extends the domain of parameter $\Re(\nu) > -1$ to $\nu \in \mathbb{C} \setminus \mathbb{Z}^-$. In view of the relation (2.3.3), it leads to a series representation for the BE functions. This in turn extends the domain of the real parameter x involved in the definition of these classical functions to a complex z. In the particular case when $x = 0$, (2.3.5) yields

$$\Psi_\nu(s;0) = \zeta(s, \nu+1) \qquad (\Re(s) > 1; \Re(\nu) > -1), \qquad (2.3.7)$$

which shows that $\Psi_\nu(s;0)$ has a meromorphic continuation [1, p. 254] to the whole complex s-plane. Furthermore, expanding the exponential function $e^{-\nu t}$ in (2.3.4) (with $x = 0$), inverting the order of summation and integration and evaluating the resulting integral by using the well-known integral representation (1.2.6) for $\zeta(s)$ leads to a series representation for

$$\Psi_\nu(s;0) = \sum_{n=0}^{\infty} \frac{(-1)^n (s)_n \zeta(s+n) \nu^n}{n!} \quad (\Re(s) > 1; s \neq 1; \mid \nu \mid < 1), \qquad (2.3.8)$$

which in the light of the summation formula (1.2.30), is the same as the relation (2.3.7).

In terms of the Bernoulli polynomials $B_n(x)$ defined by the generating function (1.4.1), one can see from (2.3.7) and (1.4.7) that

$$\Psi_\nu(-n;0) = -\frac{B_{n+1}(\nu+1)}{n+1} \qquad (n \in \mathbb{N}_0). \tag{2.3.9}$$

The series representations (2.2.14) and (2.3.6) also converge for $(s \in \mathbb{C},\ x > 0)$ and show that the eFD and eBE functions are meromorphic functions of the complex variable s with simple pole at $s = 1$ when $x = 0$. Obviously, it leads to the fact that the original FD and BE functions have meromorphic continuation to the whole complex s-plane.

Theorem 2.3.1. *The eBE has the following series representation*

$$\Psi_\nu(s;x) = q^{-s}\sum_{j=1}^{q} \Psi_{\frac{\nu+j-q}{q}}(s;qx) \quad (\Re(s) > 1; x \geq 0; \Re(\nu) > -1). \tag{2.3.10}$$

Proof. The result (2.3.10) can be derived by suitably combining (2.3.5), (1.2.45) and using the elementary series identity

$$\sum_{k=0}^{\infty}\Lambda(k) = \sum_{j=1}^{q}\left(\sum_{k=0}^{\infty}\Lambda(qk+j-1)\right) \qquad (q \in \mathbb{N}), \tag{2.3.11}$$

where $\{\Lambda(k)\}_{k\in\mathbb{N}_0}$ is a suitably bounded sequence of complex numbers. ■

Corollary 2.3.1. *The following identity holds true for the eBE function* $\Psi_\nu(s;0)$*:*

$$\Psi_\nu(s;0) = q^{-s}\sum_{j=1}^{q} \Psi_{\frac{\nu+j-q}{q}}(s;0) \quad (\Re(s) > 1; \Re(\nu) > -1). \tag{2.3.12}$$

Proof. This follows from (2.3.10) in the special case when $x = 0$. ■

Remark 2.3.1. *Identity* (2.3.12) *can be rewritten in terms of the Hurwitz (or the generalized) zeta function* $\zeta(s,a)$*, as the following well-known result:*

$$\zeta(s,a) = q^{-s}\sum_{j=1}^{q}\zeta\left(s, \frac{a+j-1}{q}\right) \quad (q \in \mathbb{N}), \tag{2.3.13}$$

which is a well-known identity. In particular, for $a = 1$, (2.3.13) *immediately yields another well-known result*

$$\zeta(s) = \zeta(s,1) = q^{-s}\sum_{j=1}^{q}\zeta\left(s,\frac{j}{q}\right) \qquad (q \in \mathbb{N}), \tag{2.3.14}$$

which in the special case $q = 2$ *corresponds to* (1.2.29).

Remark 2.3.2. *By setting* $x = \frac{p}{q}; p \in \mathbb{R}_0^+; q \in \mathbb{N}$, (2.3.10) *leads to*

$$\Psi_\nu(s;\frac{p}{q}) = q^{-s}\sum_{j=1}^{q}\Psi_{\frac{\nu+j-q}{q}}(s;p) \quad (\Re(s) > 1; p \in \mathbb{R}_0^+; \Re(\nu) > -1), \tag{2.3.15}$$

which in the special case when $q = 2$, *yields the following potentially useful relationship:*

$$\Psi_\nu(s;\frac{p}{2}) = 2^{-s}[\Psi_{\frac{\nu-1}{2}}(s;p) + \Psi_{\frac{\nu}{2}}(s;p)] \quad (\Re(s) > 1; p \in \mathbb{R}_0^+; \Re(\nu) > -1). \tag{2.3.16}$$

Further, by setting $p = 0$ *in* (2.3.16), *one can obtain*

$$\Psi_\nu(s;0) = 2^{-s}[\Psi_{\frac{\nu-1}{2}}(s;0) + \Psi_{\frac{\nu}{2}}(s;0)] \quad (\Re(s) > 1; \Re(\nu) > -1), \tag{2.3.17}$$

which for $\nu = a - 1$ *and in view of the relationship* (2.3.7), *can easily be rewritten, in terms of the Hurwitz (or generalized) zeta function* $\zeta(s,a)$ *as follows*

$$\zeta(s,a) = 2^{-s}[\zeta(s,\frac{a}{2}) + \zeta(s,\frac{a+1}{2})] \quad (\Re(s) > 1; \Re(a) > 0). \tag{2.3.18}$$

In its special case when $a = 1$, (2.3.18) *immediately yields one of the relations in* (1.2.29) *in the form:*

$$\zeta(s) = 2^{-s}[\zeta(s,\frac{1}{2}) + \zeta(s)] \quad (\Re(s) > 1). \tag{2.3.19}$$

Remark 2.3.3. *For* $q = 2$ *and* $\nu \mapsto 2\nu$, *the identity* (2.3.12) *can readily be put as follows:*

$$\Psi_{\frac{2\nu-1}{2}}(s;2x) = 2^s\Psi_{2\nu}(s;x) - \Psi_\nu(s;2x) \quad (\Re(s) > 1; \Re(\nu) > -1; x \geq 0), \tag{2.3.20}$$

which for $x = \nu - a = 0$ *and in view of the relationship* (2.3.7)*, can easily be rewritten as the following equivalent form of the relation* (2.3.18)*:*

$$\zeta(s, \frac{a+1}{2}) = 2^s\zeta(s, 2a+1) - \zeta(s, a+1) \quad (\Re(s) > 1; \Re(a) \geq 0). \tag{2.3.21}$$

Clearly, the relation (2.3.21) *with* $a \mapsto 2a + 1$ *is precisely the same as the relation* (2.3.18)*. Conversely, the relationship* (2.3.18) with $a \mapsto \frac{1}{2}(a-1)$ *is precisely the same as the relation* (2.3.21)*. Moreover, in its special case when* $a = 0$*, this last relation* (2.3.21) also *yields one of the relationships in* (1.2.29) *in the form:*

$$\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s), \tag{2.3.22}$$

which obviously is the same as (2.3.19) *above.*

Remark 2.3.4. *By putting* $\nu = 0$ *in (2.3.10), one can get the following series representation for the BE function*

$$\mathfrak{B}_{s-1}(-x) = q^{-s}\sum_{j=1}^{q} \Psi_{\frac{j-q}{q}}(s; qx) \quad (\Re(s) > 1; x \geq 0). \tag{2.3.23}$$

For $x = \frac{p}{q}; p \in \mathbb{R}_0^+; q \in \mathbb{N}$

$$\mathfrak{B}_{s-1}(-\frac{p}{q}) = q^{-s}\sum_{j=1}^{q} \Psi_{\frac{j-q}{q}}(s; p) \quad (\Re(s) > 1; p \in \mathbb{R}_0^+), \tag{2.3.24}$$

which in the special case when $q = 2$*, yields the following relationship:*

$$2^s\mathfrak{B}_{s-1}(-\frac{p}{2}) - \mathfrak{B}_{s-1}(-p) = \Psi_{-\frac{1}{2}}(s; p) \quad (\Re(s) > 1; p \in \mathbb{R}_0^+), \tag{2.3.25}$$

which expresses the eBE function in terms of BE function.

2.4 Miscellaneous results associated with the eFD and eBE functions

Among other miscellaneous results presented in this section, it is proved that the two extended functions $\Theta_\nu(s; x)$ and $\Psi_\nu(s; x)$ have a simple relationship with each other.

A connection between the FD and the BE function is also derived here as a special case.

Theorem 2.4.1. *The eFD and eBE functions are related by*

$$\Theta_{2\nu}(s;x) = \Psi_{2\nu}(s;x) - 2^{1-s}\Psi_{\nu}(s;2x) \quad (x \geq 0; \Re(s) > 1; \Re(\nu) > -1). \tag{2.4.1}$$

Proof. Considering the identity

$$\frac{e^{-2\nu t}}{e^{2t}-1} = \frac{1}{2}\left(\frac{e^{-2\nu t}}{e^{t}-1} - \frac{e^{-2\nu t}}{e^{t}+1}\right) \tag{2.4.2}$$

and making use of the definitions, (2.2.1) and (2.3.1), the following relationship between the functions $\vartheta(t;\nu)$ and $\psi(t;\nu)$ holds

$$\psi(2t;\nu) = \frac{1}{2}[\psi(t;2\nu) - \vartheta(t;2\nu)]. \tag{2.4.3}$$

By taking the Weyl transform of each member in (2.4.3) and using the fact that

$$\mathcal{W}^{-s}[\psi(2t;\nu)](x) = 2^{-s}\Psi_{\nu}(s;2x), \tag{2.4.4}$$

one can obtain the following relation

$$2^{-s}\Psi_{\nu}(s;2x) = \frac{1}{2}[\Psi_{2\nu}(s;x) - \Theta_{2\nu}(s;x)] \quad (x \geq 0; \Re(s) > 1; \Re(\nu) > -1), \tag{2.4.5}$$

which is the same as assertion (2.4.1). ■

Corollary 2.4.1. *The FD and BE functions are related as follows*

$$\mathfrak{F}_{s-1}(x) = \mathfrak{B}_{s-1}(x) - 2^{1-s}\mathfrak{B}_{s-1}(2x) \quad (x \geq 0; \Re(s) > 1). \tag{2.4.6}$$

Proof. This follows from (2.4.1), by putting $\nu = 0$ and replacing $x \mapsto -x$. ■

Theorem 2.4.2. *The following simple relation holds between the eFD and the eBE functions*

$$\Theta_{\nu}(s;x+\pi i) = e^{-i(\nu+1)\pi}\Psi_{\nu}(s;x) \quad (x \geq 0; \Re(s) > 1; \Re(\nu) > -1). \tag{2.4.7}$$

Proof. The relation (2.4.7) can be derived by applying the series representations (2.2.14) and (2.3.6)

$$\Theta_\nu(s; x+\pi i) = e^{-(\nu+1)(x+\pi i)} \sum_{n=0}^{\infty} \frac{(-1)^n e^{-n(x+\pi i)}}{(n+\nu+1)^s} = e^{-i(\nu+1)\pi} \Psi_\nu(s; x). \qquad (2.4.8)$$

■

Note that the eFD and eBE functions are dual to each other in the sense that the relation (2.4.7) can easily be inverted. This fact is of relevance for providing a function for *anyons* [10] corresponding to the FD and BE functions.

Corollary 2.4.2. *The FD and BE functions are related by*

$$\mathfrak{F}_{s-1}(x+\pi i) = e^{-i\pi} \mathfrak{B}_{s-1}(x) \quad (\Re(s) > 1; x \geq 0). \qquad (2.4.9)$$

Proof. It follows from (2.4.7) by putting $\nu = 0$. ■

Theorem 2.4.3. *The eFD and the eBE function are related as follows:*

$$\Theta_{\nu+1}(s; x) = 2^{-s}[\Psi_{\frac{\nu}{2}}(s; 2x) - \Psi_{\frac{\nu+1}{2}}(s; 2x)]$$
$$(\Re(s) > 1; x \geq 0; \Re(\nu) > -1). \qquad (2.4.10)$$

Proof. Upon replacing $\Psi_{\frac{\nu}{2}}(s; 2x)$ and $\Psi_{\frac{\nu+1}{2}}(s; 2x)$ by their Weyl integral representations given by (2.3.2), one can get

$$\Psi_{\frac{\nu}{2}}(s; 2x) - \Psi_{\frac{\nu+1}{2}}(s; 2x) = \frac{1}{\Gamma(s)} \int_{2x}^{\infty} \frac{(e^{-\frac{\nu t}{2}} - e^{-\frac{(\nu+1)t}{2}})(t-2x)^{s-1}}{e^t - 1} dt$$
$$= \frac{1}{\Gamma(s)} \int_{2x}^{\infty} \frac{e^{-\frac{(\nu+1)t}{2}}(e^{\frac{t}{2}} - 1)(t-2x)^{s-1}}{(e^{\frac{t}{2}} - 1)(e^{\frac{t}{2}} + 1)} dt = \frac{1}{\Gamma(s)} \int_{2x}^{\infty} \frac{e^{-\frac{(\nu+1)t}{2}}(t-2x)^{s-1}}{(e^{\frac{t}{2}} + 1)} dt$$
$$(x \geq 0; \Re(s) > 1; \Re(\nu) > -1). \quad (2.4.11)$$

The transformation $t = 2\tau$ in (2.4.11) leads to

$$\Psi_{\frac{\nu}{2}}(s; 2x) - \Psi_{\frac{\nu+1}{2}}(s; 2x) = \frac{2^s}{\Gamma(s)} \int_{x}^{\infty} \frac{e^{-(\nu+1)\tau}(\tau - x)^{s-1}}{e^\tau + 1} d\tau, \qquad (2.4.12)$$

which in view of the Weyl fractional integral representation in (2.3.2), yields an obviously transposed form of the relationship asserted by Theorem (2.4.3). ■

Corollary 2.4.3. *The following relation holds true:*

$$\Theta_\nu(s;0) = 2^{-s}\left[\zeta\left(s;\frac{\nu+1}{2}\right) - \zeta\left(s;\frac{\nu}{2}+1\right)\right]$$

$$(\Re(s) > 1; \Re(\nu) > 0). \qquad (2.4.13)$$

Proof. Upon setting $x = 0$ and $\nu \mapsto \nu - 1$ and using $\Psi_a(s;0) = \zeta(s, a+1)$ in (2.4.10), result (2.4.13) can be obtained. ■

Remark 2.4.1. *By applying the series representation* (2.3.6) *(with* $x = 0$ *and* $\nu = a - 1$*) on the left-hand side of the relationship* (2.4.13)*, Corollary* (2.4.3) *can at once be rewritten in the following known form (see, for example,* $[61, p.89, Eq.2.2(5)]$*):*

$$[\zeta(s;\frac{a}{2}) - \zeta(s;\frac{a+1}{2})] = 2^s\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^s}. \qquad (2.4.14)$$

Corollary 2.4.4. *The BE function can be written in terms of the eFD and eBE functions*

$$2^s\Theta_1(s;x) + \Psi_{\frac{1}{2}}(s;2x) = \mathfrak{B}_{s-1}(-2x) \quad (x \geq 0; \Re(s) > 1). \qquad (2.4.15)$$

Proof. It is the special case of (2.4.10) for $\nu = 0$. ■

Functional relations and difference equations are important for the study of special functions. For example, the Bernoulli polynomials (1.4.1) $B_n(x)$ satisfy the difference equation

$$B_n(x+1) - B_n(x) = nx^{n-1} \qquad (n \in \mathbb{N}). \qquad (2.4.16)$$

One would, therefore, like to know if the eFD and eBE functions also satisfy such relations. It turns out that this is indeed the case.

Theorem 2.4.4. *The eFD function* $\Theta_\nu(s;x)$ *satisfies the difference equation*

$$\Theta_{\nu+1}(s;x) + \Theta_\nu(s;x) = (\nu+1)^{-s}e^{-(\nu+1)x} \quad (\Re(s) > 0; x \geq 0; \Re(\nu) > -1). \qquad (2.4.17)$$

Proof. Making use of the elementary identity

$$\frac{e^{-(\nu+1)t}}{e^t+1}+\frac{e^{-\nu t}}{e^t+1}=e^{-(\nu+1)t} \tag{2.4.18}$$

and the definition (2.2.1) leads to

$$\vartheta(t;\nu+1)+\vartheta(t;\nu)=e^{-(\nu+1)t} \tag{2.4.19}$$

However, it is known that

$$\mathcal{W}^{-s}[e^{-(\nu+1)t}](x)=(\nu+1)^{-s}e^{-(\nu+1)x} \quad (\Re(s)>0;x\geq 0;\Re(\nu)>-1). \tag{2.4.20}$$

By applying the Weyl transform on both sides in (2.4.18) and using (2.2.2) as well as (2.4.20), the difference equation (2.4.17) asserted by Theorem (2.4.4) is derived. ■

Corollary 2.4.5. *The following difference equation holds true:*

$$\Theta_\nu(s;0)+\Theta_{\nu-1}(s;0)=\nu^{-s} \quad (\Re(s)>0;\Re(\nu)>0). \tag{2.4.21}$$

Proof. This follows from (2.4.17) by putting $x=0$ and replacing ν by $\nu-1$. ■

Corollary 2.4.6. *The Hurwitz-Lerch zeta function satisfies the following difference equation*

$$\Phi(z,s,\nu)-z\Phi(z,s,\nu+1)=\nu^{-s} \qquad (\Re(s)>0;\Re(\nu)>1). \tag{2.4.22}$$

Proof. The difference equation (2.4.22) follows easily from (2.2.13) and the assertion (2.4.17). ■

Remark 2.4.2. *By putting $\nu=0$ in (2.4.17), the following identity holds*

$$\Theta_1(s;x)=e^{-x}-\mathfrak{F}_{s-1}(-x) \quad (\Re(s)>0;x\geq 0). \tag{2.4.23}$$

This equation along with (2.4.17) yields the following representation

$$\Theta_m(s;x)=\sum_{n=1}^{m}n^{-s}e^{-nx}-\mathfrak{F}_{s-1}(-x) \quad (\Re(s)>0;x\geq 0;m=1,2,\cdots), \tag{2.4.24}$$

which by further specializing $x = 0$ yields the following simple relationship

$$\Theta_1(s;0) = 1 - \mathfrak{F}_{s-1}(0) = 1 - \zeta(s)(1 - 2^{1-s}). \tag{2.4.25}$$

By using the relationship (2.4.7) in (2.4.17), one can get the following difference equation for the eBE function

$$\Psi_\nu(s;x) - \Psi_{\nu+1}(s;x) = (\nu+1)^{-s}e^{-(\nu+1)x} \quad (\Re(s) > 1; x \geq 0; \Re(\nu) > -1). \tag{2.4.26}$$

It leads to the following representation

$$\Psi_m(s;x) = \mathfrak{B}_{s-1}(-x) - \sum_{n=1}^{m} n^{-s}e^{-nx} \quad (\Re(s) > 1; x \geq 0; m = 1, 2, \cdots). \tag{2.4.27}$$

This expression along with (2.3.5) can give similar expression for Hurwitz-Lerh zeta function, which by further specializing $x = 0$ gives

$$\zeta(s, m+1) = \zeta(s) - \sum_{n=1}^{m} n^{-s} \quad (\Re(s) > 1; m = 1, 2, \cdots). \tag{2.4.28}$$

For $m = 1$, equation (1.2.29) can be obtained as a special case of (2.4.28).

Remark 2.4.3. *The eFD function and the Bernoulli polynomials $B_n(x)$ are related as follows:*

$$\Theta_\nu(-n;\pi i) = e^{-i\pi\nu}\frac{B_{n+1}(\nu+1)}{n+1} \quad (n \in \mathbb{N}_0), \tag{2.4.29}$$

which can be derived by putting $x = \pi i$ in (2.2.14) *and then suitably combining it with the relations* (1.2.27) *and* (1.4.7). *Thus, by setting $s = -n$ and $x = \pi i$ in* (2.4.17) *and using the relation* (2.4.29), *the following classical result for the Bernoulli polynomials $B_n(x)$ is obtained*

$$\frac{B_{n+1}(\nu+1)}{n+1} - \frac{B_{n+1}(\nu)}{n+1} = \nu^n \qquad (n \in \mathbb{N}_0). \tag{2.4.30}$$

This shows that the difference equation (2.4.17), *which can be rewritten as follows*

$$\Theta_\nu(s;x) + \Theta_{\nu-1}(s;x) = \nu^{-s}e^{-\nu x} \quad (\Re(s) > 0; x \geq 0; \Re(\nu) > 1), \tag{2.4.31}$$

is the most general form of the difference equation satisfied by the family of the zeta functions. Similarly, by putting $z = 1$ *in* (2.4.22) *and using the relation* $\zeta(s, a) = \Phi(1, s, a)$*, the following familiar difference equation:*

$$\zeta(s, \nu) - \zeta(s, \nu + 1) = \nu^{-s} \tag{2.4.32}$$

satisfied by the Hurwitz (or generalized) zeta function is recovered. However, for $\nu = 0$, (2.4.29) *yields particular values of FD function in terms of Bernoulli polynomials*

$$\mathfrak{F}_{-n}(-\pi i) = \frac{B_n(1)}{n}. \tag{2.4.33}$$

Chapter 3

Further applications of the representation theorem and the generalized Riemann zeta function

A general representation theorem is proved in Chapter 2 to get a series representation for the eFD and some related functions. In this Chapter, an extension of this representation theorem is given by using the linearity property of the Weyl transform. The basic motivation behind this approach is to get series representation for a larger class of functions. To achieve the purpose, the integral representation of the Hurwitz zeta function is analytically continued in the critical strip.

By following the method of analytic continuation as for the Hurwitz zeta function, the gRZ function is introduced in Section (3.2). It is closely related with the eFD and eBE functions. By using these relations, series representation for the gRZ, eBE and the Hurwitz-Lerch zeta function is obtained here. It is the most general form of the series obtained in Section (3.1). It has removable singularities at $s \in \mathbb{N}$, while the earlier obtained series has simple poles at these points.

The eFD and eBE functions have simple relations with other functions of zeta family (for details, see, Chapter 2) but they are not related with one particular representation of the Riemann zeta function in the critical strip. A special case of this new generalization converges uniformly to the Riemann zeta function in the critical strip. A difference equation satisfied by this new generalization is also studied here. Note that, to prove several results the parameter ν is restricted between 0 and 1 but the integral representation for the gRZ function holds for $\Re(\nu) > -1$. The gRZ function also give a new insight that the FD and BE functions are related with the Riemann zeta function in the critical strip.

3.1 Further applications of the representation theorem

The series representation for the Hurwitz-Lerch zeta function is obtained in [21, pp. 28-29] by expressing the function as a contour integral, applying the Cauchy residue theorem and then the Hurwitz formula

$$\zeta(s,\nu) = 2(2\pi)^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty}\frac{\sin(2\pi n\nu + \pi s/2)}{n^{1-s}} \quad (\Re(s) < 0; 0 < \nu \leq 1). \qquad (3.1.1)$$

Here, the series representation for the Hurwitz-Lerch zeta function is given by an alternate method. This method depends on the Weyl transform of a particular function, which is obtained here by using the analytic continuation of the Hurwitz zeta function and representation theorem. A series representation for the BE function is also obtained here as a simple consequence of the representation theorem.

The following general result will be applied to the BE and the Hurwitz-Lerch zeta functions.

Theorem 3.1.1. *Let* $\omega \in \mathcal{H}(k;0)$ *and*

$$\pi(t) := at^{-\rho} + \omega(t) \qquad (\rho \geq 0). \qquad (3.1.2)$$

Then

$$\Pi(s;x) = a\frac{\Gamma(\rho-s)}{\Gamma(\rho)}x^{s-\rho} + \sum_{n=0}^{\infty}\frac{(-1)^n\Omega(s-n;0)x^n}{n!}$$
$$(0 \leq \Re(s) < \min(k;\rho); x > 0). \tag{3.1.3}$$

Proof. Since Weyl's transform is a linear operator, an application of it to (3.1.2) leads to

$$\Pi(s;x) := \mathcal{W}^{-s}[\pi(t)](x) = a\mathcal{W}^{-s}[t^{-\rho}](x) + \Omega(s;x)$$
$$((0 \leq \Re(s) < \min(k;\rho); x > 0). \tag{3.1.4}$$

However, using [53, p. 249(7.10)]

$$\mathcal{W}^{-s}[t^{-\rho}](x) = \frac{\Gamma(\rho-s)}{\Gamma(\rho)}x^{s-\rho} \quad (0 < \Re(s) < \rho; x > 0) \tag{3.1.5}$$

along with (2.1.1) in (3.1.4) leads to the result (3.1.3). ■

Corollary 3.1.1. *Let* $\omega \in \mathcal{H}(k;0)$ *and*

$$\pi(t) := at^{-\rho} + \omega(t) \qquad (\rho \geq 0), \tag{3.1.6}$$

then

$$\Pi(0;x) = ax^{-\rho} + \sum_{n=0}^{\infty}\frac{(-1)^n\Omega(-n;0)x^n}{n!} \qquad (0 \leq \Re(s) < \min(k;\rho); x > 0). \tag{3.1.7}$$

Proof. Upon setting $s = 0$ in theorem (3.1.1) and applying the definition (1.5.18) lead to (3.1.7). ■

Example 3.1.1. *Consider the function defined as*

$$\omega(t) := \frac{1}{e^t - 1} - \frac{1}{t} \qquad (t > 0). \tag{3.1.8}$$

Note that $\omega \in \mathcal{H}(1;0)$ *and*

$$\Omega(s;0) = \zeta(s) \qquad (0 < \Re(s) < 1). \tag{3.1.9}$$

Hence, the following classical expansion in terms of the Bernoulli numbers

$$\Omega(0;x) = \frac{1}{e^x - 1} - \frac{1}{x} = \zeta(0) + \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(-n) x^n}{n!} \quad (x > 0) \tag{3.1.10}$$

is obtained, where

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \qquad (n = 0, 1, 2, 3, \ldots). \tag{3.1.11}$$

3.1.1 Analytic continuation of the Hurwitz zeta function and applications of the representation theorem

The integral representation of the Hurwitz zeta function can be extended to the critical strip by following the method of analytic continuation of the zeta function [70, p. 22, Equation (2.7)]. For $\Re(s) > 1$, the integral representation of the Hurwitz zeta function (1.2.28) can be written as follows:

$$\Gamma(s)\zeta(s, \nu + 1) = \int_0^1 \left(\frac{e^{-\nu t}}{e^t - 1} - \frac{1}{t} \right) t^{s-1} dt + \frac{1}{s-1} + \int_1^{\infty} \frac{e^{-\nu t}}{e^t - 1} t^{s-1} dt, \tag{3.1.12}$$

which holds by analytic continuation for $(\Re(s) > 0, s \neq 1)$, [70, p. 37]. Further, for $0 < \Re(s) < 1$, the following relation holds

$$\frac{1}{s-1} = -\int_1^{\infty} \frac{t^{s-1}}{t} dt. \tag{3.1.13}$$

Hence

$$\Gamma(s)\zeta(s, \nu + 1) = \int_0^{\infty} \left(\frac{e^{-\nu t}}{e^t - 1} - \frac{1}{t} \right) t^{s-1} dt \qquad (0 < \Re(s) < 1; 0 \leq \nu < 1). \tag{3.1.14}$$

Upon putting $\nu = 0$ in the above equation the classical representation (1.2.14)

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{s-1} dt \quad (0 < \Re(s) < 1), \tag{3.1.15}$$

for the Riemann zeta function is recovered.

Theorem 3.1.2. *By making use of representation theorem, the following result holds true*

$$\mathcal{W}^{-s}\left[\frac{e^{-\nu t}}{e^t-1}-\frac{1}{t}\right](x)=\zeta(s,\nu+1)+\sum_{n=1}^{\infty}\frac{(-1)^n\zeta(s-n,\nu+1)x^n}{n!}$$

$$(0\leq\Re(s)<1;0\leq\nu<1;x\geq 0). \tag{3.1.16}$$

Proof. Consider the following function

$$\omega_\nu(t):=\left(\frac{e^{-\nu t}}{e^t-1}-\frac{1}{t}\right)\qquad(t>0;0\leq\nu<1). \tag{3.1.17}$$

Since $\omega_\nu(t)\in\mathcal{H}(1;0)$ therefore, by using (1.5.18), one can get

$$\mathcal{W}^{-s}\left[\frac{e^{-\nu t}}{e^t-1}-\frac{1}{t}\right](0)=\Gamma(s)\zeta(s,\nu+1). \tag{3.1.18}$$

An application of Corollary (2.1.1) leads to the classical series representation in terms of the Bernoulli polynomials as

$$\frac{e^{-\nu t}}{e^t-1}-\frac{1}{t}=\zeta(0,\nu+1)+\sum_{n=1}^{\infty}\frac{(-1)^n\zeta(-n,\nu+1)t^n}{n!}$$

$$(0\leq\nu<1;t>0), \tag{3.1.19}$$

where by (1.4.7)

$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1}\qquad(n=0,1,2,3,\ldots). \tag{3.1.20}$$

Now by applying the representation theorem (2.1.1), general assertion (3.1.16) follows. ■

Remark 3.1.1 *Theorem* (3.1.2) *will prove crucial in the proof of series representation for the BE and the Hurwitz-Lerch zeta functions.*

Theorem 3.1.3. *The BE function has the series representation*

$$\mathfrak{B}_{s-1}(-x)=\Gamma(1-s)x^{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(s-n)x^n}{n!}$$

$$(x\geq 0;s\neq 1,2,\cdots). \tag{3.1.21}$$

Proof. The BE function (1.3.5) can be rewritten as

$$\begin{aligned}\mathfrak{B}_{s-1}(-x) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^{t+x}-1}dt = \frac{1}{\Gamma(s)}\int_x^\infty \frac{(t-x)^{s-1}}{e^t-1}dt \\ &= \frac{1}{\Gamma(s)}\int_x^\infty \left\{\left(\frac{1}{e^t-1}-\frac{1}{t}\right)+\frac{1}{t}\right\}(t-x)^{s-1}dt.\end{aligned} \tag{3.1.22}$$

Making use of (1.5.16) in the above equation gives

$$\mathfrak{B}_{s-1}(-x) = \mathcal{W}^{-s}\left[\frac{1}{e^t-1}-\frac{1}{t}\right](x) + \mathcal{W}^{-s}\left[\frac{1}{t}\right](x). \tag{3.1.23}$$

Since the Weyl transforms

$$\mathcal{W}^{-s}\left[\frac{1}{e^t-1}-\frac{1}{t}\right](x) = \zeta(s) + \sum_{n=1}^{\infty}\frac{(-1)^n\zeta(s-n)x^n}{n!}$$

$$(x \geq 0; s \neq 1, 2, \cdots) \tag{3.1.24}$$

and

$$\mathcal{W}^{-s}\left[\frac{1}{t}\right](x) = \Gamma(1-s)x^{s-1} \qquad (x \geq 0; s \neq 1, 2, \cdots) \tag{3.1.25}$$

are well defined. Above equations (3.1.22-3.1.25) lead to the series representation (3.1.21). ■

Theorem 3.1.4. *The eBE function has a series representation*

$$\Psi_\nu(s;x) = \Gamma(1-s)x^{s-1} + \sum_{n=0}^{\infty}\zeta(s-n,\nu+1)\frac{(-x)^n}{n!}$$

$$(x \geq 0; 0 < \nu < 1; s \neq 1, 2, 3, \ldots). \tag{3.1.26}$$

Proof. The integral representation (2.3.4) for the eBE function can be rewritten as

$$\Psi_\nu(s;x) = \mathcal{W}^{-s}\left[\frac{e^{-\nu t}}{e^t-1}-\frac{1}{t}\right](x) + \mathcal{W}^{-s}\left[\frac{1}{t}\right](x), \tag{3.1.27}$$

which in turn along with (3.1.16) and (3.1.25) gives the following series representation

$$\Psi_\nu(s;x) = \Gamma(1-s)x^{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n\zeta(s-n,\nu+1)x^n}{n!} \tag{3.1.28}$$

for the eBE function. ■

Upon replacing ν by $\nu - 1$ and $x = -\log z$ in Theorem (3.1.4) and applying the relation (2.3.5) lead to the following corollary.

Corollary 3.1.2. *The Hurwitz-Lerch zeta function has a series representation*

$$\Phi(z,s,\nu) = \frac{\Gamma(1-s)}{z^{\nu}}(\log 1/z)^{s-1} + z^{-\nu}\sum_{n=0}^{\infty}\zeta(s-n,\nu)\frac{(\log z)^n}{n!}$$

$$(z \leq 1; 0 < \nu < 1; s \neq 1,2,3,\ldots). \tag{3.1.29}$$

3.2 The generalized Riemann zeta function $\Xi_\nu(s;x)$

In this section I introduce a closely related function to the eFD and the eBE functions defined by (2.2.2) and (2.3.2). These extensions have provided results for the original FD, BE and other functions of the zeta family. The weighted function $\Gamma(s)(1 - 2^{1-s})\Theta_\nu(s;0)$ converges uniformly to $\Gamma(s)(1-2^{1-s})\zeta(s)$ as $\nu \to 0^+$ in every sub-strip $0 < \sigma_1 \leq \sigma \leq \sigma_2 < 1$ of the critical strip $0 < \sigma < 1$. However, the function $\Gamma(s)\Psi_\nu(s;0)$ is not even defined in the critical strip as the integral representation (2.3.2) is divergent in $0 < \sigma < 1$. It is desirable to have a function that converges uniformly to the Riemann zeta function and connects the eFD and eBE functions. The gRZ function is introduced (by using the method of analytic continuation given in section (3.1)) as follows:

$$\Xi_\nu(s;x) := \frac{1}{\Gamma(s)}\int_x^{\infty}(t-x)^{s-1}\left(\frac{1}{e^t-1} - \frac{1}{t}\right)e^{-\nu t}dt$$

$$(0 < \nu < 1; \Re(s) > 0, x > 0; 0 < \Re(s) < 1, \nu = x = 0). \tag{3.2.1}$$

For $x = 0$ and $\nu = 0$ in (3.2.1), (1.2.14) is obtained

$$\zeta(s) \equiv \Xi_0(s;0) := \frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\left(\frac{1}{e^t-1} - \frac{1}{t}\right)dt \quad (0 < \Re(s) < 1). \tag{3.2.2}$$

Theorem 3.2.1. *The gRZ function $\Xi_\nu(s;0)$ is well defined and the weighted function $\Gamma(s)\Xi_\nu(s;0)$ converges uniformly to the weighted Riemann zeta function $\Gamma(s)\zeta(s)$ as $\nu \to 0^+$ in every sub-strip $0 < \sigma_1 \le \sigma \le \sigma_2 < 1$ of the critical strip $0 < \sigma < 1$.*

Proof. First, note that

$$\begin{aligned}|\Gamma(s)\Xi_\nu(s;0)| &= \left|\int_0^\infty t^{s-1}(\tfrac{1}{e^t-1} - \tfrac{1}{t})e^{-\nu t}dt\right| \le \left|\int_0^\infty t^{\sigma-1}(\tfrac{1}{t} - \tfrac{1}{e^t-1})e^{-\nu t}dt\right| \\ &\le \int_0^\infty t^{\sigma-1}(\tfrac{1}{t} - \tfrac{1}{e^t-1})dt = -\Gamma(\sigma)\Xi_0(\sigma) = -\Gamma(\sigma)\zeta(\sigma)\,,\end{aligned} \tag{3.2.3}$$

which shows that the gRZ function $\Xi_\nu(s;0)$ is well defined. Second, that the difference integral representation (as $1 - e^{-\nu t} \le 1$, $0 \le \nu < 1; 0 \le t < \infty$),

$$\begin{aligned}|\Gamma(s)(\Xi_\nu(s;0) - \zeta(s))| &= \left|\int_0^\infty t^{s-1}(\tfrac{1}{e^t-1} - \tfrac{1}{t})(e^{-\nu t} - 1)dt\right| \le \int_0^\infty t^{\sigma-1}(\tfrac{1}{t} - \tfrac{1}{e^t-1})(1 - e^{-\nu t})dt \\ &\le \int_0^\infty t^{\sigma-1}(\tfrac{1}{t} - \tfrac{1}{e^t-1})dt = -\Gamma(\sigma)\Xi_0(\sigma) = -\Gamma(\sigma)\zeta(\sigma) \quad (0 \le \nu < 1; 0 < \sigma_1 \le \sigma \le \sigma_2 < 1),\end{aligned} \tag{3.2.4}$$

is absolutely convergent shows that the limit as $\nu \to 0^+$ and the integral in (3.2.4) are reversible. Letting $\nu \to 0^+$ in (3.2.4), yields that the convergence

$$|\Gamma(s)(\Xi_\nu(s;0) - \zeta(s))| \to 0 \quad (\nu \to 0^+; 0 < \sigma_1 \le \sigma \le \sigma_2 < 1), \tag{3.2.5}$$

is uniform. ∎

Theorem 3.2.2. *The gRZ function is related to the Hurwitz-zeta function as follows:*

$$\Xi_\nu(s;0) = \zeta(s,\nu+1) - \frac{\Gamma(s-1)}{\Gamma(s)}\nu^{1-s} = \zeta(s,\nu+1) - \frac{1}{s-1}\nu^{1-s}$$
$$(0 < \nu < 1; \sigma > 1; \nu = 0; 0 < \sigma < 1). \tag{3.2.6}$$

Proof. For $0 < \nu < 1$ and $\sigma > 1$, (3.2.1) leads to

$$\begin{aligned}\Xi_\nu(s;0) &:= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\frac{1}{e^t-1} - \frac{1}{t})e^{-\nu t}dt = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-\nu t}}{e^t-1}dt - \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}e^{-\nu t}dt \\ &= \zeta(s,\nu+1) - \frac{\Gamma(s-1)}{\Gamma(s)}\nu^{1-s} = \zeta(s,\nu+1) - \frac{\Gamma(s-1)}{(s-1)\Gamma(s-1)}\nu^{1-s} \\ &= \zeta(s,\nu+1) - \frac{1}{s-1}\nu^{1-s} \qquad (0 < \nu < 1; \sigma > 1).\end{aligned} \tag{3.2.7}$$

Note that the RHS in (3.2.7) remains well defined for $0 < \sigma < 1$ and $0 < \nu < 1$. Moreover, for $\nu = 0$, the well-known integral representation (3.2.2) is recovered. Hence the proof. ■

Remark 3.2.1. *The representation* (3.2.6) *of the gRZ function shows that the function is meromorphic. For* $\nu \in (0,1)$ *the function has a removable singularity at* $s = 1$ *as the residue of the function is zero. However, for* $\nu = 0$ *the function has a simple pole at* $s = 1$ *with residue* 1. *Equation* (3.2.6) *can be rewritten as*

$$\Xi_\nu(s;0) = \frac{1}{s-1}[(s-1)\zeta(s,\nu+1) - \nu^{1-s}] \quad (0 < \nu < 1; \nu = 0, 0 < \sigma < 1). \quad (3.2.8)$$

Putting $s = -n$ *and using* (1.4.7)

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1} \quad (n = 0,1,2,...), \quad (3.2.9)$$

it is observed that the function is related to the Bernoulli polynomials as follows

$$\Xi_\nu(-n;0) = \frac{\nu^{n+1} - B_{n+1}(\nu+1)}{n+1} \quad (0 < \nu < 1, n = 0,1,2,3,.......). \quad (3.2.10)$$

Using the relations (see $[51, pp.26-28]$*)*

$$B_{2n+1}(\nu+1) = B_{2n+1}(\nu) + (2n+1)\nu^{2n} \quad , \quad (3.2.11)$$

$$B_{2n+1}(\nu) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_k \nu^{2n+1-k} \quad , \quad (3.2.12)$$

$$B_{2n+1}(0) =: B_{2n+1} = (2n+1)\zeta(-2n) = 0, \quad (3.2.13)$$

$$B_{2n}(0) =: B_{2n} = -2n\zeta(1-2n) \quad (n = 1,2,3,...), \quad (3.2.14)$$

and

$$B_{2n}(0) =: B_{2n} = -2n\zeta(1-2n) \sim (-1)^{n+1}\frac{(4n)!}{(2\pi)^{2n}}(1+2^{-2n}) \quad (n = 3,4,5,...), \quad (3.2.15)$$

one can obtain the closed form

$$\Xi_\nu(-2n;0) = \frac{\nu^{2n+1} - B_{2n+1}(\nu+1)}{2n+1} = \frac{\nu^{2n+1} - (B_{2n+1}(\nu) + (2n+1)\nu^{2n})}{2n+1}$$

$$= \frac{\nu^{2n+1} - \{\sum_{k=0}^{2n+1} \left(\begin{array}{c} 2n+1 \\ k \end{array} \right) B_k \nu^{2n+1-k} + (2n+1)\nu^{2n}\}}{2n+1} \quad (n = 1,2,3,...), \quad (3.2.16)$$

which shows that

$$\Xi_\nu(-2n;0) = -B_{2n}\nu + O(\nu^3) = 2n\zeta(1-2n)\nu + O(\nu^3) \quad (\nu \to 0^+, n = 1,2,3,...). \tag{3.2.17}$$

Thus the gRZ function approximates the trivial zeros ($s = -2, -4, -6, \ldots$) *of the Riemann zeta function as* $\nu \to 0^+$. *The relation* (3.2.17) *gives the rate at which these zeros are approached. One needs to see if all the zeros can be approximated uniformly. Since* $|2n\zeta(1-2n)| \to \infty$ *as* $n \to \infty$, *by setting*

$$\nu_{n,k} = \frac{2^{-k}}{|2n\zeta(1-2n)|} \quad (n = 1,2,3,....), \tag{3.2.18}$$

one can get

$$\sup_{1 \le n < \infty} |\Xi_{\nu_{n;k}}(-2n;0)| = \circ(1) \quad (k \to \infty). \tag{3.2.19}$$

which shows that all the non-trivial zeros can, indeed, be approximated uniformly.

Remark 3.2.2. *It is worth visualizing the behaviour of the function near* $\nu = 0$ *for large* n *more generally. Though* $\Xi_\nu(-n;0)$ *is a function of one continuous and one discrete variable, conceive it as if it were a sheet over the strip* $\nu \in (0,1), \quad n \in (0,\infty)$ *in the* (ν, n)*-plane. At every* n *the sheet approaches the* n*-axis arbitrarily closely, but it does not do so for all* N, *since the sheet rises increasingly more sharply for larger values of* n. *The asymptotic formula for* $\zeta(1-2n)$ *(see* [51, pp.26 − 28]*) can be used in conjunction with Stirling's formula to give the coefficient of* ν *(for small* ν*)*

$$B_{2n}(0) \sim (-1)^{n+1}\frac{(4n)!}{(2\pi)^{2n}}(1 + 2^{-2n}) \sim (-1)^{n+1}\sqrt{8\pi n}(8n^2/\pi e^2)^{2n}. \tag{3.2.20}$$

The function of the discrete variable can be thought of as the parts of the sheet lying over the grid lines of the integer values of n. The sequence where the curve intersects the grid lines gives a path. The non-trivial zeros are then clearly uniformly approximated by paths approaching $\nu = 0$ lying between $\nu = 1/|2n\zeta(1-2n)|$ and the n-axis.

3.3 Connection with the eFD and eBE functions

Theorem 3.3.1. *The gRZ function is related to the eBE function and the incomplete gamma function via*

$$\Psi_\nu(s;x) = \Xi_\nu(s;x) + \Gamma(1-s,\nu x)x^{s-1}$$
$$(x > 0; \sigma > 0, \nu > 0; \nu = 0, 0 < \sigma < 1). \tag{3.3.1}$$

Proof. Considering the identity

$$\frac{e^{-\nu t}}{e^t - 1} = (\frac{1}{e^t - 1} - \frac{1}{t})e^{-\nu t} + \frac{e^{-\nu t}}{t} \tag{3.3.2}$$

and taking the Weyl transform of both sides, lead to

$$\Psi_\nu(s;x) = \Xi_\nu(s;x) + \mathcal{W}^{-s}[\frac{e^{-\nu t}}{t}](x). \tag{3.3.3}$$

However, by using [20, pp. 255-266]

$$\mathcal{W}^{-s}[\frac{e^{-\nu t}}{t}](x) = x^{s-1}\Gamma(1-s,\nu x) \quad (\sigma > 0; \nu > 0; x > 0) \tag{3.3.4}$$

in (3.3.3), one can arrive at (3.3.1). ■

Corollary 3.3.1. *The BE function is related to the gRZ function by*

$$\mathfrak{B}_{s-1}(-x) = \Xi_0(s;x) + \Gamma(1-s)x^{s-1} \quad (0 < \sigma < 1; x > 0). \tag{3.3.5}$$

Proof. This follows from (3.3.1), by taking $\nu = 0$ and using (2.3.3). ■

Theorem 3.3.2. *The gRZ function has a representation*

$$\Xi_\nu(s;x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Xi_\nu(s-n;0)x^n}{n!}$$

$$(x > 0, \sigma > 0, 0 < \nu < 1; x = \nu = 0, 0 < \sigma < 1). \tag{3.3.6}$$

Proof. Note that

$$\Xi_\nu(0;x) := (\frac{1}{e^x - 1} - \frac{1}{x})e^{-\nu x} \in \mathcal{H}(1;0). \tag{3.3.7}$$

Therefore, an application of the general expansion result (2.1.1) leads to (3.3.6). ■

Remark 3.3.1. *The relation* (3.2.6) *can be used to write* (3.3.6) *in terms of the Hurwitz zeta function*

$$\Xi_\nu(s;x) = \sum_{n=0}^{\infty} (-1)^n \Big[\zeta(s-n, \nu+1) + \frac{\nu^{n-s+1}}{n-s+1}\Big] \frac{x^n}{n!}$$

$$(x \geq 0; 0 < \nu < 1; \nu = 0, s \neq 1, 2, \cdots). \tag{3.3.8}$$

This explicit representation tells us that the gRZ function has removable singularities at $s \in \mathbb{N}$ for $0 < \nu < 1$ and simple poles for $\nu = 0$. Further, the relation of the gRZ function with eBE and Hurwitz-Lerch zeta function yields new series representation for these functions.

Corollary 3.3.2. *The eBE function has a series representation*

$$\Psi_\nu(s;x) = \Gamma(1-s, \nu x)x^{s-1} + \sum_{n=0}^{\infty} (-1)^n \Big[\zeta(s-n, \nu+1) + \frac{\nu^{n-s+1}}{n-s+1}\Big] \frac{x^n}{n!}$$

$$(x \geq 0; 0 < \nu < 1; \nu = 0, s \neq 1, 2, \cdots). \tag{3.3.9}$$

Proof. This follows by using (3.3.1) in (3.3.8). ■

Corollary 3.3.3. *The Hurwitz-Lerch zeta function has a series representation*

$$\Phi(z, s, \nu+1) = \frac{\Gamma(1-s, \log 1/z^\nu)}{z^{\nu+1}} (\log 1/z)^{s-1} + z^{-(\nu+1)}$$
$$\sum_{n=0}^{\infty} (-1)^n \Big[\zeta(s-n, \nu+1) + \frac{\nu^{n-s+1}}{n-s+1}\Big] \frac{(\log z)^n}{n!}$$

$$(z \leq 1; 0 < \nu < 1; \nu = 0, s \neq 1, 2, \cdots). \tag{3.3.10}$$

Proof. The Hurwitz-Lerch zeta function and the eBE functions are related by (2.3.5) and by making use of this relationship in (3.3.9), one can obtain the result. ■

Remark 3.3.2. *Note that* (3.1.21), (3.1.26) *and* (3.1.29) *are the special cases of* (3.3.9). *Therefore the series representation obtained in this section is the most general form of the series for the BE, eBE and the Hurwitz-Lerch zeta functions. The following interesting special case of* (3.3.6) *arises when* $\nu = s = 0$

$$\Xi_0(0; x) = \frac{1}{e^x - 1} - \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \zeta(-n) x^n}{n!} \quad (x \geq 0). \tag{3.3.11}$$

Remark 3.3.3. *The series representation (3.1.29) for the Hurwitz-Lerch zeta function has simple poles at* $s \in \mathbb{N}$ *while the series representation obtained in this section along with the remark (3.2.1) shows that it has removable singularities at these points.*

Theorem 3.3.3. *The gRZ and the eFD functions are related by*

$$2^{1-s}\Xi_\nu(s; 2x) = \Xi_{2\nu}(s; x) - \Theta_{2\nu}(s; x)$$
$$(\sigma > 0, \nu > 0, x > 0; x = \nu = 0, 0 < \sigma < 1). \tag{3.3.12}$$

Proof. Considering the identity

$$2\left(\frac{e^{-2\nu t}}{e^{2t} - 1} - \frac{e^{-2\nu t}}{2t}\right) = \left(\frac{1}{e^t - 1} - \frac{1}{t}\right)e^{-2\nu t} - \frac{e^{-2\nu t}}{e^t + 1} \tag{3.3.13}$$

and taking the Weyl transform of both sides in (3.3.13), one can get

$$\begin{aligned} &2\mathcal{W}^{-s}[\tfrac{e^{-2\nu t}}{e^{2t}-1} - \tfrac{e^{-2\nu t}}{2t}](x) \\ &= \mathcal{W}^{-s}[\tfrac{e^{-2\nu t}}{e^t-1} - \tfrac{e^{-2\nu t}}{t}](x) - \mathcal{W}^{-s}[\tfrac{e^{-2\nu t}}{e^t+1}](x) = \Xi_{2\nu}(s; x) - \Theta_{2\nu}(s; x). \end{aligned} \tag{3.3.14}$$

However, by considering the representation

$$2\mathcal{W}^{-s}[\frac{e^{-2\nu t}}{e^{2t} - 1} - \frac{e^{-2\nu t}}{2t}](x) = \frac{2}{\Gamma(s)} \int_x^\infty (t - x)^{s-1} \left(\frac{1}{e^{2t} - 1} - \frac{1}{2t}\right) e^{-2\nu t} dt. \tag{3.3.15}$$

and making the substitution $t = \tau/2$ in (3.3.15) leads to

$$\begin{aligned} &2\mathcal{W}^{-s}[\tfrac{e^{-2\nu t}}{e^{2t}-1} - \tfrac{e^{-2\nu t}}{2t}](x) = \tfrac{1}{\Gamma(s)} \int\limits_{2x}^{\infty} (\tau/2 - x)^{s-1}(\tfrac{1}{e^{\tau}-1} - \tfrac{1}{\tau})e^{-\nu\tau}d\tau \\ &= \tfrac{2^{1-s}}{\Gamma(s)} \int\limits_{2x}^{\infty} (\tau - 2x)^{s-1}(\tfrac{1}{e^{\tau}-1} - \tfrac{1}{\tau})e^{-\nu\tau}d\tau = 2^{1-s}\Xi_{\nu}(s; 2x) \end{aligned} . \tag{3.3.16}$$

From (3.3.14-3.3.16), one can arrive at (3.3.12). ■

Remark 3.3.4. *It is useful to write* (3.3.12) *in the form*

$$\begin{aligned} &\Theta_{2\nu}(s, x) = \Xi_{2\nu}(s; x) - 2^{1-s}\Xi_{\nu}(s; 2x) \\ &\qquad (\sigma > 0, \nu > 0, x > 0; x = \nu = 0, 0 < \sigma < 1). \end{aligned} \tag{3.3.17}$$

Putting $v = x = 0$ *in* (3.3.17), *the classical integral representation*

$$\Theta_0(s, 0) = (1 - 2^{1-s})\zeta(s) = \frac{1}{\Gamma(s)} \int\limits_0^{\infty} \frac{t^{s-1}}{e^t + 1} dt \quad (\sigma > 0) \tag{3.3.18}$$

is recovered for the weighted Riemann zeta function. Note that the simple pole of the zeta function at $s = 1$ *is canceled by the (simple) zero of the factor* $1 - 2^{1-s}$ *such that the product* $\Theta_0(s, 0) = (1-2^{1-s})\zeta(s)$ *remains well defined in the sense of the Riemann removable singularity theorem. Moreover using* (2.2.13), *the relation* (3.3.12) *can be written in terms of the Hurwitz-Lerch zeta function as*

$$\begin{aligned} &\Xi_{2\nu}(s; x) - 2^{1-s}\Xi_{\nu}(s; 2x) = e^{-(2\nu+1)x}\Phi(-e^{-x}, s, 2\nu + 1) \\ &\qquad (\sigma > 0, \nu > 0, x > 0; x = \nu = 0, 0 < \sigma < 1). \end{aligned} \tag{3.3.19}$$

Corollary 3.3.4. *The gRZ function is connected with the FD function as follows:*

$$\Xi_0(s;x) - 2^{1-s}\Xi_0(s;2x) = \mathfrak{F}_{s-1}(-x) \quad (0 < \sigma < 1). \tag{3.3.20}$$

Proof. This follows from (2.2.3) and (3.3.12), by putting $\nu = 0$. ■

3.4 Difference equation for the gRZ function

Functional relations arising from difference equations are useful for the study of special functions. For example, the Hurwitz zeta function satisfies the difference equation

$$\zeta(s, v) - \zeta(s, v+1) = v^{-s}. \tag{3.4.1}$$

Do the generalized Riemann-zeta functions also satisfy such relations? The next theorem gives the answer to this question.

Theorem 3.4.1. *The gRZ function $\Xi_\nu(s;x)$ satisfies the difference equation*

$$\begin{aligned}&\Xi_\nu(s;x) - \Xi_{\nu+1}(s;x)\\ &= (\nu+1)^{-s}e^{-(\nu+1)x} - x^{s-1}\left(\Gamma(1-s,\nu x) - \Gamma(1-s,(\nu+1)x)\right)\\ &\qquad (x > 0; \nu > 0, \sigma > 0; \nu = 0, 0 < \sigma < 1).\end{aligned} \tag{3.4.2}$$

Proof. Since the following identity holds true

$$\frac{e^{-\nu t}}{e^t - 1} - \frac{e^{-\nu t}}{t} - \frac{e^{-(\nu+1)t}}{e^t - 1} + \frac{e^{-(\nu+1)t}}{t} = e^{-(\nu+1)t} - \frac{e^{-\nu t}}{t} + \frac{e^{-(\nu+1)t}}{t} \quad (\nu \geq 0), \tag{3.4.3}$$

therefore applying the Weyl transform on both sides in (3.4.3) yields

$$\begin{aligned}&\mathcal{W}^{-s}[\tfrac{e^{-\nu t}}{e^t-1} - \tfrac{e^{-\nu t}}{t}](x) - \mathcal{W}^{-s}[\tfrac{e^{-(\nu+1)t}}{e^t-1} - \tfrac{e^{-(\nu+1)t}}{t}](x)\\ &= \mathcal{W}^{-s}[e^{-(\nu+1)t}](x) - \mathcal{W}^{-s}[\tfrac{e^{-\nu t}}{t}](x) + \mathcal{W}^{-s}[\tfrac{e^{-(\nu+1)t}}{t}](x)\end{aligned} \tag{3.4.4}$$

Next, using [20, Vol. II, p. 202],

$$\mathcal{W}^{-s}[e^{-at}](x) = a^{-s}e^{-ax} \tag{3.4.5}$$

and [21, p. 262(6.9.2)(21)]

$$\mathcal{W}^{-s}[\frac{1}{t}e^{-at}](x) = x^{s-1}\Gamma(1-s, ax), \tag{3.4.6}$$

in (3.4.4) leads to (3.4.2). ■

Chapter 4

Fourier transform and distributional representations of the eFD and the eBE functions

In this chapter I present the Fourier transform and distributional representations of the eFD and eBE functions. This leads to similar representations for the polylogarithm, FD, BE, Hurwitz, Lerch and Riemann zeta functions. Applications of these representations have been shown in evaluation of some integrals of products of these functions.

This chapter consists of 7 sections. New identities involving the eFD and eBE functions are proved in Section (4.1). Fourier transform representation of the pair of eFD and eBE functions is given in Section (4.2). Functional relations involving the integrals of products of gamma function with these extended functions and other zeta-related functions are obtained in Section (4.3). By using the duality property, Fourier transforms of these functions are obtained in Section (4.4). Special cases of these Fourier transforms provide some more interesting integral formulae involving the family of zeta function. The distributional representation of the pair of eFD and eBE functions is given in Section (4.5). This leads to a distributional representation for the zeta and related functions by selecting the values of some of the parameters. Using the elementary properties of the delta function, some applications of distributional representation are given in Section (4.6). The consistency of the results obtained by the distributional representation is also checked against the results obtained by Fourier transform representation. It leads to formulae for the integrals of the product of the Riemann zeta, Hurwitz zeta, Hurwitz-Lerch zeta, FD and BE functions with the gamma function, which are not included in [11, 15, 20, 21, 25, 31, 74].

Again note that the eFD and eBE functions fail to produce any result for the Riemann zeta function in the critical strip. For this reason Fourier transform and distributional representations of the gRZ function are considered in Section (4.7). By doing so, some integrals involving the generalized Riemann zeta function and its special cases have been obtained.

4.1 New identities involving the eFD and eBE functions

The following identity is a generalization of [7, 2.5] and will prove useful to evaluate the integrals of products of functions by using Parseval's identity of Fourier transforms.

Theorem 4.1.1. *The eFD function* $\Theta_\nu(\eta; x)$ *satisfies the following equation*

$$e^{(\nu+2)x}\Gamma(\eta)[(\nu+1)\Theta_\nu(\eta;x) - \Theta_\nu(\eta-1;x)] = \int_0^\infty \frac{e^{-\nu t}t^{\eta-1}}{(e^t+e^{-x})^2}dt$$

$$(\Re(\nu) > -1; x \geq 0; \eta > 1). \qquad (4.1.1)$$

Proof. Differentiating

$$f(t,x,\nu) := \frac{e^{-\nu t}}{e^{t+x}+1} \qquad (t > 0; x \geq 0; \Re(\nu) > -1), \qquad (4.1.2)$$

one obtains the differential equation

$$f'(t,x,\nu)+(\nu+1)f(t,x,\nu)=\frac{e^{-\nu t}}{(e^{t+x}+1)^2}. \qquad (4.1.3)$$

Taking the Mellin transform (see, for details, Section (1.5)) of both sides in the real variable η in (4.1.2-4.1.3) and making use of (2.2.12) leads to

$$\mathcal{M}[f(t,x,\nu);\eta]=\Theta_\nu(\eta;x)\Gamma(\eta)e^{\nu x}, \qquad (4.1.4)$$

and

$$\mathcal{M}[f'(t,x,\nu);\eta]=-(\nu+1)\mathcal{M}[f(t,x,\nu);\eta]+\mathcal{M}[\frac{e^{-\nu t}}{(e^{t+x}+1)^2};\eta]. \qquad (4.1.5)$$

However, $\mathcal{M}[f(t,x,\nu);\eta]$ and $\mathcal{M}[f'(t,x,\nu);\eta]$ are related by

$$\mathcal{M}[f'(t,x,\nu);\eta]=-(\eta-1)\mathcal{M}[f(t,x,\nu);\eta-1], \qquad (4.1.6)$$

provided $t^{\eta-1}f(t,x,\nu)$ vanishes at zero and infinity. Hence, from (4.1.4-4.1.6), one gets

$$e^{\nu x}\Gamma(\eta)[(\nu+1)\Theta_\nu(\eta;x)-\Theta_\nu(\eta-1;x)]=\int_0^\infty\frac{e^{-\nu t}t^{\eta-1}}{(e^{t+x}+1)^2}, \qquad (4.1.7)$$

which after simplification gives the result (4.1.1). ■

Corollary 4.1.1. *The following identity involving the FD function holds*

$$e^{-2x}\Gamma(\eta)[\mathfrak{F}_{\eta-1}(x)-\mathfrak{F}_{\eta-2}(x)]=\int_0^\infty\frac{t^{\eta-1}}{(e^t+e^x)^2}dt \qquad (\eta>1;x\geq 0). \qquad (4.1.8)$$

Proof. Upon putting $\nu=0$, replacing $x\mapsto -x$ in (4.1.1) and making use of (2.2.3) one gets (4.1.8). ■

Corollary 4.1.2. *The Riemann zeta function* (1.2.16) *satisfies* [7, *p.*6]

$$\Gamma(\eta)[(1-2^{1-\eta})\zeta(\eta)-(1-2^{2-\eta})\zeta(\eta-1)]=\int_0^\infty\frac{t^{\eta-1}}{(e^t+1)^2}dt \qquad (\eta>1). \qquad (4.1.9)$$

Proof. Upon putting $x = \nu = 0$ in (4.1.1) and making use of (2.2.17) one can obtain (4.1.9). ■

Theorem 4.1.2. *The eBE function satisfies*

$$e^{(\nu+2)x}\Gamma(\eta)[\Psi_\nu(\eta-1;x)-(\nu+1)\Psi_\nu(\eta;x)] = \int_0^\infty \frac{e^{-\nu t}t^{\eta-1}}{(e^t-e^{-x})^2}dt$$

$$(\Re(\nu) > -1; \eta > 1 \textit{ when } x > 0; \eta > 2 \textit{ when } x = 0). \qquad (4.1.10)$$

Proof. This follows on similar steps as in Theorem (4.1.1) by taking

$$f_1(t,x,\nu) = \frac{e^{-\nu t}}{e^{t+x}-1} \quad (t > 0; x \geq 0; \Re(\nu) > -1) \qquad (4.1.11)$$

in place of $f(t,x,\nu)$ and using (2.3.4). ■

Corollary 4.1.3. *The following identity involving the Hurwitz-Lerch zeta function holds*

$$\frac{\Gamma(\eta)}{z}[\Phi(z,\eta-1,\nu)-\nu\Phi(z,\eta,\nu)] = \int_0^\infty \frac{e^{-(\nu-1)t}t^{\eta-1}}{(e^t-z)^2}dt$$

$$(\Re(\nu) > 0; \eta > 1 \textit{ when } 0 < z < 1; \eta > 2 \textit{ when } z = 1). \qquad (4.1.12)$$

Proof. This follows by replacing $\nu \mapsto \nu - 1$ and using (2.3.5) in (4.1.10) . ■

Corollary 4.1.4. *The polylogarithm satisfies the following identity*

$$\frac{\Gamma(\eta)}{z^2}[Li_{\eta-1}(z)-Li_\eta(z)] = \int_0^\infty \frac{t^{\eta-1}}{(e^t-z)^2}dt$$

$$(\eta > 1 \textit{ when } 0 < z < 1; \eta > 2 \textit{ when } z = 1). \qquad (4.1.13)$$

Proof. Upon putting $\nu = 0, z = e^{-x}$ in (4.1.10) and using (1.2.33), one gets (4.1.13). ■

Corollary 4.1.5. *The following identity involving the BE function holds*

$$\Gamma(\eta)e^{-2x}[\mathfrak{B}_{\eta-2}(x)-\mathfrak{B}_{\eta-1}(x)] = \int_0^\infty \frac{t^{\eta-1}}{(e^t-e^x)^2}dt \qquad (\eta > 1, x \geq 0). \qquad (4.1.14)$$

Proof. Upon putting $\nu = 0$, $x \mapsto -x$ in (4.1.10) and making use of (2.3.3) one can get (4.1.14). ■

Corollary 4.1.6. *The Hurwitz zeta function satisfies* [20, $Vol.I, p.313$]

$$\Gamma(\eta)[\zeta(\eta-1,\nu)-\nu\zeta(\eta,\nu)] = \int_0^\infty \frac{e^{-(\nu-1)t}t^{\eta-1}}{(e^t-1)^2}dt \quad (\Re(\nu)>0; \eta>2). \tag{4.1.15}$$

Proof. Upon putting $x = 0$, $\nu \mapsto \nu - 1$ in (4.1.10) and making use of (2.3.7) one can obtain (4.1.15). ■

Corollary 4.1.7. *The Riemann zeta function* (1.2.6) *satisfies* [20, $Vol.I, p.313$]

$$\Gamma(\eta)[\zeta(\eta-1)-\zeta(\eta)] = \int_0^\infty \frac{t^{\eta-1}}{(e^t-1)^2}dt \qquad (\eta>2). \tag{4.1.16}$$

Proof. Upon putting $x = \nu = 0$ in (4.1.10) and suitably combining (2.3.7) with (1.2.29) one gets (4.1.16). ■

4.2 Fourier transform representation of the eFD and eBE functions

Now I proceed to the Fourier transform representation of the eFD, FD, eBE, BE, Hurwitz-Lerch zeta, polylogarithm, Hurwitz and Riemann zeta functions.

The eBE function has the Mellin transform representation (2.3.4) and a substitution $t = e^y$ in this representation yields

$$\Gamma(\sigma+i\tau)\Psi_\nu(\sigma+i\tau;x) = e^{-(\nu+1)x}\sqrt{2\pi}\mathcal{F}[\frac{e^{\sigma y}\exp(-\nu e^y)}{\exp(e^y)-e^{-x}};\tau]$$

$$(\Re(\nu)>-1; \sigma>1 \text{ when } x=0; \sigma>0 \text{ when } x>0). \tag{4.2.1}$$

This is the Fourier transform representation of the eBE function, where the Fourier transform is defined by (1.5.13). Similarly for the Hurwitz-Lerch zeta and the polylogarithm functions, one gets the following

$$\Gamma(\sigma+i\tau)\Phi(z,\sigma+i\tau,\nu)=\sqrt{2\pi}\mathcal{F}[\frac{e^{\sigma y}\exp(-(\nu-1)e^{y})}{\exp(e^{y})-z};\tau]$$

$$(\Re(\nu)>0;\text{ and either }|z|\le 1;z\ne 1;\sigma>0\text{ or }z=1;\sigma>1) \quad (4.2.2)$$

$$\Gamma(\sigma+i\tau)\mathrm{Li}_{\sigma+i\tau}(z)=\sqrt{2\pi}z\mathcal{F}[\frac{e^{\sigma y}}{\exp(e^{y})-z};\tau]$$

$$(|z|\le 1-\delta,\delta\in(0,1)\text{ and }\sigma>0;z=1\text{ and }\sigma>1). \quad (4.2.3)$$

The BE function can be written as

$$\Gamma(\sigma+i\tau)\mathfrak{B}_{\sigma+i\tau-1}(x)=\sqrt{2\pi}e^{x}\mathcal{F}[\frac{e^{\sigma y}}{\exp(e^{y})-e^{x}};\tau]$$

$$(x\ge 0;\sigma>1). \quad (4.2.4)$$

The Hurwitz and the Riemann zeta (1.2.6) functions have the Fourier transform representations

$$\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau,\nu)=\sqrt{2\pi}\mathcal{F}[\frac{e^{\sigma y}\exp(-(\nu-1)e^{y})}{\exp(e^{y})-1};\tau]$$

$$(\Re(\nu)>0;\sigma>1). \quad (4.2.5)$$

$$\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)=\sqrt{2\pi}\mathcal{F}[\frac{e^{\sigma y}}{\exp(e^{y})-1};\tau] \quad (\sigma>1). \quad (4.2.6)$$

The Fourier transform representation of the eFD function is

$$\Gamma(\sigma+i\tau)\Theta_{\nu}(\sigma+i\tau;x)=e^{-(\nu+1)x}\sqrt{2\pi}\mathcal{F}[\frac{e^{\sigma y}\exp(-\nu e^{y})}{\exp(e^{y})+e^{-x}};\tau]$$

$$(\Re(\nu)>-1;x\ge 0;\sigma>0). \quad (4.2.7)$$

Similarly, the FD and Riemann zeta (1.2.16) functions have the following Fourier transform representations respectively

$$\Gamma(\sigma+i\tau)\mathfrak{F}_{\sigma+i\tau-1}(x)=\sqrt{2\pi}e^{x}\mathcal{F}[\frac{e^{\sigma y}}{\exp(e^{y})+e^{x}};\tau]$$

$$(x\geq 0;\sigma>1). \qquad (4.2.8)$$

$$C(\sigma+i\tau)\zeta(\sigma+i\tau)=\sqrt{2\pi}\mathcal{F}[\frac{e^{\sigma y}}{\exp(e^{y})+1};\tau] \quad (\sigma>0), \qquad (4.2.9)$$

where $C(s)$ is defined in (1.2.17).

4.3 Some applications of the Fourier transform representation by using Parseval's identity

Using Parseval's identity (1.5.14) for equation (4.2.1), one gets

$$\int_{-\infty}^{+\infty}\Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\Psi_{\nu}(\sigma+i\tau;x)\Psi_{\nu}(\rho-i\tau;x)d\tau=2\pi\int_{0}^{\infty}\frac{e^{-2(\nu+1)x}e^{-2\nu t}t^{\sigma+\rho-1}}{(e^{t}-e^{-x})^{2}}dt$$

$$(\Re(\nu)>-1;\sigma+\rho>1 \text{ when } x=0;\sigma+\rho>0 \text{ when } x>0). \qquad (4.3.1)$$

The integral on the right hand side of (4.3.1) can be evaluated by replacing $\eta\mapsto\sigma+\rho$ and $\nu\mapsto 2\nu$ in (4.1.10). This gives the following result for the *eBE function*

$$\int_{-\infty}^{+\infty}\Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\Psi_{\nu}(\sigma+i\tau;x)\Psi_{\nu}(\rho-i\tau;x)d\tau$$

$$=2\pi\Gamma(\sigma+\rho)[\Psi_{2\nu}(\sigma+\rho-1;x)-(2\nu+1)\Psi_{2\nu}(\sigma+\rho;x)]$$

$$(\Re(\nu)>-1;\sigma+\rho>1 \text{ when } x>0;\sigma+\rho>2 \text{ when } x=0) \qquad (4.3.2)$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma + i\tau)\Psi_\nu(\sigma + i\tau; x)|^2 d\tau = 2\pi\Gamma(2\sigma)[\Psi_{2\nu}(2\sigma - 1; x) - (2\nu + 1)\Psi_{2\nu}(2\sigma; x)]$$

$$(\Re(\nu) > -1; \sigma > 1/2 \text{ when } x > 0; \sigma > 1 \text{ when } x = 0). \tag{4.3.3}$$

Using (2.3.5) and (4.1.12) in (4.3.1), one obtains the following identity for the *Hurwitz-Lerch zeta function*

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\Phi(z, \sigma + i\tau, \nu)\Phi(z, \rho - i\tau, \nu)d\tau$$

$$= \frac{2\pi\Gamma(\sigma + \rho)}{z}[\Phi(z, \sigma + \rho - 1, 2\nu - 1) - (2\nu - 1)\Phi(z, \sigma + \rho, 2\nu - 1)]$$

$$(\Re(\nu) > 0; \sigma + \rho > 1 \text{ when } 0 < z < 1; \sigma + \rho > 2 \text{ when } z = 1) \tag{4.3.4}$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma + i\tau)\Phi(z, \sigma + i\tau, \nu)|^2 d\tau$$

$$= \frac{2\pi\Gamma(2\sigma)}{z}[\Phi(z, 2\sigma - 1, 2\nu - 1) - (2\nu - 1)\Phi(z, 2\sigma, 2\nu - 1)]$$

$$(\Re(\nu) > 1/2; \sigma > 1/2 \text{ when } 0 < z < 1; \sigma > 1 \text{ when } z = 1). \tag{4.3.5}$$

Similarly by using (1.2.33) and (4.1.13) in (4.3.1), one can arrive at the following result for the *polylogarithm function*

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\mathrm{Li}_{\sigma+i\tau}(z)\mathrm{Li}_{\rho-i\tau}(z)d\tau = 2\pi\Gamma(\sigma + \rho)[\mathrm{Li}_{\sigma+\rho-1}(z) - \mathrm{Li}_{\sigma+\rho}(z)]$$

$$(\sigma + \rho > 1 \text{ when } 0 < z < 1; \sigma + \rho > 2 \text{ when } z = 1) \tag{4.3.6}$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma + i\tau)\mathrm{Li}_{\sigma+i\tau}(z)|^2 d\tau = 2\pi\Gamma(2\sigma)[\mathrm{Li}_{2\sigma-1}(z) - \mathrm{Li}_{2\sigma}(z)]$$

$$(\sigma > 1/2 \text{ when } 0 < z < 1; \sigma > 1 \text{ when } z = 1). \tag{4.3.7}$$

Upon using (2.3.3) and (4.1.14) in (4.3.1), one gets the following results for the *BE function*

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\mathfrak{B}_{\sigma+i\tau-1}(x)\mathfrak{B}_{\rho-i\tau-1}(x)d\tau$$
$$= 2\pi\Gamma(\sigma+\rho)[\mathfrak{B}_{\sigma+\rho-2}(x)-\mathfrak{B}_{\sigma+\rho-1}(x)] \quad (x \geq 0; \sigma+\rho>2) \tag{4.3.8}$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma+i\tau)\mathfrak{B}_{\sigma+i\tau-1}(x)|^2 d\tau = 2\pi\Gamma(2\sigma)[\mathfrak{B}_{2\sigma-2}(x)-\mathfrak{B}_{2\sigma-1}(x)] \quad (x \geq 0; \sigma > 1). \tag{4.3.9}$$

Using (2.3.7) and (4.1.15) in (4.3.1), one can arrive at the following identity for the *Hurwitz zeta function*

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\zeta(\sigma+i\tau,\nu)\zeta(\rho-i\tau,\nu)d\tau$$
$$= 2\pi\Gamma(\sigma+\rho)[\zeta(\sigma+\rho-1,2\nu-1)-(2\nu-1)\zeta(\sigma+\rho,2\nu-1)]$$
$$(\Re(\nu) > 1/2; \sigma+\rho>2) \tag{4.3.10}$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau,\nu)|^2 d\tau = 2\pi\Gamma(2\sigma)[\zeta(2\sigma-1,2\nu-1)-(2\nu-1)\zeta(2\sigma,2\nu-1)]$$
$$(\Re(\nu) > 1/2; \sigma > 1). \tag{4.3.11}$$

Similarly by suitably combining (2.3.7) with (1.2.29) and using (4.1.16) in (4.3.1), one can arrive at the following identity for the *Riemann zeta function*

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\zeta(\sigma+i\tau)\zeta(\rho-i\tau)d\tau = 2\pi\Gamma(\sigma+\rho)[\zeta(\sigma+\rho-1)-\zeta(\sigma+\rho)]$$
$$(\sigma+\rho>2) \tag{4.3.12}$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma + i\tau)\zeta(\sigma + i\tau)|^2 d\tau = 2\pi\Gamma(2\sigma)[\zeta(2\sigma - 1) - \zeta(2\sigma)] \quad (\sigma > 1). \qquad (4.3.13)$$

Now Parseval's identity for the Fourier transform representation (4.2.7) of the *eFD function* produces the following identity

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\Theta_\nu(\sigma + i\tau; x)\Theta_\nu(\rho - i\tau; x)d\tau = 2\pi \int_0^\infty \frac{e^{-2(\nu+1)x}e^{-2\nu t}t^{\sigma+\rho-1}}{(e^t + e^{-x})^2}dt$$

$$(\Re(\nu) > 0; x \geq 0; \sigma, \rho > 0). \qquad (4.3.14)$$

The integral on the right hand side of (4.3.14) can be evaluated by $\eta \mapsto \sigma+\rho$ and $\nu \mapsto 2\nu$ in (4.1.1). This gives following result for the *eFD function*

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\Theta_\nu(\sigma + i\tau; x)\Theta_\nu(\rho - i\tau; x)d\tau$$

$$= 2\pi\Gamma(\sigma + \rho)[(2\nu + 1)\Theta_{2\nu}(\sigma + \rho; x) - \Theta_{2\nu}(\sigma + \rho - 1; x)]$$

$$(\Re(\nu > 0; x \geq 0; \sigma + \rho > 1) \qquad (4.3.15)$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma + i\tau)\Theta_\nu(\sigma + i\tau; x)|^2 d\tau = 2\pi\Gamma(2\sigma)[(2\nu + 1)\Theta_{2\nu}(2\sigma; x) - \Theta_{2\nu}(2\sigma - 1; x)]$$

$$(\Re(\nu) > 0; x \geq 0; \sigma > 1/2). \qquad (4.3.16)$$

Similarly by using (1.3.4) and (4.1.8) in (4.3.16) one gets the following result for the *FD function*

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\mathfrak{F}_{\sigma+i\tau-1}(x)\mathfrak{F}_{\rho-i\tau-1}(x)d\tau$$

$$= 2\pi\Gamma(\sigma + \rho)[\mathfrak{F}_{\sigma+\rho-1}(x) - \mathfrak{F}_{\sigma+\rho-2}(x)] \quad (x \geq 0; \sigma + \rho > 1) \qquad (4.3.17)$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma + i\tau)\mathfrak{F}_{\sigma+i\tau-1}(x)|^2 d\tau = 2\pi\Gamma(2\sigma)[\mathfrak{F}_{2\sigma-1}(x) - \mathfrak{F}_{2\sigma-2}(x)] \qquad (x \geq 0; \sigma > 1/2). \tag{4.3.18}$$

Upon using (2.2.17) and (4.1.9) in (4.3.16), one can arrive at the following identity for the *Riemann zeta function*

$$\int_{-\infty}^{+\infty} C(\sigma + i\tau)C(\rho - i\tau)\zeta(\sigma + i\tau)\zeta(\rho - i\tau)d\tau$$
$$= 2\pi\Gamma(\sigma + \rho)[(1 - 2^{1-\sigma-\rho})\zeta(\sigma + \rho) - (1 - 2^{2-\sigma-\rho})\zeta(\sigma + \rho - 1)] \quad (\sigma + \rho > 1). \tag{4.3.19}$$

and special case $\rho = \sigma$ leads to

$$\int_{-\infty}^{+\infty} |C(\sigma + i\tau)\zeta(\sigma + i\tau)|^2 d\tau = 2\pi\Gamma(2\sigma)[(1 - 2^{1-2\sigma})\zeta(2\sigma) - (1 - 2^{2-2\sigma})\zeta(2\sigma - 1)]$$
$$(\sigma > 1/2). \tag{4.3.20}$$

By making use of Parseval's identity for (4.2.2) and (4.2.4) one gets

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\Phi(e^{-x}, \sigma + i\tau, \nu)\mathfrak{B}_{\rho-i\tau-1}(-x)d\tau$$
$$= 2\pi\Gamma(\sigma + \rho)[\Phi(e^{-x}, \sigma + \rho - 1, \nu) - \nu\Phi(e^{-x}, \sigma + \rho, \nu)]$$
$$(\Re(\nu) > 0; \sigma + \rho > 1 \text{ when } x > 0; \sigma + \rho > 2 \text{ when } x = 0) \tag{4.3.21}$$

Here the left hand side is obtained by using (4.1.12), however, for $\nu = 1$ in the above relation can lead to (4.3.9). Now by making use of (4.1.14) and Parseval's identity for (4.2.3) and (4.2.4), one can get

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\mathfrak{B}_{\sigma+i\tau-1}(-x)\mathrm{Li}_{\rho-i\tau}(e^{-x})d\tau$$
$$= 2\pi\Gamma(\sigma + \rho)[\mathfrak{B}_{\sigma+\rho-2}(-x) - \mathfrak{B}_{\sigma+\rho-1}(-x))]$$
$$(\sigma + \rho > 1 \text{ when } x > 0; \sigma + \rho > 2 \text{ when } x = 0) \tag{4.3.22}$$

By making use of (4.1.13) in place of (4.1.14), the left hand side of the above equation can also be written in terms of the polylogarirhm function.

Using (4.1.15) and Parseval's identity for (4.2.5) and (4.2.6), one gets

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\zeta(\sigma+i\tau,\nu)\zeta(\rho-i\tau)d\tau$$
$$= 2\pi\Gamma(\sigma+\rho)[\zeta(\sigma+\rho-1,\nu)-\nu\zeta(\sigma+\rho,\nu)]$$
$$(\Re(\nu)>0;\sigma+\rho>2). \tag{4.3.23}$$

For $\nu = 1$ the above identity reduces to (4.3.13). Now an application of Parseval's identity for (4.2.7) and (4.2.8) leads to

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\Theta_\nu(\sigma+i\tau;x)\mathfrak{F}_{\rho-i\tau-1}(-x)d\tau$$
$$= 2\pi\Gamma(\sigma+\rho)[(\nu+1)\Theta_\nu(\sigma+\rho;x)-\Theta_\nu(\sigma+\rho-1;x)]$$
$$(\Re(\nu)>-1;x\geq 0;\sigma+\rho>1). \tag{4.3.24}$$

However for $\nu = 0$, this reduces to (4.3.18). Similarly more results can be obtained by making use of Parseval's identity for Fourier transform representations of different functions obtained in Section 3. This yields

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\Theta_\nu(\sigma+i\tau;x)\Psi_\nu(\rho-i\tau;x)d\tau = 2^{1-\sigma-\rho}\pi\Psi_\nu(\sigma+\rho;2x)$$
$$(\Re(\nu)>-1;x\geq 0;\sigma+\rho>1); \tag{4.3.25}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\mathfrak{F}_{\sigma+i\tau-1}(-x)\Phi(e^{-x},\rho-i\tau,\nu)d\tau$$
$$= e^{-x}2^{1-\sigma-\rho}\pi\Phi(e^{-2x},\sigma+\rho,\frac{\nu+1}{2}) \qquad (\Re(\nu)>-1;x\geq 0;\sigma+\rho>1); \tag{4.3.26}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\mathfrak{F}_{\sigma+i\tau-1}(-x)\mathrm{Li}_{\rho-i\tau}(e^{-x})d\tau = 2^{1-\sigma-\rho}\pi\mathrm{Li}_{\sigma+\rho}(e^{-2x})$$

$$(x \geq 0; \sigma+\rho > 1); \quad (4.3.27)$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)\mathfrak{F}_{\sigma+i\tau-1}(-x)\mathfrak{B}_{\rho-i\tau-1}(-x))d\tau = 2^{1-\sigma-\rho}\pi e^{-2x}\mathfrak{B}_{\sigma+\rho-1}(-2x)$$

$$(x \geq 0; \sigma+\rho > 1); \quad (4.3.28)$$

$$\int_{-\infty}^{+\infty} C(\sigma+i\tau)\Gamma(\rho-i\tau)\zeta(\sigma+i\tau)\zeta(\rho-i\tau)d\tau = 2^{1-\sigma-\rho}\pi\zeta(\sigma+\rho) \quad (\sigma+\rho > 1). \quad (4.3.29)$$

4.4 Some applications of the Fourier transform representation by using the duality property

Equation (1.5.15) leads to Fourier transform of the *eBE* function by taking Fourier transform of both sides of (4.2.1)

$$\mathcal{F}[\Gamma(\sigma+i\tau)\Psi_\nu(\sigma+i\tau;x);\omega] = \frac{\sqrt{2\pi}e^{-(\nu+1)x}e^{-\sigma\omega}\exp(-\nu e^{-\omega})}{\exp(e^{-\omega})-e^{-x}}$$

$$(\Re(\nu) > -1; \sigma > 0 \text{ when } x > 0; \sigma > 1 \text{ when } x = 0). \quad (4.4.1)$$

Similarly for the *eFD* function we have

$$\mathcal{F}[\Gamma(\sigma+i\tau)\Theta_\nu(\sigma+i\tau;x);\omega] = \frac{\sqrt{2\pi}e^{-(\nu+1)x}e^{-\sigma\omega}\exp(-\nu e^{-\omega})}{\exp(e^{-\omega})+e^{-x}}$$

$$(\Re(\nu) > -1; x \geq 0; \sigma > 0). \quad (4.4.2)$$

For the *Hurwitz-Lerch zeta* function

$$\mathcal{F}[\Gamma(\sigma+i\tau)\Phi(z,\sigma+i\tau,\nu);\omega] = \frac{\sqrt{2\pi}e^{-\sigma\omega}\exp(-(\nu-1)e^{-\omega})}{\exp(e^{-\omega})-z}$$

$$(\Re(\nu) > 0;\ \text{and either } |z| \leq 1; z \neq 1; \sigma > 0 \text{ or } z = 1; \sigma > 1). \quad (4.4.3)$$

Similarly for the *Polylogarithm* function

$$\mathcal{F}[\Gamma(\sigma+i\tau)\mathrm{Li}_{\sigma+i\tau}(z);\omega]=\frac{\sqrt{2\pi}ze^{-\sigma\omega}}{\exp(e^{-\omega})-z}$$

$$(|z|\leq 1-\delta,\delta\in(0,1)\text{ and }\sigma>0;z=1\text{ and }\sigma>1).\quad(4.4.4)$$

For the *BE* function

$$\mathcal{F}[\Gamma(\sigma+i\tau)\mathfrak{B}_{\sigma+i\tau-1}(x);\omega]=\frac{\sqrt{2\pi}e^{x}e^{-\sigma\omega}}{\exp(e^{-\omega})-e^{x}}\qquad(x\geq 0;\sigma>1).\qquad(4.4.5)$$

Similarly for the *FD function*

$$\mathcal{F}[\Gamma(\sigma+i\tau)\mathfrak{F}_{\sigma+i\tau-1}(x);\omega]=\frac{\sqrt{2\pi}e^{x}e^{-\sigma\omega}}{\exp(e^{-\omega})+e^{x}}\qquad(x\geq 0;\sigma>0).\qquad(4.4.6)$$

For the *Hurwitz zeta function*

$$\mathcal{F}[\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau,\nu);\omega]=\frac{\sqrt{2\pi}e^{-\sigma\omega}\exp(-(\nu-1)e^{-\omega})}{\exp(e^{-\omega})-1}\quad(\Re(\nu)>0;\sigma>1).\qquad(4.4.7)$$

Similarly the following formulae for the *Riemann zeta* function hold true

$$\mathcal{F}[\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau);\omega]=\frac{\sqrt{2\pi}e^{-\sigma\omega}}{\exp(e^{-\omega})-1}\quad(\sigma>1).\qquad(4.4.8)$$

$$\mathcal{F}[C(\sigma+i\tau)\zeta(\sigma+i\tau);\omega]=\frac{\sqrt{2\pi}e^{-\sigma\omega}}{\exp(e^{-\omega})+1}\quad(\sigma>0).\qquad(4.4.9)$$

Remark 4.4.1. *Note that special cases of these Fourier transforms can give interesting integral formulae. A substitution* $\omega=0$ *in* (4.4.1 − 4.4.9) *leads to the following*

$$\int_{-\infty}^{+\infty}\Gamma(\sigma+i\tau)\Psi_{\nu}(\sigma+i\tau;x)d\tau=\frac{2\pi e^{-\nu(x+1)}}{e^{x+1}-1}$$

$$(\Re(\nu)>-1;\sigma>0\textit{ when }x>0;\sigma>1\textit{ when }x=0).\qquad(4.4.10)$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Theta_\nu(\sigma + i\tau; x)d\tau = \frac{2\pi e^{-\nu(x+1)}}{e^{x+1} + 1}$$

$$(\Re(\nu) > -1; \sigma > 0 \textit{ when } x > 0; \sigma > 1 \textit{ when } x = 0). \tag{4.4.11}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Phi(\sigma + i\tau; z, \nu)d\tau = \frac{2\pi e^{1-\nu}}{e - z}$$

$$(\Re(\nu) > 0; \textit{ and either } |z| \leq 1; z \neq 1; \sigma > 0 \textit{ or } z = 1; \sigma > 1). \tag{4.4.12}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)Li_{\sigma+i\tau}(z)d\tau = \frac{2\pi z}{e - z}$$

$$(|z| \leq 1 - \delta, \delta \in (0, 1) \textit{ and } \sigma > 0; z = 1 \textit{ and } \sigma > 1); \tag{4.4.13}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\mathfrak{B}_{\sigma+i\tau-1}(x)d\tau = \frac{2\pi}{e^{1-x} - 1} \qquad (x \geq 0; \sigma > 1); \tag{4.4.14}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\mathfrak{F}_{\sigma+i\tau-1}(x)d\tau = \frac{2\pi}{e^{1-x} + 1} \qquad (x \geq 0; \sigma > 0); \tag{4.4.15}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau, \nu)d\tau = \frac{2\pi e^{1-\nu}}{e - 1} \quad (\Re(\nu) > 0; \sigma > 1); \tag{4.4.16}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau)d\tau = \frac{2\pi}{e - 1} \quad (\sigma > 1). \tag{4.4.17}$$

$$\int_{-\infty}^{+\infty} C(\sigma + i\tau)\zeta(\sigma + i\tau)d\tau = \frac{2\pi}{e + 1} \quad (\sigma > 0). \tag{4.4.18}$$

It can be noted that the integrals of the FD and BE functions again lead to the special cases of the FD and BE distribution functions. This fact is more obvious from equations (4.4.14-4.4.15).

4.5 Distributional representation of the eFD and eBE functions

I will proceed to obtain a representation of the eFD and eBE functions in terms of delta functions. In view of the relationships of these extended functions with the zeta family and other related functions (discussed in Chapter 2), similar representations for the FD, BE, polylogarithm and zeta functions are also obtained here.

Theorem 4.5.1. *The eBE function has the distributional representation*

$$\Psi_\nu(\sigma+i\tau;x)\Gamma(\sigma+i\tau)=2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m(n+\nu+1)^m e^{-(n+\nu+1)x}}{m!}\delta(\tau-i(\sigma+m))$$

$$(\Re(\nu)>-1;\sigma>1 \textit{ when } x=0;\sigma>0 \textit{ when } x>0). \tag{4.5.1}$$

Proof. Define a function $t\to t_+$ by setting

$$t_+=t \quad \text{If } t>0, \quad t_+=0 \quad \text{If } t\leq 0. \tag{4.5.2}$$

If $\alpha\in\mathbb{C}$ and $\Re(\alpha)>0$ then $t\to t_+^{\alpha-1}$ is locally integrable and the eBE function can be represented by

$$\Gamma(\sigma+i\tau)\Psi_\nu(\sigma+i\tau;x)=<t_+^{\sigma+i\tau-1},\frac{e^{-(\nu+1)t}e^{-(\nu+1)x}}{1-e^{-(t+x)}}>$$

$$(\Re(\nu)>-1;\sigma>1 \text{ when } x=0;\sigma>0 \text{ when } x>0). \tag{4.5.3}$$

Replacing t by e^y in (4.5.3) gives

$$\Gamma(\sigma+i\tau)\Psi_\nu(\sigma+i\tau;x)=<e^{i\tau y},\frac{e^{\sigma y}e^{-(\nu+1)x}\exp(-(\nu+1)e^y)}{1-\exp(-(e^y+x))}>$$

$$(\Re(\nu)>-1;\sigma>1 \text{ when } x=0;\sigma>0 \text{ when } x>0), \tag{4.5.4}$$

where y lies between $-\infty$ and ∞. The expansion of the denominator in (4.5.4) yields

$$[1-\exp(-(e^y+x))]^{-1}=\sum_{n=0}^{\infty}\exp\left(-n(e^y+x)\right)$$

$$(x\geq 0,-\infty<y<+\infty) \tag{4.5.5}$$

then, also

$$\exp(-(n+\nu+1)e^y) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu+1)^m}{m!} e^{my}$$

$$(\Re(\nu) > -1, -\infty < y < +\infty). \qquad (4.5.6)$$

Using (4.5.5-4.5.6) in (4.5.4) leads to the following double sum, denoted by $\psi_{(\nu,\sigma)}(x;y)$,

$$\psi_{(\nu,\sigma)}(x;y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} e^{(\sigma+m)y}$$

$$(\Re(\nu) > -1; \sigma > 1 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0). \qquad (4.5.7)$$

Now term by term application of the Fourier transform and use of the relationship [75, p. 253]

$$\mathcal{F}[e^{\omega y}; \tau] = \sqrt{2\pi}\delta(\tau - i\omega) \qquad (4.5.8)$$

in (4.5.7), yields (4.5.1), a series of delta functions. ■

Corollary 4.5.1. *The Hurwitz-Lerch zeta function $\Phi(z, \sigma+i\tau, \nu)$ has a distributional representation*

$$\Phi(z, \sigma+i\tau, \nu)\Gamma(\sigma+i\tau) = 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu)^m z^n}{m!} \delta(\tau - i(\sigma+m))$$

$$(\Re(\nu) > 0; \sigma > 0 \textit{ when } 0 < z < 1; \sigma > 1 \textit{ when } z = 1). \qquad (4.5.9)$$

Proof. This follows by (4.5.1) and (2.3.5) , when one uses $z = e^{-x}$ and replaces $\nu \mapsto \nu - 1$. ■

Corollary 4.5.2. *The polylogarithm function has a distributional representation*

$$Li_{\sigma+i\tau}(z)\Gamma(\sigma+i\tau) = 2\pi z \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+1)^m z^n \delta(\tau - i(\sigma+m))}{m!}$$

$$(\sigma > 0 \textit{ when } 0 < z < 1, \sigma > 1 \textit{ when } z = 1). \qquad (4.5.10)$$

Proof. This follows by using (2.3.5), (4.5.1) and putting $z = e^{-x}$, $\nu = 0$. ■

Corollary 4.5.3. *The BE function has the distributional representation*

$$\mathfrak{B}_{\sigma+i\tau-1}(x)\Gamma(\sigma+i\tau) = 2\pi e^{x} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+1)^m e^{nx} \delta(\tau - i(\sigma+m))}{m!}$$

$$(x \geq 0; \sigma > 1). \qquad (4.5.11)$$

Proof. This follows from (4.5.1) and (2.3.3) by putting $\nu = 0$ and replacing $x \mapsto -x$. ■

Corollary 4.5.4. *The Hurwitz zeta function has a distributional representation*

$$\zeta(\sigma+i\tau,\nu)\Gamma(\sigma+i\tau) = 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu)^m}{m!} \delta(\tau - i(\sigma+m))$$

$$(\Re(\nu) > 0; \sigma > 1). \qquad (4.5.12)$$

Proof. Upon setting $x = 0$ and replacing $\nu \mapsto \nu - 1$ in (4.5.1), If one makes use of the relation (2.3.7) then (4.5.12) can be obtained. ■

Corollary 4.5.5. *The Riemann zeta function* (1.2.6) *has a distributional representation*

$$\zeta(\sigma+i\tau)\Gamma(\sigma+i\tau) = 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+1)^m}{m!} \delta(\tau - i(\sigma+m)) \quad (\sigma > 1). \quad (4.5.13)$$

Proof. Upon setting $\nu = x = 0$ in (4.5.1) and using (1.2.29), one can arrive at (4.5.13). ■

Theorem 4.5.2. *The eFD function $\Theta_\nu(s;x)$ has the distributional representation*

$$\Theta_\nu(\sigma+i\tau;x)\Gamma(\sigma+i\tau) = 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} \delta(\tau - i(\sigma+m))$$

$$(\Re(\nu) > -1; x \geq 0; \sigma > 0). \qquad (4.5.14)$$

Proof. Proof follows on similar steps as in Theorem (4.5.1) by using the Fourier transform representation of $\Theta_\nu(\sigma+i\tau)\Gamma(\sigma+i\tau)$ given by

$$\mathcal{F}[\theta_{(\nu,\sigma)}(x;y)=\frac{e^{\sigma y}e^{-(\nu+1)x}\exp(-(\nu+1)e^{y})}{1+\exp(-e^{y})e^{-x}};\tau]=\Gamma(\sigma+i\tau)\Theta_\nu(\sigma+i\tau)$$

$$(\Re(\nu)>-1;x\geq 0;-\infty<y<+\infty). \tag{4.5.15}$$

■

Remark 4.5.1. *Alternatively,* (4.5.15) *can also be obtained by* $x\mapsto x+\pi i$ *in* (4.5.1) *and using* (2.4.7).

Corollary 4.5.6. *The FD function has the distributional representation*

$$\mathfrak{F}_{\sigma+i\tau-1}(x)\Gamma(\sigma+i\tau)=2\pi e^{x}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}(n+1)^{m}e^{nx}\delta(\tau-i(\sigma+m))}{m!}$$

$$(x\geq 0;\sigma>1). \tag{4.5.16}$$

Proof. This follows from (4.5.14), by putting $\nu=0$, replacing $x\mapsto -x$ and making use of (1.3.4). ■

Corollary 4.5.7. *The Riemann zeta function* (1.2.16) *has a distributional representation*

$$\zeta(\sigma+i\tau)C(\sigma+i\tau)=2\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}(n+1)^{m}}{m!}\delta(\tau-i(\sigma+m))\quad(\sigma>0). \tag{4.5.17}$$

Proof. Upon setting $\nu=x=0$ in (4.5.14) and making use of the relation (2.2.11) leads to (4.5.17). ■

4.6 Some applications of the distributional representation

Distributional representation of eFD, eBE and related functions is obtained as a series of delta functions, which is convergent in the distributional sense if its inner product with a test function converges. In this section, I will give some applications of the distributional representation.

In general, for all functions $\Lambda(\rho + i\tau)$ $(\rho \in \mathbb{R})$, for which delta function along imaginary axis is defined, (4.5.1) directly yields

$$\Big\langle \Gamma(\sigma+i\tau)\Psi_\nu(\sigma+i\tau;x), \Lambda(\rho+i\tau)\Big\rangle$$
$$= 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} \Big\langle \delta(\tau - i(\sigma+m)), \Lambda(\rho+i\tau)\Big\rangle \tag{4.6.1}$$

Similar representation for the eFD function is

$$\Big\langle \Gamma(\sigma+i\tau)\Theta_\nu(\sigma+i\tau;x), \Lambda(\rho+i\tau)\Big\rangle$$
$$= 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{m+n} (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} \Big\langle \delta(\tau - i(\sigma+m)), \Lambda(\rho+i\tau)\Big\rangle. \tag{4.6.2}$$

These inner products of the extended FD and BE functions are well defined for all those functions for which these infinite series converge. By using the shifting property of the delta function, one can get the following equation

$$\big\langle \delta(\tau - i(\sigma+m)), \Lambda(\rho+i\tau)\big\rangle = \Lambda(\rho-\sigma-m) \quad (m = 0, 1, 2, \cdots). \tag{4.6.3}$$

Further, note that the series of coefficients

$$\sum_{m=0}^{\infty} a_{m,n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} = \frac{e^{-\nu(x+1)}}{e^{x+1}-1} \tag{4.6.4}$$

$$\sum_{m=0}^{\infty} a'_{m,n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n} (n+\nu+1)^m e^{-(n+\nu+1)x}}{m!} = \frac{e^{-\nu(x+1)}}{e^{x+1}+1} \tag{4.6.5}$$

are convergent.

Before obtaining new results, one may check that the results obtained by using the distributional representation are consistent with the results obtained by using the Fourier transform representation. As an example, let

$$\Lambda_u(\tau) = e^{-i\tau u} \text{ where } u \in \mathbb{C} \tag{4.6.6}$$

then as a special case of it, for $u = 0 + 0i$, it is an identity function. Its inner product with the Riemann zeta functions (4.5.13) and (4.5.17) leads to the following convergent series of numbers respectively

$$< \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau), 1 >= 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}(n+1)^m \quad (\sigma > 1) \tag{4.6.7}$$

and

$$< C(\sigma + i\tau)\zeta(\sigma + i\tau), 1 >= 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{m!}(n+1)^m \quad (\sigma > 0). \tag{4.6.8}$$

These equations can be rewritten as

$$\int_{-\infty}^{\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau)d\tau = \frac{2\pi}{e-1} \quad (\sigma > 1) \tag{4.6.9}$$

and

$$\int_{-\infty}^{\infty} C(\sigma + i\tau)\zeta(\sigma + i\tau)d\tau = \frac{2\pi}{e+1} \quad (\sigma > 0). \tag{4.6.10}$$

These two results are the same as (4.4.17) and (4.4.18) respectively, which were obtained by using the Fourier transform representation. Similarly one can obtain more results by using the distributional forms of other functions obtained in this section. This will reproduce (4.4.10-4.4.18).

Next consider the inner product of these extended functions on the particular set of functions

$$\{a^{(\rho+i\tau)u}\}_{u\in\mathbb{C}} \quad (a > 0). \tag{4.6.11}$$

It gives

$$\begin{aligned} < \Gamma(\sigma + i\tau)\Psi_\nu(\sigma + i\tau; x), a^{(\rho+i\tau)u} >&= 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m (n+\nu+1)^m e^{-(n+\nu+1)x} a^{(\rho-\sigma-m)u}}{m!} \\ &= \frac{2\pi a^{(\rho-\sigma)u} \exp(-\nu(x + a^{-u}))}{\exp(x + a^{-u}) - 1} \qquad (\Re(\nu) > -1; x \geq 0; \sigma > 0). \end{aligned} \tag{4.6.12}$$

Similarly, the inner product for the *eFD* function yields

$$< \Gamma(\sigma+i\tau)\Theta_\nu(\sigma+i\tau;x), a^{(\rho+i\tau)u} >= 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{m+n}(n+\nu+1)^m e^{-(n+\nu+1)x} a^{(\rho-\sigma-m)u}}{m!}$$

$$= \frac{2\pi a^{(\rho-\sigma)u}\exp(-\nu(x+a^{-u}))}{\exp(x+a^{-u})+1} \qquad (\Re(\nu)>-1; x\geq 0; \sigma>0) \tag{4.6.13}$$

For the *Hurwitz-Lerch zeta* function

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\Gamma(\sigma+i\tau)\Phi(z,\sigma+i\tau,\nu)d\tau = \frac{2\pi a^{(\rho-\sigma)u}\exp(-(\nu-1)a^{-u})}{\exp(a^{-u})-z}$$

$$(\Re(\nu)>0; \sigma>0 \text{ when } 0<z<1; \sigma>1 \text{ when } z=1) \tag{4.6.14}$$

Similarly for the *Polylogarithm* function

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\mathrm{Li}_{\sigma+i\tau}(z)d\tau = \frac{2\pi z a^{(\rho-\sigma)u}}{\exp(a^{-u})-z}$$

$$(\sigma>0 \text{ when } 0<z<1; \sigma>1 \text{ when } z=1). \tag{4.6.15}$$

For the *BE* function

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\Gamma(\sigma+i\tau)\mathfrak{B}_{\sigma+i\tau-1}(x)d\tau = \frac{2\pi e^x a^{(\rho-\sigma)u}}{\exp(a^{-u})-e^x} \qquad (x\geq 0; \sigma>1). \tag{4.6.16}$$

Similarly for the *FD function*

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\Gamma(\sigma+i\tau)\mathfrak{F}_{\sigma+i\tau-1}(x)d\tau = \frac{2\pi e^x a^{(\rho-\sigma)u}}{\exp(a^{-u})+e^x} \qquad (x\geq 0; \sigma>0). \tag{4.6.17}$$

The following expression involving the *Hurwitz zeta function* holds

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau,\nu)d\tau = \frac{2\pi a^{(\rho-\sigma)u}\exp(-(\nu-1)a^{-u})}{\exp(a^{-u})-1}$$

$$(\Re(\nu)>0; \sigma>1). \tag{4.6.18}$$

Similarly one can get the following formulae for the *Riemann zeta* function

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)d\tau = \frac{2\pi a^{(\rho-\sigma)u}}{\exp(a^{-u})-1} \quad (\sigma>1). \tag{4.6.19}$$

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u} C(\sigma+i\tau)\zeta(\sigma+i\tau)d\tau = \frac{2\pi a^{(\rho-\sigma)u}}{\exp(a^{-u})+1} \quad (\sigma > 0). \tag{4.6.20}$$

Remark 4.6.1. *These results can be used further to obtain the analytic extensions of the Fourier transforms of the product of the eFD, eBE, Hurwitz-Lerch zeta, Hurwitz zeta, polylogarithm, FD, BE and Riemann zeta functions with the gamma function. Fourier transforms of all these functions are given in Section* 4 *when the transform variable* ω *is real. By using the distributional representations these results can be extended to complex* u*, which can be achieved by putting* $a = e$ *in* $(4.6.12 - 4.6.20)$. *Note that the transformed variable will be* $u \in \mathbb{C}$ *instead of* $\omega \in \mathbb{R}$*, for example,* (4.6.19) *yields*

$$\mathcal{F}_E[\Gamma(\sigma+i\tau)\zeta(\sigma+i\tau); u] = \frac{2\pi e^{-\sigma u}}{\exp(e^{-u})-1} \qquad (\sigma > 1), \tag{4.6.21}$$

Remark 4.6.2. *By putting* $a = e, \Im(u) = 0$ *in* $(4.6.12 - 4.6.20)$ *these analytic extensions give Fourier transforms* $(4.4.1 - 4.4.9)$ *or*

$$\mathcal{F}_E(u) = \mathcal{F}(\omega) \quad \forall \quad u = \omega + 0i. \tag{4.6.22}$$

4.7 Fourier transform and distributional representations of the gRZ function

It was discussed in Chapter 2 that the eFD and eBE functions not only generalize the FD and BE functions but also are related to the Lerch, Hurwitz and Riemann zeta functions. Several interesting relations and properties of these functions are also discussed here but not a single result or connection of these functions was found for the Riemann zeta function in the critical strip. However, a new generalization of the Riemann zeta function is discussed in Chapter 3, which generalizes the Riemann zeta function in the critical strip. In this chapter I have discussed two representations of the eFD and eBE functions, which lead to various integral formulae for them. In the light of their relations with Hurwitz, Lerch and Riemann zeta functions the integrals for these functions have obtained in previous sections. Of course the integrals for the FD and BE functions are also obtained as special cases.

Note that it is not possible to obtain a single integral for the Riemann zeta function in the critical strip (as was discussed in section (3.2)). Therefore, I find the Fourier transform and distributional representations of the gRZ function, which is related to Hurwitz zeta function by (3.2.6). Hence the Fourier transform representation for these special cases is also given here and it leads to similar representations for the Riemann zeta function in the critical strip.

The gRZ has the Mellin transform representation (3.2.1) and a substitution $t = e^y$ leads to the following Fourier transform representation

$$\Gamma(\sigma+i\tau)\Xi_\nu(\sigma+i\tau;x) = \sqrt{2\pi}e^{-(\nu+1)x}\mathcal{F}[e^{\sigma y}\exp(-\nu e^y)\Big[\frac{1}{\exp(e^y)-e^{-x}} - \frac{e^x}{e^y+x}\Big];\tau]$$
$$(\Re(\nu) > -1; 0 < \sigma < 1 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0). \tag{4.7.1}$$

On substituting $\nu = 0$ in (4.7.1), one gets

$$\Gamma(\sigma+i\tau)\Xi_0(\sigma+i\tau;x) = \sqrt{2\pi}e^{-x}\mathcal{F}[e^{\sigma y}\Big[\frac{1}{\exp(e^y)-e^{-x}} - \frac{e^x}{e^y+x}\Big];\tau]$$
$$(0 < \sigma < 1 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0). \tag{4.7.2}$$

and a substitution $x = 0$ in (4.7.1) leads to

$$\Gamma(\sigma+i\tau)\Xi_\nu(\sigma+i\tau;0) = \sqrt{2\pi}\mathcal{F}[e^{\sigma y}\exp(-\nu e^y)\Big[\frac{1}{\exp(e^y)-1} - \frac{1}{e^y}\Big];\tau]$$
$$(\Re(\nu) > -1; 0 < \sigma < 1). \tag{4.7.3}$$

The most important special case $x = \nu = 0$ in (4.7.1) yields

$$\Gamma(\sigma + i\tau)\zeta(\sigma + i\tau) = \sqrt{2\pi}\mathcal{F}[e^{\sigma y}\Big[\frac{1}{\exp(e^y) - 1} - \frac{1}{e^y}\Big]; \tau]$$

$$(0 < \sigma < 1), \tag{4.7.4}$$

which gives the Fourier transform representation of the Riemann zeta function in the critical strip.

By using the duality property (1.5.15), the Fourier transform of the gRZ function is given below

$$\mathcal{F}[\Gamma(\sigma + i\tau)\Xi_\nu(\sigma + i\tau; x); \omega]$$
$$= \frac{\sqrt{2\pi}e^{-(\nu+1)x}e^{-\sigma\omega}\exp(-\nu e^{-\omega})}{\exp(e^{-\omega}) - e^{-x}} - \frac{\sqrt{2\pi}e^{-\nu x}e^{-\sigma\omega}\exp(-\nu e^{-\omega})}{e^{-\omega} + x}$$

$$(\Re(\nu) > -1; 0 < \sigma < 1 \text{ when } x = 0; \sigma > 0 \text{ when } x > 0). \tag{4.7.5}$$

For $\nu = 0$ it leads to

$$\mathcal{F}[\Gamma(\sigma + i\tau)\Xi_0(\sigma + i\tau; x); \omega] = \frac{\sqrt{2\pi}e^{-x}e^{-\sigma\omega}}{\exp(e^{-\omega}) - e^{-x}} - \frac{\sqrt{2\pi}e^{-\sigma\omega}}{e^{-\omega} + x}$$

$$(x = 0, 0 < \sigma < 1; x > 0, \sigma > 0) \tag{4.7.6}$$

and for $x = 0$ in (4.7.5) one can get the following result

$$\mathcal{F}[\Gamma(\sigma + i\tau)\Xi_\nu(\sigma + i\tau; 0); \omega] = \frac{\sqrt{2\pi}e^{-\sigma\omega}\exp(-\nu e^{-\omega})}{\exp(e^{-\omega}) - 1} - \sqrt{2\pi}e^{(1-\sigma)\omega}\exp(-\nu e^{-\omega})$$

$$(\Re(\nu) > -1; 0 < \sigma < 1), \tag{4.7.7}$$

A substitution $x = \nu = 0$ in (4.7.5) yields the following result

$$\mathcal{F}[\Gamma(\sigma + i\tau)\zeta(\sigma + i\tau); \omega] = \frac{\sqrt{2\pi}e^{-\sigma\omega}}{\exp(e^{-\omega}) - 1} - \sqrt{2\pi}e^{(1-\sigma)\omega} \quad (0 < \sigma < 1). \tag{4.7.8}$$

Further specializing the variable $\omega = 0$ in (4.7.5-4.7.8) leads to the following interesting formulae respectively:

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Xi_\nu(\sigma + i\tau; x)d\tau = \frac{2\pi\exp(-\nu(x+1))}{\exp(x+1) - 1} - \frac{2\pi e^{-\nu x}\exp(-\nu)}{1 + x}$$

$$(\Re(\nu) > -1; x = 0, 0 < \sigma < 1; x > 0, \sigma > 0); \tag{4.7.9}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Xi_0(\sigma+i\tau;x)d\tau = \frac{2\pi}{\exp(x+1)-1} - \frac{2\pi}{1+x}$$

$$(x=0, 0<\sigma<1; x>0, \sigma>0); \tag{4.7.10}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\Xi_\nu(\sigma+i\tau;0)d\tau = \frac{2\pi e^{-\nu}}{e-1} - 2\pi e^{-\nu}$$

$$(\Re(\nu)>-1; 0<\sigma<1); \tag{4.7.11}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma+i\tau)\zeta(\sigma+i\tau)d\tau = \frac{2\pi}{e-1} - 2\pi \quad (0<\sigma<1). \tag{4.7.12}$$

Next, I obtain the distributional representation for the gRZ function $\Xi_\nu(\sigma+i\tau;0)$ and evaluate some integrals.

Theorem 4.7.1 *The generalized Riemann zeta function $\Xi_\nu(\sigma+i\tau;0)$ has a distributional representation*

$$\begin{aligned}\Xi_\nu(\sigma+i\tau;0)\Gamma(\sigma+i\tau) = 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m(n+\nu+1)^m}{m!}\delta(\tau-i(\sigma+m))\\ -2\pi\sum_{m=0}^{\infty}\frac{(-1)^m\nu^m}{m!}\delta(\tau-i(\sigma+m-1))\\ (\Re(\nu)>-1; 0<\sigma<1).\end{aligned} \tag{4.7.13}$$

Proof. Note that this representation consists of two sums and the first one follows by similar steps as in Theorem (4.5.1). To obtain the second, one can consider

$$I_2 = \int_0^\infty \frac{e^{-\nu t}t^{s-1}}{t}dt \tag{4.7.14}$$

and put $t=e^y$ to get

$$I_2 = < e^{i\tau y}, e^{(\sigma-1)y}\exp(-\nu e^y) > . \tag{4.7.15}$$

The following expansion

$$\exp(-\nu e^y) = \sum_{m=0}^{\infty}\frac{(-1)^m\nu^m}{m!}e^{my}$$

$$(\Re(\nu)>-1; -\infty<y<+\infty). \tag{4.7.16}$$

leads to

$$I_2 = \sum_{m=0}^{\infty} \frac{(-1)^m \nu^m}{m!} \delta(\tau - i(\sigma + m - 1))$$

$$(\Re(\nu) > -1; 0 < \sigma < 1), \tag{4.7.17}$$

where the Fourier transform is applied term by term and (4.5.8) is used. Now replacing $\nu \mapsto \nu + 1$ in (4.5.12) and then combining it with (4.7.17) yields the assertion of Theorem (4.7.1). ■

Corollary 4.7.1. *The Riemann zeta function* (1.2.14) *has the distributional representation*

$$\zeta(\sigma + i\tau)\Gamma(\sigma + i\tau) = 2\pi \Big[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (n+1)^m}{m!} \delta(\tau - i(\sigma + m)) - \delta(\tau - i(\sigma - 1)) \Big]$$

$$(0 < \sigma < 1). \tag{4.7.18}$$

Proof. This follows by putting $\nu = 0$ in (4.7.13). ■

This representation can be used to obtain some integrals of these functions with the set of functions (4.6.11) as were obtained for the eFD and eBE functions in Section (4.6). It will lead to the following integral formulae

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u} \Gamma(\sigma + i\tau) \Xi_\nu(\sigma + i\tau; 0) d\tau$$

$$= \frac{2\pi a^{(\rho-\sigma)u} \exp(-\nu a^{-u})}{\exp(a^{-u}) - 1} - 2\pi a^{(\rho-\sigma+1)u} \exp(-\nu a^{-u})$$

$$(\Re(\nu) > -1; 0 < \sigma < 1). \tag{4.7.19}$$

Further $\nu = 0$, in (4.7.19) leads to the following result for the Riemann zeta in the critical strip

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u} \Gamma(\sigma + i\tau) \zeta(\sigma + i\tau) d\tau = \frac{2\pi a^{(\rho-\sigma)u}}{\exp(a^{-u}) + 1} - 2\pi a^{(\rho-\sigma+1)u} \qquad (0 < \sigma < 1). \tag{4.7.20}$$

Remark 4.7.1 *Again it is interesting to note that a substitution* $a = e$ *in (4.7.19-4.7.20) leads to the analytic extensions of the classical Fourier transforms and further specializing* $\Im(u) = 0$ *gives exactly (4.7.7-4.7.8).*

Chapter 5

Further applications of the Fourier transform and distributional representations

The Fourier transform representation of various functions has been used to get integrals of product of these functions. It is achieved in Chapter 4 for the gRZ, eFD, eBE, FD, BE, Riemann zeta, Hurwitz zeta and Hurwitz-Lerch zeta functions. But there are other functions which have a Fourier transform representation and by taking the combination of both, one can get more integrals. As an example here, I consider the Fourier transform representation of the generalized gamma function and some closely related functions.

This chapter consists of 4 sections. Fourier transform representation of the generalized gamma function is given in Section (4.1), which leads to a similar representation for the Macdonald and gamma functions. Some applications of this representation are given in Section (4.2), where I have obtained some integrals of a product of generalized gamma function and its special cases with the functions of the zeta family. Distributional representation for the generalized gamma function is discussed in Section (4.3) and some applications are given in Section (4.4). The basic motivation behind this Chapter is to combine different classes of special functions to get integrals of products of these functions.

5.1 Fourier transform representation of the generalized gamma function

The *generalized gamma function* has an integral representation (1.1.7) and a substitution $t = e^y$ in it yields the Fourier transform representation

$$\Gamma_b(\sigma + i\tau) = \sqrt{2\pi}\mathcal{F}[e^{\sigma\omega}\exp(-e^{\omega} - be^{-\omega}); \tau] \qquad (\Re(b) > 0; b = 0, \sigma > 0). \quad (5.1.1)$$

For $b = 0$, it leads to the Fourier transform representation of *gamma function*

$$\Gamma(\sigma + i\tau) = \sqrt{2\pi}\mathcal{F}[e^{\sigma\omega}\exp(-e^{\omega}); \tau]. \quad (5.1.2)$$

By making use of the relation (1.1.8) in (5.1.1) yields

$$b^{i\tau/2}K_{\sigma+i\tau}(2\sqrt{b}) = \sqrt{\frac{\pi}{2}}b^{-\sigma/2}\mathcal{F}[e^{\sigma\omega}\exp(-e^{\omega} - be^{-\omega}); \tau] \qquad (\Re(b) > 0; b = 0, \sigma > 0). \quad (5.1.3)$$

A closely related integral to the *gamma function* is

$$\frac{\Gamma(\sigma + i\tau)}{(\tilde{a})^{\sigma+i\tau}} = \int_0^{\infty} e^{-\tilde{a}t}t^{\sigma+i\tau-1}dt \quad (\tilde{a} > 0), \quad (5.1.4)$$

which leads to

$$\frac{\Gamma(\sigma + i\tau)}{(\tilde{a})^{\sigma+i\tau}} = \sqrt{2\pi}\mathcal{F}[e^{\sigma\omega}\exp(-\tilde{a}e^{\omega}); \tau] \quad (\tilde{a} > 0). \quad (5.1.5)$$

For the *generalized gamma function* one can obtain

$$\frac{\Gamma_{\tilde{a}b}(\sigma + i\tau)}{(\tilde{a})^{\sigma+i\tau}} = \sqrt{2\pi}\mathcal{F}[e^{\sigma\omega}\exp(-\tilde{a}e^{\omega} - be^{-\omega}); \tau] \quad (\tilde{a} > 0) \quad (5.1.6)$$

and the relation (1.1.8) can be used to get following expression for the *Macdonald function*

$$(\frac{b}{\tilde{a}})^{i\tau/2}K_{\sigma+i\tau}(2\sqrt{\tilde{a}b}) = \sqrt{\frac{\pi}{2}}(\frac{\tilde{a}}{b})^{\sigma/2}\mathcal{F}[e^{\sigma\omega}\exp(-\tilde{a}e^{\omega}-be^{-\omega});\tau] \quad (\tilde{a}>0). \tag{5.1.7}$$

By using (1.5.15) these Fourier transform representations further yield the following formulae for the Fourier transforms of the *generalized gamma* and some related functions

$$\mathcal{F}[\Gamma_b(\sigma+i\tau);\omega] = \sqrt{2\pi}e^{-\sigma\omega}\exp(-e^{-\omega}-be^{\omega}) \qquad (\Re(b)>0; b=0,\sigma>0). \tag{5.1.8}$$

This equation along with (1.1.8) gives

$$\mathcal{F}[b^{i\tau/2}K_{\sigma+i\tau}(2\sqrt{b});\omega] = \sqrt{\frac{\pi}{2}}e^{-\sigma\omega}b^{-\sigma/2}\exp(-e^{-\omega}-be^{\omega}) \qquad (\Re(b)>0) \tag{5.1.9}$$

and in particular, for b=0 in (5.1.8), one can get

$$\mathcal{F}[\Gamma(\sigma+i\tau);\omega] = \sqrt{2\pi}e^{-\sigma\omega}\exp(-e^{-\omega}) \qquad (\sigma>0). \tag{5.1.10}$$

By following a similar pattern, the following formulae hold true:

$$\mathcal{F}[(\tilde{a})^{-i\tau}\Gamma_{\tilde{a}b}(\sigma+i\tau);\omega] = \sqrt{2\pi}(\tilde{a})^{\sigma}e^{-\sigma\omega}\exp(-\tilde{a}e^{-\omega}-be^{\omega}) \tag{5.1.11}$$

$$\mathcal{F}[(\frac{b}{\tilde{a}})^{i\tau/2}K_{\sigma+i\tau}(2\sqrt{\tilde{a}b});\omega] = \sqrt{\frac{\pi}{2}}e^{-\sigma\omega}(\frac{\tilde{a}}{b})^{\sigma/2}\exp(-\tilde{a}e^{-\omega}-be^{\omega}) \tag{5.1.12}$$

$$\mathcal{F}[(\tilde{a})^{-i\tau}\Gamma(\sigma+i\tau);\omega] = \sqrt{2\pi}(\tilde{a})^{\sigma}e^{-\sigma\omega}\exp(-\tilde{a}e^{-\omega}). \tag{5.1.13}$$

5.2 Some applications of the Fourier transform representation

By using Parseval' identity (1.5.14), for Fourier transform representations (5.1.4) and (4.2.6), one can obtain the following integral

$$\int_{-\infty}^{+\infty}(\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\zeta(\rho+i\tau)d\tau = 2\pi\zeta(\rho+\sigma,\tilde{a}+1). \tag{5.2.1}$$

By substituting $\tilde{a} = e^{\omega}, \quad \omega \in \mathbb{R}$, it leads to

$$\int_{-\infty}^{+\infty} e^{i\tau\omega}\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi e^{\sigma\omega}\zeta(\rho + \sigma, e^{\omega} + 1), \tag{5.2.2}$$

which is the Fourier transform of $\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)$. For $\omega = 0$, one can get following integral

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\zeta(\rho + \sigma, 2), \tag{5.2.3}$$

For the *Riemann zeta function* (4.2.9), one can get the following formulae

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma - i\tau)C(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\Theta_{\tilde{a}}(\rho + \sigma; 0) \tag{5.2.4}$$

$$\mathcal{F}[\Gamma(\sigma - i\tau)C(\rho + i\tau)\zeta(\rho + i\tau); \omega] = 2\pi e^{\sigma\omega}\Theta_{e^{\omega}}(\rho + \sigma; 0), \tag{5.2.5}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)C(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\Theta_1(\rho + \sigma; 0). \tag{5.2.6}$$

For the *Riemann zeta function* (4.7.4) the following formulae hold true

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\Xi_{\tilde{a}}(\rho + \sigma; 0) \tag{5.2.7}$$

$$\mathcal{F}[\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau); \omega] = 2\pi e^{\sigma\omega}\Xi_{e^{\omega}}(\rho + \sigma; 0), \tag{5.2.8}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\Xi_1(\rho + \sigma; 0). \tag{5.2.9}$$

Note that a closed form of these integrals of Riemann zeta function is expressible as a special cases of the eFD and the gRZ functions, which gives another application of these new generalizations.

For *FD function* one can get the following closed form in terms of the eFD function

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathfrak{F}_{\rho+i\tau-1}(-x)d\tau = 2\pi\Theta_{\tilde{a}}(\rho + \sigma; x), \tag{5.2.10}$$

$$\mathcal{F}[\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathfrak{F}_{\rho+i\tau-1}(-x); \omega] = 2\pi e^{\sigma\omega}\Theta_{e^{\omega}}(\rho + \sigma; x), \tag{5.2.11}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathfrak{F}_{\rho+i\tau-1}(-x)d\tau = 2\pi\Theta_1(\rho + \sigma; x). \tag{5.2.12}$$

Similarly for the *BE function* following integral formulae holds true

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathfrak{B}_{\rho+i\tau-1}(-x)d\tau = 2\pi\Psi_{\tilde{a}}(\rho + \sigma; x) \tag{5.2.13}$$

$$\mathcal{F}[\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathfrak{B}_{\rho+i\tau-1}(-x); \omega] = 2\pi e^{\sigma\omega}\Psi_{e^{\omega}}(\rho + \sigma; x), \tag{5.2.14}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathfrak{B}_{\rho+i\tau-1}(-x)d\tau = 2\pi\Psi_1(\rho + \sigma; x), \tag{5.2.15}$$

One can get following integral formulae for the *Hurwitz zeta function*

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau, \nu)d\tau = 2\pi\zeta(\rho + \sigma, \nu + \tilde{a}) \tag{5.2.16}$$

$$\mathcal{F}[\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau, \nu); \omega] = 2\pi e^{\sigma\omega}\zeta(\rho + \sigma, \nu + e^{\omega}), \tag{5.2.17}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau, \nu)d\tau = 2\pi\zeta(\rho + \sigma, 1 + \nu), \tag{5.2.18}$$

Following integral formulae for the *polylogarithm function* also holds true

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathrm{Li}_{\rho+i\tau}(z)d\tau = 2\pi\Phi(z, \rho + \sigma, \tilde{a} + 1) \tag{5.2.19}$$

$$\mathcal{F}[\Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathrm{Li}_{\rho+i\tau}(z); \omega] = 2\pi e^{\sigma\omega}\Phi(z, \rho + \sigma, e^{\omega} + 1), \tag{5.2.20}$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\mathrm{Li}_{\rho+i\tau}(z)d\tau = 2\pi\Phi(z, \rho + \sigma, 2), \tag{5.2.21}$$

For the *Hurwitz-Lerch zeta function*, one can get

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Phi(z,\rho+i\tau,\nu)d\tau = 2\pi\Phi(z,\rho+\sigma,\nu+\tilde{a}) \quad (5.2.22)$$

$$\mathcal{F}[\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Phi(z,\rho+i\tau,\nu);\omega] = 2\pi e^{\sigma\omega}\Phi(z,\rho+\sigma,e^{\omega}+\nu), \quad (5.2.23)$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Phi(z,\rho+i\tau,\nu)d\tau = 2\pi\Phi(z,\rho+\sigma,\nu+1). \quad (5.2.24)$$

For the *eFD function*

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Theta_{\nu}(\rho+i\tau;x)d\tau = 2\pi\Theta_{\nu+\tilde{a}}(\rho+\sigma;x) \quad (5.2.25)$$

$$\mathcal{F}[\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Theta_{\nu}(\rho+i\tau;x);\omega] = 2\pi e^{\sigma\omega}\Theta_{e^{\omega}+\nu}(\rho+\sigma;x), \quad (5.2.26)$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Theta_{\nu}(\rho+i\tau;x)d\tau = 2\pi\Theta_{\nu+1}(\rho+\sigma;x). \quad (5.2.27)$$

For the *eBE Function*

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Psi_{\nu}(\rho+i\tau;x)d\tau = 2\pi\Psi_{\nu+\tilde{a}}(\rho+\sigma;x) \quad (5.2.28)$$

$$\mathcal{F}[\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Psi_{\nu}(\rho+i\tau;x);\omega] = 2\pi e^{\sigma\omega}\Psi_{e^{\omega}+\nu}(\rho+\sigma;x), \quad (5.2.29)$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Psi_{\nu}(\rho+i\tau;x)d\tau = 2\pi\Psi_{\nu+1}(\rho+\sigma;x). \quad (5.2.30)$$

For the *gRZ function*

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Xi_{\nu}(\rho+i\tau;x)d\tau = 2\pi\Xi_{\nu+\tilde{a}}(\rho+\sigma;x) \quad (5.2.31)$$

$$\mathcal{F}[\Gamma(\sigma-i\tau)\Gamma(\rho+i\tau)\Xi_{\nu}(\rho+i\tau;x);\omega] = 2\pi e^{\sigma\omega}\Xi_{e^{\omega}+\nu}(\rho+\sigma;x), \quad (5.2.32)$$

$$\int_{-\infty}^{+\infty} \Gamma(\sigma - i\tau)\Gamma(\rho + i\tau)\Xi_\nu(\rho + i\tau; x)d\tau = 2\pi\Xi_{\nu+1}(\rho + \sigma; x). \quad (5.2.33)$$

Similarly by using the Parseval' identity for the *generalized gamma function* (5.1.6) and *zeta function* (4.2.6), one can get the following integrals for generalized gamma function

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma_{\tilde{a}b}(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\zeta_b(\rho + \sigma, \tilde{a} + 1). \quad (5.2.34)$$

By putting $\tilde{a} = e^\omega \quad (\omega \in \mathbb{R})$ in the above equation, one can get the following

$$\int_{-\infty}^{+\infty} e^{i\tau\omega}\Gamma_{be^\omega}(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi e^{\sigma\omega}\zeta_b(\rho + \sigma, e^\omega + 1), \quad (5.2.35)$$

which is Fourier transform of $\Gamma_{be^\omega}(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)$. Further for $\omega = 0$, it leads to

$$\int_{-\infty}^{+\infty} \Gamma_b(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\zeta_b(\rho + \sigma, 2). \quad (5.2.36)$$

For the *Riemann zeta function* (4.2.9)

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma_{\tilde{a}b}(\sigma - i\tau)C(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\zeta_b^*(\rho + \sigma, \tilde{a} + 1) \quad (5.2.37)$$

$$\mathcal{F}[\Gamma_{be^\omega}(\sigma - i\tau)C(\rho + i\tau)\zeta(\rho + i\tau); \omega] = 2\pi e^{\sigma\omega}\zeta_b^*(\rho + \sigma, e^\omega + 1), \quad (5.2.38)$$

$$\int_{-\infty}^{+\infty} \Gamma_b(\sigma - i\tau)C(\rho + i\tau)\zeta(\rho + i\tau)d\tau = 2\pi\zeta_b^*(\rho + \sigma, 2), \quad (5.2.39)$$

where ζ_b^* is defined by (1.2.49).

For the *Hurwitz zeta function*

$$\int_{-\infty}^{+\infty} (\tilde{a})^{-\sigma+i\tau}\Gamma_{\tilde{a}b}(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau, \nu)d\tau = 2\pi\zeta_b(\rho + \sigma, \nu + \tilde{a}) \quad (5.2.40)$$

$$\mathcal{F}[\Gamma_{be^\omega}(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau, \nu); \omega] = 2\pi e^{\sigma\omega}\zeta_b(\rho + \sigma, \nu + e^\omega), \quad (5.2.41)$$

$$\int_{-\infty}^{+\infty} \Gamma_b(\sigma - i\tau)\Gamma(\rho + i\tau)\zeta(\rho + i\tau, \nu)d\tau = 2\pi\zeta_b(\rho + \sigma, 1 + \nu), \tag{5.2.42}$$

Further by using the relation (1.1.8), equations (5.2.34-5.2.42) readily yield similar integrals for Macdonald function. However for $b = 0$, (5.2.34-5.2.42) reproduces (5.2.1-5.2.6) and (5.2.16-5.2.18).

By using the Fourier transform representation of the generalized gamma function, several integrals of products involving the gamma and the generalized gamma functions are obtained with the zeta family. However the Fourier transform representation of the generalized gamma function and Macdonald function is itself important to get integrals of generalized gamma and Macdonald functions. Some of these are presented here by using Parseval's identity and convolution for Fourier transforms.

Using (1.5.14) for the Fourier transform representation (5.1.1) leads to

$$\int_{-\infty}^{+\infty} \Gamma_b(\sigma + i\tau)\Gamma_b(\rho - i\tau)d\tau = 2^{1-\sigma-\rho}\pi\Gamma_{4b}(\sigma + \rho) \qquad (\Re(b) > 0; b = 0, \sigma, \rho > 0), \tag{5.2.43}$$

which can be rewritten by using (1.1.8) as

$$\int_{-\infty}^{+\infty} K_{\sigma+i\tau}(2\sqrt{b})K_{\rho-i\tau}(2\sqrt{b})d\tau = \pi K_{\sigma+\rho}(4\sqrt{b}) \qquad (\Re(b) > 0). \tag{5.2.44}$$

Similarly by using Parseval's identity for gamma and generalized gamma functions, one can get

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma_b(\rho - i\tau)d\tau = 2^{1-\sigma-\rho}\pi\Gamma_{2b}(\sigma + \rho) \qquad (\Re(b) > 0; b = 0, \sigma, \rho > 0) \tag{5.2.45}$$

and

$$\int_{-\infty}^{+\infty} b^{i\tau/2}K_{\sigma+i\tau}(2\sqrt{b})\Gamma(\rho - i\tau)d\tau = \pi 2^{1-(\frac{\sigma+\rho}{2})}b^{\rho/2}K_{\sigma+\rho}(2\sqrt{2b}) \qquad (\Re(b) > 0). \tag{5.2.46}$$

The problem of finding the integrals of products of special functions with their derivatives can also be resolved by using the Fourier transform representation. For example differentiating (5.2.43) w.r.t ρ and using (1.1.11), one can get the following simplified form

$$\int_{-\infty}^{+\infty} \Gamma_b(\sigma+i\tau)\Gamma_b(\rho-i\tau)\psi_b(\rho-i\tau)d\tau = 2^{1-\sigma-\rho}\pi\Gamma_{4b}(\sigma+\rho)[\psi_{4b}(\sigma+\rho)-\ln 2]$$

$$(\Re(b)>0; b=0,\sigma,\rho>0). \qquad (5.2.47)$$

However, by using (1.1.8), the following identity involving Macdonald function is obtained

$$\int_{-\infty}^{+\infty} K_{\sigma+i\tau}(2\sqrt{b})K_{\rho-i\tau}(2\sqrt{b})\psi_b(\rho-i\tau)d\tau = \pi K_{\sigma+\rho}(4\sqrt{b})[\psi_{4b}(\sigma+\rho)-\ln 2] \quad (\Re(b)>0).$$

$$(5.2.48)$$

In particular choosing $\sigma = 1-\rho$, (5.2.47) and (5.2.48) give

$$\int_{-\infty}^{+\infty} \Gamma_b(1-\rho+i\tau)\Gamma_b(\rho-i\tau)\psi_b(\rho-i\tau)d\tau = \Gamma_{4b}(1)\pi[\psi_{4b}(1)-\ln 2]$$

$$(\Re(b)>0; b=0,\rho>0) \qquad (5.2.49)$$

and

$$\int_{-\infty}^{+\infty} K_{1-\rho+i\tau}(2\sqrt{b})K_{\rho-i\tau}(2\sqrt{b})\psi_b(\rho-i\tau)d\tau = \pi K_1(4\sqrt{b})[\psi_{4b}(1)-\ln 2] \quad (\Re(b)>0).$$

$$(5.2.50)$$

Note that the right side in both equations do not depend on ρ. For $b=0$ in (5.2.49), one can recover [11, p. 4, Equation (4.5)].

Equation (5.1.1) can be rewritten as

$$\Gamma_b(\sigma+i\tau) = \sqrt{2\pi}\mathcal{F}[f_b^\sigma(x);\tau] \qquad (5.2.51)$$

and by using Fourier convolution [75, p. 236], one can get

$$\sqrt{2\pi}(f_b^\sigma(x) * f_b^\rho(x)) = e^{\sigma x}\int_{-\infty}^{+\infty} e^{(\rho-\sigma)t}\exp[-(1+be^{-x})e^t - \frac{b+e^x}{e^t}]dt. \qquad (5.2.52)$$

Upon setting $(1+be^{-x})e^{t} = \tau$, (5.2.52) can be written again in terms of the generalized gamma function

$$\sqrt{2\pi}(f_b^{\sigma}(x) * f_b^{\rho}(x)) = e^{\rho x}(b+e^{x})^{\sigma-\rho}\Gamma(\rho-\sigma; \frac{(b+e^{x})^2}{e^{x}}). \tag{5.2.53}$$

Note that for convenience, we have written $\Gamma_{\frac{(b+e^{x})^2}{e^{x}}}(\rho-\sigma)$ as $\Gamma(\rho-\sigma; \frac{(b+e^{x})^2}{e^{x}})$. Since the Fourier transform of the convolution is the product of the Fourier transforms

$$\sqrt{2\pi}\mathcal{F}[e^{\rho x}(b+e^{x})^{\sigma-\rho}\Gamma(\rho-\sigma; \frac{(b+e^{x})^2}{e^{x}}); \tau] = \Gamma_b(\sigma+i\tau)\Gamma_b(\rho+i\tau). \tag{5.2.54}$$

This relates the Fourier transform of the generalized gamma function to the product of generalized gamma functions. By using (1.5.15), the above equation can be written as

$$\sqrt{2\pi}e^{-\rho\omega}(b+e^{-\omega})^{\sigma-\rho}\Gamma(\rho-\sigma; \frac{(b+e^{-\omega})^2}{e^{-\omega}}) = \mathcal{F}[\Gamma_b(\sigma+i\tau)\Gamma_b(\rho+i\tau); \omega]. \tag{5.2.55}$$

The above equation gives the Fourier transform of the product of the generalized gamma functions, which can be further used to get the Fourier transforms of the product of the Macdonald function and the product of the Euler's gamma function. By using Parseval's identity for (5.2.55), one can get

$$\int_{-\infty}^{+\infty} |\Gamma_b(\sigma+i\tau)\Gamma_b(\rho+i\tau)|^2 d\tau = 2\pi \int_{-\infty}^{+\infty} e^{-2\rho\omega}(b+e^{-\omega})^{2\sigma-2\rho}\Gamma^2(\rho-\sigma; \frac{(b+e^{-\omega})^2}{e^{-\omega}}d\omega. \tag{5.2.56}$$

Putting t in place of $e^{-\omega}$ and using (1.1.8) in the above equation leads to

$$\int_{-\infty}^{+\infty} |4b^{\frac{\sigma+\rho}{2}+i\tau}K_{\sigma+i\tau}(2\sqrt{b})K_{\rho+i\tau}(2\sqrt{b})|^2 d\tau = 8\pi.\mathcal{M}[K_{\rho-\sigma}^2(2\frac{b+t}{\sqrt{t}}); \sigma+\rho] \tag{5.2.57}$$

and

$$b^{2\sigma}\int_{-\infty}^{+\infty} |K_{\sigma+i\tau}(2\sqrt{b})|^4 d\tau = \frac{\pi}{2}\mathcal{M}[K_0^2(2\frac{b+t}{\sqrt{t}}); 2\sigma], \tag{5.2.58}$$

where $\mathcal{M}$ denotes the Mellin transform. For $b = 0$ the right hand side in (5.2.55) is the standard integral [20, Vol. I, p. 334, (45)], that evaluates the following limits

$$\lim_{b\to 0^+} b^{\sigma+\rho}\int_{-\infty}^{+\infty} |K_{\sigma+i\tau}(2\sqrt{b})K_{\rho+i\tau}(2\sqrt{b})|^2 d\tau = \frac{\pi}{8}\Gamma^2(\rho+\sigma)B(2\sigma, 2\rho) \qquad (\rho, \sigma > 0) \tag{5.2.59}$$

and for $\rho = \sigma$, it leads to

$$\lim_{b \to 0^+} b^{2\sigma} \int_{-\infty}^{+\infty} |K_{\sigma+i\tau}(2\sqrt{b})|^4 d\tau = \frac{\pi\Gamma^4(2\sigma)}{8\Gamma(4\sigma)} \qquad (\sigma > 0). \tag{5.2.60}$$

5.3 Distributional representation of the generalized gamma function

Here, analogously to the derivation of a distributional representation of the gamma function [11], I obtain a distributional representation of the generalized gamma function.

Theorem 5.3.1. *The generalized gamma function has a distributional representation*

$$\Gamma_b(\sigma + i\tau) = 2\pi \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m} b^n}{m!n!} \delta(\tau - i(\sigma + m - n)) \tag{5.3.1}$$
$$(\Re(b) > 0; b = 0, \sigma > 0).$$

Proof. The generalized gamma function can be represented by the inner product of two functions, relative to the weight factor 1, over the domain $(-\infty, \infty)$ as follows:

$$\Gamma_b(\sigma + i\tau) =< t_+^{\sigma+i\tau-1}, e^{-t-\frac{b}{t}} >, \tag{5.3.2}$$

where t_+^α is used to denote the function t^α for $t > 0$ and 0 for $t \leq 0$. By making the substitution $t = e^x$ and denoting

$$e^{\sigma x} \exp(-e^x - be^{-x}) = f_b^\sigma(x) \qquad (\sigma > 0), \tag{5.3.3}$$

the generalized gamma function can be regarded as the Fourier transform, $\mathcal{F}$, of $f_b^\sigma(x)$. Using the series expansion for $f_b^\sigma(x)$:

$$e^{\sigma x} \exp(-e^x - be^{-x}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m} b^n}{m!n!} e^{(\sigma+m-n)x} \tag{5.3.4}$$

and using (4.5.8), one can rewrite (5.3.2) as a series of delta functions, namely (5.3.1).

∎

Corollary 5.3.1. *The Macdonald function has a representation*

$$K_{\sigma+i\tau}(2\sqrt{b}) = \frac{\pi}{b^{\sigma+i\tau/2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{n+m}b^n}{m!n!}\delta(\tau - i(\sigma+m-n)) \qquad (\Re(b)>0). \quad (5.3.5)$$

Proof. This follows simply by using (1.1.8) in (5.3.1). ■

Corollary 5.3.2. *The representation* (5.3.1) *gives the distributional representation for the Euler gamma function*

$$\Gamma(\sigma+i\tau) = 2\pi \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\delta(\tau - i(\sigma+m)) \qquad (\sigma>0). \quad (5.3.6)$$

Proof. Upon putting $b = 0$ in (5.3.1), the sum over n disappears, the only term surviving being $n = 0$, and it leads to (5.3.6). ■

5.4 Some applications of the distributional representation

Taking the inner product of the generalized gamma function with the set of functions (4.6.11) leads to

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\Gamma_b(\sigma+i\tau)d\tau = 2\pi a^{(\rho-\sigma)u}\exp(-a^{-u} - ba^u)$$
$$(u \in \mathbb{C}; \Re(b) > 0; \sigma > 0 \text{ when } b = 0). \quad (5.4.1)$$

Now using (1.1.8), one can get

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}b^{i\tau/2}K_{\sigma+i\tau}(2\sqrt{b})d\tau = \pi a^{(\rho-\sigma)u}b^{-\sigma/2}\exp(-a^{-u} - ba^u) \quad (\Re(b)>0). \quad (5.4.2)$$

For $b = 0$ in (5.4.1) leads to the following integral for gamma function

$$\int_{-\infty}^{+\infty} a^{(\rho+i\tau)u}\Gamma(\sigma+i\tau)d\tau = 2\pi a^{(\rho-\sigma)u}\exp(-a^{-u}). \quad (5.4.3)$$

Note that for $a = e$, these equations give the analytic extensions of some of the Fourier transforms obtained in Section 1.

Finally consider the action of the generalized gamma function on the Riemann zeta function (1.2.14), which gives

$$< \Gamma_b(\sigma + i\tau), \zeta(\rho + i\tau) >= 2\pi \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{n+m} b^n}{m!n!} \zeta(\rho - \sigma - m + n). \tag{5.4.4}$$

For $b = 0$ and $\sigma = \rho$ this yields

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau) d\tau = 2\pi \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \zeta(-m) \quad (0 < \sigma < 1). \tag{5.4.5}$$

Next by evaluating the sum on the LHS of (5.4.5), one can get

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau) d\tau = 2\pi \big[\zeta(0) - \sum_{m=0}^{\infty} \frac{\zeta(1-2m)}{(2m-1)!}\big] \quad (0 < \sigma < 1), \tag{5.4.6}$$

where I have used the fact that zeta function vanishes at even negative integers. Further by using the relations (1.4.7) and (1.4.2)

$$\zeta(1 - 2m) = -\frac{B_{2m}}{2m}, \tag{5.4.7}$$

(5.4.6) can be rewritten as

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau) d\tau = 2\pi \Big[\zeta(0) + \sum_{m=0}^{\infty} \frac{B_{2m}}{(2m)!}\Big] \quad (0 < \sigma < 1). \tag{5.4.8}$$

Upon substituting $\zeta(0) = -1/2, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30$ and so on (see, [14, Chapter 7]) yields

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau) d\tau \simeq -2.64. \tag{5.4.9}$$

However by (4.7.12)

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\zeta(\sigma + i\tau) d\tau = \frac{2\pi}{e-1} - 2\pi \simeq -2.64. \quad (0 < \sigma < 1). \tag{5.4.10}$$

Hence from (5.4.9-5.4.10), one can write the following closed form of the sum

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\zeta(-m) = \frac{2\pi}{e-1} - 2\pi. \tag{5.4.11}$$

These are just some identities that can be obtained by using the distributional representation of the generalized gamma function. More identities can be obtained involving other special functions.

Special functions satisfy certain recurrence relations, which can also be used to evaluate more integral formulae. For example the eFD function satisfies (2.4.16)

$$\Theta_{\nu}(s;x) + \Theta_{\nu-1}(s;x) = \nu^{-s}e^{-\nu x} \quad (x \geq 0, \nu \geq 1), \tag{5.4.12}$$

which along with (4.4.11) gives

$$\int_{-\infty}^{+\infty} \nu^{-\sigma-i\tau}\Gamma(\sigma+i\tau)d\tau = 2\pi e^{-\nu} \quad (\nu \geq 1, \sigma > 0), \tag{5.4.13}$$

which can also be obtained from (5.4.3) by putting $a = \frac{1}{\nu}$, $\rho = \sigma$ and $u = 1$. It is hoped that other properties of Fourier transform and Dirac delta functions will be useful to get more results involving these extended and other related functions.

Chapter 6

Conclusion and future directions

The eFD and eBE functions are defined in Chapter 1 by introducing an extra parameter. These extensions not only provide new insights into the familiar FD and BE functions but their relationship to the family of zeta functions. It is worth noting that the extensions have led to new relationships between the well known FD and BE functions that had not been seen before. It provided insights into the original function that led to advances in our understanding of the FD and BE functions, which are closely related to the family of zeta functions. The eFD and eBE functions also highlighted this close relationship, which becomes more manifest and can be explored further. Due to the extension of the real variable x to complex $z = x + iy$, these new extensions have been put forward as possible candidates for the anyon function as they interpolate very naturally between the BE and FD functions [10]. This extension may led to the proof of many other problems which remained unsolvable due to the restriction of the parameter to the real domain. The author is particularly interested in the mathematical properties of the eFD and eBE functions. The discussion presented in this thesis can be useful for a Quantum Statistician by considering the eFD distribution function

$$f_{\mathfrak{F}}(x;\nu) = \frac{e^{-\nu(t+x)}}{e^{t+x}+1} \tag{6.0.1}$$

and the eBE distribution function

$$f_{\mathfrak{B}}(x;\nu) = \frac{e^{-\nu(t+x)}}{e^{t+x}-1}. \tag{6.0.2}$$

The eFD and eBE functions have useful connections with the Lerch, Hurwitz and Riemann zeta functions but they are not related with the Riemann zeta function in the critical strip. In Chapter 2 a new generalization of the Riemann zeta function is defined, which not only generalizes the Riemann zeta function but has simple connections with the eFD and eBE functions. These connections proved useful to obtain a series representation of the function in the critical strip. It provided relations between different functions, generalizing them into special cases of a single "special function". It turned out that the new function is also related to the Bernoulli polynomials and approximates the non-trivial zeros of the Riemann zeta function as well.

A very special case of the generalized Riemann zeta function appeared earlier in the work of Tchebychev in the study of the Riemann zeta function and the location of the non-trivial zeros. According to Bombieri [5], the formula

$$\zeta(s) + 1 - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) e^{-t} dt, \tag{6.0.3}$$

was proved by Tchebychev, from which he deduced that $(s-1)\zeta(s)$ has limit one as $s \to 1$. He used the above formula in his first memoir to prove the asymptotic formula for the number of primes less than a given number. Putting $\nu = 1$ in (3.2.1), one can get

$$\Xi_1(s;0) = \zeta(s,2) - \frac{1}{s-1} = \zeta(s) + 1 - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) e^{-t} dt, \tag{6.0.4}$$

which is exactly Tchebychev's formula. However, the general case of the function and its relation with the zeta family does not seem to have been realized so far. It was studied that the new function satisfy interesting properties and functional relations. It also achieved the desired simplification of the cumbersome proofs of elegant properties of the Hurwitz-Lerch zeta function. This simplification can be expected to lead to other results that may have remained unproven due to the complexity of their proofs.

A representation theorem for a class of *good functions* was proved to get a series representation for the eFD and the gRZ functions. An elegant application of the representation theorem led to an alternate method to derive the series representation for the Hurwitz-Lerch zeta functions by using the Weyl transform. Here, one alternate proof is presented. There may be many but the remarkable simplicity of the approach used here is the extremely elegant derivation of results without locating the poles of the functions and without using the contour integration. This cumbersome procedure is replaced by the use of fractional calculus. It also leads us to hope that the representation theorem will provide simpler proofs of other properties of the Hurwitz-Lerch zeta functions.

To get this alternate proof, the domain of the Hurwitz zeta function was extended into the critical strip. The domain of any function can be extended in different ways but the same approach was followed as already used for the Riemann zeta function. It also led to the definition of the gRZ function. The extended definition of the Hurwitz zeta function is used to find the Weyl transform of a good function in terms of the series of the Hurwitz zeta function by using representation theorem. It is expected that the representation theorem may prove useful to find the Weyl transform of other functions in this class.

The Fourier transform representation of the eFD and eBE functions was obtained in Chapter 4 to get some integral formulae involving these functions. Using their connections with the zeta family and other related functions similar formulae for these functions were also obtained. To give further applications two *new identities* involving the eFD and eBE functions were proved. This led to new results involving the FD, BE and Hurwitz-Lerch zeta functions. Results for Riemann and Hurwitz zeta functions have been deduced as special cases. Parseval's identity of the Fourier transform proved crucial in obtaining the functional relations of the integrals of the product of all these functions with the gamma function by using these new identities. However, these identities can be used to get inequalities involving the eFD and eBE functions and their integrals. It can be very useful to estimate the values of the FD and BE functions. Parseval's identity for the Mellin transform along with these identities can lead to more general results involving zeta related functions.

To solve many useful and interesting problems, the *classical theory* of Fourier transforms provided a number of mathematical tools. In Chapter 4 and 5, the *duality property* was used to find the Fourier transforms of the product of the gamma function and the generalized gamma function with the higher transcendental functions (related to the zeta family). It led to some interesting special cases, which gave integral formulae involving zeta related functions. It explored the simplicity of the results obtained by using the Fourier transform representation. This approach will hopefully enhance the applicability of these functions in various physical and engineering problems.

The Fourier transform representation of the eFD and eBE functions led to new results about FD, BE, Hurwitz, Lerch and Riemann zeta functions but it failed to produce a single result involving the Riemann zeta function in the critical strip. To evaluate the integrals of the Riemann zeta function in the critical strip the Fourier transform representation of the gRZ function was used. It gives new insights that other special functions having Mellin and hence Fourier transform representations can be used to get more integral formulae.

On the one hand Dirac delta function has played an important role in Physics and mathematics. On the other hand various representations of famous Riemann, Hurwitz and Lerch zeta functions are given in the literature, for example, integral, series and asymptotic representations. In Chapter 4 a new representation of all these classical functions was given in terms of the Dirac delta functions. Some properties of the Dirac delta function were used to get new results. It was verified that the results obtained from here are consistent with the results obtained by using the Fourier transform representation, where possible. It is hoped that this new representation can help to develop the elements of a generalized theory for the zeta and related functions by using Dirac delta functions, which though not truly functions in the classical sense can, with some precautions, be treated as functions [33].

To get more integrals of the Riemann zeta function in the critical strip, distributional representation of the generalized Riemann zeta function was also discussed in Chapter 4. It led to a similar representation for its special cases as well. By doing so a new representation for the Riemann zeta function was obtained in the critical strip. This proved useful to evaluate new integrals of the Riemann zeta function in this domain as well.

The Fourier transform representations of the eFD and eBE functions proved very useful to obtain new formulae for integrals of products of these functions. It therefore seemed worth while to obtain new formulae for these functions by considering the Fourier transform representations of other functions. This proved to be the case as we obtained what appeared to be new results for products of Macdonald functions and even of the Hurwitz and Riemann zeta functions with the gamma function. It will doubtless be possible to find many more new formulae by the application of this technique. For example, by using Parseval's identity for the Fourier transform representations of Riemann zeta function and Euler gamma function led to some special cases of the eFD, eBE and gRZ functions. This yields another application of these new extensions, to solve problems other than the one these were constructed for.

Generalizations of some special functions are given by inserting a regularizer $e^{\frac{-b}{t}}$ in the original functions [14] . Closed form of some integrals of the generalized gamma function with the Riemann zeta function is expressed in terms of such extensions. To get more integrals of the generalized gamma function with the eFD and eBE functions one need to define new extensions of the Hurwitz-Lerch zeta functions by inserting a regularizer $e^{-b/t}$ in their integral representations. This will lead to integrals of the product of zeta related functions with Macdonald functions, which occurs as a special case of Bessel functions.

To get the integrals of the powers of the Riemann zeta function, one again need to solve the integrals of the extended Riemann zeta functions. It can be achieved by using the Fourier transform and distributional representations of the extended zeta functions. The use of distributional representations appears to be a powerful tool. It has proved possible to obtain the distributional representation of the gamma, the generalized gamma, eFD, eBE, FD, BE, Hurwitz, Lerch and Riemann zeta functions by using transform techniques. It might therefore, be possible to extend the method to other special functions such as extended Riemann and Hurwitz zeta functions. It will lead to the integrals of powers of zeta functions.

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