Some Exact Solutions of the Einstein–Maxwell Field Equations in Spherical Geometry

by

Ayesha Mahmood
Regn. No. NUST201390124TPSNS7113S

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Supervised by
Prof. Azad Akhter Siddiqui

Department of Mathematics
School of Natural Sciences
National University of Sciences and Technology
Islamabad, Pakistan

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Student Name: **Ms. Ayesha Mahmood** Signature: ____________

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	Designation	Associate Professor	
	Official Address	COMSATS Institute of Information Technology, Abbottabad, Pakistan	Signature

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	Name	Dr. Muhammad Sharif	
	Designation	Professor	
	Official Address	Department of Mathematics, University of the Punjab, Lahore-54590, Pakistan.	Signature

c)	**Internal Examiner:**		
	Name	Dr. Tooba Feroze	
	Designation	Professor	
	Official Address	NUST-SNS	Signature

Supervisor Name: **Prof. Azad A. Siddiqui** Signature: ____________

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Dedicated to my sweet

little

daughter Hareem Fatima

and

my husband

Muhammad Younis.

Abstract

In this thesis, the aim is to present some new classes of non–static and static, spherically symmetric solutions of the Einstein–Maxwell field equations representing compact objects with negative pressure. Throughout this thesis the space–time geometry is spherical, the radial pressure is negative, and the matter density equals the negative value of the radial pressure (either it is considered or it comes out as a consequence of the calculations). Several non–static solutions are found by taking an ansatz for the components of the metric tensor and on the square of electric field intensity. The solutions are shown to satisfy physical boundary conditions associated with the exact solutions of the Einstein–Maxwell field equations. Due to negative pressure, these solutions can model physical systems such as expanding compact objects containing negative pressure. Petrov and Segré classifications that these obtained solutions admit are also discussed in detail. Two static solutions of the field equations are also obtained with the ansatz similar to that for the non–static cases in order to have a look how the solutions behave for these kind of ansatz in static geometry. All the physical conditions are shown to be satisfied for the static solutions and it is shown that these solutions describe compact objects with negative pressure.

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I'm grateful to ALMIGHTY ALLAH for His blessings, my parents for their prayers, my husband for his tremendous support, my supervisor Prof. Azad A. Siddiqui and Dr. Tooba Feroze for their exceptional support in completing the research and thesis; both academic and moral. Of course, I am grateful to my GEC members Prof. Asghar Qadir and Dr. Ibrar Hussain for their time to time comments and guidance. I would also like to express my thanks to all faculty and staff of SNS, NUST for providing a good environment to study here.

Ayesha Mahmood

Contents

List of Abbreviations and Publications

List of Abbreviations:

Einstein–Maxwell field equations is abbreviated as EMFEs
Field equations is abbreviated as FEs

List of Publications:

Publications from this thesis are:

- A. Mahmood, Azad A. Siddiqui, and T. Feroze. A class of non-static exact solutions of the Einstein-Maxwell field equations. *International Journal of Modern Physics D*, **26**:1741008-1, (2017).
- A. Mahmood, Azad A. Siddiqui, and T. Feroze. Non–static spherically symmetric exact solution of the Einstein–Maxwell field equations. *Journal of the Korean Physical Society*, **71**:396, (2017).

Chapter 2 is based on the first paper and chapter 3 is based on the second paper. Chapter 4 is based on the paper under process and chapter 5 is based on the submitted research paper.

Chapter 1

Introduction and Preliminaries

In this chapter, first of all a brief account of the general relativity is given. Then some literature is cited that can be found on the solutions of EMFEs. It also contains some basic definitions that are required to understand the FEs.

1.1 Introduction

Relativity is the twentieth century development of the theory of motion for macroscopic objects. The theory of relativity generally refers to the two theories of Albert Einstein, Special theory of relativity of (1905) explained with his famous paper "On the Electrodynamics of Moving Bodies", and General theory of relativity of (1916) presented in his paper "Die Grundlage der allgemeinen Relativit*ä*tstheorie". Relativity theory is one of the main foundations of modern physics.

Special relativity deals with uniform, unaccelarated motion of macroscopic objects. Quite often it gives a satisfactory description of microscopic objects. General relativity attempts to deal with arbitrary motions. It succeeds for macroscopic objects in a gravitational field. At a fundamental level, the limitations of the theory are felt where a quantum description is required. To date there is no theory which can satisfactorily describe the motion of both macroscopic and microscopic objects. There are also problems in dealing with motion in which acceleration is due to fields other than pure gravity.

The ″Special″ or ″Restricted″ theory is limited to the study of uniform, unaccelerated, motion. In other words it deals with motion in a straight line with constant speed. On the other hand, ″General″ or ″Unrestricted″ theory deals with general motions. The general theory of relativity owes its existence to Einstein's physical insight and his insistence on logical and aesthetic requirements in the formulation of the theory. The mathematics developed owes a lot to Minkowski, Grossmann, Hilbert and others. The early solutions to Einstein's FEs were found by Schwarzschild, Friedmann, Reissner, Nordstr*ö*m, De Sitter, Lemaitre and others. However, it remains essentially Einstein's theory. Detailed history is given very elegantly by A. Qadir [1].

In general theory of relativity the gravitational force is expressed as a curved four–dimensional space–time. The Einstein theory considers the gravitational forces and the forces of acceleration to be equivalent and there cannot be any object in the universe that can travel faster than the speed of light, yet the gravitational pull between two objects is strong when they have smaller distance. Its central premise states that the space–time curvature is characterised through the matter and energy distribution held inside it and in turn the matter and energy distribution is characterised through the space–time curvature. In short, the geometry of space–time is influenced by the existence of mass and energy, and the motion of mass and energy is influenced by the geometry of space–time. There have been a number of productive experiments in support of this theory. All these physical phenomena are explained through Einstein's famous FEs

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = \kappa T_{ab}. \qquad (1.1.1)$$

Here R_{ab} is the Ricci tensor, g_{ab} the metric tensor, R the Ricci scalar, Λ the cosmological constant, κ the Einstein's gravitational constant, T_{ab} the stress energy tensor and F_{ab} the electromagnetic tensor.

Most of the physical problems generally involve some mathematical model that is defined through a set of differential equations. Exact solution of a problem in general denotes that solution which acquires the whole of the physical and mathematical features of a problem as opposed to an approximate or perturbed solution. For the solution of a physical problem; at first, the set of differential equations are analysed in order to find as many exact solutions as possible and then these solutions are analysed mathematically and physically. In dealing with general relativity, global analysis of the solutions is required rather than the local solution of a differential equation.

Due to nonlinearity, finding exact solutions for the FEs is a difficult task. For obtaining solutions it is convenient to make assumptions on different physical quantities and/or symmetries of the space–time. Those classes of exact solutions which exhibit some gravitational phenomena, like rotating black holes and/or the expanding universe, etc., are studied frequently. These equations become more simple under the assumption that the space–time is flat having little deviation which leads the FEs to the linearised ones. The phenomena like the gravitational waves are studied using these equations.

The study of high energy and gravitational physics usually require Maxwell's equations for the formalism of space–time. Such formalisms of space–time are also evidently harmonious with special and general relativity. In quantum and analytical mechanics the preferable forms of Maxwell's equations are those which involve potentials (both electric and magnetic). These partial differential equations associate electric field with the magnetic field as well as the electric charges and currents with the fields.

Similar to other differential equations, unique solution of these equations can only be obtained if some initial and boundary conditions are specified. For suitable boundary conditions on a finite region of space, these equations can be solved. These are given by

$$F^{ab}_{;b} = J^a, \tag{1.1.2}$$

$$F_{[ab;c]} = 0, \tag{1.1.3}$$

where $a, b, c = 0, 1, 2, 3$, j^a is the four current.

These equations together with the Einstein FEs are known as the Einstein–Maxwell FEs denoted in this thesis by EMFEs.

1.2 Some Known Exact Static Solutions

The EMFEs describe gravity together with electromagnetism. Numerous solutions of these equations have been obtained so far; the Reissner–Nordstr*ö*m [2] solution is the first exact solution of these equations.

Most of the solutions are obtained by taking assumptions on symmetries and idealised physical problems. Some well–known solutions include [3]; static, spherically symmetric solutions of Schwarzschild, Reissner–Nordstr*ö*m, Tolman and Friedmann (now known as Robertson–Walker metric), the axisymmetric electromagnetic, vacuum solutions of Weyl, the plane wave solutions, and the Kerr solution of rotating black holes.

Physicists are often interested in identifying exact solutions of the EMFEs for the problems in general relativity that describe charged perfect fluid with static, spherically symmetric distributions of matter. These types of solutions are helpful in describing the collapse of a spherical matter distribution that is collapsing due to the electrical repulsion of charges to a point singularity. The exterior of these charged spherical distributions of fluids is a region that is represented by the metric of Reissner–Nordstr*ö*m.

Lovelock [4] solved the source–free EMFEs with spherical geometry describing a static massless charged particle. But it does not match the zero mass solution of the Reissner–Nordstr*ö*m. Despite of the fact that the uncharged test particles have repulsive force between them due to both of the metrics.

Krori and Chaudhury [5] using the conformally flat Einstein's equations developed a technique to solve EMFEs both for static and non–static cases. Humi and Mansour [6] obtained several solutions of this set of equations. Pressure distribution is considered to be proportional to the mass density and space–time geometry to be spherically / plane symmetric.

Tikekar [7] solved coupled EMFEs. The solution describes the interior gravitational field of a sphere comprised of charged matter. Muller–Hoissen and sippel [8] solved the EMFEs in four–dimensions. The related FEs for static case with spherical symmetry of matter distribution are studied. Then classification of solutions is made in accordance to their properties.

Abramyan and Gutsunaev [9] presented a new series of static solutions of the EMFEs and obtained particular solutions which reduced to the Schwarzschild metric in the case of vanishing magnetic field. Melfo and Rago [10] investigated the solution of these equations in case of a fluid sphere possessing charge and having anisotropic distribution of pressure that act as a source of Reissner–Nordstr*ö*m metric when the interior metric is supposed to be conformally flat. They found that this case lead to the static configuration asymptotically while the isotropic non–static solutions are not compatible with charged models.

Roy, Rangwala, and Rana [11] studied the general relativistic FEs and obtained a solution for a charged anisotropic fluid sphere without any particular radial distribution for the mass of the sphere. Abbasi [12] obtained asymptotically flat, static solutions for vacuum FEs of Einstein and found that the Schwarzschild metric was a special and simplest form of them.

Guilfoyl [13] obtained solutions of these EMFEs for static configuration of space that demonstrate a functional relationship between the gravitational and the electric potentials. By taking spherical charged distribution of matter, several solutions are discussed generalising the interior Schwarzschild solution. All the non–dust solutions are matched to the solution by Reissner–Nordstr*ö*m showing that they are well–behaved and the constants of integration are expressed in terms of the source's total mass, total charge and radius.

Hernández and Nú*ñ*ez [14] have shown that there can be physically acceptable solutions with nonlocal type of equations of state. Space–time configuration is taken to be static spherically symmetric. The pressure distribution is taken to be anisotropic, focusing the special cases where the radial pressure gets vanished, and the other case where the tangential pressures vanish.

Mak and Harko [15] presented a class of exact solutions for the FEs of Einstein with gravitation involved for space–time geometry to be static and spherically symmetric and pressure distribution for the stellar configuration taken to be anisotropic. The solutions are obtained for a specific form of anisotropy factor. The physical quantities like radial pressure, tangential pressure, and energy density come out to be positive and finite inside the star. Mak and Harko [16] found an exact solution representing a strange quark star containing charge. Their solution describes the interior of the star when the space–time configuration is assumed to be spherically symmetric.

G*ü*rses and Himmeto*ğ*lu [17] constructed exact solutions of the EMFEs with both static and non–static space–time geometries of thin shells containing extremely charged dust on the boundary. Komathiraj and Maharaj [18] obtained two classes of exact solutions of EMFEs. They assumed an equation of state (linear) that is suitable for quark matter, and a specific type of one of the gravitational potentials is also supposed to obtain solutions.

Barrow [19] through these equations made up an analysis of the cosmological development of the sources of matter having low anisotropic pressures. The calculation of their evolution is made under the assumption that the universe was nearly isotropic during the radiation and the dust eras.

Thirukkanesh and Maharaj [20], and Maharaj and Thirukkanesh [21] considered a general compact relativistic object having pressure distribution to be anisotropic while the electromagnetic field is present. They considered a linear equation of state, a specific form of one of the gravitational potentials as well as that of the electric field intensity for obtainig exact solutions.

Arraut, Batic, and Nowakowski [22] have analysed the Einstein FEs under static and spherically symmetric distribution of perfect fluid for solutions. It is shown that the solution describes a mini black hole with the geometry outside the event horizon represented by the Schwarzschild geometry.

Bashar *et al.* [23] have discussed few formal features of these FEs and have taken the space–time configuration to be static, spherically symmetric containing charge and the fluid is perfect. They have obtained a new class of analytic solutions for the FEs.

Mammadov [24] gave derivation for the Reissner–Nordstr*ö*m metric by solving the set of FEs that correspond to a black hole that is not rotating, contains charge, and is spherically symmetric. The charge is considered to be static so that the magnetic field is not there due to the presence of electric charges.

N. Pant, Mehta, and M. J. Pant [25] presented a class of exact solutions for the FEs under the assumption that the space–time geometry has spherical symmetry and the pressure distribution is isotropic. The solutions are well behaved and regular describing balls comprising of perfect fluid having positive and finite values of pressure and density at their center.

Spruck and Yang [26] considered the model of a space that is occupied by an outstretched distribution of dust that is extremely charged. They show through their solutions of FEs that for a matter having finite amount of mass that is distributed smoothly, there exist a family of solutions that depend on space metric asymptotically and are continuous and smooth.

Varela, *et al.* [27] solved the EMFEs for self–gravitating, charged, anisotropic fluids both for isotropic and anisotropic pressure distributions. Feroze and Siddiqui [28] considered a form of the equation of state that is quadratic (that is pressure relates to the squared power of matter dansity) for the distribution of matter and have studied exact solutions of the FEs that describe a compact relativistic object's general situation. They further assumed the presence of electromagnetic field and have considered the pressure distribution to be anisotropic. They have obtained some classes of relativistic star models with static space–time configuration and spherical symmetry. There are many other solutions in addition to those given in this section and the references there in.

1.3 Some Known Exact Non–Static Solutions

Most of the solutions found in literature deal with the static space–time geometry. But in general the space–time geometry is non–static. Some non–static solutions of the EMFEs are mentioned below.

Radhakrishna [29], and Sharan and Tiwari [30] obtained non–static cylindrically symmetric solutions of the EMFEs in vacuum. Tiwari [31] obtained a class of exact solutions for non–static, cylindrically symmetric, zero–mass electromagnetic fields. A non–static solution for the EMFEs is found by Carminati and McIntosh [32] by considering the metric of the form

$$ds^2 = -\exp(2h)dt^2 + \exp(2A)(dx^2 + dy^2) + \exp(2B)dz^2, \qquad (1.3.1)$$

where h, A and B depend only on t forming thus a non–static case and the electromagnetic field is taken to be non–null.

Einstein's equations containing both fluid and a magnetic field are of cosmological interest. Hajj–Boutros and Sfeila [33] derived a plane symmetric solution of these equations in the case of non–static charged dust under the assumptions: (i) the source for the gravitational field is a charged dust, (ii) the space–time is plane symmetric and (iii) the metric is of the form

$$ds^2 = -dt^2 + \exp[2u(t,z)]dz^2 + Z^2(z)T^2(t)(dx^2 + dy^2). \qquad (1.3.2)$$

Chamorro and Virbhadra [34] gave an exact solution of these equations for non–static geometry of space–time (with null fluid). It describes the gravitational and the electromagnetic fields of a non–rotating massive radiating dyon. Gharanfoli and Abbasi [35] and Abbasi [36], obtained solutions of the Einstein FEs for the case when the cosmological constant Λ is present.

Sharif and Iqbal [37] obtained solutions of the equations of Einstein for the case when the space–time is non–static and its symmetry is spherical, and contains perfect fluid under different assumptions on the equation of state. They obtained three solutions, one of which is a dust solution and the remaining two solutions are for stiff matter. D. Shee, *et al.* [38] have described a modeling for an object that corresponds to a relativistic compact star having anisotropic distribution of pressure. Exact analytical solutions are obtained that satisfy spherical symmetry for the interior of the dense star that admit a conformal symmetry that is non–static. A large number of static and non–static solutions of the equations of Einstein are presented in [39].

The main objectives of different cosmological models include the description of different phases of the universe. It may concern the time evolution of the acceleration field of the universe. It is now well known that the universe is dominated by the so–called dark energy but the nature of this dark energy is still unknown. It is also believed that the dark energy has large negative pressure that leads to the accelerated expansion of the universe. Due to this fact much importance is given to the study of the dark energy models by many authors. The simplest example of the dark energy is a cosmological constant, introduced by Einstein in 1917 [40]. A. Cappi [41] has discussed different cosmological models with the equation of state of the form $\omega = P/\rho c^2$ (where P is pressure, ρ is mass density, and c is speed of light) and has discussed different models for $\omega = -1, -1 < \omega < 0, \omega < -1$. B. Saha [42] has solved the Einstein FEs for a system of gravitational field and a binary mixture of perfect fluid and the dark energy given by a cosmological constant and negative pressure. It is to be noticed that these solutions are obtained for static space–time structure and in view of [43, 44] the configurations of stars may not be static. Keeping this fact in mind, main aim of this thesis is to obtain solutions of the EMFEs for non–static space–time geometry that also represent a negative pressure model. Some cases for static space–time configurations are also presented. Following is the scheme of this thesis.

1.4 Scheme of the Thesis

In the remaining part of this chapter, there are given some basic definitions and notations that are necessary to understand the subject. Most prominent are the tensors and tensor algebra, the EMFEs and the Petrov and the Segré classifications of space–times.

In Chapter 2, a new class of non–static solutions of the EMFEs for a compact object with negative pressure is presented when the pressure distribution is anisotropic. Physical acceptability of solutions is also discussed.

In Chapter 3, a new class of non–static, exact solutions for the EMFEs representing an object with negative pressure is obtained in case of the isotropic pressure distribution. Conditions for the solutions to be physically acceptable are also discussed in detail.

In Chapter 4, several non–static solutions are obtained for isotropic spherically symmetric space–time geometry. All represent compact objects with negative pressure.

In Chapter 5, new classes of solutions of the EMFEs are obtained for an object with negative pressure where the space–time configuration is considered to be static. The solutions are obtained both for isotropic and anisotropic pressure distributions. The solutions are shown to satisfy the physical acceptability criterion.

1.5 Preliminaries

To work effectively in the theory of relativity, one needs to be aware of the language of tensors. It helps to summarize sets of equations compactly and solving problems more promptly, and revealing structure in the equations. Therefore, tensors and its formalisms are discussed at first in the following section (definitions in the next sections have been taken from [45]).

1.6 Tensors and Tensor Algebra

In the theory of relativity, the results are expressed in terms of a space–time coordinate system relative to an observer. An event is specified by one time coordinate and three spatial coordinates, given by

$$x^a = (ct, x^\alpha) = (ct, x, y, z), \qquad x_a = (-ct, x_\alpha) = (-ct, x, y, z), \tag{1.6.1}$$

where $a = 0, 1, 2, 3$ and $\alpha = 1, 2, 3$. The two types of coordinates are related by means of a matrix η, which can be used to raise and lower indices:

$$x^a = \eta^{ab} x_b, \qquad x_a = \eta_{ab} x^b \ , \tag{1.6.2}$$

$$\eta_{ab} = \eta^{ab} = diag(-1, 1, 1, 1), \quad \eta^a_b = \delta^a_b \ . \tag{1.6.3}$$

A line element, denoted as ds^2, in these coordinates is defined as

$$ds^2 = dx^a dx_a = -(cdt)^2 + dx^2 + dy^2 + dz^2, \tag{1.6.4}$$

and is called the **Minkowski metric**. **Lorentz transformations** are such transformations of the coordinates that leave the line element invariant i.e., $dx^a dx_a = ds^2 = dx^{a'} dx_{a'}$, given by

$$x^{a'} = L^{a'}{}_a x^a \ , \quad x_{b'} = L_{b'}{}^b x_b \ , \quad L_{b'}{}^b = \eta_{b'a'} \eta^{ab} L^{a'}{}_a \ . \tag{1.6.5}$$

For invariance of the line element, the transformations (1.6.5) must satisfy

$$L^{a'}{}_a L_{a'}{}^b = \delta^b_a \ . \tag{1.6.6}$$

A 4–vector $a_n = (a_t, a_1, a_2, a_3) = (a_t, \mathbf{a})$ is a vector that transforms like the components of the position vector and is classified to be null, spacelike, and timelike for

$$a^n a_n = -{a_t}^2 + \mathbf{a}^2 = \begin{cases} > 0 & \textit{spacelike} \\ = 0 & \textit{null} \\ < 0 & \textit{timelike} \end{cases} \tag{1.6.7}$$

Now we can define a tensor. A quadratic 4×4 matrix A^{ab} is a **tensor** of **rank** 2 if it transforms like a product of two 4–vectors under Lorentz transformations

$$A^{a'b'} = L^{a'}{}_a L^{b'}{}_b A^{ab} \quad . \tag{1.6.8}$$

A scalar function ϕ is said to be zero rank tensor if it is invariant under Lorentz transformations,

$$\phi' = \phi, \tag{1.6.9}$$

for example, rest mass of an object. The 4–vectors $\mathbf{a} = (a^n)$ are called the **rank one tensors**, for example, force and velocity. a^n and A^{ab} are the **contravariant components**, $A^a{}_b = \eta_{bc} A^{ac}$ are the **mixed components**, and a_n and $A_{ab} = \eta_{ac}\eta_{bd}A^{cd}$ the **covariant components** of respective tensors. The quantities $A^{a_1}{}_{a_2 a_3}{}^{a_4 \cdots}{}_{\cdots a_n}$ are said to be the components of a tensor of rank n if its contravariant components transform like the contravariant vectors and the covariant components transform like the covariant vectors.

Two tensors A^{ab} and S^{ab} of same rank can be **added** componentwise as

$$A^{ab} + S^{ab} = W^{ab} \quad . \tag{1.6.10}$$

$S^a + A^{ab}$ or $S^a{}_b + A^{ab}$ are meaningless expressions. The **Product** of an n^{th} rank tensor with an m^{th} rank tensor produces an $(n+m)^{th}$ rank tensor, for example

$$A^{ab}{}_c S^d{}_e{}^f = W^{abdf}{}_{ce} \quad . \tag{1.6.11}$$

The **contraction** or summing over a covariant and a contravariant index of a tensor gives an other tensor with a rank reduced by 2:

$$A^{ab}{}_{cd} \rightarrow A^{ab}{}_{ad} = S^b{}_d \quad . \tag{1.6.12}$$

The **trace** $T = A^a{}_a$ of a second rank tensor is the simplest example of the contraction. A tensor is said to be **symmetric** with respect to its two components (either both of the indices are contravariant or both of the indices are covariant) if changing their order leaves the tensor invariant, for example $A^{ab}{}_{cd}$ is symmetric with respect to c and d if

$$A^{ab}{}_{cd} = A^{ab}{}_{dc} \ . \tag{1.6.13}$$

It is called **anti–symmetric** if it changes sign on interchanging the components i.e.,

$$A^{ab}{}_{cd} = -A^{ab}{}_{dc} \ . \tag{1.6.14}$$

An arbitrary second rank tensor with components, A_{ab}, can be decomposed into its symmetric and anti–symmetric parts as

$$A_{ab} = A_{(ab)} + A_{[ab]} \ , \tag{1.6.15}$$

where

$$A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba}) \ , \tag{1.6.16}$$

is the symmetric part and

$$A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba}) \ , \tag{1.6.17}$$

is the anti–symmetric part of the tensor. The second rank tensor η_{ab} given in equation (1.6.3) is a symmetric tensor as $\eta_{ab} = \eta_{ba}$ and is called Minkowski's metric tensor. Usually a metric tensor other than the Minkowski's is denoted by g_{ab} in terms of which the line element is defined as

$$ds^2 = g_{ab}dx^a dx^b. \tag{1.6.18}$$

The contravariant of the metric tensor is g^{ab} which is the inverse of g_{ab} i.e.,

$$g_{ab}g^{ab} = \delta^a_a \ . \tag{1.6.19}$$

The **Partial derivative** of a tensor $A^{ab}{}_{cd}$ is given by

$$A^{ab}{}_{cd,p} = \frac{\partial A^{ab}{}_{cd}}{\partial x^p} \quad , \tag{1.6.20}$$

where $,p$ in the subscript represents the partial derivative with respect to the p^{th} component of x^a. Similarly, a second order derivative is given by

$$A^{ab}{}_{cd,pq} = \frac{\partial^2 A^{ab}{}_{cd}}{\partial x^p \partial x^q} \quad . \tag{1.6.21}$$

The **covariant derivative** of an arbitrary tensor $A^{ab}{}_{cd}$ is defined as

$$A^{ab}{}_{cd;p} = A^{ab}{}_{cd,p} + \Gamma^a{}_{qp} A^{qb}{}_{cd} + \Gamma^b{}_{qp} A^{aq}{}_{cd} - \Gamma^q{}_{cp} A^{ab}{}_{qd} - \Gamma^q{}_{dp} A^{ab}{}_{cq} \quad , \tag{1.6.22}$$

where $\Gamma^a{}_{bc}$ are called **Christoffel symbols** associated with the metric tensor, g_{ab}, as

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{bc,d}) \quad , \tag{1.6.23}$$

and are symmetric in lower indices. The **Lie derivative** of an arbitrary tensor $A^{ab}{}_{cd}$ with respect to $\mathbf{v}$ is defined as

$$\mathcal{L}_{\mathbf{v}} A^{ab}{}_{cd} = v^m A^{ab}{}_{cd,m} - A^{mb}{}_{cd} v^a{}_{,m} - A^{am}{}_{cd} v^b{}_{,m} + A^{ab}{}_{md} v^m{}_{,c} + A^{ab}{}_{cm} v^m{}_{,d} \quad , \tag{1.6.24}$$

that can be extended to any higher ranked tensor.

The **curvature tensor (Riemann tensor), R** with components $R^a{}_{bcd}$ is defined as

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} - \Gamma^e{}_{bc} \Gamma^a{}_{ed} \quad . \tag{1.6.25}$$

The components of the curvature tensor satisfy the symmetry relations

$$\begin{aligned} R^a{}_{bcd} &= -R^a{}_{bdc} \quad , \\ R^a{}_{[bcd]} &= 0 = R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} \quad . \end{aligned} \tag{1.6.26}$$

The covariant derivatives of the curvature tensor obey the **Bianchi identities**

$$R^a{}_{b[cd;e]} = 0. \tag{1.6.27}$$

Contracting $R^a{}_{bcd}$ by the first and the third index gives the components R_{bd} of the **Ricci tensor** defined as

$$R_{bd} = R^a{}_{bad} \quad . \tag{1.6.28}$$

1.6.1 Tetrads and Dual Bivectors

Tetrads and bivectors are very important in the study of the theory of relativity (definitions and syntax here and in the remaining chapter are taken from [46]).

An **orthonormal tetrad** $\{\mathbf{E}_a\}$ consists of one timelike vector $\mathbf{t}$ and three spacelike vectors $\mathbf{E}_\alpha$, such that

$$\{\mathbf{E}_a\} = \{\mathbf{t}, \mathbf{E}_\alpha\} = \{\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}, \quad \mathbf{E}_\alpha.\mathbf{t} = 0, \quad \mathbf{t}.\mathbf{t} = -1, \quad \mathbf{E}_\alpha.\mathbf{E}_\beta = \delta_{\alpha\beta} \quad . \tag{1.6.29}$$

The components of the metric tensor, g_{ab}, with respect to the orthonormal tetrad are given as

$$g_{ab} = -t_a t_b + x_a x_b + y_a y_b + z_a z_b \quad . \tag{1.6.30}$$

The **complex null tetrad** $\{\mathbf{e}_a\}$, consists of two real null vectors $\mathbf{l}, \mathbf{k}$ and two complex conjugate null vectors $\mathbf{m}, \bar{\mathbf{m}}$, such that

$$\{\mathbf{e}_a\} = \{\mathbf{m}, \bar{\mathbf{m}}, \mathbf{l}, \mathbf{k}\} \quad , \tag{1.6.31}$$

where the non vanishing scalar products of tetrad components are

$$k^a l_a = -1, \qquad m^a \bar{m}_a = 1 \quad . \tag{1.6.32}$$

The components of the metric tensor g_{ab} with respect to the null tetrad are given as

$$g_{ab} = m_a \bar{m}_b + m_b \bar{m}_a - k_a l_b - k_b l_a = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1.6.33}$$

The orthonormal and the null tetrads may be related as

$$\sqrt{2}\mathbf{m} = \mathbf{E}_1 - \iota\mathbf{E}_2, \quad \sqrt{2}\bar{\mathbf{m}} = \mathbf{E}_1 + \iota\mathbf{E}_2, \tag{1.6.34}$$

$$\sqrt{2}\mathbf{l} = \mathbf{t} - \mathbf{E}_3, \quad \sqrt{2}\mathbf{k} = \mathbf{t} + \mathbf{E}_3. \tag{1.6.35}$$

A **bivector** is a second rank anti–symmetric tensor. If A^{ab} denote the components of a second rank anti–symmetric tensor, its **dual**, $\widetilde{A}^{ab}$, is defined as

$$\widetilde{A}^{ab} = \frac{1}{2}\epsilon^{abcd}A_{cd} \ , \tag{1.6.36}$$

where ϵ^{abcd} are the components of the ϵ–tensor, defined to be

$$\epsilon^{abcd} = \begin{cases} 1 & \text{for all even permutations of } a,b,c,d, \\ -1 & \text{for all odd permutations of } a,b,c,d, \\ 0 & \text{otherwise} \end{cases} \tag{1.6.37}$$

and in terms of complex null tetrad

$$\epsilon_{abcd}m^a\bar{m}^bl^ck^d = \iota. \tag{1.6.38}$$

The dual of $\widetilde{A}^{ab}$ is given by

$$\widetilde{\widetilde{A}}^{ab} = \frac{1}{4}\epsilon^{abcd}\epsilon_{cdef}\widetilde{A}^{ef} = -A^{ab} \ . \tag{1.6.39}$$

A bivector is called null if

$$A^{ab}A_{ab} = 0 = A^{ab}\widetilde{A}_{ab} \ . \tag{1.6.40}$$

A complex bivector is defined by

$$A^*_{ab} = A_{ab} + \iota\widetilde{A}_{ab} \ , \tag{1.6.41}$$

and is self–dual because of the property

$$(A^*_{ab})^{\sim} = -\iota A^*_{ab} \ . \tag{1.6.42}$$

A general self–dual bivector can be expanded in terms of the basis $\mathbf{Z}^{\mu} = (\mathbf{U}, \mathbf{V}, \mathbf{W})$ constructed from the complex null tetrad as

$$U_{ab} = \bar{m}_a l_b - l_a \bar{m}_b \ , \tag{1.6.43}$$

$$V_{ab} = k_a m_b - m_a k_b \ , \tag{1.6.44}$$

$$W_{ab} = m_a \bar{m}_b - \bar{m}_a m_b - k_a l_b + l_a k_b \ . \tag{1.6.45}$$

All the contractions of these basis bivectors vanish except

$$U_{ab}V^{ab} = 2, \qquad W_{ab}W^{ab} = -4. \tag{1.6.46}$$

1.6.2 Weyl's Tensor and Ψ's

Weyl's tensor C_{abcd} is defined through the unique decomposition of the curvature tensor R_{abcd} as

$$R_{abcd} = E_{abcd} + G_{abcd} + C_{abcd} \ , \tag{1.6.47}$$

where R_{abcd} is defined in equation (1.6.25) and

$$E_{abcd} = \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}) \ , \tag{1.6.48}$$

$$G_{abcd} = \frac{1}{12}R(g_{ac}g_{bd} - g_{ad}g_{bc}) \ , \tag{1.6.49}$$

$$S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}, \qquad R = R^a{}_a \ . \tag{1.6.50}$$

R is the trace and S_{ab} is the traceless part of the Ricci tensor R_{ab}. $C_{abcd}, E_{abcd}, G_{abcd}$ have the same symmetries as that of the curvature tensor, also

$$C^a{}_{bad} = 0, \quad E^a{}_{bad} = S_{bd} \ , \quad G^a{}_{bad} = \frac{1}{4}Rg_{bd} \ . \tag{1.6.51}$$

This decomposition of the curvature tensor is just like the decomposition of an arbitrary matrix into symmetric and anti–symmetric parts. It is more convenient to find an algebraic classification of a matrix through its symmetric or anti–symmetric part rather than the real matrix itself. So, the curvature tensor is decomposed into its parts in order to simplify the procedure of its algebraic classification discussed later.

In free space the only part of the curvature tensor that exists is the Weyl's tensor. This part of the curvature tensor is responsible for the symmetry or the curvature of the space–time and it encodes the information about the gravitational field in vacuum [47]. By the Einstein equations if the stress-energy tensor is zero, the Riemann tensor is identical to the Weyl tensor. Hence the Weyl tensor gives the purely gravitational field in the absence of any source. It will thus give the static or dynamic gravitational field in itself. In the case it is dynamic this gives gravitational waves. Also, the Weyl tensor is invariant under any conformal transformation. So, if a space–time metric is conformally equal to a flat metric then the Weyl tensor is zero and the space–time is said to be **conformally flat**. The Petrov classification of space–times is actually done using Weyl's tensor. This importance of the Weyl tensor leads to give some more definitions.

The left and the right duals of C_{abcd} are given as

$$\tilde{}C_{abcd} = \frac{1}{2}\epsilon_{abef}C^{ef}{}_{cd} \ , \qquad C\tilde{}_{abcd} = \frac{1}{2}\epsilon_{cdef}C_{ab}{}^{ef} \ , \tag{1.6.52}$$

where $\tilde{}C_{abcd} = C\tilde{}_{abcd}$. The components of the complex tensor are defined as

$$C^{*}_{abcd} = C_{abcd} + \iota C\tilde{}_{abcd} \ , \tag{1.6.53}$$

which in terms of the basis $\mathbf{Z}^{\mu}$ can be expanded as

$$\begin{aligned}\tfrac{1}{2}C^{*}_{abcd} = {} & \Psi_0 U_{ab}U_{cd} + \Psi_1(U_{ab}W_{cd} + W_{ab}U_{cd}) + \Psi_2(U_{ab}V_{cd} + V_{ab}U_{cd} + \\ & W_{ab}W_{cd}) + \Psi_3(V_{ab}W_{cd} + W_{ab}V_{cd}) + \Psi_4 V_{ab}V_{cd} \ ,\end{aligned} \tag{1.6.54}$$

where the complex coefficients $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ are given by

$$\Psi_0 = C_{abcd}k^a m^b k^c m^d, \qquad \Psi_1 = C_{abcd}k^a l^b k^c m^d \ , \tag{1.6.55}$$
$$\Psi_2 = \frac{1}{2}C_{abcd}k^a l^b (k^c l^d - m^c \bar{m}^d) \ , \tag{1.6.56}$$
$$\Psi_3 = C_{abcd}k^a l^b \bar{m}^c l^d, \qquad \Psi_4 = C_{abcd}\bar{m}^a l^b \bar{m}^c l^d \ . \tag{1.6.57}$$

These complex coefficients define a symmetric complex 3×3 matrix $\mathbf{Q}$ as

$$\mathbf{Q} = \begin{pmatrix} \Psi_2 - \frac{1}{2}(\Psi_0 + \Psi_4) & \frac{1}{2}\iota(\Psi_4 - \Psi_0) & \Psi_1 - \Psi_3 \\ \frac{1}{2}\iota(\Psi_4 - \Psi_0) & \Psi_2 + \frac{1}{2}(\Psi_0 + \Psi_4) & \iota(\Psi_1 + \Psi_3) \\ \Psi_1 - \Psi_3 & \iota(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix} \tag{1.6.58}$$

1.7 The EMFEs

In the study of the general theory of relativity the EMFEs are of fundamental importance. Most of the physical problems in relativity are described using the exact solutions of these equations. Einstein's equations determine the space–time geometry with respect to a given arrangement of mass and energy of that space–time, and Maxwell's equations relate electromagnetic fields to charges and currents. These equations are given as

$$T_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \tag{1.7.1}$$
$$(\sqrt{-g}(F^{ab}))_{,b} = \sqrt{-g}j^a, \tag{1.7.2}$$

where the units are taken so that the Einstein's gravitational constant κ given in (1.1.1) is unity, T_{ab} is the stress energy tensor, F_{ab} is the electromagnetic tensor, $j^0 = \sigma$ is the charge density, and $\mathbf{J} = (j^1, j^2, j^3)$ is the electric current density. F_{ab} is an anti–symmetric tensor with components

$$F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{1.7.3}$$

where **E** and **B** are the electric and the magnetic fields, respectively. The electromagnetic part of the stress energy tensor, $T^{(em)}_{ab}$, is associated with the electromagnetic tensor, F_{ab}, as

$$T^{(em)}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right). \tag{1.7.4}$$

The matter part of T_{ab} is given as

$$T^{(m)}_{ab} = (\rho + p)u_a u_b + p g_{ab}, \qquad u_a u^a = -1, \tag{1.7.5}$$

where ρ is the mass density, p is the pressure and $\mathbf{u}$ is the 4–velocity of the fluid. Thus the components of the stress energy tensor in the presence of both fluid and the electromagnetic field are given by

$$T_{ab} = T^{(em)}_{ab} + T^{(m)}_{ab} \ . \tag{1.7.6}$$

1.7.1 Curvature Invariants

The curvature invariants are the qunatities that are unchanged under coordinate transformations. The curvature invariants are as given in [48]

1. $\mathcal{R}_1 = R = g^{ab}R_{ab}$,
2. $\mathcal{R}_2 = R_{ab}R^{ab}$,
3. $\mathcal{R}_3 = R_{ab}{}^{cd}R_{cd}{}^{ab}$,
4. $\mathcal{R}_4 = R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab}$,. . . .

These invariants are useful in determining the nature of singularities of the solutions of FEs. If there is a singularity in the solution and the curvature invariants are finite at that point then the singularity is said to be removable or coordinate singularity and can be removed by suitable coordinate transformations. If any of the curvature invariants is undefined then the singularity is essential or geometric.

1.8 The Petrov and the Segré Classifications

The algebraic classification of the Weyl part of the curvature tensor is called the **Petrov classification** and the algebraic classification of the trace free part of the Ricci tensor is known as the **Segré classification**. These classifications are of interest in at least the following three contexts:

- They help in comprehending geometrical features of space–times.
- They are useful for classification and interpretation of the distribution of matter field (energy–momentum tensor)
- In checking whether different space–times are in fact locally the same or not up to coordinate transformations.

Useful material and information about the Petrov and the Segré classifications has been extracted from [49–53] given in next sections.

1.8.1 The Petrov Classification

The classification of gravitational fields developed by A. Z. Petrov is of great importance in the theory of relativity. In [54], Petrov has represented his renowned space's classification by studying the algebraic structure of the curvature tensor or Weyl's tensor. This algebraic structure then decides which of the classes of the gravitational fields are allowed therein. Now this classification of gravitational fields or that of the spaces is termed as Petrov's classification after the name of A. Z. Petrov. It has provided the basis for the mathematical development of the exact solutions of Einstein's FEs and for the physical interpretation of the general theory of relativity. For the Petrov classification of a space–time, we need to define

- Weyl's tensor (via decomposition of the curvature tensor)
- The complex coefficients of Weyl's tensor
- The complex matrix **Q**.

The classification of the Weyl tensor involves the eigenvalue problem for the matrix **Q** given by

$$Q_{ab}X^b = \lambda X_a, \tag{1.8.1}$$

which in 3–dimensional vector notation can be written as

$$\mathbf{Qr} = \lambda \mathbf{r}. \tag{1.8.2}$$

The equation (1.8.2) leads to the characteristic equation

$$|\mathbf{Q} - \lambda \mathbf{I}| = 0, \tag{1.8.3}$$

with eigenvalues $\lambda_1, \ldots, \lambda_k$ of multiplicities $m_1, \ldots, m_k$ where $m_1 + \ldots + m_k = 3$.

Distinct algebraic structures are characterised by the elementary divisors $(\lambda - \lambda_1)^{m_1}, \ldots, (\lambda - \lambda_k)^{m_k}$ and the multiplicities of the eigenvalues. The gravitational field is characterised by the algebraic type of the matrix **Q** as given in Table 1.1

Petrov types	Order of the elementary divisors $[m_1, \ldots, m_k]$	Matrix criterion
I	[111]	$(Q - \lambda_1 I)(Q - \lambda_2 I)(Q - \lambda_3 I) = 0$
D	[(11)1]	$(Q + \frac{1}{2}\lambda I)(Q - \lambda I) = 0$
II	[21]	$(Q + \frac{1}{2}\lambda I)^2(Q - \lambda I) = 0$
N	[(21)]	$Q^2 = 0$
III	[3]	$Q^3 = 0$
O		$Q = 0$

Table 1.1: The Petrov Types

Here [111] means $\lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (from the trace free condition of Q_{ab}), [(11)1] means $\lambda_1 = \lambda_2 \neq \lambda_3$ and $2\lambda_1 + \lambda_3 = 0$, [21] means the eigenvalues are $-\lambda, 2\lambda$, [(21)] and [3] mean that all three eigenvalues are equal and the trace free condition of Q_{ab} implies that the eigenvalues are zero.

In addition to the above mentioned matrix criteria, the Petrov classification can be obtained by characterization of the Weyl tensor in terms of principal null directions (eigen directions). They have the following properties: if

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0, \tag{1.8.4}$$

then $\mathbf{k}$ is a principal null direction of multiplicity 1 and there can be at most four such null vectors. If

$$C_{abc[d}k_{f]}k^b k^c = 0, \tag{1.8.5}$$

then $\mathbf{k}$ is a principal null direction of multiplicity 2 and there can be at most two such null vectors. If

$$C_{abc[d}k_{f]}k^c = 0, \tag{1.8.6}$$

then $\mathbf{k}$ is a principal null direction of multiplicity 3 and there can be at most one such null vector. If

$$C_{abcd}k^c = 0, \tag{1.8.7}$$

then **k** is a principal null direction of multiplicity 4. The Petrov types are characterized by the multiplicities of the principal null directions. For example, type D is characterized by the existence of two principal null directions **k** and **l** that is

$$C_{abc[d}k_{f]}k^bk^c = 0, \qquad \text{and} \tag{1.8.8}$$

$$C_{abc[d}l_{f]}l^bl^c = 0. \tag{1.8.9}$$

Following table 1.2 gives relation between multiplicities of the principal null directions, equations satisfied by Weyl's tensor and the components of Ψs.

Multiplicity of principal null direction **k**	Equation satisfied by C_{abcd}	Conditions on Ψs
1	$k_{[e}C_{a]bc[d}k_{f]}k^bk^c = 0$	$\Psi_0 = 0, \Psi_1 \neq 0$
2	$C_{abc[d}k_{f]}k^bk^c = 0$	$\Psi_0 = \Psi_1 = 0, \Psi_2 \neq 0$
3	$C_{abc[d}k_{f]}k^c = 0$	$\Psi_0 = \Psi_1 = \Psi_2 = 0, \Psi_3 \neq 0$
4	$C_{abcd}k^c = 0$	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \Psi_4 \neq 0$

Table 1.2: Multiplicities of the Principal Null Directions and Conditions on C_{abcd} and Ψs

Now the next table 1.3 gives relation between multiplicities of principal null directions and the corresponding Petrov types.

Multiplicities of principal null directions	Petrov type	Description
[1111]	*I*	4 distinct principal null directions each of multiplicity 1
[211]	*II*	3 distinct principal null directions one with multiplicity 2 and other with multiplicity 1 each
[22]	*D*	2 distinct principal null directions each of multiplicity 2
[31]	*III*	2 distinct principal null direction one of multiplicity 3 and other of multiplicity 1
[4]	*N*	1 distinct principal null direction of multiplicity 4
	O	Weyl's tensor vanish identically

Table 1.3: Multiplicities of the Principal Null Directions and the Petrov Types

It is evident from the table 1.3 that the Petrov type *I* is degenerate type *II* if one of the principal null directions is of multiplicity 2 and also of degenerate type *D* if two principal null directions are each of multiplicity 2. Also the Petrov type *II* is degenerate type *D* if there are two principal null directions each of multiplicity 2, it is of degenerate type *III* if one principal null direction is of multiplicity 3 and other that of 1 and of degenerate type *N* if multiplicity of principal null direction is 4. Similarly, Petrov type *III* is degenerate type *N*. It is well illustrated by the Penrose diagram 1.1. In the Penrose diagram the arrows point in the direction of increasing multiplicity of the principal null directions; every arrow indicates one additional degeneration (a principal null direction is said to be degenerate if it has multiplicity greater than 1).

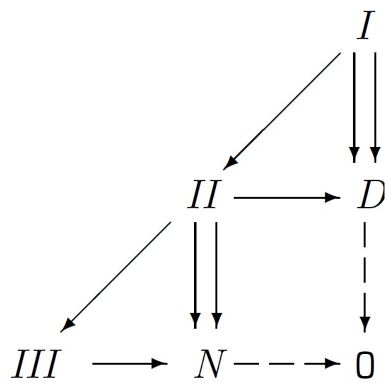

Figure 1.1: The Penrose Diagram

1.8.2 The Jordan Canonical Form and the Segré Classification

The Segré classification is basically the classification of the energy–momentum tensor. Mostly this classification is used in studying exact solutions of the EMFEs. This classifcation arises from the eigenvalue problem $(S^a_b - \lambda\delta^a_b)V^b = 0$ constructed with the trace–free Ricci tensor S_{ab}. By virtue of Einstein's equations, the Segré types of S_{ab} and that of the energy–momentum tensor are the same. In order to find the Segré classification of a space–time, we need to find

- The symmetric 3×3 matrix Φ_{ab} (through the trace–free part of Ricci tensor i.e., S_{ab})
- The eigenvalues of S^a_b
- The Jordan canonical form of S^a_b

First of all we give a method to find the Jordan canonical form of a matrix. Given a square matrix A, its Jordan canonical form can be obtained by following the steps given below.

- Find eigenvalues of A (e.g., $\lambda_1, \lambda_2, \ldots, \lambda_n$ with multiplicities $m_1, m_2, \ldots, m_n$, respectively).

- Find the nullity of $(A - \lambda_i)^{r_i}$ for $r_i = 1, 2, \ldots$ until the nullity of $(A - \lambda_i)^{r_i} = m_i$, denoting these nullities by d_{r_i} we get a sequence $d_0 < d_1 < d_2 < \ldots < d_{r_i} = m_i$.

- Now from this sequence determine the number of blocks and their sizes corresponding to λ_i. d_0 is taken to be zero. The difference $d_1 - d_0$ indicates total number of blocks that correspond to λ_i with non–zero size. Then, the difference $d_2 - d_1$ indicate the number of blocks corresponding to λ_i whose size is at least greater than one. Then, the difference $d_3 - d_2$ indicates total number of blocks corresponding to λ_i whose size must be at least greater than 2. We go on repeating this process until we know the exact number of blocks of each size.

For example, consider

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$$

Eigenvalues of A are $\lambda_1 = 1 = \lambda_2 = \lambda_3$ (1 is an eigenvalue of A of multiplicity 3). Now for $\lambda_1 = 1$ the rank of $(A - I)$ is one that implies dimension of the null space of $(A - I) = 2 = d_1$ that means that there are at least two Jordan blocks of sizes at least one (or the total number of the Jordan blocks corresponding to this eigenvalue of A is two) in the Jordan canonical form of A. Next, the matrix $(A - I)^2 = 0$, so that the rank of it is zero implying the dimension of the null space of $(A - I)^2 = 3 = d_2$ which means that the number of blocks of sizes at least two is $d_2 - d_1 = 1$. We will stop here because the dimension of the null space of the powers of the matrix $(A - I)$ is equal to the multiplicity of the eigenvalue. Now in Jordan canonical form there are two Jordan blocks one of size 2 and one of size 1 and the Jordan form of A can be written as

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with two Jordan blocks $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and (1) on the main diagonal of the Jordan matrix. In general, consider a matrix A of order n having eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ each of multiplicity $m_1, m_2, \ldots, m_r$, respectively ($m_1 + m_2 + \ldots + m_r = n$). The Jordan canonical form of A is then given as

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

where J_i is a matrix of order $m_i \times m_i$ with λ_i in every diagonal position and some arrangement of 0's and 1's in each superdiagonal position. All other entries in J_i and in J are zero. Further each matrix J_i can be written in the form

$$J_i = \begin{pmatrix} J_{i1} & & & \\ & J_{i2} & & \\ & & \ddots & \\ & & & J_{ik(i)} \end{pmatrix}$$

where J_{ij} is a matrix of order $p_{ij} \times p_{ij}$ whose diagonal entries are each equal to λ_i and whose superdiagonal elements are each equal to one and all other entries are zero. Also $p_{i1} \geq p_{i2} \geq \ldots \geq p_{ik(i)}$ and such that $p_{i1} + p_{i2} + \ldots + p_{ik(i)} = m_i$. The procedure to find $p_{ik(i)}$s is explained in the example given above. Each J_{ij} is called the basic Jordan block.

The Segré classification of the trace free ricci tensor S^a_b is just obtained from the Jordan canonical form of S^a_b by putting the sizes of the Jordan blocks corresponding to all the eigenvalues in square brackets and that of a repeated eigenvalue in the parentheses within the square brackets separately and if there is one Jordan block corresponding to an eigenvalue then the parentheses can be omitted. In general, for the Jordan canonical form of a matrix of order n discussed above with certain ordering of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ can be encoded in the symbol

$$\Big[\Big(p_{11}, \ldots, p_{1k(1)}\Big)\Big(p_{21}, \ldots, p_{2k(2)}\Big)\ldots(p_{r1}, \ldots, p_{rk(r)})\Big]$$

The Segré type of the above matrix A can be written as [(21)]. The Segré type [1(11)23] stands for the case when λ_1 is of multiplicity 1, λ_2 is of multiplicity 2 and corresponding to it there are two Jordan blocks each of size 1, λ_3, λ_4 are of multiplicities 2 and 3, respectively. There is only one Jordan block of size 2 corresponding to λ_3 and one Jordan block of size 3 corresponding to λ_4.

For the matrix S^a_b the symmetric matrix ϕ_{ab} is defined as

$$\Phi_{00} = \frac{1}{2}S_{ab}k^a k^b, \qquad (1.8.10)$$

$$\Phi_{01} = \frac{1}{2}S_{ab}k^a m^b, \qquad (1.8.11)$$

$$\Phi_{02} = \frac{1}{2}S_{ab}m^a m^b, \qquad (1.8.12)$$

$$\Phi_{11} = \frac{1}{4}S_{ab}(k^a l^b + m^a \bar{m}^b), \qquad (1.8.13)$$

$$\Phi_{12} = \frac{1}{2}S_{ab}l^a m^b, \qquad (1.8.14)$$

$$\Phi_{22} = \frac{1}{2}S_{ab}l^a l^b, \qquad (1.8.15)$$

where k^a, m^a, l^a, n^a are the components of the null tetrad. Then the segré type of the matrix is calculated through the non–zero components of ϕ_{ab}. The relation between different Segré types, ϕ_{ab} and the Petrov types is given in Table 1.4

Segré Characteristic	Set of Non–Zero ϕ_{ab}	Petrov Classification
[1111]	$\phi_{00} = \phi_{22}, \phi_{11}, \phi_{02} = \phi_{20}$	I
[(11)11]	$\phi_{11}, \phi_{02} = \phi_{20}$	D
[1(11)1]	$\phi_{00} = \phi_{22}, \phi_{11}$	D
[(11)(11)]	ϕ_{11}	D
[(111)1]	$2\phi_{11} = \phi_{02}$	O
[1(111)]	$\phi_{00} = \phi_{22} = 2\phi_{11}$	O
[(1111)]		O
[$Z\bar{Z}$11]	$\phi_{00} = -\phi_{22}, \phi_{11}, \phi_{02} = \phi_{20}$	I
[$Z\bar{Z}$(11)]	$\phi_{00} = -\phi_{22}, \phi_{11}$	D
[211]	$\phi_{11}, \phi_{22}, \phi_{02} = \phi_{20}$	II
[2(11)]	ϕ_{11}, ϕ_{22}	D
[(12)1]	$\phi_{22}, 2\phi_{11} = \phi_{02}$	N
[(112)]	ϕ_{22}	O
[31]	$2\phi_{11} = \phi_{02}, \phi_{01} \neq \phi_{10}$	III
[(13)]	ϕ_{02}	N

Table 1.4: The Segré types, non–zero elements of ϕ_{ab} and the Petrov types

Chapter 2

A Class of Anisotropic, Non–Static Exact Solutions of the EMFEs with $\mu = a_0\left(1 + \left(\frac{t+r}{a}\right)^2\right)$

In the study of charged matter distribution, the EMFEs are of fundamental importance. This system of equations possesses more unknowns than the number of equations. Therefore, different conditions/ansatz on the pressure distribution, the electric field intensity, the gravitational potentials, the equation of state, etc., are assumed to solve this system.

In this chapter, the aim is to find a class of exact solutions of the EMFEs for non–static, spherically symmetric space–times. The pressure distribution is taken to be anisotropic and particular forms are considered for the first and the third components of the metric tensor, and for the square of the electric field intensity. In the following section, the EMFEs are obtained for a general non–static, spherically symmetric space–time metric. In Section 2.2, a new class of solutions of the FEs is presented. In Section 2.3, the physical analysis of the obtained solution is given and in Section 2.4, a brief conclusion is presented.

2.1 The EMFEs for Spherical Geometry and the Algebraic Classification

A general non–static, spherically symmetric space–time has the metric of the form

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + \mu^2(t,r)d\Omega^2, \tag{2.1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The coefficients of the metric are called the gravitational potentials. In spherically symmetric space–times, only radial components of electric and magnetic fields survive. Therefore, the non–zero components of the electromagnetic tensor are $F_{01} = E = -F_{10}$ and $F_{23} = -B = F_{32}$, where E is the electric field intensity and B is the strength of the magnetic field. For spherically symmetric space–times $B = 0$. The components of the electromagnetic part of the stress energy tensor $T_{ab}^{(em)}$ are related to the components of the electromagnetic tensor F_{ab} by the expression

$$T_{ab}^{(em)} = F_a{}^c F_{bc} - \frac{1}{4} g_{ab} F_{cd} F^{cd}. \tag{2.1.2}$$

Using the above expression and adding the matter part given by

$$T_{ab}^{(m)} = diag(e^{\nu}\rho, e^{\lambda}p_r, \mu^2 p_t, \mu^2 \sin^2\theta p_t) \tag{2.1.3}$$

where ρ is the mass density, p_r is the radial pressure, and p_t is the tangential pressure (a matter distribution is called isotropic when radial and tangential pressures are equal to each other and anisotropic when they are not equal). We obtain the stress energy tensor in the presence of both matter and charge for the metric (2.1.1) as

$$T_{ab} = diag\left(e^{\nu}\rho + \frac{1}{2}e^{-\lambda}E^2, e^{\lambda}p_r - \frac{1}{2}e^{-\nu}E^2, \left(p_t + \frac{1}{2}e^{-(\nu+\lambda)}E^2\right)\mu^2, \left(p_t + \frac{1}{2}e^{-(\nu+\lambda)}E^2\right)\mu^2 \sin^2\theta\right). \tag{2.1.4}$$

For the metric (2.1.1) and the stress energy tensor given by equation (2.1.4), the EMFEs given in Chapter 1 by equations (1.7.1) and (1.7.2) for spherical, non–static geometry reduce to the following independent set of equations

$$\rho + \frac{1}{2}e^{-(\nu+\lambda)}E^2 = e^{-\lambda}\left(\frac{\lambda'\mu'}{\mu} - 2\frac{\mu''}{\mu} - \frac{\mu'^2}{\mu^2}\right) + e^{-\nu}\left(\frac{\dot{\lambda}\dot{\mu}}{\mu} + \frac{\dot{\mu}^2}{\mu^2}\right) + \frac{1}{\mu^2}, \tag{2.1.5}$$

$$p_r - \frac{1}{2}e^{-(\nu+\lambda)}E^2 = e^{-\lambda}\left(\frac{\nu'\mu'}{\mu} + \frac{\mu'^2}{\mu^2}\right) + e^{-\nu}\left(\frac{\dot{\nu}\dot{\mu}}{\mu} - 2\frac{\ddot{\mu}}{\mu} - \frac{\dot{\mu}^2}{\mu^2}\right) - \frac{1}{\mu^2}, \tag{2.1.6}$$

$$p_t + \frac{1}{2}e^{-(\nu+\lambda)}E^2 = \frac{e^{-\lambda}}{4}\left(4\frac{\mu''}{\mu} - 2\frac{\lambda'\mu'}{\mu} + 2\frac{\nu'\mu'}{\mu} - \nu'\lambda' + 2\nu'' + \nu'^2\right) + \frac{e^{-\nu}}{4}\left(-4\frac{\ddot{\mu}}{\mu} + 2\frac{\dot{\nu}\dot{\mu}}{\mu} - 2\frac{\dot{\lambda}\dot{\mu}}{\mu} + \dot{\nu}\dot{\lambda} - 2\ddot{\lambda} - \dot{\lambda}^2\right), \tag{2.1.7}$$

$$-2\dot{\mu}' + \nu'\dot{\mu} + \dot{\lambda}\mu' = 0, \tag{2.1.8}$$

$$j_0 = \frac{1}{\mu^2}e^{-\lambda}\left(E\mu^2\right)', \tag{2.1.9}$$

$$j_1 = \frac{1}{\mu^2}e^{-\nu}\left(E\mu^2\right)^{\cdot}. \tag{2.1.10}$$

Here ‘ . ’ and ‘ , ’ represent derivatives with respect to t and r respectively. Here it can be noticed that in all the independent EMFEs (2.1.5)–(2.1.10), the radial pressure, p_r, and the tangential pressure, p_t, both are functions of the variables t and r only (not of θ or ϕ), which represents that the spherically symmetric geometry has pressures independent of angular variables but there is no condition on the radial and tangential pressures that they must be equal or not (that is, isotropic case where $p_r = p_t$ and the anisotropic case where $p_r \neq p_t$, do not disturb the spherical symmetry).

The trace, T, of the stress energy tensor is

$$T = -\frac{2}{\mu^2} + e^{-\lambda}\left(-2\frac{\lambda'\mu'}{\mu} + 4\frac{\mu''}{\mu} + 2\frac{\mu'^2}{\mu^2} + 2\frac{\nu'\mu'}{\mu} - \frac{1}{2}\nu'\lambda' + \nu'' + \frac{1}{2}\nu'^2\right) \\ + e^{-\nu}\left(-2\frac{\dot\lambda\dot\mu}{\mu} - 4\frac{\ddot\mu}{\mu} - 2\frac{\dot\mu^2}{\mu^2} + 2\frac{\dot\nu\dot\mu}{\mu} + \frac{1}{2}\dot\nu\dot\lambda - \ddot\lambda - \frac{1}{2}\dot\lambda^2\right). \tag{2.1.11}$$

The curvature and the Ricci tensors, have the following non–zero components for the metric (2.1.1),

$$R_{0101} = \frac{1}{4}e^{\lambda}(2\ddot\lambda + \dot\lambda^2 - \dot\nu\dot\lambda) - \frac{1}{4}e^{\nu}(2\nu'' + \nu'^2 - \nu'\lambda'), \tag{2.1.12}$$

$$R_{0202} = \frac{1}{2}(2\mu\ddot\mu - \mu\dot\nu\dot\mu) - \frac{1}{2}\mu\nu'\mu' e^{\nu-\lambda}, \tag{2.1.13}$$

$$R_{1212} = \frac{1}{2}(2\mu\mu'' - \mu\lambda'\mu') - \frac{1}{2}\mu\dot\lambda\dot\mu e^{\lambda-\nu}, \tag{2.1.14}$$

$$R_{0212} = \frac{1}{2}(2\mu\dot\mu' - \mu\nu'\dot\mu - \mu\dot\lambda\mu'), \tag{2.1.15}$$

$$R_{2323} = 4\mu^2\sin^2\theta(\mu'^2 e^{-\lambda} - \dot\mu^2 e^{-\nu} - 1), \tag{2.1.16}$$

$$R_{0303} = \sin^2\theta R_{0202}, \quad R_{1313} = \sin^2\theta R_{1212}, \quad R_{0313} = \sin^2\theta R_{0212}, \tag{2.1.17}$$

$$R_{00} = \frac{1}{4}e^{\nu-\lambda}(4\frac{\nu'\mu'}{\mu} + 2\nu'' + \nu'^2 - \nu'\lambda') - \frac{1}{4}(8\frac{\ddot\mu}{\mu} - 4\frac{\dot\nu\dot\mu}{\mu} + 2\ddot\lambda + \dot\lambda^2 - \dot\nu\dot\lambda), \tag{2.1.18}$$

$$R_{01} = -2\frac{\dot\mu'}{\mu} + \frac{\nu'\dot\mu}{\mu} + \frac{\dot\lambda\mu'}{\mu}, \tag{2.1.19}$$

$$R_{11} = \frac{1}{4}(-8\frac{\mu''}{\mu} + 4\frac{\lambda'\mu'}{\mu} - 2\nu'' - \nu'^2 + \nu'\lambda') + \frac{1}{4}e^{\lambda-\nu}(4\frac{\dot\lambda\dot\mu}{\mu} + 2\ddot\lambda + \dot\lambda^2 - \dot\nu\dot\lambda), \tag{2.1.20}$$

$$R_{22} = \frac{1}{2}\mu^2 e^{-\lambda}(-2\frac{\mu'^2}{\mu^2} - 2\frac{\mu''}{\mu} - \frac{\nu'\mu'}{\mu} + \frac{\lambda'\mu'}{\mu}) + \frac{1}{2}\mu^2 e^{-\nu}(2\frac{\dot\mu^2}{\mu^2} + 2\frac{\ddot\mu}{\mu} - \frac{\dot\nu\dot\mu}{\mu} + \frac{\dot\lambda\dot\mu}{\mu}) + 1, \tag{2.1.21}$$

$$R_{33} = \sin^2\theta R_{22}. \tag{2.1.22}$$

The procedure to find the Petrov and the Segré classification of a space–time is discussed in Chapter 1 in detail. The components of complex null tetrad, in view of [55], can be given as

$$k^a = -\frac{1}{\sqrt{2}}(e^{-\nu/2}\frac{\partial}{\partial t} - e^{-\lambda/2}\frac{\partial}{\partial r}), \tag{2.1.23}$$

$$l^a = -\frac{1}{\sqrt{2}}(e^{-\nu/2}\frac{\partial}{\partial t} + e^{-\lambda/2}\frac{\partial}{\partial r}), \tag{2.1.24}$$

$$m^a = \frac{1}{\sqrt{2}\mu}(\frac{\partial}{\partial \theta} + \frac{\iota}{\sin\theta}\frac{\partial}{\partial \phi}), \tag{2.1.25}$$

$$\bar{m}^a = \frac{1}{\sqrt{2}\mu}(\frac{\partial}{\partial \theta} - \frac{\iota}{\sin\theta}\frac{\partial}{\partial \phi}). \tag{2.1.26}$$

Complex coefficients are now obtained as

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \tag{2.1.27}$$

$$\begin{aligned}
\Psi_2 &= \frac{1}{2}e^{-(\nu+\lambda)}C_{0101} \\
&= \frac{e^{-\lambda}}{24}(-20\frac{\mu''}{\mu} - 4\frac{\mu'^2}{\mu^2} - 10\frac{\nu'\mu'}{\mu} + 10\frac{\lambda'\mu'}{\mu} \\
&- 14\nu'' - 7\nu'^2 + 7\nu'\lambda') \\
&+ \frac{e^{-\nu}}{24}(20\frac{\ddot{\mu}}{\mu} + 4\frac{\dot{\mu}^2}{\mu^2} - 10\frac{\dot{\nu}\dot{\mu}}{\mu} + 10\frac{\dot{\lambda}\dot{\mu}}{\mu} + 14\ddot{\lambda} \\
&+ 7\dot{\lambda}^2 - 7\dot{\nu}\dot{\lambda}) + \frac{1}{6\mu^2}.
\end{aligned} \tag{2.1.28}$$

The matrix **Q**, is obtained as

$$\mathbf{Q} = \begin{pmatrix} \Psi_2 & 0 & 0 \\ 0 & \Psi_2 & 0 \\ 0 & 0 & -2\Psi_2 \end{pmatrix}.$$

The characteristic equation of **Q** is given by

$$(\Psi_2 - \lambda)^2(-2\Psi_2 - \lambda) = 0. \tag{2.1.29}$$

Now the eigenvalues, λ_1, λ_2 and λ_3, are obtained as

$$\lambda_1 = \lambda_2 = \Psi_2, \quad \text{and} \quad \lambda_3 = -2\Psi_2. \tag{2.1.30}$$

So that, the non–static, spherically symmetric geometry satisfies the matrix criterion

$$(\mathbf{Q} + \frac{1}{2}\lambda\mathbf{I})(\mathbf{Q} - \lambda\mathbf{I}) = \mathbf{0}. \tag{2.1.31}$$

It is Segré type [(11)1] that is

$$\lambda_1 = \lambda_2 \neq \lambda_3 \tag{2.1.32}$$

and corresponds to the Petrov type D. If $\Psi_2 = 0$ then the Petrov type is O.

The non–zero components of S_{ab} for the metric (2.1.1), are obtained as

$$\begin{aligned} S_{00} = {} & \frac{1}{8}e^{\nu-\lambda}(8\frac{\mu''}{\mu} + 4\frac{\mu'^2}{\mu^2} + 12\frac{\nu'\mu'}{\mu} - 4\frac{\lambda'\mu'}{\mu} + 6\nu'' + 3\nu'^2 - 3\nu'\lambda') \\ & +\frac{1}{8}(-24\frac{\ddot{\mu}}{\mu} - 4\frac{\dot{\mu}^2}{\mu^2} + 12\frac{\dot{\nu}\dot{\mu}}{\mu} - 4\frac{\dot{\lambda}\dot{\mu}}{\mu} - 6\ddot{\lambda} - 3\dot{\lambda}^2 + 3\dot{\nu}\dot{\lambda}) - \frac{e^{\nu}}{2\mu^2}, \end{aligned} \tag{2.1.33}$$

$$\begin{aligned} S_{11} = {} & \frac{1}{8}(-24\frac{\mu''}{\mu} - 4\frac{\mu'^2}{\mu^2} - 4\frac{\nu'\mu'}{\mu} + 12\frac{\lambda'\mu'}{\mu} - 6\nu'' - 3\nu'^2 + 3\nu'\lambda') \\ & +\frac{1}{8}e^{\lambda-\nu}(8\frac{\ddot{\mu}}{\mu} + 4\frac{\dot{\mu}^2}{\mu^2} - 4\frac{\dot{\nu}\dot{\mu}}{\mu} + 12\frac{\dot{\lambda}\dot{\mu}}{\mu} + 6\ddot{\lambda} + 3\dot{\lambda}^2 - 3\dot{\nu}\dot{\lambda}) + \frac{e^{\lambda}}{2\mu^2}, \end{aligned} \tag{2.1.34}$$

$$\begin{aligned} S_{22} = {} & \frac{1}{8}\mu^2 e^{-\lambda}(-16\frac{\mu''}{\mu} - 4\frac{\mu'^2}{\mu^2} - 8\frac{\nu'\mu'}{\mu} + 8\frac{\lambda'\mu'}{\mu} - 2\nu'' - \nu'^2 + \nu'\lambda') \\ & +\frac{1}{8}\mu^2 e^{-\nu}(16\frac{\ddot{\mu}}{\mu} + 4\frac{\dot{\mu}^2}{\mu^2} - 8\frac{\dot{\nu}\dot{\mu}}{\mu} + 8\frac{\dot{\lambda}\dot{\mu}}{\mu} + 2\ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda}) - 1, \end{aligned} \tag{2.1.35}$$

$$S_{33} = \sin^2\theta S_{22}, \quad S_{01} = R_{01} = -2\frac{\dot{\mu}'}{\mu} + \frac{\nu'\dot{\mu}}{\mu} + \frac{\dot{\lambda}\mu'}{\mu}. \tag{2.1.36}$$

So that, the non–zero components of Φ_{ab}, are

$$\begin{aligned} \Phi_{00} = {} & \frac{1}{4}e^{-\lambda}(-2\frac{\mu''}{\mu} + \frac{\nu'\mu'}{\mu} + \frac{\lambda'\mu'}{\mu}) + \frac{1}{4}e^{-\nu}(-2\frac{\ddot{\mu}}{\mu} + \frac{\dot{\nu}\dot{\mu}}{\mu} + \frac{\dot{\lambda}\dot{\mu}}{\mu}) \\ & +\frac{e^{-\frac{\nu+\lambda}{2}}}{2}(2\frac{\dot{\mu}'}{\mu} - \frac{\nu'\dot{\mu}}{\mu} - \frac{\dot{\lambda}\mu'}{\mu}), \end{aligned} \tag{2.1.37}$$

$$\Phi_{11} = \frac{1}{16}e^{-\lambda}(2\nu'' + \nu'^2 - \nu'\lambda') - \frac{1}{8}e^{-\nu}(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda}) - \frac{3}{8\mu^2}, \qquad (2.1.38)$$

$$\Phi_{22} = \frac{1}{4}e^{-\lambda}(-2\frac{\mu''}{\mu} + \frac{\nu'\mu'}{\mu} + \frac{\lambda'\mu'}{\mu}) + \frac{1}{4}e^{-\nu}(-2\frac{\ddot{\mu}}{\mu} + \frac{\dot{\nu}\dot{\mu}}{\mu} + \frac{\dot{\lambda}\dot{\mu}}{\mu}) - \frac{e^{-\frac{\nu+\lambda}{2}}}{2}(2\frac{\dot{\mu}'}{\mu} - \frac{\nu'\dot{\mu}}{\mu} - \frac{\dot{\lambda}\mu'}{\mu}). \qquad (2.1.39)$$

Eigenvalues of S_{ab} are $\lambda_1 = -e^{-\nu}S_{00}, \lambda_2 = e^{-\lambda}S_{11}, \lambda_3 = \lambda_4 = \frac{1}{\mu^2}S_{22}$.

Now we find the Jordan canonical form of the matrix S^a_b. Here λ_1 has multiplicity 1 so there is one Jordan block corresponding to λ_1 of size 1 and similarly for λ_2. λ_3 has multiplicity 2 and $d_1 = dimN(S^a_b - \lambda_3\delta^a_b) = 2$ that shows there are two Jordan blocks corresponding to λ_3 so each would be of size 1. Hence the Jordan canonical form of S^a_b is

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

So, as described in Chapter 1 and from the Jordan canonical form of S^a_b, the Segré type of the metric (2.1.1) is [1(11)1] if $\lambda_1 \neq \lambda_2$.

The only non–zero components of Φ_{ab} are $\Phi_{00}, \Phi_{11}, \Phi_{22}$. So, there are six possible Segré types in the table 1.4 given in Chapter 1 corresponding to it.

- Using the FE (2.1.8) in equations (2.1.37) and (2.1.39), it is obtained that $\Phi_{00} = \Phi_{22}$. So that the non–zero components of Φ_{ab} satisfy the case $\Phi_{00} = \Phi_{22}$ and $\Phi_{11} \neq 0$, so the Segré type is [1(11)1] that corresponds to the Petrov type D.

- If Φ_{11} is the only non–zero component then the Segré type is [(11)(11)] which is possible only when $\lambda_1 = \lambda_2$ which implies the case that $-e^{-\nu}S_{00} = e^{-\lambda}S_{11}$ which is possible when λ, ν , and μ

satisfy the relation,

$$e^{-\lambda}\left(2\mu'' - \nu'\mu' - \lambda'\mu'\right) + e^{-\nu}\left(2\ddot{\mu} - \dot{\nu}\dot{\mu} - \dot{\lambda}\dot{\mu}\right) = 0. \tag{2.1.40}$$

Its corresponding Petrov type is D.

- The third case that $\Phi_{00} = \Phi_{22} = 2\Phi_{11}$ is possible if ν, μ, and λ satisfy the following equation,

$$\begin{aligned}&\frac{1}{8}e^{-\lambda}\left(-4\frac{\mu''}{\mu} + 2\frac{\nu'\mu'}{\mu} + 2\frac{\lambda'\mu'}{\mu} - 2\nu'' - \nu'^2 + \nu'\lambda'\right)\\ &+ \frac{1}{4}e^{-\nu}\left(-2\frac{\ddot{\mu}}{\mu} + \frac{\dot{\nu}\dot{\mu}}{\mu} + \frac{\dot{\lambda}\dot{\mu}}{\mu} + \ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda}\right) + \frac{3}{4\mu^2} = 0.\end{aligned} \tag{2.1.41}$$

It corresponds to the Segré type [1(111)] and the Petrov type O.

- Since $\Phi_{00} = \Phi_{22}$ so the case that $\Phi_{00} = -\Phi_{22}$, is not possible.

- The case Φ_{11}, Φ_{22} are the only non–zero components is not possible as if $\Phi_{00} = 0$ then so is Φ_{22} as they are equal. Similarly the case that Φ_{22} is the only non–zero component is also not possible.

2.2 Solution of the FEs

In six partial differential equations (2.1.5)–(2.1.10) there are nine unknowns namely, $\nu, \lambda, \mu, \rho, p_r, p_t, E, j_0$, and j_1. Therefore, in order to obtain solution we take ansatz on three unknowns. We take ansatz on two of the gravitational potentials and the square of the electric field intensity as follows.

A. Qadir and M. Ziad [56], while classifying the spherically symmetric space–times using the symmetries of the Einstein equations found that for non–static solutions the third component of the metric is of the type $e^{f(t\pm r)}$. Motivated from this, the following assumption on the third metric component, μ, is taken

$$\mu = a_0\left(1+\left(\frac{t+r}{a}\right)^2\right), \tag{2.2.1}$$

where a and a_0 are non–zero constants and have length dimensions. Using the form of μ given by equation (2.2.1), equation (2.1.8) gives

$$\dot{\lambda} = -\nu' + \frac{2}{t+r}. \tag{2.2.2}$$

Since μ is taken to be a function of $t+r$, so consider a similar type of function for ν as well and take an ansatz

$$\nu = -\frac{1}{1+\left(\frac{t+r}{a}\right)^2}. \tag{2.2.3}$$

Using this expression of ν in equation (2.2.2), leads to

$$\lambda = \frac{1}{1+\left(\frac{t+r}{a}\right)^2} + \ln\left(\frac{t+r}{a}\right)^2. \tag{2.2.4}$$

As assumed in [57,58] the form of the square of the electric field intensity, E^2, for static solutions, a similar form is considered and taken to be a function of $t+r$ as follows

$$E^2 = \frac{k\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{2.2.5}$$

where k is a positive constant. Using values of λ, ν, μ and E^2 in equations (2.1.5)–(2.1.10), the mass density, radial and tangential pressures and the non–zero components of electric current density are obtained as

$$\rho = -4e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right) + 4e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{1+\left(\frac{t+r}{a}\right)^2} + \frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} - \frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right) + \frac{1/{a_0}^2 - k/2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{2.2.6}$$

$$p_r = 4e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}+\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right)-4e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{1+\left(\frac{t+r}{a}\right)^2}+\frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}-\frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right)-\frac{1/{a_0}^2-k/2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{2.2.7}$$

$$p_t = 2e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\right)+e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(-\frac{4/a^2}{1+\left(\frac{t+r}{a}\right)^2}+\frac{4/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}-\frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\right)-\frac{k/2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{2.2.8}$$

$$j_0 = e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\frac{\frac{\sqrt{k}}{a}\left(1+3\left(\frac{t+r}{a}\right)^2\right)}{\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{2.2.9}$$

$$j_1 = e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\frac{\frac{\sqrt{k}}{a}\left(1+3\left(\frac{t+r}{a}\right)^2\right)}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}. \tag{2.2.10}$$

From equations (2.2.6) and (2.2.7), the equation of state is obtained as

$$\rho + p_r = 0. \tag{2.2.11}$$

The metric of the obtained solution is

$$ds^2 = -e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}dt^2+\left(\frac{t+r}{a}\right)^2 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}dr^2+{a_0}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2 d\Omega^2. \tag{2.2.12}$$

2.3 Physical Analysis of the Solution

In the previous section, a class of non–static, spherically symmetric exact solutions of the EMFEs with anisotropic pressure is obtained. In this section, the analysis of the solution shows that the solution is physically acceptable:

• For a solution to be physically meaningful there should be no singularity in the solution. It means that none of the gravitational potentials vanish or become undefined and all the physical variables like ρ, p_r, p_t, E^2 must be defined. In the solution obtained in previous section, there is no singularity except at $t = -r$ where the second component of the metric, $e^{\lambda} = \left(\frac{t+r}{a}\right)^2 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}$, is zero. In this case it should be checked that the singularity must be coordinate/removable not geometrical/essential. For this, the curvature invariants are to be obtained for the metric (2.2.12).

The curvature invariants are now given by

$$\begin{aligned}
\mathcal{R}_1 = R = {} & \frac{1}{a_0{}^2 a^8 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} (1+\left(\frac{t+r}{a}\right)^2)^4} \Bigg(2a^8 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} \\
& - 20a^6 a_0{}^2 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} + 8a^6 a_0{}^2 + 4a^8 \left(\frac{t+r}{a}\right)^2 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} \\
& - 24a^6 a_0{}^2 \left(\frac{t+r}{a}\right)^2 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} + 36a^6 a_0{}^2 \left(\frac{t+r}{a}\right)^2 \\
& + 2a^8 \left(\frac{t+r}{a}\right)^4 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} - 8a^6 a_0{}^2 \left(\frac{t+r}{a}\right)^4 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} \\
& + 48a^6 a_0{}^2 \left(\frac{t+r}{a}\right)^4 + 24a^6 a_0{}^2 \left(\frac{t+r}{a}\right)^6 \Bigg),
\end{aligned} \tag{2.3.1}$$

$$
\begin{aligned}
\mathcal{R}_2 = R^{ab}_{cd}R^{cd}_{ab} = \frac{16}{a^{20}{a_0}^4(1+\left(\frac{t+r}{a}\right)^2)^8}\Bigg(& 4a^{20}\left(\frac{t+r}{a}\right)^8 + 128a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^8 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} \\
& + 24a^{20}\left(\frac{t+r}{a}\right)^4 + 16a^{20}\left(\frac{t+r}{a}\right)^2 + 4a^{20} + 6a^{16}{a_0}^4 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
& + 67a^{16}{a_0}^4 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} - 128a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{10} - 516a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 \\
& - 780a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 - 532a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 - 150a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^2 \\
& - 32a^{18}{a_0}^2 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} + 32a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^{10} e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} + 16a^{20}\left(\frac{t+r}{a}\right)^6 \\
& + 192a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^6 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} + 128a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^4 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} \\
& - 12a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} + 32a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^2 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} - \\
& 8a^{16}{a_0}^4 + 4a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^2 + 2a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 + 2a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} \\
& + 4a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} + 2a^{16}{a_0}^4 + 2a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} \\
& - 32a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^8 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} - 128a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^6 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} \\
& - 192a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^4 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} - 128a^{18}{a_0}^2\left(\frac{t+r}{a}\right)^2 e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}} \\
& + 66a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{12} e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} + 264a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{10} e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
& + 400a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} + 64a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}}
\end{aligned}
$$

$$
\begin{aligned}
&+ 276a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} + 256a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 81a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} + 390a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 12a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^2 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} + 264a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^2 e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 2a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{12} e^{\frac{4}{1+\left(\frac{t+r}{a}\right)^2}} + 2a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{16} e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&- 4a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} - 4a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{12} - 12a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{10} \\
&- 16a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} - 16a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 - 4a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 \\
&- 12a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^2 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} - 12a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 - 4a^{16}{a_0}^4 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 8a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{10} e^{\frac{4}{1+\left(\frac{t+r}{a}\right)^2}} + 8a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{14} e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 16a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 e^{\frac{4}{1+\left(\frac{t+r}{a}\right)^2}} + 16a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{12} e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 20a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 e^{\frac{4}{1+\left(\frac{t+r}{a}\right)^2}} + 20a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^{10} e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 16a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 e^{\frac{4}{1+\left(\frac{t+r}{a}\right)^2}} + 16a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^8 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&+ 8a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^2 e^{\frac{4}{1+\left(\frac{t+r}{a}\right)^2}} + 8a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^6 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \\
&\left. +2a^{16}{a_0}^4 e^{\frac{4}{1+\left(\frac{t+r}{a}\right)^2}} + 2a^{16}{a_0}^4\left(\frac{t+r}{a}\right)^4 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}} \right).
\end{aligned}
\tag{2.3.2}
$$

Since the curvature invariants are defined and non–zero at $t = -r$, so it is a coordinate singularity that can be removed by a suitable choice of coordinate transformation.

• For a meaningful solution, the square of the electric field intensity, E^2, should be a non–negative, continuous, bounded, and smooth function of both t and r. It can be seen that the expression taken for the squared power of electric field intensity satisfies all these properties for positive values of the parameter k and all the values of parameter a. The Figure (2.1) shows the graph of E^2 with respect to r for three different values of t with specific values of parameters.

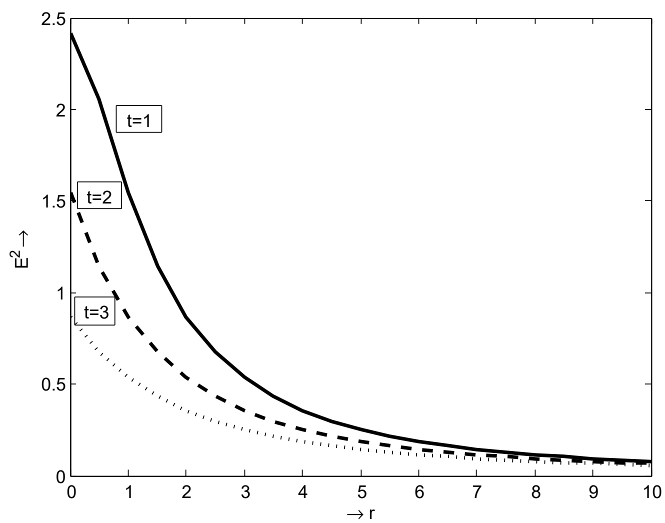

Figure 2.1: The square of electric field intensity, E^2, is shown with respect to r for $t = 1, 2$ and 3. For all cases $a = 1$ and $k = 9.65$.

• The mass density, ρ, should be a non–negative, continuous, bounded, decreasing, and smooth function. From the expression obtained in previous section, it is obvious that ρ is bounded and continuous function of both t and r for all parameter values. For ρ to be non–negative we have following conditions on parameters. At origin non–negativity of ρ requires

$$k \geq \frac{2a^2 - 15.8602{a_0}^2}{a^2{a_0}^2}, \tag{2.3.3}$$

for all other values of t and r, the non–negativity of ρ requires

$$k \geq \frac{2}{ea^2{a_0}^2}\left(ea^2 + 4{a_0}^2\left(2 - e^2 - \frac{2}{27}e^{2/3}\right)\right). \tag{2.3.4}$$

Further, the mass density, ρ, must be a decreasing function of r. A function is decreasing with respect to some parameter if its first derivative with respect to that parameter is negative. Now

$$\frac{d\rho}{dr} = \frac{8\left(\frac{t+r}{a}\right)e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}}}{a^3\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\left(2+\frac{2}{1+\left(\frac{t+r}{a}\right)^2}-\frac{1}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\right)$$
$$-\frac{8\left(\frac{t+r}{a}\right)e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}}{a^3\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\left(2-\frac{2\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}-\frac{\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right) \tag{2.3.5}$$
$$+\left(\frac{1}{{a_0}^2}-\frac{k}{2}\right)\frac{4\left(\frac{t+r}{a}\right)}{a\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}.$$

For $\frac{d\rho}{dr} < 0$, the parameters must have the relation

$$k > \frac{2a^2 - 13.0680{a_0}^2}{a^2{a_0}^2}. \tag{2.3.6}$$

From the inequalities (2.3.3), (2.3.4), and (2.3.6) we have

$$k > \frac{2a^2 - 13.0680{a_0}^2}{a^2{a_0}^2}. \tag{2.3.7}$$

Figure (2.2) shows the graph of the mass density, ρ, with respect to r for three different values of t with specific values of the parameters.

• For a physically acceptable solution the radial pressure, p_r, must be a positive, continuous, bounded, decreasing, and smooth function. But in view of our equation of state, that is, $p_r = -\rho$ and that mass density is a positive function, the radial pressure is a negative, continuous, bounded and a smooth function of both t and r that increases to zero asymptotically. The negative value of the radial pressure indicates that the solution obtained represents a compact object with negative pressure. The graph of the radial pressure, p_r, with respect to r is shown

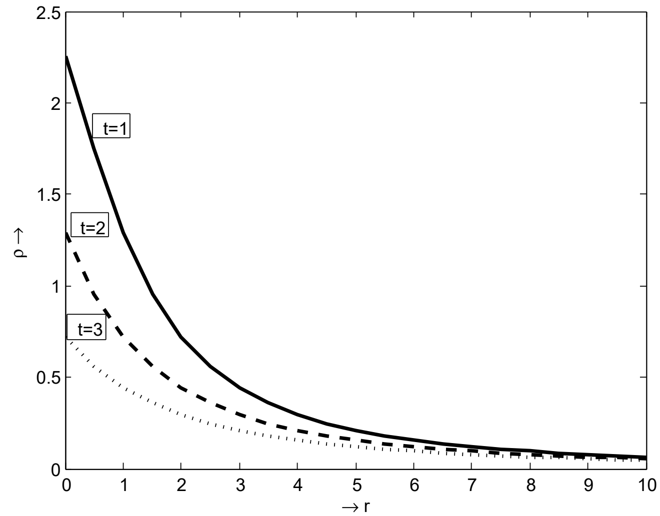

Figure 2.2: The mass density, ρ, is shown with respect to r for $t = 1, 2$ and 3. For all cases $a = 1 = a_0$ and $k = 9.65$.

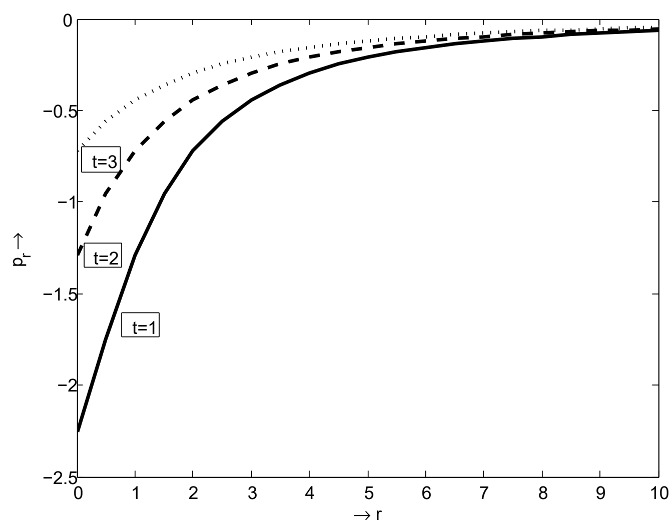

Figure 2.3: The radial pressure, p_r, is shown with respect to r for $t = 1, 2$ and 3. For all cases $a = 1 = a_0$ and $k = 9.65$.

in Figure (2.3) for three different values of t with fixed values of the parameters.

• Tangential pressure, p_t, must also satisfy the properties required to be satisfied by the radial pressure. But in the case being discussed here the tangential pressure is a negative, continuous bounded, and smooth function of both r and t and its value asymptotically approaches to zero. The graph of the tangential pressure is given in Figure (2.4) with respect to r for three different values of t and the fixed suitable values of all the three parameters.

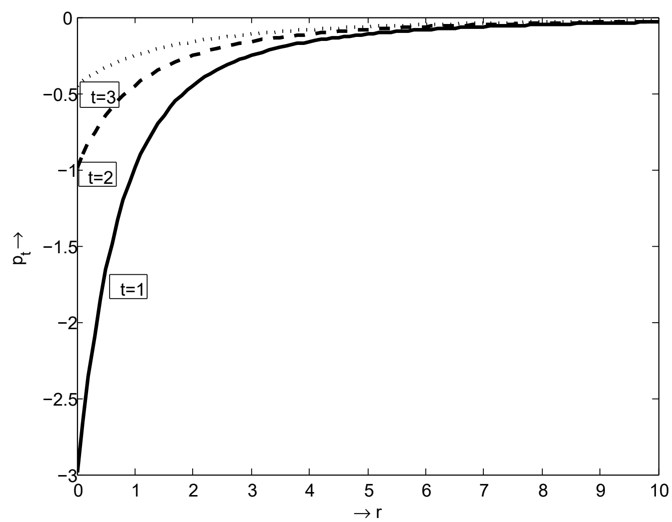

Figure 2.4: The tangential pressure, p_t, is shown with respect to r for $t = 1, 2$ and 3. For all cases $a = 1 = a_0$ and $k = 9.65$.

• The radial pressure, p_r, and the tangential pressure, p_t, must have same values at the origin, that is, $p_r = p_t$ at $t = 0 = r$. This relation is satisfied for all values of the constants k, a and a_0 satisfying the relation

$$k = \frac{a^2 + 8.6659{a_0}^2}{a^2{a_0}^2}. \tag{2.3.8}$$

Using equation (2.3.8) in the inequality (2.3.7), it can be written that

$$a^2 < 21.7339{a_0}^2. \tag{2.3.9}$$

Measure of anisotropy of the solution, denoted $\Delta = p_t - p_r$, is shown in Figure 2.5. Where the graph is plotted with respect to r for three different values of t and the fixed suitable values of all the three parameters.

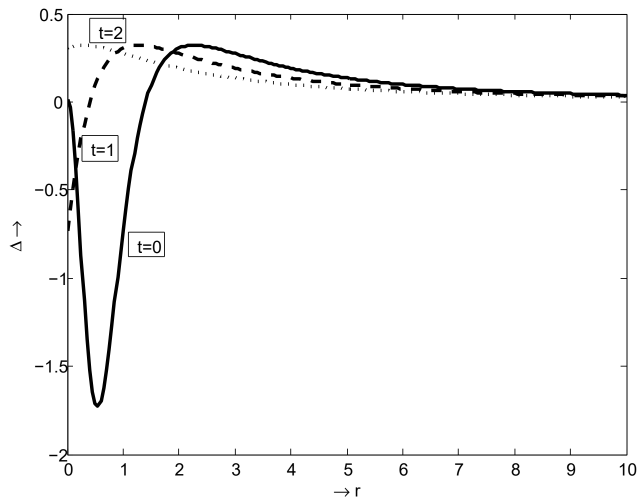

Figure 2.5: The Anisotropy, $\Delta = p_t - p_r$, is shown with respect to r for $t = 0, 1$ and 2. For all cases $a = 1 = a_0$ and $k = 9.65$.

• The weak energy conditions are given as, $\rho \geq 0$ and $\rho + p_r \geq 0$. For solution obtained above, first condition is satisfied for all values of the parameters satisfying the relation

$$k \geq \frac{2a^2 - 13.0680{a_0}^2}{a^2{a_0}^2}, \tag{2.3.10}$$

and the second condition is satisfied by the equation of state (2.2.11). The dominant energy condition, that is, $\rho \geq |p_r|$, is also satisfied as is evident from equation of state (2.2.11).

• The causality condition states that the speed of sound is less than the speed of light, that is, $0 < \frac{dp_r}{d\rho} \leq 1$. In this case it is not satisfied as the pressure is negative so speed of sound comes out to be $\frac{dp_r}{d\rho} = -1$ but still it is less than the speed of light that is taken to be unity.
• The non–zero components, j_0 and j_1, of the electric current density are also continuous and decreasing functions of both r and t and are shown in Figures (2.6) and (2.7), respectively.

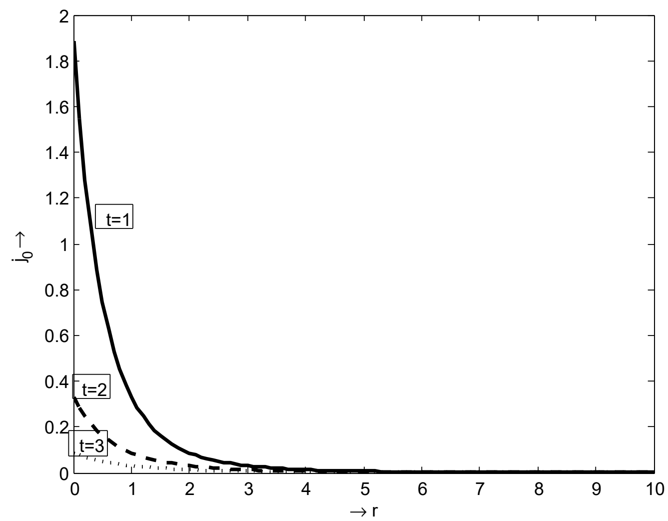

Figure 2.6: The component, j_0, of the current density is shown with respect to r for $t = 0.2, 0.3$ and 0.4. For all cases $a = 1 = a_0$ and $k = 9.65$.

2.4 Conclusion

In this chapter, we have obtained a class of non–static, spherically symmetric exact solutions of the EMFEs. We have assumed pressure distribution to be anisotropic. The mass density, ρ, is a positive decreasing function of both r and t, the radial pressure, p_r, is a negative function of both r and t and asymptotically approaches to zero. Also,

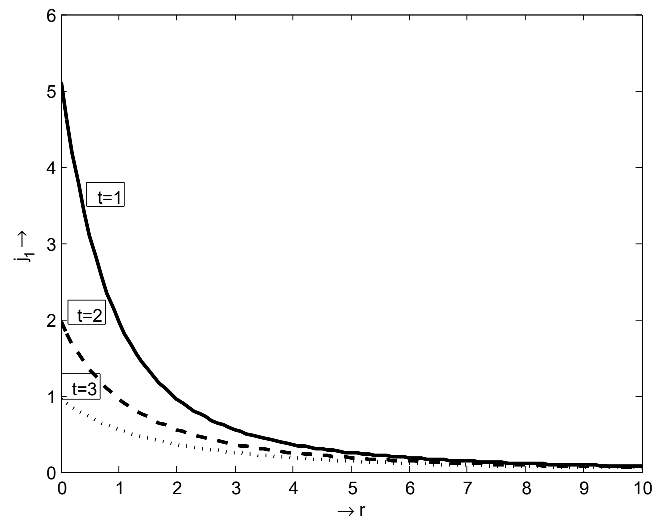

Figure 2.7: The component, j_1, of the current density is shown with respect to r for $t = 0.2, 0.4$ and 0.6. For all cases $a = 1 = a_0$ and $k = 9.65$.

the square of the electric field intensity, E^2, is a positive function of both r and t and is bounded. All these conditions deduce that the expressions obtained for all the physical variables ρ, p_r, p_t, E^2 etc., are physically meaningful. There is one singularity at $t = -r$ that is removable so the solution has no geometric singularity and is physically acceptable. Notice that ρ, p_r, p_t and E^2 are all symmetric with respect to t and r.

The solution obtained may be thought to represent an expanding compact object due to negative pressure and that the causality condition is not satisfied by the solution, as is expected for an objesct with negative pressure. In the case of static solutions we have two basic solutions of the FEs with spherical geometry: charged solution of Reissner–Nordstr*ö*m and the solution without charge of Schwarzschild. The Kerr solution is non–static but it represents a rotating object. So, there is no basic solution with which the current solution should be matched to fix the parameter values. It is just a successful attempt to find a general non–static solution of the EMFEs.

Chapter 3

A Class of Isotropic, Non-Static Exact Solutions of the EMFEs with $\mu = a_0\left(1 + \left(\frac{t+r}{a}\right)^2\right)$

Astrophysical systems such as stellar interiors can be modeled by solutions of the EMFEs. Many attempts have been made to solve this set of differential equations both for static and non–static conditions.

In this chapter a new class of solutions of the EMFEs for non–static space–time geometry is obtained that also represents a negative pressure model. The pressure distribution is assumed to be isotropic and ansatz are taken on the first and the third metric components. The solutions admit negative pressure.

In Section 3.1, a new class of solutions of the FEs is presented. In Section 3.2, the analysis for the solution to be physically acceptable is briefly discussed. In Section 3.4, a brief conclusion is given and the types of physical systems that this solution can model is also identified.

3.1 Solution of the FEs

The FEs are already given in Chapter 2 (equations (2.1.5)–(2.1.10)). In this chapter an isotropic fluid distribution i.e. $p_r = p_t = p$ is considered and ansatz on metric coefficients are taken.

After the implementation of the isotropy of pressure distribution on the FEs, there are six partial differential equations with eight unknowns namely $\nu, \lambda, \mu, \rho, p, E^2, j_0$, and j_1. So in order to solve this system of equations only two ansatz are required. The detailed discussion about the motivation of the ansatz taken on the gravitational potentials μ and ν is given in Chapter 2. Here the solutions are obtained with the same ansatz on μ and ν. So that we have

$$\nu = -\frac{1}{1+\left(\frac{t+r}{a}\right)^2}, \tag{3.1.1}$$

$$\lambda = \frac{1}{1+\left(\frac{t+r}{a}\right)^2} + \ln\left(\frac{t+r}{a}\right)^2, \tag{3.1.2}$$

$$\mu = a_0\left(1+\left(\frac{t+r}{a}\right)^2\right). \tag{3.1.3}$$

Using values of λ, ν and μ in equations (2.1.5)–(2.1.7), we get

$$-4e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right) + 4e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{1+\left(\frac{t+r}{a}\right)^2} + \frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} - \frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right) + \frac{1}{a_0{}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} = \rho + \frac{E^2}{2\left(\frac{t+r}{a}\right)^2}, \tag{3.1.4}$$

$$4e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}+\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right)-4e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{1+\left(\frac{t+r}{a}\right)^2}+\right.$$
$$\left.\frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}-\frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right)-\frac{1}{{a_0}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}=p-\frac{E^2}{2\left(\frac{t+r}{a}\right)^2}, \tag{3.1.5}$$

$$2e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\right)+e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(-\frac{4/a^2}{1+\left(\frac{t+r}{a}\right)^2}+\frac{4/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\right.$$
$$\left.-\frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\right)=p+\frac{E^2}{2\left(\frac{t+r}{a}\right)^2}. \tag{3.1.6}$$

Solving equations (3.1.4)–(3.1.6) simultaneously, the mass density, the pressure and the electric field intensity are given as

$$\rho=-e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{2/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}+\frac{2/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}+\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\right)+e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\times$$
$$\left(\frac{4/a^2}{1+\left(\frac{t+r}{a}\right)^2}-\frac{2/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}+\frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}-\frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right.$$
$$\left.+\frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\right)+\frac{1}{2{a_0}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{3.1.7}$$

$$p = e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} \left(\frac{2/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{2/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3} + \frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} \right) + e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} \times$$
$$\left(-\frac{4/a^2}{1+\left(\frac{t+r}{a}\right)^2} + \frac{2/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} - \frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3} \right.$$
$$\left. - \frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} \right) - \frac{1}{2{a_0}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{3.1.8}$$

and

$$E^2 = e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} \left(-\frac{\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} - \frac{\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3} + \frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} \right) +$$
$$e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} \left(\frac{\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2}{1+\left(\frac{t+r}{a}\right)^2} - \frac{\frac{4}{a^2}\left(\frac{t+r}{a}\right)^4}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3} - \frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^4}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} \right) \tag{3.1.9}$$
$$+ \frac{\left(\frac{t+r}{a}\right)^2}{{a_0}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}.$$

From equations (3.1.7) and (3.1.8), we have

$$\rho + p = 0, \tag{3.1.10}$$

which is the equation of state. Using values of λ, μ, ν and E^2 in equations (2.1.9) and (2.1.10), the non–zero components of the current density are obtained as

$$
\begin{aligned}
j_0 = {}& \frac{1}{2}\frac{e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}}{\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\left[e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left\{-\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2-\right.\right. \\
& \left.\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)+\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2\right\}+e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\times \\
& \left\{\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^3-\frac{4}{a^2}\left(\frac{t+r}{a}\right)^4\left(1+\left(\frac{t+r}{a}\right)^2\right)\right. \\
& \left.\left.-\frac{2}{a^2}\left(\frac{t+r}{a}\right)^4\right\}+\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2\right]^{-\frac{1}{2}}\left[e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left\{-\frac{\frac{8}{a^3}\left(\frac{t+r}{a}\right)^3}{1+\left(\frac{t+r}{a}\right)^2}\right.\right. \\
& -\frac{8}{a^3}\left(\frac{t+r}{a}\right)\left(1+\left(\frac{t+r}{a}\right)^2\right)\left(2+\left(\frac{t+r}{a}\right)^2\right)+\frac{4}{a^3}\left(\frac{t+r}{a}\right)-\frac{16}{a^3}\left(\frac{t+r}{a}\right)^3 \\
& \left.-\frac{16}{a^3}\left(\frac{t+r}{a}\right)^3\left(1+\left(\frac{t+r}{a}\right)^2\right)+\frac{\frac{4}{a^3}\left(\frac{t+r}{a}\right)^3}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\right\}+e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left\{\frac{\frac{8}{a^3}\left(\frac{t+r}{a}\right)^5}{1+\left(\frac{t+r}{a}\right)^2}\right. \\
& +\frac{8}{a^3}\left(\frac{t+r}{a}\right)\left(1+\left(\frac{t+r}{a}\right)^2\right)^3+\frac{24}{a^3}\left(\frac{t+r}{a}\right)^3\left(1+\left(\frac{t+r}{a}\right)^2\right)^2+ \\
& \left.\frac{\frac{4}{a^3}\left(\frac{t+r}{a}\right)^5}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\right\}-\frac{32}{a^3}\left(\frac{t+r}{a}\right)^3\left(1+\left(\frac{t+r}{a}\right)^2\right) \\
& \left.+\frac{2}{a}\left(\frac{t+r}{a}\right)\left(1+\left(\frac{t+r}{a}\right)^2\right)\left(1+3\left(\frac{t+r}{a}\right)^2\right)\right],
\end{aligned}
\tag{3.1.11}
$$

and

$$
\begin{aligned}
j_1 = \frac{1}{2}\frac{e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\Bigg[& e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\Bigg\{-\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2+\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2- \\
& \frac{4}{a^2}\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)\Bigg\}+e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\Bigg\{\frac{4}{a^2}\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^3- \\
& \frac{4}{a^2}\left(\frac{t+r}{a}\right)^4\left(1+\left(\frac{t+r}{a}\right)^2\right)-\frac{2}{a^2}\left(\frac{t+r}{a}\right)^4\Bigg\}+\left(\frac{t+r}{a}\right)^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2\Bigg]^{-\frac{1}{2}} \\
& \Bigg[e^{-\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\Bigg\{\frac{4}{a^3}\left(\frac{t+r}{a}\right)-\frac{8}{a^3}\left(\frac{t+r}{a}\right)\left(1+\left(\frac{t+r}{a}\right)^2\right)\left(2+\left(\frac{t+r}{a}\right)^2\right) \\
& -\frac{16}{a^3}\left(\frac{t+r}{a}\right)^3-\frac{16}{a^3}\left(\frac{t+r}{a}\right)^3\left(1+\left(\frac{t+r}{a}\right)^2\right)-\frac{\frac{8}{a^3}\left(\frac{t+r}{a}\right)^3}{1+\left(\frac{t+r}{a}\right)^2}+\frac{\frac{4}{a^3}\left(\frac{t+r}{a}\right)^3}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\Bigg\} \\
& +e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\Bigg\{\frac{\frac{4}{a^3}\left(\frac{t+r}{a}\right)^5}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}-\frac{32}{a^3}\left(\frac{t+r}{a}\right)^3\left(1+\left(\frac{t+r}{a}\right)^2\right) \\
& +\frac{8}{a^3}\left(\frac{t+r}{a}\right)\left(1+\left(\frac{t+r}{a}\right)^2\right)^3+\frac{24}{a^3}\left(\frac{t+r}{a}\right)^3\left(1+\left(\frac{t+r}{a}\right)^2\right)^2+\frac{\frac{8}{a^3}\left(\frac{t+r}{a}\right)^5}{1+\left(\frac{t+r}{a}\right)^2}\Bigg\} \\
& +\frac{2}{a}\left(\frac{t+r}{a}\right)\left(1+\left(\frac{t+r}{a}\right)^2\right)\left(1+3\left(\frac{t+r}{a}\right)^2\right)\Bigg]\ .
\end{aligned}
\tag{3.1.12}
$$

The metric of our solution is same as given by equation (2.2.12).

3.2 Physical Analysis of the Solution

In the previous section, a class of exact solutions of the EMFEs for charged isotropic, non–static, spherically symmetric geometry is obtained. The analysis below shows that our solution is physically acceptable.

• For a solution to be physically meaningful there should be no singularity in the solution. It means that none of the gravitational potentials vanish or become undefined and all the physical variables like ρ, p_r, p_t, E^2 must be defined. In the solution obtained in previous section, there is no singularity except at $t = -r$ where one of the gravitational potentials is undefined. We need to check that the singularity is coordinate/removable and not geometrical/essential. For this, the curvature invariants are to be discussed. The metric here for the isotropic case is same as that of anisotropic case obtained in Chapter 2. A detailed discussion is made about the type of the singularity there that this is a coordinate singularity, removable by making some suitable coordinate transformations. Thus this solution is free from any geometric singularity.

• For a meaningful solution, the square of the electric field intensity, E^2, should be a non–negative, continuous, bounded, and smooth function of both t and r. For $E^2 \geq 0$ the parameters must satisfy the relation

$$\frac{a^2 + 16{a_0}^2}{a^2{a_0}^2} \geq 0, \tag{3.2.1}$$

which is satisfied for all values of a and a_0. Further, it can be easily seen from the expression obtained for the square of the electric field intensity, E^2, that it is continuous, bounded and a smooth function of both r and t. Graph of E^2 with respect to r for three different values of t is given in Figure (3.1) with specific values of the parameters.

• The mass density, ρ, must be a non–negative, continuous, bounded, decreasing, and smooth function. From the expression obtained in previous section, it is obvious that ρ is bounded and continuous function of both t and r for all parameter values. For ρ to be non–negative we have following conditions on parameters. At origin non–negativity of

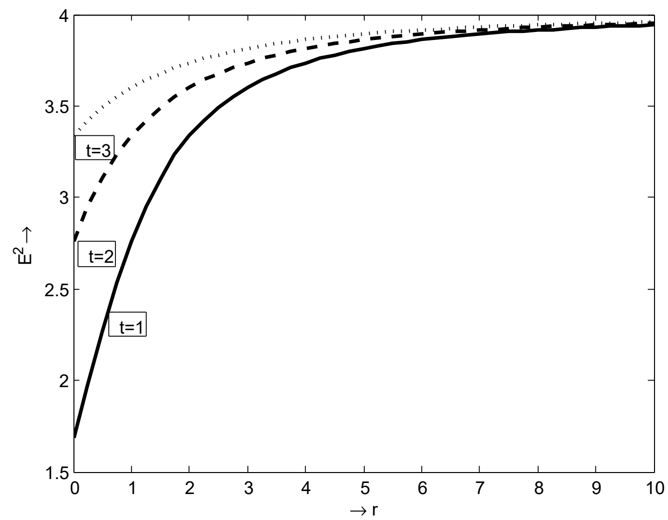

Figure 3.1: The square of electric field intensity, E^2, is shown with respect to r for $t = 0.5, 1$ and 1.5. For all cases $a = a_0 = 1$.

ρ requires that all values of the parameters satisfy the relation

$$\frac{a^2 + 7.1943{a_0}^2}{a^2{a_0}^2} \geq 0, \tag{3.2.2}$$

for all other values of t and r, the parameters must satisfy

$$\frac{a^2 + 7.2142{a_0}^2}{a^2{a_0}^2} \geq 0. \tag{3.2.3}$$

Both of the above conditions given by inequalities (3.2.2) and (3.2.3) are satisfied for all values of the parameters. Further, it can be seen that all the terms involved in the expression of ρ have decreasing values with respect to both the variables t and r but still consider the first derivative of ρ given by

$$\frac{d\rho}{dr} = \frac{-1}{a^3}\left(\frac{t+r}{a}\right)e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}}\left(\frac{2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^6} - \frac{4}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^5} - \frac{8}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\right.$$
$$\left. - \frac{8}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right) + \frac{1}{a^3}\left(\frac{t+r}{a}\right)e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\left(-\frac{8}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} - \frac{4\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}\right.$$
$$\left. + \frac{6}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} + \frac{8\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} - \frac{4\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^5} - \frac{2\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^6}\right)$$
$$- \frac{2\left(\frac{t+r}{a}\right)}{a{a_0}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^3}. \tag{3.2.4}$$

For $\frac{d\rho}{dr} < 0$ the parameters must satisfy the relation

$$-\frac{0.5176a^2 + 2.03{a_0}^2}{a^3{a_0}^2} < 0. \tag{3.2.5}$$

This relation is satisfied for all values of both of the parameters a and a_0 (further $a \geq 0$ and $a_0 \neq 0$). Thus, ρ is a positive and decreasing function of both r and t. It is shown in Figure (3.2).

• The pressure, p, is required to be positive, continuous, bounded and a smooth function. But in view of our equation of state, that is, $p_r = -\rho$ and that mass density is a positive function, the radial pressure is a negative, continuous, bounded and a smooth function of both t and r that increases to zero asymptotically. The negative value of the radial pressure indicates that the solutions obtained represent a compact object with negative pressure. It is shown in Figure (3.3).

• The weak energy condition i.e., $\rho \geq 0$, $\rho + p \geq 0$ and the dominant

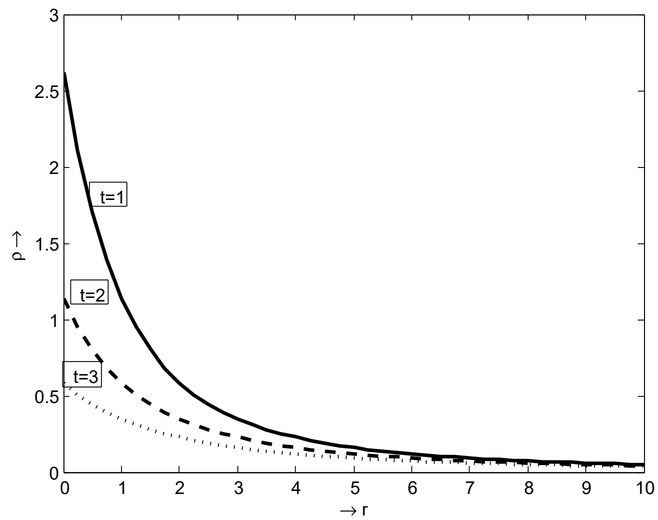

Figure 3.2: The mass density, ρ, is shown with respect to r for $t = 0, 1$ and 2. For all cases $a = a_0 = 1$.

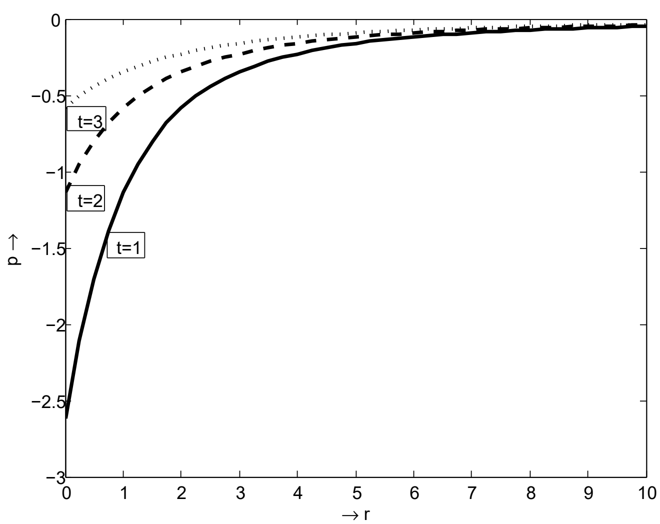

Figure 3.3: The pressure, p, is shown with respect to r for $t = 0, 1$ and 2. For all cases $a = a_0 = 1$.

energy condition i.e., $\rho \geq |p|$ are satisfied for all parameter values.

• The causality condition states that the speed of sound is less than the speed of light, that is, $0 < \frac{dp_r}{d\rho} \leq 1$. In this case it is not satisfied as the pressure is negative so speed of sound comes out to be $\frac{dp_r}{d\rho} = -1$ but still it is less than the speed of light that is taken to be unity.

• The non–zero components of the electric current density j_0 and j_1 are continuous and decreasing functions of both r and t and are shown in Figures (3.4) and (3.5), respectively.

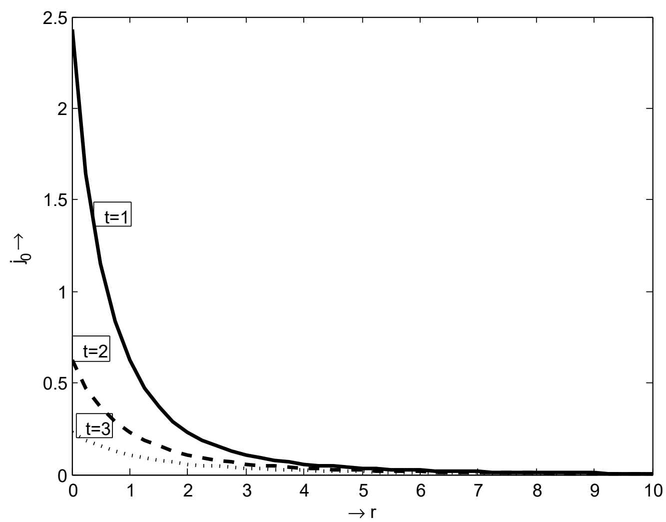

Figure 3.4: The component, j_0, of the current density is shown with respect to r for $t = 0, 0.1$ and 0.2. For all cases $a = a_0 = 1$.

3.3 The Petrov and the Segré Classification of the Solution

The procedure to find the Petrov and the Segré classification of a space–time is discussed in Chapter 1 in detail and generally for the metric

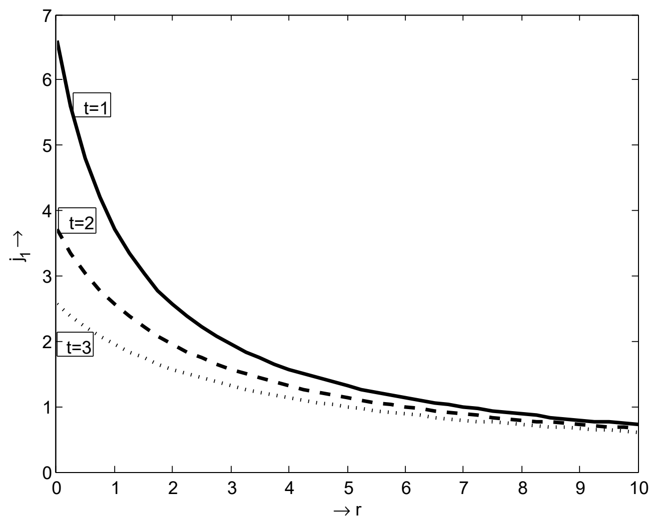

Figure 3.5: The component, j_1, of the current density is shown with respect to r for $t = 0, 0.25$ and 0.5. For all cases $a = a_0 = 1$.

(2.1.1) in Chapter 2. In this section, classification of the obtained metric (2.2.12) is discussed.

The components of complex null tetrad given by equations (2.1.23)–(2.1.26) for the solution metric (2.2.12), can be given as

$$k^a = -\frac{1}{\sqrt{2}}\left(e^{\frac{1}{2\left(1+\left(\frac{t+r}{a}\right)^2\right)}}\frac{\partial}{\partial t} - \frac{a}{t+r}e^{\frac{-1}{2\left(1+\left(\frac{t+r}{a}\right)^2\right)}}\frac{\partial}{\partial r}\right), \qquad (3.3.1)$$

$$\ell^a = -\frac{1}{\sqrt{2}}\left(e^{\frac{1}{2\left(1+\left(\frac{t+r}{a}\right)^2\right)}}\frac{\partial}{\partial t} + \frac{a}{t+r}e^{\frac{-1}{2\left(1+\left(\frac{t+r}{a}\right)^2\right)}}\frac{\partial}{\partial r}\right), \qquad (3.3.2)$$

$$m^a = \frac{1}{\sqrt{2}a_0\left(1+\left(\frac{t+r}{a}\right)^2\right)}\left(\frac{\partial}{\partial\theta} + \frac{\iota}{\sin\theta}\frac{\partial}{\partial\phi}\right), \qquad (3.3.3)$$

$$\bar{m}^a = \frac{1}{\sqrt{2}a_0\left(1+\left(\frac{t+r}{a}\right)^2\right)}\left(\frac{\partial}{\partial\theta} - \frac{\iota}{\sin\theta}\frac{\partial}{\partial\phi}\right). \qquad (3.3.4)$$

The non–zero complex coefficient for the metric (2.2.12) is now obtained as

$$\Psi_2 = \frac{1}{3a^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\left(\left(19-12\left(\frac{t+r}{a}\right)^4\right)\left(\frac{t+r}{a}\right)^2 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} - \left(5+2\left(\frac{t+r}{a}\right)^4\right)e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}}\right). \tag{3.3.5}$$

So that, as discussed in Chapter 2 the Petrov type is D.

For the metric (2.2.12), the surviving components of the Ricci tensor are

$$R_{00} = \frac{\frac{4}{a^2}}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{2}{a^2}e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}}\frac{1}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} - \frac{\frac{2}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}, \tag{3.3.6}$$

$$R_{11} = \frac{4}{a^2}\left(\frac{t+r}{a}\right)^2 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}}\frac{1}{1+\left(\frac{t+r}{a}\right)^2} - \frac{4}{a^2}\left(\frac{t+r}{a}\right)^2 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}}\frac{1}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} - \frac{2}{a^2}\left(\frac{t+r}{a}\right)^2\frac{1}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4} + \frac{2}{a^2}\left(\frac{t+r}{a}\right)^4 e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}}\frac{1}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}, \tag{3.3.7}$$

$$R_{22} = 1 + \frac{4a_0^2}{a^2}\left(1+\left(\frac{t+r}{a}\right)^2\right)e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}} - \frac{4a_0^2}{a^2}e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}}\frac{1}{1+\left(\frac{t+r}{a}\right)^2} - \frac{4a_0^2}{a^2}\left(\frac{t+r}{a}\right)^2 e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\frac{1}{1+\left(\frac{t+r}{a}\right)^2}, \tag{3.3.8}$$

$$R_{33} = R_{22}\sin^2\theta. \tag{3.3.9}$$

The non–zero components of S_{ab}, are obtained as

$$S_{00} = -\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\left(11+18\left(\frac{t+r}{a}\right)^2+14\left(\frac{t+r}{a}\right)^4+4\left(\frac{t+r}{a}\right)^6\right)$$
$$\frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}e^{\frac{-2}{1+\left(\frac{t+r}{a}\right)^2}}\left(7+6\left(\frac{t+r}{a}\right)^2+2\left(\frac{t+r}{a}\right)^4\right) \quad (3.3.10)$$
$$-\frac{1}{2a_0{}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2},$$

$$S_{11} = \frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}e^{\frac{2}{1+\left(\frac{t+r}{a}\right)^2}}\left(11+18\left(\frac{t+r}{a}\right)^2+14\left(\frac{t+r}{a}\right)^4+4\left(\frac{t+r}{a}\right)^6\right)$$
$$-\frac{\frac{1}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\left(7+6\left(\frac{t+r}{a}\right)^2+2\left(\frac{t+r}{a}\right)^4\right)+\frac{\left(\frac{t+r}{a}\right)^2e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}}{2a_0{}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2},$$
(3.3.11)

$$s_{22} = -1+\frac{a_0{}^2/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\left(\left(-5-2\left(\frac{t+r}{a}\right)^2+2\left(\frac{t+r}{a}\right)^4\right)e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}}\right.$$
$$\left.+\left(-2+\left(\frac{t+r}{a}\right)^2+4\left(\frac{t+r}{a}\right)^4+2\left(\frac{t+r}{a}\right)^6\right)e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}\right), \quad (3.3.12)$$

$$S_{33} = \sin^2\theta S_{22}, \quad S_{01} = R_{01} = 0. \quad (3.3.13)$$

Now for the metric (2.2.12), the only surviving component of Φ_{ab}, is obtained as

$$\Phi_{11} = \frac{1/a^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^4}\left(\left(2+\left(\frac{t+r}{a}\right)^2\right)e^{\frac{1}{1+\left(\frac{t+r}{a}\right)^2}}-\frac{1}{2}\left(1+2\left(\frac{t+r}{a}\right)^2\right)e^{\frac{-1}{1+\left(\frac{t+r}{a}\right)^2}}\right)$$
$$-\frac{3}{8a_0{}^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}.$$
(3.3.14)

Eigenvalues of S_{ab} are $\lambda_1 = -e^{-\nu}S_{00}, \lambda_2 = e^{-\lambda}S_{11}, \lambda_3 = \lambda_4 = \frac{1}{\mu^2}S_{22}$.

Now we find the Jordan canonical form of the matrix S^a_b. Here λ_1 has multiplicity 1 so there is one Jordan block corresponding to λ_1 of size 1 and similarly for λ_2. λ_3 has multiplicity 2 and $d_1 = dimN(S^a_b - \lambda_3\delta^a_b) = 2$ that shows there are two Jordan blocks corresponding to λ_3 so each would be of size 1. Hence the Jordan canonical form of S^a_b is

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

So, as described in Chapter 1 and from the Jordan canonical form of S^a_b, the Segré type of the metric (2.2.12) is [1(11)1] if $\lambda_1 \neq \lambda_2$. It can be easily seen from equations (3.3.10) and (3.3.11) that $\lambda_1 = -e^{-\nu}S_{00} = e^{-\lambda}S_{11} = \lambda_2$, hence the possible Segré type of the metric (2.2.12) is [(11)(11)] that corresponds to the Petrov type D from the table 1.4 with only non–zero component Φ_{11}. The other case that may be possible is [(1111)]. It is possible only when $\Phi_{11} = 0$ then the Petrov type would be O but $\Phi_{11} \neq 0$ identically, so the given solution is of Petrov type D and Segre type [(11)(11)].

3.4 Conclusion

In this chapter, we have obtained a class of non–static spherically symmetric solutions of the EMFEs. We have assumed the pressure distribution to be isotropic. The mass density, ρ, is a positive decreasing function of both r and t. The square of the electric field intensity, E^2, is a positive–definite and bounded function of both r and t. The pressure, p, turns out to be a negative function of both r and t and asymptotically approaches to zero. Here ρ, p and E^2 are all symmetric with respect to t and r.

The solution obtained may be thought to represent a moving compact object with negative pressure. The causality condition is not satisfied by the solution, as is expected for an object with negative pressure. All other physical conditions are shown to be satisfied.

There are numerous static solutions of the EMFEs to model objects with negative pressure. However, there is hardly any literature for non–static case. Keeping in view that the configuration of such objects may not be static [59, 60], we have made a successful attempt to find a class of non–static solutions of the EMFEs representing objects with negative pressure.

Chapter 4

Some Further Classes of Non-Static Exact Solutions of the EMFEs

The main purpose of this thesis is to find some new non–static solutions of the EMFEs. In previous two chapters non–static solutions are obtained for a specific choice of the third metric component μ. In this chapter several choices for μ are taken with combination of common choices for the first/second metric components.

4.1 Case $\mu = r$

For a spherically symmetric space–time the very first choice for μ is being equal to the radial parameter r. So, in this section the EMFEs are solved for three different cases with $\mu = r$: first $\nu = \nu(t, r)$ and $\lambda = \lambda(t, r)$, second $\nu = 0$ and $\lambda = \lambda(t, r)$, third $\nu = \nu(t, r)$ and $\lambda = 0$.

For $\mu = r$, equation (2.1.8) gives

$$\dot{\lambda} = 0 \Rightarrow \lambda = f(r). \tag{4.1.1}$$

Putting these values of λ and μ, equations (2.1.5)–(2.1.10) become

$$e^{-f(r)}(\frac{f'}{r}-\frac{1}{r^2})+\frac{1}{r^2} = \rho+\frac{1}{2}e^{-(\nu+f(r)}E^2, \quad (4.1.2)$$

$$e^{-f(r)}(\frac{\nu'}{r}+\frac{1}{r^2})-\frac{1}{r^2} = p_r-\frac{1}{2}e^{-(\nu+f(r))}E^2, \quad (4.1.3)$$

$$\frac{1}{4}e^{-f(r)}(-\frac{2f'}{r}+\frac{2\nu'}{r}-\nu' f'+2\nu''+\nu'^2) = p_t+\frac{1}{2}e^{-(\nu+f(r))}E^2, \quad (4.1.4)$$

$$j_0 = e^{-f(r)}(\frac{2E}{r}+E'), \qquad j_1 = e^{-\nu}\dot{E}. \quad (4.1.5)$$

Taking simplifying assumptions that the pressure distribution is isotropic, that is, $p_r = p_t = p$ and the equation of state is given as $\rho + p = 0$, lead to

$$e^{-f(r)}(\frac{\nu'+f'}{r}) = 0, \quad (4.1.6)$$

$$\Rightarrow \nu'+f' = 0, \quad (4.1.7)$$

$$\Rightarrow \nu(t,r) = -f(r)+g(t), \quad (4.1.8)$$

where $g(t)$ is the constant of integration. Using this value of ν in above equations (4.1.2)–(4.1.5) the expressions of ρ, p, E^2, j_0, and j_1 are obtained as

$$\rho = -e^{-f(r)}(\frac{f'}{2r}+\frac{f''}{4}+\frac{1}{2r^2})+\frac{1}{2r^2}, \quad (4.1.9)$$

$$p = e^{-f(r)}(\frac{f'}{2r}+\frac{f''}{4}+\frac{1}{2r^2})-\frac{1}{2r^2}, \quad (4.1.10)$$

$$E^2 = e^{-f(r)+g(t)}(\frac{f''}{2}-\frac{f'}{r}-\frac{1}{r^2})+\frac{1}{r^2}, \quad (4.1.11)$$

$$j_0 = e^{-f(r)}(\frac{2E}{r}+E'), \qquad j_1 = e^{f(r)-g(t)}\dot{E}. \quad (4.1.12)$$

It can be easily seen that $\rho = \rho(r), p = p(r)$ only and E^2, j_0, and j_1 can be functions of both t and r only if $g(t) \neq 0$. The corresponding metric is given as

$$ds^2 = -e^{-f(r)+g(t)}dt^2 + e^{f(r)}dr^2 + r^2 d\Omega^2. \quad (4.1.13)$$

The first curvature invariant, $\mathcal{R}_1 = R$, is obtained as

$$\mathcal{R}_1 = R = e^{-f(r)}\left(\frac{4f''}{r}+f''-f'^2-\frac{2}{r^2}\right)+\frac{2}{r^2}. \quad (4.1.14)$$

Further, if the first component of the metric is unity, then $\nu = 0$. Equation (2.1.8) gives

$$\dot{\lambda} = 0 \Rightarrow \lambda = f(r). \tag{4.1.15}$$

Then the EMFEs (2.1.5)–(2.1.10) are obtained as

$$\begin{aligned}
\rho + \frac{1}{2}e^{-f(r)}E^2 &= e^{-f(r)}\left(\frac{f'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, & (4.1.16)\\
p_r - \frac{1}{2}e^{-f(r)}E^2 &= e^{-f(r)}\left(\frac{1}{r^2}\right) - \frac{1}{r^2}, & (4.1.17)\\
p_t + \frac{1}{2}e^{-f(r)}E^2 &= e^{-f(r)}\frac{f'}{2r}, & (4.1.18)\\
j_0 &= e^{-f(r)}\left(\frac{2E}{r} + E'\right), & (4.1.19)\\
j_1 &= \dot{E}. & (4.1.20)
\end{aligned}$$

Taking pressure distribution to be isotropic and adding equations (4.1.17) and (4.1.18), the pressure distribution is obtained as

$$p = e^{-f(r)}\left(\frac{-f'}{4r} + \frac{1}{2r^2}\right) - \frac{1}{2r^2}. \tag{4.1.21}$$

Subtracting equation (4.1.17) from the equation (4.1.18), the expression for the square of the electric field intensity is obtained as

$$E^2 = \frac{e^{f(r)}}{r^2} - \frac{f'}{2r} - \frac{1}{r^2}, \tag{4.1.22}$$

which on using in equation (4.1.16) gives the expression for the mass density, ρ, as

$$\rho = e^{-f(r)}\left(\frac{5f'}{4r} - \frac{1}{2r^2}\right) + \frac{1}{2r^2}. \tag{4.1.23}$$

Adding equations (4.1.21) and (4.1.23) gives

$$\rho + p = e^{-f(r)}\frac{f'}{r}, \tag{4.1.24}$$

and if the equation of state is to be $\rho + p = 0$, then

$$f' = 0 \Rightarrow f(r) = \text{constant} \Rightarrow \lambda = \text{constant (say c)}. \tag{4.1.25}$$

Now the mass density, ρ, and the pressure, p, are given as

$$\rho = -p = \frac{1}{2r^2}(1 - e^{-c}), \tag{4.1.26}$$

the electric field intensity and electric current densities are given as

$$E^2 = \frac{1}{r^2}(e^c - 1), \tag{4.1.27}$$
$$j_0 = \frac{1}{r^2}e^{-c}\sqrt{e^c - 1}, \tag{4.1.28}$$
$$j_1 = 0. \tag{4.1.29}$$

For the mass density and the square of the electric field intensity to be positive

$$e^c > 1. \tag{4.1.30}$$

The metric of the solution is

$$ds^2 = -dt^2 + e^c dr^2 + r^2 d\Omega^2, \tag{4.1.31}$$

which is not the non–static one. Also the first curvature invariant, $\mathcal{R}_1 = R$, is given by

$$\mathcal{R}_1 = R = \frac{2}{r^2}\left(1 - e^{-c}\right), \tag{4.1.32}$$

which is not defined at $r = 0$.

Furthermore, if the second component of the metric is unity instead of the first, then $\lambda = 0$ and the equation (2.1.8) gives no information. The EMFEs (2.1.5)–(2.1.10) are obtained as

$$\rho + \frac{1}{2}e^{-\nu}E^2 = 0, \tag{4.1.33}$$
$$p_r - \frac{1}{2}e^{-\nu}E^2 = \frac{1}{r}\nu', \tag{4.1.34}$$
$$p_t + \frac{1}{2}e^{-\nu}E^2 = \frac{1}{2r}\nu' + \frac{1}{2}\nu'' + \frac{1}{4}\nu'^2, \tag{4.1.35}$$
$$j_0 = \frac{2E}{r} + E', \tag{4.1.36}$$
$$j_1 = e^{-\nu}\dot{E}. \tag{4.1.37}$$

Assuming pressure distribution to be isotropic and adding equations (4.1.34) and (4.1.35), the pressure, p, is obtained as

$$p = \frac{3}{4r}\nu' + \frac{1}{4}\nu'' + \frac{1}{8}\nu'^2. \tag{4.1.38}$$

Subtracting equation (4.1.34) from equation (4.1.35), the expression for the square of the electric field intensity, E^2, is obtained as

$$E^2 = e^{\nu}\left(\frac{1}{2}\nu'' + \frac{1}{4}\nu'^2 - \frac{1}{2r}\nu'\right). \tag{4.1.39}$$

Using equation (4.1.39) in equation (4.1.33), the expression for the mass density, ρ, is obtained as

$$\rho = \frac{1}{4r}\nu' - \frac{1}{4}\nu'' - \frac{1}{8}\nu'^2. \tag{4.1.40}$$

Adding equations (4.1.38) and (4.1.40) gives

$$\rho + p = e^{-\nu}\frac{1}{r^2}. \tag{4.1.41}$$

For equation of state to be $\rho + p = 0$, it is required that

$$\nu' = 0 \Rightarrow \nu = g(t). \tag{4.1.42}$$

Corresponding metric is given as

$$ds^2 = -e^{g(t)}dt^2 + dr^2 + r^2 d\Omega^2, \tag{4.1.43}$$

and the first curvature invariant, $\mathcal{R}_1 = R$, is obtained as

$$\mathcal{R}_1 = R = 0. \tag{4.1.44}$$

So, the solution matches to the flat Minkowski's metric after coordinate transformation and is static.

4.2 Case $\mu = t$

The purpose is to find non–static solutions of the EMFEs. So, parallel to the choice of $\mu = r$, the most simple choice is taking $\mu = t$ for a non–static solution. In this section the EMFEs are solved with the choice for μ being t and three different cases: first $\nu = \nu(t, r)$ and $\lambda = \lambda(t, r)$, second $\nu = 0$ and $\lambda = \lambda(t, r)$, and third $\nu = \nu(t, r)$ and $\lambda = 0$.

For $\mu = t$, equation (2.1.8) gives

$$\nu' = 0 \Rightarrow \nu = g(t). \tag{4.2.1}$$

Using it in equations (2.1.5)–(2.1.10), we get

$$\begin{aligned}
e^{-g(t)}(\frac{\dot{\lambda}}{t} + \frac{1}{t^2}) + \frac{1}{t^2} &= \rho + \frac{1}{2}e^{-(g(t)+\lambda)}E^2, && (4.2.2)\\
e^{-g(t)}(\frac{\dot{g}}{t} - \frac{1}{t^2}) - \frac{1}{t^2} &= p_r - \frac{1}{2}e^{-(g(t)+\lambda)}E^2, && (4.2.3)\\
\frac{1}{4}e^{-\nu}(\frac{2\dot{\nu}}{t} - \frac{2\dot{\lambda}}{t} + \dot{g}\dot{\lambda} - 2\ddot{\lambda} - \dot{\lambda}^2) &= p_t + \frac{1}{2}e^{-(g(t)+\lambda)}E^2, && (4.2.4)\\
j_0 = e^{-\lambda}E', \qquad j_1 &= e^{-g(t)}(\frac{2E}{t} + \dot{E}). && (4.2.5)
\end{aligned}$$

We make the simplifying assumptions that the pressure distribution is isotropic, that is, $p_r = p_t = p$ and the equation of state is $\rho + p = 0$. Using these assumptions

$$\begin{aligned}
\frac{e^{-g(t)}}{t}(\dot{g} + \dot{\lambda}) &= 0, && (4.2.6)\\
\Rightarrow \dot{g} + \dot{\lambda} &= 0, && (4.2.7)\\
\Rightarrow \lambda(t, r) &= -g(t) + f(r), && (4.2.8)
\end{aligned}$$

where $f(r)$ is arbitrary constant of integration. Using this value of λ in the equations (4.2.2)–(4.2.5) the expressions of ρ, p, E^2, j_0, and j_1 are obtained as

$$\begin{aligned}
\rho &= \frac{1}{4}e^{-g(t)}(\frac{2}{t^2} - \frac{4g(\dot{t})}{t} - g(\ddot{t}) - g(\dot{t})^2) + \frac{1}{2t^2}, &(4.2.9)\\
p &= \frac{1}{4}e^{-g(t)}(-\frac{2}{t^2} + \frac{4g(\dot{t})}{t} + g(\ddot{t}) + g(\dot{t})^2) - \frac{1}{2t^2}, &(4.2.10)\\
E^2 &= \frac{1}{2}e^{-g(t)+f(r)}(g(\ddot{t}) - g(\dot{t})^2 + \frac{2}{t^2}) + \frac{1}{t^2}e^{-f(r)}, &(4.2.11)\\
j_0 &= e^{g(t)-f(r)}E', \qquad j_1 = e^{-g(t)}(\frac{2E}{t} + \dot{E}). &(4.2.12)
\end{aligned}$$

It can be easily seen that $\rho = \rho(t)$, $p = p(t)$ only and E^2, j_0, and j_1 can be functions of both t and r if $f(r) \neq 0$ the metric (2.1.1) takes the form

$$ds^2 = -e^{g(t)}dt^2 + e^{-g(t)+f(r)} + t^2 d\Omega^2.$$

The first curvature invariant, $\mathcal{R}_1 = R$, is given as

$$\mathcal{R}_1 = R = -e^{-g}\left(\frac{4\dot{g}}{t} - \ddot{g} + \frac{2}{t^2}\right) + \frac{2}{t^2}. \qquad (4.2.13)$$

Further, if the first component of the metric is unity, then $\nu = 0$. In this case, equation (2.1.8) gives no information and the EMFEs (2.1.5)–(2.1.10) take the form

$$\begin{aligned}
\rho + \frac{1}{2}e^{-\lambda}E^2 &= \frac{\dot{\lambda}}{t} + \frac{2}{t^2}, &(4.2.14)\\
p_r - \frac{1}{2}e^{-\lambda}E^2 &= \frac{-2}{t^2}, &(4.2.15)\\
p_t + \frac{1}{2}e^{-\lambda}E^2 &= \frac{-2\dot{\lambda}}{2t} - \frac{\ddot{\lambda}}{2} - \frac{\dot{\lambda}^2}{4}, &(4.2.16)\\
j_0 &= e^{-\lambda}E', \qquad j_1 = \frac{2E}{t} + \dot{E}. &(4.2.17)
\end{aligned}$$

Assuming pressure distribution to be isotropic and adding equations (4.2.15) and (4.2.16), the pressure, p, is obtained as

$$p = -\left(\frac{\ddot{\lambda}}{4} + \frac{\dot{\lambda}^2}{8} + \frac{\dot{\lambda}}{4t} + \frac{1}{t^2}\right). \qquad (4.2.18)$$

Subtracting equation (4.2.15) from equation (4.2.16), the expression for the square of the electric field intensity, E^2, is obtained as

$$E^2 = -e^{\lambda}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} + \frac{\dot{\lambda}}{2t} - \frac{2}{t^2}\right), \tag{4.2.19}$$

which on using in equation (4.2.14), gives the expression for the mass density, ρ, to be

$$\rho = \left(\frac{\ddot{\lambda}}{4} + \frac{\dot{\lambda}^2}{8} + \frac{5\dot{\lambda}}{4t} + \frac{1}{t^2}\right). \tag{4.2.20}$$

Adding equations (4.2.18) and (4.2.20), the equation of state is obtained as

$$\rho + p = \frac{\dot{\lambda}}{t}. \tag{4.2.21}$$

If the equation of state is taken to be

$$\rho + p = 0, \tag{4.2.22}$$

then λ must satisfy

$$\dot{\lambda} = 0 \Rightarrow \lambda = f(r), \tag{4.2.23}$$

so that the expressions for ρ, p, E^2, j_0, and that of j_1 are given as

$$\rho = \frac{1}{t^2}, \qquad p = \frac{-1}{t^2}, \tag{4.2.24}$$

$$E^2 = \frac{2e^{f(r)}}{t^2}, \tag{4.2.25}$$

$$j_0 = \frac{f' e^{\frac{-f(r)}{2}}}{\sqrt{2}t}, \tag{4.2.26}$$

$$j_1 = \frac{\sqrt{2}e^{\frac{f(r)}{2}}}{t^2}. \tag{4.2.27}$$

The metric of the solution is obtained as

$$ds^2 = -dt^2 + e^{f(r)}dr^2 + t^2 d\Omega^2, \tag{4.2.28}$$

and the curvature invariant, $\mathcal{R}_1 = R$, is obtained as

$$\mathcal{R}_1 = R = \frac{4}{t^2}, \tag{4.2.29}$$

which is not defined at $t = 0$.

Furthermore, If the assumption on the metric is taken such that the second component of the matric is unity then $\lambda = 0$ and from equation (2.1.8),

$$\nu' = 0 \Rightarrow \nu = g(t). \tag{4.2.30}$$

Now the EMFEs (2.1.5)–(2.1.10) are obtained as

$$\rho + \frac{1}{2}e^{-g(t)}E^2 = e^{-g(t)}\left(\frac{1}{t^2}\right) + \frac{1}{t^2}, \tag{4.2.31}$$

$$p_r - \frac{1}{2}e^{-g(t)}E^2 = e^{-g(t)}\left(\frac{\dot{g}}{t} - \frac{1}{t^2}\right) - \frac{1}{t^2}, \tag{4.2.32}$$

$$p_t + \frac{1}{2}e^{-g(t)}E^2 = e^{-g(t)}\left(\frac{\dot{g}}{2t}\right), \tag{4.2.33}$$

$$j_0 = E', \qquad j_1 = e^{-g(t)}\left(\frac{2E}{t} + \dot{E}\right). \tag{4.2.34}$$

Assume the pressure distribution to be isotropic, that is, $p_r = p_t = p$. Then addition of equations (4.2.32) and (4.2.33), leads to

$$p = e^{-g(t)}\left(\frac{3\dot{g}}{4t} - \frac{1}{2t^2}\right) - \frac{1}{2t^2}. \tag{4.2.35}$$

Now subtracting equation (4.2.32) from equation (4.2.33), the square of the electric field intensity is obtained as

$$E^2 = \frac{-\dot{g(t)}}{2t} + \frac{1}{t^2} + e^{g(t)}\left(\frac{1}{t^2}\right). \tag{4.2.36}$$

Using equation (4.2.36) in equation (4.2.31), the mass density, ρ, is obtained as

$$\rho = e^{-g(t)}\left(\frac{\dot{g}}{4t} + \frac{1}{2t^2}\right) + \frac{1}{2t^2}. \tag{4.2.37}$$

Comparing equations (4.2.35) and (4.2.37), the equation of state comes out to be

$$\rho + p = e^{-g(t)}\frac{\dot{g}}{t}. \tag{4.2.38}$$

If the equation of state is considered to be

$$\rho + p = 0, \tag{4.2.39}$$

then it is required that,

$$\dot{g} = 0 \Rightarrow g(t) = \text{constant}. \tag{4.2.40}$$

It is possible only when

$$\nu(t, r) = constant. \tag{4.2.41}$$

The expressions of ρ, p, E^2, j_0 and j_1 are obtained as

$$\rho = -p = \frac{1}{t^2}(1 + e^{-c}), \tag{4.2.42}$$

$$E^2 = \frac{1}{t^2}(1 + e^{c}), \tag{4.2.43}$$

$$j_0 = 0 = j_1. \tag{4.2.44}$$

Corresponding metric is given as

$$ds^2 = -dt^2 + dr^2 + t^2 d\Omega^2, \tag{4.2.45}$$

and the first curvature invariant, $\mathcal{R}_1 = R$, is obtained as

$$\mathcal{R}_1 = R = \frac{2}{t^2}(1 + e^{-c}), \tag{4.2.46}$$

which is not defined for $t = 0$.

From all the discussion in this section it can be easily seen that the solutions obtained with either $\lambda = 0$ or $\nu = 0$ are singular. All the physical quantities contain singularity at $t = 0$. Further, the curvature invariants are also undefined there, leading to say that the singularity is essential and can not be removed by any means. While the solution obtained with no assumptions on λ and ν seems to be meaningful in the sense that all the physical quantities like ρ, p, and E^2 are continuous functions. But they too are undefined at $t = 0$. Curvature invariant is also undefined there but it can be thought that this singularity may be removed using some suitable value for ν. Also this solution is non–static only when $f(r) \neq 0$ otherwise through a suitable coordinate transformation it can be converted to the static one.

4.3 When $\mu = t + r$

For obtaining a non–static solution it is desirable that all the metric components as well as all the physical parameters may be functions (or implicit functions) of both t and r. For this purpose parallel to the choices for μ in above two sections, the most simple choice seems to be $\mu = t + r$. In this section the EMFEs are solved for $\mu = t + r$ with three different cases: first $\nu = \nu(t, r)$ and $\lambda = \lambda(t, r)$, second $\nu = 0$ and $\lambda = \lambda(t, r)$, and third $\nu = \nu(t, r)$ and $\lambda = 0$.

For the value of μ to be $t + r$, equation (2.1.8) takes the form

$$\nu' = -\dot{\lambda}. \tag{4.3.1}$$

If $\lambda(t, r) = g'(t, r)$ for some function $g(t, r)$, then

$$\nu = -\dot{g}(t, r) + f(t), \tag{4.3.2}$$

where $f(t)$ is the constant of integration. Now the FEs (2.1.5)–(2.1.10) take the form

$$\rho + \frac{1}{2}e^{\dot{g}-g'-f}E^2 = e^{-g'}\left(\frac{g''}{t+r} - \frac{1}{(t+r)^2}\right) + e^{\dot{g}-f}\left(\frac{\dot{g}'}{t+r} + \frac{1}{(t+r)^2}\right) + \frac{1}{(t+r)^2}, \tag{4.3.3}$$

$$p_r - \frac{1}{2}e^{\dot{g}-g'-f}E^2 = e^{-g'}\left(-\frac{\dot{g}'}{t+r} + \frac{1}{(t+r)^2}\right) + e^{\dot{g}-f}\left(-\frac{\ddot{g}}{t+r} + \frac{\dot{f}}{t+r} - \frac{1}{(t+r)^2}\right) - \frac{1}{(t+r)^2}, \tag{4.3.4}$$

$$p_t + \frac{1}{2}e^{\dot{g}-g'-f}E^2 = \frac{1}{4}e^{-g'}\left(-\frac{2g''}{t+r} - \frac{2\dot{g}'}{t+r} - g''\dot{g}' - 2\dot{g}'' + \dot{g}'^2\right) + \frac{1}{4}e^{\dot{g}-f}\left(-\frac{2\ddot{g}}{t+r} - \frac{2\dot{g}'}{t+r} + \frac{2\dot{f}}{t+r} - \ddot{g}\dot{g}' + \dot{g}'\dot{f} - \ddot{g}' - \dot{g}'^2\right), \tag{4.3.5}$$

$$j_0 = e^{-g'}\left(\frac{2E}{t+r} + E'\right), \tag{4.3.6}$$

$$j_1 = e^{\dot{g}-f}\left(\frac{2E}{t+r} + \dot{E}\right). \tag{4.3.7}$$

Under the simplifying assumptions that the pressure distribution is isotropic, that is, $p_r = p_t = p$ and the equation of state is $\rho + p = 0$, above equations give

$$g'' = \dot{g}', \quad \dot{g}' = \ddot{g}, \quad \text{and} \quad \dot{f} = 0, \tag{4.3.8}$$

$$\Rightarrow \dot{g} = g', \tag{4.3.9}$$

$$\Rightarrow g(t,r) = g(t+r), \quad \text{and} \quad \nu = -\lambda = -g' = -h(t+r). \tag{4.3.10}$$

Using this value of ν and λ the expressions of the mass density, ρ, the pressure, p, the electric field intensity, E, and the non–zero components of the electric current density are obtained as

$$\rho = e^{-h}(\frac{h'}{t+r} + \frac{h''}{4} - \frac{1}{2(t+r)^2}) + e^{h}(\frac{\dot{h}}{t+r} + \frac{\ddot{h}}{4} + \frac{1}{2(t+r)^2}) + \frac{1}{2(t+r)^2}, \quad (4.3.11)$$

$$p = e^{-h}(-\frac{h'}{t+r} - \frac{h''}{4} + \frac{1}{2(t+r)^2}) + e^{h}(-\frac{\dot{h}}{t+r} - \frac{\ddot{h}}{4} - \frac{1}{2(t+r)^2}) - \frac{1}{2(t+r)^2}, \quad (4.3.12)$$

$$E^2 = -\frac{e^{-h}}{2}(h'' + \frac{1}{2(t+r)^2}) - \frac{e^{h}}{2}(\ddot{h} - \frac{1}{2(t+r)^2}) + \frac{1}{(t+r)^2}, \quad (4.3.13)$$

$$j_0 = e^{-h}(\frac{2E}{t+r} + E'), \quad (4.3.14)$$

$$j_1 = e^{h}(\frac{2E}{t+r} + \dot{E}), \quad (4.3.15)$$

and the corresponding metric is

$$ds^2 = -e^{-h(t+r)}dt^2 + e^{h(t+r)}dr^2 + (t+r)^2 d\Omega^2. \quad (4.3.16)$$

The first curvature invariant, $\mathcal{R}_1 = R$, is obtained as

$$\mathcal{R}_1 = R = \frac{2(2e^h + e^{-h})h'}{t+r} + \frac{2(e^h - e^{-h})}{(t+r)^2} + e^h h'' + (e^h - e^{-h})h'^2 + \frac{2}{(t+r)^2}. \quad (4.3.17)$$

Further consider the first component of the metric to be unity, that is, $\nu = 0$, then the equation (2.1.8) produces

$$\dot{\lambda} = 0 \Rightarrow \lambda = f(r), \quad (4.3.18)$$

which on subtituting in the EMFEs (2.1.5)–(2.1.10), gives

$$\rho + \frac{1}{2}e^{-f(r)}E^2 = e^{-f(r)}\left(\frac{f'}{t+r} - \frac{1}{(t+r)^2}\right) + \frac{2}{(t+r)^2}, \tag{4.3.19}$$

$$p_r - \frac{1}{2}e^{-f(r)}E^2 = e^{-f(r)}\frac{1}{(t+r)^2} - \frac{2}{(t+r)^2}, \tag{4.3.20}$$

$$p_t + \frac{1}{2}e^{-f(r)}E^2 = e^{-f(r)}\frac{-f'}{2(t+r)}, \tag{4.3.21}$$

$$j_0 = e^{-f(r)}\left(\frac{2E}{t+r} + E'\right), \tag{4.3.22}$$

$$j_1 = \frac{2E}{t+r} + \dot{E}. \tag{4.3.23}$$

Assuming pressure distribution to be isotropic and adding equations (4.3.20) and (4.3.21), the expression for p is obtained as

$$p = e^{-f(r)}\left(\frac{-f'}{4(t+r)} + \frac{1}{2(t+r)^2}\right) - \frac{1}{(t+r)^2}, \tag{4.3.24}$$

subtracting equation (4.3.20) from equation (4.3.21), the expression for the squared electric field intensity is obtained as

$$E^2 = \frac{2e^{f(r)}}{(t+r)^2} - \frac{f'}{2(t+r)} - \frac{1}{(t+r)^2}, \tag{4.3.25}$$

Using equation (4.3.25) in equation (4.3.19), the expression for the mass density, ρ, is obtained as

$$\rho = e^{-f(r)}\left(\frac{5f'}{4(t+r)} - \frac{1}{2(t+r)^2}\right) + \frac{1}{(t+r)^2}. \tag{4.3.26}$$

Adding expressions of p and ρ gives

$$p + \rho = \frac{e^{-f(r)}f'}{t+r}, \tag{4.3.27}$$

for the equation of state to be $p + \rho = 0$

$$f' = 0 \Rightarrow f(r) = \text{constant (say c)}. \tag{4.3.28}$$

Now the expressions for ρ, p, E^2, j_0, and j_1 are obtained as

$$\rho = -p = \frac{2 - e^{-c}}{2(t+r)^2}, \quad (4.3.29)$$

$$E^2 = \frac{2e^c - 1}{(t+r)^2}, \quad (4.3.30)$$

$$j_0 = e^{-c}\frac{\sqrt{2e^c - 1}}{(t+r)^2}, \quad (4.3.31)$$

$$j_1 = \frac{\sqrt{2e^c - 1}}{(t+r)^2}. \quad (4.3.32)$$

Notice that both , the mass density, ρ, and the square of the electric field intensity, E^2, are positive for all

$$e^c > \frac{1}{2}. \quad (4.3.33)$$

The corresponding metric is given as

$$ds^2 = -dt^2 + e^c dr^2 + (t+r)^2 d\Omega^2, \quad (4.3.34)$$

and the first curvature invariant, $\mathcal{R}_1 = R$, is obtained as

$$\mathcal{R}_1 = R = \frac{2}{(t+r)^2}(1 - e^{-c}). \quad (4.3.35)$$

Furthermore, if the second component of the metric is taken to be unity instead of the first one, then $\lambda = 0$ and equation (2.1.8) produces

$$\nu' = 0 \Rightarrow \nu = g(t). \quad (4.3.36)$$

Substituting value of ν in EMFEs (2.1.5)–(2.1.10) lead s to

$$\rho + \frac{1}{2}e^{-g(t)}E^2 = \frac{e^{-g(t)}}{(t+r)^2}, \quad (4.3.37)$$

$$p_r - \frac{1}{2}E^{-g(t)}E^2 = e^{-g(t)}\left(\frac{\dot{g}}{t+r} - \frac{1}{(t+r)^2}\right), \quad (4.3.38)$$

$$p_t + \frac{1}{2}e^{-g(t)}E^2 = \frac{e^{-g(t)}\dot{g}}{2(t+r)}, \quad (4.3.39)$$

$$j_0 = \frac{2E}{t+r} + E', \quad (4.3.40)$$

$$j_1 = e^{-g(t)}\left(\frac{2E}{t+r} + \dot{E}\right). \quad (4.3.41)$$

Assuming pressure distribution to be isotropic and adding equations (4.3.38) and (4.3.39) leads to

$$p = e^{-g(t)}\left(\frac{3\dot{g}}{4(t+r)} - \frac{1}{2(t+r)^2}\right), \tag{4.3.42}$$

subtracting the two equations leads to

$$E^2 = \frac{-\dot{g}}{2(t+r)} + \frac{1}{(t+r)^2}. \tag{4.3.43}$$

Using expression of E^2 in equation (4.3.37), the mass density, ρ, is obtained as

$$\rho = e^{-g(t)}\left(\frac{\dot{g}}{4(t+r)} + \frac{1}{2(t+r)^2}\right). \tag{4.3.44}$$

Adding equations (4.3.42) and (4.3.44) leads to

$$\rho + p = \frac{e^{-g(t)}\dot{g}}{t+r}, \tag{4.3.45}$$

for equation of state to be $\rho + p = 0$, we must have

$$\dot{g} = 0 \Rightarrow g(t) = \text{constant (say c)}, \tag{4.3.46}$$

which implies that $\nu = c$ and the expressions for ρ, p, E^2, j_0, and j_1 are given as

$$\rho = -p = \frac{e^{-c}}{2(t+r)^2}, \tag{4.3.47}$$

$$E^2 = \frac{1}{(t+r)^2}, \tag{4.3.48}$$

$$j_0 = \frac{1}{(t+r)^2}, \tag{4.3.49}$$

$$j_1 = \frac{e^{-c}}{(t+r)^2}. \tag{4.3.50}$$

Corresponding metric and the first curvature invariant are given as

$$ds^2 = -e^c dt^2 + dr^2 + (t+r)^2 d\Omega^2 \tag{4.3.51}$$

and

$$\mathcal{R}_1 = R = \frac{2}{(t+r)^2}(1+e^{-c}), \tag{4.3.52}$$

respectively. All of the physical variables ρ, p, E, j_0, j_1, and the metric itself seem to be interesting because of being implicit functions of t and r, but there is a problem in the solution that all the physical parameters have singularity at $t + r = 0$. One condition for a solution to be physically meaningful is that, the mass density, ρ, is defined, bounded, decreasing and positive function everywhere on and inside the boundary. At $t = -r$, $\rho(t, r)$ is undefined also the curvature invariant too is undefined and the singularity is essential for all of the cases discussed for $\mu = t + r$.

4.4 When $\mu = a_0(1 + (\frac{t+r}{a})^2)$

This is the same choice that is taken in Chapter 2 and Chapter 3. In previous two chapters the choices for first two metric components are just $\nu = \nu(t, r)$ and $\lambda = \lambda(t, r)$. Here in this section two possible cases for the first two metric components are discussed: first $\nu = 0$ and $\lambda = \lambda(t, r)$ and second $\nu = \nu(t, r)$ and $\lambda = 0$.

Consider that the first component of the metric is unity , that is, $\nu = 0$. Then equation (2.1.8) gives

$$\dot{\lambda} = \frac{2}{t+r} \Rightarrow \lambda = \ln\left(\frac{t+r}{a}\right)^2. \tag{4.4.1}$$

Substituting values of μ and λ in EMFEs (2.1.5)–(2.1.7) gives

$$\rho + \frac{1}{2\left(\frac{t+r}{a}\right)^2}E^2 = \frac{\frac{8}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{1}{a_0^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{4.4.2}$$

$$p_r - \frac{1}{2\left(\frac{t+r}{a}\right)^2}E^2 = \frac{\frac{-8}{a^2}\left(\frac{t+r}{a}\right)^2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} - \frac{1}{a_0^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}, \tag{4.4.3}$$

$$p_t + \frac{1}{2\left(\frac{t+r}{a}\right)^2}E^2 = \frac{\frac{-4}{a^2}}{1+\left(\frac{t+r}{a}\right)^2}. \tag{4.4.4}$$

Adding equations (4.4.2) and (4.4.3), the equation of state comes out to be

$$\rho + p = 0, \tag{4.4.5}$$

also consider pressure distribution to be isotropic then adding equations (4.4.3) and (4.4.4) , the expression for the pressure distribution and the mass density is obtained as

$$p = -\rho = -\left(\frac{2+6\left(\frac{t+r}{a}\right)^2}{a^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{1}{2a_0^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\right). \tag{4.4.6}$$

Using equation (4.4.6) in equation (4.4.2), the expression for the square of electric field intensity is obtained as

$$E^2 = \left(\frac{t+r}{a}\right)^2\left(\frac{4\left(\left(\frac{t+r}{a}\right)^2-1\right)}{a^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2} + \frac{1}{a_0^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\right). \tag{4.4.7}$$

The metric for this solution takes the form

$$ds^2 = -dt^2 + \left(\frac{t+r}{a}\right)^2 dr^2 + a_0^2\left(1+\left(\frac{t+r}{a}\right)^2\right)^2 d\Omega^2. \tag{4.4.8}$$

Since the second component of the metric is zero for $t = -r$, so there is a singularity in the solution at $t = -r$. It is required to check curvature invariants in this case. The first curvature invariant is given as

$$\mathcal{R}_1 = R = \frac{2}{\left(1 + \left(\frac{t+r}{a}\right)^2\right)^2} \left(\frac{1}{a_0^2} + \frac{8}{a^2}\left(1 + \left(\frac{t+r}{a}\right)^2\right) - \frac{4}{a^2}\left(1 - \left(\frac{t+r}{a}\right)^2\right)\right). \tag{4.4.9}$$

Since the curvature invariant is defined at $t = -r$ hence it is a coordinate singularity not the essential singularity.

Now consider that the second component of the metric is unity, that is, $\lambda = 0$. Equation (2.1.8) gives

$$\nu' = \frac{2/a}{(t+r)/a}, \tag{4.4.10}$$

as in the previous case ν can be taken as a function of $t + r$ so that $\nu' = \dot{\nu}$ and that

$$\nu = \ln\left(\frac{t+r}{a}\right)^2. \tag{4.4.11}$$

The FEs (2.1.5)–(2.1.10) take the form

$$\begin{aligned}\rho + \frac{1}{2}\left(\frac{a}{t+r}\right)^2 E^2 = -\frac{4/a^2}{1 + (\frac{t+r}{a})^2} - \frac{\frac{4}{a^2}(\frac{t+r}{a})^2}{(1 + (\frac{t+r}{a})^2)^2} + \frac{4/a^2}{(1 + (\frac{t+r}{a})^2)^2} \\ + \frac{1}{{a_0}^2(1 + (\frac{t+r}{a})^2)^2},\end{aligned} \tag{4.4.12}$$

$$\begin{aligned}p - \frac{1}{2}\left(\frac{a}{t+r}\right)^2 E^2 = \frac{4/a^2}{1 + (\frac{t+r}{a})^2} + \frac{\frac{4}{a^2}(\frac{t+r}{a})^2}{(1 + (\frac{t+r}{a})^2)^2} - \frac{4/a^2}{(1 + (\frac{t+r}{a})^2)^2} \\ - \frac{1}{{a_0}^2(1 + (\frac{t+r}{a})^2)^2},\end{aligned} \tag{4.4.13}$$

$$p_t + \frac{1}{2}\left(\frac{a}{t+r}\right)^2 E^2 = \frac{4/a^2}{1 + (\frac{t+r}{a})^2}, \tag{4.4.14}$$

$$j_0 = E' + \frac{2E}{1 + (\frac{t+r}{a})^2}, \tag{4.4.15}$$

$$j_1 = \left(\frac{a}{t+r}\right)^2\left(\dot{E} + \frac{2E}{1 + (\frac{t+r}{a})^2}\right). \tag{4.4.16}$$

From the first two equations the equation of state come out to be $\rho + p_r = 0$. Taking pressure distribution to be isotropic, that is, $p_r = p_t = p$, the expressions for ρ, p, and E^2 are obtained as

$$\rho = -\frac{4/a^2}{1+(\frac{t+r}{a})^2} + \frac{1}{2{a_0}^2(1+(\frac{t+r}{a})^2)^2}, \tag{4.4.17}$$

$$p = \frac{4/a^2}{1+(\frac{t+r}{a})^2} - \frac{1}{2{a_0}^2(1+(\frac{t+r}{a})^2)^2}, \tag{4.4.18}$$

$$E^2 = \frac{(\frac{t+r}{a})^2(\frac{1}{{a_0}^2} + \frac{4}{a^2} - \frac{4}{a^2}(\frac{t+r}{a})^2)}{(1+(\frac{t+r}{a})^2)^2}, \tag{4.4.19}$$

the mass density, ρ, must be a positive function. This condition requires that

$$a^2 \geq 8{a_0}^2(1+(\frac{t+r}{a})^2), \tag{4.4.20}$$

and the corresponding metric is

$$ds^2 = -(\frac{t+r}{a})^2 dt^2 + dr^2 + {a_0}^2(1+(\frac{t+r}{a})^2)^2 d\Omega^2. \tag{4.4.21}$$

All the functions seem to be physically acceptable and metric is non–static except the singularity at $t+r = 0$ where the first metric component vanishes but this singularity can be checked either it is removable or not. The first curvature invariant, $\mathcal{R}_1 = R$, is given as

$$\mathcal{R}_1 = R = \frac{2}{\left(1+\left(\frac{t+r}{a}\right)^2\right)^2}\left(\frac{1}{a_0^2} - \frac{8}{a^2}\left(1+\left(\frac{t+r}{a}\right)^2\right) + \frac{4}{a^2}\left(1-\left(\frac{t+r}{a}\right)^2\right)\right), \tag{4.4.22}$$

which is defined at $t = -r$, thus the singularity is removeable not essential.

In this chapter, different assumptions on the angular component, μ^2, of the metric are considered. μ is taken to be function of t only, function of r only and the mixed function of both t and r too. If the radial variable r is taken to be the function of t, that is, if $r = r(t)$ is considered, the geometry will be no longer the four dimensional spherically symmetric with which we aim to deal in the thesis.

4.5 Conclusion

In this chapter several non–static solutions are obtained for the EMFEs with different choices on the metric components. For choices of μ being r, t, and $t + r$ the cases $\nu = \nu(t, r)$ and $\lambda = \lambda(t, r)$ give reasonable, non–static solutions and their singularity is removeable for suitable choice of unknown functions involved. But the solutions obtained for remaining two choices, that is, $\nu = 0$, $\lambda = \lambda(t, r)$ and $\nu = \nu(t, r)$, $\lambda = 0$ produce singular solutions (because the singularity is not removeable as can be seen from corresponding curvature invariants). While the solutions obtained for $\mu = a_0\left(1 + \left(\frac{t+r}{a}\right)^2\right)$ with the same choices, that is, $\nu = 0$, $\lambda = \lambda(t, r)$ and $\nu = \nu(t, r)$, $\lambda = 0$ are nonsingular. All the physical conditions as discussed in previous two chapters can be easily verified to be fulfilled by the solutions obtained in this chapter.

Equation of state either considered or obtained is $\rho + p = 0$ that makes the solutions to represent compact objects with negative pressure.

Chapter 5

Some Classes of Exact Static Solutions of the EMFEs with $\mu = a_0\left(1 + \left(\frac{r}{a}\right)^2\right)$

Mostly, the charged compact objects are described with the help of spherically symmetric, static, exact solutions of the EMFEs. This is the reason behind the interest of physicists in finding exact solutions of this set of equations. A number of solutions of the EMFEs have appeared in the literature with different conditions/assumptions.

In this chapter, the aim is to find classes of exact solutions of the EMFEs for charged, static, spherically symmetric space–times both with anisotropic and isotropic pressure distributions. Ansatz are taken on the gravitational potentials, electric field intensity (in anisotropic case), and the equation of state. In the following Section 5.1, the EMFEs for static, spherically symmetric geometry are discussed. In Section 5.2, a new class of solutions of the FEs with anisotropic pressure distribution is presented and in Section 5.3, a class of solutions with isotropic pressure distribution is obtained. In Section 5.4, the physical analysis of the solutions both for anisotropic and isotropic cases is made and in Section 5.5, a brief conclusion is presented.

5.1 The EMFEs for Static Spherical Geometry

A general static, spherically symmetric space–time has the metric of the form given by (2.1.1) and the energy–momentum tensor is taken of the form (2.1.4), where now μ, ν, and λ depend only on r. The corresponding EMFEs are

$$\rho + \frac{1}{2}e^{-(\nu+\lambda)}E^2 = e^{-\lambda}\left(\frac{\lambda'\mu'}{\mu} - 2\frac{\mu''}{\mu} - \frac{\mu'^2}{\mu^2}\right) + \frac{1}{\mu^2}, \tag{5.1.1}$$

$$p_r - \frac{1}{2}e^{-(\nu+\lambda)}E^2 = e^{-\lambda}\left(\frac{\nu'\mu'}{\mu} + \frac{\mu'^2}{\mu^2}\right) - \frac{1}{\mu^2}, \tag{5.1.2}$$

$$p_t + \frac{e^{-(\nu+\lambda)}E^2}{2} = \frac{1}{4}e^{-\lambda}\left(4\frac{\mu''}{\mu} - 2\frac{\lambda'\mu'}{\mu} + 2\frac{\nu'\mu'}{\mu} - \nu'\lambda' + 2\nu'' + \nu'^2\right), \tag{5.1.3}$$

$$j_0 = \frac{1}{\mu^2}e^{-\lambda}(E\mu^2)'. \tag{5.1.4}$$

Here ' , ' represents derivative with respect to r. The trace, T, of the stress energy tensor is

$$T = -\frac{2}{\mu^2} + e^{-\lambda}\left(-2\frac{\lambda'\mu'}{\mu} + 4\frac{\mu''}{\mu} + 2\frac{\mu'^2}{\mu^2} + 2\frac{\nu'\mu'}{\mu} - \frac{1}{2}\nu'\lambda' + \nu'' + \frac{1}{2}\nu'^2\right). \tag{5.1.5}$$

5.2 Anisotropic Solutions

We first consider the pressure distribution to be anisotropic i.e., $p_r \neq p_t$. In this case, there are four equations (5.1.1)–(5.1.4) in eight unknowns. So, in order to solve the system of equations we need four ansatz. Here we take ansatz on the metric coefficients μ and ν, and the square of the electric field intensity as

$$\mu = a_0(1 + r^2/a^2), \tag{5.2.1}$$
$$\nu = \frac{1}{1 + r^2/a^2}, \tag{5.2.2}$$
$$E^2 = \frac{kr^2/a^2}{(1 + r^2/a^2)^2}, \tag{5.2.3}$$

where a, a_0, and k are constants. In order to look for the negative pressure models, we assume the equation of state to be

$$\rho + p_r = 0. \tag{5.2.4}$$

Using equations (5.2.1), (5.2.2), and (5.2.4) in equations (5.1.1) and (5.1.2), we get

$$\lambda = -\frac{1}{1 + r^2/a^2} + \ln r^2/a^2. \tag{5.2.5}$$

Inserting these values of λ, ν, μ and E^2 in equations (5.1.1)–(5.1.4), we get the expressions for density, radial and tangential pressures, and the non–zero component of the current density as

$$\rho = -\frac{4r^2 e^{\frac{1}{1+r^2/a^2}}}{a^4(1 + r^2/a^2)^3} + \frac{1/a_0^2 - k/2}{(1 + r^2/a^2)^2}, \tag{5.2.6}$$
$$p_r = \frac{4r^2 e^{\frac{1}{1+r^2/a^2}}}{a^4(1 + r^2/a^2)^3} - \frac{1/a_0^2 - k/2}{(1 + r^2/a^2)^2}, \tag{5.2.7}$$
$$p_t = \frac{2e^{\frac{1}{1+r^2/a^2}}}{a^2(1 + r^2/a^2)^4} - \frac{k/2}{(1 + r^2/a^2)^2}, \tag{5.2.8}$$
$$j_0 = \frac{\sqrt{k}/a e^{\frac{1}{1+r^2/a^2}}(1 + 3r^2/a^2)}{r^2/a^2(1 + r^2/a^2)^2}. \tag{5.2.9}$$

5.3 Isotropic Solutions

We now consider the case when the pressure distribution is taken to be isotropic i.e., $p_r = p_t = p$. In this case we have four equation (5.1.1)–(5.1.4) in seven unknowns, so we take three ansatz. Here, we take ansatz on the equation of state and the metric coefficients μ and ν as in the case of anisotropic pressures in Section 5.2. These ansatz on inserting in equations (5.1.1)–(5.1.4), lead us to the following expressions for ρ, p, E^2 and j_0:

$$\rho = -\frac{e^{\frac{1}{1+r^2/a^2}}}{a^2(1+r^2/a^2)^4}(1+2r^2/a^2(1+r^2/a^2)) + \frac{1}{2a_0^2(1+r^2/a^2)^2}, \tag{5.3.1}$$

$$p = \frac{e^{\frac{1}{1+r^2/a^2}}}{a^2(1+r^2/a^2)^4}(1+2r^2/a^2(1+r^2/a^2)) - \frac{1}{2a_0^2(1+r^2/a^2)^2}, \tag{5.3.2}$$

$$E^2 = \frac{2r^2 e^{\frac{1}{1+r^2/a^2}}}{a^4(1+r^2/a^2)^4}(1-2r^2/a^2(1+r^2/a^2)) + \frac{r^2/a^2}{a_0^2(1+r^2/a^2)^2}, \tag{5.3.3}$$

$$\begin{aligned} j_0 = {} & \frac{1}{r^2/a^2(1+r^2/a^2)^3}\left(e^{\frac{1}{1+r^2/a^2}}\left(2-4r^2/a^2-4r^4/a^4\right)+a^2a_0^2\right)^{-1/2} \\ & \left(2/a^2 e^{\frac{2}{1+r^2/a^2}}\left\{1+r^2/a^2\right)^2\left(1+r^2/a^2-3r^4/a^4-2r^6/a^6+\right. \\ & \left. 2r^8/a^8\right)+1/a_0^2 e^{\frac{1}{1+r^2/a^2}}\left(1+4r^2/a^2+3r^4/a^4\right)\right\}. \end{aligned} \tag{5.3.4}$$

The metric of our solutions is

$$ds^2 = -e^{\frac{1}{1+r^2/a^2}}dt^2 + r^2/a^2 e^{-\frac{1}{1+r^2/a^2}}dr^2 + {a_0}^2(1+r^2/a^2)^2 d\Omega^2. \tag{5.3.5}$$

5.4 Physical Analysis

In Sections 5.2 and 5.3, we obtained classes of exact solutions of the EMFEs for charged, static, spherically symmetric space–times. In the following, we analyze our solutions to be physically acceptable, for both isotropic and anisotropic cases.

(i) *There should be no physical or geometrical singularity in the solution*. We observe that there is no singularity in both isotropic and anisotropic solutions except at $r = 0$ where one of the metric coefficients tends to zero. It is a coordinate singularity (that is the curvature invariants have finite values at $r = 0$) as is evident from the curvature invariants for the metric (2.1.1) in static case given as follows:

$$\mathcal{R}_1 = R = \frac{1}{a_0^2(1+r^2/a^2)^4}(2r^4/a^4 + 4r^2/a^2 + 2 - (8a_0^2/a^2 + 20a_0^2r^2/a^4 + 8a_0^2r^4/a^6)e^{-\frac{1}{1+r^2/a^2}}), \tag{5.4.1}$$

$$\mathcal{R}_2 = R_{ab}R^{ab} = \frac{1}{a_0^4(1+r^2/a^2)^6}(2r^4/a^4 + 4r^2/a^2 + 2 + 64a_0^4/a^4 + (96a_0^4r^4/a^8 + 64a_0^4r^2/a^6 + 32a_0^4/a^4)e^{-\frac{2}{1+r^2/a^2}} - (16a_0^2r^4/a^6 + 32a_0^2r^2/a^4 + 16a_0^2/a^2)e^{-\frac{1}{1+r^2/a^2}}), \tag{5.4.2}$$

$$\mathcal{R}_3 = R^{cd}_{ab}R^{ab}_{cd} = \frac{1}{a_0^4(1+r^2/a^2)^6}(64r^4/a^4 + 128r^2/a^2 + 64 + 128a_0^4/a^4 + (1152a_0^4r^4/a^8 + 2048a_0^4r^2/a^6 + 1024a_0^4/a^4)e^{-\frac{2}{1+r^2/a^2}} - 512(a_0^2r^4/a^6 + 2a_0^2r^2/a^4 + a_0^2/a^2)e^{-\frac{1}{1+r^2/a^2}}), \tag{5.4.3}$$

$$\mathcal{R}_4 = R^{cd}_{ab}R^{ef}_{cd}R^{ab}_{ef} = \frac{1}{a_0^6(1+r^2/a^2)^9}(-512r^6/a^6 - 1536r^4/a^4 - 1536r^2/a^2 - 512 + 512a_0^6/a^6 + (33280a_0^6r^6/a^{12} + 98304a_0^6r^4/a^{10} + 98304a_0^6r^2/a^8 + 32768a_0^6/a^6)e^{-\frac{3}{1+r^2/a^2}} - 24576(a_0^4r^6/a^{10} + 3a_0^4r^4/a^8 + 3a_0^4r^2/a^6 + a_0^4/a^4)e^{-\frac{2}{1+r^2/a^2}} + (24576a_0^2r^6/a^8 + 18432a_0^2r^2/a^4 + 6144a_0^2/a^2)e^{-\frac{1}{1+r^2/a^2}}). \tag{5.4.4}$$

(*ii*) *On the boundary $r = R$ the solutions must match the exterior Reissner–Nordström metric* given by

$$ds^2 = -(1 - 2M/r + Q^2/r^2)dt^2 + (1 - 2M/r + Q^2/r^2)^{-1}dr^2 + r^2d\Omega^2, \tag{5.4.5}$$

where M and Q denote the mass and charge enclosed by the sphere. At $r = R$, we get

$$e^{\frac{1}{1+R^2/a^2}} = 1 - 2M/R + Q^2/R^2, \tag{5.4.6}$$

$$R^2/a^2e^{-\frac{1}{1+R^2/a^2}} = (1 - 2M/R + Q^2/R^2)^{-1}, \tag{5.4.7}$$

$$a_0(1 + R^2/a^2) = R. \tag{5.4.8}$$

From equations (5.4.6) and (5.4.7), we get

$$R^2 = a^2. \tag{5.4.9}$$

Using equation (5.4.9) in (5.4.8), we obtain

$$a = 2a_0. \tag{5.4.10}$$

(*iii*) *The causality condition is* $0 < dp_r/d\rho < 1$. In our case we have considered negative pressure and $dp_r/d\rho = -1$. We check the stability of our anisotropic solution by the difference of the radial sound velocity

$$V_r^2 = \frac{dp_r}{d\rho} = -1, \tag{5.4.11}$$

and the transverse sound velocity

$$\begin{aligned} V_t^2 = \frac{dp_t}{d\rho} = \frac{1}{(a^2+r^2)} & \left[a^2\left\{e^{\frac{1}{1+r^2/a^2}}\left(4a^4+16\left(a^4+a^2r^2\right)\right)-2k\left(a^2+r^2\right)^3\right\}\right] \\ & \left[e^{\frac{1}{1+r^2/a^2}}\left(-8\left(a^2+r^2\right)^2+8a^2r^2+24r^2\left(a^2+r^2\right)\right)\right. \\ & \left. -4a^2\left(1/a_0^2-k/2\right)\left(a^2+r^2\right)^2\right]^{-1}. \end{aligned} \tag{5.4.12}$$

The region where this difference $V_r^2 - V_t^2$ is positive is called the region of stability [61,62]. Different regions of stability are shown for different values of R, in Figures 5.1–5.3.

(*iv*) *The mass density,* ρ*, and the radial pressure,* p_r*, must be continuous and decreasing from the center to the boundary.*

(*a*) *For anisotropic solutions*, the mass density, ρ, is a positive function of r for all the parameter values satisfying the relation

$$k \leq 2\frac{4a_0^2 - a^2}{a^2a_0^2}, \tag{5.4.13}$$

and

$$k \leq 2/a_0^2. \tag{5.4.14}$$

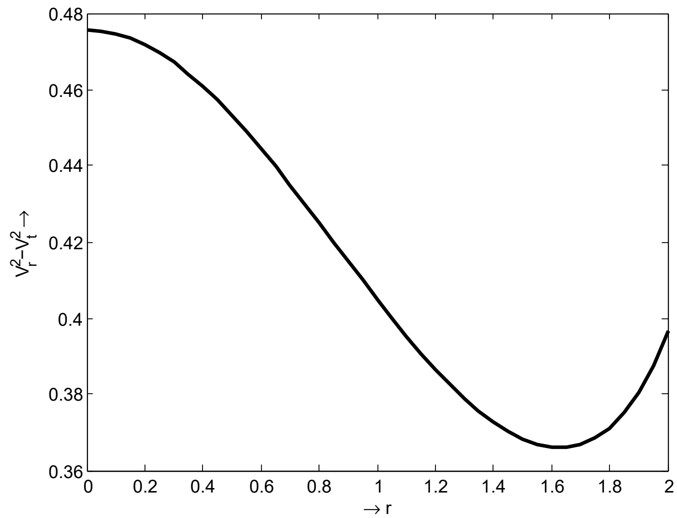

Figure 5.1: Regions of Stability when $R = 2$

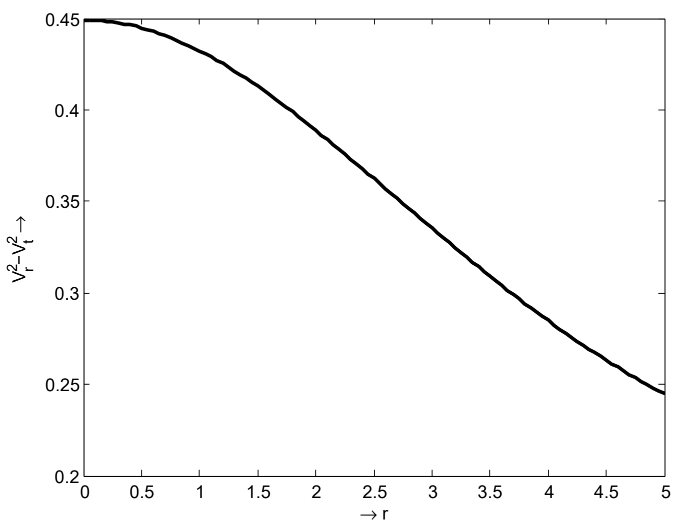

Figure 5.2: Regions of Stability when $R = 5$

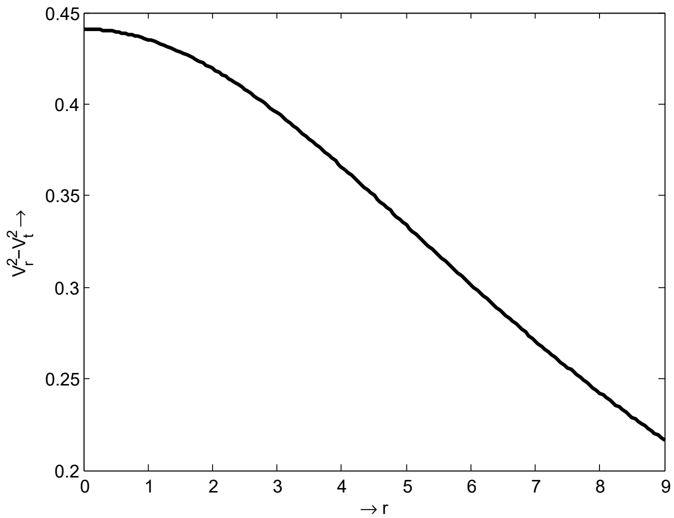

Figure 5.3: Regions of Stability when $R = 9$

Now, to check where the mass density is a decreasing function, consider

$$\frac{d\rho}{dr} = -\frac{8re^{\frac{1}{1+r^2/a^2}}}{a^6(1+r^2/a^2)^5}\left(a^2(1+r^2/a^2)^2 - r^2 - 3r^2(1+r^2/a^2)\right) - \left(\frac{1}{{a_0}^2} - \frac{k}{2}\right)\frac{4r}{a^2(1+r^2/a^2)^3}. \tag{5.4.15}$$

Now $\frac{d\rho}{dr} < 0$ for the parameter values satisfying the relation

$$a > 0 \qquad \text{and} \qquad k < \frac{6.3756{a_0}^2 + 2a^2}{{a_0}^2 a^2}. \tag{5.4.16}$$

However, the radial pressure, p_r, is a continuous, negative, and increasing function that increases to zero at the boundary $r = R$ for all the parameter values satisfying the condition

$$k = \frac{2a^2 - 4\sqrt{e}a_0^2}{a^2 a_0^2}. \tag{5.4.17}$$

(*b*) *For isotropic solutions*, the mass density, ρ, is a positive function of r for all the parameter values satisfying the relation

$$a^2 \geq 5.5a_0^2. \tag{5.4.18}$$

The mass density should be a decreasing function, for this consider

$$\frac{d\rho}{dr} = -\frac{2re^{\frac{1}{1+r^2/a^2}}}{a^4(1+r^2/a^2)^6}\Big(2(1+2r^2/a^2)(1+r^2/a^2)^2 - (1+2r^2/a^2+2r^4/a^4)$$
$$-4(1+2r^2/a^2+2r^4/a^4)(1+r^2/a^2)\Big) - \frac{r}{a^2{a_0}^2(1+r^2/a^2)^3}, \tag{5.4.19}$$

which is negative for all parameter values satisfying the inequality

$$a > 0 \qquad \text{and} \qquad \frac{3.3{a_0}^2 - 0.2588a^2}{a^2{a_0}^2} < 0. \tag{5.4.20}$$

However, the pressure, p, is a continuous, negative, and increasing function that increases to zero at the boundary $r = R$ for the parameter values satisfying

$$a^2 = 2.5\sqrt{e}a_0^2. \tag{5.4.21}$$

(v) *The radial and the tangential pressures must have same values at the origin.*

(a) *For anisotropic solutions*, at the center $r = 0$, $p_r = p_t$ for

$$k = \frac{a^2 + 2ea_0^2}{a^2a_0^2}. \tag{5.4.22}$$

(b) *For isotropic solutions*, $p_r = p_t$ everywhere.

(vi) *The electric field intensity, E, the tangential pressure, p_t, and the non–zero component of the current density, j_0, must be a continuous and bounded functions.*

(a) *For anisotropic solutions*, E^2, p_t, and j_0 are all continuous.

(b) *For isotropic solutions*, E^2 and j_0 are continuous.

(vii) *At the boundary, $r = R$, radial pressure must be zero and $E = Q/R^2$.*

(a) *For anisotropic solutions*, the radial pressure is zero at the boundary for

$$k = \frac{2a^2 - 4\sqrt{e}a_0^2}{a^2a_0^2}, \tag{5.4.23}$$

and the boundary condition on the electric field intensity requires

$$Q^2/R^4 = k/4 \tag{5.4.24}$$

which on using in equation (5.4.6) gives

$$2M/R = 1 - \sqrt{e} + ka^2/4. \tag{5.4.25}$$

(*b*) *For isotropic solutions*, the pressure, p, is zero at the boundary for

$$a^2 = 2.5\sqrt{e}a_0^2, \tag{5.4.26}$$

and the boundary condition on the electric field intensity gives

$$Q^2/R^4 = \frac{2a^2 - 3\sqrt{e}a_0^2}{8a^2a_0^2}, \tag{5.4.27}$$

which on using in equation (5.4.6) gives

$$2M/R = 1 - 11/8\sqrt{e} + a^2/4a_0^2. \tag{5.4.28}$$

For anisotropic solutions, Figures 5.4–5.9 show plots of ρ, p_r, p_t, E^2, j_0, and the measure of anisotropy $\Delta = p_t - p_r$ where the parameter values in all the cases are taken to be $a = 3$, $a_0 = 1.5$, and $k = .15$. Similarly, for isotropic solutions, Figures 5.10–5.13 show plots of ρ, p, E^2, and j_0, where the parameter values in all the cases are taken to be $a = 3$, $a_0 = 1.5$.

Following Tables 5.1 and 5.2 give different anisotropic and isotropic models, respectively.

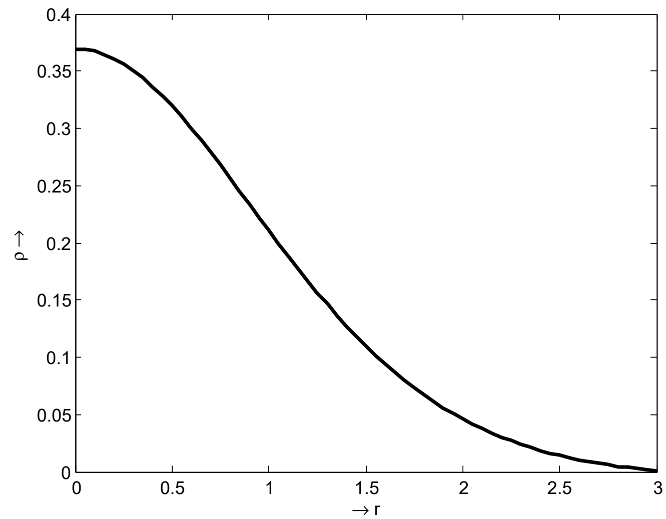

Figure 5.4: The mass density, ρ, for anisotropic case is shown. Where the parameter values are taken to be $a = 3$, $a_0 = 1.5$, and $k = .15$

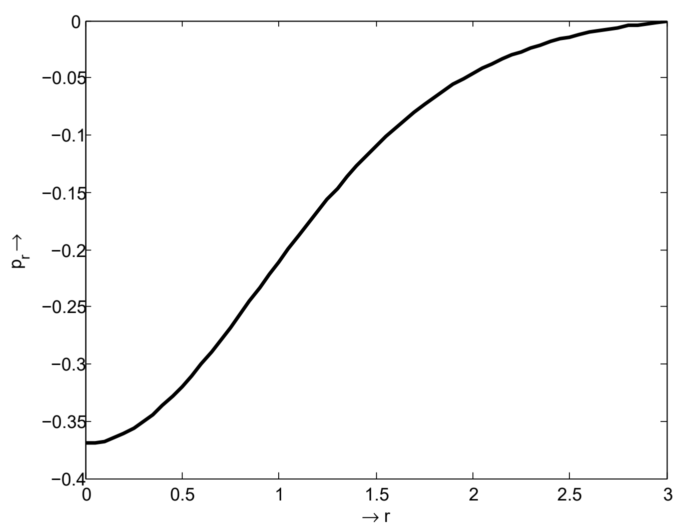

Figure 5.5: The radial pressure, p_r, for anisotropic case is shown. Where the parameter values are taken to be $a = 3$, $a_0 = 1.5$, and $k = .15$

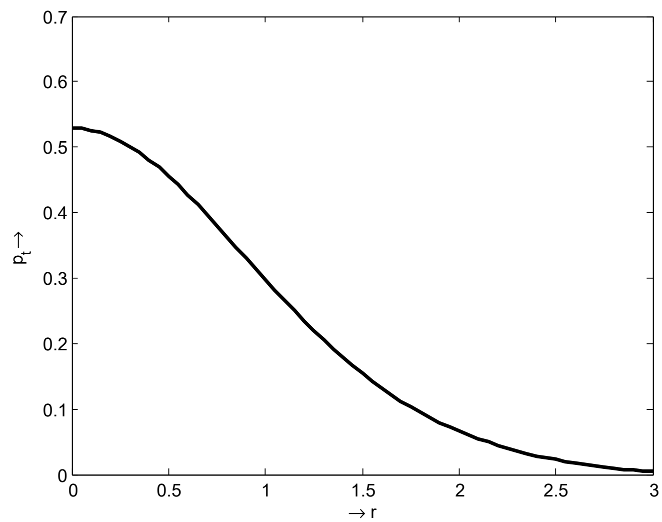

Figure 5.6: The tangential pressure, p_t, for anisotropic case is shown. Where the parameter values are taken to be $a = 3$, $a_0 = 1.5$, and $k = .15$

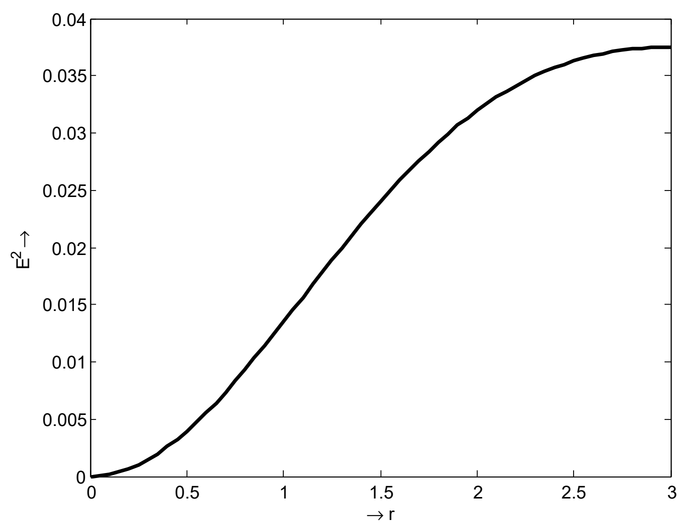

Figure 5.7: The square of electric field intensity, E^2, for anisotropic case is shown. Where the parameter values are taken to be $a = 3$, $a_0 = 1.5$, and $k = .15$

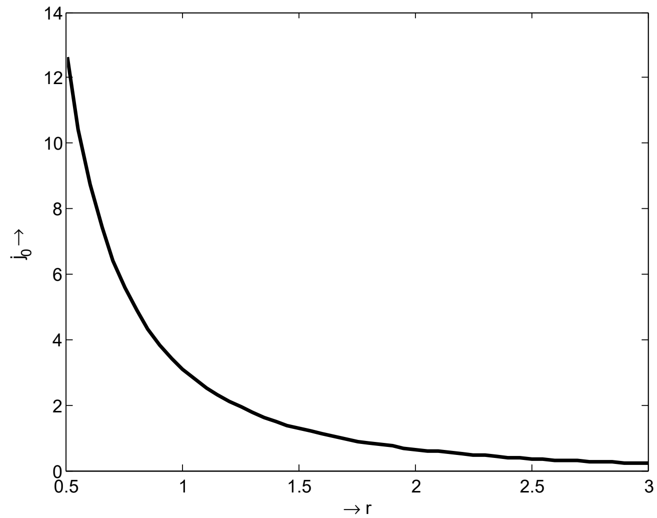

Figure 5.8: The current density, j_0, for anisotropic case is shown. Where the parameter values are taken to be $a = 3$, $a_0 = 1.5$, and $k = .15$

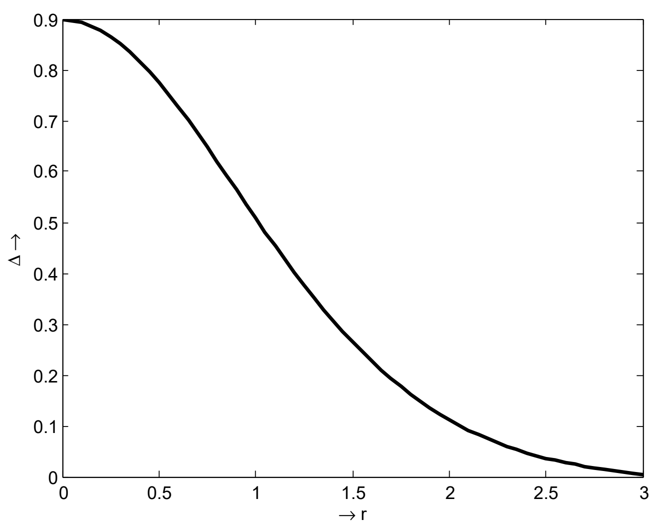

Figure 5.9: The Measure of Anisotropy, Δ, for anisotropic case is shown. Where the parameter values are taken to be $a = 3$, $a_0 = 1.5$, and $k = .15$

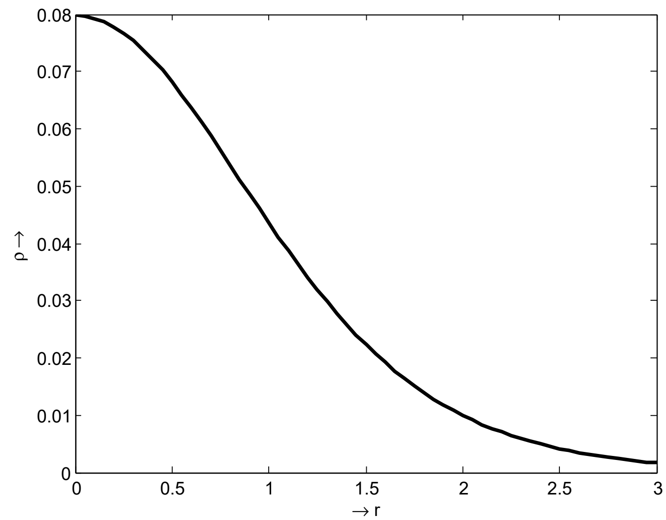

Figure 5.10: The mass density, ρ, for isotropic case is shown. Where the parameter values are taken to be $a = 3$ and $a_0 = 1.5$

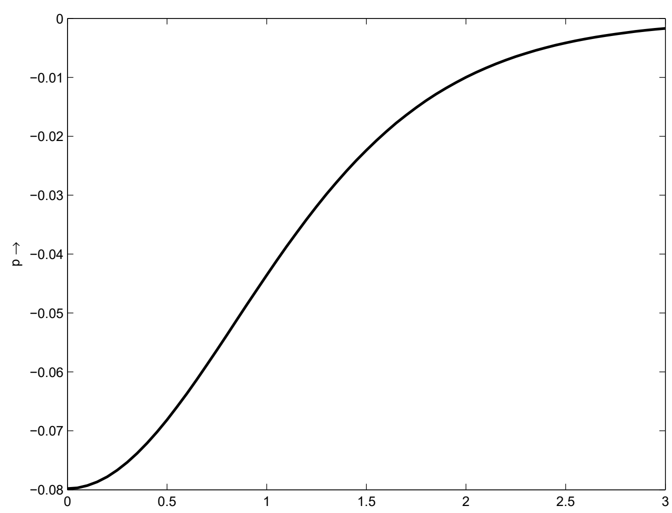

Figure 5.11: The pressure, p, for isotropic case is shown. Where the parameter values are taken to be $a = 3$ and $a_0 = 1.5$

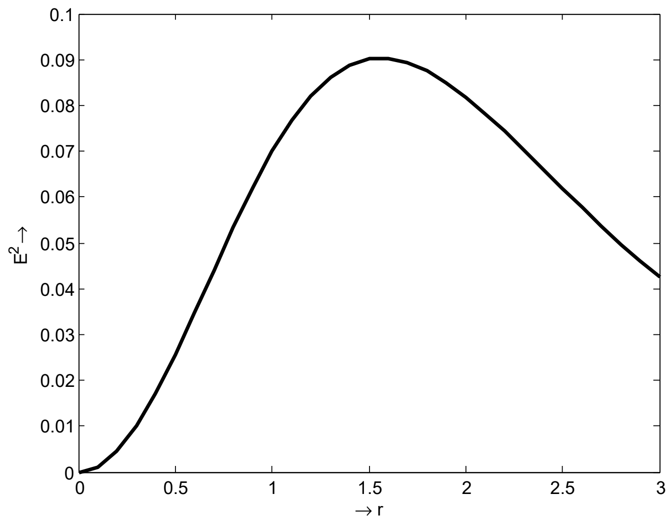

Figure 5.12: The square of the electric field intensity, E^2, for isotropic case is shown. Where the parameter values are taken to be $a = 3$ and $a_0 = 1.5$

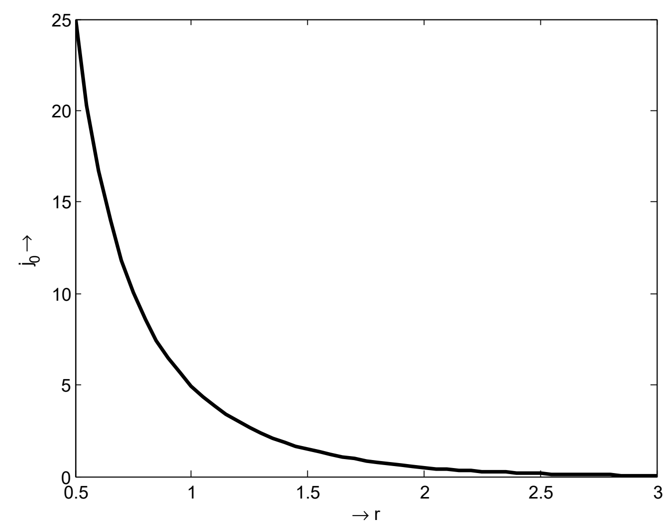

Figure 5.13: The component of current density, j_0, for isotropic case is shown. Where the parameter values are taken to be $a = 3$ and $a_0 = 1.5$

		$a = 2, a_0 = 1, k = 0.35$						$a = 3, a_0 = 1.5, k = 0.15$			
r	r/R	ρ	p_r	p_t	Δ	r	r/R	ρ	p_r	p_t	Δ
0	0	0.8250	−0.8250	1.1841	2.0091	0	0	0.3694	−0.3694	0.5291	0.8985
0.4	0.2	0.6697	−0.6697	0.9562	1.6259	0.6	0.2	0.3002	−0.3002	0.4275	0.7278
0.8	0.4	0.3704	−0.3704	0.5239	0.8942	1.2	0.4	0.1667	−0.1667	0.2349	0.4016
1.2	0.6	0.1475	−0.1475	0.2103	0.3578	1.8	0.6	0.0671	−0.0671	0.0950	0.1620
1.6	0.8	0.0398	−0.0398	0.0621	0.1019	2.4	0.8	0.0187	−0.0187	0.0286	0.0473
2	1	0.0002	−0.0002	0.0078	0.0079	3	1	0.0008	−0.0008	0.0286	0.0049

Table 5.1: Different Anisotropic Models

		$a = 3$				$a = 4$	
r	r/R	p	ρ	r	r/R	p	ρ
0	0	−0.0798	0.0798	0	0	−0.0449	0.0449
0.6	0.2	−0.0636	0.0636	0.8	0.2	−0.0358	0.0358
1.2	0.4	−0.0341	0.0341	1.6	0.4	−0.0192	0.0192
1.8	0.6	−0.0140	0.0140	2.4	0.6	−0.0078	0.0078
2.4	0.8	−0.0050	0.0050	3.2	0.8	−0.0028	0.0028
3	1	−0.0017	0.0017	4	1	−0.0010	0.0010

Table 5.2: Different Isotropic Models

5.5 Conclusion

In this chapter, we have obtained exact solutions of the EMFEs for charged, static, spherically symmetric space–time. We have obtained solutions for both anisotropic and isotropic pressure distributions. The equation of state considered represents a compact object with negative values of the radial pressure (here it is to be noticed that the negative pressure does not mean expanding solution just like in the case of Reissner–Nordstr*ö*m's metric where the radial pressure is non–zero and positive but it is a well known static solution, not a contracting one). So, these solutions represent a compact object with negative pressure. Mass–radius and the charge–radius ratios for the solutions are computed. We have shown that our solutions satisfy all the physical conditions that are required to describe a relativistic compact object except for the causality condition, that is not satisfied in case of negative pressure. In case when the causality condition is not satisfied one can check the stability of an anisotropic solution by calculating difference of the radial sound velocity and the tangential sound velocity. We have checked the stability of our anisotropic solution and it is shown that our solution is stable near the boundary for $R = 2$ and is stable everywhere inside and on the boundary for larger values of R as is shown in Figures 5.1–5.3.

Bibliography

[1] A. Qadir. *Relativity: An Introduction to the Special Theory*. World Scientific, (1989).

[2] H. Reissner. *Ü*ber Die Eigengravitation Des Elektrischen Feldes Nach Der Einsteinschen Theorie. *Annalen Der Physik (in German)*, **50**:106, (1916).
G. Nordstr*ö*m. On the Energy of the Gravitational Field in Einsteins Theory. *Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam*, **26**:1201, (1918).

[3] H. Stephani, *et al. Exact Solutions to Einstein's Field Equations*. Cambridge University Press, (2003).

[4] D. Lovelock. A Spherically Symmetric Solution of the Maxwell–Einstein Equations. *Commun. Math. Phys.*, **5**:257, (1967).

[5] K. D. Krori and T. Chaudhury. Generation of Conformally Flat Solutions of Einstein–Maxwell Scalar Equations. *Phys. Rev. D*, **22**:3138, (1980).

[6] M. Humi and J. Mansour. Interior Solutions to Einstein–Maxwell–Equations in Spherical and Plane Symmetry when $p = n\rho$. *Phys. Rev. D*, **29**:1076, (1984).

[7] R. Tikekar. A Source for the Reissner–Nordstr*ö*m Metric. *J. Astrophys. Astr.*, **5**:273, (1984).

[8] F. Muller–Hoissen and R. Sippel. Spherically Symmetric Solutions of the Non–Minimally Coupled Einstein–Maxwell Equations. *Class. Quantum Grav.*, **5**:1473, (1988).

[9] S. M. Abramyan and Ts. I. Gutsunaev. A Class of Asymptotically Flat Solutions of the Static Einstein–Maxwell Equations. *Phys. Lett. A*, **144**:437, (1990).

[10] A. Melfo and H. Rago. Conformally Flat Solutions to the Einstein–Maxwell Equations. *Astrophys. Space Sci.*, **193**:9, (1992).

[11] R. Roy, A. A. Rangwala, and N. C. Rana. Charged Anisotropic Fluid Spheres in General Relativity. *Indian J. Pure App. Math.*, **27**:1119, (1996).

[12] A. H. Abbasi. General Solutions of the Spherically Symmetric Vacuum Einstein Field Equations. *arXiv:gr–qc/9812081v2*, (1999).

[13] B. S. Guilfoyle. Interior Weyl–type Solutions of the Einstein–Maxwell Field Equations. *Gen. Rel. Gravit.*, **31**:1645, (1999).

[14] H. Hernández and L. A. Nú*ñ*ez. Nonlocal Equation of State in Anisotropic Static Fluid Spheres in General Relativity. *Can. J. Phys.*, **82**:29, (2004).

[15] M. K. Mak and T. Harko. Anisotropic Stars in General Relativity. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, **459**:393, (2003).

[16] M. K. Mak and T. Harko. Quark Stars Admitting a One–Parameter Group of Conformal Motions. *Int. J. Modern Phys. D*, **13**:149, (2004).

[17] M. G*ü*rses and B. Himmeto*ḡ*lu. Multishell Model for Majumdar–Papapetrou Space–times. *Phys. Rev. D*, **72**:024032, (2005).

[18] K. Komathiraj and S. D. Maharaj. Analytical Models for Quark Stars. *Int. J. Modern Phys. D*, **16**:1803, (2007).

[19] J. D. Barrow. Cosmological Limits on Slightly Skew Stresses. *Phys. Rev. D*, **55**:7451, (1997).

[20] S. Thirukkanesh and S. D. Maharaj. Charged Anisotropic Matter with Linear Equation of State. *Class. Quantum Grav.*, **25**:235001, (2008).

[21] S. D. Maharaj and S. Thirukkanesh. Some New Static Charged Spheres. *Nonlinear Analysis: Real World Applications*, **10**:3396, (2009).

[22] I. Arraut, D. Batic, and M. Nowakowski. A Non Commutative Model for a Mini Black Hole. *Class. Quantum Grav.*, **26**:245006, (2009).

[23] M. K. Bashar, *et al.* Spherical Charged Fluid Distributions in General Relativity. *ASA Univ. Rev.*, **3**:141, (2009).

[24] G. Mammadov. Reissner–Nördstrom Metric. *Dep. of Phys., Syracuse Univ.*, (2009).

[25] N. Pant, R. N. Mehta, and M. J. Pant. New Class of Regular and Well Behaved Exact Solutions in General Relativity. *Astrophys. Space Sci.*, **330**:353, (2010).

[26] J. Spruck and Y. Yang. *Charged Cosmological Dust Solutions of the Coupled Einstein and Maxwell Equations*, arXiv:gr–qc/1001.0949v2, (2010).

[27] V. Varela, *et al.* Charged Anisotropic Matter with Linear or Nonlinear Equation of State. *Phys. Rev. D*, **82**:044052, (2010).

[28] T. Feroze and A. A. Siddiqui. Charged Anisotropic Matter with Quadratic Equation of State. *Gen. Rel. Gravit.*, **43**:1025, (2011).

[29] L. Radhakrishna. Some Exact Non–Static Cylindrically Symmetric Electrovac Universes. *Proc. Nat. Inst. Sci. India A*, **29**:588, (1963).

[30] R. Sharan and R. N. Tiwari. Cylindrically Symmetric Electromagnetic Fields of Class One. *Proc. Nat. Inst. Sci. India A*, **31**:465, (1965).

[31] R. N. Tiwari. A Class of Non–Static Cylindrically Coupled Zero–Mass and Electromagnetic Fields–I. *Indian J. Pure App. Math.*, **10**:339, (1979).

[32] J. Carminati and C. B. G. McIntosh. A Non–Static Einstein–Maxwell Solution. *J. Phys. A: Math. Gen.*, **13**:953, (1980).

[33] J. Hajj–Boutros and J. Safeila. Non–static Plane–Symmetric Solutions for Einstein–Maxwell Equations. *Lett. Al Nuovo Cimento*, **44**:388, (1985).

[34] A. Chamorro and K. S. Virbhadra. A Radiating Dyon Solution. *Pramana*, **45**:181, (1995).

[35] S. Gharanfoli and A. H. Abbasi, *General Non–Static Spherically Symmetric Solutions of Einstein Vacuum Field Equations with* Λ, arXiv:gr–qc/9906049v2, (1999).

[36] A. H. Abbasi. Non–Static Spherically Symmetric Solution of Einstein Vacuum Field Equations with Λ. *J. High Energy Phys.*, **1999**:011, (1999).

[37] M. Sharif and T. Iqbal. Non–Static Spherically Symmetric Perfect Fluid Solutions. *Chin. J. Phys.*, **40**:242, (2002).

[38] D. Shee, *et al.* Anisotropic stars with non–static conformal symmetry. *Astrophys. Space Sci.*, **361**:1, (2016).

[39] H. Stephani, *et al. Exact Solutions to Einstein's Field Equations*. *Cambridge University Press*, (2003).

[40] A. Einstein. The Collected Papers of Albert Einstein. *Sitzungsber. Preuss. Akad. Wiss.*, **1**:142, (1997).

[41] A. Cappi. Testing Cosmological Models with Negative Pressure. *Astronomy, Cosmol. Fund. Phys.*, 421, (2003).

[42] B. Saha. Anisotropic Cosmological Models with Perfect Fluid and dark Energy. *Chin. J. Phys.*, **43**:1035, (2005).

[43] B. P. Gerasimovic. Non–Static Hydrogen Chromospheres and the Problem of Be Stars. *Monthly Notices of the Royal Astronomical Society*, **94**:737, (1934).

[44] F. Hoyle and R. A. Lyttleton. Note On Non–Static Stellar Models. *Monthly Notices of the Royal Astronomical Society*, **107**:246, (1947).

[45] H. Stephani. *Relativity: An Introduction to Special and General Relativity*. Cambridge University Press, (2004).

[46] H. Stephani, *et al. Exact Solutions to Einstein's Field Equations*. Cambridge University Press, (2003).

[47] M. Blau, *Lecture Notes on General Relativity*. Albert Einstein Center for Fundamental Physics Bern Germany, (2011).

[48] A. Qadir, *Einstein's General Theory of Relativity*, Unpublished.

[49] G. S. Hall. *Symmetries and Curvature Structure in General Relativity*. World Scientific, (2004).

[50] F. M. Paiva, *et al.* Limits of the Energy–Momentum Tensor in General Relativity. *Class. Quantum Grav.*, **15**:1031, (1998).

[51] J. M. M. Senovilla and R. Vera. Segre Decomposition of Space–times. *Class. Quantum Grav.*, **16**:1185, (1999).

[52] A. Z. Petrov. The Classification of Spaces Defining Gravitational Fields. *Gen. Rel. Gravit.*, **32**:1665, (2000).

[53] C. B. G. McIntosh, J. M. Foyster, and A. W. C. Lun. The Classification of the Ricci and Plebanski Tensors in General Relativity Using Newman–Penrose Formalism. *J. Math. Phys.*, **22**:2620, (1981).

[54] A. Z. Petrov. The Classification of Spaces Defining Gravitational Fields. *Gen. Rel. Gravit.*, **32**:1665, (2000).

[55] T. Feroze, A. Qadir, and M. Ziad. Uniqueness of the McVittie Solution as Plane Symmetric Sourceless Electromagnetic Field Spacetimes. *Modern Physics Letters A*, **23**:11, (2008).

[56] A. Qadir and M. Ziad. The Classification of Spherically Symmetric Space–Times. *IL Nuovo Cimento*, **110 B**:317, (1995).

[57] T. Feroze and A. A. Siddiqui. Charged Anisotropic Matter with Quadratic Equation of State. *Gen. Rel. Gravit.*, **43**:1025, (2011).

[58] T. Feroze. Exact Solutions of the Einstein–Maxwell Equations with Linear Equation of State. *Can. J. Phys.*, **90**:1179, (2012).

[59] B. P. Gerasimovic. Non–Static Hydrogen Chromospheres and the Problem of Be Stars. *Monthly Notices of the Royal Astronomical Society*, **94**:737, (1934).

[60] F. Hoyle and R. A. Lyttleton. Note On Non–Static Stellar Models. *Monthly Notices of the Royal Astronomical Society*, **107**:246, (1947).

[61] R. Bibi, T. Feroze, and Azad A. Siddiqui. Solution of the Einstein–Maxwell Equations with Anisotropic Negative Pressure as a Potential Model of a Dark Energy Star. *Can. J. Phys.*, **94**:758, (2016).

[62] S.K. Maurya, *et al.* New Model for Charged Anisotropic Compact Star. *Astrophys. Space Sci.*, **361**:163, (2016).