USE OF APPROXIMATE SYMMETRY METHODS TO DEFINE ENERGY OF GRAVITATIONAL WAVES

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Dedicated to

To My Parents and to My Late Grand Mother

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Abstract

In this thesis approximate Lie symmetry methods for differential equations are used to investigate the problem of energy in general relativity and in particular in gravitational waves. For this purpose second-order approximate symmetries of the system of geodesic equations for the Reissner-Nordström (RN) spacetime are studied. It is shown that in the second-order approximation, energy must be rescaled for the RN spacetime.

Then the approximate symmetries of a Lagrangian for the geodesic equations in the Kerr spacetime are investigated. Taking the Minkowski spacetime as the exact case, it is shown that the symmetry algebra of the Lagrangian is 17 dimensional. This algebra is related to the 15 dimensional algebra of conformal isometries of the Minkowski spacetime. First introducing spin angular momentum per unit mass as a small parameter first-order approximate symmetries of the Kerr spacetime as a first perturbation of the Schwarzschild spacetime are considered. We then investigate the second-order approximate symmetries of the Kerr spacetime as a second perturbation of the Minkowski spacetime.

Next, second-order approximate symmetries of the system of geodesic equations for the charged-Kerr spacetime are investigated. A rescaling of the arc length parameter for consistency of the trivial second-order approximate symmetries of the geodesic equations indicates that the energy in the charged-Kerr spacetime has to be rescaled.

Since gravitational wave spacetimes are time-varying vacuum solutions of Einstein's field equations, there is no unambiguous means to define their energy content. Here a definition, using slightly broken Noether symmetries is proposed. A problem is noted with the use of the proposal for plane-fronted gravitational waves. To attain a better understanding of the implications of this proposal we also use an artificially constructed time-varying non-vacuum plane symmetric metric and evaluate its Weyl and stress-energy tensors so as to obtain the gravitational and matter components separately and compare them with the energy content obtained by our proposal. The procedure is also used for cylindrical gravitational wave solutions. The usefulness of the definition is demonstrated by the fact that it leads to a result on whether gravitational waves suffer self-damping.

Contents

Chapter 1

Preliminaries

1.1 Introduction

Among all the theories of gravitation Einstein's theory of General Relativity (GR) is the most generally accepted. According to GR the gravitating matter alters the geometry of its surroundings and thus the behavior of nearby bodies. GR gives correct results even for strong gravitational forces (where Newton's theory fails) and agrees with Newton's theory for weak gravitational forces. It is expressed in terms of pseudo-Riemannian geometry. Here the four dimensional spacetime is represented by a Lorentzian manifold M, having signature $(+,-,-,-)$ with metric tensor g_{ab} and the stress-energy tensor T_{ab} $(a,b=0,1,2,3)$. The curvature of the spacetime is given by the Riemann curvature tensor $R^a_{\ bcd}$. Einstein's field equations (EFEs)

$$R_{ab}-\frac{1}{2}Rg_{ab}+\Lambda g_{ab}=\kappa T_{ab}, \tag{1.1}$$

provide a relation between the geometry and the distribution of matter in spacetime, where R is the trace of the Ricci tensor R_{ab}, which itself is the trace of the curvature tensor $R^a_{\ bcd}$. The stress-energy tensor corresponds to the real distribution of matter. The gravitational coupling constant $\kappa=8\pi G/c^4$, where G is Newton's gravitational constant, c is the speed of light and Λ, called the cosmological constant, is negligible in all non-cosmological situations. Equations (1.1) constitute a system of ten non-linear partial differential equations to determine the twenty unknown functions, ten g_{ab} and ten T_{ab} (as both are symmetric tensors). Due to non-linearity this system is very difficult to solve analytically unless some constraints or geometric symmetries are imposed on the metric tensor. A metric tensor is called an exact solution [1] of (1.1) if it defines a physically acceptable T_{ab}.

The first exact solution of the EFEs was obtained by Schwarzschild in 1916, just after the formulation of the theory. This is a spherically symmetric static vacuum solution, i.e. for which $R_{ab} = 0$. This spacetime admits 4 *Killing Vectors* (KVs) which gives the conservation laws of energy, angular momentum and azimuthal angular momentum. Another important spherically symmetric static solution of EFEs is the Reissner-Nordström (RN) solution, which represents the field of a point massive electric charge at rest at the origin. For this spacetime $R_{ab} \neq 0$. This spacetime also admits the same four conservation laws. Yet another well-known spacetime of GR is the Kerr spacetime. This spacetime is an axially symmetric, stationary solution of the vacuum EFEs. This spacetime admits 2 KVs which give the conservation of energy and azimuthal angular momentum. Besides, there are non-static solutions of vacuum EFEs which represent gravitational waves (GWs). These are fluctuations in spacetime. In Maxwell's theory of electromagnetism accelerated charges emit electromagnetic waves. In a similar way in Einstein's theory of GR accelerated masses produce GWs. The first exact cylindrical wave solution was given by Einstein and Rosen in 1937. Then Bondi and Robinson gave the exact plane wave solution in 1957. Linearization of GR naturally leads to the prediction of GWs. They have never been directly detected but efforts are now underway to detect them from astrophysical sources, which will bring the researchers an additional tool to study the universe (see for example [2, 3]). Newton's theory of gravitation implies that the binary period of two point masses (e.g., two stars) moving in a bound orbit is strictly a constant quantity. However, GR predicts that two stars revolving around each other in a bound orbit suffer accelerations and as a result gravitational radiation is emitted.

The definition of energy has been one of the most thorny and important problems in GR. In contrast to Newton's theory of gravity, energy is not a well defined concept in GR. In the context of Classical Mechanics the Hamiltonian in the Poisson bracket acts as a time derivative for a conservative system. Thus energy is a conserved and well defined quantity. Therefore it is clear that for energy to be conserved in GR, the spacetime must have a time-like KV, so as to allow time-translational invariance. If the spacetime is static there is a time-like isometry or

KV, which can be used to define the energy of a test-particle. Namely if the KV is $\mathbf{k}$ and the momentum of the test-particle is $\mathbf{p}$, the energy of the test-particle is given by $E = \mathbf{k}.\mathbf{p}$. Further energy conservation in the spacetime is guaranteed in the frame using $\mathbf{k}$ to define time direction. However if there does not exist a time-like KV, energy is not conserved and hence energy of a test particle can also not be defined. Since GWs must be given by non-static spacetimes the problem of defining the energy content of GWs is particularly severe.

There have been several attempts [4, 5, 6] to obtain a well defined expression for local or quasi-local energy and momentum in GR. However, there is still no generally accepted definition known. As a result, different people have different points of view. Cooperstock [7] argued that in GR, energy and momentum are localized in regions of the non-vanishing energy and momentum tensor and consequently GWs are not carriers of energy and momentum in vacuum. By definition GWs, have zero stress-energy tensor. Therefore the existence of these waves was questioned. However, GR indicates the existence of GWs as solutions of EFEs [1].

The problem for GWs was attempted by Weber and Wheeler [8] and Ehlers and Kundt [9]. They considered a sphere of test particles in the path of the waves. Weber and Wheeler gave an approximate formula for momentum imparted to test particles by cylindrical GWs. Ehlers and Kundt showed that plane waves impart a constant momentum to the test particles in their path. Qadir and Sharif [10] presented an operational procedure, embodying the same principle, that gave a closed formula for the momentum imparted by GWs to the test particles. Rosen used the energy-momentum pseudo-tensors of Einstein [11] and Landau-Lifshitz [12] and carried out calculations in cylindrical polar coordinates [13]. He concluded that the energy and momentum density components vanish for cylindrical GWs. These results supported the Scheidegger's conjecture [14] that a physical system cannot radiate gravitational energy. Later, Rosen pointed out [15] that if the calculations are performed in Cartesian coordinates the energy and momentum densities turn out to be non-vanishing and reasonable. Rosen and Virbhadra [16] used Einstein's prescription by using Cartesian coordinates and found these quantities finite and well defined. Then Virbhadra [17] used the prescriptions of Tolman, Landau-Lifshitz and Papapetrou to evaluate the energy and momentum densities and showed that the same results hold in all these prescriptions.

Energy and momentum conservation are described by the requirement that the divergence of the stress-energy tensor is zero. In GR, the partial derivative in the usual conservation equation $T^b_{a,b} = 0$, is replaced by a covariant derivative. The tensor T^b_a then represents the energy and momentum of matter and all non-gravitational fields and no longer satisfies $T^b_{a,b} = 0$. A contribution from the gravitational field must be added to obtain an energy-momentum expression with zero divergence. Following Einstein and Landau-Lifshitz, Papapetrou gave similar prescriptions [18]. Comparatively recently Weinberg gave another similar prescription [19]. The expressions they gave are called energy-momentum complexes because they can be expressed as a combination of T^b_a and a pseudo-tensor, which is interpreted to represent the energy and momentum of the gravitational field. These complexes have been criticized because they are coordinate dependent and hence non-tensorial. One can get physically meaningful results for the Einstein, Landau-Lifshitz, Papapetrou, Weinberg (ELLPW) energy-momentum complexes, only in Cartesian coordinates [12, 20, 21]. Due to the coordinate dependence many others, including Møller [22], Bondi [23], Komar [24], Ashtekar-Hansen [25] and Penrose [4], have proposed coordinate independent definitions. Møller realized that the use of a tetrad as the field variable, instead of a metric, makes it possible to introduce a first order Lagrangian for the EFEs. Bondi studied the isolated sources of radiative spacetime and observed that the 2-surface integral of a certain expansion coefficient of the line element of that spacetime in an asymptotically retarded spherical coordinate system (u, r, θ, ϕ) behaves as the energy of the system at the retarded time u. Komar introduced a tensorial super-potential which is independent of any background structure and is unique property. Ashtekar and Hansen defined the angular momentum in their specific conformal model of the spatial infinity as a certain 2-surface integral near infinity. Penrose defined quasi-local energy-momentum and angular momentum using twistor-theoretical idea. However, each of these, has its own drawbacks [22, 26, 27]. Here we mention some prescriptions relating to the evaluation of energy, some of which are also discussed in [28].

1.1.1 Different Prescriptions to Evaluate Energy and Momentum in GR

As mentioned above, all stress-energy pseudo-tensors defined satisfy the local conservation law with the partial derivative.

(a) Einstein's Prescription

First of all Einstein attempted the problem of energy and momentum in GR. He gave the energy-momentum complex [11]

$$\Psi_a^b = \frac{1}{16\pi} L_{a,c}^{bc} , \tag{1.2}$$

where

$$L_a^{bc} = \frac{g_{ad}}{\sqrt{-g}}[-g(g^{bd}g^{ce} - g^{cd}g^{be})]_{,e} \tag{1.3}$$

and we have used the Einstein summation convention here and hereafter.

(b) Landau-Lifshitz's Prescription

After Einstein Landau and Lifshitz tried to resolve the problem of energy and momentum in GR. They gave the energy-momentum complex [12]

$$\Theta^{ab} = \frac{1}{16\pi} P^{acbd}_{\ \ \ \ ,cd} , \tag{1.4}$$

where

$$\Theta^{ab} = -g(T^{ab} + t^{ab}), \tag{1.5}$$

$$P^{abcd} = -g[-g(g^{ab}g^{cd} - g^{ac}g^{bd})]. \tag{1.6}$$

Here Θ^{ab} is symmetric in indices a and b while P^{abcd} has symmetries of the Riemann curvature tensor, t^{ab} is known as the Landau-Lifshitz pseudo-tensor. The locally conserved quantity Θ^{ab} contains contributions from the matter, non-gravitational fields and gravitational fields as well.

(c) Papapetrou's Prescription

The energy-momentum complex of Papapetrou [18] is given by

$$\Phi^{ab} = \frac{1}{16\pi} K^{abcd}_{\ \ \ \ ,cd}, \tag{1.7}$$

where

$$K^{abcd} = \sqrt{-g}[g^{ab}\eta^{cd} - g^{ac}g^{bd} + g^{cd}\eta^{ab} - g^{bd}g^{ac}], \tag{1.8}$$

and η^{ab} is the Minkowski metric.

(d) Weinberg's Prescription

Weinberg energy-momentum complex is given by [19]

$$W^{ab} = \frac{1}{16\pi}\Omega^{abc}_{\ \ \ ,c}\ , \tag{1.9}$$

where

$$\Omega^{abc} = h^{e,a}_{e}\eta^{bc} - h^{e,b}_{e}\eta^{ac} - h^{ea}_{,e}\eta^{bc} + h^{eb}_{,e}\eta^{ac} + h^{ac,b} - h^{bc,a}, \tag{1.10}$$

$$\text{and } h_{ab} = g_{ab} - \eta_{ab}. \tag{1.11}$$

(e) Møller's Prescription

The energy-momentum complex of Møller is given by [21]

$$M^{b}_{a} = \frac{1}{8\pi}\Upsilon^{bc}_{a,c}\ , \tag{1.12}$$

where

$$\Upsilon^{bc}_{a} = \sqrt{-g}(g_{ad,e} - g_{ae,d})g^{be}g^{cd}. \tag{1.13}$$

Beside the idea of pseudo-tensor there were some other attempts to resolve the problem of energy in GR. Below we discuss few of them which are more relevant for our further discussion.

(f) Komar's Integral

Komar using his definition of approximate symmetry [29], wrote down an integral for the mass (energy) in a spacetime [24]

$$M = \frac{1}{8\pi}\int_{S^2} *d\widetilde{\xi}, \tag{1.14}$$

where $\widetilde{\xi}$ is the time-like Killing 1-form for the exact symmetry, $*d\widetilde{\xi}$ the dual of the 2-form $d\widetilde{\xi}$ and S^2 is the 2-surface [30]. First Cohen and de Felice [31] and then Chellathurai and Dadhich [32] used this integral to calculate the effective mass of RN, Kerr and charged-Kerr spacetimes. For our purpose, this formulation will be discussed in more detail in chapter 4.

(g) Qadir-Sharif 's formula for momentum imparted to test particles by GW

This prescription [10] does not give the energy-momentum tensor in the field but the momentum imparted to test particles. The momentum 4-vector whose proper time derivative is F_a, is given by

$$p_a = \int F_a dt, \tag{1.15}$$

where

$$F_0 = M[\{\ln \frac{A}{\sqrt{g_{00}}}\}_{,0} - \frac{g_{\alpha\beta,0}g^{\alpha\beta}_{,0}}{4A}],\ F_i = M(\ln g_{00})_{,i}, \tag{1.16}$$

$$\text{and } A = (\ln\sqrt{-g})_{,0}. \tag{1.17}$$

The spatial components of p_a give the momentum imparted to test particles as defined in the preferred frame (in which $g_{0\alpha} = 0$).

(h) Christodoulou-Thorne's memory effect

The "memory" of GW burst is the permanent displacement of the test masses of a laser interferometer detector after a wave train passes through it [33, 34]. This memory in general equals the change from before the burst to afterward, in the transverse traceless part of the $1/r$ Coulomb-type gravitational field generated by the four-momenta of the source's various independent pieces. Christodoulou [35], pointed out that the previous linearized theory calculations missed the gravitational self-interaction effect. The nonlinear memory due to the cumulative contribution of the effective stress of the gravitational waves themselves gives a measurable correction. Thorne [36] discussed this idea in a more physically intelligible way and argued that the contribution due to the nonlinear effects are already included in the expression given in [33, 34].

(i) Isaacson's stress-energy tensor

Isaacson [37] assumed that the wavelength of the gravitational wave is much smaller than the radius of curvature of the background geometry. This led to a gauge-invariant first-order approximation procedure. It was argued that the gravitational field is remarkably similar to the electromagnetic field in the behavior of its amplitude, frequency and polarization. Then the results of this linear approximation were extended to incorporate some of the essential features of the EFEs. It was found that in the high-frequency limit the gravitational field has a natural gauge-invariant stress (true) tensor. Like the Maxwell stress tensor, this stress tensor for gravitational waves involves only first derivatives of the field. This gives the freedom to introduce a Poynting vector to describe the flow of energy and momentum, and acts as a source generating curvature of spacetime. The formalism was then applied to a spherical shell of radiation expending in a spherically symmetric background geometry. For this purpose a spherically symmetric solution of Vaidya [38, 39] was used. There the source was found to lose exactly the energy and momentum contained in the radiation field.

In [28] it is shown that the different prescriptions (a) - (e), can provide the same result for different cosmological models. There it is also shown that the problem becomes very complicated and the results obtained will not be the same when rotation is included in the spacetime.

The lack of a good definition of energy, also leads to problems with the definition of mass [40]. Since energy conservation is related to time translation symmetry [41], therefore, one needs to use some concept of time symmetry that allows for slight deviations away from exact symmetry to define energy in GR and in particular in GWs. This approach was attempted earlier by various people. There have been a number of different definitions of "approximate symmetry". One idea was to assume that conservation of energy holds asymptotically [29] and to examine whether it would work for gravitational radiation and to define a positive definite energy. This seems unsatisfactory as the gravitational energy should then reach infinity. There may then be problems with orders of approximation being consistent. An altogether different approach was taken by providing a measure of the extent of break-down of symmetry. The integral of the square of the symmetrized derivative of a vector field was divided by its mean square norm [42, 37]. This led to what was called an almost symmetric space and the corresponding vector field an almost KV [43]. This measure of "non-symmetry" in a given direction was applied to the Taub cosmological solution [44] and to study gravitational radiation. It provides a choice of gauge that makes calculations simpler and was used for this purpose [45]. Essentially based on the almost symmetry, the concept of an "approximate symmetry group" was presented [46]. In [47] a method for computing approximate KVs on closed 2-surfaces was established and used to study the distortion of the horizon geometry of black holes. This latter work was related to the earlier proposal of Matzner [42] to calculate the approximate Killing fields using an eigenvalue approach, so as to define a meaningful spin for non-symmetric black holes in GR [48]. However it has not been unequivocally successful either. The approach of a slightly broken symmetry seems promising, but merely providing simplicity of calculations is not physically convincing. Other approaches need to be tried, to find one that seems significantly better than others. In this thesis we will apply the approximate symmetry methods for ordinary differential equations (ODEs) first to some static spacetimes and then to the GW spacetimes to look at the energy content in these spacetimes.

Many relativists (notably including Roger Penrose [49]), believe that the invariants of the Weyl tensor should give the gravitational radiation field. It is not, a priori, so clear *which* (or which combination) of the scalar invariants should be used. A proposal for a "radiation scalar" was provided [50] and used to extract gauge-independent (or coordinate-independent) information particularly characterizing gravitational fields for numerical relativity by encoding it in the numerical variables [51, 52, 53, 54]. However, this radiation scalar is physically meaningful only in special regions of spacetime and may not have general applicability. Further, the actual energy has not been evaluated and there are no unambiguous physical predictions coming from it. Till these are extracted one cannot be sure that this proposal will *actually* give such results.

The Weyl tensor $C^a_{\ bcd}$ which is conformally invariant [55] represents a pure gravitational field and in some sense tells us about the gravitational energy of the spacetime, but it does not give a direct measure of the gravitational energy. On the other hand the stress-energy tensor T_{ab} gives the matter content of the spacetime [40]. As for exact GWs the stress-energy tensor is zero, we calculate the Weyl and stress-energy tensors for perturbed gravitational wave spacetimes discussed here, to obtain the gravitational and matter components separately and compare them with the energy content obtained from the definition of second-order approximate symmetries of the geodesic equations.

In the literature [56] the Weyl tensor is usually defined with valence (1, 3). In spinors it is naturally given as a tensor of valence (0, 4) [57]. For usual purposes the form does not matter, but for differential symmetries of the tensor the form is crucial [58]. The (0, 4) form has physical significance for our purpose that is it relates to our definition of energy which will be seen in chapters 5 and 6.

Minkowski spacetime is maximally symmetric having 10 KVs which form the Poincarè algebra $so(1,3) \oplus_s \mathbb{R}^4$ (where $\oplus_s$ denotes semi direct sum) [59]. The generators of this algebra give conservation laws for energy, spin angular momentum and linear momentum. When one goes from Minkowski to non flat spacetimes like Schwarzschild, RN and Kerr spacetimes some of the conservation laws are lost. To recover the Lorentz covariance or lost conservation laws in the Schwarzschild spacetime approximate symmetry methods for ODEs were used by Kara *et. al.* [60]. They considered the Schwarzschild spacetime as a first perturbation of the Minkowski spacetime. In the first-order approximate symmetries of the (approximate) geodesic equations they recovered no non-trivial approximate symmetry. They recovered the lost conservation laws of linear and spin angular momentum as trivial first-order approximate conservation laws. In this thesis we will use not only first-order but also second-order approximate symmetries of the geodesic equations as well as of the Lagrangians. In contrast to Kara *et. al.* we obtain energy re-scaling in different spacetimes from the application of second-order approximate symmetry of the geodesic equations. This is further discussed in the subsequent chapters.

Because of its non-tensorial nature the idea of a pseudo-tensor (discussed above) is not good as it violates the basic spirit of GR. The way we propose for defining gravitational energy, by the use of approximate Lie symmetry methods [61] avoids the pseudo-tensor and hence does not violate GR. For the case of GWs spacetimes (to be discussed in detail in chapters 5 and 6) we plot the scaling factors by using Mathematica 5. For the artificially constructed examples of wave-like spacetimes (given in chapter 5 and 6) these plots show that the energy increases indefinitely with time. For the physical example of cylindrical GWs (given in chapter 6) the plots show that the energy oscillates between positive and negative values and asymptotically goes to zero.

Another interesting question about GWs is "whether there is the analogue of Landau-damping of electromagnetic waves for GWs". Since Maxwell's theory of electromagnetism is linear, electromagnetic waves do not interact with the field but are damped due to their interaction with matter. On the other hand GR is non-linear and so GWs can undergo self-interaction. This gives rise to the possibility of "Landau self-damping" of GWs. On the other hand, the Khan-Penrose [62] and Szekeres [63] solutions of colliding plane GWs suggest that there could even be *enhancement* of the waves, as they lead to curvature singularities after the collision. The problem of definition of energy in GR makes it very difficult to answer the question posed. Using Wheeler's "poor man's approach", we can ask whether "the mass equivalent to the energy of the GWs attracts and hence damps the waves", or like the black hole, "the energy enhances the mass and hence the energy equivalent to it in the wave". With the use of approximate Lie symmetry methods the question seems to be answerable. The cylindrical waves get damped by self-interaction. This will be discussed in chapter 6.

The plan of the thesis is as follows. The next section briefly reviews some basic definitions to be used later. In the last section of this chapter we will give a review of the approximate symmetries of the geodesic equations for the Schwarzschild spacetime. In chapter 2 we will discuss second-order approximate symmetries of the geodesic equations and of the orbital equation for the RN spacetime. Chapter 3 deals with the approximate symmetries of a Lagrangian for the Kerr and charged-Kerr spacetimes. In chapter 4 second-order approximate symmetries of the geodesic equations for the charged-Kerr spacetime are considered. In chapter 5 we will investigate the approximate symmetries of geodesic equations for plane-fronted (pp) wave and plane symmetric wave-like spacetimes. In the same chapter approximate symmetries of Lagrangians for these plane wave spacetimes are considered. In chapter 6 we will study approximate symmetries of geodesic equations and of Lagrangians for cylindrically symmetric exact wave and wave-like spacetimes. A summary and discussion are given in chapter 7.

1.2 Basics

In this section we will provide some basic definitions that will be of great use subsequently.

1.2.1 Lie Groups

A differentiable manifold G is called a Lie group if $\forall\ g, h \in G$, the map $(g,\ h) \longrightarrow gh^{-1}$ is differentiable [64]. The following are examples of Lie groups:

(i) The group of all $n \times n$, non-singular real matrices

$$GL(n,\ \mathbb{R}) = \{M_{n\times n},\ \ |M| \neq 0\}, \tag{1.18}$$

is an n dimension Lie group;

(ii) A group of scalings in the plane

$$x^* = ax,\ y^* = a^2y,\ 0 < a < \infty, \tag{1.19}$$

is a Lie group;

(iii) The circle, S^1, consisting of angles mod 2π under addition or complex numbers with absolute value 1 under multiplication is a one-dimensional compact connected abelian Lie group.

1.2.2 The Lie Bracket and Lie Algebra

Let $\mathbf{X}$ and $\mathbf{Y}$ be differentiable vector fields on M. Let $p \in M$ and $x : U \longrightarrow M$ be a parametrization at p where M is a differentiable manifold and

$$\mathbf{X} = a_i\frac{\partial}{\partial x_i},\ \mathbf{Y} = b_j\frac{\partial}{\partial x_j}, \tag{1.20}$$

be the expressions for $\mathbf{X}$ and $\mathbf{Y}$ in these parametrization. Let D be the set of all differentiable functions on M then for all $f \in D$ we have

$$[\mathbf{X}, \mathbf{Y}]f = (\mathbf{XY} - \mathbf{YX})f = \mathbf{XY}f - \mathbf{YX}f = (a_i\frac{\partial b_j}{\partial x_i} - b_j\frac{\partial a_j}{\partial x_i})\frac{\partial f}{\partial x_j}, \tag{1.21}$$

which is again a differentiable vector field. The vector field $[\mathbf{X}, \mathbf{Y}]$ obtained by the Lie product of two vector fields $\mathbf{X}$ and $\mathbf{Y}$ is called Lie bracket of $\mathbf{X}$ and $\mathbf{Y}$.

A Lie algebra L is a vector space over some field F equipped with the Lie bracket satisfying the properties

(i) $[\alpha\mathbf{X}+\beta\mathbf{Y},\mathbf{Z}]=\alpha[\mathbf{X},\mathbf{Z}]+\beta[\mathbf{Y},\mathbf{Z}]$,

$[\mathbf{X},\ \alpha\mathbf{Y}+\beta\mathbf{Z}]=\alpha[\mathbf{X},\mathbf{Y}]+\beta[\mathbf{X},\mathbf{Z}]$ linearity for all $\alpha,\ \beta\in F$;

(ii) $[\mathbf{X},\mathbf{Y}]=-[\mathbf{Y},\mathbf{X}]$ (skew commutativity);

(iii) $[\mathbf{X},[\mathbf{Y},\mathbf{Z}]]+[\mathbf{Y},[\mathbf{Z},\mathbf{X}]]+[\mathbf{Z},[\mathbf{X},\mathbf{Y}]]=0$ (Jacobi identity).

A Lie algebra is said to be real if F is the field of real numbers and complex if F is the field of complex number. The vector fields $\mathbf{X}$ and $\mathbf{Y}$ are called the generators of the Lie algebra. If the basis (generators) of the Lie algebra are finite (say n) then it is known as a finite dimensional (n dimensional) Lie algebra. Otherwise it is known as infinite dimensional Lie algebra. Now if $\langle\mathbf{X}_1,\ \mathbf{X}_2,...,\mathbf{X}_r\rangle$ is an r-dimensional Lie algebra, then we can represent the infinitesimal generator $\mathbf{X}_j(x)$ by

$$\mathbf{X}_j(x)=a_j^i(x)\frac{\partial}{\partial x^i}. \tag{1.22}$$

The Lie algebra of the generators $\mathbf{X}_j(x)$ is known from the Lie product of the two operators $[\mathbf{X}_i,\mathbf{X}_j]$. This can be simplified by using the structure constants defined by

$$[\mathbf{X}_i,\mathbf{X}_j]=C_{ij}^s\mathbf{X}_s, \tag{1.23}$$

where

$$C_{ij}^s=-C_{ji}^s \tag{1.24}$$

and the Jacobi identity gives

$$C_{ij}^\beta C_{\beta s}^\alpha+C_{js}^\beta C_{\beta i}^\alpha+C_{si}^\beta C_{\beta j}^\alpha=0. \tag{1.25}$$

To every Lie group, one can associate a Lie algebra, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. Informally one can think of elements of the Lie algebra as elements of the group that are "infinitesimally close" to the identity. For example, the Lie algebra of the general linear group $GL(n,\ \mathbb{R})$ is the vector space $M(n,\mathbb{R})$ of square matrices with the Lie bracket given by

$$[A,\ B]=AB-BA,\ \ A,\ \ B\in M. \tag{1.26}$$

Since a Lie algebra determines the local structure of the group, therefore two groups will be locally isomorphic if and only if their Lie algebras are isomorphic. To every Lie algebra there can be associated a unique simply connected Lie group, but there can be other multiply connected Lie groups.

1.2.3 Approximate Lie Algebra

Here we give the definition of an approximate Lie algebra [65]. A class of first-order differential operators

$$\mathbf{X} = \xi^i(x,\epsilon)\frac{\partial}{\partial x^i}, \tag{1.27}$$

such that

$$\xi^i(x,\epsilon) \approx \xi_0^i(x) + \epsilon\xi_1^i(x) + ... + \epsilon^k\xi_k^i(x),\ \ i = 1, ..., n, \tag{1.28}$$

with some fixed functions $\xi_0^i(x)$, $\epsilon\xi_1^i(x), ..., \epsilon^k\xi_k^i(x), (\ i = 1, ..., n)$ is called an approximate operator. An approximate Lie bracket of the approximate operators $\mathbf{X}$ and $\mathbf{Y}$ is an approximate operator denoted by $[\mathbf{X}, \mathbf{Y}]$ and is

$$[\mathbf{X}, \mathbf{Y}] \approx \mathbf{XY} - \mathbf{YX}. \tag{1.29}$$

An approximate Lie algebra L is a vector space over some field F equipped with the approximate Lie bracket satisfying the properties, namely:

(i) $[\alpha\mathbf{X} + \beta\mathbf{Y},\ \mathbf{Z}] \approx \alpha[\mathbf{X}, \mathbf{Z}] + \beta[\mathbf{Y}, \mathbf{Z}]$

$[\mathbf{X},\ \alpha\mathbf{Y} + \beta\mathbf{Z}] \approx \alpha[\mathbf{X}, \mathbf{Y}] + \beta[\mathbf{X}, \mathbf{Z}]$ linearity for all $\alpha,\ \beta \in F$

(ii) $[\mathbf{X}, \mathbf{Y}] \approx -[\mathbf{Y}, \mathbf{X}]$ (skew commutativity)

(iii) $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] \approx 0$ (Jacobi identity).

Here the approximate Lie bracket $[\mathbf{X}, \mathbf{Z}]$ is calculated to the precision indicated. We illustrate this by the following example.

Example: Consider the approximate (up to $O(\epsilon^2)$) operators [65]

$$\mathbf{X} = \frac{\partial}{\partial x} + \epsilon x\frac{\partial}{\partial y},\ \ \mathbf{Y} = \frac{\partial}{\partial y} + \epsilon y\frac{\partial}{\partial x}. \tag{1.30}$$

Their exact Lie bracket is

$$[\mathbf{X}, \mathbf{Y}] = \epsilon^2 (x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}). \tag{1.31}$$

Therefore the linear span of **X** and **Y** is not a Lie algebra in the usual (exact) sense. However, these operators span an approximate Lie algebra in the first-order of precision.

1.2.4 Exact and Approximate Symmetries of ODEs

First we will define exact symmetries of ODEs and then we will define approximate symmetries of ODEs.

According to Noether's theorem [66] for a system arising from a variational principle, conservation laws of that system come from a symmetry property. This theorem gives a procedure which relates the constants of motion of a given Lagrangian system to its symmetry transformation [67]. There is a connection between the symmetries of a manifold and its isometries [68]. Symmetry generators of a Lagrangian for the geodesic equations of a manifold form a Lie algebra which always include the generator $\partial/\partial s$ [69]. From the geometric point of view symmetries of a manifold are characterized by its isometries or KVs, which always form a finite dimensional Lie algebra [1].

In general a manifold does not possess exact symmetry but approximately does so. It would be of interest to look at the approximate symmetries of the manifold. These approximate symmetry may give us much more information.

A symmetry transformation or symmetry of a DE is that transformation which leaves the form of the equation invariant. The symmetries of an ODE [70]

$$\mathbf{E}(s; \mathbf{x}(s), \mathbf{x}'(s), \mathbf{x}''(s), ..., \mathbf{x}^{(n)}(s)) = \mathbf{0}, \tag{1.32}$$

under point transformations

$$(s, \mathbf{x}) \longrightarrow (\xi(s, \mathbf{x}), \eta(s, \mathbf{x})), \tag{1.33}$$

are given by

$$\mathbf{X}^{[k]} = \boldsymbol{\xi}(s, \mathbf{x})\frac{\partial}{\partial s} + \eta(s, \mathbf{x})\frac{\partial}{\partial \mathbf{x}} + \eta^1(s, \mathbf{x}, \mathbf{x}')\frac{\partial}{\partial \mathbf{x}'} + ... + \eta^{(k)}(s, \mathbf{x}, \mathbf{x}'..., \mathbf{x}^{(k)})\frac{\partial}{\partial \mathbf{x}^{(k)}}, \tag{1.34}$$

such that on the solution of

$$\mathbf{E} = 0, \tag{1.35}$$

we have

$$\mathbf{X}^{[k]} \left(\mathbf{E}\right)|_{\mathbf{E}=\mathbf{0}} = 0. \tag{1.36}$$

The operator $\mathbf{X}$ is called the infinitesimal generator, group operator or Lie operator and $\mathbf{X}^{[k]}$ is called the *kth prolongation* of the infinitesimal generator

$$\mathbf{X} = \xi(s,\mathbf{x})\frac{\partial}{\partial s} + \boldsymbol{\eta}(s,\mathbf{x})\frac{\partial}{\partial \mathbf{x}}, \tag{1.37}$$

where the prolongation coefficients are given by

$$\eta_{,s} = \frac{d\eta}{ds} - \mathbf{x}'\frac{d\xi}{ds}, \tag{1.38}$$

$$\eta_{,(k)} = \frac{d\eta_{(k-1)}}{ds} - \mathbf{x}^{(k)}\frac{d\xi}{ds}, k \geq 2. \tag{1.39}$$

Henceforth we shall drop the index k for the prolongation and leave it to the context to clarify which one is being used. For determining the symmetries of a system of ODEs we use the invariance criterion. The system of ODEs of order n

$$E_r(s;\mathbf{x}(s),\mathbf{x}'(s),\mathbf{x}''(s),...,\mathbf{x}^{(n)}(s)) = 0, \ (r = 1, 2..., p), \tag{1.40}$$

admits a symmetry algebra with generator

$$\mathbf{X} = \xi(s,\mathbf{x})\frac{\partial}{\partial s} + \eta^{\alpha}(s,\mathbf{x})\frac{\partial}{\partial \mathbf{x}^{\alpha}}, \tag{1.41}$$

if and only if

$$\mathbf{X}^{[n]} \left(E_r\right)|_{E_r=\mathbf{0}} = \mathbf{0}, \tag{1.42}$$

holds, where $\mathbf{x}$ is a point in the underlying m-dimensional space and $\mathbf{x}'$ is the first derivative of $\mathbf{x}$ and $\mathbf{x}^{(n)}$ is the nth-order derivative with respect to s and

$$\mathbf{X}^{[n]} = \xi(s,\mathbf{x})\frac{\partial}{\partial s} + \eta^{\alpha}(s,\mathbf{x})\frac{\partial}{\partial \mathbf{x}^{\alpha}} + \eta^{\alpha}_{,s}(s,\mathbf{x},\mathbf{x}')\frac{\partial}{\partial \mathbf{x}^{\alpha\prime}} + ... + \eta^{\alpha}_{,(n)}(s,\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(n)})\frac{\partial}{\partial \mathbf{x}^{\alpha(n)}}, \tag{1.43}$$

the prolongation coefficients are

$$\eta^{\alpha}_{,s} = \frac{d\eta^{\alpha}}{ds} - \mathbf{x}^{\alpha'}\frac{d\xi}{ds}, \tag{1.44}$$

$$\eta^{\alpha}_{,(n)} = \frac{d\eta_{\alpha(n-1)}}{ds} - \mathbf{x}^{\alpha(n)}\frac{d\xi}{ds},\ n \geq 2. \tag{1.45}$$

Now we define the kth-order approximate symmetries of a system of ODEs [61]. If

$$\mathbf{E} = \mathbf{E}_0 + \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2 + ... + \epsilon^k\mathbf{E}_k + O(\epsilon^{k+1}) \tag{1.46}$$

and

$$\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2 + ... + \epsilon^k\mathbf{X}_k, \tag{1.47}$$

so that

$$\begin{aligned}\mathbf{XE} :=& [(\mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2 + ... + \epsilon^k\mathbf{X}_k)(\mathbf{E}_0 + \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2 \\ &+ ... + \epsilon^k\mathbf{E}_k)]_{E=E_0+\epsilon E_1+...+\epsilon^k E_k} = O(\epsilon^{k+1}),\end{aligned} \tag{1.48}$$

then (1.47) is called a kth-order approximate symmetry of (1.46). Here E_0 is the exact equation E_1 is called the first-order approximate part and E_2 is called the second-order approximate part of the perturbed equation and so on. The $\mathbf{X}_0$ is the exact symmetry generator, $\mathbf{X}_1$ as the first-order approximate part, $\mathbf{X}_2$ the second-order approximate part of the symmetry generator and so on, where

$$\begin{aligned}\mathbf{X}_0 =\ & \xi_0(s,\mathbf{x})\frac{\partial}{\partial s} + \eta_0(s,\mathbf{x})\frac{\partial}{\partial \mathbf{x}} + \eta^1_0(s,\mathbf{x},\mathbf{x}')\frac{\partial}{\partial \mathbf{x}'} + \eta^2_0(s,\mathbf{x},\mathbf{x}',\mathbf{x}'')\frac{\partial}{\partial \mathbf{x}''} + ... \\ &+\eta^{(k)}_0(s,\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(k)})\frac{\partial}{\partial \mathbf{x}^{(k)}},\end{aligned} \tag{1.49}$$

$$\begin{aligned}\mathbf{X}_1 =\ & \xi_1(s,\mathbf{x})\frac{\partial}{\partial s} + \eta_1(s,\mathbf{x})\frac{\partial}{\partial \mathbf{x}} + \eta^1_1(s,\mathbf{x},\mathbf{x}')\frac{\partial}{\partial \mathbf{x}'} + \eta^2_1(s,\mathbf{x},\mathbf{x}',\mathbf{x}'')\frac{\partial}{\partial \mathbf{x}''} + ... \\ &+\eta^{(k)}_1(s,\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(k)})\frac{\partial}{\partial \mathbf{x}^{(k)}},\end{aligned} \tag{1.50}$$

$$\begin{aligned}\mathbf{X}_2 =\ & \xi_2(s,\mathbf{x})\frac{\partial}{\partial s} + \eta_2(s,\mathbf{x})\frac{\partial}{\partial \mathbf{x}} + \eta^1_2(s,\mathbf{x},\mathbf{x}')\frac{\partial}{\partial \mathbf{x}'} + \eta^2_2(s,\mathbf{x},\mathbf{x}',\mathbf{x}'')\frac{\partial}{\partial \mathbf{x}''} + ... \\ &+\eta^{(k)}_2(s,\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(k)})\frac{\partial}{\partial \mathbf{x}^{(k)}},\end{aligned} \tag{1.51}$$

and so on. The kth-order approximate symmetry is called non-trivial if at least one of the lower order symmetries are non-zero for it, that is if (at least) any one of $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$,..., $\mathbf{X}_{k-1}$ is non-zero. In the case of trivial symmetries it is also possible that lower order symmetries cancel out in the determining equations.

There are some alternate methods for defining approximate symmetries of differential equations (DEs). In this regard a definition was given by Fushchich and Shtelen [71]. They interchange the order of approximation and take the limit between the parameter of the symmetry generator of the algebra on the one hand and the approximation parameter on the other hand. This method is compared with that of Baikov *et al.* in [72, 73]. A generalization of the approximate Lie symmetry methods for DEs to include conditional symmetries was developed in [74]. Comparatively recently another notion of approximate symmetries of DEs has been developed by Burde [75]. This latter method does not find solutions of a DE directly, but provide transformations between different DEs. At this point the approach completely differs from standard perturbation methods that involve a straightforward expansion of the dependent variables, which is inserted into the perturbed DE. This method was aimed at finding transformations from the perturbed equation to the unperturbed equation: which variables are transformed (and in what way) was determined by the requirement that the transformations form a Lie group. These transformations naturally define an approximate solution of the perturbed equation that has the solution of the unperturbed equation as a zero-order part. This approach is then compared with the other approaches of approximate symmetries of DEs [75]. We will follow the method of Baikov *et al.* [61].

1.2.5 Exact and Approximate Symmetries of Lagrangians

Symmetries (and approximate symmetries) of the system of the geodesic equations for a spacetime gives us conserved quantities but in addition there are also non-Noether symmetries which are not related to conservation laws and therefore of no interest for our purpose. Further, the Lie symmetries need second prolongation (for second-order Euler-Lagrange equations) of the symmetry generator, while the symmetries of the Lagrangian give us directly conserved quantities in which we are interested and only need the first prolongation of the symmetry generator. Methods for obtaining exact symmetries and first-order approximate symmetries of a Lagrangian are available in the literature [70, 76, 77]. Here we extend to the second-order approximate case.

Symmetries of a Lagrangian also known as *Noether symmetry* [70] are defined as follows. Consider a vector field given by (1.41) whose first prolongation is

$$\mathbf{X}^{[1]} = \mathbf{X} + (\eta^j_{,s} + \eta^j_{,x^i} x^{i'} - \xi_{,s} x^{j'} - \xi_{,x^i} x^{i'} x^{j'}) \frac{\partial}{\partial x^{i'}}, \tag{1.52}$$

where i, $j = 0, 1, 2, 3$. Now consider a set of second-order ODEs (Euler-Lagrange equations)

$$x^{i''} = g(s, x^i, x^{i'}), \tag{1.53}$$

which has a Lagrangian $L(s, x^i, x^{i'})$. Then $\mathbf{X}$ is a Noether point symmetry of the Lagrangian $L(s, x^i, x^{i'})$ if there exists a function $A(s, x^i)$ such that

$$\mathbf{X}^{[1]} L + (D_s \xi) L = D_s A, \tag{1.54}$$

where the total derivative operator is

$$D_s = \frac{\partial}{\partial s} + x^{i'} \frac{\partial}{\partial x^i}. \tag{1.55}$$

For more general considerations see [70, 78]. The significance of Noether symmetries is clear from the following theorem [66].

Theorem 1.1. If $\mathbf{X}$ is a Noether point symmetry corresponding to a Lagrangian $L(s, x^i, x^{i'})$ of (1.53), then

$$I = \xi L + (\eta^i - x^{i'} \xi) L_{x^{i'}} - A, \tag{1.56}$$

is a first integral of (1.53) associated with $\mathbf{X}$. For the proof of this theorem see [79].

We define second-order approximate symmetries of the Lagrangian via the following theorem.

Theorem 1.2. If

$$L(s, x^i, x^{i'}, \epsilon, \epsilon^2) = L_0(s, x^i, x^{i'}) + \epsilon L_1(s, x^i, x^{i'}) + \epsilon^2 L_2(s, x^i, x^{i'}) + O(\epsilon^2), \tag{1.57}$$

is a first-order perturbed (up to second-order in ϵ) Lagrangian corresponding to a second-order perturbed system of equations

$$\mathbf{E} = \mathbf{E}_0 + \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2 + O(\epsilon^3) = 0, \tag{1.58}$$

and the functional $\int_V Lds$ is invariant under the one-parameter group of transformations with approximate Lie symmetry generator

$$\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2 + O(\epsilon^3), \tag{1.59}$$

up to gauge

$$A = A_0 + \epsilon A_1 + \epsilon^2 A_2, \tag{1.60}$$

where

$$\mathbf{X}_j = \xi_j \frac{\partial}{\partial s} + \eta_j^i \frac{\partial}{\partial x^i}, \ (j = 0, 1, 2 \text{ and } i = 0, 1, 2, 3), \tag{1.61}$$

then

$$\mathbf{X}_0^{[1]} L_0 + (D_s\xi) L_0 = D_s A_0, \tag{1.62}$$

$$\mathbf{X}_1^{[1]} L_0 + \mathbf{X}_0^{[1]} L_1 + (D_s\xi_1) L_0 + (D_s\xi_0) L_1 = D_s A_1 \tag{1.63}$$

and

$$\mathbf{X}_2^{[1]} L_0 + \mathbf{X}_1^{[1]} L_1 + \mathbf{X}_0^{[1]} L_2 + (D_s\xi_2) L_0 + (D_s\xi_1) L_1 + (D_s\xi_0) L_2 = D_s A_2. \tag{1.64}$$

Proof: For the unperturbed case (1.62) and first-order perturbed case (1.63), see for example [70, 77] respectively, for the second-order perturbed case (1.64), proof follows from there.

Here L_0 is the exact Lagrangian corresponding to the exact equations $\mathbf{E}_0 = 0$, $L_0 + \epsilon L_1$ the first-order approximate Lagrangian corresponding to the first-order perturbed equations $\mathbf{E}_0 + \epsilon\mathbf{E}_1 = 0$. The perturbed equations (1.63) and (1.64) always have the approximate symmetry $\epsilon\mathbf{X}_0$ which are known as trivial approximate symmetries. For a 4-dimensional spacetime (1.62) - (1.64) give 19 determining equations.

1.2.6 Approximate First Integrals

The first-order approximate first integrals are defined by setting I by $I_0 + \epsilon I_1$, ξ by $\xi_0 + \epsilon\xi_1$, $\eta_0+\epsilon\eta_1$, by $L_0+\epsilon L_1$, and A by $A_0+\epsilon A_1$ in the definition of first integral (1.56) and equating the coefficients of like powers of ϵ on both sides. This gives the zeroth (exact part) and first-order approximate part of the first-order approximate first integrals

$$I_0 = \xi_0 L_0 + (\eta_0^i - \xi_0 \dot{x}^i)\frac{\partial L_0}{\partial \dot{x}^i} - A_0, \tag{1.65}$$

$$I_1 = \xi_0 L_1 + \xi_1 L_0 + (\eta_0^i - \xi_0 \dot{x}^i)\frac{\partial L_1}{\partial \dot{x}^i} + (\eta_1^i - \xi_1 \dot{x}^i)\frac{\partial L_0}{\partial \dot{x}^i} - A_1. \tag{1.66}$$

If I_0 vanishes, then I is called an *unstable approximate first integral* and otherwise called *stable*. A detailed discussion on the approximate first integrals for Hamiltonian dynamical system is given in [80].

1.2.7 Weyl and Stress-energy Tensors

Here we define some useful tensors which will be used in the subsequent discussion.

The Riemann curvature tensor $R_{abcd} = g_{aa}R^a_{\ bcd}$ can be uniquely decomposed into three parts [1] given by

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd}, \tag{1.67}$$

where

$$E_{abcd} = \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}), \tag{1.68}$$

$$G_{abcd} = \frac{1}{12}R(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{1.69}$$

and

$$S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}, \tag{1.70}$$

denotes the traceless part of the Ricci tensor R_{ab}. The decomposition given by (1.67) defines the Weyl conformal tensor C_{abcd} given by

$$C_{abcd} = R_{abcd} - \frac{1}{2}(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}) + \frac{1}{6}R(g_{ad}g_{bc} - g_{ac}g_{bd}), \tag{1.71}$$

or

$$C^a{}_{bcd} = R^a{}_{bcd} - \frac{1}{2}(\delta^a_c R_{bd} - \delta^a_d R_{bc} + g_{bd}R^a{}_c - g_{bc}R^a{}_d) + \frac{1}{6}R(\delta^a_d g_{bc} - \delta^a_c g_{bd}). \tag{1.72}$$

The Weyl tensor has manifestly all the symmetries of the Riemann curvature tensor i.e.

$$C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab}, \tag{1.73}$$

but [55]

$$g^{bd}C_{abcd} = 0, \tag{1.74}$$

in contrast to $g^{bd}R_{abcd} = R_{ac}.$ A further distinction is that, while the Riemann tensor can be defined in a manifold endowed only with a connection, the Weyl tensor can be defined only if a metric is also defined as it is essential for defining the Ricci scalar. The Weyl tensor is also known as the conformal curvature tensor. Let $\widetilde{g}_{ab} = \Omega^2(x)g_{ab}$ be a conformal transformation of g where Ω is a smooth positive real function on M. The Weyl tensor is invariant under conformal transformations of the metric [55]. Due to the symmetry property defined by (1.74) it can be checked that the Weyl tensor is that part of the Riemann curvature tensor for which all contractions vanish. Because of its symmetry properties the Weyl tensor has at most $20-10=10$ independent components in a four dimensional spacetime. The importance of the Weyl tensor for the deeper problem of GR is the conformal invariance of $C^a{}_{bcd}$ [1, 55]. If the Weyl tensor vanishes in a neighborhood of a spacetime, the neighborhood is locally conformally equivalent to the Minkowski spacetime. Thus the Weyl tensor has geometric meaning independent of any physical interpretation.

The stress-energy tensor T_{ab} gives the matter content of a spacetime [40]. This is a symmetric tensor and can be calculated from the EFEs

$$T_{ab} = \frac{1}{\kappa}(R_{ab} - \frac{1}{2}Rg_{ab}). \tag{1.75}$$

For a 4-dimensional spacetime this tensor has 10 independent components. At each event of the spacetime this tensor gives the energy density, momentum density and stress as measured by observers at that event. Since for GW spacetimes T_{ab} is always zero and $C^a{}_{bcd}$ may be non-zero, there is no stress, energy or momentum. If there is no mass or energy at a given event, the

Ricci tensor vanishes through the EFEs. If it were not for the Weyl tensor, this would mean that matter here could not have gravitational influence on distant matter separated by a void. Thus the Weyl tensor represents that part of spacetime curvature which can propagate across and curve up a void.

1.3 Review of the Approximate Symmetries of the Schwarzschild Spacetime

In this section we review the exact and approximate symmetries of the orbital equation and of the geodesic equations for the Schwarzschild metric [60]. The field of a point gravitational source at the origin is given by the Schwarzschild metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{-\nu(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.76}$$

where

$$e^{\nu} = 1 - \frac{2GM}{c^2 r} \tag{1.77}$$

and M is the mass of the point gravitational source at the origin.

In the isometry algebra of Minkowski spacetime, $so(1,3)$ is isomorphic to $so(3) \oplus so(3)$. This algebra corresponds to the conservation of angular momentum (one of the $so(3)s$), "spin angular momentum" (the other $so(3)$) and the (linear) energy-momentum ($\mathbb{R}^4$). The symmetry generators are

$$\mathbf{Y}_0 = \frac{\partial}{\partial t},\ \mathbf{Y}_1 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \tag{1.78}$$

$$\mathbf{Y}_2 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi},\ \mathbf{Y}_3 = \frac{\partial}{\partial\phi}, \tag{1.79}$$

with the symmetry algebra $so(3) \oplus \mathbb{R}$ corresponding to the conservation of energy and angular momentum and

$$\mathbf{Y}_4 = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\csc\theta\sin\phi}{r}\frac{\partial}{\partial\phi}, \tag{1.80}$$

$$\mathbf{Y}_5 = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial\phi}, \tag{1.81}$$

$$\mathbf{Y}_6 = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \tag{1.82}$$

which give the conservation of linear momentum as well as

$$\mathbf{Y}_7 = \frac{r\sin\theta\cos\phi}{c}\frac{\partial}{\partial t} + ct(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\csc\theta\sin\phi}{r}\frac{\partial}{\partial\phi}), \tag{1.83}$$

$$\mathbf{Y}_8 = \frac{r\sin\theta\sin\phi}{c}\frac{\partial}{\partial t} + ct(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial\phi}), \tag{1.84}$$

$$\mathbf{Y}_9 = \frac{r\cos\theta}{c}\frac{\partial}{\partial t} + ct(\cos\theta\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\theta}), \tag{1.85}$$

which give the conservation of spin angular momentum due to Lorentz invariance.

The geodesic equations for the Schwarzschild metric are given by

$$\ddot{t} + \nu'\dot{t}\dot{r} = 0, \tag{1.86}$$

$$\ddot{r} + \frac{1}{2}(e^{\nu})'(e^{\nu}c^2\dot{t}^2 - e^{-\nu}\dot{r}^2) - re^{\nu}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0, \tag{1.87}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{1.88}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{1.89}$$

with

$$\nu' = \frac{2GM/c^2r^2}{1 - 2GM/c^2r}. \tag{1.90}$$

Applying the definition for the symmetries of ODEs these equations have the (exact) symmetries given by the above isometry algebra $so(3) \oplus \mathbb{R}$ (1.78) and (1.79) added to the dilatation algebra

$$d_2 = \left\langle \frac{\partial}{\partial s},\ s\frac{\partial}{\partial s} \right\rangle \tag{1.91}$$

generated by the re-parametrization allowed for the geodetic parameter. These are the exact symmetry generators. Note that here conservation of linear momentum is lost as a test particle put at a finite distance from the gravitational source will start to move. Further, the "spin angular momentum" conservation is also lost, as the motion is no longer be Lorentz invariant in the field of gravitational source.

Now for investigating the approximate symmetries of the geodesic equations for this metric Kara *et. al.* first looked at the approximate symmetries of the orbital equation for this metric. The orbital equation of motion is given by

$$\frac{d^2v}{d\phi^2} + v = \frac{GM}{h^2} + \frac{3GM}{c^2}v^2, \tag{1.92}$$

where h is the classical angular momentum per unit mass and $v = \frac{1}{r}$. In the classical limit $c \to \infty$ it gives the classical orbital equation. Applying the definition for the symmetries of ODEs to (1.92), this yields the following (exact) symmetry generators

$$\mathbf{Y}_1 = v\cos\phi\frac{\partial}{\partial\phi} - v^2\sin\phi\frac{\partial}{\partial v},\ \mathbf{Y}_2 = v\sin\phi\frac{\partial}{\partial\phi} + v^2\cos\phi\frac{\partial}{\partial v}, \tag{1.93}$$

$$\mathbf{Y}_3 = v\frac{\partial}{\partial v},\ \mathbf{Y}_4 = \cos\phi\frac{\partial}{\partial v},\ \mathbf{Y}_5 = \sin\phi\frac{\partial}{\partial v},\ \mathbf{Y}_6 = \frac{\partial}{\partial\phi}, \tag{1.94}$$

$$\mathbf{Y}_7 = \cos 2\phi\frac{\partial}{\partial\phi} - v\sin 2\phi\frac{\partial}{\partial v},\ \mathbf{Y}_8 = \sin 2\phi\frac{\partial}{\partial\phi} + v\cos 2\phi\frac{\partial}{\partial v}. \tag{1.95}$$

Considering the definition of approximate symmetries (only up to first-order) with ϵ defined to be $2GM/c^2$, (1.92) has two stable approximate symmetries

$$\mathbf{Y}_{a1} = \sin\phi\frac{\partial}{\partial v} + \epsilon(2\sin\phi\frac{\partial}{\partial\phi} + v\cos\phi\frac{\partial}{\partial v}), \tag{1.96}$$

$$\mathbf{Y}_{a2} = \cos\phi\frac{\partial}{\partial v} - \epsilon(2\cos\phi\frac{\partial}{\partial\phi} - v\sin\phi\frac{\partial}{\partial v}). \tag{1.97}$$

At best, only some of the exact symmetries lost in going from Minkowski to Schwarzschild space have been recovered. Since the orbital equation had been derived by using the symmetries to restrict the motion to an (arbitrarily chosen) equatorial plane, it could be expected that all of them will re-appear as approximate symmetries in the full system of geodesic equations. The perturbed geodesic equations for Schwarzschild metric with small term ϵ, defined above, are given by

$$\ddot{t} + \epsilon(\frac{\dot{t}\dot{r}}{r^2}) = 0, \tag{1.98}$$

$$\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \epsilon[\frac{1}{2r^2}(c^2\dot{t}^2 - \dot{r}^2) + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2] = 0, \tag{1.99}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{1.100}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{1.101}$$

Applying the definition of approximate symmetries (1.48) to these equations (1.98)-(1.101) and using (1.78) - (1.85) as exact symmetry generators, Kara *et. al.* have obtained the new approximate symmetry generators, which are exactly the same as the exact symmetry generators that were lost due to the gravitational field. Note that Lorentz invariance is recovered as an approximate symmetry in the gravitational field. So the trivial (in the sense that they are epsilon multiples of the exact symmetries) approximate symmetries provide the "stability" of all the known conservation laws. That is, the conservation laws are inherited by the perturbed geodesic equations of the Schwarzschild spacetime.

Chapter 2

Second-Order Approximate Symmetries of the Geodesic Equations and re-scaling of Energy in the Reissner-Nordström Spacetime

In this chapter we will investigate second-order approximate symmetries of the geodesic equations for the RN spacetime. A *re-scaling* of the arc length parameter s, for consistency of the trivial second-order approximate symmetries of the geodesic equations indicates that the energy in the RN spacetime has to be re-scaled [81]. Here we will also provide the second-order approximate symmetries of the orbital equation for the same spacetime.

2.1 Approximate Symmetries of the RN Spacetime

It had been pointed out [60] that there is a difference between the conservation laws obtained for the system of geodesic equations and the single orbital equation for the Schwarzschild spacetime. It was further remarked that it should be checked if this difference also holds for other spacetimes. We investigate this question for the orbital equation in the RN spacetime. Therefore, in this section we first discuss the second-order approximate symmetries of the orbital equation of the RN spacetime and then we will discuss the second-order approximate symmetries of the geodesic equations for the RN metric.

The RN spacetime is given by (1.76) with

$$e^{\nu} = 1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{c^4 r^2}, \tag{2.1}$$

where Q is the electric charge of the point gravitational source. Electromagnetism is the only long range force in Nature other than gravity and this is the only spherically symmetric, static exact solution of the "sourceless" Einstein-Maxwell equations. In the chargeless case ($Q = 0$) it reduces to the Schwarzschild metric. It is of interest to look at the symmetry structure of this metric and the corresponding symmetries, and approximate symmetries of the geodesic equations.

For determining approximate symmetries we take the same small parameter ϵ, as before. However, we have another small parameter to include

$$\alpha = \frac{GQ^2}{c^4}. \tag{2.2}$$

There is no way to meaningfully deal with two small parameters. As such we restrict our attention to RN black holes, for which $\alpha \leq \epsilon^2$. Thus we can put

$$\alpha = k\epsilon^2 \text{ with } 0 < k \leq \frac{1}{4}. \tag{2.3}$$

Hence retaining only ϵ^2 and neglecting its higher powers (2.1) gives

$$(e^{\nu})' = \frac{\epsilon}{r^2} - \frac{2k}{r^3}\epsilon^2,\ \nu' = \frac{\epsilon}{r^2} + \frac{1-2k}{r^3}\epsilon^2 \text{ and } e^{-\nu} = 1 + \frac{\epsilon}{r} + \frac{(1-k)}{r^2}\epsilon^2, \tag{2.4}$$

where "$'$" denotes the derivative with respect to r. In the limit of $\epsilon \longrightarrow 0$, this spacetime reduces to the Minkowski spacetime and when $\epsilon^2 \longrightarrow 0$ and $\epsilon \neq 0$, then we obtain the perturbed Schwarzschild spacetime discussed in chapter 1.

2.1.1 Approximate Symmetries of the Orbital Equation for the RN Spacetime

The orbital equation of motion for the RN metric up to second-order in ϵ, is given by

$$E : v'' + v - \epsilon\frac{1}{2}(3v^2 + \frac{c^2}{h^2}) + \epsilon^2(2kc^2v^3 + \frac{k}{h^2}c^2v) = 0, \tag{2.5}$$

where h is the classical angular momentum per unit mass and $v = 1/r$. The exact and first order approximate symmetry generators for this equation are given by (1.93) - (1.95) and (1.96), (1.97) respectively. The zeroth-order (exact) and first-order approximate symmetries can also be written as

$$\xi_0 = v[c_1 \cos\phi + c_2 \sin\phi] + c_4 + c_5 \cos 2\phi + c_6 \sin 2\phi, \tag{2.6}$$

$$\eta_0 = v^2[c_2 \cos\phi - c_1 \sin\phi] + v[c_3 + c_6 \cos 2\phi - c_5 \sin 2\phi] + c_7 \cos\phi + c_8 \sin\phi, \tag{2.7}$$

and

$$\xi_1 = v[a_1 \cos\phi + a_2 \sin\phi] + a_6 + a_7 \cos 2\phi + a_8 \sin 2\phi + 2(c_7 \sin\phi - c_8 \cos\phi), \tag{2.8}$$

$$\begin{aligned}\eta_1 \quad &= \quad v^2[a_2 \cos\phi - a_1 \sin\phi] + v[a_3 + a_8 \cos 2\phi - a_7 \sin 2\phi + c_7 \cos\phi + c_8 \sin\phi] \\ &\quad + a_4 \cos\phi + a_5 \sin\phi.\end{aligned} \tag{2.9}$$

We apply the second prolongation

$$\begin{aligned}\mathbf{X}^{[2]} \quad &= \quad \mathbf{X_0^{[2]}} + \epsilon\mathbf{X_1^{[2]}} + \epsilon^2\mathbf{X_2^{[2]}} = \xi(\phi, v)\frac{\partial}{\partial\phi} + \eta(\phi, v)\frac{\partial}{\partial v} + \eta_{,\phi}(\phi, v, v')\frac{\partial}{\partial v'} + \\ &\quad \eta_{,\phi\phi}(\phi, v, v', v'')\frac{\partial}{\partial v''},\end{aligned} \tag{2.10}$$

of the second-order approximate symmetry generator $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X_1} + \epsilon^2\mathbf{X_2}$, to the second-order perturbed ODE (2.5), where $\mathbf{X}_0$ is the exact part given by (2.6), (2.7) and $\mathbf{X}_1$ is the first-order approximate part given by (2.8) and (2.9) of the second-order approximate symmetry generator. We have to find the second-order approximate part $\mathbf{X}_2$ of the approximate symmetry generator.

In (2.10)

$$\xi = \xi_0 + \epsilon\xi_1 + \epsilon^2\xi_2 \text{ and } \eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2, \tag{2.11}$$

where each of the ξ_i and η_i $(i = 0, 1, 2)$, is a function of ϕ, v; $\eta_{i,\phi}$ are functions of ϕ, v, ϕ'; and $\eta_{i,\phi\phi}$ are functions of ϕ, v, ϕ', ϕ''. Applying the operator $\mathbf{X}^{[2]}$ given by (2.10) on (2.5)

$$(\mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2)[v'' + v - \epsilon\frac{1}{2}(3v^2 + \frac{c^2}{h^2}) + \epsilon^2(2kc^2v^3 + \frac{k}{h^2}c^2v)]_{(2.5)} = 0, \tag{2.12}$$

or

$$[(\eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2)\{1 - 3\epsilon v + \epsilon^2(6kc^2v^2 + \frac{k}{h^2}c^2)\} + \eta_{0,\phi\phi} + \epsilon\eta_{1,\phi\phi} + \epsilon^2\eta_{2,\phi\phi}]_{(2.5)} = 0, \tag{2.13}$$

where

$$\eta_{i,\phi\phi} = \eta_{i_{\phi\phi}} + (2\eta_{i_{\phi v}} - \xi_{i_{\phi\phi}})v' + (\eta_{i_{vv}} - 2\xi_{i_{\phi v}})v'^2 + (\eta_{i_v} - 2\xi_{i_\phi})v'' - \xi_{i_{vv}}v'^3 - 3\xi_{i_v}v'v''. \tag{2.14}$$

In (2.14) in subscripts $i, \phi\phi$ denotes second prolongation and $i_{\phi\phi}$ etc. denote second derivatives. Using (2.14) and (2.5) in (2.13) and comparing coefficients of the different powers of v', only for the terms involving ϵ^2, we obtain the following set of determining equations

$$\xi_{2_{vv}} = 0, \tag{2.15}$$

$$\eta_{2_{vv}} - 2\xi_{2_{\phi v}} = 0, \tag{2.16}$$

$$2\eta_{2_{\phi v}} - \xi_{2_{\phi\phi}} + 3v\xi_{2_v} - \frac{3}{2}\xi_{1_v}(3v^2 + \frac{c^2}{h^2}) + 3\xi_{0_v}(2kc^2v^3 + \frac{k}{h^2}c^2v) = 0, \tag{2.17}$$

$$\begin{aligned}\eta_{2_{\phi\phi}} - v\eta_{2_v} + 2v\xi_{2_\phi} + \frac{1}{2}(\eta_{1_v} - 2\xi_{1_\phi})(3v^2 + \frac{c^2}{h^2}) - (\eta_{0_v} - 2\xi_{0_\phi})(2kc^2v^3 \\ + \frac{k}{h^2}c^2v) + \eta_0(6kc^2v^2 + \frac{kc^2}{h^2}) - 3v\eta_1 + \eta_2 = 0.\end{aligned} \tag{2.18}$$

Integration of (2.15) twice with respect to v gives

$$\xi_2 = vf(\phi) + h(\phi). \tag{2.19}$$

We use (2.19) in (2.16) and then integrate it twice with respect to v we obtain

$$\eta_2 = v^2 f_\phi(\phi) + vg(\phi) + k(\phi). \tag{2.20}$$

Use of (2.6), (2.7) in (2.17) and then use of (2.19), (2.20) in the resulting equation we get

$$\begin{aligned} c_1 \cos\phi + c_2 \sin\phi &= 0, & (2.21) \\ a_1 \cos\phi + a_2 \sin\phi &= 0 & (2.22) \end{aligned}$$

and

$$\begin{aligned} f_{\phi\phi}(\phi) + f(\phi) &= 0, & (2.23) \\ 2g_\phi(\phi) - h_{\phi\phi}(\phi) &= 0. & (2.24) \end{aligned}$$

In (2.21) and (2.22), c_1, c_2, correspond to the exact symmetry generators and a_1, a_2 correspond to the first-order approximate symmetry generators. Equations (2.21) and (2.22) give us

$$a_1 = 0, a_2 = 0 \text{ and } c_1 = 0,\ c_2 = 0.$$

Integration of (2.23) yields

$$f(\phi) = b_1 \cos\phi + b_2 \sin\phi \tag{2.25}$$

and integration of (2.24) gives

$$g(\phi) = \frac{1}{2} h_\phi(\phi) + b_3. \tag{2.26}$$

Therefore,

$$\begin{aligned} \xi_2 &= v(b_1 \cos\phi + b_2 \sin\phi) + h(\phi), & (2.27) \\ \eta_2 &= v^2(b_2 \cos\phi - b_1 \sin\phi) + v(\frac{1}{2} h_\phi(\phi) + b_3) + k(\phi). & (2.28) \end{aligned}$$

Substituting the values of ξ_0, η_0 and ξ_1, η_1 from (2.6), (2.7), and (2.8), (2.9) in (2.18) and using (2.27) and (2.28) we obtain the following equations in which a_4 and a_5 correspond to first-order approximate symmetry generators.

$$h_{\phi\phi\phi}(\phi)+4h(\phi) = 6(a_4\cos\phi+a_5\sin\phi), \quad (2.29)$$

$$k_{\phi\phi}(\phi)+k(\phi) = 0. \quad (2.30)$$

Solving the non-homogeneous 3rd order ODE (2.29) we obtain

$$h(\phi)=b_6+b_7\cos 2\phi+b_8\sin 2\phi+2(a_4\sin\phi-a_5\cos\phi). \quad (2.31)$$

Integration of (2.30) yields

$$k(\phi)=b_4\cos\phi+b_5\sin\phi. \quad (2.32)$$

With the use of (2.31) and (2.32) from (2.27) and (2.28) we have

$$\xi_2=v(b_1\cos\phi+b_2\sin\phi)+b_6+b_7\cos 2\phi+b_8\sin 2\phi+2(a_4\sin\phi-a_5\cos\phi), \quad (2.33)$$

$$\eta_2=v^2(b_2\cos\phi-b_1\sin\phi)+v(b_8\cos 2\phi-b_7\sin 2\phi+a_4\cos\phi+a_5\sin\phi+b_3) +b_4\cos\phi+b_5\sin\phi. \quad (2.34)$$

Here b_i $(i=1,...,8)$ are arbitrary constants of integration.

In the second approximation, i.e. when we retain terms quadratic in ϵ, the orbital equation (2.5) possesses *no* non-trivial second-order approximate symmetry generators, but the first-order approximate symmetry generators (non-trivial in the case of the Schwarzschild spacetime) are still retained. Thus there is no new approximate conservation law but only the previous conservation laws that have been recovered. The second-order trivial approximate symmetry generators are the same as given by (1.93) - (1.97).

2.1.2 Approximate Symmetries of the System of Geodesic Equations for the RN Spacetime

A better idea of what is actually required comes from the full system of geodesic equations. The geodesic equations for the RN metric are given by

$$E_1 : \ddot{t} + \epsilon(\frac{\dot{t}\dot{r}}{r^2}) + \epsilon^2[\frac{(1-2k)}{r^3}\dot{t}\dot{r}] = 0, \tag{2.36}$$

$$E_2 : \ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \epsilon[\frac{1}{2r^2}(c^2\dot{t}^2 - \dot{r}^2) + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2]\ddot{r}$$

$$-\epsilon^2\frac{1}{2r^3}[(1+2k)c^2\dot{t}^2 + (1-2k)\dot{r}^2 + 2rk(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)] = 0, \tag{2.37}$$

$$E_3 : \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{2.38}$$

$$E_4 : \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{2.39}$$

where "$\cdot$" denotes the derivative with respect to s. Since this is the system of second-order ODEs, with second-order perturbation term, we apply to it the second prolongation of the approximate symmetry generator $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2$, where the exact part $\mathbf{X}_0$ and first-order approximate part $\mathbf{X}_1$ are given by (1.78) - (1.85), the second-order approximate part $\mathbf{X}_2$ of the symmetry generators to be determined. The second prolongation of the symmetry generator is given by

$$\mathbf{X}^{[2]} = \xi\frac{\partial}{\partial s} + \eta^0\frac{\partial}{\partial t} + \eta^1\frac{\partial}{\partial r} + \eta^2\frac{\partial}{\partial \theta} + \eta^3\frac{\partial}{\partial \phi} + \eta^0_{,s}\frac{\partial}{\partial \dot{t}} + \eta^1_{,s}\frac{\partial}{\partial \dot{r}} + \eta^2_{,s}\frac{\partial}{\partial \dot{\theta}}$$

$$+ \eta^3_{,s}\frac{\partial}{\partial \dot{\phi}} + \eta^0_{,ss}\frac{\partial}{\partial \ddot{t}} + \eta^1_{,ss}\frac{\partial}{\partial \ddot{r}} + \eta^2_{,ss}\frac{\partial}{\partial \ddot{\theta}} + \eta^3_{,ss}\frac{\partial}{\partial \ddot{\phi}} \quad . \tag{2.40}$$

Note that in (2.40) $\xi = \xi_0 + \epsilon\xi_1 + \epsilon^2\xi_2$ and $\eta^\mu = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2$ etc. ($\mu = 0, 1, 2, 3$). Here ξ, η^μ are all functions of s, t, r, θ and ϕ; $\eta^\mu_{,s}$ are all functions of $s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}$ and $\dot{\phi}$; and $\eta^\mu_{,ss}$ are all functions of $s, t, r, \theta, \phi, \dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}, \ddot{t}, \ddot{r}, \ddot{\theta}$ and $\ddot{\phi}$. We apply this symmetry generator to the geodesic equations (2.36) - (2.39). We obtain the following equations

$$\mathbf{X}^{[2]}E_1 = [\eta^0_{0,ss} + \epsilon\eta^0_{1,ss} + \epsilon^2\eta^0_{2,ss} + \{\frac{\epsilon}{r^2} + \frac{1-2k}{r^3}\epsilon^2\}\{(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s})\dot{r} +$$

$$(\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s})\dot{t}\} - \dot{t}\dot{r}(\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2)(\frac{2\epsilon}{r^3} + \frac{3(1-2k)}{r^4}\epsilon^2)]_{E_j=0} = 0, \tag{2.41}$$

$$\mathbf{X}^{[2]}E_2=[\eta^1_{0,ss}+\epsilon\eta^1_{1,ss}+\epsilon^2\eta^1_{2,ss}+c^2\dot{t}(\frac{\epsilon}{r^2}+\frac{1+2k}{r^3}\epsilon^2)(\eta^0_{0,s}+\epsilon\eta^0_{1,s}+\epsilon^2\eta^0_{2,s})$$
$$-\{\frac{c^2\dot{t}^2}{2}(\frac{2\epsilon}{r^3}-\frac{3(1+2k)}{r^4}\epsilon^2)-\frac{\dot{r}^2}{2}(\frac{2\epsilon}{r^3}+\frac{3(1-2k)}{r^4}\epsilon^2)+(1-\frac{k\epsilon^2}{r^2})(\dot{\theta}^2$$
$$+\sin^2\theta\dot{\phi}^2)\}(\eta^1_0+\epsilon\eta^1_1+\epsilon^2\eta^1_2)-\dot{r}(\frac{\epsilon}{r^2}+\frac{1-2k}{r^3}\epsilon^2)(\eta^1_{0,s}+\epsilon\eta^1_{1,s}+$$
$$\epsilon^2\eta^1_{2,s})-2(r-\epsilon+\frac{k\epsilon^2}{r})\{(\eta^2_{0,s}+\epsilon\eta^2_{1,s}+\epsilon^2\eta^2_{2,s})\dot{\theta}+\sin\theta\cos\theta\dot{\phi}^2(\eta^2_0$$
$$+\epsilon\eta^2_1+\epsilon^2\eta^2_2)+\dot{\phi}\sin^2\theta(\eta^3_{0,s}+\epsilon\eta^3_{1,s}+\epsilon^2\eta^3_{2,s})\}]_{E_j=0}=0, \tag{2.42}$$

$$\mathbf{X}^{[2]}E_3=[\eta^2_{0,ss}+\epsilon\eta^2_{1,ss}+\epsilon^2\eta^2_{2,ss}-\frac{2\dot{r}\dot{\theta}}{r^2}(\eta^1_0+\epsilon\eta^1_1+\epsilon^2\eta^1_2)+\frac{2\dot{\theta}}{r}(\eta^1_{0,s}+\epsilon\eta^1_{1,s}$$
$$+\epsilon^2\eta^1_{2,s})+\frac{2\dot{r}}{r}(\eta^2_{0,s}+\epsilon\eta^2_{1,s}+\epsilon^2\eta^2_{2,s})-\dot{\phi}^2(\cos^2\theta-\sin^2\theta)(\eta^2_0+$$
$$\epsilon\eta^2_1+\epsilon^2\eta^2_2)-2\sin\theta\cos\theta\dot{\phi}(\eta^3_{0,s}+\epsilon\eta^3_{1,s}+\epsilon^2\eta^3_{2,s})]_{E_j=0}=0, \tag{2.43}$$

$$\mathbf{X}^{[2]}E_4=[\eta^3_{0,ss}+\epsilon\eta^3_{1,ss}+\epsilon^2\eta^3_{2,ss}-\frac{2\dot{r}\dot{\phi}}{r^2}(\eta^1_0+\epsilon\eta^1_1+\epsilon^2\eta^1_2)+\frac{2\dot{\theta}}{r}(\eta^1_{0,s}+\epsilon\eta^1_{1,s}$$
$$+\epsilon^2\eta^1_{2,s})+2(\frac{\dot{r}}{r}+\dot{\theta}\cot\theta)(\eta^3_{0,s}+\epsilon\eta^3_{1,s}+\epsilon^2\eta^3_{2,s})-2\dot{\theta}\dot{\phi}\csc^2\theta(\eta^2_0+$$
$$\epsilon\eta^2_1+\epsilon^2\eta^2_2)+2\dot{\phi}\cot\theta(\eta^2_{0,s}+\epsilon\eta^2_{1,s}+\epsilon^2\eta^2_{2,s})]_{E_j=0}=0, \tag{2.44}$$

where $(j=1,2,3,4)$. The prolongation coefficients are

$$\eta^0_{i,s}=\eta^0_{is}+\dot{t}(\eta^0_{it}-\xi_{is})+\dot{r}\eta^0_{ir}+\dot{\theta}\eta^0_{i\theta}+\dot{\phi}\eta^0_{i\phi}-\dot{t}^2\xi_{it}-\dot{t}\dot{r}\xi_{ir}-\dot{t}\dot{\theta}\xi_{i\theta}-\dot{t}\dot{\phi}\xi_{i\phi}, \tag{2.45}$$

$$\eta^1_{i,s}=\eta^1_{is}+\dot{t}\eta^1_{it}+\dot{r}(\eta^1_{ir}-\xi_{is})+\dot{\theta}\eta^1_{i\theta}+\dot{\phi}\eta^1_{i\phi}-\dot{t}\dot{r}\xi_{it}-\dot{r}^2\xi_{ir}-\dot{r}\dot{\theta}\xi_{i\theta}-\dot{r}\dot{\phi}\xi_{i\phi}, \tag{2.46}$$

$$\eta^2_{i,s}=\eta^2_{is}+\dot{t}\eta^2_{it}+\dot{r}\eta^2_{ir}+\dot{\theta}(\eta^2_{i\theta}-\xi_{is})+\dot{\phi}\eta^2_{i\phi}-\dot{t}\dot{\theta}\xi_{it}-\dot{r}\dot{\theta}\xi_{ir}-\dot{\theta}^2\xi_{i\theta}-\dot{\theta}\dot{\phi}\xi_{i\phi}, \tag{2.47}$$

$$\eta^3_{i,s}=\eta^3_{is}+\dot{t}\eta^3_{it}+\dot{r}\eta^3_{ir}+\dot{\theta}\eta^3_{i\theta}+\dot{\phi}(\eta^3_{i\phi}-\xi_{is})-\dot{t}\dot{\phi}\xi_{it}-\dot{r}\dot{\phi}\xi_{ir}-\dot{\theta}\dot{\phi}\xi_{i\theta}-\dot{\phi}^2\xi_{i\phi}, \tag{2.48}$$

$$
\begin{aligned}
\eta^0_{i,ss} = {} & \eta^0_{iss} + \dot{t}(2\eta^0_{ist} - \xi_{iss}) + \dot{t}^2(\eta^0_{itt} - \xi_{ist}) - \dot{t}^3\xi_{itt} + 2\dot{r}\eta^0_{isr} + \dot{r}^2\eta^0_{irr} + 2\dot{\theta}\eta^0_{is\theta} + \\
& \dot{\theta}^2\eta^0_{i\theta\theta} + 2\dot{\phi}\eta^0_{is\phi} + \dot{\phi}^2\eta^0_{i\phi\phi} + 2\dot{t}\dot{r}(\eta^0_{itr} - \xi_{isr}) + 2\dot{t}\dot{\theta}(\eta^0_{it\theta} - \xi_{is\theta}) + 2\dot{t}\dot{\phi}(\eta^0_{it\phi} - \xi_{is\phi}) \\
& + 2\dot{r}\dot{\theta}\eta^0_{ir\theta} + 2\dot{r}\dot{\phi}\eta^0_{ir\phi} + 2\dot{\theta}\dot{\phi}\eta^0_{i\theta\phi} - 2\dot{t}^2\dot{r}\xi_{itr} - 2\dot{t}^2\dot{\theta}\xi_{it\theta} - 2\dot{t}^2\dot{\phi}\xi_{it\phi} - \dot{t}\dot{r}^2\xi_{irr} - \\
& \dot{t}\dot{\theta}^2\xi_{i\theta\theta} - \dot{t}\dot{\phi}^2\xi_{i\phi\phi} - 2\dot{t}\dot{r}\dot{\theta}\xi_{ir\theta} - 2\dot{t}\dot{r}\dot{\phi}\xi_{ir\phi} - 2\dot{t}\dot{\theta}\dot{\phi}\xi_{i\theta\phi} + \ddot{t}(\eta^0_{it} - 2\xi_{is} - 3\dot{t}\xi_{it} - \\
& 2\dot{r}\xi_{ir} - 2\dot{\theta}\xi_{i\theta} - 2\dot{\phi}\xi_{i\phi}) + \ddot{r}(\eta^0_{ir} - \dot{t}\xi_{ir}) + \ddot{\theta}(\eta^0_{i\theta} - \dot{t}\xi_{i\theta}) + \ddot{\phi}(\eta^0_{i\phi} - \dot{t}\xi_{i\phi}),
\end{aligned}
\tag{2.49}
$$

$$
\begin{aligned}
\eta^1_{i,ss} = {} & \eta^1_{iss} + 2\dot{t}\eta^1_{ist} + \dot{t}^2\eta^1_{itt} + \dot{r}(2\eta^1_{isr} - \xi_{iss}) + \dot{r}^2(\eta^1_{irr} - 2\xi_{isr}) - \dot{r}^3\xi_{irr} + 2\dot{\theta}\eta^1_{is\theta} \\
& + \dot{\theta}^2\eta^1_{i\theta\theta} + 2\dot{\phi}\eta^1_{is\phi} + \dot{\phi}^2\eta^1_{i\phi\phi} + 2\dot{t}\dot{r}(\eta^1_{itr} - \xi_{ist}) + 2\dot{t}\dot{\theta}\eta^1_{it\theta} + 2\dot{t}\dot{\phi}\eta^1_{it\phi} + 2\dot{r}\dot{\theta}(\eta^1_{ir\theta} \\
& - \xi_{is\theta}) + 2\dot{r}\dot{\phi}(\eta^1_{ir\phi} - \xi_{is\phi}) + 2\dot{\theta}\dot{\phi}\eta^1_{i\theta\phi} - \dot{t}^2\dot{r}\xi_{itt} - \dot{r}\dot{\theta}^2\xi_{i\theta\theta} - \dot{r}\dot{\phi}^2\xi_{i\phi\phi} - 2\dot{t}\dot{r}^2\xi_{itr} \\
& - 2\dot{r}^2\dot{\theta}\xi_{ir\theta} - 2\dot{r}^2\dot{\phi}\xi_{ir\phi} - 2\dot{t}\dot{r}\dot{\theta}\xi_{it\theta} - 2\dot{t}\dot{r}\dot{\phi}\xi_{it\phi} - 2\dot{r}\dot{\theta}\dot{\phi}\xi_{i\theta\phi} + \ddot{t}(\eta^1_{it} - \dot{r}\xi_{it}) + \ddot{r}(\eta^1_{ir} \\
& - 2\xi_{is} - 3\dot{r}\xi_{ir} - 2\dot{t}\xi_{it} - 2\dot{\theta}\xi_{i\theta} - 2\dot{\phi}\xi_{i\phi}) + \ddot{\theta}(\eta^1_{i\theta} - \dot{r}\xi_{i\theta}) + \ddot{\phi}(\eta^1_{i\phi} - \dot{r}\xi_{i\phi}),
\end{aligned}
\tag{2.50}
$$

$$
\begin{aligned}
\eta^2_{i,ss} = {} & \eta^2_{iss} + 2\dot{t}\eta^2_{ist} + \dot{t}^2\eta^2_{itt} + 2\dot{r}\eta^2_{isr} + \dot{r}^2\eta^2_{irr} + \dot{\theta}(2\eta^2_{is\theta} - \xi_{iss}) + \dot{\theta}^2(\eta^2_{i\theta\theta} - 2\xi_{is\theta}) - \\
& \dot{\theta}^3\xi_{i\theta\theta} + 2\dot{\phi}\eta^2_{is\phi} + \dot{\phi}^2\eta^2_{i\phi\phi} + 2\dot{t}\dot{r}\eta^2_{itr} + 2\dot{t}\dot{\theta}(\eta^2_{it\theta} - \xi_{ist}) + 2\dot{t}\dot{\phi}\eta^2_{it\phi} + 2\dot{r}\dot{\theta}(\eta^2_{ir\theta} - \xi_{isr}) \\
& + 2\dot{r}\dot{\phi}\eta^2_{ir\phi} + 2\dot{\theta}\dot{\phi}(\eta^2_{i\theta\phi} - \xi_{is\phi}) - \dot{t}^2\dot{\theta}\xi_{itt} - \dot{r}^2\dot{\theta}\xi_{irr} - \dot{\theta}\dot{\phi}^2\xi_{i\phi\phi} - 2\dot{t}\dot{\theta}^2\xi_{it\theta} - 2\dot{r}\dot{\theta}^2\xi_{ir\theta} \\
& - 2\dot{\theta}^2\dot{\phi}\xi_{i\theta\phi} - 2\dot{t}\dot{r}\dot{\theta}\xi_{itr} - 2\dot{t}\dot{\theta}\dot{\phi}\xi_{it\phi} - 2\dot{r}\dot{\theta}\dot{\phi}\xi_{ir\phi} + \ddot{t}(\eta^2_{it} - \dot{\theta}\xi_{it}) + \ddot{r}(\eta^2_{ir} - \dot{\theta}\xi_{ir}) + \\
& \ddot{\theta}(\eta^2_{i\theta} - 2\xi_{is} - 3\dot{\theta}\xi_{i\theta} - 2\dot{t}\xi_{it} - 2\dot{r}\xi_{ir} - 2\dot{\phi}\xi_{i\phi}) + \ddot{\phi}(\eta^2_{i\phi} - \dot{\theta}\xi_{i\phi}),
\end{aligned}
\tag{2.51}
$$

$$
\begin{aligned}
\eta^3_{i,ss} = {} & \eta^3_{iss} + 2\dot{t}\eta^3_{ist} + \dot{t}^2\eta^3_{itt} + 2\dot{r}\eta^3_{isr} + \dot{r}^2\eta^3_{irr} + 2\dot{\theta}\eta^3_{is\theta} + \dot{\theta}^2\eta^3_{i\theta\theta} + \dot{\phi}(2\eta^3_{is\phi} - \xi_{iss}) + \\
& \dot{\phi}^2(\eta^3_{i\phi\phi} - 2\xi_{is\phi}) - \dot{\phi}\xi_{i\phi\phi} + 2\dot{t}\dot{r}\eta^3_{itr} + 2\dot{t}\dot{\theta}\eta^3_{it\theta} + 2\dot{t}\dot{\phi}(\eta^3_{it\phi} - \xi_{ist}) + 2\dot{r}\dot{\theta}\eta^3_{ir\theta} + 2\dot{r}\dot{\phi}(\eta^3_{ir\phi} \\
& - \xi_{isr}) + 2\dot{\theta}\dot{\phi}(\eta^3_{i\theta\phi} - \xi_{is\theta}) - 2\dot{t}\dot{r}\dot{\phi}\xi_{itr} - 2\dot{t}\dot{\theta}\dot{\phi}\xi_{it\theta} - 2\dot{r}\dot{\theta}\dot{\phi}\xi_{ir\theta} - 2\dot{t}\dot{\phi}^2\xi_{it\phi} - 2\dot{r}\dot{\phi}^2\xi_{ir\phi} \\
& - \dot{t}^2\dot{\phi}\xi_{itt} - \dot{r}^2\dot{\phi}\xi_{irr} - \dot{\theta}^2\dot{\phi}\xi_{i\theta\theta} - 2\dot{\theta}\dot{\phi}^2\xi_{i\theta\phi} + \ddot{t}(\eta^3_{it} - \dot{\phi}\xi_{it}) + \ddot{r}(\eta^3_{ir} - \dot{\phi}\xi_{ir}) + \ddot{\theta}(\eta^3_{i\theta} - \\
& \dot{\phi}\xi_{i\theta}) + \ddot{\phi}(\eta^3_{i\phi} - 2\xi_{is} - 3\dot{\phi}\xi_{i\phi} - 2\dot{t}\xi_{it} - 2\dot{r}\xi_{ir} - 2\dot{\theta}\xi_{i\theta}),
\end{aligned}
\tag{2.52}
$$

(where $i = 0, 1, 2$ denoting the exact, first-order approximate and second-order approximate part respectively). Substituting these values, the exact and first-order approximate symmetry generators $\mathbf{X}_0$ and $\mathbf{X}_1$ given by (1.78) - (1.85) in (2.41) - (2.44) and using the geodesic equations (2.36) - (2.39), we get the following set of determining equations.

$$2r^2\xi_{2_{tr}} - a_2 = 0,\ \xi_{2_{t\theta}} = 0,\ \xi_{2_{t\phi}} = 0,\ r\xi_{2_{r\theta}} - \xi_{2_\theta} = 0,$$
$$r\xi_{2_{r\phi}} - \xi_{2_\phi} = 0,\ r\xi_{2_{\theta\phi}} - \cot\theta\xi_{2_\phi} = 0,$$
$$2r^2\xi_{2_{tt}} - c^2[\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] = 0,$$
$$2r^2\xi_{2_{rr}} - [\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] = 0,$$
$$\xi_{2_{\theta\theta}} + r\xi_{2_r} - [\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] = 0,$$
$$\xi_{2_{\phi\phi}} + r\sin^2\theta\xi_{2_r} + \sin\theta\cos\theta\xi_{2_\theta} - \sin^2\theta[\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] = 0, \qquad (2.53)$$

$$2r^2(\eta^0_{2_{tt}} - 2\xi_{2_{st}}) + c(a_2\sin\theta\cos\phi + a_3\sin\theta\sin\phi + a_4\cos\theta) = 0, \qquad (2.54)$$
$$r^3\eta^1_{2_{tt}} - c^2(a_5\sin\theta\cos\phi + a_6\sin\theta\sin\phi + a_7\cos\theta + a_2ct\sin\theta\cos\phi +$$
$$a_3ct\sin\theta\sin\phi + a_4ct\cos\theta) = 0, \qquad (2.55)$$
$$2r^4\eta^2_{2_{tt}} + c^2(a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi - a_7\sin\theta + a_2ct\cos\theta\cos\phi$$
$$+a_3ct\cos\theta\sin\phi - a_4ct\sin\theta) = 0, \qquad (2.56)$$
$$2r^4\eta^3_{2_{tt}} + c^2(a_6\csc\theta\cos\phi - a_5\csc\theta\sin\phi - a_2ct\csc\theta\sin\phi + a_3ct\csc\theta\cos\phi) = 0, \qquad (2.57)$$
$$2cr^2\eta^0_{2_{rr}} + 3(a_2\sin\theta\cos\phi + a_3\sin\theta\sin\phi + a_4\cos\theta) = 0, \qquad (2.58)$$
$$r^3(\eta^1_{2_{rr}} - 2\xi_{2_{sr}}) + a_5\sin\theta\cos\phi + a_6\sin\theta\sin\phi + a_7\cos\theta + a_2ct\sin\theta\cos\phi +$$
$$a_3ct\sin\theta\sin\phi + a_4ct\cos\theta = 0, \qquad (2.59)$$
$$2r^3(r\eta^2_{2_{rr}} + 2\eta^2_{2_r}) - a_5\cos\theta\cos\phi - a_6\cos\theta\sin\phi + a_7\sin\theta - a_2ct\cos\theta\cos\phi$$
$$-a_3ct\cos\theta\sin\phi + a_4ct\sin\theta = 0, \qquad (2.60)$$
$$2r^3(r\eta^3_{2_{rr}} + 2\eta^3_{2_r}) + a_5\csc\theta\sin\phi - a_6\csc\theta\sin\cos + a_7\sin\theta + a_2ct\csc\theta\sin\phi$$
$$-a_3ct\csc\theta\cos\phi = 0, \qquad (2.61)$$
$$c(\eta^0_{2_{\theta\theta}} + r\eta^0_{2_r}) - a_2\sin\theta\cos\phi - a_3\sin\theta\sin\phi - a_4\cos\theta = 0, \qquad (2.62)$$

$$r(\eta^1_{2_{\theta\theta}} + r\eta^1_{2_r} - \eta^1_2 - 2r\eta^2_{2_\theta}) - 2(a_5\sin\theta\cos\phi + a_6\sin\theta\sin\phi + a_7\cos\theta$$

$$+a_2ct\sin\theta\cos\phi + a_3ct\sin\theta\sin\phi + a_4ct\cos\theta) = 0, \tag{2.63}$$

$$r(r\eta^2_{2_{\theta\theta}} + r^2\eta^2_{2_r} + 2r\eta^1_{2_\theta} - 2r\xi_{2_{s\theta}}) + a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi - a_7\sin\theta$$

$$+a_2ct\cos\theta\cos\phi + a_3ct\cos\theta\sin\phi - a_4ct\sin\theta = 0, \tag{2.64}$$

$$r^2(\eta^3_{2_{\theta\theta}} + r\eta^3_{2_r} + 2r^2\cot\theta\eta^3_{2_\theta}) - a_5\csc\theta\sin\phi + a_6\csc\theta\cos\phi - a_2ct\csc\theta\sin\phi$$

$$+a_3ct\csc\theta\cos\phi = 0, \tag{2.65}$$

$$c(\sin\theta\cos\phi\eta^0_{2_\theta} + \eta^0_{2_{\phi\phi}} + r\sin^2\theta\eta^0_{2_r}) - \sin^2\theta(a_2\sin\theta\cos\phi + a_3\sin\theta\sin\phi$$

$$+a_4\cos\theta) = 0, \tag{2.66}$$

$$r(\eta^1_{2_{\phi\phi}} + r\sin^2\theta\eta^1_{2_r} + \sin\theta\cos\theta\eta^1_{2_\theta} - \sin^2\theta\eta^1_2 - 2r\sin\theta\cos\theta\eta^2_2 - 2r\sin\theta\eta^3_{2_\phi}) -$$

$$2\sin^2\theta(a_5\sin\theta\cos\phi + a_6\sin\theta\sin\phi + a_7\cos\theta + a_2ct\sin\theta\cos\phi + a_3ct\sin\theta\sin\phi$$

$$+a_4ct\cos\theta) = 0, \tag{2.67}$$

$$r^2(\eta^2_{2_{\phi\phi}} + r\sin^2\theta\eta^2_{2_r} - \cos 2\theta\eta^2_2 - 2\sin\theta\cos\theta\eta^3_{2_\phi} + \sin\theta\cos\theta\eta^2_{2_\theta}) + \sin^2\theta(a_5\sin\theta$$

$$\cos\phi + a_6\sin\theta\sin\phi - a_7\cos\theta + a_2ct\cos\theta\cos\phi + a_3ct\cos\theta\sin\phi - a_4ct\sin\theta) = 0, \tag{2.68}$$

$$r(r\eta^3_{2_{\phi\phi}} - 2r\xi_{2_{s\phi}} + r^2\sin^2\theta\eta^3_{2_r} + r\sin\theta\cos\theta\eta^3_{2_\theta} + 2r\eta^1_{2_\phi} + 2r^2\cot\theta\eta^2_{2_\phi}) +$$

$$\sin^2\theta(a_6\csc\theta\cos\phi - a_5\csc\theta\sin\phi - a_2ct\csc\theta\sin\phi + a_3ct\csc\theta\cos\phi) = 0, \tag{2.69}$$

$$2\eta^0_{2_{st}} - \xi_{2_{ss}} = 0,\ \eta^1_{2_{st}} = 0,\ \eta^2_{2_{st}} = 0,\ \eta^3_{2_{st}} = 0, \tag{2.70}$$

$$\eta^0_{2_{sr}} = 0,\ 2\eta^1_{2_{sr}} - \xi_{2_{ss}} = 0,\ r\eta^2_{2_{sr}} + \eta^2_{2_s} = 0,\ r\eta^3_{2_{sr}} + \eta^3_{2_s} = 0, \tag{2.71}$$

$$\eta^0_{2_{s\theta}} = 0,\ \eta^1_{2_{s\theta}} - r\eta^2_{2_s} = 0,\ r(2\eta^2_{2_{s\theta}} - \xi_{2_{ss}}) + 2\eta^1_{2_s} = 0,\ \eta^3_{2_{s\theta}} + \cot\theta\eta^3_{2_s} = 0, \tag{2.72}$$

$$\eta^0_{2_{s\phi}} = 0,\ \eta^1_{2_{s\phi}} - r\sin^2\theta\eta^3_{2_s} = 0,\ \eta^2_{2_{s\phi}} - \sin\theta\cos\theta\eta^3_{2_s} = 0, \tag{2.73}$$

$$r(2\eta^3_{2_{s\phi}} - \xi_{2_{ss}}) + 2\eta^1_{2_s} + 2r\cot\theta\eta^2_{2_s} = 0, \tag{2.74}$$

$$\eta^0_{2_{ss}} = 0,\ \eta^1_{2_{ss}} = 0,\ \eta^2_{2_{ss}} = 0,\ \eta^3_{2_{ss}} = 0, \tag{2.75}$$

$$r^3(\eta^0_{2_{tr}} - \xi_{2_{sr}}) - (a_5\sin\theta\cos\phi + a_6\sin\theta\sin\phi + a_7\cos\theta + a_2ct\sin\theta\cos\phi +$$

$$a_3ct\sin\theta\sin\phi + a_4ct\cos\theta) = 0, \tag{2.76}$$

$$2r^2(\eta^1_{2_{tr}} - \xi_{2_{st}}) - c(a_2ct\sin\theta\cos\phi + a_3ct\sin\theta\sin\phi + a_4ct\cos\theta) = 0, \tag{2.77}$$

$$2r^2(r\eta^2_{2_{tr}} + \eta^2_{2_t}) - c(a_2\cos\theta\cos\phi + a_3\cos\theta\sin\phi - a_4\sin\theta) = 0, \tag{2.78}$$

$$2r^2(r\eta^3_{2_{tr}} + \eta^3_{2_t}) - c(a_3\csc\theta\cos\phi - a_2\csc\theta\sin\phi) = 0, \tag{2.79}$$

$$2r^2(\eta^0_{2_{t\theta}} - \xi_{2_{s\theta}}) + a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi - a_7\sin\theta + a_2ct\cos\theta\cos\phi +$$

$$a_3ct\cos\theta\sin\phi - a_4ct\sin\theta = 0, \tag{2.80}$$

$$2r(\eta^1_{2_{t\theta}} - r\eta^2_{2_t}) + 3c(a_2\cos\theta\cos\phi + a_3\cos\theta\sin\phi - a_4\sin\theta) = 0, \tag{2.81}$$

$$r(\eta^2_{2_{t\theta}} - \xi_{2_{st}}) + \eta^1_{2_t} = 0, \tag{2.82}$$

$$\eta^3_{2_{t\theta}} + \cot\theta\eta^3_{2_t} = 0, \tag{2.83}$$

$$2r^2(\eta^0_{2_{t\phi}} - \xi_{2_{s\phi}}) + a_6\sin\theta\cos\phi - a_5\sin\theta\sin\phi - a_2ct\sin\theta\sin\phi$$

$$+a_3ct\sin\theta\cos\phi = 0, \tag{2.84}$$

$$2r(\eta^1_{2_{t\phi}} - r\sin^2\theta\eta^3_{2_t}) + 3c(a_3\sin\theta\cos\phi - a_2\sin\theta\sin\phi) = 0, \tag{2.85}$$

$$\eta^2_{2_{t\phi}} - \sin\theta\cos\theta\eta^3_{2_t} = 0, \tag{2.86}$$

$$r(\eta^3_{2_{t\phi}} - \xi_{2_{st}}) + \eta^1_{2_t} + r\cot\theta\eta^2_{2_t} = 0, \tag{2.87}$$

$$2c(r\eta^0_{2_{r\theta}} - \eta^0_{2_\theta}) + a_2\cos\theta\cos\phi + a_3\cos\theta\sin\phi - a_4\sin\theta = 0,$$

$$2r(r\eta^1_{2_{r\theta}} - r\xi_{2_{s\theta}} - \eta^1_{2_\theta} - r^2\eta^2_{2_r}) - 3(a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi - a_7\sin\theta \tag{2.88}$$

$$+a_2ct\cos\theta\cos\phi + a_3ct\cos\theta\sin\phi - a_4ct\sin\theta) = 0, \tag{2.89}$$

$$r^2(\eta^2_{2_{r\theta}} - \xi_{2_{sr}}) - \eta^1_2 + r\eta^1_{2_r} = 0, \tag{2.90}$$

$$\eta^3_{2_{r\theta}} + \cot\theta\eta^3_{2_r} = 0, \tag{2.91}$$

$$2c(r\eta^0_{2_{r\phi}} - \eta^0_{2_\phi}) + a_3\sin\theta\cos\phi - a_2\sin\theta\sin\phi = 0, \tag{2.92}$$

$$2r(r\eta^1_{2_{r\phi}} - r\xi_{2_{s\phi}} - \eta^1_{2_\phi} - r^2\sin^2\theta\eta^3_{2_r}) + 3(a_5\sin\theta\cos\phi - a_6\sin\theta\cos\phi +$$

$$a_2ct\sin\theta\sin\phi - a_3ct\sin\theta\cos\phi) = 0, \tag{2.93}$$

$$\eta^2_{2_{r\phi}} - \sin\theta\cos\theta\eta^3_{2_r} = 0, \tag{2.94}$$

$$r^2(\eta^3_{2_{r\phi}} - \xi_{2_{sr}} + \cot\theta\eta^2_{2_r}) + r\eta^1_{2_r} - \eta^1_2 = 0, \tag{2.95}$$

$$\eta^0_{2_{\theta\phi}} - \cot\theta\eta^0_{2_\phi} = 0, \tag{2.96}$$

$$\eta^1_{2_{\theta\phi}} - \cot\theta\eta^1_{2_\phi} - r\eta^2_{2_\phi} - r\sin^2\theta\eta^3_{2_\theta} = 0, \tag{2.97}$$

$$r(\eta^2_{2_{\theta\phi}} - \xi_{2_{s\phi}} - \cot\theta\eta^2_{2_\phi} - \sin\theta\cos\theta\eta^3_{2_\theta}) + \eta^1_{2_\phi} = 0, \tag{2.98}$$

$$r(\eta^3_{2_{\theta\phi}} - \xi_{2_{s\theta}} - \csc^2\theta\eta^2_2 + \cot\theta\eta^2_{2_\theta}) + \eta^1_{2_\theta} = 0. \tag{2.99}$$

It should be noted that the symmetry algebra of the exact case i.e. of the geodesic equations of the Minkowski spacetime (maximally symmetric) is $sl(6,\mathbb{R})$, (35 symmetry generators) [68], which has many symmetries that do not correspond to conservation laws, arising from the mixing of the geodesic re-parametrization generators with the Noether symmetry generators of the geodesic equations. Since we are looking for non-trivial approximate conservation laws which can come from Noether symmetries, therefore we only use the 10 KVs (which are also Noether symmetry generators) and d_2, as exact symmetry generators in the construction of the above determining equations for second-order approximate symmetries of the geodesic equations of the RN spacetime. These were also used in the construction of the determining equations of the first-order approximate symmetries of the geodesic equation for the Schwarzschild spacetime [60].

First integrating (2.53-i) with respect to r and then with respect to t we get

$$\xi_2 = -\frac{a_2 t}{2r} + \int f_1(s,t,\theta,\phi)dt + f_2(s,r,\theta,\phi), \tag{2.100}$$

where f_1 and f_2 are arbitrary functions of integration.

Equation (2.53-ii) implies that

$$f_1 = f_3(s,t,\phi) \tag{2.101}$$

and (2.53-iii) gives

$$f_3 = f_4(s,t). \tag{2.102}$$

Substituting the value of ξ_2 from (2.100) in (2.53-vii) and then integration twice with respect to t yields

$$f_4 = \frac{c^2 t}{2r^2}[\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] + f_5(s).$$

Integration of f_4 with respect to t and then substitution in (2.100) gives

$$\begin{aligned}\xi_2 \quad = \quad & -\frac{a_2 t}{2r} + \frac{c^2 t^2}{4r^2}[\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] + tf_5(s) \\ & + f_2(s,r,\theta,\phi) + b_0,\end{aligned} \tag{2.103}$$

where b_0 is constant of integration.

Equation (2.53-x) gives us

$$\cos\theta(a_3\sin\phi - a_4\cos\phi) - a_5\sin\phi = 0,$$

which implies that

$$a_3 = 0,\ \ a_4 = 0 \text{ and } a_5 = 0.$$

Therefore from (2.103) we have

$$\xi_2 = -\frac{a_2 t}{2r} + tf_5(s) + f_2(s, r, \theta, \phi) + b_0. \tag{2.104}$$

Now use of (2.104) into (2.53-viii) yields

$$a_2 = 0$$

and thus

$$\xi_2 = tf_5(s) + f_2(s, r, \theta, \phi) + b_0. \tag{2.105}$$

Also (2.53-viii) implies that

$$f_2 = rf_6(s, \theta, \phi) + f_7(s, \theta, \phi).$$

Hence (2.105) becomes

$$\xi_2 = tf_5(s) + rf_6(s, \theta, \phi) + f_7(s, \theta, \phi) + b_0. \tag{2.106}$$

Use of (2.106) in (2.53-ix) gives

$$f_7 = \theta f_8(s, \phi) + f_9(s, \phi).$$

Therefore

$$\xi_2 = tf_5(s) + rf_6(s, \theta, \phi) + \theta f_8(s, \phi) + f_9(s, \phi) + b_0. \tag{2.107}$$

The use of (2.107) in (2.53-x) implies

$$f_8 = 0 \text{ and } f_9 = \phi f_{10}(s) + f_{11}(s).$$

From (2.107) we have

$$\xi_2 = t f_5(s) + r f_6(s, \theta, \phi) + \phi f_{10}(s) + f_{11}(s) + b_0. \tag{2.108}$$

Now from (2.53-vi) with the use of (2.108) we get

$$f_6 = \sin\theta \int f_{12}(s, \phi) d\phi + f_{13}(s, \theta) \text{ and } f_{10} = 0.$$

Thus

$$\xi_2 = t f_5(s) + r[\sin\theta \int f_{12}(s, \phi) d\phi + f_{13}(s, \theta)] + f_{11}(s) + b_0. \tag{2.109}$$

Integration of (2.75-i), (2.75-ii), (2.75-iii) and (2.75-iv) twice with respect to s give us respectively

$$\eta_2^0 = g_1(t, r, \theta, \phi)s + g_2(t, r, \theta, \phi), \tag{2.110}$$

$$\eta_2^1 = h_1(t, r, \theta, \phi)s + h_2(t, r, \theta, \phi), \tag{2.111}$$

$$\eta_2^2 = l_1(t, r, \theta, \phi)s + l_2(t, r, \theta, \phi), \tag{2.112}$$

$$\eta_2^3 = m_1(t, r, \theta, \phi)s + m_2(t, r, \theta, \phi). \tag{2.113}$$

Equation (2.70-i) yield

$$g_1 = \frac{1}{4} t^2 f_{5_{ss}}(s) + t g_3(r, \theta, \phi) + g_4(r, \theta, \phi),$$

therefore

$$\eta_2^0 = [\frac{1}{4} t^2 f_{5_{ss}}(s) + t g_3(r, \theta, \phi) + g_4(r, \theta, \phi)]s + g_2(t, r, \theta, \phi). \tag{2.114}$$

Now (2.70-i) implies

$$g_3 = b_1$$

and

$$\begin{aligned} f_{11} &= b_1 s^2 + b_2 s + b_3, \\ f_{12} &= s f_{14}(\phi) + f_{15}(\phi), \\ f_{13} &= s f_{16}(\theta) + f_{17}(\theta). \end{aligned}$$

Therefore

$$\begin{aligned} \xi_2 &= t f_5(s) + r[\sin\theta\{s\int f_{14}(\phi)d\phi + \int f_{15}(\phi)d\phi\} + s f_{16}(\theta) + f_{17}(\theta)] \\ &\quad + b_1 s^2 + b_2 s + b_4, \end{aligned} \tag{2.115}$$

where $b_4 = b_0 + b_3$ and

$$\eta_2^0 = [\frac{1}{4}t^2 f_{5_{ss}}(s) + t b_1 + g_4(r,\theta,\phi)]s + g_2(t,r,\theta,\phi) \tag{2.116}$$

Equations (2.70-ii), (2.75-iii) and (2.75-iv) give respectively

$$h_1 = h_3(r,\theta,\phi),\ \ l_1 = l_3(r,\theta,\phi) \text{ and } m_1 = m_3(r,\theta,\phi).$$

Thus

$$\eta_2^1 = h_3(r,\theta,\phi)s + h_2(t,r,\theta,\phi), \tag{2.117}$$

$$\eta_2^2 = l_3(r,\theta,\phi)s + l_2(t,r,\theta,\phi), \tag{2.118}$$

$$\eta_2^3 = m_3(r,\theta,\phi)s + m_2(t,r,\theta,\phi). \tag{2.119}$$

From (2.71-i) we obtain

$$g_4 = g_5(\theta,\phi).$$

From (2.116) we get

$$\eta_2^0 = [\frac{1}{4}t^2 f_{5_{ss}}(s) + t b_1 + g_5(\theta,\phi)]s + g_2(t,r,\theta,\phi). \tag{2.120}$$

Now (2.71-ii) yields

$$h_3 = \frac{b_1}{2} r + h_4(\theta, \phi) \text{ and } f_5 = b_5 s + b_6.$$

Thus

$$\begin{aligned} \xi_2 &= t(b_5 s + b_6) + r[\sin\theta\{s\int f_{14}(\phi)d\phi + \int f_{15}(\phi)d\phi\} + sf_{16}(\theta) + f_{17}(\theta)] \\ & +b_1 s^2 + b_2 s + b_4, \end{aligned} \tag{2.121}$$

$$\eta_2^0 = [tb_1 + g_5(\theta, \phi)]s + g_2(t, r, \theta, \phi) \tag{2.122}$$

and

$$\eta_2^1 = [\frac{b_1}{2} r + h_4(\theta, \phi)]s + h_2(t, r, \theta, \phi). \tag{2.123}$$

Equations (2.71-iii) and (2.71-iv) gives us respectively

$$l_3 = \frac{1}{r} l_4(\theta, \phi) \text{ and } m_3 = \frac{1}{r} m_4(\theta, \phi).$$

Therefore

$$\eta_2^2 = \frac{s}{r} l_4(\theta, \phi) + l_2(t, r, \theta, \phi), \tag{2.124}$$

$$\eta_2^3 = \frac{s}{r} m_4(\theta, \phi) + m_2(t, r, \theta, \phi). \tag{2.125}$$

From (2.72-i) we obtain

$$g_5 = g_6(\phi).$$

Equation (2.122) gives

$$\eta_2^0 = [tb_1 + g_6(\phi)]s + g_2(t, r, \theta, \phi). \tag{2.126}$$

Equation (2.72-ii) yields

$$l_4 = h_{4_\theta}(\theta, \phi)$$

and (2.72-iv) gives

$$m_4 = m_5(\phi) \csc\theta.$$

Therefore

$$\begin{aligned} \eta_2^2 &= \frac{s}{r} h_{4_\theta}(\theta, \phi) + l_2(t, r, \theta, \phi), && (2.127) \\ \eta_2^3 &= \frac{s}{r} m_5(\phi) \csc\theta + m_2(t, r, \theta, \phi). && (2.128) \end{aligned}$$

Equation (2.73-i) yields

$$g_6 = b_7.$$

From (2.126) we get

$$\eta_2^0 = [tb_1 + b_7]s + g_2(t, r, \theta, \phi). \quad (2.129)$$

Equation (2.73-ii) implies that

$$h_4 = \sin\theta \int m_5(\phi) d\phi + h_5(\phi),$$

therefore

$$\eta_2^1 = [\frac{b_1}{2} r + \sin\theta \int m_5(\phi) d\phi + h_5(\theta)]s + h_2(t, r, \theta, \phi) \quad (2.130)$$

and

$$\eta_2^2 = \frac{s}{r}[\cos\theta \int m_5(\phi) d\phi + h_{5_\theta}(\theta)] + l_2(t, r, \theta, \phi). \quad (2.131)$$

Now from (2.73-iv) we obtain

$$m_5(\phi) = b_8 \cos\phi + b_9 \sin\phi.$$

From (2.130), (2.131) and (2.128) we get respectively

$$\begin{aligned} \eta_2^1 &= [\frac{b_1}{2} r + \sin\theta(b_8 \sin\theta\phi - b_9 \cos\phi + b_{10}) + h_5(\theta)]s + h_2(t, r, \theta, \phi), && (2.132) \\ \eta_2^2 &= \frac{s}{r}[\cos\theta(b_8 \sin\phi - b_9 \cos\phi + b_{10}) + h_{5_\theta}(\theta)] + l_2(t, r, \theta, \phi) && (2.133) \end{aligned}$$

and

$$\eta_2^3 = \frac{s}{r}(b_8 \sin\phi - b_9 \cos\phi + b_{10}) \csc\theta + m_2(t, r, \theta, \phi). \quad (2.134)$$

From (2.72-iii) we have

$$h_5 = b_{11}\cos\theta + b_{12}\sin\theta,$$

thereby

$$\begin{aligned}\eta_2^1 &= [\frac{b_1}{2}r + \sin\theta(b_8\sin\theta\phi - b_9\cos\phi + b_{10}) + b_{11}\cos\theta + b_{12}\sin\theta]s \\ &\quad + h_2(t,r,\theta,\phi) \end{aligned} \tag{2.135}$$

and

$$\eta_2^2 = \frac{s}{r}[\cos\theta(b_8\sin\phi - b_9\cos\phi + b_{10}) - b_{11}\sin\theta + b_{12}\cos\theta] + l_2(t,r,\theta,\phi). \tag{2.136}$$

Equation (2.73-iv) yields

$$b_{10} = -b_{12}.$$

Using this in (2.135) and (2.136) we get respectively

$$\eta_2^1 = [\frac{b_1}{2}r + \sin\theta(b_8\sin\theta\phi - b_9\cos\phi) + b_{11}\cos\theta]s + h_2(t,r,\theta,\phi), \tag{2.137}$$

and

$$\eta_2^2 = \frac{s}{r}[\cos\theta(b_8\sin\phi - b_9\cos\phi) - b_{11}\sin\theta] + l_2(t,r,\theta,\phi). \tag{2.138}$$

Now going back to (2.53-ix) and (2.53-x) give us respectively

$$\begin{aligned} f_{16} &= b_{13}\cos\theta + b_{14}\sin\theta, \\ f_{17} &= b_{15}\cos\theta + b_{16}\sin\theta, \end{aligned}$$

and

$$\begin{aligned} f_{14} &= b_{17}\cos\phi + b_{18}\sin\phi, \\ f_{15} &= b_{19}\cos\phi + b_{20}\sin\phi, \end{aligned}$$

thus

$$\begin{aligned}\xi_2 &= t(b_5 s + b_6) + r[\sin\theta\{s(b_{17}\sin\phi - b_{18}\cos\phi + b_{21}) + b_{19}\sin\phi - \\ &\quad b_{20}\cos\phi + b_{22}\} + s(b_{13}\cos\theta + b_{14}\sin\theta) + b_{15}\cos\theta + b_{16}\sin\theta] \\ &\quad + b_1 s^2 + b_2 s + b_4. \qquad (2.139)\end{aligned}$$

Equation (2.53-x) implies

$$b_{21} = -b_{14} \text{ and } b_{22} = -b_{16}.$$

Use of this in (2.121) yields

$$\begin{aligned}\xi_2 &= t(b_5 s + b_6) + r[\sin\theta\{s(b_{17}\sin\phi - b_{18}\cos\phi) + b_{19}\sin\phi - b_{20}\cos\phi\} \\ &\quad + s b_{13}\cos\theta + b_{15}\cos\theta] + b_1 s^2 + b_2 s + b_4. \qquad (2.141)\end{aligned}$$

From (2.54) we get

$$g_2 = t^2 b_5 + g_7(r,\theta,\phi)t + g_8(r,\theta,\phi). \qquad (2.142)$$

Therefore

$$\eta_2^0 = [tb_1 + b_7]s + t^2 b_5 + g_7(r,\theta,\phi)t + g_8(r,\theta,\phi). \qquad (2.143)$$

Equation (2.55) gives us

$$h_2 = \frac{c^2 t^2}{2r^3}(a_6 \sin\theta\sin\phi + a_7\cos\theta) + h_5(r,\theta,\phi)t + h_6(r,\theta,\phi).$$

From (2.137) we have

$$\begin{aligned}\eta_2^1 &= [\frac{b_1}{2}r + \sin\theta(b_8\sin\theta\phi - b_9\cos\phi) + b_{11}\cos\theta]s + \frac{c^2t^2}{2r^3}(a_6\sin\theta\sin\phi \\ &\quad + a_7\cos\theta) + h_5(r,\theta,\phi)t + h_6(r,\theta,\phi). \qquad (2.144)\end{aligned}$$

Equation (2.56) yields

$$l_2 = -\frac{c^2t^2}{4r^4}(a_6\cos\theta\sin\phi - a_7\sin\theta) + l_5(r,\theta,\phi)t + l_6(r,\theta,\phi).$$

Therefore

$$\eta_2^2 = \frac{s}{r}[\cos\theta(b_8\sin\phi - b_9\cos\phi) - b_{11}\sin\theta] + -\frac{c^2t^2}{4r^4}(a_6\cos\theta\sin\phi - a_7\sin\theta)$$
$$+ l_5(r,\theta,\phi)t + l_6(r,\theta,\phi). \qquad (2.145)$$

From (2.57) we get

$$m_2 = -\frac{c^2t^2}{4r^4}(a_6\csc\theta\cos\phi) + m_6(r,\theta,\phi)t + m_7(r,\theta,\phi),$$

so

$$\eta_3^2 = \frac{s}{r}[\csc\theta(b_8\cos\phi + b_9\sin\phi)] - \frac{c^2t^2}{4r^4}(a_6\csc\theta\cos\phi)$$
$$+ m_6(r,\theta,\phi)t + m_7(r,\theta,\phi). \qquad (2.146)$$

Now (2.58) gives

$$g_7 = g_9(\theta,\phi)r + g_{10}(\theta,\phi) \text{ and } g_8 = g_{11}(\theta,\phi)r + g_{12}(\theta,\phi).$$

Hence

$$\eta_2^0 = [tb_1 + b_7]s + t^2b_5 + [g_9(\theta,\phi)r + g_{10}(\theta,\phi)]t + g_{11}(\theta,\phi)r + g_{12}(\theta,\phi). \qquad (2.147)$$

From (2.59) we obtain

$$a_6 = 0, \ a_7 = 0,$$

$$h_5 = h_7(\theta,\phi)r + h_8(\theta,\phi)$$

and

$$h_6 = r^2[\sin\theta(b_{17}\sin\phi - b_{18}\cos\phi) + b_{13}\cos\theta] + h_9(\theta,\phi)r + h_{10}(\theta,\phi).$$

Thus

$$\eta_2^1 = [\frac{b_1}{2}r + \sin\theta(b_8\sin\theta\phi - b_9\cos\phi) + b_{11}\cos\theta]s + [h_7(\theta,\phi)r + h_8(\theta,\phi)]t$$
$$+r^2[\sin\theta(b_{17}\sin\phi - b_{18}\cos\phi) + b_{13}\cos\theta] + h_9(\theta,\phi)r + h_{10}(\theta,\phi), \quad (2.148)$$
$$\eta_2^2 = \frac{s}{r}[\cos\theta(b_8\sin\phi - b_9\cos\phi) - b_{11}\sin\theta] + l_5(r,\theta,\phi)t + l_6(r,\theta,\phi) \quad (2.149)$$

and

$$\eta_3^2 = \frac{s}{r}[\csc\theta(b_8\cos\phi + b_9\sin\phi)] + m_6(r,\theta,\phi)t + M_7(r,\theta,\phi). \quad (2.150)$$

With the disappearance of $a_2, ..., a_7$, the above system of determining equations (2.53) - (2.100) becomes homogeneous (i.e. reduces to that of the Minkowski spacetime) and its solution is given in chapter 1.

Check

We make an easy check for the verification of our calculations. That is to check whether all the constants appearing in the determining equations for the second-order approximation vanish or not. We do not take the linear combination of all the symmetry generators (1st approximate) but only take one of them, for example the following one with exact symmetry generator equal to zero. That is

$$\mathbf{Y}_1 = a_1(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}) \text{ and } \mathbf{Y}_0 = 0. \quad (2.151)$$

We assume that η_2^0 is function of r, t only and does not depend on θ and ϕ. Thus we get the following set of equations

$$\eta^0_{2_{tt}} = 0, \quad (2.152)$$
$$\eta^0_{2_{rr}} = 0, \quad (2.153)$$
$$\eta^0_{2_{\theta\theta}} + r\eta^0_{2_r} = 0, \quad (2.154)$$
$$\eta^0_{2_{\phi\phi}} + r\sin^2\theta\eta^0_{2_r} + \sin\theta\cos\theta\eta^0_{2_\theta} = 0, \quad (2.155)$$
$$\eta^0_{2_{tr}} = \frac{a_1}{r^3}\cos\theta, \quad (2.156)$$

$$2\eta^0_{2_{t\theta}} = \frac{a_1}{r^2}\sin\theta, \tag{2.157}$$

$$\eta^0_{2_{t\phi}} = 0, \tag{2.158}$$

$$r\eta^0_{2_{r\theta}} - \eta^0_{2_\theta} = 0, \tag{2.159}$$

$$r\eta^0_{2_{r\phi}} - \eta^0_{2_\phi} = 0, \tag{2.160}$$

$$\eta^0_{2_{\theta\phi}} - \cot\theta\eta^0_{2_\phi} = 0. \tag{2.161}$$

Now integration of (2.156) first with respect to t and then with respect to r gives

$$\eta^0_2 = -\frac{a_1 t}{2r^2}\cos\theta + \int f(r,\theta,\phi)dr + g(\theta,\phi). \tag{2.162}$$

Equation (2.153) yields

$$a_1 = 0 \text{ and } f = f_1(\theta,\phi). \tag{2.163}$$

Therefore

$$\eta^0_2 = rf_1(\theta,\phi) + g(\theta,\phi) + c_0.$$

From (2.154) we obtain

$$g = \theta g_1(\phi) + g_2(\phi). \tag{2.164}$$

Hence

$$\eta^0_2 = rf_1(\theta,\phi) + \theta g_1(\phi) + g_2(\phi) + c_0. \tag{2.165}$$

Now (2.155) gives us

$$g_1 = 0 \text{ and } g_2 = c_1\phi + c_2. \tag{2.166}$$

Therefore

$$\eta^0_2 = rf_1(\theta,\phi) + c_1\phi + c_2 + c_0. \tag{2.167}$$

From (2.160) we deduce

$$c_1 = 0, \tag{2.168}$$

so

$$\eta^0_2 = rf_1(\theta,\phi) + c_3, \tag{2.169}$$

where

$$c_3 = c_0 + c_2. \tag{2.170}$$

Equation (2.161) yields

$$f_1(\theta, \phi) = \sin\theta \int h(\phi) d\phi + l(\phi). \tag{2.171}$$

Therefore

$$\eta_2^0 = r[\sin\theta \int h(\phi) d\phi + l(\phi)] + c_3. \tag{2.172}$$

Equation (2.154) implies

$$l = c_4 \cos\theta + c_5 \sin\theta. \tag{2.173}$$

Thus

$$\eta_2^0 = r[\sin\theta \int h(\phi) d\phi + c_4 \cos\theta + c_5 \sin\theta] + c_3. \tag{2.174}$$

From (2.155) we have

$$h = c_6 \cos\phi + c_7 \sin\phi. \tag{2.175}$$

Therefore

$$\eta_2^0 = r[\sin\theta\{c_6 \sin\phi - c_7 \cos\phi + c_8\} + c_4 \cos\theta + c_5 \sin\theta] + c_3. \tag{2.176}$$

Again (2.155) implies

$$c_5 = 0 \text{ and } c_8 = 0. \tag{2.177}$$

Hence

$$\eta_2^0 = r[\sin\theta\{c_6 \sin\phi - c_7 \cos\phi_8\} + c_4 \cos\theta] + c_3. \tag{2.178}$$

In this check we see that the constant a_1 corresponding to the first-order approximate symmetry disappears for the consistency of the determining equations. Hence there is no non-trivial symmetry.

In the construction of the determining equations for the second-order approximate symmetries of the geodesic equations (2.36) - (2.39) for the RN spacetime, we used (1.78) and (1.79) as the 4 exact symmetry generators and (1.80) - (1.85) as the 6 first-order approximate symmetry generators. Of the 4 exact generators 2 did not appear in the new determining equations and the other 2 canceled out. The 6 generators of the first-order approximate symmetry had to be eliminated for consistency of the new determining equations, (as seen above) making them homogeneous. The resulting system was the same as for the Minkowski spacetime, yielding 12 second-order trivial approximate symmetry generators i.e. 10 KVs and $\partial/\partial s$, $s\partial/\partial s$. Four of them are again the exact symmetry generators used earlier, and hence simply add into them, making no difference. The other 6 replace the lost first-order approximate symmetry generators. The full set has the Poincarè algebra $so(1,3)\oplus_s \mathbb{R}^4$ apart from d_2. In conclusion there are no non-trivial second-order approximate symmetries of the geodesic equations for the RN spacetime as was the case for the first-order approximate symmetries of the Schwarzschild spacetime.

2.2 Energy (Mass) in the RN Spacetime

It is worth remarking that for the first-order approximate symmetries of the geodesic equations for the Schwarzschild spacetime it did not matter whether we used the full system $\mathbf{E}=\mathbf{E_0}+\epsilon\mathbf{E_1}=\mathbf{0}$, or the unperturbed system $\mathbf{E}_0=0$, in the construction of the determining equations. However, for the second-order approximate symmetries it *does* make a difference. One needs to use the full system $\mathbf{E}=\mathbf{E_0}+\epsilon\mathbf{E_1}+\epsilon^2\mathbf{E_2}=\mathbf{0}$, and *not* the unperturbed system, in the determining equations to obtain the solution.

The exact symmetry generators i.e. for the Minkowski spacetime, include not only (1.78) - (1.85), but also the generators of the dilation algebra, $\partial/\partial s$, $s\partial/\partial s$ corresponding to

$$\xi(s)=c_0 s+c_1. \tag{2.179}$$

In the determining equations for the first-order approximate symmetries of the geodesic equations for the Schwarzschild spacetime the terms involving $\xi_s=c_0$, cancel out. However, for the second-order approximate symmetries of the geodesic equations for the RN spacetime the terms involving ξ_s do *not* automatically cancel out but collect a scaling factor of $(1-2k)$ so as to cancel out. (This factor comes from the application of the perturbed system, rather than the unperturbed one in the determining equations). Since energy conservation comes from time translational invariance and ξ is the coefficient of $\partial/\partial s$ in the point transformations given by (1.34), where s is the proper time, the scaling factor $(1-2k)$ corresponds to a *re-scaling* of energy. Thus, whereas there was no energy re-scaling needed for the first-order approximate symmetries of the geodesic equations of the Schwarzschild spacetime, it arises naturally in the second-order approximate symmetries of the RN spacetime. Using (2.3) we get the energy re-scaling factor (taking $G = 1$, $c = 1$)

$$(1 - 2k) = (1 - \frac{Q^2}{2M^2}). \tag{2.180}$$

Thus, even though there are no non-trivial second-order approximate symmetries for the geodesic equations of the RN spacetime, we get the *non-trivial* result of energy re-scaling from the definition of second-order approximate symmetries of ODEs which will be further discussed in chapter 7.

It is worth remarking that when some symmetries are *lost* at one order (exact or first-order approximate) they are *recovered* at the next (at least to second-order) as *trivial* approximate symmetries.

We give the main results of this chapter in the form of the following theorem.

Theorem 2.1. *The energy in the RN spacetime is re-scaled by the factor (2.180).*

Chapter 3

Second-Order Approximate Noether Symmetries for the Kerr Spacetime

In this chapter we study second-order approximate Noether symmetries for the Kerr spacetime. First we consider the Kerr spacetime as a first perturbation of the Schwarzschild spacetime by introducing the spin angular momentum per unit mass as a small parameter ϵ. For a first-order Lagrangian of this perturbed spacetime there does not exist any non-trivial approximate symmetry. Then we consider the Kerr spacetime as a second perturbation of the Minkowski spacetime, taking mass of the order of ϵ and spin angular momentum per unit mass of the order of ϵ^2. The Noether symmetry algebra of the exact case i.e. of the Minkowski spacetime is 17 dimensional which properly contains the conformal conformal Killing vectors (CKVs) of this spacetime [82]. Like the first-order approximate case there is no non-trivial approximate symmetry in the second-order approximation.

3.1 Symmetries and Approximate Symmetries of the Lagrangian for the Kerr Spacetime

We first discuss the exact symmetries of the Lagrangian for the Kerr spacetime.

3.1.1 Exact Noether Symmetries of the Kerr Spacetime

The line element for this spacetime in Boyer-Lindqust coordinates is given by [40]

$$ds^2 = (1 - \frac{2GMr}{\rho^2 c^2})c^2 dt^2 - (\frac{\rho^2}{\Delta})dr^2 - \rho^2 d\theta^2 - \Lambda\frac{\sin^2\theta}{\rho^2}d\phi^2 + (\frac{2GMra\sin^2\theta}{\rho^2 c^2})dt d\phi, \tag{3.1}$$

where

$$\rho^2 = r^2 + \frac{a^2}{c^2}\cos^2\theta,\ \Lambda = (r^2 + \frac{a^2}{c^2})^2 - a^2\Delta\sin^2\theta \text{ and } \Delta = r^2 + \frac{a^2}{c^2} - 2GMr,$$

M is the mass and a the angular momentum per unit mass of the gravitating source. This metric reduces to the Schwarzschild metric when $a = 0$. This spacetime has two KVs, $\partial/\partial t$ and $\partial/\partial\phi$ and a non-trivial Killing tensor [1]. Thus the only conserved quantities are energy and angular momentum for this spacetime.

A Lagrangian for the geodesic equations is in general defined by

$$L[x^a, \dot{x}^a] = g_{ab(x^c)}\frac{dx^a}{ds}\frac{dx^b}{ds}, \tag{3.2}$$

and for the metric given by (3.1) we get

$$L = (1 - \frac{2GMr}{\rho^2 c^2})c^2\dot{t}^2 - \frac{\rho^2}{\Delta}\dot{r}^2 - \rho^2\dot{\theta}^2 - \Lambda\frac{\sin^2\theta}{\rho^2}\dot{\phi}^2 + \frac{2GMra\sin^2\theta}{\rho^2 c^2}\dot{t}\dot{\phi}, \tag{3.3}$$

Using (3.3) in (1.54) we obtain a set of determining equations for 6 unknown functions ξ, η_i $(i = 0, 1, 2, 3)$ and A, where each of these is a function of 5 variables, i.e. s, t, r, θ and ϕ.

$$\xi_t = 0,\ \xi_r = 0,\ \xi_\theta = 0,\ \xi_\phi = 0,\ A_s = 0, \tag{3.4}$$

$$\eta^0_s(\frac{2GMr}{c^2\rho^2} - 1) + \eta^3_s 2a(1 - \frac{GMr}{c^2\rho^2})\sin^2\theta = A_t, \tag{3.5}$$

$$\eta^0_s 2a(1 - \frac{GMr}{c^2\rho^2})\sin^2\theta - \eta^3_s\frac{\Lambda}{\rho^2}\sin^2\theta = A_\phi, \tag{3.6}$$

$$\eta^1_s\frac{\rho^2}{\Delta} = -A_r,\ \eta^2_s\rho^2 = -A_\theta,\ \eta^1_\theta + \Delta\eta^2_r = 0, \tag{3.7}$$

$$\eta^1 r - \eta^2\frac{a^2}{2}\sin 2\theta + \rho^2(\eta^2_\theta - \frac{1}{2}\xi_s) = 0, \tag{3.8}$$

$$\eta^0_r(\frac{2GMr}{c^2\rho^2}-1)-\eta^1_t\frac{\rho^2}{\Delta}+\eta^3_r 2a(1-\frac{GMr}{c^2\rho^2})\sin^2\theta=0, \tag{3.9}$$

$$\eta^0_\theta(\frac{2GMr}{c^2\rho^2}-1)-\eta^2_t\rho^2+\eta^3_\theta 2a(1-\frac{GMr}{c^2\rho^2})\sin^2\theta=0, \tag{3.10}$$

$$\eta^0_r 2a(1-\frac{GMr}{c^2\rho^2})\sin^2\theta-\eta^1_\phi\frac{\rho^2}{\Delta}-\eta^3_r\frac{\Lambda}{\rho^2}\sin^2\theta=0, \tag{3.11}$$

$$\eta^1\frac{1}{\Delta}\{r\Delta-\rho^2(r-\frac{GM}{c^2})\}-\eta^2\frac{a^2}{2}\sin 2\theta+\rho^2(\eta^1_r-\frac{1}{2}\xi_s)=0, \tag{3.12}$$

$$\eta^0_\theta 2a(1-\frac{GMr}{c^2\rho^2})\sin^2\theta-\eta^2_\phi\rho^2-\eta^3_\theta\frac{\Lambda}{\rho^2}\sin^2\theta+\rho^2(\eta^2_\theta-\frac{1}{2}\xi_s)=0, \tag{3.13}$$

$$\begin{aligned}\eta^1\frac{M}{\rho^4}(\rho^2-2r^2)+\eta^2\frac{GMa^2r}{c^2\rho^4}\sin 2\theta+\eta^0_t(\frac{2GMr}{c^2\rho^2}-1)+\eta^3_t 2a(1\\ -\frac{GMr}{c^2\rho^2})\sin^2\theta-\frac{1}{2}\xi_s(\frac{2GMr}{c^2\rho^2}-1)=0,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\eta^1\frac{1}{\rho^4}[\rho^2\{2a(a^2+r^2)-a^2(r-\frac{GM}{c^2})\sin^2\theta\}-r\Lambda]\sin^2\theta+\eta^2\frac{1}{\rho^2}[\Lambda+\frac{a^2}{\rho^2}(\Lambda\\ -\rho^2\Delta)\sin^2\theta]\sin 2\theta-\eta^0_\phi 2a(1-\frac{GMr}{c^2\rho^2})\sin^2\theta+\frac{\Lambda\sin^2\theta}{\rho^2}(\eta^3_\phi-\frac{1}{2}\xi_s)=0,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\eta^1\frac{aGM}{c^2\rho^4}(2r^2-\rho^2)\sin^2\theta-\eta^2\frac{aGMr}{c^2\rho^2}(\frac{a^2}{\rho^2}\sin^2\theta+1)\sin 2\theta+\eta^0_\phi(\frac{2GMr}{c^2\rho^2}-1)\\ +2a(\eta^0_t+\eta^3_\phi)(1-\frac{GMr}{c^2\rho^2})\sin^2\theta-\eta^3_t\frac{\Lambda}{\rho^2}\sin^2\theta-\xi_s 3a(1-\frac{GMr}{c^2\rho^2})\sin^2\theta=0.\end{aligned} \tag{3.16}$$

Solving these equations by the same method used in chapter 2, i.e. back and forth substitution, we get the symmetry generators i.e.

$$\mathbf{Y}_0=\frac{\partial}{\partial t},\ \mathbf{Y}_3=\frac{\partial}{\partial\phi},\ \mathbf{Z}_0=\frac{\partial}{\partial s}\text{ with }A=c\text{ (constant)}. \tag{3.17}$$

Thus here we see that the isometries form a subalgebra of the symmetries of the Lagrangian. With this information, from (1.56) one can obtain the first integrals of the geodesic equations for the Kerr metric.

3.1.2 Approximate Noether Symmetries of the Kerr Spacetime

In this section we investigate the approximate symmetries of the Kerr spacetime in two different ways. First we consider the Kerr spacetime as a first perturbation of the Schwarzschild spacetime and then we take it as a second perturbation of the Minkowski spacetime.

Kerr Spacetime as a First Perturbation of the Schwarzschild Spacetime

For the first-order approximate symmetries of a Lagrangian for the Kerr metric we introduce the spin angular momentum per unit mass

$$a = \epsilon, \tag{3.18}$$

as a small parameter. In this case the first-order perturbed Lagrangian is given by

$$L = (1 - \frac{2GM}{rc^2})c^2\dot{t}^2 - (1 - \frac{2GM}{rc^2})^{-1}c^2\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \epsilon\frac{2GM}{rc^2}\sin^2\theta\dot{t}\dot{\phi}. \tag{3.19}$$

For $\epsilon = 0$, the above Lagrangian reduces to that of the Schwarzschild spacetime. For the exact (unperturbed) Schwarzschild spacetime the Noether symmetry algebra is 5 dimensional, given by $so(3) \oplus \mathbb{R} \oplus d_1$ with symmetry generators (3.20) and (3.21), which properly contains the isometry algebra, and the gauge function is just a constant [83]. Using the Lagrangian (3.19) and the exact Noether symmetries of the Schwarzschild spacetime

$$\mathbf{Y}_0 = \frac{\partial}{\partial t},\ \mathbf{Y}_1 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \tag{3.20}$$

$$\mathbf{Y}_2 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi},\ \mathbf{Y}_3 = \frac{\partial}{\partial\phi},\ \mathbf{Z}_0 = \frac{\partial}{\partial s}, \tag{3.21}$$

in the first-order approximate Noether symmetry conditions (1.63) we obtain the following set of determining equations

$$\xi_t = 0,\ \xi_r = 0,\ \xi_\theta = 0,\ \xi_\phi = 0,\ A_s = 0, \tag{3.22}$$

$$2\eta^0_s(1 - \frac{2GM}{rc^2}) = A_t,\ 2(\frac{rc^2}{rc^2 - 2GM})\eta^1_s = -A_r, \tag{3.23}$$

$$2\eta^2_s r^2 = -A_\theta,\ 2\eta^3_s r^2\sin^2\theta = -A_\phi, \tag{3.24}$$

$$2\eta^1 + 2r\eta^2_\theta - r\xi_s = 0,\ \eta^0_r(1 - \frac{2GM}{rc^2})^2 - \eta^1_t = 0, \tag{3.25}$$

$$\eta^1_\theta(\frac{rc^2}{rc^2 - 2GM}) + \eta^3_r r^2\sin^2\theta = 0,\ \eta^2_\phi + \eta^3_\theta\sin^2\theta = 0, \tag{3.26}$$

$$\eta^1_\theta(\frac{rc^2}{rc^2 - 2GM}) + \eta^2_r r^2 = 0,\ \eta^1(\frac{2GM}{rc^2 - 2GM}) - r(2\eta^1_r + \xi_s) = 0, \tag{3.27}$$

$$\eta^1\frac{2GM}{rc^2}+(1-\frac{2GM}{rc^2})(2\eta^0_t-\xi_s)=0, \tag{3.28}$$

$$2\eta^2 r\cos\theta+(2\eta^1+2r\eta^3_\phi-r\xi_s)\sin\theta=0, \tag{3.29}$$

$$\eta^0_\theta(1-\frac{2GM}{rc^2})-\eta^2_t r^2+\frac{GM}{rc^2}(c_1\sin\phi-c_2\cos\phi)\sin^2\theta=0, \tag{3.30}$$

$$\eta^0_\phi(1-\frac{2GM}{rc^2})-\eta^3_t r^2\sin^2\theta+\frac{GM}{2rc^2}(c_1\cos\phi+c_2\sin\phi)\sin 2\theta=0. \tag{3.31}$$

In the above equations (3.30) and (3.31) two constants c_1 and c_2 corresponding to the exact Noether symmetry generators of the Schwarzschild spacetime appear.

Equation (3.22) yields

$$\xi=f_1(s),\ \ A=A(t,r,\theta,\phi). \tag{3.32}$$

From (3.23) and (3.24) we obtain

$$\eta^0 = sf_2(t,r,\theta,\phi)+f_3(t,r,\theta,\phi), \tag{3.33}$$

$$\eta^1 = sf_4(t,r,\theta,\phi)+f_5(t,r,\theta,\phi), \tag{3.34}$$

$$\eta^2 = sf_6(t,r,\theta,\phi)+f_7(t,r,\theta,\phi), \tag{3.35}$$

$$\eta^3 = sf_8(t,r,\theta,\phi)+f_9(t,r,\theta,\phi). \tag{3.36}$$

Use of (3.34) in the second of (3.27) and then differentiation twice with respect to s gives

$$\xi_{sss}=0. \tag{3.37}$$

On integration (3.37) yields

$$\xi=\frac{1}{2}b_0 s^2+b_1 s+b_2. \tag{3.38}$$

Use of (3.33), (3.34) and (3.38) in (3.28) and then separation by the powers of s gives

$$\frac{2M}{r^2}f_4(t,r,\theta,\phi)+(1-\frac{2GM}{rc^2})[2f_{2_t}(t,r,\theta,\phi)-b_0] = 0, \tag{3.39}$$

$$\frac{2M}{r^2}f_5(t,r,\theta,\phi)+(1-\frac{2GM}{rc^2})[2f_{3_t}(t,r,\theta,\phi)-b_1] = 0. \tag{3.40}$$

Differentiate (3.33) with respect to s, substitute it in the first of (3.23) and then integrate the resulting equation with respect to t, we determine

$$A = 2(1 - \frac{2GM}{rc^2}) \int f_2(t, r, \theta, \phi)dt + f_{10}(r, \theta, \phi). \quad (3.41)$$

Use of (3.34) and (3.41) in the second of (3.23) gives

$$\begin{aligned} f_4(t, r, \theta, \phi) &= -\frac{1}{2}(1 - \frac{2GM}{rc^2})[\frac{4GM}{r^2c^2} \int f_2(t, r, \theta, \phi)dt + \\ & 2(1 - \frac{2GM}{rc^2}) \int f_{2_r}(t, r, \theta, \phi)dt + f_{10_r}(r, \theta, \phi)]. \end{aligned} \quad (3.42)$$

Use of (3.42) in (3.39) and then differentiate with respect to t we obtain

$$(1 - \frac{2GM}{rc^2})f_{2_r}(t, r, \theta, \phi) + (\frac{2G^2M^2}{r^4c^4} - 1)f_2(t, r, \theta, \phi) = 0. \quad (3.43)$$

The use of (3.35) and (3.41), in the first of (3.24) and the use of (3.36) and (3.41), in the second of (3.24) yields

$$f_6(t, r, \theta, \phi) = -\frac{1}{r^2}[(1 - \frac{2GM}{rc^2}) \int f_{2_\theta}(t, r, \theta, \phi)dt + \frac{1}{2}f_{10_\theta}(r, \theta, \phi)], \quad (3.44)$$

$$f_8(t, r, \theta, \phi) = -\frac{\csc^2\theta}{r^2}[(1 - \frac{2GM}{rc^2}) \int f_{2_\phi}(t, r, \theta, \phi)dt + \frac{1}{2}f_{10_\phi}(r, \theta, \phi)]. \quad (3.45)$$

Therefore

$$\eta^2 = -\frac{s}{r^2}[(1 - \frac{2GM}{rc^2}) \int f_{2_\theta}(t, r, \theta, \phi)dt + \frac{1}{2}f_{10_\theta}(r, \theta, \phi)] + f_7(t, r, \theta, \phi), \quad (3.46)$$

$$\eta^3 = -\frac{s\csc^2\theta}{r^2}[(1 - \frac{2GM}{rc^2}) \int f_{2_\phi}(t, r, \theta, \phi)dt + \frac{1}{2}f_{10_\phi}(r, \theta, \phi)] + f_9(t, r, \theta, \phi). \quad (3.47)$$

Use (3.33) and (3.34) in the second of (3.25) and then separate by the powers of s we deduce

$$f_2(t, r, \theta, \phi) = 0. \quad (3.48)$$

With the use of (3.48) from (3.39) we obtain

$$f_4(t, r, \theta, \phi) = \frac{r^2}{2M}(1 - \frac{2GM}{rc^2})b_0. \quad (3.49)$$

Thus

$$\eta^0 = f_3(t, r, \theta, \phi), \tag{3.50}$$
$$\eta^1 = \frac{sr^2}{2M}(1 - \frac{2GM}{rc^2})b_0 + f_5(t, r, \theta, \phi), \tag{3.51}$$
$$\eta^2 = \frac{-s}{2r^2} f_{10_\theta}(r, \theta, \phi) + f_7(t, r, \theta, \phi), \tag{3.52}$$
$$\eta^3 = \frac{-s \csc^2\theta}{2r^2} f_{10_\phi}(r, \theta, \phi) + f_9(t, r, \theta, \phi), \tag{3.53}$$
$$A = f_{10}(r, \theta, \phi). \tag{3.54}$$

We differentiate (3.51) with respect to s and (3.54) with respect to r and substitute them in the second of (3.23), we determine

$$f_{10_r}(r, \theta, \phi) = \frac{-r^2}{M} b_0. \tag{3.55}$$

Equation (3.55) on integration with respect to r gives

$$f_{10}(r, \theta, \phi) = -\frac{r^3}{3M} b_0 + g_1(\theta, \phi). \tag{3.56}$$

First we substitute (3.56) in (3.52). Then differentiate (3.51) with respect to θ and (3.52) with respect to r and substitute them in the first of (3.27) we obtain

$$(\frac{rc^2}{rc^2 - 2GM}) f_{5_\theta}(t, r, \theta, \phi) + \frac{s}{r^2} g_{1_\theta}(\theta, \phi) + r f_{7_r}(t, r, \theta, \phi) = 0. \tag{3.57}$$

Separation of (3.57) for the different powers of s yields

$$(\frac{rc^2}{rc^2 - 2GM}) f_{5_\theta}(t, r, \theta, \phi) + r f_{7_r}(t, r, \theta, \phi) = 0, \tag{3.58}$$
$$g_1(\theta, \phi) = g_2(\phi). \tag{3.59}$$

Therefore

$$\eta^2 = f_7(t, r, \theta, \phi), \tag{3.60}$$
$$\eta^3 = -\frac{s \csc^2\theta}{2r^2} g_{2_\phi}(\phi) + f_9(t, r, \theta, \phi), \tag{3.61}$$

$$A = -\frac{r^3}{3M}b_0 + g_2(\phi). \tag{3.62}$$

We use (3.60) and (3.61) in the second of (3.26) and then separate by the powers of s we obtain

$$f_{7_\phi}(t,r,\theta,\phi) + \sin^2\theta f_{9_\theta}(t,r,\theta,\phi) = 0, \tag{3.63}$$

$$g_2(\phi) = b_3. \tag{3.64}$$

Therefore

$$\eta^3 = f_9(t,r,\theta,\phi), \tag{3.65}$$

$$A = b_3 - \frac{r^3}{3M}b_0. \tag{3.66}$$

We differentiate (3.38) with respect to s and (3.51) with respect to r and use them in the second of (3.27). Then separate by the powers of s we get

$$b_0 = 0. \tag{3.67}$$

Differentiate (3.50) with respect to r and (3.51) with respect to t and then the use of them in the second of (3.25) yields

$$(1 - \frac{2GM}{rc^2})^2 f_{3_r}(t,r,\theta,\phi) - f_{5_t}(t,r,\theta,\phi) = 0. \tag{3.68}$$

Integration of (3.68) with respect to t gives

$$f_5(t,r,\theta,\phi) = (1 - \frac{2GM}{rc^2})^2 \int f_{3_r}(t,r,\theta,\phi)dt + g_4(r,\theta,\phi). \tag{3.69}$$

Thus

$$\xi = sb_1 + b_2, \tag{3.70}$$

$$\eta^1 = (1 - \frac{2GM}{rc^2})^2 \int f_{3_r}(t,r,\theta,\phi)dt + g_4(r,\theta,\phi), \tag{3.71}$$

$$A = b_3. \tag{3.72}$$

We differentiate (3.50) with respect to θ and (3.60) with respect to t. Substitution of these in (3.30) yields

$$(1-\frac{2GM}{rc^2})f_{3_\theta}(t,r,\theta,\phi)-r^2f_{7_t}(t,r,\theta,\phi)+\frac{GM}{rc^2}(c_1\sin\phi-c_2\cos\phi)\sin^2\theta=0. \tag{3.73}$$

Integration of (3.73) with respect to t yields

$$\begin{aligned}\eta^2 \quad = \quad & \frac{1}{r^2}(1-\frac{2GM}{rc^2})\int f_{3_\theta}(t,r,\theta,\phi)dt+\frac{GMt}{r^3c^2}(c_1\sin\phi-c_2\cos\phi)\sin^2\theta \\ & +g_5(r,\theta,\phi).\end{aligned} \tag{3.74}$$

Similarly from (3.31) we obtain

$$\begin{aligned}\eta^3 \quad = \quad & \frac{\csc^2\theta}{r^2}(1-\frac{2GM}{rc^2})\int f_{3_\phi}(t,r,\theta,\phi)dt+\frac{GMt}{r^3c^2}(c_1\cos\phi+c_2\sin\phi)\cot\theta \\ & +g_6(r,\theta,\phi).\end{aligned} \tag{3.75}$$

Differentiation of (3.74) with respect to θ and of (3.75) with respect to ϕ and then the use of them in (3.26) gives

$$c_1\cos\phi+c_2\sin\phi=0, \tag{3.76}$$

this yields

$$c_1=0 \text{ and } c_2=0. \tag{3.77}$$

This makes the system of the determining equations (3.23) - (3.31) homogeneous. Hence there is no non-trivial first-order approximate symmetry for this case. We only recover the 5 exact symmetry generators given by (3.20) and (3.21), as trivial first-order approximate Noether symmetry generators for this perturbed Kerr spacetime.

Kerr Spacetime as a Second Perturbation of the Minkowski Spacetime

Now we consider the Kerr spacetime as a second perturbation of the Minkowski spacetime. For this we use the same small parameter ϵ, defined for the approximate Schwarzschild spacetime in chapter 1 and

$$a=k_1\epsilon. \tag{3.78}$$

For the Kerr black hole (see for example [84]) we have $0 < k_1 \leq 1/4$. Here the second-order perturbed Lagrangian is given by

$$\begin{aligned} L &= \dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 - \frac{2}{r}\epsilon(\dot{t}^2 + \dot{r}^2) - \epsilon^2[\frac{1}{r^2}(1 - \frac{k_1^2}{4}\sin^2\theta)\dot{r}^2 \\ &\quad + k_1^2\cos^2\theta\dot{\theta}^2 + k_1^2\sin^2\theta\dot{\phi}^2 - \frac{\sqrt{k_1}}{r}\sin^2\theta\dot{t}\dot{\phi}] + O(\epsilon^3). \end{aligned} \tag{3.79}$$

For the exact case we take $\epsilon = 0$, which means that there is no mass or angular momentum per unit mass. In this case the Lagrangian (3.79) reduces to that of the Minkowski spacetime.

Noether Symmetries of the Minkowski Spacetime

Here we calculate the Noether symmetries of the exact case, i.e. of the Minkowski spacetime. Using (1.54) we get the following set of 19 determining equations,

$$\xi_t = 0,\ \xi_r = 0,\ \xi_\theta = 0,\ \xi_\phi = 0,\ A_s = 0, \tag{3.80}$$

$$2\eta^0_s = A_t,\ -2\eta^1_s = A_r,\ -2r^2\eta^2_s = A_\theta,\ -2r^2\sin^2\theta\eta^3_s = A_\phi \tag{3.81}$$

$$2\eta^0_t - \xi_s = 0,\ 2\eta^1_r - \xi_s = 0,\ \eta^0_r - \eta^1_t = 0,\ \eta^0_\theta - r^2\eta^2_t = 0, \tag{3.82}$$

$$\eta^0_\phi - r^2\sin^2\theta\eta^3_t = 0,\ \eta^1_\theta + r^2\eta^2_r = 0,\ \eta^1_\phi + r^2\sin^2\theta\eta^3_r = 0, \tag{3.83}$$

$$\eta^2_\phi + \sin^2\theta\eta^3_\theta = 0,\ 2(\eta^1 + r\eta^2_\theta) - r\xi_s = 0, \tag{3.84}$$

$$2\eta^1\sin\theta + 2r\eta^2\cos\theta + 2r\eta^3_\phi\sin\theta - r\sin\theta\xi_s = 0. \tag{3.85}$$

Equation (3.80) yields

$$\xi = f_1(s),\ A = A(t, r, \theta, \phi). \tag{3.86}$$

On integration with respect to t the first of (3.82) gives

$$\eta^0 = \frac{t}{2}f_{1_s}(s) + f_2(s, r, \theta, \phi). \tag{3.87}$$

We use (3.87) in the first of (3.81) and differentiate with respect to s, we determine

$$f_{1_{sss}}(s) = 0,\ f_{2_{ss}}(s, r, \theta, \phi) = 0. \tag{3.88}$$

Integration of (3.88) gives

$$f_1(s) = \xi = \frac{1}{2}c_0 s^2 + c_1 s + c_2 \tag{3.89}$$

and

$$f_2(s, r, \theta, \phi) = s f_3(r, \theta, \phi) + f_4(r, \theta, \phi). \tag{3.90}$$

Therefore

$$\eta^0 = \frac{t}{2}(c_0 s + c_1) + s f_3(r, \theta, \phi) + f_4(r, \theta, \phi). \tag{3.91}$$

Use of (3.91) in the first of (3.81) gives

$$A = \frac{c_0}{2}t^2 + 2t f_3(r, \theta, \phi) + f_5(r, \theta, \phi). \tag{3.92}$$

Differentiation of (3.89) with respect to s and substitution in the second of (3.82) yields

$$\eta^1_r = \frac{1}{2}(c_0 s + c_1). \tag{3.93}$$

Integration of (3.93) with respect to r determine

$$\eta^1 = \frac{r}{2}(c_0 s + c_1) + f_6(s, t, \theta, \phi). \tag{3.94}$$

Use of (3.92) and (3.94) in the second of (3.81) yields

$$f_{6_{ss}}(s, t, \theta, \phi) = 0. \tag{3.95}$$

Integration of (3.95) gives

$$f_6(s, t, \theta, \phi) = s f_7(t, \theta, \phi) + f_8(t, \theta, \phi). \tag{3.96}$$

Thus

$$\eta^1 = \frac{r}{2}(c_0 s + c_1) + s f_7(t, \theta, \phi) + f_8(t, \theta, \phi). \tag{3.97}$$

Differentiation of (3.91) with respect to r and of (3.97) with respect to t and then the insertion of these in the third of (3.82) yields

$$f_7(t,\theta,\phi) = tf_9(\theta,\phi) + f_{10}(\theta,\phi), \tag{3.98}$$

$$f_3(r,\theta,\phi) = -rf_9(\theta,\phi) + g_1(\theta,\phi). \tag{3.99}$$

We use (3.98) in (3.97) and (3.99) in (3.92). Then the substitution of these in the second of (3.81) yields

$$f_5(r,\theta,\phi) = -\frac{c_0}{2}r^2 - 2rf_{10}(\theta,\phi) + g_2(\theta,\phi).$$

Therefore, (3.92), (3.91) and (3.97) become

$$A = \frac{c_0}{2}(t^2 - r^2) + 2t[g_1(\theta,\phi) - rf_9(\theta,\phi)] + g_2(\theta,\phi) - 2rf_{10}(\theta,\phi), \tag{3.100}$$

$$\eta^0 = \frac{t}{2}(c_0 s + c_1) + s[g_1(\theta,\phi) - rf_9(\theta,\phi)] + f_4(r,\theta,\phi), \tag{3.101}$$

$$\eta^1 = \frac{r}{2}(c_0 s + c_1) + s[tf_9(\theta,\phi) + f_{10}(\theta,\phi)] + f_8(t,\theta,\phi). \tag{3.102}$$

From (3.89), (3.102) and the second of (3.84) we obtain

$$\eta^2_\theta = -\frac{1}{r}[s\{tf_9(\theta,\phi) + f_{10}(\theta,\phi)\} + f_8(t,\theta,\phi)]. \tag{3.103}$$

Integration of (3.103) with respect to θ, gives

$$\eta^2 = -\frac{1}{r}\int [s\{tf_9(\theta,\phi) + f_{10}(\theta,\phi)\} + f_8(t,\theta,\phi)]d\theta + g_3(s,t,r,\phi). \tag{3.104}$$

Differentiate (3.100) with respect to θ and (3.104) with respect to s, then from the substitution of them in the third of (3.81) we have

$$g_{3_{ss}}(s,t,r,\phi) = 0. \tag{3.105}$$

Integration of (3.105) yields

$$g_3(s,t,r,\phi) = sg_4(t,r,\phi) + g_5(t,r,\phi). \tag{3.106}$$

Insert (3.104) along with (3.106) in the third of (3.81) and then differentiate it with respect to t we obtain

$$g_{4_{tt}}(t,r,\phi)=0. \qquad (3.107)$$

On integration (3.107) yields

$$g_4(t,r,\phi)=tg_6(r,\phi)+g_7(r,\phi). \qquad (3.108)$$

We put (3.108) in (3.106) and then (3.106) in (3.104). Then we differentiate (3.104) with respect to s and (3.100) with respect to θ. With the use of these from the third (3.81) we have

$$f_{9_{\theta\theta}}(\theta,\phi)+f_9(\theta,\phi) = 0, \qquad (3.109)$$

$$g_{1_{\theta\theta}}(\theta,\phi) = 0. \qquad (3.110)$$

Integration of (3.110) gives

$$g_1(\theta,\phi)=\theta g_8(\phi)+g_9(\phi). \qquad (3.111)$$

Again from the third of (3.81) we obtain

$$[r^2 g_6(r,\phi)]_{rr}=0. \qquad (3.112)$$

Integration of (3.112) yields

$$g_6(r,\phi)=\frac{1}{r}g_{10}(\phi)+\frac{1}{r^2}h_1(\phi). \qquad (3.113)$$

Substitute (3.113) in the same equation we obtain

$$h_1(\phi)=-g_8(\phi). \qquad (3.114)$$

Differentiation of (3.101) with respect to r and of (3.102) with respect to t and substitution of them in the third of (3.82) gives

$$f_9(\theta,\phi) = 0, \qquad (3.115)$$

$$f_{8_{tt}}(t,\theta,\phi) = 0. \qquad (3.116)$$

Integration of (3.116) yields

$$f_8(t,\theta,\phi) = th_2(\theta,\phi) + h_3(\theta,\phi). \tag{3.117}$$

From the use of (3.117) in (3.102) and then from the third of (3.82) we get

$$f_4(r,\theta,\phi) = rh_2(\theta,\phi) + h_4(\theta,\phi). \tag{3.118}$$

Now (3.100), (3.101), (3.102) and (3.104) become

$$A = \frac{c_0}{2}(t^2 - r^2) + 2t[\theta g_8(\phi) + g_9(\phi)] + g_2(\theta,\phi) - 2rf_{10}(\theta,\phi), \tag{3.119}$$

$$\eta^0 = \frac{t}{2}(c_0 s + c_1) + s[\theta g_8(\phi) + g_9(\phi)] + rh_2(\theta,\phi) + h_4(\theta,\phi), \tag{3.120}$$

$$\eta^1 = \frac{r}{2}(c_0 s + c_1) + sf_{10}(\theta,\phi) + th_2(\theta,\phi) + h_3(\theta,\phi), \tag{3.121}$$

$$\eta^2 = -\frac{1}{r}\int[sf_{10}(\theta,\phi) + th_2(\theta,\phi) + h_3(\theta,\phi)]d\theta + s[t\{\frac{1}{r}g_{10}(\phi) - \frac{1}{r^2}g_8(\phi)\} + g_7(r,\phi)] + g_5(t,r,\phi). \tag{3.122}$$

We differentiate (3.120) with respect to θ and (3.122) with respect to t, substitute these in the fourth of (3.82). Then differentiate the resulting equation with respect to s we obtain

$$g_8(\phi) = 0, \text{ and } g_{10}(\phi) = 0. \tag{3.123}$$

Use of (3.123) back in the fourth of (3.82) and then differentiation of this with respect to t we get

$$g_{5_{tt}}(t,r,\phi) = 0. \tag{3.124}$$

On integration (3.124) yields

$$g_5(t,r,\phi) = th_5(r,\phi) + h_6(r,\phi). \tag{3.125}$$

Substitute (3.125), in the fourth of (3.82) and differentiate twice with respect to r, we obtain

$$\begin{aligned} h_{2_{\theta\theta}}(\theta,\phi) + h_2(\theta,\phi) &= 0, && (3.126)\\ [r^2 h_5(r,\phi)]_{rr} &= 0. && (3.127)\end{aligned}$$

Integration of (3.127) gives

$$h_5(r,\phi) = \frac{1}{r}h_7(\phi) + \frac{1}{r^2}h_8(\phi). \quad (3.128)$$

Use of (3.128) in (3.122) and then use of this and (3.120) in fourth of (3.82) yields

$$h_4(\theta,\phi) = \theta h_8(\phi) + h_9(\phi). \quad (3.129)$$

Therefore (3.199), (3.120) and (3.122) become

$$\begin{aligned} A &= \frac{c_0}{2}(t^2 - r^2) + 2tg_9(\phi) + g_2(\theta,\phi) - 2rf_{10}(\theta,\phi), && (3.130)\\ \eta^0 &= \frac{t}{2}(c_0 s + c_1) + sg_9(\phi) + rh_2(\theta,\phi) + \theta h_8(\phi) + h_9(\phi), && (3.131)\\ \eta^2 &= -\frac{1}{r}\int [sf_{10}(\theta,\phi) + th_2(\theta,\phi) + h_3(\theta,\phi)]d\theta + sg_7(r,\phi) + \frac{t}{r}[h_7(\phi)\\ &\quad + \frac{1}{r}h_8(\phi)] + h_6(r,\phi). && (3.132)\end{aligned}$$

We differentiate (3.121) with respect to θ and (3.132) with respect to r. Then we put them in the second of (3.83). From the differentiation of the resulting equation with respect to θ, we have

$$f_{10_{\theta\theta}}(\theta,\phi) + f_{10}(\theta,\phi) = 0, \quad (3.133)$$

$$[g_7(r,\phi)]_r = 0, \ \Rightarrow \ g_7(r,\phi) = p_1(\phi). \quad (3.134)$$

Use of (3.122) in the second of (3.83) with (3.128) and then differentiation of the resulting equation once with respect to t and then with respect to r, determine

$$h_8(\phi) = 0. \quad (3.135)$$

Now again the second of (3.83) yields

$$h_{3_{\theta\theta}}(\theta,\phi) + h_3(\theta,\phi) = 0, \tag{3.136}$$

$$[r^2 h_{6_r}(r,\phi)]_r = 0. \tag{3.137}$$

Integration of (3.137) gives

$$h_6(r,\phi) = \frac{-1}{r} h_{10}(\phi) + p_2(\phi). \tag{3.138}$$

Therefore (3.132) becomes

$$\eta^2 = -\frac{1}{r}\int [sf_{10}(\theta,\phi) + th_2(\theta,\phi) + h_3(\theta,\phi)]d\theta + sp_1(\phi) + \frac{1}{r}[th_7(\phi) - h_{10}(\phi)] + p_2(\phi). \tag{3.139}$$

We use (3.130) and (3.139) in the third of (3.81) which gives

$$g_{2_{\theta\theta}}(\theta,\phi) = 0, \tag{3.140}$$

$$p_1(\phi) = 0. \tag{3.141}$$

On integration (3.140) yields

$$g_2(\theta,\phi) = \theta p_3(\phi) + p_4(\phi). \tag{3.142}$$

Now with the use of (3.130) and (3.139) the third of (3.81) gives

$$p_3(\phi) = 0. \tag{3.143}$$

Use of (3.131) and (3.139) in fourth of (3.82) deduce

$$h_8(\phi) = 0. \tag{3.144}$$

Thus

$$A = \frac{c_0}{2}(t^2 - r^2) + 2tg_9(\phi) + p_4(\phi) - 2rf_{10}(\theta,\phi), \tag{3.145}$$

$$\eta^0 = \frac{t}{2}(c_0 s + c_1) + s g_9(\phi) + r h_2(\theta,\phi) + h_9(\phi), \tag{3.146}$$

$$\eta^2 = -\frac{1}{r}\int [s f_{10}(\theta,\phi) + t h_2(\theta,\phi) + h_3(\theta,\phi)]d\theta + \frac{1}{r}[t h_7(\phi) - h_{10}(\phi)] + p_2(\phi). \tag{3.147}$$

Differentiation of (3.121) with respect to θ and (3.147) with respect to r, then use of them in the second of (3.83) yields

$$h_7(\phi) = 0, \;\; h_{10}(\phi) = 0. \tag{3.148}$$

Therefore (3.147) becomes

$$\eta^2 = -\frac{1}{r}\int [s f_{10}(\theta,\phi) + t h_2(\theta,\phi) + h_3(\theta,\phi)]d\theta + p_2(\phi). \tag{3.149}$$

Use of (3.145) in the fourth of (3.81) gives

$$\eta^3 = -\frac{s}{2r^2}\csc^2\theta[2t g_{9_\phi}(\phi) + p_{4_\phi}(\phi) - 2r f_{10_\phi}(\theta,\phi)] + p_5(t,r,\theta,\phi). \tag{3.150}$$

From (3.146), (3.150) and the first of (3.83) we have

$$g_{9_\phi}(\phi) = 0, \;\; \Rightarrow \;\; g_9(\phi) = c_3, \tag{3.151}$$

$$p_{5_{tt}}(t,r,\theta,\phi) = 0. \tag{3.152}$$

Integration of (3.152) yields

$$p_5(t,r,\theta,\phi) = t p_6(r,\theta,\phi) + p_7(r,\theta,\phi). \tag{3.153}$$

From the use of (3.153) in the first of (3.83) we obtain

$$p_6(r,\theta,\phi) = \frac{1}{r}[h_{2_\phi}(\theta,\phi) + \frac{1}{r}h_{9_\phi}(\phi)]\csc^2\theta. \tag{3.154}$$

We use (3.151) in (3.120) and (3.153), (3.154) in (3.150) and then from the third of (3.83) we obtain

$$p_{4_\phi}(\phi) = 0, \;\; \Rightarrow \;\; p_4(\phi) = c_4. \tag{3.155}$$

Substitution of (3.155) in the first of (3.83) gives

$$h_{9_\phi}(\phi) = 0, \quad \Rightarrow \quad h_9(\phi) = c_5 \tag{3.156}$$

and

$$p_7(r,\theta,\phi) = \frac{1}{r} h_{3_\phi}(\theta,\phi)\csc^2\theta + p_8(\theta,\phi). \tag{3.157}$$

Thus

$$\begin{aligned} A &= \frac{c_0}{2}(t^2 - r^2) - 2rf_{10}(\theta,\phi) + 2tc_3 + c_4, & (3.158)\\ \eta^0 &= \frac{t}{2}(c_0 s + c_1) + rh_2(\theta,\phi) + sc_3 + c_5, & (3.159)\\ \eta^3 &= \frac{1}{r}[sf_{10_\phi}(\theta,\phi) + h_{3_\phi}(\theta,\phi) + th_{2_\phi}(\theta,\phi)]\csc^2\theta + p_8(\theta,\phi). & (3.160) \end{aligned}$$

We differentiate (3.147) with respect to ϕ and (3.160) with respect to θ. Then use them in the first of (3.84) we obtain

$$\begin{aligned} \int f_{10_\phi}(\theta,\phi)d\theta + 2f_{10_\phi}(\theta,\phi)\cot\theta - f_{10_{\theta\phi}}(\theta,\phi) &= 0, & (3.161)\\ \int h_{2_\phi}(\theta,\phi)d\theta + 2h_{2_\phi}(\theta,\phi)\cot\theta - h_{2_{\theta\phi}}(\theta,\phi) &= 0, & (3.162)\\ \int h_{3_\phi}(\theta,\phi)d\theta + 2h_{3_\phi}(\theta,\phi)\cot\theta - h_{3_{\theta\phi}}(\theta,\phi) &= 0 & (3.163) \end{aligned}$$

and

$$p_8(\theta,\phi) = p_{2\phi}(\phi)\cot\theta + p_9(\phi). \tag{3.164}$$

Therefore (3.160) becomes

$$\eta^3 = \frac{1}{r}[sf_{10_\phi}(\theta,\phi) + h_{3_\phi}(\theta,\phi) + th_{2_\phi}(\theta,\phi)]\csc^2\theta + p_{2\phi}(\phi)\cot\theta + p_9(\phi). \tag{3.165}$$

From (3.85) we have

$$\int f_{10}(\theta,\phi)d\theta - f_{10}(\theta,\phi)\tan\theta - f_{10_{\phi\phi}}(\theta,\phi)\sec\theta\csc\theta = 0, \tag{3.166}$$

$$\int h_2(\theta,\phi)d\theta - h_2(\theta,\phi)\tan\theta - h_{2_{\phi\phi}}(\theta,\phi)\sec\theta\csc\theta = 0, \quad (3.167)$$

$$\int h_3(\theta,\phi)d\theta - h_3(\theta,\phi)\tan\theta - h_{3_{\phi\phi}}(\theta,\phi)\sec\theta\csc\theta = 0 \quad (3.168)$$

and

$$p_{2_{\phi\phi}}(\phi) + p_2(\phi) + p_{9_\phi}(\phi)\tan\theta = 0. \quad (3.169)$$

Equation (3.169) yields

$$p_{9_\phi}(\phi) = 0 \Rightarrow p_9 = c_6. \quad (3.170)$$

Use of (3.170) in (3.169) gives

$$p_2(\phi) = c_7\cos\phi + c_8\sin\phi. \quad (3.171)$$

Now solve (3.126), (3.133) and (3.136) along with (3.161) - (3.163) and (3.166) - (3.108) we obtain

$$h_2(\theta,\phi) = c_9\sin\theta\cos\phi + c_{10}\sin\theta\sin\phi + c_{11}\cos\theta, \quad (3.172)$$

$$f_{10}(\theta,\phi) = c_{15}\sin\theta\cos\phi + c_{16}\sin\theta\sin\phi + c_{17}\cos\theta, \quad (3.173)$$

$$h_3(\theta,\phi) = c_{12}\sin\theta\cos\phi + c_{13}\sin\theta\sin\phi + c_{14}\cos\theta, \quad (3.174)$$

Thus

$$A = \frac{c_0}{2}(t^2 - r^2) - 2r(c_{15}\sin\theta\cos\phi + c_{16}\sin\theta\sin\phi + c_{17}\cos\theta) + 2tc_3 + c_4, \quad (3.175)$$

$$\xi = \frac{1}{2}c_0 s^2 + c_1 s + c_2, \quad (3.176)$$

$$\eta^0 = \frac{t}{2}(c_0 s + c_1) + r(c_9\sin\theta\cos\phi + c_{10}\sin\theta\sin\phi + c_{11}\cos\theta) + sc_3 + c_5, \quad (3.177)$$

$$\eta^1 = \frac{r}{2}(c_0 s + c_1) + s(c_{15}\sin\theta\cos\phi + c_{16}\sin\theta\sin\phi + c_{17}\cos\theta) + t(c_9\sin\theta\cos\phi + c_{10}\sin\theta\sin\phi + c_{11}\cos\theta) + (c_{12}\cos\phi + c_{13}\sin\phi)\sin\theta + c_{14}\cos\theta, \quad (3.178)$$

$$
\begin{aligned}
\eta^2 &= \frac{1}{r}[s(c_{15}\cos\theta\cos\phi + c_{16}\cos\theta\sin\phi - c_{17}\sin\theta) + t(c_9\cos\theta\cos\phi + \\
&\quad c_{10}\cos\theta\sin\phi - c_{11}\sin\theta) + c_{12}\sin\theta\cos\phi + c_{13}\sin\theta\sin\phi + c_{14}\cos\theta] \\
&\quad + c_7\cos\phi + c_8\sin\phi, \qquad (3.179) \\
\eta^3 &= \frac{1}{r}[s(c_{16}\sin\theta\cos\phi - c_{15}\sin\theta\sin\phi) + t(c_{10}\sin\theta\cos\phi - c_9\sin\theta\sin\phi) \\
&\quad + c_{13}\sin\theta\cos\phi - c_{12}\sin\theta\sin\phi]\csc^2\theta + c_8\cos\phi - c_7\sin\phi + c_6. \qquad (3.180)
\end{aligned}
$$

Here 17 constants of integration appear in (3.175) - (3.180). Thus the Noether symmetries of the Minkowski spacetime form a 17 dimensional Lie algebra. The symmetry generators apart from the 10 KVs (Poincarè algebra $so(1,3) \oplus_s \mathbb{R}^4$) given in (1.78) - (1.85) are

$$
\mathbf{Z}_0 = \frac{\partial}{\partial s}, \; \mathbf{Z}_1 = s\frac{\partial}{\partial s} + \frac{1}{2}(t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}), \; \mathbf{Z}_2 = s\mathbf{Y}_0, \qquad (3.181)
$$

$$
\mathbf{Z}_3 = \frac{1}{2}[s^2\frac{\partial}{\partial s} + s(t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r})], \; \mathbf{Z}_4 = s\mathbf{Y}_7, \; \mathbf{Z}_5 = s\mathbf{Y}_8, \; \mathbf{Z}_6 = s\mathbf{Y}_9. \qquad (3.182)
$$

The $\mathbf{Z}_0$ is translation in s, $\mathbf{Z}_1$ is scaling symmetry in s, t, r and $[\mathbf{Z}_0, \mathbf{Z}_3] = \mathbf{Z}_1$.

This is reasonable as symmetries of a Lagrangian always form a subalgebra of the symmetries of the Euler-Lagrange (geodesic) equations [64] and the algebra of the Euler-Lagrange equations for Minkowski space is $sl(6,\mathbb{R})$, which is 35 dimensional [68]. Using $\mathbf{Z}_1$ we can write $s = t^2$ or $s = r^2$ and

$$
\mathbf{Z}_3 = \frac{r^2}{4}[\frac{1}{t}(r^2 + 2t^2)\frac{\partial}{\partial t} + 3r\frac{\partial}{\partial r}]. \qquad (3.183)
$$

Now, every flat spacetime is conformally flat, i.e. for which all components of the Weyl tensor are zero [1]. The Lie algebra of the Conformal Killing Vectors (CKVs) for a conformally flat spacetime is 15 dimensional [59]. Therefore for the Minkowski spacetime we already know that there are 15 CKVs. The 5 symmetry generators, i.e. $\mathbf{Z}_i$ for $i = 2, ..., 6$ given in (3.181) and (3.182), are proper CKVs with conformal factor

$$
\psi = \frac{1}{2}(c_0 t^2 + c_1). \qquad (3.184)
$$

Thus we see that not only the KVs but *also* the CKVs form a subalgebra of the symmetries of the Lagrangian for the Minkowski spacetime.

Schwarzschild Spacetime as a First Perturbation of the Minkowski Spacetime

Retaining terms of first-order in ϵ and neglecting $O(\epsilon^2)$, the Lagrangian (3.79) becomes a first-order perturbed Lagrangian for the Schwarzschild spacetime.

$$L = \dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 - \frac{2}{r}\epsilon(\dot{t}^2 + \dot{r}^2). \tag{3.185}$$

Using (1.56) and the exact symmetry generators given by (1.78) - (1.85) and (3.181) - (3.182) we get a set of 19 equations. In these equations only 12 of the 17 constants corresponding to exact symmetry generators appear.

$$\xi_t = 0,\ \xi_r = 0,\ \xi_\theta = 0,\ \xi_\phi = 0,\ A_s = 0, \tag{3.186}$$

$$\frac{-4}{r}(\frac{rc_0}{2} + c_{14}\sin\theta\cos\phi + c_{15}\sin\theta\sin\phi + c_{16}\cos\theta) - 2\eta^1_s = A_r, \tag{3.187}$$

$$-2r^2\eta^2_s = A_\theta,\ \ -2r^2\sin^2\theta\eta^3_s = A_\phi,\ 2(c_0t + 2c_3 - r\eta^0_s) = -rA_t, \tag{3.188}$$

$$4(c_9\sin\theta\cos\phi + c_{10}\sin\theta\sin\phi + c_{11}\cos\theta) - r(\eta^0_r - \eta^1_t) = 0, \tag{3.189}$$

$$2(c_9\cos\theta\cos\phi + c_{10}\cos\theta\sin\phi - c_{11}\sin\theta) - \eta^0_\theta + r^2\eta^2_t = 0, \tag{3.190}$$

$$2(c_9\cos\theta\cos\phi - c_{10}\sin\theta\cos\phi) + \eta^0_\phi - r^2\sin^2\theta\eta^3_t = 0, \tag{3.191}$$

$$2\eta^1\sin\theta + 2r\eta^2\cos\theta + 2r\eta^3_\phi\sin\theta - r\sin\theta\xi_s = 0, \tag{3.192}$$

$$\eta^2_\phi + \sin^2\theta\eta^3_\theta = 0,\ 2\eta^1 - 2r\eta^2_\theta - r\xi_s = 0, \tag{3.193}$$

$$\begin{aligned}&2[s(c_{14}\sin\theta\cos\phi + c_{15}\cos\theta\sin\phi - c_{16}\sin\theta) + t(c_9\cos\theta\cos\phi + \\ &c_{10}\cos\theta\sin\phi - c_{11}\sin\theta) + c_{12}\cos\theta\cos\phi + c_{13}\cos\theta\sin\phi - c_{17}\sin\theta] \\ &+r\eta^1_\theta + r^3\eta^2_r = 0,\end{aligned} \tag{3.194}$$

$$\begin{aligned}&2[s(c_{14}\sin\theta\sin\phi - c_{15}\sin\theta\cos\phi) + t(c_9\sin\theta\sin\phi - c_{10}\sin\theta\cos\phi) \\ &+c_{12}\sin\theta\sin\phi - c_{13}\sin\theta\cos\phi] - r\eta^1_\phi - r^3\sin^2\theta\eta^3_r = 0,\end{aligned} \tag{3.195}$$

$$\begin{aligned}&[s(c_{14}\sin\theta\cos\phi + c_{15}\cos\theta\sin\phi + c_{16}\cos\theta) + t(c_9\sin\theta\cos\phi + c_{10}\sin\theta\sin\phi \\ &+c_{11}\cos\theta) + c_{12}\sin\theta\cos\phi + c_{13}\sin\theta\sin\phi + c_{17}\cos\theta] + \frac{r^2}{2}(2\eta^0_t - \xi_s) \\ &+\frac{r}{2}(c_0s + c_1) = 0,\end{aligned} \tag{3.196}$$

$$2[s(c_{14}\sin\theta\cos\phi + c_{15}\cos\theta\sin\phi + c_{16}\cos\theta) + t(c_9\sin\theta\cos\phi + c_{10}\sin\theta\sin\phi$$
$$+c_{11}\cos\theta) + c_{12}\sin\theta\cos\phi + c_{13}\sin\theta\sin\phi + c_{17}\cos\theta] - r^2(2\eta_r^1 - \xi_s)$$
$$+r(c_0 s + c_1) = 0. \tag{3.197}$$

Solving this system by the same method used earlier in this thesis, the 12 generators of the exact symmetry have to be eliminated for consistency of these new determining equations, making them homogeneous. The resulting system is the same as for the Minkowski space, yielding 17 first-order approximate symmetry generators given by (1.78) - (1.85) and (3.181) - (3.182). Thus for the Schwarzschild spacetime as a first perturbation of the Minkowski spacetime, there is no non-trivial approximate Noether symmetry. We recover all the lost conservation laws as trivial first-order approximate conservation laws. Beside energy and angular momentum which always remain conserved for the Schwarzschild spacetime (for both exact and perturbed) we see the approximate conservation of linear momentum and spin angular momentum. This was also seen in the first-order approximate symmetries of the geodesic equations for Schwarzschild metric [60].

Kerr Spacetime as a Second Perturbation of the Minkowski Spacetime

Though we have recovered all the lost conservation laws as trivial first-order approximate conservation laws in the case of Schwarzschild spacetime as a first perturbation of the Minkowski spacetime. There is no non-trivial approximate symmetry in the first approximation. In hope of finding some non-trivial approximate Noether symmetry from the definition of second-order approximate symmetry of the Lagrangian we take the Kerr spacetime as a second perturbation of the Minkowski spacetime. In the second approximation, that is when we retain terms quadratic in ϵ, we have the Lagrangian given by (3.79). From (1.64) we have a new system of 19 determining equations. In these equations now 14 of the 17 constants corresponding to the exact (also first-order approximate) symmetry generators appear.

$$\xi_t = 0,\ \xi_r = 0,\ \xi_\theta = 0,\ \xi_\phi = 0,\ A_s = 0, \tag{3.198}$$
$$2k_1\sin\theta(c_{15}\cos\phi - c_{14}\sin\phi) - 2r(c_0 t + 2c_3) - 2r^2\eta_s^0 = r^2 A_t, \tag{3.199}$$

$$[(k_1^2\sin^2\theta-4)-2][\frac{c_0 r}{2}+c_{14}\sin\theta\cos\phi+c_{15}\cos\theta\sin\phi+c_{16}\cos\theta]$$
$$-r^2\eta^1_s=\frac{r^2}{2}A_r, \tag{3.200}$$
$$k_1^2\cos^2\theta(c_{14}\cos\theta\cos\phi+c_{15}\cos\theta\sin\phi-c_{16}\sin\theta)+r^3\eta^2_s=\frac{-r}{2}A_\theta, \tag{3.201}$$
$$k_1(c_0t+2c_3)-2k_1^2\csc^2\theta(c_{15}\cos\phi-c_{14}\sin\phi)-r^3\eta^3_s=\frac{r}{2}\csc^2\theta A_\phi, \tag{3.202}$$
$$\frac{1}{r^2}(k_1^2\sin^2\theta-5)(c_9\sin\theta\cos\phi+c_{10}\sin\theta\sin\phi+c_{11}\cos\theta)$$
$$-\frac{k_1}{r^3}\sin\theta[s(c_{15}\cos\phi-c_{14}\sin\phi)-t(c_9\sin\phi-c_{10}\cos\phi)-c_{12}\sin\phi+$$
$$c_{13}\cos\phi]+r(\eta^0_r-\eta^1_t)=0, \tag{3.203}$$
$$\frac{1}{r}(k_1^2\cos^2\theta+2)(c_9\cos\theta\cos\phi+c_{10}\cos\theta\sin\phi-c_{11}\sin\theta)+\frac{k_1}{r^2}\sin\theta[s\cot\theta$$
$$(c_{15}\cos\phi-c_{14}\sin\phi)-\cot\theta(c_{12}\sin\phi-c_{13}\cos\phi)-t\cot\theta(c_9\sin\phi-$$
$$c_{10}\cos\phi)+r\csc\theta(a_8\cos\phi-a_7\sin\phi)]-\eta^0_\theta+r^2\eta^2_t=0, \tag{3.204}$$
$$-\frac{k_1}{r^2}\sin^2\theta[s(c_{14}\sin\theta\cos\phi+c_{15}\sin\theta\sin\phi+c_{16}\cos\theta)+t(c_9\sin\theta\cos\phi+$$
$$c_{10}\sin\theta\sin\phi+c_{11}\cos\theta)+c_{12}\sin\theta\cos\phi+c_{13}\sin\theta\sin\phi+c_{17}\cos\theta]+$$
$$\frac{k_1}{r^2}\sin 2\theta[s(c_{14}\cos\theta\cos\phi+c_{15}\cos\theta\sin\phi-c_{16}\sin\theta)+t(c_9\cos\theta\cos\phi$$
$$+c_{10}\cos\theta\sin\phi-c_{11}\sin\theta)+c_{12}\cos\theta\cos\phi+c_{13}\cos\theta\sin\phi-c_{17}\sin\theta+$$
$$r(a_7\cos\phi+a_8\sin\phi)]-\frac{k_1}{r^2}\sin\theta[s(c_{15}\sin\phi+c_{14}\cos\phi)+c_{12}\cos\phi+c_{13}\sin\phi$$
$$+t(c_9\cos\phi+c_{10}\sin\phi)+r\cos\theta(a_7\cos\phi+a_8\sin\phi)]+\frac{k_1^2}{r^2}\sin^2\theta(c_9\sin\phi$$
$$-c_{10}\cos\phi)+2(c_9\sin\theta\sin\phi+c_{10}\sin\theta\cos\phi)+\eta^0_\phi-r^2\sin^2\theta\eta^3_t=0, \tag{3.205}$$
$$\frac{1}{r^2}(k_1^2-6)[s(c_{14}\cos\theta\cos\phi+c_{15}\cos\theta\sin\phi-c_{16}\sin\theta)+t(c_9\cos\theta\cos\phi$$
$$+c_{10}\cos\theta\sin\phi-c_{11}\sin\theta)+c_{12}\cos\theta\cos\phi+c_{13}\cos\theta\sin\phi-c_{17}\sin\theta]-$$
$$\eta^1_\theta-r^2\eta^2_r=0, \tag{3.206}$$
$$\frac{1}{r}[k_1^2(1+\sin^2\theta)-6][s(c_{15}\sin\theta\cos\phi-c_{14}\sin\theta\sin\phi)-t(c_9\sin\theta\sin\phi-$$
$$c_{10}\sin\theta\cos\phi)-c_{12}\sin\theta\sin\phi+c_{13}\sin\theta\cos\phi]+k_1\sin^2\theta(c_9\sin\theta\cos\phi$$
$$+c_{10}\sin\theta\sin\phi+c_{11}\cos\theta)-r\eta^1_\phi-r^3\sin^2\theta\eta^3_r=0, \tag{3.207}$$

$$\frac{k_1^2}{r}\sin^2\theta[s(c_{15}\cos\theta\cos\phi - c_{14}\cos\theta\sin\phi) - t(c_9\cos\theta\sin\phi - c_{10}\cos\theta\cos\phi) -$$
$$c_{12}\cos\theta\sin\phi + c_{13}\cos\theta\cos\phi + r(a_8\cos\phi - a_7\sin\phi)] + k_1\sin^2\theta(c_9\cos\theta\cos\phi$$
$$+c_{10}\cos\theta\sin\phi - c_{11}\sin\theta) - r^2(\eta^2_\phi + \sin^2\theta\eta^3_\theta) = 0, \tag{3.208}$$
$$-\frac{k_1^2}{r}\sin 2\theta[s(c_{14}\cos\theta\cos\phi + c_{15}\cos\theta\sin\phi - c_{16}\sin\theta) + t(c_9\cos\theta\cos\phi +$$
$$c_{10}\cos\theta\sin\phi - c_{11}\sin\theta) + c_{12}\cos\theta\cos\phi + c_{13}\cos\theta\sin\phi - c_{17}\sin\theta] -$$
$$2k_1\sin^2\theta(c_9\sin\theta\sin\phi - c_{10}\sin\theta\cos\phi) + \frac{k_1^2}{r}\sin\theta[s(c_{15}\sin\phi + c_{14}\cos\phi)$$
$$+t(c_{10}\sin\phi + c_9\cos\phi) + c_{12}\cos\phi + c_{13}\sin\phi)] - 2r\eta^1\sin^2\theta - r^2\eta^2\sin 2\theta -$$
$$r^2\sin^2\theta(2\eta^3_\phi - \xi_s) - k_1^2\sin^2\theta(c_0 s + c_1) = 0, \tag{3.209}$$
$$\frac{k_1^2}{r}\sin 2\theta[s(c_{14}\cos\theta\cos\phi + c_{15}\cos\theta\sin\phi - c_{16}\sin\theta) + t(c_9\cos\theta\cos\phi +$$
$$c_{10}\cos\theta\sin\phi - c_{11}\sin\theta) + c_{12}\cos\theta\cos\phi + c_{13}\cos\theta\sin\phi - c_{17}\sin\theta +$$
$$r(a_7\cos\phi + a_8\sin\phi)] + \frac{2k_1^2}{r}\cos^2\theta[s(c_{14}\sin\theta\cos\phi + c_{15}\sin\theta\sin\phi +$$
$$c_{16}\cos\theta) + t(c_9\sin\theta\cos\phi + c_{10}\sin\theta\sin\phi + c_{11}\sin\theta) + c_{12}\sin\theta\cos\phi +$$
$$c_{13}\sin\theta\sin\phi + c_{17}\cos\theta] + k_1^2\cos^2\theta(c_0 s + c_1) - 2r\eta^1 - 2r^2\eta^2_\theta + r^2\xi_s = 0, \tag{3.210}$$
$$\frac{1}{r^2}(5 - k_1^2\sin^2\theta)[s(c_{14}\sin\theta\cos\phi + c_{15}\sin\theta\sin\phi + c_{16}\cos\theta) + t(c_9\sin\theta\cos\phi$$
$$+c_{10}\sin\theta\sin\phi + c_{11}\cos\theta) + c_{12}\sin\theta\cos\phi + c_{13}\sin\theta\sin\phi + c_{17}\cos\theta +$$
$$r(c_0 s + c_1)] + \frac{k_1^2}{r^3}\sin 2\theta[s(c_{14}\cos\theta\cos\phi + c_{15}\cos\theta\sin\phi - c_{16}\sin\theta) +$$
$$t(c_9\cos\theta\cos\phi + c_{10}\cos\theta\sin\phi - c_{11}\sin\theta) + +c_{12}\cos\theta\cos\phi + c_{13}\cos\theta\sin\phi$$
$$-c_{17}\sin\theta + r(a_7\cos\phi + a_8\sin\phi)] - 2\eta^1_r + \xi_s = 0, \tag{3.211}$$
$$\frac{2}{r^2}[s(c_{14}\sin\theta\cos\phi + c_{15}\sin\theta\sin\phi + c_{16}\cos\theta) + t(c_9\sin\theta\cos\phi +$$
$$c_{10}\sin\theta\sin\phi + c_{11}\cos\theta) + c_{12}\sin\theta\cos\phi + c_{13}\sin\theta\sin\phi + c_{17}\cos\theta]$$
$$+\frac{2k_1}{r^2}\sin^2\theta(c_{10}\cos\phi - c_9\sin\phi) + 2\eta^0_t - \xi_s + \frac{1}{r}(c_0 s + c_1) = 0. \tag{3.212}$$

In the above set of determining equations corresponding to the exact (also first-order approximate) symmetry generators 14 constants appear. The 2 constants corresponding to $\mathbf{Y}_1$ and $\mathbf{Y}_2$, given in (1.78) and (1.79), appear here and did not appear in the first-order approximation. At first sight it seems that these 2 new constants may give some non-trivial second-order approximate symmetries. But for the consistency of the determining equations all of these 14 constants have to be eliminated and the system again becomes homogeneous. The resulting system is again the same as for the Minkowski spacetime, yielding 17 second-order approximate symmetry generators given by(1.78) - (1.85) and (3.181) - (3.182). Thus there is no non-trivial second-order approximate symmetry generator for the Kerr spacetime taken as a second perturbation of the Minkowski spacetime.

Going from Minkowski to the Kerr spacetime we are left with only two Killing vectors, that is only with energy and angular momentum conservations. Also for the Lagrangian of the exact Kerr metric there are only three symmetry generators given by (3.17). Here we recover all the lost conservation laws as trivial second-order approximate conservation laws for the Kerr spacetime.

The results of this chapter are summarised in the following theorem.

Theorem 3.1. *The Noether symmetries of the Minkowski spacetime form a 17 dimensional Lie algebra which properly contains the 15 dimensional algebra of the CKVs for this spacetime.*

Chapter 4

Second-Order Approximate Symmetries of the Geodesic Equations and re-scaling of Energy in the Charged-Kerr Spacetime

The re-scaling of energy for a test particle in the RN spacetime [81] was seen in the second-order approximate symmetries of the geodesic equations. A problem arises in the search for a scaling factor for the energy of test particles in the Kerr spacetime. Whereas, in the RN-case the energy re-scaling was by $(1 - Q^2/2M^2)$, there is a simple multiplicative factor for the Kerr spacetime. In the absence of the constant (unity in this case), it is not clear what significance to attach to the re-scaling. So as to relate that factor to the factor arising in the RN-case, in this chapter we investigate second-order approximate symmetries of the geodesic equations for the charged-Kerr spacetime [82].

The line element for the charged-Kerr spacetime is given by [40, 85]

$$\begin{aligned} ds^2 &= [1 - \frac{G(2c^2Mr - Q^2)}{\rho^2 c^4}]c^2dt^2 - (\frac{\rho^2}{\Delta})dr^2 - \rho^2 d\theta^2 - \Lambda\frac{\sin^2\theta}{\rho^2}d\phi^2 + \\ &\quad \frac{Ga}{\rho^2 c^2}(2Mr - \frac{Q^2}{c^2})\sin^2\theta dt d\phi, \end{aligned} \tag{4.1}$$

where

$$\Delta = \frac{a^2}{c^2} + r^2 - \frac{G}{c^2}(2Mr - \frac{Q^2}{c^2}). \tag{4.2}$$

Using the same ϵ defined for the Schwarzschild spacetime in chapter 1 and setting

$$\frac{GQ^2}{c^4} = k\epsilon^2,$$

we have the second-order perturbed (in ϵ) geodesic equation for the charged-Kerr spacetime

$$E_1 : \ddot{t} + \epsilon\frac{1}{r^2}\dot{t}\dot{r} + \epsilon^2[\frac{1}{r^3}(1-2k)\dot{t}\dot{r} - \frac{2\sqrt{k_1}}{r^2}\sin^2\theta\dot{r}\dot{\phi}] + O(\epsilon^3) = 0, \tag{4.3}$$

$$\begin{aligned} E_2 : \ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \epsilon[\frac{1}{2r^2}(\dot{t}^2 - \dot{r}^2) + (\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)] - \\ \epsilon^2[\frac{1}{2r^3}(1+2k)\dot{t}^2 + \frac{\sqrt{k_1}}{r^2}\sin^2\dot{t}\dot{\phi} - \frac{\dot{r}^2}{r^3}\{2(k_1\sin\theta + k) - 1\} \\ + \frac{k_1}{r^2}\sin^2\theta\dot{r}\dot{\theta} + \frac{1}{r}(k_1\sin^2\theta + k)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)] + O(\epsilon^3) = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} E_3 : \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \epsilon^2[\frac{\sqrt{k_1}}{r^3}\sin 2\theta\dot{t}\dot{\phi} - \frac{k_1}{2r^4}\sin 2\theta\dot{r}^2 \\ - \frac{2k_1}{r^3}\cos^2\theta\dot{r}\dot{\theta} - \frac{k_1}{2r^2}\sin 2\theta(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)] + O(\epsilon^3) = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} E_4 : \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} + \epsilon^2[\frac{\sqrt{k_1}}{r^4}\dot{t}\dot{r} - \frac{2\sqrt{k_1}}{r^3}\cot\theta\dot{t}\dot{\theta} - \frac{2k_1}{r^3}\dot{r}\dot{\phi}] \\ + O(\epsilon^3) = 0. \end{aligned} \tag{4.6}$$

If $\epsilon = 0$, then these equations reduces to that of the Minkowski spacetime and when we retain terms only up to order ϵ and neglect higher orders, then these equations reduces to the first-order perturbed geodesic equation of the Schwarzschild spacetime.

4.1 Second-Order Approximate Symmetries of the Geodesic Equations for the Charged-Kerr Spacetime

In this section we discuss second-order approximate symmetries of the geodesic equations for the charged-Kerr spacetime. Since the geodesic equations (4.3) - (4.6) are second-order ODEs, with second-order perturbation term, we apply to it the second prolongation $\mathbf{X}^{[2]}$ of the generator

$\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2$ defined in (2.40), which yields

$$\begin{aligned}\mathbf{X}^{[2]}E_1 = [&\eta^0_{0,ss} + \epsilon\eta^0_{1,ss} + \epsilon^2\eta^0_{2,ss} + \{\frac{\epsilon}{r^2} + \frac{1-2k}{r^3}\epsilon^2\}\{(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s})\dot{r} + \\ &(\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s})\dot{t}\} - \dot{t}\dot{r}(\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2)(\frac{2\epsilon}{r^3} + \frac{3(1-2k)}{r^4}\epsilon^2) \\ &- \frac{2\sqrt{k_1}\epsilon^2}{r^2}\{\frac{2\sin^2\theta}{r}(\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2) + (\eta^2_0 + \epsilon\eta^2_1 + \epsilon^2\eta^2_2)\sin 2\theta\}\dot{r}\dot{\phi} \\ &\frac{2\sqrt{k_1}\epsilon^2}{r^2}\sin^2\theta(\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s})\dot{\phi} + (\eta^3_{0,s} + \epsilon\eta^3_{1,s} + \epsilon^2\eta^3_{2,s})\dot{r}]_{E_j=0} = 0, \quad (4.7)\end{aligned}$$

$$\begin{aligned}\mathbf{X}^{[2]}E_2 = [&\eta^1_{0,ss} + \epsilon\eta^1_{1,ss} + \epsilon^2\eta^1_{2,ss} + c^2\{\frac{\epsilon}{r^2}\dot{t} + \epsilon^2(\frac{1+2k}{r^3}\dot{t} + \frac{\sqrt{k_1}\epsilon^2}{r^2}\sin^2\theta\dot{\phi})\} \\ &(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s}) - \{\frac{c^2\dot{t}^2}{2}(\frac{2\epsilon}{r^3} - \frac{3(1+2k)}{r^4}\epsilon^2) - \frac{\dot{r}^2}{2}(\frac{2\epsilon}{r^3} + \\ &\frac{3(1-2k)}{r^4}\epsilon^2) + (1 - \frac{k\epsilon^2}{r^2})(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \epsilon^2\frac{k_1}{r^2}\sin^2\theta(\frac{2}{r\sqrt{k_1}}\dot{t}\dot{\theta} - \\ &\frac{3}{r^2}\dot{r}^2 + \frac{2}{r}\dot{r}\dot{\theta} + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\}(\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2) - \frac{1}{r^2}\{\epsilon\dot{r} + \\ &\epsilon^2(\frac{1-2k}{r}\dot{r} + \frac{2k_1}{r}\sin^2\theta\dot{r} + k_1\sin^2\theta\dot{\theta})\}(\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s}) - \\ &\{r\dot{\phi}^2 - \epsilon\dot{\phi}^2 + \epsilon^2(\frac{k}{r}\dot{\phi}^2 + \frac{\sqrt{k_1}}{r^2}\dot{t}\dot{\phi} + \frac{2k_1}{r^3}\dot{r}^2 - \frac{k_1}{r^2}\dot{r}\dot{\theta} - \frac{k_1}{r}\dot{\theta}^2)\} \\ &(\eta^2_0 + \epsilon\eta^2_1 + \epsilon^2\eta^2_2)\sin 2\theta - \{r\dot{\theta} - \epsilon\dot{\theta} + \frac{\epsilon^2}{r}(k\dot{\theta} + \frac{k_1}{r}\sin^2\theta\dot{r} - \\ &2k_1\sin^2\theta\dot{\theta})\}\{(\eta^2_{0,s} + \epsilon\eta^2_{1,s} + \epsilon^2\eta^2_{2,s}) - \{r\dot{\phi} - \epsilon\dot{\phi} + \frac{\epsilon^2}{r}(k\dot{\phi} + \\ &\frac{\sqrt{k_1}}{r}\dot{t} + 2k_1\sin^2\theta\dot{\phi})\}\sin^2\theta(\eta^3_{0,s} + \epsilon\eta^3_{1,s} + \epsilon^2\eta^3_{2,s})\}]_{E_j=0} = 0, \quad (4.8)\end{aligned}$$

$$\begin{aligned}\mathbf{X}^{[2]}E_3 = [&\eta^2_{0,ss} + \epsilon\eta^2_{1,ss} + \epsilon^2\eta^2_{2,ss} - \{\frac{2\dot{r}\dot{\theta}}{r^2} + \epsilon^2\frac{k_1}{r^3}(\frac{3}{r\sqrt{k_1}}\dot{t}\dot{\phi} - \frac{2}{r^2}\dot{r}^2 + \\ &\frac{3\cot\theta}{r}\dot{r}\dot{\theta} - \dot{\theta}^2 - \sin^2\theta\dot{\phi}^2)\sin 2\theta\}(\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2) + \{\frac{2\dot{\theta}}{r} - \\ &\epsilon^2\frac{k_1}{r^3}(\frac{\dot{r}}{r} + \cot\theta\dot{\theta})\sin 2\theta\}(\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s}) + \{\frac{2\dot{r}}{r} - \epsilon^2\frac{k_1}{r^2}(\dot{\theta} \\ &+ \frac{1}{r}\cot\theta\dot{r})\}\sin 2\theta(\eta^2_{0,s} + \epsilon\eta^2_{1,s} + \epsilon^2\eta^2_{2,s}) - \{\dot{\phi}^2(\cos^2\theta - \sin^2\theta)\end{aligned}$$

$$
- \epsilon^2 \frac{k_1}{r^2}(\frac{2}{r\sqrt{k_1}}\dot{t}\dot{\phi} - \frac{1}{r^2}\dot{r}^2 + \frac{2\tan 2\theta}{r}\dot{r}\dot{\theta} - 2(\dot{\theta}^2 + (3\sin^2\theta -
4\sin^4\theta)\dot{\phi}^2)\cos 2\theta\}(\eta_0^2 + \epsilon\eta_1^2 + \epsilon^2\eta_2^2) + \frac{\sqrt{k_1}\sin 2\theta}{r^3}\dot{\phi}(\eta_{0,s}^0 + \epsilon\eta_{1,s}^0
+ \epsilon^2\eta_{2,s}^0) - \{2\sin\theta\cos\theta\dot{\phi} - \epsilon^2\frac{\sqrt{k_1}}{r^2}(\frac{\dot{t}}{r} - \sqrt{k_1}\sin^2\theta\dot{\phi})\}(\eta_{0,s}^3 + \epsilon\eta_{1,s}^3
+\epsilon^2\eta_{2,s}^3)]_{E_j=0} = 0, \quad (4.9)
$$

$$
\mathbf{X}^{[2]}E_4 = [\eta_{0,ss}^3 + \epsilon\eta_{1,ss}^3 + \epsilon^2\eta_{2,ss}^3 - \{\frac{2}{r^2}\dot{r}\dot{\phi} + \epsilon^2\frac{2\sqrt{k_1}}{r^4}(\frac{2}{r}\dot{t}\dot{r} + 3\cot\theta\dot{t}\dot{\theta}
- 3\dot{r}\dot{\phi})\}(\eta_0^1 + \epsilon\eta_1^1 + \epsilon^2\eta_2^1) + \{\frac{2\dot{\theta}}{r} + \epsilon^2\frac{\sqrt{k_1}}{r^3}(\frac{\dot{t}}{r} - \dot{\phi})\}(\eta_{0,s}^1 + \epsilon\eta_{1,s}^1
+ \epsilon^2\eta_{2,s}^1) + 2\{\frac{\dot{r}}{r} + \dot{\theta}\cot\theta - \epsilon^2\frac{2k_1}{r^3}\dot{r}\}(\eta_{0,s}^3 + \epsilon\eta_{1,s}^3 + \epsilon^2\eta_{2,s}^3) - (2\dot{\theta}\dot{\phi}
+ \epsilon^2\dot{t}\dot{\theta})\csc^2\theta(\eta_0^2 + \epsilon\eta_1^2 + \epsilon^2\eta_2^2) + \epsilon^2\frac{\sqrt{k_1}}{r^3}(\frac{\dot{r}}{r} + 2\cot\theta\dot{\theta})(\eta_{0,s}^0 + \epsilon\eta_{1,s}^0
+ \epsilon^2\eta_{2,s}^0) + 2\{\dot{\phi} + \epsilon^2\frac{\sqrt{k_1}}{r^3}\dot{t}\}\cot\theta(\eta_{0,s}^2 + \epsilon\eta_{1,s}^2 + \epsilon^2\eta_{2,s}^2)]_{E_j=0} = 0, \quad (4.10)
$$

where $(j = 1, 2, 3, 4)$. We use the prolongation coefficients defined in (2.45) - (2.52), the exact (for the Minkowski spacetime) and the first-order approximate (for the Schwarzschild spacetime) symmetry generators $\mathbf{Y}_0$ and $\mathbf{Y}_1$ given by (1.78) - (1.85) and the second-order perturbed geodesic equations in above equations (4.7) - (4.10). We get the following set of determining equations.

$$
2r^2\xi_{2_{tr}} - a_2 = 0, \ \xi_{2_{t\theta}} = 0, \ \xi_{2_{t\phi}} = 0, \ r\xi_{2_{r\theta}} - \xi_{2_\theta} = 0,
$$
$$
r\xi_{2_{r\phi}} - \xi_{2_\phi} = 0, \ r\xi_{2_{\theta\phi}} - \cot\theta\xi_{2_\phi} = 0,
$$
$$
2r^2\xi_{2_{tt}} - c^2[\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] = 0,
$$
$$
2r^2\xi_{2_{rr}} - [\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] = 0,
$$
$$
\xi_{2_{\theta\theta}} + r\xi_{2_r} - [\sin\theta(a_3\sin\phi - a_4\cos\phi) + a_5\cos\phi] = 0,
$$
$$
\xi_{2_{\phi\phi}} + r\sin^2\theta\xi_{2_r} + \sin\theta\cos\theta\xi_{2_\theta} - \sin^2\theta[\sin\theta(a_3\sin\phi - a_4\cos\phi) +
a_5\cos\phi] = 0, \quad (4.11)
$$

$$2r^2(\eta^0_{2_{tt}} - 2\xi_{2_{st}}) + c(a_2 \sin\theta\cos\phi + a_3 \sin\theta\sin\phi + a_4\cos\theta) = 0, \tag{4.12}$$

$$r^3\eta^1_{2_{tt}} - c^2(a_5 \sin\theta\cos\phi + a_6\sin\theta\sin\phi + a_7\cos\theta + a_2ct\sin\theta\cos\phi + a_3ct\sin\theta\sin\phi + a_4ct\cos\theta) = 0, \tag{4.13}$$

$$2r^4\eta^2_{2_{tt}} + c^2(a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi - a_7\sin\theta + a_2ct\cos\theta\cos\phi + a_3ct\cos\theta\sin\phi - a_4ct\sin\theta) = 0, \tag{4.14}$$

$$2r^4\eta^3_{2_{tt}} + c^2(a_6\cos\phi - a_5\sin\phi - a_2ct\sin\phi + a_3ct\cos\phi)\csc\theta = 0, \tag{4.15}$$

$$2cr^2\eta^0_{2_{rr}} + 3(a_2\sin\theta\cos\phi + a_3\sin\theta\sin\phi + a_4\cos\theta) = 0, \tag{4.17}$$

$$r^3(\eta^1_{2_{rr}} - 2\xi_{2_{sr}}) + (a_5\cos\phi + a_6\sin\phi)\sin\theta + a_7\cos\theta + ct\sin\theta(a_2\cos\phi + a_3\sin\phi) + a_4ct\cos\theta = 0, \tag{4.18}$$

$$2r^3(r\eta^2_{2_{rr}} + 2\eta^2_{2_r}) - a_5\cos\theta\cos\phi - a_6\cos\theta\sin\phi + a_7\sin\theta - a_2ct\cos\theta\cos\phi - a_3ct\cos\theta\sin\phi + a_4ct\sin\theta = 0, \tag{4.19}$$

$$2r^3(r\eta^3_{2_{rr}} + 2\eta^3_{2_r}) + (a_5\sin\phi - a_6\sin\theta\cos\theta)\csc\theta + a_7\sin\theta + ct\csc\theta(a_2\sin\phi - a_3\cos\phi) = 0, \tag{4.20}$$

$$c(\eta^0_{2_{\theta\theta}} + r\eta^0_{2_r}) - \sin\theta(a_2\cos\phi - a_3\sin\phi) - a_4\cos\theta = 0, \tag{4.21}$$

$$r(\eta^1_{2_{\theta\theta}} + r\eta^1_{2_r} - \eta^1_2 - 2r\eta^2_{2_\theta}) - 2(a_5\cos\phi + a_6\sin\phi)\sin\theta + 2a_7\cos\theta + 2ct\sin\theta(a_2\cos\phi + a_3\sin\phi) + 2a_4ct\cos\theta = 0, \tag{4.22}$$

$$r(r\eta^2_{2_{\theta\theta}} + r^2\eta^2_{2_r} + 2r\eta^1_{2_\theta} - 2r\xi_{2_{s\theta}}) + (a_5\cos\phi + a_6\sin\phi)\cos\theta - a_7\sin\theta + (a_2\cos\phi + a_3\sin\phi)ct\cos\theta - a_4ct\sin\theta = 0, \tag{4.23}$$

$$r^2(\eta^3_{2_{\theta\theta}} + r\eta^3_{2_r} + 2r^2\cot\theta\eta^3_{2_\theta}) - (a_5\sin\phi + a_6\cos\phi - a_2ct\sin\phi + a_3ct\cos\phi)\csc\theta = 0, \tag{4.24}$$

$$c(\sin\theta\cos\phi\eta^0_{2_\theta} + \eta^0_{2_{\phi\phi}} + r\sin^2\theta\eta^0_{2_r}) - \sin^2\theta(a_2\sin\theta\cos\phi + a_3\sin\theta\sin\phi + a_4\cos\theta) = 0, \tag{4.25}$$

$$r(\eta^1_{2_{\phi\phi}}\csc\theta + r\sin\theta\eta^1_{2_r} + \cos\theta\eta^1_{2_\theta} - \sin\theta\eta^1_2 - 2r\cos\theta\eta^2_2 - 2r\eta^3_{2_\phi})\sin\theta - 2\sin^3\theta(a_5\cos\phi + a_6\sin\phi + a_7\cot\theta + a_2ct\cos\phi + a_3ct\sin\phi + a_4ct\cot\theta) = 0, \tag{4.26}$$

$$r^2(\eta^2_{2_{\phi\phi}} + r\sin^2\theta\eta^2_{2_r} - \cos 2\theta\eta^2_2 - \sin 2\theta\eta^3_{2_\phi} + \frac{1}{2}\sin 2\theta\eta^2_{2_\theta}) + \sin^3\theta(a_5\cos\phi$$

$$+a_6\sin\phi - a_7\cot\theta + a_2ct\cot\theta\cos\phi + a_3ct\cot\theta\sin\phi - a_4ct) = 0, \tag{4.27}$$

$$r(r\eta^3_{2_{\phi\phi}} - 2r\xi_{2_{s\phi}} + r^2\sin^2\theta\eta^3_{2_r} + r\sin\theta\cos\theta\eta^3_{2_\theta} + 2r\eta^1_{2_\phi} + 2r^2\cot\theta\eta^2_{2_\phi})$$

$$+\sin^2\theta(a_6\csc\theta\cos\phi - a_5\csc\theta\sin\phi - a_2ct\csc\theta\sin\phi + a_3ct\csc\theta\cos\phi) = 0, \tag{4.28}$$

$$2\eta^0_{2_{st}} - \xi_{2_{ss}} = 0,\ \eta^1_{2_{st}} = 0,\ \eta^2_{2_{st}} = 0,\ \eta^3_{2_{st}} = 0, \tag{4.29}$$

$$\eta^0_{2_{sr}} = 0,\ 2\eta^1_{2_{sr}} - \xi_{2_{ss}} = 0,\ r\eta^2_{2_{sr}} + \eta^2_{2_s} = 0,\ r\eta^3_{2_{sr}} + \eta^3_{2_s} = 0, \tag{4.30}$$

$$\eta^0_{2_{s\theta}} = 0,\ \eta^1_{2_{s\theta}} - r\eta^2_{2_s} = 0,\ r(2\eta^2_{2_{s\theta}} - \xi_{2_{ss}}) + 2\eta^1_{2_s} = 0,\ \eta^3_{2_{s\theta}} + \cot\theta\eta^3_{2_s} = 0, \tag{4.31}$$

$$\eta^0_{2_{s\phi}} = 0,\ \eta^1_{2_{s\phi}} - r\sin^2\theta\eta^3_{2_s} = 0,\ \eta^2_{2_{s\phi}} - \sin\theta\cos\theta\eta^3_{2_s} = 0, \tag{4.32}$$

$$r(2\eta^3_{2_{s\phi}} - \xi_{2_{ss}}) + 2\eta^1_{2_s} + 2r\cot\theta\eta^2_{2_s} = 0, \tag{4.33}$$

$$\eta^0_{2_{ss}} = 0,\ \eta^1_{2_{ss}} = 0,\ \eta^2_{2_{ss}} = 0,\ \eta^3_{2_{ss}} = 0, \tag{4.34}$$

$$r^3(\eta^0_{2_{tr}} - \xi_{2_{sr}}) - (a_5\sin\theta\cos\phi + a_6\sin\theta\sin\phi + a_7\cos\theta + a_2ct\sin\theta\cos\phi +$$

$$a_3ct\sin\theta\sin\phi + a_4ct\cos\theta) = 0, \tag{4.35}$$

$$2r^2(\eta^1_{2_{tr}} - \xi_{2_{st}}) - c(a_2ct\sin\theta\cos\phi + a_3ct\sin\theta\sin\phi + a_4ct\cos\theta) = 0, \tag{4.36}$$

$$2r^2(r\eta^2_{2_{tr}} + \eta^2_{2_t}) - c(a_2\cos\theta\cos\phi + a_3\cos\theta\sin\phi - a_4\sin\theta) = 0, \tag{4.37}$$

$$2r^2(r\eta^3_{2_{tr}} + \eta^3_{2_t}) - c(a_3\csc\theta\cos\phi - a_2\csc\theta\sin\phi) = 0, \tag{4.38}$$

$$2r^2(\eta^0_{2_{t\theta}} - \xi_{2_{s\theta}}) + a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi - a_7\sin\theta + a_2ct\cos\theta\cos\phi$$

$$+a_3ct\cos\theta\sin\phi - a_4ct\sin\theta = 0, \tag{4.39}$$

$$2r(\eta^1_{2_{t\theta}} - r\eta^2_{2_t}) + 3c(a_2\cos\theta\cos\phi + a_3\cos\theta\sin\phi - a_4\sin\theta) = 0, \tag{4.40}$$

$$r(\eta^2_{2_{t\theta}} - \xi_{2_{st}}) + \eta^1_{2_t} = 0, \tag{4.41}$$

$$\eta^3_{2_{t\theta}} + \cot\theta\eta^3_{2_t} = 0, \tag{4.42}$$

$$2r^2(\eta^0_{2_{t\phi}} - \xi_{2_{s\phi}}) + a_6\sin\theta\cos\phi - a_5\sin\theta\sin\phi - a_2ct\sin\theta\sin\phi$$

$$+a_3ct\sin\theta\cos\phi = 0, \tag{4.43}$$

$$2r(\eta^1_{2_{t\phi}} - r\sin^2\theta\eta^3_{2_t}) + 3c(a_3\sin\theta\cos\phi - a_2\sin\theta\sin\phi) = 0, \tag{4.44}$$

$$\eta^2_{2_{t\phi}} - \sin\theta\cos\theta\eta^3_{2_t} = 0, \tag{4.45}$$

$$r(\eta^3_{2_{t\phi}} - \xi_{2_{st}}) + \eta^1_{2_t} + r\cot\theta\eta^2_{2_t} = 0, \tag{4.46}$$

$$2c(r\eta^0_{2_{r\theta}} - \eta^0_{2_\theta}) + a_2\cos\theta\cos\phi + a_3\cos\theta\sin\phi - a_4\sin\theta = 0, \tag{4.47}$$

$$2r(r\eta^1_{2_{r\theta}} - r\xi_{2_{s\theta}} - \eta^1_{2_\theta} - r^2\eta^2_{2_r}) - 3(a_5\cos\theta\cos\phi + a_6\cos\theta\sin\phi - a_7\sin\theta$$

$$+a_2ct\cos\theta\cos\phi + a_3ct\cos\theta\sin\phi - a_4ct\sin\theta) = 0, \tag{4.48}$$

$$r^2(\eta^2_{2_{r\theta}} - \xi_{2_{sr}}) - \eta^1_2 + r\eta^1_{2_r} = 0, \tag{4.49}$$

$$\eta^3_{2_{r\theta}} + \cot\theta\eta^3_{2_r} = 0, \tag{4.50}$$

$$2c(r\eta^0_{2_{r\phi}} - \eta^0_{2_\phi}) + a_3\sin\theta\cos\phi - a_2\sin\theta\sin\phi = 0, \tag{4.51}$$

$$2r(r\eta^1_{2_{r\phi}} - r\xi_{2_{s\phi}} - \eta^1_{2_\phi} - r^2\sin^2\theta\eta^3_{2_r}) + 3(a_5\sin\theta\cos\phi - a_6\sin\theta\cos\phi +$$

$$a_2ct\sin\theta\sin\phi - a_3ct\sin\theta\cos\phi) = 0, \tag{4.52}$$

$$\eta^2_{2_{r\phi}} - \sin\theta\cos\theta\eta^3_{2_r} = 0, \tag{4.53}$$

$$r^2(\eta^3_{2_{r\phi}} - \xi_{2_{sr}} + \cot\theta\eta^2_{2_r}) + r\eta^1_{2_r} - \eta^1_2 = 0, \tag{4.54}$$

$$\eta^0_{2_{\theta\phi}} - \cot\theta\eta^0_{2_\phi} = 0, \tag{4.55}$$

$$\eta^1_{2_{\theta\phi}} - \cot\theta\eta^1_{2_\phi} - r\eta^2_{2_\phi} - r\sin^2\theta\eta^3_{2_\theta} = 0, \tag{4.56}$$

$$r(\eta^2_{2_{\theta\phi}} - \xi_{2_{s\phi}} - \cot\theta\eta^2_{2_\phi} - \sin\theta\cos\theta\eta^3_{2_\theta}) + \eta^1_{2_\phi} = 0, \tag{4.57}$$

$$r(\eta^3_{2_{\theta\phi}} - \xi_{2_{s\theta}} - \csc^2\theta\eta^2_2 + \cot\theta\eta^2_{2_\theta}) + \eta^1_{2_\theta} = 0. \tag{4.58}$$

The above set of determining equations (4.11) - (4.58) is exactly the same to that for the RN spacetime. Only 6 constants corresponding to the first-order approximate symmetry appear and the other 4 constants corresponding to the exact symmetry do not appear as explained in the RN-case. These 6 constants disappear (by the same method adopted in the RN-case) for consistency of the above equations (4.11) - (4.58) making them homogeneous. Thus there is no non-trivial approximate symmetry for the second-order perturbed geodesic equations of the charged-Kerr spacetime. We only recover the exact (also first-order approximate) symmetry generators as trivial second-order approximate symmetry generators which form the Poincarè algebra $so(1,3)\oplus_s \mathbb{R}^4$ apart from d_2.

4.2 Energy (Mass) in the Charged-Kerr Spacetime

The exact symmetry algebra i.e. (when $\epsilon = 0$), as well as the first-order approximate symmetry algebra when $\epsilon^2 = 0$ and $\epsilon \neq 0$, of the geodesic equations for the charged-Kerr spacetime include the generators of dilation algebra, $\partial/\partial s$, $s\partial/\partial s$ corresponding to

$$\xi(s) = c_0 s + c_1. \tag{4.59}$$

Like the RN spacetime, for the second-order approximate symmetries of the geodesic equations for the charged-Kerr spacetime in the determining equations the terms involving $\xi_s = c_0$ do *not* automatically cancel out but collect a scaling factor of

$$\frac{1}{r^3}(1-2k)\dot{t} - \frac{2\sqrt{k_1}}{r^2}\sin^2\theta\dot{\phi}, \tag{4.60}$$

so as to cancel out. (As mentioned earlier this factor comes from the application of the perturbed system, rather than the unperturbed one in the determining equations.) The scaling factor (4.60) corresponds to a *re-scaling* of energy (mass) as explained for the RN spacetime. This scaling factor for the charged-Kerr spacetime involves the derivatives of the coordinates t and ϕ and derivatives only apply to the paths of the particles. To get the energy in the spacetime field these derivatives can be replaced by the exact first integrals of the geodesic equations and involve a constant that has units of mass. As such, we write it as M

$$\dot{t} = 2M,\ \dot{\phi} = \frac{-M^2}{2r^2\sin^2\theta}. \tag{4.61}$$

Using these first integrals and

$$k = \frac{Q^2}{4GM^2},\ k_1 = (\frac{ac}{2GM})^2, \tag{4.62}$$

in (4.60) and denoting it by M_{c-K} we get the energy re-scaling factor for the charged-Kerr spacetime (taking $G = 1$, $c = 1$)

$$M_{c-K} = M - \frac{Q^2}{2M} + \frac{Ma}{r}. \tag{4.63}$$

For $a = 0$, the expression (4.63) reduces to M-times of the expression for the RN spacetime [81].

The energy (mass) in the charged-Kerr spacetime has been defined by many authors. We discuss some of these definitions and then compare our definition (4.63) with them.

Komar, using his definition of approximate symmetry [29], wrote down an integral (1.14) for the mass in a spacetime. Using the Komar integral (1.14) Cohen and de Felice considered ξ as the stationary Killing 1-form over a charged-Kerr background metric [31]. They obtained a formula for the effective mass (and hence energy) for the charged-Kerr spacetime

$$M_{c-K} = M - \frac{Q^2}{r} - \frac{Q^2(r^2+a^2)}{ar^2}\tan^{-1}(\frac{a}{r}). \tag{4.64}$$

In the above expression (4.64) a does not appear explicitly and only appears in a product with Q. When $Q \longrightarrow 0$ in the above expression (4.64) the effects of rotation also disappear. This does not seem reasonable. In the limit of $a \longrightarrow 0$ expression (4.64) reduces to that of the RN spacetime given in [86, 87].

Chellathurai and Dadhich modified the Komar integral and obtained an expression for the effective mass of the charged-Kerr black hole [32]

$$M_{c-K} = M - \frac{Q^2}{r} - \frac{(12M^2+Q^2)a^2}{3r^3} + \frac{14MQ^2a^2}{3r^4} + ... \quad . \tag{4.65}$$

This expression (4.65) reduces to that of the RN spacetime in the limit $a \longrightarrow 0$ and in the limit $Q \longrightarrow 0$ reduces to that for the Kerr spacetime [88]. However, it is not clear that this modification satisfactorily adjusts for the approximate symmetry of Komar.

Qadir and Quamar [89] obtained an expression for the ψN-potential of the charged-Kerr spacetime,

$$\varphi = -\frac{Mr - Q^2/2}{(r^2+a^2\cos^2\theta)}. \tag{4.66}$$

In the limit $a \longrightarrow 0$ (4.66) reduces to that for the RN spacetime [90, 91]. This yields the approximate modification of the mass to be

$$M_{c-K} = M - \frac{Q^2}{2r} - \frac{Ma^2\cos^2\theta}{r^2} + \frac{Q^2a^2\cos^2\theta}{2r^3} + ... \quad . \tag{4.67}$$

The significance and comparison of our expression (4.63), with the other three expressions (4.64), (4.65) and (4.67) will be discussed in more detail in chapter 7.

The main result of this chapter is given in the form of the following theorem.

Theorem 4.1. *For the charged-Kerr spacetime the energy is re-scaled by the factor (4.63).*

Chapter 5

Approximate Noether Symmetries of the Geodesic Equations for Plane Symmetric Gravitational Wave Spacetimes and the Definition of Energy

In this chapter we give first-order approximate Noether symmetries of an artificially constructed example of a plane symmetric "wave-like" spacetime, which represents a gravitational wave interacting with matter and of the pp-wave spacetime. For the wave-like spacetime a non-trivial approximate Noether symmetry is found. We use this non-trivial approximate Noether symmetry to contract the energy momentum vector that gives the conserved quantity for the wave-like spacetime. To look at the energy content of the plane wave spacetimes we then investigate second-order approximate symmetries of the geodesic equations for the pp-wave and the wave-like spacetimes. Since ϵ^2 does not appear in the geodesic equations for perturbed pp-waves, there is a problem in applying the definition of second-order approximate symmetries of ODEs, which gives the scaling factor mentioned earlier, to them. To obtain a better understanding of the energy re-scaling in plane GWs, the wave-like spacetime is then investigated. We give the non-zero components of the Weyl and stress-energy tensors for the spacetimes discussed in this chapter.

5.1 First-Order Approximate Noether Symmetries of Plane Wave Spacetimes

In this section we first discuss the wave-like spacetime and then we study the pp-wave spacetime.

5.1.1 First-Order Approximate Noether Symmetries of the Plane Wave-Like Spacetime

To study first-order approximate symmetries of the Lagrangian for plane symmetric spacetime we consider the following line element of a static spacetime [92]

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - e^{2\mu(x)}(dy^2 + dz^2), \tag{5.1}$$

with

$$\mu(x) = \nu^2(x) = (x/X)^2, \tag{5.2}$$

where X is a constant having the same dimensions as x.

The Lagrangian for (5.1) is

$$L = e^{2x/X}\dot{t}^2 - \dot{x}^2 - e^{2(x/X)^2}(\dot{y}^2 + \dot{z}^2). \tag{5.3}$$

Using this Lagrangian in (1.54) we get the following set of determining equations.

$$\xi_t = 0,\ \xi_x = 0,\ \xi_y = 0,\ \xi_z = 0,\ A_s = 0, \tag{5.4}$$

$$2e^{2x/X}\eta^0_s = A_t,\ \ -2\eta^1_s = A_x,\ \ -2e^{2(x/X)^2}\eta^2_s = A_y,\ \ -2e^{2x/X}\eta^3_s = A_z, \tag{5.5}$$

$$2\eta^1_x - \xi_s = 0,\ \eta^2_z + \eta^3_y = 0,\ \eta^1_y + e^{2(x/X)^2}\eta^2_x = 0,\ \eta^1_z + e^{2(x/X)^2}\eta^3_x = 0, \tag{5.6}$$

$$e^{2x/X}\eta^0_x - \eta^1_t = 0,\ e^{2x/X}\eta^0_y - e^{2(x/X)^2}\eta^2_t = 0,\ e^{2x/X}\eta^0_z - e^{2(x/X)^2}\eta^3_t = 0, \tag{5.7}$$

$$2\eta^1 + X(2\eta^0_t - \xi_s) = 0,\ 4x\eta^1 + X^2(2\eta^2_y - \xi_s) = 0,\ 4x\eta^1 + X^2(2\eta^3_z - \xi_s) = 0. \tag{5.8}$$

Solving the above set of equations by back and forth substitution we get

$$\mathbf{Y}_0 = \frac{\partial}{\partial t},\ \mathbf{Y}_1 = \frac{\partial}{\partial y},\ \mathbf{Y}_2 = \frac{\partial}{\partial z},\ \mathbf{Y}_3 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ \mathbf{Z}_0 = \frac{\partial}{\partial s},\ A = c, \tag{5.9}$$

where c is a constant, $\mathbf{Y}_0$ corresponds to energy conservation, $\mathbf{Y}_1$ and $\mathbf{Y}_2$ correspond to linear momentum conservation along y and z, while $\mathbf{Y}_3$ corresponds to angular momentum conservation in the yz plane.

Since GWs are non-static spacetimes therefore the static spacetime (5.3) is perturbed with a time-dependent small parameter (for definiteness ϵt) to make it slightly non-static. For this in (5.1) we take [93]

$$\nu(x) = \frac{x}{X} + \epsilon\frac{t}{T} \text{ and } \mu(x) = (\frac{x}{X})^2 + \epsilon\frac{t}{T}, \tag{5.10}$$

where T is a constant having dimensions of t. Its first-order perturbed Lagrangian is

$$L = e^{2x/X}\dot{t}^2 - \dot{x}^2 - e^{2(x/X)^2}(\dot{y}^2 + \dot{z}^2) + \frac{2\epsilon t}{T}[e^{2x/X}\dot{t}^2 - e^{2(x/X)^2}(\dot{y}^2 + \dot{z}^2)] + O(\epsilon^2).$$

Using this perturbed Lagrangian and the exact symmetry generators given in (5.9), we obtain the following set of determining equations

$$\xi_t = 0,\ \xi_x = 0,\ \xi_y = 0,\ \xi_z = 0,\ A_s = 0, \tag{5.11}$$

$$2e^{2x/X}\eta^0_s = A_t,\ -2\eta^1_s = A_x,\ -2e^{2(x/X)^2}\eta^2_s = A_y,\ -2e^{2x/X}\eta^3_s = A_z, \tag{5.12}$$

$$2\eta^1_x - \xi_s = 0,\ \eta^2_z + \eta^3_y = 0,\ \eta^1_y + e^{2(x/X)^2}\eta^2_x = 0,\ \eta^1_z + e^{2(x/X)^2}\eta^3_x = 0, \tag{5.13}$$

$$e^{2x/X}\eta^0_x - \eta^1_t = 0,\ e^{2x/X}\eta^0_y - e^{2(x/X)^2}\eta^2_t = 0,\ e^{2x/X}\eta^0_z - e^{2(x/X)^2}\eta^3_t = 0, \tag{5.14}$$

$$\frac{2a_0}{T} + \frac{2}{X}\eta^1 + 2\eta^0_t - \xi_s = 0,\ -\frac{2a_0}{T} - \frac{4x}{X^2}\eta^1 - 2\eta^2_y + \xi_s = 0, \tag{5.15}$$

$$-\frac{2a_0}{T} - \frac{4x}{X^2}\eta^1 - 2\eta^3_z + \xi_s = 0. \tag{5.16}$$

In these equations only one constant, i.e. a_0 corresponding to the exact symmetry generator $\mathbf{Y}_0$, appears. Equation (5.11) yields

$$\xi = f_1(s),\ A = A(t, x, y, z). \tag{5.17}$$

Integrating (5.12) with respect to s we get

$$\begin{aligned}
\eta^0 &= \frac{s}{2}e^{-2x/X}A_t + f_2(t,x,y,z), &(5.18)\\
\eta^1 &= -\frac{s}{2}A_x + f_3(t,x,y,z), &(5.19)\\
\eta^2 &= -\frac{s}{2}e^{-2(x/X)^2}A_y + f_4(t,x,y,z), &(5.20)\\
\eta^3 &= -\frac{s}{2}e^{-2(x/X)^2}A_z + f_5(t,x,y,z), &(5.21)
\end{aligned}$$

We use (5.18) and (5.19) in the first of (5.15). Then we differentiate it twice with respect to s. Integrating the resulting equation we obtain

$$\xi = \frac{1}{2}b_0s^2 + b_1s + b_2. \tag{5.22}$$

Again from the first of (5.15) we obtain

$$f_3(t,x,y,z) = f_6(x,y,z) - Xf_{2_t}(t,x,y,z). \tag{5.23}$$

We substitute (5.23) in (5.19). Then we put (5.19) along with (5.22), in the first of (5.15), from this we obtain

$$f_6 = \frac{X}{2}(b_1 - \frac{2a_0}{T}). \tag{5.24}$$

We use (5.19) and (5.22) in the second of (5.13) and then separate the resulting equation for different powers of s. This give two equations which on integration yields

$$A = -\frac{1}{2}b_0x^2 + xf_7(t,y,z) + f_8(t,y,z) \tag{5.25}$$

and

$$f_2(t,x,y,z) = -\frac{b_1xt}{2X} + \int f_9(t,y,z)dt + f_{10}(x,y,z). \tag{5.26}$$

From the second of (5.15) and (5.16), with the use of (5.19), (5.20) and (5.22) we obtain

$$f_4(t,x,y,z) = \frac{2x}{X}\int f_9(t,y,z)dy + yb_1(\frac{1}{2} - \frac{x^2}{X^2} - \frac{x}{X}) + y\frac{a_0}{T}(\frac{2x}{X} - 1) + g_1(t,y,z), \tag{5.27}$$

$$f_5(t,x,y,z) = \frac{2x}{X}\int f_9(t,y,z)dz + zb_1(\frac{1}{2} - \frac{x^2}{X^2} - \frac{x}{X}) + z\frac{a_0}{T}(\frac{2x}{X} - 1) + g_2(t,y,z), \tag{5.28}$$

respectively. Use of (5.18) and (5.19) in the first of (5.14) gives

$$f_7(t,y,z) = 0 \text{ and } f_8 = g_3(y,z). \tag{5.29}$$

Now second of (5.15) yields

$$b_0 = 0, \tag{5.30}$$

$$g_3(y,z) = yg_4(z) + g_5(z), \tag{5.31}$$

From (5.16) we obtain

$$g_4(z) = b_3z + b_4 \text{ and } g_5(z) = b_5z + b_6. \tag{5.32}$$

The first of (5.13) yields

$$b_1 = 0. \tag{5.33}$$

From the third of (5.13) we get

$$b_3 = 0, \text{ and } b_4 = 0, \tag{5.34}$$

$$f_9 = yg_6(t) + g_7(t). \tag{5.35}$$

Now the third of (5.13) gives

$$g_6(t) = 0 \text{ and } g_7(t) = -\frac{a_0}{T}, \tag{5.36}$$

$$g_1 = g_9(t,z). \tag{5.37}$$

From the fourth of (5.13) we obtain

$$b_5 = 0 \text{ and } g_2 = g_{10}(t, y). \tag{5.38}$$

The second of (5.14) yields

$$g_9 = h_1(z)t + h_2(z), \tag{5.39}$$
$$f_{10} = yh_1(z)e^{2x/X(x/X-1)} + h_3(x, z). \tag{5.40}$$

From the third of (5.14) we have

$$h_1 = b_7 z + b_8, \tag{5.41}$$
$$g_{10} = (b_7 y + b_9)t + h_4(z), \tag{5.42}$$
$$h_3 = h_5(x) + zb_9 e^{2x/X(x/X-1)}. \tag{5.43}$$

From the second of (5.13) we get

$$b_7 = 0. \tag{5.44}$$

The first of (5.14) yields

$$b_8 = 0, \; b_8 = 0 \text{ and } h_5 = b_{10}. \tag{5.45}$$

From the second of (5.13) we obtain

$$h_2 = b_{11} z + b_{12}, \tag{5.46}$$
$$h_4 = -b_{11} z + b_{13}. \tag{5.47}$$

Therefore

$$\xi = b_2, \; A = b_6, \tag{5.48}$$
$$\eta^0 = -\frac{a_0 t}{T} + b_{10}, \; \eta^1 = 0, \tag{5.49}$$
$$\eta^2 = -\frac{a_0 y}{T} + b_{11} z + b_{12}, \tag{5.50}$$
$$\eta^3 = -\frac{a_0 z}{T} - b_{11} z + b_{13}. \tag{5.51}$$

Thus the non-trivial first-order approximate symmetry generator is

$$\mathbf{Y}_a = \frac{\partial}{\partial t} - \frac{\epsilon}{T}(t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}). \tag{5.52}$$

Using (1.65) and (1.66) we obtain the corresponding stable (non-trivial) approximate first integral

$$I = e^{2x/X}\dot{t} + \frac{2\epsilon}{T}[e^{2x/X}t\dot{t} - e^{2(x/X)^2}(y\dot{y} + z\dot{z})]. \tag{5.53}$$

Using the time-like non-trivial first-order approximate Noether symmetry generator (5.52) we contract the energy momentum vector. This gives the corresponding conserved quantity

$$Q = E - \frac{\epsilon}{T}(tE + yp_y + zp_z), \tag{5.54}$$

where E is the energy and p is the momentum. This gives the energy imparted to the test particles with energy and momentum given by (5.54). However this does not give the energy in the spacetime field.

5.1.2 First-Order Approximate Noether Symmetries of the PP-Wave Spacetime

To check the conserved quantity Q, in the pp-wave spacetime, we investigate the first-order approximate Noether symmetries for this spacetime.

The line element for the pp-waves [94] is

$$\begin{aligned} ds^2 &= h\omega^2[(x^2-y^2)\sin(\omega(t-z)) + 2xy\cos(\omega(t-z))](dt^2 - 2dtdz - dz^2) \\ &\quad +dt^2 - dx^2 - dy^2 - dz^2, \end{aligned} \tag{5.55}$$

where h is the amplitude of the wave and ω is the frequency.

The Lagrangian for the exact pp-waves (5.55) is

$$\begin{aligned} L &= h\omega^2[(x^2-y^2)\sin(\omega(t-z)) + 2xy\cos(\omega(t-z))](\dot{t}^2 + \dot{z}^2 - 2\dot{t}\dot{z}) \\ &\quad +\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \end{aligned} \tag{5.56}$$

For the Lagrangian (5.55), from (1.54), we get the following set of determining equations

$$\xi_t = 0,\ \xi_x = 0,\ \xi_y = 0,\ \xi_z = 0,\ A_s = 0, \tag{5.57}$$

$$2h\omega^2[(x^2 - y^2)\sin(\omega(t-z)) + 2xy\cos(\omega(t-z))]\eta^0_s = A_t,\ \ -2\eta^1_s = A_x, \tag{5.58}$$

$$-2\eta^2_s = A_y,\ \eta^0_s = A_z,\ \eta^0_z = 0,\ 2\eta^1_x - \xi_s = 0,\ 2\eta^2_y - \xi_s = 0, \tag{5.59}$$

$$\eta^1_y + \eta^2_x = 0,\ \eta^0_x - 2\eta^1_z = 0,\ \eta^0_y - 2\eta^2_z = 0,\ \eta^0_t + \eta^3_z - \xi_s = 0, \tag{5.60}$$

$$2h\omega^2[(x^2 - y^2)\sin(\omega(t-z)) + 2xy\cos(\omega(t-z))]\eta^0_x - 2\eta^1_t + \eta^3_x = 0, \tag{5.61}$$

$$2h\omega^2[(x^2 - y^2)\sin(\omega(t-z)) + 2xy\cos(\omega(t-z))]\eta^0_y - 2\eta^2_t + \eta^3_y = 0, \tag{5.62}$$

$$\begin{aligned}&\omega[(x^2 - y^2)\cos(\omega(t-z)) - 2xy\sin(\omega(t-z))](\eta^0 - \eta^3) + 2[x\sin(\omega(t-z))\\ &+y\cos(\omega(t-z))]\eta^1 + 2[x\cos(\omega(t-z)) - y\sin(\omega(t-z))]\eta^2 +\\ &[(x^2 - y^2)\sin(\omega(t-z)) + 2xy\cos(\omega(t-z))](2\eta^0_t - \xi_s) + \frac{1}{h\omega^2}\eta^3_t = 0.\end{aligned} \tag{5.63}$$

Solving the above system of determining equations (5.57) - (5.63) by the usual method, we get the following symmetry generator and the gauge function as a constant

$$\mathbf{Y}_0 = \frac{\partial}{\partial t} + \frac{\partial}{\partial z},\ \mathbf{Z}_0 = \frac{\partial}{\partial s},\ A = c. \tag{5.64}$$

To investigate the approximate symmetries of pp-waves we first remove the t-dependent part of (5.55) and putting $h = 1$ to define a static spacetime [95]

$$ds^2 = \omega^2[(x^2 - y^2) + 2xy](dt^2 - 2dtdz - dz^2) + dt^2 - dx^2 - dy^2 - dz^2. \tag{5.65}$$

For this static spacetime we obtain the Lagrangian

$$L = \omega^2[(x^2 - y^2) + 2xy](\dot{t}^2 + \dot{z}^2 - 2\dot{t}\dot{z}) + \dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{5.66}$$

For the Lagrangian (5.66) the set of determining equations is

$$\xi_t = 0,\ \xi_x = 0,\ \xi_y = 0,\ \xi_z = 0,\ A_s = 0,\ \ -2\eta^1_s = A_x, \tag{5.67}$$

$$-2\eta^2_s = A_y,,\ 2\eta^1_x - \xi_s = 0,\ 2\eta^2_y - \xi_s = 0,\ \eta^1_y + \eta^2_x = 0, \tag{5.68}$$

$$\omega^2(x^2 - y^2 + xy)(\eta^0_x - \eta^3_x) + \eta^0_x - \eta^1_t = 0, \tag{5.69}$$

$$\omega^2(x^2 - y^2 + xy)(\eta^0_y - \eta^3_y) + \eta^0_y - \eta^2_t = 0, \tag{5.70}$$

$$\omega^2(x^2 - y^2 + xy)(\eta^3_y - \eta^0_y) - \eta^2_z - \eta^3_y = 0, \tag{5.71}$$

$$2\omega^2[(x+y)\eta^1 + (x-y)\eta^2] + 2\omega^2(x^2 - y^2 + xy)(\eta^0_t - \eta^3_t) +$$
$$2\eta^0_t - [\omega^2(x^2 - y^2 + xy) + 1]\xi_s = 0, \tag{5.72}$$

$$2\omega^2[(x+y)\eta^1 + (x-y)\eta^2] - 2\omega^2(x^2 - y^2 + xy)(\eta^0_z - \eta^3_z) -$$
$$2\eta^3_z - [\omega^2(x^2 - y^2 + xy) - 1]\xi_s = 0, \tag{5.73}$$

$$2\omega^2[(x+y)\eta^1 + (x-y)\eta^2] - \omega^2(x^2 - y^2 + xy)(\eta^0_z - \eta^3_z$$
$$-\eta^0_t + \eta^3_t) - \eta^0_z + \eta^3_t - 2\omega^2(x^2 - y^2 + xy)\xi_s = 0, \tag{5.74}$$

$$\omega^2(x^2 - y^2 + xy)(\eta^3_x - \eta^0_x) - \eta^1_z - \eta^3_z = 0, \tag{5.75}$$

$$2\omega^2(x^2 - y^2 + xy)(\eta^0_s - \eta^3_s) + 2\eta^0_s = A_t, \tag{5.76}$$

$$2\omega^2(x^2 - y^2 + xy)(\eta^3_s - \eta^0_s) + 2\eta^3_s = -A_z. \tag{5.77}$$

Solving the above system of equations by the same method used earlier, we obtain the following symmetry generators with a constant gauge function

$$\mathbf{Y}_0 = \frac{\partial}{\partial t},\ \mathbf{Y}_1 = \frac{\partial}{\partial z},\ \mathbf{Z}_0 = \frac{\partial}{\partial s},\ A = c. \tag{5.78}$$

To obtain the approximate Noether symmetries of the pp-waves the exact pp-wave spacetime (5.55) is considered as a perturbation on the static spacetime (5.56). For this purpose the amplitude $h = \epsilon$, is taken as a small parameter and the line element of the perturbed pp-waves is

$$\begin{aligned} ds^2 &= \omega^2[(x^2 - y^2) + 2xy + \epsilon\{(x^2 - y^2)\sin(\omega(t-z)) + 2xy\cos(\omega(t-z))\}](dt^2 \\ &\quad -2dtdz - dz^2) + dt^2 - dx^2 - dy^2 - dz^2. \end{aligned} \tag{5.79}$$

The Lagrangian for the above perturbed spacetime (5.79) is

$$L = \omega^2[(x^2-y^2)+2xy+\epsilon\{(x^2-y^2)\sin(\omega(t-z))+2xy\cos(\omega(t-z))\}](\dot{t}^2+ \dot{z}^2-2\dot{t}\dot{z})+\dot{t}^2-\dot{x}^2-\dot{y}^2-\dot{z}^2. \tag{5.80}$$

The set of determining equations for the Lagrangian (5.80) is

$$\xi_t=0,\ \xi_x=0,\ \xi_y=0,\ \xi_z=0,\ A_s=0,\ -2\eta^1_s=A_x, \tag{5.81}$$

$$-2\eta^2_s=A_y,,\ 2\eta^1_x-\xi_s=0,\ 2\eta^2_y-\xi_s=0,\ \eta^1_y+\eta^2_x=0, \tag{5.82}$$

$$\omega^2(x^2-y^2+xy)(\eta^0_x-\eta^3_x)+\eta^0_x-\eta^1_t=0, \tag{5.83}$$

$$\omega^2(x^2-y^2+xy)(\eta^0_y-\eta^3_y)+\eta^0_y-\eta^2_t=0, \tag{5.84}$$

$$\omega^2(x^2-y^2+xy)(\eta^3_y-\eta^0_y)-\eta^2_z-\eta^3_y=0, \tag{5.85}$$

$$2\omega^2[(x+y)\eta^1+(x-y)\eta^2]+2\omega^2(x^2-y^2+xy)(\eta^0_t-\eta^3_t)+ 2\eta^0_t-[\omega^2(x^2-y^2+xy)+1]\xi_s+\omega^3[(x^2-y^2)\cos(\omega(t-z)) -2xy\sin(\omega(t-z))](a_0-a_1)=0, \tag{5.86}$$

$$2\omega^2[(x+y)\eta^1+(x-y)\eta^2]-2\omega^2(x^2-y^2+xy)(\eta^0_z-\eta^3_z)- 2\eta^3_z-[\omega^2(x^2-y^2+xy)-1]\xi_s+\omega^3[(x^2-y^2)\cos(\omega(t-z)) -2xy\sin(\omega(t-z))](a_0-a_1)=0, \tag{5.87}$$

$$2\omega^2[(x+y)\eta^1+(x-y)\eta^2]-\omega^2(x^2-y^2+xy)(\eta^0_z-\eta^3_z -\eta^0_t+\eta^3_t)-\eta^0_z+\eta^3_t-2\omega^2(x^2-y^2+xy)\xi_s-2\omega^3[(x^2- y^2)\cos(\omega(t-z))-2xy\sin(\omega(t-z))](a_0-a_1)=0, \tag{5.88}$$

$$\omega^2(x^2-y^2+xy)(\eta^3_x-\eta^0_x)-\eta^1_z-\eta^3_z=0, \tag{5.89}$$

$$2\omega^2(x^2-y^2+xy)(\eta^0_s-\eta^3_s)+2\eta^0_s=A_t, \tag{5.90}$$

$$2\omega^2(x^2-y^2+xy)(\eta^3_s-\eta^0_s)+2\eta^3_s=-A_z. \tag{5.91}$$

For $\epsilon=0$, the above Lagrangian (5.80) reduces to (5.66). Using this first-order perturbed Lagrangian and the three exact symmetry generators given by (5.78), in (1.63), in the resulting system of determining equations two constants corresponding to the exact symmetry generators

$\mathbf{Y}_0$ and $\mathbf{Y}_1$, given in (5.78), appear. Solving the system of equations (5.81) - (5.91), by the same back and forth substitution method used earlier the two constants have to be eliminated for consistency of the determining equations, making them homogeneous. The resulting system is the same as for the static (exact) spacetime (5.65). Thus there is no non-trivial approximate symmetry for this perturbed Lagrangian and the gauge function is a constant. Hence we can not obtain the conserved quantity in the case of perturbed pp-wave spacetime. Only the three exact symmetry generators are recovered as trivial first-order approximate Noether symmetries which give trivial first-order approximate conservation laws for energy and linear momentum along z.

5.2 Second-Order Approximate Symmetries of the Geodesic Equations for Plane Wave Spacetimes

In this section we first analyze pp-waves and then we study the plane wave-like spacetime.

5.2.1 Approximate Symmetries of the Geodesic Equations for the PP-Wave Spacetime

For the perturbed pp-wave spacetime (5.79) we have the system of first-order perturbed geodesic equations and ϵ^2 does not appear [95],

$$\begin{aligned} E_1 : \ddot{t} + \omega^2(\dot{t} - \dot{z})[(x+y)\dot{x} + (x-y)\dot{y}] + \epsilon[\frac{\omega^3}{2}\{(x^2-y^2)\cos(\omega(z-t))+ \\ 2xy\sin(\omega(z-t))\}(\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) - \omega^2(\dot{t}-\dot{z})\{x\sin(\omega(z-t)) - y\cos(\omega(z \\ -t))\}\dot{x} + \omega^2\{y\sin(\omega(z-t)) + x\cos(\omega(z-t))\}\dot{y}] = 0, \end{aligned} \quad (5.92)$$

$$E_2 : \ddot{x} + [\omega^2(x+y) - \epsilon\omega^2\{x\sin(\omega(z-t)) - y\cos(\omega(z-t))\}](\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) = 0, \quad (5.93)$$

$$E_3 : \ddot{y} + [\omega^2(x+y) - \epsilon\omega^2\{x\cos(\omega(z-t)) - y\sin(\omega(z-t))\}](\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) = 0, \quad (5.94)$$

$$\begin{aligned} E_4 : \ddot{z} + \omega^2(\dot{t} - \dot{z})[(x+y)\dot{x} + (x-y)\dot{y}] + \epsilon[\frac{\omega^3}{2}\{(x^2-y^2)\cos(\omega(z-t))+ \\ 2xy\sin(\omega(z-t))\}(\dot{t}^2 + \dot{z}^2 - \dot{t}\dot{z}) - \omega^2(\dot{t}-\dot{z})\{x\sin(\omega(z-t)) - y\cos(\omega(z \\ -t))\}\dot{x} + \omega^2\{y\sin(\omega(z-t)) + x\cos(\omega(z-t))\}\dot{y}] = 0. \end{aligned} \quad (5.95)$$

Since there is no quadratic term in ϵ, in the above geodesic equations (5.92) - (5.95), we cannot apply the definition of second-order approximate symmetries, which gives us the energy re-scaling factor to them. This behavior is consistent with the pp-wave geometry in which the wave front moves as parallel planes and the spacetime curvature is absolutely zero before the pp-wave pulse arrives and after it has passed [40]. There is no region where there is a *slight* shift from the flat geometry as required for obtaining an approximate symmetry. Thus the proposal of approximate Lie symmetries for determining the energy content of GWs cannot be checked for pp-waves.

5.2.2 Approximate Symmetries of the Geodesic Equations for the Plane Wave-Like Spacetime

Retaining ϵ^2 in (5.10) and neglecting its higher powers, second-order perturbed geodesic equations are obtained [95]

$$E_1 : \ddot{t} + \frac{2}{X}\dot{t}\dot{x} - \frac{\epsilon}{T}[\dot{t}^2 - (\dot{y}^2 + \dot{z}^2)e^{2((x/X)^2 - x/X)}] + \frac{\epsilon^2 t}{T^2}[\dot{t}^2 + (\dot{y}^2 + \dot{z}^2)e^{2((x/X)^2 - x/X)}] + O(\epsilon^3) = 0, \tag{5.96}$$

$$E_2 : \ddot{x} + \frac{e^{2x/X}}{X}\dot{t}^2 - \frac{2x}{X^2}e^{2(x/X)^2}(\dot{y}^2 + \dot{z}^2) + \frac{2t\epsilon}{TX}[e^{2x/X}\dot{t}^2 - \frac{2x}{X}e^{2(x/X)^2}(\dot{y}^2 + \dot{z}^2)] + \frac{t^2\epsilon^2}{T^2X}[e^{2x/X}\dot{t}^2 - \frac{2x}{X}e^{2(x/X)^2}(\dot{y}^2 + \dot{z}^2)] + O(\epsilon^3) = 0, \tag{5.97}$$

$$E_3 : \ddot{y} + \frac{4x^2}{X^2}\dot{x}\dot{y} + \frac{2\epsilon}{T}\dot{t}\dot{y} - \frac{2t\epsilon^2}{T^2}\dot{t}\dot{y} + O(\epsilon^3) = 0, \tag{5.98}$$

$$E_4 : \ddot{z} + \frac{4x^2}{X^2}\dot{x}\dot{z} + \frac{2\epsilon}{T}\dot{t}\dot{z} - \frac{2t\epsilon^2}{T^2}\dot{t}\dot{z} + O(\epsilon^3) = 0. \tag{5.99}$$

We apply the second prolongation $\mathbf{X}^{[2]}$ of the generator $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2$ defined in (2.40), to (5.96) - (5.99), which yields

$$\begin{aligned}\mathbf{X}^{[2]}E_1 \quad = \quad & [\eta^0_{0,ss} + \epsilon\eta^0_{1,ss} + \epsilon^2\eta^0_{2,ss} + (\eta^0_0 + \epsilon\eta^0_1 + \epsilon^2\eta^0_2)\frac{\epsilon^2}{T^2}\{\dot{t}^2 + (\dot{y}^2 + \dot{z}^2) \\ & e^{2((x/X)^2 - x/X)}\} + (\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2)(\frac{\epsilon}{T} + \frac{\epsilon^2 t}{T^2})\{\frac{2}{X}(\frac{x}{X} - 1)(\dot{y}^2 + \\ & \dot{z}^2)e^{2((x/X)^2 - x/X)}\} + 2(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s})(\frac{\dot{x}}{X} - \frac{\epsilon\dot{t}}{T} + \frac{\epsilon^2 t\dot{t}}{T^2}) +\end{aligned}$$

$$2(\eta^1_{0,s}+\epsilon\eta^1_{1,s}+\epsilon^2\eta^1_{2,s})(\frac{\dot{t}}{X})+\frac{2}{T}\{\dot{y}(\eta^2_{0,s}+\epsilon\eta^2_{1,s}+\epsilon^2\eta^2_{2,s})+\dot{z}(\eta^3_{0,s}$$
$$+\epsilon\eta^3_{1,s}+\epsilon^2\eta^3_{2,s})\}(\epsilon+\frac{\epsilon^2 t}{T})e^{2((x/X)^2-x/X)}]_{E_j=0}=0, \tag{5.100}$$

$$\begin{aligned}\mathbf{X}^{[2]}E_2 \quad = \quad & [\eta^1_{0,ss}+\epsilon\eta^1_{1,ss}+\epsilon^2\eta^1_{2,ss}+\frac{2}{TX}(\eta^0_0+\epsilon\eta^0_1+\epsilon^2\eta^0_2)(\epsilon+\frac{\epsilon^2 t}{T})\{\dot{t}^2 e^{2x/X}\\ & -(\dot{y}^2+\dot{z}^2)e^{2(x/X)^2}\}+\frac{2}{X^2}(\eta^1_0+\epsilon\eta^1_1+\epsilon^2\eta^1_2)\{\dot{t}^2 e^{2x/X}-(\dot{y}^2+\dot{z}^2)(1\\ & +\frac{4x^2}{X^2})e^{2(x/X)^2}\}(1+\frac{2\epsilon t}{T}+\frac{\epsilon^2 t^2}{T^2})+\frac{2\dot{t}}{X}(\eta^0_{0,s}+\epsilon\eta^0_{1,s}+\epsilon^2\eta^0_{2,s})e^{2x/X}(1\\ & +\frac{4\epsilon t}{T}+\frac{2\epsilon^2 t^2}{T^2})+-\frac{4x}{X^2}\{\dot{y}(\eta^2_{0,s}+\epsilon\eta^2_{1,s}+\epsilon^2\eta^2_{2,s})+\dot{z}(\eta^3_{0,s}+\epsilon\eta^3_{1,s}+\\ & \epsilon^2\eta^3_{2,s})\}(1+\frac{2\epsilon t}{T}+\frac{\epsilon^2 t^2}{T^2})e^{2(x/X)^2}]_{E_j=0}=0, \end{aligned} \tag{5.101}$$

$$\begin{aligned}\mathbf{X}^{[2]}E_3 \quad = \quad & [\eta^2_{0,ss}+\epsilon\eta^2_{1,ss}+\epsilon^2\eta^2_{2,ss}-\frac{2}{T^2}\dot{t}\dot{y}(\eta^0_0+\epsilon\eta^0_1+\epsilon^2\eta^0_2)+\frac{4}{X^2}\dot{x}\dot{y}(\eta^1_0+\epsilon\eta^1_1\\ & +\epsilon^2\eta^1_2)+\frac{2}{T}\dot{y}(\epsilon-\frac{\epsilon^2 t}{T})(\eta^0_{0,s}+\epsilon\eta^0_{1,s}+\epsilon^2\eta^0_{2,s})+\frac{4x}{X^2}\dot{y}(\eta^1_{0,s}+\epsilon\eta^1_{1,s}+\\ & \epsilon^2\eta^1_{2,s})+2(\eta^2_{0,s}+\epsilon\eta^2_{1,s}+\epsilon^2\eta^2_{2,s})(\frac{2x}{X^2}\dot{x}+\frac{\epsilon}{T}\dot{t}-\frac{\epsilon^2 t}{T^2}\dot{t})]_{E_j=0}=0, \end{aligned} \tag{5.102}$$

$$\begin{aligned}\mathbf{X}^{[2]}E_4 \quad = \quad & [\eta^3_{0,ss}+\epsilon\eta^3_{1,ss}+\epsilon^2\eta^3_{2,ss}-\frac{2}{T^2}\dot{t}\dot{z}(\eta^0_0+\epsilon\eta^0_1+\epsilon^2\eta^0_2)+\frac{4}{X^2}\dot{x}\dot{z}(\eta^1_0+\epsilon\eta^1_1\\ & +\epsilon^2\eta^1_2)+\frac{2}{T}\dot{z}(\epsilon-\frac{\epsilon^2 t}{T})(\eta^0_{0,s}+\epsilon\eta^0_{1,s}+\epsilon^2\eta^0_{2,s})+\frac{4x}{X^2}\dot{z}(\eta^1_{0,s}+\epsilon\eta^1_{1,s}+\\ & \epsilon^2\eta^1_{2,s})+2(\eta^3_{0,s}+\epsilon\eta^3_{1,s}+\epsilon^2\eta^3_{2,s})(\frac{2x}{X^2}\dot{x}+\frac{\epsilon}{T}\dot{t}-\frac{\epsilon^2 t}{T^2}\dot{t})]_{E_j=0}=0, \end{aligned} \tag{5.103}$$

where $(j=1,2,3,4)$. For $\epsilon=0$, (5.96) - (5.99) yield the equations for the exact case and only retaining first power of ϵ, neglecting its higher powers will give the equations for first-order approximate case. For the exact ($\epsilon=0$) the geodesic equations (5.96) - (5.99) admit the symmetry generators

$$\mathbf{Y}_0=\frac{\partial}{\partial t},\ \mathbf{Y}_1=\frac{\partial}{\partial y},\ \mathbf{Y}_2=\frac{\partial}{\partial z},\ \mathbf{Y}_3=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\ \mathbf{Z}_0=\frac{\partial}{\partial s},\ \mathbf{Z}_1=s\frac{\partial}{\partial s}, \tag{5.104}$$

with the gauge function as a constant.

We use the prolongation coefficients defined in (2.45) - (2.52), the exact symmetry generators given by (5.104), the first-order approximate symmetry generators (which are consist of the exact symmetry generators and the non-trivial symmetry generator given by (5.52)), and the second-order perturbed geodesic equations, in above equations (5.100) - (5.103). We get the following set of determining equations

$$\begin{gathered}\xi_{2_{tt}} - \frac{e^{2(x/X)}}{X}\xi_{2_x} = 0,\ \xi_{2_{xx}} = 0,\ \xi_{2_{yy}} + \frac{2xe^{2(x/X)^2}}{X^2}\xi_{2_x} = 0,\\ \xi_{2_{zz}} + \frac{2xe^{2(x/X)^2}}{X^2}\xi_{2_x} = 0,\ \xi_{2_{tx}} - \frac{1}{X}\xi_{2_t} = 0,\ \xi_{2_{ty}} = 0,\\ \xi_{2_{tz}} = 0,\ \xi_{2_{xy}} - \frac{2x}{X^2}\xi_{2_y} = 0,\ \xi_{2_{xz}} - \frac{2x}{X^2}\xi_{2_z} = 0,\ \xi_{2_{yz}} = 0,\end{gathered} \tag{5.105}$$

$$\eta^0_{2_{tt}} + \frac{2e^{2(x/X)}}{X}\eta^0_{2_x} + \frac{2}{X}\eta^1_{2_t} + \frac{a_0}{T^2} - 2\xi_{2_{st}} = 0, \tag{5.106}$$

$$\eta^1_{2_{tt}} + \frac{2e^{2(x/X)}}{X}(\eta^0_{2_t} - \frac{1}{2}\eta^1_{2_x} - \frac{2t}{T^2}a_0) + \frac{2}{X^2}\eta^1_2 = 0, \tag{5.107}$$

$$\eta^2_{2_{tt}} - \frac{e^{2(x/X)}}{X}\eta^2_{2_x} = 0,\ \eta^3_{2_{tt}} - \frac{e^{2(x/X)}}{X}\eta^3_{2_x} = 0, \tag{5.108}$$

$$\eta^0_{2_{xx}} + \frac{2}{X}\eta^0_{2_x} = 0,\ \eta^1_{2_{xx}} - 2\xi_{2_{sx}} = 0, \tag{5.109}$$

$$\eta^2_{2_{xx}} + \frac{4x}{X^2}\eta^2_{2_x} = 0,\ \eta^3_{2_{xx}} + \frac{4x}{X^2}\eta^3_{2_x} = 0, \tag{5.110}$$

$$\eta^0_{2_{yy}} + \frac{2xe^{2(x/X)^2}}{X^2}\eta^0_{2_x} + \frac{a_0}{T^2}e^{2[(x/X)^2 - x/X]} = 0, \tag{5.111}$$

$$\eta^1_{2_{yy}} + \frac{2e^{2(x/X)^2}}{X^2}[x\eta^1_{2_x} - (1 + \frac{4x^2}{X^2})\eta^1_2 - 2x(\eta^2_{2_y} - \xi_{2_x})] = 0, \tag{5.112}$$

$$\eta^2_{2_{yy}} + \frac{2x}{X^2}(2\eta^1_{2_y} + \eta^2_{2_x}e^{2(x/X)^2}) - 2\xi_{2_{sy}} = 0, \tag{5.113}$$

$$\eta^3_{2_{yy}} + \frac{2x}{X^2}\eta^3_{2_x}e^{2(x/X)^2} = 0, \tag{5.114}$$

$$\eta^0_{2_{zz}} + \frac{2xe^{2(x/X)^2}}{X^2}\eta^0_{2_x} + \frac{a_0}{T^2}e^{2[(x/X)^2 - x/X]} = 0, \tag{5.115}$$

$$\eta^1_{2_{zz}} + \frac{2e^{2(x/X)^2}}{X^2}[x\eta^1_{2_x} - (1 + \frac{4x^2}{X^2})\eta^1_2 - 2x(\eta^2_{2_y} - \xi_{2_x})] = 0, \tag{5.116}$$

$$\eta^2_{2_{yy}} + \frac{2x}{X^2}\eta^2_{2_x}e^{2(x/X)^2} = 0, \tag{5.117}$$

$$\eta^3_{2_{zz}} + \frac{2x}{X^2}(2\eta^1_{2_z} + \eta^3_{2_x}e^{2(x/X)^2}) - 2\xi_{2_{sz}} = 0, \tag{5.118}$$

$$\frac{2}{X}\eta^1_{2_s} + 2\eta^0_{2_{st}} - \xi_{2_{ss}} = 0,\ \frac{e^{2(x/X)}}{X}\eta^0_{2_s} + \eta^1_{2_{st}} = 0,\ \eta^2_{2_{st}} = 0, \tag{5.119}$$

$$\eta^3_{2_{st}} = 0,\ \frac{1}{X}\eta^0_{2_s} + \eta^0_{2_{sx}} = 0,\ \eta^1_{2_{sx}} - \xi_{2_{ss}} = 0, \tag{5.120}$$

$$\frac{2x}{X^2}\eta^2_{2_s} + \eta^2_{2_{sx}} = 0,\ \frac{2x}{X^2}\eta^3_{2_s} + \eta^3_{2_{sx}} = 0,\ \eta^0_{2_{sy}} = 0, \tag{5.121}$$

$$\frac{2xe^{2(x/X)}}{X}\eta^2_{2_s} - \eta^1_{2_{sy}} = 0,\ \frac{4x}{X^2}\eta^1_{2_s} + 2\eta^2_{2_{sy}} - \xi_{2_{ss}} = 0, \tag{5.122}$$

$$\eta^3_{2_{sy}} = 0,\ \eta^0_{2_{sz}} = 0,\ \frac{2xe^{2(x/X)}}{X}\eta^3_{2_s} - \eta^1_{2_{sz}} = 0, \tag{5.124}$$

$$\eta^2_{2_{sz}} = 0,\ \frac{4x}{X^2}\eta^1_{2_s} + 2\eta^3_{2_{sz}} - \xi_{2_{ss}} = 0, \tag{5.125}$$

$$\eta^0_{2_{st}} + \eta^1_{2_x} - \eta^0_{2_t} + X(\eta^0_{2_{xt}} - \xi_{2_{sx}}) = 0, \tag{5.126}$$

$$\eta^0_{2_x}e^{2(x/X)} - \eta^1_{2_t} - \eta^0_{2_t} + X(\eta^1_{2_{xt}} - \xi_{2_{st}}) = 0, \tag{5.127}$$

$$(2x - X)\eta^2_{2_t} + X^2\eta^2_{2_{xt}} = 0,\ (2x - X)\eta^3_{2_t} + X^2\eta^3_{2_{xt}} = 0, \tag{5.128}$$

$$\frac{1}{X}\eta^1_{2_y} + \eta^0_{2_{ty}} - \xi_{2_{sy}} = 0,\ \frac{1}{X}\eta^0_{2_y} + \eta^1_{2_{ty}} - \frac{2xe^{2(x/X)^2}}{X^2}\eta^2_{2_t} = 0, \tag{5.129}$$

$$\eta^2_{2_{ty}} - \frac{2x}{X^2}\eta^1_{2_t} - \xi_{2_{st}} + \frac{a_0}{T^2} = 0,\ \eta^3_{2_{ty}} + \frac{a_0}{T^2} = 0, \tag{5.130}$$

$$\frac{1}{X}\eta^1_{2_z} + \eta^0_{2_{tz}} - \xi_{2_{sz}} = 0,\ \eta^2_{2_{tz}} + \frac{a_0}{T^2} = 0, \tag{5.131}$$

$$\eta^1_{2_{tz}} + \frac{1}{X}(e^{2(x/X)}\eta^0_{2_z} - \frac{2xe^{2(x/X)^2}}{X}\eta^3_{2_t}) = 0, \tag{5.132}$$

$$\frac{4x}{X^2}\eta^1_{2_t} + 2\eta^3_{2_{tz}} - \xi_{2_{st}} - \frac{a_0}{T^2} = 0, \tag{5.133}$$

$$\frac{1}{X}(1 - \frac{2x}{X})\eta^0_{2_y} + \eta^0_{2_{xy}} = 0,\ \eta^3_{2_{xy}} = 0, \tag{5.134}$$

$$\eta^1_{2_{xy}} - \frac{2x}{X^2}(e^{2(x/X)^2}\eta^2_{2_x} + \eta^1_{2_y}) - \xi_{2_{sy}} = 0, \tag{5.135}$$

$$\eta^2_{2_{xy}} - \xi_{2_{sx}} + \frac{2}{X^2}(2x\eta^1_{2_x} + \eta^1_2) = 0, \tag{5.136}$$

$$\eta^0_{2_{xz}} + \frac{1}{X}(1 - \frac{2x}{X})\eta^0_{2_z} = 0,\ \eta^2_{2_{xz}} = 0, \tag{5.137}$$

$$\eta^1_{2_{xz}} - \frac{2x}{X^2}(e^{2(x/X)^2}\eta^3_{2_x} + \eta^1_{2_z}) - \xi_{2_{sz}} = 0, \tag{5.138}$$

$$\eta^2_{2_{xy}} - \xi_{2_{sx}} + \frac{2}{X^2}(2x\eta^1_{2_x} + \eta^1_2) = 0, \tag{5.139}$$

$$\eta^2_{2_{yz}} = 0,\ \eta^1_{2_{yz}} - \frac{2xe^{2(x/X)^2}}{X^2}(\eta^2_{2_z} + \eta^3_{2_y}) = 0, \tag{5.140}$$

$$\eta^2_{2_{yz}} + \frac{2x}{X^2}\eta^1_{2_z} - \xi_{2_{sz}} = 0,\ \eta^3_{2_{yz}} + \frac{2x}{X^2}\eta^1_{2_y} - \xi_{2_{sz}} = 0, \tag{5.141}$$

$$\eta^0_{2_{ss}} = 0,\ \eta^1_{2_{ss}} = 0,\ \eta^2_{2_{ss}} = 0,\ \eta^3_{2_{ss}} = 0. \tag{5.142}$$

Like the first-order approximate case for this wavelike spacetime only one constant corresponding to the time translation exact symmetry generator appears in the above equations (5.105) - (5.142). Solving these equations by the same method used earlier in this thesis we get no non-trivial symmetry generator. The first-order approximate symmetry generator given by (5.52) along with the exact symmetry generators given in (5.9) are obtained as trivial approximate symmetry generators and the gauge function is a constant. Hence in the second-order approximation there is non non-trivial approximate symmetry for the plane wave-like spacetime.

It should be noted that for the first-order approximate case we obtain the same set of determining equations (5.105) - (5.142). The solution of these equations is given above.

5.3 Energy (Mass) in a Plane Wave-Like Spacetime

The Lie symmetry algebra for the exact or unperturbed (when $\epsilon = 0$) (as well as the first-order approximate symmetry algebra i.e. when $\epsilon^2 = 0$ and $\epsilon \neq 0$) geodesic equations for the plane wave-like spacetime include the generators of dilation algebra, $\partial/\partial s$, $s\partial/\partial s$ corresponding to

$$\xi(s) = c_0 s + c_1. \tag{5.143}$$

Like the RN [81] and charged-Kerr [82] spacetimes, for the second-order approximate symmetries of the geodesic equations in the determining equations the terms involving $\xi_s = c_0$, do *not* automatically cancel out but collect a scaling factor of [95]

$$\frac{t}{T^2}[\dot{t}^2 + (\dot{y}^2 + \dot{z}^2)e^{2((x/X)^2 - x/X)}]. \tag{5.144}$$

so as to cancel out. The scaling factor (5.144) corresponds to a *re-scaling* of energy (mass) as explained earlier. This scaling factor for the wave-like spacetime involve the derivatives of the coordinates t and ϕ. Since derivatives only apply to the paths of the particles, to get the energy in the spacetime field we replace these derivatives by the exact first integrals of the geodesic equations

$$\dot{t} = \frac{c_1}{2}e^{-2(x/X)}, \; \dot{y} = c_2 e^{-2((x/X)^2 - x/X)} = \dot{z}. \tag{5.145}$$

Using these first integrals (5.145) in (5.144) we obtain the scaling factor (5.146) for the plane wave-like spacetime [93]

$$\frac{t}{4T}[e^{-4(x/X)} + 2e^{-2((x/X)^2+x/X)}]. \tag{5.146}$$

This energy expression is plotted below for different values of t and x, using Mathematica 5.0. The values of X and T are arbitrary. The above scaling factor for this wave-like spacetime depends linearly on t and in both diagrams below the energy in the gravitational field increases linearly with time. In Fig. 5-1 the energy is seen to decrease along x and disappears sharply close to $x = 0$. To see the variation with x we enlarge the diagram by reducing the range of x in Fig. 5-2. As we move along x the increase in energy with time becomes gradual. Since the small parameter ϵ (which is considered as the strength of the wave) is arbitrary the units of energy are arbitrarily chosen. Throughout this thesis gravitational units are used and space, time and mass are given in seconds.

Admittedly, in this artificial example of the wave-like spacetime energy increases linearly without limit. This is because of the (nonphysical) choice of a linearly increasing component of the metric tensor for convenience of computation, leading to a corresponding increase in the scaling factors (5.146).

5.4 The Weyl and Stress-Energy Tensors for Plane Symmetric Spacetimes

Though the Weyl tensor gives information about the gravitational energy of the spacetime, it is not clear how to obtain a *measure* of the energy in it. For the pure gravitational part and matter part here we give the independent nonzero components of Weyl and stress-energy tensors for the perturbed pp-wave and plane wave-like spacetimes.

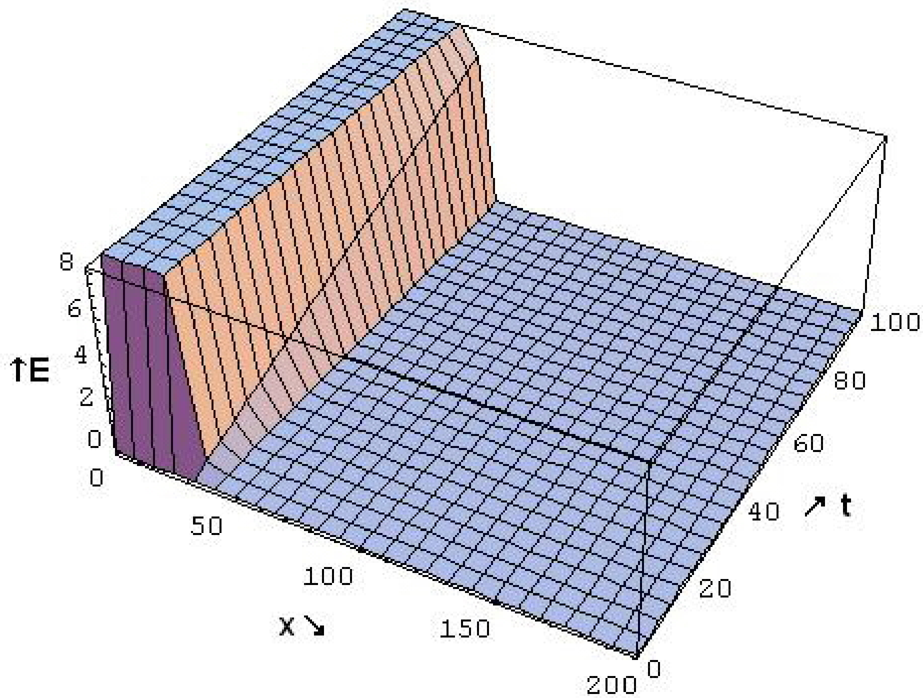

Figure 5-1: Plane symmetric gravitational wave-like spacetime. The energy increases indefinitely in time close to $x = 0$ and then disappears suddenly after some distance. The small parameter ϵ, (considered as strength of the wave) is arbitrary in all the spacetimes discussed in this thesis. Thus the units of energy are chosen arbitrarily. Throughout this thesis gravitational units are adopted and space, time and mass are given in seconds

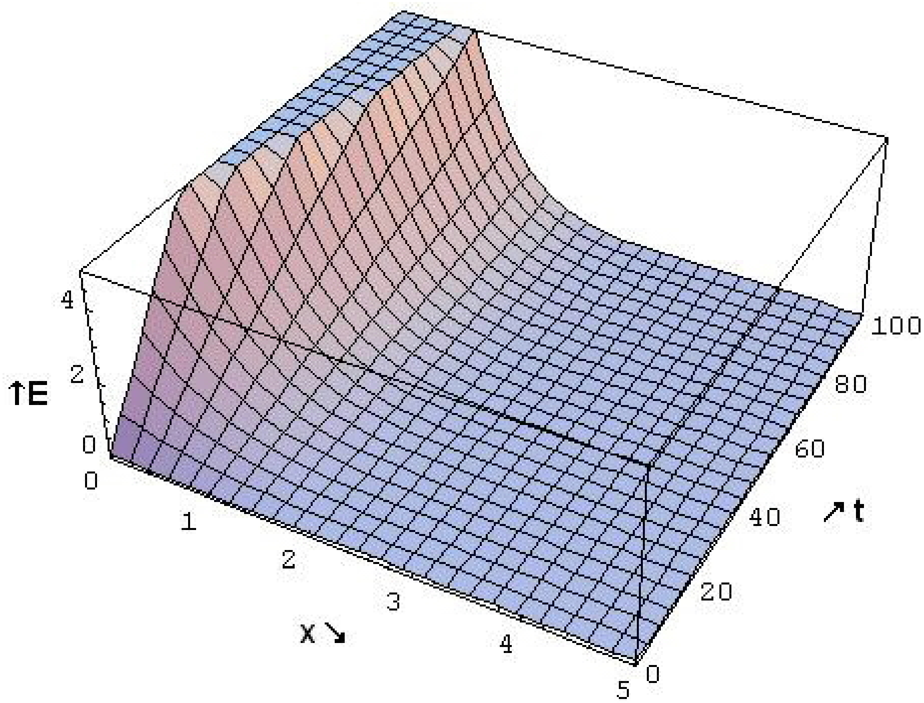

Figure 5-2: This is a expanded version of Fig. 5-1. Here the range of x is shrunk and it is seen that the energy decreases smoothly with distance.

5.4.1 The Weyl and Stress-Energy Tensors for the PP-Wave Spacetime

Following are the independent nonzero components of the Weyl tensor for the perturbed pp-wave spacetime (5.79) [95]

$$
\begin{aligned}
C^{0}{}_{101} &= -\omega^2[\omega^2(x^2-y^2-2xy)+1+\epsilon\{2\omega^2(x^2-y^2-xy)\sin\omega(z-t)- \\
&\quad 2\omega^2 xy\cos\omega(z-t)-\sin\omega(z-t)\}]+O(\epsilon^2), \\
C^{0}{}_{113} &= C^{0}{}_{101} = C^{1}{}_{313} = -C^{0}{}_{202} = C^{0}{}_{223} = C^{2}{}_{323}, \\
C^{0}{}_{102} &= -\omega^2[\omega^2(x^2-y^2-2xy)+1+\epsilon\{2\omega^2(x^2-y^2)\sin\omega(z-t)-\omega^2(x^2 \\
&\quad -y^2+4xy)\cos\omega(z-t)+\cos\omega(z-t)]+O(\epsilon^2), \\
C^{0}{}_{123} &= C^{0}{}_{102} = C^{0}{}_{213} = C^{1}{}_{323}.
\end{aligned} \tag{5.147}
$$

As given in the first section of chapter 1, for usual purposes the form of the Weyl tensor does not matter, but for differential symmetries of the tensor the form is crucial [58]. In covariant form the components of the Weyl tensor are

$$
\begin{aligned}
C_{0101} &= -\omega^2[1-\epsilon\sin\omega(z-t)] = C_{0113} = C_{1313} = -C_{0202} = C_{0223} = C_{2323}, \\
C_{0102} &= -\omega^2[1-\epsilon\cos\omega(z-t)] = C_{0123} = C_{0213} = C_{1323}.
\end{aligned} \tag{5.148}
$$

From here it appears that the (0, 4) form may give the physically relevant quantities as the space dependence in (5.147) does not seem to correspond to the geometry of the pp-wave, while (5.148) does. Here the pure gravitational field which "curves up the void" is seen to be sinusoidal. For this spacetime there is obviously no nonzero component of the stress-energy tensor.

5.4.2 The Weyl and Stress-Energy Tensors for the Plane Wave-like Spacetime

The independent nonzero components of the Weyl tensor for the plane wave-like spacetime (5.10) are

$$
\begin{aligned}
C^0{}_{101} &= \frac{1}{3X^3}(2x+X)+O(\epsilon^2),\\
C^0{}_{202} &= C^0{}_{303}=\frac{e^{2(x/X)^2}}{6X^3}(1+\epsilon\frac{2t}{T})(2x+X)+O(\epsilon^2),\\
C^1{}_{212} &= C^1{}_{313}=-C^0{}_{202},\ C^2{}_{323}=-2C^0{}_{202}.
\end{aligned} \tag{5.149}
$$

From Figs. 5-1 and 5-2 (where the wave is along the x direction), it is clear that the energy in the gravitational field of the plane wave-like spacetime increases with time. Therefore the first component of the Weyl tensor must depend on t linearly which corresponds to the covariant form (given below) and not the mixed form.

$$
\begin{aligned}
C_{0101} &= \frac{e^{2x/X}}{3X^3}(1+\epsilon\frac{2t}{T})(2x+X)+O(\epsilon^2),\\
C_{0202} &= C_{0303}=\frac{e^{2(x/X)^2+2x/X}}{6X^3}(1+\epsilon\frac{4t}{T})(2x+X)+O(\epsilon^2),\\
C_{1212} &= C_{1313}=-C_{0202},\ C_{2323}=-2C_{0202}.
\end{aligned} \tag{5.150}
$$

The nonzero components of the stress-energy tensor for this wave-like spacetime are

$$
\begin{aligned}
T_{00} &= \frac{4e^{2x/X}}{\kappa X^2}(1+\epsilon\frac{2t}{T})(1+\frac{3x^2}{X^2})+O(\epsilon^2),\\
T_{11} &= \frac{-8x}{\kappa X^4}(x+X)+O(\epsilon^2),\\
T_{22} &= T_{33}=\frac{e^{2x^2/X^2}}{\kappa X^4}(1+\epsilon\frac{2t}{T})(2xX+4x^2+3X^2)+O(\epsilon^2),\\
T_{01} &= \frac{\epsilon}{\kappa T X^2}(x-X)+O(\epsilon^2).
\end{aligned} \tag{5.151}
$$

It is worth noting that the x-direction stress has no approximate part of first-order and the approximate part of the energy increases linearly with time and quadratically at large distances. More interestingly there is an approximate momentum in the x-direction that increases linearly with the value of x. This linear increase in energy was built into the metric and it entails the momentum in the x-direction.

Now we give the fraction of energy density imparted to the matter field, by the following expression

$$E_{imp} = \frac{(T_{00})_P}{(T_{00})_E}, \tag{5.152}$$

where $(T_{00})_E$ and $(T_{00})_P$ are the energy densities of the exact (i.e. when $\epsilon = 0$) and first-order approximate spacetime respectively. For the plane wave-like spacetime we have

$$E_{imp} = \epsilon \frac{2t}{T}. \tag{5.153}$$

Here we give the result of this chapter in the form of the following theorem.

Theorem 5.1. *The energy of plane gravitational wave in a plane gravitational wave-like spacetime is the classical energy re-scaled by the factor (5.146).*

Chapter 6

Approximate Noether Symmetries of the Geodesic Equations for Cylindrically Symmetric Gravitational Wave Spacetimes and the Definition of Energy

In this chapter we study first-order approximate Noether symmetries of the cylindrical analogue of the artificially constructed example of a plane "wave-like" spacetime discussed in chapter 5, and of cylindrically symmetric GW spacetime. To obtain the energy content of the cylindrical wave spacetimes we investigate second-order approximate symmetries of the geodesic equations for these spacetimes. Here we also give the non-zero components of the Weyl and stress-energy tensors for the cylindrical GW spacetimes discussed in this chapter.

6.1 First-Order Approximate Noether Symmetries of Cylindrical Wave Spacetimes

In this section we first study the wave-like spacetime and then we investigate the cylindrically symmetric GW spacetime.

6.1.1 First-Order Approximate Noether Symmetries of the Cylindrical Wave-like Spacetime

To investigate first-order approximate symmetries of the Lagrangian for cylindrically symmetric spacetime we consider the following line element of a static spacetime [96]

$$ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - e^{2\mu(\rho)}(a^2 d\phi^2 + dz^2), \tag{6.1}$$

with

$$\nu(\rho) = (\rho/R)^2 \text{and } \mu(\rho) = (\rho/R)^3, \tag{6.2}$$

where R is a constant having the same dimensions as of ρ.

A Lagrangian for the static spacetime (6.1) is

$$L = e^{(\rho/R)^2}\dot{t}^2 - \dot{\rho}^2 - e^{(\rho/R)^3}(a^2\dot{\phi}^2 + \dot{z}^2). \tag{6.3}$$

Using this Lagrangian in (1.54) we get the following set of determining equations.

$$\xi_t = 0,\ \xi_\rho = 0,\ \xi_\phi = 0,\ \xi_z = 0,\ A_s = 0, \tag{6.4}$$

$$2e^{(\rho/R)^2}\eta^0_s = A_t,\ \ -2\eta^1_s = A_\rho,\ \ -2a^2 e^{(\rho/R)^3}\eta^2_s = A_\phi,\ \ -2e^{(\rho/R)^3}\eta^3_s = A_z, \tag{6.5}$$

$$2\eta^1_\rho - \xi_s = 0,\ a^2\eta^2_z + \eta^3_\phi = 0,\ \eta^1_\phi + a^2 e^{(\rho/R)^3}\eta^2_\rho = 0,\ \eta^1_z + e^{(\rho/R)^3}\eta^3_\rho = 0, \tag{6.6}$$

$$e^{(\rho/R)^2}\eta^0_\rho - \eta^1_t = 0,\ e^{(\rho/R)^2}\eta^0_\phi - e^{(\rho/R)^3}\eta^2_t = 0,\ e^{(\rho/R)^2}\eta^0_z - e^{(\rho/R)^3}\eta^3_t = 0, \tag{6.7}$$

$$\frac{2\rho}{R^2}\eta^1 + 2\eta^0_t - \xi_s = 0,\ \frac{3\rho^2}{R^3}\eta^1 + 2\eta^2_\phi - \xi_s = 0,\ \frac{3\rho^2}{R^3}\eta^1 + 2\eta^3_z - \xi_s = 0. \tag{6.8}$$

Solving the above set of equations (6.4) - (6.8) by the same method as for the plane symmetric case we get the following symmetry generators

$$\mathbf{Y}_0=\frac{\partial}{\partial t},\ \mathbf{Y}_1=\frac{\partial}{\partial \phi},\ \mathbf{Y}_2=\frac{\partial}{\partial z},\ \mathbf{Y}_3=z\frac{\partial}{\partial \phi}-a^2\phi\frac{\partial}{\partial z},\ \mathbf{Z}_0=\frac{\partial}{\partial s},\ A=c. \tag{6.9}$$

where c is a constant, $\mathbf{Y}_0$ corresponds to energy conservation, $\mathbf{Y}_1$ corresponds to azimuthal angular momentum conservation and $\mathbf{Y}_2$ to linear momentum conservation along z, while $\mathbf{Y}_3$ corresponds to angular momentum conservation.

For the approximate symmetries of the Lagrangian for the cylindrical wave-like spacetime we consider [95]

$$\nu(\rho)=(\rho/R)^2+\epsilon\frac{2t}{T} \text{ and } \mu(x)=(\rho/R)^3+\epsilon\frac{2t}{T}, \tag{6.10}$$

in the metric (6.1), where T is a constant having dimensions of t. Its first-order perturbed Lagrangian is

$$\begin{aligned} L &= e^{(\rho/R)^2}\dot{t}^2-\dot{\rho}^2-e^{(\rho/R)^3}(a^2\dot{\phi}^2+\dot{z}^2)+\frac{2\epsilon t}{T}[e^{(\rho/R)^2}\dot{t}^2-e^{(\rho/R)^3}(a^2\dot{\phi}^2 \\ &\quad +\dot{z}^2)]+O(\epsilon^2). \end{aligned} \tag{6.11}$$

Using this first-order perturbed Lagrangian (6.8) and the exact symmetry generators (6.9), in (1.63), we obtain the following set of equations (6.12) - (6.17), in which only one constant a_0 corresponding to the exact symmetry generator $\mathbf{Y}_0$, given in (6.9), appears.

$$\xi_t=0,\ \xi_\rho=0,\ \xi_\phi=0,\ \xi_z=0,\ A_s=0, \tag{6.12}$$

$$2e^{(\rho/R)^2}\eta^0_s=A_t,\ \ -2\eta^1_s=A_\rho,\ \ -2a^2e^{(\rho/R)^3}\eta^2_s=A_\phi,\ \ -2e^{(\rho/R)^3}\eta^3_s=A_z, \tag{6.13}$$

$$2\eta^1_\rho-\xi_s=0,\ a^2\eta^2_z+\eta^3_\phi=0,\ \eta^1_\phi+a^2e^{(\rho/R)^3}\eta^2_\rho=0,\ \eta^1_z+e^{(\rho/R)^3}\eta^3_\rho=0, \tag{6.14}$$

$$e^{(\rho/R)^2}\eta^0_\rho-\eta^1_t=0,\ e^{(\rho/R)^2}\eta^0_\phi-e^{(\rho/R)^3}\eta^2_t=0,\ e^{(\rho/R)^2}\eta^0_z-e^{(\rho/R)^3}\eta^3_t=0, \tag{6.15}$$

$$\frac{2a_0}{T}+\frac{2\rho}{R^2}\eta^1+2\eta^0_t-\xi_s=0,\ \ -\frac{2a_0}{T}-\frac{3\rho^2}{R^3}\eta^1-2\eta^2_y+\xi_s=0, \tag{6.16}$$

$$-\frac{2a_0}{T}-\frac{3\rho^2}{R^3}\eta^1-2\eta^3_z+\xi_s=0. \tag{6.17}$$

Solving these equations by the same method as for the plane symmetric case we get the following non-trivial approximate symmetry generator (6.18) along with the trivial approximate symmetry generators given by (6.9)

$$\mathbf{Y}_a = \frac{\partial}{\partial t} - \epsilon(t\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial \phi} + z\frac{\partial}{\partial z}). \tag{6.18}$$

The corresponding first-order approximate (stable) first integral is

$$I = e^{(\rho/R)^2}\dot{t} + \frac{2\epsilon}{T}[e^{(\rho/R)^2}t\dot{t} - e^{(\rho/R)^3}(a^2\phi\dot{\phi} + z\dot{z})]. \tag{6.19}$$

Like the plane symmetric case the conserved quantity for this cylindrical wave-like case is

$$Q = E - \frac{\epsilon}{T}(tE + \phi p_\phi + zp_z). \tag{6.20}$$

6.1.2 First-Order Approximate Noether Symmetries of the Cylindrically Symmetric GW Spacetime

To check the conserved quantity Q, in the cylindrical GW spacetime, we investigate the first-order approximate Noether symmetries for this spacetime.

The line element for the cylindrical wave spacetime [97] is

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi}d\phi^2 - e^{2\psi}dz^2, \tag{6.21}$$

where γ and ψ are arbitrary functions of t and ρ, subject to the vacuum EFEs

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0,\ \gamma' = \rho(\psi'^2 + \dot{\psi}^2),\ \dot{\gamma} = 2\rho\dot{\psi}\psi', \tag{6.22}$$

where dot denotes differentiation with respect t and prime with respect to ρ. The solution of (6.22) is given by

$$\psi = AJ_0(\omega\rho)\cos(\omega t) + BY_0(\omega\rho)\sin(\omega t), \tag{6.23}$$

$$\gamma = \frac{\omega\rho}{2}[(A^2J_0J_0' - B^2Y_0Y_0')\cos(2\omega t) - AB\{(J_0Y_0' + Y_0J_0')\sin(2\omega t) - 2(J_0Y_0' - Y_0J_0')\omega t\}]. \tag{6.24}$$

This metric has two KVs $\partial/\partial\phi$ and $\partial/\partial z$ [1]; this means that there is only azimuthal angular momentum conservation and linear momentum conservation along z.

The Lagrangian for (6.22),

$$L = e^{2(\gamma-\psi)}(\dot{t}^2 - \dot{\rho}^2) - \rho^2 e^{-2\psi}\dot{\phi}^2 - e^{2\psi}\dot{z}^2, \tag{6.25}$$

admits the symmetry generator $\partial/\partial s$, along with the two KVs and the gauge function is a constant.

To discuss the approximate symmetries of cylindrical GWs first a static spacetime is defined as follows. We remove the t-dependent part in (6.21) and put the strength of the wave $A = 1$ and $B = 0$. Since Y_0 is badly behaved at $\rho = 0$, we choose $B = 0$ [95].

$$ds^2 = e^{2(\gamma_0-\psi_0)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi_0}d\phi^2 - e^{2\psi_0}dz^2, \tag{6.26}$$

where

$$\psi_0 = J_0(\omega\rho) \text{ and } \gamma_0 = \frac{\omega\rho}{2}J_0(\omega\rho)J_0'(\omega\rho). \tag{6.27}$$

A Lagrangian for the spacetime (6.25) is

$$L = e^{2(\gamma_0-\psi_0)}(\dot{t}^2 - \dot{\rho}^2) - \rho^2 e^{-2\psi_0}\dot{\phi}^2 - e^{2\psi_0}\dot{z}^2. \tag{6.28}$$

Using this Lagrangian (6.27) in (1.54) we obtain the following set equations

$$\xi_t = 0,\ \xi_r = 0,\ \xi_\theta = 0,\ \xi_\phi = 0,\ A_s = 0, \tag{6.29}$$

$$2e^{2(\gamma_0-\psi_0)}\eta^0_s = A_t,\ \ -2e^{2(\gamma_0-\psi_0)}\eta^1_s = A_\rho, \tag{6.30}$$

$$-2\rho^2 e^{-2\psi_0}\eta^2_s = A_\phi,\ \ -2e^{2\psi_0}\eta^3_s = A_z, \tag{6.31}$$

$$2\eta^1_\rho - \xi_s + 2\eta^1(\gamma_0' - \psi_0') = 0, \tag{6.32}$$

$$2\eta^0_t - \xi_s + 2\eta^1(\gamma_0' - \psi_0') = 0, \tag{6.33}$$

$$(2\eta^2_\phi - \xi_s)\rho + 2\eta^1(1 - \psi_0') = 0, \tag{6.34}$$

$$2\eta^1\psi_0' + 2\eta^3_z - \xi_s = 0, \tag{6.35}$$

$$e^{2\gamma_0}\eta^1_\phi + \rho^2\eta^2_z = 0,\ e^{2(\gamma_0-\psi_0)}\eta^1_z + e^{2\psi_0}\eta^3_\rho = 0, \tag{6.36}$$

$$\rho^2\eta^2_z + e^{2\psi_0}\eta^3_\phi = 0,\ \eta^0_\rho - \eta^1_t = 0, \tag{6.38}$$

$$e^{2\gamma_0}\eta^0_\phi - \rho^2\eta^2_t = 0,\ e^{2(\gamma_0-\psi_0)}\eta^0_z - e^{2\psi_0}\eta^3_t = 0, \tag{6.39}$$

where the prime over γ_0 and ψ_0 denote derivative with respect to $\omega\rho$. Solving the above equations (6.29) - (6.38) by the same back and forth substitution method used earlier in the thesis we get the following symmetry generators with the gauge function as a constant

$$\mathbf{Y}_0 = \frac{\partial}{\partial t},\ \mathbf{Y}_1 = \frac{\partial}{\partial \phi},\ \mathbf{Y}_2 = \frac{\partial}{\partial z},\ \mathbf{Z}_0 = \frac{\partial}{\partial s},\ A = c.$$

For the approximate case we put the strength of the wave as a small parameter, i.e. $A = \epsilon$, and take the exact wave as a perturbation on the static metric (6.25) in the following way.

$$\psi = J_0(\omega\rho)(1 + \epsilon\cos(\omega t)) = \psi_0 + \epsilon\psi_1, \tag{6.40}$$

$$\gamma = \frac{\omega\rho}{2}J_0(\omega\rho)J_0'(\omega\rho)(1 + \epsilon^2\cos(\omega t)) = \gamma_0 + \epsilon^2\gamma_1. \tag{6.41}$$

For this perturbed cylindrically symmetric GW spacetime we obtain the following first-order perturbed Lagrangian

$$\begin{aligned} L = {} & e^{2(\gamma_0-\psi_0)}(\dot{t}^2 - \dot{\rho}^2) - \rho^2 e^{-2\psi_0}\dot{\phi}^2 - e^{2\psi_0}\dot{z}^2 - 2\epsilon\psi_1[e^{2(\gamma_0-\psi_0)}(\dot{t}^2 - \dot{\rho}^2) \\ & -\rho^2 e^{-2\psi_0}\dot{\phi}^2 + e^{2\psi_0}\dot{z}^2] + O(\epsilon^2). \end{aligned} \tag{6.42}$$

Using this perturbed Lagrangian (3.42) and the exact symmetry generators (6.39) in (1.63) we obtain the following set of determining equations

$$\xi_t = 0,\ \xi_r = 0,\ \xi_\theta = 0,\ \xi_\phi = 0,\ A_s = 0, \tag{6.43}$$

$$2e^{2(\gamma_0-\psi_0)}\eta^0_s = A_t,\ \ -2e^{2(\gamma_0-\psi_0)}\eta^1_s = A_\rho, \tag{6.44}$$

$$-2\rho^2 e^{-2\psi_0}\eta^2_s = A_\phi,\ \ -2e^{2\psi_0}\eta^3_s = A_z, \tag{6.45}$$

$$2\eta^1_\rho - \xi_s + 2\eta^1(\gamma_0' - \psi_0') - 2a_0\dot{\psi}_1 = 0, \tag{6.46}$$

$$2\eta^0_t - \xi_s + 2\eta^1(\gamma_0' - \psi_0') - 2a_0\dot{\psi}_1 = 0, \tag{6.47}$$

$$(2\eta^2_\phi - \xi_s)\rho + 2\eta^1(1 - \psi_0') - 2a_0\rho\dot{\psi}_1 = 0, \tag{6.48}$$

$$2\eta^1\psi_0' + 2\eta^3_z - \xi_s - 2a_0\dot{\psi}_1 = 0, \tag{6.49}$$

$$e^{2\gamma_0}\eta^1_\phi + \rho^2\eta^2_z = 0,\ e^{2(\gamma_0-\psi_0)}\eta^1_z + e^{2\psi_0}\eta^3_\rho = 0, \tag{6.50}$$

$$\rho^2\eta^2_z + e^{2\psi_0}\eta^3_\phi = 0,\ \eta^0_\rho - \eta^1_t = 0, \tag{6.51}$$

$$e^{2\gamma_0}\eta^0_\phi - \rho^2\eta^2_t = 0,\ e^{2(\gamma_0-\psi_0)}\eta^0_z - e^{2\psi_0}\eta^3_t = 0. \tag{6.52}$$

In the above equations (6.43) - (6.52) only one constant a_0 corresponding to the exact symmetry generator $\mathbf{Y}_0$, given in (6.39), appears. Solving these equations by the same method used earlier this constant has to be eliminated for the consistency of the above determining equations making them homogeneous. Thus there is no non-trivial approximate symmetry for this perturbed cylindrical GW spacetime. We only recover the exact symmetry generators given in (6.39), as trivial first-order approximate Noether symmetry generators. Hence energy conservation, azimuthal angular momentum conservation and linear momentum conservation along the axis of the cylinder are obtained as trivial first-order approximate conservation laws. The conserved quantity Q cannot be found here which was obtained for the case of the wave-like spacetime (6.10).

6.2 Second-Order Approximate Symmetries of the Geodesic Equations for Cylindrical Wave Spacetimes

In this section we first discuss the cylindrical wave-like spacetime and then we study the perturbed cylindrically symmetric GW spacetime.

6.2.1 Approximate Symmetries of the Geodesic Equations for the Cylindrical Wave-Like Spacetime

Retaining ϵ^2 in (6.10) and neglecting its higher powers, we obtain second-order perturbed geodesic equations for this wave-like spacetime [95]

$$\ddot{t} + \frac{2}{R}\dot{t}\dot{\rho} - \frac{\epsilon t}{T}[\dot{t}^2 - e^{(\rho/R)^2(\rho/R-1)}(a^2\dot{\phi}^2 + \dot{z}^2)] + \frac{\epsilon t^2}{T^2}[\dot{t}^2 + e^{(\rho/R)^2(\rho/R-1)}(a^2\dot{\phi}^2 + \dot{z}^2)] + O(\epsilon^3) = 0, \tag{6.53}$$

$$\ddot{\rho} + \frac{e^{(\rho/R)^2}}{R}\dot{t}^2 - \frac{2\rho}{R^2}e^{(\rho/R)^3}(a^2\dot{\phi}^2 + \dot{z}^2) + \frac{2t\epsilon}{TR}[\dot{t}^2 e^{(\rho/R)^2} -$$
$$\frac{2\rho}{R}e^{(\rho/R)^3}(a^2\dot{\phi}^2 + \dot{z}^2)] + \frac{t^2\epsilon^2}{T^2 R}[\dot{t}^2 e^{(\rho/R)^2} - \frac{2\rho}{R}e^{(\rho/R)^3}(a^2\dot{\phi}^2$$
$$+\dot{z}^2)] + O(\epsilon^3) = 0, \tag{6.54}$$

$$\ddot{\phi} + \frac{4\rho}{R^2}\dot{\rho}\dot{\phi} + \frac{2\epsilon}{T}\dot{t}\dot{\phi} - \frac{2t\epsilon^2}{T^2}\dot{t}\dot{\phi} + O(\epsilon^3) = 0, \tag{6.55}$$

$$\ddot{z} + \frac{4\rho}{R^2}\dot{\rho}\dot{z} + \frac{2\epsilon}{T}\dot{t}\dot{z} - \frac{2t\epsilon^2}{T^2}\dot{t}\dot{z} + O(\epsilon^3) = 0. \tag{6.56}$$

These equations are exactly the same as those for the plane wave-like spacetime discussed in chapter 5, in cylindrical coordinates. Applying the second prolongation $\mathbf{X}^{[2]}$ of the generator $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2$ defined in (2.40), to (6.53) - (6.56), and using the prolongation coefficients defined in (2.45) - (2.52), we get the same set of determining equations (5.105) - (5.142) in cylindrical coordinates. For these equations there does not exist any non-trivial second-order approximate symmetry. We only recover the exact and first-order approximate symmetries as trivial second-order approximate symmetry generators.

6.3 Energy in a Cylindrical Wave-like Spacetime

Like the plane wave-like spacetime (5.10) we obtain the following scaling factor for this cylindrical wave-like spacetime (6.10) [95]

$$\frac{t}{4T}[e^{-2(\rho/R)^2} + 2e^{-(\rho/R)^2(\rho/R-1)}]. \tag{6.57}$$

This factor is exactly the same as that for the plane wave-like spacetime. Here it is in cylindrical coordinates. This gives the energy re-scaling in this wave-like spacetime. The plots of this factor (6.57) are same as those for the plane wave-like spacetime given in chapter 5.

6.3.1 Approximate Symmetries of the Geodesic Equations for the Perturbed Cylindrically Symmetric GW Spacetime

For the perturbed cylindrically symmetric GW spacetime (6.21) for which γ and ψ are defined by (6.40) and (6.41), we have the following system of second-order perturbed geodesic equations

[95]

$$\ddot{t} + 2(\gamma_0' - \psi_0')\dot{t}\dot{\rho} - \epsilon[\dot{\psi}_1(\dot{t}^2 + \dot{\rho}^2) + \rho^2 e^{-2\gamma_0}\dot{\psi}_1\dot{\phi}^2 - \dot{\psi}_1 e^{2(2\psi_0-\gamma_0)}\dot{z}^2 +$$
$$2\psi_1'\dot{t}\dot{\rho}] + \epsilon^2[\dot{\gamma}_1(\dot{t}^2 + \dot{\rho}^2) + 4\dot{\psi}_1\psi_1 e^{2(2\psi_0-\gamma_0)}\dot{z}^2 - 2\gamma_1'\dot{t}\dot{\rho}] + O(\epsilon^3) = 0, \tag{6.58}$$
$$\ddot{\rho} + (\gamma_0' - \psi_0')(\dot{t}^2 + \dot{\rho}^2 + \dot{t}\dot{\rho}) + \rho^2 e^{-2\gamma_0}(\psi_0' - 1)\dot{\phi}^2 + \psi_0' e^{2(2\psi_0-\gamma_0)}\dot{z}^2 -$$
$$\epsilon[\psi_1'(\dot{t}^2 + \dot{\rho}^2) - \rho^2 e^{-2\gamma_0}\dot{\psi}_1\dot{\phi}^2 - e^{2(2\psi_0-\gamma_0)}(4\psi_0'\psi_1 - \psi_1')\dot{z}^2 + \psi_1'\dot{t}\dot{\rho}]$$
$$+\epsilon^2[\gamma_1'(\dot{t}^2 + \dot{\rho}^2) - \rho^2 e^{-2\gamma_0}\gamma_1(\psi_0' - 1)\dot{\phi}^2 + 2e^{2(2\psi_0-\gamma_0)}(4\psi_0'\psi_1^2 - \psi_0'\gamma_1 -$$
$$\psi_1\psi_1')\dot{z}^2 + \gamma_1'\dot{t}\dot{\rho}] + O(\epsilon^3) = 0, \tag{6.59}$$
$$\ddot{\phi} + \frac{1}{\rho}(1 - \psi_0')\dot{\rho}\dot{\phi} - \epsilon\frac{1}{\rho}(\dot{\psi}_1\dot{t} + \psi_1'\dot{\rho})\dot{\phi} - \epsilon^2\frac{1}{\rho}(1 - \psi_0')\psi_1^2\dot{\rho}\dot{\phi} + O(\epsilon^3) = 0, \tag{6.60}$$
$$\ddot{z} + \psi_0')\dot{\rho}\dot{z} + \epsilon[\dot{\psi}_1\dot{t} + \psi_1'\dot{\rho}]\dot{z} - \epsilon^2 2\psi_0'\psi_1^2\dot{\rho}\dot{z} + O(\epsilon^3) = 0, \tag{6.61}$$

where dot over γ_1 and ψ_1 denotes differentiation with respect to ωt.

We apply the second prolongation $\mathbf{X}^{[2]}$ of the generator $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2$ defined in (2.40), to (6.58) - (6.61), which yields

$$\mathbf{X}^{[2]}E_1 = [\eta^0_{0,ss} + \epsilon\eta^0_{1,ss} + \epsilon^2\eta^0_{2,ss} + (\eta^0_0 + \epsilon\eta^0_1 + \epsilon^2\eta^0_2)\{-\epsilon(\ddot{\psi}_1(\dot{t}^2 + \dot{\rho}^2) + \rho^2 e^{-2\gamma_0}\ddot{\psi}_1\dot{\phi}^2$$
$$- \ddot{\psi}_1 e^{2(2\psi_0-\gamma_0)}\dot{z}^2 + 2\dot{\psi}_1'\dot{t}\dot{\rho}) + \epsilon^2(\ddot{\gamma}_1(\dot{t}^2 + \dot{\rho}^2) + 4(\ddot{\psi}_1\psi_1 + \dot{\psi}_1^2)e^{2(2\psi_0-\gamma_0)}\dot{z}$$
$$- 2\dot{\gamma}_1'\dot{t}\dot{\rho})\} + 2(\gamma_0'' - \psi_0'')\dot{t}\dot{\rho}(\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2) + 2(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s})$$
$$\{(\gamma_0' - \psi_0')\dot{\rho} - \epsilon(\dot{\psi}_1\dot{t} + \psi_1'\dot{\rho}) + \epsilon^2(\dot{\gamma}_1\dot{t} + \gamma_1'\dot{\rho})\} + 2(\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s})$$
$$\{(\gamma_0' - \psi_0')\dot{t} - \epsilon(\dot{\psi}_1\dot{\rho} + \psi_1'\dot{t}) + \epsilon^2(\dot{\gamma}_1\dot{\rho} + \gamma_1'\dot{t})\} - 4\epsilon(\eta^2_{0,s} + \epsilon\eta^2_{1,s} + \epsilon^2\eta^2_{2,s})$$
$$(\dot{\psi}_1\rho^2 e^{-2\gamma_0}\dot{\phi}) + 2\dot{\psi}_1 e^{2(2\psi_0-\gamma_0)}(\eta^3_{0,s} + \epsilon\eta^3_{1,s} + \epsilon^2\eta^3_{2,s})(\epsilon + 4\psi_1\epsilon^2)\dot{z}]_{E_j=0} = 0, \tag{6.62}$$

$$\mathbf{X}^{[2]}E_2 = [\eta^1_{0,ss} + \epsilon\eta^1_{1,ss} + \epsilon^2\eta^1_{2,ss} + (\eta^0_0 + \epsilon\eta^0_1 + \epsilon^2\eta^0_2)\{-\epsilon(\dot{\psi}'_1(\dot{t}^2 + \dot{\rho}^2)- \rho^2 e^{-2\gamma_0}\ddot{\psi}_1\dot{\phi}^2 - e^{2(2\psi_0-\gamma_0)}(4\psi'_0\dot{\psi}_1 - \dot{\psi}'_1)\dot{z}^2 + \dot{\psi}'_1\dot{t}\dot{\rho}) + \epsilon^2(\dot{\gamma}'_1(\dot{t}^2 + \dot{\rho}^2) - \rho^2\dot{\gamma}_1 e^{-2\gamma_0}(\psi'_0 - 1)\dot{\phi}^2 + 2(8\psi'_0\dot{\psi}_1\psi_1 - \dot{\gamma}_1\psi'_0 - \dot{\psi}_1\psi'_1 - \psi_1\dot{\psi}'_1)e^{2(2\psi_0-\gamma_0)}\dot{z}^2 + \dot{\gamma}'_1\dot{t}\dot{\rho})\} + (\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2)\{(\gamma''_0 - \psi''_0)(\dot{t}^2 + \dot{\rho}^2 + \dot{t}\dot{\rho}) + \rho^2 e^{-2\gamma_0}(2(1-\gamma'_0) + \psi''_0)(\psi'_0 - 1)\dot{\phi}^2 + (e^{2(2\psi_0-\gamma_0)}\psi''_0 + 2\psi'_0(2\psi'_0 - \gamma'_0))\dot{z}^2\} + \{(\psi'_0 - \gamma'_0) - \epsilon\psi'_1 + \epsilon^2\gamma'_1\}(2\dot{t} + \dot{\rho})(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s}) + (\dot{t} + 2\dot{\rho})(\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s})\{(\gamma'_0 - \psi'_0) - \epsilon\psi'_1 + \epsilon^2\gamma'_1\} + 2\rho^2 e^{-2\gamma_0}\{(\psi'_0 - 1) + \epsilon\dot{\psi}_1 - \epsilon^2(\psi'_0 - 1)\}\dot{\phi}(\eta^2_{0,s} + \epsilon\eta^2_{1,s} + \epsilon^2\eta^2_{2,s}) + 2e^{2(2\psi_0-\gamma_0)}(\eta^3_{0,s} + \epsilon\eta^3_{1,s} + \epsilon^2\eta^3_{2,s})\{1 + \epsilon(4\psi'_0\psi_1 - \psi'_1) + 2\epsilon^2(4\psi'_0\psi_1^2 - \psi'_0\gamma_1 - \psi'_1\psi_1)\}\dot{z}]_{E_j=0} = 0, \quad (6.63)$$

$$\mathbf{X}^{[2]}E_3 = [\eta^2_{0,ss} + \epsilon\eta^2_{1,ss} + \epsilon^2\eta^2_{2,ss} + (\eta^0_0 + \epsilon\eta^0_1 + \epsilon^2\eta^0_2)\rho\{-\epsilon(\dot{\psi}'_1\dot{\rho} + \ddot{\psi}_1\dot{t})\dot{\phi} - \epsilon^2\frac{2}{\rho}\psi_1\dot{\psi}_1(1-\psi'_0)\dot{\rho}\dot{\phi}\} - \frac{1}{\rho}(\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2)\{(1-\psi'_0) + \psi''_0\}\dot{\rho}\dot{\phi} - \epsilon\dot{\psi}_1(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s})\rho\dot{\phi} + (\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s})\{(1-\psi'_0) - \epsilon\psi'_1 - \epsilon^2(1-\psi'_0)\psi_1^2\}\rho\dot{\phi} + (\eta^2_{0,s} + \epsilon\eta^2_{1,s} + \epsilon^2\eta^2_{2,s})\{(1-\psi'_0)\dot{\rho} - \epsilon(\psi'_1\dot{\rho} + \dot{\psi}_1\dot{t}) - \epsilon^2(1-\psi'_0)\dot{\rho}\}]_{E_j=0} = 0, \quad (6.64)$$

$$\mathbf{X}^{[2]}E_4 = [\eta^3_{0,ss} + \epsilon\eta^3_{1,ss} + \epsilon^2\eta^3_{2,ss} + (\eta^0_0 + \epsilon\eta^0_1 + \epsilon^2\eta^0_2)\rho\{\epsilon(\dot{\psi}'_1\dot{\rho} + \ddot{\psi}_1\dot{t})\dot{z} - 4\epsilon^2\psi'_0\psi_1\dot{\psi}_1\dot{\rho}\dot{z}\} + (\eta^1_0 + \epsilon\eta^1_1 + \epsilon^2\eta^1_2)\{\psi''_0\dot{\rho}\dot{z} + \epsilon(\psi''_1\dot{\rho} + \dot{\psi}'_1\dot{t})\dot{z}\}\dot{\rho}\dot{\phi} - 4\epsilon^2\psi'_0\psi_1\psi'_1\dot{\rho}\dot{z}\} + \epsilon\dot{\psi}_1(\eta^0_{0,s} + \epsilon\eta^0_{1,s} + \epsilon^2\eta^0_{2,s})\dot{z} + (\eta^1_{0,s} + \epsilon\eta^1_{1,s} + \epsilon^2\eta^1_{2,s})\{\psi'_0 + \epsilon\psi'_1 - 2\epsilon^2\psi'_0\psi_1^2\}\dot{z} + (\eta^3_{0,s} + \epsilon\eta^3_{1,s} + \epsilon^3\eta^2_{2,s})\{\psi'_0\dot{\rho} + \epsilon(\psi'_1\dot{\rho} + \dot{\psi}_1\dot{t}) - 2\epsilon^2\psi'_0\psi_1^2\dot{\rho}\}]_{E_j=0} = 0, \quad (6.65)$$

where $(j = 1, 2, 3, 4)$. For $\epsilon = 0$, (6.58) - (6.61) yield equations for the exact case and only retaining first power of ϵ, neglecting its higher powers, will give equations for the first-order approximate case. For the exact ($\epsilon = 0$) the geodesic equations (6.58) - (6.61) admit the symmetry generators

$$\mathbf{Y}_0 = \frac{\partial}{\partial t},\ \mathbf{Y}_1 = \frac{\partial}{\partial \phi},\ \mathbf{Y}_2 = \frac{\partial}{\partial z},\ \mathbf{Z}_0 = \frac{\partial}{\partial s},\ \mathbf{Z}_1 = s\frac{\partial}{\partial s}, \tag{6.66}$$

with the gauge function as a constant.

We use the prolongation coefficients defined in (2.45) - (2.52), the exact symmetry generators given by (6.104), the first-order approximate symmetry generators which are same to those of the exact case and the second-order approximate geodesic equations (6.58) - (6.61) in above equations (6.62) - (6.65). We get the following set of determining equations

$$\begin{aligned}
&\xi_{2_{tt}} + (\psi_0' - \gamma_0')\xi_{2_\rho} = 0,\ \xi_{2_{xx}} + (\psi_0' - \gamma_0')\xi_{2_\rho} = 0,\\
&\xi_{2_{\phi\phi}} - \rho^2 e^{-2\gamma_0}\xi_{2_\rho} = 0,\ \xi_{2_{zz}} - e^{2(2\psi_0 - \gamma_0)}\xi_{2_\rho} = 0,\\
&2\xi_{2_{t\rho}} - (\psi_0' - \gamma_0')(2\xi_{2_t} - \xi_{2_\rho}) = 0,\ \xi_{2_{t\phi}} = 0,\\
&2\xi_{2_{\rho\phi}} + (2\gamma_0' - 3\psi_0' - 1)\xi_{2_\phi} = 0,\ \xi_{2_{\rho\phi}} - \psi_0'\xi_{2_\phi} = 0,\\
&\xi_{2_{tz}} = 0,\ \xi_{2_{\phi z}} = 0,
\end{aligned} \tag{6.67}$$

$$\eta^0_{2_{tt}} + (\psi_0' - \gamma_0')(\eta^0_{2_\rho} + 2\eta^1_{2_t}) - a_0(\ddot{\psi}_1 - \ddot{\gamma}_1) - 2\xi_{2_{st}} = 0, \tag{6.68}$$

$$\eta^1_{2_{tt}} - (\psi_0' - \gamma_0')(2\eta^0_{2_t} + \eta^1_{2_t} - \eta^1_{2_\rho}) - a_0(\dot{\psi}_1' - \dot{\gamma}_1') + (\psi_0'' - \gamma_0'')\eta^1_2 = 0, \tag{6.69}$$

$$\eta^2_{2_{tt}} - (\psi_0' - \gamma_0')\eta^2_{2_\rho} = 0,\ \eta^3_{2_{tt}} - (\psi_0' - \gamma_0')\eta^3_{2_\rho} = 0, \tag{6.70}$$

$$\eta^0_{2_{\rho\rho}} - \eta^0_{2_\rho}(\psi_0' - \gamma_0') + a_0(\ddot{\psi}_1 - \ddot{\gamma}_1) = 0, \tag{6.71}$$

$$\eta^1_{2_{\rho\rho}} - a_0(\dot{\psi}_1' - \dot{\gamma}_1') + (\psi_0'' - \gamma_0'')\eta^1_2 + (\psi_0' - \gamma_0')(\eta^0_{2_\rho} + 2\eta^1_{2_\rho}) - 2\xi_{2_{ss}} = 0, \tag{6.72}$$

$$\eta^2_{2_{\rho\rho}} + (1 - \gamma_0')\eta^2_{2_\rho} = 0,\ \eta^3_{2_{\rho\rho}} + (2\psi_0' - \gamma_0')\eta^3_{2_\rho} = 0, \tag{6.73}$$

$$\eta^0_{2_{\phi\phi}} - a_0\rho^2 e^{-2\gamma_0}\ddot{\psi}_1 - \rho^2 e^{-2\gamma_0}(\psi_0' - 1)\eta^0_{2_\rho} = 0, \tag{6.74}$$

$$\begin{aligned}
\eta^1_{2_{\phi\phi}} + a_0\rho^2 e^{-2\gamma_0}(\dot{\gamma} - \dot{\gamma}\psi_0' - \ddot{\psi}_1) + \rho^2 e^{-2\gamma_0}[2(\psi_0' - 1)(1 - \gamma_0') +\\
\psi_0'']\eta^1_2 + \rho^2 e^{-2\gamma_0}(\psi_0' - 1)(2\eta^2_{2_\phi} - \eta^1_{2_\rho}) = 0,
\end{aligned} \tag{6.75}$$

$$\eta^2_{2_{\phi\phi}} - (\psi_0' - 1)(\eta^1_{2_\phi} + \rho^2 e^{-2\gamma_0}\eta^2_{2_\rho}) - 2\xi_{2_{s\phi}} = 0, \tag{6.76}$$

$$\eta^0_{2_{zz}} + a_0 e^{2(2\psi_0-\gamma_0)}[4(\dot{\psi}_1^2 + \psi_1\ddot{\psi}_1) + \ddot{\psi}_1] - \psi_0' e^{2(2\psi_0-\gamma_0)}\eta^0_{2_\rho} = 0, \tag{6.77}$$

$$\eta^1_{2_{zz}} + a_0 e^{2(2\psi_0-\gamma_0)}[2(8\psi_0'\psi_1\dot{\psi}_1 - \dot{\gamma}_1\psi_0' - \dot{\psi}_1\psi_1' - \psi_1\dot{\psi}_1')+$$

$$4\psi_0'\dot{\psi}_1 - \dot{\psi}_1'] + e^{2(2\psi_0-\gamma_0)}[2\psi_0'(2\psi_0'-\gamma_0') + \psi_0'']\eta^1_2 + e^{2(2\psi_0-\gamma_0)}\psi_0'$$

$$(2\eta^3_{2_z} - \eta^1_{2_\rho}) = 0,\ \eta^3_{2_{\phi\phi}} - \rho^2 e^{-2\gamma_0}(\psi_0'-1)\eta^3_{2_\phi} = 0, \tag{6.78}$$

$$\eta^2_{2_{zz}} - e^{2(2\psi_0-\gamma_0)}\psi_0'\eta^2_{2_\rho} = 0,\ \eta^3_{2_{zz}} - e^{2(2\psi_0-\gamma_0)}\psi_0'\eta^3_{2_\rho} + \psi_0'\eta^1_{2_z} - 2\xi_{2_{sz}} = 0, \tag{6.79}$$

$$2\eta^0_{2_{st}} - 2(\psi_0'-\gamma_0')\eta^1_{2_s} - \xi_{2_{ss}} = 0,\ 2\eta^1_{2_{st}} - (\psi_0'-\gamma_0')(2\eta^0_{2_s} + \eta^1_{2_s}) = 0, \tag{6.80}$$

$$\eta^2_{2_{st}} = 0,\ \eta^3_{2_{st}} = 0,\ \eta^0_{2_{s\rho}} - (\psi_0'-\gamma_0')\eta^0_{2_s} = 0, \tag{6.81}$$

$$2\eta^1_{2_{s\rho}} - \xi_{2_{ss}} - (\psi_0'-\gamma_0')(\eta^0_{2_s} + 2\eta^1_{2_s}) = 0, \tag{6.82}$$

$$2\eta^2_{2_{s\rho}} + (1-\psi_0')\eta^2_{2_s} = 0,\ 2\eta^3_{2_{s\rho}} + \psi_0'\eta^2_{2_s} = 0,\ \eta^0_{2_{s\phi}} = 0, \tag{6.83}$$

$$\eta^1_{2_{s\rho}} - (1-\psi_0')\rho^2 e^{-2\gamma_0}\eta^2_{2_s} = 0, 2\eta^2_{2_{s\rho}} + (1-\psi_0')\eta^1_{2_s} - \xi_{2_{ss}} = 0, \tag{6.84}$$

$$\eta^3_{2_{s\phi}} = 0,\ \eta^0_{2_{sz}} = 0,\ \eta^1_{2_{sz}} + \psi_0' e^{2(2\psi_0-\gamma_0)}\eta^3_{2_s} = 0,\ \eta^2_{2_{sz}} = 0, \tag{6.85}$$

$$\eta^3_{2_{sz}} + \psi_0'\eta^1_{2_s} - \xi_{2_{ss}} = 0, \tag{6.86}$$

$$\eta^0_{2_{t\rho}} - (\psi_0'-\gamma_0')(\frac{1}{2}\eta^0_{2_\rho} - \eta^1_{2_\rho}) - \xi_{2_{s\rho}} - (\psi_0''-\gamma_0'')\eta^1_2 - a_0(\dot{\psi}_1' + \dot{\gamma}_1') = 0, \tag{6.87}$$

$$2\eta^1_{2_{t\rho}} - (\psi_0'-\gamma_0')(\eta^0_{2_t} + 2\eta^0_{2_\rho}) - 2\xi_{2_{s\rho}} - (\psi_0''-\gamma_0'')\eta^1_2 - a_0(\dot{\psi}_1' - \dot{\gamma}_1') = 0, \tag{6.88}$$

$$2\eta^2_{2_{t\rho}} + (1-\psi_0')\eta^2_{2_t} + (\psi_0'-\gamma_0')(2\eta^2_{2_t} + \eta^2_{2_\rho}) = 0, \tag{6.89}$$

$$2\eta^3_{2_{t\rho}} + \psi_0'\eta^3_{2_t} + (\psi_0'-\gamma_0')(2\eta^3_{2_t} + \eta^3_{2_\rho}) = 0,\ \eta^0_{2_{t\phi}} - \xi_{2_{s\phi}} - (\psi_0'-\gamma_0')\eta^1_{2_\phi} = 0, \tag{6.90}$$

$$2\eta^1_{2_{t\phi}} - 2(1-\psi_0')\rho^2 e^{-2\gamma_0}\eta^2_{2_t} - (\psi_0'-\gamma_0')(2\eta^0_{2_\phi} + \eta^1_{2_\phi}) = 0, \tag{6.91}$$

$$2\eta^2_{2_{t\phi}} + (1-\psi_0')\eta^1_{2_t} - 2\xi_{2_{st}} - a_0\rho\ddot{\psi}_1 = 0,\ \eta^3_{2_{t\phi}} = 0, \tag{6.92}$$

$$\eta^0_{2_{tz}} - \xi_{2_{sz}} - (\psi_0'-\gamma_0')\eta^1_{2_z} = 0,\ \eta^2_{2_{tz}} = 0, \tag{6.93}$$

$$2\eta^1_{2_{tz}} - (\psi_0'-\gamma_0')(2\eta^0_{2_z} + \eta^1_{2_z}) - 2\psi_0' e^{2(2\psi_0-\gamma_0)}\eta^3_{2_t} = 0, \tag{6.94}$$

$$2\eta^3_{2_{tz}} + \psi_0'\eta^1_{2_t} - 2\xi_{2_{st}} - a_0\ddot{\psi}_1 = 0,\ 2\eta^0_{2_{\rho\phi}} + (3\psi_0' + \gamma_0' - 1)\eta^0_{2_\phi} = 0, \tag{6.95}$$

$$2\eta^1_{2_{\rho\phi}} - (\psi_0'-\gamma_0')(\eta^0_{2_\phi} + 2\eta^1_{2_\phi}) + 2\rho^2(\psi_0'-1)e^{-2\gamma_0}\eta^2_{2_\rho} - (1-\psi_0')\eta^1_{2_\phi}$$

$$-2\xi_{2_{s\phi}} = 0,\ 2\eta^3_{2_{\rho\phi}} + (2\psi_0'-1)\eta^3_{2_\phi} = 0, \tag{6.96}$$

$$2\eta^2_{2_{\rho\phi}} + (1-\psi_0')\eta^1_{2_\rho} + (\psi_0'' + \psi_0' - 1)\eta^1_2 + a_0[2(\psi_0'-1)\psi_1\dot{\psi}_1 - \dot{\psi}_1']$$

$$-2\xi_{2_{s\rho}} = 0,\ \eta^0_{2_{\rho z}} - (2\psi_0'-\gamma_0')\eta^0_{2_z} = 0,\ 2\eta^2_{2_{\rho z}} - (2\psi_0'-1)\eta^2_{2_z} = 0, \tag{6.97}$$

$$2\eta^1_{2_{\rho z}} - (\psi_0'-\gamma_0')(\eta^0_{2_z} + 2\eta^1_{2_z}) + \psi_0'(2e^{2(2\psi_0-\gamma_0)}\eta^3_{2_\rho} - \eta^1_{2_z}) - 2\xi_{2_{sz}} = 0, \tag{6.98}$$

$$2\eta^1_{2_{\rho z}} + \psi_0''\eta^1_2 + \psi_0'\eta^1_{2_\rho} - 2\xi_{2_{s\rho}} + a_0(\dot{\psi}_1' - 4\psi_0'\psi_1\dot{\psi}_1) = 0,\ \eta^0_{2_{\phi z}} = 0, \tag{6.99}$$

$$\eta^1_{2_{\phi z}} + e^{-2\gamma_0}[\rho^2(\psi_0' - 1)\eta^2_{2_z} + \psi_0'\eta^3_{2_\phi}] = 0,\ 2\eta^3_{2_{\phi z}} + \psi_0'\eta^1_{2_\phi} - 2\xi_{2_{s\phi}} = 0, \tag{6.100}$$

$$2\eta^2_{2_{\phi z}} - (\psi_0' - 1)\eta^1_{2_z} - 2\xi_{2_{s\phi}} = 0,\ \eta^0_{2_{ss}} = 0,\ \eta^1_{2_{ss}} = 0,\ \eta^2_{2_{ss}} = 0,\ \eta^3_{2_{ss}} = 0. \tag{6.101}$$

In these equations only one constant a_0 corresponding to the exact symmetry generator $\mathbf{Y}_0$ given in (6.66) appears. Solving these equations by the same method as used before this constant a_0 disappears for the consistency of the above set of determining equations (6.67) - (6.101). Thus these equations become homogenous and there is no non-trivial second-order approximate symmetry. We only recover the exact symmetry generators given by (6.66), as trivial second-order approximate symmetry generators.

Note that for the first-order approximate symmetries we get the same set of determining equations (6.67) - (6.101), which is obtained in the second-order approximate case. The solution of these equations is discussed above. Hence like the second-order approximate case there is no non-trivial first-order approximate symmetry for this perturbed cylindrical wave spacetime.

6.4 Energy in the Perturbed Cylindrical Wave Spacetime

We obtain the following scaling factor for the perturbed cylindrical wave spacetime in the same way as for RN, charged-Kerr and wave-like spacetimes. The scaling factor is [95]

$$\dot{\gamma}_1(\dot{t}^2 + \dot{\rho}^2) + 4\dot{\psi}_1\psi_1 e^{2(2\psi_0 - \gamma_0)}\dot{z}^2 - 2\gamma_1'\dot{t}\dot{\rho}. \tag{6.102}$$

To replace the derivative of the coordinates t, z we use the exact first integrals

$$\dot{t} = \frac{1}{2}e^{2(\psi_0 - \gamma_0)},\ \dot{z} = -\frac{1}{2}e^{-2\psi_0}. \tag{6.103}$$

Further it is assumed that there is no initial velocity in the z and ϕ directions. Hence $\dot{z}$ and $\dot{\phi}$ vanish. To replace $\dot{\rho}$ we use the Lagrangian (6.28) of the exact (unperturbed) case, i.e.

$$\dot{\rho} = e^{(\psi_0 - \gamma_0)}(e^{3(\psi_0 - \gamma_0)} - 1)^{\frac{1}{2}}. \tag{6.104}$$

Using (6.103) and (6.104) in (6.102) the following scaling factor is obtained

$$\dot{\gamma}_1 e^{2(\psi_0-\gamma_0)}(e^{2(\psi_0-\gamma_0)}+e^{3(\psi_0-\gamma_0)}-1)-2\gamma_1' e^{3(\psi_0-\gamma_0)}(e^{3(\psi_0-\gamma_0)}-1)^{\frac{1}{2}}, \tag{6.105}$$

where γ_1 is given in (6.41). This scaling factor involves the Bessel function of the first kind and its derivatives. The asymptotic representation of the Bessel function of the first kind for large value of the argument is given in [98]. Using that asymptotic representation of the Bessel function in (6.105), we obtain an asymptotic representation of the scaling factor as follows

$$\frac{3\times 2^{\frac{11}{4}}}{\pi^{\frac{3}{2}}}[(|\cos(\omega\rho)|)^{\frac{3}{2}}\sin(2\omega t)](\omega\rho)^{\frac{-1}{2}}+O([\omega\rho]^{\frac{-3}{2}}). \tag{6.106}$$

In the above factor (6.106) the magnitude of the coefficient of $(\omega\rho)^{-1/2}$ is greater than the magnitude of the coefficient of $(\omega\rho)^{-3/2}$. Therefore the contribution of the second term is very small and is neglected.

Thus the energy in this perturbed spacetime field is re-scaled by the factor (6.106). It is plotted below for different values of t, ρ and ω (in radians per second), in which the energy oscillates between positive and negative values and goes to zero as ρ tends to infinity. Here the behavior is much more recognizably wave-like. Since the strength of the wave, $A=\epsilon$, is arbitrary the energy is given in arbitrarily chosen units.

As we mentioned in the first chapter as to "whether there is the analogue of Landau-damping of electromagnetic waves for GWs". With our present proposal, i.e. the use of approximate Lie symmetry methods the question seems to be answerable. Classically the energy density in cylindrical waves reduces by the factor $1/2\pi\rho$. From (6.106) the energy density decreases by a further factor of $3\times 2^{\frac{11}{4}}/\sqrt{\pi^3\times(\omega\rho)}$. Hence for sufficiently large ρ the scaling factor $\sim 1/\sqrt{\omega\rho^3}$ is a significant *self-damping* of the waves.

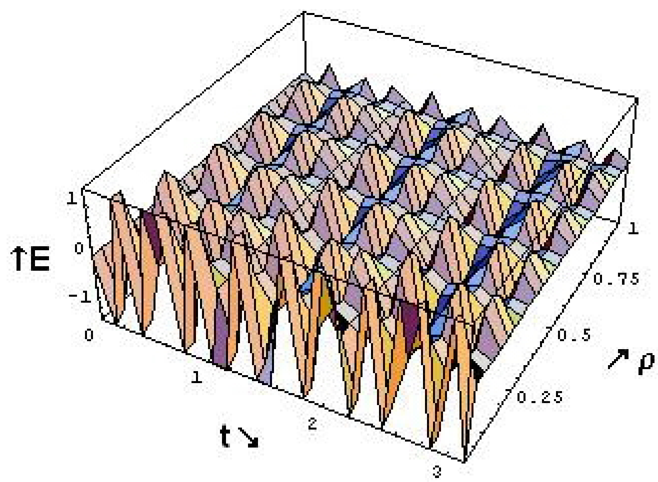

Figure 6-1: Cylindrically symmetric GWs with $\omega = 15$ the gravitational energy oscillates between positive and negative values and disappears as ρ approaches very large values. The units of energy are arbitrary in all diagrams.

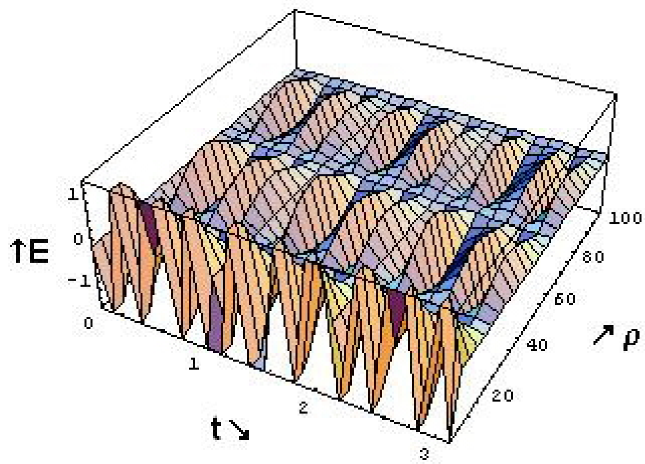

Figure 6-2: To see the behaviour of energy for comparatively large distance, the range of ρ is extended to 100 units.

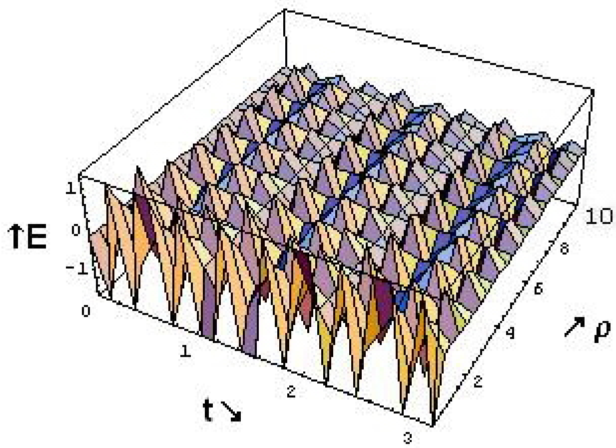

Figure 6-3: To see a further extended version of the above Fig. 6-2 the range of ρ is given in units of 10^5.

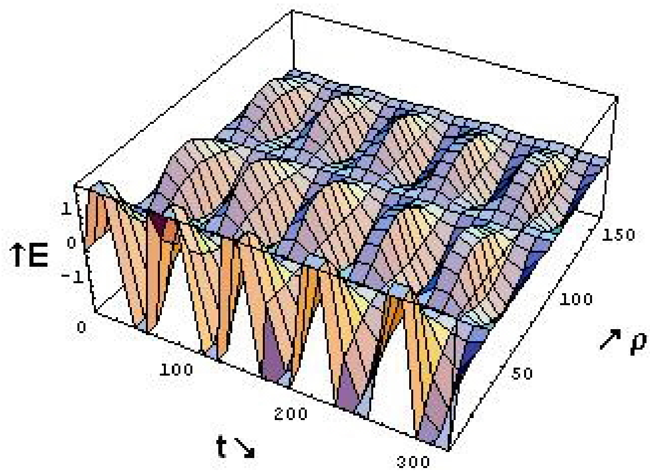

Figure 6-4: Here the value of the frequency is comparatively small i.e. $\omega = 0.05$. To see the variation along time, the range of t is keept larger.

6.5 The Weyl and Stress-Energy Tensors for Cylindrically Symmetric GW Spacetimes

In this section, for the pure gravitational part and matter part, we give the independent nonzero components of Weyl and stress-energy tensors for the perturbed cylindrically symmetric GW and cylindrical wave-like spacetimes.

6.5.1 The Weyl and Stress-Energy Tensors for the Perturbed Cylindrically Symmetric GW Spacetime

Following are the independent nonzero components of the Weyl tensor for the perturbed cylindrical wave spacetime discussed above [95]

$$\begin{aligned}
C^0{}_{101} &= \frac{-1}{3}[\psi_0'' - \gamma_0'' + 2\psi_0'(\psi_0' - 1) + \epsilon\{4\psi_0'(\psi_1' + \psi_1 - \psi_0'\psi_1) + \psi_1'' - \ddot{\psi}_1 \\
&\quad -2\psi_1(\psi_0'' - \gamma_0'')\}] + O(\epsilon^2), \\
C^0{}_{202} &= \frac{\rho^2}{6}[4\psi_0'' - \gamma_0'' + 2\psi_0'(4\psi_0' - 3\gamma_0' - 1) + 3\gamma_0' + 2\epsilon\{2\psi_1'' + \ddot{\psi}_1 + \psi_1'(8\psi_0' \\
&\quad -3\gamma_0' - 1)\}] + O(\epsilon^2), \\
C^0{}_{303} &= \frac{e^{2(2\psi_0-\gamma_0)}}{6}[2\psi_0'' + \gamma_0'' + 2\psi_0'(2\psi_0' - 3\gamma_0' + 1) + 3\gamma_0' + 2\epsilon\{\psi_1'' + 2\ddot{\psi}_1 + \\
&\quad \psi_1'(4\psi_0' - 3\gamma_0' + 1)\}] + O(\epsilon^2), \\
C^1{}_{212} &= -e^{-4\psi_0}C^0_{303},\ \rho C^1_{313} = e^{2(2\psi_0-\gamma_0)}C^0_{202}, \\
C^2{}_{323} &= \frac{e^{2(2\psi_0-\gamma_0)}}{3}[\gamma_0'' - \psi_0'' + 2\psi_0'(1 - \psi_0') - 2\epsilon\{\psi_1'' - \ddot{\psi}_1 + \psi_1'(2\psi_0' - 1)\}] + O(\epsilon^2), \\
C^0{}_{212} &= -\epsilon\rho^2 e^{-2\gamma_0}[(2\psi_0' + \gamma_0')\dot{\psi}_1 + \dot{\psi}_1'] + O(\epsilon^2), \\
C^0{}_{313} &= \epsilon e^{2(2\psi_0-\gamma_0)}[(2\psi_0' - \gamma_0')\dot{\psi}_1 + \dot{\psi}_1'] + O(\epsilon^2).
\end{aligned} \tag{6.107}$$

This yields the pure gravitational field for the above discussed cylindrical perturbed wave spacetime. As is evident from Figs. 6-1 to 6-4, the energy in the gravitational field oscillates and then vanishes for large ρ, here all the components of the Weyl tensor also depend on the Bessel function of the first kind and its derivatives which oscillates and goes to zero as ρ approaches very large value. The last two components only appear for the approximate part of the spacetime. In this case the components of the Weyl tensor in the covariant form are not very different from those in the mixed form given above and therefore we do not give them separately.

The non-vanishing components of the stress-energy tensor are

$$\begin{aligned}
T_{00} &= T_{11} = \frac{1}{\kappa}(\psi_0'^2 - \gamma_0' + 2\epsilon\psi_0'\psi_1') + O(\epsilon^2),\\
T_{22} &= \frac{\rho^2}{\kappa}e^{2(\gamma_0-2\psi_0)}(\psi_0'^2 - \gamma_0'' + 2\epsilon\psi_0'\psi_1') + O(\epsilon^2),\\
T_{33} &= \frac{1}{\kappa}e^{2(\gamma_0-2\psi_0)}[2\psi_0'' - \gamma_0'' - \psi_0'(\psi_0'-2) + 2\epsilon\{\psi_1'' - \ddot{\psi}_1 + 2\psi_1(\psi_0''\\
&\quad -\gamma_0'' - \psi_0'^2 + 3\psi_0') - \psi_0'\psi_1' + 2(\psi_1' - 2\psi_0'\psi_1)\}] + O(\epsilon^2),\\
T_{01} &= \frac{\epsilon}{\kappa}\psi_0'\dot{\psi}_1 + O(\epsilon^2). \qquad (6.108)
\end{aligned}$$

Like the components of the Weyl tensor the above components of the stress-energy tensor also depend on the Bessel function of the first-kind and its derivatives. In this case of cylindrical perturbed waves the fraction of energy density imparted to the matter field is

$$E_{imp} = 2\epsilon\frac{\psi_0'\psi_1'}{\psi_0'^2 - \gamma_0'}. \qquad (6.109)$$

This fraction of energy goes to zero as $\epsilon \longrightarrow 0$.

6.5.2 The Weyl and Stress-Energy Tensors for the Cylindrical Wave-Like Spacetime

The non-vanishing components of the Weyl tensor for the cylindrical wave-like spacetime are

$$\begin{aligned}
C^0{}_{101} &= \frac{1}{3R^5}(3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3) + O(\epsilon^2),\\
C^0{}_{202} &= a^2C^0{}_{303} = -(1+\epsilon\frac{2t}{T})\frac{a^2e^{\rho^3/R^3}}{6R^5}(3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3) + O(\epsilon^2),\\
C^1{}_{212} &= a^2C^1{}_{313} = -C^0{}_{202},\ C^2{}_{323} = -2C^0{}_{202}. \qquad (6.110)
\end{aligned}$$

In covariant form the components of the Weyl tensor are

$$
\begin{aligned}
C_{0101} &= \frac{1}{3R^5}(1+\epsilon\frac{2t}{T})(3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3) + O(\epsilon^2), \\
C_{0202} &= a^2 C^0_{303} = -\frac{a^2 e^{\rho^3/R^3+\rho^2/R^2}}{6R^5}(1+\epsilon\frac{4t}{T})(3R^2\rho - 2R\rho^2 + 3\rho^3 - R^3) + O(\epsilon^2), \\
C_{1212} &= a^2 C_{1313} = -C_{0202},\ C_{2323} = -2C_{2202}. \qquad (6.111)
\end{aligned}
$$

The components in the covariant form are physically reasonable as they follow the geometry of the constructed metric and the energy defined by approximate symmetry.

The nonzero components of stress-energy tensor are

$$
\begin{aligned}
T_{00} &= \frac{3e^{\rho^2/R^2}}{2\kappa R^6}(1+\epsilon\frac{2t}{T})(9\rho^3 + 4R^3) + O(\epsilon^2),\ T_{11} = \frac{3}{2\kappa R^6}(3\rho + 4R) + O(\epsilon^2), \\
T_{22} &= a^2 T_{33} = -\frac{a^2 e^{\rho^3/R^3}}{2\kappa R^2}(1+\epsilon\frac{2t}{T})(6R^3\rho + 4R^2\rho^2 + 9\rho^4 - 6R\rho^3 + 6R^4) + O(\epsilon^2), \\
T_{01} &= \epsilon\frac{\rho}{\kappa T R^3}(3\rho - 2R) + O(\epsilon^2). \qquad (6.112)
\end{aligned}
$$

Here the momentum density is along the radius of the cylinder. For this case we have the same relative energy density imparted to the matter field, as given by (5.152).

The results of this chapter are given in the form of the following theorems.

Theorem 6.1. *The energy in the cylindrical gravitational wave is the classical energy of a cylindrical wave re-scaled by the factor (6.106).*

Theorem 6.2. *The energy of cylindrical gravitational wave in a cylindrical gravitational wave-like spacetime is the classical energy re-scaled by the factor (6.57).*

Chapter 7

Summary and Discussion

In this thesis we addressed the problem of energy in GR, using slightly broken or approximate Lie symmetry methods. These methods were first applied to the Schwarzschild spacetime [60], where only first-order approximate symmetries of the geodesic equations were investigated. Following the method adopted in [60], in this work we first considered second-order approximate symmetries of the geodesic equations and first-order approximate symmetries of the Lagrangians of some static spacetimes, i.e. RN, Kerr and charged-Kerr spacetimes. Then we applied these methods to non-static spacetimes where the problem of definition of energy is more severe. For these spacetimes we obtained scaling factors. These scaling factors give the re-scaling of energy in these spacetimes. For the cylindrical wave spacetime we also obtained that the wave has to be damped in self-interaction. Here the approximation or the breaking involves a small parameter whose powers, higher than some chosen value, is neglected. The scaling factors obtained here, are independent of the strength of the perturbation parameter. This means that we do not need a finite perturbation and can take the limit as it goes to zero. This is reminiscent of the d' Alembert principle for statics [99]. The d' Alembert principle is an extension of the principle of virtual work from statics to dynamics, i.e. the work done by a force is along a virtual displacement and not along the actual displacement. The limit of this virtual displacement can be taken zero, to attain the staticity and the total work done is independent of it.

In this chapter we give a summary and discussion of chapters 2 to 6 in two separate sections, i.e. in the first section we discuss static spacetimes and in the second section we discuss non-static spacetimes.

7.1 Static Spacetimes.

RN Spacetime

In chapter 2 we studied the approximate symmetries of the RN spacetime. First we investigated second-order approximate symmetries of the orbital equation for this spacetime. For this orbital equation there does not exist any non-trivial approximate symmetry. We only recovered the exact and first-order approximate symmetries as trivial second-order approximate symmetries. This spacetime has isometry algebra $so(3) \oplus \mathbb{R}$ with generators (1.78) and (1.79). The symmetry algebra of the geodesic equations for this metric is $so(3) \oplus \mathbb{R} \oplus d_2$. We then explored the second-order approximate symmetries of the RN spacetime. Neglecting terms containing ϵ^2, in the geodesic equations (2.36) - (2.39), this spacetime has the same first-order approximate symmetries as those of the Schwarzschild spacetime [60]. Again we get no non-trivial approximate symmetry generator in the second approximation. We only recover the lost conservation laws as approximate conservation laws. As for the Schwarzschild spacetime, where there is a difference between the conservation laws obtained for the system of geodesic equations and for the single orbital equation, the difference also holds for the RN spacetime.

Importantly from the consistency of the trivial second-order approximate symmetries of the perturbed geodesic equations for the RN spacetime we have obtained the scaling factor (2.180). This gives the re-scaling of energy in the RN spacetime field. This scaling factor for the RN spacetime, which does not appear for the Schwarzschild spacetime is of special interest. The pseudo-Newtonian formalism [90, 91, 100, 101] gives re-scaling of force by $(1 - Q^2/rMc^2)$. The reduction is by the ratio of the electromagnetic potential energy at a distance r to the rest energy of the gravitational source. It is position dependent. The scaling factor $(1 - Q^2/2GM^2)$ obtained here from the use of approximate Lie symmetry methods is more reasonable as relating the electromagnetic self-energy to the gravitational self-energy, with the radial parameter, r, canceled out.

Kerr Spacetime

In the third chapter we discussed exact and approximate symmetries of a Lagrangian for the geodesic equations in the Kerr spacetime. The unperturbed Lagrangian for the geodesic equations in the Kerr spacetime has an additional symmetry $\partial/\partial s$, along with the two KVs. The unperturbed Lagrangian for the Schwarzschild spacetime has a 5 dimensional algebra which contains the four KVs of this metric and $\partial/\partial s$. Taking the Kerr spacetime as a first perturbation of the Schwarzschild spacetime with spin as a small parameter we recovered the conservation laws as trivial first-order approximate conservation laws which were lost in going from the Schwarzschild spacetime to the Kerr spacetime.

Retaining terms of $O(\epsilon^2)$ in the Kerr spacetime we have a second-order perturbed Lagrangian given by (3.79). This Lagrangian reduces to that of Minkowski spacetime if $\epsilon = 0$ and if we retain terms of first-order in ϵ and neglecting $O(\epsilon^2)$, we get a Lagrangian for the perturbed Schwarzschild spacetime which is a first perturbation of the Minkowski spacetime. For the exact case (Minkowski spacetime) symmetries of the Lagrangian form a 17 dimensional Lie algebra, which also holds in Cartesian coordinates and thus there is no coordinate dependence.

Remark. It is mentioned here that the symmetries of the Minkowski spacetime Lagrangian were first discussed in [83], where the metric taken was $ds^2 = \cos h(x/a)dt^2 - dx^2 - dy^2 - dz^2$, which is not Minkowski, as it has $R^0{}_{101} \neq 0$. The calculation was left incomplete, giving an impression that the algebra is infinite dimensional, and it was shown that the isometry algebra is a subalgebra of the symmetries of the Lagrangian. This problem was revisited in [102], with the correct metric, but the symmetry algebra of the Lagrangian was given as 12 dimensional and the gauge function as zero, which was again erroneous.

For the first-order approximate case (perturbed Schwarzschild) there is no non-trivial first-order approximate symmetry of the Lagrangian. However all the exact 17 symmetry generators are recovered as first-order approximate symmetry generators. In the second-order approximate case, i.e. when we retain terms quadratic in ϵ, which is the second perturbation of the Minkowski spacetime, we again have no non-trivial second-order approximate symmetry of the Lagrangian and only 17 symmetry generators of the exact case are recovered as second-order approximate symmetry generators. Thus we see that in going from Minkowski to Schwarzschild and Kerr metrics the conservation laws which were lost are now recovered as approximate conservation laws. It was shown [69] that a Lagrangian for the geodesic equations possesses at least one additional symmetry generator, $\partial/\partial s$, apart from the isometry algebra. This is verified for the Schwarzschild and Kerr spacetimes. As in the case of the Minkowski metric the CKVs form a subalgebra of the symmetries of the Lagrangian which include $\partial/\partial s$. Therefore we make the following conjecture

Conjecture. *The CKVs form a subalgebra of the symmetries of the Lagrangian that minimize the arc length, for any spacetime.*

For both the non-flat spacetimes, i.e. Schwarzschild and Kerr spacetimes the unperturbed Lagrangian has only the one additional symmetry generator $\partial/\partial s$ and the gauge function A is a constant. In Minkowski spacetime there are 7 additional symmetry generators and the gauge function A is a function of 4 variables t, r, θ and ϕ. The significance of these additional 7 symmetry generators of the Minkowski spacetime Lagrangian, which are also recovered as first-order and second-order approximate symmetries generators for the Schwarzschild and Kerr spacetimes respectively, is discussed in detail in chapter 3.

Charged-Kerr Spacetime

In chapter 4 we studied the second-order approximate symmetries of the geodesic equations for the charged-Kerr spacetime. For this spacetime we obtained the scaling factor (4.63). In the RN spacetime, the re-scaling was independent of r (discussed in chapter 2) while for the charged-Kerr spacetime the re-scaling factor given by (4.63) consists of two parts - one is due to charge and the other is due to spin of the gravitating source which depends on r. The charge comes in *quadratically* compared to unity in one term. The spin comes in *linearly.* It does not come with a constant term to compare. However, taken as a whole, we see that the spin has an effectively *lower order* effect.

In all three expressions (4.64), (4.65) and (4.67), the charge and spin appear at the same order (quadratically). The last one comes with a θ-dependent part, which arises from the θ-dependence of the "force" experienced by a body in the Fermi-Walker frame [101]. As mentioned earlier, (4.64) seems unreasonable as the rotational effect depends on the presence of a charge and disappears with it! In (4.63) in the absence of charge, the effect is to *enhance* the mass. This seems reasonable as the frame-dragging effect also appears to lead to an enhanced mass -"friction" of the rotating mass with the background spacetime, as it were. Recall that one can extract rotational energy from a rotating black hole and hence the rotation should *add* into the mass. As would be expected, this effect decreases with r. The other three expressions give a *reduction* of the rotating mass. Also notice that (4.63) gives a change in the mass due to charge that is position independent. That this should be so is not so clear to us. However, nor is it clear to us that it *should* be position dependent. The force experience by a particle in the field of a charged gravitational source would be position dependent, but this does not say that the mass should be modified by a position dependent expression. It might be that in (4.63) the modification is due to the electromagnetic self-energy to the gravitational self-energy. As such, we conclude that the other expressions have definite drawbacks of which (4.63) seems to be free.

It would be of interest to analyze the Kerr-AdS and other solutions using approximate Noether symmetries. One could use references [103] and [104] and those cited therein for the purpose.

7.2 Non-Static Spacetimes

In this section we summaries chapter 5 and 6 where we have addressed the problem of energy in non-static spacetimes, using approximate Lie symmetry methods for DEs.

Plane Wave Spacetimes

In chapter 5, first we investigated Noether symmetries of the plane wave spacetimes. In this regard first of all we studied the plane wave-like spacetime [93] for which there exists a non-trivial first-order approximate Noether symmetry. We used this non-trivial approximate Noether symmetry to contract the energy-momentum vector that gives a conserved quantity (5.54) which gives the energy imparted to the test particles. Then we investigated first-order approximate Noether symmetries for the perturbed pp-wave spacetime for which there is no non-trivial approximate Noether symmetry.

To resolve the problem of energy in GW spacetimes we used the second-order approximate symmetries of the geodesic equations for perturbed gravitational wave spacetimes discussed here. First the pp-wave spacetime is investigated. Since there is no ϵ^2 in the geodesic equations for the perturbed pp-waves, the definition of second-order approximate symmetries of ODEs which gives the scaling factor, cannot be applied to them. This is similar to the result of Qadir and Sharif's work [10], using the pseudo-Newtonian formalism, which just gave a constant momentum imparted to test particles in the path of the waves and no determinable value for it. For a better understanding of the implication of the definition of second-order approximate symmetries of ODEs, in plane symmetric waves this definition has applied to the artificially constructed time-varying non-vacuum plane symmetric spacetime [93], for which the scaling factor (5.146) is obtained. It is seen from the plots 5-1 and 5-2, of the plane wave-like spacetime that the energy increases without limit, with time close to the origin for x and then disappears. The reason for this increase in energy is the (nonphysical) choice of a linearly increasing component of the metric tensor for convenience of computation, this leads to a corresponding increase in the scaling factors (5.146).

Cylindrical Wave Spacetimes

We then considered a cylindrical analog of the plane wave-like spacetime [93]. A non-trivial first-order approximate Noether symmetry exists for this wave-like spacetime. Like the plane wave-like spacetime, this non-trivial Noether symmetry gives the conserved quantity (6.20). Then we studied first-order approximate Noether symmetries of the perturbed cylindrical wave spacetime for which there is no non-trivial approximate Noether symmetry.

To obtain the energy in the cylindrical wave spacetimes we investigated the second-order approximate symmetries of the perturbed geodesic equations for these spacetimes. The scaling factors (6.57) and (6.106) are obtained for these spacetimes. In the factor (6.106) the magnitude of the coefficient of $(\omega\rho)^{-1/2}$ is greater than the magnitude of the coefficient of $(\omega\rho)^{-3/2}$. Therefore the contribution of the second term is very small and is neglected. In Figs. 6-1 to 6-4 the re-scaling factor for the energy oscillates between positive and negative values along t and ρ. It disappears as ρ tends to infinity.

Using the idea of pseudo-tensors [30], different people have claimed that the gravitational energy should be positive at large scales as well as at small scales [105, 106, 107, 108, 109]. The positivity of gravitational energy does not seem convincing because the total energy of the universe is zero [110], which suggests that the gravitational energy must fluctuate between positive and negative values to be able to give the net energy of a given spacetime zero.

Our definition of gravitational energy, obtained from approximate Lie symmetry methods, avoids the pseudo-tensor and hence does not violate GR. The radiation scalar [50] does not violate the spirit of GR either. However, nor does it give a measure of the energy of GWs. Our expression of energy *does* give such a measure and is also reasonable as the gravitational energy oscillates over positive and negative values, as it should. Admittedly, in the artificial example we constructed the energy increased linearly without limit. For the physical example of cylindrical exact waves, the Bessel function of the first kind goes to zero asymptotically for large values of the argument [98]. Correspondingly, because of our scaling factor for cylindrical waves they die out asymptotically.

There is a problem with Isaacson's work [37], which in fact deals with the perturbation, h, on the background Minkowski metric to give the total metric, g. We do not have a guarantee that the perturbation series converges for the vacuum spacetime. Where it can be applied it will give the higher order terms as the nonlinear source of the linearized gravitational waves. This is the energy they claim. Since there is no good reason to take the flat Minkowski space as the relevant one, it is not at all clear that this would be the physically sound way to work out the energy. In fact, Minkowski spacetime is unstable under such perturbations. Essentially, there is nothing with respect to which the energy is to be small.

The "Christodoulou memory effect" [35], deals with the energy imparted to test particles. Hence it does not address the issue of the energy in the field. Consequently, when applied to the exact cylindrical GW solution for the first-order approximation, we get the Weber-Wheeler first-order energy approximation and when the Christodoulou and Thorne procedure is applied it should yield the second-order approximation given by Weber and Wheeler. The general treatment should, then, yield the formula obtained by Qadir and Sharif [10].

We also addressed the question "whether there is the analogue of Landau-damping of electromagnetic waves for gravitational waves". The problem of definition of energy in GR makes it very difficult to answer the above question. With the use of approximate Lie symmetry methods for DEs the above question seems to be answerable. In the Newtonian theory the energy density in cylindrical waves reduces by the factor $1/2\pi\rho$. From the expression (6.106) the energy density in cylindrical waves, in curved spacetimes, decreases by a further factor of $3\times 2^{11/4}/\sqrt{\pi^3 \times (\omega\rho)}$. Hence for sufficiently large distance from the origin of the source of wave the scaling factor $\sim 1/\sqrt{\omega\rho^3}$ is a significant *self-damping* of the waves! This enhanced asymptotic attenuation of gravitational waves will obviously have profound observational significance [111].

It would be of great interest to apply this approximate symmetry analysis to the Khan-Penrose [62] and Szekeres [63] solutions to see whether they suffer self-damping or enhancement according to our definition. Of course, it may be that the procedure will be inapplicable for those plane wave solutions as well. Also, the analysis should be applied to "spherical solutions" like those of Nutku [112].

7.3 Weyl and Stress-Energy Tensors

The Weyl tensor represents the pure gravitational field and the stress-energy tensor gives the matter part of a spacetime. Therefore to obtain the pure gravitational field and the matter field the approximate Weyl and stress-energy tensors for the GW spacetimes, discussed in the thesis are calculated. The components of the Weyl tensor are given in the (0, 4) (covariant) form as well. For the perturbed pp-wave spacetime it appears that the (0, 4) form gives the physically relevant quantities as the space dependence in the (1, 3) (mixed) form of the Weyl tensor does not seem to correspond to the geometry of the pp-wave while the covariant form does. For the wave-like spacetimes the components in the covariant form are physically reasonable as they follow the geometry of the constructed metrics and the energy defined by approximate symmetry. The stress-energy tensor density imparted to the matter field in the wave-like and perturbed cylindrical wave spacetimes was obtained.

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Chapter 8

Work Already Published from the Thesis