On Means and Inequalities for C_0-Semigroup of Operators

by

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Dedicated To

My Parents

For earning an honest living for us, for loving,

supporting and encouraging me to

believe in myself.

My Parents-in-law

For their complete support and

lots of love.

My Husband

A strong and gentle soul who adopted my dreams

and lived his best to

make them true.

My Siblings

For their love...

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Abstract

The strongly continuous semigroups of operators are of great significance due to the fact that these have numerous applications in various areas of science. The qualitative understanding of a complicated deterministic system can be acquired by analyzing the solutions of a related differential equations in terms of these operators. Moreover, the theory of convexity, means, Cauchy means and inequalities has a huge impact in everyday science. Therefore, the pronounced nature of the means and inequalities defined on semigroups of operators, can not be contradicted.

This dissertation is a staunch effort to generalize the theory of means and inequalities to the operator semigroups. A new theory of power means is introduced on a C_0-group of continuous linear operators. A mean value theorem is proved. Moreover, the Cauchy-type power means on a C_0-group of continuous linear operators, are obtained systematically.

A Jessen's type inequality for normalized positive C_0-semigroups is obtained. An adjoint of Jessen's type inequality has also been derived for the corresponding adjoint semigroup, which does not give the analogous results but the behavior is still interesting. Moreover, it is followed by some results regarding exponential convexity of complex structures involving operators from a semigroup. Few applications of Jessen's type inequality are also presented, yielding the Hölder's type and Minkowski's type inequalities for corresponding semigroup. Moreover, a Dresher's type inequality for two-parameter family of means, is also proved. A Jensen's and Hermite-Hadamard's type inequalities are also obtained for a semigroup of positive linear operators and a superquadratic mapping defined on a Banach lattice algebra. The corresponding mean value theorems conduct us to find a new sets of Cauchy's type means.

List of publications

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  About Hermite-Hadamard Inequalities and Cauchy's Type Means For Positive C_0-Semigroups, *(Submitted)*.

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Chapter 1

Introduction

The chapter intents to highlight the development and significance of all the major constituents of this thesis and also takes their mathematical background into its account. It aims to reveal that the idea of the main contents of this thesis is not very unaccustomed but is the active area of research in the present decade.

1.1 Historical Development and Mathematical Structure of Operator-Semigroups

Semigroups of operators on a Banach space offers very general models and tools in the analysis of the phenomena of time evolution and dynamic systems. They have a long history in mathematics and have been studied in a number of areas, such as functional analysis and mathematical physics, probability theory, Reimannian geometry, Lie groups, etc.

The theory of one-parameter semigroups of linear operators on Banach spaces established by the first half of 19th century, earned its core in 1948 with the Hille-Yosida generation theorem. The 1957 edition of *Semigroups and Functional Analysis* by E. Hille and R.S. Phillips was the apogee of this theory. Due to the efforts of many scientists, the theory was at its zenith in 1970s and 80s. It reached a certain level of perfection which is well presented in the monographs by E.B. Davies [33], J.A. Goldstein [41] and A. Pazy [65] and many others.

The basic underlying philosophy in introducing operator-semigroups is to generalize the famous *Cauchy functional equation.*

Theorem 1.1.1. *[36] Let $\phi : [0, \infty) \to \mathbb{R}$ be such that*

(i) $\phi(s+t) = \phi(s)\phi(t)$ *for all* $s, t \geq 0$*,*

(ii) $\phi(0) = 1$*,*

(iii) ϕ *is continuous on* $[0, \infty)$ *(on the right at 0).*

Then ϕ *has the form;* $\phi(t) = e^{at}$*, for some constant* $a \in \mathbb{R}$*.*

□

The first two conditions for strongly continuous semigroup of operators are the reminiscent of the basic properties (i) and (ii) of the exponential functions, while the continuity has been established as an analogue of condition (iii) [62]. Therefore, a strongly continuous one-parameter semigroup (C_0-semigroup) can be somehow regarded as an operator-analogue of the exponential function.

Just as exponential functions provide solutions of scalar linear constant coefficient ordinary differential equations, strongly continuous semigroups provide solutions of linear constant coefficient ordinary differential equations in Banach spaces. Such differential equations in Banach spaces arise from e.g. delay differential equations and partial differential equations.

For a Banach space X, such initial value problem for an X-valued function u and an operator A with $D(A) \subset X$, usually referred to as an *Abstract Cauchy Problem* is given by;

$$\frac{du}{dt} = Au \quad (t > 0); \quad u(0) = u_0, \tag{1.1}$$

where the derivative on the left is the strong derivative of u calculated by using norm on X. One may now conjecture that the operators $Z = \{Z(t)\}_{t \geq 0}$ forming a C_0-semigroup have the form;

$$Z(t) := \exp(tA), \quad \text{for some operator A} \tag{1.2}$$

This conjecture leads to many questions which form the basis of semigroup theory of operators. In [62], A.B. Morante and A. C. McBride has presented the answer to these questions in a very elegant and sophisticated way.

A one-parameter semigroup can be seen as the mathematical formulation of autonomous, deterministic motion. It is located in the center of the debate on the representation of Nature by mathematical terms. A large proportion of the present-day science is engaged in the investigation of the motion of systems. "Motion " will designate , here and in what follows all forms of temporary change and therefore a much more general term than just a change of location. According to the approach of G. Nickel [epilogue [36]], the mathematical framework for this study can be summarized as follows;

1. The aim of the study is the movement of the system in time . Time is represented by the additive group of real numbers $\mathbb{R}$ (or additive semigroup $\mathbb{R}_+$).

2. The considered system is characterized by a set Z (the state space) of individual states $z \in Z$, whose temporary change is to be determined. The set of all possible states of the system is thus established from the beginning.

3. The system 's mobility (*motion*) is represented by the temporal change of states or mathematically by a function $\mathbb{R} \ni t \mapsto z(t) \in Z$. It assigns a unique position (or state) $z(t) \in Z$ to each instant $t \in \mathbb{R}$.

   So far , we have described a motion by function $Z(.)$. In this sense , the *motion* is already being deterministic. This perspective amplifies if all possible motions are taken up.

4. For every instant $t_0 \in \mathbb{R}$ and each initial state $z_0 \in Z$, there is a unique motion $z_{t_0,z_0} : \mathbb{R} \to Z$ satisfying $z_{t_0,z_0}(t_0) = z_0$.

   By assumption 4 , the initial state $z_0 \in Z$ at time t_0 can be varied to obtain a uniquely determined state $z_{t_0,z_0}(t_1)$ of the given system at target time t_1. It therefore describes the mapping

$$\Psi_{t_1,t_0} : Z \to Z, \quad \Psi_{t_1,t_0}(z_0) := z_{t_0,z_0}(t_1).$$

   In the next stage , by taking a new initial state $z_1 = \Psi_{t_1,t_0}(z_0)$ with a new initial time t_1 and by using the assumption 4, one may obtain a unique state given by

$\Psi_{t,t_1}(z_1) = \Psi_{t,t_1}(\Psi_{t_1,t_0}(z_0))$ at time t. This state coincides with the one reached by the original motion $z_{t_0,z_0}(.)$ at time t, which also passes through z_1 at time t_1, i.e.

$$\Psi_{t,t_1}(\Psi_{t_1,t_0}(z_0)) = \Psi_{t,t_0}(z_0), \quad \forall\, z_0 \in Z \quad and \quad t, t_1, t_0 \in \mathbb{R}.$$

Therefore, for the family of mappings $\{\Psi_{t,s} : t, s \in \mathbb{R}\}$ it means;

$$\Psi_{t,s} \circ \Psi_{s,r} = \Psi_{t,r}, \quad and \quad \Psi_{t,t} = I, \quad t, s, r \in \mathbb{R}, \tag{1.3}$$

where I is the identity map $I : z \mapsto z$. Such a system with the state space Z, the time space $\mathbb{R}$ (or $\mathbb{R}_+$), and a family of mappings $\{\Psi_{t,s} : t, s \in \mathbb{R}\}$ satisfying (1.3) is usually known as a *deterministic system.*

In many cases , for instance when some physical laws do not change with respect to the time and no external force acts on the system , the following assumption is appropriate.

5. At the time t , the state $z_1 = \Psi_{t,s} z_0$ is depending on the initial condition z_0 at time s and the time difference $\tau = t - s$. Such systems is referred to as *autonomous.* For the family $\{\Psi_{t,s} : t, s \in \mathbb{R}\}$ this is $\Psi_{t,s} = \Psi_{r,u}$ whenever $t - s = r - u$. By taking $T(t) = \Psi_{r,r-t}$, a one-parameter family of $\{T(t) : t \in \mathbb{R}\}$ mappings on a state space is obtained, which satisfies;

$$T(s+t) = T(s)T(t), \quad \forall\, s, t \in \mathbb{R}. \tag{1.4}$$

Since by (1.4) the family of mappings $\{T(t) : t \in \mathbb{R}\}$ is closed under the composition function, so it forms a one-parameter group (or a one-parameter semigroup if $s, t \in \mathbb{R}_+$). It is the mathematical model of autonomous, deterministic motion of a system in time.

1.2 Historical Development of Inequalities

Since last many decades, mathematical inequalities are essential to the study of Mathematics. It plays a vital role in many other fields and their uses are extensive. The database of the American Mathematical Society includes more than 63,000 references of inequalities and their applications. While the concept is a simple one, some of the most famous and significant results in Mathematics are inequalities.

The history of inequalities has been well written by A. M. Fink [39]. Among many others, the book "Inequalities" by Hardy et al. [45] organizes and gives the proper attribution to the source. Others include by Beck Bach and Bellman [28] and the one of Mitrinović [57] . Each of these carefully defines the origin of discussed inequalities. However , as a crowning his mathematical career, D.S. Mitrinoviċ promised to publish and document *all* inequalities. This led to many of designed volumes [31, 59–61], to be complete and to give an idea of development in applications of the inequalities.

Only a few inequalities are from old traditions. Grabiner [42] argues that people were largely not very interested in mathematics beyond the applicable areas in the 18th century. This also applies to the first years of the 19th century. Therefore, not much of abstract mathematics was done; in particular , no one was interested in an "inequality" for his own good. Four drifts of inequalities transpired in the first eight decades of the 19th century.

In the last two decades, a number of inequalities were proved having names affiliated to them and were considered as a part of analyst's knowledge-base. In addition, some documents whose sole purpose was to prove an inequality were published. From the third decade of this century, Hardy and his colleagues began to develop a systematic study inequalities. This effort ended with the publication of [45]. It became respectable to do research and write articles on "inequalities" and there were lots of them [30, 37, 38].

The most important named inequalities are those of Hölder [46] and Minkowski [56]. Hadamard [43] wrote an article on the determinants and their inequalities, which now bears his name. He did not think the result was overly important but he was wrong. It was the basic instrument of Fredholm's theory of integral equations. In [44], Hadamard wrote "I had been attracted by a question on determinants in 1893. When solving it, I had no suspicion of any definite use it might have, only feeling that it deserved interest; then in 1900 appeared Fredholm's theory, for which the result obtained in 1893 happens to be essential. This is the theory I failed to discover. It has been a consolation for my self-esteem to have brought a necessary link to Fredholm's argument".

According to few historical considerations [58,66], Hermite sent a letter to the journal *Mathesis* on November 22, 1881 and an extract from it got published in *Mathesis* **3** (1883, p. 82). It says "*Sur deux limites d'une intégrale définie. Soit $f(x)$ une fonction qui varie toujours dans le même sens de $x = a$, $x = b$. On aura les relations*

$$(b-a)f(\frac{a+b}{2}) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2} \tag{1.5}$$

ou bien

$$(b-a)f(\frac{a+b}{2}) > \int_a^b f(x)dx > (b-a)\frac{f(a)+f(b)}{2}$$

suivant que la courbe $y = f(x)$ tourne sa convexité ou sa concavité vers l'axe des abcisses. En faisant dans ces formules $f(x) = 1/(1+x)$, $a = 0$, $b = x$ il ient

$$x - \frac{x^2}{2+x} < \log(1+x) < x - \frac{x^2}{2(1+x)}."$$

It is beguiling to know that this brief note of Hermite isn't credited anywhere in mathematical literature. His note is neither recorded in any authoritative journal nor collected in Hermite's papers which were published *"sous les auspices de l'Académie des sciences de Paris par* Emile Picard (1905-1917), *membre de l'Institut"*. Hermite's captivating result in (1.5) establishes the necessary and sufficient condition for f to be convex on (a, b) but after more then twenty years of publication of Hermites's note, Jensen (1906) defined convex function (J-convex functions) using;

$$f(\frac{a+b}{2}) \leq \frac{f(a)+f(b)}{2}. \tag{1.6}$$

It is easily seen that (1.5) is an interpolating inequality for (1.6). Perhaps the best known and most commonly used inequality is named after Jensen [47].

$$\sum_{i=1}^{n} f(\alpha_i x_i) \leq \sum_{i=1}^{n} \alpha_i f(x_i), \tag{1.7}$$

where the weights α_i's are positive with $\sum_{i=1}^{n} \alpha_i = 1$ and f is a convex function.

Jensen's type inequalities in their diverse parameters ranging from discrete to continuous case, play a vital role in various branches of modern mathematics [7,8,34]. A simple search in MathSciNet database of American Mathematical Society with the keywords "Jensen's inequality" in the title indicates that there are more than 300 articles devoted closely to this famous inequality. While the number of articles involving the applications of Jensen's inequality, is a lot larger.

1.3 Cauchy's Type Means

In recent years, Cauchy's type means is an active area of research from the field of probability analysis. A significant theory of Cauchy's type means has been developed [12–17], which is both extensive and elegant. Some mean-value theorems of the Cauchy type, which are connected with Jensen's inequality, are given in the papers by Mercer [55] and Pečarić et al. [67]. In [55] there is given the discrete form and in [67] the integral form, and there were also given some applications of these results on power means. While some further generalizations of some of the aforementioned results are given in [68]. The idea has been generalized for many inequalities and some operator versions are also obtained.

For real and continuous functions φ, χ on a closed interval $K := [k_1, k_2]$, such that φ, χ are differentiable in the interior of K and $\chi' \neq 0$, through out the interior of K. A very well know Cauchy mean value theorem guarantees the existence of a number $\zeta \in (k_1, k_2)$, such that

$$\frac{\varphi'(\zeta)}{\chi'(\zeta)} = \frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}.$$

Now, if the function $\frac{\varphi'}{\chi'}$ is invertible, then the number ζ is unique and

$$\zeta := \Big(\frac{\varphi'}{\chi'}\Big)^{-1}\Big(\frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}\Big).$$

The number ζ is called *Cauchy's mean value* of numbers k_1, k_2. It is possible to define such mean for several variables, in terms of divided difference. Which is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{[k_1, k_2, ..., k_n]\varphi}{[k_1, k_2, ..., k_n]\chi}\Big).$$

This mean value was first defined and examined by Leach and Sholander [53]. The integral representation of Cauchy mean is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{\int_{E_{n-1}} \varphi^{n-1}(k.u)du}{\int_{E_{n-1}} \chi^{n-1}(k.u)du}\Big)',$$

where $E_{n-1} := \{(u_1, u_2, ..., u_n) : u_i \geq 0, 1 \leq i \leq n-1, \sum_{i=1}^{n-1} u_i \leq 1\}$, is an $(n-1)$-dimensional simplex, $u = (u_1, u_2, ..., u_n)$, $u_n = 1 - \sum_{i=1}^{n-1} u_i$, $du = du_1 du_2 ... du_n$ and $k.u = \sum_{i=1}^{n} u_i k_i$.

1.4 Development in the Theory of Semigroups and Inequalities

The intention of this section is to highlight recent advances in the blend of theory of semigroups of operators and inequalities. The theory of inequalities is not only limited to the real numbers or the basic real functions. Since past few years the generalization of known inequalities to the operators, has become the topic of active research in the field of applied analysis. While in recent years, there has been considerable interest in the generalization of "functional-inequalities" and "type-inequalities" to the semigroups of operators defined on a Banach space. Due to the mathematical structure and physical applications of operator semigroups, it is significant to find the new expressions and relations among them. A brief account of few important results is given next.

In 1989, J. A. Siddiqi and A. Elkoutri [73] developed a norm-inequality for the infinitesimal generator A of a bounded analytic semigroup in a sector $\{z \in \mathbb{C} : |argz| \leq (\alpha\pi)/2\}$ of bounded linear operators on a Banach space. This inequality related the norms of the images of powers of the infinitesimal generator A.

In 2004, D. Bakry [23] developed some of the most interesting inequalities related to Markov semigroups, namely spectral gap inequalities, Logarithmic Sobolev inequalities and Sobolev inequalities. He showed different aspects of their meanings and applications and then described some tools used to establish them in various situations.

In 2005, Feng-Yu Wang [77] introduced the functional inequalities to describe the spectrum of the generator, the essential and discrete spectrums, high order eigenvalues, the principle eigenvalue, and the spectral gap. The semigroup properties were also described including the uniform intergrability, the compactness, the convergence rate, and the existence of density. The reference measure and the intrinsic metric is discussed such as, the concentration, the isoperimetic inequality, and the transportation cost inequality.

In 2006, S. K. Chua and R. L. Wheeden [32] proved that the 1-dimensional Poincaré inequality holds for all Lipschitz continuous functions f. For $1 < p < \infty$, $-\infty < a < b < \infty$

$$\left\| f - \frac{\int_a^b f(t)dt}{b-a} \right\|_{L^1([a,b])} \leq C(b-a)^{2-\frac{1}{p}} \|f'\|_{L^p([a,b])}$$

and the best constant C (independent of a, b) is $C = \frac{1}{2}(1+p')^{1/p'}$, where $p' > 1 : \frac{1}{p} + \frac{1}{p'} = 1$. In 2008, W. Trebelsa and U. Westphalban [74] proved an abstract Ulyanov type inequality between the (modified) K-functionals with respect to $(X, D_X((A)^\alpha))$ and $(Y, D_Y((A)^\alpha))$, $\alpha > 0$, where A is the infinitesimal generator of $\{T(t)\}$. Where X, Y are the Banach spaces and $\{T(t) : t \geq 0\}$ be a consistent, equibounded semigroup of linear operators on X as well as on Y. It was assumed that $\{T(t)\}$ satisfies a Nikolskii type inequality with respect to X and Y:

$$\|T(2t)f\|_Y \lesssim \phi(t)\|T(t)f\|_X.$$

In 2009, G. A. Anastassiou [10] got motivated from [32] and presented Poincaré-type general L_p inequalities regarding semigroups, cosine and sine operator functions.

In 2012, H. Mustafayev [63] proved the difference inequalities for the groups and semigroups of operators on Banach spaces. For $T = \{T(t)\}_{t\in\mathbb{R}}$ be a $\sigma(X, F)$-continuous group of isometries on a Banach space X with generator A, where $\sigma(X, F)$ is an appropriate local convex topology on X induced by functionals from $F \subset X^*$. Let $\sigma_A(x)$ be the local spectrum of A at $x \in X$ and $r_A(x) := \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$, the local spectral radius of A at x. Then for every $x \in X$ and $\tau \in \mathbb{R}$,

$$\|T(\tau)x - x\| \leq |\tau| r_A(x)\|x\|.$$

Moreover if $0 \leq \tau r_A(x) \leq \frac{\pi}{2}$, then it holds that;

$$\|T(\tau)x - T(-\tau)x\| \leq 2\sin(\tau r_A(x))\|x\|.$$

Asymptotic versions of these results for C_0-semigroup of contractions are also obtained there. If $T = \{T(t)\}_{t\geq 0}$ is a C_0-semigroup of contractions, then for every $x \in X$ and $\tau \geq 0$,

$$\lim_{t\to\infty} \|T(t+\tau)x - T(t)x\| \leq \tau \sup\{|\lambda| : \lambda \in \sigma_A(x) \cap i\mathbb{R}\}\|x\|.$$

This is perhaps a very brief history of development done in last few years but for more recent results one may refer to [11, 24, 78].

1.5 Outline of Thesis

The main aim of this thesis is to continue the effort mentioned in the last section. The theory of inequalities and Cauchy's type means, is generalized to the semigroups of operators defined on a Banach space. The concept of arithmetic mean of C_0-semigroups of operators, already exists in literature and is very active topic of research, as it forms the basis of the *Mean Ergodic Theory.* But as per our knowledge, the concept of power means is generalized to the C_0-semigroups of operators, for the very first time. The approach is also adapted to produce a new family of means and new relations involving the operators from positive semigroups defined on a Banach lattice algebra.

Our plan of campaign is as follows.

In Chapter 2, we give a brief introduction of those topics which will be required later. Such as the basic theory of Banach spaces and bounded linear operators thereon. We also present an introduction to the theory of Banach lattice algebra.

In Chapter 3, we turn our attention to the detail study of the theory of strongly continuous semigroups of linear operators, adjoint semigroups and strongly continuous positive semigroups of operators.

In Chapter 4, a new theory of power means is introduced on a C_0-group of continuous linear operators defined on Banach space. A systematic procedure is produced to prove mean value theorems, which build the basis of the strategy to obtain new Cauchy-type power means on a C_0-group of continuous linear operators.

In Chpater 5, we introduce the notion of normalized semigroups analogous to normalized positive linear functionals to generalize one of the very essential inequality in the field of inequalities. The Jessen's type inequality. This inequality is obtained for normalized positive C_0-semigroups defined on a Banach lattice algebra. An adjoint of Jessen's type inequality has also been derived for the corresponding adjoint-semigroup, which does not give the analogous results but the behavior is still interesting. Moreover, it is followed by some results regarding positive definiteness and exponential convexity of complex structures involving operators from a semigroup. In this chapter, we also present few applications of Jessen's inequality, yielding Hölder's type and Minkowski's type inequalities for corresponding semigroup. Moreover, a Dresher's type inequality for two-parameter family of means, is also proved.

In Chapter 6, we consider the superquadratic mappings and use their characteristics to obtain new inequalities. A Jensen's type inequality for a semigroup of positive linear operators and a superquadratic mapping, defined on a Banach lattice algebra, is obtained. The corresponding mean value theorems, conduct the authors to find a new set of Cauchy's type means. The same strategy is adapted to obtain a Hermite-Hadamard's type inequality for a semigroup of positive linear operators and a superquadratic mapping defined on a Banach lattice algebra. The corresponding mean value theorems, Cauchy's type means and related results are also given.

In Chapter 7, we conclude the thesis and briefly identify some ideas on which further work can be done.

Chapter 2

Preambles

This chapter is intended to give a brief exposition to those topics from functional analysis and lattices theory, which are indispensable for the understanding of subsequent chapters. We do not want to demonstrate the results we present, since they can be easily found in the literature (e.g. [33, 62, 64]). We rather want to give the readers who are not familiar with these terminologies, a necessary information for an intelligent reading of the main contents of this thesis. Since a very fundamental knowledge of functional analysis is required, we firstly go through the brief introduction of some notions from functional analysis.

2.1 Functional Analysis

Here we gather some simple ideas from functional analysis which are extensively required. Proofs are omitted and can be found in standard text books. We assume that the readers are familiar with most basic ideas from the theory of vector spaces (or linear spaces). In what follows, X will denote the real vector space (i.e., a vector space over the field $\mathbb{R}$ of real numbers).

Definition 2.1.1. A norm on X is a mapping $\|.\| : X \to \mathbb{R}$, such that

(i) $\|x\| \geq 0$ for all $x \in X$.

(ii) $\|x\| = 0$ if and only if $x = \mathbf{0}$ (the zero vector in X).

(iii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$, $\lambda \in \mathbb{R}$.

(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (the triangular inequality).

The pair $(X, \|.\|)$ is then referred to as a (real) normed vector (or linear) space.

Introduction of a norm on X imparts a framework within which analysis can be associated to linear algebra.

Definition 2.1.2. Let $(X, \|.\|)$ be a real normed space.

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to $x \in X$ if $\|x_n - x\| \to 0$ as $n \to \infty$. i.e. for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$ for all $n \geq N_\epsilon$.

(ii) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is a Cauchy sequence if, for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n \geq N_\epsilon$.

Definition 2.1.3. A real normed vector space $(X, \|.\|)$ is complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to a unique limit $x \in X$. The pair $(X, \|.\|)$ is then called a (real) *Banach space.*

Example 2.1.4. For $1 \leq p < \infty$, we define the p-norm on $\mathbb{R}^n$ (or $\mathbb{C}^n$) by

$$\|(x_1, x_2, ..., x_n)\|^p = [|x_1|^p + |x_2|^p + ... + |x_n|^p]^{1/p}.$$

For $p = \infty$, we define the ∞ (or maximum) norm by

$$\|(x_1, x_2, ..., x_n)\|_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\}.$$

Then $\mathbb{R}^n$ equipped with the p-norm is a finite-dimensional Banach space for $1 \leq p \leq \infty$.

$\square$

Throughout this section we let $(X, \|.\|)$ be a real Banach space and $T : X \to X$ is a linear operator. T is linear in the usual sense and has the whole of X as its domain.

Definition 2.1.5. Let $T : X \to X$ be a linear operator.

- If there exists a constant $C \geq 0$ such that $\|T(x)\| \leq C\|x\|$ for all $x \in X$. The operator T is said to be *bounded.*

- Let $\{x_n\}_{n=1}^{\infty} \subseteq X$ be a sequence converging (w.r.t. $\|.\|$) to $x \in X$. If the sequence $\{T(x_n)\}_{n=1}^{\infty} \subseteq X$ converges to $T(x)$, the operator T is said to be *continuous* at $x \in X$.

- The operator T is continuous on X if it is continuous on all $x \in X$.

Notation 2.1.6. Let $(X, \|.\|)$ be a Banach space.

- The set of all bounded (i.e. continuous) linear operators defined from X into itself, is denoted by $B(X)$. $B(X)$ is a vector space with respect to the usual operations of addition and scalar multiplication. i.e. for $T_1, T_2, T \in B(X)$, $\lambda \in \mathbb{R}$ and $x \in X$;

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \quad (\lambda T)(x) = \lambda(T(x)).$$

- For $T \in B(X)$, $\|T\|$ denotes the smallest positive constant K such that,

$$\|T(x)\| \leq K\|x\|,$$

  for all $x \in X$.

Theorem 2.1.7. *With the above notation, $\|.\|$ defines a norm on $B(X)$,with respect to which $B(X)$ is a Banach space. For $T \in B(X)$*

$$\begin{aligned} \|T\| &= \sup\{\|T(x)\| : x \in X \textit{ with } \|x\| = 1\} \\ &= \sup\{\frac{\|T(x)\|}{\|x\|} : x \in X, x \neq 0\} \end{aligned} \tag{2.1}$$

$\|T(x)\| \leq \|T\|\|x\|$ for all $T \in B(X)$, $x \in X$.

Note: No confusion should arise because of the use of the same symbol for the norm on X and the norm on $B(X)$.

Example 2.1.8. For $-\infty < a < b < \infty$, let $C[a, b]$ denotes the set of all real-valued continuous (hence bounded)linear functions on $[a, b]$ (on the right at a and on the left at b). For $x \in X = C[a, b]$, let

$$\|x\|_\infty = \sup_{s\in[a,b]} |x(s)|. \tag{2.2}$$

$C[a, b]$ is real vector space with respect to the usual pointwise operations of addition and scalar-multiplication of functions. Moreover, $\|.\|_\infty$ defines a norm on $C[a, b]$ with respect to which $C[a,b]$ is complete. Therefore, $(C[a,b], \|.\|_\infty)$ is a Banach space.

For $x \in X = C[a,b]$, define $T(x)$ on $[a,b]$ by;

$$(Tx)(s) = \int_a^s x(t)dt, \quad a \leq s \leq b, \quad x \in X.$$

From the standard properties of Riemann integral, T maps $C[a,b]$ into $C[a,b]$. Consider

$$|(Tx(s))| \leq \int_a^s |x(t)|dt \leq \int_a^b |x(t)|dt \leq \|x\|_\infty \int_a^b 1dt.$$

This gives

$$\|T(x)\|_\infty \leq (b-a)\|x\|_\infty.$$

Hence for $T \in B(X)$, with respect to $\|.\|_\infty$ we have;

$$\|T\| \leq b-a. \tag{2.3}$$

Particularly for a constant map $x(s) \equiv 1$ for all $x \in [a,b]$, $(Tx(s)) = s-a$. Since $\|x\|_\infty = 1$ and $\|T(x)\|_\infty = b-a$ (at s=b), by (2.1) we know $\|T\| \geq b-a$, which in conjunction with (2.3) shows that $\|T\| = b-a$.

Definition 2.1.9. A linear operator $T : X \supseteq D(T) \to X$ is said to be *closed* if for a sequence $(x_n)_{n\in\mathbb{N}} \in D(T)$, such that $x_n \to x \in X$ and $T(x_n) \to y$ implies $x \in D(T)$ and $T(x) = y$.

In words, if $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ are both convergent in X, then the limit of $(x_n)_{n\in\mathbb{N}}$ is in $D(T)$ and image of the limit is the limit of the images. Or in other words, $D(T)$ contains all of its limit points.

Remark 2.1.10. For $T \in B(X)$, so that $D(T) = X$ then convergence of the sequence (x_n) to $x \in X$ automatically ensure the convergence of $T(x_n)$ to $T(x)$. Therefore a bounded (continuous) operator on X is closed. While the converse is false. But for any unbounded operator, being close is the next best thing. The differential operator on the space of continuously differentiable functions is the famous example in this regard, since this operator is not bounded but closed.

Definition 2.1.11. A numerical function $x^*(x) = (x^*, x)$ defined on a linear topological space X, is said to be a *continuous linear functional* if the following conditions are satisfied.

(a) For any elements $x, y \in X$ and numbers α_1, α_2, we have

$$(x^*, \alpha_1 x + \alpha_2 y) = \alpha_1 x^*(x) + \alpha_2 x^*(y). \quad (linearity)$$

In particular $(x^*, 0) = 0$.

(b) For any $\epsilon > 0$ there is a neighborhood U of zero such that

$$|(x^*, x)| < \epsilon, \quad for\ x \in U. \quad (continuity)$$

This in fact defines the continuity of x^* at the point $x = 0$. But a linear functional which is continuous at 0 is continuous at any point x_0, since

$$(x^*, x) - (x^*, x_0) = (x^*, x - x_0).$$

The space of all linear continuous functionals X^* on a linear topological space X may, in turn to be again a linear space according to the usual definition of addition and scalar multiplication. For $x^*, y^* \in X^*$, $x \in X$ and numbers α_1, α_2;

$$(\alpha_1 x^* + \alpha_2 y^*, x) = \alpha_1 (x^*, x) + \alpha_2 (y^*, x).$$

Evidently the functional $\alpha_1 x^* + \alpha_2 y^*$, defined above, is linear and continuous if x^* and y^* are continuous. The space X^* is known as the *dual (or conjugate) space* of X. If X is a normed space, then X^* is a complete normed (Banach) space with the norm

$$\|x^*\| = \sup_{\|x\| \leq 1} |(x^*, x)|.$$

For details we refer to [40, 71].

Definition 2.1.12. Given two Banach spaces X and Y and a bounded linear operator $L : X \to Y$, recall that the adjoint $L^* : Y^* \to X^*$ is defined by

$$(L^* y^*)x := y^*(Lx), \quad y^* \in Y^*, x \in X. \tag{2.4}$$

Or

$$(L^* y^*, x) := (y^*, Lx), \quad y^* \in Y^*, x \in X. \tag{2.5}$$

In [9], some kind of adjoint has been associated to a nonlinear operator F. In fact, this is possible for Lipschitz continuous operators only. Consider the Banach space $\mathfrak{Lip}_0(X,Y)$ of all Lipschitz continuous operators $F : X \to Y$ satisfying $F(\theta_X) = \theta_Y$, equipped with the norm

$$[F]_{Lip} = sup_{x_1 \neq x_2} \frac{\|F(x_1) - F(x_2)\|}{\|x_1 - x_2\|}, \quad x_1, x_2 \in X.$$

It is easy to see that the space $B(X,Y)$ of all bounded linear operators from X to Y is a closed subspace of $\mathfrak{Lip}_0(X,Y)$. In particular, we set

$$\mathfrak{Lip}_0(X,\mathbb{K}) := X^\sharp$$

where $\mathbb{K}$ is field. The space $X^\sharp$ is called as the pseudo-dual space of X, which contains the usual dual space X^* as closed subspace.

Definition 2.1.13. For $F \in \mathfrak{Lip}_0(X,Y)$, the pseudo-adjoint $F^\sharp : Y^\sharp \to X^\sharp$ of F is defined by;

$$F^\sharp(y^\sharp)(x) := y^\sharp(F(x)), \quad y^\sharp \in Y^\sharp, x \in X. \tag{2.6}$$

This is of course a straightforward generalization of (2.5); in fact, for linear operators L we have $L^\sharp|_{Y^*} = L^*$. i.e. the restriction of the pseudo-adjoint to the dual space is the classical adjoint. Next, we talk about the algebras on vector and Banach spaces, which is essential to relate some important notions from next section.

Definition 2.1.14. [71] A *vector algebra* is a vector space X over the field $\mathbb{K}$, in which multiplication is defined that satisfies

1. $x(yz) = (xy)z$,
2. $(x + y)z = xz + yz$,
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

for all $x, y, z \in X$ and scalar $\alpha \in \mathbb{K}$. Moreover, if X is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\|\|y\|, \quad (x, y \in X)$$

and if X contains a unit element e such that $xe = ex = x$ for $x \in X$ and $\|e\| = 1$, then X is called a *Unital Banach algebra*.

Relatively few general results on ordered Banach spaces are required. Therefore, the next section deals about Banach lattice.

2.2 Banach Lattices

The notion of Banach lattice was introduced to get a common abstract setting, within which one could talk about the ordering of elements. Therefore, the phenomena related to positivity can be generalized. It had mostly been studied in various types to spaces of real-valued functions, e.g. the space $C(K)$ of continuous functions over a compact topological space K, the Lebesgue space $L^1(\mu)$ or even more generally the space $L^p(\mu)$ constructed over measure space $(X, \sum, \mu)$ for $1 \leq p \leq \infty$.

Definition 2.2.1. [64] Any (real) vector space V with an ordering satisfying;

O_1**:** $u \leq v$ implies $u + w \leq v + w$ for all $u, v, w \in V$,

O_2**:** $v \geq 0$ implies $\lambda v \geq 0$ for all $v \in V$ and $\lambda \geq 0$,

is called an *ordered vector space*.

The axiom O_1, expresses the translation invariance and therefore implies that the ordering of an ordered vector space V is completely determined by the positive part $V_+ = \{v \in V : v \geq 0\}$ of V. In other words, $u \leq v$ if and only if $v - u \in V_+$. Moreover, the other property O_2, reveals that the positive part of V is a convex set and a cone with vertex 0 (mostly called the *positive cone* of V).

Definition 2.2.2. An ordered vector space V is called a *vector lattice*, if any two elements $u, v \in V$ have a supremum, which is denoted by $\sup(u, v)$ and an infimum denoted by $\inf(u, v)$.

It is trivially understood that the existence of supremum of any two elements in an ordered vector space implies the existence of supremum of finite number of elements in V.

Furthermore, $u \geq v$ implies $-u \leq -v$, so the existence of finite infima therefore implied. Few important quantities are defined as follows

$$\begin{aligned} \sup(v, -v) &= |v| \quad \textit{(absolute value of } v\textit{)} \\ \sup(v, 0) &= v^+ \quad \textit{(positive part of } v\textit{)} \\ \sup(-v, 0) &= v^- \quad \textit{(negative part of } v\textit{)}. \end{aligned}$$

Some compatibility axiom is required, between norm and order. This is given in the following short way:

$$|u| \leq |v| \quad \textit{implies} \quad \|u\| \leq \|v\|. \tag{2.7}$$

The norm defined on a vector lattice is called a lattice norm. Now, we are in a position, to give formal definitions.

Definition 2.2.3. (a) A *Banach lattice* is a Banach space V endowed with an ordering $\leq$, such that $(V, \leq)$ is a vector lattice with a lattice norm defined on it.

(b) A Banach lattice is said to be *Banach lattice algebra*, provided $u, v \in V_+$ implies $uv \in V_+$.

(c) For the multiplicative identity element "e", a Banach lattice algebra V, with $e \in V$, is called *unital Banach lattice algebra* (UBLA).

Below are the elementary, yet very important formulas valid in any vector lattice.

$$\begin{aligned} v = v^+ - v^-, &\qquad |u + v| \leq |u| + |v| \\ |v| = v^+ + v^-, &\qquad u + v = \sup(u, v) + \inf(u, v) \end{aligned}$$

Remark 2.2.4. (i) The lattice operations $(u, v) \to \sup(u, v)$ and $(u, v) \to \inf(u, v)$ and the mappings $v \to v^+, v \to v^-, v \to |v|$ are uniformly continuous.

(ii) The positive cone is closed.

(iii) A linear functional $v^* \in V^*$ on a vector lattice V is *positive* (i.e. $v^* \geq 0$) , if $(v^*, v) \geq 0$ for all $v \in V_+$.

(iv) The dual space V^* of a Banach lattice V is a Banach lattice with respect to the natural ordering among the elements. i.e. $v^* \leq u^*$ if and only if $v^*(v) \leq u^*(v)$ for all $v \in V_+$.

Example 2.2.5. The vector space $M_2(\mathbb{R})$ of 2×2 real matrices is a vector lattice with the order relation defined below;

$$A \leq B \quad \text{if and only if} \quad B - A \in {M_2}^+(\mathbb{R}),$$

where ${M_2}^+(\mathbb{R})$ is the set of all matrices with positive real entries. Moreover, it is also complete with respect to the norm;

$$\|A\| = \|(\alpha_{ij})\| = \max_j\{\sum_i |\alpha_{ij}| : i, j = 1, 2\}.$$

Therefore, $M_2(\mathbb{R})$ is a Banach lattice.

Example 2.2.6. The space of all continuous real valued functions $C[a, b]$ on the closed interval $[a, b]$ form a Banach lattice with the ordering $\leq$ defined by $u \leq v$ if and only $(v - u)(x) \geq 0$ for all $x \in [a, b]$ and with the norm defined in (2.2).

Definition 2.2.7. A linear mapping ψ from an ordered Banach space V into itself is *positive* (denoted by: $\psi \geq 0$) if $\psi v \in V_+$, for all $v \in V_+$.

The set of all positive linear mappings forms a convex cone in the space $L(V)$ of all linear mappings from V into itself, defining the natural ordering of $L(V)$. The absolute value of ψ, if it exists, is given by

$$|\psi|(v) = sup\{\psi w : |w| \leq v\}, \quad (v \in V_+).$$

For details we refer to [64].

Chapter 3

Semigroup Theory of Linear Operators

An ordered pair $(A, *)$ consisting of a non-empty set A together with an associative binary operation $*$, is called a semigroup if A is closed with respect to $*$, i.e., $a, b \in A \Rightarrow a * b \in A$. Unlike the group, a semigroup may or may not have the identity element and the inverse of given element of A.

We are not concerned with the theory of semigroups in general. Instead we mainly focus on particular semigroups, consisting of a family of bounded linear operators defined on a Banach space. The binary operation among these is simply the composition of operators, denoted by juxtaposition of the operators. It worths mentioning that a theory of semigroups of non-linear operators has been developed as well. However, this theory is by no means complete. While the semigroup theory of linear operators is both elegant and extensive.

3.1 Semigroup Theory

The set $B(V)$ of all bounded linear operators defined on a Banach space V, inherits much of analytic structure from that of V. For instance, $B(V)$ is a Banach space by Theorem 2.1.7. Therefore, the concept of convergence and continuity can be studied therein. We begin with the axiomatic definition of the strongly continuous semigroup of operators.

Definition 3.1.1. A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space V is a family $\{Z(t)\}_{t\geq 0} \subset B(V)$ such that

(i) $Z(s)Z(t) = Z(s+t)$ for all $s, t \in \mathbb{R}^+$.

(ii) Z(0)=I, the identity operator on V.

(iii) for each fixed $f \in V$, $Z(t)f \to f$ (with respect to the norm on V) as $t \to 0^+$.

By the notation $\{Z(t)\}_{t\geq 0}$ we mean there is one operator $Z(t)$ for every non-negative real number t. The axioms (i) and (ii) in the above definition should be interpreted as operator equations in $B(V)$. Here, (i) ensures that the family $\{Z(t)\}_{t\geq 0}$ is closed under the operation of composition of operators and therefore forms a commutative semigroup. Since

$$Z(s)Z(t) = Z(s+t) = Z(t+s) = Z(t)Z(s), \quad \text{for all } s, t \in \mathbb{R}.$$

Combining (i) and (ii) we have that the semigroup $\{Z(t)\}_{t\geq 0}$ has an identity element $T(0)$. The condition (iii) defines the analytic structure so that, for each fixed $v \in V$ the mapping $f_v : [0, \infty) \to V$ defined by $f_v(t) = Z(t)v$ should be right-continuous at 0, since $Z(0)v = v$ by (i). By the use of algebraic structure, this condition leads to something more interesting. For any fixed positive number t_0;

$$\begin{aligned} Z(t_0 + t)v = Z(t + t_0)v &= Z(t)[Z(t_0)v] \\ &\to Z(t_0)v \quad \text{as } t \to 0_+ \text{ by (iii)} \end{aligned}$$

So that the mapping f_v is right-continuous at any $t_0 > 0$.

Example 3.1.2. The space $V = C[0, \infty)$ denotes the set of all complex-valued continuous functions on $[0, \infty)$ (on the right at 0) such that $v \in V$ tends to a finite limit as $v \to \infty$. V is a Banach space with respect to the norm defined by (2.2). For a given function $v \in V$, define $Z(t)v$ on $[0, \infty)$ by

$$[Z(t)v](x) = v(x+t), \quad (x > 0).$$

The operator $Z(t)$ is a translation operator as it moves the graph of v, t-units to the left. The family $\{Z(t)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on $(C[0, \infty), \|.\|_\infty)$. For details of the proof see [54].

Definition 3.1.3. A (one parameter) C_0-group (or strongly continuous group) of operators on a Banach space V is a family $\{Z(t)\}_{t\in\mathbb{R}} \subset B(V)$ such that

(i) $Z(s)Z(t) = Z(s+t)$ for all $s,t \in \mathbb{R}$.

(ii) Z(0)=I, the identity operator on V.

(iii) for each fixed $v \in V$, $Z(t)v \to v$(with respect to the norm on V) as $t \to 0$.

The axioms (i) and (ii) in the above definition shows that we do have a group. Since $Z(0)$ is the identity element and for any $t \in \mathbb{R}$, $Z(t)Z(-t) = Z(-t)Z(t) = Z(-t+t) = Z(t+(-t)) = Z(0) = I$. Therefore, $\{Z(t)\}_{t<0}$ gives the inverses of the family $\{Z(t)\}_{t>0}$. All the properties and characteristics of C_0-Semigroup are also possessed by C_0-group.

Example 3.1.4. By analogy with the previous example, let $V = C(-\infty,\infty)$ be the space of all continuous linear functions on $\mathbb{R}$ such that $v(x)$ tends to a finite limit as $x \to -\infty$ and as $x \to \infty$. V again becomes a Banach space with the norm (2.2) for $x \in \mathbb{R}$. As before define $Z(t)$ on V by

$$[Z(t)v](x) = v(x+t), \quad x,t \in \mathbb{R}.$$

The family $\{Z(t)\}_{t\in\mathbb{R}}$ is a C_0-group on $C[-\infty,\infty]$.

3.2 Generator of the C_0-semigroup

For a strongly continuous semigroups, the analogue of the constant "a" in Theorem 1.1.1, will be called the generator of the semigroup. It will be a linear, but generally unbounded, operator defined only on a dense subspace $D(A)$ of the Banach space V. The solution of the algebraic *functional equation* that is continuous, must already be differentiable (even analytic) and therefore solves (DE).

$$\frac{d}{dt}\phi(t) = A\phi(t). \tag{3.1}$$

The strong continuity of a semigroup $\{Z(t)\}_{t\geq 0}$ also imply some differentiability of the *orbit maps*

$$\zeta_v : t \to Z(t)v \in V.$$

Lemma 3.2.1 (Lemma 1.1, [36])**.** *Let* $\{Z(t)\}_{t\geq 0}$ *be the strongly continuous semigroup on a Banach space* V*. For an element* $v \in V$ *and the orbit map* $\zeta_v : t \to Z(t)v$*, the following properties are equivalent.*

(a) $\zeta_v(.)$ *is differentiable on* $\mathbb{R}^+$*.*

(b) $\zeta_v(.)$ *is right differentiable at* $t = 0$*.*

The desired operator A is obtained by the right derivative at $t = 0$, on the subspace of V consisting of all those elements $v \in V$ for which the orbit maps ζ_v are differentiable.

Definition 3.2.2. The (infinitesimal) generator of $\{Z(t)\}_{t\geq 0}$ is the densely defined closed linear operator $A : V \supseteq D(A) \to R(A) \subseteq V$ such that

$$\begin{aligned} D(A) &= \{v : v \in V, \lim_{t\to 0^+} A_t(v) \text{ exists in } V\} \\ A(v) &= \lim_{t\to 0^+} A_t(v), \quad (v \in D(A)) \end{aligned}$$

where, for $t > 0$,

$$A_t(v) = \frac{[Z(t)v - v]}{t} \quad (v \in V).$$

The domain $D(A)$, which is a linear dense subspace of V, is an essential part of the definition of the generator A. By the next result we can obtain the operators $Z(t)$ as the "exponentials e^{tA}".

Theorem 3.2.3 ([36], p.52)**.** *Let* $\{Z(t)\}_{t\geq 0}$ *be a strongly continuous semigroup on a Banach space* V *with generator* $(A, D(A))$*. If the generator* A *is bounded, i.e., there exists* $M > 0$ *such that* $\|A(v)\| \leq M\|v\|$ *for all* $v \in D(A)$ *with the* $D(A)$ *be closed in* V*. The semigroup is given by*

$$Z(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \quad t \geq 0.$$

Theorem 3.2.4 ([65],p.4)**.** *Let* $\{Z(t)\}_{t\geq 0}$ *be a strongly continuous semigroup on a Banach space* V*. Then there exists constants* $\omega \geq 0$ *and* $M \geq 1$ *such that*

$$\|Z(t)\| \leq Me^{\omega t}, \quad (t \geq 0).$$

That is all we need to know about the generator of the C_0-semigroup, for the understanding of the remaining theory. For the detail study we refer to [36, 54, 76].

3.3 Adjoint Semigroups

The adjoint of a bounded linear operator has already been defined in Definition 2.5. For a strongly continuous positive semigroup $\{Z(t)\}_{t\geq 0}$ on a Banach space V, by defining $Z^*(t) = (Z(t))^*$ for every t, we get a corresponding adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ on the dual space V^*. In [75], it is obtained that, the adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ fails in general to be strongly continuous. The investigation [69], shows that $\{Z^*(t)\}_{t\geq 0}$ acts in a strongly continuous way on;

$$V^{\odot} := \{v^* \in V^* : \lim_{t\to 0^+} \|Z^*(t)v^* - v^*\| = 0\}.$$

This is the maximal such subspace on V^*. The space $V^{\odot}$ was introduced by Philips in 1955, and latter has been studied extensively by various authors. The semigroup given by the restricted operators

$$Z^{\odot}(t) := Z^*(t)|_{V^{\odot}}, \quad (t \geq 0)$$

is called the *sun dual semigroup*. Even though the adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ is not necessarily strongly continuous on V^*, it is still possible to associate a "generator" to it (see p.61, [36]). While the generator $(A^{\odot}, D(A^{\odot}))$ of the strongly continuous semigroup $\{Z^{\odot}(t)\}_{t\geq 0}$ is the part of $(A^*, D(A^*))$ in V^*, i.e.,

$$A^{\odot}(v^*) = A^*(v^*), \quad for \quad v \in D(A^{\odot}) = \{v^* \in D(A^*) : A^*(v^*) \in V^{\odot}\}.$$

For the study of the subsequent chapters, we do not necessarily require the strong continuity of the adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ on V^*.

3.4 Positive Semigroups on Banach Lattice

Since the theory of positive semigroups on Banach lattices is far more complicated and advanced in comparison of the theory of C_0-semigroups, it is not possible for us to give the detail introduction here. It is sufficient for us to highlight the existence of the required positive semigroups and to give few important basic results showing the conditions on a generator A of the semigroup $\{Z(t)\}_{t\geq 0}$ which are equivalent to the positivity of the semigroup. While we refer the reader to [64] for the details, since it contains a very elegant literature regarding this theory.

Let $\{Z(t)\}_{t\geq 0}$ be the strongly continuous positive semigroup, defined on a Banach lattice V. The positivity of the semigroup is equivalent to

$$|Z(t)v| \leq Z(t)|v|, \quad t \geq 0, \quad v \in V.$$

Moreover, for positive contraction semigroups $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V we have;

$$\|(Z(t)f)^+\| \leq \|f^+\|, \quad \text{for all } v \in V.$$

Let V be a real Banach lattice, the canonical half-norm $N^+ : V \to \mathbb{R}$ is given by

$$N^+(v) = \|v^+\|, \quad (v \in V). \tag{3.2}$$

It is easy to note that N^+ is a half-norm. The sub-differential of N^+ is given by;

$$dN^+(v) = \{v^* \in V_+^* : \|v^*\| \leq 1, (v, v^*) = \|v^+\|\}.$$

We call an operator A on V *dispersive*, if it is N^+-dissipative i.e. for ever $v \in D(A)$ we have $(A(v), v^*) \leq 0$ for some $v^* \in dN^+(v)$.

Theorem 3.4.1 (Theorem 1.2, [64])**.** *Let A be a densely defined operator on a real Banach lattice V. Then the following assertions are equivalent.*

(i) *A is the generator of a positive contraction semigroup.*

(ii) *A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$.*

Next, we present a very important tool, namely the *positive minimum principle*. Its importance is due to the fact, it is the necessary condition for an operator A to generate a positive semigroup. However in special case (for example if A is bounded) the positive minimum principle is sufficient for the positivity of the semigroup.

Definition 3.4.2. An operator A on V satisfies the *positive minimum principle* if for all $v \in D(A)_+ = D(A) \cap V_+$, $v^* \in V_+^*$

$$(v, v^*) = 0 \quad \text{implies} \quad (A(v), v^*) \geq 0. \tag{3.3}$$

Proposition 3.4.3 ([64], p.253)**.** *The generator of the strongly continuous positive semigroup satisfies the positive minimum principle (3.3).*

Yet the positive minimum principle is not the sufficient condition in general, the following case is of interest anyways.

Theorem 3.4.4. *[64] Let* $\{Z(t)\}_{t\geq 0}$ *be the* C_0*-semigroup on a Banach space* V *and* A *be its generator. Assume that*

(i) *There exists* $\omega \in \mathbb{R}$ *such that* $\|Z(t)\| \leq e^{\omega t}$ *for all* $t \geq 0$*.*

(ii) *There exists a core* $D_0 \subseteq D(A)$ *of* A *such that* $v \in D_0$ *implies* $|v| \in D_0$*.*

If the restriction of the operator A *to* D_0 *satisfies the positive minimum principle, then the semigroup is positive.*

The literature presented in [64], guarantees the existence of the strongly continuous positive semigroups and positive contraction semigroups on Banach lattice V, with some conditions imposed on the generator. The *Kato's inequality* is important among these but the very important is, that it must always satisfy (3.3). For the study of the adjoint of positive semigroups we refer to [75].

Chapter 4

Cauchy Type Means On One-Parameter C_0-Group Of Operators*

The theory of means and inequalities has been restricted to the space of real functions for very long. Whereas the Cauchy's type means have been studied in recent years on real linear functionals and a significant theory of Cauchy type means has been developed [12–17], which is both extensive and elegant.

In this chapter, our idea is to generalize this concept for the C_0-semigroup of bounded linear operators which also contains the inverses and hence forming a C_0-group. We define the power means on C_0-group and prove some mean value theorems. These theorems lead us to obtain a new set of means, called Cauchy's type means.

4.1 Power Means For C_0-Semigroups

Previous chapters gave the detailed exposition to all the important definitions and results in the theory of strongly continuous groups(semigroups) of bounded linear operators defined on a Banach space X. These are indispensable for an understanding of the proceeding sections. The *strongly continuous one-parameter group (or C_0-group)* on a Banach space X is defined

*This chapter is based on the following publication:

1.Gul I Hina Aslam, Matloob Anwar, *Cauchy Type Means On One-Parameter C_0-Group Of Operators*, Journal of Mathematical Inequalities. VOL 9, No. 2, 631-639. 2015.

by the Definition 3.1.3. For $\{Z(t)\}_{t\geq 0} \subseteq B(X)$ a C_0-semigroup on the Banach space X, there exists constants $M > 0$ and $\omega \geq 0$ such that $\|Z(t)\| \leq Me^{\omega t}$, for all $t \geq 0$. See ([62], Theorem 2.14). In case $M = 1$ and $\omega = 0$, we obtain a C_0-semigroup (correspondingly group) of contractions.

The arithmetic mean on the C_0-semigroup of operators is defined as [66],

$$m(Z, f, t) = \frac{1}{t}\int_0^t Z(\tau)f d\tau. \tag{4.1}$$

Means of C_0-semigroups of operators have great importance and form the basis of *Mean Ergodic Theory*, which has been a center of interest in research for decades (see e.g. [29,35,79]). In order to define power means on C_0-semigroup of operators, things need little more concentration. As the real-powers are involved, $\{Z(t)\}_{t\geq 0}$ should also contain the inverse operators (to define the powers like $r < 0$). One can observe that when r is any integer (positive or negative), the C_0-group property implies that $Z(t)^r = Z(rt)$. While we can generalize it for $r \in \mathbb{R}$. e.g. take $Z(\frac{1}{2}t)Z(\frac{1}{2}t) = Z(t)$ and thus we get $Z(t)^{1/2} = Z(\frac{1}{2}t)$. For $r \in \mathbb{R}$, the generator of $\{Z(rt)\}_{t\geq 0}$ is $(rA, D(A))$. Such semigroups are often called *rescaled semigroups.* (See e.g. [64]). For $f \in X$ and $t > 0$, a C_0-semigroup(group) $\{Z(t)\}_{t\geq 0}$ generated by an operator A, has the form $Z(t)f = exp[tA]f$. For reference see [62]. Hence $\ln[Z(t)f]$ makes sense.

In correspondence with the usual definition of power integral means, we can define the power means for C_0-group of operators.

Definition 4.1.1. Let X be a Banach space and $\{Z(t)\}_{t\in\mathbb{R}}$ the C_0-group of linear operators on X. For $f \in X$ and $t \in \mathbb{R}$, the power mean is defined as follows

$$M_r(Z, f, t) = \begin{cases} \left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}^{1/r}, & r \neq 0 \\ \\ exp[\frac{1}{t}\int_0^t ln[Z(\tau)]f d\tau], & r = 0 \end{cases} \tag{4.2}$$

□

For $t > 0$ and $r \in \mathbb{R}^+$, $Z(t)^r = Z(-t)^{-r}$. Therefore the integral domain is taken to be non-negative. Moreover for $r = 1$, $M_r(Z, f, t) = m(Z, f, t)$, the arithmetic mean, for $r = 0$

it defines the geometric mean and for $r = -1$ it defines the harmonic mean on C_0-group of operators (and hence satisfying the property of power-mean). For $r > 0$, $M_{-r}(Z, f, t)$ gives the inverse of the mean of inverse of $Z(t)^r$.

A brief introduction of Cauchy's type means has been given in Section 1.3. The purpose of our work is to introduce new means of Cauchy's type based on power means, defined on C_0-group of operators.

4.2 A Set of Main Results

This section includes the sequence of results, which at a long run proves the power mean on C_0-group of linear operators to be of Cauchy type.

Theorem 4.2.1. *[62] Let $(X, \|.\|)$ be a Banach space and let $\phi : [a, b] \to X$ be continuous. Then, for each $t \in [a, b]$, the strong Riemann integral $\int_a^t \phi$ exists in X and*

$$\frac{d}{dt}\Big[\int_a^t \phi\Big] = \frac{d}{dt}\Big[\int_a^t \phi(s)ds\Big] = \phi(t) \tag{4.3}$$

Lemma 4.2.2. *Let $\{Z(t)\}_{t\geq 0} \subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X. For $f \in X$ and $t > 0$,*

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} Z(u)f du = Z(t)f. \tag{4.4}$$

Proof. By Theorem 4.2.1 we have

$$\frac{d}{dt}\int_0^t Z(s)f ds = Z(t)f. \tag{4.5}$$

Therefore, if we consider;

$$\begin{aligned}
\frac{d}{dt}\int_0^t Z(s)f ds &= \lim_{h\to 0^+} \frac{1}{h}\Big[\int_0^{t+h} Z(s)ds - \int_0^t Z(s)ds\Big] \\
&= \lim_{h\to 0^+} \frac{1}{h}\Big[\int_0^{t+h} Z(s)ds + \int_t^0 Z(s)ds\Big] \\
&= \lim_{h\to 0^+} \frac{1}{h}\Big[\int_t^{t+h} Z(s)ds\Big].
\end{aligned}$$

By using (4.5), the assertion (4.11) follows directly. □

The lemma above, will help us to prove that the arithmetic mean of a C_0-semigroup defined by (4.1) belongs to the domain of the generator of the semigroup and therefore belongs to the Banach space X. And so are the power means (4.2).

Lemma 4.2.3. *Let $\{Z(t)\}_{t\geq 0} \subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X. And let A be the generator of the semigroup. Then, for $f \in X$ and $t > 0$,*

$$m(Z, f, t) = \frac{1}{t}\int_0^t Z(\tau) f d\tau \in D(A) \subset X \tag{4.6}$$

Proof. Let $h > 0$ and consider

$$\begin{aligned}
\frac{Z(h)-I}{h}\Big\{\int_0^t Z(u) f du\Big\} &= \frac{1}{h}\int_0^t \{Z(u+h)f - Z(u)f\} du \\
&= \frac{1}{h}\int_0^t Z(u+h) f du - \frac{1}{h}\int_0^t Z(u) f du \\
&= \frac{1}{h}\int_h^{t+h} Z(u) f du - \frac{1}{h}\int_0^t Z(u) f du \\
&= \frac{1}{h}\int_h^{t} Z(u) f du + \frac{1}{h}\int_t^{t+h} Z(u) f du - \frac{1}{h}\int_0^t Z(u) f du \\
&= \frac{1}{h}\int_t^{t+h} Z(u) f du - \frac{1}{h}\int_0^{t} Z(u) f du - \frac{1}{h}\int_t^h Z(u) f du \\
&= \frac{1}{h}\int_t^{t+h} Z(u) f du - \frac{1}{h}\int_0^{h} Z(u) f du.
\end{aligned}$$

On letting $h \to 0^+$ and using the Lemma 4.2.1, we get

$$\lim_{h\to 0^+} \frac{Z(h)-I}{h}\Big\{\int_0^t Z(u) f du\Big\} = Z(t)f - f = [Z(t) - I]f \in D(A).$$

Hence

$$\int_0^t Z(\tau) f d\tau \in D(A).$$

Since $D(A)$ is a vector subspace of X, therefore $m(Z, f, t) \in D(A)$. Also $D(A) \subset \overline{D(A)} = X$. Hence the result follows. □

Corollary 4.2.4. *Let $\{Z(t)\}_{t\geq 0} \subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X. For $f \in X$ and $t > 0$,*

$$M_r(Z, f, t) \in X$$

Where $M_r(Z, f, t)$ is defined by (4.2).

Proof. For $\{Z(t)\}_{t\in\mathbb{R}} \subseteq B(X)$, by group-law we have, for $\tau, r \in \mathbb{R}$,

$$[Z(\tau)]^r f = Z(r\tau)f = Z(s)f$$

Where $r\tau = s$, then $Z(s) \in \{Z(t)\}_{t\in\mathbb{R}}$. By above stated Lemma, we finally get that $M_r(Z, f, t) \in X$. □

Lemma 4.2.5. *Let X be a Banach algebra. Let the fractional powers $x^{1/r} \in X$ such that $r \in \mathbb{Z}$. Then*

$$\|x^{1/r}\| \geq \|x\|^{1/r}, \quad r \in \mathbb{Z}_+. \tag{4.7}$$

and

$$\|x^{1/r}\| \leq \|x\|^{1/r}, \quad r \in \mathbb{Z}_-. \tag{4.8}$$

Proof. Since by the definition of a Banach algebra we know

$$\|x_1.x_2\| \leq \|x_1\|\|x_2\|, \quad x_1, x_2 \in X,$$

thats how we can have such relation for any positive power. s.t.

$$\|x^n\| \leq \|x\|^n, \quad x \in X, n \in \mathbb{N}.$$

For $r \in \mathbb{Z}_+$, we have $(x^{1/r})^r = x$ and hence

$$\|x\| = \|(x^{1/r})^r\| \leq \|x^{1/r}\|^r.$$

This gives the assertion (4.7).

$$\|x\|^{1/r} \leq \|x^{1/r}\|.$$

Since we know

$$1 = \|e\| = \|x.x^{-1}\| \leq \|x\|\|x^{-1}\|$$

and therefore $\|x^{-1}\| \geq \frac{1}{\|x\|} = \|x\|^{-1}$.

Moreover, for $r \in \mathbb{Z}_-$, $-r \in \mathbb{Z}_+$. Consider for $x \in X$ and $r \in \mathbb{Z}_-$;

$$\begin{aligned} \|x^r\| = \|(x^{-r})^{-1}\| &\geq \|x^{-r}\|^{-1} \\ &= \frac{1}{\|x^{-r}\|} \\ &\geq \frac{1}{\|x\|^{-r}} = \|x\|^r. \end{aligned}$$

Therefore the assertion (4.8) follows directly. □

Proposition 4.2.6. *For a C_0-semigroup of contractions $\{Z(t)\}_{t\geq 0} \subseteq B(X)$, we have $\|Z(t)\| \leq 1$, for all $t \in \mathbb{R}^+$. For such C_0-groups*

(i) *For arithmetic mean defined in (4.1)*

$$\|m\| = Sup_{f\in X}\frac{\|m(Z,f,t)\|}{\|f\|} \leq 1, \quad for\ t > 0.$$

(ii) *The power mean $M_r(Z,f,t)$ is defined by (4.2). For $r > 0$,*

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \geq \eta, \quad f \in X, t > 0,$$

and for $r < 0$,

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \leq \eta, \quad f \in X, t > 0,$$

where $\eta = \|f\|^{-(r+1)}$. Moreover for $r = 0$,

$$\|M_0\| = Sup_{f\in X}\frac{\|M_0(Z,f,t)\|}{\|f\|} \leq 1, \quad f \in X, t > 0.$$

(iii) *Let $\{f_n\}_{n=0}^{\infty} \subset X$, such that $f_n \to f \in X$, then for $r \leq 0$, $M_r(Z,f_n,t) \to M_r(Z,f,t)$.*

Proof. **(i)** Consider the mean $m(Z,f,t)$;

$$\begin{aligned}\|m(Z,f,t)\| &= \|\frac{1}{t}\int_0^t [Z(t)f]dt\| \\ &\leq \frac{1}{t}\int_0^t \|Z(t)f\|dt \\ &\leq \|Z(t)\|\|f\| \\ &\leq \|f\|.\end{aligned}$$

Therefore

$$\|m\| = Sup_{f\in X}\frac{\|m(Z,f,t)\|}{\|f\|} \leq 1, \quad for\, t > 0.$$

(ii) Consider the power mean $M_r(Z,f,t)$. Using the Lemma 4.2.5 for $r>0$,

$$\begin{aligned}
\|M_r(Z,f,t)\| &= \left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}^{1/r}\right\| \\
&\geq \left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}\right\|^{1/r} \\
&= \left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}\right\|^{r^{-1}} \\
&\geq \frac{1}{\left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}\right\|^{r}} \\
&\geq \frac{1}{\left\{\frac{1}{t}\int_0^t \|[Z(\tau)]^r f\| d\tau\right\}^{r}} \\
&\geq \frac{1}{\|f\|^r} = \|f\|^{-r}.
\end{aligned}$$

Therefore

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \geq \frac{\|f\|^{-r}}{\|f\|} = \|f\|^{-(r+1)} = \eta, \quad f\in X, t>0.$$

Similarly by using the Lemma 4.2.5 for $r<0$,

$$\begin{aligned}
\|M_r(Z,f,t)\| &= \left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}^{1/r}\right\| \\
&\leq \left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}\right\|^{1/r} \\
&= \left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}\right\|^{r^{-1}} \\
&\leq \frac{1}{\left\|\left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}\right\|^{r}} \\
&\leq \frac{1}{\left\{\frac{1}{t}\int_0^t \|[Z(\tau)]^r f\| d\tau\right\}^{r}} \\
&\leq \frac{1}{\|f\|^r} = \|f\|^{-r}.
\end{aligned}$$

Therefore

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \leq \frac{\|f\|^{-r}}{\|f\|} = \|f\|^{-(r+1)} = \eta, \quad f\in X, t>0.$$

Now consider $(M_r(Z,f,t))$ for $r=0$,

$$\begin{aligned} \|M_0(Z,f,t)\| &= \left\| \exp\left[\frac{1}{t}\int_0^t \ln[Z(\tau)]fd\tau\right]\right\| \\ &\le \exp\left\|\left[\frac{1}{t}\int_0^t \ln[Z(\tau)]fd\tau\right]\right\| \\ &\le \exp\left[\frac{1}{t}\int_0^t \ln\|[Z(\tau)]f\|d\tau\right] \\ &\le \|f\|. \end{aligned}$$

Therefore

$$\|M_0\| = Sup_{f\in X}\frac{\|M_0(Z,f,t)\|}{\|f\|} \le \frac{\|f\|}{\|f\|} = 1, \quad f\in X, t>0.$$

(iii) Let $\{f_n\}_{n=0}^{\infty}\subset X$, such that $f_n \to f\in X$. Then we have

$$\|M_r(Z,f_n,t) - M_r(Z,f,t)\| \le \|M_r\|\|f_n - f\|, \quad r\le 0.$$

By (ii) we have;

$$\|M_r(Z,f_n,t) - M_r(Z,f,t)\| \le \eta\|f_n - f\|, \quad r\le 0.$$

Therefore, $\|M_r(Z,f_n,t) - M_r(Z,f,t)\| \to 0$ as $n\to\infty$.

Thus, $M_r(Z,f_n,t)\to M_r(Z,f,t)$ as $n\to\infty$.

□

Next, we prove a very interesting mean value theorem. It actually forms the basis of rest of the theory and can be somehow regarded as the analogue of (Theorem 1, [68]) to Banach spaces.

Theorem 4.2.7. *Let X be a Banach space and $\{Z(t)\}_{t\ge 0}\subset B(X)$ be a C_0-semigroup of operators on X. For $\phi,\psi\in C^2(X)$ and some $\xi\in X$*

$$\frac{\frac{1}{t}\int_0^t \phi[Z(\tau)]fd\tau - \phi[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau]}{\frac{1}{t}\int_0^t \psi[Z(\tau)]fd\tau - \psi[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau]} = \frac{\phi''(\xi)}{\psi''(\xi)}. \tag{4.9}$$

Proof. For the sake of simplicity throughout the proof, we shall denote $m(Z,f,t)$ by m_t. For $\rho\in X$, define

$$(Q\phi)(\rho) := \frac{1}{t}\int_0^t \phi[\rho[Z(\tau)f] + (1-\rho)m_t]d\tau - \phi(m_t)$$

similarly, for the operator ψ, we define $(Q\psi)(\rho)$. It is observed that,

$$(Q\phi)'(\rho) := \frac{1}{t}\int_0^t [Z(\tau)f - m_t]\phi'[\rho[Z(\tau)f] + (1-\rho)m_t]d\tau$$

and

$$(Q\phi)''(\rho) := \frac{1}{t}\int_0^t [Z(\tau)f - m_t]^2\phi''[\rho[Z(\tau)f] + (1-\rho)m_t]d\tau.$$

Here, $(.)'$ denotes the Gateaux derivative. Let us define an other operator $W(\rho)$, as follows

$$W(\rho) = (Q\psi)(1)(Q\phi)(\rho) - (Q\phi)(1)(Q\psi)(\rho).$$

It can be easily seen that

$$W(0) = W(1) = W'(0) = 0$$

where 0,1 are the zero, identity elements of X, respectively.

By applying the Mean Value theorem [49] twice, we may conjecture that there exists an element $\eta \in X$ such that

$$W''(\eta) = 0.$$

Hence

$$\frac{1}{t}\int_0^t [Z(\tau)f - m_t]^2\{(Q\psi)(1)\phi''[\eta[Z(\tau)f] + (1-\eta)m_t] - (Q\phi)(1)\psi''[\eta[Z(\tau)f] + (1-\eta)]\}m_t d\tau = 0 \tag{4.10}$$

A mapping $\varphi_f : [0,\infty) \to X$ defined by

$$\varphi_f(t) = Z(t)f, \quad f \in X$$

is continuous on $[0,\infty)$. See ([62],Lemma 2.4). Hence for any fixed $\eta \in X$, the expression in the braces in (4.10) is a continuous function of τ, so it vanishes for some value of $\tau \geq 0$. Corresponding to that value of $\tau \geq 0$, we get an element $\xi \in X$, s.t.

$$\xi = \eta[Z(\tau)f] + (1-\eta)m_t, \quad f \in X.$$

So that

$$(Q\psi)(1)\phi''(\xi) - (Q\phi)(1)\psi''(\xi) = 0$$

The assertion (4.9) follows directly. □

Corollary 4.2.8. *Let X be a Banach space and $\{Z(t)\}_{t\geq 0} \subset B(X)$ be a C_0-semigroup of operators on X. For $\phi \in C^2(X)$ and some $\xi \in X$*

$$\frac{1}{t}\int_0^t \phi[Z(\tau)f]d\tau - \phi\Big[\frac{1}{t}\int_0^t [Z(\tau)f]d\tau\Big] = \frac{\phi''(\xi)}{2}\Big\{\frac{1}{t}\int_0^t [Z(\tau)]^2 f d\tau - \Big[\frac{1}{t}\int_0^t [Z(\tau)]f d\tau\Big]^2\Big\}. \tag{4.11}$$

Proof. By setting $\psi(f) = f^2$ for $f \in X$, in Theorem 4.2.7, we get the assertion (4.11). □

Next, let G be the group of invertible bounded linear operators from a Banach space X to itself. For $\{Z(t)\}_{t\geq 0} \subset B(X)$ a C_0-semigroup of operators defined on X and $H \in G$, the quasi-arithmetic mean is defined as

$$M'_H(Z, f, t) = H^{-1}\Big\{\frac{1}{t}\int_0^t H[Z(\tau)f]d\tau\Big\}, \quad f \in X, t \geq 0. \tag{4.12}$$

By (Lemma 1.85, [62]) $B(X)$ is closed under composition of operators so the above expressions exists and belongs to X. For the sake of simplicity, the set of all elements of G, whose second order derivative (in Gateaux's sense) exits, is denoted by $C^2G(X)$.

Theorem 4.2.9. *Let X be a Banach space and let $H, F, K \in C^2G(X)$. Then*

$$\frac{H(M'_H(Z,f,t)) - H(M'_F(Z,f,t))}{K(M'_K(Z,f,t)) - K(M'_F(Z,f,t))} = \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)}. \tag{4.13}$$

For some $\xi \in X$, provided that the denominator on the left hand side of (4.13) is non-zero.

Proof. By choosing the operators ϕ and ψ in Theorem 4.2.7, such that

$$\phi = H \circ F^{-1}, \quad \psi = K \circ F^{-1} \quad \textit{and} \quad Z(\tau)f = F[Z(\tau)f]$$

where $H, F, K \in C^2G(X)$. We find that there exists $\xi \in X$, such that

$$\frac{H(M'_H(Z,f,t)) - H(M'_F(Z,f,t))}{K(M'_K(Z,f,t)) - K(M'_F(Z,f,t))} = \frac{H''(F^{-1}(\xi))F'(F^{-1}(\xi)) - H'(F^{-1}(\xi))F''(F^{-1}(\xi))}{K''(F^{-1}(\xi))F'(F^{-1}(\xi)) - K'(F^{-1}(\xi))F''(F^{-1}(\xi))}.$$

Therefore, by setting $F^{-1}(\xi) = \eta$, we find that there exists $\eta \in X$, such that

$$\frac{H(M'_H(Z,f,t)) - H(M'_F(Z,f,t))}{K(M'_K(Z,f,t)) - K(M'_F(Z,f,t))} = \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)},$$

which completes the proof. □

Remark 4.2.10. For $(X, \|.\|)$ a Banach space, it follows from Theorem 4.2.9 that

$$m \leq \Big\| \frac{H(M'_H(Z,f,t)) - H(M'_F(Z,f,t))}{K(M'_K(Z,f,t)) - K(M'_F(Z,f,t))} \Big\| \leq M,$$

Where m and M are respectively, the minimum and maximum values of

$$\Big\| \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)} \Big\|, \;\; \eta \in X.$$

By next very simple, yet important result is to show that the defined power means $M_r(Z,f,t)$ on C_0-group of operators are of Cauchy type.

Corollary 4.2.11. *Let $r, s, l \in \mathbb{R}$ and $\{Z(t)\}_{t\in\mathbb{R}} \subset B(X)$ be a C_0-semigroup of operators on a Banach space X. Then*

$$\frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)} = \frac{r(r-s)}{l(l-s)}\eta^{r-l}, \quad \eta \in X. \tag{4.14}$$

Here $M_r(Z,f,t)$ is defined by (4.2).

Proof. For $r, s, l \in \mathbb{R}$ and $f \in X$, if we set

$$H(f) = f^r, \;\; F(f) = f^s, \;\; K(f) = f^l$$

in Theorem 4.2.9, the assertion in (4.14) follows directly. □

Remark 4.2.12. It follows from Corollary 4.2.11 that

$$\Big|\frac{r(r-s)}{l(l-s)}\Big| m \leq \Big\| \frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)} \Big\| \leq \Big|\frac{r(r-s)}{l(l-s)}\Big| M.$$

Here m and M are respectively, the minimum and maximum values of $\|\eta^{r-l}\|$, $\eta \in X$.

In the next definition we have defined means of the Cauchy type on C_0-group of linear operators.

Definition 4.2.13. Let $r, s, l \in \mathbb{R}$ and $\{Z(t)\}_{t\in\mathbb{R}} \subset B(X)$ be a C_0-semigroup of operators on a Banach space X. Then

$$\mathfrak{M}_r^{l,s}(Z,f,t) = \Big(\frac{l(l-s)}{r(r-s)} \frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)} \Big)^{\frac{1}{r-l}}, \tag{4.15}$$

is a mean of the Cauchy type on C_0-group of operators. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases, as in [17].

Chapter 5

Jessen's Type Inequality and Its Applications for Positive C_0-Semigroups*

A significant theory regarding inequalities and exponential convexity for real valued functions, has been developed [6, 12]. The intention to generalize such concepts for the C_0-semigroup of operators, is a motivation from [18].
In the present chapter, our goal is to derive a Jessen's type inequality and the corresponding adjoint-inequality, for a C_0-semigroup and the adjoint-semigroup, respectively [19]. Then, to apply the derived Jessen's type inequality to obtain the expressions called the Hölder's type inequality, the Minkowski's type inequality and Dresher's type inequality [20].

5.1 Jessen's Type Inequality

In 1931, Jessen [48] gave the generalization of the Jensen's inequality for a convex function and positive linear functionals. See ([66], p.47). We intent to prove this inequality for a normalized positive C_0-semigroup and convex operator, defined on a Banach lattice.

*This chapter is based on the following publications:

1. Gul I Hina Aslam, Matloob Anwar, *Jessen's Type Inequality and Exponential Convexity For Operator Semigroups*, Journal of Inequalities and Applications. 353, 2015.

2.Gul I Hina Aslam, Matloob Anwar, *Application Of Jessen's Type Inequality For Positive C_0-Semigroup Of Operators*, Journal of Statistical Science and Applications., Vol. 3, No. 7-8, 122-129. 2015.

Throughout the section, V denotes a unital Banach lattice algebra endowed with an ordering $\leq$, unless otherwise stated.

Definition 5.1.1. Let U be a nonempty open convex subset of V. An operator $F : U \to V$ is convex if it satisfies

$$F(\beta u + (1-\beta)v) \leq \beta F(u) + (1-\beta)F(v), \tag{5.1}$$

whenever $u, v \in U$ and $0 \leq \beta \leq 1$.

Definition 5.1.2.

- A Banach algebra X, with the multiplicative identity element e, is called the *unital Banach algebra.*
- The strongly continuous semigroup $\{Z(t)\}_{t\geq 0}$ defined on X, is said to be a *normalized semigroup*, whenever it satisfies

$$Z(t)(e) = e, \quad for\,all \quad t > 0. \tag{5.2}$$

The notion of normalized semigroup is inspired from normalized functionals [12]. Let $\mathfrak{D}_c(V)$ denotes the set of all differentiable convex operators $\phi : V \to V$. From these further, we develop a theory for this class of operators along with the normalized semigroups of positive linear operators defined on a Banach lattice V.

Theorem 5.1.3. ***Jessen's Type Inequality [19]*** *Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup on V such that it satisfies (5.2). For an operator $\phi \in \mathfrak{D}_c(V)$ and $t \geq 0$;*

$$\phi(Z(t)f) \leq Z(t)(\phi f), \quad f \in V. \tag{5.3}$$

Proof. Since $\phi : V \to V$ is convex and differentiable, by considering an operator-analogue of [Theorem A, p.98, [70]], we have for any $f_0 \in V$, there is a fixed vector $m = m(f_0) = \phi'(f_0)$ such that

$$\phi(f) \geq \phi(f_0) + m(f - f_0), \quad f \in V.$$

Using the property (5.2) along with the linearity and positivity of operators in a semigroup, we obtain

$$Z(t)(\phi(f)) \geq \phi(f_0) + m(Z(t)f - f_0), \quad f \in V, t \geq 0.$$

In this inequality, set $f_0 = Z(t)f$ and the assertion (5.3) follows. □

The existence of an identity element and normalization (5.2), are the necessary imposed conditions for the above theorem. We now prove the said, by following examples.

Example 5.1.4. Let $X := C_0(\mathbb{R})$, $\{Z(t)\}_{t\geq 0}$ be the left shift semigroup defined on X and ϕ taking the mirroring along y-axis. The identity function does not contain a compact support and therefore is not in X. If we now take a bell-shaped curve like $f(x) := e^{-x^2}$, $x \in \mathbb{R}$. Then f is positive, $\phi f = f$, and $Z(t)(\phi f) = e^{-(x-t)^2}$ has maximum at $x = t$, and it is between 0 and 1 elsewhere. On the other hand, $\phi(Z(t)f) = e^{-(x+t)}$ has a maximum at $s = t$ and it is immediate that we cannot compare the two functions in the usual ordering. See figure 5.1.

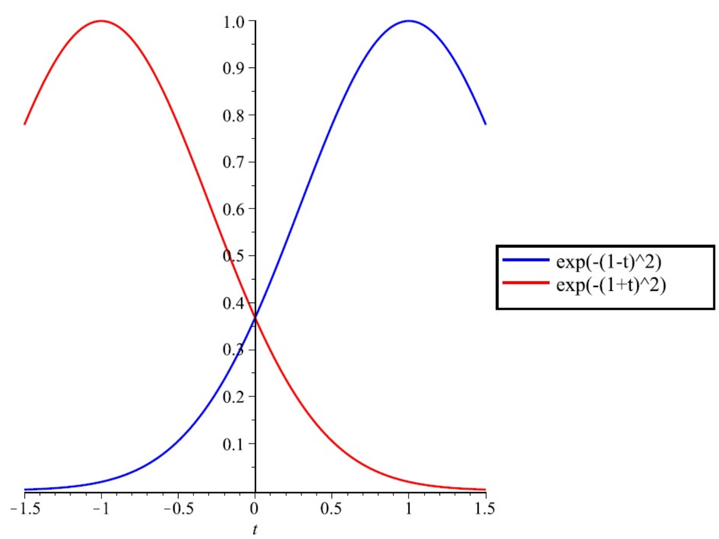

Figure 5.1: The behaviour of the Jessen's type inequality in Example 5.1.4 for $x = 1$.

Example 5.1.5. Let $\Gamma := \{s \in \mathbb{C} : |s| = 1\}$, and $X = C(\Gamma)$. The rotation semigroup $\{Z(t)\}_{t\geq 0}$ is defined as,

$$Z(t)f(s) = f(e^{it} \cdot s), \quad f \in X. \tag{5.4}$$

Let ϕ be the mirroring along y-axis. The identity element $E \in X$, s.t. for all $s \in \Gamma$, $E(s) = s$. Then $Z(t)E(s) = E(e^{it} \cdot s) = e^{it} \cdot s$. Or we can say that any complex number $s = e^{ix}$ is mapped to $e^{i(x+t)}$. $Z(t)$ satisfies (5.2), only when t is a multiple of 2π. Let $f(s) = \Re(s)+1 > 0$, then $(Z(t)f)(e^{is}) = f(e^{i(t+s)}) = cos(t+s)+1$, hence $\phi[(Z(t)f)(e^{is})] = cos(t-s)+1$. On the other hand, $\phi(f) = f$, and $Z(t)[(\phi f)(e^{is})] = (Z(t)f)(e^{is}) = cos(t+s)+1$. Hence, the equality holds in (5.3) when t is a multiple of 2π, but the two sides are not comparable in general.

It can be easily seen that $Z' = \{Z(2\pi t)\}_{t=0}^{\infty}$ is a subgroup of $Z = \{Z(t)\}_{t=0}^{\infty}$, since $Z(2\pi t)Z(2\pi s) = Z(2\pi(t+s))$. Therefore Z' is a normalized semigroup. See figure 5.2.

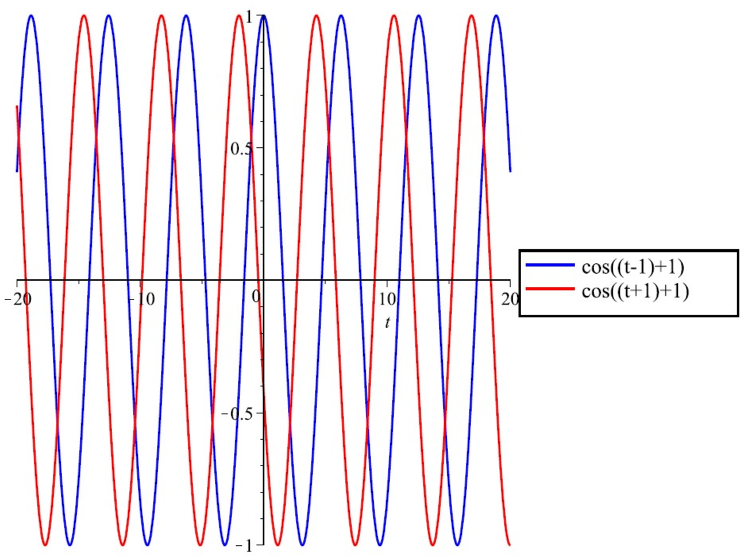

Figure 5.2: The behaviour of the Jessen's type inequality in Example 5.1.5 for $s = 1$.

5.2 Adjoint-Jessen's Type Inequality

In previous section, a Jessen's type inequality has been derived, for a normalized positive C_0-semigroup $\{Z(t)\}_{t\geq 0}$. This gives us the motivation towards knowing the behaviour of this inequality, for its corresponding adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ defined on V^*. As the theory for dual spaces gets more complicated, we do not expect to have the analogous results.

The detail introduction of the dual spaces, adjoint operators and the pseudo-adjoint operators is given in the chapter 2, which gives us the sufficient knowledge towards knowing a part of the dual space V^*, for which an Adjoint of Jessen's type inequality makes sense.

The pseudo-adjoint operator is of course a straightforward generalization of the classic-adjoint (5.1). In fact, for linear operators L we have $L^\sharp|_{Y^*} = L^*$. i.e. the restriction of the pseudo-adjoint to the dual space is the classical adjoint. See Chapter 2 for details.

For the sake of convenience, we denote the adjoint of the operator F by F^*, throughout the present section. Either it's a classical adjoint or the Pseudo-adjoint (depending upon the operator F).
Similarly, the considered dual space of the vector lattice algebra V will be denoted by V^*, which can be the intersection of the pseudo-dual and classical dual spaces in case of a nonlinear convex operator.

Lemma 5.2.1. *Let F be the convex operator on a Banach space X, then the adjoint operator F^* on the dual space X^* is also convex.*

Proof. For $x \in X$ and $0 \le \lambda \le 1$

$$\begin{aligned}(F^*(\lambda x_1^* + (1-\lambda)x_2^*), x) &= (\lambda x_1^* + (1-\lambda)x_2^*, F(x)), \\ &= \lambda(x_1^*, F(x)) + (1-\lambda)(x_2^*, F(x)),\end{aligned}$$

where $x_1^*, x_2^* \in X^*$. By putting $x = \mu x + (1-\mu)x$, for $0 \le \mu \le 1$ and using the convexity of the operator F we finally get

$$F^*(\lambda x_1^* + (1-\lambda)x_2^*) \le \lambda F^*(x_1^*) + (1-\lambda)F^*(x_2^*).$$

Hence, F^* is convex on X^*. □

Theorem 5.2.2. ***Adjoint-Jessen's Inequality*** *[19] Let $\{Z^*(t)\}_{t\ge0}$ be the adjoint semigroup on V^* such that the original semigroup $\{Z(t)\}_{t\ge0}$, the operator ϕ and the space V are same as in Theorem 5.1.3. For a convex operator $\phi^* : V^* \to V^*$ and $t \ge 0$*

$$\phi^*(Z^*(t)f^*) \ge Z^*(t)(\phi^* f^*), \quad f^* \in V^*. \tag{5.5}$$

Proof. For $f \in V$ and $t \geq 0$, consider

$$\begin{aligned}(\phi^*[Z^*(t)f^*], f) &= (Z^*(t)f^*, \phi(f)) \\ &= (f^*, Z(t)(\phi f)) \\ &\geq (f^*, \phi(Z(t)f)) \\ &= (\phi^*(f^*), Z(t)f) \\ &= (Z^*(t)[\phi^* f^*], f)\end{aligned}$$

Therefore, the assertion (5.5) is satisfied. □

5.3 Exponential Convexity

In this section we define the exponential convexity of an operator. Moreover, few complex structures, involving the operators from a semigroup, will be proved to be exponentially convex.

Definition 5.3.1. Let V be a Banach lattice endowed with ordering $\leq$. An operator $H : I \to V$ is exponentially convex if it is continuous and for all $n \in \mathbb{N}$

$$\sum_{i,j=1}^{n} \xi_i \xi_j H(x_i + x_j) f \geq 0, \quad f \in V, \tag{5.6}$$

where $\xi_i \in \mathbb{R}$ such that $x_i + x_j \in I \subseteq \mathbb{R}$, $1 \leq i, j \leq n$.

Proposition 5.3.2. *Let V be a Banach lattice endowed with ordering $\leq$. For an operator $H : I \to V$, the following propositions are equivalent.*

(i) *H is exponentially convex.*

(ii) *H is continuous and for all $n \in \mathbb{N}$*

$$\sum_{i,j=1}^{n} \xi_i \xi_j H\Big(\frac{x_i + x_j}{2}\Big) f \geq 0, \quad f \in V, \tag{5.7}$$

where $\xi_i \in \mathbb{R}$ and $x_i \in I \subseteq \mathbb{R}$, $1 \leq i \leq n$.

Proof. $(i) \Rightarrow (ii)$

Take any $\xi_i \in \mathbb{R}$ and $x_i \in I$, $1 \leq i \leq n$. Since the interval $I \subseteq \mathbb{R}$ is convex, the midpoints, $\frac{x_i+x_j}{2} \in I$. Now set $y_i = \frac{x_i}{2}$, for $1 \leq i \leq n$. Then we have, $y_i + y_j = \frac{x_i+x_j}{2} \in I$, for all $1 \leq i, j \leq n$. Therefore, for all $n \in \mathbb{N}$, we can apply (i) to get,

$$\sum_{i,j=1}^{n} \xi_i \xi_j H(y_i + y_j) f = \sum_{i,j=1}^{n} \xi_i \xi_j H\big(\frac{x_i + x_j}{2}\big) f \geq 0, \quad f \in V.$$

$(ii) \Rightarrow (i)$

Let $\xi_i, x_i \in \mathbb{R}$, such that $x_i + x_j \in I$, for $1 \leq i, j \leq n$. Define $y_i = 2x_i$, so that $x_i + x_j = \frac{y_i+y_j}{2} \in I$. Therefore, for all $n \in \mathbb{N}$, we can apply (ii) to get,

$$\sum_{i,j=1}^{n} \xi_i \xi_j H\big(\frac{y_i + y_j}{2}\big) f = \sum_{i,j=1}^{n} \xi_i \xi_j H(x_i + x_j) f \geq 0, \quad f \in V.$$

□

Remark 5.3.3. Let H be an exponentially convex operator. Writing down the fact for $n = 1$, in (5.6), we get that $H(x)f \geq 0$, for $x \in I$ and $f \in V$. For $n = 2$, we have

$$\xi_1^2 H(x_1) f + 2\xi_1 \xi_2 H\big(\frac{x_1 + x_2}{2}\big) f + \xi_2^2 H(x_2) f \geq 0.$$

Hence, for $\xi_1 = -1$ and $\xi_2 = 1$, we have

$$H\big(\frac{x_1 + x_2}{2}\big) f \leq \frac{H(x_1)f + H(x_2)f}{2},$$

i.e. $H : I \to V$, does indeed satisfy the condition of convexity.

□

Let $\mathfrak{L}(V)$ denotes the space of all linear transformations from V into itself. For $U \subseteq V$, let us assume that $F : U \to V$ is continuously differentiable on U. i.e. the mapping $F' : U \to \mathfrak{L}(V)$, is continuous. Moreover $F''(f)$, will be a continuous linear transformation from V to $\mathfrak{L}(V)$. A bilinear transformation B defined on $V \times V$ is symmetric if $B(f, g) = B(g, f)$ for all $f, g \in V$. Such a transformation is **positive definite [nonnegative definite]**, if for every nonzero $f \in V$, $B(f, f) > 0$ $[B(f, f) \geq 0]$. Then, $F''(f)$ is symmetric wherever it exists. See [[70], p.69].

Theorem 5.3.4 ([70], p.100). *Let F be continuously differentiable and suppose that second derivative exists throughout an open convex set $U \subseteq V$. Then F is convex on U if and only if $F''(f)$ is nonnegative definite for each $f \in U$. And if $F''(f)$ is positive definite on U, then F is strictly convex.*

□

Definition 5.3.5. [[72]] Let V be a Banach algebra with unit e. For $f \in V$, we define a function $log(f)$ from V to V.

$$log(f) = -\sum_{n=1}^{\infty} \frac{(e-f)^n}{n} = -(e-f) - \frac{(e-f)^2}{2} - \frac{(e-f)^3}{3} - ...$$

for $||(e-x)|| \leq 1$.

Lemma 5.3.6. *Let V be a unital Banach algebra. For $f \in V$, a family of operators F_t is defined as*

$$F_t(f) = \begin{cases} \frac{f^t}{t(t-1)}, & t \neq 0, 1; \\ -\log f, & t = 0; \\ f \log f, & t = 1. \end{cases} \tag{5.8}$$

Then $D^2 F_t(f) := f^{t-2}$. Whenever, $f \in V_+$, $D^2 F_t(f) \in V_+$, therefore by Theorem 5.3.4, the mapping $f \to F_t(f)$ is convex.

Theorem 5.3.7. *Let $\{Z(t)\}_{t \geq 0}$ be the positive C_0-semigroup, defined on a unital Banach lattice algebra V, such that it satisfies (5.2). Let $f \in V$, such that $f^r \in V$, for $r \in I \setminus \{0, 1\} \subset \mathbb{R}$, $logf \in V$ if $r = 0$ and $flogf \in V$, if $r = 1$. Let us define*

$$\Lambda_t := Z(t)(F_t(f)) - F_t(Z(t)f) \tag{5.9}$$

Then

(i) *for every $n \in \mathbb{N}$ and for every $p_k \in I$, $k = 1, 2, ..., n$,*

$$\left[\Lambda_{\frac{p_i+p_j}{2}}\right]_{i,j=1}^{n} \geq 0. \tag{5.10}$$

(ii) *If the mapping $f \to \Lambda_t$ is continuous on I, then it is exponentially convex on I.*

Proof. Consider the operator

$$G(f) = \sum_{i,j=1}^{n} u_i u_j F_{p_{ij}}(f)$$

for $f > 0$, $u_i \in \mathbb{R}$ and $p_{ij} \in I$ where $p_{ij} = \frac{p_i+p_j}{2}$. Then

$$D^2G(f) := \sum_{i,j=1}^{n} u_i u_j f^{p_{ij}-2} = \big(\sum_{i=1}^{1} u_i f^{\frac{p_i}{2}-1}\big)^2 \geq 0, \quad f > 0.$$

So, $G(f)$ is a convex operator. Therefore by applying (5.3) we get

$$\sum_{i,j=1}^{n} u_i u_j \Lambda_{p_{ij}} \geq 0,$$

and the assertion (5.10) follows. Assuming the continuity and using the Proposition 5.3.2, we have also exponential convexity of the operator $f \to \Lambda_t$. □

Lemma 5.3.8. *Let V be a unital Banach algebra, for $f \in V$, let us define the following family of operators*

$$H_t(f) = \begin{cases} \frac{e^{tf}}{t^2}, & t \neq 0; \\ \\ \frac{f^2}{2}, & t = 0. \end{cases}$$

Then, $D^2H_t(f) = e^{tf}$. By Theorem 5.3.4, the mapping $f \to H_t(f)$, is convex on V.

Theorem 5.3.9. *For $\Lambda_t := Z(t)(H_t(f)) - H_t(Z(t)f)$, (i) and (ii) from Theorem 5.3.7, holds.*

Proof. Similar to the proof of Theorem 5.3.7. □

5.4 Applications of Jessen's Type Inequality

Now we present some consequences of the Jessen's type inequality for normalized positive C_0-semigroup defined on a Banach lattice algebra V [20]. The motivation for this paper is from [27], where such results are proved for isotonic linear functionals. These results take the form of Hölder's type, Minkowski's type and Dresher's type inequalities.

For a strongly continuous semigroup of linear operators $\{Z(t)\}_{t\geq 0}$ defined on a Banach lattice X and strictly monotonic continuous operator $\psi : X \to X$, we define the generalized mean:

$$M_{\psi}(Z, f, t) := \psi^{-1}\{Z(t)\psi(f)\}, \quad f \in X. \tag{5.11}$$

Theorem 5.4.1 ([20]). *For a normalized semigroup of positive linear operators $\{Z(t)\}_{t\geq 0}$ defined on (UBLA) V and strictly monotonic continuous operators $\psi, \chi : V \to V$*

$$M_{\psi}(Z, f, t) \leq M_{\chi}(Z, f, t), \quad f \in V, \tag{5.12}$$

provided either χ is increasing and $\phi = \chi \circ \psi^{-1}$ is convex or χ is decreasing and ϕ is concave.

Proof. For $f \in V$, we have $\psi(f), \chi(f) \in V$ and therefore, $\phi(\psi(f)) = \chi(f) \in V$. Thus, if ϕ is convex, by Jessen's type inequality (5.3) we have for $f \in V$;

$$\begin{aligned} \phi(Z(t)(\psi(f))) &\leq Z(t)(\phi(\psi(f))) \\ &= Z(t)(\chi(f)). \end{aligned}$$

Hence, if χ is increasing then χ^{-1} is also increasing and we finally obtain

$$\chi^{-1}[\phi(Z(t)(\psi(f)))] \leq \chi^{-1}[Z(t)(\chi(f))]$$

and the assertion (5.12) follows. If ϕ is concave then $-\phi$ is convex and one can obtain the required inequality similarly. □

In correspondence with the usual definition of generalized power means for isotonic functionals [5], we shall define the generalized power means for semigroup of operators, as follows.

Definition 5.4.2. Let X be a Banach space and $\{Z(t)\}_{t\in\mathbb{R}}$ the C_0-semigroup of linear operators on X. For $f \in X$ and $t \in \mathbb{R}_+$, the genralized power mean is defined as;

$$M_{G_r}(Z,f,t) = \begin{cases} \Big(Z(t)[f^r]\Big)^{1/r}, & r \neq 0 \\ \\ exp[Z(t)[log(f)]], & r = 0. \end{cases} \tag{5.13}$$

□

As an application of Theorem 5.4.1, we obtain the following theorem as a special case.

Theorem 5.4.3. *Let V be a unital Banach lattice algebra and $\{Z(t)\}_{t\in\mathbb{R}}$ the C_0-semigroup of positive linear operators on V. For $f \in V_+$ and $t \in \mathbb{R}_+$, we have;*

$$M_{G_r}(Z,f,t) \leq M_{G_s}(Z,f,t), \quad -\infty \leq r \leq s \leq \infty. \tag{5.14}$$

Proof. Consider the following function;

$$G_r(f) = \begin{cases} f^r, & r \neq 0 \\ log(f), & r = 0. \end{cases} \tag{5.15}$$

By setting;

$$\psi = G_r, \quad and \quad \chi = G_s, \quad -\infty \leq r \leq s \leq \infty.$$

in Theorem 5.4.1. The assertion (5.14) follows, since

$$\phi(f) = \chi \circ \psi(f) = f^{s/r}, \quad -\infty < 0 \neq r \leq s \neq 0 < \infty$$

is convex by Theorem 5.3.4. □

5.4.1 Hölder's Type Inequality

Furthermore, our aim is to obtain an expression for the Hölder's type inequality for the strongly continuous semigroup of positive operators defined on a Banach lattice algebra.

Lemma 5.4.4. *Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup defined on V such that it satisfies (5.2). For a convex operator $\phi : V \to V$ and $t \geq 0$, we have;*

$$\phi\Big[\frac{[Z(t)[f_1 h_1]]}{Z(t)[f_1]}\Big] \leq \frac{Z(t)[f_1\phi[h_1]]}{Z(t)[f_1]}, \quad f_1, h_1 \in V_+. \tag{5.16}$$

Proof. For $f \in V_+$ we have $\phi[f] \in V$. Since V is a lattice algebra, $f, k \in V_+$ implies $fk \in V_+$, therefore the set of operators defined by;

$$F_k(t) := \frac{Z(t)[fk]}{Z(t)[f]}, \quad f \in V_+, t \geq 0,$$

is a semigroups of positive linear operators satisfying $F_k(t)[e] = e$. Thus the assertion (5.16) follows from (5.3). □

One can observe that when r is any integer (positive or negative), the C_0-semigroup property implies that $Z(t)^r = Z(rt)$. While we can generalize it for $r \in \mathbb{R}_+$. For example take $Z(1/2t)Z(1/2t) = Z(t)$ and thus we get $Z(t)^{1/2} = Z(1/2t)$. For $r \in \mathbb{R}_+$, the generator of $\{Z(rt)\}_{t\geq 0}$ is $(rA, D(A))$. Such semigroups are often called *rescaled semigroups.* (See e.g. [64]).

The above argument motivate us next to prove a Hölder's type inequality for positive C_0-semigroup of operators, assuming the fractional powers of elements in Banach algebra exist.

Theorem 5.4.5 (**Hölder's Type Inequality For C_0-semigroups**). *Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup defined on V. If $p > 1$ and $q = \frac{p}{p-1}$ so $p^{-1} + q^{-1} = 1$, then if $f, g, h \in V_+$ and $fg^p, fh^q, fgh \in V_+$, we have for $t \geq 0$;*

$$Z(t)[fgh] \leq [Z(t)]^{1/p}[fg^p].[Z(t)]^{1/q}[fh^q] \tag{5.17}$$

Proof. Since $fh^q \in V_+$, we have for $t \geq 0$, $Z(t)[fh^q] \in V_+$. For $p > 1$, by doing the following substitution in (5.16);

$$\phi(x) = x^p, \quad h_1 = gh^{-q/p}, \quad f_1 = fh^q.$$

We have;

$$\begin{aligned} \Big[\frac{Z(t)[fh^q gh^{-q/p}]}{Z(t)[fh^q]}\Big]^p &\leq \frac{Z(t)[fh^q[gh^{-q/p}]^p]}{Z(t)[fh^q]} \\ \Rightarrow \qquad \frac{[Z(t)[fgh]]^p}{[Z(t)[fh^q]]^p} &\leq [Z(t)[fg^p]][Z(t)[fh^q]]^{-1} \\ \Rightarrow \qquad Z(t)[fgh] &\leq [[Z(t)][fg^p]]^{1/p}[Z(t)[fh^q]]^{\frac{1}{p}-1} \end{aligned}$$

The assertion (5.17) follows directly. □

5.4.2 Minkowski's Type Inequality

The theory developed in the previous section, readily leads us to another result, the Minkowski's type inequality.

Theorem 5.4.6 (Minkowski's Type Inequality For C_0-semigroups). *Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup defined on V. If $p > 1$ and $f, g, h \in V_+$ such that $hf^p, hg^p, h(f+g)^p \in V_+$, then;*

$$Z(t)[h(f+g)^p] \leq Z(t)^{1/p}[hf^p] + Z(t)^{1/p}[hg^p], \quad f \geq 0. \tag{5.18}$$

Proof. For $f, g, h \in V_+$ and $p > 1$, we have

$$h(f+g)^p = hf(f+g)^{p-1} + hg(f+g)^{p-1}$$

The assertion (5.18) follows by using (5.17). □

5.4.3 Dresher's Type Inequality

In the flow of defining classical-type inequalities for C_0-semigroup of operators on a Banach lattice, we ultimately reach the final result of this chapter called the Dresher's type inequality. For this purpose, we firstly introduce two-parameter family of means in the following way.

Definition 5.4.7. Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous semigroup defined on a Banach algebra X. Then the two-parameter family of means $B_{r,s}(Z, f, t)$ for $r, s \in \mathbb{R}$ is defined by;

$$B_{r,s} = \begin{cases} \left[\frac{Z(t)[f^r]}{Z(t)[f^s]}\right]^{\frac{1}{r-s}}, & r \neq s \\ \exp\left[\frac{Z(t)[f^r logf]}{Z(t)[f^r]}\right], & r = s \end{cases} \tag{5.19}$$

Remark 5.4.8.

- Hölder's inequality for a strongly continuous semigroup defined on a Banach lattice algebra, can also be stated in the following way. Let $\{Z(t)\}_{t\geq 0}$ be the positive normalized C_0-semigroup defined on V. If $0 < \lambda < 1$ and $g, h, g^\lambda, h^{1-\lambda} \in V_+$, we have for $t \geq 0$;

  $$Z(t)[g^\lambda h^{1-\lambda}] \leq Z(t)[g^\lambda] Z(t)[h^{1-\lambda}] \tag{5.20}$$

  This result can be obtained by applying Theorem 5.4.5 for $f = f_1^{1/\lambda}$ and $g = g_1^{1/1-\lambda}$.

- If $f^r \in V_+$ for all $r \in \mathbb{R}_+$, then the assertion (5.20) yields the following expression;

$$Z(t)[f^{\lambda r+(1-\lambda)s}] \leq Z(t)[f^r]^{\lambda} Z(t)[g^s]^{1-\lambda}, \quad 0 < \lambda < 1 \tag{5.21}$$

- For $\phi(x) = \log Z(t)[f^x]$, the convexity follows from (5.21).

Theorem 5.4.9 (Dresher's Type Inequality). *Let $\{Z(t)\}_{t\geq 0}$ be a positive C_0-semigroup defined on a Banach lattice algebra V. Then for $f \in V_+$ and $p, q, r, s \in \mathbb{R}$, we have;*

$$B_{r,s}(Z, f, t) \leq B_{p,q}(Z, f, t) \quad r \leq p,\ s \leq q \quad \text{and} \quad r \neq s,\ p \neq q. \tag{5.22}$$

Proof. Let $p, q, r, s \in \mathbb{R}$ such that $r \leq p,\ s \leq q$ and $r \neq s,\ p \neq q$. When applying the known result for convex functions

$$\frac{\phi(r) - \phi(s)}{r - s} \leq \frac{\phi(p) - \phi(q)}{p - q}, \tag{5.23}$$

to the convex operator $\phi(x) = \log Z(t)[f^x]$, we can obtain (5.22).

We now show that (5.22) holds even if $r = s$ or $p = q$. To prove this we use the fact that $M_{G_r}(Z, f, t)$ is increasing function of $r \in \mathbb{R}$. In particular for $f \in V_+$;

$$(Z(t)[f^{s-r}])^{\frac{1}{s-r}} \leq \exp[Z(t)logf] \leq (Z(t)[f^{r-s}])^{\frac{1}{r-s}}, \quad s < r. \tag{5.24}$$

Apply (5.24) to the positive semigroup (see Lemma 5.4.4) $Z_m(t)g := \frac{Z(t)[f^m g]}{Z(t)[f^m]}$. By taking $m = s$ the right-hand inequality (5.24) reduces to

$$B_{s,s}(Z, f, t) \leq B_{r,s}(Z, f, t), \quad s < r.$$

Similarly, by taking $m = r$ the left-hand inequality of (5.24) reduces to

$$B_{r,s}(Z, f, t) \leq B_{r,r}(Z, f, t), \quad s < r.$$

By these two inequalities we conclude that the inequality (5.22) holds for $r = s$ or $p = q$. □

Chapter 6

Superquadratic Mappings and Cauchy's Type Means For Positive C_0-Semigroups*

The notion of superquadratic functions was introduced in [1]. Some basic properties of this class of functions were given in the same paper (see also [3]). Since then, it has been of high interest to develop and generalize the theory of inequalities for superquadratic functions [2, 25, 50–52].

In the running chapter, we intent to introduce the concept of superquadratic operators analogously and to develop the "type" expressions for two of very famous classical inequalities. Where the major aim is to obtain the new set of Cauchy's type means.

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*This chapter is based on the following publications:

1. Gul I Hina Aslam, Matloob Anwar, *About Jensen's Inequality and Cauchy's Type Means for Positive C_0-Semigroups*, Journal of Semigroup Theory and Applications. 2015:6 ,2015.

2. Gul I Hina Aslam, Matloob Anwar, *About Hermite-Hadamard Inequalities and Cauchy's Type Means For Positive C_0-Semigroups*, (Submitted).

6.1 Jensen's Type Inequality and Corresponding Means

The Jensen's type inequality for superquadratic function on isotonic linear functionals, is given in [[26], Theorem 10]. In [5], the corresponding Cauchy type means are defined.
In what follows, we firstly prove the Jensen's type inequality for semigroup of positive linear operators defined on a Banach lattice algebra. Result will be followed by some generalized mean value theorems, bringing in a new set of Cauchy type means.

Definition 6.1.1. Let V be a Banach lattice algebra. A mapping $\phi : V_+ \to V$ is superquadratic, provided that for all $v \geq 0$ there exists a constant vector $C(v)$ such that

$$\phi(u) - \phi(v) - \phi(|u-v|) \geq C(v)(u-v) \tag{6.1}$$

for all $u \geq 0$. We say that the mapping ϕ is *subquadratic* if $-\phi$ is superquadratic.

Theorem 6.1.2. *Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on a Banach lattice algebra V. Then for $g \in V_+$ and the continuous superquadratic mapping $\phi : V_+ \to V$, we have;*

$$\phi\Big([Z(t)g]^{-1}[Z(t)(gf)]\Big) \leq \frac{Z(t)[g\phi(f)] - Z(t)\Big[g\phi\Big(\Big|f - [Z(t)g]^{-1}[Z(t)(gf)]\Big|\Big)\Big]}{[Z(t)g]}, \quad f \in V_+. \tag{6.2}$$

If ϕ is subquadratic then a reversed inequality in (6.2) holds.

Proof. Since the mapping ϕ is superquadratic, inequality (6.1) holds for all $u, v \geq 0$. As $f, g \geq 0$ and the operator $Z(t)$ is positive for all $t \geq 0$, we have $[Z(t)g]^{-1}[Z(t)(gf)] \geq 0$. Setting $u = f$ and $v = [Z(t)g]^{-1}[Z(t)(gf)]$ in (6.1), we obtain;

$$\begin{aligned} \phi(f) \;\geq\; & \phi\Big([Z(t)g]^{-1}[Z(t)(gf)]\Big) + C\Big[[Z(t)g]^{-1}[Z(t)(gf)]\Big]\Big[f - [Z(t)g]^{-1}[Z(t)(gf)]\Big] \\ & + \phi\Big(\Big|f - [Z(t)g]^{-1}[Z(t)(gf)]\Big|\Big), \end{aligned}$$

for all $t \geq 0$. Multiplying the above inequality by $g \in V_+$, we get

$$\begin{aligned} g\phi(f) \;\geq\; & g\phi\Big([Z(t)g]^{-1}[Z(t)(gf)]\Big) + C\Big[[Z(t)g]^{-1}[Z(t)(gf)]\Big]\Big[gf - g[Z(t)g]^{-1}[Z(t)(gf)]\Big] \\ & + g\phi\Big(\Big|f - [Z(t)f]^{-1}[Z(t)(gf)]\Big|\Big). \end{aligned}$$

By applying the operator $Z(t)$ on both sides, we get for all $t \geq 0$;

$$\begin{aligned} Z(t)[g\phi(f)] \geq\ & Z(t)[g]\phi\Big([Z(t)g]^{-1}[Z(t)(gf)]\Big) \\ & +C\Big[[Z(t)g]^{-1}[Z(t)(gf)]\Big]\Big[Z(t)(gf) - Z(t)[g][Z(t)g]^{-1}[Z(t)(gf)]\Big] \\ & +Z(t)\Big[g\phi\Big(\Big|f - [Z(t)f]^{-1}[Z(t)(gf)]\Big|\Big)\Big]. \end{aligned}$$

The assertion (6.2) follows directly. □

Throughout the remaining section, V shall denote the (real) unital Banach lattice algebra with identity element e, until and unless stated otherwise.

Theorem 6.1.3 ([21]). *Let $\{Z(t)\}_{t\geq 0}$ be a normalized strongly continuous positive semigroup of operators defined on V; then for a continuous superquadratic operator $\phi : V_+ \to V$, we have*

$$\phi[Z(t)f] \leq Z(t)[\phi(f)] - Z(t)[\phi(|f - Z(t)f|)], \quad f \in V_+. \tag{6.3}$$

If the mapping ϕ is subquadratic, then the inequality above is reversed.

Proof. Since $\{Z(t)\}_{t\geq 0}$ is a normalized semigroup it must satisfy (5.2). By taking $g \equiv e$ in Theorem 6.1.2, we obtain (6.3). □

Definition 6.1.4. Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous normalized positive semigroup of operators defined on V; then for a continuous operator $\phi : V_+ \to V$, we define an other operator $\Lambda_\phi : V_+ \to V$;

$$\Lambda_\phi := Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)], \quad f \in V_+. \tag{6.4}$$

If ϕ is continuous superquadratic mapping then, $\Lambda_\phi \geq 0$.

□

Below we give an operator analogue of ([1], Lemma 3.1).

Lemma 6.1.5. *Suppose $\phi : V_+ \to V$ is continuously differentiable and $\phi(0) \leq 0$. If ϕ' is superadditive or $f \to \frac{\phi'(f)}{f}, \quad f \in V_+$, is increasing, then ϕ is superquadratic.*

□

Lemma 6.1.6. *Let $\phi \in C^2[V_+]$ and $u, U \in V$ be such that*

$$u \leq \Big(\frac{\phi'(f)}{f}\Big)' = \frac{f\phi''(f) - \phi'(f)}{f^2} \leq U, \quad \forall f > 0. \tag{6.5}$$

Consider the operators $\phi_1, \phi_2 : V_+ \to V$ defined as:

$$\phi_1(f) = \frac{Uf^3}{3} - \phi(f), \quad \phi_2 = \phi(f) - \frac{uf^3}{3}$$

Then the mappings $f \to \frac{\phi_1'(f)}{f}$ and $f \to \frac{\phi_2'(f)}{f}$ are increasing. If also $\phi_i(0) = 0, i = 1, 2$, then these are superquadratic mappings.

Proof. By using the inequality (6.3), it can be easily seen that the mappings $f \to \frac{\phi_1'(f)}{f}$ and $f \to \frac{\phi_2'(f)}{f}$ are increasing. Moreover, if $\phi_i(0) = 0, i = 1, 2$, Lemma 6.1.5 implies these to be superquadratic. □

Theorem 6.1.7. *Let $\{Z(t)\}_{t\geq 0}$ be a positive normalized C_0-semigroup of operators defined on V and $\frac{\phi'}{f} \in C^1(V_+)$ and $\phi(0) = 0$, then the following inequality holds*

$$\Lambda_\phi = \frac{\xi\phi''(\xi) - \phi'(\xi)}{3\xi^2}\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)(|f - Z(t)f|^3)\}, \quad f \in V_+. \tag{6.6}$$

Proof. Suppose that $u = min_{f\in V_+}(\frac{\phi'(f)}{f})'$ and $U = max_{f\in V_+}(\frac{\phi'(f)}{f})$ exists. Taking ϕ_1 instead of ϕ in (6.3), we get for $f \in V_+$;

$$Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)] \leq \frac{U}{3}\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\}.$$

Similarly, by taking ϕ_2 instead of ϕ in (6.3), we get for $f \in V_+$;

$$Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)] \geq \frac{u}{3}\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\}.$$

Since, $\phi = f^3$ is superquadratic and $Z(t) \in \{Z(t)\}_{t\geq 0}$ is the positive operator, therefore

$$Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|] \geq 0, \quad f \in V_+.$$

By combining the above two inequalities and using (6.5), we obtain that, there exists $\xi \in V_+$, such that the assertion (6.6) holds. □

Theorem 6.1.8. *Let $\{Z(t)\}_{t\geq 0}$ be a positive normalized C_0-semigroup of operators defined on V and $\frac{\phi'}{f}, \frac{\psi'}{f} \in C^1(V_+)$ such that, $\phi(0) = \psi(0) = 0$, we have*

$$\frac{\Lambda_\phi}{\Lambda_\psi} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = K(\xi), \quad \xi \in V_+, \tag{6.7}$$

provided the denominators do not vanish. If K is invertible, we have the following new mean;

$$\xi = K^{-1}\Big(\frac{\Lambda_\phi}{\Lambda_\psi}\Big), \quad \Lambda_\psi \neq 0, \tag{6.8}$$

Proof. Lets consider a function $\Omega = c_1\phi - c_2\psi$, where

$$c_1 = \Lambda_\psi, \quad c_2 = \Lambda_\phi.$$

Then for $f \in V_+$;

$$\frac{\Omega'}{f} = c_1\frac{\phi'}{f} - c_2\frac{\psi'}{f} \in C^1(V_+).$$

One may calculate that $\Lambda_\Omega = 0$ and using Lemma 6.1.6 with $\phi = \Omega$ we obtain;

$$[c_1(\xi\phi''(\xi) - \phi'(\xi)) - c_2(\xi\psi''(\xi) - \psi'(\xi))]\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\} = 0, \quad f \in V_+.$$

Since $\phi = f^3$ is superquadratic and $\{Z(t)\}_{t\geq 0}$ is semigroup of positive operators, therefore we may conclude that

$$\frac{c_2}{c_1} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \frac{\Lambda_\phi}{\Lambda_\psi}, \quad \xi \in V_+,$$

providing the denominator do not vanish. This completes the proof. □

We shall denote the set of all invertible strictly monotone continuous operators, defined from V to itself, by $G_M(V)$.

Definition 6.1.9. For a positive normalized C_0-semigroup $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V and $F \in G_M(V)$, we define the generalized mean:

$$M_F(Z, f, t) := F^{-1}\{Z(t)F(f)\}, \quad f \in X. \tag{6.9}$$

For the sake of simplicity, the set of all elements of G_M, whose second order derivative (in Gateaux's sense) exits, shall be denoted by $C^2G_M(V)$.

Theorem 6.1.10. *Let $\{Z(t)\}_{t\geq 0}$ be a positive normalized C_0-semigroup defined on V and $H, F, K \in C^2G_M(V)$. Let for $f \in V_+$, $\frac{H\circ F^{-1}(f)}{f}, \frac{K\circ F^{-1}(f)}{f} \in C^1(V)$ with $H \circ F^{-1}(0) = 0 = K \circ F^{-1}(0)$, then for $f \in V_+$ and $t \geq 0$;*

$$\begin{aligned}&\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(\tau)f] - FM_F(Z,f,t)|, f, t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(\tau)f] - FM_F(Z,f,t)|, f, t))}\\ = \;&\frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}},\end{aligned} \tag{6.10}$$

holds for some $\eta \in V_+$, provided the denominator do not vanish.

Proof. By choosing the operators ϕ and ψ in Theorem 6.1.8, such that

$$\phi = H \circ F^{-1}, \quad \psi = K \circ F^{-1} \quad \textit{and} \quad Z(t)f = F[Z(t)f], \quad f \in V_+,$$

where $H, F, K \in C^2G_M(V)$. We find that there exists $\xi \in V_+$, such that

$$\begin{aligned}&\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f, t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f, t))}\\ = \;&\frac{\xi\{H''(F^{-1}\xi)F'(F^{-1}\xi) - H'(F^{-1}\xi)F''(F^{-1}\xi) - H'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\}}{\xi\Big\{K''(F^{-1}\xi)F'(F^{-1}\xi) - K'(F^{-1}\xi)F''(F^{-1}\xi) - K'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\Big\}}.\end{aligned}$$

Therefore, by setting $F^{-1}(\xi) = \eta$, we find that there exists $\eta \in X$, such that the assertion (6.10) follows directly. □

The above theorem accredit us to define new means. Set

$$L(\eta) = \frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}},$$

and when $F \in G(V)$;

$$\eta = L^{-1}\Big(\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f, t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FMF(Z,f,t)|, f, t))}\Big)$$

Remark 6.1.11. For $(V, \|.\|)$ a Banach lattice algebra, it follows from Theorem 6.1.10 that

$$m \leq \Big\|\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f, t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f, t))}\Big\| \leq M,$$

Where m and M are respectively, the minimum and maximum values of

$$\Big\|\frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}}\Big\|, \quad \eta \in V.$$

For a Banach algebra V with the unit element e and $f \in V$, the function $log(f)$ from V to V has been stated in Definition 5.3.5. While in correspondence with the usual definition of generalized power means for isotonic functionals [5], the generalized power means for semigroup of operators has been defined by 5.13.

Now, we prove an important result which is going to lead us, to define the Cauchy's type means on C_0-semigroup of operators.

Corollary 6.1.12. *Let all the conditions of Theorem 6.1.10 are satisfied. For $r, s, l \in \mathbb{R}_+$ such that $r \neq l; l \neq 2s$, we have*

$$\frac{M^r_{G_r}(Z,f,t) - M^r_{G_s}(Z,f,t) - M^r_{G_r}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}}, f, t)}{M^l_{G_l}(Z,f,t) - M^l_{G_s}(Z,f,t) - M^l_{G_l}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}}, f, t)} = \frac{r(r-2s)}{l(l-2s)}\eta^{r-l} \tag{6.11}$$

The assertion (6.11) holds for some η, provided that the denominators do not vanish.

Proof. For $r, s, l \in \mathbb{R}_+$ and $f \in V_+$, if we set

$$H(f) = f^r, \quad F(f) = f^s, \quad K(f) = f^l$$

in Theorem 6.1.10, the assertion in (6.11) follows directly. □

Ultimately, we define means of the Cauchy's type on C_0-semigroup of positive linear operators defined on Banach lattice algebra V.

Definition 6.1.13 ([21]). Let $r, s, l \in \mathbb{R}_+$ and $\{Z(t)\}_{t\geq 0} \subset B(V)$ be a normalized C_0-semigroup of positive linear operators on a unital Banach lattice algebra V. Then

$$\mathfrak{M}^{l,s}_{G_r}(Z,f,t) = \Big(\frac{l(l-2s)}{r(r-2s)} \frac{M^r_{G_r}(Z,f,t) - M^r_{G_s}(Z,f,t) - M^r_{G_r}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}}, f, t)}{M^l_{G_l}(Z,f,t) - M^l_{G_s}(Z,f,t) - M^l_{G_l}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}}, f, t)}\Big)^{\frac{1}{r-l}}.$$

is a mean of the Cauchy's type. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases, as in [17].

6.2 Hermite-Hadamard Type Inequality and Corresponding Means

The Hermite-Hadamard type inequality for positive linear functionals is proved in [26]. In [4], the corresponding mean value theorems are given, which ultimately lead to define new means of Cauchy's type.

In the present section, we prove the Hermite-Hadamard type inequality for semigroup of positive linear operators defined on a Banach lattice algebra. We also prove some generalized mean value theorems and define related Cauchy's type means.

Theorem 6.2.1 ([22]). *Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on V; then for an integrable superquadratic operator $\phi : V_+ \to V$, we have*

$$\phi\Big[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big] + \frac{1}{t}\int_0^t \phi\Big[\Big|[Z(\tau)]f - \frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big|\Big]d\tau \leq \frac{1}{t}\int_0^t \phi[Z(\tau)]fd\tau, \quad f \in V_+. \tag{6.12}$$

Proof. Let ϕ be a superquadratic mapping, then (6.1) holds for all $u, v \in V_+$. Choosing $u = [Z(\tau)]f$ and $v = \frac{1}{t}\int_0^t [Z(\tau)]fd\tau$ in (6.1) we get

$$\begin{aligned}\phi[[Z(\tau)]f] \;\; \geq \;\; & \phi\Big[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big] + C\Big[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big]\Big[[Z(\tau)]f \\ & -\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big] + \phi\Big[\Big|[Z(\tau)]f - \frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big|\Big]\end{aligned}$$

By integrating from $0 \to t$ we obtain;

$$\begin{aligned}\int_0^t \phi[[Z(\tau)]f]d\tau \;\; \geq \;\; & t.\phi\Big[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big] + C\Big[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big]\Big[\int_0^t [Z(\tau)]fd\tau - t\Big\{\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big\}\Big] \\ & + \int_0^t \phi\Big[\Big|[Z(\tau)]f - \frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big|\Big]d\tau,\end{aligned}$$

or

$$\int_0^t \phi[[Z(\tau)]f]d\tau \geq t.\phi\Big[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big] + \int_0^t \phi\Big[\Big|[Z(\tau)]f - \frac{1}{t}\int_0^t [Z(\tau)]fd\tau\Big|\Big]d\tau.$$

Multiplication by $1/t$, finally yields the assertion (6.12). $\square$

Definition 6.2.2. Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on V; then for an integrable operator $\phi : V_+ \to V$, we define an other operator $\Lambda_\phi : V_+ \to V$

$$\Lambda_\phi := \frac{1}{t}\int_0^t \phi[Z(\tau)]f d\tau - \phi\Big[\frac{1}{t}\int_0^t [Z(\tau)]f d\tau\Big] - \frac{1}{t}\int_0^t \phi\Big[\Big|[Z(\tau)]f - \frac{1}{t}\int_0^t [Z(\tau)]f d\tau\Big|\Big]d\tau, \quad f \in V_+. \tag{6.13}$$

If ϕ is continuous superquadratic mapping then by (6.12), $\Lambda_\phi \geq 0$.

For the sake of simplicity of expressions throughout the article, we denote $\frac{1}{t}\int_0^t [Z(\tau)]f d\tau$ by $M_1(t)$. Therefore, Λ_ϕ can be written as;

$$\Lambda_\phi := \frac{1}{t}\int_0^t \phi[Z(\tau)]f d\tau - \phi[M_1(t)] - \frac{1}{t}\int_0^t \phi\Big[\Big|[Z(\tau)]f - M_1(t)\Big|\Big]d\tau$$

Theorem 6.2.3. *Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on V and $\frac{\phi'}{f} \in C^1(V_+)$ and $\phi(0) = 0$, then the following inequality holds*

$$\Lambda_\phi = \frac{\xi\phi''(\xi) - \phi'(\xi)}{3\xi^2}\{(M_3(t))^3 - (M_1(t))^3 - \frac{1}{t}\int_0^t |Z(\tau)f - m_t|^3 d\tau\} \tag{6.14}$$

Proof. Suppose the conditions in Lemma 6.1.6 holds for all $f \in V_+$. Using ϕ_1 instead of ϕ in (6.12), we get;

$$\begin{aligned}\frac{1}{t}\int_0^t \phi[Z(\tau)]f d\tau - \phi[M_1(t)] - \frac{1}{t}\int_0^t \phi\Big[\Big|[Z(\tau)]f - M_1(t)\Big|\Big]d\tau \;&\leq\; \frac{U}{3}\{(M_3(t))^3 - (M_1(t))^3 \\ &\quad -\frac{1}{t}\int_0^t |Z(\tau)f - m_t|^3 d\tau\}\end{aligned}$$

Similarly, using ϕ_2 instead of ϕ in (6.12), we get;

$$\begin{aligned}\frac{1}{t}\int_0^t \phi[Z(\tau)]f d\tau - \phi[M_1(t)] - \frac{1}{t}\int_0^t \phi\Big[\Big|[Z(\tau)]f - M_1(t)\Big|\Big]d\tau \;&\geq\; \frac{u}{3}\{(M_3(t))^3 - (M_1(t))^3 \\ &\quad -\frac{1}{t}\int_0^t |Z(\tau)f - m_t|^3 d\tau\}\end{aligned}$$

By combining the above two inequalities and using intermediate value theorem [49], we have existence of $\xi \in V_+$ such that (6.14) holds. □

Theorem 6.2.4. *Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on V and $\frac{\phi'}{f}, \frac{\psi'}{f} \in C^1(V_+)$ such that, $\phi(0) = \psi(0) = 0$, we have*

$$\frac{\Lambda_\phi}{\Lambda_\psi} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = K(\xi), \quad \xi \in V_+, \tag{6.15}$$

provided the denominators do not vanish. If K is invertible, we have the following new mean;

$$\xi = K^{-1}\Big(\frac{\Lambda_\phi}{\Lambda_\psi}\Big), \quad \Lambda_\psi \neq 0, \tag{6.16}$$

Proof. Consider a function $\Omega = c_1\phi - c_2\psi$, where

$$c_1 = \Lambda_\psi, \quad c_2 = \Lambda_\phi.$$

Then for $f \in V_+$;

$$\frac{\Omega'}{f} = c_1\frac{\phi'}{f} - c_2\frac{\psi'}{f} \in C^1(V_+).$$

One may calculate that $\Lambda_\Omega = 0$ and using Lemma 6.1.6 with $\phi = \Omega$ we obtain;

$$[c_1(\xi\phi''(\xi)-\phi'(\xi))-c_2(\xi\psi''(\xi)-\psi'(\xi))]\Big[(M_3(t))^3-(M_1(t))^3-\frac{1}{t}\int_0^t |Z(\tau)f-m_t|^3 d\tau\Big] = 0, \quad f \in V_+.$$

Since $\phi = f^3$ is superquadratic mapping and $\{Z(t)\}_{t\geq 0}$ is semigroup of positive operators, therefore we conclude that

$$\frac{c_2}{c_1} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \frac{\Lambda_\phi}{\Lambda_\psi}, \quad \xi \in V_+,$$

providing the denominator do not vanish. This completes the proof. □

Next, let G be the set of invertible bounded linear operators from a Banach lattice algebra V to itself. For $\{Z(t)\}_{t\geq 0} \subset B(V)$ a C_0-semigroup of positive operators defined on V and $H \in G$, the quasi-arithmetic mean is defined as [18];

$$M_H^\circ(Z,f,t) = H^{-1}\Big\{\frac{1}{t}\int_0^t H[Z(\tau)f]d\tau\Big\}, \quad f \in V_+, t \geq 0. \tag{6.17}$$

By ([62], Lemma 1.85), $B(V)$ is closed under composition of operators so the above expressions exists and belongs to V. For the sake of simplicity, the set of all elements of G, whose second order derivative (in Gateaux's sense) exits, is denoted by $C^2G(V)$.

Theorem 6.2.5. *Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on V and $H, F, K \in C^2G(V)$. Let for $f \in V_+, \frac{H\circ F^{-1}(f)}{f}, \frac{K\circ F^{-1}(f)}{f} \in C^1(V)$ with $H \circ F^{-1}(0) = 0 = K \circ F^{-1}(0)$, then for $f \in V_+$*

$$\begin{aligned} &\frac{H(M_H^\circ(Z,f,t)) - H(M_F^\circ(Z,f,t)) - H(M_H^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|, f, t))}{K(M_H^\circ(Z,f,t)) - K(M_F^\circ(Z,f,t)) - K(M_K^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|, f, t))} \\ = \; &\frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}}, \end{aligned} \tag{6.18}$$

holds for some $\eta \in V_+$, provided the denominator do not vanish.

Proof. By choosing the operators ϕ and ψ in Theorem 6.2.4, such that

$$\phi = H \circ F^{-1}, \quad \psi = K \circ F^{-1} \quad and \quad Z(\tau)f = F[Z(\tau)f], \quad f \in V_+,$$

where $H, F, K \in C^2G(V)$. We find that there exists $\xi \in V_+$, such that

$$\frac{H(M_H^\circ(Z,f,t)) - H(M_F^\circ(Z,f,t)) - H(M_H^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|,f,t))}{K(M_H^\circ(Z,f,t)) - K(M_F^\circ(Z,f,t)) - K(M_K^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|,f,t))}$$
$$= \frac{\xi\{H''(F^{-1}\xi)F'(F^{-1}\xi) - H'(F^{-1}\xi)F''(F^{-1}\xi) - H'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\}}{\xi\Big\{K''(F^{-1}\xi)F'(F^{-1}\xi) - K'(F^{-1}\xi)F''(F^{-1}\xi) - K'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\Big\}},$$

Therefore, by setting $F^{-1}(\xi) = \eta$, for some $\eta \in X$, such that the assertion (6.18) follows directly. □

The above theorem suggests us to define new means. Set

$$L(\eta) = \frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}},$$

and when $F \in G(V)$;

$$\eta = L^{-1}\Big(\frac{H(M_H^\circ(Z,f,t)) - H(M_F^\circ(Z,f,t)) - H(M_H^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|,f,t))}{K(M_H^\circ(Z,f,t)) - K(M_F^\circ(Z,f,t)) - K(M_K^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|,f,t))}\Big)$$

Remark 6.2.6. For $(V, \|.\|)$ a Banach lattice algebra, it follows from Theorem 6.2.5 that

$$m \leq \Big\|\frac{H(M_H^\circ(Z,f,t)) - H(M_F^\circ(Z,f,t)) - H(M_H^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|,f,t))}{K(M_H^\circ(Z,f,t)) - K(M_F^\circ(Z,f,t)) - K(M_K^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(Z,f,t)|,f,t))}\Big\| \leq M,$$

Where m and M are respectively, the minimum and maximum values of

$$\Big\|\frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}}\Big\|, \quad \eta \in V.$$

Now, we prove an important result which lead us to define the Cauchy's type means on C_0-group of operators.

Corollary 6.2.7. *Let all the conditions of Theorem 6.2.5 are satisfied. For $r, s, l \in \mathbb{R}_+$ such that $r \neq l; l \neq 2s$, we have*

$$\frac{M_r^r(Z,f,t) - M_s^r(Z,f,t) - M_r^r(|[Z(\tau)f]^s - M_s^s(Z,f,t)|^{\frac{1}{s}},f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t) - M_l^l(|[Z(\tau)f]^s - M_s^s(Z,f,t)|^{\frac{1}{s}},f,t)} = \frac{r(r-2s)}{l(l-2s)}\eta^{r-l} \tag{6.19}$$

where $M_r(Z,f,t)$ is defined by (4.2). The assertion (6.19) holds for some η, provided that the denominators do not vanish.

Proof. For $r, s, l \in \mathbb{R}_+$ and $f \in V_+$, if we set

$$H(f) = f^r, \quad F(f) = f^s, \quad K(f) = f^l$$

in Theorem (6.2.5), the assertion in (6.19) follows directly. □

Next, we define means of the Cauchy's type on C_0-semigroup of positive linear operators defined on Banach lattice algebra V.

Definition 6.2.8 ([22]). Let $r, s, l \in \mathbb{R}_+$ and $\{Z(t)\}_{t\geq 0} \subset B(V)$ be a C_0-semigroup of positive operators on a Banach lattice algebra V. Then

$$\mathfrak{M}_r^{l,s}(Z,f,t) = \Big(\frac{l(l-2s)}{r(r-2s)}\frac{M_r^r(Z,f,t) - M_s^r(Z,f,t) - M_r^r(|[Z(\tau)f]^s - M_s^s(Z,f,t)|^{\frac{1}{s}}, f, t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t) - M_l^l(|[Z(\tau)f]^s - M_s^s(Z,f,t)|^{\frac{1}{s}}, f, t)}\Big)^{\frac{1}{r-l}}.$$

is a mean of the Cauchy's type on C_0-semigroup of positive operators. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases, as in [17].

Chapter 7

Conclusion

Due to the vastness of the theory of inequalities and the advancement of the theory of strongly continuous semigroups of operators, the fusion is exceptionally vital subject of this span. This thesis is a contribution towards this concept. The main idea of this dissertation has two dimensions. Firstly, to obtain the expressions for operators from strongly continuous semigroups, which apparently take the form of classical inequalities. So that we name them as "type" inequalities. Secondly, to acquire the set of means which are inspired from Cauchy's means and therefore called the Cauchy type means.
Since the theory of strongly continuous semigroups has many applications in ODEs and a deterministic system can be expressed in terms of these operators, these expressions might be very useful while handling the physical problems.

In this dissertation, a new set of power means is defined on a C_0-group of continuous linear operators. A mean value theorem is proved, which builds the basis of the procedure to obtain the Cauchy-type power means on a C_0-group of continuous linear operators.

The Jessen's type inequality for normalized positive C_0-semigroups is obtained in chapter 5. An adjoint of Jessen's type inequality has also been derived for the corresponding adjoint-semigroup, which does not give the analogous results but the behavior is still interesting. Moreover, it is followed by some results regarding positive definiteness and exponential convexity of complex structures involving operators from a semigroup. We also present few applications of the Jessen's type inequality for normalized positive C0- semigroups. In the same section, we present few results of this inequality, yielding Hölder's type and Minkowski's type inequalities for corresponding semigroup. Moreover, a Dresher's type inequality for two-parameter family of means, is also proved.

The chapter 6 has genuinely been divided into two parts. In the first half, we prove a Jensen's type inequality for a normalized semigroup of positive linear operators and a superquadratic mapping, defined on a Banach lattice algebra. A systematic procedure has been adopted to prove the corresponding mean value theorems, which lead us to a new set of means. These means are Cauchy's type means for the mentioned operators.

In the other half of chapter 6, we follow the same lines and gave a Hermite-Hadamard's type inequality for a semigroup of positive linear operators and a superquadratic mapping, defined on a Banach lattice algebra. The similar procedure has been systematically used to prove the corresponding mean value theorems, which lead us to a new set of means. These means are Cauchy's type means for the mentioned operators.

The constructive nature of our approach suggests that we can extend this theory to produce more expressions for the operators from a strongly continuous semigroup. By following the similar procedure as in chapter 6, many functional inequalities can be generalized for the operator semigroups and corresponding means can be obtained. We also suggest to investigate the solutions of Cauchy abstract problems using the means and inequalities defined for strongly continuous semigroup on Banach spaces.

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CAUCHY TYPE MEANS ON ONE–PARAMETER C_0–GROUP OF OPERATORS

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Dedicated to our parents

(*Communicated by A. Meskhi*)

Abstract. A new theory of power means is introduced on a C_0-group of continuous linear operators. A mean value theorem is proved, which builds the basis of the procedure to obtain Cauchy-type power means on a C_0-group of continuous linear operators.

1. Introduction

A significant theory of Cauchy type means has been developed [2, 3, 4, 5, 6, 7], which is both extensive and elegant. In this paper we define new means on the C_0-semigroup of bounded linear operators which also contains the inverses and hence forming a C_0-group. Later on, these means are shown to be of Cauchy-type.

This section is actually intended to give a brief exposition to few definitions and results in the theory of uniformly continuous groups(semigroups) of bounded linear operators defined on a Banach space X, which are indispensable for an understanding of the next section. Let $B(X)$ denotes the space of bounded linear operators defined on a Banach space X. A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space X is a family $\{Z(t)\}_{t\geqslant 0}\subset B(X)$ such that

(i) $Z(s)Z(t)=Z(s+t)$ for all $s,t\geqslant 0$.

(ii) $Z(0)=I$, the identity operator on X.

(iii) for each fixed $f\in X$, $Z(t)f\to f$ (with respect to the norm on X) as $t\to 0^+$.

If the above mentioned properties hold for $\mathbb{R}$ instead of $\mathbb{R}^+$, we call $\{Z(t)\}_{t\in\mathbb{R}}$ a *strongly continuous (one parameter) group (or C_0-group)* on X, where for $f\in X$, $Z(t)Z(-t)f=Z(0)f=f$. Therefore, $\{Z(t)\}_{t<0}$ gives the inverses of $\{Z(t)\}_{t>0}$. All the properties and characteristics of C_0-Semigroup are also possessed by C_0-group, so we shall be considering only C_0-semigroups at the moment.

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The (infinitesimal) generator of $\{Z(t)\}_{t\geqslant 0}$ is the closed linear operator $A : X \supseteq D(A) \to R(A) \subseteq X$ defined by

$$D(A) = \{f : f \in X, \lim_{t\to 0^+} A_t f \text{ exists in } X\}$$

$$Af = \lim_{t\to 0^+} A_t f \ \ (f \in D(A))$$

where, for $t > 0$,

$$A_t f = \frac{[Z(t) - I]f}{t} \ \ (f \in X).$$

Moreover $D(A)$ is a dense vector subspace of X. For $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$ a C_0-semigroup on the Banach space X, there exists constants $M > 0$ and $\omega \geqslant 0$ such that $\|Z(t)\| \leqslant Me^{\omega t}$, for all $t \geqslant 0$. See ([8], Theorem 2.14). In case $M = 1$ and $\omega = 0$, we obtain a C_0-semigroup (correspondingly group) of contractions.

The arithmetic mean on the C_0-semigroup of operators is defined as [10],

$$m(Z, f, t) = \frac{1}{t}\int_0^t Z(\tau) f d\tau.$$

Means of C_0-semigroups of operators have great importance and form the basis of *Mean Ergodic Theory*, which has been a center of interest in research for decades. (See e.g. [16, 9, 14]).

To define power-means on C_0-semigroup of operators, things need little more concentration. As the real-powers are involved, $\{Z(t)\}_{t\geqslant 0}$ should also contain the inverse operators (to define the powers like $r < 0$). One can observe that when r is any integer (positive or negative), the C_0-group property implies that $Z(t)^r = Z(rt)$. While we can generalize it for $r \in \mathbb{R}$. For example take $Z(1/2t)Z(1/2t) = Z(t)$ and thus we get $Z(t)^{1/2} = Z(1/2t)$. For $r \in \mathbb{R}$, the generator of $\{Z(rt)\}_{t\geqslant 0}$ is $(rA, D(A))$. Such semigroups are often called *rescaled semigroups*. (See e.g. [11, 13]). For $f \in X$ and $t > 0$, a C_0-semigroup(group) $\{Z(t)\}_{t\geqslant 0}$ generated by an operator A, has the form $Z(t)f = \exp[tA]f$ (see [8]). Hence $\ln[Z(t)f]$ makes sense.

In correspondence with the usual definition of power integral means, we define the power means for C_0-group of operators.

DEFINITION 1. Let X be a Banach space and $\{Z(t)\}_{t\in\mathbb{R}}$ the C_0-group of linear operators on X. For $f \in X$ and $t \in \mathbb{R}$, the power mean is defined as follows

$$M_r(Z, f, t) = \begin{cases} \left\{\frac{1}{t}\int_0^t [Z(\tau)]^r f d\tau\right\}^{1/r}, & r \neq 0 \\ \exp[\frac{1}{t}\int_0^t \ln[Z(\tau)] f d\tau], & r = 0. \end{cases} \tag{1}$$

For $t > 0$ and $r \in \mathbb{R}^+$, $Z(t)^r = Z(-t)^{-r}$. Therefore the integral domain is taken to be non-negative. Moreover for $r = 1$, $M_r(Z, f, t) = m(Z, f, t)$, the arithmetic mean, for $r = 0$ it defines the geometric mean and for $r = -1$ it defines the harmonic mean on C_0-group of operators (and hence satisfying the property of power-mean). For $r > 0$, $M_{-r}(Z, f, t)$ gives the inverse of the mean of inverse of $Z(t)^r$.

For real and continuous functions φ,χ on a closed interval $K:=[k_1,k_2]$, such that φ,χ are differentiable in the interior of I and $\chi'\neq 0$, throughout the interior of I. A very well know Cauchy mean value theorem guarantees the existence of a number $\zeta\in(k_1,k_2)$, such that

$$\frac{\varphi'(\zeta)}{\chi'(\zeta)}=\frac{\varphi(k_1)-\varphi(k_2)}{\chi_(k_1)-\chi(k_2)}.$$

Now, if the function $\frac{\varphi'}{\chi'}$ is invertible, then the number ζ is unique and

$$\zeta:=\Big(\frac{\varphi'}{\chi'}\Big)^{-1}\Big(\frac{\varphi(k_1)-\varphi(k_2)}{\chi_(k_1)-\chi(k_2)}\Big).$$

The number ζ is called *Cauchy's mean value* of numbers k_1,k_2. It is possible to define such a mean for several variables, in terms of divided difference. Which is given by

$$\zeta:=\Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{[k_1,k_2,...,k_n]\varphi}{[k_1,k_2,...,k_n]\chi}\Big).$$

This mean value was first defined and examined by Leach and Sholander [12]. The integral representation of Cauchy mean is given by

$$\zeta:=\Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{\int_{E_{n-1}}\varphi^{n-1}(k.u)du}{\int_{E_{n-1}}\chi^{n-1}(k.u)du}\Big)',$$

where $E_{n-1}:=\{(u_1,u_2,...,u_n):u_i\geqslant 0,\ 1\leqslant i\leqslant n,\ \sum_{i=1}^{n-1}u_i\leqslant 1\}$, is (n-1) dimensional simplex, $u=(u_1,u_2,...,u_n)$, $u_n=1-\sum_{i=1}^{n-1}u_i$, $du=du_1du_2...du_n$ and $k.u=\sum_{i=1}^{n}u_ik_i$.

A mean which can be expressed in the similar form as of Cauchy mean, is called *Cauchy type mean*. The purpose of our work is to introduce new means of Cauchy type defined on C_0-group of operators.

2. Main results

The present section includes a chain of results. Two mean value theorems are proved. As applications of these mean value theorems we have defined new means for C_0-group of linear operators.

LEMMA 1. *Let $\{Z(t)\}_{t\geqslant 0}\subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X. For $f\in X$ and $t>0$,*

$$m(Z,f,t)=\frac{1}{t}\int_0^t Z(\tau)fd\tau\in X. \tag{2}$$

Proof. Let $h > 0$ and consider

$$\begin{aligned}
\frac{Z(h)-I}{h}\left\{\int_0^t Z(u)fdu\right\} &= \frac{1}{h}\int_0^t \{Z(u+h)f - Z(u)f\}du \\
&= \frac{1}{h}\int_0^t Z(u+h)fdu - \frac{1}{h}\int_0^t Z(u)fdu \\
&= \frac{1}{h}\int_h^{t+h} Z(u)fdu - \frac{1}{h}\int_0^t Z(u)fdu \\
&= \frac{1}{h}\int_h^t Z(u)fdu + \frac{1}{h}\int_t^{t+h} Z(u)fdu - \frac{1}{h}\int_0^t Z(u)fdu \\
&= \frac{1}{h}\int_t^{t+h} Z(u)fdu - \frac{1}{h}\int_0^t Z(u)fdu - \frac{1}{h}\int_t^h Z(u)fdu \\
&= \frac{1}{h}\int_t^{t+h} Z(u)fdu - \frac{1}{h}\int_0^t Z(u)fdu
\end{aligned}$$

on letting $h \to 0^+$ and using the fundamental theorem of calculus

$$\lim_{h\to 0^+} \frac{Z(h)-I}{h}\left\{\int_0^t Z(u)fdu\right\} = Z(t)f - f = [Z(t)-I]f \in D(A)$$

hence

$$\int_0^t Z(\tau)fd\tau \in D(A)$$

and since $D(A)$ is a vector subspace of X, therefore $m(Z,f,t) \in D(A)$. Also $D(A) \subset \overline{D(A)} = X$. Hence the result follows. $\square$

COROLLARY 1. *Let $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X. For $f \in X$ and $t > 0$,*

$$M_r(Z,f,t) \in X,$$

where $M_r(Z,f,t)$ is defined by (1).

Proof. For $\{Z(t)\}_{t\in\mathbb{R}} \subseteq B(X)$, by group-law we have, for $\tau, r \in \mathbb{R}$,

$$[Z(\tau)]^r f = Z(r\tau)f = Z(s)f$$

where $r\tau = s$, then $Z(s) \in \{Z(t)\}_{t\in\mathbb{R}}$. By Lemma 1, we finally get that $M_r(Z,f,t) \in X$. $\square$

REMARK 1. For a C_0-semigroup of contractions $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$, we have $\|Z(t)\| \leqslant 1$, for all $t \in \mathbb{R}$. For such C_0-groups

- The mean $m(Z,f,t)$, satisfies

$$\|m\| = Sup_{f\in X}\frac{\|m(Z,f,t)\|}{\|f\|} \leqslant 1, \text{ for } t > 0.$$

- The power mean $M_r(Z,f,t)$ is defined by (1.1). For $r>0$,

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \geqslant \eta, \quad f\in X,\ t>0,$$

  and for $r<0$,

$$\|M_r\| = Sup_{f\in X}\frac{\|M_r(Z,f,t)\|}{\|f\|} \leqslant \eta, \quad f\in X,\ t>0,$$

  where $\eta = \|f\|^{-(r+1)}$. Moreover for $r=0$,

$$\|M_0\| = Sup_{f\in X}\frac{\|M_0(Z,f,t)\|}{\|f\|} \leqslant 1, \quad f\in X,\ t>0.$$

- Let $\{f_n\}_{n=0}^{\infty}\subset X$, such that $f_n\to f\in X$, and $\|Z(t)\|\leqslant 1$ for $t\in\mathbb{R}$,

$$\|M_r(Z,f_n,t)-M_r(Z,f,t)\| \leqslant \|M_r\|\|f_n-f\|, \quad r\leqslant 0.$$

  Therefore, for $r\leqslant 0$, $M_r(Z,f_n,t)\to M_r(Z,f,t)$.

Next, we shall prove a mean value theorem which actually forms the basis of rest of the theory and somehow, can be regarded as the analogue of ([15],Theorem 1) to Banach spaces.

THEOREM 1. *Let X be a Banach space and $\{Z(t)\}_{t\geqslant 0}\subset B(X)$ be a C_0-semigroup of operators on X. For $\phi,\psi\in C^2(X)$ there exists some $\xi\in X$ such that*

$$\frac{\frac{1}{t}\int_0^t \phi[Z(\tau)]f d\tau - \phi[\frac{1}{t}\int_0^t [Z(\tau)]f d\tau]}{\frac{1}{t}\int_0^t \psi[Z(\tau)]f d\tau - \psi[\frac{1}{t}\int_0^t [Z(\tau)]f d\tau]} = \frac{\phi''(\xi)}{\psi''(\xi)}. \tag{3}$$

Proof. For the sake of simplicity throughout the proof, we shall denote $m(Z,f,t)$ by m_t. For $\rho\in X$, define

$$(Q\phi)(\rho) := \frac{1}{t}\int_0^t \phi[\rho[Z(\tau)f]+(1-\rho)m_t]d\tau - \phi(m_t)$$

similarly, for the operator ψ, we define $(Q\psi)(\rho)$. It is observed that,

$$(Q\phi)'(\rho) := \frac{1}{t}\int_0^t [Z(\tau)f-m_t]\phi'[\rho[Z(\tau)f]+(1-\rho)m_t]d\tau$$

and

$$(Q\phi)''(\rho) := \frac{1}{t}\int_0^t [Z(\tau)f-m_t]^2\phi''[\rho[Z(\tau)f]+(1-\rho)m_t]d\tau.$$

Here, $(.)'$ denotes the Gateaux derivative. Let us define an other operator $W(\rho)$, as follows

$$W(\rho) = (Q\psi)(1)(Q\phi)(\rho) - (Q\phi)(1)(Q\psi)(\rho).$$

It can be easily seen that

$$W(0) = W(1) = W'(0) = 0$$

where 0, 1 are the zero, identity elements of X, respectively.

After two applications of Mean Value theorem [1], we conclude that there exists an element $\eta \in X$ such that

$$W''(\eta) = 0.$$

Hence

$$\frac{1}{t}\int_0^t [Z(\tau)f - m_t]^2\{(Q\psi)(1)\phi''[\eta[Z(\tau)f] + (1-\eta)m_t] \\ -(Q\phi)(1)\psi''[\eta[Z(\tau)f] + (1-\eta)]\}m_t d\tau = 0 \tag{4}$$

A mapping $\varphi_f : [0,\infty) \to X$ defined by

$$\varphi_f(t) = Z(t)f, \quad f \in X$$

is continuous on $[0,\infty)$. See ([8], Lemma 2.4). Hence for any fixed $\eta \in X$, the expression in the braces in (4) is a continuous function of τ, so it vanishes for some value of $\tau \geqslant 0$. Corresponding to that value of $\tau \geqslant 0$, we get an element $\xi \in X$, such that

$$\xi = \eta[Z(\tau)f] + (1-\eta)m_t, \quad f \in X.$$

So that

$$(Q\psi)(1)\phi''(\xi) - (Q\phi)(1)\psi''(\xi) = 0.$$

The assertion (3) follows directly. □

COROLLARY 2. *Let X be a Banach space and $\{Z(t)\}_{t\geqslant 0} \subseteq B(X)$ be a C_0-semigroup of operators on X. For $\phi,\psi \in C^2(X)$ such that $\frac{\phi''}{\psi''}$ is invertible. Then there exists a unique $\xi \in X$ which is the mean of the Cauchy type that is*

$$\xi = \left(\frac{\phi''}{\psi''}\right)^{-1}\left(\frac{\frac{1}{t}\int_0^t \phi[Z(\tau)]fd\tau - \phi[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau]}{\frac{1}{t}\int_0^t \psi[Z(\tau)]fd\tau - \psi[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau]}\right). \tag{5}$$

COROLLARY 3. *Let X be a Banach space and $\{Z(t)\}_{t\geqslant 0} \subset B(X)$ be a C_0-semigroup of operators on X. For $\phi \in C^2(X)$ and some $\xi \in X$*

$$\frac{1}{t}\int_0^t \phi[Z(\tau)f]d\tau - \phi\left[\frac{1}{t}\int_0^t [Z(\tau)f]d\tau\right] = \frac{\phi''(\xi)}{2}\left\{\frac{1}{t}\int_0^t [Z(\tau)]^2 fd\tau - \left[\frac{1}{t}\int_0^t [Z(\tau)]fd\tau\right]^2\right\}. \tag{6}$$

Proof. By setting $\psi(f) = f^2$ for $f \in X$, in Theorem 1, we get the assertion (6). □

Next, let G be the group of invertible bounded linear operators from a Banach space X to itself. For $\{Z(t)\}_{t\geqslant 0} \subset B(X)$ a C_0-semigroup of operators defined on X and $H \in G$, the quasi-arithmetic mean is defined as

$$M_H^{\circ}(Z,f,t) = H^{-1}\Big\{\frac{1}{t}\int_0^t H[Z(\tau)f]d\tau\Big\}, \quad f \in X, t \geqslant 0. \tag{7}$$

By ([8], Lemma 1.85), $B(X)$ is closed under composition of operators so the above expressions exists and belongs to X. For the sake of simplicity, the set of all elements of G, whose second order derivative (in Gateaux's sense) exits, is denoted by $C^2G(X)$.

THEOREM 2. *Let X be a Banach space and let $H,F,K \in C^2G(X)$. Then*

$$\frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))} = \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)}. \tag{8}$$

For some $\xi \in X$, provided that the denominator on the left hand side of (8) is non-zero.

Proof. By choosing the operators ϕ and ψ in Theorem 1, such that

$$\phi = H \circ F^{-1}, \quad \psi = K \circ F^{-1} \quad \textit{and} \quad Z(\tau)f = F[Z(\tau)f]$$

where $H,F,K \in C^2G(X)$. We find that there exists $\xi \in X$, such that

$$\frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))} = \frac{H''(F^{-1}(\xi))F'(F^{-1}(\xi)) - H'(F^{-1}(\xi))F''(F^{-1}(\xi))}{K''(F^{-1}(\xi))F'(F^{-1}(\xi)) - K'(F^{-1}(\xi))F''(F^{-1}(\xi))}.$$

Therefore, by setting $F^{-1}(\xi) = \eta$, we find that there exists $\eta \in X$, such that

$$\frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))} = \frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)},$$

which completes the proof. □

REMARK 2. For $(X, \|.\|)$ a Banach space, it follows from Theorem 2 that

$$m \leqslant \Big\|\frac{H(M_H^{\circ}(Z,f,t)) - H(M_F^{\circ}(Z,f,t))}{K(M_K^{\circ}(Z,f,t)) - K(M_F^{\circ}(Z,f,t))}\Big\| \leqslant M,$$

Where m and M are respectively, the minimum and maximum values of

$$\Big\|\frac{H''(\eta)F'(\eta) - H'(\eta)F''(\eta)}{K''(\eta)F'(\eta) - K'(\xi)F''(\xi)}\Big\|, \ \eta \in X.$$

Next, we prove an important result which lead us to define the Cauchy type means on C_0-group of operators.

COROLLARY 4. *Let $r,s,l \in \mathbb{R}$ and $\{Z(t)\}_{t\in\mathbb{R}} \subset B(X)$ be a C_0-semigroup of operators on a Banach space X. Then*

$$\frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)} = \frac{r(r-s)}{l(l-s)}\eta^{r-l}, \quad \eta \in X. \tag{9}$$

Where $M_r(Z,f,t)$ is defined by (1).

Proof. For $r,s,l \in \mathbb{R}$ and $f \in X$, if we set

$$H(f) = f^r,\ F(f) = f^s,\ K(f) = f^l$$

in Theorem 2, the assertion in (9) follows directly. □

REMARK 3. It follows from Corollary (4) that

$$\left|\frac{r(r-s)}{l(l-s)}\right| m \leqslant \left\|\frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)}\right\| \leqslant \left|\frac{r(r-s)}{l(l-s)}\right| M.$$

Where m and M are respectively, the minimum and maximum values of $\|\eta^{r-l}\|$, $\eta \in X$.

In the next definition we have defined means of the Cauchy type on C_0-group of linear operators.

DEFINITION 2. Let $r,s,l \in \mathbb{R}$ and $\{Z(t)\}_{t\in\mathbb{R}} \subset B(X)$ be a C_0-semigroup of operators on a Banach space X. Then

$$\mathfrak{M}_r^{l,s}(Z,f,t) = \left(\frac{l(l-s)}{r(r-s)} \frac{M_r^r(Z,f,t) - M_s^r(Z,f,t)}{M_l^l(Z,f,t) - M_s^l(Z,f,t)}\right)^{\frac{1}{r-l}}. \tag{10}$$

is a mean of the Cauchy type on C_0-group of operators. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases, as in [6].

3. Conclusion

Firstly, we have proved two mean value theorems. A systematic procedure has been used to define means on C_0-group of linear operators. These means are Cauchy type means on C_0-group of linear operators. Moreover, it can be easily proved that these means are monotonic.

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Jessen's type inequality and exponential convexity for positive C_0-semigroups

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Abstract

In this paper a Jessen's type inequality for normalized positive C_0-semigroups is obtained. An adjoint of Jessen's type inequality has also been derived for the corresponding adjoint semigroup, which does not give the analogous results but the behavior is still interesting. Moreover, it is followed by some results regarding positive definiteness and exponential convexity of complex structures involving operators from a semigroup.

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Keywords: positive semigroups; Jessen's inequality; Banach lattice algebra; exponential convexity

1 Introduction and preliminaries

A significant theory regarding inequalities and exponential convexity for real-valued functions has been developed [1, 2]. The intention to generalize such concepts for the C_0-semigroup of operators is motivated from [3].

In the present article, we shall derive a Jessen type inequality and the corresponding adjoint inequality, for some C_0-semigroup and the adjoint semigroup, respectively.

The notion of Banach lattice was introduced to get a common abstract setting, within which one could talk about the ordering of elements. Therefore, the phenomena related to positivity can be generalized. It had mostly been studied in various types of spaces of real-valued functions, *e.g.* the space $C(K)$ of continuous functions over a compact topological space K, the Lebesque space $L^1(\mu)$ or even more generally the space $L^p(\mu)$ constructed over measure space (X, Σ, μ) for $1 \le p \le \infty$. We shall use without further explanation the terms: order relation (ordering), ordered set, supremum, infimum.

First, we shall go through the definition of a vector lattice.

Definition 1 [4] Any (real) vector space V with an ordering satisfying:

O_1: $f \le g$ implies $f + h \le g + h$ for all $f, g, h \in V$,
O_2: $f \ge 0$ implies $\lambda f \ge 0$ for all $f \in V$ and $\lambda \ge 0$,

is called an *ordered vector space.*

The axiom O_1, expresses the translation invariance and therefore implies that the ordering of an ordered vector space V is completely determined by the positive part

$V_+ = \{f \in V : f \geq 0\}$ of V. In other words, $f \leq g$ if and only if $g - f \in V_+$. Moreover, the other property, O_2, reveals that the positive part of V is a convex set and a cone with vertex 0 (mostly called the *positive cone* of V).

- An ordered vector space V is called a *vector lattice*, if any two elements $f, g \in V$ have a supremum, which is denoted by $\sup(f, g)$ and an infimum denoted by $\inf(f, g)$.

  It is trivially understood that the existence of supremum of any two elements in an ordered vector space implies the existence of supremum of finite number of elements in V. Furthermore, $f \geq g$ implies $-f \leq -g$, so the existence of finite infima is therefore implied.
- Some important quantities are defined as follows:

$$\sup(f, -f) = |f| \quad (\textit{absolute value of } f),$$
$$\sup(f, 0) = f^+ \quad (\textit{positive part of } f),$$
$$\sup(-f, 0) = f^- \quad (\textit{negative part of } f).$$

- Some compatibility axiom is required between norm and order. This is given in the following short way:

$$|f| \leq |g| \quad \text{implies} \quad \|f\| \leq \|g\|. \tag{1}$$

  The norm defined on a vector lattice is called a lattice norm.

Now, we are in a position to define a Banach lattice in a formal way.

Definition 2 A *Banach lattice* is a Banach space V endowed with an ordering $\leq$, such that $(V, \leq)$ is a vector lattice with a lattice norm defined on it.

A Banach lattice transforms to *Banach lattice algebra*, provided $u, v \in V_+$ implies $uv \in V_+$.

A linear mapping ψ from an ordered Banach space V into itself is *positive* (denoted $\psi \geq 0$) if $\psi f \in V_+$, for all $f \in V_+$. The set of all positive linear mappings forms a convex cone in the space $L(V)$ of all linear mappings from V into itself, defining the natural ordering of $L(V)$. The absolute value of ψ, if it exists, is given by

$$|\psi|(f) = \sup\{\psi h : |h| \leq f\} \quad (f \in V_+).$$

Thus $\psi : V \to V$ is positive if and only if $|\psi f| \leq \psi|f|$ holds for any $f \in V$.

Lemma 1 ([4], p.249) *A bounded linear operator ψ on a Banach lattice V is a positive contraction if and only if $\|(\psi f)^+\| \leq \|f^+\|$ for all $f \in V$.*

An operator A on V satisfies the positive minimum principle if for all $f \in D(A)_+ = D(A) \cap V_+$, $\phi \in V'_+$

$$\langle f, \phi\rangle = 0 \quad \text{implies} \quad \langle Af, \phi\rangle \geq 0. \tag{2}$$

Definition 3 A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space V is a family $\{Z(t)\}_{t\geq 0} \subset B(V)$ such that:

(i) $Z(s)Z(t) = Z(s+t)$ for all $s, t \in \mathbb{R}^+$.
(ii) $Z(0) = I$, the identity operator on V.
(iii) For each fixed $f \in V$, $Z(t)f \to f$ (with respect to the norm on V) as $t \to 0^+$.

Here $B(V)$ denotes the space of all bounded linear operators defined on a Banach space V.

Definition 4 The (infinitesimal) generator of $\{Z(t)\}_{t\geq 0}$ is the densely defined closed linear operator $A : V \supseteq D(A) \to R(A) \subseteq V$ such that

$$D(A) = \Big\{f : f \in V, \lim_{t\to 0^+} A_t f \text{ exists in } V\Big\},$$
$$Af = \lim_{t\to 0^+} A_t f \quad \big(f \in D(A)\big),$$

where, for $t > 0$,

$$A_t f = \frac{[Z(t) - I]f}{t} \quad (f \in V).$$

Let $\{Z(t)\}_{t\geq 0}$ be the strongly continuous positive semigroup, defined on a Banach lattice V. The positivity of the semigroup is equivalent to

$$\big|Z(t)f\big| \leq Z(t)|f|, \quad t \geq 0, f \in V.$$

Here, for positive contraction semigroups $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V, we have

$$\big\|\big(Z(t)f\big)^+\big\| \leq \big\|f^+\big\|, \quad \text{for all } f \in V.$$

Reference [4] guarantees the existence of the strongly continuous positive semigroups and positive contraction semigroups on a Banach lattice V, with some conditions imposed on the generator. A very important among them is that it must always satisfy (2).

A Banach algebra X, with the multiplicative identity element e, is called the *unital Banach algebra*. We shall call the strongly continuous semigroup $\{Z(t)\}_{t\geq 0}$ defined on X a *normalized semigroup* whenever it satisfies

$$Z(t)(e) = e, \quad \text{for all } t > 0. \tag{3}$$

The notion of normalized semigroup is inspired by normalized functionals [2]. The theory presented in the next section is defined on such semigroups of positive linear operators defined on a Banach lattice V.

2 Jessen's type inequality

In 1931, Jessen [5] gave the generalization of the Jensen's inequality for a convex function and positive linear functionals. See [6], p.47. We shall prove this inequality for a normalized positive C_0-semigroup and a convex operator defined on a Banach lattice.

Throughout the present section, V will always denote a unital Banach lattice algebra, endowed with an ordering $\leq$.

Definition 5 Let U be a nonempty open convex subset of V. An operator $F : U \to V$ is convex if it satisfies

$$F\big(tu + (1-t)v\big) \leq tF(u) + (1-t)F(v), \tag{4}$$

whenever $u, v \in U$ and $0 \leq t \leq 1$.

Let $\mathfrak{D}_c(V)$ denotes the set of all differentiable convex functions $\phi : V \to V$.

Theorem 1 (Jessen's type inequality) *Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup on V such that it satisfies* (3). *For an operator $\phi \in \mathfrak{D}_c(V)$ and $t \geq 0$,*

$$\phi\big(Z(t)f\big) \leq Z(t)(\phi f), \quad f \in V. \tag{5}$$

Proof Since $\phi : V \to V$ is convex and differentiable, by considering an operator analog of (Theorem A, p.98, [7]), we see, for any $f_0 \in V$, that there is a fixed vector $m = m(f_0) = \phi'(f_0)$ such that

$$\phi(f) \geq \phi(f_0) + m(f - f_0), \quad f \in V.$$

Using the property (3) along with the linearity and positivity of operators in a semigroup, we obtain

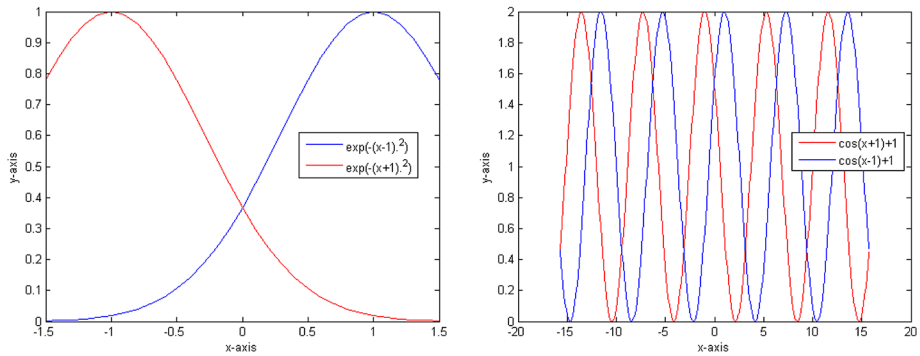
$$Z(t)\big(\phi(f)\big) \geq \phi(f_0) + m\big(Z(t)f - f_0\big), \quad f \in V, t \geq 0.$$

In this inequality, set $f_0 = Z(t)f$ and the assertion (5) follows. □

The existence of an identity element and condition (3), imposed in the hypothesis of the above theorem, is necessary. We shall elaborate all this by the following examples.

Example 1 Let $X := C_0(\mathbb{R})$, $\{Z(t)\}_{t\geq 0}$ be the left shift semigroup defined on X and ϕ taking the mirroring along y-axis. The identity function does not contain a compact support and therefore is not in X. We now take a bell-shaped curve like $f(x) := e^{-x^2}$, $x \in \mathbb{R}$. Then f is positive, $\phi f = f$, and $Z(t)(\phi f)$ has maximum at $x = -t$, and it is between 0 and 1 elsewhere. On the other hand, $\phi(Z(t)f)$ has a maximum at $s = t$ and it is immediate that we cannot compare the two functions in the usual ordering. See Figure 1(a).

Example 2 Let $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$, and $X = C(\Gamma)$. The rotation semigroup $\{Z(t)\}_{t\geq 0}$ is defined as $T(t)f(z) = f(e^{it} \cdot z), f \in X$. The identity element $E \in X$, s.t. for all $z \in \Gamma$, $E(z) = z$. Then $Z(t)E(z) = E(e^{it} \cdot z) = e^{it} \cdot z$. Or we can say that any complex number $z = e^{ix}$ is mapped to $e^{i(x+t)}$. $Z(t)$ satisfies (3), only when t is a multiple of 2π. Let $f(z) = \Re(z) + 1 > 0$, then $(Z(t)f)(e^{iz}) = f(e^{i(t+z)}) = \cos(t + z) + 1$, hence $\phi(Z(t)f)(e^{iz}) = \cos(t - z) + 1$. On the other hand, $\phi(f) = f$, and $Z(t)(\phi f)(e^{iz}) = (Z(t)f)(e^{iz}) = \cos(t + z) + 1$. Hence, equality holds in (5) when t is a multiple of 2π, but the two sides are not comparable in general. It can easily be verified that $Z' = \{Z(2\pi t)\}_{t=0}^{\infty}$ is a subgroup of $Z = \{Z(t)\}_{t=0}^{\infty}$, as $Z(2\pi t)Z(2\pi s) = Z(2\pi(t+s))$. Therefore Z' is a normalized semigroup. See Figure 1(b).



(a) (b)

Figure 1 Visualization of the behavior of Jessen's inequality for examples 1 & 2.

3 Adjoint Jessen's type inequality

In the previous section, a Jessen type inequality has been derived for a normalized positive C_0-semigroup $\{Z(t)\}_{t\geq 0}$. This gives us the motivation toward finding the behavior of its corresponding adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ on V^*. As the theory for dual spaces gets more complicated, we do not expect to have the analogous results. One may ask for a detailed introduction toward a part of the dual space V^*, for which an adjoint of Jessen's type inequality makes sense.

Definition 6 Given two Banach spaces X and Y and a bounded linear operator $L : X \to Y$, recall that the adjoint $L^* : Y^* \to X^*$ is defined by

$$(L^*y^*)x := y^*(Lx), \quad y^* \in Y^*, x \in X. \tag{6}$$

For a strongly continuous positive semigroup $\{Z(t)\}_{t\geq 0}$ on a Banach space X, by defining $Z^*(t) = (Z(t))^*$ for every t, we get a corresponding adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ on the dual space X^*. In [8], the result is obtained that the adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ fails in general to be strongly continuous. The investigation [9], shows that $\{Z^*(t)\}_{t\geq 0}$ acts in a strongly continuous way on

$$X^{\odot} := \left\{x^* \in X^* : \lim_{t\to 0}\left\|Z^*(t)x^* - x^*\right\| = 0\right\}.$$

This is the maximal such subspace on X^*. The space $X^{\odot}$ was introduced by Philips in 1955 and later has been studied extensively by various authors. At the present moment, we do not necessarily require the strong continuity of the adjoint semigroup $\{Z^*(t)\}_{t\geq 0}$ on X^*.

If X is an ordered vector space, we say that a functional x^* on X is positive if $x^*(x) \geq 0$, for each $x \in X$. By the linearity of x^*, this is equivalent to x^* being order preserving; *i.e.* $x \leq y$ implies $x^*(x) \leq x^*(y)$. The set P of all positive linear functionals on X is a cone in X^*.

We are mainly interested in the study of the space V^*, where in our case V is a Banach lattice algebra. Let us consider the regular ordering among the elements of V^*, *i.e.* $v_1^* \geq v_2^*$, whenever $v_1^*(v) \geq v_2^*(v)$, for each $v \in V$.

Consider the convex operator (4). In the case of equality, F is simply a linear operator and the adjoint F can be defined as above. But how can it be defined in the other case? This question has already been answered.

In [10], some kind of adjoint has been associated to a nonlinear operator F. In fact, this is possible for Lipschitz continuous operators only. Consider the Banach space $\mathfrak{Lip}_0(X,Y)$ of all Lipschitz continuous operators $F: X \to Y$ satisfying $F(\theta) = \theta$, equipped with the norm

$$[F]_{\mathrm{Lip}} = \sup_{x_1 \neq x_2} \frac{\|F(x_1) - F(x_2)\|}{\|x_1 - x_2\|}, \quad x_1, x_2 \in X.$$

Here $\theta \in X$ is the identity. It is easy to see that the space $L(X,Y)$ of all bounded linear operators from X to Y is a closed subspace of $\mathfrak{Lip}_0(X,Y)$. In particular, we set

$$\mathfrak{Lip}_0(X,\mathbb{K}) := X^\sharp$$

and call $X^\sharp$ the pseudo-dual space of X; this space contains the usual dual space X^* as a closed subspace.

Definition 7 For $F \in \mathfrak{Lip}_0(X,Y)$, the pseudo-adjoint $F^\sharp : Y^\sharp \to X^\sharp$ of F is defined by

$$F^\sharp\bigl(y^\sharp\bigr)(x) := y^\sharp\bigl(F(x)\bigr), \quad y^\sharp \in Y^\sharp, x \in X. \tag{7}$$

This is, of course, a straightforward generalization of (6); in fact, for linear operators L we have $L^\sharp|_{Y^*} = L^*$; *i.e.* the restriction of the pseudo-adjoint to the dual space is the classical adjoint.

For the sake of convenience, we shall denote the adjoint of the operator F by F^* throughout the present section. Either it is a classical adjoint or the pseudo-adjoint (depending upon the operator F).

Similarly, the considered dual space of the vector lattice algebra V will be denoted by V^*, which can be the intersection of the pseudo-dual and classical dual spaces in the case of a nonlinear convex operator.

Lemma 2 *Let F be the convex operator on a Banach space X, then the adjoint operator F^* on the dual space X^* is also convex.*

Proof For $x \in X$ and $0 \leq \lambda \leq 1$

$$\begin{aligned}\bigl(F^*\bigl(\lambda x_1^* + (1-\lambda)x_2^*\bigr), x\bigr) &= \bigl(\lambda x_1^* + (1-\lambda)x_2^*, F(x)\bigr), \\ &= \lambda\bigl(x_1^*, F(x)\bigr) + (1-\lambda)\bigl(x_2^*, F(x)\bigr),\end{aligned}$$

where $x_1^*, x_2^* \in X^*$. By putting $x = \mu x + (1-\mu)x$, for $0 \leq \mu \leq 1$ and using the convexity of the operator F we finally get

$$F^*\bigl(\lambda x_1^* + (1-\lambda)x_2^*\bigr) \leq \lambda F^*\bigl(x_1^*\bigr) + (1-\lambda)F^*\bigl(x_2^*\bigr).$$

Hence, F^* is convex on X^*. □

Theorem 2 (Adjoint Jessen's inequality) *Let $\{Z^*(t)\}_{t\geq 0}$ be the adjoint semigroup on V^* such that the original semigroup $\{Z(t)\}_{t\geq 0}$, the operator ϕ and the space V are same as in*

Theorem 1. *For a convex operator $\phi^* : V^* \to V^*$ and $t \geq 0$*

$$\phi^*\bigl(Z^*(t)f^*\bigr) \geq Z^*(t)\bigl(\phi^* f^*\bigr), \quad f^* \in V^*. \tag{8}$$

Proof For $f \in V$ and $t \geq 0$, consider

$$\begin{aligned}
\bigl(\phi^*\bigl[Z^*(t)f^*\bigr], f\bigr) &= \bigl(Z^*(t)f^*, \phi(f)\bigr) \\
&= \bigl(f^*, Z(t)(\phi f)\bigr) \\
&\geq \bigl(f^*, \phi\bigl(Z(t)f\bigr)\bigr) \\
&= \bigl(\phi^*\bigl(f^*\bigr), Z(t)f\bigr) \\
&= \bigl(Z^*(t)\bigl[\phi^* f^*\bigr], f\bigr).
\end{aligned}$$

Therefore, the assertion (8) is satisfied. □

4 Exponential convexity

In this section we shall define the exponential convexity of an operator. Moreover, some complex structures, involving the operators from a semigroup, will be proved to be exponentially convex.

Definition 8 Let V be a Banach lattice endowed with ordering $\leq$. An operator $H : I \to V$ is exponentially convex if it is continuous and for all $n \in \mathbb{N}$

$$\sum_{i,j=1}^{n} \xi_i \xi_j H(x_i + x_j) f \geq 0, \quad f \in V, \tag{9}$$

where $\xi_i \in \mathbb{R}$ such that $x_i + x_j \in I \subseteq \mathbb{R}$, $1 \leq i, j \leq n$.

Proposition 1 *Let V be a Banach lattice endowed with ordering $\leq$. For an operator $H : I \to V$, the following propositions are equivalent*:

(i) *H is exponentially convex.*

(ii) *H is continuous and for all $n \in \mathbb{N}$*

$$\sum_{i,j=1}^{n} \xi_i \xi_j H\left(\frac{x_i + x_j}{2}\right) f \geq 0, \quad f \in V, \tag{10}$$

where $\xi_i \in \mathbb{R}$ and $x_i \in I \subseteq \mathbb{R}$, $1 \leq i \leq n$.

Proof (i) $\Rightarrow$ (ii). Take any $\xi_i \in \mathbb{R}$ and $x_i \in I$, $1 \leq i \leq n$. Since the interval $I \subseteq \mathbb{R}$ is convex, the midpoints $\frac{x_i + x_j}{2} \in I$. Now set $y_i = \frac{x_i}{2}$, for $1 \leq i \leq n$. Then we have $y_i + y_j = \frac{x_i + x_j}{2} \in I$, for all $1 \leq i, j \leq n$. Therefore, for all $n \in \mathbb{N}$, we can apply (i) to get

$$\sum_{i,j=1}^{n} \xi_i \xi_j H(y_i + y_j) f = \sum_{i,j=1}^{n} \xi_i \xi_j H\left(\frac{x_i + x_j}{2}\right) f \geq 0, \quad f \in V.$$

(ii) ⇒ (i). Let $\xi_i, x_i \in \mathbb{R}$, such that $x_i + x_j \in I$, for $1 \le i,j \le n$. Define $y_i = 2x_i$, so that $x_i + x_j = \frac{y_i+y_j}{2} \in I$. Therefore, for all $n \in \mathbb{N}$, we can apply (ii) to get

$$\sum_{i,j=1}^{n} \xi_i \xi_j H\left(\frac{y_i + y_j}{2}\right) f = \sum_{i,j=1}^{n} \xi_i \xi_j H(x_i + x_j) f \ge 0, \quad f \in V. \qquad \square$$

Remark 1 Let H be an exponentially convex operator. Writing down the fact for $n = 1$, in (10), we get $H(x)f \ge 0$, for $x \in I$ and $f \in V$. For $n = 2$, we have

$$\xi_1^2 H(x_1) f + 2\xi_1 \xi_2 H\left(\frac{x_1 + x_2}{2}\right) f + \xi_2^2 H(x_2) f \ge 0.$$

Hence, for $\xi_1 = -1$ and $\xi_2 = 1$, we have

$$H\left(\frac{x_1 + x_2}{2}\right) f \le \frac{H(x_1)f + H(x_2)f}{2},$$

i.e. $H : I \to V$ does indeed satisfy the condition of convexity.

For $U \subseteq V$, let us assume that $F : U \to V$ is continuously differentiable on U, *i.e.* the mapping $F' : U \to \mathfrak{L}(V)$, is continuous. Moreover, $F''(f)$, will be a continuous linear transformation from V to $\mathfrak{L}(V)$. A bilinear transformation B defined on $V \times V$ is symmetric if $B(f,g) = B(g,f)$ for all $f, g \in V$. Such a transformation is *positive definite (nonnegative definite)*, if for every nonzero $f \in V$, $B(f,f) > 0$ ($B(f,f) \ge 0$). Then $F''(f)$ is symmetric wherever it exists. See [7], p.69.

Theorem 3 ([7], p.100) *Let F be continuously differentiable and suppose that the second derivative exists throughout an open convex set $U \subseteq V$. Then F is convex on U if and only if $F''(f)$ is nonnegative definite for each $f \in U$. If $F''(f)$ is positive definite on U, then F is strictly convex.*

Definition 9 [11] Let V be a Banach algebra with unit e. For $f \in V$, we define a function $\log(f)$ from V to V,

$$\log(f) = -\sum_{n=1}^{\infty} \frac{(e-f)^n}{n} = -(e-f) - \frac{(e-f)^2}{2} - \frac{(e-f)^3}{3} - \cdots$$

for $\|(e-x)\| \le 1$.

Lemma 3 *Let V be a unital Banach algebra. For $f \in V$, a family of operators F_t is defined as*

$$F_t(f) = \begin{cases} \frac{f^t}{t(t-1)}, & t \ne 0, 1; \\ -\log f, & t = 0; \\ f \log f, & t = 1. \end{cases} \tag{11}$$

Then $D^2 F_t(f) := f^{t-2}$. Whenever $f \in V_+$, $D^2 F_t(f) \in V_+$, therefore by Theorem 3, the mapping $f \to F_t(f)$ is convex.

Theorem 4 *Let* $\{Z(t)\}_{t\geq 0}$ *be the positive* C_0*-semigroup, defined on a unital Banach lattice algebra* V*, such that it satisfies* (3). *Let* $f \in V$*, such that* $f^r \in V$*, for* $r \in \mathbb{R} \supseteq I\{0,1\}$, $\log f \in V$*, if* $r = 0$ *and* $f \log f \in V$*, if* $r = 1$. *Let us define*

$$\Lambda_t := Z(t)\bigl(F_t(f)\bigr) - F_t\bigl(Z(t)f\bigr). \tag{12}$$

Then:

(i) *For every* $n \in \mathbb{N}$ *and for every* $p_k \in I$, $k = 1, 2, \ldots, n$,

$$[\Lambda_{\frac{p_i+p_j}{2}}]_{i,j=1}^{n} \geq 0. \tag{13}$$

(ii) *If the mapping* $f \to \Lambda_t$ *is continuous on* I*, then it is exponentially convex on* I.

Proof Consider the operator

$$G(f) = \sum_{i,j=1}^{n} u_i u_j F_{p_{ij}}(f)$$

for $f > 0$, $u_i \in \mathbb{R}$ and $p_{ij} \in I$ where $p_{ij} = \frac{p_i+p_j}{2}$. Then

$$D^2 G(f) := \sum_{i,j=1}^{n} u_i u_j f^{p_{ij}-2} = \left(\sum_{i=1}^{1} u_i f^{\frac{p_i}{2}-1}\right)^2 \geq 0, \quad f > 0.$$

So, $G(f)$ is a convex operator. Therefore by applying (5) we get

$$\sum_{i,j=1}^{n} u_i u_j \Lambda_{p_{ij}} \geq 0,$$

and the assertion (13) follows. Assuming the continuity and using Proposition 1 we have also exponential convexity of the operator $f \to \Lambda_t$. □

Lemma 4 *Let* V *be a unital Banach algebra, for* $f \in V$*, let us define the following family of operators*:

$$H_t(f) = \begin{cases} \frac{e^{tf}}{t^2}, & t \neq 0; \\ \frac{f^2}{2}, & t = 0. \end{cases}$$

Then $D^2 H_t(f) = e^{tf}$. *By Theorem* 3*, the mapping* $f \to H_t(f)$ *is convex on* V.

Theorem 5 *For* $\Lambda_t := Z(t)(H_t(f)) - H_t(Z(t)f)$, (i) *and* (ii) *from Theorem* 4 *hold*.

Proof Similar to the proof of Theorem 4. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Application Of Jessen's Type Inequality For Positive C_0 -Semigroup Of Operators

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Recently in [4], the Jessen's type inequality for normalized positive C_0-semigroups is obtained. In this note, we present few results of this inequality, yielding Hölder's Type and Minkowski's type inequalities for corresponding semigroup. Moreover, a Dresher's type inequality for two-parameter family of means, is also proved.

Keywords: Mean inequalities, Positive semigroup of operators, Hölder's Type Inequality, Minkowski's Type Inequality, Dresher's Type Inequality.

Introduction

In last few years the "Type" functional inequalities and their applications have been addressed extensively by several authors like [2, 6, 9]. Researchers have great interest in this field due to vast applications of these inequalities. In tejti, the authors have derived a Jessen's type inequality for normalized positive C_0-semigroup of operators. The classical Jessen's inequality has a wide theory of its applications in the field of inequalities and analysis.

In the presented note the authors established certain applications of Jessen's type inequality to obtain mean-inequalities and functional inequalities for normalized positive C_0-semigroup of operators defined on a Banach lattice algebra. These resultstake the form of Hölder's type and Minkowski's type inequalities. Then finally in the last section a Dresher's type inequality is established for two-parameter family of means.

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Preliminaries and Definitions

In this section, we will present some definitions that will be used in the proof of our main results.

Definition 1. A (real) vector space V endowed with an ordering $\geq$, such that it satisfies

O_1: $v \leq w$ implies $v+u \leq w+u$ for all $u, v, w \in V$,

O_2: $v \geq 0$ implies $\lambda v \geq 0$ for all $v \in V$ and $\lambda \geq 0$,

is known as an *ordered vector space* (see [8]).

It can be readily seen that O_1 expresses the translation invariance. Therefore, it implies that the ordering of an ordered vector space V can be completely determined by the positive part $V_+ = \{v \in V : v \geq 0\}$ of V. In other words, $v \leq qw$ if and only if $w - v \in V_+$.

The other property O_2, shows that the positive part of V is a convex set and a cone with vertex 0 (mostly called the *positive cone* of V).

If for any two elements $v, w \in V$, a supremum $sup(v,w)$ and an infumum $inf(v,w)$ can be defined, an ordered vector space V becomes a *vector lattice*. It is understood that the existence of supremum of any two elements in an ordered vectorspace implies the existence of supremum of finite number of elements in V. Moreover, $v \geq w$ implies $-v \leq -w$, so the existence of finite infima thus implied.

Here are a few important number of definitions

$$sup(v,-v) = |v| \quad (\textbf{absolutevalueofv})$$
$$sup(v,0) = v^+ \quad (\textbf{positivepartofv})$$
$$sup(-v,0) = v^- \quad (\textbf{negativepartofv}).$$

Remark 2. Some compatibility axiom between norm and order is required to move from a vector lattice to a Banach lattice. It is considered in the following short way:

$$|v| \leq |w| \quad implies \quad \|v\| \leq \|w\|. \tag{1}$$

The norm defined on a vector lattice is called as a lattice norm.

Now, we are in position to define a Banach lattice in a formal way.

Definition 3

A *Banach lattice* is a Banach space V endowed with an ordering $\leq$, such that $(V, \leq)$ is a vector lattice with a lattice norm defined on it.

A Banach lattice transforms to *Banach lattice algebra*, provided $u, v \in V_+$ implies $uv \in V_+$.

A linear mapping T from an ordered Banach space V into itself is *positive* (denoted by: $T \geq 0$) if $T(v) \in V_+$, for all $v \in V_+$. The set of all positive linear mappings forms a convex cone in the space $L(V)$ of all linear mappings from V into itself, defining the natural ordering of $L(V)$. The absolute value of T, if it exists, is given by

$$|T|(v) = sup\{T(u) : |u| \le v\}, \quad (v \in V_+).$$

Thus $T : V \to V$ is positive if and only if $|T|(v) \le T(|v|)$ holds for any $v \in V$.

Lemma 4. [8], *PP-249* A bounded linear operator T on a Banach lattice V is a positive contraction if and only if $\|(Tv)^+\| \le \|v^+\|$ for all $v \in V$.

An operator A on V satisfies the positive minimum principle if for all $v \in D(A)_+ = D(A) \cap V_+$, $\varphi \in V_+$,

$$\langle v, \varphi \rangle = 0 \quad implies \quad \langle Av, \varphi \rangle \ge 0. \tag{2}$$

Definition 5. A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space X is a family $\{Z(t)\}_{t\ge 0} \subset B(X)$ such that

(i) $Z(s)Z(t) = Z(s+t)$ for all $s, t \in R^+$.

(ii) Z(0)=I, the identity operator on X.

(iii) for each fixed $f \in X$, $Z(t)f \to f$ (with respect to the norm on X) as $t \to 0^+$.

Where $B(X)$ denotes the space of all bounded linear operators defined on a Banach space X.

Definition 6. The (infinitesimal) generator of $\{Z(t)\}_{t\ge 0}$ is the densely defined closed linear operator $A : X \supseteq D(A) \to R(A) \subseteq X$ such that

$$D(A) = \{f : f \in X, lim_{t\to 0^+} A_t f \ exists\, in\, X\}$$

$$Af = lim_{t\to 0^+} A_t f \quad (f \in D(A))$$

where, for $t > 0$,

$$A_t f = \frac{[Z(t) - I]f}{t} \quad (f \in X).$$

A Banach algebra X, with the multiplicative identity element e is called the *unital Banach algebra*.

We shall call the strongly continuous semigroup $\{Z(t)\}_{t\ge 0}$ defined on X, a *normalized semigroup*, whenever it satisfies

$$Z(t)(e) = e, \quad for\, all \quad t > 0. \tag{3}$$

The notion of normalized semigroup is inspired from normalized functionals [7].

Let $\{Z(t)\}_{t\ge 0}$ be a strongly continuous positive semigroup, defined on a Banach lattice V. The positivity of the semigroup is equivalent to

$$|Z(t)v| \le Z(t)|v|, \quad t \ge 0, \quad v \in V.$$

Where for positive contraction semigroups $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V we have;

$$\|(Z(t)v)^{+}\|\leq\|v^{+}\|, \quad for\, all\, v\in V.$$

The literature presented in [8], guarantees the existence of the strongly continuous positive semigroups and positive contraction semigroups on Banach lattice V with some conditions imposed on the generator of the strongly continuous positive semigroup and the very important amongst them is, that it must always satisfy (2).

A Banach lattice V is said to be *Banach Lattice Algebra* whenever for $u,v\in V_{+}$, $uv\in V_{+}$ and $\|uv\|\leq\|u\|\|v\|$.

The theory presented in next section, is defined on normalized semigroups of positive linear operators defined on a unital Banach lattice algebra (UBLA) V.

Hölder's Type and Minkowski's Type Inequalities

In this section, we present several consequences of the Jessen's type inequality for normalized positive C_0-semigroup defined on a Banach lattice algebra V [4]. The motivation for this paper is from [3], where such results are proved forisotonic linear functionals. These results take the form of Hölder's type and Minkowski's type inequalities.

Let $D_c(V)$ denotes the set of all differentiable convex operators $\varphi : V\to V$.

Theorem 1. *[4]* Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup on V such that it satisfies (3). For an operator $\varphi\in D_c(V)$ and $t\geq 0$;

$$\varphi(Z(t)f)\leq Z(t)(\varphi f), \quad f\in V. \tag{4}$$

For a strongly continuous semigroup of linear operators $\{Z(t)\}_{t\geq 0}$ defined on a Banach lattice X and strictly monotonic continuous operator $\psi : X\to X$, we define the generalized mean:

$$M_{\psi}(Z,f,t):=\psi^{-1}\{Z(t)\psi(f)\}, \quad f\in X. \tag{5}$$

Theorem 2. For a normalized semigroup of positive linear operators $\{Z(t)\}_{t\geq 0}$ defined on (UBLA) V and strictly monotonic continuous operators $\psi,\chi : V\to V$

$$M_{\psi}(Z,f,t)\leq M_{\chi}(Z,f,t), \quad f\in V, \tag{6}$$

provided either χ is increasing and $\varphi=\chi\circ\psi^{-1}$ is convex or χ is decreasing and φ is concave.

Proof: For $f\in V$, we have $\psi(f),\chi(f)\in V$ and therefore, $\varphi(\psi(f))=\chi(f)\in V$. Thus, if φ is convex, by Jessen's type inequality (4) we have for $f\in V$;

$$\varphi(Z(t)(\psi(f))) \leq Z(t)(\varphi(\psi(f))) \\ = Z(t)(\chi(f)).$$

Hence, if χ is increasing then χ^{-1} is also increasing and we finally obtain

$$\chi^{-1}[\varphi(Z(t)(\psi(f)))] \leq \chi^{-1}[Z(t)(\chi(f))]$$

and the assertion (6) follows. If φ is concave then $-\varphi$ is convex and one can obtain the required inequality similarly.

Definition 3. [10] Let V be a Banach algebra with unit e. For $f \in V$, we define a function $log(f)$ from V to V;

$$log(f) = -\sum_{n=1}^{\infty} \frac{(e-f)^n}{n} = -(e-f) - \frac{(e-f)^2}{2} - \frac{(e-f)^3}{3} - \ldots$$

for $\| (e-f) \| \leq 1$.

In correspondence with the usual definition of generalized power means for isotonic functionals [1], we shall define the generalized power means for semigroup of operators, as follows.

Definition 4. Let X be a Banach space and $\{Z(t)\}_{t \in R}$ the C_0-semigroup of linear operators on X. For $f \in X$ and $t \in R_+$, the genralized power mean is defined as;

$$M_{G_r}(Z,f,t) = \{(Z(t)[f^r])^{1/r}, r \neq 0 exp[Z(t)[log(f)]], r = 0.(7)$$

As an application of Theorem (2), it follows as a special case that;

$$M_{G_r}(Z,f,t) \leq M_{G_s}(Z,f,t), \quad -\infty \leq r \leq s \leq \infty.$$

Lemma 5. Let $\{Z(t)\}_{t \geq 0}$ be the positive C_0-semigroup defined on V such that it satisfies (3). For a convex operator $\varphi : V \to V$ and $t \geq 0$, we have;

$$\frac{\varphi[Z(t)[f_1 h_1]]}{Z(t)[f_1]} \leq \frac{Z(t)[f_1 \varphi[h_1]]}{Z(t)[f_1]}, \quad f_1, h_1 \in V_+. \quad (8)$$

Proof: For $f \in V_+$ we have $\varphi[f] \in V$. Since V is a lattice algebra, $f, k \in V_+$ implies $fk \in V_+$, therefore the set of operators defined by;

$$F_f(t) := \frac{Z(t)[fk]}{Z(t)[f]}, \quad f \in V_+, t \geq 0,$$

is a semigroups of positive linear operators satisfying $F_f(t)[e]=e$. Thus the assertion (8) follows from (4).

One can observe that when r is any integer (positive or negative), the C_0-semigroup property implies that $Z(t)^r = Z(rt)$. While we can generalize it for $r \in R_+$. For example take $Z(1/2t)Z(1/2t) = Z(t)$ and thus we get $Z(t)^{1/2} = Z(1/2t)$ .For $r \in R_+$, the generator of $\{Z(rt)\}_{t\geq 0}$ is $(rA, D(A))$. Such semigroups are often called *rescaled semigroups*. (See e.g. [4,8]).

Next, we prove a Hölder's type inequality for positive C_0-semigroup of operators, assuming the fractional powers of elements in Banach algebra exist.

Theorem 6. *Hölder's Type Inequality For C_0-semigroups* Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup defined on V. If $p>1$ and $q=\frac{p}{p-1}$ so $p^{-1}+q^{-1}=1$, then if $f,g,h \in V_+$ and $fg^p, fh^q, fgh \in V_+$, we have for $t \geq 0$;

$$Z(t)[fgh] \leq [Z(t)]^{1/p}[gf^p].[Z(t)]^{1/q}[fh^q] \tag{9}$$

Proof: Since $fh^q \in V_+$, we have for $t \geq 0$, $Z(t)[fh^q] \in V_+$. For $p>1$, (9) follows from (8) by substituting;

$$\varphi(f) = f^p, \quad h_1 = gh^{-q/p}, \quad f_1 = fh^q.$$

Theorem 7. *Minkowski's Type Inequality For C_0-semigroups* Let $\{Z(t)\}_{t\geq 0}$ be the positive C_0-semigroup defined on V. If $p>1$ and $f,g,h \in V_+$ such that $hf^p, hg^p, h(f+g)^p \in V_+$, then;

$$Z(t)[h(f+g)^p] \leq Z(t)^{1/p}[hf^p] + Z(t)^{1/p}[hg^p], \quad f \geq 0. \tag{10}$$

Proof: For $f,g,h \in V_+$ and $p>1$, we have

$$h(f+g)^p = hf(f+g)^{p-1} + hg(f+g)^{p-1}$$

The assertion (10) follows by using (9).

Dresher's Type Inequality

First, we introduce two-parameter family of means in the following way.

Definition 1. Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous semigroup defined on a Banach algebra X. Then the two-parameter family of means $B_{r,s}(Z,f,t)$ for $r,s \in R$ is defined by;

$$B_{r,s} = \{\frac{Z(t)[f^r]}{Z(t)[f^s]}\}^{\frac{1}{r-s}}, \quad r \neq s,$$

$$B_{r,r} = exp\{\frac{Z(t)[f^r logf]}{Z(t)[f^r]}\} \tag{11}$$

Theorem 2. *Dresher's Type Inequality* Let $\{Z(t)\}_{t\geq 0}$ be a positive C_0-semigroup defined on a Banach lattice algebra V. Then for $f \in V_+$ and $p,q,r,s \in R$, we have;

$$B_{r,s}(Z,f,t) \leq B_{p,q}(Z,f,t) \quad r \leq p, s \leq q \quad and \quad r \neq s, p \neq q. \tag{12}$$

Proof: Let $p,q,r,s \in R$ such that $r \leq p, s \leq q$ and $r \neq s, p \neq q$. When applying the known result for convex functions

$$\frac{\varphi(r)-\varphi(s)}{r-s} \leq \frac{\varphi(p)-\varphi(q)}{p-q}, \tag{13}$$

to the convex operator $\varphi(x) = \log Z(t)[f^x]$, we can obtain (12).

We now show that (12) holds even if $r = s$ or $p = q$. To prove this we use the fact that $M_{G_r}(Z,f,t)$ is increasing function of $r \in R$. In particular for $f \in V_+$;

$$(Z(t)[f^{s-r}])^{\frac{1}{s-r}} \leq \exp[Z(t)logf] \leq (Z(t)[f^{r-s}])^{\frac{1}{r-s}}, \quad s < r. \tag{14}$$

Apply (14) to the positive semigroup (see Lemma 5) $Z_m(t)g := \frac{Z(t)[f^m g]}{Z(t)[f^m]}$. By taking $m = s$ the right-hand inequality (14) reduces to

$$B_{s,s}(Z,f,t) \leq B_{r,s}(Z,f,t), \quad s < r.$$

Similarly, by taking $m = r$ the left-hand inequality of (14) reduces to

$$B_{r,s}(Z,f,t) \leq B_{r,r}(Z,f,t), \quad s < r.$$

By these two inequalities we conclude that the inequality (12) holds for $r = s$ or $p = q$.

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ABOUT JENSEN'S INEQUALITY AND CAUCHY'S TYPE MEANS FOR POSITIVE C_0-SEMIGROUPS

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Abstract. A Jensen's type inequality is obtained for a semigroup of positive linear operators and a superquadratic mapping defined on a Banach lattice algebra. The corresponding mean value theorems conduct the authors to find a new set of Cauchy's type means.

Keywords: Jensen's type inequalities; Positive semigroup of operators; Cauchy's type means; Superquadratic mappings.

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1. Introduction and preliminaries

A consequential theory of Cauchy type means has been developed [4, 5, 6, 7, 8, 9], which is both substantial and elegant. In this paper we shall define new means on the C_0-semigroup of bounded linear positive operators, defined on a Banach lattice algebra. The intention to generalize the concept of Cauchy's type means for operator-semigroups, is not very unaccustomed. As recently in [10], a new theory of power means is introduced on a C_0-group of continuous linear operators and Cauchy's type mean are obtained.

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The notion of Banach lattice was introduced to get a common abstract setting. Within this framework, one can talk about the ordering of elements. Therefore, the phenomena related to positivity can be generalized. It had mostly been studied in various types to spaces of real-valued functions, e.g. the space $C(K)$ of continuous functions over a compact topological space K, the Lebesque space $L^1(\mu)$ or even more generally the space $L^p(\mu)$ constructed over measure space $(X, \sum, \mu)$ for $1 \leq p \leq \infty$. We shall use without further explanation the terms order relation (ordering), ordered set, supremum, infimum.

Before moving on, to define Banach lattice, we shall firstly go through the definition of vector lattice.

Definition 1.1. A (real) vector space V endowed with an ordering $\geq$, such that it satisfies

O_1**::** $v \leq w$ implies $v+u \leq w+u$ for all $u, v, w \in V$,
O_2**::** $v \geq 0$ implies $\lambda v \geq 0$ for al $v \in V$ and $\lambda \geq 0$,

is known as an *ordered vector space* (see [11]).

It can be easily noted that, O_1 expresses the translation invariance and thus implies that the ordering of an ordered vector space V can be completely determined by the positive part $V_+ = \{v \in V : v \geq 0\}$ of V. In other words, $v \leq w$ if and only if $w - v \in V_+$.

Moreover, the other property O_2, shows that the positive part of V is a convex set and a cone with vertex 0 (mostly called the *positive cone* of V).

If for any two elements $v, w \in V$, a supremum $sup(v, w)$ and thus an infumum $inf(v, w)$ can be defined, an ordered vector space V turns into a *vector lattice*. It is trivially understood that the existence of supremum of any two elements in an ordered vector space implies the existence of supremum of finite number of elements in V. Moreover, $v \geq w$ implies $-v \leq -w$, so the existence of finite infima therefore implied.

Below are few importantly defined quantities;

$$sup(v, -v) = |v|, \quad \textbf{(absolute value of v)}$$
$$sup(v, 0) = v^+, \quad \textbf{(positive part of v)}$$
$$sup(-v, 0) = v^-. \quad \textbf{(negative part of v)}$$

Some compatibility axiom between norm and order is required to move from vector lattice to a Banach lattice. Which is given in the following short way:

$$|v| \leq |w| \quad implies \quad \|v\| \leq \|w\|. \tag{1}$$

The norm defined on a vector lattice is called a lattice norm.

Now, we are in position to define a Banach lattice in a formal way.

Definition 1.2.

- A *Banach lattice* is a Banach space V endowed with an ordering $\leq$, such that $(V, \leq)$ is a vector lattice with a lattice norm defined on it.
- A Banach lattice with the property that, $u, v \in V_+$, implies $uv \in V_+$, is called *Banach lattice algebra*. If the multiplicative identity element $e \in V$, it ultimately turns to unital Banach lattice algebra.

A linear mapping T from an ordered Banach space V into itself is *positive* (denoted by: $T \geq 0$) if $T(v) \in V_+$, for all $v \in V_+$. The set of all positive linear mappings forms a convex cone in the space $L(V)$ of all linear mappings from V into itself, defining the natural ordering of $L(V)$. The absolute value of T, if it exists, is given by

$$|T|(v) = sup\{T(u) : |u| \leq v\}, \quad (v \in V_+).$$

Thus $T : V \to V$ is positive if and only if $|T(v)| \leq T(|v|)$ holds for any $v \in V$.

Lemma 1.1. **[[11], P-249]** *A bounded linear operator T on a Banach lattice V is a positive contraction if and only if $\|(Tv)^+\| \leq \|v^+\|$ for all $v \in V$.* □

An operator A on V satisfies the positive minimum principle if for all $v \in D(A)_+ = D(A) \cap V_+$, $\phi \in V'_+$

$$\langle v, \phi \rangle = 0 \quad implies \quad \langle Av, \phi \rangle \geq 0. \tag{2}$$

Definition 1.3. A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space X is a family $\{Z(t)\}_{t \geq 0} \subset B(X)$ such that

(i): $Z(s)Z(t) = Z(s+t)$ for all $s, t \in \mathbb{R}^+$.

(ii): Z(0)=I, the identity operator on X.

(iii): for each fixed $f \in X, Z(t)f \to f$(with respect to the norm on X) as $t \to 0^+$,

where $B(X)$ denotes the space of all bounded linear operators defined on a Banach space X.

Definition 1.4. The (infinitesimal) generator of $\{Z(t)\}_{t\geq 0}$ is the densely defined closed linear operator $A : X \supseteq D(A) \to R(A) \subseteq X$ such that

$$D(A) = \{f : f \in X, lim_{t\to 0^+} A_t f \; exists \, in X\}$$

$$Af = lim_{t\to 0^+} A_t f \;\; (f \in D(A)),$$

where, for $t > 0$,

$$A_t f = \frac{[Z(t) - I]f}{t} \;\; (f \in X).$$

□

Let $\{Z(t)\}_{t\geq 0}$ be the strongly continuous positive semigroup, defined on a Banach lattice V. The positivity of the semigroup is equivalent to

$$|Z(t)f| \leq Z(t)|f|, \quad t \geq 0, \quad f \in V,$$

where for positive contraction semigroups $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V we have;

$$\|(Z(t)f)^+\| \leq \|f^+\|, \quad for\, all\, f \in V.$$

The literature presented in [11], guarantees the existence of the strongly continuous positive semigroups and positive contraction semigroups on Banach lattice V with some conditions imposed on the generator of the strongly continuous positive semigroup and the very important amongst them is, that it must always satisfy (2).

For X be a unital Banach algebra with identity element e. We shall call the strongly continuous semigroup $\{Z(t)\}_{t\geq 0}$ defined on X, a *normalized semigroup*, whenever it satisfies

$$Z(t)(e) = e, \quad for\, all \quad t > 0. \tag{3}$$

The notion of normalized semigroup is inspired from normalized functionals [13].

Example 1.1. *Let* $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$*, and* $X = C(\Gamma)$*. The rotation semigroup* $\{Z(t)\}_{t\geq 0}$ *is defined as,* $Z(t)f(z) = f(e^{it} \cdot z)$*,* $f \in X$*. The identity element* $E \in X$*, s.t. for all* $z \in \Gamma$*,* $E(z) = z$*.*

Then $Z(t)E(z) = E(e^{it} \cdot z) = e^{it} \cdot z$. Or we can say that any complex number $z = e^{ix}$ is mapped to $e^{i(x+t)}$. $Z(t)$ satisfies (3) only when t is a multiple of 2π. It can easily be verified that $Z' = \{Z(2\pi t)\}_{t\geq 0}$ is a subgroup of $Z = \{Z(t)\}_{t\geq 0}$, as $Z(2\pi t)Z(2\pi s) = Z(2\pi(t+s))$. Therefore $Z' = \{Z(2\pi t)\}_{t\geq 0}$ is a normalized semigroup. □

For real and continuous functions φ, χ on a closed interval $K := [k_1, k_2]$, such that φ, χ are differentiable in the interior of K and $\chi' \neq 0$, throughout the interior of K. A very well know Cauchy mean value theorem guarantees the existence of of a number $\zeta \in (k_1, k_2)$, such that

$$\frac{\varphi'(\zeta)}{\chi'(\zeta)} = \frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}.$$

Now, if the function $\frac{\varphi'}{\chi'}$ is invertible, then the number ζ is unique and

$$\zeta := \Big(\frac{\varphi'}{\chi'}\Big)^{-1}\Big(\frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}\Big).$$

The number ζ is called *Cauchy's mean value* of numbers k_1, k_2. It is possible to define such a mean for several variables, in terms of divided difference. Which is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{[k_1, k_2, ..., k_n]\varphi}{[k_1, k_2, ..., k_n]\chi}\Big).$$

This mean value was first defined and examined by Leach and Sholander [**?**]. The integral representation of Cauchy mean is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{\int_{E_{n-1}} \varphi^{n-1}(k.u)du}{\int_{E_{n-1}} \chi^{n-1}(k.u)du}\Big)',$$

where $E_{n-1} := \{(u_1, u_2, ..., u_n) : u_i \geq 0, 1 \leq i \leq n,\ , \sum_{i=1}^{n-1} u_i \leq 1\}$, is (n-1) dimensional simplex, $u = (u_1, u_2, ..., u_n), u_n = 1 - \sum_{i=1}^{n-1} u_i, du = du_1 du_2 ... du_n$ and $k.u = \sum_{i=1}^{n} u_i k_i$.

A mean which can be expressed in the similar form as of Cauchy mean, is called *Cauchy type mean*. The purpose of our work is to introduce new means of Cauchy type defined on aC_0-semigroup of positive operators.

2. Jensen's type inequality and corresponding means

The Jensen type inequality for superquadratic function on isotonic linear functionals, is given in [[13], Theorem 10]. In [1], the corresponding Cauchy type means are defined.

In the present paper, we shall firstly prove the Jensen's type inequality for semigroup of positive linear operators defined on a Banach lattice algebra. Result will be followed by some generalized mean value theorems, bringing in a new set of Cauchy type means.

Definition 2.1. Let V be a Banach lattice algebra. A mapping $\phi : V_+ \to V$ is superquadratic, provided that for all $v \geq 0$ there exists a constant vector $C(v)$ such that

$$\phi(u) - \phi(v) - \phi(|u-v|) \geq C(v)(u-v) \tag{4}$$

for all $u \geq 0$. We say that the mapping ϕ is *subquadratic* if $-\phi$ is superquadratic.

Theorem 2.1. *Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on Banach lattice algebra V. Then for $g \in V_+$ and the continuous superquadratic mapping $\phi : V_+ \to V$, we have;*

$$\phi\left([Z(t)g]^{-1}[Z(t)(gf)]\right) \leq \frac{Z(t)[g\phi(f)] - Z(t)\left[g\phi\left(\left|f-[Z(t)g]^{-1}[Z(t)(gf)]\right|\right)\right]}{[Z(t)g]}, \quad f \in V_+. \tag{5}$$

If ϕ is subquadratic then a reversed inequality in (5) holds.

Proof. Since the mapping ϕ is superquadratic, inequality (4) holds for all $u, v \geq 0$. As $f, g \geq 0$ and the operator $Z(t)$ is positive for all $t \geq 0$, we have $[Z(t)g]^{-1}[Z(t)(gf)] \geq 0$. Setting $u = f$ and $v = [Z(t)g]^{-1}[Z(t)(gf)]$ in (4), we obtain;

$$\begin{aligned} \phi(f) \geq\ & \phi\left([Z(t)g]^{-1}[Z(t)(gf)]\right) + C\left[[Z(t)g]^{-1}[Z(t)(gf)]\right]\left[f - [Z(t)g]^{-1}[Z(t)(gf)]\right] \\ & + \phi\left(\left|f - [Z(t)g]^{-1}[Z(t)(gf)]\right|\right), \end{aligned}$$

for all $t \geq 0$. Multiplying the above inequality by $g \in V_+$, we get

$$\begin{aligned} g\phi(f) \geq\ & g\phi\left([Z(t)g]^{-1}[Z(t)(gf)]\right) + C\left[[Z(t)g]^{-1}[Z(t)(gf)]\right]\left[gf - g[Z(t)g]^{-1}[Z(t)(gf)]\right] \\ & + g\phi\left(\left|f - [Z(t)f]^{-1}[Z(t)(gf)]\right|\right). \end{aligned}$$

By applying the operator $Z(t)$ on both sides, we get for all $t \geq 0$;

$$\begin{aligned} Z(t)[g\phi(f)] &\geq Z(t)[g]\phi\Big([Z(t)g]^{-1}[Z(t)(gf)]\Big) \\ &+C\Big[[Z(t)g]^{-1}[Z(t)(gf)]\Big]\Big[Z(t)(gf)-Z(t)[g][Z(t)g]^{-1}[Z(t)(gf)]\Big] \\ &+Z(t)\Big[g\phi\Big(\Big|f-[Z(t)f]^{-1}[Z(t)(gf)]\Big|\Big)\Big]. \end{aligned}$$

The assertion (5) follows directly. □

Throughout the remaining article, V shall denote the (real) unital Banach lattice algebra with identity element e, until and unless stated otherwise.

Theorem 2.2. *Let $\{Z(t)\}_{t\geq 0}$ be a normalized strongly continuous positive semigroup of operators defined on V; then for a continuous superquadratic operator $\phi : V_+ \to V$, we have*

$$\phi[Z(t)f] \leq Z(t)[\phi(f)] - Z(t)[\phi(|f - Z(t)f|)], \quad f \in V_+. \tag{6}$$

If the mapping ϕ is subquadratic, then the inequality above is reversed.

Proof. Since $\{Z(t)\}_{t\geq 0}$ is a normalized semigroup it must satisfy (3). By taking $g \equiv e$ in Theorem (2.1), we obtain (6). □

Definition 2.2. Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous normalized positive semigroup of operators defined on V; then for a continuous operator $\phi : V_+ \to V$, we define an other operator $\Lambda_\phi : V_+ \to V$;

$$\Lambda_\phi := Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)], \quad f \in V_+. \tag{7}$$

If ϕ is continuous superquadratic mapping then, $\Lambda_\phi \geq 0$.

Below we give an operator analogue of [[2], Lemma 3.1].

Lemma 2.1. *Suppose $\phi : V_+ \to V$ is continuously differentiable and $\phi(0) \leq 0$. If ϕ' is superadditive or $f \to \frac{\phi'(f)}{f}$, $\quad f \in V_+$, is increasing, then ϕ is superquadratic.*

Lemma 2.2. *Let $\phi \in C^2[V_+]$ and $u, U \in V$ be such that*

$$u \leq \Big(\frac{\phi'(f)}{f}\Big)' = \frac{f\phi''(f) - \phi'(f)}{f^2} \leq U, \quad \forall f > 0. \tag{8}$$

Consider the operators $\phi_1, \phi_2 : V_+ \to V$ *defined as:*

$$\phi_1(f) = \frac{Uf^3}{3} - \phi(f), \quad \phi_2 = \phi(f) - \frac{uf^3}{3}.$$

Then the mappings $f \to \frac{\phi_1'(f)}{f}$ *and* $f \to \frac{\phi_2'(f)}{f}$ *are increasing. If also* $\phi_i(0) = 0, i = 1, 2$, *then these are superquadratic mappings.*

Proof. By using the inequality (6), it can be easily seen that the mappings $f \to \frac{\phi_1'(f)}{f}$ and $f \to \frac{\phi_2'(f)}{f}$ are increasing. Moreover, if $\phi_i(0) = 0, i = 1, 2$, Lemma (2.1) implies these to be superquadratic. □

Theorem 2.3. *Let* $\{Z(t)\}_{t\geq 0}$ *be a positive normalized* C_0*-semigroup of operators defined on* V *and* $\frac{\phi'}{f} \in C^1(V_+)$ *and* $\phi(0) = 0$, *then the following inequality holds*

$$\Lambda_\phi = \frac{\xi\phi''(\xi) - \phi'(\xi)}{3\xi^2}\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)(|f - Z(t)f|^3)\}, \quad f \in V_+. \tag{9}$$

Proof. Suppose that $u = min_{f\in V_+}(\frac{\phi'(f)}{f})'$ and $U = max_{f\in V_+}(\frac{\phi'(f)}{f})$ exists. Taking ϕ_1 instead of ϕ in (6), we get for $f \in V_+$;

$$Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)] \leq \frac{U}{3}\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\}.$$

Similarly, by taking ϕ_2 instead of ϕ in (6), we get for $f \in V_+$;

$$Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)] \geq \frac{u}{3}\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\}.$$

Since, $\phi = f^3$ is superquadratic and $Z(t) \in \{Z(t)\}_{t\geq 0}$ is the positive operator, therefore

$$Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|] \geq 0, \quad f \in V_+.$$

By combining the above two inequalities and using (8), we obtain that, there exists $\xi \in V_+$, such that the assertion (9) holds.

Theorem 2.4. *Let* $\{Z(t)\}_{t\geq 0}$ *be a positive normalized* C_0*-semigroup of operators defined on* V *and* $\frac{\phi'}{f}, \frac{\psi'}{f} \in C^1(V_+)$ *such that,* $\phi(0) = \psi(0) = 0$, *we have*

$$\frac{\Lambda_\phi}{\Lambda_\psi} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = K(\xi), \quad \xi \in V_+, \tag{10}$$

provided the denominators do not vanish. If K is invertible, we have the following new mean;

$$\xi = K^{-1}\Big(\frac{\Lambda_\phi}{\Lambda_\psi}\Big), \quad \Lambda_\psi \neq 0, \tag{11}$$

Proof. Lets consider a function $\Omega = c_1\phi - c_2\psi$, where

$$c_1 = \Lambda_\psi, \quad c_2 = \Lambda_\phi.$$

Then for $f \in V_+$;

$$\frac{\Omega'}{f} = c_1\frac{\phi'}{f} - c_2\frac{\psi'}{f} \in C^1(V_+).$$

One may calculate that $\Lambda_\Omega = 0$ and using Lemma (2.2) with $\phi = \Omega$ we obtain;

$$[c_1(\xi\phi''(\xi) - \phi'(\xi)) - c_2(\xi\psi''(\xi) - \psi'(\xi))]\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\} = 0, f \in V_+.$$

Since $\phi = f^3$ is superquadratic and $\{Z(t)\}_{t\geq 0}$ is semigroup of positive operators, therefore we may conclude that

$$\frac{c_2}{c_1} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \frac{\Lambda_\phi}{\Lambda_\psi}, \quad \xi \in V_+,$$

providing the denominator do not vanish. This completes the proof. □

We shall denote the set of all invertible strictly monotone continuous operators, defined from V to itself, by $G_M(V)$.

Definition 2.3. For a positive normalized C_0-semigroup $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V and $F \in G_M(V)$, we define the generalized mean:

$$M_F(Z, f, t) := F^{-1}\{Z(t)F(f)\}, \quad f \in X. \tag{12}$$

For the sake of simplicity, the set of all elements of G_M, whose second order derivative (in Gateaux's sense) exits, shall be denoted by $C^2G_M(V)$.

Theorem 2.5. *Let $\{Z(t)\}_{t\geq 0}$ be a positive normalized C_0-semigroup defined on V and $H, F, K \in C^2G_M(V)$. Let for $f \in V_+, \frac{H\circ F^{-1}(f)}{f}, \frac{K\circ F^{-1}(f)}{f} \in C^1(V)$ with $H\circ F^{-1}(0) = 0 = K\circ F^{-1}(0)$, then for $f \in V_+$ and $t \geq 0$;*

$$\begin{aligned} & \frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(\tau)f] - FM_F(Z,f,t)|, f, t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(\tau)f] - FM_F(Z,f,t)|, f, t))} \\ (13) \quad = \; & \frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}}, \end{aligned}$$

holds for some $\eta \in V_+$, provided the denominator do not vanish.

Proof. By choosing the operators ϕ and ψ in Theorem 2.4, such that

$$\phi = H \circ F^{-1}, \quad \psi = K \circ F^{-1} \quad and \quad Z(t)f = F[Z(t)f], \quad f \in V_+,$$

where $H, F, K \in C^2G_M(V)$. We find that there exists $\xi \in V_+$, such that

$$\begin{aligned} &\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|,f,t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|,f,t))} \\ = \; &\frac{\xi\{H''(F^{-1}\xi)F'(F^{-1}\xi) - H'(F^{-1}\xi)F''(F^{-1}\xi) - H'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\}}{\xi\Big\{K''(F^{-1}\xi)F'(F^{-1}\xi) - K'(F^{-1}\xi)F''(F^{-1}\xi) - K'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\Big\}}. \end{aligned}$$

Therefore, by setting $F^{-1}(\xi) = \eta$, we find that there exists $\eta \in X$, such that the assertion (13) follows directly. □

The above theorem accredit us to define new means. Set

$$L(\eta) = \frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}},$$

and when $F \in G(V)$;

$$\eta = L^{-1}\Big(\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|,f,t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FMF(Z,f,t)|,f,t))}\Big)$$

Remark 2.1. For $(V, \|.\|)$ a Banach lattice algebra, it follows from Theorem 2.5 that

$$m \leq \Big\|\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|,f,t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|,f,t))}\Big\| \leq M,$$

Where m and M are respectively, the minimum and maximum values of

$$\Big\|\frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}}\Big\|, \quad \eta \in V.$$

□

Definition 2.4. [12] Let V be a Banach algebra with unit e. For $f \in V$, we define a function $log(f)$ from V to V;

$$log(f) = -\sum_{n=1}^{\infty} \frac{(e-f)^n}{n} = -(e-f) - \frac{(e-f)^2}{2} - \frac{(e-f)^3}{3} - \ldots$$

for $||(e-x)|| \leq 1$. □

In correspondence with the usual definition of generalized power means for isotonic functionals [1], we shall define the generalized power means for semigroup of operators, as follows.

Definition 2.5. Let X be a Banach space and $\{Z(t)\}_{t\in\mathbb{R}}$ the C_0-semigroup of linear operators on X. For $f \in X$ and $t \in \mathbb{R}_+$, the genralized power mean is defined as;

$$M_{G_r}(Z,f,t) = \begin{cases} \Big(Z(t)[f^r]\Big)^{1/r}, & r \neq 0, \\ \\ exp[Z(t)[log(f)]], & r = 0. \end{cases} \tag{14}$$

Now, we shall prove an important result which is going to lead us, to define the Cauchy's type means on C_0-semigroup of operators.

Corollary 2.1. *Let all the conditions of Theorem 2.5 are satisfied. For $r,s,l \in \mathbb{R}_+$ such that $r \neq l; l \neq 2s$, we have*

$$\frac{M^r_{G_r}(Z,f,t) - M^r_{G_s}(Z,f,t) - M^r_{G_r}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)}{M^l_{G_l}(Z,f,t) - M^l_{G_s}(Z,f,t) - M^l_{G_l}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)} = \frac{r(r-2s)}{l(l-2s)}\eta^{r-l} \tag{15}$$

The assertion (15) holds for some η, provided that the denominators do not vanish.

Proof. For $r,s,l \in \mathbb{R}_+$ and $f \in V_+$, if we set

$$H(f) = f^r, \quad F(f) = f^s, \quad K(f) = f^l$$

in Theorem (2.5), the assertion in (15) follows directly.

Ultimately, we shall define means of the Cauchy's type on C_0-semigroup of positive linear operators defined on Banach lattice algebra V.

Definition 2.6. Let $r,s,l \in \mathbb{R}_+$ and $\{Z(t)\}_{t\geq 0} \subset B(V)$ be a normalized C_0-semigroup of positive linear operators on a unital Banach lattice algebra V. Then

(16)

$$\mathfrak{M}^{l,s}_{G_r}(Z,f,t) = \Big(\frac{l(l-2s)}{r(r-2s)}\frac{M^r_{G_r}(Z,f,t) - M^r_{G_s}(Z,f,t) - M^r_{G_r}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)}{M^l_{G_l}(Z,f,t) - M^l_{G_s}(Z,f,t) - M^l_{G_l}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)}\Big)^{\frac{1}{r-l}}.$$

is a mean of the Cauchy's type. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases, as in [8].

3. Conclusion

Firstly, we have proved a Jensen's type inequality for a normalized semigroup of positive linear operators and a superquadratic mapping, defined on a Banach lattice algebra. A systematic procedure has been used to prove the corresponding mean value theorems, which lead us to a new set of means. These means are Cauchy's type means for the mentioned operators. By following the similar procedure, many functional inequalities can be generalized for the operator semigroups and corresponding means can be obtained.

Conflict of Interests

The authors declare that there is no conflict of interests.

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Hermite-Hadamard's Inequality and Cauchy's Mean-Operators For Positive C_0-Semigroups

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Abstract The characteristics of superquadratic mappings are used to obtain the Hermite-Hadamard's type inequality for a semigroup of positive linear operators defined on a Banach lattice algebra. The corresponding mean value theorems, Cauchy's type mean-operators and related results are also discussed.

Keywords Hermite-Hadamard type inequalities, Positive semigroup of operators, Cauchy's type means, Superquadratic Mappings.

Mathematics Subject Classification (2010) 47D03 · 46B42 · 43A35 · 43A17

1 Introduction and Preliminaries

Due to the vastness of the theory of inequalities and the advancement of the theory of strongly continuous semigroups of operators, the fusion is exceptionally vital subject of this span. Recently in [6–9], a new theory of means and inequalities has been introduced on C_0-semigroups of continuous linear operators. This article is a contribution towards this concept.

In this note, we define new means on the C_0-semigroup of bounded linear positive operators defined on a Banach lattice algebra. The intention to generalize the concept of Cauchy's type means for operator-semigroups, is not very new. Since in [9], a Jensen's type inequality for a semigroup of positive linear operators and a superquadratic mapping, defined on a Banach lattice algebra, is obtained.
The concept of network Banach was introduced to obtain an abstract framework where one could talk about the order of the elements. Therefore , phenomena related

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to the positivity may be generalized. This mechanism was usually studied in different types of real-function spaces, e.g. the space $C(K)$ of continuous functions over a compact topological space K, the Lebesque space $L^1(\mu)$ or even more generally the space $L^p(\mu)$ constructed over measure space $(X,\sum,\mu)$ for $1 \leq p \leq \infty$. We use without further explanation, the terminologies as order relation (ordering), ordered set, supremum, infimum.
In the first place, we will go through the formal definition of a vector lattice.

Definition 1 ([11]) A (real) vector space E equipped with an ordering $\geq$ so that the satisfied

O_1: $f \leq g$ implies $f+h \leq g+h$ for all $f,g,h \in E$,
O_2: $f \geq 0$ implies $\lambda f \geq 0$ for al $f \in E$ and $\lambda \geq 0$,

is called an *ordered vector space*.

It is easily seen that O_1 expresses the translation invariance. It thus implies that the ordering of an ordered vector space E, is completely determined by the positive part $E_+ = \{f \in V : f \geq 0\}$ of E. In other words, $f \leq g$ if and only if $g - f \in E_+$. In addition, the other axiom O_2 shows that the positive part of E is a convex set and a cone with vertex 0 (usually called the *positive cone* of E).
If for two elements $f,g \in E$, a supremum $\sup(f,g)$ and infumum $\inf(f,g)$ can be defined , ordered vector space E is a *vector lattice*. It is simple to note that, the existence of supremum of two elements in an ordered vector space E implies the existence of supremum of finite number of elements in E. Since $f \geq g$ means that $-f \leq -g$, therefore the existence of finite infima is promised. Here are some terminologies to be defined

$$\begin{aligned} \sup(f,-f) &= |f| \quad \text{(absolute value of f)} \\ \sup(f,0) &= f^+ \quad \text{(positive part of f)} \\ \sup(-f,0) &= f^- \quad \text{(negative part of f).} \end{aligned}$$

The Compatibility axiom between the norm and order is given in the following manner;

$$|f| \leq |g| \quad \text{implies} \quad \|f\| \leq \|g\|. \tag{1}$$

Therefore, the norm defined on a vector lattice is said to be a *lattice norm*.
Now we are able to define a Banach lattice formally.

Definition 2
- A *Banach lattice* is a Banach space E equipped with a lattice-norm, where $(E,\leq)$ is a vector lattice.
- If a Banach lattice E satisfies the property that,

$$f,g \in E_+ \quad \text{implies} \quad fg \in E_+,$$

then E is called a *Banach lattice algebra*.

□

A linear mapping $L : E \to E$, is *positive* (denoted by: $L \geq 0$), if $L(f) \in E_+$, for all $f \in E_+$. The set of all positive linear mappings forms a convex cone in the space $\mathfrak{L}(E)$ of all linear mappings from E into itself, defining the natural ordering of $\mathfrak{L}(E)$. If the absolute value of L exists, it is given by

$$|L|(f) = sup\{L(g) : |g| \leq f\}, \quad (f \in E_+).$$

Thus $L : E \to E$ is positive if and only if $|L(f)| \leq L(|f|)$ holds for any $f \in E$.

Lemma 1 ([11], P.249) *A bounded linear operator L on a Banach lattice E is a positive contraction if and only if* $\|(Lf)^+\| \leq \|f^+\|$ *for all* $f \in E$.

□

An operator A on E satisfies the *positive minimum principle* if for all $f \in D(A)_+ = D(A) \cap E_+$, $g' \in E'_+$

$$\langle f, g' \rangle = 0 \quad \text{implies} \quad \langle A(f), g' \rangle \geq 0. \tag{2}$$

Definition 3 A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space X is a family $\{T(t)\}_{t\geq 0} \subset B(X)$ such that

(i) $T(s)T(t) = T(s+t)$ for all $s,t \in \mathbb{R}^+$.
(ii) T(0)=I, the identity operator on X.
(iii) for each fixed $f \in X$, $T(t)f \to f$(with respect to the norm on X) as $t \to 0^+$.

Where $B(X)$ is the space of all bounded linear operators defined on a Banach space X into itself.

Definition 4 The (infinitesimal) generator of $\{T(t)\}_{t\geq 0}$ is the densely defined closed linear operator $A : X \supseteq D(A) \to R(A) \subseteq X$ such that

$$D(A) = \{f : f \in X, \lim_{t\to 0^+} A_t f \quad \text{exists in } X\}$$

$$Af = \lim_{t\to 0^+} A_t f \quad (f \in D(A))$$

where, for $t > 0$,

$$A_t f = \frac{[T(t) - I]f}{t} \quad (f \in X).$$

For a strongly continuous positive semigroup $\{T(t)\}_{t\geq 0}$ defined on a Banach lattice E, the positivity of the semigroup is equivalent to

$$|T(t)f| \leq T(t)|f|, \quad t \geq 0, \quad f \in E.$$

Where for positive contraction semigroups $\{T(t)\}_{t\geq 0}$, defined on a Banach lattice E we have;

$$\|(T(t)f)^+\| \leq \|f^+\|, \quad for all\, f \in E.$$

For detail information we refer to [11], which guarantees the existence of the strongly continuous positive semigroups and positive contraction semigroups on Banach lattice E. Certain conditions are imposed on the generator , while the most important of them is that it must still meet (2).

In accordance with the usual definition of power integral means , the power means for C_0-group of operators are defined.

Definition 5 ([6]) Let X be a Banach space and $\{T(t)\}_{t\in\mathbb{R}}$ the C_0-group of linear operators on X. For $f \in X$ and $t \in \mathbb{R}$, the power mean is defined as follows

$$M_r(T,f,t) = \begin{cases} \left\{\frac{1}{t}\int_0^t [T(\tau)]^r f d\tau\right\}^{1/r}, & r \neq 0 \\ \\ exp[\frac{1}{t}\int_0^t ln[T(\tau)]f d\tau], & r = 0. \end{cases} \tag{3}$$

□

For real and continuous functions ϕ, ξ on a closed interval $\Im := [a_1, a_2]$, such that ϕ, ξ are differentiable in the interior of $\Im$ and $\xi' \neq 0$, throughout the interior of $\Im$. The well-known Cauchy mean value theorem guarantees the existence of a number $\varepsilon \in (a_1, a_2)$, such that

$$\frac{\phi'(\varepsilon)}{\xi'(\varepsilon)} = \frac{\phi(a_1) - \phi(a_2)}{\xi_(a_1) - \xi(a_2)}.$$

Now, if the function $\frac{\phi'}{\xi'}$ is invertible, then the number ε is unique and

$$\varepsilon := \Big(\frac{\phi'}{\xi'}\Big)^{-1}\Big(\frac{\phi(a_1) - \phi(a_2)}{\xi_(a_1) - \xi(a_2)}\Big).$$

The number ε is called *Cauchy's mean value* of numbers a_1, a_2. It is possible to define such a mean for several variables, in terms of divided difference. Which is given by

$$\varepsilon := \Big(\frac{\phi^{n-1}}{\xi^{n-1}}\Big)^{-1}\Big(\frac{[a_1, a_2, ..., a_n]\phi}{[a_1, a_2, ..., a_n]\xi}\Big).$$

This mean value was first defined and examined by Leach and Sholander [10]. The integral representation of Cauchy mean is given by

$$\varepsilon := \Big(\frac{\phi^{n-1}}{\xi^{n-1}}\Big)^{-1}\Big(\frac{\int_{S_{n-1}} \phi^{n-1}(a.s)ds}{\int_{S_{n-1}} \xi^{n-1}(a.s)ds}\Big)',$$

where $S_{n-1} := \{(s_1, s_2, ..., s_n) : s_i \geq 0, 1 \leq i \leq n, \ , \sum_{i=1}^{n-1} s_i \leq 1\}$, is (n-1) dimensional simplex, $s = (s_1, s_2, ..., s_n), s_n = 1 - \sum_{i=1}^{n-1} s_i, \ ds = ds_1 ds_2 ... ds_n$ and $a.s = \sum_{i=1}^{n} s_i a_i$.

A mean which can be expressed as in the above equation is called ***Cauchy's type mean***. The purpose of our work is to introduce new means of Cauchy's type defined on a C_0-semigroup of positive operators.

2 Hermite-Hadamard Type Inequality and Corresponding Means

The Hermite-Hadamard type inequality for positive linear functionals is proved in [5]. In [1], the corresponding mean-value theorems are given, that ultimately lead to new means of Cauchy's type.
In this section , we derive the Hermite-Hadamard type inequality for semigroup of positive linear operators defined on a Banach lattice algebra. We also prove some generalized mean value theorems and define related Cauchy's type means.
Throughout the article, E denotes the real Banach lattice algebra, until and unless stated otherwise.

Definition 6 Let E be a Banach lattice algebra. A mapping $\psi : E_+ \to E$ is superquadratic, provided that for all $g_1 \geq 0$ there exists a fixed vector $C(g_1)$ such that

$$\psi(f_1) - \psi(g_1) - \psi(|f_1 - g_1|) \geq C(g_1)(f_1 - g_1) \tag{4}$$

for all $f_1 \geq 0$.

Theorem 1 *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on E; then for an integrable superquadratic operator $\psi : E_+ \to E$, we have*

$$\psi\Big[\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big] + \frac{1}{t}\int_0^t \psi\Big[\Big|[T(\tau)]f - \frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big|\Big] d\tau \leq \frac{1}{t}\int_0^t \psi[T(\tau)]f d\tau, \quad f \in E_+. \tag{5}$$

Proof: Let ψ be a superquadratic mapping, then (4) holds for all $f, g \in E_+$. Choosing $f_1 = [T(\tau)]f$ and $g_1 = \frac{1}{t}\int_0^t [T(\tau)]f d\tau$ in (4) we get

$$\psi[[T(\tau)]f] \geq \psi\Big[\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big] + C\Big[\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big]\Big[[T(\tau)]f - \frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big]$$
$$+\psi\Big[\Big|[T(\tau)]f - \frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big|\Big]$$

By integrating from $0 \to t$ we obtain;

$$\int_0^t \psi[[T(\tau)]f]d\tau \geq t.\psi\Big[\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big] + C\Big[\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big]\Big[\int_0^t [T(\tau)]f d\tau - t\Big\{\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big\}\Big]$$
$$+ \int_0^t \psi\Big[\Big|[T(\tau)]f - \frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big|\Big] d\tau,$$

or

$$\int_0^t \psi[[T(\tau)]f]d\tau \geq t.\psi\Big[\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big] + \int_0^t \psi\Big[\Big|[T(\tau)]f - \frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big|\Big] d\tau.$$

By multiplying $1/t$, we finally get the assertion (5).

Definition 7 Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on E; then for an integrable operator $\psi : E_+ \to E$, we define an other operator $\Lambda_\psi : E_+ \to E$

$$\Lambda_\psi := \frac{1}{t}\int_0^t \psi[T(\tau)]f d\tau - \psi\Big[\frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big] - \frac{1}{t}\int_0^t \psi\Big[\Big|[T(\tau)]f - \frac{1}{t}\int_0^t [T(\tau)]f d\tau\Big|\Big] d\tau, \quad f \in E_+. \tag{6}$$

If ψ is continuous superquadratic mapping then by (5), $\Lambda_\psi \geq 0$.

□

For simplicity throughout the expressions, we write $\frac{1}{t}\int_0^t [T(\tau)]fd\tau$ by $M_1(t)$. Therefore, Λ_ψ can be written as;

$$\Lambda_\psi := \frac{1}{t}\int_0^t \psi[T(\tau)]fd\tau - \psi[M_1(t)] - \frac{1}{t}\int_0^t \psi\Big[\Big|[T(\tau)]f - M_1(t)\Big|\Big]d\tau$$

Below we give an operator analogue of [[2], Lemma 3.1]

Lemma 2 *Suppose $\psi : E_+ \to E$ is continuously differentiable and $\psi(0) \leq 0$. If ψ' is super-additive or $f \to \frac{\psi'(f)}{f}$, $\quad f \in E_+$, is increasing, then ψ is superquadratic.*

□

Lemma 3 *Let $\psi \in C^2[E_+]$ and $k, K \in E$ be such that*

$$k \leq \Big(\frac{\psi'(f)}{f}\Big)' = \frac{f\psi''(f) - \psi'(f)}{f^2} \leq K, \quad \forall f > 0. \tag{7}$$

Consider the operators $\psi_1, \psi_2 : E_+ \to E$ defined as:

$$\psi_1(f) = \frac{Kf^3}{3} - \psi(f), \quad \psi_2 = \psi(f) - \frac{kf^3}{3}$$

Then the mappings $f \to \frac{\psi_1'(f)}{f}$ and $f \to \frac{\psi_2'(f)}{f}$ are increasing. If also $\psi_i(0) = 0, i = 1,2$, then these are superquadratic mappings.

Proof: By using the inequality (7), it can be easily seen that the mappings $f \to \frac{\psi_1'(f)}{f}$ and $f \to \frac{\psi_2'(f)}{f}$ are increasing. Moreover, if $\psi_i(0) = 0, i = 1,2$, Lemma (2) implies these to be superquadratic. □

Theorem 2 *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on E and $\frac{\psi'}{f} \in C^1(E_+)$ and $\psi(0) = 0$, then the following inequality holds*

$$\Lambda_\psi = \frac{\xi\psi''(\xi) - \psi'(\xi)}{3\xi^2}\{(M_3(t))^3 - (M_1(t))^3 - \frac{1}{t}\int_0^t |T(\tau)f - m_t|^3 d\tau\} \tag{8}$$

Proof: Suppose the conditions in Lemma 3 holds for all $f \in E_+$. Using ψ_1 instead of ψ in (5), we get;

$$\frac{1}{t}\int_0^t \psi[T(\tau)]fd\tau - \psi[M_1(t)] - \frac{1}{t}\int_0^t \psi\Big[\Big|[T(\tau)]f - M_1(t)\Big|\Big]d\tau \leq \frac{K}{3}\{(M_3(t))^3 - (M_1(t))^3 - \frac{1}{t}\int_0^t |T(\tau)f - m_t|^3 d\tau\}$$

Similarly, using ψ_2 instead of ψ in (5), we get;

$$\frac{1}{t}\int_0^t \psi[T(\tau)]fd\tau - \psi[M_1(t)] - \frac{1}{t}\int_0^t \psi\Big[\Big|[T(\tau)]f - M_1(t)\Big|\Big]d\tau \geq \frac{k}{3}\{(M_3(t))^3 - (M_1(t))^3 - \frac{1}{t}\int_0^t |T(\tau)f - m_t|^3 d\tau\}$$

By combining the above two inequalities and using intermediate value theorem [3], we have existence of $\xi \in E_+$ such that (8) holds. □

Theorem 3 *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on E and $\frac{\psi'}{f}, \frac{\phi'}{f} \in C^1(E_+)$ such that, $\psi(0) = \phi(0) = 0$, we have*

$$\frac{\Lambda_\psi}{\Lambda_\phi} = \frac{\xi\psi''(\xi) - \psi'(\xi)}{\xi\phi''(\xi) - \phi'(\xi)} = F(\xi), \quad \xi \in E_+, \tag{9}$$

provided the denominators do not vanish. If F is invertible, we have the following new mean;

$$\xi = F^{-1}\left(\frac{\Lambda_\psi}{\Lambda_\phi}\right), \quad \Lambda_\phi \neq 0, \tag{10}$$

Proof: Consider a function $\Omega = c_1\psi - c_2\phi$, where

$$c_1 = \Lambda_\phi, \quad c_2 = \Lambda_\psi.$$

Then for $f \in E_+$;

$$\frac{\Omega'}{f} = c_1\frac{\psi'}{f} - c_2\frac{\phi'}{f} \in C^1(E_+).$$

One may calculate that $\Lambda_\Omega = 0$ and using Lemma (3) with $\psi = \Omega$ we obtain;

$$[c_1(\xi\psi''(\xi) - \psi'(\xi)) - c_2(\xi\phi''(\xi) - \phi'(\xi))]\Big[(M_3(t))^3 - (M_1(t))^3 - \frac{1}{t}\int_0^t |T(\tau)f - m_t|^3 d\tau\Big] = 0, \quad f \in E_+.$$

Since $\psi = f^3$ is superquadratic mapping and $\{T(t)\}_{t\geq 0}$ is semigroup of positive operators, therefore we conclude that

$$\frac{c_2}{c_1} = \frac{\xi\psi''(\xi) - \psi'(\xi)}{\xi\phi''(\xi) - \phi'(\xi)} = \frac{\Lambda_\psi}{\Lambda_\phi}, \quad \xi \in E_+,$$

providing the denominator do not vanish. This completes the proof. □

Let$G$ denotes the set of invertible bounded linear operators $H : E \to E$. For a C_0-semigroup of positive operators $\{T(t)\}_{t\geq 0} \subset B(E)$ defined on E and $H \in G$, the quasi-arithmetic mean is given as [6];

$$M_H^\circ(T, f, t) = H^{-1}\left\{\frac{1}{t}\int_0^t H[T(\tau)f]d\tau\right\}, \quad f \in E_+, t \geq 0. \tag{11}$$

By ([4], Lemma 1.85), $B(E)$ is closed under composition of operators, therefore the above expressions exists and $M_H^\circ(T, f, t) \in E$. For simplicity, the set of all elements of G, whose second order derivative (in Gateaux's sense) exits, is denoted by $C^2G(E)$.

Theorem 4 *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on E and $H, F, K \in C^2G(E)$. Let for $f \in E_+$, $\frac{H\circ F^{-1}(f)}{f}, \frac{K\circ F^{-1}(f)}{f} \in C^1(E)$ with $H \circ F^{-1}(0) = 0 = K \circ F^{-1}(0)$, then for $f \in E_+$*

$$\begin{aligned}&\frac{H(M_H^\circ(T,f,t)) - H(M_F^\circ(T,f,t)) - H(M_H^\circ(F^{-1}|F[Z(\tau)f] - FM_F^\circ(T,f,t)|, f, t))}{K(M_H^\circ(T,f,t)) - K(M_F^\circ(T,f,t)) - K(M_K^\circ(F^{-1}|F[T(\tau)f] - FM_T^\circ(T,f,t)|, f, t))}\\ &= \frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}},\end{aligned} \tag{12}$$

holds for some $\eta \in E_+$, provided the denominator do not vanish.

Proof: By choosing the operators ψ and ϕ in Theorem 3, such that

$$\psi = H \circ F^{-1}, \quad \phi = K \circ F^{-1} \quad and \quad T(\tau)f = F[T(\tau)f], \quad f \in E_+,$$

where $H, F, K \in C^2G(E)$. We find that there exists $\xi \in E_+$, such that

$$\frac{H(M_H^\circ(T,f,t)) - H(M_F^\circ(T,f,t)) - H(M_H^\circ(F^{-1}|F[T(\tau)f] - FM_F^\circ(T,f,t)|,f,t))}{K(M_H^\circ(T,f,t)) - K(M_F^\circ(T,f,t)) - K(M_K^\circ(F^{-1}|F[T(\tau)f] - FM_F^\circ(T,f,t)|,f,t))}$$
$$= \frac{\xi\{H''(F^{-1}\xi)F'(F^{-1}\xi) - H'(F^{-1}\xi)F''(F^{-1}\xi) - H'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\}}{\xi\left\{K''(F^{-1}\xi)F'(F^{-1}\xi) - K'(F^{-1}\xi)F''(F^{-1}\xi) - K'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\right\}},$$

Therefore, by setting $F^{-1}(\xi) = \mu$, for some $\mu \in E$, such that the assertion (12) follows directly. □

The above theorem allows us to define new means. Set

$$L(\mu) = \frac{F(\mu)\{H''(\mu)F'(\mu) - H'(\mu)F''(\mu) - H'(\mu)[F'(\mu)]^2\}}{F(\mu)\{K''(\mu)F'(\mu) - K'(\mu)F''(\mu) - K'(\mu)[F'(\mu)]^2\}},$$

and when $F \in G(E)$;

$$\mu = L^{-1}\Big(\frac{H(M_H^\circ(T,f,t)) - H(M_F^\circ(T,f,t)) - H(M_H^\circ(F^{-1}|F[T(\tau)f] - FM_F^\circ(T,f,t)|,f,t))}{K(M_H^\circ(T,f,t)) - K(M_F^\circ(T,f,t)) - K(M_K^\circ(F^{-1}|F[T(\tau)f] - FM_F^\circ(T,f,t)|,f,t))}\Big)$$

Remark 1 For $(E, \|.\|)$ a Banach lattice algebra, it follows from Theorem 4 that

$$m \leq \Big\|\frac{H(M_H^\circ(T,f,t)) - H(M_F^\circ(T,f,t)) - H(M_H^\circ(F^{-1}|F[T(\tau)f] - FM_F^\circ(T,f,t)|,f,t))}{K(M_H^\circ(T,f,t)) - K(M_F^\circ(T,f,t)) - K(M_K^\circ(F^{-1}|F[T(\tau)f] - FM_F^\circ(T,f,t)|,f,t))}\Big\| \leq M,$$

Where m and M are respectively, the minimum and maximum values of

$$\Big\|\frac{F(\mu)\{H''(\mu)F'(\mu) - H'(\mu)F''(\mu) - H'(\mu)[F'(\mu)]^2\}}{F(\mu)\{K''(\mu)F'(\mu) - K'(\mu)F''(\mu) - K'(\mu)[F'(\mu)]^2\}}\Big\|, \quad \eta \in E.$$

□

Further we prove a significant result which lead us to define the Cauchy's type means on C_0-group of operators.

Corollary 1 *Let all the conditions of Theorem 4 are satisfied. For $r, s, l \in \mathbb{R}_+$ such that $r \neq l; l \neq 2s$, we have*

$$\frac{M_r^r(T,f,t) - M_s^r(T,f,t) - M_r^r(|[T(\tau)f]^s - M_s^s(T,f,t)|^{\frac{1}{s}},f,t)}{M_l^l(T,f,t) - M_s^l(T,f,t) - M_l^l(|[T(\tau)f]^s - M_s^s(T,f,t)|^{\frac{1}{s}},f,t)} = \frac{r(r-2s)}{l(l-2s)}\mu^{r-l} \tag{13}$$

where $M_r(T,f,t)$ is defined by (3). The assertion (13) holds for some μ, provided that the denominators do not vanish.

Proof: For $r,s,l \in \mathbb{R}_+$ and $f \in E_+$, if we set

$$H(f) = f^r, \quad F(f) = f^s, \quad K(f) = f^l$$

in Theorem (4), the assertion in (13) follows directly.

Ultimately we define mean-operators of the Cauchy's type on C_0-semigroup of positive linear operators defined on Banach lattice algebra E.

Definition 8 Let $r,s,l \in \mathbb{R}_+$ and $\{T(t)\}_{t\geq 0} \subset B(E)$ be a C_0-semigroup of positive operators on a Banach lattice algebra E. Then

$$\mathscr{M}_r^{l,s}(T,f,t) = \Big(\frac{l(l-2s)}{r(r-2s)} \frac{M_r^r(T,f,t) - M_s^r(T,f,t) - M_r^r(|[T(\tau)f]^s - M_s^s(T,f,t)|^{\frac{1}{s}},f,t)}{M_l^l(T,f,t) - M_s^l(T,f,t) - M_l^l(|[T(\tau)f]^s - M_s^s(T,f,t)|^{\frac{1}{s}},f,t)}\Big)^{\frac{1}{r-l}}. \tag{14}$$

is a mean-operator of the Cauchy's type on C_0-semigroup of positive operators. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases.

3 Conclusion

First, we gave a Hermite-Hadamard type inequality for a semigroup of positive linear operators and a superquadratic mapping, defined on a Banach lattice algebra. A systematic procedure has been used to prove the corresponding mean value theorems, which lead us to a new set of mean-operators. These mean-operators are Cauchy's type means for the mentioned operators. By following the similar procedure, many functional inequalities can be generalized for the operator semigroups and corresponding means can be obtained.

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