Resolvability in Wheel Related Graphs and Nanostructures

Hafiz Muhammad Afzal Siddiqui

School of Natural Sciences
National University of Sciences and Technology
Pakistan

PhD Thesis

2015

Resolvability in Wheel Related Graphs and Nanostructures

by

Hafiz Muhammad Afzal Siddiqui

Supervised by
Dr. Muhammad Imran

School of Natural Sciences,
National University of Sciences and Technology,
Islamabad, Pakistan

A thesis submitted for the degree of
Doctor of Philosophy

Abstract

Resolvability in graphs has appeared in numerous applications of graph theory, e.g. in pattern recognition, image processing, robot navigation in networks, computer sciences, combinatorial optimization, mastermind games, coin-weighing problems, etc. It is well known fact that computing the metric dimension for an arbitrary graph is an NP-complete problem. Therefore, a lot of research has been done in order to compute the metric dimension of several classes of graphs. Apart from calculating the metric dimension of graphs, it is natural to ask for the characterization of graph families with respect to the nature of their metric dimension.

In this thesis, we study two important parameters of resolvability, namely the metric dimension and partition dimension. Partition dimension is a natural generalization of metric dimension as well as a standard graph decomposition problem where we require that distance code of each vertex in a partition set is distinct with respect to the other partition sets.

The main objective of this thesis is to study the resolving properties of wheel related graphs, certain nanostructures and to characterize these classes of graphs with respect to the nature of their metric dimension. We prove that certain wheel related graphs and convex polytopes generated by wheel related graphs have unbounded metric dimension and an exact value of their metric dimension is determined in most of the cases.

We also study the metric dimension and partition dimension of 2-dimensional lattices of certain nanotubes generated by the tiling of the plane and prove that these 2-dimensional lattices of nanotubes have discrepancies between their metric dimension and partition dimension. We also compute the exact value of metric dimension for an infinite class of generalized Petersen networks denoted by $P(n, 3)$ by giving answer to an open problem raised by Imran *et al.* in 2014, which complete the study of metric dimension for the class of generalized Petersen networks $P(n, 3)$.

Acknowledgements

In the name of Almighty Allah, the creator of everything, I am thankful to Allah for blessing me and inspiring me with courage to do this work.

I would like to express my deepest gratitude to my parents (specially my father), elder brothers (Muhammad Ashraf, Akhtar Ali and Safdar Hussain), younger brother Amjid Ali and sisters for their sincere prayers, support (financial and moral), encouragement, caring and patience during my studies. I also owe much to my wife, without whose love, encourgement and understanding I would not have completed this work.

I would like to express my special thanks and gratitude to my supervisor **Dr. Muhammad Imran** for his wonderful support, guidance and encouragement to complete my Ph.D. research. Without his constant support and guidance, completion of this tough journey would have not been possible at all. His kind and illumines methodology of teaching has inspired my whole research idealogy, indeed it has opened new research horizons for me. Hopefully, it would be helpful for academic world as well. It would be very unjust of me, if I don't pay thanks to principal **Prof. Azad Akhter Siddiqui**, who gave me the golden opportunity to do this wonderful dissertation. His countless efforts for all PhD scholars are worth great applause.

Secondly, I would also like to pay my regards to my GEC committee members, Dr. Matloob Anwar, Dr. Muhammad Ishaq and Dr. Tayyab Kamran. They have always encouraged me for quality of work. Their helpful comments have improved my work day by day. Their help in preparation of seminars is unforgettable as well. My special thanks goes to Dr. Matloob Anwar for his kind support in each aspect of my studies.

I am also grateful to the Dr. Rashid Farooq (HOD) and all faculty members of Mathematics department, for helping me during my whole Ph.D programm. I would also like to the thank my friends (Muhammad Yasir Hayat Malik, Mehar Ali Malik, Lt. Cdr. Muhammad Afzal Awan, Adnan Khaliq And Muhammad Shoib), my colleagues, my seniors (Farhad Ali, Dr. Takasar Hussain, Muhammad

Ozair), for their valuable assistance, my juniors (Rishi Naeem, Akhter Munir, Muhammad Shoib, Khadeeja Afzal), for their help and prayers, and the staff of School of Natural sciences (SNS).

Finally, I would like to thank National University of Sciences and Technology (NUST) for providing me special scholarship for one year to support my studies, and also for awarding me the financial assistance to support my research work.

Hafiz Muhammad Afzal Siddiqui

List of Publications and Manuscripts

1. H. M. A. Siddiqui, M. Imran, Computation of metric dimension and partition dimension of nanotubes, *J. Comput. Theor. Nanosci.*, 12 (2015) 199–203.

2. H. M. A. Siddiqui, M. Imran, Computing metric and partition dimension of 2-dimensional lattices of certain nanotubes, *J. Comput. Theor. Nanosci.*, 11 (2014) 2419–2423.

3. H. M. A. Siddiqui, M. Imran, Computing the metric dimension of wheel related graphs, *Appl. Math. Comput.*, 242 (2014) 624–632.

4. H. M. A. Siddiqui, M. Imran, Computing the metric and partition dimension of H-Naphtalenic and VC_5C_7 nanotubes, *J. Optoelectron. Adv. Mater.*, 5–6 (2015), to appear.

5. M. Imran, H. M. A. Siddiqui, Computing the metric dimension of convex polytopes generated by wheel related graphs, *Period. Math. Hung.*, (Accepted 2015), in press.

6. M. Imran, H. M. A. Siddiqui, Computing the metric dimension of a class of generalized Petersen networks, submitted.

Contents

List of Tables

List of Figures

Introduction

A fundamental problem in chemistry is to give mathematical representations to a set of chemical compounds in such a manner that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations to the vertices of a graph in such a manner that distinct vertices have distinct representations.

This dissertation is divided into six chapters. The first chapter deals with the basic concepts, notion and terminologies of the graphs, distances in the graphs and related notions. In the second chapter, the concept of resolvability in graphs, characterization of graphs with respect to the metric dimension and some well known results of the metric dimension and partition dimension are presented. The concept of discrepancy between metric dimension and partition dimension of graphs is also discussed in this chapter.

In the third chapter, we define m-level wheel and antiweb-gear graphs generated by wheel W_n of order $n+1$ and prove that their metric dimension is unbounded by providing an exact formula. The metric dimension of convex polytopes generated by some wheel related graphs is also presented and we prove that the metric dimension is unbounded by presenting a precise formula which gives a negative answer to an open problem raised in [30]. Therefore, it shows that not all the convex polytopes are the graphs with constant metric dimension.

In the fourth chapter, we compute the exact value of the metric dimension of the class of generalized Petersen graphs $P(n, 3)$ for $n \equiv 2, 3, 4, 5$ $(mod\ 6)$ which gives the answer to an open problem raised by Imran *et al.* in [28] and the partial answer to an open problem raised by Javaid *et al.* in [36]. Thus, we complete the study of a class of generalized Petersen graphs $P(n, 3)$. We prove that the metric dimension of $P(n, 3)$ is 3 when $n \equiv 1$ $(mod\ 6)$ and 4 otherwise.

In the fifth chapter, we study the metric dimension and partition dimension of some infinite regular graphs generated by the tiling of the plane, e.g. 2-dimensional lattices of some carbon nanotubes. We prove that the metric dimension of these carbon nanotubes is no more finite but the partition dimension is finite. Therefore, these nanotubes have discrepancies between their metric dimension and partition dimension.

In the sixth chapter, we present the conclusion of the research work which has been done so far and included in this dissertation. Moreover, some open problems for further study in this area of research are also included.

Chapter 1

Preliminaries and Basic Concepts

In this chapter, some basic concepts related to graph theory will be presented. It incorporates the concept of distances in graphs and some pertinent definitions. In the first part, some basic notions, definitions and terminologies in graph theory are introduced that will be used throughout this thesis. In the second section, the planarity of graphs is presented. In the third section, some distance related definitions and properties are presented.

1.1 Preliminaries

A *graph* G consists of a nonempty set $V(G)$ of objects called *vertices* (the singular is vertex) and a set $E(G)$ of 2-element subsets of V called *edges*. The sets $V(G)$ and $E(G)$ are the *vertex set* and *edge set* of a graph G, respectively. So a graph G is an ordered pair of the vertex set $V(G)$ and the edge set $E(G)$. For this reason, a graph is symbolically represented as $G = (V(G), E(G))$. Two graphs G and K are equal if they have same vertex sets and edge sets, i.e., $V(G) = V(K)$ and $E(G) = E(K)$. The *order* of a graph G is the number of its vertices and the *size* is the number of its edges denoted by $|V(G)|$ and $|E(G)|$, respectively. If x and y are two vertices of a graph and if the unordered pair $\{x, y\}$ is an edge denoted by $e = xy$, we say that e joins x and y or that it is an edge between x and y. So the vertices x and y are said to be incident on e and e is incident to both x and y. Since the vertex set of a graph G is nonempty, the order of every graph is at least 1. A graph with exactly one vertex is called a *trivial graph*, implying that the order of a *nontrivial graph* is at least 2. Two or more edges that joins the same pair of distinct vertices are called *parallel edges*. An edge represented by an unordered pair in which two elements are not distinct is known as a loop. A *simple graph* is a graph having no parallel edges and loops. Throughout this dissertation we are considering simple, connected and undirected graphs only.

If u and v are two distinct adjacent vertices in a graph G then in this case, u is said to be a *neighbor* of v, and vice versa. The set of neighbors of a vertex u in a graph G is denoted by $N_G(u)$ and is called neighbourhood of u in G. The *degree* of a vertex is the number of its neighbors and is denoted as $d_G(u)$ or simply $d(u)$. More generally, the degree of a vertex u in a graph G is the number of edges incident on u in G. The *minimum degree* of a graph G is the minimum degree among the vertices of G and is denoted by $\delta(G)$; the *maximum degree* of a graph G is the maximum degree among the vertices of G and is denoted by $\Delta(G)$ and the *average degree* of a graph G is denoted by $d(G)$ and is defined as $d(G) = \frac{\sum_{i=1}^{n} d(v_i)}{n}$, where n is the order of the graph. So if G is a graph of order n and u is any vertex of the graph G, then

$$0 \leq \delta(G) \leq d(u) \leq \Delta(G) \leq n - 1.$$

An isolated and an end vertex have degrees 0 and 1, respectively. A vertex is said to be an even or odd vertex if its degree is even or odd, respectively. The *complement* of a simple graph $G = (V, E)$ is the simple graph $\overline{G} = (V, \overline{E})$, where the edge set $\overline{E}$ contains all those edges which are not in G. A graph which is isomorphic to its complement is known as *self-complementary graph*. If G is a self-complementary graph on n vertices, then $|E(G)| = \frac{n(n-1)}{4}$, and $n \equiv 0, 1$ (mod 4). The

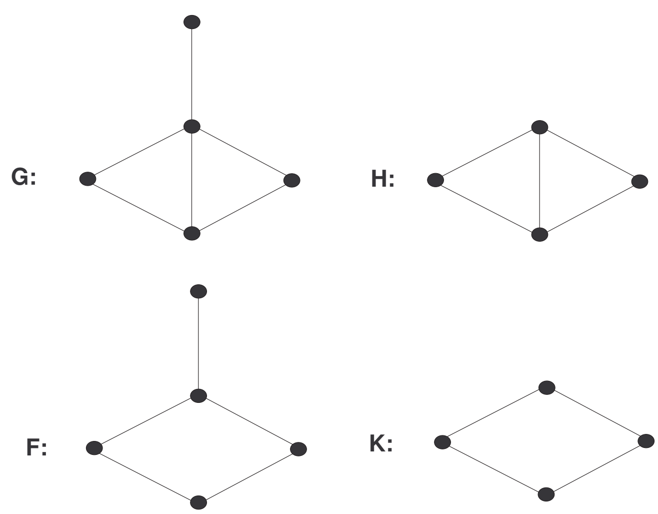

Figure 1.1: A graph G and some of its subgraphs

graph $K = (U, F_1)$ is a *subgraph* of the graph $G = (V, E)$ if U and F_1 are the subsets of V and E, respectively. So the graph K of Fig. 1.1 is a subgraph of the graph G. A subgraph H of a graph G is said to be an *induced subgraph* (or full) if, for any pair of vertices u and v of H, uv is an edge of H if and only if uv is an edge of G. In other words, H is an induced subgraph of G if it has exactly the edges that appear in G over the same vertex set. Therefore, the graph H of Fig. 1.1 is an induced subgraph of the graph G. If a subgraph F of a graph G has the same vertex set as G, then F is said to be a spanning subgraph of G. Therefore, the graph F of Fig. 1.1 is a spanning subgraph of the graph G. A spanning subgraph with at least one edge is known as a factor of a graph. Any two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic* if there are bijections $\varphi : V(G) \to V'(G')$ and $\psi : E(G) \to E'(G')$ such that vertex u and edge e are incident in G if and only if vertex $\varphi(u)$ and edge $\psi(e)$ are incident in G'. The pair of mappings (φ, ψ) is an isomorphism from G to G' and is written as $G \cong G'$. A *matching* in a graph G is the set M of edges in which no two edges have a vertex in common. The vertices that are incident to an edge of M are said be *matched*. A *perfect matching* in a graph G is a matching which matches all the vertices of G. A *clique* of a graph G is a complete subgraph of G. A *maximum clique* of a graph G is a clique, such that there is no clique with more vertices. The number of vertices in a maximum clique in a graph G is known as *clique number* and is denoted by $\omega(G)$.

1.1.1 Well-known classes of graphs

A *walk* from u to v in a graph G is a finite alternating sequence $v_0e_1v_1e_2v_2 \cdots v_{n-1}e_nv_n$ of vertices and edges, where v_i's and e_i's represents vertices and edges, respectively, $v_0 = u$ and $v_n = v$. The vertices and edges of the walk need not to be distinct. A walk in which no edge is repeated is called a *trail*. A closed walk in which no edge is repeated is called a *circuit*.

A *path* from u to v is a walk from u to v in which no edge and no vertex is repeated. The length of walk or a path is the number of edges traversed. Moreover, A path of length $n-1$ is denoted by P_n. A *cycle* is a circuit in which no vertex is repeated. The length of a cycle is the number of edges traversed. A cycle of length n is denoted by C_n. In a simple graph G, a cycle containing k vertices is called a k-cycle in G. The cycle is even or odd accordingly as k is even or odd.

The *complete graph* denoted by K_n is a graph with n vertices in which every pair of vertices is joined by a single edge. The graph K_1 is known as trivial graph having single vertex and no edge.

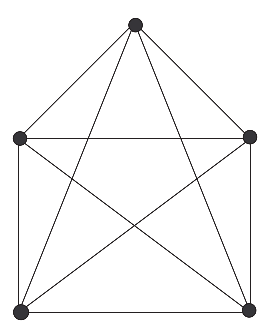

Figure 1.2: Complete graph K_5

K_n has the maximum possible size for a graph with n vertices. Since every two distinct vertices are joined by an edge, the number of pair of vertices in K_n is $\binom{n}{2}$ and the size of the complete graph K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$.

A graph G is a *bipartite graph* if $V(G)$ can be partitioned into two subsets U and W, called partite sets, such that every edge of G joins a vertex of U and a vertex of W. It is represented by (U, W, E). The *complete bipartite* graph $K_{m,n}$ is a graph having m and n vertices in the partite sets U and W,

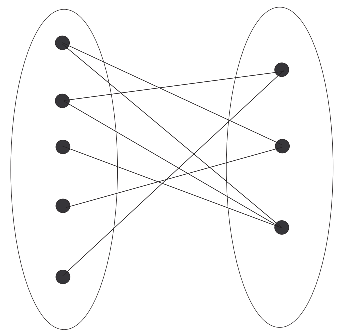

Figure 1.3: A bipartite graph

respectively and in which very vertex of U is adjacent to every vertex in W. The complete bipartite graph $K_{1,n}$ is known as a *star graph.* The following theorem provides a necessary and sufficient condition for a graph to be bipartite.

Theorem 1.1.1. [15] *A nontrivial graph G is a bipartite graph if and only if G contains no odd cycle.*

Bipartite graphs belongs to a more general class of graphs. A graph G is a *k-partite graph* if $V(G)$ can be partitioned into k sets $U_1, U_2, \cdots, U_k$ (once again known as partite sets) such that xy is an edge of G if x and y are from different partite sets. Moreover, if every pair of vertices in different partite sets is joined by an edge, then G is *complete k-partite graph.* If $|U_i| = n_i$ for $1 \leq i \leq k$, then this complete graph is denoted as $K_{n_1,n_2,\cdots,n_k}$. The complete k-partite graphs are also known as *complete multipartite graphs.*
A graph G is called *regular* if the degree of each of its vertex is same. A graph G is called a *k-regular* if the degree of each vertex in G is k.

Theorem 1.1.2. [15] *Every r-regular bipartite graph ($r \geq 1$) has a perfect matching.*

The *hypercube or n-cube graph Q_n* is defined as the graph whose vertex set is the set of ordered n-tuples of 0s and 1s (commonly called n-bit strings) and where two vertices are adjacent if their ordered n-tuples differ in exactly one position (coordinate). The n-cubes for $n = 1, 2, 3$ are given in the Fig. 1.4, where the vertices are labeled by n-bit strings.

Let n, m and $a_1, a_2, ..., a_m$ be positive integers, $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$ and $a_i \neq a_j$ for all $1 \leq i, j \leq m$ and $i \neq j$. An undirected graph with set of vertices $V = \{v_1, v_2, ..., v_n\}$ and the set of edges $E = \{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo n, is called a *circulant graph* and it is denoted by $C_n(a_1, a_2, ..., a_m)$. The numbers $a_1, a_2, ..., a_m$ are called the *generators* and we say that the edge $v_i v_{i+a_j}$ is of type a_j.
It is easy to see that the circulant graph $C_n(a_1, a_2, ..., a_m)$ is a regular graph of degree r, where

$$r = \begin{cases} 2m - 1, & \text{if } \frac{n}{2} \in \{a_1, a_2, ..., a_m\}; \\ 2m, & \text{otherwise.} \end{cases}$$

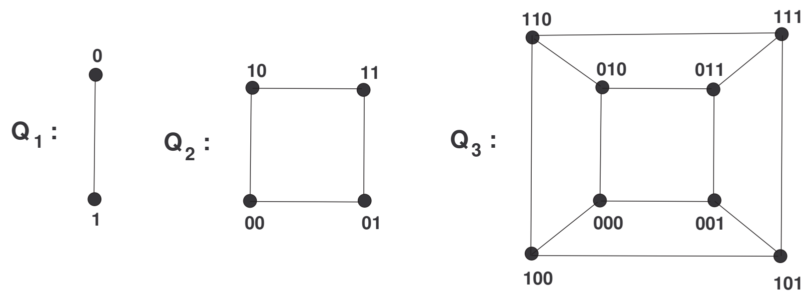

Figure 1.4: The n-cubes for $1 \leq n \leq 3$

The circulant graph $C_n(a_1, a_2, ..., a_m)$ is connected if and only if $gcd(a_1, a_2, ..., a_m, n) = 1$.

The *Harary graph* $H_{n,k}$ is a graph on the n vertices $\{v_1, v_2, \cdots, v_n\}$ defined by the following construction:

- If k is even, then each vertex v_i is adjacent to $v_{i\pm1}, v_{i\pm2}, \cdots, v_{i\pm\frac{k}{2}}$, where the indices are subjected to the wraparound convention that $v_i \equiv v_{i+n}$ (e.g. v_{n+3} represents v_3).
- If k is odd and n is even, then $H_{n,k}$ is $H_{n,k-1}$ with additional adjacencies between each v_i and $v_{i+\frac{n}{2}}$ for each i.
- If k and n are both odd, then $H_{n,k}$ is $H_{n,k-1}$ with additional adjacencies $\{v_1, v_{1+\frac{n-1}{2}}\}$, $\{v_1, v_{1+\frac{n+1}{2}}\}$, $\{v_2, v_{2+\frac{n+1}{2}}\}$, $\{v_3, v_{3+\frac{n+1}{2}}\}, \cdots, \{v_{\frac{n-1}{2}}, v_n\}$.

1.1.2 Graph operations

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs of order n_1 and n_2, respectively:

- The *union* of G_1 and G_2 is denoted by $G_1 \cup G_2$ and is defined as the graph with set of vertices and edges $V_1 \cup V_2$ and $E_1 \cup E_2$, respectively.
- The *intersection* of G_1 and G_2 is $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.
- The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is the graph having vertex set
$$V(G_1 + G_2) = V_1(G_1) \cup V_2(G_2)$$
and the edge set
$$E(G_1 + G_2) = E_1(G_1) \cup E_2(G_2) \cup \{xy :\ x \in V_1(G_1), y \in V_2(G_2)\}.$$
- The *cartesian product* $G_1 \Box G_2$ of graphs G_1 and G_2 is a graph with vertex set of $G_1 \Box G_2$ is $V(G_1) \times V(G_2)$; and any two vertices (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \Box G_2$ if and only if either $x_1 = y_1$ and x_2 is adjacent with y_2 in G_2, or $x_2 = y_2$ and x_1 is adjacent with y_1 in

G_1. The *Hamming graphs*, *Hypercubes*, the *grid graphs* and the *torus graphs* are some simple cases of cartesian product graphs. The cartesian product of graphs is connected if and only if each of its factor is connected.

- The *direct product* $G_1 \times G_2$ of two graphs G_1 and G_2 is a graph with vertex set
$V(G_1 \times G_2) = V(G_1) \times V(G_2) = \{(x_1, x_2)|\ x_1 \in G_1 \text{ and } x_2 \in G_2\}$
and edge set
$E(G_1 \times G_2) = \{(x_1, x_2)(y_1, y_2)|\ x_1y_1 \in E(G_1) \text{ and } x_2y_2 \in E(G_2)\}$
The direct product is also named in literature as *kronecker product*, the *categorical product*, the *tensor product*, the *cardinal product*, the *cross product*, the *conjunction*, the *relational product* and the *weak direct product*. This product is associative and commutative in a natural fashion. The direct product of nontrivial connected graphs is connected if and only if each of its factor is connected and at least one of them is nonbipartite.

- The *strong product* of two graphs G_1 and G_2 is the graph $G_1 \boxtimes G_2$, such that $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and two vertices (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \boxtimes G_2$ if and only if either

  1. $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or
  2. $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or
  3. $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$.

  The strong product is also known as the *strong direct product* or the *symmetric composition.* Also note that $G_1 \Box G_2$ and $G_1 \times G_2$ are subgraphs of the strong product $G_1 \boxtimes G_2$. A graph obtained from the strong product is connected if and only if each of its factor is connected.

- The *laxicographic product* of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$, such that $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$ and two vertices (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \circ G_2$ if and only if either

  1. $x_1y_1 \in E(G_1)$, or
  2. $x_1 = y_1$ and $x_2y_2 \in E(G_2)$.

  The lexicographic product also named as *composition* or the *substitution.* A lexicographic product $G_1 \circ G_2$ of the graphs G_1 and G_2 is connected if and only, if G_1 is connected.

- The *corona product* $G_1 \odot G_2$ of two graphs G_1 and G_2 is the graph obtained from G_1 and G_2 by considering one copy of G_1 and n_1 copies of G_2 and joining by an edge each vertex from the j^{th}-copy of G_2 with j^{th} vertex of G_1. The vertex set of G_1 is denoted as $V(G_1) = u_1, u_2, \cdots, u_{n_1}$

while $G_{2j} = (V_j, E_j)$ denotes the j^{th}-copy of G_2 such that $x_j \sim x$ for every $x \in V_j$. The product graph $G_1 \odot G_2$ is connected if and only if G_1 is connected. Moreover, this product is neither an associative nor a commutative operation.

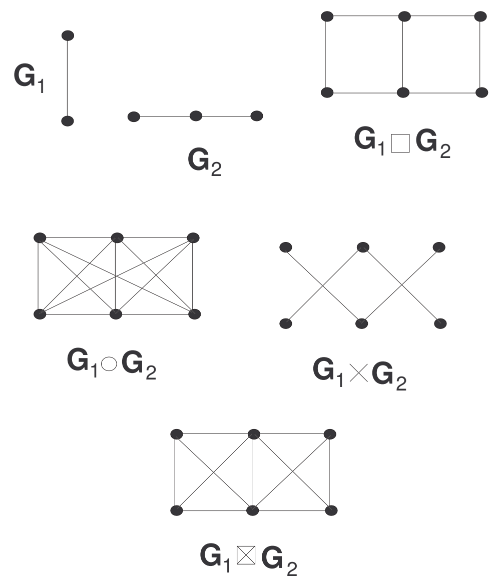

Figure 1.5: Some product graphs

- The *Cartesian sum* $G_1 \bigoplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \bigoplus G_2) = V(G_1) \times V(G_2)$ and any two vertices (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \bigoplus G_2$ if and only if either

  1. $x_1y_1 \in E(G_1)$, or
  2. $x_2y_2 \in E(G_2)$.

  The cartesian sum is also referred in literature as the *disjunctive product* and the *inclusive product*. This product graph is associative and commutative operation. Moreover, $G_1 \times G_2$, $G_1 \Box G_2$, and $G_1 \boxtimes G_2$ are subgraphs of $G_1 \bigoplus G_2$.

1.1.3 Connectivity

Any two vertices u and v are said to be connected in a graph G if there is a path between them. A graph G is a *connected graph* if there is a path between every pair of vertices of the graph G; otherwise, it is a disconnected graph. A maximal connected subgraph of G is known as component of the graph G. A graph is connected if and only if it has only one component. Let S be a set of edges in a graph $G = (V, E)$, then the graph $G - S$ can be obtained from G by deleting all the edges of S. If S is a singleton set having edge e then $G - S$ can be written as $G - e$. A set S is said to be a *disconnecting set* of the graph G if $G - S$ has more than one component. A singleton edge disconnecting set S is known as *bridge*. The following theorem provides a necessary and sufficient condition for an edge of a graph G to be bridge.

Theorem 1.1.3. [15] *An edge e of a graph G is a bridge if and only if e lies on no cycle of G.*

If every disconnecting set of a graph G has at least k elements then the graph is said to be k-edge connected. The edge disconnecting set having minimum cardinality is known as *edge connectivity number* of the graph G and is denoted by $\lambda(G)$. A disconnecting set S is said to be a *cut set* if no proper subset of S is a disconnecting set. The concept of vertex disconnecting set can be defined analogously, if H is a set of vertices of a graph $G = (V, E)$, then the graph obtained by deleting all the vertices belonging to H as well as the edges incident to the vertices in H is denoted by $G - H$. If H is a singleton set with vertex v, then graph $G - H$ is denoted by $G - v$. A set H of vertices in a connected graph G is known as *separating set* (also known as the *vertex cut*) in G if $G - H$ has more than one component. A separating set having single vertex v is known as *cut vertex.* The cardinality of a separating set of minimum size is the *connectivity number* $\kappa(G)$ of a graph G. Since a complete graph has no separating set, we say by convention that the connectivity number of the complete graph of order n is $n - 1$ for all n. The following theorem provides us the relation in terms of inequalities concerning the vertex connectivity, edge-connectivity and minimum degree of a graph.

Theorem 1.1.4. [15] *For any graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Some facts about cut-vertex are established in the following results.

Theorem 1.1.5. [15] *Let w be a vertex incident with a bridge in a connected graph G. Then w is a cut-vertex of G if and only if $d(w) \geq 2$.*

Theorem 1.1.6. [15] *Let w be a cut-vertex in a connected graph G, and let v and u be vertices in distinct components of $G - w$. Then w lies on every $v - u$ path in G.*

A nontrivial connected graph having no cut-vertex is known as *nonseparable graph.*

Theorem 1.1.7. [15] *A graph of order at least 3 is nonseparable if and only if every two vertices lie on a common cycle.*

A graph G is said to be k-connected if $\kappa(G) \geq k$. Thus K_n is $(n-1)$-connected for all n, and a graph that is not complete is k-connected if and only if every separating set in it has at least k vertices. The graph G has connectivity number zero if and only if G is either the trivial graph K_1 or is a disconnected graph. A cyclic graph is 2-connected.

Both the inequalities in Theorem 1.1.4 can be strict as the graph G in the Figure 1.6 shows.
The following theorem shows the equality relation between edge connectivity and vertex connectivity.

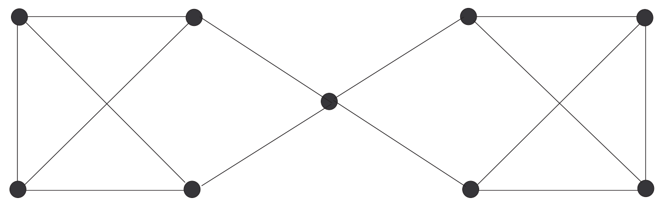

Figure 1.6: A graph G with $k(G) = 1$, $\lambda(G) = 2$, and $\delta(G) = 3$

Theorem 1.1.8. [15] *If G is a cubic graph, then $\lambda(G) = k(G)$.*

The following theorem provides a sharp upper bound for $\kappa(G)$.

Theorem 1.1.9. [15] *Let G be a graph of order n and size $m \geq n-1$, then*

$$\kappa(G) \leq \lfloor \tfrac{2m}{n} \rfloor.$$

A vertex set H of a graph G is said to *separate* two vertices w and v of a graph G if $G - H$ is no more a connected graph and the vertices w and v belong to different components of $G - H$. Thus, if H separates w and v, then w and v are nonadjacent vertices and H is a vertex cut of the graph G. The cardinality of the set H must be as at least as large as $\kappa(G)$. Such a vertex set H is known as $w - v$ *Separating Set*. A $w - v$ *Separating Set* of minimum cardinality is known as *minimum* $w - v$ *Separating Set*. An *internal vertex* of a $w - v$ path Q is a vertex of Q different from w and v. A collection $\{Q_1, Q_2, \cdots, Q_k\}$ of $w - v$ paths is known as *internally disjoint* if every two of these paths have no vertices in common other than w and v.

There are many theorems in mathematics which state the minimum number of elements in some set equals the maximum number of elements in some other set. The following theorem is such a "min-max" theorem. It is referred to as *Menger's Theorem.*

Theorem 1.1.10. [15] *Let w and v be two nonadjacent vertices in a graph G. The minimum number of vertices in a $w - v$ separating set equals the maximum number of internally disjoint $w - v$ paths in G.*

With the aid of Menger's Theorem, Hassler Whitney was able to give a characterization of k-connected graphs.

Theorem 1.1.11. [15] *A nontrivial graph G is k-connected for some integer $k \geq 2$ if and only if for each pair w, v of distinct vertices of G there are at least k internally disjoint $w - v$ paths in G.*

The following theorem is a generalization of the Theorem 1.1.7 for k-connected graphs.

Theorem 1.1.12. [15] *If G is a k-connected graph, $k \geq 2$, then every k vertices of G lie on a common cycle of G*

There are also edge-connectivity analogues of both Theorems 1.2.1 and 1.2.2. We state the followings.

Theorem 1.1.13. [15] *For distinct vertices v and w in a graph G, the minimum number of edges of G that separate v and w equals the maximum number of pairwise edge-disjoint $v - w$ paths in G.*

Theorem 1.1.14. [15] *A nontrivial graph G is k-edge-connected if and only if G contains k pairwise edge-disjoint $v - w$ paths for each pair v, w of distinct vertices of a graph G.*

1.1.4 Trees and spanning trees

Trees find applications in many diverse fields, including computer science, the enumeration of saturated hydrocarbons, the study of electrical circuits, road networks and communication networks. The spanning trees are used to find minimum cost. The spanning trees are also used to find the shortest roots among the cities in a road network problem.
A cycle free graph is known as *acyclic* graph. A *tree* is a connected acyclic graph. If the spanning subgraph K of a connected graph G is a tree, then K is called a *spanning tree.* In a tree the vertices of degree 1 are called *end vertices* or *leaves* (singular is leaf).
The following theorem gives some properties of a tree.

Theorem 1.1.15. [17] *The following statements are equivalent for a graph G of order n.*

(i) *G is a tree.*

(ii) *There is a unique path between every pair of distinct vertices in G.*

(iii) *G is connected and every edge in G is a bridge.*

(iv) *G is connected, and has $n - 1$ edges.*

(v) *G is acyclic, and has $n - 1$ edges.*

(vi) *G is acyclic, and whenever any two arbitrary nonadjacent vertices in G are joined by an edge, the resulting enlarged graph has a unique cycle.*

The following theorem gives a characterization of connected graphs in terms of its spanning trees.

Theorem 1.1.16. [17] *A graph is connected if and only if it has a spanning tree.*

All the nontrivial trees of a connected graph G have the following property.

Theorem 1.1.17. [15] *Every nontrivial tree has at least two end-vertices.*

If T is a tree of order k, then it should be clear that T is isomorphic to a subgraph of K_k. Of course, $\delta(K_k) = k - 1$. Not only is T isomorphic to a subgraph of K_k, the tree T is isomorphic to a subgraph of every graph having minimum degree at least $k - 1$.

Theorem 1.1.18. [17] *Let T be a tree of order k. If G is a graph with $\delta(G) \geq k - 1$, then T is isomorphic to some subgraph of G.*

1.2 Planarity

There is a well known problem that has appeared in many practical situations. There are three different utilities (water, gas, and electricity) that need to be connected to three different houses by water mains, gas lines, and electricity lines. Can this be done without any of the lines or mains crossing each other? This problem is known as *Three Houses and Three Utilities Problem.* This situation can be modeled by the graph as shown in Figure 1.7, which, in fact, the complete bipartite graph $K_{3,3}$.

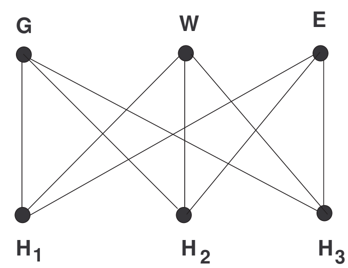

Figure 1.7: The Three Houses and Three Utilities Problem

The concept of the planar graphs is used to provide the answer of such kind of problems.
A graph G is called a *planar graph* if it can be drawn on the plane without edge crossing as shown in Figure 1.8.

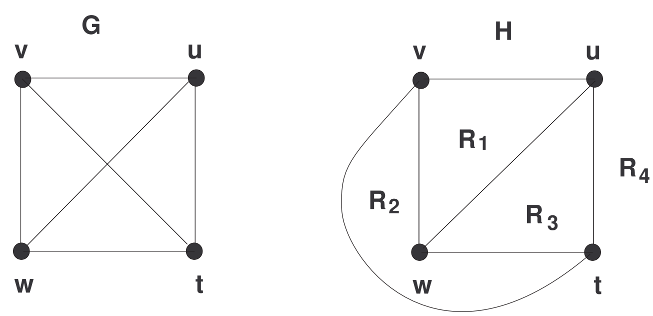

Figure 1.8: The graph G is planar and H is its plane drawing

A graph that is not planar is known as *nonplanar.* A graph G is called a *plane graph* if it is drawn in the plane so that no two edges of G cross. Thus, while a graph may be planar, as drawn it may not be a planar graph, such as Figure 1.8, graph G. A plane graph divides the plane into different connected pieces called *regions.* For example, in the case of plane graph H of Figure 1.8, there are four different regions labelled as R_1, R_2, R_3 (interior regions) and R_4 (exterior region). These regions also known as *faces*. Letting $n, m,\ f$ denote the order, size and the number of regions or faces, respectively. So in this case, $n - m + f = 2$. Leonhard Euler observed that this is always true. The following property for a planar graph is known as the Euler Identity.

Theorem 1.2.1. *(The Euler Identity)* [15] *If G is a connected graph of order n, size m and having f regions, then $n - m + f = 2$.*

The Euler Identity has many useful and interesting consequences. One of these (which will allow us to prove that some graphs are not planar) tells us that the planar graphs cannot have too many edges.

Theorem 1.2.2. [15] *If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n - 6$.*

Theorem 1.2.2 provides a necessary condition for a graph to be planar and so provides a sufficient condition for a graph to be nonplanar. In particular, the contrapositive of Theorem 1.2.2 gives the following:
If G is a graph of order $n \geq 3$ and size m such that $m > 3n - 6$, then G is nonplanar.
A graph G is called a *maximal planar* if G is planar but the addition of an edge between any two nonadjacent vertices of G results in a nonplanar graph. A graph G' is known as a *subdivision* of a graph G if one or more vertices of degree two are inserted into one or more edges of the graph G.

Theorem 1.2.3. *(Kuratowski's Theorem)* [15] *A graph G is planar if and only if G does not contain K_5, $K_{3,3}$ or a subdivision of K_5 or $K_{3,3}$ as subgraph.*

The *crossing number* of a graph G is the smallest number of pairwise crossings of edges among all drawings of G in the plane. In the last decade, there has been significant progress on a true theory of crossing numbers. There are now many theorems on the crossing number of a general graph and the structure of crossing-critical graphs, whereas in the past, most results were about the crossing numbers of either individual graphs or the members of special families of graphs.

The study of crossing numbers began during the Second World War with Paul Turán. He tells the story of working in a brickyard and wondering about how to design an efficient rail system from the 'kilns'to the 'storage yards'. For each kiln and each storage yard, there was a track directly connecting them. The problem he considered was how to lay the rails to reduce the number of crossings, where the cars tended to fall off the tracks, requiring the workers to reload the bricks onto the cars. This is the problem of finding the crossing number of the complete bipartite graph.

1.3 Distances in Graphs

Let us suppose that we are building a city. We must be interested in placing emergency facilities where the response time should be minimum furthermore, we can not have all facilities at one place.

Providing we account for all possible emergency situations simultaneously, we must locate all possible points where we can approach to any place of city in minimum time. In this section, we give the definitions of different kinds of centers and others.

Let's review the definition of distance in a connected graph G. The *distance* $d(x, y)$ between any two vertices x and y in a connected graph G is the length of a shortest path between them. The term distance that we just defined satisfies the following properties in a connected graph G.

1. $d(x, y) \geq 0$ for all $x, y \in V(G)$.

2. $d(x, y) = 0$ if and only if $x = y$.

3. $d(x, y) = d(y, x)$ for all $x, y \in V(G)$.

4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in V(G)$.

The ordered pair $(V(G), d)$ is a metric space as the distance d satisfies the properties $1 - 4$.
The *eccentricity* $e(x)$ of a vertex $x \in V(G)$ is the distance to a vertex farthest from x. Thus $e(x) = max\{d(x, y) : y \in V\}$. The *radius* $rad(G)$ is the minimum eccentricity among the vertices of G while the *diameter* $diam(G)$ is the maximum eccentricity among the vertices of G.
A vertex x is a *central vertex* of G if $e(x) = rad(G)$ and the *center* denoted by $Cen(G)$ of G is a subgraph induced by all the central vertices. Thus the center contains all vertices of G having minimum eccentricity. If every vertex of a graph G is a central vertex, then G is known as *self-centered.* A vertex $x \in V(G)$ is called *peripheral vertex* if $e(x) = diam(G)$ and periphery $Per(G)$ is the subgraph induced by the set of all such vertices.
The following theorem describe the relation between radius and the diameter of the graph.

Theorem 1.3.1. [15] *If G be a nontrivial connected graph then*

$$rad(G) \leq diam(G) \leq 2rad(G).$$

The set of all vertices at distances $e(x)$ from x are called *eccentric vertices.* For a given vertex x in a graph G, we have studied seeking a vertex x such that $d(x, y) = e(x)$, that is, the vertex x is farthest from y. Such a vertex x is known as an eccentric vertex of y. A vertex x is an *eccentric vertex of the graph* G if x is an eccentric vertex of some vertex of G. If every vertex of a graph G is an eccentric vertex then G is called an *eccentric graph. The eccentric subgraph* $E_{cc}(G)$ of G is the subgraph of G induced by the set of eccentric vertices of G. The following theorem [15] gives a necessary and sufficient condition for a connected graph G to be an eccentric subgraph of some graph.

Theorem 1.3.2. [15] *A nontrivial graph G is an eccentric subgraph of some graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1.*

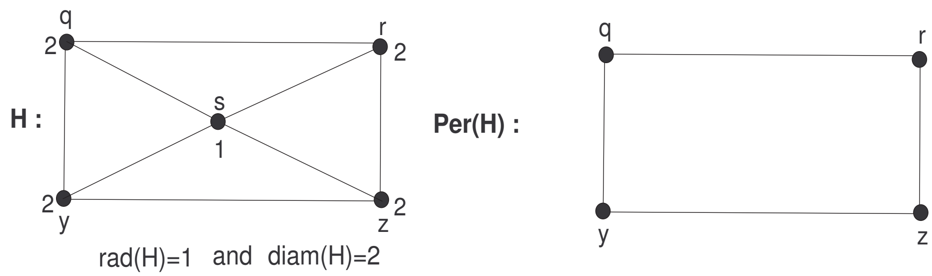

Figure 1.9: The eccentricities of the vertices of a graph

A tree is known as a *central tree* or *bicentral* if it contains one vertex or two central vertices, respectively. A graph G is a unique eccentric vertex graph if each vertex in G has only one eccentric vertex.
The following theorem gives a property for the center of a tree.

Theorem 1.3.3. [6] *The center of a tree consists of either a single vertex or a pair of adjacent vertices.*

Theorem 1.3.4. [6] *A unique eccentric vertex graph is self-centered if and only if each node of G is eccentric.*

The center of a graph is an important factor in the applications involving emergency facilities where response time (distance) to each single location (vertex) in the region (graph) is critical. Suppose instead we consider a service facility such as shopping mall, post office, bank, or power station. When deciding where to locate a post office, we are interested to minimize the average distance that a person being serviced by the post office must travel. This is equivalent to minimizing the total distance traveled by all people within the city. In this situations, the concept of *median* is described.
Let G be a simple connected graph. The *status* $s(u)$ of a vertex u in a graph G is the sum of the distances from u to each other vertex in a graph G. This concept was introduced by Harary [22]. The set of vertices having minimum status is known as the median $M(G)$ of a graph G. The *minimum status* $ms(G)$ of a graph G is the value of the minimum status; the sum of all the status values is known as *total status* $ts(G)$ of a graph G.

Chapter 2

Resolvability in graphs

In this chapter, we discuss the definition of resolving sets, metric dimension (metric generator), resolving partition, and partition dimension which are the major notions of this dissertation. We discuss some known results of these resolvability parameters and relation between them. We also write about the applications of these parameters in the different branches of applied sciences.

2.1 Resolving Sets and Metric Dimension

Suppose a facility consists of seven rooms $R_1, R_2, R_3, R_4, R_5, R_6$ and R_7 as shown in Fig. 2.1. The

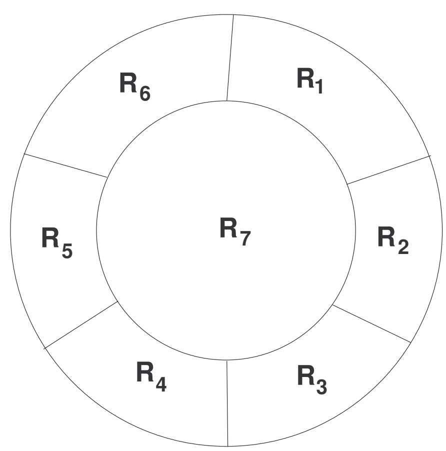

Figure 2.1: A facility consists of seven rooms

distance between rooms R_1 and R_3, R_4 or R_5 is 2, and the distance between R_2 and R_4, R_5 or R_6 is 2. The distance between rooms R_3 and R_5 or R_6 is 2, and distance between R_4 and R_6 is also 2. The distance between a room and itself is 0. The distance between all other pairs of distinct rooms is 1. Suppose that a certain red sensor is placed in one of the rooms. If a fire takes place in one of the rooms, then the sensor is able to detect the distance from the room with the red sensor to the room containing the fire. Suppose, for example, that the red sensor is placed in R_1. If fire takes place in R_4, then the red sensor alerts us that a room at distance 2 from R_1 is on fire; that is, any of the room R_3, R_4 or R_5 is on fire since these are the only rooms at distance 2 from R_1. If the room R_1 is on fire, then the red sensor indicates that the fire has occurred in a room at distance 0 from R_1; that is the fire is in R_1. However, if the fire is in any of the rooms R_2, R_6 or R_7 then the sensor tells us that there is fire in a room at distance 1 from R_1. But with this information we can not tell exactly in which room the fire has occurred. In fact, there is no room in which the red sensor can be placed to identify the exact location of a fire in every instance.

Similarly, If we place the red sensor in R_1 and a blue senor in any of the neighboring room R_2, R_6 or R_7, then there are pair of rooms $\{R_4, R_5\}$, $\{R_3, R_4\}$ and $\{R_2, R_6\}$ with the same distance, respectively. If we place the red sensor in R_1 and a blue senor in any of the room R_3, R_4 or R_5, then the pair of neighboring rooms of one of these rooms have the same distance. On the other hand, if we place the red sensor in R_1, a blue sensor in R_3, and a yellow sensor in R_5, and if the room R_4 is on fire, then the red sensor tells us that the fire is in a room at distance 2 from R_1, while the blue sensor and yellow senor tell us that the fire is in a room at a distance 1 from R_3 and from room R_5, that is, R_4 has the code (2,1,1). Since the codes are distinct for all rooms, the minimum number of sensors required to detect the exact location of the fire in any room is three. Even though three is the answer, care must be taken as to where the three sensors are placed. For example, we can not place sensors in R_1, R_2, and in R_7 since, in this case, the codes of R_4 and R_5 are $(2, 2, 1)$ and we can not distinguish the precise location of the fire. The facility that we have just described can be represented by a graph, whose vertices are the rooms and such that two vertices in this graph are adjacent if the corresponding two rooms are adjacent. This gives rise to a problem involving graphs.

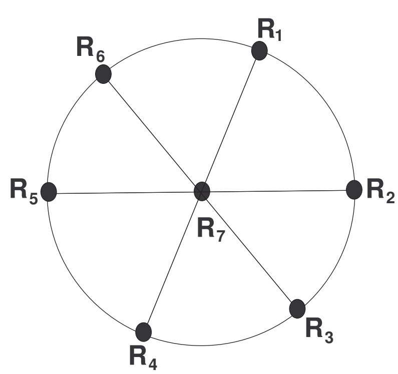

Figure 2.2: A graph model of seven rooms

Let G be a simple connected, and undirected graph, the distance $d(x, y)$ between any two vertices x, y of the graph G is the length of the shortest path between them. Any two vertices $x, y \in V(G)$ of a simple connected graph G are said to be resolved or distinguished by a vertex $z \in V(G)$ if the distance

between x and z is not same as between y and z, i.e., $d(x,z) \neq d(y,z)$. A *resolving set* or *metric generator* for G is the set $W \subseteq V(G)$, if any two distinct vertices of G are distinguished by some elements of W. A minimum resolving set is known as *metric basis* and its cardinality is known as *metric dimension* of the graph G, denoted by $dim(G)$. Let $W = \{w_1, w_2, \cdots, w_k\}$ be an ordered set of $V(G)$, we mention to the k-vector (ordered k-tuple) $code(z|W) = (d(z,w_1), d(z,w_2)), \cdots, d(z,w_k)$ as the code (or representation) of z with respect to W. We have another equivalent definition. We can say that W is a metric generator or resolving set if distinct vertices of G have distinct codes with respect to the ordered set W and its cardinality is the metric dimension of the graph. Moreover, the metric dimension is also known as *location number* and denoted by $loc(G)$.
For an ordered set W of a graph G, the j-th component of $code(z|W)$ is 0 if and only if $z = w_j$. Thus, to show that W is a resolving set it is suffices to verify that $code(u|W) \neq code(v|W)$ for every pair of vertices $u, v \in V(G) \setminus W$. For example, consider the graph G of Figure 2.3. The set $W = \{v, v_1\}$

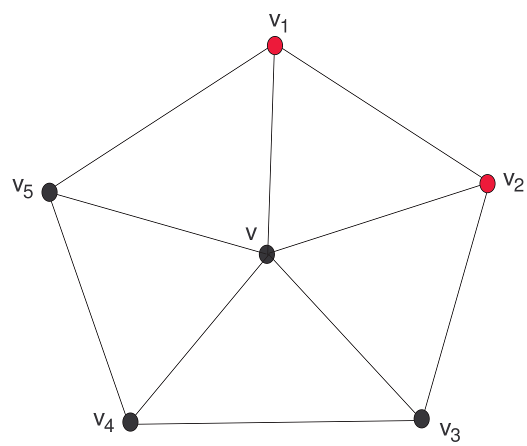

Figure 2.3: A graph with $dim(G) = 2$

is not a resolving set of the graph G since the $code(v_2|W) = (1,1) = code(v_5|W)$. But if we consider the set $W_1 = \{v, v_1, v_2\}$ then W_1 is the resolving set of G with codes $code(v|W_1) = (0,1,1)$, $code(v_1|W_1) = (1,0,1)$, $code(v_2|W_1) = (1,1,0)$, $code(v_3|W_1) = (1,2,1)$, $code(v_4|W_1) = (1,2,2)$, and $code(v_5|W_1) = (1,1,2)$. However, W_1 is not a minimum resolving set since $W_2 = \{v_1, v_2\}$ is also a resolving set having only two vertices. As there is no single vertex which resolves the graph G. Therefore, it follows that W_2 is a minimum resolving set for the graph G and hence $dim(G) = 2$.
In the following lemma, a useful property to find the metric dimension of G is presented:

Lemma 2.1.1. [59] *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

Let $\mathcal{F}$ denotes a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n \geq 1}$ depending on n as follows: the order $|V(G_n)| = \varphi(n)$ and $\lim_{n \to \infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G_n) \leq C$

for every $n \geq 1$, then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension. If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension.
A model application of these distance related parameters to robot navigation in networks [39], the robot proceed from node to node of a graph and can locate itself throughout a uniquely exclusive labelled "landmarks" node set of the graph. It is assume that a robot moving over the graph is using the distances to the landmarks to "knows" its position in each moment, i.e., if a robot knows the distances to the vertices of the node set, then its position on the graph is uniquely determined. With this purpose, the set of landmarks is the metric generator for the graph modeling the network topology. A very important goal is then to minimize the number of landmarks needed, and to determine where they should be located, so that the distances to the landmarks uniquely determine the robot's position on the graph. Solutions to these questions are produced by the metric dimension and some metric basis of the graph, respectively.
The concept of metric dimension was first introduced by Slater in [55,56], and then studied independently by Harary and Melter in [21]. Slater refereed to the metric dimension of a simple connected graph as its "location number" and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set.

2.2 Resolving Partitions and Partition Dimension

The concept of metric dimension was further generalized in the following fashion. Let $S \subset V(G)$ be a proper subset of the set of vertices $V(G)$ of the graph G and $w \in V(G)$ be any vertex of the graph G, then we define the distance $d(w, S)$ between the vertex w and the set S, by $d(w, S) = min\{d(w, v)|v \in S\}$. If the set $\Pi = \{S_1, S_2, \cdots, S_k\}$ is an ordered k-partition of vertices of G and let w be any vertex of G. The representation or code $code(w|\Pi)$ of w with respect to Π is the k-tuple $(d(w, S_1), d(W, S_2), \cdots, d(w, S_k))$. If the distinct vertices of the graph G have distinct codes or representation with respect to the set Π, then Π is called a *resolving partition* for $V(G)$ and the minimum cardinality of the resolving partition of $V(G)$ is called partition dimension of G, denoted by $pd(G)$. This concept was introduced in [13,14]. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be an ordered k-partition of $V(G)$. If $v \in S_i,\ w \in S_j$ where $1 \leq i, j \leq k$ and $i \neq j$, then $code(v|\Pi) \neq code(w|\Pi)$ since $d(w, S_j) = 0$ but $d(v, S_j) \neq 0$. Thus, in order to determine whether a given partition Π of $V(G)$ is a resolving partition for $V(G)$, it is suffices to verify that if the vertices of G belonging to the same class of Π have distinct codes with respect to Π. When $d(v, S_j) \neq d(w, S_j)$ we can say that the class S_j distinguishes vertices w and v.
To illustrate these concepts, consider the graph G of order 5 in Figure 2.4.

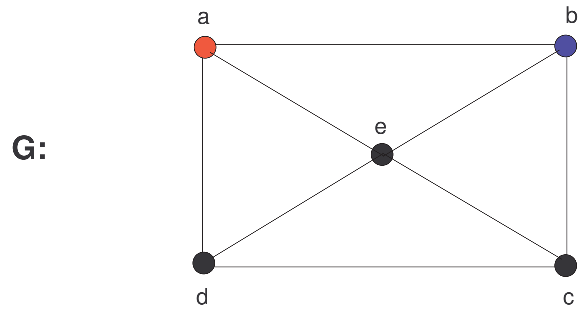

Figure 2.4: A graph with $pd(G) = 3$

Let $\Pi = \{S_1, S_2, S_3\}$ be an ordered 3-partition of G, Where $S_1 = \{a, b\}$, $S_2 = \{d, e\}$, and $S_3 = \{c\}$. Then the five codes (3-vectors) are $code(b|\Pi) = (0, 1, 1)$, $code(d|\Pi) = (1, 0, 1)$, $code(a|\Pi) = (0, 1, 2)$, $code(e|\Pi) = (1, 0, 1)$, $code(c|\Pi) = (1, 1, 0)$.
Since $code(d|\Pi) = (1, 0, 1) = code(e|\Pi) = (1, 0, 1)$, this shows that Π is not a resolving partition of the graph G. Next, let $\Pi_1 = \{S_1, S_2, S_3, S_4\}$ be an ordered 4-partition of G, Where $S_1 = \{a, b\}$, $S_2 = \{d\}$, $S_3 = \{e\}$, and $S_4 = \{c\}$. Then the five codes (4-vectors) are $code(b|\Pi_1) = (0, 2, 1, 1)$, $code(d|\Pi_1) = (1, 0, 1, 1)$, $code(a|\Pi_1) = (0, 1, 1, 2)$, $code(e|\Pi_1) = (1, 1, 0, 1)$, $code(c|\Pi_1) = (1, 1, 1, 0)$.
Since the five codes are distinct, Π_1 is a resolving partition of G. However, Π_1 is not a minimum resolving partition of the graph G. To find the minimum resolving partition of G, let $\Pi_2 = \{S_1, S_2, S_3\}$ be an ordered 3-partition of G, Where $S_1 = \{a\}$, $S_2 = \{b\}$, and $S_3 = \{c, d, e\}$. Then the corresponding five codes (3-vectors) are $code(b|\Pi_2) = (1, 0, 1)$, $code(d|\Pi_2) = (1, 2, 0)$, $code(a|\Pi_2) = (0, 1, 1)$, $code(e|\Pi_2) = (1, 1, 0), code(c|\Pi_2) = (2, 1, 0)$.
It follows that Π_2 is a resolving partition of G. Since, there is no 2-partition of G which resolves the graph G, hence Π_2 is a minimum resolving partition of the graph G and so $pd(G) = 3$
A useful property to determine the $pd(G)$ is the following lemma [14].

Lemma 2.2.1. [14] *Let Π be a resolving partition of $V(G)$ and $v, w \in V(G)$. If $d(v, u) = d(w, u)$ for all vertices $u \in V(G) \setminus \{v, w\}$, then v and w belong to different classes of* Π.

These concepts have some applications in chemistry for representing chemical compounds in a way that gives distinct representations to distinct compounds [37] as well as in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [40].

2.3 Some Known Results on Metric Dimension and Partition Dimension

If G is a nontrivial simple connected k-dimensional graph of order n, then $1 \leq k \leq n-1$.
Some important results are presented over here which have been established in the literature. The following result states that for every $1 \leq k \leq n-1$, there exists a k-dimensional connected graph of order n.

Theorem 2.3.1. [11] *For every pair n, k of integers with $1 \leq k \leq n-1$, there exists a k-dimensional connected graph of order n.*

The following theorems characterize simple connected graphs of order n having dimension $1, n-1$ or $n-2$ and give the dimensions of some well-known classes of graphs.
For two vertex-disjoint connected graphs G and K, $G \cup K$ is a disconnected graph having vertex set $V(G) \cup V(K)$ and edge the set $E(G) \cup E(K)$. The join $G + K$ consists of all vertices of $G \cup K$ and all edges joining a vertex of G and a vertex of K.

Theorem 2.3.2. [11] *Let G be a simple connected graph of order $n \geq 2$. Then*
(a) $dim(G) = 1$ if and only if $G = P_n$.
(b) $dim(G) = n-1$ if and only if $G = K_n$.
(c) for $n \geq 4$, $dim(G) = n-2$ if and only if $G = K_{r,s}$,$r, s \geq 1$, $G = K_r + \bar{K_s}, r \geq 1, s \geq 2$ or $G = K_r + (K_1 \cup K_s), r, s \geq 1$.

There is no characterization available for the graphs with metric dimension 2. The next two theorems [57] gives properties of simple connected graphs with metric dimension 2.

Theorem 2.3.3. [57] *Let G be a simple connected graph with metric dimension 2 and let $\{u_1, u_2\} \subseteq V(G)$ be a metric basis in G, then the degree of both u_1 and u_2 is at most 3 and there exists a unique path between u_1 and u_2.*

Theorem 2.3.4. [57] *A simple connected graph G with $dim(G) = 2$ can not have following:*
- *K_5 as a subgraph.*
- *$K_5 - e$ as a subgraph, where e is an edge.*
- *$K_{3,3}$ as a subgraph.*
- *The Petersen graph as a subgraph.*

Metric dimension of some families of circulant graphs denoted by $G \cong C(n, \pm\{1, 2, \cdots, j\})$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 3$ is stated in the following theorem.

Theorem 2.3.5. [20] *Let $G \cong C(n, \pm\{1, 2, \cdots, j\})$, then*

$$dim(G) \begin{cases} = j+1, & when\ n \equiv i (mod\ 2j);\ 2 \leq i \leq j+2; \\ \leq i-1, & when\ n \equiv i (mod\ 2j);\ 2+j \leq i \leq 2j+1. \end{cases}$$

Metric dimension of some families of Harary graphs denoted by $G \cong H(r,n) \cong C(n, \pm\{1, 2, \cdots, \frac{r-1}{2}, \frac{n}{2}\})$, when r is odd, n is even and $j = \frac{r-1}{2}$ is stated in the following theorem.

Theorem 2.3.6. [20] *Let $G \cong C(n, \pm\{1, 2, \cdots, j, \frac{n}{2}\})$, then*

$$dim(G) \begin{cases} = j+2, & when\ n \equiv 2j+2i(mod\ 4j);\ 2 \leq i \leq j+1; \\ \leq i-1, & when\ n \equiv 2j+2i(mod\ 4j);\ 2+j \leq i \leq 2j+1. \end{cases}$$

We would like to use the terminology given in [11], [47] and [50]. A vertex of degree at least 3 will be known as a major vertex. An end-vertex w of a graph T is said to be a terminal vertex of a major vertex u of T if $d(w,u) < d(w,v)$ for every other major vertex v of T. The terminal degree $ter(u)$ of a major vertex u is the number of terminal vertices of u. A major vertex u of T is an exterior major vertex of T if it has a positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of T and let $ex(T)$ denote the number of exterior major vertices of T.

Theorem 2.3.7. [21] *If T is a tree that is not a path, then*

$$dim(T) = \sigma(T) - ex(T).$$

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. The dimension of a simple connected graph G is also affected by the addition of an edge. It has been established in [11] that the dimension of a tree is also affected by the addition of an edge. The result suggests that the dimension can increase by at most 1 or decrease by at most 2. We state the result in the following theorem.

Theorem 2.3.8. [11] *Let T be a tree of order at least 3 and let f be an edge in its complement $\bar{T}$ of T. Then*

$$dim(T) - 2 \leq dim(T+f) \leq dim(T) + 1.$$

Moreover, for each integer j with $-2 \leq j \leq 1$, there exists a tree T_j and an edge f_j in $\bar{T}_j$ such that

$$dim(T_j + f_j) = dim(T_j) + j.$$

The relationship between the dimension of cartesian product $H \Box K_2$ of a simple connected graph H and complete graph K_2 and the dimension of H was also proved in [11].

Theorem 2.3.9. [11] *For every nontrivial simple connected graph H,*

$$dim(H) \leq dim(H \Box K_2) \leq dim(H) + 1.$$

The metric dimension of cartesian product of connected graphs has been studied in [9] and [43]. The following result in [11] provides bounds for the dimension of a graph in terms of its order and diameter.

Theorem 2.3.10. [11] *For any positive integers $D = diam(G)$ and n with $D < n$, define $f(n, D)$ as the least positive integer k such that $k + D^k \geq n$. Then for a simple connected graph G of order $n \geq 2$ and diameter D,*

$$f(n, D) \leq dim(G) \leq n - D.$$

A sharp lower bound for the metric dimension of a simple connected graph G in terms of its maximum degree $\Delta(G)$ is presented in the following theorem.

Theorem 2.3.11. [12] *Let G be a nontrivial simple connected graph. Then*

$$dim(G) \geq \lceil log_3(\Delta(G) + 1) \rceil$$

and this bound is sharp.

In fact, for each pair $k, \triangle$ of integers such that $3^k = \triangle + 1$, there exists a simple connected graph $G_{k,\triangle}$ such that $dim(G_{k,\triangle}) = k$ and $\triangle(G_{k,\triangle}) = \triangle$.
Now we present some results about the partition dimension of simple connected graphs. If G is a simple connected graph of order $n \geq 2$, then certainly $2 \leq pd(G) \leq n$. It was also proved in [14] that every pair k, n of integers with $2 \leq k \leq n$ is realizable as the partition dimension and order of some simple connected graph.

Theorem 2.3.12. [14] *For each pair k, n of integers with $2 \leq k \leq n$, there exists a simple connected graph of order n with partition dimension k.*

The metric dimension and partition dimension of a simple connected graph are related. In [14], the following theorem has been established.

Theorem 2.3.13. [14] *If G is a nontrivial simple connected graph, then*

$$pd(G) \leq dim(G) + 1.$$

Moreover, for each pair b, a of positive integers with $\lceil \frac{a}{2} \rceil + 1 \leq b \leq a + 1$, there exists a simple connected graph G such that $pd(G) = b$ and $dim(G) = a$.

However, the metric dimension may be much larger than the partition dimension and this phenomena is known as discrepancy between metric dimension and partition dimension of simple connected graphs. The discrepancies between metric dimension and partition dimension of a connected graph G have already been subject of the following papers [40, 58, 59]. The metric dimension of infinite graphs is studied in [10]. The detailed discussion about the discrepancies between metric dimension and partition dimension is given in chapter 5.

In [14], an open problem was proposed focusing the Theorem 2.3.13. Is it the case that $pd(G) \geq \lceil \frac{dim(G)}{2} \rceil + 1$ for every nontrivial simple connected graph G? Tomescu provided a negative answer to this question in [58].

Chartrand and Zhang established an improved upper bound for $pd(G)$ in terms of order and diameter of the connected graph G in [14].

Theorem 2.3.14. [14] *If G is a simple connected graph of order $n \geq 3$ and diameter D, then*

$$pd(G) \leq n - D + 1.$$

Chartrand, Salehi and Zhang computed the partition dimension of some well known classes of simple connected graphs in [14], where the simple connected graphs of order n with $pd(G) = 2, n-1, n$ are characterized.

Theorem 2.3.15. [14] *Let G be a nontrivial simple connected graph of order n. Then*

(a) $pd(G) = 2$ *if and only if* $G = P_n$

(b) $pd(G) = n$ *if and only if* $G = K_n$ *and*

(c) for $n \geq 3$, $pd(G) = n - 1$ *if and only if*

$$G \in \{K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-1})\}.$$

Tomescu [58] characterized all the graphs of order $n \geq 9$ with partition dimension $n - 2$, thus completing the characterization of graphs of order n with partition dimension $2, n-2, n-1$ or n given by Chartrand, Salehi and Zhang. The list of such graphs includes 23 members.

Chartrand, Salehi and Zhang also studied the partition dimension of a tree in [16]. Although, the partition dimension of some special types of trees, such as stars, paths, double stars and caterpillars have been computed, but a general formula for the partition dimension of a tree is still an open question. However, it was proved that there is no tree of order n having partition dimension $n - 2$. The partition dimension as well as the connected partition dimension of the wheel W_n with n spokes has been computed in [61]. The metric dimension and partition dimension of Cayley digraphs have been determined in [18] and [19].

A connected graph G is called *unicyclic graph* if it contains only one cycle. The following theorem provides the upper bound for partition dimension of unicyclic graphs in terms of the partition dimension of spanning trees.

Theorem 2.3.16. [48] *Let T be a spanning tree of a unicyclic graph G, then $pd(G) \leq pd(T) + 3$.*

The partition dimension of some products of graphs are presented in the following theorems. In the following theorem, the upper bound and lower bound for the partition dimension of strong product of graphs is presented.

Theorem 2.3.17. [63] *For any two nontrivial connected graphs G and H, we have*

$$4 \leq pd(G \boxtimes H) \leq pd(G).pd(H).$$

In the following theorem the lower bound for the partition dimension of strong product of graphs is presented.

Theorem 2.3.18. [63] *For any connected non-complete graph G of order $t \geq 3$ and any integer $n \geq 2$, we have*

$$pd(G \boxtimes K_n) \geq n + 2.$$

In the following theorem the upper bound for the partition dimension of cartesian product of graphs is presented.

Theorem 2.3.19. [63] *For any two nontrivial connected graphs G and H, we have*

$$pd(G \square H) \leq pd(G) + pd(H) - 1.$$

In the following theorem the upper bound for partition dimension of corona product of graphs in terms of the metric dimension of the corona product of graphs is presented.

Theorem 2.3.20. [49] *For any two nontrivial connected graphs G and H, we have*

$$pd(G \odot H) \leq dim(G \odot H) + 1.$$

Theorem 2.3.21. [49] *If G and H are two simple connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. If $D(H) \leq 2$, then*

$$pd(G \odot H) \leq pd(G) + pd(H).$$

Chapter 3

Metric Dimension of Wheel Related Graphs

In this chapter, we study the metric dimension of certain wheel related graphs, namely m-level wheel, an infinite class of convex polytopes defined in [2] and antiweb-gear graphs denoted by $W_{n,m}$, $\mathbb{Q}_n$ and AWJ_{2n}, respectively. The study of this infinite class of convex polytopes gives a negative answer to an open problem proposed in [30].

Open Problem [30]: *Is it the case that graph of every convex polytope has constant metric dimension?* We prove that these infinite classes of wheel related graphs have unbounded metric dimension. Moreover, we extend this study to infinite classes of convex polytopes $\mathbb{Q}_n^m$, $\mathbb{D}_n$, and $\mathbb{B}_n$ generated by wheel related graphs.

We prove that these infinite classes of convex polytopes generated by wheel related graphs have unbounded metric dimension which further supports the negative answer to the open problem raised in [30]. It is natural to ask for the characterization of graphs with unbounded metric dimension.

3.1 Introduction and preliminary results

A *polytope* is a geometric object with flat sides and may exist in any general number of dimensions d as a d-dimensional polytope or d-polytope. For example, a 2-dimensional polygon is a 2-polytope and a 3-dimensional polyhedron is a 3-polytope. A d-polytope is called *simple polytope* if each vertex is contained in exactly d faces. A *convex polytope* is a special case of polytopes, having the additional property that it is also a convex set of points in the d-dimensional space $\mathbb{R}^d$. Convex polytopes play an important role in various branches of mathematics, applied sciences and most notably in linear programming.

The families of graphs with constant metric dimension were discussed previously in [24, 26, 30, 36]. The metric dimension of several classes of convex polytopes has been studied in [24, 27, 29] and it was proved that the metric dimension of those convex polytopes is constant.

By denoting $G + H$ the join of two graphs G and H, a *fan* is defined as $f_n = K_1 + P_n$, for $n \geq 1$. Caceres et al. [8] determined the metric dimension of fan graph as given in the following theorem.

Theorem 3.1.1. [8] *Let f_n be a fan of order $n \geq 1$. Then*

$$dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor, \text{ for } n \geq 7.$$

The *helm graph* H_n is a graph obtained by adding a pendant edge on each rim vertex of the wheel graph W_n. In [34], it was proved that the metric dimension of the helm graph is unbounded, i.e. $dim(H_n) = \lfloor \frac{2n+2}{5} \rfloor$.

In this chapter, the metric dimension of certain *wheel related graphs* has been computed. Moreover, we extend this study to infinite classes of convex polytopes generated by wheel or gear graphs. These results add further support to the negative answer of an open problem raised in [30].

We show that these infinite classes of convex polytopes generated by wheel related graphs have unbounded metric dimension.

3.2 Metric dimension of m-level wheels

Denoting by $G + H$ as join of two graphs, a wheel graph denoted by $W_{n,1}$ is defined as $W_{n,1} \cong C_{n,1} + K_1$, where $C_{n,1} : v_1, v_2, \cdots, v_n, v_1$ for $n \geq 3$ is a cycle of length n. For our convenience, we denote the outer cycle of the wheel by $C_{n,1}$. Moreover, the vertices lying on cycle(s) are known as *rim* vertices and the edges incident on the central vertex of a wheel are known as *spokes*. It was proved in [7] that $dim(W_{n,1}) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$, implying that wheels have unbounded metric dimension.

Suppose $C_{n,1}$ is an outer cycle of length n of $W_{n,1}$. If B is a basis of $W_{n,1}$ then it contains $r \geq 2$ vertices on $C_{n,1}$ for $n \geq 3$ and we can order the vertices of $B = \{v_{i_1}, v_{i_2}, \cdots, v_{i_r}\}$ so that $i_1 < i_2 < \cdots, i_r$. We shall say that the pairs of vertices $\{v_{i_a}, v_{i_{a+1}}\}$ for $1 \leq a \leq r-1$ and $\{v_{i_r}, v_{i_1}\}$ are pairs of neighboring vertices. Given such an ordering, as in [7], we will define the *gap* g_a for $1 \leq a \leq r-1$ as the set of vertices $\{v_j | i_a < j < i_{a+1}\}$ and $g_r = \{v_j | 1 \leq j < i_1 \text{ or } i_r < j \leq n\}$. Thus we have *r gaps*, some of which may be empty. We will say that gaps g_a and g_b are *neighboring gaps* when $|a - b| = 1$ or $r - 1$.

It was shown in [7] that if B is a basis for $W_{n,1}$ then B consists of only those vertices of $C_{n,1}$ that satisfy the following properties:

(a) *Every gap of B contains at most three vertices.*

(b) *At most one gap of B contains three vertices.*

(c) *If a gap of B contains at least two vertices, then both of its neighboring gaps contain at most one vertex.*

Definition 3.2.1. A *double-wheel* graph $W_{n,2}$ can be obtained as a join of $2C_n + K_1$, and inductively we can construct an m-level wheel graph denoted by $W_{n,m}$ as $W_{n,m} \cong mC_n + K_1$.

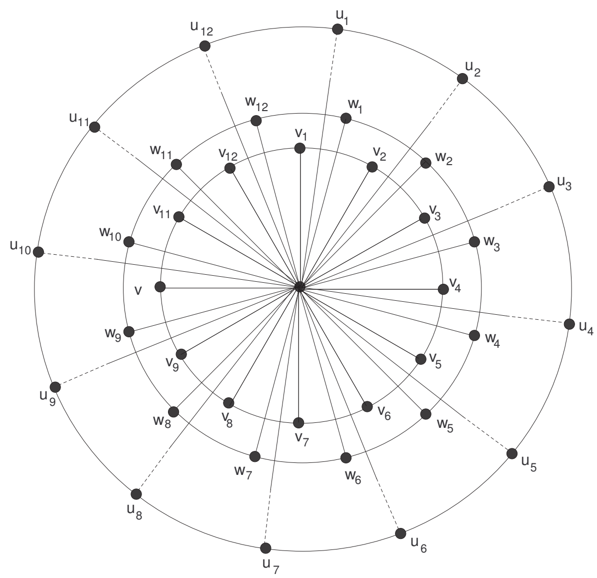

Figure 3.1: An m-level wheel $W_{12,m}$

Let $C_{n,1}, \cdots, C_{n,m}$ represent the cycles of $W_{n,m}$ at levels $1, \cdots, m$, respectively, as shown in Fig. 3.1. We want to compute the metric dimensions of $W_{n,2}, \cdots, W_{n,m}$. For this, we first study the metric dimension of $W_{n,2}$.

Suppose that $W_{n,2} \cong 2C_{n,1} + K_1$ for $n \geq 3$, then the central vertex v does not belong to any basis. Since $diam(W_{n,2}) = 2$, so if v belongs to any metric basis, say B, then there must exist two distinct vertices v_i and v_j, for $1 \leq i \neq j \leq n$ such that $code(v_i|B) = code(v_j|B)$. Consequently, the basis vertices belong to the rim vertices of $W_{n,2}$ only. If B is a basis of $W_{n,2}$, then B contains only vertices from the cycle induced by $C_{n,1}$ and $C_{n,2}$. We have the following gap conditions for the selection of basis vertices:

(i) *Every gap of B for the vertices of $C_{n,1}$ must satisfy conditions (a)-(c) presented for $W_{n,1}$.*

(ii) *Every gap of B may have at most three vertices of $C_{n,1}$ or $C_{n,2}$.* Otherwise, there may be a gap having three vertices, say, w_i, w_{i+1}, w_{i+2} $(1 \leq i \leq n)$ of $C_{n,2}$ and addition performed modulo n such that $code(w_{i+1}|B) = code(v_{i+1}|B)$, where v_i, v_{i+1}, v_{i+2} are the vertices of the gap of $C_{n,1}$. In other words, we can say that at most one gap of B have three vertices.

(iii) *If a gap of B have two vertices; then its neighboring gap contains at most one vertex.* Otherwise, there exists five consecutive vertices, say, $w_i, w_{i+1}, w_{i+2}, w_{i+3}, w_{i+4}$ such that $w_{i+2} \in B$ $(1 \leq i \leq n)$. However, then we have $code(w_{i+1}|B) = code(w_{i+3}|B)$.

Now suppose that B is any set of vertices of $C_{n,1}$ and $C_{n,2}$ that satisfies conditions (i)-(iii) and let $y \in V(W_{n,2}) \setminus B$. There are following possibilities to be discussed:

(1) If y belongs to a gap of B of vertices of $C_{n,1}$ then it must satisfy the following conditions:
(a') *y belongs to a gap of size one of B.* Suppose v_i and v_j be the neighboring vertices of B that determine this gap. Then y is adjacent to v_i and v_j and has distance two from all other vertices of B. Since $n \geq 7$, no other vertex of $W_{n,1}$ has this property and so $code(y|B) \neq code(x|B)$ for $x \neq y$, where $x, y \in V(G) \setminus B$.
(b') *y belongs to a gap of size two of B.* Then we may assume that $v_j, v_{j+1} = y, v_{j+2}, v_{j+3}$ are vertices of $C_{n,1}$, where $v_{j+1}, v_{j+3} \in B$ and $v_{j+2} \notin B$. Then y is adjacent to v_j and has distance 2 from all other vertices of B. By property (c), only y has this property and so $code(y|B) \neq code(x|B)$ for $x \neq y$, where $x, y \in V(G) \setminus B$.
(c') *y belongs to a gap of size three of B.* Then there exists vertices v_j, v_{j+1}, v_{j+2},
v_{j+3}, v_{j+4} of $C_{n,1}$, where only $v_{j+1}, v_{j+4} \in B$. Assume first that $y = v_{j+1}$. Then y is adjacent to v_j and has distance 2 from all other vertices of B. By property (c), y is the only vertex of $W_{n,1}$ with this property and so $code(y|B) \neq code(x|B)$ for $x \neq y$, where $x, y \in V(G) \setminus B$.
Next, we assume that $y = v_{j+2}$. Then $code(y|B) = (2, 2, \cdots, 2)$. By properties (a) and (b), no other vertex of $W_{n,1}$ has this representation.
(d') *$y = v$ be a central vertex.* Then $code(y|B) = (1, 1, \cdots, 1)$ and y is the only vertex of $W_{n,1}$ with this representation.
(2) Similarly, one can show that if either y belongs to a gap of size one, two, or three of B of vertices of cycle induced by $C_{n,2}$ or if y is a central vertex of $W_{n,2}$; then we have $code(y|B) \neq code(x|B)$ for $x \neq y$; $x, y \in V(W_{n,2})$.
Therefore, any set B having properties (i)-(iii) is a resolving set for $W_{n,2}$.
In the next theorem, we give a precise formula for computing the metric dimension of double wheel $W_{n,2}$ for $n \geq 7$. This result provides a base for extending the result to the metric dimension of m-level wheels.

Theorem 3.2.2. *If $n \geq 7$, then we have $dim(W_{n,2}) = dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$.*

Proof. Let $W_{n,1} \cong C_{n,1} + K_1$ and $W_{n,2} \cong 2C_{n,1} + K_1$, where v is the central vertex of $W_{n,2}$ and $C_{n,1} : v_1, \cdots, v_n, v_1$ and $C_{n,2} : w_1, \cdots, w_n, w_1$ be the outer cycles of $W_{n,2}$ at levels 1 and 2, respectively. First we prove that $dim(W_{n,2}) \leq dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$ by constructing a resolving set in $W_{n,2}$ having $dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$ vertices. We assume the following cases according to the residue class modulo 5 to which n belongs.

Case 1: When $n \equiv 0 \ (mod\ 5)$, then we may write $n = 5k$, where $k \geq 2$, and $dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor = 4k$. Since $B = \{v_{5i+1}, v_{5i+4}, w_{5j+1}, w_{5j+4} : 0 \leq i, j \leq k-1\}$, it is a resolving set having $4k$ vertices as it satisfies conditions (i)-(iii).

Case 2: When $n \equiv 1 \ (mod\ 5)$, then we may write $n = 5k+1$, where $k \geq 2$, and $dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor =$

$4k+1$. Since $B = \{v_{5i+1}, v_{5i+4} : 0 \leq i \leq k-2\} \cup \{v_{5k-4}, v_{5k}\} \cup \{w_{5j+1}, w_{5j+4} : 0 \leq j \leq k-1\} \cup \{w_{5k+1}\}$, it is a resolving set having $4k+1$ vertices as it satisfies conditions (i)-(iii).

Case 3: When $n \equiv 2\ (mod\ 5)$, then we may write $n = 5k+2$, where $k \geq 1$, and $dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor = 4k+2$. Since $B = \{v_{5i+1}, v_{5i+4}, w_{5j+1}, w_{5j+4} : 0 \leq i, j \leq k-1\} \cup \{v_{5k+1}, w_{5k+1}\}$, it is a resolving set having $4k+2$ vertices as it satisfies conditions (i)-(iii).

Case 4: When $n \equiv 3\ (mod\ 5)$, then we may write $n = 5k+3$, where $k \geq 1$, and $dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor = 4k+3$. Since $B = \{v_{5i+1}, v_{5i+4} : 0 \leq i \leq k-2\} \cup \{v_{5k-4}, v_{5k, v_{5k+2}}\} \cup \{w_{5j+4}, w_{5j+6} : 0 \leq j \leq k-1\} \cup \{w_1, w_{5k+3}\}$, it is a resolving set having $4k+3$ vertices as it satisfies conditions (i)-(iii).

Case 5: When $n \equiv 4\ (mod\ 5)$, then we may write $n = 5k+4$, where $k \geq 1$, and $dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor = 4k+4$. Since $B = \{v_{5i+1}, v_{5i+4}, w_{5j+1}, w_{5j+4} : 0 \leq i, j \leq k\}$, it is a resolving set having $4k+4$ vertices as it satisfies conditions (i)-(iii).
Hence, it follows from above discussion that $dim(W_{n,2}) \leq dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$.
Next, we show that $dim(W_{n,2}) \geq dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$. Let B be a basis for $W_{n,2}$. We consider the following cases:

Case(a). subcase(a_1): $\mid B_1 \mid= 2l$ for some integer $l \geq 1$, where B_1 is the basis for $W_{n,1}$ as obtained in [7], i.e. $\mid B_1 \mid \geq dim(W_{n,1}) = \lfloor \frac{2n+2}{5} \rfloor$.
subcase(a_2): $\mid B_2 \mid= 2t$ for some integer $t \geq 1$, where B_2 represents the resolving vertices lying on $C_{n,2}$ in presence of vertices of B_1. The conditions (i)-(iii) imply that at most t gaps of B_2 contain two vertices. So the number of vertices that belong to different gaps of B_2 are at most $3t$. Therefore we get, $n-2t \leq 3t$ which implies that $\mid B_2 \mid= 2t \geq \lceil \frac{2n}{5} \rceil \geq \lfloor \frac{2n+4}{5} \rfloor$.
subcase(a_3): $\mid B_2 \mid= 2t+1$ for some integer $t \geq 1$, where B_2 represents the resolving vertices lying on $C_{n,2}$ in presence of vertices B_1. Condition (i)-(iii) imply that at most t gaps of B_2 contain two vertices. So the number of vertices that belong to different gaps of B_2 are at most $3t+1$. Therefore we get, $n-2t-1 \leq 3t+1$ which implies that $\mid B_2 \mid= 2t+1 \geq \lceil \frac{2n+1}{5} \rceil \geq \lfloor \frac{2n+4}{5} \rfloor$. Hence by combining subcase(a_1) with subcase(a_2) or subcase(a_3), we obtain that $\mid B \mid=\mid B_1 \mid + \mid B_2 \mid \geq 2l+2t \geq dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$.

Case(b). subcase(b_1): $\mid B_1 \mid= 2l+1$ for some integer $l \geq 1$, where B_1 is the basis for $W_{n,1}$ as obtained in [7], i.e. $\mid B_1 \mid \geq dim(W_{n,1})$.

subcase(b_2): $\mid B_2 \mid= 2t+1$ for some integer $t \geq 1$, where B_2 represents the resolving vertices lying on $C_{n,2}$ in presence of vertices B_1. Condition (i)-(iii) imply that at most t gaps of B_2 contain two vertices. So the number of vertices that belong to different gaps of B_2 are at most $3t+1$.

Therefore we get, $n-2t-1 \leq 3t+1$ which implies that $\mid B_2 \mid = 2t+1 \geq \lceil \frac{2n+1}{5} \rceil \geq \lfloor \frac{2n+4}{5} \rfloor$
subcase(b_3): This case is similar to the subcase(a_2). Hence by combining subcase(b_1) with subcase(b_2) or subcase(b_3), we obtain that $\mid B \mid = \mid B_1 \mid + \mid B_2 \mid \geq 2l+1+2t+1 \geq dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$, which completes the proof. □

We now extend our result to m-level wheel denoted by $W_{n,m}$. In the next theorem, we apply mathematical induction on the number of levels of wheel to prove the result.

Theorem 3.2.3. *We have $dim(W_{n,m}) = dim(W_{n,1}) + (m-1)\lfloor \frac{2n+4}{5} \rfloor$ for every integer $n \geq 7$ and $m \geq 3$.*

Proof. We will prove this result by induction on the number of levels of wheel denoted by m. When $m = 1$, then $dim(W_{n,1}) = \lfloor \frac{2n+2}{5} \rfloor$ is obtained in [7]. When $m = 2$, then $dim(W_{n,2}) = dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$ by Theorem 3.2.2. Now we assume that the assertion is true for $m = k$, i.e.,

$$dim(W_{n,k}) = dim(W_{n,1}) + (k-1)\lfloor \frac{2n+4}{5} \rfloor. \tag{3.2.1}$$

We will show that it is true for $m = k+1$. Suppose $dim(W_{n,k+1}) = dim(W_{n,k}) + \lfloor \frac{2n+4}{5} \rfloor$, then by using equation (3.2.1), we have $dim(W_{n,k+1}) = \{dim(W_{n,1}) + (k-1)\lfloor \frac{2n+4}{5} \rfloor\} + \lfloor \frac{2n+4}{5} \rfloor = dim(W_{n,1}) + (k)\lfloor \frac{2n+4}{5} \rfloor$. Hence the result is true for all positive integers $m \geq 3$. □

3.3 Metric dimension of antiweb-gear graphs

The gear graph denoted by J_{2n} is defined as follows: Consider an even cycle $C_{2n} : v_1, v_2, \cdots, v_{2n}, v_1$, where $n \geq 2$ and a new vertex v is adjacent to n vertices of $C_{2n} : v_2, v_4, \cdots, v_{2n}$. The gear graph J_{2n} can be obtained from the wheel W_{2n} by alternately deleting n spokes. Tomescu et al. [60] proved that $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ for $n \geq 4$.
The square of a cycle C_n is denoted by C_n^2 and it is isomorphic to circulant graph $C_n(1,2)$, i.e. $C_n^2 \cong C_n(1,2)$. An antiweb-wheel denoted by AWW_n can be defined as $AWW_n \cong C_n^2 + K_1$. We have $V(AWW_n) = V(W_n)$ and $E(AWW_n) = E(W_n) \cup \{v_i v_{i+2} : 0 \leq i \leq n\}$, where the indices are taken modulo n. In [41], it was proved that

$$dim(AWW_n) = \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

The antiweb-gear graph can be obtained from gear graph J_{2n} by replacing C_{2n} by C_{2n}^2 and is denoted by AWJ_{2n}. We have $V(AWJ_{2n}) = V(J_{2n})$ and $E(AWJ_{2n}) = E(J_{2n}) \cup \{v_i v_{i+2} : 0 \leq i \leq n\}$, where the indices are taken modulo n. In this section, we study the metric dimension of antiweb-gear graphs and we prove that this class has unbounded metric dimension.
Suppose that AWJ_{2n} for $n \geq 3$, then the central vertex v does not belong to any basis. Since $diam(AWJ_{2n}) = 4$, if v belongs to any metric basis, say B, then there must exist two distinct

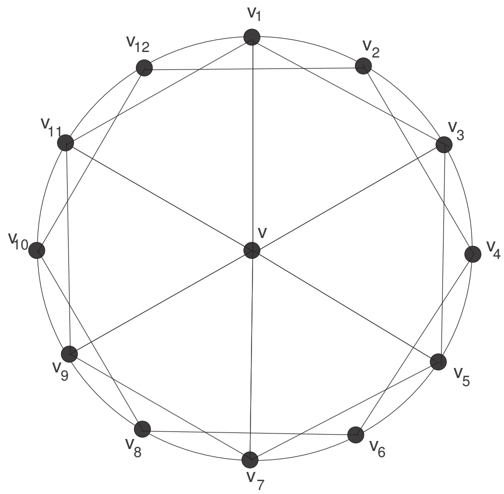

Figure 3.2: An antiweb-gear graph AWJ_{12}

vertices v_i and v_j, for $1 \leq i \neq j \leq n$ such that $code(v_i|B) = code(v_j|B)$. Consequently, the basis vertices belong to the rim vertices of AWJ_{2n} only.

A gap determined by neighboring vertices v_i and v_j will be called an $\alpha - \beta$ gap with $\alpha \geq \beta$ when $d(v_i) = \alpha$ and $d(v_j) = \beta$ or when $d(v_i) = \beta$ and $d(v_j) = \alpha$. Hence we have three kinds of gaps in AWJ_{2n}, i.e. $4-4$, $5-4$ and $5-5$ gaps.

Lemma 3.3.1. *Let B be a basis of AWJ_{2n} $n \geq 6$, then every $4-4$, $5-4$ and $5-5$ gap of B contains at most 9, 8 and 7 vertices, respectively.*

Proof. On contrary, suppose that there is a $4-4$ gap of B having 11 vertices $v_1, \cdots, v_{11}$ of C_{2n} such that $deg(v_1) = deg(v_{11}) = 5$. For this case, $code(v_5|B) = code(v_7|B)$, a contradiction. Similarly, if there is a $5-4$ having 10 vertices of C_{2n} say, $v_1, \cdots, v_{10}$ such that $deg(v_1) = 5$ and $deg(v_{10}) = 4$. In this case, we get $code(v_5|B) = code(v_7|B)$, a contradiction. If there is a $5-5$ gap having 9 vertices say, $v_1, \cdots, v_9$ such that $deg(v_1) = deg(v_9) = 4$, then $code(v_5|B) = code(v_7|B)$, a contradiction. ☐

From now on, the $4-4$, $5-4$ and $5-5$ gaps having 9, 8 and 7 vertices, respectively will be referred as *major gaps*, while the rest of all will be referred as *minor gaps*. The vertices having degree 5 and 4 are known as *major* (labeled by star) and *minor* vertices, respectively.

Lemma 3.3.2. *Any basis B of AWJ_{2n} $(n \geq 6)$ contains at most one major $4-4$ or $5-4$ gap.*

Proof. On contrary, suppose that B contains two distinct major gaps of kind $4-4$ or $5-4$, then we have the following cases:

• $4-4$ and $4-4$ gaps: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9$ and $\overset{*}{u}_1, u_2, \overset{*}{u}_3, u_4, \overset{*}{u}_5, u_6, \overset{*}{u}_7, u_8, \overset{*}{u}_9$; in this case we have $code(\overset{*}{v}_5|B) = code(\overset{*}{v}_7|B)$.

• $4-4$ and $5-4$ gaps: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9$ and $u_1, \overset{*}{u}_2, u_3, \overset{*}{u}_4, u_5, \overset{*}{u}_6, u_7, \overset{*}{u}_8$; in this case we have $code(\overset{*}{v}_5|B) = code(\overset{*}{u}_4|B)$.
• $5-4$ and $5-4$ gaps: $u_1, \overset{*}{u}_2, u_3, \overset{*}{u}_4, u_5, \overset{*}{u}_6, u_7, \overset{*}{u}_8$ and $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8$; in this case we have $code(\overset{*}{v}_4|B) = code(\overset{*}{u}_4|B)$. □

In the next lemma, we will prove that any two neighboring gaps, one of which being *major gap* may contain together at most 12 vertices.

Lemma 3.3.3. *For any basis B of AWJ_{2n} $(n \geq 6)$, any two neighboring gaps, one of which being major gap of kind $4-4$, $5-4$ or $5-5$ contain together at most 12 vertices.*

Proof. If the major gap is a $4-4$ gap (with 9) vertices, then by Lemma 3.3.2 its neighboring gap can neither be a $4-4$ gap having 5 vertices nor be a $5-4$ gap having 4 vertices. If it is true, consider a path: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9, v_{10}, \overset{*}{v}_{11}, v_{12}, \overset{*}{v}_{13}, v_{14}, \overset{*}{v}_{15}$ on C_{2n} where $v_{10} \in B$ such that $code(v_9|B) = code(v_{11}|B)$, a contradiction. If the major gap is a $5-4$ gap (with 8) vertices; then by Lemma 4.3.2, its neighboring gap can't be a $4-4$ gap having 5 vertices. If it is true, consider a path $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}, v_{13}, \overset{*}{v}_{14}$ on C_{2n} where $\overset{*}{v}_9 \in B$ such that $code(v_8|B) = code(v_{10}|B)$, a contradiction. If the major gap is a $5-5$ gap having 7 vertices then its neighboring gap can't be a minor $5-5$ gap having 5 vertices. If it is true; consider a path: $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}, v_{13}$ on C_{2n} where $\overset{*}{v}_8 \in B$ such that $code(\overset{*}{v}_6|B) = code(\overset{*}{v}_{10}|B)$, a contradiction. □

In the next lemma, we will prove that any two *minor* neighboring gaps may contain together at most ten vertices.

Lemma 3.3.4. *If B is any basis of AWJ_{2n} $(n \geq 6)$, then any two minor neighboring gaps contain together at most 10 vertices.*

Proof. By Lemma 3.3.1, any minor $4-4$, $5-4$, and $5-5$ gap contains 7, 6 and 5 vertices, respectively. It suffices to prove the following cases:
• *Any minor $4-4$ gap having 5 or 7 vertices has a neighboring $4-4$ or $5-4$ gaps with at most 3 and 2 vertices, respectively.* Otherwise, there is a neighboring $4-4$ or $5-4$ gap having 5 and 4 vertices, respectively. In this case, consider a path: $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9, v_{10}, \overset{*}{v}_{11}, v_{12}, \overset{*}{v}_{13}$ on C_{2n}, where $v_8 \in B$ such that $code(\overset{*}{v}_7|B) = code(\overset{*}{v}_9|B)$, a contradiction.
• *Any minor $5-4$ gap having 6 or 4 vertices has a neighboring $4-4$ or $5-4$ gaps with at most 3 and 4 vertices, respectively.* Otherwise, there is a neighboring $4-4$ or $5-4$ gap having 5 and 6 vertices, respectively. In this case, consider a path: $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}$ on C_{2n}, where $v_7 \in B$ such that $code(\overset{*}{v}_6|B) = code(\overset{*}{v}_8|B)$, or $\overset{*}{v}_1, v_2, \overset{*}{v}_3, v_4, \overset{*}{v}_5, v_6, \overset{*}{v}_7, v_8, \overset{*}{v}_9, v_{10}, \overset{*}{v}_{11}, v_{12}, \overset{*}{v}_{13}$ on C_{2n}, where $\overset{*}{v}_7 \in B$ such that $code(v_6|B) = code(v_8|B)$, a contradiction.
• *Any $5-5$ gap having 5 vertices has a neighboring $5-5$ or $5-4$ gap with at most 3 and 4 vertices,*

respectively. Otherwise, the neighboring gaps may contain 5 and 6 vertices, respectively. In this case, consider a path: $v_1, \overset{*}{v}_2, v_3, \overset{*}{v}_4, v_5, \overset{*}{v}_6, v_7, \overset{*}{v}_8, v_9, \overset{*}{v}_{10}, v_{11}, \overset{*}{v}_{12}, v_{13}$ on C_{2n}, where $\overset{*}{v}_7 \in B$ such that $code(v_5|B) = code(v_7|B)$, a contradiction. □

In the next theorem, we compute the exact value of metric dimension for antiweb-gear graphs.

Theorem 3.3.5. *For every integer $n \geq 15$, we have $dim(AWJ_{2n}) = \lceil \frac{2n+1}{6} \rceil$.*

Proof. Consider the antiweb-gear graphs AWJ_{2n}, then we have $dim(AWJ_{2n}) = 3$, for all $2 \leq n \leq 8$ and $W_1 = \{v_1, v_2, v_3\}$ and $W_2 = \{v_1, v_4, v_9\}$ being metric basis for all $2 \leq n \leq 6$ and $n = 7,\ 8$, respectively. $dim(AWJ_{2n}) = 4$, for all $9 \leq n \leq 12$ and $W_3 = \{v_1, v_4, v_8, v_{15}\}$, $W_4 = \{v_1, v_2, v_{10}, v_{12}\}$, $W_5 = \{v_1, v_2, v_{10}, v_{14}\}$ and $W_6 = \{v_1, v_4, v_{12}, v_{16}\}$ being metric basis for $n = 9,\ 10,\ 11,\ 12$, respectively. $dim(AWJ_{2n}) = 5$ and $W_7 = \{v_1, v_5, v_{12}, v_{15}, v_{20}\}$ and $W_8 = \{v_1, v_4, v_{12}, v_{16}, v_{20}\}$ being metric basis for $n = 13,\ 14$, respectively. However for $n \geq 15$, the dimension of AWJ_{2n} increases with number of vertices n. We also know that the central vertex cannot belong to any basis of AWJ_{2n}. For $n \geq 15$, we prove the result by double inequality. First we show that $dim(AWJ_{2n}) \leq \lceil \frac{2n+1}{6} \rceil$ by constructing a resolving set M in AWJ_{2n} having $\lceil \frac{2n+1}{6} \rceil$ vertices. For this we consider the following cases:

Case 1: When $n \equiv 0(mod\ 6)$, then we may write $2n = 6k$, where $k \geq 6$ and $\lceil \frac{2n+1}{6} \rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18} : 0 \leq i \leq \frac{k-4}{2}\} \cup \{v_{2n-2}\}$.

Case 2: When $n \equiv 1(mod\ 6)$, then we may write $2n = 6k + 2$, where $k \geq 6$ and $\lceil \frac{2n+1}{6} \rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18} : 0 \leq i \leq \frac{k-4}{2}\} \cup \{v_{2n-2}\}$ is a resolving set.

Case 3: When $n \equiv 2(mod\ 6)$, then we may write $2n = 6k + 4$, where $k \geq 6$ and $\lceil \frac{2n+1}{6} \rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18} : 0 \leq i \leq \frac{k-4}{2}\} \cup \{v_{2n-2}\}$ is a resolving set.

Case 4: When $n \equiv 3(mod\ 6)$, then we may write $2n = 6k$, where $k \geq 5$ and $\lceil \frac{2n+1}{6} \rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18} : 0 \leq i \leq \frac{k-5}{2}\} \cup \{v_{2n-4}, v_{2n}\}$ is a resolving set.

Case 5: When $n \equiv 4(mod\ 6)$, then we may write $2n = 6k + 2$, where $k \geq 5$ and $\lceil \frac{2n+1}{6} \rceil = k + 1$. In this case, $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18} : 0 \leq i \leq \frac{k-3}{2}\}$ is a resolving set.

Case 6: When $n \equiv 5(mod\ 6)$, then we may write $2n = 6k + 4$, where $k \geq 5$ and $\lceil \frac{2n+1}{6} \rceil = k + 1$. We define $M = \{v_1, v_{10}\} \cup \{v_{12i+14}, v_{12i+18} : 0 \leq i \leq \frac{k-3}{2}\}$ is a resolving set.

The set M contains only one major vertex, rest of the vertices are all minor vertices. So there is a unique $5-4$ major and $5-4$ minor gap, and rest of all are minor $4-4$ gaps containing seven

and three, one and three or three and three vertices alternatively. M is a resolving set of AWJ_{2n}, since any two minor or any two major vertices, respectively, lying in different gaps (neighboring or not) are separated by at least one vertex in the set of three or four vertices of M determining these two gaps. This property is true for the vertices lying in the same gap. Also we note that $code(v|S) = (1, 2, 2, \cdots, 2)$ and $code(v|S) \neq code(x|S)$, for every $x \in V(AWJ_{2n})$ where v is a central vertex and $x \neq v$.
To prove that $dim(AWJ_{2n}) \geq \lceil \frac{2n+1}{6} \rceil$, let B be a basis of AWJ_{2n} and $\mid B \mid = l$. Then B induces l gaps on C_{2n}, namely $g_1, \cdots, g_l$ such that g_j and g_{j+1} are neighboring gaps for every $1 \leq j \leq l-1$, and also g_1 and g_l are neighboring gaps. By Lemma 3.3.2, at most one of the gaps is major, say g_1. By Lemma 3.3.3, and Lemma 3.3.4, we can write

$$\mid g_1 \mid + \mid g_2 \mid \leq 11;$$

$$\mid g_2 \mid + \mid g_3 \mid \leq 6;$$

$$\mid g_j \mid + \mid g_{j+1} \mid \leq 10;$$

for every $j = 3, \cdots, l-2$

$$\mid g_{l-1} \mid + \mid g_l \mid \leq 9$$

and

$$\mid g_l \mid + \mid g_1 \mid \leq 12.$$

By adding these inequalities, we get

$$2(2n - l) = 2\sum_{j=1}^{l} \mid g_j \mid \leq 10l - 2.$$

It follows that $l \geq \lceil \frac{2n+1}{6} \rceil$. Since l is an integer, for each $2n \equiv 0, 2, 4 (mod\ 6)$, we have $l \geq \lceil \frac{2n+1}{6} \rceil$, which completes the proof. □

3.4 Metric Dimension of an Infinite Class of Convex Polytopes $\mathbb{Q}_n$

Let $I = \{1, \cdots, n\}$ be an indexed set and Q_n be the graph of an antiprism. The antiprism Q_n, $n \geq 3$ is defined in [36] as a plane regular graph. Let us denote the vertex set of Q_n by $V(Q_n) = \{y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_n\}$ and the edge set by $E(Q_n) = \{y_i y_{i+1} : i \in I\} \cup \{z_i z_{i+1} : i \in I\} \cup \{y_i z_{i+1} : i \in I\}$. We make the convention that $y_{n+1} = y_1$ and $z_{n+1} = z_1$ to simplify later notations. The face set $F(Q_n)$ contains $2n$ 3-sided faces and two n-sided faces (internal and external). We insert exactly one vertex x (t) into the internal (external) n-sided face of Q_n and consider the graph $\mathbb{Q}_n$ with the vertex set $V(\mathbb{Q}_n) = V(Q_n) \cup \{x, t\}$ and the edge set $E(\mathbb{Q}_n) = E(Q_n) \cup \{xy_i : i \in I\} \cup \{z_i t : i \in I\}$. The $\mathbb{Q}_n$ is the plane graph consisting of 3-sided faces and constitutes an infinite class of convex polytopes.

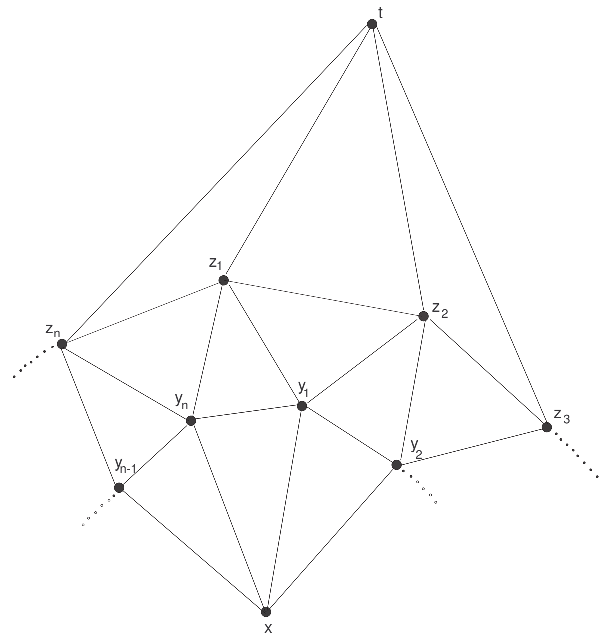

Figure 3.3: An infinite class of convex polytopes $\mathbb{Q}_n$

The metric dimension of several classes of graphs was studied in [24–27,29–31,34,36], and was proved that the classes of convex polytopes have constant metric dimension. The following open problem was proposed in [30].

Open Problem [30]: *Is it the case that graph of every convex polytope has constant metric dimension?*

In this section, we study the metric dimension of the class of convex polytopes denoted by $\mathbb{Q}_n$ and we prove that this class of graph has unbounded metric dimension, thus giving a negative answer to the open problem proposed in [30]. Let $y_1, y_2, \cdots, y_n$ and $z_1, z_2, \cdots, z_n$ represents the vertices of inner cycle $C_{n,1}$ and outer cycle $C_{n,2}$ of $\mathbb{Q}_n$, respectively, as shown in Fig. 3.3. Suppose that $\mathbb{Q}_n$ for $n \geq 3$ be an infinite class of convex polytopes, then the central vertices x and t do not belong to any basis. Since $diam(\mathbb{Q}_n) = 3$, so if one of x and t belongs to any metric basis, say B, then there must exist two distinct vertices v_i and v_j, for $1 \leq i \neq j \leq n$ such that $code(v_i|B) = code(v_j|B)$. consequently, the basis vertices belong to the rim vertices of $\mathbb{Q}_n$ only.

If B is a basis of $\mathbb{Q}_n$, then B contains only vertices of inner cycle of $\mathbb{Q}_n$. We have the following gap conditions for the selection of the basis vertices:

(i) *Every gap of B may have at most two vertices of $C_{n,1}$.* Otherwise there exist a gap of B having three vertices y_p, y_{p+1}, y_{p+2}, y_{p+3} and y_{p+4} with y_p, $y_{p+4} \in B$ such that $code(y_{p+2}|B) = code(t|B) = (2, 2, \cdots, 2)$.

(ii) *If a gap of B contains two vertices of $C_{n,1}$, then its neighboring gaps may contain at most one* *vertex*. Otherwise, there exist five consecutive vertices y_p, y_{p+1}, y_{p+2}, y_{p+3} and y_{p+4}, with $y_{p+2} \in B$ such that $code(y_{p+1}|B) = code(y_{p+3}|B)$.

Now we assume that B is any set of vertices of $C_{n,1}$ that satisfies condition (i) and (ii) and let $y \in V(\mathbb{Q}_n)$. There are following possibilities to be discussed:

• If y belongs to a gap of size two of B with vertices y_p, $y_{p+1} = y$, y_{p+2}, y_{p+3} such that $y_p, y_{p+3} \in B$, then $code(y|B) = (1, 2, \cdots, 2)$.

• If y belongs to a gap of size one of B with vertices y_p, $y_{p+1} = y$, y_{p+2} such that $y_p, y_{p+2} \in B$, then $code(y|B) = (1, 1, 2, \cdots, 2)$.

• If $y = t$, then $code(y|B) = (2, 2, \cdots, 2)$.

• If $y = x$, then $code(y|B) = (1, 1, \cdots, 1)$.

• If $y = z_p \in V(C_{n,2})$ and y is adjacent to y_p and y_{p+n-1} with $y_p, y_{p+3}, y_{p+n-1} \in B$, then $code(y|B) = (1, 3, \cdots, 3, 1)$.

• If $y = z_p \in V(C_{n,2})$ and y is adjacent to y_p and y_{p+n-1} with $y_p, y_{p+3}, y_{p+n-2} \in B$, then $code(y|B) = (1, 3, \cdots, 3, 2)$.

• If $y = z_p \in V(C_{n,2})$ and y is adjacent to y_p and y_{p+n-1} with $y_{p+1}, y_{p+n-2} \in B$, then $code(y|B) = (2, 3, \cdots, 3, 2)$.

• If $y = z_p \in V(C_{n,2})$ and y is adjacent to y_p and y_{p+n-1} with $y_{p+1}, y_{p+n-3}, y_{p+n-1} \in B$, then $code(y|B) = (2, 3, \cdots, 3, 1)$.

Therefore, any set B having properties (i) and (ii) is a resolving set of $\mathbb{Q}_n$. We now present an exact formula for computing the metric dimension of $\mathbb{Q}_n$ for every integer $n \geq 6$.

Theorem 3.4.1. *If $n \geq 6$, then we have $dim(\mathbb{Q}_n) = \lfloor \frac{2n+4}{5} \rfloor$.*

Proof. We prove this result by double inequality. First we prove that $dim(\mathbb{Q}_n) \leq \lfloor \frac{2n+4}{5} \rfloor$ by constructing a resolving set in $\mathbb{Q}_n$ with $\lfloor \frac{2n+4}{5} \rfloor$ vertices. We consider the following cases according to the residue class modulo 5 to which n belongs.

Case 1: When $n \equiv 0 \ (mod\ 5)$, then we may write $n = 5k$, where $k \geq 2$, and $\lfloor \frac{2n+4}{5} \rfloor = 2k$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\}$, it is a resolving set having $2k$ vertices as it satisfies conditions (i) and (ii).

Case 2: When $n \equiv 1 \ (mod\ 5)$, then we may write $n = 5k+1$, where $k \geq 1$, and $\lfloor \frac{2n+4}{5} \rfloor = 2k+1$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\} \cup \{y_{5k+1}\}$, it is a resolving set having $2k+1$ vertices as it satisfies conditions (i) and (ii).

Case 3: When $n \equiv 2 \ (mod\ 5)$, then we may write $n = 5k+2$, where $k \geq 1$, and $\lfloor \frac{2n+4}{5} \rfloor = 2k+1$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\} \cup \{w_{5k+1}\}$, it is a resolving set having $2k+1$ vertices as it satisfies conditions (i) and (ii).

Case 4: When $n \equiv 3 \ (mod\ 5)$, then we may write $n = 5k + 3$, where $k \geq 1$, and $\lfloor \frac{2n+4}{5} \rfloor = 2k + 2$. Since $B = \{y_{5i+4}, y_{5i+6} : 0 \leq i \leq k-1\} \cup \{y_1, y_{5k+3}\}$, it is a resolving set having $2k + 2$ vertices as it satisfies conditions (i) and (ii).

Case 5: When $n \equiv 4 \ (mod\ 5)$, then we may write $n = 5k + 4$, where $k \geq 1$, and $\lfloor \frac{2n+4}{5} \rfloor = 2k + 2$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k\}$, it is a resolving set having $2k + 2$ vertices as it satisfies conditions (i) and (ii). Hence, from above it follows that $dim(\mathbb{Q}_n) \leq \lfloor \frac{2n+4}{5} \rfloor$.
Next we show that $dim(\mathbb{Q}_n) \geq \lfloor \frac{2n+4}{5} \rfloor$. Let B be a basis of $\mathbb{Q}_n$. We consider the following cases:

Case (a): $|B| = 2t$ for some integer $t \geq 1$. The conditions (i) and (ii) imply that at most t gaps of B contains two vertices. So the number of vertices that belong to different gaps of B are at most $3t$. Therefore $n - 2t \leq 3t$, which implies that $|B| = 2t \geq \lceil \frac{2n}{5} \rceil \geq \lfloor \frac{2n+4}{5} \rfloor$.

Case (b): $|B| = 2t + 1$ for some integer $t \geq 1$. The conditions (i) and (ii) imply that at most t gaps of B contain two vertices. So the number of vertices that belong to different gaps of B are at most $3t + 1$. Therefore $n - 2t - 1 \leq 3t + 1$, which implies that $|B| = 2t + 1 \geq \lceil \frac{2n+1}{5} \rceil \geq \lfloor \frac{2n+4}{5} \rfloor$, which complete the proof. $\square$

3.5 Metric Dimension of an Infinite Class of Convex Polytopes $\mathbb{Q}_n^m$

The antiprism Q_n, $n \geq 3$ is defined in [36], is a 4-regular plane graph and also known as Archimedean convex polytope. Let $I = \{1, \cdots, n\}$ and $J = \{1, \cdots, m\}$ be index sets. For $n \geq 3$ and $m \geq 1$ we denote by Q_n^m the plane graph of convex polytope, which is obtained as a combination of m antiprisms Q_n. Let us denote the vertex set of Q_n^m by $V(Q_n^m) = \{y_{j,i} : i \in I \text{ and } j \in J \cup \{m+1\}\}$ and the edge set by $E(Q_n^m) = \{y_{j,i}y_{j,i+1} : i \in I \text{ and } j \in J \cup \{m+1\}\} \cup \{y_{j,i}y_{j+1,i} : i \in I \text{ and } j \in J\} \cup \{y_{j,i+1}y_{j+1,i} : i \in I \text{ and } j \in J, j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1} : i \in I \text{ and } j \in J, j \text{ even}\}$. We make the convention that $y_{j,n+1} = y_{j,1}$ for $j \in J \cup \{m+1\}$. The face set $F(Q_n^m)$ contains $2mn$ 3-sided faces, an internal n-sided face and an external n-sided face. We insert exactly one vertex x (z) into the internal (external) n-sided face of Q_n^m and connect the vertex x (z) with the vertices $y_{1,i}$ $(y_{m+1,i})$, $i \in I$. Thus, we obtain the plane graph $\mathbb{Q}_n^m$ [5] (labelled as in Fig. 3.4), consisting of 3-sided faces with the vertex set $V(\mathbb{Q}_n^m) = V(Q_n) \cup \{x, z\}$ and the edge set $E(\mathbb{Q}_n^m) = E(Q_n) \cup \{xy_{1,i} : i \in I\} \cup \{y_{m+1,i}z : i \in I\}$ where $|V(\mathbb{Q}_n^m)| = (m+1)n + 2$, $|E(\mathbb{Q}_n^m)| = 3n(m+1)$ and $|F(\mathbb{Q}_n^m)| = 2n(m+1)$.
The metric dimension of several classes of graphs was studied in [24, 26, 27, 29, 30, 34, 36], and it was proved that these classes of convex polytopes have constant metric dimension. The metric dimension of convex polytope $\mathbb{Q}_n$ has been computed in the Theorem 3.4.1 in the section 3.4.

In this section, we extend the result proved in 3.4 for the convex polytope $\mathbb{Q}_n^m$ for $m \geq 2$

and we prove that this class of graph has unbounded metric dimension. Let $y_{1,1}, y_{1,2}, \cdots, y_{1,n}$ and

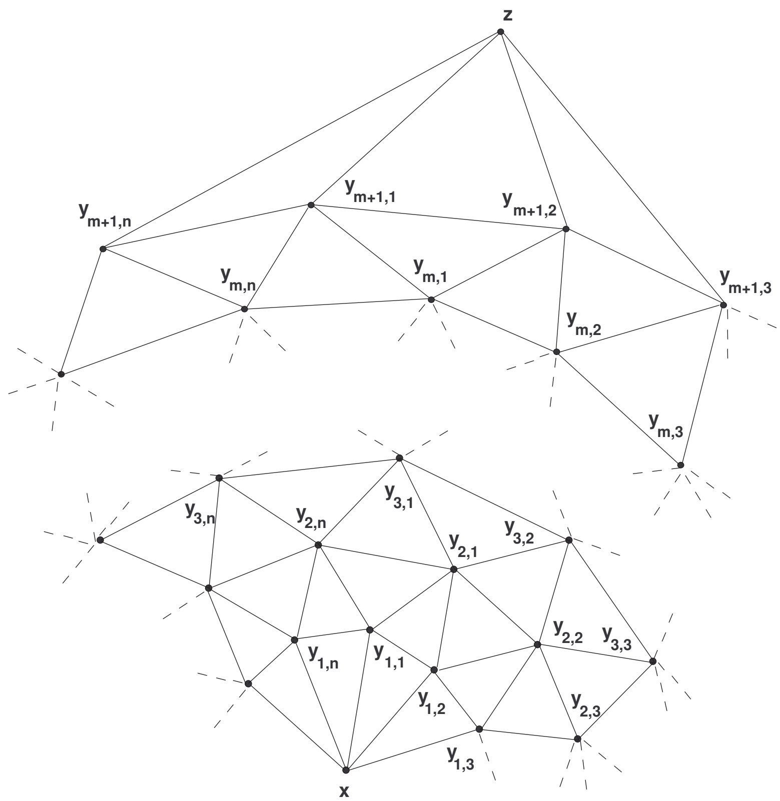

Figure 3.4: An infinite class of convex polytopes $\mathbb{Q}_n^m$

$y_{m+1,1}, y_{m+1,2}$,
$\cdots, y_{m+1,n}$ represents the vertices of inner cycle $C_{n,1}$ and outer cycle $C_{n,m+1}$ of $\mathbb{Q}_n^m$, respectively, as shown in Fig. 3.4. Suppose that $\mathbb{Q}_n^m$ for $n \geq 3$ and $m \geq 1$ be an infinite class of convex polytopes, then the central vertices x and z do not belong to any basis. Since $diam(\mathbb{Q}_n^m) = m + 2$, so if one of x and z belongs to any metric basis, say B, then there must exist two distinct vertices $y_{j,i}$ and $y_{j,k}$ $(1 \leq i \neq k \leq n)$ and $(1 \leq j \leq m + 1)$ such that $code(y_{j,i}|B) = code(y_{j,k}|B)$, a contradiction. Consequently, the basis vertices belong to the rim vertices of the convex polytope graph $\mathbb{Q}_n^m$ only.

Suppose B be a basis of $\mathbb{Q}_n^m$ and if it contains $s(\geq 2)$ vertices on $C_{n,1}$. Consider the inner cycle $C_{n,1}$ of $\mathbb{Q}_n^m$ with n vertices. We can order the vertices of $B = \{y_{1,j_1}, y_{1,j_2}, \cdots, y_{1,j_s}\}$ on $C_{n,1}$ so that $j_1 < j_2 < \cdots < j_s$. We will say that the pairs of vertices $\{y_{1,j_a}, y_{1,j_{a+1}}\}$ for $1 \leq a \leq s-1$ and $\{y_{1,j_s}, y_{1,j_1}\}$ are pairs of neighboring vertices. For such type of ordering, we can define the gap g_a for $1 \leq a \leq s-1$ as the set of vertices $\{y_{1,j}|j_a < j < j_{a+1}\}$ and $g_s = \{y_{1,j}|1 \leq j < j_1$ or $j_s < j \leq n\}$. Thus we have s gaps on $C_{n,1}$, some of which may be empty. We will say that g_t and g_r are neighboring gaps when $|t - r| = 1$ or $s - 1$.
Now, if B is a basis of $\mathbb{Q}_n^2$, then B contains only vertices of inner cycles of $\mathbb{Q}_n^2$. We have the following

gap conditions for the selection of the basis vertices:

(i) *Every gap of B may have at most two vertices of $C_{n,1}$.* Otherwise, there exist a gap of B having three vertices, say, $y_{1,p}$, $y_{1,p+1}$, $y_{1,p+2}$, $y_{1,p+3}$ and $y_{1,p+4}$ with $y_{1,p}$, $y_{1,p+4} \in B$ such that $code(y_{3,p+2}|B) = code(z|B) = (3, 3, \cdots, 3)$.

(ii) *If a gap of B contains two vertices of $C_{n,1}$, then its neighboring gaps may contain at most one vertex.* Otherwise, there exist five consecutive vertices, say, $y_{1,p+1}$, $y_{1,p+2}$, $y_{1,p+3}$, $y_{1,p+4}$ and $y_{1,p+5}$, with $y_{1,p+3} \in B$ such that $code(y_{1,p+2}|B) = code(y_{1,p+4}|B)$.

(iii) *If a gap of B contains two vertices of $C_{n,1}$ and $n \equiv 4(mod\ 5)$, then one of its neighboring gaps may not be empty gap.* Otherwise, there is an empty gap with $y_{1,p+n-1}, y_{1,p}, y_{1,p+3} \in B$ such that $code(y_{3,p}|B) = code(y_{3,p+n-1}|B)$.

Now we assume that B is any set of vertices of $C_{n,1}$ that satisfies condition (i)-(iii) and let $y \in V(\mathbb{Q}_n^2)$. There are following possibilities to be discussed:

- If y belongs to a gap of size two of B with vertices $y_{1,p}$, $y_{1,p+1} = y$, $y_{1,p+2}$, $y_{1,p+3}$ such that $y_{1,p}, y_{1,p+3} \in B$, then $code(y|B) = (1, 2, \cdots, 2)$.
- If y belongs to a gap of size one of B with vertices $y_{1,p}$, $y_{1,p+1} = y$, $y_{1,p+2}$ such that $y_{1,p}, y_{1,p+2} \in B$, then $code(y|B) = (1, 1, 2, \cdots, 2)$.
- If $y = z$, then $code(y|B) = (3, 3, \cdots, 3)$.
- If $y = x$, then $r(y|B) = (1, 1, \cdots, 1)$.
- If $y = y_{2,p+n-1} \in V(C_{n,2})$ and y is adjacent to $y_{1,p}$ and $y_{1,p+n-1}$ with $y_{1,p}, y_{1,p+3}, y_{1,p+n-1} \in B$, then $code(y|B) = (1, 3, \cdots, 3, 1)$.
- If $y = y_{2,p+n-1} \in V(C_{n,2})$ and y is adjacent to $y_{1,p}$ and $y_{1,p+n-1}$ with $y_{1,p}, y_{1,p+3}, y_{1,p+n-2} \in B$, then $code(y|B) = (1, 3, \cdots, 3, 2)$.
- If $y = y_{2,p+n-1} \in V(C_{n,2})$ and y is adjacent to $y_{1,p}$ and $y_{1,p+n-1}$ with $y_{1,p+1}, y_{1,p+n-2} \in B$, then $code(y|B) = (2, 3, \cdots, 3, 2)$.
- If $y = y_{2,p+n-1} \in V(C_{n,2})$ and y is adjacent to $y_{1,p}$ and $y_{1,p+n-1}$ with $y_{1,p+1}, y_{1,p+n-3}, y_{1,p+n-1} \in B$, then $code(y|B) = (2, 3, \cdots, 3, 1)$.
- If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+3}, y_{1,p+n-2} \in B$, then $code(y|B) = (2, 4, \cdots, 4, 3)$.
- If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+1}, y_{1,p+n-2} \in B$, then $code(y|B) = (2, 2, 4, \cdots, 4, 3)$.
- If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+2}, y_{1,p+n-3} \in B$, then $code(y|B) = (2, 3, 4, \cdots, 4, 4)$.
- If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+1}, y_{1,p+n-3} \in B$, then $code(y|B) = (2, 2, 4, \cdots, 4, 4)$.
- If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p+1}, y_{1,p+n-3}, y_{1,p+n-1} \in B$, then $code(y|B) = (2, 4, \cdots, 4, 2)$.
- If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+2}, y_{1,p+n-1} \in B$,

then $code(y|B) = (2, 3, 4, \cdots, 4, 2)$.
• If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p+2}, y_{1,p+n-1}, y_{1,p+n-2} \in B$, then $code(y|B) = (3, 4, 4, \cdots, 4, 3, 2)$.
• If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p+2}, y_{1,p+n-1}, y_{1,p+n-3} \in B$, then $code(y|B) = (2, 4, 4, \cdots, 4, 2)$.
Therefore, any set B having properties (i)-(iii) is a resolving set of $\mathbb{Q}_n^2$. We now present an exact formula for computing the metric dimension of $\mathbb{Q}_n^2$ for every integer $n \geq 6$. In fact, this formula is same as we computed for $\mathbb{Q}_n^1$ in Theorem 3.4.1, with a small change in the choice of basis vertices when $n \equiv 4\ (mod\ 5)$.

Theorem 3.5.1. *If $n \geq 6$, then we have $dim(\mathbb{Q}_n^2) = \lfloor \frac{2n+4}{5} \rfloor$.*

Proof. We prove this result by double inequality. First we prove that $dim(\mathbb{Q}_n^2) \leq \lfloor \frac{2n+4}{5} \rfloor$ by constructing a resolving set in $\mathbb{Q}_n^2$ with $\lfloor \frac{2n+4}{5} \rfloor$ vertices. We consider the following cases according to the residue class modulo 5 to which n belongs.

Case 1: When $n \equiv 0\ (mod\ 5)$, then we may write $n = 5k$, where $k \geq 2$ and $\lfloor \frac{2n+4}{5} \rfloor = 2k$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\}$, it is a resolving set having $2k$ vertices as it satisfies conditions (i)-(iii).

Case 2: When $n \equiv 1, 2\ (mod\ 5)$, then we may write $n = 5k+1, 5k+2$, respectively, where $k \geq 1$, and $\lfloor \frac{2n+4}{5} \rfloor = 2k+1$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\} \cup \{y_{5k+1}\}$, it is a resolving set having $2k+1$ vertices as it satisfies conditions (i)-(iii).

Case 3: When $n \equiv 3, 4\ (mod\ 5)$, then we may write $n = 5k+3, 5k+4$, respectively, where $k \geq 1$, and $\lfloor \frac{2n+4}{5} \rfloor = 2k+2$. Since $B = \{y_{5i+4}, y_{5i+6} : 0 \leq i \leq k-1\} \cup \{y_1, y_{5k+3}\}$, it is a resolving set having $2k+2$ vertices as it satisfies conditions (i)-(iii).

The proof of the reverse inequality $dim(\mathbb{Q}_n^2) \geq \lfloor \frac{2n+4}{5} \rfloor$ follows exactly on the same line as proof of the Theorem 3.4.1, and is therefore omitted. □

Now we will study the metric dimension of the convex polytope $\mathbb{Q}_n^3$. If B is a basis of $\mathbb{Q}_n^3$, then B contains only vertices of inner cycles $C_{n,1}$ and $C_{n,4}$ of $\mathbb{Q}_n^3$. We have the following gap conditions for the selection of the basis vertices:
$(\acute{i})$ The gap conditions (i)-(iii) must be satisfied.
The following conditions are only for the vertices of $C_{n,4}$:
$(\acute{ii})$ *Every gap of B may have at most thirteen vertices of $C_{n,4}$.* Otherwise, there is a gap having fourteen vertices $y_{4,1}, y_{4,2}, \cdots, y_{4,15}$ with $y_{4,1} \in B$ such that $code(y_{4,9}|B) = code(y_{4,10}|B)$.
$(\acute{iii})$ *At most one gap of B may have thirteen vertices.*

$(\acute{iv})$ *If a gap of B have thirteen vertices, then its neighboring gaps may have at most five vertices.*
$(\acute{v})$ *Any gap of B may have thirteen vertices only if $n \equiv 1,2,3,4 (mod\ 5)$.* Otherwise, any gap of B may have at most five vertices.
$(\acute{vi})$ *If $n \equiv 0 (mod\ 5)$ and a gap of B have five vertices, then any of its neighboring gap may have at most five vertices.* Otherwise, there is a path of $C_{n,4}$ having eleven vertices $y_{4,1}, y_{4,2}, \cdots, y_{4,11}$ with $y_{4,5} \in B$ such that $code(y_{4,9}|B) = code(y_{4,10}|B)$.
Now we assume that B is any set of vertices of $C_{n,1}$ and $C_{n,4}$ that satisfies condition $(\acute{i})$-$(\acute{vi})$ and let $y \in V(\mathbb{Q}_n^3)$. There are following possibilities to be discussed:

- If y belongs to a gap of size two of B with vertices $y_{1,p}$, $y_{1,p+1} = y$, $y_{1,p+2}$, $y_{1,p+3}$ such that $y_{1,p}, y_{1,p+3}, y_{4,p+4}, y_{4,p+n-1} \in B$, then $code(y|B) = (1,2,\cdots,2,\overline{5,\cdots,5,3})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.
- If y belongs to a gap of size two of B with vertices $y_{1,p}$, $y_{1,p+1}$, $y_{1,p+2} = y$, $y_{1,p+3}$ such that $y_{1,p}, y_{1,p+3}, y_{4,p+4}, y_{4,p+n-1} \in B$, then $code(y|B) = (2,1,2,\cdots,2,\overline{4,5,\cdots,5,4})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.
- If y belongs to a gap of size one of B with vertices $y_{1,p}$, $y_{1,p+1} = y$, $y_{1,p+2}$ such that $y_{1,p}, y_{1,p+2}, y_{4,p+4}$, $y_{4,p+n-1} \in B$, then $code(y|B) = (1,1,2,\cdots,2,\overline{5,\cdots,5,3})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.
- If $y = z$, then $code(y|B) = (4,4,\cdots 4,\overline{1,1,\cdots,1})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property.
- If $y = x$, then $code(y|B) = (1,1,\cdots 1,\overline{4,4,\cdots,4})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property.
- If $y = y_{2,p+n-1} \in V(C_{n,2})$ and y is adjacent to $y_{1,p}$ and $y_{1,p+n-1}$ with $y_{1,p}, y_{1,p+3}, y_{1,p+n-2}, y_{4,p+n-1} \in B$, then $code(y|B) = (1,3,\cdots,3,2,\overline{4,\cdots,4,2})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.
- If $y = y_{2,p+n-1} \in V(C_{n,2})$ and y is adjacent to $y_{1,p}$ and $y_{1,p+n-1}$ with $y_{1,p}, y_{1,p+3} y_{1,p+n-1}, y_{1,p+n-3}$, $y_{4,p+n-2} \in B$, then $code(y|B) = (1,3,\cdots,3,\overline{4,\cdots,4,2})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property.
- If $y = y_{2,p+1} \in V(C_{n,2})$ and y is adjacent to $y_{1,p+1}$ and $y_{1,p+2}$ with $y_{1,p}, y_{1,p+3}, y_{4,5p} \in B$, then $code(y|B) = (2,2,3,\cdots,3,\overline{4,\cdots,4})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property.
- If $y = y_{3,p} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+3}, y_{1,p+n-2}, y_{4,p+n-1} \in B$, then $code(y|B) = (2,4,\cdots,4,3,\overline{3,\cdots,3,1})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property.
- If $y = y_{3,p} \in V(C_{n,3})$ and y is adjacent to $y_{2,p}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+n-1}, y_{1,p+n-3}, y_{4,p+n-2} \in B$, then $code(y|B) = (2,4,\cdots,4,2,\overline{3,\cdots,3,2})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex

in $\mathbb{Q}_n^3$ with this property.

• If $y = y_{3,p+n-1} \in V(C_{n,3})$ and y is adjacent to $y_{2,p+n-2}$ and $y_{2,p+n-1}$ with $y_{1,p}, y_{1,p+n-2}, y_{4,p+n-1} \in B$, then $code(y|B) = (2,4,\cdots,4,2,\overline{3,\cdots,3,1})$. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property.

• If $y = y_{4,p}, y_{4,p+1}, y_{4,p+2}$ or $y_{4,p+3}$ belongs to a gap of size four of B on $C_{n,4}$ with $y_{4,p+4}, y_{4,p+n-1} \in B$, then $code(y|B) = (3,4,5,\cdots,5,4,\overline{1,2,\cdots,2})$, $code(y|B) = (3,3,5,\cdots,5,\overline{2,,\cdots,2})$, $code(y|B) = (4,3,4,5,\cdots,5,\overline{2,,\cdots,2})$, $code(y|B) = (5,3,3,5,\cdots,5,\overline{2,1,2,\cdots,2})$, respectively. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.

• If $y = y_{4,p}, y_{4,p+1}, y_{4,p+2}, y_{4,p+3}, y_{4,p+n-1}, y_{4,p+n-2}, y_{4,p+n-3}, y_{4,p+n-4}, y_{4,p+n-5}$ or $y_{4,p+n-6}$ belongs to a gap of size ten of B on $C_{n,4}$ with $y_{4,p+4}, y_{4,p+n-7} \in B$, then $code(y|B) = (3,4,5,\cdots,5,3,\overline{2,\cdots,2})$, $code(y|B) = (3,3,5,\cdots,5,4,\overline{2,2,\cdots,2})$, $code(y|B) = (4,3,4,5,\cdots,5,\overline{2,2,\cdots,2})$, $code(y|B) = (5,3,3,5,\cdots,5,\overline{1,2,\cdots,2})$, $code(y|B) = (3,5,\cdots,4,3,\overline{2,2,\cdots,2})$, $code(y|B) = (3,5,\cdots,5,3,3,\overline{2,2,\cdots,2})$, $code(y|B) = (4,5,\cdots,5,3,3,\overline{2,2,\cdots,2})$, $code(y|B) = (5,5,\cdots,5,4,3,4,\overline{2,2,\cdots,2})$, $code(y|B) = (5,\cdots,5,3,3,5,\overline{2,\cdots,2})$ and $code(y|B) = (5,\cdots,5,4,3,4,5,\overline{2,\cdots,2,1})$, respectively. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.

• If $y = y_{4,p}, y_{4,p+1}, y_{4,p+2}, y_{4,p+3}, y_{4,p+n-1}, y_{4,p+n-2}, y_{4,p+n-3}, y_{4,p+n-4}, y_{4,p+n-5}$ $y_{4,p+n-6}$ or $y_{4,p+n-7}$ belongs to a gap of size eleven of B on $C_{n,4}$ with $y_{4,p+4}$, $y_{4,p+n-8} \in B$, then $code(y|B) = (3,4,5,\cdots,5,4,\overline{2,2,\cdots,2})$, $code(y|B) = (3,3,5,\cdots,5,\overline{2,2,\cdots,2})$, $code(y|B) = (4,3,4,5,\cdots,5,\overline{2,2,\cdots,2})$, $code(y|B) = (5,3,3,5,\cdots,5,\overline{1,2,\cdots,2})$, $code(y|B) = (3,5,\cdots,5,,3,\overline{2,2,\cdots,2})$, $code(y|B) = (3,5,\cdots,5,4,3,\overline{2,2,\cdots,2})$, $code(y|B) = (4,5,\cdots,5,3,3,\overline{2,2,\cdots,2})$, $code(y|B) = (5,\cdots,5,3,3,\overline{2,2,\cdots,2,2})$, $code(y|B) = (5,5,\cdots,5,4,3,4,\overline{2,2,\cdots,2})$, $code(y|B) = (5,\cdots,5,3,3,5,\overline{2,2,\cdots,2})$ and $code(y|B) = (5,\cdots,5,4,3,4,5,\overline{2,2,\cdots,2,1})$, respectively. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.

• If $y = y_{4,p}, y_{4,p+1}, y_{4,p+2}, y_{4,p+3}, y_{4,p+n-1}, y_{4,p+n-2}, y_{4,p+n-3}, y_{4,p+n-4}, y_{4,p+n-5}$ $y_{4,p+n-6}, y_{4,p+n-7}$ or $y_{4,p+n-8}$ belongs to a gap of size twelve of B on $C_{n,4}$ with $y_{4,p+4}, y_{4,p+n-9} \in B$, then $code(y|B) = (3,4,5,\cdots,5,3,\overline{2,2,\cdots,2})$, $code(y|B) = (3,3,5,\cdots,5,3,\overline{2,\cdots,2})$, $code(y|B) = (4,3,4,5,\cdots,5,\overline{2,\cdots,2})$, $code(y|B) = (5,3,3,5,\cdots,5,\overline{1,2,\cdots,2})$, $code(y|B) = (3,5,\cdots,5,4,3,\overline{2,\cdots,2})$, $code(y|B) = (3,5,\cdots,5,3,3,\overline{2,\cdots,2})$, $code(y|B) = (4,5,\cdots,5,4,3,3,\overline{2,\cdots,2})$, $code(y|B) = (5,5,\cdots,5,3,3,4,\overline{2,\cdots,2})$, $code(y|B) = (5,\cdots,5,3,3,5,\overline{2,\cdots,2})$, $code(y|B) = (5,5,\cdots,5,4,3,4,5,\overline{2,\cdots,2})$, $code(y|B) = (5,\cdots,5,3,3,5,5,\overline{2,2,\cdots,2})$ and $code(y|B) = (5,\cdots,5,4,3,4,5,5,\overline{2,2,\cdots,2,1})$, respectively. By properties $(\acute{i}) - (\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.

• If $y = y_{4,p}, y_{4,p+1}, y_{4,p+2}, y_{4,p+3}, y_{4,p+n-1}, y_{4,p+n-2}, y_{4,p+n-3}, y_{4,p+n-4}, y_{4,p+n-5}$ $y_{4,p+n-6}, y_{4,p+n-7}$, $y_{4,p+n-8}$ or $y_{4,p+n-9}$ belongs to a gap of size thirteen of B on $C_{n,4}$ with $y_{4,p+4}, y_{4,p+n-10} \in B$, then

$code(y|B) = (3,4,5,\cdots,5,4,\overline{2,2,\cdots,2})$, $code(y|B) = (3,3,5,\cdots,5,5,\overline{2,2,\cdots,2})$, $code(y|B) = (4,3,4,5,\cdots,5,\overline{2,2,\cdots,2})$, $code(y|B) = (5,3,3,5,\cdots,5,\overline{1,2,\cdots,2})$, $code(y|B) = (3,5,\cdots,5,3,\overline{2,2,\cdots,2})$, $code(y|B) = (3,5,\cdots,5,4,3,\overline{2,2,\cdots,2})$, $code(y|B) = (4,5,\cdots,5,3,3,\overline{2,2,\cdots,2})$, $code(y|B) = (5,5,\cdots,5,4,3,3,\overline{2,2,\cdots,2})$, $code(y|B) = (5,\cdots,5,3,3,4,\overline{2,\cdots,2,2})$, $code(y|B) = (5,5,\cdots,5,3,3,5,\overline{2,2,\cdots,2})$, $code(y|B) = (5,\cdots,5,4,3,4,5,\overline{2,2,\cdots,2})$, $code(y|B) = (5,\cdots,5,3,3,5,5,\overline{2,2,\cdots,2,2})$ and $code(y|B) = (5,\cdots,5,4,3,4,5,5,\overline{2,2,\cdots,2,1})$, respectively. By properties $(\acute{i})-(\acute{vi})$, there exist no other vertex in $\mathbb{Q}_n^3$ with this property. The overlined codes represent vertices of B on $C_{n,4}$.

Therefore, any set B having properties $(\acute{i})-(\acute{vi})$ is a resolving set of the convex polytope graph $\mathbb{Q}_n^3$. We now present a precise formula for computing the metric dimension of the convex polytope graph $\mathbb{Q}_n^3$ for every integer $n \geq 7$.

Theorem 3.5.2. *If $n \geq 7$, then we have*

$$dim(\mathbb{Q}_n^3) = \begin{cases} \lfloor \frac{3n}{5} \rfloor & ; n \equiv 0 (mod\ 5), \\ \\ \lfloor \frac{3n-3}{5} \rfloor & ; otherwise \end{cases}$$

Proof. We prove this result by double inequality. First we prove that

$$dim(\mathbb{Q}_n^3) \leq \begin{cases} \lfloor \frac{3n}{5} \rfloor & ; n \equiv 0 (mod\ 5), \\ \\ \lfloor \frac{3n-3}{5} \rfloor & ; otherwise \end{cases}$$

by constructing a resolving set in $\mathbb{Q}_n^3$ with $\lfloor \frac{3n}{5} \rfloor$, $\lfloor \frac{3n-3}{5} \rfloor$ vertices for $n \equiv 0 (mod\ 5)$ and $n \equiv 1,2,3,4 (mod\ 5)$, respectively. We consider the following cases according to the residue class modulo 5 to which n belongs.

Case 1: When $n \equiv 0\ (mod\ 5)$, then we may write $n = 5k$, where $k \geq 2$, and $\lfloor \frac{3n}{5} \rfloor = 3k$. Since $B = \{y_{1,5i+1}, y_{1,5i+4} : 0 \leq i \leq k-1\} \cup \{y_{4,10i+5}, y_{4,10i+10} : 0 \leq i \leq \frac{k-2}{2}\}$ and $B = \{y_{1,5i+1}, y_{1,5i+4} : 0 \leq i \leq k-1\} \cup \{y_{4,10i+5}, y_{4,10i+10} : 0 \leq i \leq \frac{k-3}{3}\} \cup \{y_{4,5k}\}$ for k even and odd, respectively, it is a resolving set having $3k$ vertices as it satisfies conditions $(\acute{i})-(\acute{vi})$.

Case 2:

subcase(a) When $n \equiv 1,2\ (mod\ 5)$, then we may write $n = 5k+1, 5k+2$, respectively, where $k \geq 3$, and $\lfloor \frac{3n-3}{5} \rfloor = 3k$. Since $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\} \cup \{y_{5k+1}\} \cup \{y_{4,10i+5}, y_{4,10i+10} : 0 \leq i \leq \frac{k-3}{2}\}$ and $B = \{y_{5i+1}, y_{5i+4} : 0 \leq i \leq k-1\} \cup \{y_{5k+1}\} \cup \{y_{4,10i+5}, y_{4,10i+10} : 0 \leq i \leq \frac{k-4}{2}\} \cup \{y_{4,5k}\}$ for k odd and even, respectively, it is a resolving set having $3k$ vertices as it satisfies conditions $(\acute{i})-(\acute{vi})$.

subcase(b) When $n \equiv 3,4\ (mod\ 5)$, then we may write $n = 5k+3, 5k+4$, respectively, where $k \geq 3$, and $\lfloor\frac{3n-3}{5}\rfloor = 3k+1$. Since $B = \{y_{1,5i+4}, y_{1,5i+6} : 0 \le i \le k-1\} \cup \{y_{1,1}, y_{1,5k+3}\} \cup \{y_{4,10i+5}, y_{4,10i+10} : 0 \le i \le \frac{k-3}{2}\}$ and $B = \{y_{1,5i+4}, y_{1,5i+6} : 0 \le i \le k-1\} \cup \{y_{1,1}, y_{5k+3}\} \cup \{y_{4,10i+5}, y_{4,10i+10} : 0 \le i \le \frac{k-4}{2}\} \cup \{y_{4,5k}\}$ for k odd and even, respectively, it is a resolving set having $3k+1$ vertices as it satisfies conditions $(\acute{i}) - (\acute{vi})$.

Next we have to show that

$$dim(\mathbb{Q}_n^3) \ge \begin{cases} \lfloor\frac{3n}{5}\rfloor & ; n \equiv 0(mod\ 5), \\ \\ \lfloor\frac{3n-3}{5}\rfloor & ; otherwise \end{cases}$$

Let B be a basis of $\mathbb{Q}_n^3$. We consider the following cases:

Case (a): $|B| = 3t$ for some integers $t \ge 1$. The conditions $(\acute{i}) - (\acute{vi})$ imply that at most t gaps of B contains more than one vertex, and all of them contains two vertices on $C_{n,1}$ and at most t gaps of B contains four vertices on $C_{n,4}$. So the number of vertices that belong to different gaps of B are at most $3t + 4t$. Therefore $2n - 3t \le 3t + 4t$, this implies that $3n \le 15t$. Hence $|B| = 3t \ge \frac{3n}{5} \ge \lfloor\frac{3n}{5}\rfloor$.

Case (b): $|B| = 3t+1$ for some integers $t \ge 1$. The conditions $(\acute{i}) - (\acute{vi})$ imply that at most t gaps of B contains more than one vertex, and all of them contains two vertices on $C_{n,1}$ and at most $t-1$ gaps of B contains four vertices on $C_{n,4}$. So the number of vertices that belong to different gaps of B are at most $3t+4t-3$. Therefore $2n-3t-1 \le 3t+4t-3$, this implies that $3n+3 \le 15t$. Hence $|B| = 3t+1 \ge \frac{3n+8}{5} \ge \lfloor\frac{3n-3}{5}\rfloor$, which completes the proof. □

We believe that the above result is also true for every $m \ge 4$ but we are not able to give a rigorous proof. Here, we propose the following open problem.

Open Problem: Is it the case that the metric dimension of convex polytope $\mathbb{Q}_n^m$ is given by the following formula

$$dim(\mathbb{Q}_n^m) = \begin{cases} \lfloor\frac{3n}{5}\rfloor & ; n \equiv 0(mod\ 5), \\ \\ \lfloor\frac{3n-3}{5}\rfloor & ; otherwise \end{cases}$$

for all $n \ge 7$ and $m \ge 4$?

3.6 Metric dimension of convex polytopes $\mathbb{D}_n$ and $\mathbb{B}_n$

The gear graph denoted by J_n (n is even) is defined as follows: Consider an even cycle C_n : $v_1, v_2, \cdots, v_n, v_1$, where $n \ge 4$ and a new vertex v is adjacent to n vertices of $C_n : v_2, v_4, \cdots, v_{2m}$. The gear graph J_n can be obtained from the wheel W_n by alternately deleting $\frac{n}{2}$ spokes. Tomescu and Javaid [60] proved that $dim(J_n) = \lfloor\frac{n}{3}\rfloor$ for $n \ge 8$.

Let $I = \{1, 2, \cdots, n\}$ and $J = \{1, 2\}$ be indexed sets and D_n be the graph of a prism. The prism D_n, ($n \ge 3$), is a trivalent graph which can be defined as the Cartesian product $P_2 \square C_n$ of a path on two vertices with a cycle on n vertices, embedded in the plane. Let us denote the vertex set

of D_n by $V(D_n) = \{x_{j,i} : j \in J$ and $i \in I\}$ and edge set by $E(D_n) = \{x_{j,i}x_{j,i+1} : j \in J$ and $i \in I\} \cup \{x_{1,i}x_{2,i} : i \in I\}$. We make the convention that $x_{j,n+1} = x_{j,1}$ $x_{j,n+2} = x_{j,2}$ for $j \in J$.
The face set $F(D_n)$ contains n 4-sided faces and two n-sided faces (internal and external). We insert exactly one vertex $y(z)$ into the internal (external) n-sided faces of D_n. Suppose that n is even, $n \geq 4$, and consider the graph $\mathbb{D}_n$ with the vertex set $V(\mathbb{D}_n) = V(D_n) \cup \{y, z\}$ and the edge set $E(\mathbb{D}_n) = E(D_n) \cup \{x_{1,2k-1}y : k = 1, 2, \cdots, \frac{n}{2}\} \cup \{x_{2,2k}z : k = 1, 2, \cdots, \frac{n}{2}\}$. The $\mathbb{D}_n$, $n \geq 4$, is the plane graph on $|V(\mathbb{D}_n)| = 2n + 2$ vertices, $|E(\mathbb{D}_n)| = 4n$ edges and consisting of $|F(\mathbb{D}_n)| = 2n$ faces [3]. Let its vertices be labelled as in Fig. 3.5.

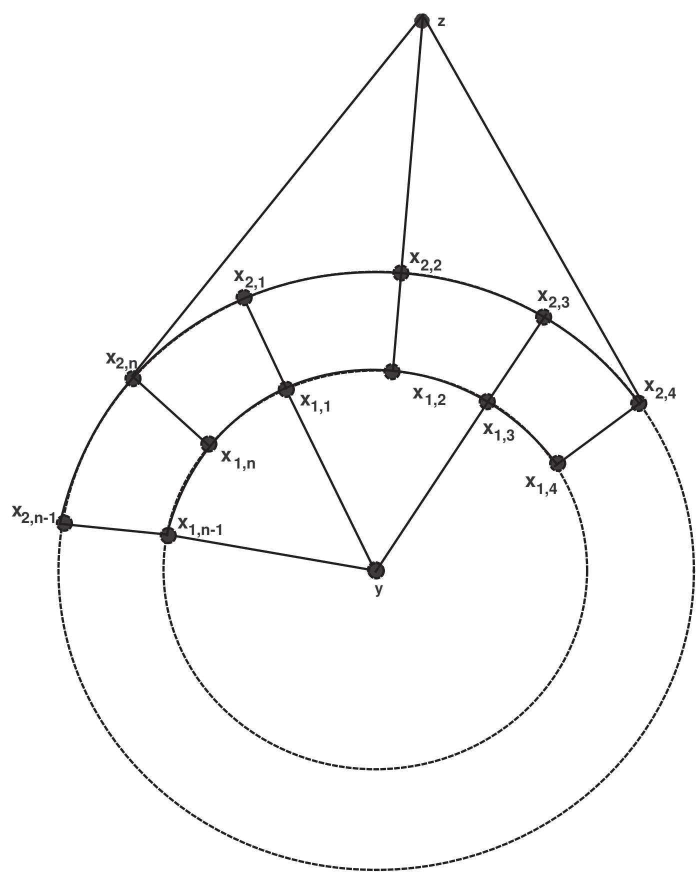

Figure 3.5: The convex polytope graph $\mathbb{D}_n$

In this section, we study the metric dimension of convex polytope graphs $\mathbb{D}_n$ and we prove that this class of convex polytopes has unbounded metric dimension.
Suppose that $\mathbb{D}_n$, for $n \geq 4$, be an infinite class of convex polytopes, then any of the central vertices z or y do not belong to any basis. Since $diam(\mathbb{D}_n) = 4$, if z or y belong to any metric basis, say B, then there must exist two distinct vertices $v_{j,i}$ and $v_{j,k}$, for $1 \leq i \neq k \leq n$ and $j \in \{1, 2\}$ such that $code(x_{j,i}|B) = code(x_{j,k}|B)$. Consequently, the basis vertices belong to the rim vertices of $\mathbb{D}_n$ only.
A gap determined by neighboring vertices $x_{j,i}$ and $x_{j,k}$ will be called an $\alpha - \beta$ gap with $\alpha \leq \beta$ when

$d(x_{j,i}) = \alpha$ and $d(x_{j,k}) = \beta$ or when $d(x_{j,i}) = \beta$ and $d(x_{j,k}) = \alpha$. Hence we have three kinds of gaps in $\mathbb{D}_n$, i.e. $3-3$, $3-4$ and $4-4$ gaps.

Lemma 3.6.1. *Let B be a basis of $\mathbb{D}_n$ for n even, $n \geq 12$, then every $3-3$, $3-4$ and $4-4$ gap of B contains at most 5, 4 and 3 vertices of $C_{n,1}$, respectively.*

Proof. On contrary, suppose that there is a $3-3$ gap of B having seven consecutive vertices $x_{1,1}, x_{1,2}, \cdots, x_{1,7}$ of $C_{n,1}$ such that $deg(x_{1,1}) = deg(x_{1,7}) = 4$. For this case, $code(x_{1,3}|B) = code(x_{1,5}|B)$, a contradiction. Similarly, if there is a $3-4$ gap having six vertices of $C_{n,1}$ say, $x_{1,1}, \cdots, x_{1,6}$ such that $deg(x_{1,1}) = 4$ and $deg(x_{1,6}) = 3$. In this case, we get $code(x_{1,3}|B) = code(x_{1,5}|B)$, a contradiction. If there is a $4-4$ gap of B having five consecutive vertices of $C_{n,1}$ say, $x_{1,2}, \cdots, x_{1,6}$ such that $deg(x_{1,2}) = deg(x_{1,6}) = 3$, then $code(x_{1,3}|B) = code(x_{1,5}|B)$, a contradiction. □

From now on, the $3-3$, $3-4$ and $4-4$ gaps of $C_{n,1}$ having 5, 4 and 3 vertices, respectively will be referred as *major gaps*, while the rest of all will be referred as *minor gaps*. The vertices having degrees 4 and 3 are known as major (labelled by star) and minor vertices, respectively.

Lemma 3.6.2. *Any basis B of $\mathbb{D}_n$ for n even $(n \geq 12)$ contains at most one major $3-3$, $3-4$ or $4-4$ gap of $C_{n,1}$.*

Proof. On contrary, suppose that B contains two distinct major gaps, then we have the following cases:

- $4-4$ and $4-4$ gaps: $x_{1,1}, \overset{*}{x}_{1,2}, x_{1,3}$ and $y_{1,1}, \overset{*}{y}_{1,2}, y_{1,3}$; in this case we have $code(\overset{*}{x}_{1,2}|B) = code(\overset{*}{y}_{1,2}|B)$, a contradiction.
- $4-4$ and $3-3$ gaps: $x_{1,1}, \overset{*}{x}_{1,2}, x_{1,3}$ and $\overset{*}{y}_{1,1}, y_{1,2}, \overset{*}{y}_{1,3}, y_{1,4}, \overset{*}{y}_{1,5}$; in this case we have $code(\overset{*}{x}_{1,2}|B) = code(\overset{*}{y}_{1,3}|B)$, a contradiction.
- $4-4$ and $3-4$ gaps: $x_{1,1}, \overset{*}{x}_{1,2}, x_{1,3}$ and $\overset{*}{y}_{1,1}, y_{1,2}, \overset{*}{y}_{1,3}, y_{1,4}$; in this case we have $code(\overset{*}{x}_{1,2}|B) = code(\overset{*}{y}_{1,3}|B)$, a contradiction.
- $3-3$ and $3-3$ gaps: $\overset{*}{x}_{1,1}, x_{1,2}, \overset{*}{x}_{1,3}, x_{1,4}, \overset{*}{x}_{1,5}$ and $\overset{*}{y}_{1,1}, y_{1,2}, \overset{*}{y}_{1,3}, y_{1,4}, \overset{*}{y}_{1,5}$; in this case we have $code(\overset{*}{x}_{1,3}|B) = code(\overset{*}{y}_{1,3}|B)$, a contradiction.
- $3-3$ and $3-4$ gaps: $\overset{*}{x}_{1,1}, x_{1,2}, \overset{*}{x}_{1,3}, x_{1,4}, \overset{*}{x}_{1,5}$ and $\overset{*}{y}_{1,1}, y_{1,2}, \overset{*}{y}_{1,3}, y_{1,4}$; in this case we have $code(\overset{*}{x}_{1,3}|B) = code(\overset{*}{y}_{1,3}|B)$, a contradiction.
- $3-4$ and $3-4$ gaps: $\overset{*}{x}_{1,1}, x_{1,2}, \overset{*}{x}_{1,3}, x_{1,4}$ and $\overset{*}{y}_{1,1}, y_{1,2}, \overset{*}{y}_{1,3}, y_{1,4}$; in this case we have $code(\overset{*}{x}_{1,3}|B) = code(\overset{*}{y}_{1,3}|B)$, which contradicts the hypothesis. □

In the next lemma, we will prove that any two neighboring gaps, one of which being *major* may contain together at most six vertices.

Lemma 3.6.3. *For any basis B of $\mathbb{D}_n$ for n even $(n \geq 12)$, any two neighboring gaps, one of which being major gap contain together at most six vertices of $C_{n,1}$.*

Proof. If the major gap is a $4-4$ gap (with three vertices), then by Lemma 4.3.2 its neighboring gap may be a minor $3-3$ or $3-4$ gap having at most three vertices, which concludes the proof in this case.

If the major gap is a $3-3$ gap with five vertices, then we need to show that its neighboring gap can neither be a minor $3-3$ gap having three vertices, nor a minor $3-4$ gap having two vertices. If a major $3-3$ gap has a neighboring $3-3$ gap with three vertices, we have the following path consisting of consecutive vertices of $C_{n,1}$: $\overset{*}{x}_{1,1}, x_{1,2}, \overset{*}{x}_{1,3}, x_{1,4}, \overset{*}{x}_{1,5}, x_{1,6}, \overset{*}{x}_{1,7}, x_{1,8}, \overset{*}{x}_{1,9}$, where $x_{1,4} \in B$. In this case $code(\overset{*}{x}_{1,3}|B) = code(\overset{*}{x}_{1,5}|B)$, a contradiction. A similar conclusion can be obtained if a major $3-3$ gap has a neighboring $3-4$ gap with two vertices. If the major gap is a $3-4$ gap with four vertices, by Lemma 4.3.2, it is sufficient to show that its neighboring gap cannot be a minor $3-3$ gap with three vertices. If this is true, we consider the following path: $\overset{*}{x}_{1,1}, x_{1,2}, \overset{*}{x}_{1,3}, x_{1,4}, \overset{*}{x}_{1,5}, x_{1,6}, \overset{*}{x}_{1,7}, x_{1,8}$, where $x_{1,4} \in B$. In this case $code(\overset{*}{x}_{1,3}|B) = code(\overset{*}{x}_{1,5}|B)$, a contradiction. □

In the next lemma, we will prove that any two *minor* neighboring gaps may contain together at most four vertices of $C_{n,1}$.

Lemma 3.6.4. *If B is any basis of $\mathbb{D}_n$ for n even ($n \geq 12$), then any two minor neighboring gaps contain together at most four vertices.*

Proof. By Lemma 4.3.1, any minor $3-3$, $3-4$, and $4-4$ gap contains three, two and one vertex, respectively. It suffices to prove the following cases:

(1) A $3-3$ gap with three vertices has a neighboring $3-3$ gap with three vertices.

(2) A $3-3$ gap with three vertices has a neighboring $3-4$ gap with two vertices cannot occur.

If the case (1) is true, then there exists a path: $\overset{*}{x}_{1,1}, x_{1,2}, \overset{*}{x}_{1,3}, x_{1,4}, \overset{*}{x}_{1,5}, x_{1,6}, \overset{*}{x}_{1,7}$, on $C_{n,1}$, where $x_{1,4} \in B$ such that $code(\overset{*}{x}_{1,3}|B) = code(\overset{*}{x}_{1,5}|B)$; If the case (2) is true, then there exists a path: $\overset{*}{x}_{1,1}, x_{1,2}, \overset{*}{x}_{1,3}, x_{1,4}, \overset{*}{x}_{1,5}, x_{1,6}$, on $C_{n,1}$, where $x_{1,4} \in B$, which implies that $code(\overset{*}{x}_{1,3}|B) = code(\overset{*}{x}_{1,5}|B)$, a contradiction. □

For the selection of basis B of $\mathbb{D}_n$, we also make the following claims.

Claim 1: The vertex $x_{2,2} = x_{2,p+1} \in B$. Otherwise, there are vertices $x_{1,3} = x_{1,p+2}, x_{1,p+1}, x_{1,p+n-1}$, $x_{1,p+7}$ on cycle $C_{n,1}$ with $x_{1,p+1}, x_{1,p+n-1}, x_{1,p+7} \in B$, which implies that $code(x_{2,p+1}|B) = code(x_{1,p+2}|B)$.

Claim 2: There must exist a $4-4$ gap of size three of B on $C_{n,2}$. Otherwise, there is a $4-4$ gap of B on $C_{n,2}$ containing five vertices, namely $x_{2,3} = x_{2,p+2}, x_{2,p+3}, x_{2,p+4}, x_{2,p+5}, x_{2,p+6}$ with $x_{1,p+1}, x_{1,p+7}, x_{1,p+9}, x_{2,p+1}, x_{2,p+7} \in B$, which implies that $code(x_{2,p+3}|B) = code(x_{2,p+4}|B) = (3, 3, \cdots, 3, 2, 2)$.

Claim 3: One of the neighboring $4-4$ gaps of B on $C_{n,2}$ contains at most five vertices and the other one contains at most seven vertices. Otherwise, there are $4-4$ neighboring gaps of B on $C_{n,2}$ containing seven and nine vertices. Consider a path: $x_{2,3}, \overset{*}{x}_{2,4}, x_{2,5}, \overset{*}{x}_{2,6}, x_{2,7}, \overset{*}{x}_{2,8}, x_{2,9}, \overset{*}{x}_{2,10}, x_{2,11}, \overset{*}{x}_{2,12}, x_{2,13}$ having eleven vertices with $x_{1,2}, x_{1,8}, x_{1,10}, x_{1,14}, \overset{*}{x}_{2,2}, \overset{*}{x}_{2,6}, \overset{*}{x}_{2,14} \in B$, which implies that $code(x_{2,4}|B) = code(x_{2,12}|B)$, a contradiction. Now consider a path: $x_{2,n-7}, \overset{*}{x}_{2,n-6}, x_{2,n-5}, \overset{*}{x}_{2,n-4}, x_{2,n-3}, \overset{*}{x}_{2,n-2}, x_{2,n-1}, \overset{*}{x}_{2,n}, x_{2,1}, \overset{*}{x}_{2,22}, x_{2,3}, \overset{*}{x}_{2,4}, x_{2,5}$ having thirteen vertices with $x_{1,2}$, $x_{1,8}, x_{1,n}, x_{1,n-2}, x_{1,n-4}, x_{1,n-6}, x_{1,n-8}, \overset{*}{x}_{2,2}, \overset{*}{x}_{2,6}, \overset{*}{x}_{2,n-8} \in B$, which implies that $code(x_{2,4}|B) = code(x_{2,n-6}|B)$, a contradiction.

Claim 4: At most one $4-4$ gap of B have seven vertices on $C_{n,2}$.

• If y belongs to a gap of size three of B on $C_{n,2}$ and $y = x_{2,1}, \overset{*}{x}_{2,n}$ or $x_{2,n-1}$ with $x_{1,2}, x_{1,8}, x_{1,n-4}, x_{1,n}, \overset{*}{x}_{2,2}, \overset{*}{x}_{2,6}, \overset{*}{x}_{2,n-2} \in B$, then $code(y|B) = (2, 4, \cdots, 4, 2, \overline{1, 3, \cdots, 3})$, $code(y|B) = (3, 3, \cdots, 3, 1, \overline{2, \cdots, 2})$ and
$code(y|B) = (4, 4, \cdots, 4, 2, \overline{3, \cdots, 3, 1})$, respectively. Where overlined codes represent the vertices of B on $C_{n,2}$. By Lemma 4.3.1 to Lemma 4.3.3 and by claims $1-4$, there is no other vertex of $\mathbb{D}_n$ having this property.

• If y belongs to a gap of size three of B on $C_{n,2}$ and $y = x_{2,3}, \overset{*}{x}_{2,4}$ or $x_{2,5}$ with $x_{1,2}, x_{1,8}, x_{1,10}, x_{1,n}, \overset{*}{x}_{2,2}, \overset{*}{x}_{2,6}, \overset{*}{x}_{2,12} \in B$, then $code(y|B) = (2, 4, \cdots, 4, \overline{1, 3, \cdots, 3})$, $code(y|B) = (3, 3, \cdots, 3, \overline{2, \cdots, 2})$ and $code(y|B) = (4, 4, \cdots, 4, \overline{3, 1, 3, \cdots, 3})$, respectively. Where overlined codes represent the vertices of B on $C_{n,2}$. By Lemmas 4.3.1-4.3.3 and by claims $1-4$, there is no other vertex of $\mathbb{D}_n$ having this property.

• If y belongs to a gap of size five of B on $C_{n,2}$ and $y = x_{2,7}, \overset{*}{x}_{2,8}, x_{2,9}, \overset{*}{x}_{2,10}$ or $x_{2,11}$ with $x_{1,2}, x_{1,8}, x_{1,10}, x_{1,14}, \overset{*}{x}_{2,2}, \overset{*}{x}_{2,6}, \overset{*}{x}_{2,12} \in B$, then $code(y|B) = (4, 2, \cdots, 4\overline{3, 1, 3, \cdots, 3})$, $code(y|B) = (3, 1, 3, \cdots, 3, \overline{2, \cdots, 2})$, $code(y|B) = (4, 2, 2, 4, \cdots, 4, \overline{3, \cdots, 3})$, $code(y|B) = (3, 3, 1, 3, \cdots, 3, \overline{2, \cdots, 2})$ and $code(y|B) = (4, 4, 2, 3, 4 \cdots, 4, \overline{3, 3, 1, 3, \cdots, 3})$, respectively. Where overlined codes represent the vertices of B on $C_{n,2}$. Similarly, for other gaps of size five of B. By Lemmas 4.3.1-4.3.3 and by claims $1-4$, there is no other vertex of $\mathbb{D}_n$ having this property.

• If y belongs to a gap of size seven of B on $C_{n,2}$ and $y = x_{2,1}, \overset{*}{x}_{2,n}, x_{2,n-1}, \overset{*}{x}_{2,n-2}, x_{2,n-3}, \overset{*}{x}_{2,n-4}$ or $x_{2,n-5}$, with $x_{1,2}, x_{1,8}, x_{1,n}, x_{1,n-2}, x_{1,n-4}, x_{1,n-8}\overset{*}{x}_{2,2}, \overset{*}{x}_{2,n-6}, \overset{*}{x}_{2,n-12} \in B$, then $code(y|B) = (2, 4, \cdots, 4, 2, \overline{1, 3, \cdots, 3})$, $r(y|B) = r(3, 3, \cdots, 3, 1, \overline{2, \cdots, 2})$, $code(y|B) = (4, 4, \cdots, 4, 2, 2, \overline{3, \cdots, 3})$, $r(y|B) = r(3, 3, \cdots, 3, 1, 3, \overline{2, \cdots, 2})$, $code(y|B) = (4, 4, \cdots, 4, 2, 2, 4, \overline{3, \cdots, 3})$, $code(y|B) = (3, 3, \cdots, 3, 1, 3, 3, \overline{2, \cdots, 2})$ and $code(y|B) = (4, 4 \cdots, 4, 2, 4, 4, \overline{3, 3, \cdots, 3, 1})$, respectively. Where overlined codes represent the vertices of B on $C_{n,2}$. By Lemmas 4.3.1, 4.3.2, 3.6.3 and 4.3.3 and by claims $1-4$, there is no other vertex of $\mathbb{D}_n$ having this property.

In the next theorem, we compute the exact value of metric dimension for the convex polytope graphs $\mathbb{D}_n$ for n even and $n \geq 8$.

Theorem 3.6.5. *For every even integer $n \geq 8$, we have $dim(\mathbb{D}_n) = \frac{n}{2}$.*

Proof. Consider the convex polytope graphs $\mathbb{D}_n$, then we have $dim(\mathbb{D}_n) = 4$, for all $4 \leq n \leq 8$ and $W_2 = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}\}$ and $W_3 = \{x_{1,2}, x_{1,4}, x_{1,6}, x_{2,3}\}$ and $W_4 = \{x_{1,2}, x_{1,4}, x_{2,2}, x_{2,6}\}$ being metric basis for all $4 \leq n \leq 8$, respectively. $dim(\mathbb{D}_n) = 5$ and $dim(\mathbb{D}_n) = 4$ being metric basis for $n = 10$ and $n = 12$ $W_5 = \{x_{1,2}, x_{1,4}, x_{1,6}, x_{2,2}, x_{2,8}\}$ and $W_6 = \{x_{1,2}, x_{1,8}, x_{1,10}, x_{1,12}, x_{2,2}, x_{2,6}\}$, respectively. However for $n \geq 14$, the dimension of $\mathbb{D}_n$ increases with number of vertices n. We also know that the central vertex can't belong to any basis of $\mathbb{D}_n$. For $n \geq 14$, we prove the result by double inequality. First we show that $dim(\mathbb{D}_n) \leq \frac{n}{2}$ by constructing a resolving set B in $\mathbb{D}_n$ having $\frac{n}{2}$ vertices. For this consider $n = 2m$, then we have the following cases to be discussed:

Case 1: $m \equiv 0(mod\ 3)$. Let $2m = 3k$, where k is even, $k \geq 4$ and $m = \frac{3k}{2}$. In this case, $B = \{x_{1,2}, x_{1,n}\} \cup \{x_{1,6i+2}, x_{1,6i+4} : 1 \leq i \leq \frac{k}{2} - 1\} \cup \{x_{2,10i+2}, x_{2,12i+6} : 0 \leq i \leq \frac{k-4}{4}$ for m even$\} \cup [\{x_{2,2}\} \cup \{x_{2,12i+6}, x_{2,12i+12} : 0 \leq i \leq \frac{k-6}{6}\}$ for m odd$]$.

Case 2: $m \equiv 1(mod\ 3)$. Let $2m = 3k + 2$, where k is even, $k \geq 4$ and $m = \frac{3k+2}{2}$. In this case, $B = \{x_{1,2}, x_{1,n}\} \cup \{x_{1,6i+2}, x_{1,6i+4} : 1 \leq i \leq \frac{k}{2} - 1\} \cup \{x_{2,10i+2}, x_{2,12i+6} : 0 \leq i \leq \frac{k-2}{4}$ for m even$\} \cup [\{x_{2,2}\} \cup \{x_{2,12i+6}, x_{2,12i+12} : 0 \leq i \leq \frac{k-4}{4}\}$ for m odd$]$.

Case 3: $m \equiv 2(mod\ 3)$. Let $2m = 3k + 1$, where k is odd, $k \geq 5$ and $m = \frac{3k+1}{2}$. In this case, $B = \{x_{1,2}\} \cup \{x_{1,6i+2}, x_{1,6i+4} : 1 \leq i \leq \frac{k-1}{2}\} \cup \{x_{2,10i+2}, x_{2,12i+6} : 0 \leq i \leq \frac{k-3}{4}$ for m odd$\} \cup [\{x_{2,2}\} \cup \{x_{2,12i+6}, x_{2,12i+12} : 0 \leq i \leq \frac{k-5}{4}\}$ for m even$]$.

To prove that $dim(\mathbb{D}_n) \geq \frac{n}{2}$, let B be a basis of $\mathbb{D}_n$ and $|B| = l + r$, $(l, r \geq 1)$, where l and r represent the number of vertices of B lying on $C_{n,1}$ and $C_{n,2}$, respectively. Then B induces l gaps on $C_{n,1}$, namely $g_1, \cdots, g_l$ such that g_j and g_{j+1} are neighboring gaps for every $1 \leq j \leq l-1$, and also g_1 and g_l are neighboring gaps. By Lemma 4.3.2, at most one of the gaps is major, say g_1. By Lemma 3.6.3, and Lemma 4.3.3, we can write

$$| g_1 | + | g_2 | \leq 6;$$

$$| g_{l-1} | + | g_l | \leq 6;$$

and

$$| g_j | + | g_{j+1} | \leq 4,$$

for every $j = 2, \cdots, l-2$. Also B induces r gaps on $C_{n,2}$. By claims $1-4$, at most $r-1$ gaps containing five vertices on $C_{n,2}$. By adding these inequalities and by the conditions for the gaps on $C_{n,2}$, we get

$$2(n-l) + n - r = 2\sum_{j=1}^{l} | g_j | + n - r \leq 4l + 4 + 5(r-1).$$

$$\Rightarrow 3n + 1 \leq 6l + 6r$$

$$\Rightarrow \frac{n}{2} \leq \frac{n}{2} + \frac{1}{6} \leq l + r.$$

It follows that $l+r \geq \frac{n}{2}$. Since $l+r$ is an integer, for each $m \equiv 0, 1, 2 (mod\ 3)$, we have $l+r \geq \frac{n}{2}$,which completes the result. $\square$

Let $I = \{1, 2, \cdots, n\}$ and $J = \{1, 2, 3\}$ be indexed sets and B_n be the graph of a biprism B_n ($n \geq 3$), which can be defined as the Cartesian product $P_3 \square C_n$ of a path on three vertices with a cycle on n vertices, embedded in the plane. Let us denote the vertex set of B_n by $V(B_n) = \{z_{j,i} : j \in J$ and $i \in I\}$ and the edge set by $E(B_n) = \{z_{j,i}z_{j,i+1} : j \in J$ and $i \in I\} \cup \{z_{1,i}z_{2,i} : i \in I\} \cup \{z_{2,i}z_{3,i} : i \in I\}$. We make the convention that $z_{j,n+1} = z_{j,1}$ for $j \in J$.

The face set $F(B_n)$ contains $2n$ 4-sided faces and two n-sided faces (internal and external). We insert exactly one vertex x into the internal n-sided face of B_n and exactly one vertex y into the external n-sided face of B_n. Suppose that n is even, $n \geq 4$, and consider the graph $\mathbb{B}_n$ with vertex set $V(\mathbb{B}_n) = V(B_n) \cup \{x, y\}$ and the edge set $E(\mathbb{B}_n) = E(B_n) \cup \{z_{1,2k-1}x : k = 1, 2, \cdots, \frac{n}{2}\} \cup \{z_{3,2k-1}y : k = 1, 2, \cdots, \frac{n}{2}\}$. Then $\mathbb{B}_n$ ($n \geq 4$) is a graph of the convex polytope on $|V(\mathbb{B}_n)| = 3n + 2$ vertices, $|E(\mathbb{B}_n)| = 6n$ edges and consisting of $F(\mathbb{B}_n) = 3n$ 4-sided faces [4]. Let the vertices of $\mathbb{B}_n$ be labelled as in Fig. 3.6.

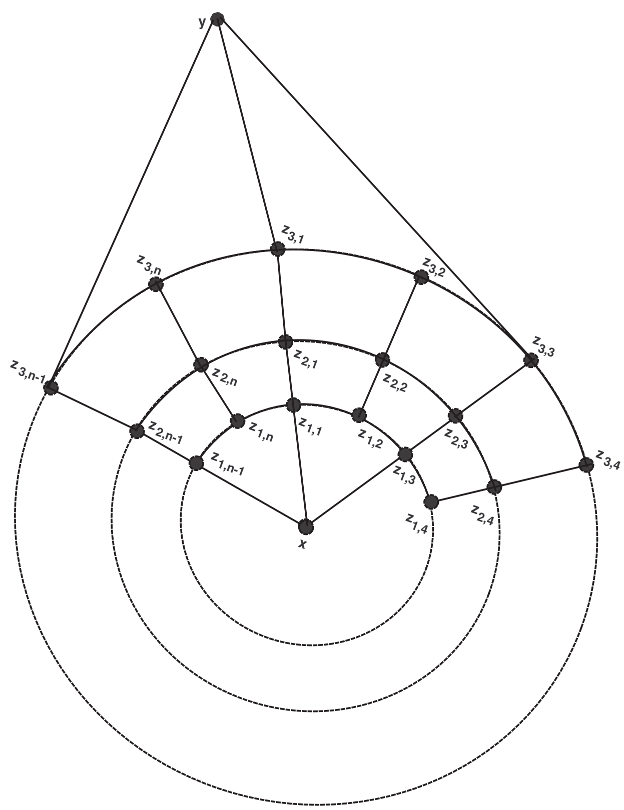

Figure 3.6: The convex polytope graph $\mathbb{B}_n$

In the next Lemma, we will show that there must be a $4-4$ gap of B on $C_{n,2}$.

Lemma 3.6.6. *Let B be a basis of $\mathbb{B}_n$ for even integer $n \geq 12$, then there must be a $4-4$ gap of B on $C_{n,2}$ having at most two vertices.*

Proof. On contrary, suppose that B has a $4-4$ gap on $C_{n,2}$ having three vertices. Consider a path: $\overset{*}{z}_{2,2}, \overset{*}{z}_{2,3}, \overset{*}{z}_{2,4}, \overset{*}{z}_{2,5}, \overset{*}{z}_{2,6}$, where $\overset{*}{z}_{2,2}, \overset{*}{z}_{2,6} \in B$, which implies that $code(\overset{*}{z}_{2,3}|B) = code(\overset{*}{z}_{3,2}|B)$ and $code(y|B) = code(\overset{*}{z}_{2,4}|B)$, a contradiction. □

Remark 3.6.7. The Lemmas 4.3.1-4.3.3 can also be translated for the convex polytope $\mathbb{B}_n$.

In the following theorem, we computed the metric dimension of convex polytope graph $\mathbb{B}_n$. We proved that this class of convex polytope has unbounded metric dimension.

Theorem 3.6.8. *For every even integer $n \geq 12$, we have $dim(\mathbb{B}_n) = dim(J_n) + 2 = \lfloor \frac{n}{3} \rfloor + 2$*

Proof. Consider the convex polytope graphs $\mathbb{B}_n$, then we have $dim(\mathbb{B}_n) = 4$, for all $4 \leq n \leq 8$ and $W_2 = \{z_{1,1}, z_{1,2}, z_{1,4}, z_{2,1}\}$ and $W_3 = \{z_{1,2}, x_{1,4}, z_{2,3}, x\}$ and $W_4 = \{z_{1,2}, z_{1,4}, z_{2,2}, z_{2,7}\}$ being metric basis for all $4 \leq n \leq 8$, respectively. $dim(\mathbb{B}_n) = 5$ being metric basis for $n = 10$ and $W_5 = \{z_{1,2}, z_{1,4}, z_{1,6}, z_{2,1}, z_{3,7}\}$. However, for $n \geq 12$, the dimension of $\mathbb{B}_n$ increases with number of vertices n. We also know that the central vertex can't belong to any basis of $\mathbb{B}_n$. For $n \geq 12$, we prove the result by double inequality. First we show that $dim(\mathbb{B}_n) \leq dim(J_n) + 2$ by constructing a resolving set W in $\mathbb{B}_n$ having $\lfloor \frac{n}{3} \rfloor + 2$ vertices. For this we consider $n = 2m$, then there are following cases to be discussed:

Case 1: $m \equiv 0(mod\ 3)$. Let $2m = 3k$, where k is even, $k \geq 4$ and $\lfloor \frac{n}{3} \rfloor + 2 = k + 2$. In this case, $B = \{z_{1,2}, z_{1,n}\} \cup \{z_{1,6i+2}, z_{1,6i+4} : 1 \leq i \leq \frac{k}{2} - 1\} \cup \{z_{2,2}, z_{2,5}\}$.

Case 2: $m \equiv 1(mod\ 3)$. Let $2m = 3k + 2$, where k is even, $k \geq 4$ and $\lfloor \frac{n}{3} \rfloor + 2 = k + 2$. In this case, $B = \{z_{1,2}, z_{1,n}\} \cup \{z_{1,6i+2}, z_{1,6i+4} : 1 \leq i \leq \frac{k}{2} - 1\} \cup \{z_{2,2}, z_{2,5}\}$.

Case 3: $m \equiv 2(mod\ 3)$. Let $2m = 3k + 1$, where k is odd, $k \geq 5$ and $\lfloor \frac{n}{3} \rfloor + 2 = k + 2$. In this case, $B = \{z_{1,2}\} \cup \{z_{1,6i+2}, z_{1,6i+4} : 1 \leq i \leq \frac{k-1}{2}\} \cup \{z_{2,2}, z_{2,5}\}$.

The set B contains only three major vertices and the rest of the vertices are all minor vertices. So there is a unique $3-3$ major gap on $C_{n,1}$ and all other gaps are $3-3$ minor gaps alternately having one and three vertices on $C_{n,1}$. All the vertices contained in a $3-3$ minor gap with one vertex on $C_{n,1}$ are major vertices. The set B contains only two $4-4$ gap on $C_{n,2}$ having two and $n-6$ vertices. B is a resolving set of $\mathbb{B}_n$, since any two minor or any two major vertices, respectively, lying in different gaps (neighboring or not) are separated by at least one vertex in the set of three or four vertices of B determining these two gaps (neighboring or not). This property is true for the vertices lying in the same gap. Also, we note that $r(x|B) = (2, 2, \cdots, 2, 2, 2)$, $r(y|B) = (4, \cdots, 4, 3, 2)$ and $r(y|B) \neq r(x|B) \neq r(z_{j,i}|B)$, for every $z_{j,i} \in V(\mathbb{B}_n)$, where x, y are central vertices and $x \neq y \neq z_{j,i}$. To prove that $dim(\mathbb{B}_n) \geq \lfloor \frac{n}{3} \rfloor + 2 = dim(J_n) + 2$, let B be a basis of $\mathbb{B}_n$ and $|B| = l + 2$. Then B induces l gaps on $C_{n,1}$, namely $g_1, \cdots, g_l$ such that g_j and g_{j+1} are neighboring gaps for every $1 \leq j \leq l-1$ and also g_1 and g_l are neighboring gaps, and there are two gaps on $C_{n,2}$ of size two and $n-4$. By Lemma 4.3.2, at most one of the gaps is major, say g_1. By Lemma 3.6.3, and Lemma 4.3.3, we can write

$$|g_1| + |g_2| \leq 6;$$

$$|g_{l-1}| + |g_l| \leq 6;$$

and

$$|g_j| + |g_{j+1}| \leq 4,$$

for every $j = 2, \cdots, l-2$. By adding these inequalities and by Lemma 3.6.6 for the gaps on $C_{n,2}$, we get

$$2(2n - l) + n - 2 = 2\sum_{j=1}^{l} |g_j| + n - 2 \leq 4l + 4 + n - 4 + 2;$$

$$\Rightarrow \lfloor \frac{n}{3} \rfloor + \frac{8}{6} \leq \lfloor \frac{n}{3} \rfloor + 2 \leq l + 2.$$

It follows that $l + 2 \geq \lfloor \frac{n}{3} \rfloor + 2$. Since l is an integer, for each $n = 2m$ and $m \equiv 0, 1, 2 (mod\ 3)$, we have $l + 2 \geq \lfloor \frac{n}{3} \rfloor + 2$, which completes the result. $\square$

Chapter 4

Further Results On Metric Dimension of Generalized Petersen Networks

The Petersen graph has attracted a large number of graph theorists over the years because of its appearance as a counter example in many places. Because of its ubiquity, it seemed a natural graph to be used in many places. The Petersen graph is named after Julius Petersen, who in 1898 constructed it to be the smallest bridgeless cubic graph with no three-edge-coloring. In 1950, H. S. M. Coxeter introduced a family of graphs generalizing the Petersen graph. The generalized Petersen graph is the most efficient small network in terms of node degree, diameter, and network size. Due to its unique and optimal properties, several network topologies based on the generalized Petersen graph have been proposed and investigated in the literature [44].

In 2005, Cáceres et al. [8] studied the metric dimension of generalized Petersen networks $P(n, m)$, for $m = 1$, and proved that this class of graphs have metric dimension 2 when n is odd and 3 otherwise. In 2008, Javaid et al. [36] studied the metric dimension of generalized Petersen networks $P(n, m)$, for $m = 2$, and proved that this class of graphs have constant metric dimension. In 2012, Javaid et al. [35] also studied the metric dimension of $P(n, m)$, for $n = 2m + 1$ and $m \geq 1$. In 2013, Ahmad et al. [1] studied the metric dimension of $P(n, m-1)$, for $n = 2m$. In 2014, Imran et al. [28] computed the metric dimension of a class of generalized Petersen networks $P(n, m)$, for $m = 3$. In 2014, Naz et al. [42] studied the metric dimension of $P(n, m)$, for $m = 4$.

In this chapter, we study the metric dimension of an infinite class of generalized Petersen networks. We compute the exact value of the metric dimension for generalized Petersen networks $P(n, m)$, for $m = 3$ and when $n \equiv 2, 3, 4, 5 \ (mod\ 6)$, thus completing the study of a class of generalized Petersen networks $P(n, m)$ for $m = 3$. We improve the upper bound for the metric dimension of $P(n, 3)$ when $n \equiv 2 \ (mod\ 6)$ by providing answer to the open problem proposed in [28] and which also gives a partial answer to an open problem raised in [36]: "*Is it the case that the metric dimension of generalized Petersen networks $P(n, m)$, for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ a class of networks with* *constant metric dimension?*"

We prove that the generalized Petersen network $P(n,m)$, for $m = 3$, has metric dimension 3 when $n \equiv 1\ (mod\ 6)$ and 4 otherwise.

4.1 Introduction and preliminary results

The metric dimension of graphs has found key importance in the evolution of cooperation [45, 46]. The families of graphs with constant metric dimension were discussed previously in [24, 26, 28, 36]. The metric dimension of generalized Petersen networks $P(n,m)$ has been subject of the many papers in the literature. In [36], it was proved that the class of generalized Petersen networks $P(n,2)$ have constant metric dimension and only 3 vertices suffices to resolve $V(P(n,2))$, and raised an open problem.

Open Problem [36]: *Is $P(n,m)$, for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of networks with constant metric dimension?*

In [35], the metric dimension of generalized Petersen network $P(n,m)$ for $n = 2m+1$ and $m \geq 1$ was studied and proved that this class has metric dimension 3, which also provides a partial answer of an open problem raised in [36]. Ahmad et al. [1] studied the metric dimension of the generalized Petersen networks $P(n,m-1)$ for $n = 2m$ and gave a partial answer to an open problem raised in [36]. They proved that this class of graphs has constant metric dimension and only 3 vertices are suffice to resolve $V(P(2m,m-1))$ for all odd $m \geq 3$ and 4 vertices are suffices for all even $m \geq 3$. In [42], Naz et al. considered the family of generalized Petersen Networks $P(n,4)$ and proved that this family also has constant metric dimension, i.e.,

$$dim(P(n,4)) = \begin{cases} 3 & ; for\ n \equiv 0\ (mod\ 4); \\ 4 & ; for\ n = 4k+3\ and\ k\ is\ even; \\ \leq 4 & ; for\ n \equiv 1,2\ (mod\ 4)\ and\ n = 4k+3\ (k\ odd). \end{cases}$$

In this paper, we improve the bound for the metric dimension of $P(n,3)$ when $n \equiv 2\ (mod\ 6)$ and provide answer to the open problem proposed in [28] and give a partial answer to an open problem proposed in [36] and prove that the generalized Petersen networks $P(n,3)$ is a family of regular networks having constant metric dimension and only 4 vertices appropriately chosen suffices to resolve all the vertices of the generalized Petersen networks $P(n,3)$ when $n \equiv 2,3,4,5\ (mod\ 6)$. Thus we conclude that each network in the family of generalized Petersen networks $P(n,3)$ is a network with constant metric dimension, which is

$$dim(P(n,3)) = \begin{cases} 3 & ; for\ n \equiv 1 (mod\ 6)\ and\ n \geq 25; \\ 4 & ; otherwise,\ \ for\ n \geq 24. \end{cases}$$

4.2 Upper bound for metric dimension of the generalized Petersen networks $P(n, 3)$ when $n \equiv 2(mod\ 6)$

The generalized Petersen network denoted by $P(n, m)$, where $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, is a cubic graph having vertex set

$$V(P(n, m)) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$$

and edge set

$$E(P(n, m)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+m} : 1 \leq i \leq n\}.$$

Generalized Petersen networks were first defined by Watkins [62]. For $m = 1$, the generalized Petersen network $P(n, 1)$ is called prism, denoted by D_n. In [8], it was shown that

$$dim(P_m \square C_n) = \begin{cases} 2 & ; if\ n\ is\ odd \\ 3 & ; otherwise. \end{cases}$$

Since the prism D_n is actually the cross product of path P_2 with a cycle C_n, this implies that

$$dim(D_n) = \begin{cases} 2 & ; if\ n\ is\ odd \\ 3 & ; otherwise. \end{cases}$$

So, prisms constitute a family of 3-regular graphs having bounded metric dimension. In [36], this study was extended and proved that $dim(P(n, 2)) = 3$ for every $n \geq 5$ and following open problem was raised.

Open Problem [36]: *Is $P(n, m)$, for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of networks with constant metric dimension?*

Imran et al. [28] by giving a partial answer to the above open problem showed that

$$dim(P(n, 3)) = \begin{cases} 3 & ; for\ n \equiv 1\ (mod\ 6)\ and\ n \geq 25; \\ 4 & ; for\ n \equiv 0\ (mod\ 6)\ and\ n \geq 24; \\ \leq 4 & ; for\ n \equiv 3, 4, 5\ (mod\ 6)\ and\ n \geq 17; \\ \leq 5 & ; for\ n \equiv 2\ (mod\ 6)\ and\ n \geq 8. \end{cases} \tag{4.2.1}$$

They also proposed the following open problem in [28].

Open Problem [28]: *Find the exact value of the metric dimension for generalized Petersen network $P(n, 3)$ when $n \equiv 2, 3, 4, 5\ (mod\ 6)$.*

When $m = 3$, $\{u_1, u_2, \cdots, u_n\}$ induces a cycle in $P(n, 3)$ with $u_i u_{i+1} (1 \leq i \leq n)$, as edges. If $n = 3l$ ($l \geq 3$), then $\{v_1, v_2, \cdots, v_n\}$ induces 3 cycles of length l, otherwise it induces a cycle of length n with $v_i v_{i+3}$ ($1 \leq i \leq n$), as edges. For example, $P(8, 3)$ is the *Möbius-Kantor* graph.

In the sequel, we will compute the exact value of the metric dimension of $P(n, 3)$ when $n \equiv 2, 3, 4, 5\ (mod\ 6)$.

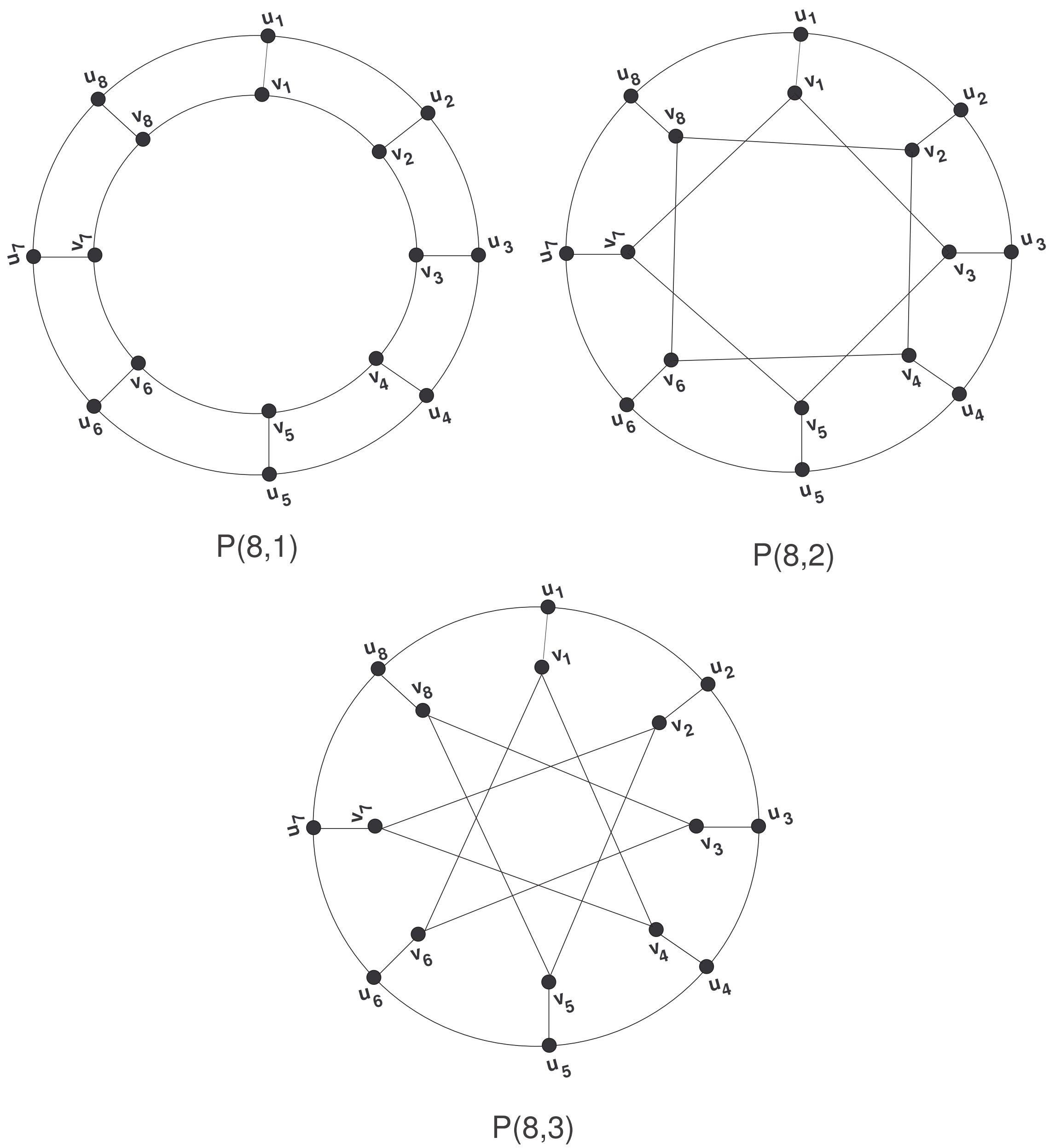

Figure 4.1: The generalized Petersen graphs P(8,1), P(8,2) and P(8,3)

Since generalized Petersen networks form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges, it is desirable to find their metric dimensions. For our convenience, we call the cycle induced by $\{u_1, u_2, \cdots, u_n\}$ as outer cycle and cycle(s) induced by $\{v_1, v_2, \cdots, v_n\}$ as inner cycle(s). Note that the choice of appropriate basis vertices (also known as landmarks) is the core of the problem.

In this chapter, we solve the open problem proposed in [28] and give a partial answer to an open problem proposed in [36], thus completing the study of the metric dimension of the generalized Petersen network $P(n, 3)$.

In the following lemma, we improve the bound given in equation (4.2.1) for the metric dimension of the generalized Petersen networks $P(n, 3)$ when $n \equiv 2(mod\ 6)$ by providing a resolving set of cardinality 4.

Lemma 4.2.1. *Let $P(n, 3)$ be the generalized Petersen network, then $dim(P(n, 3)) \leq 4$ when $n \equiv 2\ (mod\ 6)$ and $n \geq 24$.*

Proof. When $n \equiv 2\ (mod\ 6)$ and $n \geq 24$, we can write as $n = 6k + 2$ where $k \geq 4$. Suppose that $W = \{v_1, v_{3k-2}, u_{3k-1}, u_n\}$ be the subset of set of vertices of $P(n, 3)$. We show that the set W resolves the vertices of $P(n, 3)$. The codes of the vertices in $V(P(n, 3)) \setminus W$ with respect W are described in the following cases:
Case (i): Codes for the outer cycle vertices of $P(n, 3)$, when $n \equiv 2\ (mod\ 6)$ and $n \geq 24$ are given as $code(u_1|W) = (1, k, k+2, 1)$, $code(u_2|W) = (2, k+1, k+1, 2)$, $code(u_3|W) = (2, k, k+2, 3)$ and those which are given in Table 4.1 and Table 4.2:

d(.,.)	v_1	v_{3k-2}	u_{3k-1}	u_n	
u_{3i+1}	$i+1$	$k-i$	$k-i+2$	$i+3$	$1 \leq i \leq k-2$
	$2k-i+3$	$i-k+2$	$i-k+4$	$2k-i+3$	$k+1 \leq i \leq 2k-2$
u_{3i+2}	$i+2$	$k-i+1$	$k-i+1$	$i+4$	$1 \leq i \leq k-2$
	$2k-i+2$	$i-k+3$	$i-k+3$	$2k-i+2$	$k+1 \leq i \leq 2k-2$
u_{3i+3}	$i+3$	$k-i$	$k-i+2$	$i+3$	$1 \leq i \leq k-3$
	$2k-i+1$	$i-k+4$	$i-k+4$	$2k-i+3$	$k+1 \leq i \leq 2k-2$

Table 4.1: Distinct codes of outer cycle vertices

Case (ii): Codes for the inner cycle(s) vertices are given as $code(v_2|W) = (3, k+2, k, 3)$, $code(v_3|W) = (4, k+1, k+1, 2)$ and those which are given in Table 4.3 and Table 4.4.
We observe that there are no two vertices on the inner cycle(s) with same representations. Also there are no two vertices on the inner cycle(s) and on the outer cycle having same representations and no two vertices on the outer cycle having same representations. This implies that $W = \{v_1, v_{3k-2}, u_{3k-1}, u_n\}$ is a resolving set for $V(P(n, 3))$ when $n \equiv 2\ (mod\ 6)$ implying that $dim(P(n, 3)) \leq 4$, which completes the proof. □

4.3 Metric dimension of $P(n, 3)$ for $n \equiv 2, 3, 4, 5\ (mod\ 6)$

In this section, we study the lower bound for the metric dimension of the generalized Petersen networks $P(n, 3)$. We prove that $dim(P(n, 3)) \geq 4$ for $n \equiv 2, 3, 4, 5\ (mod\ 6)$ and $n \geq 24$, giving exact value of $dim(P(n, 3))$ in this case by Lemma 4.2.1 and equation (4.2.1). For this purpose we

d(.,.)	v_1	v_{3k-2}	u_{3k-1}	u_n
u_{3k-3}	$k+1$	2	2	$k+1$
u_{3k-2}	k	1	1	$k+2$
u_{3k}	$k+2$	3	1	$k+2$
u_{3k+1}	$k+1$	2	2	$k+3$
u_{3k+2}	$k+2$	3	3	$k+2$
u_{3k+3}	$k+1$	4	4	$k+3$
u_{6k-2}	4	$k+1$	$k+3$	4
u_{6k-1}	3	$k+2$	$k+2$	3
u_{6k}	2	$k+1$	$k+3$	2
u_{6k+1}	3	$k+2$	$k+2$	1

Table 4.2: Distinct codes of outer cycle vertices

d(.,.)	v_1	v_{3k-2}	u_{3k-1}	u_n	
v_{3i+1}	i	$k-i-1$	$k-i+1$	$i+2$	$1 \leq i \leq k-2$
	i	$i-k+1$	$i-k+3$	$2k-i+2$	$k \leq i \leq 2k-4$
	$2k-i+4$	$i-k+1$	$i-k+3$	$2k-i+2$	$2k-3 \leq i \leq 2k-1$
v_{3i+2}	$i+3$	$k-i+2$	$k-i$	$i+3$	$1 \leq i \leq k-1$
	$2k-i+3$	$i-k+4$	$i-k+2$	$2k-i+1$	$k \leq i \leq 2k-1$
v_{3i+3}	$i+4$	$k-i+1$	$k-i+1$	$i+2$	$1 \leq i \leq k-2$
	$2k-i$	$i-k+5$	$i-k+3$	$2k-i+2$	$k \leq i \leq 2k-3$

Table 4.3: Distinct codes of inner cycle(s) vertices

d(.,.)	v_1	v_{3k-2}	u_{3k-1}	u_n
v_{3k}	$k+1$	4	2	$k+1$
v_{6k-3}	2	$k+1$	$k+1$	4
v_{6k}	1	$k+1$	$k+2$	3
v_{6k+1}	4	$k+1$	$k+1$	2
v_{6k+2}	3	$k+2$	$k+2$	1

Table 4.4: Distinct codes of inner cycle(s) vertices

need few more notions and definitions. As we specified in section 4.2 that the vertices of the outer cycle are labeled by $u_1, u_2, \cdots, u_n$ in the clockwise direction. For any two distinct vertices u_i and u_j we would like to define the *"clockwise distance"* from u_i to u_j, denoted by $d^*(u_i, u_j)$ the distance

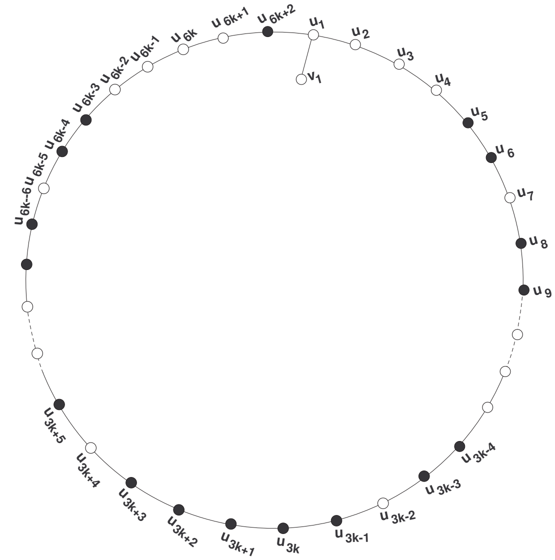

Figure 4.2: Good vertices for $u_1 (n = 6k + 2)$

measured in clockwise direction, from u_i to u_j in the subgraph induced by the outer cycle. For instance, $d^*(u_1, u_j) = j - 1$ and $d^*(u_j, u_1) = 1$; in general we have $d^*(u_1, u_j) + d^*(u_j, u_1) = j$. This definition can be extended to any two vertices of the generalized Petersen network $P(n, 3)$ for $i \neq j$ by: $d^*(u_i, v_j) = d^*(v_i, u_j) = d^*(v_i, v_j) = d^*(u_i, u_j)$.

Suppose that u_1 is a vertex on outer cycle of $P(n, 3)$. A vertex u_i is said to be a *good vertex* for u_1 if $d(u_1, u_i) = d(u_1, u_{i+2})$; otherwise u_i is referred as a *bad vertex* for u_1. This can be extended for inner cycle(s) vertices as well: u_i is a good vertex for v_1 if $d(v_1, u_i) = d(v_1, u_{i+2})$ and bad otherwise. In Fig. 4.2, we labeled by black dots all the good vertices for u_1 when $n = 6k + 2 \geq 24$.

It can be observed that the set of good vertices for v_1 is deduced from the set of good vertices for u_1 by adding 4 new vertices on the outer cycle, namely u_2, u_3, u_{6k-1} and u_{6k}. Similarly, a vertex u_k is known to be a good vertex for the pair $\{u_1, u_j\}$ if $d(u_1, u_k) = d(u_1, u_{k+2})$ and $d(u_j, u_k) = d(u_j, u_{k+2})$. If u_i is a good vertex for the pairs $\{u_1, u_j\}$ and $\{u_1, u_k\}$ then u_i is also known as a good vertex for the triplet $\{u_1, u_j, u_k\}$, i.e., $d(u_1, u_i) = d(u_1, u_{i+2}), d(u_j, u_i) = d(u_j, u_{i+2})$ and $d(u_k, u_i) = d(u_k, u_{i+2})$. Due to rotational symmetry of $P(n, 3)$, we deduce the following results.

Lemma 4.3.1. *If u_i and u_j are any two vertices on the outer cycle of $P(n, 3)$ then we have $d(u_i, u_j) = d(u_{i+l}, u_{j+l})$ for any $1 \leq l \leq n - 1$.*

The following lemma is quite useful to find the good vertices for the pair of vertices on the outer cycle of $P(n, 3)$.

Lemma 4.3.2. *If u_i is a good vertex for u_1 and u_{i-j} is also a good vertex for any u_1 for $1 \leq j \leq n-5$, then u_i is also a good vertex for the pair $\{u_1, u_{j+1}\}$.*

Proof. By definition of good vertices we have $d(u_1, u_i) = d(u_1, u_{i+2})$ and $d(u_1, u_{i-j}) = d(u_1, u_{i-j+2})$. By Lemma 4.3.1, we have from the last equality that $d(u_{j+1}, u_i) = d(u_{j+1}, u_{i+2})$. □

In the following lemma, we will prove that the $dim(P(n,3))$ is bounded below by 4.

Lemma 4.3.3. *If* $n = 6k+2$ *and* $n \geq 24$ *then* $dim(P(n,3)) \geq 4$.

Proof. We shall prove that $dim(P(n,3)) \geq 4$, or there is no resolving set of $V(P(n,3))$ having three vertices X, Y and Z. It is sufficient to consider only the case when X, Y and Z belong to the outer cycle. Since we can deduce the set of good vertices for v_1 from the set of good vertices for u_1 (represented in Fig. 4.2) by adding new vertices u_2, u_3, u_{6k-1} and u_{6k}. We can see that for any three vertices X, Y and Z of $V(P(n,3))$ such that $d^*(X,Y) < d^*(X,Z)$ it is possible to find a pair of vertices at distance 2 on the outer cycle, $\{u_i, u_{i+2}\}$ having equal distances to X, Y and Z, respectively. If $n = 6k+2$ and X, Y, Z are on the outer cycle, we can assume that $X = u_1$. By denoting $(r,s) \equiv (x,y)$ $(mod\ 3)$ if $r \equiv x$ $(mod\ 3)$ and $s \equiv y$ $(mod\ 3)$, the following nine cases occur: $(d^*(u_1,Y), d^*(u_1,Z)) \equiv (x,y)$ $(mod\ 3)$, where (x,y) can take the form
(1) $(0,0)$; **(2)** $(1,1)$; **(3)** $(2,2)$; **(4)** $(0,1)$ **(5)** $(0,2)$; **(6)** $(1,0)$; **(7)** $(1,2)$; **(8)** $(2,0)$; **(9)** $(2,1)$.
Due to rotational symmetry of $P(n,3)$ some of the cases reduced to other cases e.g., from case (2) by permutation $X \to Y,\ Y \to Z,\ Z \to X$ we obtain case (4) and by permutation $X \to Z,\ Y \to X,\ Z \to Y$ we get the case (7). The reducibility between the cases is illustrated with help of graphs in the Fig. 4.3.

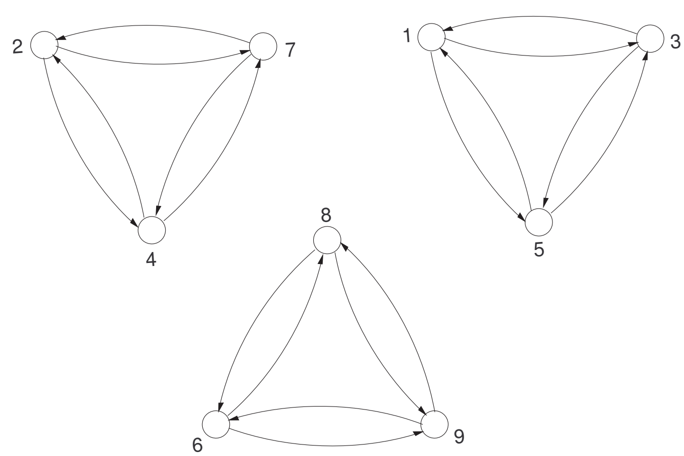

Figure 4.3: Reducibility between the cases

It follows that it is sufficient to consider only cases (1), (2) and (9).

Case 1**:** If we choose a good vertex u_6 and we go in counter clockwise direction approaching vertices $u_3, u_{6k}, u_{6k-3}, u_{6k-6}, \cdots, u_9$; we find only two bad vertices u_3 and u_{6k} for u_1 with property that $d^*(u_1,Y) \equiv 2$ $(mod\ 3)$, where $Y \in \{u_3, u_{6k}, u_{6k-3}, u_{6k-6}, \cdots, u_9\}$ (see Fig. 4.2). By Lemma 4.3.2, it follows that u_6 is a good vertex for all pairs $\{u_1, Y\}$ where $d^*(u_1,Y) \equiv 0$ $(mod\ 3)$ and $Y \notin \{u_4, u_{10}\}$. It follows that u_6 is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_4$ and $Z \in \{u_7, u_{10}, u_{13}, \cdots, u_{6k+1}\}$; $Y = u_{10}$ and $Z \in \{u_7, u_{13}, \cdots, u_{6k+1}\}$. For these triplets we must find other good vertices on the outer cycle. In a similar way, u_{3k+6} is a good vertex for u_1 since

$6k-6 \geq 3k+6$ and for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 0 (mod\ 3)$ and $Y \notin \{u_{3k+4}, u_{3k+10}\}$. Consequently, we have found a good vertex (u_6 or u_{3k+6}) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_4, u_{3k+4}\}, \{u_4, u_{3k+10}\}, \{u_{10}, u_{3k+4}\}, \{u_{10}, u_{3k+10}\}$. Finally u_9 is a good vertex for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 0\ (mod\ 3)$ and $Y \notin \{u_7, u_{13}\}$. Since $k \geq 4$ we have $3k+4 \geq 13$ and u_9 is a good vertex for the remaining triplets $\{u_1, Y, Z\}$, where $Y \in \{u_4, u_{10}\}$ and $Z \in \{u_{3k+4}, u_{3k+10}\}$.
Case 2: In a similar way we get that u_5 is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1\ (mod\ 3)$ and $Y \notin \{u_2, u_5, u_8\}$. Therefore u_5 is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_2$ and $Z \in \{u_5, u_8, u_{3k-4}, u_{3k-1}, u_{3k+2}, u_{3k+5}, u_{3k+8}, \cdots, u_{6k-4}, u_{6k-1}, u_{6k+2}\}$; $Y = u_5$ and $Z \in \{u_8, u_{3k-4}, u_{3k-1}, u_{3k+2}, u_{3k+5}, u_{3k+8}, \cdots, u_{6k-4}, u_{6k-1}, u_{6k+2}\}$; $Y = u_8$ and $Z \in \{u_{11}, u_{3k-1}, u_{3k+2}, u_{3k+5}, u_{3k+8}, \cdots, u_{6k-4}, u_{6k-1}, u_{6k+2}\}$. Since $6k-7 > 3k-1$, it follows that u_{6k-7} is a good vertex for u_1 and for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1\ (mod\ 3)$ and $Y \notin \{u_{6k-7, u_{6k-4}, u_{3k+5}}\}$. We have found a good vertex (u_5 or u_{6k-7}) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_2, u_{3k+5}\}, \{u_2, u_{6k-7}\}, \{u_2, u_{6k-4}\}, \{u_5, u_{3k+5}\}, \{u_5, u_{6k-7}\}, \{u_5, u_{6k-4}\}, \{u_8, u_{3k+5}\}, \{u_8, u_{6k-7}\}, \{u_8, u_{6k-4}\}$. Since $k \geq 4$ we find for triplets $\{u_1, u_2, u_{3k+5}\}$, $\{u_1, u_2, u_{6k-7}\}$, $\{u_1, u_2, u_{6k-4}\}$, $\{u_1, u_5, u_{3k+5}\}$, $\{u_1, u_5, u_{6k-7}\}$, $\{u_1, u_5, u_{6k-4}\}$, $\{u_1, u_8, u_{3k+5}\}$, $\{u_1, u_8, u_{6k-7}\}$, $\{u_1, u_8, u_{6k-4}\}$ good vertices u_{3k-4} for first six triplets and u_{3k-1} for last three triplets, respectively (e.g. by using Lemma 4.3.2).
Case 9: Similarly, u_5 is a good vertex for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 2\ (mod\ 3)$ and $Y \notin \{u_3, u_9\}$. It follows that u_5 is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_3$ and $Z \in u_6, u_9, u_{3k-4}, \cdots, u_{6k}$; $Y = u_9$ and $Z \in u_6, u_{3k-4}, \cdots, u_{6k}$. For these triplets we must find other good vertices on the outer cycle. Similarly, u_{3k+5} is a good vertex for u_1 since $6k-7 \geq 3k+5$ and for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 2\ (mod\ 3)$ and $Y \notin \{u_{3k+3}, u_{6k-6}\}$. Consequently, we have found a good vertex (u_5 or u_{3k+5}) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_3, u_{3k+3}\}$, $\{u_3, u_{6k-6}\}$, $\{u_9, u_{3k+3}\}$, $\{u_9, u_{6k-6}\}$. Finally, u_8 is a good vertex for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 2\ (mod\ 3)$ and $Y \notin \{u_6, u_{12}\}$. Since $k \geq 4$ we have $3k+3 > 12$ and u_8 is a good vertex for the remaining triplets $\{u_1, Y, Z\}$, where $Y \in \{u_3, u_9\}$ and $Z \in \{u_{3k+3}, u_{6k-6}\}$, which completes the proof. □

In a similar fashion as in Lemma 4.3.3, we can prove that $dim(P(n, 3)) \geq 4$ when $n \equiv 3, 4, 5\ (mod\ 6)$ and $n \geq 24$, while the set of good vertices for u_1 are shown by black dots in Fig. 4.4.
In the following theorem, we prove the exact value of $P(n, 3)$ when $n \equiv 2, 3, 4, 5\ (mod\ 6)$.

Theorem 4.3.4. *If* $n = 6k+2,\ 6k+3,\ 6k+4,\ 6k+5$ *and* $n \geq 24$ *then we have* $dim(P(n, 3)) = 4$.

Proof. To prove the theorem we use double inequality. First, we see that $dim(P(n, 3)) \leq 4$ can be followed from Lemma 4.2.1 and equation (4.2.1). The reverse inequality can be followed on the same lines as in Lemma 4.3.3, which concludes the proof. □

The following theorem can be deduced as an immediate consequence of the equation (4.2.1) and Theorem 4.3.4, which completes the study of generalized Petersen networks $P(n, 3)$.

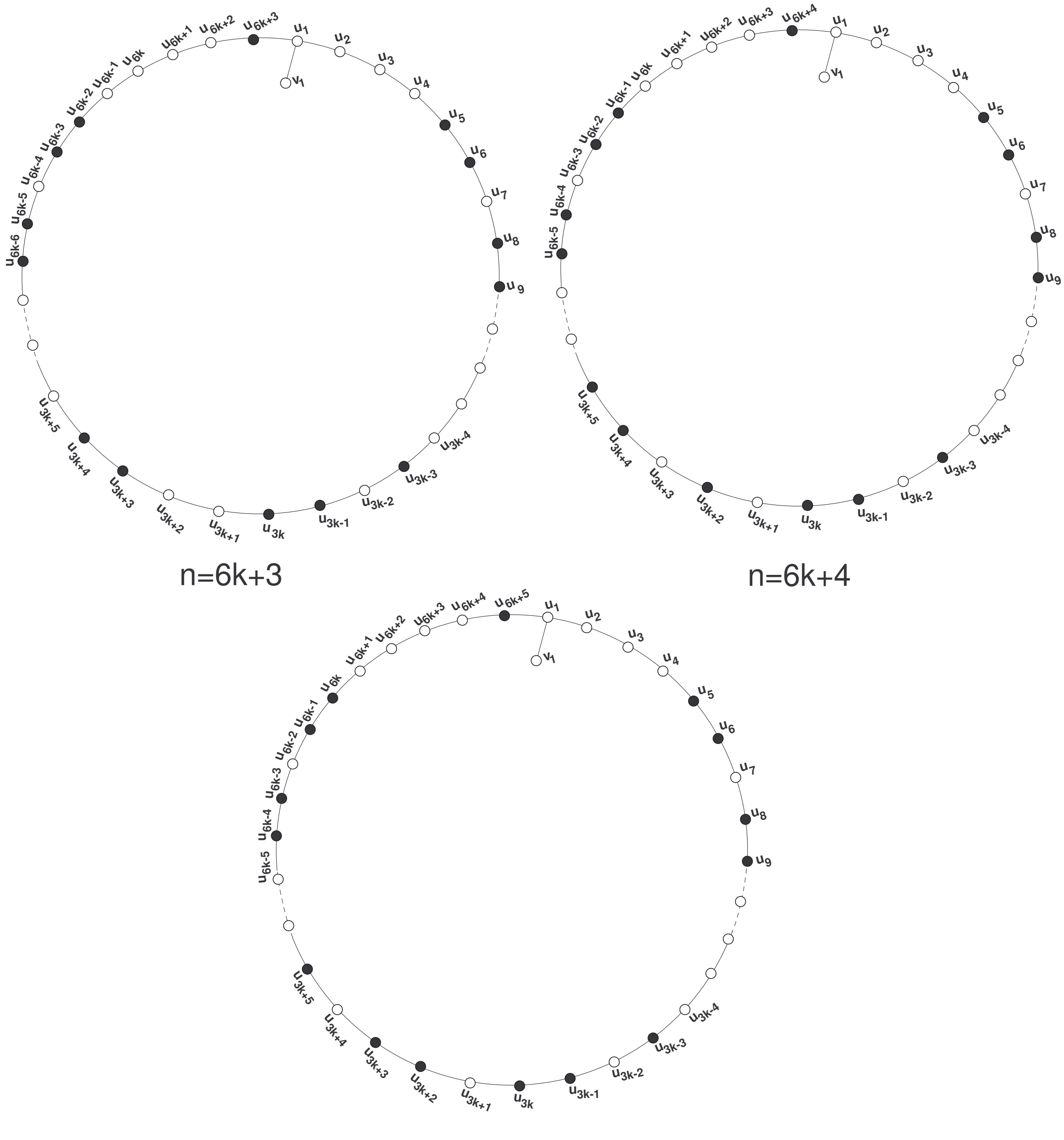

Figure 4.4: Good vertices for u_1 ($n = 6k+3, 6k+4, 6k+5$)

Theorem 4.3.5. *For the generalized Petersen networks $P(n,m)$ for $m = 3$, we have*

$$dim(P(n,3)) = \begin{cases} 3 & ; for\ n \equiv 1\ (mod\ 6)\ and\ n \geq 25; \\ 4 & ; otherwise,\ \ for\ n \geq 24. \end{cases}$$

Chapter 5

Metric Dimension and Partition Dimension of Nanostructures

The metric dimension $dim(G)$ and partition dimension $pd(G)$ of a connected graph G are related as $pd(G) \leq dim(G) + 1$.
Moreover, there are some simple connected graphs having metric dimension much larger than the partition dimension and this phenomena is called *discrepancy* between metric dimension and partition dimension. The discrepancies between metric dimension and partition dimension of a connected graph G have already been subject of the following papers [40, 58, 59]. The metric dimension of infinite graphs is studied in [10].

In this chapter, we study the metric dimension and partition dimension of 2-dimensional lattices of certain infinite nanotubes generated by the tiling of the plane. We prove that metric dimension of these infinite nanotubes is not finite but their partition dimension is finite and is evaluated, implying that these nanostructures are among the graphs having discrepancy between their metric dimension and partition dimension. It is also proved that there exist some induced subgraphs of 2-dimensional lattices of these nanostructures having unbounded metric dimension as well as having constant metric dimension.

5.1 Introduction and Preiminary Results

It is natural to think about the relation between metric dimension and partition dimension of connected graphs and the characterization of the graphs having discrepancies between their metric dimension and partition dimension. It was shown in [14] that metric dimension and partition dimension for any connected graph G are related as $pd(G) \leq 1 + dim(G)$.
Consider the infinite graphs $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$ generated by two metrics

$$d_4[(x_1, y_1), (x_2, y_2)] = |x_2 - x_1| + |y_2 - y_1|\text{: (City block distance)}$$

and

$$d_8[(x_1, y_1), (x_2, y_2)] = max(|x_2 - x_1|, |y_2 - y_1|)\text{: (Chessboard distance)}$$

on $\mathbb{Z}^2$ having the same vertex set $\mathbb{Z}^2$ and the set of edges consisting of all pairs of vertices whose city block and chess board distances are 1.

In [40], it was proved that $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$ have no finite metric bases, where $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$ are infinite graphs generated by tiling of the plane by integral lattice equipped with city block and chessboard metric, respectively. Also for any natural number $n \geq 3$ there exist induced subgraphs of $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$ having metric dimension equal to n, constant and bounded by 3. Also, in [58] it was proved that $pd(\mathbb{Z}^2, \mathcal{E}_4) = 3$ and $pd(\mathbb{Z}^2, \mathcal{E}_8) = 4$, hence proving that $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$ have discrepancy between their metric dimension and partition dimension. The discrepancies between the metric dimension and partition dimension of infinite regular graphs generated by tilings of the plane by regular hexagons or equilateral triangles are also discussed by Tomescu et al. in [59]. Some infinite regular graphs generated by tiling of the plane by an infinite hexagonal grid are also discussed in [23] and was proved that these graphs also have discrepancies between their metric dimension and partition dimension. It was also proved that their are some infinite induced subgraphs of these graphs having metric dimension equal to n and some have metric dimension equal to 3.

In what follows we shall consider certain infinite regular graphs generated by tiling of the plane by 2-dimensional lattices of certain infinite *Carbon nanotubes*. *Carbon nanotubes* are basically sheets of graphite rolled up into a tube. It is constructed from the hexagonal 2-dimensional lattice of graphite mapped on a given one-dimensional cylinder of radius R. The nanotubes HAC_5C_7 and HC_5C_7 are constructed in a similar way by a sheet covered by pentagons and heptagons. The nanotube $HAC_5C_6C_7$ is constructed by a sheet covered by pentagons, hexagons and heptagons. A *V-Phenylenic nanotube* is constructed by a sheet covered by C_4, C_6 and C_8. The nanotubes $TUC_4C_8(R)$ and $TUC_4C_8(S)$ are constructed by a sheet covered by C_4 and C_8. The nanotube H-Naphtalenic is constructed by a sheet covered by squares, hexagons and octagons. Also the nanotube VC_5C_7 is constructed by a sheet covered by pentagons and heptagons. Carbon nanotubes are allotropes carbon with a cylindrical nano-structure. These cylindrical Carbon nanotubes have unusual properties, which are valuable for nanotechnology, electrons, optics and other fields of material sciences and technology. Nanotubes are members of fullerene structure family. Their name is derived from their long, hollow structure.

5.2 Metric Dimension and Partition dimension of Nanotubes

In this section, we compute the metric dimension and partition dimension of certain infinite regular graphs generated by tiling of the plane by 2-dimensional lattices of some infinite nanotubes. We prove that these 2-dimensional lattices of nanotubes have infinite metric bases but their partition dimension is finite and evaluated. Hence, these nanotubes have discrepancy between their metric dimension and partition dimension. These results show that these 2-dimensional lattices of nanotubes are among the graphs for which the strict inequality $pd(G) < dim(G) + 1$ holds. It is also shown that there exist induced subgraphs of 2-dimensional lattices of these nanostructures having metric dimension unbounded as well as having constant metric dimension.

In the next lemma, we show that the infinite nanotubes HAC_5C_7, HC_5C_7, $HAC_5C_6C_7$, H-Naphtalenic, VC_5C_7, V-Phenylenic, $TUC_4C_8(R)$ and $TUC_4C_8(S)$ have no finite metric basis.

Lemma 5.2.1. *The 2-dimensional lattices of infinite nanotubes HAC_5C_7, HC_5C_7, $HAC_5C_6C_7$, H-Naphtalenic, VC_5C_7, V-Phenylenic, $TUC_4C_8(R)$ and $TUC_4C_8(S)$ nanotubes have no finite metric basis, i.e.,* $dim(HAC_5C_7) = dim(HC_5C_7) = dim(HAC_5C_6C_7) = dim(H - Naphtalenic) = dim(VC_5C_7) = dim(V - Phenylenic) = dim(TUC_4C_8(R)) = dim(TUC_4C_8(S)) = \infty$.

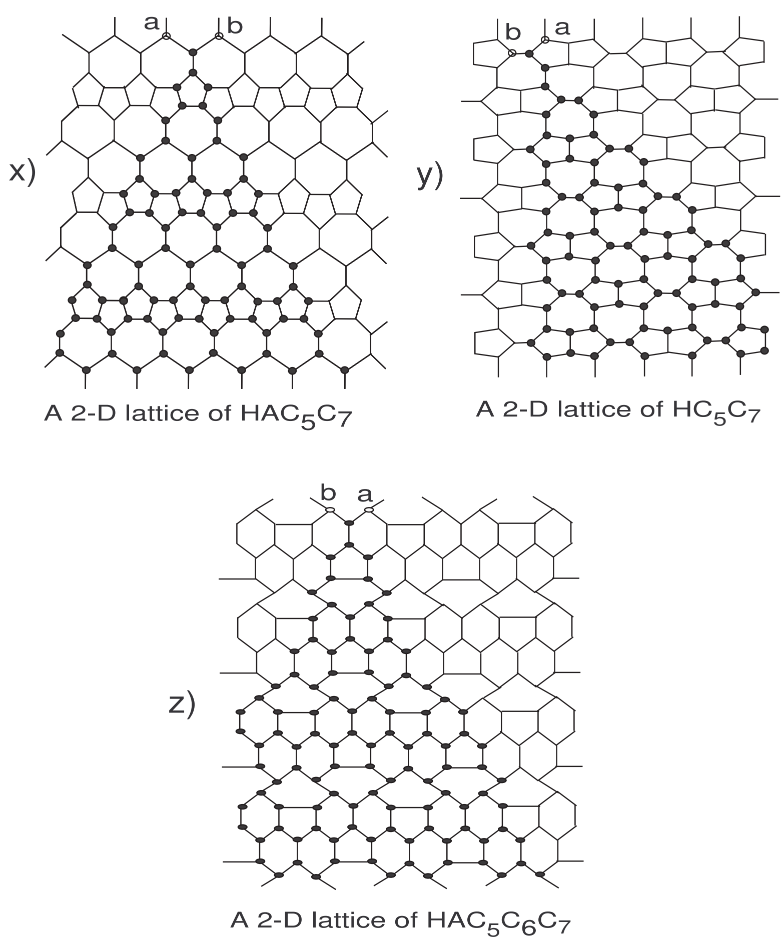

Figure 5.1: Vertices having equal distances from a and b

Proof. In the graph of 2-dimensional lattice of HAC_5C_7 as shown in Fig. 5.1 (x), we labeled two vertices by a and b and set of vertices c in this graphs such that $d(a,c) = d(b,c)$. To the contrary, suppose that this graph has a finite metric basis $\mathbf{B}$. We can find two vertices a, b and a subset $\mathbf{A}$ of this graph consisting of all vertices c such that $d(a,c) = d(b,c) \leq m$ for every positive integer m such that $\mathbf{B} \subset \mathbf{A}$. This implies that $d(a,c) = d(b,c)$ for all $c \in \mathbf{B}$, a contradiction to our assumption. The proof for the metric dimension of 2-dimensional lattices of the following nanotubes HC_5C_7, $HAC_5C_6C_7$, V-Phenylenic, $TUC_4C_8(R)$, $TUC_4C_8(S)$, H-Naphtalenic and VC_5C_7 nanotubes can be followed in the similar fashion as shown in Fig. 5.2 and Fig. 5.3. □

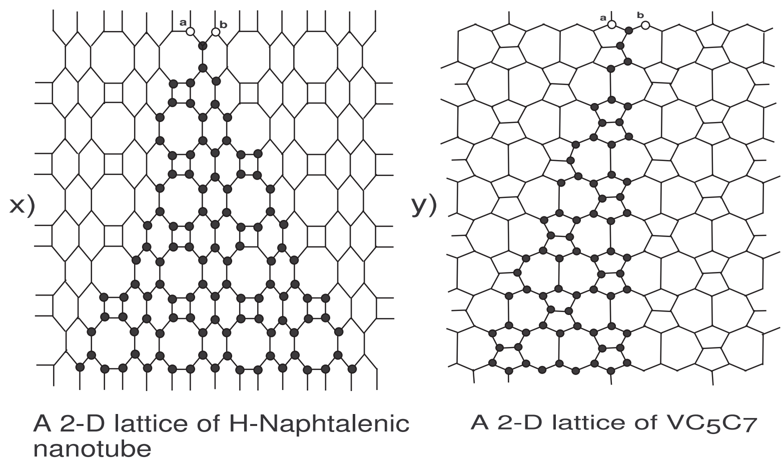

Figure 5.2: Vertices having equal distances from a and b

The partition dimension of 2-dimensional lattice of HAC_5C_7, HC_5C_7, $HAC_5C_6C_7$, H-Naphtalenic, VC_5C_7, V-Phenylenic, $TUC_4C_8(R)$, and $TUC_4C_8(S)$ nanotubes has been determined in the following lemma.

Lemma 5.2.2. *We have* $pd(HAC_5C_7) = pd(HC_5C_7) = pd(HAC_5C_6C_7) = pd(H - Naphtalenic) = pd(VC_5C_7) = pd(V - Phenylenic) = pd(TUC_4C_8(R)) = pd(TUC_4C_8(S)) = 3$.

Proof. In [13] it was proved that $pd(G) = 2$ if and only if G is path and this property holds for infinite graphs too. It follows that $pd(HAC_5C_7) \geq 3$, $pd(HC_5C_7) \geq 3$ and $pd(HAC_5C_6C_7) \geq 3$. Fig. 5.4 provides a resolving 3-partition of 2-dimensional lattices of these infinite nanotubes. It follows that $pd(HAC_5C_7) = pd(HC_5C_7) = pd(HAC_5C_6C_7) = 3$. The proof for the rest of the 2-dimensional lattices of the infinite nanotubes can be followed on the same lines and their resolving 3-partitions are shown in Fig. 5.5 and Fig. 5.6, which completes the proof. □

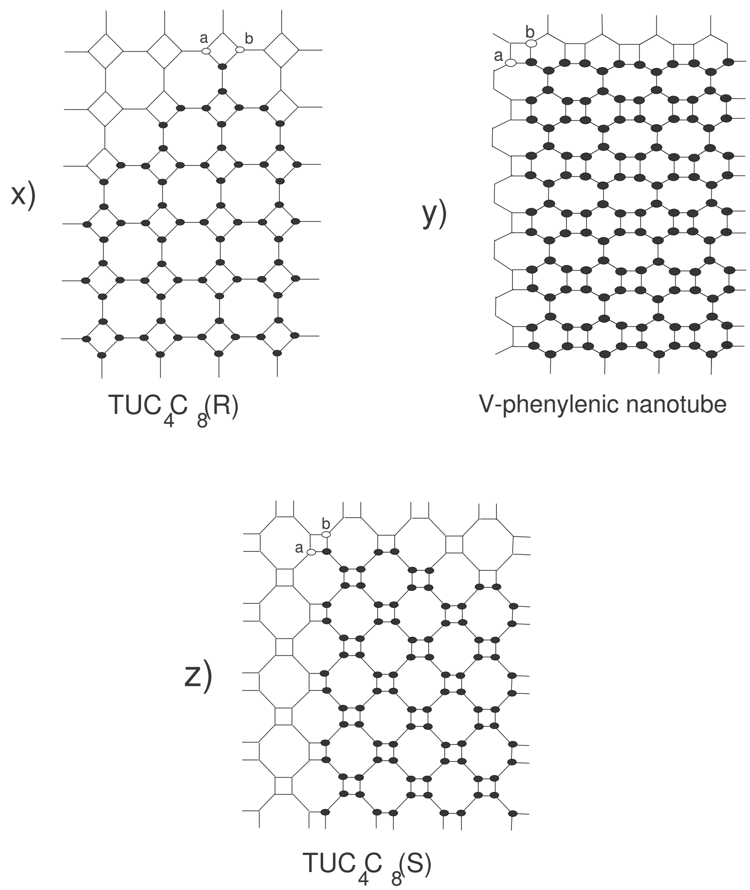

Figure 5.3: Vertices having equal distances from a and b

5.3 Metric Dimension of Induced Subgraphs of the Nanotubes

A *k-polyomino* system is a finite 2-connected plane graph such that each interior face (also called cell) is surrounded by a regular 4k-cycle of length one. In other words, it is an edge-connected union of cells as shown in Fig. 5.8.

Fig. 5.7 and Fig. 5.8 represent induced subgraphs of 2-dimensional lattices of HC_5C_7, HAC_5C_7 and $HAC_5C_6C_7$ nanotubes. The graph HP_n consists of series of n induced heptagons, I_n is an alternate edge-connected union of $\frac{n}{2}$ induced 5-cycles and $\frac{n}{2}$ induced 7-cycles, HP_n^1 is defined as an edge-connected union of $\frac{n}{2}$ pair of induced 7-cycles, E_n is an edge-connected union of $\frac{n}{2}$ pair of induced C_5, I_n^1 is an alternate edge connected union of C_5, C_7 and an edge, while the induced subgraph HH_n is define as an edge connected union of pairwise zig-zag sequence of n heptagons. Note that all these induced subgraphs are of order l.

In the next theorems, we determined the metric dimension of the induced subgraphs defined above.

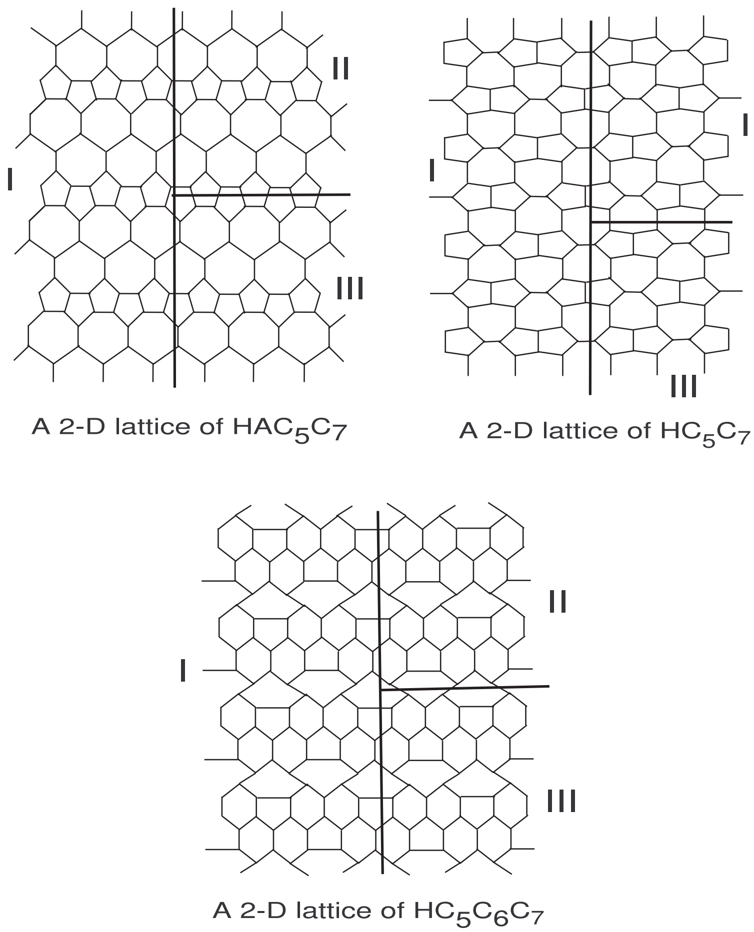

Figure 5.4: A resolving 3-partition of 2-dimensional lattices of HAC_5C_7, HC_5C_7 and $HAC_5C_6C_7$ nanotubes

We proved that there exist induced subgraphs of 2-dimensional lattices of these infinite nanotubes having metric dimension unbounded as well as having constant metric dimension.

Theorem 5.3.1. *a) For every integer $n \geq 1$ we have: $dim(HP_n) = dim(I_n) = 2$.*
b) For every integer $n \geq 3$ we have: $dim(HP_n^1) = dim(E_n) = dim(I_n^1) = \lceil \frac{n}{2} \rceil$.

Proof. a) From [11], it implies that $dim(HP_n) \geq 2$. In Fig. 5.7, the vertices of HP_n lying on the upper half and lower half of the induced 7-cycles are represented by u_i and v_i, respectively, where $1 \leq i \leq \frac{l}{2}$. To prove that $dim(HP_n) \leq 2$, we show that the set $W = \{u_2, v_1\}$ resolves $V(HP_n)$. For this, we give representations of the vertices of $V(HP_n) \setminus W$ with respect to W. We have
$code(u_1|W) = (1, 1)$, $code(u_3|W) = (1, 3)$, $code(v_2|W) = (3, 1)$.
Also

$$code(u_i|W) = (i - 2, i - 1), \quad \text{for} \quad 4 \leq i \leq \frac{l}{2},$$

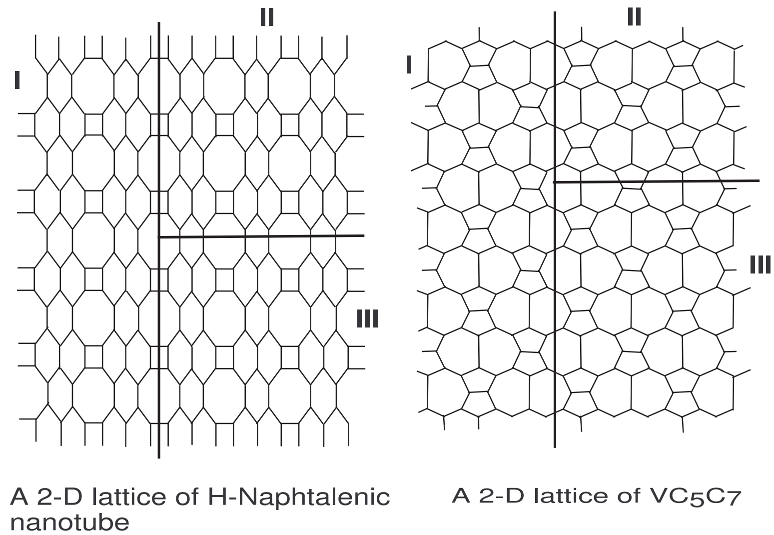

Figure 5.5: A resolving 3-partition of 2-dimensional lattice of H-Naphtalenic, and VC_5C_7 nanotubes

$$code(v_i|W) = (i, i-1), \quad \text{for} \quad 3 \leq i \leq 5$$

and

$$code(v_i|W) = (i-1, i-1), \quad \text{for} \quad 6 \leq i \leq \frac{l}{2}.$$

Since all the distinct vertices receive different codes, this implies that $dim(HP_n) = 2$. The proof follows on the same lines for I_n.

b) Suppose that there are even numbers of heptagons in HP_n^1. The vertices a and b have equal distances to all vertices of HP_n^1 except the vertices c, d, e, f, g, h, p and q, while the vertices a and b may be distinguished only by c, d, e, f, g, h, p or q if a and b do not belong to any basis of HP_n^1. It follows that at least one vertex from the set $\{a, b, c, d, e, f, g, h, p, q\}$ must belong to any basis of HP_n^1. In other words, a metric basis of HP_n^1 can be constructed by choosing exactly one vertex of degree two from each pair of consecutive heptagons and the result follows. Note that the result is also true when n is odd but the number of vertices are different. The situation is quite similar for the induced subgraphs E_n and I_n^1. □

Now we determine the exact value of the metric dimension of zig-zag chain of 7-cycles showing that the zig-zag chain of 7-cycles have unbounded metric dimension.

Theorem 5.3.2. *For every integer $n \geq 1$, we have $dim(HH_n) = \lceil \frac{n+2}{2} \rceil$.*

Proof. Suppose that heptagons of HH_n have been numbered by $1, 2, 3, ..., n$ from left to right. Fig. 5.8 shows that $dim(HH_1) = dim(HH_2) = 2$ with metric basis $\{x, y\}$ and $dim(HH_3) = 3$

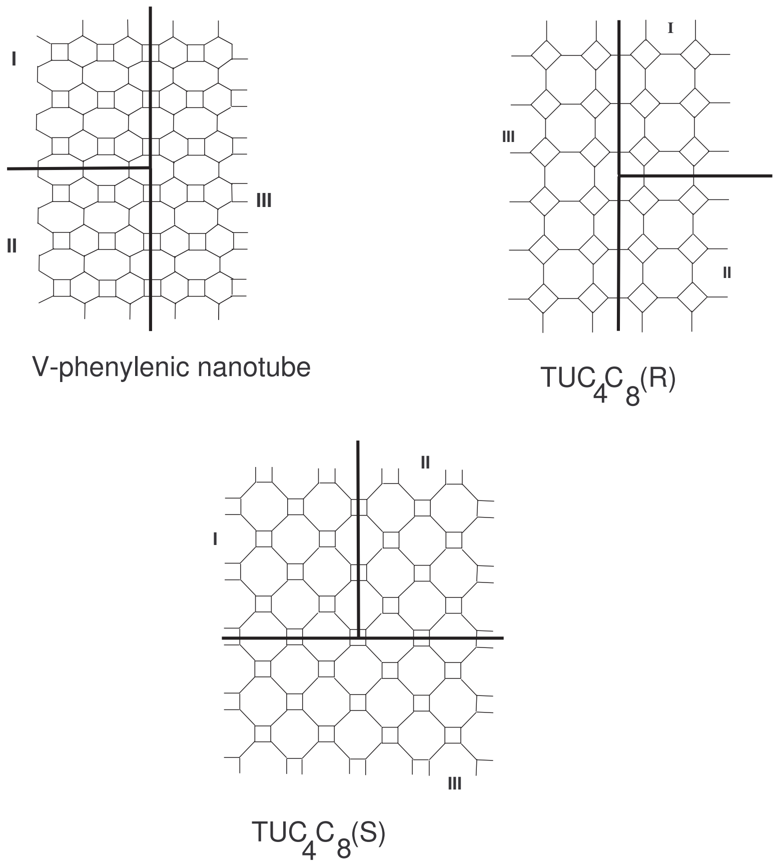

Figure 5.6: A resolving 3-partition of 2-dimensional lattices of V-Phenylenic nanotube, $TUC_4C_8(R)$, and $TUC_4C_8(S)$ nanotubes

with metric basis being $\{y, t, h\}$ containing vertices of type y, t and h. Now we will prove that $dim(HH_n) = \lceil \frac{n+2}{2} \rceil$ for $n \geq 4$. This can be shown by using double inequality.

For an upper bound, we construct a resolving set having $\lceil \frac{n+2}{2} \rceil$ vertices (in each case, whether n odd or even). Hence, a resolving set $\mathcal{A}$ can be constructed having cardinality $\lceil \frac{n+2}{2} \rceil$ as follows:

• For n odd, we can choose vertices x and t from heptagons numbered 1 and 2 respectively, and a vertex of type s from each of the heptagons numbered by $5, 7, 9, ..., n-1$ from left to right and also a vertex of type h from nth heptagon.

• When n is even, we choose vertices x and t from heptagons numbered 1 and 2 respectively, and a vertex of type s from each of the heptagons numbered by $4, 6, 8, ..., n-2$ from left to right and also a vertex of type j from nth heptagon.

One can easily verify that any two vertices of HH_n having same distances to any other vertex of $\mathcal{A}$ may be distinguished by other vertices of $\mathcal{A}$; which shows that $dim(HH_n) \leq \lceil \frac{n+2}{2} \rceil$ for $n \geq 1$.

For reverse inequality, we will first show that every resolving set of HH_n must contain at least one vertex from each of the two consecutive heptagons in the finite sequence of heptagons represented

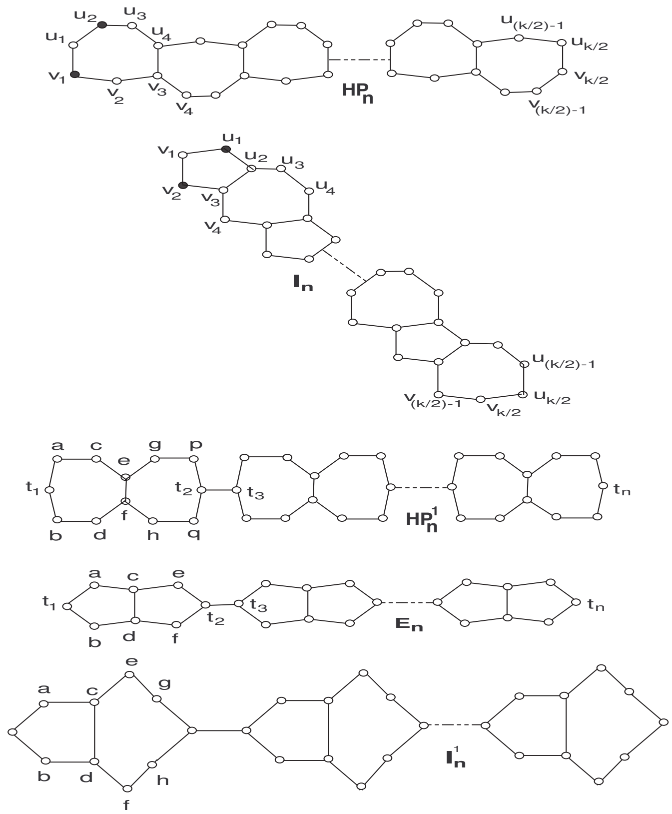

Figure 5.7: Some induced subgraphs of nanotubes

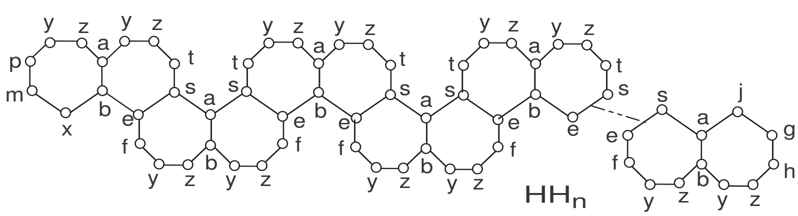

Figure 5.8: The zig-zag chain of 7-cycles

by HH_n. The vertices e and a in the third heptagon from left to right may be distinguished only by the vertices of type a, b and y in second heptagon, and by vertices of type f, y, z and b in third heptagon and all vertices in fourth octagon except vertex of type a of HH_n. A similar argument can be established for other pair of vertices of type f, b and y, z. This shows that any two consecutive heptagons of HH_n must contain at least one vertex in any metric basis. This implies that if we assign to i-th heptagon the binary variable H_i having value 1 if it has a vertex in a resolving set $\mathcal{A}$ of HH_n and 0 otherwise, we can write:

$$H_1 = 1;$$

$$\begin{aligned} H_2 + H_3 &\geq 1 \\ H_3 + H_4 &\geq 1 \\ H_4 + H_5 &\geq 1 \\ &\cdots\cdots \\ &\cdots\cdots \\ &\cdots\cdots \\ H_{n-1} + H_n &\geq 2 \end{aligned}$$

By summing up these inequalities, we get:

$$S = H_2 + 2H_3 + 2H_4 + \cdots + 2H_{n-1} + H_n \geq n.$$

Hence we get

$$2 \mid \mathcal{A} \mid = 2\sum_{i=1}^{n} H_i = S + H_2 + H_n + 2 \geq n + 2.$$

This implies that $\mid \mathcal{A} \mid \geq \lceil \frac{n+2}{2} \rceil$, hence $dim(HH_n) \geq \lceil \frac{n+2}{2} \rceil$, which completes the proof. □

Fig. 5.9 represents some induced subgraphs of $TUC_4C_8(R)$, *V-Phenylenic* and $TUC_4C_8(S)$ nanotubes. Y_n is defined as the series of n induced C_4, the graph O_n is an edge-connected union of n induced 8-cycles, B_n is defined as an edge-connected union of n pairs of induced C_8 and C_4 alternatively, D_n is an edge-connected union of n pairs of induced C_6 and C_8 alternatively, while X_n is an edge-connected union of n pairs of induced C_4 and C_6 alternatively. The metric dimension of F_n has been determined in [59]. Fig. 5.10 represents a polyomino chains of 8-cycles and we denoted it by OO_n. Note that all these induced subgraphs are of order k.

In the next theorem, we have determined the metric dimension of the induced subgraphs defined above. We prove that there exist induced subgraphs of these nanotubes having metric dimension n as well as having constant metric dimension.

Theorem 5.3.3. *a) For every integer $n \geq 2$, we have: $dim(Y_n) = n$ and Y_n has $n!2^n$ metric basis.*
b) For every integer $n \geq 1$, we have $dim(O_n) = dim(B_n) = dim(D_n) = 2$.
c) We have $dim(X_n) = 3$ for every positive integer $n \geq 2$.

Proof. a) Fig. 5.9 represents two vertices a and b having equal distances to all other vertices of Y_n implying that at least one of them must be included in any metric basis of Y_n. In this way, by choosing only one vertex of degree two in each induced square of Y_n (in 2^n ways), these sets of

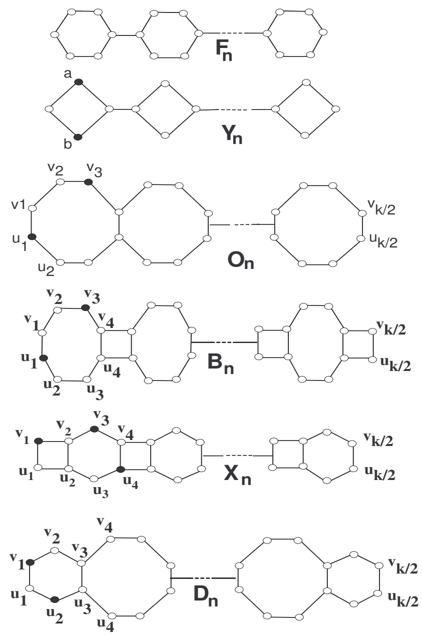

Figure 5.9: Some induced subgraphs of nanotubes

vertices form a metric basis for Y_n. These vertices can be arranged in $n!$ ways and the result follows.
b) In [11], it was proved that $dim(G) = 1$ if and only if G is a path. It follows that $dim(O_n) \geq 2$. In Fig. 5.9 the vertices lying on the upper half and lower half of the induced C_8 of O_n are represented by v_i and u_i respectively, where $1 \leq i \leq \frac{k}{2}$. Now to show that $dim(G) \leq 2$, we will show that the set $W = \{u_1, v_3\}$ resolves $V(O_n)$. For this, we give representation of the vertices in $V(O_n)\backslash W$ with respect to W.
$code(v_1|W) = (1, 2)$, $code(v_2|W) = (2, 1)$, $code(u_2|W) = (1, 4)$, $code(u_3|W) = (2, 3)$.
Also

$$code(v_i|W) = (i, i-3), \quad \text{for} \quad 4 \leq i \leq \frac{k}{2}$$

and

$$code(u_i|W) = (i-1, i-2), \quad \text{for} \quad 4 \leq i \leq \frac{k}{2}.$$

It follows that $dim(O_n) = 2$. The result for $dim(B_n) = 2$ can be followed on the same lines.
In Fig. 5.9 the vertices lying on the upper half and lower half of the C_8 or C_6 of D_n are represented by v_j and u_j respectively, where $1 \leq j \leq \frac{k}{2}$. Now to show that $dim(D_n) \leq 2$, we will show that the set $W_1 = \{v_1, u_2\}$ resolves $V(D_n)$. So for this we give representation of the vertices in $V(D_n) \setminus W_1$ with respect to W_1, i.e., $code(v_2 \mid W_1) = (1, 3)$ and $code(u_1 \mid W_1) = (1, 1)$.
Also

$$code(v_j|W_1) = (j-1, j-1), \quad \text{for} \quad 3 \leq j \leq \frac{k}{2}$$

and

$$code(u_j|W_1) = (j, j-2), \quad \text{for} \quad 3 \leq j \leq \frac{k}{2}.$$

It follows that $dim(D_n) = 2$

c) To prove that $dim(X_n) = 3$, we will first show that $dim(X_n) \neq 2$. For this we have the following three cases:

• If we take any two distinct vertices u and v from any induced C_4 of X_n, then there exist two neighbouring vertices of either u or v which have same representation.

• If we take two distinct vertices u and v from any induced C_6 such that $u, v \notin V(C_4) \cap V(C_6)$, then again there are two neighbouring vertices of u or v with same representation.

• If we select the vertex u from any induced C_6 and the vertex v from any induced C_4 of X_n and the vertices u, v are not in same induced C_4 or induced C_6, then in this case there are two vertices having same distance from u and/or v, so have the same representation. It follows that $dim(X_n) \geq 3$.

Now to show that $dim(X_n) \leq 3$, we will give representation of the vertices in $V(X_n) \setminus W$ with respect to $W = \{v_1, v_3, u_4\}$.

We have $code(v_2|W) = (1,1,3)$, $code(v_4|W) = (3,1,1)$, $code(u_1|W) = (1,3,3)$, $code(u_2|W) = (2,2,2)$, $r(v_2|W) = (3,3,1)$.

Also

$$code(v_i|W) = (i-1, i-3, i-3), \quad \text{for} \quad 5 \leq i \leq \frac{k}{2}$$

and

$$code(u_i|W) = (i, i-2, i-4), \quad \text{for} \quad 5 \leq i \leq \frac{k}{2}.$$

It follow that $dim(X_n) = 3$, which completes the proof. □

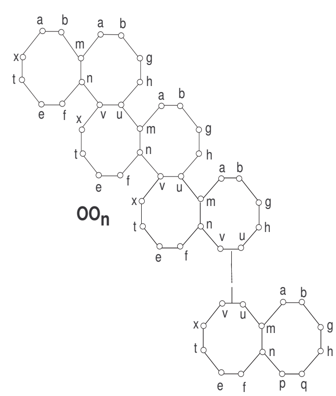

Figure 5.10: The zig-zag chain of 8-cycles

Theorem 5.3.4. *For every integer $n \geq 1$, we have $dim(OO_n) = \lceil \frac{n+2}{2} \rceil$.*

Proof. Suppose that the induced octagons of OO_n have been numbered by $1, 2, 3, ..., n$ from left to right. Fig. 5.10 shows that $dim(OO_1) = dim(OO_2) = 2$ having metric basis $\{a, e\}$ and $dim(OO_3) = 3$ being metric basis $\{b, f, h\}$. Now we will show that $dim(OO_n) = \lceil \frac{n+2}{2} \rceil$ for $n \geq 4$. This can be proved by double inequality.

For the upper bound, we construct a resolving set having $\lceil \frac{n+2}{2} \rceil$ vertices (in both cases, n odd or even). We can construct a resolving set A of cardinality $\lceil \frac{n+2}{2} \rceil$ as follows:

• When n is odd, we choose vertices f and h from octagons numbered 1 and 2 respectively, and a vertex of type v from each of the octagons numbered by $5, 7, 9, ..., n-1$ from left to right and also a vertex of type g from nth octagon.

• When n is even, we choose vertices f and h from octagons numbered by 1 and 2 respectively, and a vertex of type v from each of the octagons numbered by $4, 6, 8, ..., n-2$ and of type p of nth octagon from left to right.

It can be easily verified that any two vertices of OO_n having equal distances to any other vertex of A can be distinguished by the other vertices of A, which shows that $dim(OO_n) \leq \lceil \frac{n+2}{2} \rceil$ for $n \geq 1$.

For reverse inequality, we first prove that every resolving set must contain at least one vertex from each group of two consecutive octagons in the chain of octagons representing OO_n. The vertices v and m in third octagon from left to right can be distinguished only by the vertices of type a, m and n in second octagon and by x, t and f in third octagon and all vertices in fourth octagon except m of OO_n. Similar argument can be established for the pair of vertices x, n and t, f. It follows that any two consecutive octagons of OO_n must contain at least one vertex in any metric basis. This implies that if we assign to i-th octagon the binary variable O_i having value 1 if it has a vertex in a resolving set B of OO_n and 0 otherwise, we can write:

$$O_1 = 1;$$

$$\begin{aligned} O_2 + O_3 &\geq 1 \\ O_3 + O_4 &\geq 1 \\ O_4 + O_5 &\geq 1 \\ &\cdots\cdots \\ &\cdots\cdots \\ &\cdots\cdots \\ O_{n-1} + O_n &\geq 2 \end{aligned}$$

By summing up these inequalities, we get:

$$S = O_2 + 2O_3 + 2O_4 + \cdots + 2O_{n-1} + O_n \geq n.$$

Hence we get

$$2 \mid B \mid = 2\sum_{i=1}^{n} O_i = S + O_2 + O_n + 2 \geq n + 2.$$

Which implies that $\mid B \mid \geq \lceil \frac{n+2}{2} \rceil$, hence $dim(OO_n) \geq \lceil \frac{n+2}{2} \rceil$, which completes the proof. □

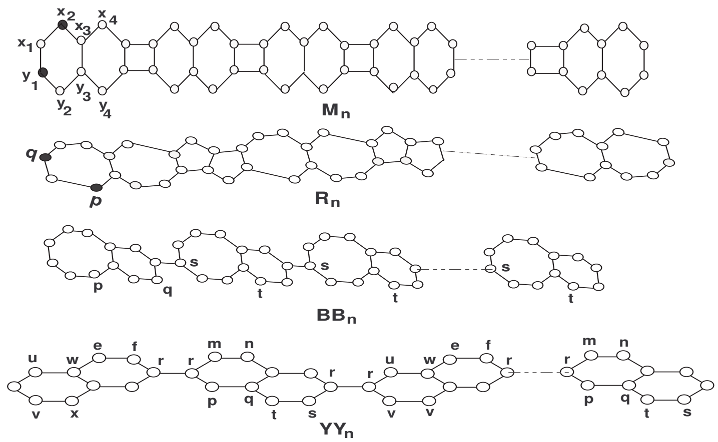

Figure 5.11: Some induced subgraphs of nanotubes

Fig. 5.11 represents some induced subgraphs of 2-dimensional lattices of H-Naphtalenic and VC_5C_7 nanotubes. The induced subgraph M_n is defined an edge-connected union of n pairs of induced 6-cycles and $n-1$ induced 4-cycles alternatively, R_n is defined as an edge-connected union of n pairs of induced 7-cycles and n pairs of induced 4-cycles alternatively, BB_n is defined as an edge-connected union of n induced pairs of C_8 and C_6, and an edge alternatively, while the induced subgraph YY_n is defined as an edge-connected union of n pairs of zig-zag C_6 and an edge alternatively. Note that all the defined induced subgraphs are of order l. In the next theorem, we determine the metric dimension of induced subgraphs defined above.

Theorem 5.3.5. *a) For every positive integer $n \geq 2$, we have: $dim(M_n) = dim(R_n) = 2$;*
b) For every positive integer $n \geq 2$ we have: $dim(BB_n) = dim(YY_n) = n$.

Proof. a) From [11], it implies that $dim(M_n) \geq 2$. In Fig. 5.11, the vertices of M_n lying on the upper and lower half of the induced 6-cycles and 4-cycles are labeled by x_i and y_i, respectively, where $1 \leq i \leq \frac{l}{2}$. To show that $dim(M_n) \leq 2$, we prove that $W_1 = \{x_2, y_1\}$ resolves $V(M_n)$. For this, we give representations of the vertices of $V(M_n) \setminus W_1$ with respect to W_1.
$code(x_1|W_1) = (1, 1)$, $code(y_2|W_1) = (3, 1)$.
Also

$$code(x_i|W_1) = (i-2, i), \text{ for } 3 \leq i \leq \tfrac{l}{2},$$

and

$$code(y_i|W_1) = (i-1, i-1), \text{ for } 3 \leq i \leq \tfrac{l}{2}.$$

This implies that $dim(M_n) = 2$. The proof of $dim(R_n) = 2$ follows on the same lines and is therefore omitted.

b) It can be seen that there are n induced pairs containing octagons and hexagons in BB_n. Suppose that these octagonal-hexagonal pairs of BB_n have been numbered by $1, 2, \cdots, n$ from left to right. Fig. 5.11 depicts that the vertices p and q of BB_n can only be distinguished by the vertices of the pair numbered by 1 in BB_n and the vertex of type s of the remaining pairs and have equal distance to all other vertices of BB_n if p or q do not belong to the metric basis of BB_n. This implies that at least one of them must be included in any metric basis of BB_n.

We can construct a metric basis of BB_n by taking only one vertex of type t from each pair numbered by $2, 3, \cdots, n$ of the induced subgraph BB_n and the result follows.

There are n pairs of zig-zag hexagons in the sequence of hexagons of the graph YY_n. The vertices u and v have equal distances to all vertices of YY_n different from w, x, e and f of first pairs of zig-zag hexagons, and the vertices w and x may be distinguished by the vertices u, v, e or f of YY_n, the situation is similar for all other pairs of hexagons. If u and v do not belong to basis of YY_n, it follows that at least one vertex from the set $\{u, v, w, x, e, f\}$ of YY_n must belong to any metric basis of YY_n.

On the other hand, by choosing exactly one vertex of degree two in each of pair of zig-zag hexagons of YY_n, these sets of vertices form a metric basis for YY_n and the result follows. □

Chapter 6

Concluding Remarks and Open Problems

In chapter 3, We studied the metric dimension of certain wheel related graphs, namely m-level wheel, an infinite class of convex polytopes and antiweb-gear graphs denoted by $W_{n,m}$, $\mathbb{Q}_n$ and AWJ_{2n}, respectively. We proved that these infinite classes of wheel related graphs have unbounded metric dimension. Moreover, we extended this study to infinite classes of convex polytopes $\mathbb{Q}_n^m$, $\mathbb{D}_n$ and $\mathbb{B}_n$ generated by wheel related graphs. We proved that these infinite classes of convex polytopes generated by wheel related graphs also have unbounded metric dimension.

In chapter 4, we have proved that the generalized Petersen networks $P(n,3)$ is a family of regular networks having constant metric dimension and only 4 vertices appropriately chosen suffices to resolve all the vertices of the generalized Petersen networks $P(n,3)$ when $n \equiv 2,3,4,5$ $(mod\ 6)$. Thus we conclude that each network in the family of generalized Petersen networks $P(n,3)$ is a network with constant metric dimension. This completes the study of metric dimension for generalized Petersen networks $P(n,3)$. However, to determine a precise formula for the whole class of generalized Petersen networks still remains a challenging for the researchers.

In chapter 5, we have studied that the metric dimension and partition dimension of 2-dimensional lattices of HAC_5C_7, HC_5C_7, $HAC_5C_6C_7$, H-Naphtalenic, VC_5C_7, V-Phenylenic, $TUC_4C_8(R)$ and $TUC_4C_8(S)$ nanotubes generated by the tiling of the plane. We prove that metric dimension of these infinite nanotubes is not finite but their partition dimension is finite and evaluated, implying that these nano-structures are among the graphs having discrepancies between their metric dimension and partition dimension. It has also been proved that there exist some induced subgraphs of 2-dimensional lattices of these nano-structures which have unbounded metric dimension while others have constant metric dimension. It seems that all 2-dimensional lattices of infinite nanotubes have discrepancy their metric dimension and partition dimension.
The future work in this direction is to characterise the graph families having discrepancies between their metric dimension and partition dimension and characterize the graph families having constant

metric dimension, bounded metric dimension and unbounded metric dimension. So, the reader is invited to work on the following open problems:

1. Characterize the convex polytopes with respect to the nature of their metric dimension.
2. Is it the case that the metric dimension of convex polytope $\mathbb{Q}_n^m$ is given by the following formula

$$dim(\mathbb{Q}_n^m) = \begin{cases} \lfloor \frac{3n}{5} \rfloor & n \equiv 0(mod\, 5), \\ \\ \lfloor \frac{3n-3}{5} \rfloor & ; otherwise \end{cases}$$

   for all $n \geq 7$ and $m \geq 4$?
3. Is it the case that all the 2-dimensional lattices of infinite nanotubes have discrepancies between their metric dimension and partition dimension?
4. Determine a precise formula for the whole class of generalized Petersen networks $P(n, m)$.

Bibliography

[1] S. Ahmad, M. A. Chaudhry, I. Javaid, M. Salman, On the metric dimension generalized Petersen graphs, *Questions Math.*, 36(2013), $421-435$.

[2] M. Bača, On face antimagic labelings of plane grahs, *JCMCC*, $53(2005), 33-38$.

[3] M. Bača, Special face numbering of plane quartic grahs, *Ars Combin.*, $57(2000), 285-292$.

[4] M. Bača, M. Miller, Antimagic face labelings of convex polytopes based on biprisms, *JCMCC*, $36(2001), 229-236$.

[5] M. Bača, M. Miller, Valuation of a certain classes of convex polytopes, *JCMCC*, $43(2002), 207-218$.

[6] F. Buckley, F. Harary, Distances in graphs, Addison-Wesley Publishing Company, 1990.

[7] P. S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hung.*, $46(1)(2003), 9-15$.

[8] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of some families of graphs, *Electronic Notes in Disc. Math.*, 22(2005), $129-133$.

[9] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of cartesian product of graphs, *SIAM J. Disc. Math.*, $2(21), (2007), 423-441$.

[10] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, On the metric dimension of infinite graphs, *Disc. Appl. Math.*, 160(2012), $2618-2626$.

[11] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and metric dimension of a graph, *Disc. Appl. Math.*, 105(2000), $99-113$.

[12] G. Chartrand, C. Poisson, P. Zhang, Resolvability and upper dimension of a graph, *J. Comput. Math. Appl.*, $39(2000), 19-28$.

[13] G. Chartrand, E. Salehi, P. Zhang, On the partition dimension of a graph, *Congress. Numer.*, 131(1998), $55-66$.

[14] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, *Aequationes Math.*, 59(2000), $45-54$.

[15] G. Chartrand, P. Zhang, Introduction to graph theory, *McGraw-Hill*, 2005.

[16] G. Chartrand, P. Zhang, The theory and applications of resolvability in graphs, *Congress. Numer.*, 160(2003), $47-68$.

[17] R. Diestel, Graph Theory, second edition. Springer-Verlag, 2000.

[18] M. Fehr, S. Gosselin, O. R. Oellermann, The metric dimension of Cayley digraphs, *Disc. Math.*, 306(2006), $31-41$.

[19] M. Fehr, S. Gosselin, O. R. Oellermann, The partition dimension of Cayley digraphs, *Aequationes Math.*, 71(2006), $1-18$.

[20] C. Grigorious, P. Manuel, M. Miller, B. Rajan, S. Stephen, On the metric dimension of circulant and Harary graphs, *Appl. Math. Comput.*, 248(2014), $47-54$.

[21] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.*, 2(1976), $191-195$.

[22] F. Harary, Status and contrastatus, *Sociometry*, 22(1959), $23-43$.

[23] M. Imran, Some results on discrepancies between metric dimension and partition dimension of a graph, *Sci. Int. (Lahore)*, 25(2013), $175-179$.

[24] M. Imran, A. Ahmad, S. A. Bokhary, A. Seminičovà-Feňovčikovà, On classes of regular graphs with constant metric dimension, *Acta Math. Sci.*, 33(2013), $187-206$.

[25] M. Imran, A. Q. Baig, An infinite class of convex polytopes with contant metric dimension, *JCMCC*, 81(2012), $135-143$.

[26] M. Imran, A. Q. Baig, A. Ahmad, Families of plane graphs with constant metric dimension, *Utilitas Math.*, 88(2012), $43-57$.

[27] M. Imran, A. Q. Baig, M. K. Shafiq, A. Seminičovà-Feňovčikovà, Classes of convex polytopes with constant metric dimension, *Utilitas Math.*, 90(2013), $85-99$.

[28] M. Imran, A. Q. Baig, M. K. Shafiq, I. Tomescu, On metric dimension of generalized Petersen graphs $P(n,3)$, *Ars Combin.*, 117(2014), $113-130$.

[29] M. Imran, F. Bashir, A. Q. Baig, S. A. Bokhary, A. Riyasat, I. Tomescu, On metric dimension of flower graphs $f_{n\times m}$ and convex polytopes, *Utilitas Math.,* $92(2013), 389-409$.

[30] M. Imran, S. A. Bokhary, A. Q. Baig, On families of convex polytopes with constant metric dimension, *Comput. Math. Appl.*, 60(2010), 2629 − 2638.

[31] M. Imran, S. A. Bokhary, A. Q. Baig, On families of rotationally symmetric plane graphs with constant metric dimension, *Southeast Asian Bull. Math.*, 36(2012), 663 − 675.

[32] M. Imran, H. M. A. Siddiqui, Computing the metric dimension of convex polytopes generated by wheel related graphs, *Period. Math. Hung.*, Accepted, in press.

[33] M. Imran, H. M. A. Siddiqui, Computing the metric dimension of a class of generalized Petersen networks, submitted.

[34] I. Javaid, On metric and partition dimension of some wheel related graphs, PhD Thesis, GC University Lahore, Pakistan, 2007.

[35] I. Javaid, S. Ahmad, M. N. Azhar, On the metric dimension generalized Petersen graphs, *Ars Combin.*, 3(2012), 171 − 182.

[36] I. Javaid, M. T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, *Utilitas Math.*, 75(2008), 21 − 33.

[37] M. A. Johnson, Structure-activity maps for visualizing the graph variable arizing in drug design, *J. Biophrm. Stat.*, 3(1993), 203 − 236.

[38] S. Kuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Disc. Appl. Math.*, 70(1996), 217 − 229.

[39] K. Liu, N. Abu-Ghazaleh, Virtual coordinate backtracking for void traversal in geometric routing, *LCNS*, 4104(2006), 46 − 59.

[40] R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing*, 25(1984), 113 − 121.

[41] R. Naeem, M. Imran , On resolvability and exchange property in antiweb-wheels, *Utilitas Math.*, in press.

[42] S. Naz, M. Salman, U. Ali, I. Javaid, On the metric dimension generalized Petersen graphs P(n,4), *Acta Math. Sin. Eng. Ser.*, 30(7), (2014), 1145 − 1160.

[43] O. R. Oellermann, J. Peters-Fransen, The metric dimension of cartesian products of graphs, *Utilitas Math.*, 69(2006), 33 − 42.

[44] S. Ohring, S. K. Das, Folded Petersen cube networks: new competitors for the hypercubes, *IEEE Trans. Parallel Distb. Sys.*, 7(2), (1996), 151 − 168.

[45] M. Perc, J. Gómez-Gardeñes, A. Szolnoki, L. M. Floría, Y. Moreno, Evolutionary dynamics of group interactions on structured populations: A review, *J. R. Soc.* Interface 10, 2012.0997(2013).

[46] M. Perc, A. Szolnoki, Coevolutionary games-A mini review, *BioSystems*, 99(2010), 109 – 125.

[47] C. Poisson, P. Zhang, The metric dimension of unicyclic graphs, *J. Combin. Math. Combin. Comput.*, 40(2002), 17 – 32.

[48] J. A. Rodríguez-Velázquez, I. G. Yero, H. Fernau, On the partition dimension of unicyclic graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, 57(105), 4(2014), 381 – 391.

[49] J. A. Rodríguez-Velázquez, I. G. Yero, D. Kuziak, The partition dimension of corona product graphs, *Ars Combin.*, to appear.

[50] V. Saenpholphat, P. Zhang, Connected partition dimension of graphs, *Discuss. Math. Graph Theory*, 22(2002), 305 – 323).

[51] H. M. A. Siddiqui, M. Imran, Computing metric and partition dimension of 2-dimensional lattices of certain nanotubes, *J. Comput. Theor. Nanosci.*, 11(2014), 2419 – 2423.

[52] H. M. A. Siddiqui, M. Imran, Computing the metric dimension of wheel related graphs, *Appl. Math. Comput.*, 242(2014), 624 – 632.

[53] H. M. A. Siddiqui, M. Imran, Computation of metric dimension and partition dimension of nanotubes, *J. Comput. Theor. Nanosci.*, 12(2015), 199 – 203.

[54] H. M. A. Siddiqui, M. Imran, Computing the metric and partition dimension of H-Naphtalenic and VC_5C_7 nanotubes, *J. Optoelectron. Adv. Mater.*, 17(2015), 790 – 794.

[55] P. J. Slater, Leaves of trees, *Congress. Numer.*, 14(1975), 549 – 559.

[56] P. J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, 22(1988), 445 – 455.

[57] G. Sudhakara, A. R. H. Kumar, Graphs with metric dimension two a characterization, *World Acad. Sci. Eng. Tech.*,3(2009), 12 – 27.

[58] I. Tomescu, Discrepancies between metric dimension and partition dimension of a connected graph, *Disc. Math.*, 22(308), (2008), 5026 – 5031.

[59] I. Tomescu, M. Imran, On metric and partition dimensions of some infinite regular graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, 52(100), 4(2009), 461 – 472.

[60] I. Tomescu, I. Javaid, On the metric dimension of Jahangir graph, *Bull. Math. Soc. Sci. Math. Roumanie*, 50(98), 4(2007), 371 – 376.

[61] I. Tomescu, I. Javaid, Slamin, On the partition dimension and connected partition dimension of wheels, *Ars Combin.*, 84(2007), 311 − 317.

[62] M. E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, *J. Combin. Theory*, 6(1969), 152 − 164.

[63] I. G. Yero, M. Jakovac, D. Kuziak, A. Taranenko, The partition dimension of strong product graphs and Cartesian product graphs, *Disc. Math.*, 331(2014), 43 − 52.

Work in Progress

1. Sakander Hayat, Hafiz Muhammad Afzal Siddiqui, "On topological indices of diamond-like networks", (Submitted).

2. Hafiz Muhammad Afzal Siddiqui, Muhammad Imran, "Resolvability in boron nanosheets", (Submitted)

3. Sakander Hayat, Hafiz Muhammad Afzal Siddiqui, Imran Nadeem, Hani Shakir, "Valency based and frustration related topological descriptors of single-walled titania nanotubes", (Submitted).

4. Sakander Hayat, Hafiz Muhammad Afzal Siddiqui, "On bipartite edge frustration of carbon and boron nanotubes", (Submitted).

5. Sakander Hayat, Hafiz Muhammad Afzal Siddiqui, "On energy and Estrada index of boron triangular nanotubes", (Submitted).