Linearization of higher order ordinary differential equations

by

Hina Munir Dutt

Supervised by **Prof. Asghar Qadir**

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H-12, Islamabad, Pakistan

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Dedicated to

My Daughters
Anaya and Noor

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Abstract

Lie's method for converting a scalar second order ordinary differential equation (ODE) to linear ODE by point transformations was already extended to third and fourth order scalar ODEs by point and contact transformations and to the systems of second order ODEs. The point symmetry group classification of linear n^{th} order scalar and second order systems of m ODEs was provided. Till recently no work on the linearization and classification has been done for higher order systems of ODEs and scalar ODEs linearizable via point, contact and higher order derivative transformations. In this work, we use Meleshko's algorithm for reducing fourth order autonomous ODEs to second and third order linearizable ODEs and then applying the Ibragimov and Meleshko linearization test for the obtained ODEs. This method can be applied to solve those nonlinear ODEs that are not linearizable by point and contact transformations.

Complex-linearization of a class of systems of second order ODEs had been studied with complex symmetry analysis. Linearization of this class had been achieved earlier by complex method, however, linearization conditions and the most general linearizable form of such systems have not been derived yet. It is shown that the general linearizable form of the complex-linearizable systems of two second order ODEs is (at most) quadratically semilinear in the first order derivatives of the dependent variables. Linearization conditions for such systems are derived in terms of coefficients of the system and their derivatives. Further, complex methods are employed to obtain the complex-linearizable form of $2-$dimensional systems of third order ODEs. This complex-linearizable form leads to a linearizable class of these systems of ODEs. The most general linearizable form and

linearization conditions for such class of 2−dimensional systems of third order ODEs are derived with complex-linearization.

A canonical form for 2−dimensional linear systems of third order ODEs is obtained by splitting the complex, scalar, third order, linear ODE. This canonical form is used for the symmetry group classification of 2−dimensional linear systems of third order ODEs. Five equivalence classes of such systems with Lie algebras of dimensions 8, 9, 10, 11, and 13 are proved to exist.

Contact and higher order derivative symmetries of scalar ODEs are related with the point symmetries of the reduced systems. Two new types of transformations that build up these relations and equivalence classes of scalar third and fourth order ODEs linearizable via these transformations are obtained. Four equivalence classes of these equations are seen to exist.

Contents

Publications from the thesis

- H.M. Dutt and A. Qadir, Reduction of fourth order ordinary differential equations to second and third order Lie linearizable forms, *Commun. Nonlinear Sci. Numer. Simulat.* **19** (2014) 2653.
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Chapter 1

Introduction

Most of the important governing equations in physics and mathematical models in engineering sciences, biology, economics, chemistry etc. are given in terms of nonlinear differential equations (DEs). In earlier days, it was easier to approximate a situation by one that led to easily solved equations. Despite enormous advances of approximation methods for solving nonlinear DEs, the key features of the phenomenon being modelled may be lost in the approximation. In other words we lose the essential part of a problem under consideration along with the nonessentials. An important and difficult aspect of nonlinear DEs is to solve these equations exactly so that their significance is not lost. One of the methods of solving them is the transformation of a given DE into another equation of a standard form. Since linear DEs are the simplest, we would like to transform nonlinear DEs into the linear form by transforming their independent and dependent variables which is called *linearization* in symmetry analysis. (However, in the literature the approximation of nonlinear equations by linear ones is also called linearization). Linearization does not only simplify a nonlinear DE but also allows us to construct its exact solutions. Therefore, linearization can play a significant role in the theory of DEs. These equations involve the behaviour of certain unknown functions, called dependent variables, at given values of independent variables and their derivatives. A DE is of order n if the highest derivative involved in it is of order n. If dependent variables are functions of a single independent variable, these are called ODEs. If these equations involve more than one independent variable, they are called *partial differential equations* (PDEs). The number of arbitrary constants that appear in the general solutions of linear ODEs is same as the order of the ODEs. To get the exact solution of a linear ODE we have to put as many initial conditions as the order of the ODE. This allows us to classify ODEs on the basis of initial conditions to be satisfied by an ODE. We also use other generalizations of linearization for the purpose of the classification of ODEs. This classification of ODEs is the main motivation of this thesis.

The concept of groups is related to the invariance or symmetry of objects under some action. In 1826 Abel [1] proved that irreducible quintic equations are not solvable by means of radicals. Independently, Galois [24], in 1830, used symmetries to prove that quintic and higher order polynomial equations are not solvable by means of radicals. This leads to the concept of *groups*. Groups used by Galois were finite and are now called *Galois groups*. Lie (1880) wanted to use similar methods to try to solve and classify DEs. Polynomial equations have (at most) as many solutions as their order, while DEs have infinitely many solutions. To deal with DEs, Lie needed to have not only infinitely continuous but differentiable groups [38, 40, 42, 45]. These groups are now called *Lie groups*. Galois groups deal with the symmetries of algebraic equations while symmetries of DEs are discussed in terms of Lie group theory. Unlike Galois groups, Lie groups deal with infinitely many transformations and depend on continuously varying parameters. The crucial idea of Lie group theory is to employ infinitesimals instead of finite transformations. Lie showed how invariance under the action of an infinitesimal generator of a symmetry can be used to reduce the number of independent variables in a PDE, or to reduce the order of an ODE. Thus, if there are enough symmetry generators any DE can be reduced to quadratures. He also classified the symmetries required for solving DEs and hence the DEs solvable.

Lie transformed DEs to the linear form by transforming their independent and dependent variables invertibly. Such transformations are called *point transformations* and the transformed DEs are called *linearized*. DEs that can be transformed to the linear form are called *linearizable*. Lie [38] proved that a nonlinear scalar second order ODE can be mapped to a linear ODE via an invertible point transformation if and only if it has eight Lie point symmetries. He used the fact that all linear scalar second order ODEs can be mapped to the free particle equation via an invertible point transformation. Hence all linearizable scalar second order ODEs can be put into one equivalence class. He proved that any nonlinear scalar second order ODE is linearizable if it is semilinear and at most cubic in the first derivative. Further, coefficients of the linearizable ODE must satisfy four conditions that involve these coefficients and two auxiliary functions and their first order derivatives. In 1894 Tressè [70, 71] reduced these four conditions to two by eliminating these auxiliary functions.

Later developments have extended in many directions, including transformations of derivatives as well (contact transformations). In 1940 Chern [14, 15] extended the linearization programme to scalar third order ODEs by using contact transformations. He obtained conditions for scalar third order ODEs to be linearizable to the equations $u''' = 0$ and $u''' + u = 0$. In 1990 Mahomed and Leach [48] showed that all linearizable scalar ODEs of order n ($n \geq 3$) can be put into three equivalence classes with $n + 1$, $n + 2$ and $n + 4$ Lie point symmetries. Grebot [26, 27], in 1996, used the restricted class of point transformations to address the linearization of scalar third order ODEs. Explicit linearization criteria for scalar third order ODEs were obtained by Neut and Petitot [59] in 2002 and independently by Ibragimov and Meleshko (IM) [29,30] in 2005. They determined the linearizability criteria and procedure for the construction of linearizing transformations for scalar third order ODEs by following Lie's original procedure [38]. The linearization problem for scalar fourth order ODEs gets more complicated and was tackled by Ibragimov, Meleshko and Suksern (IMS) [31, 69] in 2008. They used Lie's approach to obtain the explicit linearizability criteria for scalar fourth order ODEs. In 2006 Meleshko [56] provided a simple algorithm to reduce autonomous third order scalar ODEs to second order ODEs saisfying Lie linearizability criteria.

All developments mentioned above are for scalar ODEs. The extension of the classification to systems was achieved by Goringe and Leach [25] in 1988 for a limited class and generalized for all classes by Wafo Soh and Mahomed [73] in 2000. Gorringe and

Leach treated the case of systems of two linear second order ODEs with constant coefficients in the complex domain and proved that they can have 7, 8 or 15 point symmetries. Wafo Soh and Mahomed [73] extended to variable coefficients and proved that the Lie algebra for linearizable systems of two second order ODEs can only be $5-$, $6-$, $7-$, $8-$, or $15-$dimensional. In 2001 they generalized it further to $n-$dimensional systems of second order ODEs by using the group classification and found that the number of classes increases by one with each increased dimension [74]. The number of generators in the minimal case is $n+3$ and for the highest sub-maximal case is $2n+4$. There are five equivalence classes of linearizable systems of two second order ODEs and not only the one found by Lie for scalar ODEs. Algebraic linearization criteria for systems of second order ODEs via invertible point transformations were provided by Wafo Soh and Mahomed [74], Bagderina [11] and Ayub et al. [10].

A connection between the symmetries of a system of second order ODEs, that could be regarded as geodesics on a manifold and of the underlying manifold (isometries) was found by Aminova and Aminov [7] in 2000 and independently by Feroze, Mahomed and Qadir [22] in 2006. For the connection to make sense one needs to be able to determine when the system of ODEs can correspond to a system of geodesic equations (which are defined in section 2.5) and then construct the manifold on which the geodesics live. A mathematica code for this purpose was obtained by Fredericks et al. [23]. It was noted by Aminova and Aminov that the requirement for the system to be linearizable is that the curvature of the manifold be zero. The linearizability criteria for a system of second order quadratically semilinear ODEs were obtained by Mahomed and Qadir [50] in 2007. They considered a system of second order ODEs of geodesic type (quadratically semilinear in the first derivative with no linear terms) and independently proved that the conditions for the system to be linearizable are to treat coefficients of the system of ODEs as if they are Christoffel symbols and require that the curvature tensor constructed from them be zero. In other words the geodesics are straight lines if the manifold is flat. Using the projection procedure of Aminova and Aminov [8], the system of n second order ODEs of geodesic type can be reduced to a system of $(n-1)$ second order quadratically semilinear

ODEs [52]. Following this procedure of projection, the linearization criteria for a system of two cubically semilinear ODEs were derived by Mahomed and Qadir [52]. When this procedure was applied to a system of two dimensions, Lie linearization conditions for scalar second order ODEs were obtained.

Another recent development was of *complex symmetry analysis* (CSA). It deals with those systems of DEs that come from the systems of complex DEs. Complex DEs are those in which dependent variables are complex functions of complex or real independent variables. Whereas Lie considered complex DEs, he did not use the analyticity of complex functions. The fact used recently was that the dependent variables must be analytic and satisfy Cauchy-Reimann (CR) equations [2, 4]. Complex ODEs give systems of two PDEs on splitting into real and imaginary parts. A complex function of a real variable yields two real functions with CR structure on both the variables. In this way we get a system of two ODEs. We can also ask for the linearizability of complex scalar ODEs to obtain linearizable PDEs/ODEs [5]. Using CSA it has been shown that we get three of the five linearizable classes of systems of two second order ODEs [66].

In Lie's programme of linearization, no definite statement is available for the cases when ODEs are not linearizable. This gap may be filled by another development called *conditional linearization*. Conditional linearization is a totally different direction for linearization given by Mahomed and Qadir [51] in 2008. Differentiating a linearizable scalar second order ODE and then requiring that the original equation holds, gives the conditionally linearizable third order ODE [51]. The result is that the new third order conditionally linearizable ODE may contain only two arbitrary constants in its solution. This is a new class of ODEs and is not contained in IM [29] and Neut and Petitot class [59]. The same procedure was applied to a system of ODEs [49] to get a system of two third order conditionally linearizable ODEs. This method was repeated with the third order conditionally linearizable ODE to get a fourth order ODE [53].

Linearization maps a nonlinear DE to a linear one by using an invertible transformation. The inverse transformation maps the solution of the linear DE to the solution of the nonlinear DE, thus allowing us to get an exact solution of the nonlinear DE. Not only this, but we also obtain the general solution of a nonlinear DE. The general solution of a linear DE contains arbitrary constants equal to its order. So initial conditions equal in number to the order of a linearizable DE must be given in order to find its exact solution. Our motivation for considering the linearization problem is that it does not only give the general solution of a nonlinear ODE but also classifies ODEs according to the number of initial conditions to be satisfied by an ODE. Using the power of linearization, we can put DEs into one equivalence class of solvable DEs.

The thesis is organized as follows. In the second chapter we review some results on the linearization of scalar and systems of ODEs. In the third chapter we use Meleshko's algorithm to reduce fourth order ODEs by one (or two) order(s) and then apply IM's (Lie's) linearization criteria to the reduced ODEs. In this way we get the solution of a nonlinear fourth order ODE by quadrature. We give complete criteria for scalar fourth order ODEs that are reducible to the lower order linearizable equations. Fourth order ODEs that are linearizable by Meleshko's method are not necessarily contained in IMS' class of linearizable ODEs. The fourth chapter is on complex linearization of $2-$dimensional systems of second order ODEs. We first obtain the linearizable form for a $2-$dimensional complex-linearizable system of second and third order ODEs. Further, linearization conditions for such systems are derived in terms of coefficients of the system and their derivatives. The most general linearizable form and the linearization criteria for a class of $2-$dimensional systems of third order ODEs are also derived by complex-linearization. In the fifth chapter we use complex methods for the classification of $2-$dimensional linear systems of third order ODEs. We first obtain the canonical form of $2-$dimensional linear systems of third order ODEs by using complex methods. This form provides the classification of $2-$dimensional linear systems of third order ODEs that corresponds to a scalar complex third order ODE. We prove that there are five equivalence classes of such equations from eight to thirteen dimensions, excluding twelve. In the sixth chapter we relate contact and higher order derivative symmetries of scalar ODEs with point symmetries of the reduced systems. We define new types of transformations that build up these relations and obtain equivalence classes of scalar ODEs linearizable via these transformations. We first obtain canonical forms of linear scalar third and fourth order ODEs, then perform the group classification for these equations. Four equivalence classes of linear scalar third and fourth order ODEs are seen to exist. In the last chapter, we conclude the work done in the proceeding chapters along with some future directions.

1.1 Lie groups of transformations and Lie algebras

In this section, we give basic definitions with examples and necessary tools to deal with ODEs by symmetry methods that will be used subsequently.

1.1.1 Lie groups of transformations

A set G, closed under a binary operation '$\cdot$' is called a *groupoid*. If it is associative i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $\forall\ a, b, c \in G$, then it is called a *semigroup*. It is called a *monoid* if it contains an identity i.e., $\exists!\ e \in G$ such that $a \cdot e = e \cdot a = a$, $\forall\ a \in G$. Note that we can have a left identity that is not a right identity and vice versa, e.g. if $\uparrow$ is the operation of raising to the power defined in N then 1 is the right identity that is not a left identity, $n \uparrow 1 = n$ but $1 \uparrow n = 1 \neq n$. If for a monoid we have the property that $\forall\ a \in G$, $\exists!$ $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$, then it is called a *group*. A group G is called an *abelian* group if the group operation is commutative i.e., $a \cdot b = b \cdot a$, $\forall\ a, b \in G$. If there are a finite number of elements in a group, then the group is called a *finite* group. The number of elements in a group is called the *order* of the group. A basic example of a finite group is the symmetric group S_n which is the group of permutations of n objects. The set of integers $\mathbb{Z}$ is an infinite group with '+' as the group operation. These are all examples of groups with a discrete number of elements. If we consider those groups whose number of elements are a continuum, such as the space $\mathbb{R}$, then we can talk about continuous groups. A group G is continuous if there is some notion of 'continuity' imposed on the elements of the group in the sense that a small change in a or b produces a correspondingly small change in $a \cdot b$. It is defined in a precise way as:

Definition 1.1.1. A *continuous group* is a topological space, whose elements form a group and the group operation and its inverse are continuous mappings.

Remember that a map or function is continuous from one topological space to another if the inverse image of every open set is open. The most common example of a continuous group is the set of all real numbers $\mathbb{R}$ with '+' as the group operation. The same set $\mathbb{R}$ is not a group under the operation of multiplication because $\mathbb{R}$ has an element 0 which has no multiplicative inverse. The space of extended real numbers, $\bar{\mathbb{R}}$, is not a group under multiplication. This is because $1/0$ is $\infty \in \bar{\mathbb{R}}$ and when we approach 0 from the left its inverse approaches $-\infty$ and when we approach 0 from the right its inverse approaches $+\infty$. As the inverse of every element of a group has to be unique, so $\bar{\mathbb{R}}$ fails to form a group, and hence, is not a continuous group.

Definition 1.1.2. If the elements of a differentiable manifold form a group and the group operation

$$m : G \times G \longrightarrow G, \quad m(a, b) = a.b, \quad a, b \in G,$$

and the inversion

$$i : G \longrightarrow G, \quad i(a) = a^{-1}, \quad a \in G$$

are smooth mappings then it is called a *Lie group* [60] .

The sets $\mathbb{R}$ and $\mathbb{C}$ with '+' as the group operation are examples of Lie groups. If we eliminate 0 from the set of real numbers and denote the set as $\mathbb{R}^*$, then $\mathbb{R}^*$ forms a continuous group under multiplication but not a Lie group. This is because the space $\mathbb{R}^*$ can be written as a union of two disjoint subsets $(-\infty, 0)$ and $(0, \infty)$ and hence, is disconnected. Unlike $\mathbb{R}^*$, $\mathbb{C}^*$ is a Lie group under multiplication because when we remove the origin, the space is still connected.

Definition 1.1.3. A transformation that involves a change of dependent and independent variables without involving other functions i.e.,

$$\bar{x} = \bar{x}(x, u), \quad \bar{u} = \bar{u}(x, u), \tag{1.1}$$

is called a *point transformation*.

Definition 1.1.4. Suppose the point transformation (1.1) depends continuously on (at least) one parameter a, (see e.g., [68]), i.e.

$$\bar{x} = \bar{x}(x, u; a), \quad \bar{u} = \bar{u}(x, u; a). \tag{1.2}$$

Further suppose that these transformations (1.2) are defined for each $(x, u) \in D \subset \mathbb{R}^2$ and the parameter a belongs to a continuous group $S \subset \mathbb{R}$ with a law of composition $\sigma(a, b)$. The set of transformations (1.2) forms a *one-parameter Lie group of transformations* G if:

(i) the closure law holds i.e., if

$$\bar{x} = \bar{x}(x, u; a), \quad \bar{u} = \bar{u}(x, u; a),$$

then

$$\bar{\bar{x}} = \bar{\bar{x}}(\bar{x}, \bar{u}; \bar{a}), \quad \bar{\bar{u}} = \bar{\bar{u}}(\bar{x}, \bar{u}; \bar{a});$$

(ii) the associative law holds;

(iii) the set of transformations (1.2) contains the identity transformation, i.e. $\exists !\ e \in S$ such that

$$\bar{x} = \bar{x}(x, u; e) = x, \quad \bar{u} = \bar{u}(x, u; e) = u;$$

(iv) the transformations (1.2) are invertible, i.e. $\forall\ a$ in S $\exists !\ a' \in S$ such that

$$\bar{a}(a, a') = e\ .$$

Examples of Lie groups of transformations

(1) Consider the 1−dimensional transformation

$$\bar{x} = ax, \tag{1.3}$$

where a is a non-zero real number.

$$\bar{\bar{x}} = b\bar{x} = bax. \tag{1.4}$$

By writing $\bar{\bar{x}} = cx$, we have the product element $c = ba$, so the composition of two transformations is described by an analytic function that yields another transformation of the form in (1.3). This operation is clearly associative, as well as abelian, since the composition of transformations corresponds to the multiplication of real numbers. The identity is determined from $\bar{x} = x$, which clearly corresponds to the transformation (1.3) with $a = 1$. The inverse of (1.3) is seen to correspond to the transformation with $\bar{a} = 1/a$, which explains the requirement that $a \neq 0$. Hence, the transformations defined in (1.3) form a one-parameter, abelian Lie group.

(2) The set of transformations

$$\bar{x} = a_1 x + a_2, \quad a_1 \neq 0, \tag{1.5}$$

forms a two-parameter, non-abelian group. The identity element corresponds to the transformation (1.5) with $a_1 = 1$ and $a_2 = 0$. The inverse of (1.5) is determined by taking $\bar{a_1} = 1/a_1$ and $\bar{a_2} = -a_2/a_1$. The composition of two transformations is given by the product rule $c_1 = b_1 a_1$ and $c_2 = b_2 + b_1 a_2$ where $\bar{\bar{x}} = b_1 \bar{x} + b_2$.

(3) *General linear groups*:

The set of transformations

$$\begin{aligned} \bar{x} &= a_1 x + a_2 u, \\ \bar{u} &= a_3 x + a_4 u, \end{aligned} \tag{1.6}$$

with $a_1 a_4 - a_2 a_3 \neq 0$ forms a general linear group in two dimensions. If x and u denote the components of a vector $\mathbf{r}$, the transformation (1.6) can be written in the matrix notation as

$$\bar{\mathbf{r}} = \mathbf{A}\mathbf{r},$$

where

$$\bar{\mathbf{r}} = \begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

The linear group in two dimensions is isomorphic to the group of 2×2 non-singular matrices with the matrix multiplication as the law of composition. Here the identity corresponds to

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the inverse is the matrix inverse $\mathbf{A}^{-1}$ and the product element is given by the matrix multiplication $\mathbf{C} = \mathbf{BA}$, where $\bar{\bar{\mathbf{r}}} = \mathbf{B}\bar{\mathbf{r}}$. The linear group in two dimensions is a four-parameter, non-abelian group and is denoted by $GL(2,\mathbb{R})$, where $\mathbb{R}$ signifies that the entries are real. The general linear group with complex entries is denoted by $GL(2,\mathbb{C})$. In n dimensions, these transformation groups are denoted by $GL(n,\mathbb{R})$ and, with complex entries, by $GL(n,\mathbb{C})$. The number of parameters for an $n-$dimensional linear group is n^2.

(4) *Special linear group*:

This group is obtained by restricting the determinant of the transformations in example (3) to unity. This restriction provides one functional relation between the n^2 parameters. Thus we have an $(n^2-1)-$parameter group. This group is denoted by $SL(n,\mathbb{R})$ for real entries and by $SL(n,\mathbb{C})$ for complex entries.

(5) *Orthogonal groups*:

We restrict the transformations in example (3) to be length invariant:

$$\bar{x}^2 + \bar{u}^2 = (a_1 x + a_2 u)^2 + (a_3 x + a_4 u)^2 = x^2 + u^2. \tag{1.7}$$

For the above equation to hold we must have

$$a_1^2 + a_3^2 = 1, \quad a_2^2 + a_4^2 = 1, \quad a_1 a_2 + a_3 a_4 = 0. \tag{1.8}$$

We have three conditions imposed on four parameters, leaving one free parameter. Thus, we have a one-parameter group. This group of transformations is called *orthogonal group* and is denoted by $O(2)$. Its elements are rotations and combinations of rotations and reflections. This group is abelian as the angle of resultant of two transformations is the sum of the angles of the individual transformations. Orthogonal group is isomorphic to the group of 2×2 orthogonal matrices with the matrix multiplication as the law of composition. From orthogonality, $(det\hat{O})^2 = 1$ which implies that $det\hat{O} = \pm 1$, where $\hat{O}$ is an orthogonal matrix. The subset of $O(2)$ for which $det\hat{O} = 1$ forms a group known as the *special orthogonal group* (or *unimodular orthogonal group*), $SO(2)$. Since $det\hat{O} = 1$ does not impose an additional condition on the parameters, we have one independent real parameter. Geometrically, $SO(2)$ can be regarded as a group of rotations about the $z-$axis and written as

$$\bar{x} = x\cos\theta - u\sin\theta,$$
$$\bar{u} = x\sin\theta + u\cos\theta, \quad (0 \leq \theta \leq 2\pi), \tag{1.9}$$

where θ is the angle of rotation about the $z-$axis.

The important thing here is that $O(2)$ is not a Lie group while its subgroup $SO(2)$ is a Lie group. For this, consider the orthogonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ that cannot be connected to the identity by a continuous transformation. Since a continuous group is, by definition, always connected to the identity, the orthogonal matrix is not in a continuous group. The determinant of this matrix is -1 and of the identity is $+1$. Since the product of determinants is the determinant of products, all matrices continuously connected to the identity will have determinant $+1$ and will thus lie in $SO(2)$. Hence $SO(2)$ forms a Lie group while $O(2)$ does not. Note that reflections do not form a continuous group of $O(2)$. Hence there is only one connected component of $O(2)$. Further, we restrict transformations of the general linear group to those which leave $\Sigma_{i=1}^{n} x_i^2$ invariant. The n^2 parameters are subjected to $n + n(n-1)/2$ conditions, leaving $n(n-1)/2$ free parameters. This group, which is not a Lie group, is denoted by $O(n)$.

If we consider transformations in example (3) to be complex, i.e. take x_i as complex variables and a_{ij} as complex coefficients, the number of free (real) parameters is $2n^2$.

1.1.2 Infinitesimal transformations

The one-parameter group of transformations (1.2) can be characterized as the motion in the $xu-$plane. The image of an arbitrary point (x_0, u_0) under the one-parameter group of transformations moves in the $xu-$plane, when the parameter a varies. In this way, we get different curves for different initial points. Each curve represents points that can be transformed into each other under the action of the group. These curves, called *orbits* of the groups, are completely characterized by the field of its tangent vectors $\mathbf{X}$. The idea of orbits can be concisely described by considering infinitesimal transformations defined below [28].

Definition 1.1.5. Consider the one-parameter Lie group of transformations (1.2). Let the functions $\bar{x}(x, u; a)$ and $\bar{u}(x, u; a)$ satisfy the initial conditions:

$$\bar{x} = x, \quad \bar{u} = u, \quad \text{at} \quad a = 0. \tag{1.10}$$

We expand (1.2) as a Taylor series in the parameter a in a neighborhood of $a = 0$. Invoking (1.10), we arrive at what is called the *infinitesimal transformation* of the group G

$$\begin{aligned} \bar{x}(x, u; a) &\approx x + a\xi(x, u), \\ \bar{u}(x, u; a) &\approx u + a\eta(x, u), \end{aligned} \tag{1.11}$$

where the functions $\xi(x, u)$ and $\eta(x, u)$ are defined by

$$\xi(x, u) = \frac{\partial \bar{x}}{\partial a} \mid_{a=0} \quad , \quad \eta(x, u) = \frac{\partial \bar{u}}{\partial a} \mid_{a=0} \quad . \tag{1.12}$$

Definition 1.1.6. The functions $\xi(x, u)$ and $\eta(x, u)$ defined by (1.12) are components of the operator

$$\mathbf{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}. \tag{1.13}$$

The operator $\mathbf{X}$, is called an *infinitesimal generator* of the transformation (1.2).

Repeated applications of $\mathbf{X}$ generate a finite transformation. Although finite transformations are complicated and nonlinear, their infinitesimal generators are always linear operators. So instead of considering a group as a whole we will consider an infinitesimal transformation around the identity.

Geometrically, integral curves of vector fields are group orbits i.e., we obtain the finite transformation (1.2) by integrating

$$\frac{\partial \bar{x}}{\partial a} = \xi(\bar{x}, \bar{u}), \quad \frac{\partial \bar{u}}{\partial a} = \eta(\bar{x}, \bar{u}), \tag{1.14}$$

with initial conditions (1.10). It is to be mentioned here that an infinitesimal generator uniquely determines the group orbits but orbits give the generator only up to a constant factor.

Examples

(1) Consider the following group

$$\begin{aligned} \bar{x} &= ax, \\ \bar{u} &= \frac{1}{a}u. \end{aligned} \tag{1.15}$$

The identity transformation has $a = 1$. The infinitesimal transformation is given by

$$\bar{x} \approx (1 + a)x = x + ax,$$

$$\bar{u} \approx (1 - a)u = u - au.$$

From this we find

$$\frac{\partial \bar{x}}{\partial a} \mid_{a=1} = x \ , \quad \frac{\partial \bar{u}}{\partial a} \mid_{a=1} = -u \ ,$$

so that the corresponding infinitesimal generator of the given group is

$$\mathbf{X} = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{1.16}$$

(2) Take the example of the rotation group (1.9). Here the identity transformation has $\theta = 0$. We have the corresponding infinitesimal transformations

$$\bar{x} \approx x - \theta u,$$

$$\bar{u} \approx u + \theta x,$$

so that

$$\frac{\partial \bar{x}}{\partial \theta}\,|_{\theta=0} = -u\ , \quad \frac{\partial \bar{u}}{\partial \theta}\,|_{\theta=0} = x$$

and the corresponding infinitesimal generator is

$$\mathbf{X} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}. \tag{1.17}$$

Infinitesimal transformations of multi-parameter Lie groups

Transformations (1.2) can depend on more than one parameter (see e.g. [68]) as we have seen in previous examples. In this section we define multi-parameter Lie groups of transformations and their infinitesimal generators.

Definition 1.1.7. The transformations

$$\bar{x} = \bar{x}(x, u; a_r), \quad \bar{u} = \bar{u}(x, u; a_r) \quad \text{where} \quad r = 1, 2, \ldots, N, \tag{1.18}$$

form an $N-$*parameter Lie group of transformations* if:

(i) all a_r are independent of each other;

(ii) transformations (1.18) contain the identity and are invertible;

(iii) the law of composition holds; and

(iv) the law of association holds.

The $N-$parameter Lie group of transformations is denoted by G_N.

Definition 1.1.8. We can associate an infinitesimal generator $\mathbf{X}_r$ to each parameter a_r by

$$\mathbf{X}_r = \xi_r\frac{\partial}{\partial x} + \eta_r\frac{\partial}{\partial u}, \tag{1.19}$$

with

$$\xi_r(x, u) = \frac{\partial \bar{x}}{\partial a_r}, \quad \eta_r(x, u) = \frac{\partial \bar{u}}{\partial a_r},$$

with all parameters $a_s = 0,\ s = 1, 2, \ldots, N$.

Each parameter of an $N-$parameter Lie group of transformations leads to an infinitesimal generator. So there are N infinitesimal generators of an $N-$parameter Lie group of transformations.

Examples

(1) Consider the example of a group

$$\bar{x} = a_1 x + a_2.$$

The identity transformation has parameters $a_1 = 1$ and $a_2 = 0$. The infinitesimal transformations are

$$\bar{x} \approx (1 + a_1)x + a_2,$$

with the infinitesimal generators

$$\mathbf{X}_1 = x\frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x}.$$

(2) Consider the following 3$-$parameter Lie group of transformations

$$\begin{aligned} \bar{x} &= x\cos a_1 - u\sin a_1 + a_2, \\ \bar{u} &= x\sin a_1 + u\cos a_1 + a_3. \end{aligned} \tag{1.20}$$

The infinitesimal generators of the above group of transformations are

$$\mathbf{X}_1 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \tag{1.21}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial x}, \tag{1.22}$$

$$\mathbf{X}_3 = \frac{\partial}{\partial u}, \tag{1.23}$$

where (1.21) corresponds to the rotation in the $xu-$plane, while (1.22) and (1.23) are translations along the x and u axes, respectively. This group is called the *Euclidean group* or *group of rigid motions* in $\mathbb{R}^2$.

1.1.3 Lie algebras

An $N-$parameter Lie group of transformations is determined by N infinitesimal generators $\mathbf{X}_r$. These infinitesimal generators define an $N-$dimensional linear vector space. This linear vector space has an additional structure, called the commutators which is defined below [68].

Consider an $N-$parameter Lie groups of transformations (1.18) with infinitesimal generators (1.19).

Definition 1.1.9. The *commutator* of two generators $\mathbf{X}_r$ and $\mathbf{X}_s$ is defined by

$$[\mathbf{X}_r, \mathbf{X}_s] = \mathbf{X}_r\mathbf{X}_s - \mathbf{X}_s\mathbf{X}_r = (\mathbf{X}_r\xi_s - \mathbf{X}_s\xi_r)\frac{\partial}{\partial x} + \ldots. \tag{1.24}$$

From the above definition it follows that

$$[\mathbf{X}_r, \mathbf{X}_s] = -[\mathbf{X}_s, \mathbf{X}_r]. \tag{1.25}$$

Also the commutators (1.24) are bilinear, i.e.

$$[c_1\mathbf{X}_r + c_2\mathbf{X}_s, \mathbf{X}_t] = c_1[\mathbf{X}_r, \mathbf{X}_t] + c_2[\mathbf{X}_s, \mathbf{X}_t], \tag{1.26}$$

$$[\mathbf{X}_t, c_1\mathbf{X}_r + c_2\mathbf{X}_s] = c_1[\mathbf{X}_t, \mathbf{X}_r] + c_2[\mathbf{X}_t, \mathbf{X}_s]. \tag{1.27}$$

Definition 1.1.10. The *Jacobi identity* for the commutators (1.24), defined by

$$[\mathbf{X}_r, [\mathbf{X}_s, \mathbf{X}_t]] + [\mathbf{X}_s, [\mathbf{X}_t, \mathbf{X}_r]] + [\mathbf{X}_t, [\mathbf{X}_r, \mathbf{X}_s]] = 0, \tag{1.28}$$

always holds.

Theorem 1.1.11. *The commutator of any two infinitesimal generators of an $N-$parameter Lie group of transformations is again an infinitesimal generator of the same Lie group of transformations. More generally we can write it using the Einstein summation convention as*

$$[\boldsymbol{X}_r, \boldsymbol{X}_s] = C^t_{rs}\boldsymbol{X}_t, \quad \textit{with} \quad r, s, t = 1, 2, \ldots, n, \tag{1.29}$$

where C^t_{rs}, defined by (1.29)*, are called* structure constants.

Structure constants possess the following properties:

(i) Structure constants are antisymmetric in the lower indices, i.e.

$$C^t_{rs} = -C^t_{sr}.$$

(ii) Because of the Jacobi identity (1.28), structure constants satisfy *Lie identity*

$$C^t_{rs}C^n_{oq} + C^t_{sq}C^n_{or} + C^t_{qr}C^n_{os} = 0.$$

Definition 1.1.12. A *Lie algebra* is a vector space, L of generators (1.19) which are closed with respect to the commutator relations satisfying (1.25), (1.26), (1.27) and (1.28).

A Lie algebra is denoted by the same letter L. The dimension of a Lie algebra is the dimension of the vector space L. An $N-$dimensional Lie algebra is denoted by the symbol L_N. If we are given N linearly independent operators (1.19), their linear span is a Lie algebra L_N provided the relation (1.29) holds. It is convenient to use the relations (1.29) in the form of a *table of commutators* of the basis (1.19).

Example

Similitude group in $\mathbb{R}^2$ is a four-parameter Lie group of transformations that consists of uniform scaling and rigid motions in $\mathbb{R}^2$:

$$\begin{aligned} \bar{x} &= e^{a_4}(x\cos a_1 - u\sin a_1) + a_2, \\ \bar{u} &= e^{a_4}(x\sin a_1 + u\cos a_1) + a_3. \end{aligned} \tag{1.30}$$

The infinitesimal generators of the given group are

$$\mathbf{X}_1 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = \frac{\partial}{\partial u}, \quad \mathbf{X}_4 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{1.31}$$

The commutator table of the above infinitesimal generators is

	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	0	$-\mathbf{X}_3$	$\mathbf{X}_2$	0
$\mathbf{X}_2$	$\mathbf{X}_3$	0	0	$\mathbf{X}_2$
$\mathbf{X}_3$	$-\mathbf{X}_2$	0	0	$\mathbf{X}_3$
$\mathbf{X}_4$	0	$-\mathbf{X}_2$	$-\mathbf{X}_3$	0

The above table is antisymmetric with its diagonal elements all zero. The structure constants are easily read off from the table. This shows that the algebra of these generators is closed with respect to the commutation satisfying the relations (1.25), (1.26), (1.27) and (1.28). Hence, the generators (1.31) form a basis of a Lie algebra L_4.

1.2 Lie symmetry analysis of ODEs

An n^{th} order ODE

$$F(x, u; u', u'', \ldots, u^{(n)}) = 0, \tag{1.32}$$

can be viewed geometrically as a surface in an $(n+2)-$dimensional space whose coordinates are given by the independent variable, the dependent variable and their derivatives up to the order n. So the solutions of ODEs are particular curves on this surface. From this point of view, a symmetry transformation represents the motion that moves solution curves into solution curves. More precisely, a symmetry is a one-parameter group of transformations that acts on an $(n+2)-$dimensional space and maps solutions to solutions.

In this section, we show how to find infinitesimal symmetry generators admitted by an n^{th} order scalar and system of ODEs. For this purpose, we first have to prolong or extend generators up to the n^{th} order.

1.2.1 Extension of transformations and their generators

To apply a point transformation (1.1) or (1.2) to (1.32) we have to transform derivatives $u^{(n)}$ i.e., to extend the point transformation to derivatives (see e.g. [68]). This is done by defining

$$\begin{aligned}
\bar{u}' &= \frac{d\bar{u}(x,u;a)}{d\bar{x}(x,u;a)} = \frac{u'(\partial\bar{u}/\partial u) + (\partial\bar{u}/\partial x)}{u'(\partial\bar{x}/\partial u) + (\partial\bar{x}/\partial x)} = \bar{u}'(x,u,u';a), \\
\bar{u}'' &= \frac{d\bar{u}'(x,u,u';a)}{d\bar{x}(x,u;a)} = \bar{u}''(x,u,u',u'';a), \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\bar{u}^{(n)} &= \frac{d\bar{u}^{(n-1)}(x,u,u',\ldots,u^{(n-1)};a)}{d\bar{x}(x,u;a)} = \bar{u}^{(n)}(x,u,u',\ldots,u^{(n)};a),
\end{aligned} \tag{1.33}$$

which are the extended or prolonged transformations. To obtain the extension of an infinitesimal generator $\mathbf{X}$, we write

$$\begin{aligned}
\bar{x} &= x + a\xi(x,u) + \cdots = x + a\mathbf{X}x + \cdots\ , \\
\bar{u} &= u + a\eta(x,u) + \cdots = u + a\mathbf{X}u + \cdots\ , \\
\bar{u'} &= u' + a\eta^{(1)}(x,u,u') + \cdots = u' + a\mathbf{X}u' + \cdots\ , \\
&\vdots \\
\bar{u}^{(n)} &= u^{(n)} + a\eta^{(n)}(x,u,u',\cdots,u^{(n)}) + \cdots = u^{(n)} + a\mathbf{X}u^{(n)} + \cdots\ ,
\end{aligned} \tag{1.34}$$

where $\eta^{(1)},\dots,\eta^{(n)}$ are defined by

$$\eta^{(1)} = \frac{\partial \bar{u}'}{\partial a},\dots,\eta^{(n)} = \frac{\partial \bar{u}^{(n)}}{\partial a}, \quad \text{at } a = 0. \tag{1.35}$$

The expressions (1.33), on account of (1.34), take the form

$$\begin{aligned}
\bar{u}' &= u' + a\eta^{(1)} + \dots = \frac{d\bar{u}}{d\bar{x}} = \frac{du + ad\eta + \dots}{dx + ad\xi + \dots} \\
&= \frac{u' + a(d\eta/dx) + \dots}{1 + a(d\xi/dx) + \dots} = u' + a(\frac{d\eta}{dx} - u'\frac{d\xi}{x}) + \dots, \\
&\vdots \\
\bar{u}^{(n)} &= u^{(n)} + a\eta^{(n)} + \dots = \frac{d\bar{u}^{(n-1)}}{d\bar{x}} \\
&= u^{(n)} + a(\frac{d\eta^{(n-1)}}{dx} - u^{(n)}\frac{d\xi}{dx}) + \dots,
\end{aligned}$$

which on comparison with (1.34) yields

$$\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - u^{(n)}\frac{d\xi}{dx}, \quad n \geq 2 \tag{1.36}$$

and which in turn, determines the components of the extended generator $\mathbf{X}$ of order n. From (1.36) it is clear that the $\eta^{(n)}$ is not the n^{th} derivative of η.

We can summarize the above result as follows.

Definition 1.2.1. If

$$\mathbf{X} = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u},$$

is an infinitesimal generator of a point transformation, then its extension or prolongation up to the n^{th} order, denoted by $\mathbf{X}^{(n)}$, is given by

$$\mathbf{X}^{(n)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \eta^{(1)} \frac{\partial}{\partial u'} + \ldots + \eta^{(n)} \frac{\partial}{\partial u^{(n)}}, \tag{1.37}$$

where $\eta^{(n)}$ are defined by (1.36).

By using the implicit function theorem, the n^{th} order ODE (1.32) can be locally written as

$$u^{(n)}(x) = f(x, u; u', u'', \ldots, u^{(n-1)}). \tag{1.38}$$

Definition 1.2.2. The n^{th} order ODE (1.38) admits the one-parameter group of transformations (1.2) if and only if the n^{th} extension of its generator leaves the solution curve invariant.

In other words we say that the form of the ODE (1.38) remains invariant under the point transformation (1.1) or (1.2). This implies

$$\bar{u}^{(n)}(\bar{x}) = f(\bar{x}, \bar{u}; \bar{u}', \ldots, \bar{u}^{(n-1)}). \tag{1.39}$$

If the n^{th} order ODE (1.38) admits the one-parameter group of transformations (1.2) then an infinitesimal generator of the group is called an *infinitesimal symmetry generator* or *symmetry* of the ODE (1.38).

Theorem 1.2.3. *An n^{th} order ODE* (1.38) *admits an extended infinitesimal generator* (1.37) *of order n if and only if*

$$\boldsymbol{X}^{(n)}[u^{(n)}(x) - f(x, u; u', \ldots, u^{(n-1)})] = 0,$$

or

$$\eta^{(n)}(x, u; u', \ldots, u^{(n)}) = \boldsymbol{X}^{(n-1)} f(x, u; u', \ldots, u^{(n-1)}), \tag{1.40}$$

with

$$u^{(n)}(x) - f(x, u; u', \ldots, u^{(n-1)}) = 0. \tag{1.41}$$

Definition 1.2.4. Equation (1.40) with $\eta^{(n)}$ given by (1.36) are called the *symmetry conditions* for the n^{th} order ODE (1.38).

Expressions for $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$ and $\eta^{(4)}$

The third chapter of this thesis contains linearization of scalar fourth order ODEs. So we present here the derivation of the expressions for $\eta^{(i)}$ up to $i = 4$.

Equation (1.36) with $n = 1$ is

$$\eta^{(1)} = \frac{d\eta(x,u)}{dx} - u'\frac{d\xi(x,u)}{dx}, \tag{1.42}$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + \dots .$$

Applying the operator $\frac{d}{dx}$ in (1.42), we obtain

$$\eta^{(1)} = \eta_{,x} + u'(\eta_{,u} - \xi_{,x}) - u'^2\xi_{,u}. \tag{1.43}$$

Proceeding in the same way, we have

$$\eta^{(2)} = \eta_{,xx} + u'(2\eta_{,xu} - \xi_{,xx}) + u'^2(\eta_{,uu} - 2\xi_{,xu}) - u'^3\xi_{,uu} + u''(\eta_{,u} - 2\xi_{,x} - 3u'\xi_{,u}), \tag{1.44}$$

$$\begin{aligned}\eta^{(3)} = \eta_{,xxx} &+ (3\eta_{,xxu} - \xi_{,xxx})u' + 3(\eta_{,xuu} - \xi_{,xxu})u'^2 + (\eta_{,uuu} - 3\xi_{,xuu})u'^3 - \xi_{,uuu}u'^4 \\ &+3(\eta_{,xu} - \xi_{,xx})u'' + 3(\eta_{,uu} - 3\xi_{,xu})u'u'' - 6\xi_{,uu}u'^2u'' - 3\xi_{,u}u''^2 \\ &+(\eta_{,u} - 3\xi_{,x})u''' - 4\xi_{,u}u'u''',\end{aligned} \tag{1.45}$$

$$\begin{aligned}\eta^{(4)} = \eta_{,xxxx} &+ u'(4\eta_{,xxxu} - \xi_{,xxxx}) + u'^2(6\eta_{,xxuu} - 4)\xi_{,xxxu} + u'^3(4\eta_{,xuuu} - 6\xi_{,xxuu}) \\ &+u'^4(\eta_{,uuuu} - 4\xi_{,xuuu}) + u'^5(-\xi_{,uuuu}) + u''(6\eta_{,xxu} - 4\xi_{,xxx} + 12u'\eta_{,xuu} \\ &-18u'\xi_{,xxu} + 6u'^2\eta_{,uuu} - 24u'^2\xi_{,xuu} - 10u'^3\xi_{,uuu}) + u''^2(3\eta_{,uu} - 12\xi_{,xu} \\ &-15u'\xi_{,uu}) + u'''(4\eta_{,xu} - 6\xi_{,xx} + 4u'\eta_{,uu} - 16u'\xi_{,xu} - 10u'^2\xi_{,uu} \\ &-10u''\xi_{,u}) + u^{(iv)}(\eta_{,u} - 4\xi_{,x} - 5u'\xi_{,u}).\end{aligned} \tag{1.46}$$

Example

Consider the rotation group (1.9) with the infinitesimal generator given by (1.17). Substituting the values of ξ and η into the expressions (1.43) and (1.44) yields

$$\eta^{(1)} = 1 + u'^2, \quad \eta^{(2)} = 3u'u'',$$

so the prolonged generator for the rotation group (1.9) is

$$\mathbf{X}^{(2)} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u'^2)\frac{\partial}{\partial u'} + 3u'u''\frac{\partial}{\partial u''}.$$

1.2.2 Lie point symmetries of scalar ODEs

Here, we will show how to find the Lie point symmetry generators of scalar ODEs. For the first order ODE

$$u' = f(x,u),$$

the symmetry condition (1.40) is

$$\eta^{(1)} = \mathbf{X}f = (\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u})f = \xi f_{,x} + \eta f_{,u}\ ,$$

or

$$\xi f_{,x} + \xi_{,x} f + \xi_{,u} f^2 = \eta_{,x} + \eta_{,u} f - \eta f_{,u},$$

where the function $f(x,u)$ is always given. The above PDE always has a solution for the functions $\xi(x,u)$ and $\eta(x,u)$. So a first order ODE always has an infinite number of symmetries.

For the second order ODE

$$u'' = f(x,u;u'), \tag{1.47}$$

the symmetry condition (1.40) reads

$$\eta^{(2)} = \mathbf{X}^{(1)}f = (\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta'\frac{\partial}{\partial u'})f.$$

Invoking the expressions (1.43) and (1.44) for $\eta^{(1)}$ and $\eta^{(2)}$, we obtain

$$f(\eta_{,u} - 2\xi_{,x} - 3u'\xi_{,u}) - f_{,x}\xi - f_{,u}\eta + fu'[\eta_{,x} + u'(\eta_{,u} - \xi_{,x}) - u'^2\xi_{,u}] + \eta_{,xx}$$
$$+u'(2\eta_{,xu} - \xi_{,xx}) + u'^2(\eta_{,uu} - 2\xi_{,xu}) - u'^3\xi_{,uu} = 0.$$

From the above DE we have to determine ξ and η. Since ξ and η do not depend on u', so the above DE splits into several equations, called *determining PDEs*, for ξ and η.

These determining PDEs are then solved for ξ and η to obtain the point symmetries of the equation (1.47). Similarly by following this procedure we can find the point symmetries of any ODE of order $n > 2$ after calculating $\eta^{(i)},\ i = 1, 2, \ldots n$. Although the procedure becomes lengthy for higher order ODEs but the determining PDEs for ξ and η are always linear.

For linear ODEs of order n we have the following result [68].

Theorem 1.2.5. *An n^{th} order linear (systems of) ODE(s) admits at least an $n-$parameter Lie group of point symmetries.*

1.2.3 Lie point symmetry conditions for systems of ODEs

Suppose we have a system of k ODEs of order n:

$$\mathbf{u}^{(n)} = \mathbf{f}(x, \mathbf{u}; \mathbf{u}', \ldots, \mathbf{u}^{(n-1)}),$$

where $\mathbf{f} = (f_1, f_2, \ldots, f_k),\ \mathbf{u} = (u_1, u_2, \ldots, u_k),\ \mathbf{u}' = (u_1', u_2', \ldots, u_k')$ and so on. To find Lie point symmetry conditions for the above system of ODEs we first define the extended transformation and extended infinitesimal transformations for k dependent variables [12].

Consider a one-parameter Lie group of point transformations with $\mathbf{u} = (u_1, u_2, \ldots, u_k)$ as dependent variables and x as independent variable:

$$\begin{aligned}\bar{x} &= \bar{x}(x, \mathbf{u}; a),\\ \bar{\mathbf{u}} &= \bar{\mathbf{u}}(x, \mathbf{u}; a).\end{aligned}$$

The extended transformations of the above transformations up to the order n are given by

$$\begin{aligned}\bar{\mathbf{u}}' &= \frac{d\bar{\mathbf{u}}(x, \mathbf{u}; a)}{d\bar{x}(x, \mathbf{u}; a)} = \bar{\mathbf{u}}'(x, \mathbf{u}, \mathbf{u}'; a),\\ \bar{\mathbf{u}}'' &= \frac{d\bar{\mathbf{u}}'(x, \mathbf{u}, \mathbf{u}'; a)}{d\bar{x}(x, \mathbf{u}; a)} = \bar{\mathbf{u}}''(x, \mathbf{u}, \mathbf{u}', \mathbf{u}''; a),\\ &\vdots \qquad\qquad\qquad \vdots\\ \bar{\mathbf{u}}^{(n)} &= \frac{d\bar{\mathbf{u}}^{(n-1)}(x, \mathbf{u}, \mathbf{u}', \ldots, \mathbf{u}^{(n-1)}; a)}{d\bar{x}(x, \mathbf{u}; a)} = \bar{\mathbf{u}}^{(n)}(x, \mathbf{u}, \mathbf{u}', \ldots, \mathbf{u}^{(n)}; a),\end{aligned} \tag{1.48}$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + u_i' \frac{\partial}{\partial u_i} + u_i'' \frac{\partial}{\partial u_i'} + \ldots, \quad i = 1, 2, \ldots, k.$$

The infinitesimal transformations are

$$\begin{aligned}
\bar{x} &= x + a\xi(x, \mathbf{u}) + O(a^2),\\
\bar{\mathbf{u}} &= \mathbf{u} + a\boldsymbol{\eta}(x, \mathbf{u}) + O(a^2),\\
\bar{\mathbf{u}}' &= \mathbf{u}' + a\boldsymbol{\eta}^{(1)}(x, \mathbf{u}, \mathbf{u}') + O(a^2),\\
&\vdots\\
\bar{\mathbf{u}}^{(n)} &= \mathbf{u}^{(n)} + a\boldsymbol{\eta}^{(n)}(x, \mathbf{u}, \mathbf{u}', \ldots, \mathbf{u}^{(n)}) + O(a^2),
\end{aligned}$$

with the extended infinitesimals, for $n \geq 2$, given by

$$\eta_i^{(n)} = \frac{d\eta_i^{(n-1)}}{dx} - u_i^{(n)} \frac{d\xi}{x}, \quad i = 1, 2, \ldots, k \tag{1.49}$$

and

$$\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_k), \;\; \boldsymbol{\eta}^{(1)} = (\eta_1^{(1)}, \eta_2^{(1)}, \ldots, \eta_k^{(1)}), \text{ etc.}$$

Here the extended infinitesimal generator is

$$\mathbf{X}^{(n)} = \xi \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial u_i} + \eta_i^{(1)} \frac{\partial}{\partial u_i'} + \ldots + \eta_i^{(n)} \frac{\partial}{\partial u_i^{(n)}}, \tag{1.50}$$

where the summation varies from $i = 1$ to k.

Expressions for $\eta_i^{(1)}$, $\eta_i^{(2)}$ and $\eta_i^{(3)}$

Since the present work mainly revolves around the linearization and the group classification of systems of two third order ODEs, so here we derive expressions for $\eta_i^{(1)}$, $\eta_i^{(2)}$ and $\eta_i^{(3)}$, with y and z as dependent variables.

Equation (1.49) for the dependent variable y with $n = 1$ is

$$\eta_1^{(1)} = \frac{d\eta(x, y, z)}{dx} - y' \frac{d\xi(x, y, z)}{dx}. \tag{1.51}$$

Applying the operator

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z} + y''\frac{\partial}{\partial y'} + z''\frac{\partial}{\partial z'} + y'''\frac{\partial}{\partial y''} + z'''\frac{\partial}{\partial z''},$$

we obtain

$$\eta_1^{(1)} = \eta_{,x} + y'(\eta_{1,y} - \xi_{,x}) + z'(\eta_{1,z} - y'\xi_{,z}) - y'^2\xi_{,y}. \tag{1.52}$$

Proceeding in the same way, we have

$$\begin{aligned} \eta_1^{(2)} = \eta_{1,xx} &+ y'(2\eta_{1,xy} - \xi_{xx}) + 2z'\eta_{1,xz} + y''(\eta_{1,y} - 2\xi_{,x} - 3y'\xi_{,y} - 2z'\xi_{,z}) \\ &+ z''(\eta_{1,z} - y'\xi_{,z}) + y'^2(\eta_{1,yy} - 2\xi_{,xy} - 2z'\xi_{,yz}) + 2y'z'(\eta_{1,yz} - \xi_{,xz}) \\ &+ z'^2(\eta_{1,zz} - y'\xi_{,zz}) - y'^3\xi_{,yy} \ , \end{aligned} \tag{1.53}$$

$$\begin{aligned} \eta_1^{(3)} = \eta_{1,xxx} &+ y'(3\eta_{1,xxy} - \xi_{,xxx}) + z'(3\eta_{1,xxz}) + y'^2(3\eta_{1,xyy} - 3\xi_{,xxy}) \\ &+ y'z'(6\eta_{1,xyz} - 3\xi_{,xxz}) + z'^2(3\eta_{1,xzz}) + y'^3(\eta_{1,yyy} - 3\xi_{,xyy}) \\ &+ y'^2z'(3\eta_{1,yyz} - 6\xi_{,xyz}) + y'z'^2(3\eta_{1,yzz} - 3\xi_{,xzz}) + z'^3(\eta_{1,zzz}) \\ &+ y'^4(-\xi_{,yyy}) + y'^3z'(-3\xi_{,yyz}) + y'^2z'^2(-3\xi_{,yzz}) + y'z'^3(-\xi_{,zzz}) \\ &+ y''(3\eta_{1,xy} - 3\xi_{,xx}) + z''(3\eta_{1,xz}) + y'y''(3\eta_{1,yy} - 9\xi_{,xy}) \\ &+ z'y''(3\eta_{1,yz} - 6\xi_{,xz}) + y'z''(3\eta_{1,yz} - 3\xi_{,xz}) + z'z''(3\eta_{1,zz}) \\ &+ y'^2y''(-6\xi_{,yy}) + y'z'y''(-9\xi_{,yz}) + z'^2y''(-3\xi_{,zz}) + y'^2z''(-3\xi_{,yz}) \\ &+ y'z'z''(-3\xi_{,zz}) + y''^2(-3\xi_{,y}) + y''z''(-3\xi_{,z}) + y'''(\eta_{1,y} - 3\xi_{,x}) \\ &+ z'''(\eta_{1,z}) + y'y'''(-4\xi_{,y}) + z'y'''(-3\xi_{,z}) + y'z'''(-\xi_{,z}) \ . \end{aligned} \tag{1.54}$$

Similarly for the dependent variable z, we have the expressions:

$$\eta_2^{(1)} = \eta_{2,x} + z'(\eta_{2,z} - \xi_{,x}) + y'(\eta_{2,y} - z'\xi_{,y}) - z'^2\xi_{,z} \ , \tag{1.55}$$

$$\begin{aligned} \eta_2^{(2)} = \eta_{2,xx} &+ z'(2\eta_{1,xz} - \xi_{,xx}) + 2y'\eta_{2,xy} + z''(\eta_{2,z} - 2\xi_{,x} - 2y'\xi_{,y} - 3z'\xi_{,z}) \\ &+ y''(\eta_{2,y} - z'\xi_{,y}) + z'^2(\eta_{2,zz} - 2\xi_{,xz} - 2z'\xi_{,yz}) + 2y'z'(\eta_{2,yz} - \xi_{,xy}) \\ &+ y'^2(\eta_{2,zz} - z'\xi_{,yy}) - z'^3\xi_{,zz} \end{aligned} \tag{1.56}$$

and

$$\eta_2^{(3)} = \eta_{2,xxx} + y'(3\eta_{2,xxy}) + z'(3\eta_{2,xxz} - \xi_{,xxx}) + y'^2(3\eta_{2,xyy})$$
$$+y'z'(6\eta_{2,xyz} - 3\xi_{,xxy}) + z'^2(3\eta_{2,xzz} - 3\xi_{2,xxz}) + y'^3(\eta_{2,yyy})$$
$$+y'^2z'(3\eta_{2,yyz} - 3\xi_{,xyy}) + y'z'^2(3\eta_{2,yzz} - 6\xi_{,xyz})$$
$$+z'^3(\eta_{2,zzz} - 3\xi_{,xzz}) + y'^3z'(-\xi_{,yyy}) + y'^2z'^2(-3\xi_{,yyz})$$
$$+y'z'^3(-3\xi_{,yzz}) + z'^4(-\xi_{,zzz}) + y''(3\eta_{2,xy})$$
$$+z''(3\eta_{2,xz} - 3\xi_{,xx}) + y'y''(3\eta_{2,yy}) + z'y''(3\eta_{2,yz} - 3\xi_{,xy})$$
$$+y'z''(3\eta_{2,yz} - 6\xi_{,xy}) + z'z''(3\eta_{2,zz} - 9\xi_{,xz}) + y'z'y''(-3\xi_{,yy})$$
$$+z'^2y''(-3\xi_{,yz}) + y'^2z''(-3\xi_{,yy}) + y'z'z''(-9\xi_{,yz})$$
$$+z'^2z''(-6\xi_{,zz}) + y''z''(-3\xi_{,y}) + z''^2(-3\xi_{,z}) + y'''(\eta_{2,y})$$
$$+z'''(\eta_{2,z} - 3\xi_{,x}) + z'y'''(-\xi_{,y}) + y'z'''(-3\xi_{,y}) + z'z'''(-4\xi_{,z}) \ . \tag{1.57}$$

To derive the symmetry condition for a system of two third order ODEs

$$y''' = f_1(x, y, z; y', z', y'', z''),$$
$$z''' = f_2(x, y, z; y', z', y'', z''),$$

we suppose that the above system admits the symmetry generator

$$\mathbf{X}^{(3)} = \xi\frac{\partial}{\partial x} + \eta_1\frac{\partial}{\partial y} + \eta_2\frac{\partial}{\partial z} + \eta_1^{(1)}\frac{\partial}{\partial y'} + \eta_2^{(1)}\frac{\partial}{\partial z'} + \eta_1^{(2)}\frac{\partial}{\partial y''} + \eta_2^{(2)}\frac{\partial}{\partial z''}$$
$$+\eta_1^{(3)}\frac{\partial}{\partial y'''} + \eta_2^{(3)}\frac{\partial}{\partial z'''}.$$

The symmetry conditions read as

$$\mathbf{X}^{(3)}[y''' - f_1(x, y, z; y', z', y'', z'')] = 0,$$
$$\mathbf{X}^{(3)}[z''' - f_2(x, y, z; y', z', y'', z'')] = 0,$$

or

$$\eta_1^{(3)} = \mathbf{X}^{(2)}f_1(x, y, z; y', z', y'', z''),$$
$$\eta_2^{(3)} = \mathbf{X}^{(2)}f_2(x, y, z; y', z', y'', z'').$$

Applying the operator $\mathbf{X}^{(2)}$ on the above expressions, we get

$$\eta_1^{(3)} = \xi f_{1,x} + \eta_1 f_{1,y} + \eta_2 f_{1,z} + \eta_1^{(1)} f_{1,y'} + \eta_2^{(1)} f_{1,z'} + \eta_1^{(2)} f_{1,y''} + \eta_2^{(2)} f_{1,z''},$$
$$\eta_2^{(3)} = \xi f_{2,x} + \eta_1 f_{2,y} + \eta_2 f_{2,z} + \eta_1^{(1)} f_{2,y'} + \eta_2^{(1)} f_{2,z'} + \eta_1^{(2)} f_{2,y''} + \eta_2^{(2)} f_{2,z''}.$$

These are the symmetry conditions for the system of two third order ODEs with $\eta_i^{(1)}$, $\eta_i^{(2)}$ and $\eta_i^{(3)}$, $(i = 1, 2)$ given by the expressions (1.52)−(1.57).

1.2.4 Contact and Lie-B$\ddot{a}$cklund transformations and their infinitesimal generators

We will be using a generalization of contact and higher order symmetry generators, we first define the contact and Lie-B$\ddot{a}$cklund transformations and their infinitesimal generators. Some basic results related to these transformations [12] are also reviewed.

Definition 1.2.6. A transformation

$$\bar{x} = \varphi(x, u, p), \quad \bar{u} = \psi_1(x, u, p), \quad \bar{p} = \psi_2(x, u, p), \tag{1.58}$$

with $p = u'$ is called a *contact transformation* if it preserves the contact condition $du = pdx$ i.e.,

$$d\bar{u} = \bar{p} d\bar{x}. \tag{1.59}$$

The contact condition (1.59) can also be written as

$$\psi_2 = \frac{D_x \psi_1(x, u, p)}{D_x \varphi(x, u, p)}.$$

From the above equation we find that the functions φ, ψ_1 and ψ_2 are related by

$$\psi_{1,p} = \psi_2 \varphi_{,p}, \quad \psi_{1,x} + p\psi_{1,u} = (\varphi_{,x} + p\varphi_{,u})\psi_2. \tag{1.60}$$

We can write the above result in the form of the following theorem.

Theorem 1.2.7. *Equations* (1.58) *define a contact transformation if and only if* $\{\varphi, \psi_1, \psi_2\}$ *satisfies the relations* (1.60).

Example

The contact transformation

$$t = u', \quad s = -u + xu', \quad s' = x, \tag{1.61}$$

is known as the *Legendre transformation* [28]. Indeed, this transformation maps the DE of hyperbolas

$$u''' - \frac{3u''^2}{2u'} = 0,$$

to the linear equation

$$2ts''' + 3s'' = 0.$$

Definition 1.2.8. A one-parameter Lie group of contact transformations is defined by

$$\bar{x} = x + a\xi(x, u, u') + O(a^2), \tag{1.62}$$

$$\bar{u} = x + a\eta(x, u, u') + O(a^2), \tag{1.63}$$

$$\bar{p} = x + a\zeta(x, u, u') + O(a^2), \tag{1.64}$$

with the infinitesimal generator

$$\mathbf{X} = \xi(x, u, u')\frac{\partial}{\partial x} + \eta(x, u, u')\frac{\partial}{\partial u} + \zeta(x, u, u')\frac{\partial}{\partial u'},$$

provided the contact condition is preserved.

Theorem 1.2.9. *Equations* (1.62)−(1.64) *define a one-parameter Lie group of contact transformations if and only if* ξ *and* η *satisfy*

$$\frac{\partial \eta}{\partial u'} = \frac{\partial \xi}{\partial u'} u'.$$

Let the characteristic function $W = W(x, u, u')$ be defined by

$$W = \xi u' - \eta,$$

then the infinitesimal generator in terms of the characteristic function is given by

$$\xi = \frac{\partial W}{\partial u'},$$
$$\eta = u'\frac{\partial W}{\partial u'} - W,$$
$$\zeta = -\frac{\partial W}{\partial x} - u'\frac{\partial W}{\partial u}.$$

Definition 1.2.10. A transformation of the form

$$\bar{x} = \varphi(x, u, u', u'', \dots, u^{(m)}), \tag{1.65}$$

$$\bar{u} = \psi(x, u, u', u'', \dots, u^{(m)}), \tag{1.66}$$

$$\bar{u}_i = \psi_i(x, u, u', u'', \dots, u^{(m)}), \quad i = 1, 2, \dots m, \tag{1.67}$$

is called a *Lie-Bäcklund transformation* of order m if it preserves the contact conditions up to the order m:

$$u_i = \frac{du_{i-1}}{dt}, \quad i = 1, 2, \dots m. \tag{1.68}$$

Lie-Bäcklund transformations depend on independent and dependent variables and derivatives of the dependent variable up to some finite order. The following theorem shows that the contact transformations are special case of Lie-Bäcklund transformations.

Theorem 1.2.11. *Any Lie-Bäcklund transformation with an infinitesimal generator of the form*

$$\boldsymbol{X} = W(x, u, u^{(1)})\frac{\partial}{\partial u},$$

is equivalent to a contact transformation with the infinitesimal generator

$$\boldsymbol{X} = \xi(x, u, u^{(1)})\frac{\partial}{\partial x} + \eta(x, u, u^{(1)})\frac{\partial}{\partial u} + \zeta(x, u, u^{(1)})\frac{\partial}{\partial u'},$$

where $u' = u^{(1)}$.

Chapter 2

Linearization of ODEs

The study of nonlinear DEs was initially focussed on approximating them by linear DEs [58], but then the main aspects of the nonlinear DEs that are crucial for the phenomenon being modelled could be lost. Using approximations one was not clear how the essence of nonlinearity was lost. This method works only when the iteration converges. One faces the same problem with the numerical methods for solutions. To find the numerical solution of a nonlinear equation one first needs to know whether its solution exists or not. Proof of existence and convergence is furnished by functional analytic methods when they can be applied. Further, the rate of convergence is very important. If we have an adequate accuracy only after a million terms, the fact that it converges is of little help in obtaining the solution. So one needs to find the exact solution of nonlinear DEs. In the latter part of the 19^{th} century Lie developed the method of linearization to solve nonlinear DEs exactly. Linearization is an invertible mapping that converts a nonlinear DE to a linear one by the change of variables. Linearization criteria comprise the most general forms of DEs that could be the candidates of linearization and the sufficient conditions that ensure existence of invertible transformations from nonlinear to linear equations.

Suppose we have an n^{th} order nonlinear ODE and an invertible transformation that converts it into a linear ODE of the same order. As every linear n^{th} order ODE has n linearly independent solutions, by applying the inverse transformation to the solutions of the linear ODE, we obtain the n exact solutions of the given nonlinear ODE. The linearization scheme can be best displayed by Figure 2.1.

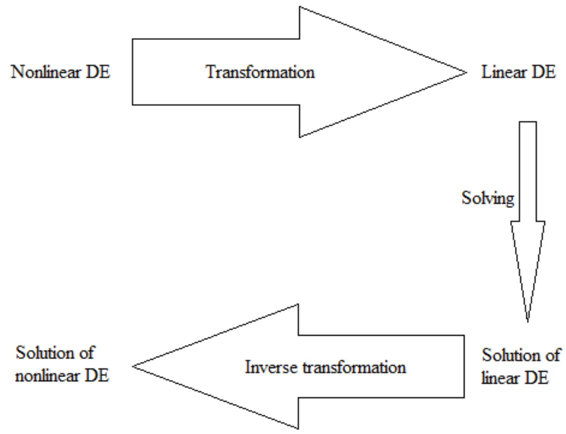

Figure 2.1: The linearization scheme. By setting all arbitrary constants but one in the linear superposition as zero, we can get the linearly independent solutions of the nonlinear ODE.

This chapter briefly reviews results on the linearization of scalar and systems of ODEs. We first review the original work of Lie and then more recent work on linearization is presented. The results on linearization based on geometry and complex analysis are given. New forms of linearization called by the authors "Meleshko linearization" and "conditional linearization", are reviewed.

2.1 Linearization of scalar ODEs

This section is devoted to the linearization of scalar ODEs by point and contact transformations. We give Lie's criteria for the scalar second order ODEs linearizable via point transformations. The necessary forms and the sufficient conditions for the linearization of scalar third and fourth order ODEs by point transformations are presented. Further, the most general forms of third and fourth order ODEs linearizable via contact transformations are provided, but first we give some well known results on the equivalence of scalar ODEs [46].

Definition 2.1.1. Two equations are said to be *equivalent* if there exists an invertible transformation which transforms one equation into the other. The problem of finding all equations which are equivalent to a given equation (called the target equation) is called the *equivalence problem.*

- **First order ODEs.** All first order scalar ODEs are equivalent to one another. In particular, an equation of the form $u' = f(x,u)$ can always be transformed to the simplest equation $s' = 0$, by a suitable point transformation

$$t = \varphi(x,u), \quad s = \psi(x,u). \tag{2.1}$$

- **Linear second order ODEs.** As proved by Lie [38], any linear, scalar, second order ODE can be transformed to the simplest equation $s'' = 0$, by the transformation (2.1). Hence all linearizable scalar second order ODEs belong to one equivalence class.

- **Linear ODEs of order $n \geq 3$.** A linear scalar ODE of order $n \geq 3$ need not be transformable into the simplest form. We now present a theorem due to Laguerre [32, 33] for linear scalar n^{th} order ODEs.

Theorem 2.1.2. *Any linear homogeneous n^{th} order scalar ODE*

$$u^{(n)} + \sum_{i=0}^{n-1} \bar{k}_i(x) u^{(i)} = 0 \ , \quad n \geq 3, \tag{2.2}$$

can be transformed by the point transformation (2.1) *to the equation*

$$s^{(n)} + \sum_{i=0}^{n-3} k_i(t) s^{(i)} = 0 \ . \tag{2.3}$$

Equation (2.3) *is called the Laguerre canonical form of the ODE* (2.2).

2.1.1 Linearization of scalar second order ODEs via point transformations

The linearization problem for DEs was first solved by Lie [38]. He found the most general linearizable form for scalar second order ODEs via point transformations and showed that any second order scalar ODE is linearizable if it is at most cubic in the first derivative with the coefficients of the nonlinear ODE satisfying four conditions. These conditions, (2.11), involve up to the second order partial derivatives of the coefficients and first order derivatives of two auxiliary functions. Tressé [70, 71] eliminated the auxiliary functions from these four conditions and reduced them to two (namely (2.12)). These conditions are in the form of DEs that need to be checked but not to be solved. Here, we give Lie's linearization criteria in detail.

To obtain the Lie linearizable form for scalar second order ODEs (see e.g. [28]), we assume that the ODE

$$u'' = f(x, u, u'), \tag{2.4}$$

comes from the simplest linear equation

$$s'' = 0, \tag{2.5}$$

by the point transformation (2.1). The derivatives s' and s'' are transformed as follows:

$$s' = \frac{D_x\psi}{D_x\varphi} = P(x, u, u')\ , \quad s'' = \frac{D_x P}{D_x\varphi},$$

where D_x is the total derivative operator with respect to x and is given by

$$D_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'}.$$

Equation (2.5), on using the above transformation, becomes

$$D_x(\varphi)D_x^2(\psi) - D_x(\psi)D_x^2(\varphi) = 0, \tag{2.6}$$

where

$$\begin{aligned} D_x\varphi &= \varphi_{,x} + u'\varphi_{,u}, \\ D_x^2\varphi &= \varphi_{,xx} + 2u'\varphi_{,xu} + u'^2\varphi_{,uu} + u''\varphi_{,u}, \end{aligned} \tag{2.7}$$

and similar expressions for $D_x\psi$ and $D_x^2\psi$. After inserting the above expressions, equation (2.6) takes the form

$$\begin{aligned} u'' &+ \frac{1}{\Delta}(\varphi_{,u}\psi_{,uu} - \psi_{,u}\varphi_{,uu})u'^3 + \frac{1}{\Delta}(\varphi_{,x}\psi_{,uu} + 2\varphi_{,u}\psi_{,xu} - \psi_{,x}\varphi_{,uu} - 2\psi_{,u}\varphi_{,xu})u'^2 \\ &+ \frac{1}{\Delta}(\varphi_{,u}\psi_{,xx} + 2\varphi_{,x}\psi_{,xu} - \psi_{,u}\varphi_{,xx} - 2\psi_{,x}\varphi_{,xu})u' + \frac{1}{\Delta}(\varphi_{,x}\psi_{,xx} - \psi_{,x}\varphi_{,xx}) = 0, \end{aligned} \tag{2.8}$$

where $\Delta = \varphi_{,x}\psi_{,u} - \psi_{,x}\varphi_{,u} \neq 0$, is the Jacobian of the transformation (2.1).
Writing

$$\begin{aligned} a &= \Delta^{-1}(\varphi_{,u}\psi_{,uu} - \psi_{,u}\varphi_{,uu}), \\ b &= \Delta^{-1}(\varphi_{,x}\psi_{,uu} + 2\varphi_{,u}\psi_{,xu} - \psi_{,x}\varphi_{,uu} - 2\psi_{,u}\varphi_{,xu}), \\ c &= \Delta^{-1}(\varphi_{,u}\psi_{,xx} + 2\varphi_{,x}\psi_{,xu} - \psi_{,u}\varphi_{,xx} - 2\psi_{,x}\varphi_{,xu}), \\ d &= \Delta^{-1}(\varphi_{,x}\psi_{,xx} - \psi_{,x}\varphi_{,xx}), \end{aligned} \tag{2.9}$$

equation (2.8) takes the form

$$u'' + a(x,u)u'^3 + b(x,u)u'^2 + c(x,u)u' + d(x,u) = 0. \tag{2.10}$$

Hence, we have the following theorem.

Theorem 2.1.3. *A scalar second order ODE* (2.4) *is linearizable by the point transformation* (2.1) *if it is at most cubic in* u' *i.e., has the form* (2.10).

Equation (2.10) has four arbitrary functions viz; $a,\ b,\ c$, and d while (2.8) involves two functions $\varphi(x,u)$ and $\psi(x,u)$. So coefficients of the equation (2.10) linearizable via point transformations must be restricted by two relations. These relations (namely (2.12)) are given in the following theorem.

Theorem 2.1.4. *The following statements are equivalent:*

(*i*) *a scalar second order ODE* (2.4) *is linearizable by the point transformation* (2.1)*;*

(*ii*) *equation* (2.4) *has an* 8*-dimensional Lie algebra;*

(*iii*) *equation* (2.4) *has the form* (2.10) *with the coefficients* $a,\ b,\ c,\ d$ *satisfying the following integrability conditions:*

$$\begin{aligned} F_{1,x} &= F_1^2 - dF_2 - cF_1 + d_{,u} + bd, \\ F_{1,u} &= -F_1F_2 + ad - \frac{1}{3}b_{,x} + \frac{2}{3}c_{,u}, \\ F_{2,x} &= F_1F_2 - ad - cF_1 - \frac{1}{3}c_{,u} + \frac{2}{3}b_{,x}, \\ F_{2,u} &= F_2^2 + bF_2 + aF_1 + a_{,x} - ac, \end{aligned} \tag{2.11}$$

where F_1 *and* F_2 *are auxiliary functions;*

(*iv*) *the coefficients* $a,\ b,\ c,\ d$ *satisfy the following set of constraints:*

$$\begin{aligned} &3a_{,xx} - 2b_{,xu} + c_{,uu} - 3a_{,x}c + 3a_{,u}d + 2b_{,x}b - 3c_{,x}a - c_{,u}b + 6d_{,u}a = 0, \\ &b_{,xx} - 2c_{,xu} + 3d_{,uu} - 6a_{,x}d + b_{,x}c + 3b_{,u}d - 2c_{,u}a - 3d_{,x}a + 3d_{,u}b = 0. \end{aligned} \tag{2.12}$$

Example

Consider the following ODE

$$u'' + u'^2 - \frac{1}{x}u' = 0. \tag{2.13}$$

Comparing (2.13) with (2.10) gives $b = 1,\ c = -\frac{1}{x},\ a = d = 0$, which satisfy constraints (2.12). Hence (2.13) is linearizable. In fact the point transformation $t = e^u,\ s = x^2$ transforms the ODE (2.13) into $s'' = 0$, whose solution is $s = c_1t + c_2$, where c_1 and c_2 are arbitrary constants. By inverting the transformation we get the solution of the nonlinear ODE (2.13) in the explicit form: $u = \ln(\frac{1}{c_1}x^2 - \frac{c_2}{c_1})$.

2.1.2 Linearization of higher order scalar ODEs via point transformations

IM [30] obtained the linearization criteria for scalar third order ODEs by following Lie's procedure of point transformations. They also obtained the linearizing transformations for these ODEs. Linearization of fourth order scalar ODEs via point transformations was studied by IMS [31]. They obtained the necessary form for linearizable scalar fourth order ODEs and also generalized the form for higher order scalar ODEs.

Linearization criteria for a scalar third order ODE

For obtaining the necessary condition for a third order ODE

$$u''' = f(x, u, u', u''), \tag{2.14}$$

to be linearizable via point transformation (2.1), we assume that (2.14) is obtained from the linear equation

$$s'''(t) + k_0(t)s(t) = 0, \tag{2.15}$$

by the transformation (2.1). Remember that (2.15) is the Laguerre canonical form for linear scalar third order ODEs. The derivatives are changed as follows:

$$\begin{aligned} s' &= \frac{D_x\psi}{D_x\varphi} = P(x, u, u'), \\ s'' &= \frac{D_xP}{D_x\varphi} = Q(x, u, u', u''), \\ s''' &= \frac{D_xQ}{D_x\varphi}, \end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + u'''\frac{\partial}{\partial u''},$$

is the total derivative operative with respect to x. Expanding these derivatives, one has

$$D_x\varphi = \varphi_{,x} + u'\psi_{,u}\ , \quad D_x\psi = \psi_{,x} + u'\psi_{,u},$$

so that

$$s''' = \frac{D_xQ}{D_x\varphi} = \frac{\Delta}{(\varphi_{,x} + u'\varphi_{,u})^5}[(\varphi_{,x} + u'\varphi_{,u})u''' - 3\varphi_{,u}(u'')^2] + \dots,$$

the omitted terms being at most linear in u'', and

$$\Delta = \varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x} \neq 0.$$

It turns out that the transformation (2.1) with $\varphi_{,u} = 0$ and $\varphi_{,u} \neq 0$ provides two distinct types of linearizable equations.

IM type I. If $\varphi_{,u} = 0$, then we obtain the first type for linearization given by

$$u''' + (a_1 u' + a_0)u'' + b_3 u'^3 + b_2 u'^2 + b_1 u' + b_0 = 0, \tag{2.16}$$

where

$$\begin{aligned}
&a_1 = 3{\psi_{,u}}^{-1}\psi_{,uu}, \quad a_0 = 3(\varphi_{,x}\psi_{,u})^{-1}(\varphi_{,x}\psi_{,xu} - \psi_{,u}\varphi_{,xx}),\\
&b_3 = (\psi_{,u})^{-1}\psi_{,uuu}, \quad b_2 = 3(\varphi_{,x}\psi_{,u})^{-1}(\varphi_{,x}\psi_{,xuu} - \psi_{,uu}\varphi_{,xx}),\\
&b_1 = (\varphi_x^2\psi_{,u})^{-1}(3\varphi_{,xx}^2\psi_{,u} - \varphi_{,xxx}\varphi_{,x}\psi_{,u} - 6\varphi_{,xx}\varphi_{,x}\psi_{,xu} + 3\varphi_{,x}^2\psi_{,xxu}),\\
&b_0 = (\varphi_{,x}^2\psi_{,x})^{-1}(3\varphi_{,xx}^2\psi_{,x} - \varphi_{,xxx}\varphi_{,x}\psi_{,x} - 3\varphi_{,xx}\varphi_{,x}\psi_{,xx} + \varphi_{,x}^2\psi_{,xxx} + k_0\psi\varphi_{,x}^5.
\end{aligned}$$

IM type II. If $\varphi_{,u} \neq 0$, set $\lambda(x,u) = \varphi_{,x}/\varphi_{,u}$, equations are of the form

$$\begin{aligned}
u''' \;&+\; \frac{1}{u'+\lambda}[-3(u'')^2 + (c_2 u'^2 + c_1 u' + c_0)u'' + d_5 u'^5 + d_4 u'^4\\
&+\; d_3 u'^3 + d_2 u'^2 + d_1 u' + d_0] = 0,
\end{aligned} \tag{2.17}$$

where $c_i = c_i(x,u)$ and $d_j = d_j(x,u)$.

Thus every linearizable third order ODE belongs either to the type I with linear dependence on the second derivative u'' or to the type II of equations that are at most quadratic in the second derivative u'' with a specific dependence on the first derivative u'. We have the following theorems [30].

Theorem 2.1.5. *Equation* (2.16) *is linearizable if and only if its coefficients satisfy the following conditions:*

$$\begin{aligned}
&a_{0,u} - a_{1,x} = 0, \qquad (3b_1 - {a_0}^2 - 3a_{0,x})_{,u} = 0,\\
&3a_{1,x} + a_0 a_1 - 3b_2 = 0, \qquad 3a_{1,u} + {a_1}^2 - 9b_3 = 0,\\
&(9b_1 - 6a_{0,x} - 2{a_0}^2)a_{1,x} + 9(b_{1,x} - a_1 b_0)_{,u} + 3b_{1,u}a_0 - 27b_{0,uu} = 0.
\end{aligned} \tag{2.18}$$

Theorem 2.1.6. *Equation* (2.17) *is linearizable if and only if its coefficients satisfy the constraints given in Appendix A.1.*

Linearization criteria for a scalar fourth order ODE

The Laguerre canonical form of linear scalar fourth order ODEs is

$$s^{(4)}(t)+k_1(t)s'(t)+k_0(t)s(t)=0. \tag{2.19}$$

IMS [31] proved that there are two disjoint forms of scalar fourth order ODEs that are linearizable to the equation (2.19) by the point transformations (2.1).

IMS type I. If $\varphi_{,u}=0$, the first candidate for linearization is

$$\begin{aligned}u^{(4)}+(a_1u'+a_0)u'''+b_0u''^2+(c_2u'^2+c_1u'+c_0)u''\\+d_4u'^4+d_3u'^3+d_2u'^2+d_1u'+d_0=0,\end{aligned}$$

where all the coefficients, being functions of x and u, satisfy *ten* long, complicated constraint equations which are not of our concern here.

IMS type II. If $\varphi_{,u}\neq 0$, then the second candidate for linearization is

$$\begin{aligned}u^{(4)}+\frac{1}{u'+\lambda}(-10u''+f_2u'^2+f_1u'+f_0)u'''+\frac{1}{(u'+\lambda)^2}[15u''^3+(h_2u'^2+h_1u'\\+h_0)u''^2+(j_4u'^4+j_3u'^3+j_2u'^2+j_1u'+j_0)u''+k_7u'^7+k_6u'^6+k_5u'^5\\+k_4u'^4+k_3u'^3+k_2u'^2+k_1u'+k_0]=0,\end{aligned}$$

where $\lambda(x,u)=\varphi_{,x}/\varphi_{,u}$. All the coefficients, being functions of x and u, satisfy certain constraint requirements.

Necessary form of a linearizable i^{th} $(i\geq 4)$ order scalar ODE

IMS [31] derived two necessary forms of linearizable i^{th} $(i\geq 4)$ scalar order ODEs.

IMS type I. If $\varphi_{,u}=0$, the first type for linearization is

$$u^{(i)}+u^{(i-1)}[a_1u'+a_0]+...=0,$$

where $a_j=a_j(x,u)$.

IMS type II. If $\varphi_{,u}\neq 0$, the second type for linearization is

$$u^{(i)}+u^{(i-1)}\frac{1}{(u'+\lambda)}[-u''\frac{i(i+1)}{2}+f_2u'^2+f_1u'+f_0]+...=0,$$

where $f_j=f_j(x,u)$ and $\lambda(x,u)=\varphi_{,x}/\varphi_{,u}$.

2.1.3 Linearization of scalar ODEs via contact transformations

IM [30] also used contact transformations to solve the linearization problem for scalar third order ODEs. They proved that any general scalar third order ODE is linearizable to the equation (2.15) via the contact transformation (1.58) if it is at most cubic in the second derivative, i.e. of the form

$$u''' + a(x,u,u')u''^3 + b(x,u,u')u''^2 + c(x,u,u')u'' + d(x,u,u') = 0. \tag{2.20}$$

The coefficients in the equation (2.20) have to satisfy certain constraints called sufficient conditions for the linearization.

Linearization of fourth order scalar ODEs via contact transformations (1.58) was studied by IMS [69]. They showed that all fourth order ODEs that are linearizable via contact transformations (1.58) are contained in the class of equations of the form

$$\begin{aligned} u^{(4)} &+ \frac{1}{u''+\mu}[-3(u''')^2 + (c_2u''^2 + c_1u'' + c_0)u''' + d_5u''^5 + d_4u''^4 \\ &+ d_3u''^3 + d_2u''^2 + d_1u'' + d_0] = 0, \end{aligned} \tag{2.21}$$

where all coefficients are functions of x, u, u' and $\mu = (\varphi_{,x} + \varphi_{,u})/\varphi_{,u'}$. They derived the sufficient conditions for the linearization, the methods for constructing the linearizing transformations as well as the coefficients of the resulting linear equations. They also formulated the linearizable form for i^{th}, $(i > 4)$ order scalar ODEs via contact transformation which is given by

$$u^{(i)} + u^{(i-1)}\frac{1}{(u''+\mu)}[-u''\frac{i(i+1)}{2} + a_2u'^2 + a_1u' + a_0] + ... = 0,$$

where all coefficients are functions of (x, u, u') and μ is the same as defined above.

2.2 Linearization and classification of systems of ODEs

In this section Lie's linearization criteria for scalar second order ODEs are extended to $2-$dimensional systems of second order ODEs. The two canonical forms of linear systems of second order ODEs of the dimension n are given with the linearization criteria developed for both the forms. We provide the equivalence classes of 2−dimensional linear systems of second order ODEs.

A general non-homogeneous $n-$dimensional system of linear second order ODEs contains $2n^2 + n$ arbitrary coefficients. This number of arbitrary coefficients makes it very difficult to address the classification problem of such systems. Since invertible point transformations preserve the number of symmetries, therefore, it is needed to convert this general system of ODEs into a simple form of ODEs that involves a fewer number of coefficients. These simpler forms are called *canonical forms* of the given ODEs. In [72] two canonical forms for $n-$dimensional systems of second order ODEs were presented which are stated in the following theorem.

Theorem 2.2.1. *Any linear non-homogeneous system of n second order ODEs*

$$\boldsymbol{u}''(x) = \boldsymbol{A}\boldsymbol{u}'(x) + \boldsymbol{B}\boldsymbol{u}(x) + \boldsymbol{c}, \tag{2.22}$$

can be mapped via a point transformation to one of the following forms: either

$$\boldsymbol{v}''(t) = \boldsymbol{F}\boldsymbol{v}'(t), \tag{2.23}$$

or

$$\boldsymbol{w}''(t) = \boldsymbol{G}\boldsymbol{w}(t), \tag{2.24}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ are $n \times n$ matrix functions and $\boldsymbol{u}$, $\boldsymbol{c}$ are vector functions of x, while $\boldsymbol{F}$, $\boldsymbol{G}$ are $n \times n$ matrix functions and $\boldsymbol{v}$, $\boldsymbol{w}$, are vector functions of t.

For the case $n = 2$, the number of arbitrary coefficients has been reduced from ten to four. Consider the linear system (2.24) with $n = 2$ having 4 arbitrary coefficients.

$$\begin{aligned} w_1'' &= g_{11}(t)w_1 + g_{12}(t)w_2, \\ w_2'' &= g_{21}(t)w_1 + g_{22}(t)w_2. \end{aligned} \tag{2.25}$$

The number of arbitrary coefficients are further reduced from four to three by the change of variables

$$y = w_1/\tau(t), \quad z = w_2/\tau(t), \quad x = \int^t \tau^{-2}(s)ds,$$

where τ satisfies

$$\tau'' - \frac{g_{11} + g_{22}}{2}\tau = 0,$$

to the linear system

$$\begin{aligned} y'' &= \bar{g}_{11}(x)y + \bar{g}_{12}(x)z, \\ z'' &= \bar{g}_{21}(x)y - \bar{g}_{11}(x)z, \end{aligned} \tag{2.26}$$

with

$$\bar{g}_{11} = \frac{\tau^3(g_{11} - g_{22})}{2}, \quad \bar{g}_{12} = \tau^3 g_{12}, \quad \bar{g}_{21} = \tau^3 g_{21}.$$

Thus we have the following theorem [73].

Theorem 2.2.2. *A* $2-$*dimensional system of linear second order ODEs can be mapped invertibly to the linear system* (2.26)*.*

The system (2.26) is the *optimal canonical form* of linear systems of two second order ODEs. This form provides five equivalence classes of linearizable systems of two second order ODEs, with $5-$, $6-$, $7-$, $8-$ and $15-$dimensional Lie algebras [73]. For the maximal symmetry class of $2-$dimensional systems of second order ODEs, we state the following theorem.

Theorem 2.2.3. *A* $2-$*dimensional system of second order ODEs can be reduced to the system of free particle equations*

$$w_1'' = 0, \quad w_2'' = 0, \tag{2.27}$$

if and only if it has a $15-$*dimensional Lie algebra.*

Following Lie's procedure [57], one uses invertible point transformations

$$t = \varphi(x, y, z), \quad w_1 = \psi_1(x, y, z), \quad w_2 = \psi_2(x, y, z), \tag{2.28}$$

to map the general system of two second order ODEs in semilinear form to the simplest form (2.27). Under (2.28) the derivatives transform as

$$\begin{aligned} w_1' &= \frac{D_x\psi_1}{D_x\varphi} = P_1(x, y, z, y', z'), \\ w_2' &= \frac{D_x\psi_2}{D_x\varphi} = P_2(x, y, z, y', z') \end{aligned}$$

and

$$w_1'' = \frac{D_x P_1}{D_x t}, \quad w_2'' = \frac{D_x P_2}{D_x t},$$

where D_x is the total derivative operator. This yields

$$\begin{aligned} &y'' + a_{11}y'^3 + a_{12}y'^2z' + a_{13}y'z'^2 + a_{14}z'^3 + b_{11}y'^2 + b_{12}y'z' + b_{13}z'^2 + c_{11}y' + c_{12}z' + d_1 = 0, \\ &z''' + a_{21}z'^3 + a_{22}y'^2z' + a_{23}y'z'^2 + a_{24}z'^3 + b_{21}y'^2 + b_{22}y'z' + b_{23}z'^2 + c_{21}y' + c_{22}z' + d_2 = 0, \end{aligned} \tag{2.29}$$

where the above twenty coefficients are arbitrary functions of $x,\ y$ and z. The system (2.29) represents the most general form of a system of two second order ODEs equivalent to the system of free particle equations (2.27).

One can also use the invertible point transformation (2.28) to map 2−dimensional systems of second order ODEs to the second canonical form (2.25) with $n = 2$ to yield the following theorem [67].

Theorem 2.2.4. *A* 2−*dimensional system of second order ODEs is equivalent to the system* (2.25) *via an invertible point transformation* (2.28) *if it is of the form*

$$\begin{aligned} &y'' + \bar{a}_{11}y'^3 + \bar{a}_{12}y'^2z' + \bar{a}_{13}y'z'^2 + \bar{b}_{11}y'^2 + \bar{b}_{12}y'z' + \bar{b}_{13}z'^2 + \bar{c}_{11}y' + \bar{c}_{12}z' + \bar{d}_1 = 0, \\ &z'' + \bar{a}_{11}y'^2z' + \bar{a}_{12}y'z'^2 + \bar{a}_{13}z'^3 + \bar{b}_{21}y'^2 + \bar{b}_{22}y'z' + \bar{b}_{23}z'^2 + \bar{c}_{21}y' + \bar{c}_{22}z' + \bar{d}_2 = 0, \end{aligned}$$

where all of the above coefficients are arbitrary functions of $x,\ y$ *and* z.

2.3 Meleshko linearization

Meleshko presented a new method to solve autonomous third order scalar ODEs and called it linearization [56]. He considers those third order ODEs that do not satisfy IM linearization criteria and reduces them to the second order ODEs and then linearizes them if they are. Since such kind of ODEs do not satisfy IM linearization criteria and hence are not linearizable as IM pointed out. We call this reduction of ODEs to the lower order linearizable equations *Meleshko linearization*. This type of linearization, which is not actually linearization but uses the base of linearization allows us to define a new class of ODEs that does not lie in IM class. We here give the essence of Meleshko's linearization for scalar third order autonomous ODEs. Consider a third order autonomous ODE

$$u''' = f(u, u', u''). \tag{2.30}$$

Since the independent variable is missing, we can take u as the new independent variable and its derivative as the new dependent variable $y(u) = u'$. By this we get a second order ODE of the form

$$y^2 y'' + yy'^2 = f(u, y, yy'),$$

which is Lie linearizable if f is of the form

$$f(u, u', u'') = a(u, u')u''^3 + b(u, u')u''^2 + c(u, u')u'' + d(x, u)$$

and the coefficients $a,\ b,\ c,\ d$ have to satisfy the following conditions:

$$\begin{aligned} b_{,uu}u'^4 + (3b_{,u}c + 3ad_{,u} + 6a_{,u}d - 2c_{,uu'})u'^3 + (2c_{,u} - 2cc_{,u'} + 3b_{,u'}d \\ +3bd_{,u'} + 3d_{,u'u'})u'^2 + (6bd - 2c^2 - 9d_{,u})u' + 9d = 0, \\ 3a_{,uu}u'^4 + (2bb_{,u} - 3a_{,u}c - 3ac_{,u} - 2b_{,uu'})u'^3 + (2b_{,u} + 3a_{,u'}d + 6ad_{,u'} \\ -bc_{,u'} + c_{,u'u'})u'^2 + (bc - 9ad - 3c_{,u'})u' + 3c = 0. \end{aligned} \tag{2.31}$$

This leads to the following theorem.

Theorem 2.3.1. *A third order autonomous ODE* (2.30) *is reducible to the second order linearizable equation (Meleshko linearizable by order one) if it is of the form*

$$u''' = a(u, u')u''^3 + b(u, u')u''^2 + c(u, u')u'' + d(x, u),$$

with the coefficients satisfying the constraints (2.31).

2.4 Complex linearization

Lie used complex DEs of complex variables, but he did not consider the analyticity of complex variables embodied in the CR-equations. A complex dependent variable splits into two dependent real variables while the complex independent variable splits into two real independent variables. In this way a scalar complex ODE splits into a system of two PDEs. The CR-equations would apply not only between the independent and dependent variables but also between the independent variables and the derivatives of the dependent variables. If we restrict the independent variable to be real we get a system of two real ODEs [2,4]. In this thesis we are only concerned with the splitting of complex scalar ODEs into systems of two real ODEs. Consider the complex ODE

$$u'' = f(x, u, u'), \tag{2.32}$$

where u is a complex function of real variable x. Now by writing $u = y+iz$ and $f = f_1+if_2$, will split the scalar complex ODE (2.32) into a system of two real ODEs

$$y'' = f_1(x, y, z; y', z'), \quad z'' = f_2(x, y, z; y', z'). \tag{2.33}$$

For the function f to be analytic, its real and imaginary parts must satisfy the CR-equations with respect to the dependent variables and their derivatives. For complete characterization of such systems, we state the following theorem [6].

Theorem 2.4.1. *A general* $2-$*dimensional system of second order ODEs* (2.33) *corresponds to a complex equation* (2.32) *if and only if* f_1 *and* f_2 *satisfy the CR-equations*

$$\begin{aligned} f_{1,y} = f_{2,z}, \quad f_{1,z} = -f_{2,y}, \\ f_{1,y'} = f_{2,z'}, \quad f_{1,z'} = -f_{2,y'}. \end{aligned} \tag{2.34}$$

When the dependent variable in a linearizable second order scalar ODE is a complex function of a real independent variable, it leads to *complex linearization.* To obtain complex linearization criteria [5] for a $2-$dimensional system of second order ODEs, we suppose u in the linearizable ODE (2.10) to be a complex function of a real variable x. Suppose there exist complex functions

$$\begin{aligned} a(x,u) &= a_1(x,y,z) + ia_2(x,y,z), \\ b(x,u) &= b_1(x,y,z) + ib_2(x,y,z), \\ c(x,u) &= c_1(x,y,z) + ic_2(x,y,z), \\ d(x,u) &= d_1(x,y,z) + id_2(x,y,z). \end{aligned} \tag{2.35}$$

Now by writing $u(x) = y(x) + iz(x)$ will split the scalar complex ODE (2.10) into a system of two second order ODEs of the form

$$\begin{aligned} y'' + a_1y'^3 - 3a_2y'^2z' - 3a_1y'z'^2 + a_2z'^3 + b_1y'^2 - 2b_2y'z' - b_1z'^2 + c_1y' - c_2z' + d_1 = 0, \\ z'' + a_2y'^3 + 3a_1y'^2z' - 3a_2y'z'^2 - a_1z'^3 + b_2y'^2 + 2b_1y'z' - b_2z'^2 + c_2y' + c_1z' + d_2 = 0. \end{aligned} \tag{2.36}$$

The sufficient conditions for linearization (2.12) now split into a set of four constraint equations:

$$\begin{gathered} 12a_{1,xx} + 12c_1a_{1,x} - 12c_2a_{2,x} - 6d_1a_{1,y} - 6d_1a_{2,z} + 6d_2a_{2,y} \\ -6d_2a_{1,z} + 12a_1c_{1,x} - 12a_2c_{2,x} + c_{1,yy} - c_{1,zz} + 2c_{2,yz} \\ -12a_1d_{1,y} - 12a_1d_{2,z} + 12a_2d_{2,y} - 12a_2d_{1,z} + 2b_1c_{1,y} + 2b_1c_{2,z} \\ -2b_2c_{2,y} + 2b_2c_{1,z} - 8b_1b_{1,x} + 8b_2b_{2,x} - 4b_{1,xy} - 4b_{2,xz} = 0, \end{gathered}$$

$$\begin{gathered} 12a_{2,xx} + 12c_2a_{1,x} + 12c_1a_{2,x} - 6d_2a_{1,y} - 6d_2a_{2,z} - 6d_1a_{2,y} \\ +6d_1a_{1,z} + 12a_2c_{1,x} + 12a_1c_{2,x} + c_{2,yy} - c_{2,zz} - 2c_{1,yz} \\ -12a_2d_{1,y} - 12a_2d_{2,z} - 12a_1d_{2,y} + 12a_1d_{1,z} + 2b_2c_{1,y} + 2b_2c_{2,z} \\ +2b_1c_{2,y} - 2b_1c_{1,z} - 8b_2b_{1,x} - 8b_1b_{2,x} - 4b_{2,xy} + 4b_{1,xz} = 0, \end{gathered}$$

$$\begin{gathered} 24d_1a_{1,x} - 24d_2a_{2,x} - 6d_1b_{1,y} - 6d_1b_{2,z} + 6d_2b_{2,y} - 6d_2b_{1,z} \\ +12a_1d_{1,x} - 12a_2d_{2,x} + 4b_{1,xx} - 4c_{1,xy} - 4c_{2,xz} - 6b_1d_{1,y} \\ -6b_1d_{2,z} + 6b_2d_{2,z} - 6b_2d_{1,z} + 3d_{1,yy} - 3d_{1,zz} + 6d_{2,yz} + 4c_1c_{1,y} \\ +4c_1c_{2,z} - 4c_2c_{2,y} + 4c_2c_{1,z} - 4c_1b_{1,x} + 4c_2b_{2,x} = 0, \end{gathered}$$

$$24d_2a_{1,x} + 24d_1a_{2,x} - 6d_2b_{1,y} - 6d_2b_{2,z} - 6d_1b_{2,y} + 6d_1b_{1,z}$$
$$+12a_2d_{1,x} + 12a_1d_{2,x} + 4b_{2,xx} - 4c_{2,xy} + 4c_{1,xz} - 6b_2d_{1,y}$$
$$-6b_2d_{2,z} - 6b_1d_{2,y} + 6b_1d_{1,z} + 3d_{2,yy} - 3d_{2,zz} - 6d_{1,yz} + 4c_2c_{1,y}$$
$$-4c_2c_{2,z} + 4c_1c_{2,y} - 4c_1c_{1,z} - 4c_2b_{1,x} - 4c_1b_{2,x} = 0. \qquad (2.37)$$

Thus we have the following theorem [5].

Theorem 2.4.2. *A* $2-$*dimensional system of second order ODEs is complex linearizable if it is of the form* (2.36) *and its coefficients satisfy the conditions* (2.37) *and CR-equations with respect to* y *and* z*.*

2.4.1 Equivalent classes of systems of ODEs obtained by complex methods

As mentioned earlier, there are five classes of $2-$dimensional linearizable systems of ODEs with 5, 6, 7, 8 or 15 Lie point symmetries. By using CSA we get three of the five linearizable classes. To obtain these classes we need a canonical form for linear systems of two second order ODEs corresponding to complex scalar ODEs [66]. We start with a general linear scalar complex second order ODE

$$u'' = a(x)u' + b(x)u + c(x). \qquad (2.38)$$

As all linear scalar second order ODEs are equivalent, so (2.38) is equivalent to the following scalar second order complex ODEs

$$u'' = d(x)u, \qquad (2.39)$$

$$u'' = e(x)u', \qquad (2.40)$$

where all these ODEs belong to one equivalence class and have 8 Lie point symmetries. So these ODEs are transformable to each other and reducible to the free particle equation. To extract systems of two linear second order ODEs from (2.39) and (2.40), we write

$u(x) = y(x) + iz(x)$, $d(x) = d_1(x) + id_2(x)$ and $e(x) = e_1(x) + ie_2(x)$ and obtain the following two forms of linear systems of second order ODEs

$$\begin{aligned} y'' &= d_1(x)y - d_2(x)z, \\ z'' &= d_2(x)y + d_1(x)z \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} y'' &= e_1(x)y' - e_2(x)z', \\ z'' &= e_2(x)y' + e_1(x)z'. \end{aligned} \tag{2.42}$$

Thus we can state the above discussion in the form of the following theorem.

Theorem 2.4.3. *If a* $2-$*dimensional system of second order ODEs is linearizable via invertible complex point transformations then it can be mapped to one of the two forms* (2.41) *and* (2.42)*.*

These two linear forms contain only two arbitrary coefficients while the minimum number of coefficients obtained earlier was three. The reason of reduction of number is that we are dealing with the special classes of systems of ODEs that correspond to the scalar complex ODEs. The number of coefficients in (2.41) can be further reduced to one by the following theorem [66].

Theorem 2.4.4. *Any linear system of two second order ODEs of the form* (2.41) *can be mapped to the simplest system of two linear ODEs*

$$\begin{aligned} \bar{y}'' &= -d(\bar{x})\bar{z}, \\ \bar{z}'' &= d(\bar{x})\bar{y}, \end{aligned} \tag{2.43}$$

via the real point transformation

$$\bar{y} = \frac{y}{\alpha(x)}, \quad \bar{z} = \frac{z}{\alpha(x)}, \quad \bar{x} = \int^x \alpha^{-2}(s)ds, \tag{2.44}$$

where $\alpha'' - d_1\alpha = 0$ *and* $d = \alpha d_2$*.*

The system (2.43) is the reduced optimal canonical form for the linear systems of two second order ODEs. It involves only one arbitrary coefficient, so three usual cases arise: (a) d is zero, (b) d is an arbitrary constant, (c) $d(\bar{x})$ is an arbitrary function. It was found [66] that (a) gives 15, (b) 7 and (c) 6 Lie point symmetry generators.

2.5 Geometric linearization

Since DEs live on manifolds, it is natural to ask about the connection between symmetries in geometry and for DEs. A connection between the symmetries of systems of geodesics equations and the underlying manifold was provided in the form of a theorem in [22]. This theorem leads us to a procedure of checking the linearizability of systems of quadratically semilinear second order ODEs.

A quadratically semilinear system of second order ODEs in the general form is given by

$$\ddot{u}^a + A^a_{bc}\dot{u}^b\dot{u}^c + B^a_b\dot{u}^b + C^a = 0. \tag{2.45}$$

The above system of ODEs is of *geodesic type* if $B^a_b = C^a = 0$. It is said to be a system of *geodesic equations* if there exists some metric tensor for which the Christoffel symbols Γ^i_{jk}, given by (2.47), satisfy $\Gamma^a_{bc} = A^a_{bc}$.

2.5.1 Linearization criteria for a quadratically semilinear system of second order ODEs

We consider the system of n geodesic equations

$$\ddot{u}^i + \Gamma^i_{jk}\dot{u}^j\dot{u}^k = 0, \quad i, j, k = 1, 2, \ldots n, \tag{2.46}$$

where the Christoffel symbols Γ^i_{jk}, are given in terms of the metric tensor g_{ij}

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}) \tag{2.47}$$

and $\dot{u}^i$ is the derivative with respect to the arc length l defined by

$$dl^2 = g_{ij} du^i du^j.$$

The Reimann tensor is defined by

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}. \tag{2.48}$$

The following theorem [50] gives the linearization criteria for systems of ODEs of the form (2.46) via invertible point transformations.

Theorem 2.5.1. *A system of n second order ODEs of the form* (2.46) *is linearizable via an invertible point transformation if and only if the curvature tensor* (2.48) *constructed by treating the coefficients of* (2.46) *as Christoffel symbols is zero and the admitted symmetry algebra is $sl(n+2,\mathbb{R})$.*

2.5.2 Linearization criteria for a cubically semilinear system of second order ODEs

Projecting the $n-$dimensional system of geodesic equations (2.46) down to $(n-1)-$dimensional, we get the linearization criteria for cubically semilinear systems of second order ODEs [52].

In fact, taking u^1 as the new dependent variable we treat all other dependent variables as functions of u^1. The projection procedure puts

$$u^{a\prime} = \frac{du^a}{du^1} u^{1\prime}, \quad (a = 2, 3, \dots, n)$$

and

$$u^{a\prime\prime} = \frac{d^2u^a}{(du^1)^2}(u^{1\prime})^2 + \frac{du^a}{du^1} u^{1\prime\prime}, \quad (a = 2, 3, \dots, n).$$

This gives

$$u^{a\prime\prime} + \alpha_{bc} u^{a\prime} u^{b\prime} u^{c\prime} + \beta^a_{bc} u^{b\prime} u^{c\prime} + \gamma^a_b u^{b\prime} + \theta^a = 0, \quad (a = 2, 3, \dots, n), \tag{2.49}$$

where prime denotes differentiation with respect to the parameter u^1 and the coefficients in terms of Christoffel symbols are

$$\alpha_{bc} = -\Gamma^1_{bc}, \quad \beta^a_{bc} = \Gamma^a_{bc} - 2\delta^a_c\Gamma^1_{b1}, \quad \gamma^a_b = 2\Gamma^a_{1b} - \delta^a_b\Gamma^1_{11},$$
$$\theta^a = \Gamma^a_{11}, \quad (a = 2, 3, \ldots, n).$$

The linearization constraints are that the curvature tensor formed from the Christoffel symbols is zero.

Taking $n = 2$ in (2.49) gives a system of two cubically semilinear second order ODEs

$$y'' + \alpha_1 y'^3 + 2\alpha_2 y'^2 z' + \alpha_3 y' z'^2 + \beta_1 y'^2 + 2\beta_2 y' z' + \beta_3 z'^2 + \gamma_1 y' + \gamma_2 z' + \theta_1 = 0,$$
$$z'' + \alpha_1 y'^2 z' + 2\alpha_2 y' z'^2 + \alpha_3 z'^3 + \beta_4 y'^2 + 2\beta_5 y' z' + \beta_6 z'^2 + \gamma_3 y' + \gamma_4 z' + \theta_2 = 0, \quad (2.50)$$

that comes from the system of three ODEs of the form (2.46) by projection.

As a by-product of the projection procedure we re-derive the Lie conditions. This result is stated in the form of the following remark [52, 62].

Remark 2.5.2. By taking geodesics equations with $n = 2$ and projecting down, we get a scalar cubically semilinear ODE (2.10) and the linearization conditions are the Lie conditions (2.12).

2.6 Conditional linearization

Conditional linearization is another form of linearization that does not actually linearize an ODE but uses the linearizable ODE as a base. Mahomed and Qadir introduced this idea [51] by giving the conditional linearizability criteria for scalar third order ODEs. It was then extended to the fourth order scalar [53] and systems of two ODEs [49]. Here we give the criteria in detail for the conditionally linearizable third order ODEs only, but first we define such equations [51].

Definition 2.6.1. By differentiating the second (third) order scalar ODEs linearizable by point transformations and then requiring that the original equation holds, is called *conditional linearizability* by point transformations of third (fourth) order scalar ODEs. The original linearizable equation is called the *root* equation.

To obtain the conditional linearization criteria for scalar third order ODEs, we differentiate the cubically semi-linear, scalar, second order ODE (2.10) with respect to the independent variable x to get third order ODE of the general form

$$u''' + (g_2 u'^2 - g_1 u' + g_0)u'' + h_4 u'^4 - h_3 u'^3 + h_2 u'^2 - h_1 u' + h_0 = 0, \tag{2.51}$$

where the coefficients are given by

$$\begin{aligned} &a = \frac{g_2}{3}, \quad b = \frac{-g_1}{2}, \quad c = g_0, \quad h_4 = \frac{g_{2,u}}{3}, \\ &h_3 = \frac{g_{1,u}}{2} - \frac{g_{1,x}}{3}, \quad h_2 = g_{0,u} - \frac{g_{1,x}}{2}, \quad d = \int h_2 dx + l(y) \end{aligned} \tag{2.52}$$

and $l(y)$ is the arbitrary function.

We can use the original equation (2.10) to replace the second order derivative and write the equation as quintically nonlinear in the first derivative

$$u''' - j_5 u'^5 + j_4 u'^4 - j_3 u'^3 + j_2 u'^2 - j_1 u' + j_0 = 0, \tag{2.53}$$

with the identifications of the coefficients

$$\begin{aligned} &a = \frac{j_1}{\sqrt{3}}, \quad b = \frac{1}{5a}(a_{,u} - j_2), \\ &c = \frac{1}{4a}(j_3 - 2b^2 + b_{,u} + a_{,x}), \quad d = \frac{1}{3a}(c_{,u} + b_{,x} - j_2 - 3bc), \\ &j_1 = 2bd + c^2 - d_{,u} - c_{,x}, \quad j_0 = d_{,x} - cd. \end{aligned} \tag{2.54}$$

This equation will have all solutions of the original ODE but may not have any more. As such, it could be a third order nonlinear ODE with only two arbitrary constants. We thus have two classes of conditionally linearizable third order ODEs, those with replacement, i.e. of the form (2.53) and those without, given by (2.51).

The above result can be stated in the form of the following theorems [51].

Theorem 2.6.2. *Equation* (2.51) *is conditionally linearizable by a point transformation with respect to the second order ODE* (2.10) *if its coefficients satisfy the linearizability criteria* (2.12) *with the identification of coefficients given by* (2.52).

Theorem 2.6.3. *Equation* (2.53) *is conditionally linearizable by a point transformation with respect to the second order ODE* (2.10) *if its coefficients satisfy the linearizability criteria* (2.12) *with the coefficients given by* (2.54)*.*

The above procedure is repeated to obtain the scalar fourth order ODEs with the second order root equation by differentiating either (2.51) or (2.53) and using the second order or one of these equations to replace the relevant derivative terms. In this way we get five types of fourth order ODEs with a second order root equation. For details we refer the reader to [53]. All of these types of equations have two arbitrary constants in their solutions. Fourth order ODEs with third order root equation is obtained by differentiating the IM type I (2.16) or IM type II (2.17) and then either replacing the derivatives terms with any of these two equations or not [54]. In this way we get four types of fourth order conditionally linearizable ODEs with the third order root equation. One can also go one step further by taking IMS' or Meleshko's linearizable ODEs as the root equations to go to the higher order ODEs, differentiating these equations and then replacing or not. The same procedure can be repeated by differentiating the equations linearizable via contact transformations. We get different classes of conditionally linearizable ODEs. This classification will then be by the number of arbitrary initial conditions that can be satisfied.

Chapter 3

Meleshko linearization of fourth order scalar ODEs

If a scalar third order ODE does not depend explicitly on the independent variable, we can reduce it to a second order ODE [56]. We can then apply the Lie linearization test to the reduced ODE. If the reduced ODE satisfies the test then after finding a linearizing transformation, the general solution of the original equation is obtained by quadrature. We call this reduction of ODEs to lower order linearizable equations *Meleshko linearization*.

In this chapter we extend Meleshko's procedure to the fourth order ODEs in the cases that the equations do not depend explicitly on the independent or the dependent variable (or both) to reduce it to third (respectively second) order equations [16]. Once the order is reduced we can apply the IM (or Lie) linearization test. If the reduced third (or second) order ODE satisfies the IM (or Lie) linearization test, then after finding a linearizing transformation, the general solution of the original equation is obtained by quadrature. So this method is effective in the sense that it reduces many ODEs, that cannot be linearized, to lower order linearizable equations. Meleshko linearization allows us to define a new class of ODEs that do not lie in IM's class, IMS'class or conditionally linearizable classes.

Meleshko had only developed the algorithm for those third order ODEs that do not involve independent variable x. We include independence of u for completeness before proceeding to the fourth order.

3.1 Third order ODEs independent of u

Consider the scalar third order ODE that do not depend explicitly on the dependent variable

$$u''' = f(x, u', u''). \tag{3.1}$$

Since the dependent variable u is missing so we take u' as the new dependent variable $y(x)$. The derivatives are transformed as

$$u'' = y', \quad u''' = y''. \tag{3.2}$$

This converts the ODE (3.1) to the second order ODE

$$y'' = f(x, y, y'). \tag{3.3}$$

The above ODE is linearizable by Lie's criteria if and only if it is at most cubically semilinear (2.10) i.e., f is of the form

$$f(x, y, y') = -a(x, y)y'^3 - b(x, y)y'^2 - c(x, y)y' - d(x, y), \tag{3.4}$$

with the coefficients satisfying the conditions (2.12). Replacing y by u' and invoking (3.2) in (3.4) and (2.12), we get what is called the Meleshko linearization criteria for the scalar third order ODEs (3.1). We write it in the following theorem.

Theorem 3.1.1. *Equation* (3.1) *is reducible to the second order linearizable equation (or Meleshko linearizable of order one) if and only if*

$$f(x, u', u'') = -a(x, u')u''^3 - b(x, u')u''^2 - c(x, u')u'' - d(x, u'), \tag{3.5}$$

with the coefficients satisfying

$$\begin{aligned} &3a_{,xx} - 2b_{,xu'} + c_{,u'u'} - 3a_{,x}c + 3a_{,u'}d + 2b_{,x}b - 3c_{,x}a - c_{,u'} + 6d_{,u'}a = 0, \\ &b_{,xx} - 2c_{,xu'} + 3d_{,u'u'} - 6a_{,x}d + b_{,x}c + 3b_{,u'}d - 2c_{u'}a - 3d_{,x}a + 3d_{,u'}b = 0. \end{aligned} \tag{3.6}$$

3.2 Fourth order ODEs

In this section we consider those fourth order ODEs that do not explicitly involve independent variable or dependent variable or both.

Case I. Fourth order ODEs independent of x

Consider the autonomous fourth order ODE

$$u^{(4)} = f(u, u', u'', u'''). \tag{3.7}$$

Here the independent variable x is missing, so we take u' as the new dependent variable of the new independent variable u:

$$y(u) = u'. \tag{3.8}$$

The derivatives are transformed as follows:

$$\begin{aligned} u'' &= \frac{du'}{dx} = \frac{du'}{du}\frac{du}{dx} = \frac{dy}{du}u' = y'y, \\ u''' &= \frac{dy'}{du}y' = y''y^2 + y'^2y, \\ u^{(4)} &= \frac{dy''}{du}y' = y'''y^3 + 4y^2y'y'' + yy'^3. \end{aligned} \tag{3.9}$$

This transforms the ODE (3.7) into the equation

$$y^3y''' + 4y^2y'y'' + yy'^3 - f(u, y, yy', y^2y'' + yy'^2) = 0, \tag{3.10}$$

which is a third order ODE in (u, y). It is linearizable of IM type I if it is of the form (2.16) i.e.,

$$\begin{aligned} f(u, y, y'y, y''y^2 + yy'^2) = -y^3[(a_1y' + a_0)y'' + b_3y'^3 + b_2y'^2 + b_1y' + b_0] \\ +4y^2y'y'' + yy'^3, \end{aligned} \tag{3.11}$$

where $a_i = a_i(u, y),\ (i = 0, 1)$ and $b_j = b_j(u, y),\ (j = 0, 1, 2, 3)$. Invoking (3.11), equation (3.10) takes the form

$$y''' + (a_1y' + a_0)y'' + b_3y'^3 + b_2y'^2 + b_1y' + b_0 = 0. \tag{3.12}$$

Now applying the inverse of the transformation (3.8), i.e.

$$y = u', \quad y' = \frac{u''}{u'}, \quad y'' = \frac{u'u'' - u''^2}{u'^3}, \quad y''' = \frac{u'^2u^{(4)} - 4u'u''u''' + 3u''^3}{u'^5}, \tag{3.13}$$

the ODE (3.12) is transformed to a fourth order ODE with x as independent variable and u as dependent variable:

$$u^{(4)} + (A_1u'' + A_0)u''' + B_3u''^3 + B_2u''^2 + B_1u'' + B_0 = 0, \tag{3.14}$$

where

$$A_i = A_i(u, u'),\ (i = 0, 1); \quad B_j = B_j(u, u'),\ (j = 0, 1, 2, 3), \tag{3.15}$$

subject to the identification of coefficients

$$\begin{aligned} a_1 &= A_1 + \frac{4}{u'}, \quad a_0 = \frac{A_0}{u'}, \quad b_3 = B_3 + \frac{A_1}{u'} + \frac{1}{u'^2}, \\ b_2 &= \frac{B_2}{u'} + \frac{A_0}{u'^2}, \quad b_1 = \frac{B_1}{u'^2}\ , \quad b_0 = \frac{B_0}{u'^3}, \end{aligned}$$

with the constraints

$$\begin{aligned} &u'^2A_{1,u} - u'A_{0,u'} + A_0 = 0, \\ &u'^2(-3A_{0,uu'}) + u'(3B_{1,u'} + 3A_{0,u} - 2A_0A_{0,u'}) + (-6B_1 + 2A_0^2) = 0, \\ &u'^2(3A_{1,u}) + u'(A_0A_1 - 3B_2) + A_0 = 0, \\ &u'^2(3A_{1,u'} - 9B_3 + A_1^2) - u'A_1 - 5 = 0, \\ &u'^4(-6A_{0,u}A_{1,u}) + u'^3(9B_1A_{1,u} - 2A_0^2A_{1,u} + 9B_{1,uu'}) + u'^2(-18B_{1,u} - 9A_1B_{0,u'} \\ &-9B_0A_{1,u'} + 3A_0B_{1,u'} - 27B_{0,u'u'}) + u'(27A_1B_0 - 6A_0B_1 + 126B_{0,u'}) - 180B_0 = 0. \end{aligned} \tag{3.16}$$

If the equation (3.10) is linearizable of IM type II, then we have to take

$$\begin{aligned} f(u, y, yy', y^2y'' + yy'^2) = &-\frac{y^3}{y' + \lambda}[-3(y'')^2 + (c_2y'^2 + c_1y' + c_0)y'' \\ &+d_5y'^5 + d_4y'^4 + d_3y'^3 + d_2y'^2 + d_1y' + d_0] + 4y^2y'y'' + yy'^3, \end{aligned} \tag{3.17}$$

where $c_i = c_i(u, y)$, $(i = 0, 1, 2)$, $d_j = d_j(u, y)$, $(j = 0, 1, 2, 3, 4, 5)$ and $\lambda = \lambda(u, y)$.

Considering the form (3.17) and converting (3.10) into the fourth order by applying the inverse transformation (3.13), we have

$$\begin{aligned} &u^{(4)} + \frac{1}{u'' + \lambda_0}[-3(u''')^2 + (C_2 u''^2 + C_1 u'' + C_0)u''' \\ &+D_5 u''^5 + D_4 u''^4 + D_3 u''^3 + D_2 u''^2 + D_1 u'' + D_0] = 0, \end{aligned} \tag{3.18}$$

where $C_i = C_i(u, u')$, $(i = 0, 1, 2)$; $D_j = D_j(u, u')$, $(j = 0, 1, 2, 3, 4, 5)$; $\lambda_0 = \lambda_0(u, u')$, subject to the identification of coefficients

$$\begin{aligned} &c_2 = C_2 - \frac{2}{u'}, \quad c_1 = C_1 + \frac{4\lambda_0}{u'}, \quad c_0 = \frac{C_0}{u'^2}, \quad d_5 = \frac{D_5}{u'^5}, \\ &d_4 = D_4 + \frac{C_2}{u'} - \frac{2}{u'^2}, \quad d_3 = \frac{D_3}{u'} + \frac{C_1}{u'} + \frac{4\lambda_0}{u'^2} - \frac{3\lambda_0}{u'^3}, \\ &d_2 = \frac{D_2}{u'^2} + \frac{C_0}{u'^3}, \quad d_1 = \frac{D_1}{u'^3}\ , \quad d_0 = \frac{D_0}{u'^4}, \quad \lambda = \frac{\lambda_0}{u'}, \end{aligned}$$

with the constraint equations (A.29)−(A.37), presented in Appendix *A*.4.

The above results can be stated in the form of the following theorems.

Theorem 3.2.1. *Equation* (3.14) *is reduced to the third order linearizable equation (Meleshko linearizable of order one) if and only if it obeys* (3.16)*.*

Theorem 3.2.2. *Equation* (3.18) *is reduced to the third order linearizable equation (Meleshko linearizable of order one) if and only if it obeys* (*A*.29) − (*A*.37)*, given in Appendix A.4.*

Case II. Fourth order ODEs independent of u

Consider the general form of a fourth order ODE independent of u

$$u^{(4)} = f(x, u', u'', u'''). \tag{3.19}$$

Here the dependent variable u is missing. By taking u' as the new dependent variable $y(x)$, the above ODE is reduced to the third order ODE

$$y''' = f(x, y, y', y''). \tag{3.20}$$

To make equation (3.20) linearizable for the IM type I (2.16), we have to take

$$f(x, y, y', y'') = -(a_1 y' + a_0)y'' - b_3 y'^3 - b_2 y'^2 - b_1 y' - b_0, \tag{3.21}$$

with the coefficients $a_i = a_i(x, y)$, $(i = 0, 1)$ and $b_j = b_j(x, y)$, $(j = 0, 1, 2, 3)$, satisfying the conditions (2.18). By applying the inverse transformation we get the fourth order ODE in (x, u) variables:

$$u^{(4)} + (a_1 u'' + a_0)u''' + b_3 u''^3 + b_2 u''^2 + b_1 u'' + b_0 = 0, \tag{3.22}$$

and the coefficients being functions of (x, u') must satisfy

$$\begin{aligned} & a_{0,u'} - a_{1,x} = 0, \quad (3b_1 - a_0{}^2 - 3a_{0,x})_{,u'} = 0, \\ & 3a_{1,x} + a_0 a_1 - 3b_2 = 0, \quad 3a_{1,u'} + a_1{}^2 - 9b_3 = 0, \\ & (9b_1 - 6a_{0,x} - 2a_0{}^2)a_{1,x} + 9(b_{1,x} - a_1 b_0)_{,u'} + 3b_{1,u'}a_0 - 27b_{0,u'u'} = 0. \end{aligned} \tag{3.23}$$

Equation (3.20) is linearizable of IM type II (2.17), if f is of the form

$$\begin{aligned} f(x, y, y', y'') = & \frac{-1}{y' + \lambda}[-3(y'')^2 + (c_2 y'^2 + c_1 y' + c_0)y'' \\ & + d_5 y'^5 + d_4 y'^4 + d_3 y'^3 + d_2 y'^2 + d_1 y' + d_0], \end{aligned} \tag{3.24}$$

and the coefficients $c_i = c_i(x, y)$, $(i = 0, 1, 2)$, $d_j = d_j(x, y)$, $(j = 0, 1, \ldots, 5)$ and $\lambda = \lambda(x, y)$ have to satisfy constraint equations presented in Appendix A.2.

Again by applying the inverse transformation to (3.20) with f of the form (3.24) we get the fourth order ODE in (x, u) variables:

$$\begin{aligned} u^{(4)} + & \frac{1}{u'' + \lambda}[-3(u''')^2 + (c_2 u''^2 + c_1 u'' + c_0)u''' \\ & + d_5 u''^5 + d_4 u''^4 + d_3 u''^3 + d_2 u''^2 + d_1 u'' + d_0] = 0. \end{aligned} \tag{3.25}$$

The coefficients are now functions of x and u' and they satisfy constraints presented in Appendix A.3.

We give the above results in the form of the following theorems.

Theorem 3.2.3. *Equation of the form* (3.22) *is reduced to the third order linearizable equation (Meleshko linearizable of order one) if and only if its coefficients satisfy the constraints* (3.23).

Theorem 3.2.4. *Equation of the form* (3.25) *is reduced to the third order linearizable form (Meleshko linearizable of order one) if and only if its coefficients satisfy the constraints given in Appendix A.3.*

Case III. Fourth order ODEs independent of x and u

Consider the following ODE

$$u^{(4)} = f(u', u'', u'''). \tag{3.26}$$

By considering u' as the independent and u'' as the dependent variable, we convert the equation (3.26) into a second order ODE:

$$y^2 y'' + yy'^2 = f(u', y, yy'). \tag{3.27}$$

For equation (3.27) to be Lie linearizable, we must have

$$f(u', y, yy') = -y^2[a(u', y)y'^3 + b(u', y)y'^2 + c(u', y)y' + d(u', y)] + yy'^2. \tag{3.28}$$

Hence equation (3.26) takes the form

$$u^{(4)} + a(u', u'')u'''^3 + b(u', u'')u'''^2 + c(u', u'')u''' + d(u', u'') = 0, \tag{3.29}$$

where a, b, c and d must satisfy the constraints:

$$\begin{aligned}
&(3a_{,u'u'})u''^4 + (2bb_{,u'} - 3ca_{,u'} - 3ac_{,u'} - 2b_{,u'u''})u''^3 + (2b_{,u'} - bc_{,u''}\\
&+3a_{,u''}d + 6ad_{,u''} - c_{,u''u''})u''^2 + (bc - 9ad - 3c_{,u''})u'' - c = 0,\\
&(b_{u'u'})u''^4 + (b_{,u'}c + 3d_{,u''}b - 3d_{,u'}a - 6a_{,u'}d - 2c_{,u'u''})u''^3 + (c_{,u'} + 3d_{,u''}\\
&-6bd + 3b_{,u''}d - 2cc_{,u''} + 3d_{,u''u''})u''^2 + (2c^2 - 6d - 12d_{,u''})u'' + 15d = 0.
\end{aligned} \tag{3.30}$$

Hence we have the following theorem.

Theorem 3.2.5. *Equation* (3.29) *is reduced to the second order linearizable equation (Meleshko linearizable of order two) if and only if it obeys* (3.30)*.*

Remark 3.2.6. It is to be remarked here that the symmetry Lie algebra admitted by Meleshko linearizable ODEs of order two is non-commutative in general. For example, the ODE

$$u''^3 u^{(4)} + u'u'''^3 = 0, \tag{3.31}$$

is of the form (3.29) with the coefficients satisfying the constraints (3.30). It has the following 4 Lie-point symmetries:

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial u}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial x}, \quad \mathbf{X}_4 = u\frac{\partial}{\partial u}, \tag{3.32}$$

that form a non-Abelian algebra.

Remark 3.2.7. If we have a fourth order ODE of the form

$$u^{(4)} = -f(x,u)u'^5 + 10\frac{u''u'''}{u'} - 15\frac{u''^3}{u'^2},$$

with $f(x,u)$ linear in x, then we can convert it to a linear ODE $x^{(4)} = f(x,u)$ by simply taking x as dependent and u as independent variables.

3.3 Illustrative examples

Example 1. The nonlinear fourth order ODE

$$u^3u'u^{(4)} - u^3u''u''' + 3u^2u'^2u''' + 3u'^5 = 0, \tag{3.33}$$

cannot be linearized by point or contact transformation and has only two Lie-point symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = u\frac{\partial}{\partial u}.$$

It is of the form (3.14) with the coefficients

$$A_1 = -\frac{1}{u'}, \quad A_0 = 3\frac{u'}{u}, \quad B_3 = B_2 = B_1 = 0, \quad B_0 = 3\frac{u'^4}{u^3}.$$

One can verify that these coefficients satisfy the conditions (3.16). The transformation $u' = y(u)$ will reduce this ODE to the third order linearizable ODE

$$u^3yy''' + 3u^2(uy' + y)y'' + 3u^2y'^2 + 3y^2. \tag{3.34}$$

By using the linearizing transformation equations given in Appendix $A.1$, we arrive at the transformation

$$t = u, \quad s = uy^2, \tag{3.35}$$

which maps (3.34) to the linear third order ODE

$$s''' + \frac{6}{t^3}s = 0,$$

whose solution is given by

$$s = c_1 t^{-1} + t^2\{c_2 \cos(\sqrt{2}\ln t) + c_3 \sin(\sqrt{2}\ln t)\},$$

where c_i are arbitrary constants. By using the inverse transformation of (3.35) we get the solution of (3.34) given by

$$y = \pm\sqrt{c_1 u^{-2} + c_2 u \cos(\sqrt{2}\ln u) + c_3 u \sin(\sqrt{2}\ln u)}.$$

Hence the general solution of (3.33) is obtained by taking the quadrature

$$\int \frac{du}{\sqrt{c_1 u^{-2} + c_2 u \cos(\sqrt{2}\ln u) + c_3 u \sin(\sqrt{2}\ln u)}} = \pm x + c_4,$$

where c_i are arbitrary constants.

Example 2. The nonlinear ODE

$$u^2 u'^2 u^{(4)} - 10u^2 u' u'' u''' - 3uu'^3 u''' + 15u^2 u''^3 + 9uu'^2 u''^2 + 3u'^4 u'' = 0, \tag{3.36}$$

is of the form (3.14) with the coefficients

$$A_1 = \frac{-10}{u'}, \quad A_0 = \frac{-3u'}{u}, \quad B_3 = \frac{15}{u'^2}, \quad B_2 = \frac{9}{u}, \quad B_1 = \frac{3u'^2}{u^2}, \quad B_0 = 0,$$

satisfying the conditions (3.16). So it is reduced to the third order linearizable ODE

$$u^2 y^2 y''' - 3uy^2 y'' - 6u^2 yy' y'' + 3y^2 y' + 6uyy'^2 + 6u^2 y'^3 = 0, \tag{3.37}$$

in (u, y) variables. The transformation

$$t = u^2, \quad s = \frac{1}{y},$$

reduces (3.37) to the linear third order ODE $s''' = 0$, whose solution is

$$s = c_1 t^2 + c_2 t + c_3.$$

Now one only needs to solve the equation

$$u' = 1/(c_1u^4 + c_2u^2 + c_3),$$

where c_i are arbitrary constants. Hence, the general solution of (3.36) is given by

$$x = c_1^*u^5 + c_2^*u^3 + c_3u + c_4,$$

where $c_1^*,\ c_2^*,\ c_4$ and c_4 are arbitrary constants with $c_1^* = \frac{c_1}{5},\ c_2^* = \frac{c_2}{3}$.

Example 3. The ODE

$$u'u''u^{(4)} - 3u'u'''^2 + 6u'^3u''^2u''' - 4u''^2u''' - u'u''^5 = 0, \tag{3.38}$$

has two Lie-point symmetries. It is of the form (3.18) with the coefficients

$$\lambda_0 = 0, \quad C_2 = 6u'^2 - \frac{4}{u'}, \quad C_1 = C_0 = 0, \quad D_5 = -1, \quad D_4 = D_3 = D_2 = D_1 = D_0 = 0,$$

obey the conditions (A.29)−(A.37). So it is reducible to the third order linearizable ODE

$$y''' + \frac{1}{y'}[-3y''^2 - uy'^5] = 0. \tag{3.39}$$

The transformation

$$t = y, \quad s = u,$$

will convert the nonlinear ODE (3.39) to the linear ODE

$$s''' + s = 0,$$

with solution

$$s = c_1e^{-t} + c_2e^{\frac{t}{2}}\cos t + c_3e^{\frac{t}{2}}\sin t.$$

Finally to find the solution of (3.38), we only need to solve

$$u = c_1e^{-u'} + c_2e^{\frac{u'}{2}}\cos u' + c_3e^{\frac{u'}{2}}\sin u'.$$

Example 4. The nonlinear ODE

$$u'^2 u'' u^{(4)} - u'^2 u'''^3 - 2u'^2 u'''^2 + u' u''^2 u''' + u''^4 = 0, \tag{3.40}$$

is of the form (3.29) and the coefficients

$$a = -\frac{1}{u''}, \quad b = -\frac{2}{u''}, \quad c = \frac{u''}{u'}, \quad d = \frac{u''^3}{u'^2},$$

satisfy the conditions (3.30). So it is reduced to the second order linearizable ODE

$$v^2 y y'' - 2y'^2 v^2 + y y' v + y^2 = 0,$$

by considering $u' = v$ as independent and $u'' = y$ as dependent variables. The transformation

$$t = v, \quad s = \frac{v}{y},$$

reduces it to linear ODE

$$s'' = 0,$$

whose solution is given by

$$s = c_1 t + c_2,$$

where c_i are arbitrary constants. So the solution of (3.40) is given by quadrature

$$x = c_1 u' + c_2 \ln u' + c_3,$$

where c_i are arbitrary constants.

Example 5. The nonlinear ODE

$$(u' + 1)u^{(4)} + 3(u' + 2)u'' u''' + (u' + 3)u''^3 + x^3(u' + 1)u'' + xu' = 0, \tag{3.41}$$

is of the form (3.22) and the coefficients satisfying the constraint requirements (3.23). So it can be reduced to the third order ODE

$$(y + 1)y''' + 3(y + 2)y' y'' + (y + 3)y'^3 + x^3(yy' + y') + xy = 0.$$

By using the transformation

$$t = x, \quad s = ye^{y}, \tag{3.42}$$

we can reduce the above third order ODE to the linear equation

$$s''' + t^3 s' + ts = 0,$$

whose solution can easily be found.

Chapter 4

Linearization of two dimensional systems of second and third order ODEs by complex methods

CSA has been employed to solve certain classes of systems of nonlinear ODEs and linear PDEs. Of particular interest here, is the linearization of systems of second order ODEs (see, e.g., [5, 6]) that is achieved by complex methods. These classes are obtained from linearizable scalar and systems of ODEs by regarding their dependent variables as complex functions of a real independent variable, which when split into the real and imaginary parts give two dependent variables. In this way, a scalar ODE produces a system of two coupled equations, with CR-structure on both the equations. These CR-equations appear as constraint equations that restrict the emerging systems of ODEs to special subclasses of the general class of such systems. These subclasses of $2-$dimensional systems of second order ODEs may trivially be studied with CSA, however, they appear to be nontrivial when viewed from real symmetry analysis. Complex-linearizable (c-linearizable) classes are characterized by complex transformations of the form

$$U : (x, u(x)) \to (t(x), s(x, u)). \tag{4.1}$$

When linearizable scalar second order ODEs are treated as complex by considering the dependent variable as a complex function of a real independent variable, they lead to the c-linearization. On splitting the complex functions involved in the associated constraint equations (2.12), we get four equations. These four equations constitute the c-linearization criteria [5], for the corresponding class of systems of two second order ODEs. The reason for calling them c-linearization, instead of linearization criteria is that, in earlier works, explicit Lie's procedure to obtain linearization conditions of this class of systems, was not performed after incorporating complex symmetry approach on scalar ODEs. The most general form of the c-linearizable, $2-$dimensional, linearizable systems of second order ODEs is obtained here by real and complex methods. This derivation shows that the general linearizable forms (obtained by real and complex procedures) of $2-$dimensional, c-linearizable systems of second order ODEs are identical. Moreover, associated linearization criteria have been derived, again by adopting both the real and complex symmetry methods. These linearization conditions are also shown to be similar whether derived from Lie's procedure developed for systems or by employing complex symmetry analysis on scalar ODEs [19]. We exploit this result to obtain linearizable form and sufficient conditions for the linearization of systems of third order ODEs [20]. This chapter is mainly divided into two sections. First section is on the linearization of c-linearizable systems of second order ODEs. The result is then employed to obtain linearization criteria for the systems of third order ODEs in the second section.

4.1 Linearization of $2-$dimensional c-linearizable systems of second order ODEs

The point transformations (2.1) yield the most general form of scalar, second order, linearizable ODEs (2.10) that is derived in section 2.1.2. Restricting these transformations to

$$t = \phi(x), \quad s = \psi(x, u), \tag{4.2}$$

i.e., assuming $\phi_{,u}=0$, leads to a quadratically semilinear, scalar, second order ODE that is derived here explicitly. Under transformations (4.2) the first and second order derivatives of $s(t)$ with respect to t read as

$$s' = \frac{D_x\psi(x,u)}{D_x\phi(x)} = p(x,u,u'),$$

and

$$s'' = \frac{D_x p(x,u,u')}{D_x\phi(x)},$$

respectively. Here

$$D_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + \cdots,$$

is the total derivative operator. Applying the total derivative operator in both the above equations leads us to the following

$$s' = \frac{\psi_{,x} + u'\psi_{,u}}{\phi_{,x}},$$

and

$$s'' = \frac{\phi_{,x}(\psi_{,xx} + 2u'\psi_{,xu} + u'^2\psi_{,uu} + u''\psi_{,u}) - \phi_{,xx}(\psi_{,x} + u'\psi_{,u})}{\phi_{,x}^3}, \tag{4.3}$$

respectively. Equating (4.3) to zero, i.e., considering $s''=0$, leaves a quadratically semilinear ODE of the form

$$u'' + a(x,u)u'^2 + b(x,u)u' + c(x,u) = 0, \tag{4.4}$$

with the coefficients

$$a(x,u) = \frac{\psi_{,uu}}{\psi_{,u}}, \quad b(x,u) = \frac{2\phi_{,x}\psi_{,xu} - \psi_{,u}\phi_{,xx}}{\phi_{,x}\psi_{,u}}, \quad c(x,u) = \frac{\phi_{,x}\psi_{,xx} - \psi_{,x}\phi_{,xx}}{\phi_{,x}\psi_{,u}}. \tag{4.5}$$

The quadratic, nonlinear (in the first derivative) equation (4.4) with three coefficients (4.5) is a subcase of the general linearizable (cubically semilinear) second order ODE (2.10).

Now for the derivation of the Lie linearization criteria of the nonlinear equation (4.4) via the point transformation (4.2), we start with the re-arrangement

$$\begin{aligned}\psi_{,uu} &= a(x,u)\psi_{,u},\\ 2\psi_{,xu} &= \phi_{,x}^{-1}\psi_{,u}\phi_{,xx} + b(x,u)\psi_{,u},\\ \psi_{,xx} &= \phi_{,x}^{-1}\psi_{,x}\phi_{,xx} + c(x,u)\psi_{,u}.\end{aligned}$$

of the relations (4.5). Equating the mixed derivatives of ψ, such that

$$(\psi_{,xu})_{,u} = (\psi_{,uu})_{,x} \quad \text{and} \quad (\psi_{,xu})_{,x} = (\psi_{,xx})_{,u}, \tag{4.6}$$

we find

$$b_{,u} - 2a_{,x} = 0, \tag{4.7}$$

and

$$\phi_{,x}^{-2}(2\phi_{,x}\phi_{,xx} - 3\phi_{,xx}^{2}) = 4(c_u + ac) - (2b_{,x} + b^2). \tag{4.8}$$

As $\phi_{,u} = 0$, differentiating (4.8) with respect to u, simplifies it to

$$c_{,uu} - a_{,xx} - a_{,x}b + a_{,u}c + c_{,u}a = 0. \tag{4.9}$$

Equations (4.7) and (4.9) constitute the linearization criteria for the scalar, second order, quadratically semilinear ODEs (4.4).

In the subsequent subsections we derive the c-linearization and Lie linearization criteria for a system of two second order ODEs.

4.1.1 c-linearization

Treat $u(x)$ in (4.4) as a complex function of a real variable x, i.e. $u(x) = y(x) + iz(x)$. Further assume that

$$\begin{aligned}a(x,u) &= a_1(x,y,z) + ia_2(x,y,z),\\ b(x,u) &= b_1(x,y,z) + ib_2(x,y,z),\\ c(x,u) &= c_1(x,y,z) + ic_2(x,y,z).\end{aligned} \tag{4.10}$$

This converts the scalar ODE (4.4) to a system of two second order ODEs of the form

$$\begin{aligned} y'' + a_1 y'^2 - 2a_2 y'z' - a_1 z'^2 + b_1 y' - b_2 z' + c_1 = 0, \\ z'' + a_2 y'^2 + 2a_1 y'z' - a_2 z'^2 + b_2 y' + b_1 z' + c_2 = 0, \end{aligned} \tag{4.11}$$

with the coefficients $a_j, b_j, c_j; (j = 1, 2)$, satisfying the CR-equations

$$\begin{aligned} a_{1,y} = a_{2,z}, \quad a_{1,z} = -a_{2,y}, \\ b_{1,y} = b_{2,z}, \quad b_{1,z} = -b_{2,y}, \\ c_{1,y} = c_{2,z}, \quad c_{1,z} = -c_{2,y}. \end{aligned} \tag{4.12}$$

Moreover, the conditions (4.7) and (4.9) can now be converted into a set of four equations

$$2a_{1,x} - b_{1,y} = 0, \tag{4.13}$$

$$2a_{2,x} + b_{1,z} = 0, \tag{4.14}$$

$$c_{1,zz} + a_{1,xx} + a_{1,x}b_1 - a_{2,x}b_2 - (a_2c_1)_{,z} - (a_1c_2)_{,z} = 0, \tag{4.15}$$

$$c_{2,yy} - a_{2,xx} - a_{2,x}b_1 - a_{1,x}b_2 + (a_2c_1)_{,y} + (a_1c_2)_{,y} = 0, \tag{4.16}$$

by splitting the complex coefficients (4.12) into the real and imaginary parts.

As evident from [5], such a (complex) procedure leads us to the c-linearization of systems of ODEs. Our claim here is that the equations (4.13)−(4.16) are actually the linearization conditions despite of being just the c-linearization conditions for the system (4.11). In order to prove this fact, we now use the Lie linearization approach in the next subsection to derive the linearization conditions for the system (4.11).

4.1.2 Lie linearization

The previous work on the c-linearizable [5, 6] and their linearizable subclass of systems [64, 66] of second order ODEs reveals that point transformations of the form

$$t = \phi(x), \quad v = \psi_1(x, y, z), \quad w = \psi_2(x, y, z), \tag{4.17}$$

where

$$\psi_{1,y} = \psi_{2,z}, \quad \psi_{2,y} = -\psi_{1,z}, \tag{4.18}$$

i.e., ψ_j, for $j = 1, 2$, satisfy the CR-equations that involve derivatives with respect to both the dependent variables, linearizes the c-linearizable systems. Notice that (4.17) are obtainable from (4.2) that is a subclass of (2.1). These transformations map the first and second order derivatives as

$$v' = \frac{D_x\psi_1}{D_x\phi} = p_1(x, y, z, y', z'), \quad (4.19)$$
$$w' = \frac{D_x\psi_2}{D_x\phi} = p_2(x, y, z, y', z'),$$

and

$$v'' = \frac{D_x p_1}{D_x\phi} = q_1(x, y, z, y', z', y'', z''), \quad (4.20)$$
$$w'' = \frac{D_x p_2}{D_x\phi} = q_2(x, y, z, y', z', y'', z''),$$

where

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z} + y''\frac{\partial}{\partial y'} + z''\frac{\partial}{\partial z'} + \cdots .$$

Inserting the total derivative operator in the above equations and substituting these in $v'' = 0,\ w'' = 0$ and simplifying, we arrive at the following 2-dimensional system

$$y'' + \alpha_1 y'^2 - 2\alpha_2 y'z' + \alpha_3 z'^2 + \beta_1 y' - \beta_2 z' + \gamma_1 = 0,$$
$$z'' + \alpha_4 y'^2 + 2\alpha_5 y'z' + \alpha_6 z'^2 + \beta_3 y' + \beta_4 z' + \gamma_2 = 0, \quad (4.21)$$

where

$$\alpha_1 = \phi_{,x}\Delta^{-1}(\psi_{2,z}\psi_{1,yy} - \psi_{1,z}\psi_{2,yy}),$$
$$\alpha_2 = \phi_{,x}\Delta^{-1}(\psi_{1,z}\psi_{2,yz} - \psi_{2,z}\psi_{1,yz}),$$
$$\alpha_3 = \phi_{,x}\Delta^{-1}(\psi_{2,z}\psi_{1,zz} - \psi_{1,z}\psi_{2,zz}),$$
$$\alpha_4 = \phi_{,x}\Delta^{-1}(\psi_{1,y}\psi_{2,yy} - \psi_{2,y}\psi_{1,yy}),$$
$$\alpha_5 = \phi_{,x}\Delta^{-1}(\psi_{1,y}\psi_{2,yz} - \psi_{2,y}\psi_{1,yz}),$$
$$\alpha_6 = \phi_{,x}\Delta^{-1}(\psi_{1,y}\psi_{2,zz} - \psi_{2,y}\psi_{1,zz}),$$
$$\beta_1 = 2\phi_{,x}\Delta^{-1}(\psi_{2,z}\psi_{1,xy} - \psi_{1,z}\psi_{2,xy}) - \frac{\phi_{,xx}}{\phi_{,x}},$$
$$\beta_2 = 2\phi_{,x}\Delta^{-1}(\psi_{1,z}\psi_{2,xz} - \psi_{2,z}\psi_{1,xz}),$$

$$\beta_3 = 2\phi_{,x}\Delta^{-1}(\psi_{1,y}\psi_{2,xy} - \psi_{2,y}\psi_{1,xy}),$$
$$\beta_4 = 2\phi_{,x}\Delta^{-1}(\psi_{1,y}\psi_{2,xz} - \psi_{2,y}\psi_{1,xz}) - \frac{\phi_{,xx}}{\phi_x},$$
$$\gamma_1 = \Delta^{-1}(\phi_{,x}\psi_{1,y}\psi_{1,xx} - \psi_{1,x}\psi_{1,y}\phi_{,xx} - \phi_{,x}\psi_{1,z}\psi_{2,xx} + \psi_{1,z}\psi_{2,x}\phi_{,xx}) \ ,$$
$$\gamma_2 = \Delta^{-1}(\phi_{,x}\psi_{1,z}\psi_{1,xx} - \psi_{1,x}\psi_{1,z}\phi_{,xx} + \phi_{,x}\psi_{1,y}\psi_{2,xx} + \psi_{1,y}\psi_{2,x}\phi_{,xx}) \ , \qquad (4.22)$$

with

$$\Delta = \phi_{,x}(\psi_{1,y}\psi_{2,z} - \psi_{1,z}\psi_{2,y}) \neq 0,$$

is the Jacobian of the transformation (4.17). The coefficients (4.5) of the scalar ODE (4.4) split into the coefficients of the corresponding $2-$dimensional system of second order ODEs. This happens due to the presence of the complex dependent function u in the coefficients (4.5). The restricted fibre preserving transformations (4.17) used to derive the linearizable form (4.21), are obtainable from the complex transformations (4.2) that are employed to deduce (4.4). Therefore, the transformations (4.17) along with (4.18) appear to be the real and imaginary parts of complex transformation (4.2), they reveal the correspondence of the linearizable forms of $2-$dimensional systems and scalar complex ODEs. The CR-equations are not yet incorporated in the linearizable form (4.21). Insertion of the CR-equations (4.18) and their derivatives

$$\psi_{1,yy} = \psi_{2,yz} = -\psi_{1,zz},$$
$$\psi_{2,zz} = \psi_{1,yz} = -\psi_{2,yy}, \qquad (4.23)$$

brings out the correspondence between the coefficients (4.5) of the complex linearizable ODEs (4.4) and the coefficients (4.22) of the system (4.21). Employing (4.18) and (4.23) the number of coefficients (4.22) reduces to six that read as

$$\alpha_1 = -\alpha_3 = \alpha_5 = a_1, \quad \alpha_2 = \alpha_4 = -\alpha_6 = a_2,$$
$$\beta_1 = \beta_4 = b_1, \quad \beta_2 = \beta_3 = b_2, \quad \gamma_1 = c_1, \quad \gamma_2 = c_2. \qquad (4.24)$$

Here the coefficients a_j, b_j and c_j are the real and imaginary parts of the complex coefficients (4.5). The linearizable form of the systems derived in this section by real method is the same as one obtains by splitting the corresponding form of the scalar complex equation (4.4). This analysis leads us to the following theorem.

Theorem 4.1.1. *The most general form of the linearizable, 2−dimensional, c-linearizable systems of second order ODEs is quadratically semilinear.*

Sufficient conditions for the linearization of a c-linearizable system

Consider the most general form of the c-linearizable, 2−dimensional systems of second order ODEs (4.11), with constraint equations (4.12). Rewriting the coefficients of the system (4.11) in the form

$$\begin{aligned}
a_1 &= \Delta^{-1}\phi_{,x}(\psi_{1,y}\psi_{1,yy} + \psi_{1,z}\psi_{1,yz}),\\
a_2 &= \Delta^{-1}\phi_{,x}(\psi_{1,z}\psi_{1,yy} + \psi_{1,y}\psi_{1,yz}),\\
b_1 &= 2\Delta^{-1}\phi_{,x}(\psi_{1,y}\psi_{1,xy} + \psi_{1,z}\psi_{1,xz}) - \frac{\phi_{,xx}}{\phi_{,x}},\\
b_2 &= 2\Delta^{-1}\phi_{,x}(\psi_{1,z}\psi_{1,xy} + \psi_{1,y}\psi_{1,xz}),\\
c_1 &= \Delta^{-1}(\phi_{,x}\psi_{1,y}\psi_{1,xx} - \psi_{1,x}\psi_{1,y}\phi_{,xx} - \phi_{,x}\psi_{1,z}\psi_{2,xx} + \psi_{1,z}\psi_{2,x}\phi_{,xx}),\\
c_2 &= \Delta^{-1}(\phi_{,x}\psi_{1,z}\psi_{1,xx} - \psi_{1,x}\psi_{1,z}\phi_{,xx} + \phi_{,x}\psi_{1,y}\psi_{2,xx} + \psi_{1,y}\psi_{2,x}\phi_{,xx}).
\end{aligned} \tag{4.25}$$

For obtaining the sufficient conditions of linearization for (4.11), we have to solve the compatibility problem, that has already been solved for the scalar equations earlier in this work, for the set of equations (4.25). It is an over determined system of PDEs for the functions ϕ, ψ_1 and ψ_2 with known a_j, b_j, c_j.

The system (4.25) gives us

$$\begin{aligned}
\psi_{1,yy} &= \psi_{1,y}a_1 + \psi_{1,z}a_2, \quad \psi_{1,yz} = \psi_{1,z}a_1 - \psi_{1,y}a_2,\\
\psi_{1,xy} &= \frac{1}{2}(\psi_{1,y}b_1 + \psi_{1,z}b_2 + \psi_{1,y}\frac{\phi_{,xx}}{\phi_{,x}}),\\
\psi_{1,xz} &= \frac{1}{2}(\psi_{1,z}b_1 - \psi_{1,y}b_2 + \psi_{1,z}\frac{\phi_{,xx}}{\phi_{x}}),\\
\psi_{1,xx} &= \psi_{1,y}c_1 + \psi_{1,z}c_2 + \psi_{1,x}\frac{\phi_{,xx}}{\phi_{,x}},\\
\psi_{2,xx} &= \psi_{1,y}c_2 - \psi_{1,z}c_1 + \psi_{2,x}\frac{\phi_{,xx}}{\phi_{,x}}.
\end{aligned}$$

The compatibility of the system (4.25) first requires to compute partial derivatives $\Delta_{,x},\ \Delta_{,y},\ \Delta_{,z}$, which are

$$\begin{aligned}\Delta_{,x} &= 2\Delta\frac{\phi_{,xx}}{\phi_{,x}}+\Delta b_1,\\ \Delta_{,y} &= 2\Delta a_1,\\ \Delta_{,z} &= -2\Delta a_2,\end{aligned}$$

of the Jacobian. Comparing the mixed derivatives $(\Delta_{,y})_{,z}=(\Delta_{,z})_{,y}$, we obtain

$$a_{1,z}+a_{2,y}=0. \tag{4.26}$$

Invoking $(\Delta_{,x})_{,y}=(\Delta_{,y})_{,x}$ gives

$$2a_{1,x}-b_{1,y}=0, \tag{4.27}$$

and $(\Delta_{,x})_{,z}=(\Delta_{,z})_{,x}$, we obtain

$$2a_{2,x}+b_{1,z}=0. \tag{4.28}$$

Equating the mixed derivatives

$$\begin{aligned}&(\psi_{1,yy})_{,z}=(\psi_{1,yz})_{,y},\quad (\psi_{1,yy})_{,x}=(\psi_{1,xy})_{,y},\quad (\psi_{1,xx})_{,y}=(\psi_{1,xy})_{,x},\\ &(\psi_{1,xx})_{,z}=(\psi_{1,xz})_{,x},\quad (\psi_{1,xy})_{,z}=(\psi_{1,xz})_{,y},\quad (\psi_{2,xx})_{,y}=(\psi_{2,xy})_{,x}\\ &\text{and}\quad (\psi_{2,xx})_{,z}=(\psi_{2,xz})_{,x},\end{aligned}$$

gives us

$$a_{1,y}-a_{2,z}=0, \tag{4.29}$$

$$b_{2,y}+b_{1,z}=0, \tag{4.30}$$

$$b_{2,z}-b_{1,y}=0, \tag{4.31}$$

$$c_{2,z}-c_{1,y}=0, \tag{4.32}$$

$$c_{2,y}+c_{1,z}=0, \tag{4.33}$$

$$c_{1,zz}+a_{1,xx}+a_{1,x}b_1-a_{2,x}b_2-(a_2c_1)_{,z}-(a_1c_2)_{,z}=0, \tag{4.34}$$

$$c_{2,yy}-a_{2,xx}-a_{2,x}b_1-a_{1,x}b_2+(a_1c_2)_{,y}-(a_2c_1)_{,y}=0, \tag{4.35}$$

respectively. Note that $(\psi_{1,yz})_{,x} - (\psi_{1,xz,})_{,y} = 0$ and $(\psi_{1,xy})_{,z} - (\psi_{1,yz})_{,x} = 0$ are satisfied. Also (4.26), (4.29), and (4.30)-(4.33) are CR-equations for the coefficients a_j, b_j, c_j. Therefore, the solution of the compatibility problem of the system (4.25), provides CR-constraints on the coefficients of (4.11) and the linearization conditions.

Theorem 4.1.2. *A $2-$dimensional, c-linearizable system of second order ODEs of the form* (4.11) *is linearizable if and only if its coefficients satisfy the CR-equations and conditions* (4.27)*,* (4.28)*,* (4.34)*,* (4.35)*.*

Note that these are the same conditions (4.13)$-$(4.16) that are already obtained, by employing complex analysis, i.e., splitting the linearization conditions associated with the base scalar equation (4.4), into the real and imaginary parts.

Corollary 4.1.3. *The c-linearization conditions for a $2-$dimensional system of quadratically semilinear, second order ODEs are the linearization conditions.*

4.1.3 Examples

We present some examples to illustrate our results.

Example 1. The 2-dimensional system of second order ODEs

$$\begin{aligned} y'' - (\frac{2y}{y^2+z^2})y'^2 - 2(\frac{2z}{y^2+z^2})y'z' + (\frac{2y}{y^2+z^2})z'^2 - \frac{2}{x}y' - \frac{2y}{x^2} = 0, \\ z'' + (\frac{2z}{y^2+z^2})y'^2 - 2(\frac{2y}{y^2+z^2})y'z' - (\frac{2z}{y^2+z^2})z'^2 - \frac{2}{x}z' - \frac{2z}{x^2} = 0. \end{aligned} \tag{4.36}$$

is of the same form as (4.11) with

$$a_1 = \frac{-2y}{y^2+z^2}, \quad a_2 = \frac{2z}{y^2+z^2}, \quad b_1 = \frac{-2}{x}, \quad b_2 = 0, \quad c_1 = \frac{-2y}{x^2}, \quad c_2 = \frac{-2z}{x^2}. \tag{4.37}$$

One can easily verify that (4.37) satisfy the conditions (4.27), (4.28), (4.34), (4.35) and CR-equations with respect to y and z. So the system of ODEs (4.36) is linearizable. The transformation

$$t = x, \quad v = \frac{y}{x(y^2+z^2)}, \quad w = \frac{-z}{x(y^2+z^2)}, \tag{4.38}$$

reduces the nonlinear system (4.36) to the linear system

$$v'' = 0, \quad w'' = 0, \tag{4.39}$$

whose solution is given by

$$v = c_1 x + c_2, \quad w = c_3 x + c_4. \tag{4.40}$$

By applying the inverse of the transformation (4.38) on solution of the linear system (4.40), we get the solution of the nonlinear system (4.36) explicitly

$$y = \frac{c_1 x + c_2}{x[(c_1 x + c_2)^2 + (c_3 x + c_4)^2]}, \quad z = -\frac{c_3 x + c_4}{x[(c_1 x + c_2)^2 + (c_3 x + c_4)^2]} \tag{4.41}$$

Example 2. Consider the following system of nonlinear ODEs

$$\begin{aligned} y'' - \frac{1}{f(y,z)}(y'^2 \cos y \sin y - z'^2 \cos y \sin y - 2y'z' \cosh z \sinh z) + \frac{2y'}{x} &= 0, \\ z'' - \frac{1}{f(y,z)}(y'^2 \cosh z \sinh z - z'^2 \cosh z \sinh z + 2y'z' \cos y \sin y) + \frac{2z'}{x} &= 0, \end{aligned} \tag{4.42}$$

where $f(y,z) = \sin^2 y \cosh^2 z + \cos^2 y \sinh^2 z$ .The coefficients satisfy the CR-equations and the linearization conditions (4.27), (4.28), (4.34), (4.35). Hence Theorem 4.1.2 guarantees that the system (4.42) is linearizable. In fact, it can be transformed to the linear system (4.39) via the linearizing transformations

$$t = x, \quad v = x \cos y \cosh z, \quad w = -x \sin y \sinh z. \tag{4.43}$$

The solution of (4.42) is given implicitly by applying the inverse of the transformation (4.43) to the solution (4.40) of the linear system (4.39)

$$\cos y \cosh z = c_1 + \frac{c_2}{x}, \quad \sin y \sinh z = -c_3 - \frac{c_4}{x}, \tag{4.44}$$

where all c_i are arbitrary constants.

Example 3. Consider the anisotropic oscillator system

$$\begin{aligned} y'' + f(x)y &= 0, \\ z'' + g(x)z &= 0. \end{aligned} \tag{4.45}$$

In [52] it is shown that the system (4.45) is reducible to the linear system (4.39) provided $f = g$. Our c-linearization criteria also lead to the same condition, i.e. $f = g$.

In the next section we use Theorems 4.1.1 and 4.1.2 to obtain the linearization criteria for the 2−dimensional systems of third order ODEs.

4.2 Linearization of the 2-dimensional systems of third order ODEs

We have shown in section 2.1.3 that a scalar third order ODE in real variables linearizes to (2.15) via the restricted real point transformations (4.2), if it is of the form (2.16). Treat u in (2.16) as a complex function of a real variable x i.e.,

$$u(x) = y(x) + iz(x), \tag{4.46}$$

and the coefficients of (2.16) as complex functions

$$\begin{aligned} a_j(x,u) &= \alpha_j(x,y,z) + i\beta_j(x,y,z); \quad (j = 0,1), \\ b_k(x,u) &= \gamma_k(x,y,z) + i\delta_k(x,y,z); \quad (k = 0,1,2,3). \end{aligned} \tag{4.47}$$

This converts the equation (2.16) in it to a system of two third order ODEs, when split into the real and imaginary parts. The system so obtained according to (4.46) and (4.47) is

$$\begin{aligned} &y''' + (\alpha_1 y' - \beta_1 z' + \alpha_0)y'' - (\beta_1 y' + \alpha_1 z' + \beta_0)z'' + \gamma_3 y'^3 - 3\delta_3 y'^2 z' \\ &\quad -3\gamma_3 y' z'^2 + \delta_3 z'^3 + \gamma_2 y'^2 - 2\delta_2 y' z' - \gamma_2 z'^2 + \gamma_1 y' - \delta_1 z' + \gamma_0 = 0, \\ &z''' + (\beta_1 y' + \alpha_1 z' + \beta_0)y'' + (\alpha_1 y' - \beta_1 z' + \alpha_0)z'' + \delta_3 y'^3 + 3\gamma_3 y'^2 z' \\ &\quad -3\delta_3 y' z'^2 - \gamma_3 z'^3 + \delta_2 y'^2 + 2\gamma_2 y' z' - \delta_2 z'^2 + \delta_1 y' + \gamma_1 z' + \delta_0 = 0, \end{aligned} \tag{4.48}$$

where the coefficients $\alpha_j,\ \beta_j;\ (j = 0,1),\ \gamma_k,\ \delta_k;\ (k = 0,1,2,3)$, are functions of (x,y,z). These coefficients are analytical functions of (x,y,z), so they satisfy the CR-equations given by

$$\alpha_{j,y} = \beta_{j,z}, \quad \alpha_{j,z} = -\beta_{j,y}; \quad (j = 0, 1)$$
$$\gamma_{k,y} = \delta_{k,z}, \quad \gamma_{k,z} = -\delta_{k,y}; \quad (k = 0, 1, 2, 3). \tag{4.49}$$

Theorem 4.2.1. *The system of ODEs* (4.48) *represents the most general form of 2-dimensional systems of third order ODEs, that can be a candidate of linearization due to CSA.*

Proof. The result follows by employing the point transformations (4.17) with the CR-structure (4.18) on the system

$$v'''(t) + k_1(t)v(t) - k_2(t)w(t) = 0,$$
$$w'''(t) + k_2(t)v(t) + k_1(t)w(t) = 0,$$

that corresponds to (2.15) due to $s(t) = v(t) + iw(t)$ and $k_0(t) = k_1(t) + ik_2(t)$, map it to (4.48). □

Theorem 4.2.2. *The sufficient conditions for a* $2-$*dimensional system of third order ODEs of the form* (4.48) *to be linearizable are that its coefficients satisfy CR-equations* (4.49) *and the following conditions*

$$\alpha_{0,y} + \beta_{0,z} - 2\alpha_{1,x} = 0,$$
$$\beta_{0,y} - \alpha_{0,z} - 2\beta_{1,x} = 0,$$
$$3\alpha_{1,x} + \alpha_0\alpha_1 - \beta_0\beta_1 - 3\gamma_2 = 0,$$
$$3\beta_{1,x} + \alpha_0\beta_1 + \alpha_1\beta_0 - 3\delta_2 = 0,$$
$$3\alpha_{1,y} + 3\beta_{1,z} + 2\alpha_1^2 - 2\beta_1^2 - 18\gamma_3 = 0,$$
$$3\beta_{1,y} - 3\alpha_{1,z} + 4\alpha_1\beta_1 - 18\delta_3 = 0,$$
$$3\gamma_{1,y} + 3\delta_{1,z} - 2(\alpha_0\beta_0)_{,z} - (\alpha_0)^2{}_{,y} + (\beta_0)^2{}_{,y} - 3\beta_{0,xz} - 3\alpha_{0,xy} = 0,$$
$$3\delta_{1,y} - 3\gamma_{1,z} - 2(\alpha_0\beta_0)_{,y} + (\alpha_0)^2{}_{,z} - (\beta_0)^2{}_{,z} - 3\beta_{0,xy} + 3\alpha_{0,xz} = 0,$$

$$
\begin{aligned}
&4(9\gamma_1 - 6\alpha_{0,x} - 2\alpha_0^2 + 2\beta_0^2)\alpha_{1,x} - 4(9\delta_1 - 6\beta_{0,x} - 4\alpha_0\beta_0)\beta_{1,x} \\
&+18\gamma_{1,xy} + 18\delta_{1,xz} - 18(\alpha_1\gamma_0)_{,y} + 18(\beta_1\delta_0)_{,y} - 18(\alpha_1\delta_0)_{,z} \\
&-18(\beta_1\gamma_0)_{,z} + 6(\gamma_{1,y}\alpha_0 + \delta_{1,z}\alpha_0 + \gamma_{1,z}\beta_0 - \delta_{1,y}\beta_0) \\
&-27(\gamma_{0,yy} + 2\delta_{0,yz} - \gamma_{0,zz}) = 0, \\
&4(9\gamma_1 - 6\alpha_{0,x} - 2\alpha_0^2 + 2\beta_0^2)\beta_{1,x} + 4(9\delta_1 - 6\beta_{0,x} - 4\alpha_0\beta_0)\alpha_{1,x} \\
&-18\gamma_{1,xz} + 18\delta_{1,xy} + 18(\alpha_1\gamma_0)_{,z} - 18(\beta_1\delta_0)_{,z} - 18(\alpha_1\delta_0)_{,y} \\
&-18(\beta_1\gamma_0)_{,y} + 6(\gamma_{1,y}\beta_0 + \delta_{1,z}\beta_0 - \gamma_{1,z}\alpha_0 + \delta_{1,y}\alpha_0) \\
&-27(\delta_{0,yy} - 2\gamma_{0,zy} - \delta_{0,zz}) = 0. \qquad (4.50)
\end{aligned}
$$

Proof. The derivation of the linearization criteria for 2-dimensional systems of second order ODEs with CR-structure, by point transformations of the form (4.17) along with the CR-constraints (4.18) has been demonstrated in section 4.1.2. There it is shown that splitting the linearization criteria associated with the base complex equations due to the complex transformations (4.2), into the real and imaginary parts, provides the sufficient conditions for the linearization of the corresponding systems. Therefore, linearization criteria for the 2-dimensional systems of third order ODEs (4.48) along with (4.49), are provided here by splitting of the linearization criteria (2.18) associated with the scalar third order ODEs. □

Scalar third order ODEs of the form (2.17) and their associated linearization criteria cannot be regarded as linearizing the corresponding systems of third order equations. This class of scalar third order ODEs (2.17) has been derived by exploiting the general point transformations (2.1), that leads to c-linearization instead of linearization. The linearizing conditions associated with this class of scalar equations reveal the complex-linearizability of the corresponding systems and do not ensure that they are transformable to a linear form.

4.2.1 Illustration of the result

Example 1. The 2−dimensional system of nonlinear third order ODEs

$$
\begin{aligned}
y''' + \left(\frac{3y'y'' - 3z'z''}{y^2+z^2}\right) y + \left(\frac{3z'y'' + 3y'z''}{y^2+z^2}\right) z = 0, \\
z''' + \left(\frac{3z'y'' + 3y'z''}{y^2+z^2}\right) y - \left(\frac{3y'z'' - 3z'y''}{y^2+z^2}\right) z = 0,
\end{aligned} \tag{4.51}
$$

where prime denotes differentiation with respect to x, has the same form as given in (4.48). Moreover, its non-zero coefficients

$$
\alpha_1 = \frac{3y}{y^2+z^2}, \quad \beta_1 = \frac{-3z}{y^2+z^2},
$$

satisfy the requirements of Theorem 4.2.2. So this system is transformable to the linear system

$$
v''' = 0, \quad w''' = 0, \tag{4.52}
$$

here prime denotes differentiation with respect to t, under the invertible point transformation

$$
t = x, \quad v = y^2 - z^2, \quad w = 2yz. \tag{4.53}
$$

The solution of (4.52) is given by

$$
v = c_1t^2 + c_2t + c_3, \quad w = c_4t^2 + c_5t + c_6,
$$

where all $c_1 \ldots, c_6$ are arbitrary constants. By using the inverse of the transformation (4.53), we get the general solution of (4.51) in the implicit form

$$
y^2 - z^2 = c_1x^2 + c_2x + c_3, \quad 2yz = c_4x^2 + c_5x + c_6.
$$

After a few calculations, we get the general solution in the explicit form

$$
\begin{aligned}
y &= \frac{\pm 1}{\sqrt{2}} \sqrt{\sqrt{(c_1x^2+c_2x+c_3)^2 + (c_4x^2+c_5x+c_6)^2} + (c_1x^2+c_2x+c_3)}, \\
z &= \frac{\pm 1}{\sqrt{2}} \sqrt{\sqrt{(c_1x^2+c_2x+c_3)^2 + (c_4x^2+c_5x+c_6)^2} - (c_1x^2+c_2x+c_3)}.
\end{aligned}
$$

Example 2. Consider the following 2-dimensional system of nonlinear third order ODEs

$$y''' + \left(3y' + \frac{3}{x}\right) y'' - 3z'z'' + y'^3 - 3y'z'^2 + \frac{3}{x}y'^2 - \frac{3}{x}z'^2 = 0,$$
$$z''' + 3z'y'' + \left(3y' + \frac{3}{x}\right) z'' + 3y'^2 z' - z'^3 + \frac{6}{x}y'z' = 0, \tag{4.54}$$

with the coefficients

$$\alpha_1 = 3, \quad \alpha_0 = \frac{3}{x}, \quad \gamma_3 = \delta_3 = 1, \quad \gamma_2 = \frac{3}{x},$$

and all others zero. The above system is of the form (4.48) and its coefficients satisfy the CR-equations and the constraints (4.50). Hence it is linearizable and transforms to the linear system

$$v''' = 0, \quad w''' = 0. \tag{4.55}$$

The linearizing transformations used in this case are

$$t = x, \quad v = xe^y \cos z, \quad w = xe^y \sin z. \tag{4.56}$$

The solution of (4.55) is

$$v = c_1 t^2 + c_2 t + c_3, \quad w = c_4 t^2 + c_5 t + c_6,$$

where all $c_1 \ldots, c_6$ are arbitrary constants. By using the inverse of the transformations (4.56), we obtain the general solution

$$y = \frac{1}{2} \ln[(c_1 x + c_2 + \frac{c_3}{x})^2 + (c_4 x + c_5 + \frac{c_6}{x})^2],$$
$$z = \arctan(\frac{c_4 x^2 + c_5 x + c_6}{c_1 x^2 + c_2 x + c_3}),$$

of the nonlinear system (4.54).

Example 3. Consider the following nonlinear system of third order ODEs

$$
\begin{aligned}
&y''' + \left(\frac{3yy'}{y^2+z^2} + \frac{3zz'}{y^2+z^2} - 3\right) y'' - \left(-\frac{3zy'}{y^2+z^2} + \frac{3yz'}{y^2+z^2}\right) z'' \\
&-\frac{3yz'^2}{y^2+z^2} - \frac{6zy'z'}{y^2+z^2} + \frac{3yz'^2}{y^2+z^2} + 2y' + 3y = 0, \\
&z''' + \left(-\frac{3zy'}{y^2+z^2} + \frac{3yz'}{y^2+z^2}\right) y'' - \left(\frac{3yz'}{y^2+z^2} + \frac{3zz'}{y^2+z^2} - 3\right) z'' \\
&+\frac{3zy'^2}{y^2+z^2} - \frac{6yy'z'}{y^2+z^2} - \frac{3zz'^2}{y^2+z^2} + 2z' + 3z = 0. \qquad (4.57)
\end{aligned}
$$

The coefficients

$$
\begin{aligned}
&\alpha_1 = \frac{3y}{y^2+z^2}, \quad \alpha_0 = -3, \quad \beta_1 = \frac{-3z}{y^2+z^2}, \quad \gamma_2 = \frac{-3y}{y^2+z^2}, \\
&\gamma_1 = 2, \quad \gamma_0 = 3y, \quad \delta_2 = \frac{3y}{y^2+z^2}, \quad \delta_0 = 3z, \\
&\gamma_3 = \delta_3 = \delta_1 = \beta_0 = 0. \qquad (4.58)
\end{aligned}
$$

satisfy the CR-equations and the conditions (4.50). Hence, (4.57) is linearizable. It is transformable to the linear system

$$
v''' + \frac{6}{t^3} v = 0, \quad w''' + \frac{6}{t^3} w = 0. \qquad (4.59)
$$

The linearizing transformations which establish the above correspondence between the nonlinear and linear systems are

$$
t = e^x, \quad v = y^2 - z^2, \quad w = 2yz. \qquad (4.60)
$$

The solution of (4.59) is

$$
\begin{aligned}
v &= c_1 t^{-1} + t^2\{c_2 \cos(\sqrt{2}\ln t) + c_3 \sin(\sqrt{2}\ln t)\}, \\
w &= c_4 t^{-1} + t^2\{c_5 \cos(\sqrt{2}\ln t) + c_6 \sin(\sqrt{2}\ln t)\}, \qquad (4.61)
\end{aligned}
$$

where all $c_1 \ldots, c_6$ are arbitrary constants. By using the inverse of the transformation

(4.60), the general solution of (4.57) is given explicitly by

$$\begin{aligned} y &= \frac{\pm 1}{\sqrt{2}}[\sqrt{[c_1 e^{-x} + e^{2x}\{c_2\cos(\sqrt{2}x) + c_3\sin(\sqrt{2}x)\}]^2 + [c_4 e^{-x} + e^{2x}\{c_5\cos(\sqrt{2}x) + c_6\sin(\sqrt{2}x)\}]^2} \\ &+ c_1 e^{-x} + e^{2x}\{c_2\cos(\sqrt{2}x) + c_3\sin(\sqrt{2}x)\}]^{\frac{1}{2}}, \\ z &= \frac{\pm 1}{\sqrt{2}}[\sqrt{[c_1 e^{-x} + e^{2x}\{c_2\cos(\sqrt{2}x) + c_3\sin(\sqrt{2}x)\}]^2 + [c_4 e^{-x} + e^{2x}\{c_5\cos(\sqrt{2}x) + c_6\sin(\sqrt{2}x)\}]^2} \\ &- c_1 e^{-x} - e^{2x}\{c_2\cos(\sqrt{2}x) + c_3\sin(\sqrt{2}x)\}]^{\frac{1}{2}}. \end{aligned}$$

Example 4. The nonlinear system of ODEs

$$\begin{aligned} &y''' + 3(1+y')y'' - z'z'' + y'^3 - 3y'z'^2 + 3y'^2 - 3z'^2 + 3y' + (1+x) = 0, \\ &z''' + 3(1+y')z'' + 3z'y'' + z'^3 + 3y'^2 z' + 6y'z' + 3z' = 0, \end{aligned} \tag{4.62}$$

satisfies the conditions of theorems 4.2.1 and 4.2.2, which guarantee its transformation to a linear form. Therefore, the following transformation is found

$$t = x, \quad y = \frac{1}{2}\ln(v^2 + w^2) - x, \quad z = \arctan(\frac{w}{v}),$$

that converts the nonlinear system (4.62) to the linear system

$$v''' + tv = 0, \quad w''' + tw = 0.$$

Chapter 5

Lie-point symmetry classification of two dimensional linear systems of third order ODEs by complex methods

A unique equivalence class (with eight infinitesimal symmetry generators) exists for second order linearizable ODEs, whereas for the third order there are three classes with four, five and seven generators [48]. As far as 2-dimensional systems of second order ODEs are concerned there are five classes with $5-$, $6-$, $7-$, $8-$ and $15-$dimensional Lie point symmetry algebras [73]. The complex procedure has been adopted to linearize a class of $2-$dimensional systems of second order ODEs that is shown to possess $6-$, $7-$ and $15-$dimensional algebras [66]. This chapter deals with the symmetry classification of the $2-$dimensional systems of linear third order ODEs obtainable from a complex scalar ODE of the same order. We provide symmetry algebras of those systems of third order ODEs that correspond to scalar linearizable third order complex ODEs. For this purpose a canonical form for the linear systems obtainable from a complex linear equation is derived. It is shown that this form provides five equivalence classes for linearizable 2-dimensional systems of third order ODEs. The dimensions of the associated Lie algebras are found to be eight to twelve and thirteen [21].

5.1 The canonical form obtained by complex methods

In order to classify linearizable ODEs and systems we need a canonical form of the corresponding linear equations. For obtaining a canonical form of $2-$dimensional linear systems of third order ODEs we need the Laguerre canonical form given by

$$u'''(x) + k_0(x)u(x) = 0. \tag{5.1}$$

Three equivalence classes arise from the Laguerre canonical form for the linear scalar third order ODEs viz; for $k_0 = 0$, $k_0 =$ non-zero constant and $k_0 =$ non-zero function of x, the associated symmetry algebras are $7-$, $5-$ and $4-$dimensional respectively.

Taking u in (5.1) as a complex function of a real variable x, i.e., $u(x) = y(x) + iz(x)$ and $k_0(x) = k_1(x) + ik_2(x)$, converts the scalar third order ODE (2.15) to a linear system

$$\begin{aligned} y'''(x) + k_1(x)y(x) - k_2(x)z(x) &= 0, \\ z'''(x) + k_2(x)y(x) + k_1(x)z(x) &= 0. \end{aligned} \tag{5.2}$$

A point transformation of the form

$$x = \rho_1(t), \quad y(x) = \rho_2(t)v(t) + \rho_3(t)w(t), \quad z(x) = \rho_4(t)v(t) + \rho_5(t)w(t), \tag{5.3}$$

is considered to map the system (5.2) to itself (with different coefficients). The transformed coefficients are analyzed and it is found that there exist no point transformation of the form (5.3) that can further reduce the number of arbitrary coefficients of (5.2) to one. Therefore, system (5.2) with two arbitrary functions $k_1(x)$ and $k_2(x)$, is the canonical form for a system of two linear third order ODEs obtained by CSA.

5.2 Group classification

Let

$$\mathbf{X}^{(3)} = \xi(x,y,z)\frac{\partial}{\partial x} + \eta_1(x,y,z)\frac{\partial}{\partial y} + \eta_2(x,y,z)\frac{\partial}{\partial z} + \eta_1^{(1)}(x,y,z)\frac{\partial}{\partial y'}$$
$$+\eta_2^{(1)}(x,y,z)\frac{\partial}{\partial z'} + \eta_1^{(2)}(x,y,z)\frac{\partial}{\partial y''} + \eta_2^{(2)}(x,y,z)\frac{\partial}{\partial z''}$$
$$+\eta_1^{(3)}(x,y,z)\frac{\partial}{\partial y'''} + \eta_2^{(3)}(x,y,z)\frac{\partial}{\partial z'''}, \tag{5.4}$$

be the symmetry of (5.2) with $\eta_i^{(1)}$, $\eta_i^{(2)}$ and $\eta_i^{(3)}, (i = 1,2)$ given by the expressions (1.52)−(1.57). According to the Lie algorithm, we have the following symmetry conditions

$$\eta_1^{(3)} = \mathbf{X}^{(2)}(-k_1(x)y(x) + k_2(x)z(x)),$$
$$\eta_2^{(3)} = \mathbf{X}^{(2)}(-k_2(x)y(x) - k_1(x)z(x)),$$

which give the following set of PDEs, i.e., the determining equations for the system (5.2)

$$\xi_{,y} = \xi_{,z} = 0,\ \eta_{1,zz} = \eta_{1,xz} = \eta_{2,yy} = \eta_{2,xy} = 0, \tag{5.5}$$

$$\eta_{1,yy} - 3\xi_{,xy} = \eta_{1,yz} - 2\xi_{,xz} = \eta_{1,xy} - \xi_{,xx} = 0, \tag{5.6}$$

$$\eta_{2,zz} - 3\xi_{,xz} = \eta_{2,yz} - 2\xi_{,xy} = \eta_{2,xz} - \xi_{,xx} = 0, \tag{5.7}$$

$$\eta_{1,xxz} + yk_1\xi_{,z} - zk_2\xi_{,z} = \eta_{2,xxy} + yk_2\xi_{,y} + zk_1\xi_{,y} = 0, \tag{5.8}$$

$$\eta_{1,xxy} - \xi_{,xxx} + 4yk_1\xi_{,y} - 4zk_2\xi_{,y} + yk_2\xi_{,z} + zk_1\xi_{,z} = 0, \tag{5.9}$$

$$\eta_{2,xxz} - \xi_{,xxx} + 4yk_2\xi_{,z} + 4zk_1\xi_{,z} + yk_1\xi_{,y} - zk_2\xi_{,y} = 0, \tag{5.10}$$

$$y\xi k_{1,x} - z\xi k_{2,x} + \eta_1 k_1 - \eta_2 k_2 + \eta_{1,xxx} + (-yk_1 + zk_2)(\eta_{1,y} - 3\xi_{,x})$$
$$-(yk_2 + zk_1)\eta_{1,z} = 0, \tag{5.11}$$

$$y\xi k_{2,x} + z\xi k_{1,x} + \eta_1 k_2 + \eta_2 k_1 + \eta_{2,xxx} - (yk_2 + zk_1)(\eta_{2,z} - 3\xi_{,x})$$
$$+(-yk_1 + zk_2)\eta_{2,y} = 0. \tag{5.12}$$

Equations (5.5)−(5.7) give the following solution

$$\xi = c_1\frac{x^2}{2} + c_2 x + c_3, \tag{5.13}$$

$$\eta_1 = (c_4 + c_2)y + c_1 xy + c_5 z + v(x), \tag{5.14}$$

$$\eta_2 = c_6 y + (c_7 - c_2)z + c_1 xz + w(x),$$

where c_j $(j = 1, 2, \ldots, 7)$ are constants and $v(x)$, $w(x)$ are arbitrary functions of their argument.

Now we start the group classification for the system (5.2). The following cases arise in the equations of the system of determining equations (5.8)−(5.12) for the values of $k_1(x)$ and $k_2(x)$. Examples for each case are provided. These examples illustrate transformations of the linear as well as nonlinear 2−dimensional systems of third order ODEs to the canonical form (5.2) via point transformations.

Case I. $k_1 = 0 = k_2$

In this case we find from equations (5.8)−(5.12) that

$$\eta_{1,xxx} = 0, \quad \eta_{2,xxx} = 0, \tag{5.15}$$

that produces

$$v(x) = c_8 \frac{x^2}{2} + c_9 x + c_{10}, \tag{5.16}$$

$$w(x) = c_{11} \frac{x^2}{2} + c_{12} x + c_{13}, \tag{5.17}$$

where c_p, $p = 8, \ldots, 13$, are arbitrary constants. This yields a 13−dimensional Lie point symmetry algebra:

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial y}, \quad \mathbf{X}_3 = \frac{\partial}{\partial z}, \quad \mathbf{X}_4 = x\frac{\partial}{\partial x}, \\
&\mathbf{X}_5 = y\frac{\partial}{\partial y}, \quad \mathbf{X}_6 = z\frac{\partial}{\partial z}, \quad \mathbf{X}_7 = x\frac{\partial}{\partial y}, \quad \mathbf{X}_8 = x\frac{\partial}{\partial z}, \\
&\mathbf{X}_9 = z\frac{\partial}{\partial y}, \quad \mathbf{X}_{10} = y\frac{\partial}{\partial z}, \quad \mathbf{X}_{11} = \frac{x^2}{2}\frac{\partial}{\partial y}, \quad \mathbf{X}_{12} = \frac{x^2}{2}\frac{\partial}{\partial z}, \qquad (5.18) \\
&\mathbf{X}_{13} = \frac{x^2}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}. \qquad (5.19)
\end{aligned}$$

As an example of this case we consider the following system of ODEs

$$\begin{aligned}
&tv''' + 3tv'v'' + 3v'' + tv'^3 + 3v'^2 = 0, \\
&ww''' + 3w'w'' = 0, \qquad (5.20)
\end{aligned}$$

where prime denotes differentiation with respect to t, which has a 13−dimensional algebra. The above system of ODEs can be converted to the simplest linear system

$$y''' = 0, \quad z''' = 0, \tag{5.21}$$

here prime denotes differentiation with respect to x, by using the transformation

$$t = x, \quad v = \ln(y/x), \quad w = \sqrt{z}. \tag{5.22}$$

The solution of the system (5.21) is given by

$$y = c_1x^2 + c_2x + c_3, \quad z = c_4x^2 + c_5x + c_6,$$

which on applying the inverse of the transformation (5.22) and a simple calculation, yields the solution of the nonlinear system (5.20)

$$v = \ln(c_1t + c_2 + \frac{c_3}{t}), \quad w = \sqrt{c_4t^2 + c_5t + c_6},$$

where all $c_i,\ i = 1, 2 \ldots, 6$ are arbitrary constants.

Case II. $k_1 = C \neq 0,\ k_2 = 0$

Keeping these choices of $k_1,\ k_2$ for the system (5.2) yields an uncoupled 2−dimensional system of linear third order ODEs. Accordingly the set of equations (5.8)−(5.12) reduces to

$$\eta_1 + \eta_{1,xxx} - y(\eta_{1,y} - 3\xi_{,x}) + z\eta_{1,z} = 0\ , \tag{5.23}$$

$$\eta_2 + \eta_{2,xxx} - z(\eta_{2,z} - 3\xi_{,x}) - y\eta_{2,y} = 0\ . \tag{5.24}$$

This gives

$$\xi = c_3, \tag{5.25}$$

$$\eta_1 = c_4y + c_5z + v(x), \tag{5.26}$$

$$\eta_2 = c_6y + c_7z + w(x), \tag{5.27}$$

where $(v(x), w(x))$ solves (5.2) with $k_1 = constant \neq 0$ and $k_2 = 0$. In this case we have the following 11−dimensional Lie algebra

$$\begin{aligned} &\mathbf{X}_j = v_j\frac{\partial}{\partial y} + w_j\frac{\partial}{\partial z}, \quad j = 1, 2, \ldots, 6, \\ &\mathbf{X}_7 = \frac{\partial}{\partial x}, \quad \mathbf{X}_8 = y\frac{\partial}{\partial y}, \quad \mathbf{X}_9 = z\frac{\partial}{\partial z}, \quad \mathbf{X}_{10} = z\frac{\partial}{\partial y}, \quad \mathbf{X}_{11} = y\frac{\partial}{\partial z}, \end{aligned} \tag{5.28}$$

where (v_j, w_j) are linearly independent solutions of (5.2).

A coupled system of nonlinear ODEs

$$2tvv''' + 6(tv' + v)v'' + 6v'^2 + atv^2 = 0,$$
$$vw''' + wv''' + 3w'v'' + 3v'w'' + avw = 0, \tag{5.29}$$

where a is a constant, is transformable to

$$y''' + ay = 0, \quad z''' + az = 0, \tag{5.30}$$

via the linearizing transformation

$$t = x, \quad v = \sqrt{y/x}, \quad w = z\sqrt{x/y}. \tag{5.31}$$

The nonlinear system (5.29) and the linear system (5.30) both have an 11−dimensional algebra.

Case III. $k_1 = \varrho(x) \neq 0,\ k_2 = 0$

In this case again an uncoupled system is obtained for which we find

$$c_1 = c_2 = c_3 = 0. \tag{5.32}$$

This leads to a 10−dimensional Lie algebra

$$\mathbf{X}_j = v_j\frac{\partial}{\partial y} + w_j\frac{\partial}{\partial z}, \quad j = 1, 2, \ldots, 6,$$
$$\mathbf{X}_7 = y\frac{\partial}{\partial y}, \quad \mathbf{X}_8 = z\frac{\partial}{\partial z}, \quad \mathbf{X}_9 = z\frac{\partial}{\partial y}, \quad \mathbf{X}_{10} = y\frac{\partial}{\partial z}, \tag{5.33}$$

where (v_j, w_j) are linearly independent solutions of (5.2).

As an example of this case we consider the following nonlinear system

$$2tv''' + 2ww''' + 6w'w'' + 6v'' + 2t^2v + tw^2 = 0,$$
$$v^4w''' - v^2v''' + 6vv'v'' - 6v'^3 + v^3 + tv^4w = 0,$$

that has a 10−dimensional algebra. This system can be converted to a linear system of the form

$$y''' + xy = 0, \quad z''' + xz = 0,$$

as it also has 10 Lie point symmetries. The point transformations that relate both the nonlinear and linear systems is

$$x = t, \quad y = w + \frac{1}{v}, \quad z = 2tv + w^2.$$

Case IV. $k_1 = 0,\ k_2 = C \neq 0$

Here we get

$$c_1 = c_2 = 0, \quad c_5 = -c_6, \quad c_4 = c_7, \tag{5.34}$$

that yields a 9−dimensional Lie algebra

$$\begin{aligned} &\mathbf{X}_j = v_j \frac{\partial}{\partial y} + w_j \frac{\partial}{\partial z}, \quad j = 1, 2, \ldots, 6\ , \\ &\mathbf{X}_7 = \frac{\partial}{\partial x}, \quad \mathbf{X}_8 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \mathbf{X}_9 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \end{aligned} \tag{5.35}$$

where (v_j, w_j) are linearly independent solutions of (5.2) with $k_1 = 0$ and $k_2 =$ non-zero constant.

As an example of nonlinear and linear systems of ODEs that have 9−dimensional Lie algebras consider

$$v''' + w^3 = 0, \quad 3w^2w''' + 18ww'w'' + 6w'^3 + v = 0, \tag{5.36}$$

and

$$y''' + z = 0, \quad z''' + y = 0.$$

Both the systems have same 9 Lie point symmetries which guarantees existence of a point transformation, which in this case is

$$x = t, \quad y = w^3, \quad z = v.$$

Case V. Both $k_1 = C_1 \neq 0,\ k_2 = C_2 \neq 0$

Substitution of k_1 and k_2 in (5.2) leads to a coupled system of linear equations. Inserting non-zero constants in place of k_1, k_2 in equations (5.8)−(5.12) gives

$$c_1 = c_2 = 0, \quad c_5 = -c_6, \quad c_4 = c_7. \tag{5.37}$$

which is the case IV, therefore yielding the same 9−dimensional Lie algebra.

The following nonlinear system can be taken as an illustration of this case

$$v^2v''' - 6vv'v'' + 6v'^3 + v^3 - v^4w = 0, \quad vw''' - vw - 1 = 0. \tag{5.38}$$

The above system is linearizable to the system

$$y''' - y - z = 0, \quad z''' + y - z = 0, \tag{5.39}$$

via the linearizing transformation

$$t = x, \quad v = \frac{1}{z}, \quad w = y. \tag{5.40}$$

The nonlinear system (5.38) and the linear system (5.39) both have a 9−dimensional Lie algebra.

Case VI. $k_1 = \varrho(x) \neq 0,\ k_2 = C \neq 0$

A coupled system is obtained here for which

$$c_1 = c_2 = c_3 = 0, \quad c_5 = -c_6, \quad c_2 = c_7, \tag{5.41}$$

that makes

$$\begin{aligned} &\xi = 0, \\ &\eta_1 = c_4 y + c_1 z + v(x), \\ &\eta_2 = -c_1 y + c_4 z + w(x), \end{aligned} \tag{5.42}$$

resulting in the following 8−dimensional Lie algebra

$$\begin{aligned} \mathbf{X}_j &= v_j\frac{\partial}{\partial y} + w_j\frac{\partial}{\partial z}, \quad j = 1, 2, \ldots, 6, \\ \mathbf{X}_7 &= y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \mathbf{X}_8 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \end{aligned} \tag{5.43}$$

where (v_j, w_j) are linearly independent solutions of (5.2).

Consider the following nonlinear system

$$tv''' + 3v'' + t^2v - v - w = 0, \quad v''' + w''' + 2tv + tw = 0,$$

that has an 8−dimensional Lie algebra. The transformation

$$t = x, \quad v = y/x, \quad w = z - y/x,$$

converts the above nonlinear system into the linear system of equations

$$y''' + xy - z = 0, \quad z''' + y + xz = 0.$$

Case VII. $k_1 = \varrho_1(x) \neq 0,\ k_2 = \varrho_2(x)$

Considering $\varrho_2(x) \neq 0$ we have the subcases: (i) $\varrho_1 = 0$, (ii) $\varrho_1 = C$ and (iii) $\varrho_1(x) = non - constant\ function\ of\ x$. All of these subcases produce

$$c_1 = c_2 = c_3 = 0, \quad c_5 = -c_6, \quad c_2 = c_7. \tag{5.44}$$

So we have an 8−dimensional Lie algebra, that is similar to the case VI.

The linear system

$$v''' + 2w''' + t^2 v + t(2t+1)w + t^3 = 0, \quad w''' - tv + t(t-2)w + t^4 = 0,$$

can be mapped into the the linear system

$$y''' + x^2 y - xz = 0, \quad z''' + xy + x^2 z = 0,$$

by using the point transformation

$$t = x, \quad v = z - 2y + 2x^2, \quad w = y - x^2,$$

Both of the above linear systems possess an algebra of 8 Lie point symmetries.

The following theorem emerges from the group classification performed in this section.

Theorem 5.2.1. *CSA provides us five equivalence classes for a linear* 2−*dimensional system of third order ODEs, namely, with* 8−, 9−, 10−, 11−, *and* 13−*dimensional Lie algebras.*

Chapter 6

Classification of scalar higher order ODEs linearizable via generalized contact and generalized Lie-Bäcklund transformations

Three equivalence classes exist [48] for scalar third order ODEs linearizable via point transformations. There is no classification provided for these equations linearizable via contact transformations. Indeed IM [29,30] obtained the necessary form of scalar third order ODEs linearizable via contact transformations, given by (2.20). However they neither discussed its contact symmetries nor provided the classification for it. Here our aim is to investigate the contact and higher order symmetries of linearizable scalar ODEs. In fact we reduce a scalar third (fourth) order ODE to a system of two second order ODEs and relate contact (higher order) symmetries of the scalar ODE with point symmetries of the reduced system. In doing so we define new types of transformations that are more general than contact and higher order tangent transformations and perform group classification of scalar third and fourth order ODEs linearizable via these transformations [17, 18].

6.1 Generalized contact transformations

Consider an n^{th} $(n \geq 3)$ order scalar ODE

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}). \tag{6.1}$$

We substitute $y' = z$ in the above ODE. This reduces the above scalar ODE to the following system of two ODEs of order $n-1$

$$\begin{aligned} y^{(n-1)} &= z^{(n-2)}, \\ z^{(n-1)} &= f(x, y, z; z', z'', \ldots, z^{(n-2)}). \end{aligned} \tag{6.2}$$

A point transformation

$$t = \varphi(x, y, z), \quad v = \psi_1(x, y, z), \quad w = \psi_2(x, y, z), \tag{6.3}$$

for the system (6.2) corresponds to a *generalized contact transformation* for the scalar n^{th} order ODE (6.1) with $z = y'$. The transformation (6.3) is actually the contact transformation without the contact condition.

The transformations (6.3) are generalized contact transformations for the scalar ODE (6.1) in the sense that they involve the change of variable y' by considering it as a separate dependent variable z.

Now consider the general form of a linear scalar third order ODE

$$y''' = \delta(x) + \sigma(x)y + \alpha(x)y' + \beta(x)y''. \tag{6.4}$$

Defining $y' = z$ we reduce the order to 2 and double the dimensions, i.e.

$$\begin{aligned} y'' &= z', \\ z'' &= \delta(x) + \sigma(x)y + \alpha(x)z + \beta(x)z'. \end{aligned} \tag{6.5}$$

Here the variable z is actually the derivative of the variable y. For the purpose of the group classification we replace the variable z by y' and compare the corresponding system of equations (6.5) with the canonical form (2.23) of the linear system of second order

ODEs. Since (2.23) only depends explicitly on y' and z' so it gives $\delta(x) = 0$ and $\sigma(x) = 0$. The system (6.5) is now of the form

$$\begin{aligned} y'' &= z', \\ z'' &= \alpha(x)y' + \beta(x)z'. \end{aligned} \tag{6.6}$$

This is the reduced form of the systems of second order ODEs that is obtained from a scalar linear, third order ODE.

6.1.1 Group classification

In this section we perform the group classification on the system (6.6). Let

$$\begin{aligned} \mathbf{X}^{(2)} = {} & \xi(x,y,z)\frac{\partial}{\partial x} + \eta_1(x,y,z)\frac{\partial}{\partial y} + \eta_2(x,y,z)\frac{\partial}{\partial z} + \eta_1^{(1)}(x,y,z)\frac{\partial}{\partial y'} \\ & + \eta_2^{(1)}(x,y,z)\frac{\partial}{\partial z'} + \eta_1^{(2)}(x,y,z)\frac{\partial}{\partial y''} + \eta_2^{(2)}(x,y,z)\frac{\partial}{\partial z''}, \end{aligned} \tag{6.7}$$

be the symmetry generators for (6.6), then symmetry conditions read as

$$\begin{aligned} \eta_1^{(2)} &= \mathbf{X}^{(1)}(z'), \\ \eta_2^{(2)} &= \mathbf{X}^{(1)}(\alpha(x)y' + \beta(x)z'), \end{aligned} \tag{6.8}$$

where $\eta_1^{(2)}$ and $\eta_2^{(2)}$ are second order extension coefficients given by (1.53) and (1.56) respectively. The symmetry conditions (6.8) give the following system of determining PDEs

$$\xi_{,yy} = \xi_{,yz} = \xi_{,zz} = 0, \quad \eta_{1,zz} - \xi_{,z} = 0, \quad \eta_{1,xx} - \eta_{2,x} = 0, \tag{6.9}$$

$$\eta_{1,yy} - 2\xi_{,xy} - \alpha\xi_{,z} = 0, \quad \eta_{2,yy} - \alpha\xi_{,y} = 0, \quad 2\eta_{2,zz} - 2\xi_{,xz} - \xi_{,y} - 2\beta\xi_{,z} = 0, \tag{6.10}$$

$$2\eta_{1,yz} - 2\xi_{,xz} - 2\xi_{,y} - \beta\xi_{,z} = 0, \quad 2\eta_{2,yz} - 2\xi_{,xy} - \beta\xi_{,y} - 2\alpha\xi_{,z} = 0, \tag{6.11}$$

$$2\eta_{1,xy} - \eta_{2,y} - \xi_{,xx} + \alpha\eta_{1,z} = 0, \; 2\eta_{2,xy} + \alpha\eta_{2,z} - \beta\eta_{2,y} - \alpha\eta_{1,y} - (\alpha\xi)_{,x} = 0, \tag{6.12}$$

$$2\eta_{1,xz} + \eta_{1,y} - \eta_{2,z} - \xi_{,x} + \beta\eta_{1,z} = 0, \; 2\eta_{2,xz} - \alpha\eta_{1,z} + \eta_{2,y} - \xi_{,xx} - (\beta\xi)_{,x} = 0, \tag{6.13}$$

$$\eta_{2,xx} - \alpha\eta_{1,x} - \beta\eta_{2,x} = 0. \tag{6.14}$$

The system of PDEs (6.9) gives the following solution

$$\xi = ya_1(x) + za_2(x) + a_3(x), \tag{6.15}$$

$$\eta_1 = \frac{1}{2}z^2 a_2 + za_4(x,y) + a_5(x,y), \tag{6.16}$$

$$\eta_2 = \frac{1}{2}z^2 a_{2,x} + za_{4,x} + a_{5,x} + a_6(y,z), \tag{6.17}$$

where $a_i,\ (i = 1, 2, \ldots, 6)$ are arbitrary functions of their arguments.

We now assume $\beta(x)$ to be zero, nonzero constant and an arbitrary function of x and consider the following cases.

Case I $\beta(x) = 0$

The system of PDEs (6.10)−(6.14) in this case takes the form

$$\eta_{1,yy} - 2\xi_{,xy} - \alpha\xi_{,z} = 0, \quad \eta_{2,yy} - \alpha\xi_{,y} = 0, \quad 2\eta_{2,zz} - 2\xi_{,xz} - \xi_{,y} = 0, \tag{6.18}$$

$$2\eta_{1,yz} - 2\xi_{,xz} - 2\xi_{,y} = 0, \quad 2\eta_{2,yz} - 2\xi_{,xy} - 2\alpha\xi_{,z} = 0, \tag{6.19}$$

$$2\eta_{1,xy} - \eta_{2,y} - \xi_{,xx} + \alpha\eta_{1,z} = 0, \quad 2\eta_{2,xy} + \alpha\eta_{2,z} - \alpha\eta_{1,y} - \xi\alpha_{,x} - \alpha\xi_{,x} = 0, \tag{6.20}$$

$$2\eta_{1,xz} + \eta_{1,y} - \eta_{2,z} - \xi_{,x} = 0, \quad 2\eta_{2,xz} - \alpha\eta_{1,z} + \eta_{2,y} - \xi_{,xx} = 0, \tag{6.21}$$

$$\eta_{2,xx} - \alpha\eta_{1,x} = 0. \tag{6.22}$$

The above system of PDEs is solved for different values of $\alpha(x)$.

Case I.1 *Both α and β are zero*

In this case the system of PDEs (6.18)−(6.22) reduces to

$$\eta_{1,yy} - 2\xi_{,xy} = 0, \quad \eta_{2,yy} = 0, \quad 2\eta_{2,zz} - 2\xi_{,xz} - \xi_{,y} = 0, \tag{6.23}$$

$$2\eta_{1,yz} - 2\xi_{,xz} - 2\xi_{,y} = 0, \quad 2\eta_{2,yz} - 2\xi_{,xy} = 0, \tag{6.24}$$

$$2\eta_{1,xy} - \eta_{2,y} - \xi_{,xx} = 0, \quad 2\eta_{2,xy} = 0, \tag{6.25}$$

$$2\eta_{1,xz} + \eta_{1,y} - \eta_{2,z} - \xi_{,x} = 0, \quad 2\eta_{2,xz} + \eta_{2,y} - \xi_{,xx} = 0, \tag{6.26}$$

$$\eta_{2,xx} = 0. \tag{6.27}$$

Solving the above system yields the following 15 Lie point symmetries:

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial y}, \quad \mathbf{X}_3 = \frac{\partial}{\partial z}, \tag{6.28}$$

$$\mathbf{X}_4 = x\frac{\partial}{\partial y}, \quad \mathbf{X}_5 = z\frac{\partial}{\partial y}, \quad \mathbf{X}_6 = \frac{1}{2}x^2\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \tag{6.29}$$

$$\mathbf{X}_7 = z\frac{\partial}{\partial x} + \frac{1}{2}z^2\frac{\partial}{\partial y}, \quad \mathbf{X}_8 = x\frac{\partial}{\partial x} + \frac{1}{2}xz\frac{\partial}{\partial y}, \tag{6.30}$$

$$\mathbf{X}_9 = \frac{1}{2}xz\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \mathbf{X}_{10} = (\frac{1}{2}xz + y)\frac{\partial}{\partial y}, \tag{6.31}$$

$$\mathbf{X}_{11} = \frac{1}{2}x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \tag{6.32}$$

$$\mathbf{X}_{12} = x^2\frac{\partial}{\partial x} + \frac{1}{2}(x^2z + xy)\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, \tag{6.33}$$

$$\mathbf{X}_{13} = -\frac{1}{2}xz\frac{\partial}{\partial x} + \frac{1}{4}z(xz + 2y)\frac{\partial}{\partial y} + \frac{1}{2}z^2\frac{\partial}{\partial z}, \tag{6.34}$$

$$\mathbf{X}_{14} = (\frac{1}{2}xz + y)\frac{\partial}{\partial x} - \frac{1}{4}z(xz - 2y)\frac{\partial}{\partial y}, \tag{6.35}$$

$$\mathbf{X}_{15} = (\frac{1}{2}x^2z - xy)\frac{\partial}{\partial x} + (\frac{1}{4}x^2z^2 - y^2)\frac{\partial}{\partial y} + (\frac{1}{2}xz^2 - yz)\frac{\partial}{\partial z}. \tag{6.36}$$

Case I.2 $\alpha = \alpha_0 \neq 0$, $\beta = 0$

The system of PDEs (6.18)$-$(6.22) in this case yields a 15$-$dimensional Lie algebra. The first three operators are $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$, given by (6.28), while the remaining 12 operators are

$$\mathbf{Y}_1 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \tag{6.37}$$

$$\mathbf{Y}_2 = z\frac{\partial}{\partial y} + \alpha_0 y\frac{\partial}{\partial z}, \tag{6.38}$$

$$\mathbf{Y}_3 = e^{\sqrt{\alpha_0}x}\frac{\partial}{\partial y} + \sqrt{\alpha_0}e^{\sqrt{\alpha_0}x}\frac{\partial}{\partial z}, \tag{6.39}$$

$$\mathbf{Y}_4 = y\frac{\partial}{\partial x} + zy\frac{\partial}{\partial y} + \frac{1}{2}(y^2\alpha_0 + z^2)\frac{\partial}{\partial z}, \tag{6.40}$$

$$\mathbf{Y}_5 = z\frac{\partial}{\partial x} + \frac{1}{2}(y^2\alpha_0 + z^2)\frac{\partial}{\partial y} + \alpha_0 zy\frac{\partial}{\partial z}, \tag{6.41}$$

$$\mathbf{Y}_6 = e^{-\sqrt{\alpha_0}x}\frac{\partial}{\partial y} - \sqrt{\alpha_0}e^{-\sqrt{\alpha_0}x}\frac{\partial}{\partial z}, \tag{6.42}$$

$$\mathbf{Y}_7 = e^{\sqrt{\alpha_0}x}\frac{\partial}{\partial x} + \sqrt{\alpha_0}e^{\sqrt{\alpha_0}x}y\frac{\partial}{\partial y} + \alpha_0 e^{\sqrt{\alpha_0}x}y\frac{\partial}{\partial z}, \tag{6.43}$$

$$\mathbf{Y}_8 = e^{-\sqrt{\alpha_0}x}\frac{\partial}{\partial x} - \sqrt{\alpha_0}e^{-\sqrt{\alpha_0}x}y\frac{\partial}{\partial y} + \alpha_0 e^{-\sqrt{\alpha_0}x}y\frac{\partial}{\partial z}, \tag{6.44}$$

$$\mathbf{Y}_9 = \frac{(z\sqrt{\alpha_0} - \alpha_0 y)e^{\sqrt{\alpha_0}x}}{\sqrt{\alpha_0}}\frac{\partial}{\partial y} + (z\sqrt{\alpha_0} - \alpha_0 y)e^{\sqrt{\alpha_0}x}\frac{\partial}{\partial z}, \tag{6.45}$$

$$\mathbf{Y}_{10} = \frac{(z\sqrt{\alpha_0} + \alpha_0 y)e^{-\sqrt{\alpha_0}x}}{\sqrt{\alpha_0}}\frac{\partial}{\partial y} - (z\sqrt{\alpha_0} - \alpha_0 y)e^{-\sqrt{\alpha_0}x}\frac{\partial}{\partial z}, \tag{6.46}$$

$$\mathbf{Y}_{11} = \frac{(\sqrt{\alpha_0}y - z)e^{\sqrt{\alpha_0}x}}{\sqrt{\alpha_0}}\frac{\partial}{\partial x} + \frac{(\alpha_0 y^2 - z^2)e^{\sqrt{\alpha_0}x}}{2\sqrt{\alpha_0}}\frac{\partial}{\partial y} + \frac{(\alpha_0 y^2 - z^2)e^{\sqrt{\alpha_0}x}}{2}\frac{\partial}{\partial z}, \tag{6.47}$$

$$\mathbf{Y}_{12} = \frac{(\sqrt{\alpha_0}y + z)e^{-\sqrt{\alpha_0}x}}{\sqrt{\alpha_0}}\frac{\partial}{\partial x} + \frac{(\alpha_0 y^2 - z^2)e^{-\sqrt{\alpha_0}x}}{2\sqrt{\alpha_0}}\frac{\partial}{\partial y} + \frac{(\alpha_0 y^2 - z^2)e^{-\sqrt{\alpha_0}x}}{2}\frac{\partial}{\partial z}. \tag{6.48}$$

Case I.3.1 $\alpha = (x \pm c)^m,\ m \neq 2$, or $e^x,\ \beta = 0$

This case produces a 5−dimensional Lie algebra. The first three operators are $\mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1$ given by (6.28) and (6.37).

Case I.3.2 $\alpha = (x \pm c)^{-2},\ \beta = 0$

In this case we obtain a 6−dimensional Lie algebra.

Case II $\beta(x) \neq 0$

The following subcases arise:

Case II.1 $\alpha = 0,\ \beta = \beta_0 \neq 0$

The system of PDEs (6.10)−(6.14) simplifies to

$$\eta_{1,yy} - 2\xi_{,xy} = 0, \quad \eta_{2,yy} = 0, \quad 2\eta_{2,zz} - 2\xi_{,xz} - \xi_{,y} - 2\beta_0\xi_{,z} = 0, \tag{6.49}$$

$$2\eta_{1,yz} - 2\xi_{,xz} - 2\xi_{,y} - \beta_0\xi_{,z} = 0, \quad 2\eta_{2,yz} - 2\xi_{,xy} - \beta_0\xi_{,y} = 0, \tag{6.50}$$

$$2\eta_{1,xy} - \eta_{2,y} - \xi_{,xx} = 0, \quad 2\eta_{2,xy} - \beta_0\eta_{2,y} = 0, \tag{6.51}$$

$$2\eta_{1,xz} + \eta_{1,y} - \eta_{2,z} - \xi_{,x} + \beta_0\eta_{1,z} = 0, \quad 2\eta_{2,xz} + \eta_{2,y} - \xi_{,xx} - \beta_0\xi_{,x} = 0, \tag{6.52}$$

$$\eta_{2,xx} - \beta_0\eta_{2,x} = 0, \tag{6.53}$$

which produces a 7−dimensional Lie algebra. The first four of these operators are $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1$ given by (6.28) and (6.37) while the remaining three are

$$\mathbf{Y}_2 = x\frac{\partial}{\partial y}, \quad \mathbf{Y}_3 = (-\beta_0 y + z)\frac{\partial}{\partial y}, \quad \mathbf{Y}_4 = \frac{e^{\beta_0 x}}{\beta_0}\frac{\partial}{\partial y} + e^{\beta_0 x}\frac{\partial}{\partial z}.$$

Case II.2.1 $\alpha = \alpha_0 = \beta_0 \neq 0$

In this subcase the system of PDEs (6.10−6.14) gives the following set of solution

$$\xi = c_1, \quad \eta_1 = c_2 y + c_3 z + c_4 + c_5 e^{\alpha_1 x} + c_6 e^{\alpha_2 x},$$
$$\eta_2 = c_5 \alpha_1 e^{\alpha_1 x} + c_6 \alpha_2 e^{\alpha_2 x} + c_3 (y + z)\alpha_0 + c_2 z + c_7,$$

where

$$\alpha_1 = \frac{1}{2}(\alpha_0 + \sqrt{\alpha_0^2 + 4\alpha_0}), \quad \alpha_2 = \frac{1}{2}(\alpha_0 - \sqrt{\alpha_0^2 + 4\alpha_0}),$$

and c_i, $(i = 1, 2, \ldots, 7)$ are arbitrary constants. This yields a 7−dimensional Lie algebra with $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{Y}_1$ given by (6.28) and (6.37). The other three operators are

$$\mathbf{Y}_2 = z\frac{\partial}{\partial y} + \alpha_0 (y + z)\frac{\partial}{\partial z},$$
$$\mathbf{Y}_3 = e^{\alpha_1 x}\frac{\partial}{\partial y} + \alpha_1 e^{\alpha_1 x}\frac{\partial}{\partial z},$$
$$\mathbf{Y}_4 = e^{\alpha_2 x}\frac{\partial}{\partial y} + \alpha_2 e^{\alpha_2 x}\frac{\partial}{\partial z}.$$

Case II.2.2 $\alpha = \alpha_0 \neq 0$ and $\beta = \beta_0 \neq 0$ with $\alpha_0 \neq \beta_0$

This case produces the following set of solution for the PDEs (6.10−6.14)

$$\xi = c_1, \quad \eta_1 = c_2 y + c_3 z + c_4 + c_5 e^{\beta_1 x} + c_6 e^{\beta_2 x},$$
$$\eta_2 = c_5 \beta_1 e^{\beta_1 x} + c_6 \beta_2 e^{\beta_2 x} + c_3 \alpha_0 y + c_3 \beta_0 z + c_2 z + c_7,$$

where

$$\beta_1 = \frac{1}{2}(\beta_0 + \sqrt{\beta_0^2 + 4\alpha_0}), \quad \text{and} \quad \beta_2 = \frac{1}{2}(\beta_0 - \sqrt{\beta^2 + 4\alpha_0}).$$

From above we get a 7−dimensional Lie algebras. The first four operators are given by (6.28) and (6.37).

Case II.3 $\beta = \beta_0,\ \alpha(x) = cx^m, (x + c)^m, m = 1, 2$

This case produces a 5−dimensional Lie algebra with first three generators $\mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1$.

Case II.4.1 $\alpha(x) = (cx \pm d)^m, \ \beta(x) = (cx \pm d)^m, \ m = 1, 2$

Here we find a algebra with 6 Lie point symmetries.

Case II.4.2 $\alpha(x) = \beta(x) = x^{(-1)}, x^{(-2)}$

This case lies in the above case.

The system of PDEs (6.9)−(6.14) provides us four equivalence classes with 5, 6, 7 and 15 symmetries. These Lie point symmetries correspond to the generalized contact symmetries for the scalar ODE (6.4). Thus we have the following theorem.

Theorem 6.1.1. *A linear, scalar, third order ODE has one of* 5*,* 6*,* 7 *and* 15 *generators of generalized contact transformations.*

For the scalar fourth order ODE we can reduce it to a system of two third order ODEs and following the same procedure we can find generalized contact symmetries of the scalar ODE. We can also reduce scalar fourth and higher order ODEs to systems of second order ODEs and investigate about its symmetries. For this purpose we generalize Lie-Bäcklund transformation in the subsequent section.

6.2 Generalized Lie-Bäcklund transformations

Consider the general form of scalar n^{th} order ODEs (6.1) with $n \geq 4$. We define m to be

$$1 < m \leq \frac{n}{2}, \text{ if } n \text{ is even}$$
$$1 < m \leq \frac{n-1}{2}, \text{ if } n \text{ is odd.}$$

We replace $y^{(m)} = z$ in (6.1) to form the following system of two ODEs

$$\begin{aligned} &y^{(m)} = z, \\ &z^{(m)} = f(x, y, z; z'z'', \ldots, z^{(m-1)}), \quad \text{if } n \text{ is even,} \end{aligned} \tag{6.54}$$

and

$$\begin{aligned} &y^{(m+1)} = z', \\ &z^{(m+1)} = f(x, y, z; z'z'', \ldots, z^{(m)}), \quad \text{if } n \text{ is odd.} \end{aligned} \tag{6.55}$$

A point transformation (6.3) for the above systems corresponds to a *generalized Lie-Bäcklund transformation* of order m for the scalar ODE (6.1) with $y^{(m)} = z$. Generalized Lie-Bäcklund transformations depend on the independent, dependent variables and the m^{th} order derivative of the dependent variable but do not require the contact conditions (1.68) to hold.

Consider the general form of a linear, scalar, fourth order ODE

$$y^{(4)} = \pi(x) + \gamma(x)y + \rho(x)y' + \lambda(x)y'' + \varrho(x)y'''. \tag{6.56}$$

By taking $y'' = z$ will convert the above equation to a system of two second order ODEs

$$\begin{aligned} y'' &= z, \\ z'' &= \pi(x) + \gamma(x)y + \rho(x)y' + \lambda(x)z + \varrho(x)z'. \end{aligned} \tag{6.57}$$

We now identify the above system with the canonical form (2.24) of the linear system of second order ODEs. The form (2.24) depend explicitly on y and z. This makes all coefficients functions zero but $\gamma(x)$ and $\lambda(x)$. Hence we have the following reduced system of linear second order ODEs

$$\begin{aligned} y'' &= z, \\ z'' &= \gamma(x)y + \lambda(x)z. \end{aligned} \tag{6.58}$$

6.2.1 Group classification

We now perform the group classification for the system of ODEs (6.58). Suppose $\mathbf{X}^{(2)}$ given by (6.7) be the symmetry generator for the system (6.58). The symmetry conditions give us the following set of determining PDEs

$$\xi_{,yy} = \xi_{,yz} = \xi_{,zz} = 0, \quad \eta_{1,zz} = \eta_{2,yy} = 0, \quad \eta_{1,yy} - 2\xi_{,xy} = 0, \tag{6.59}$$

$$\eta_{2,zz} - 2\xi_{,xz} = 0, \quad \eta_{1,yz} - \xi_{,xz} = 0, \quad \eta_{2,yz} - 2\xi_{,xy} = 0, \quad \eta_{1,xz} - z\xi_{,z} = 0, \tag{6.60}$$

$$2\eta_{1,xy} - \xi_{,xx} - 3z\xi_{,y} - \gamma y\xi_{,z} - \lambda z\xi_{,z} = 0, \quad \eta_{2,xy} - \lambda y\xi_{,y} - \gamma z\xi_{,y} = 0, \tag{6.61}$$

$$2\eta_{2,xz} - \xi_{,xx} - z\xi_{,y} - 3\gamma y\xi_{,z} - 3\lambda z\xi_{,z} = 0, \tag{6.62}$$

$$-\eta_2 + \eta_{1,xx} + z\eta_{1,y} - 2z\xi_{,x} + \gamma y\eta_{1,z} + \lambda z\eta_{1,z} = 0, \tag{6.63}$$

$$\begin{aligned}\xi y\gamma_{,x} + \xi z\lambda_{,x} + \eta_1\gamma + \eta_2\lambda - \eta_{2,xx} - z\eta_{2,y} - \gamma y\eta_{2,z} + 2\gamma y\xi_{,x}& \\ -\lambda z\eta_{2,z} + 2\lambda z\xi_{,x} = 0.&\end{aligned} \tag{6.64}$$

The PDEs (6.59)−(6.62) yield the following set of solutions

$$\xi = a_1(x), \tag{6.65}$$

$$\eta_1 = (\frac{1}{2}a_{1,x} + c_3)y + c_1z + a_2(x), \tag{6.66}$$

$$\eta_2 = c_2y + (\frac{1}{2}a_{1,x} + c_4)z + a_3(x), \tag{6.67}$$

where $c_i,\ (i = 1, 2, 3, 4)$ are arbitrary constants and $a_j,\ (j = 1, 2, 3)$ are arbitrary functions of x. We now substitute (6.65)−(6.67) into (6.63)−(6.64). After some calculations we get

$$\xi = a_1(x), \tag{6.68}$$

$$\eta_1 = (\frac{a_{1,x}}{2} + c_3)y + c_1z + v(x), \tag{6.69}$$

$$\eta_2 = c_2y + (\frac{a_{1,x}}{2} + c_4)z + w(x), \tag{6.70}$$

where (v, w) solves (6.58) and $a_1(x)$ satisfies

$$a_{1,xxx} + 2c_1\gamma - 2c_2 = 0, \tag{6.71}$$

$$2a_{1,x} - c_1\lambda - c_3 + c_4 = 0, \tag{6.72}$$

$$2\gamma a_{1,x} + a_1\gamma_{,x} + (c_3 - c_4)\gamma + c_2\lambda = 0, \tag{6.73}$$

$$a_{1,xxx} - 4\lambda a_{1,x} - 2\lambda_{,x}a_1 - c_1\gamma + c_2 = 0. \tag{6.74}$$

We now consider different cases for $\gamma(x)$ to be zero, nonzero constant and an arbitrary function of x.

Case I $\gamma(x) = 0$

With the substitution $\gamma = 0$ in (6.73), it becomes

$$c_2\lambda = 0, \tag{6.75}$$

which prompts the consideration of the following cases.

Case I.1 $\lambda = 0,\ \gamma = 0$

This makes $c_2 = 0$. From (6.72) we get

$$a_1 = (c_3 - c_4)\frac{x}{2} + c_5,$$

so that we have

$$\xi = (c_3 - c_4)\frac{x}{2} + c_5.$$

Therefore in this case we get the following 8−dimensional Lie algebra

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial y}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial y}, \quad \mathbf{X}_4 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \tag{6.76}$$

$$\mathbf{X}_5 = z\frac{\partial}{\partial y}, \quad \mathbf{X}_6 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \tag{6.77}$$

$$\mathbf{X}_7 = \frac{1}{6}x^3\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{X}_8 = \frac{1}{2}x^2\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \tag{6.78}$$

Case I.2 $\lambda = \lambda_0 \neq 0,\ \gamma = 0$

From (6.75) we get $c_2 = 0$. From (6.74) we have $a_{1,x} = 0$ which implies that $a_1 = c_5$. Hence we get 7−dimensional Lie algebra. The first four operators are given by (6.76) and the other three are

$$\mathbf{Y}_5 = z\frac{\partial}{\partial y} + \lambda_0 z\frac{\partial}{\partial z},$$
$$\mathbf{Y}_6 = e^{\sqrt{\lambda_0}x}\frac{\partial}{\partial y} + \lambda_0 e^{\sqrt{\lambda_0}x}\frac{\partial}{\partial z},$$
$$\mathbf{Y}_7 = e^{-\sqrt{\lambda_0}x}\frac{\partial}{\partial y} + \lambda_0 e^{-\sqrt{\lambda_0}x}\frac{\partial}{\partial z}.$$

Case I.3 $\lambda(x) = e^x,\ (cx+d)^m,\ (m = \pm 1, 2,),\quad \gamma = 0$

In this case we find the following 5−dimensional Lie algebra

$$\mathbf{Y}_i = v_j\frac{\partial}{\partial y} + w_j\frac{\partial}{\partial z}, \quad i = 1, 2, 3, 4, \tag{6.79}$$

$$\mathbf{Y}_5 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

where (v_j, w_j) are linearly independent solutions of (6.58).

Case I.4 $\lambda(x) = (cx+d)^{-2}, \quad \gamma = 0$

This case produces a 6−dimensional Lie algebra. The first four operators are given by (6.79). The extra two operators are

$$\mathbf{Y}_5 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \mathbf{Y}_6 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z}.$$

Case II $\gamma(x) \neq 0$

Here we have the following subcases.

Case II.1 $\gamma = \gamma_0, \ \lambda = 0$

This case produces $a_1 = c_5$ and $c_2 = \gamma c_1$. Hence we have a 7−dimensional Lie algebra. The first four generators are given by (6.79) and extending generator is

$$\mathbf{Y}_5 = \frac{\partial}{\partial x}, \quad \mathbf{Y}_6 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \mathbf{Y}_7 = z\frac{\partial}{\partial y} + \gamma_0 y\frac{\partial}{\partial z}.$$

Case II.2 $\gamma = \gamma_0, \ \lambda = \lambda_0$

In this case we have a 7−dimensional Lie algebra with $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ and $\mathbf{Y}_4$ given by (6.79) The additional two operators are given by

$$\mathbf{Y}_5 = \frac{\partial}{\partial x}, \quad \mathbf{Y}_6 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad \mathbf{Y}_7 = z\frac{\partial}{\partial y} + (\gamma_0 y + \lambda_0 z)\frac{\partial}{\partial z}.$$

Case II.3 $\gamma = \gamma(x) \neq 0, \ \lambda = \lambda_0$

Here we get a 5−dimensional Lie algebra with first four operators given by (6.79). The additional operator is

$$\mathbf{Y}_5 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Case II.4 $\gamma = \gamma(x) \neq 0, \ \lambda = x^m$

This case produces the Lie algebra of *case I.3*.

The system of PDEs (6.59)−(6.64) provides us four equivalence classes with 5, 6, 7 and 8 symmetries. These Lie point symmetries are the generalized Lie-Bäcklund symmetries of order 2 for a scalar fourth order linearizable ODE. Thus we have the following theorem.

Theorem 6.2.1. *A linear, scalar, fourth order ODE has one of* 5, 6, 7 *and* 8 *generators of generalized Lie-Bäcklund transformations of order* 2.

Chapter 7

Conclusions and future directions

7.1 Conclusions

Nonlinear ODEs are difficult to solve but, if they can be converted to linear ones by invertible transformations, they can be solved. Lie completely resolved the problem of the use of point transformations for the case of scalar second order ODEs. However, he did not extend to the higher order ODEs, to systems or to PDEs. For those he relied on his general algorithms. In the thesis, we address the issue of linearization and classification of scalar third and fourth order and systems of two third order ODEs. The classification on the basis of symmetries puts the ODEs into the equivalence classes of linearizable ODEs and hence makes them solvable. Not only this, but we get the general solution of nonlinear ODEs. In order to tackle the problem of linearization and classification we employed three types of transformations: (a) derivative independent transformations (b) derivatives dependent transformations (c) complex transformations.

We started our work by presenting criteria for fourth order autonomous ODEs to be reducible to linearizable third and second order ODEs. There are certain fourth order ODEs, not depending explicitly on the independent variable, which cannot be linearized by point or contact transformations but can be reducible to third order linearizable ODEs by Meleshko's method. The solution of the original equation is then obtained by a quadrature.

Various fourth order ODEs with fewer symmetries can be reduced to linearizable form by this procedure. The class of ODEs linearizable by Meleshko's method is not included in IM's, IMS' or conditionally linearizable classes [51,53] of the ODEs (though there can be an overlap but it is not contained in that either). The reason is that it is not linearizable but reducible to a lower order linearizable ODE. By using the concept of Meleshko linearization a new class of scalar ODEs may be defined on the basis of initial conditions to be satisfied by the ODEs.

Complex methods have been adopted to achieve linearization of the $2-$dimensional systems of ODEs with the help of the scalar, complex linearizable ODEs. By the complex ODEs we extract systems with CR-structure on both the equations and employ the fibre preserving point transformations with CR-structure, to derive the sufficient conditions of linearization. Though by going complex we deal with the subclasses of higher dimensional systems of higher order ODEs, it linearizes them in a trivial way that is impossible to attempt with the real symmetry methods yet. When both the $2-$dimensional systems of ODEs and the point transformations have a specific CR-structure, then there is no need to adopt the Lie procedure for nonlinear systems to obtain the linearization conditions. By splitting the linearization criteria associated with the base equations due to the complex fibre preserving point transformations, into the real and imaginary parts, leads to the linearization conditions for the corresponding systems.

The c-linearization of $2-$dimensional systems of second order ODEs is achieved earlier by considering the scalar, second order, linearizable ODEs as complex. Their associated linearization criteria are separated into the real and imaginary parts due to the complex functions involved. In this thesis the linearizable form of c-linearizable systems has been derived and it is shown to be (atmost) quadratically semilinear in the first order derivatives. Moreover, the c-linearization criteria are proved to be the linearization criteria for such 2-dimensional systems that are linearizable due to their correspondence with the complex scalar ODEs. Using this fact, we also provided the linearization conditions for $2-$dimensional systems of third order ODEs. The obtained linearizable form is (at most) linear in second derivative and cubically semilinear in the first derivative.

CSA is employed to tackle the problem of classification of linear systems of two third order ODEs. We used the Laguerre canonical form of a complex, linear, scalar, third order ODE to obtain the system of two linear third order ODEs. Five equivalence classes are obtained from this linear system of ODEs. The allowed symmetry Lie algebras may be $8-$, $9-$, $10-$, $11-$ or $13-$dimensional. The maximal symmetry case reduces to the simplest equations $y''' = 0$, $z''' = 0$. Further, systems with 11 Lie point symmetries are linearizable to the uncoupled systems with constant coefficients. The systems with $9-$dimensional Lie algebras are those that are reducible to linear coupled systems with constant coefficients. The systems of two third order ODEs that are linearizable to the systems with variable coefficients have 10 and 8 Lie point symmetries. The former class corresponds to uncoupled systems while the latter corresponds to coupled systems. It is worth noting that in the maximal algebra there is a sub-algebra of three translations and three scalings along the one independent and two dependent variables. There are also "cross-scalings" between the one independent and two dependent variables, giving a total of four such generators and then three generators with quadratic coefficients. The next largest class has only five generators involving the independent and dependent variables, namely one translation along the independent variable, two scalings along the dependent variables and two "cross-scalings" between them. Then there are six extra generators that involve the solutions of the equations as coefficients. For the next two cases there is the translation along the independent variable and a scaling and rotation between the dependent variables. These two correspond to the single complex scaling. For the next two we lose the translation along the independent variable and only retain the complex rotation as the real pair of scalings and rotations.

After presenting linearization criteria and classification of systems of two third order ODEs, we turn to the problem of classification of scalar ODEs linearizable via derivative dependent transformations. Though Leach and Mahomed [48] had shown that there are three equivalence classes of third order scalar ODEs linearizable via point transformations, no work on the symmetry group classification of these equations linearizable via contact transformations was done. In fact IM [30] got the necessary form of a scalar third order

ODE linearizable via contact transformations but there was no attempt to address the classification problem with their methods. We found a connection between (generalized) contact transformations of systems of order n with point transformations of the reduced systems of order $n-1$. By defining the first derivative of dependent variables to be new variables we reduce the order of a system from n to $n-1$ and increase its dimension form m to $2m$. Point transformations for the lower order system correspond to generalized contact transformations for the higher order system. We obtained the canonical form of scalar third order ODEs linearizable via generalized contact transformations. This canonical form gave us four equivalence classes for scalar third order ODEs depending on the number of infinitesimal generators. Here we obtained group classification of a scalar third order ODE by reducing it to a system of two second order ODEs. If the reduced system of ODEs is linearizable then it can be solved by using geometric linearization [61]. In reducing a scalar third order ODE to a system of two second order ODEs, the advantage of geometric linearization could be availed, where we can find the solution of the system easily by simply employing the coordinate transformations as the linearizing transformations.

A similar procedure is carried out to the fourth order scalar ODEs. We reduced a linear scalar fourth order ODE to a system of two linear second order ODEs. Any point transformation for the reduced system corresponds to a generalized Lie-Bäcklund transformation of the scalar ODE. The reduced system of two linear second order ODEs provided us four equivalence classes of such equations with 5, 6, 7 and 8 generalized Lie-Bäcklund symmetries of order 2.

7.2 Future directions

In the last we will give some future directions associated with the work presented in the thesis.

- We reduced the scalar fourth order ODEs to linearizable third and second order equations. It would be interesting to apply the procedure to systems of third order ODEs in the cases that the equations do not depend explicitly on one of the dependent variables or the independent variable. These systems can then be reduced to systems of second order ODEs to apply the power of geometry. If they satisfy the geometric linearization criteria they can then be solved easily.

- The concept of c-linearization can be generalized to the $2m-$dimensional systems of n^{th} order ODEs by splitting the complex, linearizable, $m-$dimensional systems of ODEs of the same order. There is a development [55] to obtain odd dimensional systems by splitting iteratively starting with a scalar base equation. This procedure may lead us to the linearization of $m-$dimensional system of second order ODEs, by iteratively complexifying a scalar, second order, linearizable ODE.

- In the present work, we obtained point symmetry group classification of linear systems of two third order ODEs using complex methods. CSA may lead us to the classification of the higher dimensional systems of higher order ODEs. Similarly, the classification problem of such linearizable systems is addressable with the complex methods. Presently, the symmetry classification and linearization of higher dimensional systems of higher order ODEs seems to be exploitable only with CSA.

- A scalar third order linear ODE has three classes with 4, 5 and 10 contact symmetries. The maximal symmetry class with 15 generalized contact symmetries corresponds to the maximal symmetry class of contact symmetries. It remains an open problem to study the correspondence between the other classes of generalized contact symmetries and those of contact symmetries. Also of great interest is to obtain the linearization criteria via generalized contact transformations of systems of ODEs.

- Here we reduced a scalar fourth order ODE to a system of two linear second order ODEs. We can reduce a scalar fourth order ODE to a system of two third order ODEs and following the same procedure of group classification we can find generalized contact symmetries of the scalar ODE. Similarly we can take it to higher order ODEs to get equivalence classes of these equations by simply reducing scalar ODEs to systems of ODEs. One could use the given procedure to find the equivalence classes of systems of higher order ODEs. We can also reduce a scalar fourth order ODE, $y^{(4)} = f(x, y, y', y'', y''')$, to a system of three second order ODEs by following two steps. In the first step we reduce it to a system of two third order ODEs

$$y''' = z'', \quad z''' = f(x, y, z; z', z''), \tag{7.1}$$

by defining $y' = z$. In the second step we define $z' = u$ to reduce (7.1) to system of three ODEs of order two

$$y'' = z', \quad z'' = u', \quad u'' = f(x, y, z, u; u'). \tag{7.2}$$

Similarly any system of ODEs of order $n \geq 3$ can be reduced in m steps to a system of second order ODEs to use the power of geometry. In this way we can relate the higher order symmetries of the scalar ODEs with the point symmetries of reduced systems and can find the equivalence classes of the higher order ODEs. The m steps of reduction of an ODE can be shrunk into one step by defining the second or higher order derivative to be a new dependent variable. In this way the point symmetries of the reduced system correspond to the generalized Lie B$\ddot{a}$cklund symmetries of the corresponding scalar ODE.

- This procedure could be carried out to reduce scalar n^{th} order ODEs to systems of lower order with three or more dimensions. As an example consider the scalar fifth order ODE

$$y^{(5)} = f(x, y; y', y'', y''', y^{(4)}). \tag{7.3}$$

By defining $y'' = z$ and $z' = u$ we can reduce it to a system of three ODEs of order two

$$y'' = z, \quad z'' = u', \quad u'' = f(x, y, z, u; y', u'), \tag{7.4}$$

and investigate the point symmetries of the reduced system. We can also reduce the scalar ODE (7.3) to a system of two third ODEs

$$y''' = z'', \quad z''' = f(x, y, z; y', z', z''), \tag{7.5}$$

by defining $y'' = z'$ and find the Lie point symmetries of the above system. It would be interesting to find a connection between the equivalence classes of the systems (7.4) and (7.5) and relate their point symmetries.

- In the thesis, we performed the classification of those systems that are linearizable via generalized contact and generalized Lie B*ä*cklund transformations. Also of great interest is to develop linearization criteria for such systems and relate the linearizable classes of these systems. This is the start. There is much work that needs to be done in this direction to explore new and interesting results about these transformations and their equivalence classes. The equations linearizable via these transformations may form a new class of ODEs that do not fall into IM's, IMS', Meleshko's classes of linearizable ODEs. In Lie's programme there is no definite statement available for the cases when the ODEs are not linearizable. By the recent developments this gap may be filled.

Appendix A

A.1

The linearizing transformation is provided by the third order ODE

$$6\frac{d\chi}{dx} - 3\chi^2 = 3b_1 - {a_0}^2 - 3a_{0,x}, \tag{A.1}$$

where

$$\chi = \frac{\varphi_{,xx}}{\varphi_{,x}}, \tag{A.2}$$

and by the following integrable system of partial differential equations

$$\begin{gathered} 3\psi_{,uu} = a_1\psi_{,u}, \quad 3\psi_{,xu} = (3\chi + a_0)\psi_{,u} \ , \\ \psi_{,xxx} = 3\chi\psi_{,xx} + b_0\psi_{,u} - \frac{1}{6}(3a_{0,x} + {a_0}^2 - 3b_1 + 9\chi^2)\psi_{,x} - \Omega\psi \ , \end{gathered} \tag{A.3}$$

where

$$\Omega = \frac{1}{54}(9a_{0,xx} + 18a_{0,x}a_0 + 54b_{0,u} - 27b_{1,x} + 4{a_0}^3 - 18a_0b_1 + 18a_1b_0) \ , \tag{A.4}$$

and

$$\alpha = \Omega{\varphi_{,x}}^{-3}. \tag{A.5}$$

A.2

$$c_0 = 6\lambda\lambda_{,u} - 6\lambda_{,x} + \lambda c_1 - \lambda^2 c_2 \ , \tag{A.6}$$

$$6\lambda_{,uu} = c_{2,x} - c_{1,u} + \lambda c_{2,u} + c_2\lambda_{,u}, \tag{A.7}$$

$$\begin{aligned} 18d_0 &= 3\lambda^2[\lambda c_{1,u} - 2c_{1,x} - \lambda c_{2,x} + 3\lambda^2 c_{2,u} - 12\lambda_{,xu}] - 54(\lambda_{,x})^2 \\ &+6\lambda[3\lambda_{,xx} + 15\lambda_{,x}\lambda_{,u} - 6\lambda(\lambda_{,u})^2 + (3c_1 - \lambda c_2)\lambda_{,x}] \\ &+\lambda^2[9(\lambda c_2 - 2c_1)\lambda_{,u} - 2{c_1}^2 + 2\lambda c_1 c_2 + 4\lambda^2 {c_2}^2 + 18\lambda^2 d_4 - 72\lambda^3 d_5] \ , \end{aligned} \tag{A.8}$$

$$\begin{aligned} 18d_1 &= 9\lambda^2 c_{1,u} - 12\lambda c_{1,x} - 27\lambda^2 c_{2,x} + 33\lambda^3 c_{2,u} - 36\lambda\lambda_{,xu} \\ &+18\lambda_{,xx} + 6(3c_1 + 4\lambda c_2)\lambda_{,x} - 3\lambda(6c_1 + 7\lambda c_2)\lambda_{,u} + 18\lambda(\lambda_{,y})^2 \\ &-18\lambda_{,x}\lambda_{,u} - 4\lambda c_1^2 - 2\lambda^2 c_1 c_2 + 20\lambda^3 c_2^2 + 72\lambda^3 d_4 - 270\lambda^4 d_5 \ , \end{aligned} \tag{A.9}$$

$$\begin{aligned} 9d_2 &= 3\lambda c_{1,u} - 3c_{1,x} - 21\lambda c_{2,x} + 21\lambda^2 c_{2,u} + 15c_2\lambda_x \\ &-15\lambda c_2\lambda_{,u} - {c_1}^2 - 5\lambda c_1 c_2 + 14\lambda^2 {c_2}^2 + 54\lambda^2 d_4 - 180\lambda^3 d_5 \ , \end{aligned} \tag{A.10}$$

$$3d_3 = 3\lambda c_{2,u} - 3c_{2,x} - c_1 c_2 + 2\lambda c_2^2 + 12\lambda d_4 - 30\lambda^2 d_5, \tag{A.11}$$

$$\begin{aligned} 54d_{4,x} &= 18c_{1,uu} + 3c_2 c_{1,u} - 72c_{2,xu} - 39c_2 c_{2,x} \\ &+18\lambda c_{2,uu} - 3\lambda c_2 c_{2,u} + (72c_{2,u} + 33c_2^2)\lambda_{,u} + 108d_4\lambda_{,u} \\ &+270d_5\lambda_{,x} + 378\lambda d_{5,x} - 108\lambda^2 d_{5,y} - 540\lambda d_5\lambda_{,y} \\ &+36\lambda c_1 d_5 - 8\lambda c_2^3 - 36\lambda c_2 d_4 + 108\lambda^2 c_2 d_5 + 54\lambda H \ , \end{aligned} \tag{A.12}$$

and

$$H_x = 3H\lambda_{,u} + \lambda H_{,u} \ , \tag{A.13}$$

where

$$H = d_{4,u} - 2d_{5,x} - 3\lambda d_{5,u} - 5d_5\lambda_{,u} - 2\lambda c_2 d_5$$
$$+\frac{1}{3}[c_{2,uu} + 2c_2c_{2,u} - 2c_1d_5 + 2c_2d_4] + \frac{4}{27}c_2^3 \ . \tag{A.14}$$

The functions $\varphi(x,u)$ and $\psi(x,u)$ are found by the following system of equations:

$$\varphi_{,x} = \lambda\varphi_{,y} \ , \quad \psi_{,x} = -\varphi_{,u}W + \lambda\psi_{,u}, \tag{A.15}$$

$$6\varphi_{,u}\varphi_{,uuu} = 9(\varphi_{,uu})^2 + \{15\lambda a d_5 - 3d_4 - c_2^2 - 3c_{2,u}\}(\varphi_{,u})^2, \tag{A.16}$$

$$\psi_{,uuu} = Wd_5\varphi_{,u} + \frac{1}{6}\{15\lambda d_5 - c_2^2 - 3d_4 - 3c_{2,u}\}\psi_{,u} - \frac{1}{2}H\psi$$
$$+3\varphi_{,uu}\psi_{,uu}(\varphi_{,u})^{-1} - \frac{3}{2}(\varphi_{,uu})^2\psi_{,u}(\varphi_{,u})^{-2} \ , \tag{A.17}$$

where

$$3W_{,x} = \{c_1 - \lambda c_2 + 6\lambda_{,u}\}W \ , \quad 3W_{,u} = c_2 W \ , \tag{A.18}$$

and

$$\alpha = \frac{H}{2(\varphi_{,u})^3} \ . \tag{A.19}$$

A.3

$$c_0 = 6\lambda\lambda_{,u'} - 6\lambda_{,x} + \lambda c_1 - \lambda^2 c_2 \ , \tag{A.20}$$

$$6\lambda_{,u'u'} = c_{2,x} - c_{1,u'} + \lambda c_{2,u'} + c_2\lambda'_u, \tag{A.21}$$

$$\begin{aligned} 18d_0 &= 3\lambda^2[\lambda c_{1,u'} - 2c_{1,x} - \lambda c_{2,x} + 3\lambda^2 c_{2,u'} - 12\lambda_{,xu'}] - 54(\lambda_{,x})^2 \\ &+6\lambda[3\lambda_{,xx} + 15\lambda_{,x}\lambda_{,u'} - 6\lambda(\lambda_{u'})^2 + (3c_1 - \lambda c_2)\lambda_{,x}] \\ &+\lambda^2[9(\lambda c_2 - 2c_1)\lambda_{,u'} - 2{c_1}^2 + 2\lambda c_1 c_2 + 4\lambda^2 {c_2}^2 + 18\lambda^2 d_4 - 72\lambda^3 d_5] \ , \end{aligned} \tag{A.22}$$

$$\begin{aligned} 18d_1 &= 9\lambda^2 c_{1,u'} - 12\lambda c_{1,x} - 27\lambda^2 c_{2,x} + 33\lambda^3 c_{2,u'} - 36\lambda\lambda_{x,u'} \\ &+18\lambda_{,xx} + 6(3c_1 + 4\lambda c_2)\lambda_{,x} - 3\lambda(6c_1 + 7\lambda c_2)\lambda_{,u'} + 18\lambda(\lambda_{,u'})^2 \\ &-18\lambda_{,x}\lambda_{,u'} - 4\lambda c_1^2 - 2\lambda^2 c_1 c_2 + 20\lambda^3 c_2^2 + 72\lambda^3 d_4 - 270\lambda^4 d_5 \ , \end{aligned} \tag{A.23}$$

$$\begin{aligned} 9d_2 &= 3\lambda c_{1,u'} - 3c_{1,x} - 21\lambda c_{2,x} + 21\lambda^2 c_{2,u'} + 15c_2\lambda_{,x} \\ &-15\lambda c_2\lambda_{,y'} - {c_1}^2 - 5\lambda c_1 c_2 + 14\lambda^2 {c_2}^2 + 54\lambda^2 d_4 - 180\lambda^3 d_5 \ , \end{aligned} \tag{A.24}$$

$$3d_3 = 3\lambda c_{2,u'} - 3c_{2,x} - c_1 c_2 + 2\lambda c_2^2 + 12\lambda d_4 - 30\lambda^2 d_5, \tag{A.25}$$

$$\begin{aligned} 54d_{4,x} &= 18c_{1,u'u'} + 3c_2 c_{1,u'} - 72c_{2,xu'} - 39c_2 c_{2,x} \\ &+18\lambda c_{2,u'u'} - 3\lambda c_2 c_{2,u'} + (72c_{,2u'} + 33c_2^2)\lambda_{,u'} + 108d_4\lambda_{,u'} \\ &+270d_5\lambda_{,x} + 378\lambda d_{5,x} - 108\lambda^2 d_{5,u'} - 540\lambda d_5\lambda'_u \\ &+36\lambda c_1 d_5 - 8\lambda c_2^3 - 36\lambda c_2 d_4 + 108\lambda^2 c_2 d_5 + 54\lambda H \ , \end{aligned} \tag{A.26}$$

and

$$H_{,x} = 3H\lambda_{,u'} + \lambda H_{,u'} \ , \tag{A.27}$$

where

$$H = d_{4,u'} - 2d_{5,x} - 3\lambda d_{5,u'} - 5d_5\lambda_{,u'} - 2\lambda c_2 d_5$$
$$+\frac{1}{3}[c_{2,u'u'} + 2c_2c_{2,u'} - 2c_1d_5 + 2c_2d_4] + \frac{4}{27}c_2^3 \ . \quad \text{(A.28)}$$

A.4

$$(\lambda_0 C_1 - 6\lambda_{0,u})u'^2 + (6\lambda_0\lambda_{0,u'} + 4\lambda_0^2 - \lambda_0^2 C_2 - C_0)u' - 4\lambda_0^2 = 0 \ , \quad \text{(A.29)}$$

$$(C_{2,u} - C_{1,u'})u'^3 + (\lambda_0 C_{2,u'} + C_2\lambda_{0,u'} - 4\lambda_{0,u'} - 6\lambda_{0,u'u'})u'^2$$
$$+(10\lambda_{0,u'} + 4\lambda_0 - C_2\lambda_0)u' - 8\lambda_0 = 0 \ , \quad \text{(A.30)}$$

$$(-6\lambda_0^2 C_{1,u} - 54(\lambda_{0,uy})^2 + 18\lambda_0\lambda_{0,uu} + 18\lambda_0\lambda_{0,u}C_1 - 2\lambda_0^2 C_1^2)u'^8$$
$$+(3\lambda_0^3 C_{1,u'} + 48\lambda_0^2\lambda_{0,u} - 3\lambda_0^3 C_{2,u} - 36\lambda_0^2\lambda_{0,uu'} - 6\lambda_0^2\lambda_{0,u}C_2 - 18\lambda_0^2\lambda_{0,u'}C_1$$
$$+2\lambda_0^3 C_1 C_2 - 16\lambda_0^3 C_1)u'^7 + (-60\lambda_0^3\lambda_{0,u'} + 9\lambda_0^4 C_{2,u'} - 42\lambda_0^2\lambda_{0,u}$$
$$-36\lambda_0^2(\lambda_{0,u'})^2 + 9\lambda_0^3\lambda_{0,u'}C_2 + 14\lambda_0^3 C_1 - 32\lambda_0^4 + 8\lambda_0^4 C_2 + 4\lambda_0^4 C_2^2$$
$$+18\lambda_0^4 D_4)u'^6 + (44\lambda_0^4 + 72\lambda_0^2\lambda_{0,u'} - 18\lambda_0^3\lambda_{0,u'} - 7\lambda_0^4 C_2)u'^5$$
$$+(-20\lambda_0^4)u'^4 - 72\lambda_0^5 D_5 = 0, \quad \text{(A.31)}$$

$$(-12\lambda_0 C_{1,u} + 18\lambda_{0,uu'} + 18\lambda_{0,u}C_1 - 4\lambda_0 C_1^2)u'^8 + (9\lambda_0^2 C_{1,u'} - 48\lambda_0\lambda_{0,u}$$
$$-27\lambda_0^2 C_{2,u} - 36\lambda_0\lambda_{0,uu'} - 18\lambda_{0,u} + 72\lambda_0\lambda_{0,u} + 24\lambda_0\lambda_{0,u}C_2 - 18\lambda_0\lambda_{0,u'}C_1$$
$$-18\lambda_0\lambda_{0,u'} - 32\lambda_0^2 C_1 - 2\lambda_0^2 C_1 C_2)u'^7 + (-18D_1 - 36\lambda_0^2\lambda_{0,u'} + 33\lambda_0^3 C_{2,u'}$$
$$+6\lambda_0\lambda_{0,u} + 18\lambda_0^2 C_1 - 21\lambda_0^2\lambda_{0,u'}C_2 + 18\lambda_0(\lambda_{0,u'})^2 - 64\lambda_0^3 + 4\lambda_0^2 C_1 - 8\lambda_0^3 C_2$$
$$+20\lambda_0^3 C_2^2 + 72\lambda_0^3 D_4)u'^6 + (52\lambda_0^3 + 6\lambda_0^2\lambda_{0,u'} + 13\lambda_0^3 c_2)u'^5$$
$$+(-22\lambda_0^3)u'^4 - 270\lambda_0^4 D_5 = 0 \ , \quad \text{(A.32)}$$

$$(-3C_{1,u} - C_1^2)u'^8 + (3\lambda_0 C_{1,u'} - 12\lambda_{0,u} - 21\lambda_0 C_{2,u} - 8\lambda_0 C_1$$
$$+15\lambda_{0,u}C_2 - 5\lambda_0 C_1 C_2)u'^7 + (-9d_2 + 12\lambda_0\lambda_{0,u'} + 21\lambda_0^2 C_{2,u'} - 30\lambda_{0,u}$$
$$-15\lambda_0\lambda_{0,u'}C_2 + 10\lambda_0 C_1 - 20\lambda_0^2 C_2 + 14\lambda_0^2 C_2^2 + 54\lambda_0^2 D_4$$
$$-16\lambda_0^2)u'^6 + (-9C_0 + 28\lambda_0^2 + 30\lambda_0\lambda_{0,u'} + 13\lambda_0^2 C_2)u'^5 + (-40\lambda_0^2)u'^4$$
$$-180\lambda_0^3 D_5 = 0 \; , \tag{A.33}$$

$$(-3C_{2,u} - C_1 C_2)u'^7 + (-3D_3 + 4C_1 + 3\lambda_0 C_{2,u'} - 4\lambda_0 C_2 + 2\lambda_0 C_2^2$$
$$+12\lambda_0 D_4)u'^6 + (-4\lambda_0 + 4\lambda_0 C_2)u'^5 + (-\lambda_0)u'^4 - 30\lambda_0^2 D_5 = 0 \; , \tag{A.34}$$

$$(-54D_{4,u} + 18C_{1,u'u'} + 3C_2 C_{1,u'} - 72C_{2,uu'} - 39C_2 C_{2,u})u'^8 + (24C_{2,u}$$
$$+72\lambda_{0,u'u'} + 12C_2\lambda_{0,u'} - 6C_{1,u'} + 36\lambda_0 C_{2,u'u'} - 3\lambda_0 C_2 C_{2,u'} + 72\lambda_{0,u'}C_{2,u'}$$
$$+33C_2^2\lambda_{0,u')} + 108D_4\lambda_{0,u'} + 54\lambda_0 d_{4,u'} + 36\lambda_0 C_2^2 + 18\lambda_0 C_{2u'u'})u'^7$$
$$+(-168\lambda_{0,u'} - 12\lambda_0 C_2 - 138\lambda_0 C_{2,u'} - 24C_2\lambda_{0,u'} - 33\lambda_0 C_2^2 - 36\lambda_0 D_4)u'^6$$
$$+(168\lambda_0 - 228\lambda_0 C_2 + 60\lambda_{0,u'})u'^5 + (-120\lambda_0)u'^4 + (270D_5\lambda_{0,u}$$
$$+270\lambda_0 D_{5,u})u'^2 + (54\lambda_0^2 D_{5,u'} - 810\lambda_0\lambda_{0,u'}D_5)u' + 2160\lambda_0^2 D_5 = 0 \; , \tag{A.35}$$

and

$$(-H_{,u})u'^2 + (3H\lambda_{0,u'} + \lambda_0 H_{,u'})u' - 3H\lambda_0 = 0 \; , \tag{A.36}$$

where

$$\begin{aligned} H = {} & (D_{4,u'} + \frac{1}{3}C_{2,u'u'} + \frac{2}{3}C_2 C_{2,u'} + \frac{2}{3}C_2 D_4 + \frac{4}{27}C_2^3) \\ & + \frac{1}{u'}(-\frac{4}{3}C_{2,u'} + \frac{2}{3}C_2^2 - \frac{4}{3}D_4 - \frac{8}{9}C_2^2) \\ & + \frac{1}{u'^2}(-\frac{5}{9}C_2) + \frac{1}{u'^3}(\frac{40}{27}) + \frac{1}{u'^5}(-2D_{5,u} - \frac{2}{3}C_1 D_5) \\ & + \frac{1}{u'^6}(-3\lambda_0 D_{5,u'} - 5D_5\lambda_{0,u'} - 2\lambda_0 C_2 D_5 - \frac{8}{3}\lambda_0 D_5) \\ & + \frac{1}{u'^7}(24\lambda_0 D_5) \; . \end{aligned} \tag{A.37}$$

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Reduction of fourth order ordinary differential equations to second and third order Lie linearizable forms

Hina M. Dutt *, Asghar Qadir

School of Natural Sciences, National University of Sciences and Technology, Campus H-12, 44000 Islamabad, Pakistan

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ABSTRACT

Meleshko presented a new method for reducing third order autonomous ordinary differential equations (ODEs) to Lie linearizable second order ODEs. We extended his work by reducing fourth order autonomous ODEs to second and third order linearizable ODEs and then applying the Ibragimov and Meleshko linearization test for the obtained ODEs. The application of the algorithm to several ODEs is also presented.

1. Introduction

First order ODEs can always be linearized [1] by point transformations [2]. Lie [3] showed that all linearizable second order ODEs must be cubically semi-linear, i.e.,

$$y'' + a_1(x,y)y'^3 - a_2(x,y)y'^2 + a_3(x,y)y' - a_4(x,y) = 0 \tag{1}$$

the coefficients a_1, a_2, a_3, a_4 satisfy an over-determined integrable system of four constraints involving two auxiliary functions, which Tresse wrote in more usable form [4]

$$\begin{aligned} &3(a_1a_3)_x - 3a_4a_{1y} - 6a_1a_{4y} - 2a_2a_{2x} + a_2a_{3y} - 3a_{1xx} + 2a_{2xy} - a_{3yy} = 0, \\ &3(a_4a_2)_y - 3a_1a_{4x} - 6a_4a_{1x} - 2a_3a_{3y} + a_3a_{2x} + 3a_{4yy} - 2a_{3xy} + a_{2xx} = 0. \end{aligned} \tag{2}$$

We call such equations *Lie linearizable*.

Chern [5,6] and Grebot [7,8] extended the linearization programme to the third order using contact and point transformations, respectively to obtain linearizability criteria for equations reducible to the forms $u'''(t) = 0$ and $u'''(t) + u(t) = 0$. It was shown [9] that there are three classes of third order ODEs that are linearizable by point transformations, viz. those that reduce to the above two forms or $u'''(t) + \alpha(t)u(t) = 0$. Neut and Petitot [10] dealt with the general third order ODEs. Ibragimov and Meleshko (IM) [11] used the original Lie procedure [3] of point transformation to determine the linearizability criteria for third order ODEs. They showed that any third order ODE $y''' = f(x, y, y', y'')$ obtained from a linear equation $u''' + \alpha(t)u = 0$ by means of point transformations $t = \varphi(x,y), u = \psi(x,y)$, must belong to one of the following two types of equations.

* Corresponding author. Tel.: +92 3035979229.
E-mail addresses: hinadutt@yahoo.com (H.M. Dutt), aqadirmath@yahoo.com (A. Qadir).

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Type I: If $\varphi_y = 0$ the equations that are linearizable are of the form

$$y''' + (a_1 y' + a_0)y'' + b_3 y'^3 + b_2 y'^2 + b_1 y' + b_0 = 0. \tag{3}$$

Type II: If $\varphi_y \neq 0$, set $r(x,y) = \varphi_x/\varphi_y$, equations are of the form

$$y''' + \frac{1}{y' + r}[-3(y'')^2 + (c_2 y'^2 + c_1 y' + c_0)y'' + d_5 y'^5 + d_4 y'^4 + d_3 y'^3 + d_2 y'^2 + d_1 y' + d_0] = 0, \tag{4}$$

where all coefficients a_i, b_i, c_i, d_i, being the functions of x and y, satisfy certain constraint requirements. Afterwards [12,13] used the point and contact transformations to determine the criteria for the linearizability of fourth order scalar ODEs. Meleshko [14] provided a simple algorithm to reduce third order ODEs of the form $y''' = f(y, y', y'')$ to second order ODEs. If the reduced equations satisfy the Lie linearizability criteria, they can then be solved by linearization. Meleshko showed that a third order ODE is reducible to the second order linearizable ODE if it is of the form

$$y''' + A(y,y')y''^3 + B(y,y')y''^2 + C(y,y')y'' + D(y,y'), \tag{5}$$

where the coefficients A, B, C, D satisfy certain constraints.

In the present paper we extend Meleshko's procedure to the fourth order ODEs in the cases that the equations do not depend explicitly on the independent or the dependent variable (or both) to reduce it to third (respectively second) order equations. Once the order is reduced we can apply the IM (or Lie) linearization test. If the reduced third (or second) order ODE satisfies the IM (or Lie) linearization test, then after finding a linearizing transformation, the general solution of the original equation is obtained by quadrature. So this method is effective in the sense that it reduces many ODEs, that cannot be linearized, to lower order linearizable forms. This is one of the motivations for studying this method. Another hope for the study of the linearization problem is that by using it we may be able to provide a complete classification of ODEs according to the number of arbitrary initial conditions that can be satisfied [17].

2. Equations reducible to linearizable forms

Meleshko had only treated the special case of independence of x for third order ODE. We include independence of y for completeness before proceeding to the fourth order.

2.1. Third order ODEs independent of y

Taking y' as the independent variable $u(x)$, we convert the ODE

$$y''' = f(x, y', y''), \tag{6}$$

to the second order ODE

$$u'' = f(x, u, u'), \tag{7}$$

which is linearizable by Lie's criteria if it is cubically semi-linear with the coefficients satisfying conditions (2).

Hence (7) is reducible to second order linearizable form if and only if

$$f(x, y', y'') = -c(x,y')y''^3 + g(x,y')y''^2 - h(x,y')y'' + d(x,y'), \tag{8}$$

with the coefficients satisfying

$$\begin{aligned} &3(ch)_x - 3dc_{y'} - 6cd_{y'} - 2gg_x + gh_{y'} - 3c_{xx} + 2g_{xy'} - h_{y'y'} = 0,\\ &3(dg)_{y'} - 3cd_x - 6dc_x - 2hh_{y'} + hg_x + 3d_{y'y'} - 2h_{xy'} + g_{xx} = 0. \end{aligned} \tag{9}$$

2.2. Fourth order ODEs independent of y

Since the variable y is missing, by taking y' as the new dependent variable $u(x)$, the ODE

$$y^{(4)} = f(x, y', y'', y'''), \tag{10}$$

is reduced to third order ODE

$$u''' = f(x, u, u', u''). \tag{11}$$

Eq. (11) is linearizable for the type I of Ibragimov and Meleshko's criteria if and only if

$$f(x, y', y'', y''') = -(a_1 y'' + a_0)y''' - b_3 y''^3 - b_2 y''^2 - b_1 y'' - b_0, \tag{12}$$

with the coefficients $a_i = a_i(x, y'), (i = 0, 1)$ and $b_j = b_j(x, y'), (j = 0, 1, 2, 3)$, satisfying the conditions

$$\begin{aligned}
&a_{0y'} - a_{1x} = 0,\\
&(3b_1 - a_0^2 - 3a_{0x})_{y'} = 0,\\
&3a_{1x} + a_0 a_1 - 3b_2 = 0,\\
&3a_{1y'} + a_1^2 - 9b_3 = 0,\\
&(9b_1 - 6a_{0x} - 2a_0^2)a_{1x} + 9(b_{1x} - a_1 b_0)_{y'} + 3b_{1y'}a_0 - 27b_{0y'y'} = 0.
\end{aligned} \tag{13}$$

Also the necessary and sufficient conditions for (11) to be linearizable for the type II of Ibragimov and Meleshko's criteria are

$$f(x,y',y'',y''') = \frac{-1}{y''+r}[-3(y''')^2 + (c_2 y''^2 + c_1 y'' + c_0)y''' + d_5 y''^5 + d_4 y''^4 + d_3 y''^3 + d_2 y''^2 + d_1 y'' + d_0] \tag{14}$$

and the coefficients $c_i = c_i(x,y'), (i = 0,1,2)$, $d_j = d_j(x,y'), (j = 0,1,2,3,4,5)$ and $r = r(x,y')$ have to satisfy constraint equations which can be produced simply by replacing y by y' for the type II constraint equations in [11].

2.3. Fourth order ODEs independent of x

The transformation $y' = u(y)$ will transform autonomous ODE of the fourth order

$$y^{(4)} = f(y,y',y'',y'''), \tag{15}$$

into the equation

$$u^3u''' + 4u^2u'u'' + uu'^3 - f(y,u,uu',u^2u'' + uu'^2) = 0, \tag{16}$$

which is a third order ODE in (y,u). It is linearizable by Ibragimov Meleshko's criteria if it is of the form (3) i.e.,

$$f(y,u,u'u,u''u^2 + uu'^2) = -u^3[(a_1u' + a_0)u'' + b_3u'^3 + b_2u'^2 + b_1u' + b_0] + 4u^2u'u'' + uu'^3, \tag{17}$$

where $a_i = a_i(y,u), (i = 0,1)$ and $b_j = b_j(y,u), (j = 0,1,2,3)$. With this (16) takes the form

$$u''' + (a_1u' + a_0)u'' + b_3u'^3 + b_2u'^2 + b_1u' + b_0 = 0. \tag{18}$$

Transforming (18) into a fourth order ODE with x as independent variable and y as dependent variable:

$$y^{(4)} + (A_1y'' + A_0)y''' + B_3y''^3 + B_2y''^2 + B_1y'' + B_0 = 0, \tag{19}$$

where

$$A_i = A_i(y,y'),\ (i = 0,1);\quad B_j = B_j(y,y'),\ (j = 0,1,2,3) \tag{20}$$

subject to the identification of coefficients

$$\begin{aligned}
&a_1 = A_1 + \frac{4}{y'},\quad a_0 = \frac{A_0}{y'},\quad b_3 = B_3 + \frac{A_1}{y'} + \frac{1}{y'^2},\\
&b_2 = \frac{B_2}{y'} + \frac{A_0}{y'^2},\quad b_1 = \frac{B_1}{y'^2},\quad b_0 = \frac{B_0}{y'^3},
\end{aligned} \tag{21}$$

with the constraints

$$\begin{aligned}
&y'^2A_{1y} - y'A_{0y'} + A_0 = 0,\\
&y'^2(-3A_{0yy'}) + y'(3B_{1y'} + 3A_{0y} - 2A_0A_{0y'}) + (-6B_1 + 2A_0^2) = 0,\\
&y'^2(3A_{1y}) + y'(A_0A_1 - 3B_2) + A_0 = 0,\\
&y'^2(3A_{1y'} - 9B_3 + A_1^2) - y'A_1 - 5 = 0,\\
&y'^4(-6A_{0y}A_{1y}) + y'^3(9B_1A_{1y} - 2A_0^2A_{1y} + 9B_{1yy'}) + y'^2(-18B_{1y} - 9A_1B_{0y'} - 9B_0A_{1y'} + 3A_0B_{1y'} - 27B_{0y'y'}) + y'(27A_1B_0\\
&\quad - 6A_0B_1 + 126B_{0y'}) - 180B_0\\
&\quad = 0.
\end{aligned} \tag{22}$$

Also in order to make (16) linearizable of type II of Ibragimov and Meleshko's criteria we have to take

$$\begin{aligned}
f(y,u,uu',u^2u'' + uu'^2) = &-\frac{u^3}{u'+r}[-3(u'')^2 + (c_2u'^2 + c_1u' + c_0)u'' + d_5u'^5 + d_4u'^4 + d_3u'^3 + d_2u'^2 + d_1u' + d_0]\\
&+ 4u^2u'u'' + uu'^3,
\end{aligned} \tag{23}$$

where $c_i = c_i(y, u), (i = 0, 1, 2)$, $d_j = d_j(y, u), (j = 0, 1, 2, 3, 4, 5)$ and $r = r(y, u)$.

Considering the form (23) and converting (16) into fourth order with x as independent and y as dependent variable, we have

$$y^{(4)} + \frac{1}{y'' + r_0}[-3(y''')^2 + (C_2 y''^2 + C_1 y'' + C_0)y''' + D_5 y''^5 + D_4 y''^4 + D_3 y''^3 + D_2 y''^2 + D_1 y'' + D_0] = 0, \tag{24}$$

where

$$C_i = C_i(y, y'), (i = 0, 1, 2); \quad D_j = D_j(y, y'), (j = 0, 1, 2, 3, 4, 5); \quad r_0 = r_0(y, y'),$$

subject to the identification of coefficients

$$\begin{aligned} &c_2 = C_2 - \frac{2}{y'}, \quad c_1 = C_1 + \frac{4r_0}{y'}, \quad c_0 = \frac{C_0}{y'^2}, \quad d_5 = \frac{D_5}{y'^5}, \\ &d_4 = D_4 + \frac{C_2}{y'} - \frac{2}{y'^2}, \quad d_3 = \frac{D_3}{y'} + \frac{C_1}{y'} + \frac{4r_0}{y'^2} - \frac{3r_0}{y'^3}, \\ &d_2 = \frac{D_2}{y'^2} + \frac{C_0}{y'^3}, \quad d_1 = \frac{D_1}{y'^3}, \quad d_0 = \frac{D_0}{y'^4}, \quad r = \frac{r_0}{y'}, \end{aligned} \tag{25}$$

with the constraints (43)–(51) (presented in the Appendix A).

2.4. Fourth order ODEs independent of x and y

By considering y' as independent and y'' as dependent variable, we convert the equation

$$y^{(4)} = f(y', y'', y'''), \tag{26}$$

into a second order ODE:

$$u^2 u'' + u u'^2 = f(y', u, u u'). \tag{27}$$

For (27) to be Lie-linearizable we must have

$$f(y', u, uu') = -u^2[A(y', u)u'^3 + B(y', u)u'^2 + C(y', u)u' + D(y', u)] + uu'^2. \tag{28}$$

Hence (26) takes the form

$$y^{(4)} + a(y', y'')y'''^3 + b(y', y'')y'''^2 + c(y', y'')y''' + d(y', y'') = 0, \tag{29}$$

where a, b, c and d must satisfy the constraints:

$$(3a_{y'y'})y''^4 + (2bb_{y'} - 3ca_{y'} - 3ac_{y'} - 2b_{y'y''})y''^3 + (2b_{y'} - bc_{y''} + 3a_{y''}d + 6ad_{y''} - c_{y''y''})y''^2 + (bc - 9ad - 3c_{y''})y'' - c = 0,$$

$$\begin{aligned} &(by'y')y''^4 + (b_{y'}c + 3d_{y''}b - 3d_{y'}a - 6a_{y'}d - 2c_{y'y''})y''^3 + (c_{y'} + 3d_{y''} - 6bd + 3b_{y''}d - 2cc_{y''} + 3d_{y''y''})y''^2 + (2c^2 - 6d \\ &\quad - 12d_{y''})y'' + 15d \\ &\quad = 0. \end{aligned} \tag{30}$$

Thus we have the following theorems.

Theorem 1. *Eq. (19) is reduced to the third order linearizable form if and only if it obeys (22).*

Theorem 2. *Eq. (24) is reduced to the third order linearizable form if and only if it obeys (43)–(51)* (presented in the appendix II).

Theorem 3. *Eq. (29) is reduced to the second order linearizable form if and only if it obeys (30).*

Remark. If we have a fourth order ODE of the form

$$y^{(4)} = -f(x, y)y'^5 + 10\frac{y''y'''}{y'} - 15\frac{y''^3}{y'^2}, \tag{31}$$

with $f(x, y)$ linear in x, then we can convert it to a linear ODE $x^{(4)} = f(x, y)$ by simply taking x as dependent and y as independent variables.

3. Illustrative examples

Example 1. The nonlinear fourth order ODE

$$y'y^{(4)} - y''y''' - 3y'^2y''' + 2y'^3y'' + 3y'^5 = 0, \tag{32}$$

cannot be linearized by point or contact transformation. It has the form (19) with the coefficients $A_1 = -1/y', A_0 = -3y'$, $B_3 = B_2 = 0, B_1 = 2y'^2, B_0 = +3y'^5$. One can verify that these coefficients satisfy the conditions (22). The transformation $y' = u(y)$ will reduce this ODE to the 3rd order linearizable ODE

$$u''' + \frac{3}{u}u'u'' - 3u'' - \frac{3}{u}u'^2 + 2u' + 3u = 0. \tag{33}$$

By using transformation equations in [11], we arrive at the transformation $t = e^y, s = u^2$ which maps (33) to the linear third order ODE $s''' + 6s/t^3 = 0$, whose solution is given by $s = c_1t^{-1} + c_2t^2\cos(2^{1/2}\ln t) + c_3t^2\sin(2^{1/2}\ln t)$, where c_i are arbitrary constants. By using the above transformation we get the solution of (33) given by $u = \pm\sqrt{c_1e^{-y} + c_2e^{2y}\cos(2^{1/2}y) + c_3e^{2y}\sin(2^{1/2}y)}$. Hence the general solution of (32) is obtained by taking the quadrature

$$\int \frac{dy}{\sqrt{c_1e^{-y} + c_2e^{2y}\cos(2^{1/2}y) + c_3e^{2y}\sin(2^{1/2}y)}} = \pm x + c_4, \tag{34}$$

where c_i are arbitrary constants.

Example 2. The nonlinear ODE

$$y^2y'^2y^{(4)} - 10y^2y'y''y''' - 3yy'^3y''' + 15y^2y''^3 + 9yy'^2y''^2 + 3y'^4y'' = 0, \tag{35}$$

is of the form (19) with the coefficients $A_1 = \frac{-10}{y'}, A_0 = \frac{-3y'}{y}, B_3 = \frac{15}{y'^2}, B_2 = \frac{9}{y}, B_1 = \frac{3y'^2}{y^2}, B_0 = 0$ satisfying the conditions (22). So it is reduced to the third order linearizable ODE

$$y^2u^2u''' - 3yu^2u'' - 6y^2uu'u'' + 3u^2u' + 6yuu'^2 + 6y^2u'^3 = 0, \tag{36}$$

with y as independent and u as dependent variable. The transformation $t = y^2, s = \frac{1}{u}$, reduces (36) to the linear third order ODE $s''' = 0$, whose solution is $s = c_1t^2 + c_2t + c_3$. Now one only needs to solve the equation $y' = 1/(c_1y^4 + c_2y^2 + c_3)$, where c_i are arbitrary constants. Hence, the general solution of (35) is given by

$$x = c_1y^5 + c_2y^3 + c_3y + c_4.$$

Example 3. The ODE

$$y'y''y^{(4)} - 3y'y'''^2 + 6y'^3y''^2y''' - 4y''^2y''' - y'y''^5 = 0, \tag{37}$$

has 2 symmetries. It is of the form (24) with the coefficients $r_0 = 0, C_2 = 6y'^2 - \frac{4}{y'}, C_1 = C_0 = 0, D_5 = -1, D_4 = D_3 = D_2 = D_1 = D_0 = 0$, obey the conditions (43)–(51). So it is reducible to linearizable third order ODE

$$u''' + \frac{1}{u'}[-3u''^2 - yu'^5] = 0. \tag{38}$$

The transformation $t = u, s = y$, will convert the nonlinear ODE (38) to the linear ODE $s''' + s = 0$ with solution

$$s = c_1e^{-t} + c_2e^{\frac{t}{2}}\cos t + c_3e^{\frac{t}{2}}\sin t. \tag{39}$$

Finally to find the solution of (37), we only need to solve

$$y = c_1e^{-y'} + c_2e^{\frac{y'}{2}}\cos y' + c_3e^{\frac{y'}{2}}\sin y'. \tag{40}$$

Example 4. The nonlinear ODE

$$y''y^{(4)} + y'''^3 - y'''^2 - y''y''', \tag{41}$$

is of the form (29) and the coefficients $a = \frac{1}{y''}, b = -\frac{1}{y''}, c = -y'', d = 0$, that satisfy conditions (30). So it is reduced to the linearizable second order ODE $u'' + u'^3 - u' = 0$. By using the transformation $t = u, s = e^y$, we can reduce it to linear ODE $s'' - s = 0$, whose solution is given by $s = c_1e^t + c_2e^{-t}$, where c_i are arbitrary constants. So that solution of (41) is obtained by solving the second order ODE

$$e^{y'} = c_1e^{-y''} + c_2e^{y''}, \tag{42}$$

where c_i are arbitrary constants.

4. Concluding remarks

Nonlinear ODEs are difficult to solve but, if they can be converted to linear ones by invertible transformations, they can be solved. Hence linearization plays a significant role in the theory of ODEs. In this paper we have presented criteria for fourth order autonomous ODEs to be reducible to linearizable third and second order ODEs. There are certain fourth order ODEs, not depending explicitly on the independent variable, which cannot be linearized by point or contact transformations but can be reducible to linearizable third order ODEs by Meleshko's method. The solution of the original equation is then obtained by a quadrature. Various fourth order ODEs with fewer symmetries can be reduced to linearizable form by this procedure. The class of ODEs linearizable by this method is not included in the Ibragimov and Meleshko classes or conditionally linearizable classes [15,16] of the ODEs (though there can be an overlap but it is not contained in that either). The reason is that it is not linearizable but reducible to linearizable form. In Lie's programme there is no definite statement available for the cases when the ODEs are not linearizable. By the recent developments this gap may be filled. By using the concept of Meleshko linearization a new class of scalar ODEs may be defined on the basis of initial conditions to be satisfied by ODEs.

Appendix A

$$(r_0C_1 - 6r_{0y})y'^2 + (6r_0r_{0y'} + 4r_0^2 - r_0^2C_2 - C_0)y' - 4r_0^2 = 0, \tag{43}$$

$$(C_{2y} - C_{1y'})y'^3 + (r_0C_{2y'} + C_2r_{0y'} - 4r_{0y'} - 6r_{0y'y'})y'^2 + (10r_{0y'} + 4r_0 - C_2r_0)y' - 8r_0 = 0, \tag{44}$$

$$\begin{aligned}&(-6r_0^2C_{1y} - 54(r_{0y})^2 + 18r_0r_{oyy} + 18r_0r_{0y}C_1 - 2r_0^2C_1^2)y'^8 + (3r_0^3C_{1y'} + 48r_0^2r_{0y} - 3r_0^3C_{2y} - 36r_0^2r_{0yy'} - 6r_0^2r_{0y}C_2\\&\quad - 18r_0^2r_{0y'}C_1 + 2r_0^3C_1C_2 - 16r_0^3C_1)y'^7 + (-60r_0^3r_{0y'} + 9r_0^4C_{2y'} - 42r_0^2r_{0y} - 36r_0^2(r_{oy'})^2 + 9r_0^3r_{0y'}C_2 + 14r_0^3C_1\\&\quad - 32r_0^4 + 8r_0^4C_2 + 4r_0^4C_2^2 + 18r_0^4D_4)y'^6 + (44r_0^4 + 72r_0^2r_{0y'} - 18r_0^3r_{0y'} - 7r_0^4C_2)y'^5 + (-20r_0^4)y'^4 - 72r_0^5D_5 = 0,\end{aligned} \tag{45}$$

$$\begin{aligned}&(-12r_0C_{1y} + 18r_{oyy'} + 18r_{0y}C_1 - 4r_0C_1^2)y'^8 + (9r_0^2C_{1y'} - 48r_0r_{0y} - 27r_0^2C_{2y} - 36r_0r_{0yy'} - 18r_{0y} + 72r_0r_{0y} + 24r_0r_{0y}C_2\\&\quad - 18r_0r_{0y'}C_1 - 18r_0r_{0y'} - 32r_0^2C_1 - 2r_0^2C_1C_2)y'^7 + (-18D_1 - 36r_0^2r_{0y'} + 33r_0^3C_{2y'} + 6r_0r_{0y} + 18r_0^2C_1 - 21r_0^2r_{0y'}C_2\\&\quad + 18r_0(r_{0y'})^2 - 64r_0^3 + 4r_0^2C_1 - 8r_0^3C_2 + 20r_0^3C_2^2 + 72r_0^3D_4)y'^6 + (52r_0^3 + 6r_0^2r_{oy'} + 13r_0^3c_2)y'^5\\&\quad + (-22r_0^3)y'^4 - 270r_0^4D_5 = 0,\end{aligned} \tag{46}$$

$$\begin{aligned}&(-3C_{1y} - C_1^2)y'^8 + (3r_0C_{1y'} - 12r_{0y} - 21r_0C_{2y} - 8r_0C_1 + 15r_{oy}C_2 - 5r_0C_1C_2)y'^7\\&\quad + (-9d_2 + 12r_0r_{0y'} + 21r_0^2C_{2y'} - 30r_{0y} - 15r_0r_{0y'}C_2 + 10r_0C_1 - 20r_0^2C_2 + 14r_0^2C_2^2 + 54r_0^2D_4 - 16r_0^2)y'^6\\&\quad + (-9C_0 + 28r_0^2 + 30r_0r_{0y'} + 13r_0^2C_2)y'^5 + (-40r_0^2)y'^4 - 180r_0^3D_5 = 0,\end{aligned} \tag{47}$$

$$\begin{aligned}&(-3C_{2y} - C_1C_2)y'^7 + (-3D_3 + 4C_1 + 3r_0C_{2y'} - 4r_0C_2 + 2r_0C_2^2 + 12r_0D_4)y'^6 + (-4r_0 + 4r_0C_2)y'^5\\&\quad + (-r_0)y'^4 - 30r_0^2D_5 = 0,\end{aligned} \tag{48}$$

$$\begin{aligned}&(-54D_{4y} + 18C_{1y'y'} + 3C_2C_{1y'} - 72C_{2yy'} - 39C_2C_{2y})y'^8 + (24C_{2y} + 72r_{0y'y'} + 12C_2r_{0y'} - 6C_{1y'} + 36r_0C_{2y'y'} - 3r_0C_2C_{2y'}\\&\quad + 72r_{0y'}C_{2y'} + 33C_2^2r_{0y')} + 108D_4r_{0y'} + 54r_0d_{4y'} + 36r_0C_2^2 + 18r_0C_{2y'y'})y'^7\\&\quad + (-168r_{0y'} - 12r_0C_2 - 138r_0C_{2y'} - 24C_2r_{0y'} - 33r_0C_2^2 - 36r_0D_4)y'^6 + (168r_0 - 228r_0C_2 + 60r_{0y'})y'^5\\&\quad + (-120r_0)y'^4 + (270D_5r_{0y} + 270r_0D_{5y})y'^2 + (54r_0^2D_{5y'} - 810r_0r_{0y'}D_5)y' + 2160r_0^2D_5 = 0,\end{aligned} \tag{49}$$

and

$$(-H_y)y'^2 + (3Hr_{0y'} + r_0H'_y)y' - 3Hr_0 = 0, \tag{50}$$

where

$$\begin{aligned}H = {}&\left(D_{4y'} + \frac{1}{3}C_{2y'y'} + \frac{2}{3}C_2C_{2y'} + \frac{2}{3}C_2D_4 + \frac{4}{27}C_2^3\right) + \frac{1}{y'}\left(-\frac{4}{3}C_{2y'} + \frac{2}{3}C_2^2 - \frac{4}{3}D_4 - \frac{8}{9}C_2^2\right) + \frac{1}{y'^2}\left(-\frac{5}{9}C_2\right) + \frac{1}{y'^3}\left(\frac{40}{27}\right)\\&+ \frac{1}{y'^5}\left(-2D_{5y} - \frac{2}{3}C_1D_5\right) + \frac{1}{y'^6}\left(-3r_0D_{5y'} - 5D_5r_{0y'} - 2r_0C_2D_5 - \frac{8}{3}r_0D_5\right) + \frac{1}{y'^7}(24r_0D_5).\end{aligned} \tag{51}$$

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Linearization of Two Dimensional Complex-Linearizable Systems of Second Order Ordinary Differential Equations

Hina M. Dutt

School of Natural Sciences
National University of Sciences and Technology
Islamabad, Pakistan

M. Safdar

School of Mechanical and Manufacturing Engineering
National University of Sciences and Technology
Islamabad, Pakistan

Abstract

Complex-linearization of a class of systems of second order ordinary differential equations (ODEs) has already been studied with complex symmetry analysis. Linearization of this class has been achieved earlier by complex method, however, linearization criteria and the most general linearizable form of such systems have not been derived yet. In this paper, it is shown that the general *linearizable form of the complex-linearizable systems* of two second order ODEs is (at most) quadratically semi-linear in the first order derivatives of the dependent variables. Further, linearization conditions are derived in terms of coefficients of system and their derivatives. These linearizable 2-dimensional complex-linearizable systems of second order ODEs are characterized here, by adopting both the real and complex procedures.

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1 Introduction

Most of the algorithms constructed to solve differential equations (DEs) with symmetry analysis involve an invertible change of the dependent and/or independent (point transformations) variables. For solving nonlinear DEs symmetry analysis uses a tool called linearization, which maps them to linear equations under invertible change of the variables. Linearization procedure requires the most general forms of the DEs that could be candidates of linearization and linearization criteria that ensure existence of invertible transformations from nonlinear to linear equations. Though construction of point transformations and finally getting to an analytic solution of the concerned problem are also involved in linearization process, these issues are of secondary nature as one needs to first investigate linearizability of DEs. An explicit linearizable form and linearization criteria for the scalar second order ODEs have been derived by Sophus Lie (see, e.g., [3]). Similarly, linearization of higher order scalar ODEs and systems of these equations attracted a great deal of interest and studied comprehensively over the last decade (see, e.g., [4]-[11]).

Complex symmetry analysis has been employed to solve certain classes of systems of nonlinear ODEs and linear PDEs. Of particular interest here, is linearization of systems of second order ODEs (see, e.g., [1]-[2]) that is achieved by complex methods. These classes are obtained from linearizable scalar and systems of ODEs by considering their dependent variables as complex functions of a real independent variable, which when split into the real and imaginary parts give two dependent variables. In this way, a scalar ODE produces a system of two coupled equations, with Cauchy-Riemann (CR) structure on both the equations. These CR-equations appear as constraint equations that restrict the emerging systems of ODEs to special subclasses of the general class of such systems. These subclasses of 2-dimensional systems of second order ODEs may trivially be studied with real symmetry analysis, however, they appear to be nontrivial when viewed from complex approach. Complex-linearizable (c-linearizable) classes explored earlier [1]-[2] and studied in this paper provide us means to extend linearization procedure to m-dimensional systems ($m \geq 3$), of n^{th} order ($n \geq 2$) ODEs. Though these classes are subcases of the general m-dimensional systems of n^{th} order ODEs, their linearization has not been achieved yet, with real symmetry analysis. Presently symmetry classification and solvability of higher dimensional systems of higher order ODEs seems to be exploitable *only with complex symmetry analysis.*

When linearizable scalar second order ODEs are considered complex by taking the dependent variable as a complex function of a real independent variable, they lead to c-linearization. The associated linearization criteria that consist of two equations (see, e.g., [3]) involving coefficients of the second order equations and their partial derivatives of (at most) order two, also yield four constraint equations for the corresponding system of two ODEs on splitting the complex functions involved, into the real and imaginary parts. These four equations constitute the c-linearization criteria [1], for the corresponding class of systems of two second order ODEs. The reason for calling them c-linearization instead of linearization criteria is that, in earlier works, explicit *Lie procedure* to obtain linearization conditions of this class of systems, was not performed after incorporating complex symmetry approach on scalar ODEs. The *most general form of the c-linearizable* 2*-dimensional linearizable systems* of second order ODEs is obtained here by real and complex methods. This derivation shows that the general linearizable forms (obtained by real and complex procedures) of 2-dimensional c-linearizable systems of second order ODEs are identical. Moreover, associated linearization criteria have been derived, again by adopting both the real and complex symmetry methods. These linearization conditions are also shown to be similar whether derived from real Lie procedure developed for systems or by employing complex symmetry analysis on scalar ODE. The core result obtained here is refinement of the c-linearization conditions to linearization criteria for 2-dimensional systems of second order ODEs, obtainable from linearizable complex scalar second order ODEs.

The plan of the paper is as follows. The second section presents derivation of the linearizable form for the scalar second order ODEs and Lie procedure to obtain associated linearization criteria. The subsequent section is on the linearization of 2-dimensional c-lineariable systems of second order ODEs, by real and complex symmetry methods. The fourth section contains some illustrative examples. The last section concludes the paper.

2 A subclass of linearizable scalar second order ODEs

The following point transformations

$$\tilde{x} = \phi(x,u), \quad \tilde{u} = \psi(x,u), \tag{1}$$

where ϕ and ψ are arbitrary functions of x and u, yield the most general form of linearizable scalar second order ODEs

$$u'' + \alpha(x,u)u'^3 + \beta(x,u)u'^2 + \gamma(x,u)u' + \delta(x,u) = 0, \tag{2}$$

with four arbitrary coefficients, that is cubically semi-linear in the first order derivative of the dependent variable, for derivation see [3]. Restricting these transformations to

$$\tilde{x} = \phi(x), \quad \tilde{u} = \psi(x,u), \tag{3}$$

i.e., assuming $\phi_u = 0$, leads to a quadratically semi-linear scalar second order ODE that is derived here explicitly. Under transformations (3) the first and second order derivatives of $\tilde{u}(\tilde{x})$ with respect to $\tilde{x}$ read as

$$\tilde{u}' = \frac{D\psi(x,u)}{D\phi(x)} = \lambda(x,u,u'), \tag{4}$$

and

$$\tilde{u}'' = \frac{D\lambda(x,u,u')}{D\phi(x)} = \mu(x,u,u',u''), \tag{5}$$

respectively. Here

$$D = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + \cdots, \tag{6}$$

is the total derivative operator. Inserting the total derivative operator in both the above equations leads us to the following

$$\tilde{u}' = \frac{\psi_x + u'\psi_u}{\phi_x}, \tag{7}$$

and

$$\tilde{u}'' = \frac{\phi_x(\psi_{xx} + 2u'\psi_{xu} + u'^2\psi_{uu} + u''\psi_u) - \phi_{xx}(\psi_x + u'\psi_u)}{\phi_x^3}, \tag{8}$$

respectively. Equating (8) to zero, i.e., considering $\tilde{u}'' = 0$, leaves a quadratically semi-linear ODE of the form

$$u'' + a(x,u)u'^2 + b(x,u)u' + c(x,u) = 0, \tag{9}$$

with the coefficients

$$a(x,u) = \frac{\psi_{uu}}{\psi_u}, \quad b(x,u) = \frac{2\phi_x\psi_{xu} - \psi_u\phi_{xx}}{\phi_x\psi_u}, \quad c(x,u) = \frac{\phi_x\psi_{xx} - \psi_x\phi_{xx}}{\phi_x\psi_u}. \tag{10}$$

The quadratic nonlinear (in the first derivative) equation (9) with three coefficients (10) is a subcase of the general linearizable (cubically semi-linear) second order ODE (2).

Now for the derivation of Lie linearization criteria of nonlinear equation (9), we start with a re-arrangement

$$\begin{aligned} \psi_{uu} &= a(x,u)\psi_u\ , \\ 2\psi_{xu} &= \phi_x^{-1}\psi_u\phi_{xx} + b(x,u)\psi_u\ , \\ \psi_{xx} &= \phi_x^{-1}\psi_x\phi_{xx} + c(x,u)\psi_u\ . \end{aligned} \tag{11}$$

of the relations (10). Equating the mixed derivatives of ψ, such that $(\psi_{xu})_u = (\psi_{uu})_x$ and $(\psi_{xu})_x = (\psi_{xx})_u$, we find

$$b_u - 2a_x = 0, \tag{12}$$

and

$$\phi_x^{-2}(2\phi_x\phi_{xx} - 3\phi_{xx}^2) = 4(c_u + ac) - (2b_x + b^2). \tag{13}$$

As $\phi_u = 0$, differentiating (13) with respect to u, simplifies it to

$$c_{uu} - a_{xx} - a_x b + a_u c + c_u a = 0. \tag{14}$$

Equations (12) and (14) constitute the linearization criteria for the scalar second order quadratically semi-linear ODEs.

3 Linearizable two dimensional c-linearizable systems of second order ODEs

We derive c-linearization and Lie-linearization criteria for a system of two second order ODEs.

3.1 C-linearization

Suppose $u(x)$ in (9) be complex function of a real variable x i.e., $u(x) = y(x) + iz(x)$. Further assume that

$$\begin{aligned} a(x,u) &= a_1(x,y,z) + ia_2(x,y,z)\ , \\ b(x,u) &= b_1(x,y,z) + ib_2(x,y,z)\ , \\ c(x,u) &= c_1(x,y,z) + ic_2(x,y,z)\ . \end{aligned} \tag{15}$$

This converts the scalar ODE (9) to a system of two second order ODEs of the form

$$\begin{aligned} y'' + a_1 y'^2 - 2a_2 y'z' - a_1 z'^2 + b_1 y' - b_2 z' + c_1 = 0\ , \\ z'' + a_2 y'^2 + 2a_1 y'z' - a_2 z'^2 + b_2 y' + b_2 z' + c_2 = 0\ , \end{aligned} \tag{16}$$

with the coefficients $a_j, b_j, c_j; (j = 1, 2)$, satisfying the CR-equations

$$\begin{aligned} a_{1,y} = a_{2,z}, \quad a_{1,z} = -a_{2,y}, \\ b_{1,y} = b_{2,z}, \quad b_{1,z} = -b_{2,y}, \\ c_{1,y} = c_{2,z}, \quad c_{1,z} = -c_{2,y}. \end{aligned} \tag{17}$$

Moreover, conditions (12) and (14) can now be converted into a set of four equations

$$2a_{1,x} - b_{1,y} = 0, \tag{18}$$
$$2a_{2,x} + b_{1,z} = 0, \tag{19}$$
$$c_{1,zz} + a_{1,xx} + a_{1,x}b_1 - a_{2,x}b_2 - (a_2c_1)_{,z} - (a_1c_2)_{,z} = 0, \tag{20}$$
$$c_{2,yy} - a_{2,xx} - a_{2,x}b_1 - a_{1,x}b_2 + (a_2c_1)_{,y} + (a_1c_2)_{,y} = 0, \tag{21}$$

by splitting the complex coefficients (17) into the real and imaginary parts.

As evident from [1], such a (complex) procedure leads us to c-linearization of systems of ODEs. Our claim here is that equations (18-21) are actually the linearization conditions despite of being just the c-linearization conditions for system (16). In order to prove this fact, we now use Lie linearization approach in the next subsection to derive the linearization conditions for system (16).

3.2 Lie linearization

The previous work on c-linearizable [1, 2] and their linearizable subclass of systems [9, 10] of second order ODEs reveals that point transformations of the form

$$\tilde{x} = \phi(x), \quad \tilde{y} = \psi_1(x, y, z), \quad \tilde{z} = \psi_2(x, y, z), \tag{22}$$

where

$$\psi_{1,y} = \psi_{2,z}, \quad \psi_{2,y} = -\psi_{1,z}, \tag{23}$$

i.e., ψ_j, for $j = 1, 2$, satisfy the CR-equations that involve derivatives with respect to both the dependent variables, linearizes the c-linearizable systems. Notice that (22) are obtainable from (3) that is a subclass of (1). These transformations map the first and second order derivatives as

$$\tilde{y}' = \frac{D\psi_1}{D\phi} = \lambda_1(x, y, z, y', z'), \quad \tilde{z}' = \frac{D\psi_2}{D\phi} = \lambda_2(x, y, z, y', z'), \tag{24}$$

and

$$\tilde{y}'' = \frac{D\lambda_1}{D\phi} = \mu_1(x, y, z, y', z', y'', z''), \quad \tilde{z}'' = \frac{D\lambda_2}{D\phi} = \mu_2(x, y, z, y', z', y'', z''), \tag{25}$$

where

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z} + y''\frac{\partial}{\partial y'} + z''\frac{\partial}{\partial z'} + \cdots. \tag{26}$$

Inserting the total derivative operator in the above equations and simplifying, we arrive at the following 2-dimensional system

$$\begin{aligned} y'' + \alpha_1 y'^2 - 2\alpha_2 y'z' + \alpha_3 z'^2 + \beta_1 y' - \beta_2 z' + \gamma_1 = 0, \\ z'' + \alpha_4 y'^2 + 2\alpha_5 y'z' + \alpha_6 z'^2 + \beta_3 y' + \beta_4 z' + \gamma_2 = 0, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \alpha_1 &= \phi_x \Delta^{-1}(\psi_{2,z}\psi_{1,yy} - \psi_{1,z}\psi_{2,yy}), \quad \alpha_2 = \phi_x \Delta^{-1}(\psi_{1,z}\psi_{2,yz} - \psi_{2,z}\psi_{1,yz}), \\ \alpha_3 &= \phi_x \Delta^{-1}(\psi_{2,z}\psi_{1,zz} - \psi_{1,z}\psi_{2,zz}), \quad \alpha_4 = \phi_x \Delta^{-1}(\psi_{1,y}\psi_{2,yy} - \psi_{2,y}\psi_{1,yy}), \\ \alpha_5 &= \phi_x \Delta^{-1}(\psi_{1,y}\psi_{2,yz} - \psi_{2,y}\psi_{1,yz}), \quad \alpha_6 = \phi_x \Delta^{-1}(\psi_{1,y}\psi_{2,zz} - \psi_{2,y}\psi_{1,zz}), \end{aligned}$$

$$\begin{aligned} \beta_1 &= 2\phi_x \Delta^{-1}(\psi_{2,z}\psi_{1,xy} - \psi_{1,z}\psi_{2,xy}) - \frac{\phi_{xx}}{\phi_x}, \quad \beta_2 = 2\phi_x \Delta^{-1}(\psi_{1,z}\psi_{2,xz} - \psi_{2,z}\psi_{1,xz}), \\ \beta_3 &= 2\phi_x \Delta^{-1}(\psi_{1,y}\psi_{2,xy} - \psi_{2,y}\psi_{1,xy}), \quad \beta_4 = 2\phi_x \Delta^{-1}(\psi_{1,y}\psi_{2,xz} - \psi_{2,y}\psi_{1,xz}) - \frac{\phi_{xx}}{\phi_x}, \end{aligned}$$

and

$$\begin{aligned} \gamma_1 &= \Delta^{-1}(\phi_x\psi_{1,y}\psi_{1,xx} - \psi_{1,x}\psi_{1,y}\phi_{xx} - \phi_x\psi_{1,z}\psi_{2,xx} + \psi_{1,z}\psi_{2,x}\phi_{xx}) , \\ \gamma_2 &= \Delta^{-1}(\phi_x\psi_{1,z}\psi_{1,xx} - \psi_{1,x}\psi_{1,z}\phi_{xx} + \phi_x\psi_{1,y}\psi_{2,xx} + \psi_{1,y}\psi_{2,x}\phi_{xx}) , \end{aligned} \tag{28}$$

where

$$\Delta = \phi_x(\psi_{1,y}\psi_{2,z} - \psi_{1,z}\psi_{2,y}) \neq 0 , \tag{29}$$

is the Jacobian of the transformation (22). The coefficients (10) of the scalar ODE (9) split into the coefficients of the corresponding 2-dimensional system of second order ODEs. This happens due to presence of the complex dependent function u, in the coefficients (10). The restricted fibre preserving transformations (22) used to derive the linearizable form (27), are obtainable from the complex transformations (3) that are employed to deduce (9). Therefore, transformations (22) along with (23) appear to be the real and imaginary parts of complex transformation (3), they reveal the correspondence of the linearizable forms of 2-dimensional systems and scalar complex ODEs. The CR-equations are not yet incorporated in the linearizable form (27). Insertion of the CR-equations (23) and their derivatives

$$\begin{aligned} \psi_{1,yy} = \psi_{2,yz} = -\psi_{1,zz}, \\ \psi_{2,zz} = \psi_{1,yz} = -\psi_{2,yy}, \end{aligned} \tag{30}$$

brings out the correspondence between the coefficients (10) of the complex linearizable ODEs (9) and coefficients (28) of the system (27). Employing (23) and (30) the coefficients (28) reduces to *only six arbitrary* coefficients that read as

$$\begin{aligned} \alpha_1 = -\alpha_3 = \alpha_5 = a_1, \quad \alpha_2 = \alpha_4 = -\alpha_6 = a_2, \\ \beta_1 = \beta_4 = b_1, \quad \beta_2 = \beta_3 = b_2, \quad \gamma_1 = c_1, \quad \gamma_2 = c_2. \end{aligned} \tag{31}$$

Here the coefficients a_j, b_j and c_j are the real and imaginary parts of the complex coefficients (10). The linearizable form of systems derived in this section by real method appears to be the same as one obtains by splitting the corresponding form of the scalar complex equation (9). This analysis leads us to the following theorem.

Theorem 3.1 *The most general form of the linearizable two dimensional c-linearizable systems of second order ODEs is quadratically semi-linear.*

3.2.1 Sufficient conditions for the linearization of a c-linearizable system

Consider the most general form of the c-linearizable 2-dimensional systems of second order ODEs (16), with constraint equations (17). Rewriting the coefficients of the system (16) in the form

$$\begin{aligned} a_1 &= \Delta^{-1}\phi_x(\psi_{1,y}\psi_{1,yy} + \psi_{1,z}\psi_{1,yz}), \\ a_2 &= \Delta^{-1}\phi_x(\psi_{1,z}\psi_{1,yy} + \psi_{1,y}\psi_{1,yz}), \\ b_1 &= 2\Delta^{-1}\phi_x(\psi_{1,y}\psi_{1,xy} + \psi_{1,z}\psi_{1,xz}) - \frac{\phi_{xx}}{\phi_x}, \\ b_2 &= 2\Delta^{-1}\phi_x(\psi_{1,z}\psi_{1,xy} + \psi_{1,y}\psi_{1,xz}), \\ c_1 &= \Delta^{-1}(\phi_x\psi_{1,y}\psi_{1,xx} - \psi_{1,x}\psi_{1,y}\phi_{xx} - \phi_x\psi_{1,z}\psi_{2,xx} + \psi_{1,z}\psi_{2,x}\phi_{xx}), \\ c_2 &= \Delta^{-1}(\phi_x\psi_{1,z}\psi_{1,xx} - \psi_{1,x}\psi_{1,z}\phi_{xx} + \phi_x\psi_{1,y}\psi_{2,xx} + \psi_{1,y}\psi_{2,x}\phi_{xx}). \end{aligned} \tag{32}$$

For obtaining the sufficient linearizability conditions of (16), we have to solve compatibility problem, that has already been solved for the scalar equations earlier in this work, for the set of equations (32). It is an over determined system of partial differential equations for the functions ϕ, ψ_1 and ψ_2 with known a_j, b_j, c_j.
The system (32) gives us

$$\begin{aligned} \psi_{1,yy} &= \psi_{1,y}a_1 + \psi_{1,z}a_2 \ , \\ \psi_{1,yz} &= \psi_{1,z}a_1 - \psi_{1,y}a_2 \ , \\ \psi_{1,xy} &= \frac{1}{2}(\psi_{1,y}b_1 + \psi_{1,z}b_2 + \psi_{1,y}\frac{\phi_{xx}}{\phi_x}) \ , \\ \psi_{1,xz} &= \frac{1}{2}(\psi_{1,z}b_1 - \psi_{1,y}b_2 + \psi_{1,z}\frac{\phi_{xx}}{\phi_x}) \ , \end{aligned}$$

$$\begin{aligned}\psi_{1,xx} &= \psi_{1,y}c_1 + \psi_{1,z}c_2 + \psi_{1,x}\frac{\phi_{xx}}{\phi_x}\ , \\ \psi_{2,xx} &= \psi_{1,y}c_2 - \psi_{1,z}c_1 + \psi_{2,x}\frac{\phi_{xx}}{\phi_x}\ .\end{aligned}$$

The compatibility of the system (32) first requires to compute partial derivatives

$$\begin{aligned}\Delta_x &= 2\Delta\frac{\phi_{xx}}{\phi_x} + \Delta b_1\ , \\ \Delta_y &= 2\Delta a_1\ , \\ \Delta_z &= -2\Delta a_2\ ,\end{aligned}$$

of the Jacobian. Comparing the mixed derivatives $(\Delta_y)_z = (\Delta_z)_y$, $(\Delta_x)_y = (\Delta_y)_x$ and $(\Delta_x)_z = (\Delta_z)_x$, we obtain

$$a_{1,z} + a_{2,y} = 0\ , \tag{33}$$

$$2a_{1,x} - b_{1,y} = 0\ , \tag{34}$$

$$2a_{2,x} + b_{1,z} = 0\ , \tag{35}$$

respectively. Equating the mixed derivatives $(\psi_{1,yy})_z = (\psi_{1,yz})_y$, $(\psi_{1,yy})_x = (\psi_{1,xy})_y$, $(\psi_{1,xx})_y = (\psi_{1,xy})_x$, $(\psi_{1,xx})_z = (\psi_{1,xz})_x$, $(\psi_{1,xy})_z = (\psi_{1,xz})_y$, $(\psi_{2,xx})_y = (\psi_{2,xy})_y$ and $(\psi_{2,xx})_z = (\psi_{2,xz})_x$ gives us

$$a_{1,y} - a_{2,z} = 0, \tag{36}$$

$$b_{2,y} + b_{1,z} = 0, \tag{37}$$

$$b_{2,z} - b_{1,y} = 0, \tag{38}$$

$$c_{2,z} - c_{1,y} = 0, \tag{39}$$

$$c_{2,y} + c_{1,z} = 0, \tag{40}$$

$$c_{1,zz} + a_{1,xx} + a_{1,x}b_1 - a_{2,x}b_2 - (a_2c_1)_{,z} - (a_1c_2)_{,z} = 0, \tag{41}$$

$$c_{2,yy} - a_{2,xx} - a_{2,x}b_1 - a_{1,x}b_2 + (a_1c_2)_{,y} - (a_2c_1)_{,y} = 0. \tag{42}$$

(43)

Note that $(\psi_{1,yz})_x - (\psi_{1,xz,})_y = 0$ and $(\psi_{1,xy})_z - (\psi_{1,yz})_x = 0$ are satisfied. Also (33), (36), and (37)-(40) are CR-equations for the coefficients a_j, b_j, c_j. Therefore, the solution of the compatibility problem of the system (32), provides CR-constraints on the coefficients of (16) and the linearization conditions.

Theorem 3.2 *A two dimensional c-linearizable system of second order ODEs of the form* (16) *is linearizable if and only if its coefficients satisfy the CR-equations and conditions* (34)*,* (35)*,* (41)*,* (42)*.*

These are the same conditions that are already obtained (18-21), by employing complex analysis, i.e., splitting the linearization conditions associated with the base scalar equation (9), into the real and imaginary parts.

Corollary 3.3 *The c-linearization conditions for a two dimensional system of quadratically semi-linear second order ODEs are the linearization conditions.*

4 Examples

We present some examples to illustrate our results.

1. The 2-dimensional system of second order ODEs

$$y'' - (\frac{2y}{y^2+z^2})y'^2 - 2(\frac{2z}{y^2+z^2})y'z' + (\frac{2y}{y^2+z^2})z'^2 - \frac{2}{x}y' - \frac{2y}{x^2} = 0 \ ,$$
$$z'' + (\frac{2z}{y^2+z^2})y'^2 - 2(\frac{2z}{y^2+z^2})y'z' - (\frac{2z}{y^2+z^2})z'^2 - \frac{2}{x}z' - \frac{2z}{x^2} = 0 \ . \qquad (44)$$

is of the same form as (16) with

$$a_1 = \frac{-2y}{y^2+z^2} \ , \quad a_2 = \frac{2z}{y^2+z^2} \ , \quad b_1 = \frac{-2}{x} \ , \quad b_2 = 0 \ , \quad c_1 = \frac{-2y}{x^2} \ , \quad c_2 = \frac{-2z}{x^2} \ .(45)$$

One can easily verify that (45) satisfy the conditions (34), (35), (41), (42) and CR-equations w.r.t y and z. So the system of ODEs (44) is linearizable. The transformation

$$t = x \ , \quad u = \frac{y}{x(y^2+z^2)} \ , \quad v = \frac{-z}{x(y^2+z^2)} \ , \qquad (46)$$

reduces the nonlinear system (44) to the linear system $u'' = 0 \ , v'' = 0$.

2. Consider the following system of nonlinear ODEs

$$y'' - \frac{1}{f(y,z)}(y'^2 \cos y \sin y - z'^2 \cos y \sin y - 2y'z' \cosh z \sinh z) + \frac{2y'}{x} = 0 \ ,$$
$$z'' - \frac{1}{f(y,z)}(y'^2 \cosh z \sinh z - z'^2 \cosh z \sinh z + 2y'z' \cosh y \sinh y) + \frac{2z'}{x} = 0 \ ,(47)$$

where $f(y,z) = \sin^2 y \cosh^2 z + \cos^2 y \sinh^2 y$, and the coefficients satisfy the CR-constraint and linearization conditions (34), (35), (41), (42). Hence Theorem 2 guarantees that system (47) can be transformed to system of linear equations $u'' = 0 \ , \ v'' = 0$. The linearizing transformations in this case are

$$t = x \ , \quad u = x \cos y \cosh z \ , \quad v = -x \sin y \sinh z \ . \qquad (48)$$

3. Consider the anisotropic oscillator system

$$y'' + f(x)y = 0 \ ,$$
$$z'' + g(x)z = 0 \ . \qquad (49)$$

In [7] it is shown that system (49) is reducible to the free particle system ($u'' = 0$, $v'' = 0$) provided $f = g$. Our c-linearization criteria also leads to the same condition, i.e. $f = g$.

5 Conclusion

C-linearization of 2-dimensional systems of second order ODEs is achieved earlier by considering the scalar second order linearizable ODEs as complex. Their associated linearization criteria are separated into the real and imaginary parts due to complex functions involved. In this work, the c-linearization and linearization are shown to be two different criteria for a 2-dimensional systems of second order ODEs. Linearizable form of such c-linearizable systems has been derived and it is shown to be quadratically semi-linear in the first order derivatives. Moreover, complex linearization criteria have been refined to linearization criteria for such 2-dimensional systems that are linearizable due to their correspondence with the complex scalar ODEs.

Earlier in this work, c-linearizable classes of systems of ODEs are claimed to be non-trivial, when viewed from complex approach. The reason for calling them non-trivial is that the concept of c-linearization of systems of ODEs is extendable to m-dimensional systems of n^{th} order ODEs. The simplest procedure that might lead us to linearization of m-dimensional system of second order ODEs, is to iteratively complexify a scalar second order linearizable ODE. Therefore, complex symmetry analysis needs to be extended to 2- and 3-dimensional systems of third and second order ODEs, respectively, in order to derive the general linearization results mentioned above. Likewise, complex symmetry analysis may lead us to algebraic classification of the higher dimensional systems of higher order ODEs.

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