Fixed point theorems for Banach-G type contractions

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A dissertation submitted in the partial fulfillment of the

Degree of Doctor of Philosophy

In

Mathematics

2014

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Man gets whatever he strives for...
Al-Quran 53:39

To my parents and teachers...

Contents

Preface

For the last few decades fixed point theory has been an active and thriving area of research for mathematicians. The fixed point theory is mainly concerned with obtaining conditions on the structure that the underlying space(set) X must be endowed with and on the properties of selfmapping f on X, in order to obtain the existence (and uniqueness) of fixed point. The underlying ambient space cover a variety of structures: metric space, generalized metric space, linear topological spaces, lattice and statistical metric space etc. The fundamentals of theory are attributed to the celebrated and remarkable mathematicians such as Banach, Brouwer and Tarski in metric, topological and discrete perspectives respectively.

In 1922 the famous Polish mathematician Stafen Banach laid the ground for metric fixed point theory by formulating his notable theorem, known as Banach contraction principle. His theorem carved a niche for itself as a rapidly growing area of research in mathematics. It has become the most celebrated tool in non-linear analysis to find the existence of solutions for various linear and nonlinear equations such as Volterra integral equations, integro-differential equations and existence of equilibria in game theory as well. These applications elicit the panoptic nature of Banach contraction principle.

In metric fixed point theory the results critically rely on prenominal geometric conditions of underlying spaces in conjunction with some sufficient metric constraints imposed on the behavior of mappings. Although a huge number of results have been established, there are still several interesting questions which remain to be answered regarding to what extent the theory can be nurtured and extended. Mathematicians have been intrigued and endeavored in the search of these questions which led them to the new avenues of research except for some questions which seem merely tantalizing. Thereby contributing enormously in the field of fixed point theory by finding the fixed point(s) of selfmappings or non-selfmappings defined on several ambient spaces and satisfying variety of conditions. Among these fixed point theorems, some of them have much more practical importance, i.e., they provide a constructive method for finding fixed point(s). Thus provide information on convergence rate along with error estimates. Banach contraction principle is one of such theorems wherein the proposed iterative scheme converges linearly. Commonly, the iterative procedures serve as constructive methods in fixed point theory. A number of fixed point theorems have been obtained by considering the following ways:

1. by weakening the contractive assumption and possibly by simultaneously giving to the space a sufficiently rich structure in order to compensate the relaxation of the contractiveness assumption; or

2. by enriching/extending the structure of the ambient space.

Various extensions of Banach contraction principle have also been obtained by unifying the ideas mentioned above or by adding suitable supplementary conditions. In many problems, in particular the problem of convergence of measurable functions with respect to a measure evoke the need to generalize the notion of a metric. This quest led Czerwik to the concept of b-metric. Afterwards, many mathematicians undertook further developments in this direction and established some exciting fixed points results.

On the other hand the inception of statistical metric spaces by Menger in 1942 opened a new avenue in research. He introduced the basic concepts of probabilistic geometry. Fixed point theory is a part of probabilistic analysis which has become a dynamic area of research. The introduction of fundamental notions of statistical metric space and its geometry owed a lot to Menger. Schweizer and Sklar contributed a major influence upon the development of the theory of probabilistic metric spaces.

The theory of fixed points of a self mapping on a partially ordered set is owed to Turinici. Afterwards Ran and Reurings undertook further investigations in this context. They presented some fixed point theorems wherein the mappings do not necessarily require to satisfy contraction condition for all possible pairs of points in the underlying set rather the contraction condition has to be satisfied only for those points which are related to each other with respect to the partial order defined on the set.

In this context Jachymski generalized the idea by delineating the underlying set with a graph. He introduced the notion of Banach G-contractions and obtained a very novel variant which is a hybrid of Banach contraction principle and the main results of Ran and Reurings.

The purpose of this dissertation is twofold: to generalize the notion of Banach G-contraction by weakening contractive condition; to give the underlying space a rich geometric structure so that the notion becomes more viable. In the quest of the first aspect we bring in light two new categories of mappings: integral G-contractions; weakly G-contractive mappings in the setting of metric space. In Chapter 3, by using the graph theoretic approach we obtain some significant fixed point theorems for such class of contractions. While the quest of the second objective led us to undertake some investigations in b-metric space and statistical metric space.

In Chapter 4 we contribute some recent developments in the direction of b-metric space. In first half of Chapter 4, by using the concept of φ-contractions and Hardy-Roger's contractive condition we introduce two new notions when the underlying b-metric space is endowed with a graph G. Furthermore, in the second half of Chapter 4 by using the gauge function two convergence theorems are established which give the constructive iterative process to approach the fixed point. We also calculate a priori and a posteriori estimates for the proposed iterative scheme. As an application we obtaine an existence theorem for the solution of first order differential equation where the rate of convergence of proposed iterative scheme is at leat $r \geq 1$.

Chapter 5 is devoted to fixed point theorems for mappings in probabilistic metric space endowed with a graph G. In this context we extende the concept of Banach G-contractions and put forth a new type of contractions known as probabilistic G-contractions.

After unveiling every new concept, examples are furnished to elucidate the validity of our notions and to show the degree of generality of our results over the pre-existing results. As applications of some of our results we also obtaine some fixed point theorems for cyclic contraction.

Chapter 1

Introduction and preliminaries

The purpose of this chapter is stipulated with the basic concepts, terminologies and notations which are used throughout this dissertation. We aim to write each chapter self content and pedagogical. For this reason we also add preliminaries section to the chapters where they are necessary. Subsequently let us represent with $\mathbb{R}$ the set of all real numbers, $\mathbb{N}$ the set of natural numbers, $\mathbb{R}^+ := [0, \infty)$ and $\emptyset$ the empty set.

1.1 Fixed points and weakly Picard operators

Let X and Y are nonempty sets and $f : X \to Y$ then fx represents the image of x under f. We assume that the set X is endowed with a metric d. Let $f : X \to X$ be a self-map. A point $\xi \in X$ is a fixed point of f if

$$f(\xi) = \xi.$$

Let us denote with $Fixf$, the set of all fixed points of f, i.e.,

$$Fixf = \{x \in X : f(x) = x\}.$$

Example 1.1.1. Assume $X := (-\infty, +\infty)$ and $f : X \to X$.

1. If $f(x) = x^2 + 3x + 1$ then $Fixf := \{-1\}$;

2. If $f(x) = x^2$ then $Fixf := \{0, 1\}$;

3. If $f(x) = x$ then $Fixf := \mathbb{R}$;

4. If $f(x) = x + 1$ then $Fixf := \emptyset$.

Definition 1.1.2. A mapping $f : X \to X$ is said to be Lipschtizian if there exists a constant $\kappa \geq 0$ such that

$$d(fx, fy) \leq \kappa d(x, y) \qquad \text{for all } x, y \in X. \tag{1.1}$$

One can observe that every Lipschitizian map is necessarily continuous. The smallest real number κ for which (1.1) is satisfied is known as Lipschitizian constant for f.

Definition 1.1.3. A mapping $f : X \to X$ is said to be a contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$d(fx, fy) \leq \kappa d(x, y) \qquad \text{for all } x, y \in X. \tag{1.2}$$

Let X be any nonempty set and $f : X \to X$. For a given $x \in X$ we define $f^n x$ inductively by $f^0 x = x$ and $f^{n+1} x = f(f^n x); n = 0, 1, 2, \cdots$. We call $f^n x$ the n-th iteration of x under f. For any $x_0 \in X$ the sequence $\{x_n\}; n = 0, 1, 2, \cdots$ defined by

$$x_n = fx_{n-1} = f^n x_0, \qquad n = 1, 2, \cdots,$$

is known as the sequence of successive approximation or Picard iterations starting at x_0.

Definition 1.1.4. [84] A mapping $f : X \to X$ is called a Picard operator (briefly, PO) if f has a unique fixed point $\xi \in X$ and $\lim_{n\to 0} f^n x = \xi$ for all $x \in X$.

Definition 1.1.5. [95] A mapping $f : X \to X$ is called a weakly Picard operator (briefly, WPO) if the sequence $\{f^n x\}$ converges for all $x \in X$ and the limit of the sequence is a fixed point of f.

Example 1.1.6. Let $X := [0, 1]$ and $f : X \to X$ is defined as $fx = x^2$. Then $Fixf = \{0, 1\}$ and f is a weakly Picard operator.

It is obvious that the class of weakly Picard operators subsumes the class of Picard operators.

1.2 Gauge functions

Let $\varphi : [0, \infty) \to [0, \infty)$. Consider the following properties:

$(i)_\varphi$ $t_1 \leq t_2 \implies \varphi(t_1) \leq \varphi(t_2)$, for all $t_1, t_2 \in [0, \infty)$;

$(ii)_\varphi$ $\varphi(t) < t$ for $t > 0$;

$(iii)_\varphi$ $\varphi(0) = 0$;

$(iv)_\varphi$ $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \geq 0$;

$(v)_\varphi$ $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$;

$(vi)_\varphi$ φ is continuous;

$(vii)_\varphi$ $t - \varphi(t) \to \infty$ as $t \to \infty$;

$(viii)_\varphi$ φ is subadditive.

The function $\varphi : [0, \infty) \to [0, \infty)$ satisfying at least one of the above properties is known as a gauge function. It is easily seen that: $(i)_\varphi$ and $(iv)_\varphi$ imply $(ii)_\varphi$; $(i)_\varphi$ and $(ii)_\varphi$ imply $(iii)_\varphi$.

Definition 1.2.1. A function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $(i)_\varphi$ and $(iv)_\varphi$ is said to be a comparison function.

Definition 1.2.2. [86] A function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $(i)_\varphi$ and $(v)_\varphi$ is known as a (c)-comparison function.

Example 1.2.3. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\varphi(t) = \frac{t}{1+t}; t \in \mathbb{R}^+$. Then φ is a comparison function but not a (c)-comparison function. On the other hand define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = \frac{t}{2}; 0 \leq t \leq 1$ and $\varphi(t) = t - \frac{1}{2}; t > 1$. Then φ is a (c)-comparison function.

It follows from above example that any (c)-comparison function is a comparison function but converse may not be true.

Subsequently, let J always denote an interval in $\mathbb{R}^+$ containing 0 i.e., an interval of the form $[0, R]$, $[0, R)$ or $[0, \infty)$ ($[0, 0] = \{0\}$ is a trivial interval). Let $P_n(t)$ denote a polynomial of the form $P_n(t) = 1 + t + \cdots + t^{n-1}$ and $P_0(t) = 0$. Let φ^n denote nth iterate of a function $\varphi : J \to J$.

Definition 1.2.4. [85] Let $r \geq 1$. A function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(i) $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0,1)$ and $t \in J$,

(ii) $\varphi(t) < t$ for all $t \in J \setminus \{0\}$.

The condition (i) of Definition 1.2.4 elicits $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J \setminus \{0\}$.

Example 1.2.5. (i) The function $\varphi : J \to J$ defined by $\varphi(t) = ct$ $(0 < c < 1)$ is a gauge function of first order on $J = [0, \infty)$.
(ii) The function $\varphi : J \to J$ defined by $\varphi(t) = ct^r$ $(c > 0, r > 1)$ where $R = (1/c)^{1/(r-1)}$ is a gauge function of order r on $J = [0, R)$.

1.3 Graph theory

A graph G is a mathematical model which is conveniently used to delineate many real world situations. A graph G is a pair $(V(G), E(G))$ where the vertex set $V(G)$ is a nonempty set of elements and the edge set $E(G)$ is a binary operation on $V(G)$. A graph G may be directed or undirected. A directed graph is one in which each edge is specified with the direction from one vertex to other. Let $G = (V(G), E(G))$ be a directed graph. By G^{-1} we denote the graph obtained from G by reversing the direction of edges and by letter $\widetilde{G}$ we denote the undirected graph obtained from G by ignoring the direction of edges. It will be more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric, i.e.,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Definition 1.3.1. Let x and y are vertices in a graph G. Then a path in G from x to y of length l is a sequence $\{x_i\}_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x, x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \cdots, l$.

Definition 1.3.2. A graph G is called connected if there is a path between any two vertices. The graph G is weakly connected if $\widetilde{G}$ is connected.

Definition 1.3.3. Let G be a graph such that $E(G)$ is symmetric and x is a vertex in G, the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called component of G containing x.

In such case $V(G_x) = [x]_{\widetilde{G}}$, where $[x]_{\widetilde{G}}$ is the equivalence class of a relation R defined on $V(G)$ by the rule: yRz if there is a path in G from y to z. Clearly, G_x is connected. For comprehensive study on the subject we refer the readers to [55].

1.4 b-metric space

From the last few decades fixed point theory is being rapidly evolved not only in metric structure but also in various abstract spaces. The b-metric space is one of the interesting generalizations of metric space which was initiated in some works of Bourbaki, Bakhtin, Czerwik and Heinonen. Afterwards, several articles appeared in literature which deal with the fixed point theory for single valued and multi-valued functions in a b-metric space [7, 8, 13, 17, 19, 20, 21, 22, 31, 49, 78, 81] etc.

Definition 1.4.1. [31, 8] Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a $b-$metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(d1) $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$;

(d3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a $b-$metric space with the coefficient s.

The following example show that the class of b-metric spaces is essentially larger than the class of metric spaces.

Example 1.4.2. [12, 31, 49] 1. Let $X := l_p(\mathbb{R})$ with $0 < p < 1$ where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $d : X \times X \to \mathbb{R}^+$ as:

$$d(x, y) = \big(\sum_{n=1}^{\infty} |x_n - y_n|^p\big)^{1/p},$$

where $x = \{x_n\}, y = \{y_n\}$. Then (X, d) is a b-metric space with coefficient $s = 2^{1/p}$.

2. Let $X := L_p[0,1]$ be the space of all real functions $x(t),\ t \in [0,1]$ such that $\int_0^1 |x(t)|^p < \infty$. Define $d : X \times X \to \mathbb{R}^+$ as:

$$d(x,y) = \Big(\int_0^1 |x(t) - y(t)|^p dt\Big)^{1/p}.$$

Then (X, d) is a b-metric space with coefficient $s = 2^{1/p}$.

1.5 Probabilistic metric space

In 1942 Menger [67] introduced the notion of probabilistic metric space (briefly, PM space) and since then there have been made enormous developments in the theory of probabilistic metric space in many directions [27, 44, 107]. The fundamental idea of Menger is to replace real numbers with distribution functions as values of metric. This intuitive approach was the result of the simple fact that even in measurement of ordinary length the number given as the distance between two points is often not the result of a single measurement but the average of a series of measurements. Hence in such case the implication of theory of metric space i.e., the very association of a single real number with a pair of elements becomes an over idealization. Hence in such and many similar situations it is more appropriate to look upon the distance as a statistical rather than a determinate one.

A succinct and intuitive response to fill this gap is to assign a distribution function $F_{x,y}$ instead of a real number $d(x,y)$ with every pair of elements x, y. This leads to the generalization of metric space known as statistical or probabilistic metric space. For detailed discussion on probabilistic metric spaces and their applications we refer the readers to Onicescu [75, Chapter 7], Schweizer [105, 106, 107] and Hadzić and Pap [44].

Definition 1.5.1. A mapping $F : \mathbb{R} \to [0,1]$ is called a distribution function if it is non-decreasing, left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$. In addition if $F(0) = 0$ then F is called a distance distribution function.

Let $\mathcal{D}^+$ denote the set of all distance distribution functions satisfying $\lim_{t\to\infty} F(t) = 1$. The space $\mathcal{D}^+$ is partially ordered with respect to usual pointwise ordering of functions, i.e.,

$F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The element $\epsilon_0 \in \mathcal{D}^+$ acts as the maximal element in the space and is defined by

$$\epsilon_0(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{1.3}$$

Definition 1.5.2. A statistical or probabilistic metric space (briefly, PM-space) is an ordered pair $(X, \mathscr{F})$ where X is a nonempty set and $\mathscr{F} : X \times X \to \mathcal{D}^+$ and the following conditions are satisfied ($\mathscr{F}(x, y) = F_{x,y},$ for all $(x, y) \in X \times X$):

(PM1) $F_{x,y}(t) = \epsilon_0(t) \iff x = y$ and $x, y \in X$;

(PM2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;

(PM3) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$,
for all $x, y, z \in X$ and for every $t, s \geq 0$.

Example 1.5.3. (Sehgal [104]) Let (X, d) be a metric space. Define $F_{x,y}(t) = \epsilon_0(t - d(x, y))$ for all $x, y \in X$ and $t > 0$. Then $(X, \mathscr{F})$ is a PM-space induced by the metric d.

Chapter 2

Banach contraction principle and its generalizations

In 1922 Banach presented a very interesting theorem in his doctoral dissertation which is known as Banach contraction principle. It has been extensively used to study the existence of solutions of many nonlinear differential and integral equations and to prove the convergence of algorithms in computational mathematics. The aim of this chapter is to present a brief literature review of metric fixed point theory. Furthermore, some necessary and relevant results are also refurnished and their proofs can be found in monographs in the list of references.

2.1 Banach contraction principle

Theorem 2.1.1. (Banach contraction principle [9]) Let (X, d) be a complete metric space and $f : X \to X$ be a contraction mapping with contraction constant $\kappa \in (0, 1)$. Then f has a unique fixed point $\xi \in X$. Moreover, for any $x \in X$:

1. the iterative sequence $\{f^n x\}$ converges to ξ;
2. the following priori estimate holds
$$d(f^n x, \xi) \leq \frac{\kappa^n}{1-\kappa} d(x, fx) \qquad n = 0, 1, 2, \cdots ;$$
3. the following posteriori estimate holds
$$d(f^{n+1} x, \xi) \leq d(f^{n+1} x, f^n x) \qquad n = 0, 1, 2, \cdots .$$

The strength of contraction theorem lies in the fact that it not only provides a constructive algorithm to approach the fixed point but it also provides the error bounds. Following well known theorem is due to Picard-Lindelöf which illuminates the validity of Banach contraction principle.

Theorem 2.1.2. (Picard iteration theorem [30]) Consider the following first order initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{2.1}$$

Suppose the following conditions hold:

1. f is continuous on a rectangle

$$R = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}.$$

2. f is Lipschitz function with respect to x, or equivalently

$$|f(t, x) - f(t, y)| \leq \kappa |x - y|,$$

for $(t, x), (t, y) \in R$ and $\kappa > 0$.

Then the initial value problem (2.1) has a unique solution on an interval $[t_0 - \beta, t_0 + \beta]$ where $\beta < \min\{a, \frac{b}{c}, \frac{1}{\kappa}\}$.

The direct applications of Banach principle such as the existence of solutions for differential equations, existence of equilibria in game theory etc., elucidate its significance. This is one of the most important reason why the researchers have always been intrigued and propelled to establish a variety of generalizations of Theorem 2.1.1. A lot of interesting fixed point theorems have been obtained by considering contraction condition involving not only $d(x, y)$ on right hand side of (1.2) but also the displacement of x and y under the mapping f i.e., $d(x, fx), d(y, fy), d(x, fy), d(y, fx)$.

Next subsection deals with some extensions obtained by weakening contractive condition and by imposing some supplementary conditions to the structure of metric space or the mapping f.

2.2 Weaker forms of contraction

Banach [9] showed that every contraction mapping on a complete metric space always possesses a unique fixed point. Edelstein [36] defined the notion of contractive mappings. The mapping $f : X \to X$ is said to be contractive if

$$d(fx, fy) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y. \tag{2.2}$$

In order to obtain a fixed point of a contractive map we have to add further assumptions such as there exists a point $x \in X$ for which $\{f^n x\}$ contains a convergent subsequence or the space is compact. The mappings $f : X \to X$ is said to be nonexpansive if

$$d(fx, fy) \leq d(x, y) \text{ for all } x, y \in X. \tag{2.3}$$

To obtain a fixed point of nonexpansive map we also need to impose some certain assumptions such as uniform normal structure or compactness of the space. The mapping $f : X \to X$ is said to be weakly contractive if

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for all } x, y \in X, \tag{2.4}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Alber and Guerre-Delabriere [3] introduced the notion of weakly contractive mapping when the underlying space was taken to be a Hilbert space. They proved that every weakly contractive mapping defined on a Hilbert space posses a unique fixed point point, with out any additional assumption. Later on, Rhoades [93] showed that this result also holds for metric spaces. From the definitions it is clear that weakly contractive maps lie between contraction maps and contractive maps.

The inception of Kannan's fixed point theorem in 1969 [57] has carved a niche for itself in fixed point theory like Banach principle. The mapping $f : X \to X$ is said to be a Kannan mapping if there exists $\kappa \in [0, \frac{1}{2})$ such that

$$d(fx, fy) \leq \kappa[d(x, fx) + d(y, fy)] \quad \text{for all } x, y \in X. \tag{2.5}$$

Unlike contraction, contractive, nonexpensive and weakly contractive mappings the Kannan's mappings are not necessarily continuous. By pursuing on the same lines, Chatterjea [26] proposed the following condition:

$$d(fx, fy) \leq \kappa[d(x, fy) + d(y, fx)] \quad \text{for all} \quad x, y \in X, \tag{2.6}$$

where $\kappa \in [0, \frac{1}{2})$.

The notion of Kannan's mapping was further refined by Ćirić, Reich and Rus [28, 91, 98], Zamfirescu [113] and then by Hardy and Rogers [45]. The mapping $f : X \to X$ is said to be a Čirič-Reich-Rus operator if there exists nonnegative real numbers a, b, c with $a + b + c < 1$ such that

$$d(fx, fy) \leq ad(x, y) + bd(x, fx) + cd(y, fy) \quad \text{for all} \quad x, y \in X. \tag{2.7}$$

Hardy and Rogers [45] attenuated the class of Čirič-Reich-Rus operators by defining the following condition:

$$d(fx, fy) \leq ad(x, y) + bd(x, fx) + cd(y, fy) + ed(x, fy) + fd(y, fx) \quad \text{for all} \quad x, y \in X, \tag{2.8}$$

where $a+b+c+e+f < 1$. The condition (2.8) unified the notion of Kannan's and Chatterjea's mappings and also incorporates with Čirič-Reich-Rus operators as well.

Theorem 2.2.1. (Rhoades [93, Theorm 1]) Let (X, d) be a complete metric space. Assume the mapping $f : X \to X$ is weakly contractive. Then f has a unique fixed point.

Theorem 2.2.2. (Nemytzki-Edelstein [70, 36]) Let (X, d) be a compact metric space and $f : X \to X$ be a contractive mapping. Then f has a unique fixed point.

Theorem 2.2.3. (Kannan [57]) Let (X, d) be a complete metric space and $f : X \to X$ be a Kannan mapping. Then f has a unique fixed point.

Theorem 2.2.4. (Chatterjea [26]) Let (X, d) be a complete metric space and $f : X \to X$ satisfies (2.6). Then f has a unique fixed point.

Theorem 2.2.5. (Hardy and Rogers [45]) Let (X, d) be a complete metric space and $f : X \to X$ satisfies (2.8). Then f has a unique fixed point.

Branciari [23] generalized the Banach contraction principle by proving the existence of unique fixed point of a mapping on a complete metric space satisfying a general contractive condition of integral type. Afterwards, many authors undertook further investigations in this direction (see, e.g., [6, 35, 94, 110, 112]).

Let Φ denote the class of all mappings $\phi : [0, +\infty) \to [0, +\infty)$ which are Lebesgue integrable, summable on each compact subset of $[0, +\infty)$, nonnegative and for each $\epsilon > 0$, $\int_0^\epsilon \phi(s)ds > 0$.

Theorem 2.2.6. (Branciari [23, Theorem 2.1]) Let (X, d) be a complete metric space, $\kappa \in (0, 1)$, and let $f : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \phi(s)ds \leq \kappa \int_0^{d(x,y)} \phi(s)ds, \tag{2.9}$$

where $\phi \in \Phi$. Then f has a unique fixed point $\xi \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = t$.

Clearly, when $\phi(s) = 1$ for $s \in [0, +\infty)$ Theorem 2.2.6 subsumes Banach contraction principle. But converse may not hold (see, [23, Exmaple 3.6]). Subsequently, Rhoades [94] further generalized Theorem 2.2.6 by replacing term $d(x, y)$ in (2.9) with $m(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}\}$.

2.3 φ-contractions

Matkowski [66] introduced the class of φ-contraction in metric fixed point theory. Afterwards, many authors contributed to further developments in this direction by analyzing the properties on function φ (see, [11, 76, 77, 85, 97, 99]). Most of the fixed point theorems for such class of mappings are established wherein iterative sequence converges to the fixed point. But only a few of them cater with a constructive method to approach the fixed point and also render information on convergence rate (see, e.g., [11, 85]).

Definition 2.3.1. Let $f : X \to X$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function. Then f is said to be φ-contraction if:

$$d(fx, fy) \leq \varphi(d(x, y)), \quad \text{for all } \ x, y \in X. \tag{2.10}$$

Theorem 2.3.2. (Rakotch [89]) Let (X, d) be a complete metric space and $f : X \to X$ satisfies

$$d(fx, fy) \leq \varphi(d(x, y))d(x, y), \quad \text{for all } \ x, y \in X.$$

Where $\varphi : \mathbb{R}^+ \to [0, 1)$ is nondecreasing. Then f is a Picard operator.

Theorem 2.3.3. (Browder [24]) Let (X, d) be a complete metric space and D be a bounded subset of X. Assume $f : D \to D$ satisfies

$$d(fx, fy) \leq \varphi(d(x, y)), \quad \text{for all} \ \ x, y \in X,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is monotone nondecreasing and right continuous such that $\varphi(t) < t$ for all $t > 0$. Then f is Picard operator.

In the following, Berinde [11] calculated the rate of convergence for iterative process by using (c)-comparison function.

Theorem 2.3.4. (Berinde [11]) Let (X, d) be a complete metric space and $f : X \to X$ be a φ-contraction where φ is a (c)-comparison function. Then f has a unique fixed point ξ. Moreover, for any $x \in X$:

1. the iterative sequence $\{f^n x\}$ converges to ξ;
2. the following estimates hold

$$d(x_n, \xi) \leq s(d(x_n, x_{n+1}), \quad n = 0, 1, 2, \cdots,$$

   where $s(t) = \sum_{k=0}^{\infty} \varphi^k(t)$.

Theorem 2.3.5. (Matkowski [66]) Let (X, d) be a complete metric space and $f : X \to X$ be a φ-contraction where φ is a comparison function. Then f has a unique fixed point ξ and $\{f^n x\}$ converges to ξ for any $x \in X$.

Recently, Proinov [85] came up with a very nice variant wherein the constructive iterative scheme was proposed with a higher order of convergence towards the fixed point.

Lemma 2.3.6. (Proinov [85]) Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying:

$$\varphi(t) = t\phi(t) \quad \text{for all } t \in J. \tag{2.11}$$

Then it has the following properties:

1. $0 \leq \phi(t) < 1$ for all $t \in J$;

2. $\phi(\lambda t) \leq \lambda^{r-1}\phi(t)$ for all $\lambda \in (0,1)$ and $t \in J$.

Assume that $f : D \subset X \to X$ is an operator on X and satisfies the following iterated contractive condition:

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \quad \text{for all } x \in D,\ fx \in D \text{ with } d(x, fx) \in J, \tag{2.12}$$

where φ is a gauge function of order $r \geq 1$ on an interval J. A point $x_0 \in D$ is said to be an initial point of f if $d(x_0, fx_0) \in J$ and all the iterates $x_0, x_1, \cdots$, are well defined and belong to D.

Theorem 2.3.7. (Proinov [85]) Let $f : D \subset X \to X$ be an operator on a complete metric space (X, d) such that f satisfies contractive condition (2.12) with a Bianchini Grandolfi gauge function φ on an interval J having order $r \geq 1$. Further, suppose that $x_0 \in D$ is an initial point of f, then following statements hold true.

1. The iterative sequence $\{x_n\}$ remains in $\overline{B}(x_0, \rho_0)$ and converges with rate of convergence at least $r \geq 1$ to a point ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n), n = 0, 1, \cdots$, and

$$\rho_n = d(x_n, x_{n+1}) \sum_{j=0}^{\infty} \left[\phi(d(x_n, x_{n+1}))\right] \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))}, \tag{2.13}$$

   where ϕ is a nonnegative and nondecreasing function on J satisfying (2.11).

2. Following priori estimate holds for all $n \geq 0$,

$$d(x_n, \xi) \leq d(x_0, fx_0) \sum_{j=n}^{\infty} \lambda^{P_j(r)} \leq \frac{\lambda^{P_n(r)}}{1 - \lambda^{r^n}} d(x_0, fx_0), \tag{2.14}$$

   where $\lambda = \phi(d(x_0, fx_0))$.

3. Following posteriori estimate holds for all $n \geq 1$,

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} \left[\phi(\varphi(d(x_n, x_{n-1})))\right]^{P_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \end{aligned} \tag{2.15}$$

4. For all $n \geq 1$, we have,

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1}) \leq \lambda^{P_j(r)} d(x_0, fx_0). \tag{2.16}$$

5. If $\xi \in D$ and f is continuous at ξ, then ξ is a fixed point of f.

2.4 $\alpha - \varphi$ contractive mappings

In [100] Samet et al. introduced the notion of $\alpha - \varphi$ contractive mappings. Soon after that a huge literature emerged consuming this idea in each and every possible way to extend the notion (see e.g., [51, 54, 60]).

Definition 2.4.1. (Samet et al. [100]) Let (X, d) be a metric space and $f : X \to X$. Then f is said to be an $\alpha - \varphi$ contractive mapping if

$$\alpha(x, y)d(fx, fy) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in X, \tag{2.17}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing such that $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$ and $\alpha : X \times X \to [0, \infty)$.

Theorem 2.4.2. (Samet et al. [100]) Let (X, d) be a complete metric space and $f : X \to X$ be an $\alpha - \varphi$ contractive mapping satisfying the following conditions:

(i) f is α-admissible, i.e., $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$ for every $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) f is continuous.

Then f has a fixed point.

Theorem 2.4.3. (Samet et al. [100]) Let (X, d) be a complete metric space and $f : X \to X$ be an $\alpha - \varphi$ contractive mapping satisfying the following conditions:

(i) f is α-admissible, i.e., $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$ for every $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.

Then f has a fixed point.

2.5 Contractions on an ordered metric space

Following Turinici [111], Ran and Reurings [90] studied fixed points of self mappings on a metric space (X, d) endowed with the partial ordering $\preceq$. In this context, many interesting results have been established by some authors for partially ordered set endowed with a complete metric (see e.g., [1, 34, 40, 72, 84, 87, 90]). Most of these fixed point results are hybrid of Banach principle and Knaster-Tarski's theorem (see, [41, 53]).

Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$. The mapping f is said to be nonincreasing if $x, y \in X, x \preceq y \Rightarrow f(x) \succeq f(y)$. The mapping f is nondecreasing if $x, y \in X, x \preceq y \Rightarrow f(x) \preceq f(y)$. The mapping f maps comparable elements to comparable elements if:

$$\text{for all } x, y \in X,\ x \preceq y \implies fx \preceq fy \text{ or } fy \preceq fx.$$

Theorem 2.5.1. (Ran and Reurings [90]) Let (X, d) be a complete metric space endowed with a partial ordering $\preceq$. Let $f : X \to X$ satisfies

$$d(fx, fy) \leq \kappa d(x, y) \text{ for all } x, y \in X, \text{ with } x \preceq y,$$

where $\kappa \in (0, 1)$. Moreover, if the following conditions are satisfied:

1. $\exists x_0 \in X$ with $x_0 \preceq fx_0$ or $fx_0 \preceq x_0$;

2. f is monotone and continuous;

3. every pair of elements of X has an upper *and* a lower bound.

Then f is Picard operator.

Nieto and Rodríguez-López [72] further improved Theorem 2.5.1 and as an application obtained existence result for the solution of periodic boundary value problem for ordinary differential equations.

Theorem 2.5.2. (Nieto and Rodríguez-López [72]) Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let $f : X \to X$ be nondecreasing (with respect to $\preceq$) and satisfies

$$d(fx, fy) \leq \kappa d(x, y) \quad \text{for all } x, y \in X, \text{ with } x \preceq y,$$

where $\kappa \in (0,1)$. Assume that one of the following conditions hold:

1. f is continuous and there exist some $x_0 \in X$ such that either $x_0 \preceq fx_0$ or $fx_0 \preceq x_0$;
2. for any nondecreasing sequence $\{x_n\}$, $x_n \to x$ implies $x_n \preceq x$ for $n \in \mathbb{N}$ and there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;
3. for any nonincreasing sequence $\{x_n\}$, $x_n \to x$ implies $x \preceq x_n$ and there exists $x_0 \in X$ such that $fx_0 \preceq x_0$.

Then f has a fixed point. Furthermore, if every pair of elements of X has an upper *or* a lower bound then f is a Picard operator.

Following the same direction authors in [71, 73, 84] extended above result by enfeebling continuity condition in the following way.

Theorem 2.5.3. ([71, 73, 84]) Let (X,d) be a complete metric space endowed with a partial order $\preceq$. Let $f : X \to X$ preserves comparable elements and satisfies

$$d(fx,fy) \leq \kappa d(x,y) \quad \text{for all } x,y \in X, \text{ with } x \preceq y,$$

where $\kappa \in (0,1)$. Assume that the following conditions hold:

1. either f is orbitally continuous; or

   for any sequence $\{x_n\}$ if $x_n \to x$ and every pair of elements (x_n, x_{n+1}) is comparable for $n \in \mathbb{N}$ then there exists a subsequence $\{x_{n_k}\}$ such that the pair of elements (x_{n_k}, x) are comparable for $k \in \mathbb{N}$;
2. there exists $x_0 \in X$ such that the pair (x_0, fx_0) is comparable.

Then f has a fixed point. Furthermore, if every pair of elements of X has an upper *or* a lower bound then f is a Picard operator.

Afterwards authors in [2] moved ahead and by weakening contractive condition presented the following result.

Theorem 2.5.4. (Agarwal et al. [2]) Let (X,d) be a complete metric space endowed with a partial order $\preceq$. Let $f : X \to X$ be nondecreasing and for all $x, y \in X$ with $x \preceq y$ satisfies

$$d(fx, fy) \leq \varphi\Big(\max\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,fy)+d(y,fy)]\}\Big),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing with $\lim n \to \infty \varphi^n(t) = 0$ for each $t > 0$. Assume that the following conditions hold:

1. either f is continuous; or

   for any nondecreasing sequence $\{x_n\}$ if $x_n \to x$ then $x_n \leq x$ for $n \in \mathbb{N}$;

2. there exists $x_0 \in X$ such that $x_0 \preceq fx_0$.

Then f has a fixed point.

Further developments in this direction were found in [77] by O'Regan and Petruşel. Recently, Harjani and Sadarangani [46] obtained some fixed point results for weakly contractive mappings defined on partially ordered set endowed with a complete metric space.

We state following results from Harjani and Sadarangani [46] for convenience. In Chapter 2 some generalizations of the following main results will be proposed.

Theorem 2.5.5. (Harjani and Sadarangani [46, Theorem 2]) Let (X,d) be a complete metric space endowed with a partial order $\preceq$. Let $f : X \to X$ be a continuous and nondecreasing mapping such that

$$d(fx, fy) \leq d(x,y) - \psi(d(x,y)) \text{ for all } x, y \in X \text{ with } x \succeq y, \tag{2.18}$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous nondecreasing such that ψ is positive in $(0,\infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ then f has a fixed point.

Above result was further refined by making use of the following hypothesis which was appeared in (Nieto and Rodríguez-López [72, Theorem 1])
If $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all $n \in \mathbb{N}$.

(2.19)

Theorem 2.5.6. (Harjani and Sadarangani [46, Theorem 3]) Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Assume that X satisfies (2.19). Let $f : X \to X$ be nondecreasing mapping such that

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for all } x, y \in X \text{ with } x \succeq y, \tag{2.20}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ then f has a fixed point.

It was shown in Nieto and Rodríguez-López [72] that the following two conditions are equivalent.

$$\text{For } x, y \in (X, \preceq) \text{ there exists a lower or an upper bound.} \tag{2.21}$$

$$\text{For } x, y \in (X, \preceq) \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{2.22}$$

Theorem 2.5.7. (Harjani and Sadarangani [46, Theorem 4]) Adding condition (2.22) to the hypothesis of Theorem 2.5.5 (resp. Theorem 2.5.6) we obtain the uniqueness of the fixed point.

Theorem 2.5.8. (Harjani and Sadarangani [46, Theorem 5]) Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let $(X, \preceq)$ satisfies (2.22) and $f : X \to X$ be a nonincreasing mapping such that

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for } x \succeq y, \tag{2.23}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Suppose also that either

(i) f is continuous, or

(ii) X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is comparable to the limit x.

(2.24)

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ or $x_0 \succeq fx_0$ then f has a unique fixed point.

Theorem 2.5.9. (Harjani and Sadarangani [46, Theorem 6]) Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let $(X, \preceq)$ satisfies (2.22) and $f : X \to X$ maps comparable elements to comparable elements, that is,

$$\text{for } x, y \in X, x \preceq y \Rightarrow fx \preceq fy \text{ or } fx \succeq fy.$$

Further assume that f satisfies the following condition

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for all } x, y \in X \text{ with } x \succeq y,$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Suppose that either f is continuous or X is such that condition (2.24) holds. If there exists $x_0 \in X$ with x_0 comparable to fx_0 then f has a unique fixed point ξ. Moreover, for $x \in X$, $\lim_{n\to\infty} f^n x = \xi$.

For further study on the subject we refer the readers to [2, 25, 29, 39, 71, 72, 74, 90]. In [72, 73, 90], some applications for the solutions of matrix equations and ordinary differential equations are furnished.

2.6 Contractions on a metric space endowed with a graph

Jachymski [52] established a generalized and novel variant by utilizing the graph theoretic approach instead of partial ordering and unified the results given by authors [72, 84, 90]. From then on, investigations have been carried out to obtain better and generalized results by weakening contraction condition and analyzing connectivity of graph (see, [4, 15, 101, 102]). Jachymski [52] showed that a mapping on a complete metric space still has a fixed point provided that the mapping satisfied contraction condition for pairs of points which form edges in the graph. Subsequently, Beg et al. [10] established a multivalued version of main result of Jachymski [52]. Later on, Bojor [15] obtained some results in such settings by weakening the condition of Banach G-contraction and introducing some new type of connectivity of the graph.

We state, for convenience the following definition and main result due to Jachymski [52]. First we need to recollect some definitive notations. Let X be a complete metric space with metric

d (unless specified otherwise) and Ω is the diagonal of the Cartesian product $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Omega$. Assume that G has no parallel edges. We may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 2.6.1. (Jachymski [52, Definition 2.1]) A mapping $f : X \to X$ is called a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,

$$\text{for all } x, y \in X,\ (x, y) \in E(G) \Rightarrow (fx, fy) \in E(G), \tag{2.25}$$

and f decreases weights of edges of G in the following way:

$$\exists\ \kappa \in (0, 1)\ \text{ for all } x, y \in X,\ (x, y) \in E(G) \Rightarrow d(fx, fy) \le \kappa\ d(x, y). \tag{2.26}$$

Theorem 2.6.2. (Jachymski [52, Theorem 3.2]) Let (X, d) be a complete metric space endowed with a graph G and let the triple (X, d, G) have the following property:

($\mathcal{P}$) : for any sequence $\{x_n\}$ in X, if $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

Assume $f : X \to X$ be a G-contraction and $X_f := \{x \in X : (x, fx) \in E(G)\}$. Then the following assertions hold.

1. $card\ Fixf = card\{[x]_{\widetilde{G}} : x \in X_f\}$.
   (For a set S $card\ S$ is the number of elements in S).
2. $Fixf \neq \emptyset \iff X_f \neq \emptyset$.
3. f has a unique fixed point $\iff$ there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.
4. For any $x \in X_f$, $f\mid_{[x]_{\widetilde{G}}}$ is a Picard operator.
5. If $X_f \neq \emptyset$ and G is weakly connected then f is a Picard operator.
6. If $X' := \cup\{[x]_{\widetilde{G}} : x \in X_f\}$ then $f\mid_{X'}$ is a weakly Picard operator.
7. If $f \subseteq E(G)$ then f is a weakly Picard operator.

Bojor [15] relaxed the condition $X_f \neq \emptyset$ by introducing the following notion of f-connectivity which is somewhat stronger than connectivity or weak connectivity of a graph.

Definition 2.6.3. ([15, Definition 8]) Let X be a nonempty set endowed with a graph G and $f : X \to X$. The graph G is called f-connected if for all $x, y \in V(G)$ such that $(x, y) \notin E(G)$, there exists a path in G, $\{x_i\}_{i=0}^{N}$ from x to y such that $x_0 = x, x_N = y$ and $(x_i, fx_i) \in E(G)$ for all $i = 1, 2, \cdots, N-1$. A graph G is weakly f-connected if $\widetilde{G}$ is f-connected.

Definition 2.6.4. ([15, Definition 7]) Let (X, d) be a metric space endowed with a graph G. The mapping $f : X \to X$ is said to be a G-Ciric-Reich-Rus operator if:

$$\text{for all } \ x, y \in X, (x, y) \in E(G) \implies (fx, fy) \in E(G); \tag{2.27}$$

there exist nonnegative real numbers a, b, c with $a + b + c < 1$ such that

$$\text{for all } \quad x, y \in X, (x, y) \in E(G) \implies d(fx, fy) \leq ad(x, y) + bd(x, fx) + cd(y, fy). \tag{2.28}$$

Theorem 2.6.5. (Bojor, [15, Theorem 6]) Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a G-Ciric-Reich-Rus operator. Assume that:

1. G is f-connected;
2. for any sequence $\{x_n\}$ in X, if $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ then there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$.

Then f is a Picard operator.

In this context, many authors undertook further proceedings and put forth some nice refinements of Theorem 2.6.2. For some recent developments in this direction we refer the readers to [4, 10, 15, 16, 33, 42, 101]. In [42], it has been substantiated with the help of a counter example that Theorem 2.6.2 can not be improved for any φ-contractions where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing such that $\lim_{n\to\infty} \varphi^n(t) = 0, t > 0$.

Example 2.6.6. (Gwóźdź-Łukawska and Jachymski [42]) Let (X, d) be a Euclidean metric space with $X = \{s_n : n \in \mathbb{N}\}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Then (X, d) is complete. Let the graph G consists of $V(G) = X$ and $E(G) = \{(s_n, s_{n+1}) : n \in \mathbb{N}\} \cup \{(s_n, s_n) : n \in \mathbb{N}\}$. Then G

is reflexive and connected. Set $fs_n = s_{n+1}$, for $n \in \mathbb{N}$. Note that f is edge-preserving. Let $(x, y) \in E(G)$ and $x \neq y$. Then there is $k \in \mathbb{N}$ such that $x = s_k$ and $y = s_{k+1}$. Hence, $|x - y| = 1/(k + 1)$ and $|fx - fy| = 1/(k + 2)$. Define a function $\phi : \mathbb{R} \to \mathbb{R}$ by;

$$\phi(0) = 0, \qquad \phi(t) = 1/3 \quad \text{for } t > 1/2,$$

and $\phi|_{(0,1/2]}$ is the polygonal line with nodes $(1/(n+1), 1/(n+2))$ for $n \in \mathbb{N}$. Then $|fx-fy| = \phi(|x - y|)$. It is easy to see that ϕ is nondecreasing, continuous and such that $\phi(t) < t$ for $t > 0$, hence $\lim_{n\to\infty} \phi^n(t) = 0$. Furthermore, f is nonexpansive, hence continuous. Since every convergent sequence in X is constant for sufficiently large n, which infers that all conditions Theorem 2.6.2 are satisfied but f has no fixed point.

2.7 Cyclic contractions

Kirk et al. [63] introduced the notion of cyclic representations and cyclic contractions to generalize Banach contraction principle. This theory has been further investigated and evolved by many authors (see, e.g., [5, 58, 59, 61, 62, 79, 82, 88]). We recall the definition which is primitively rooted in [63] but its succinct version is refurnished in [96].

Definition 2.7.1. Let X be a nonempty set, m a positive integer and $f : X \to X$ an operator. Then $X := \cup_{i=1}^m A_i$ is known as a cyclic representation of X with respect to f if

1. $A_i, i = 1, 2, \cdots, m$ are nonempty sets;

2. $f(A_1) \subset A_2, \cdots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$. (2.29)

Theorem 2.7.2. (Kirk et al. [63]) Let (X, d) be a complete metric space. Let m be positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that

1. $\cup_{i=1}^m A_i$ is cyclic representation of Y with respect to f;

2. $\exists \kappa \in (0, 1)$ such that $d(fx, fy) \leq \kappa\, d(x, y)$ for $x \in A_i, y \in A_{i+1}$ where $A_{m+1} = A_1$.

Then f has a unique fixed point $\xi \in \cap_{i=1}^m A_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^m A_i$.

Petric [82] refined above result by using Ćirić-Reich-Rus mappings.

Theorem 2.7.3. (Petric [82]) Let (X, d) be a complete metric space. Let m be positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that

1. $\cup_{i=1}^m A_i$ is cyclic representation of Y with respect to f;
2. there exist nonnegative real numbers a, b, c with $a + b + c < 1$ such that for $x \in A_i, y \in A_{i+1}$ where $A_{m+1} = A_1$ we have

$$d(fx, fy) \leq ad(x, y) + bd(x, fx) + cd(y, fy). \tag{2.30}$$

Then f has a unique fixed point $\xi \in \cap_{i=1}^m A_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^m A_i$.

Karapinar [58] furnished the lore by investigating cyclic weakly contractive mappings.

Theorem 2.7.4. (Karapinar [58, Theorem 6]) Let (X, d) be a complete metric space. Let m be positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that:

1. $\cup_{i=1}^m A_i$ is cyclic representation of Y with respect to f and
2. there exists $\psi : [0, \infty) \to [0, \infty)$ where ψ is continuous, nondecreasing, positive on $(0, \infty)$, $\psi(0) = 0$ and the following holds:

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for } x \in A_i, y \in A_{i+1};\ A_{m+1} = A_1.$$

Then f has a unique fixed point $\xi \in \cap_{i=1}^m A_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^m A_i$.

Păcurar and Rus [79] undertook further investigations in this context by establishing a brief study of cyclic φ-contractions in metric fixed point theory.

Theorem 2.7.5. (Păcurar and Rus [79, Theorem 2.1]) Let (X, d) be a complete metric space. Let m be positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that:

1. $\cup_{i=1}^m A_i$ is cyclic representation of Y with respect to f and
2. there exists a (c)-comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$d(fx, fy) \leq \varphi(d(x, y)) \text{ for } x \in A_i, y \in A_{i+1};\ A_{m+1} = A_1.$$

Then f has a unique fixed point $\xi \in \cap_{i=1}^m A_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^m A_i$.

Chapter 3

Fixed point theorems in metric spaces endowed with a graph

Motivation behind this chapter is to generalize the notion of Banach G-contractions by weakening contraction inequality. In this context we introduced two new notions: weakly G-contractive mappings; integral G-contractions [101, 102].

Subsequently throughout this chapter let (X, d) be a metric space. Let Ω denote the diagonal of the cartesian product $X \times X$. Let G be a directed graph such that the set of its vertices $V(G)$ coincides with X, and the set of its edges $E(G)$ contains all loops, that is, $E(G) \supseteq \Omega$. We assume that G has no parallel edges. We may treat G as a weighted graph by attributing to each edge the distance between its vertices.

Definition 3.0.6. [52] A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $\{k_n\}$ of positive integers, $f^{k_n}x \to y$ implies $f(f^{k_n}x) \to fy$ as $n \to \infty$.

Definition 3.0.7. [52] A mapping $f : X \to X$ is called orbitally G-continuous if for all $x, y \in X$ and any sequence $\{k_n\}$ of positive integers, $f^{k_n}x \to y$ and $(f^{k_n}x, f^{k_{n+1}}x) \in E(G)$ $\forall n \in \mathbb{N}$ imply $f(f^{k_n}x) \to fy$.

Definition 3.0.8. Two sequences $\{x_n\}$ and $\{y_n\}$ in a metric space (X, d) are said to be equivalent if $d(x_n, y_n) \to 0$. Moreover if each of them is Cauchy then these are called Cauchy equivalent.

In [52, 4], the authors put forth the following:

($\mathcal{C}$) for any $\{x_n\}$ in X such that $x_n \to x$ with $(x_{n+1}, x_n) \in E(G)$ for all $n \geq 1$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x, x_{n_k}) \in E(G)$ [52];

($\mathcal{H}$) for any $\{x_n\}$ in X such that $x_n \to x \in X$ with $x_n \in [x]_{\widetilde{G}}$ for all $n \geq 1$ then $r(x_n, x) \to 0$ [4].

A graph G satisfying property ($\mathcal{C}$) or ($\mathcal{H}$) is called ($\mathcal{C}$)-graph or ($\mathcal{H}$)-graph respectively. It has been shown that properties ($\mathcal{C}$) and ($\mathcal{H}$) are independent (see, [4, Examples 2.1, 2.2]). In this context we define the following properties for a graph G. Let $f : X \to X$ and the metric space (X, d) is endowed with a graph G:

($\mathcal{C}_f$) for any $\{f^n x\}$ in X such that $f^n x \to y \in X$ with $(f^{n+1}x, f^n x) \in E(G)$ there exists a subsequence $\{f^{n_k}x\}$ of $\{f^n x\}$ and $n_0 \in \mathbb{N}$ such that $(y, f^{n_k}x) \in E(G)$ for all $k \geq n_0$;

($\mathcal{H}_f$) for any $\{f^n x\}$ in X such that $f^n x \to y \in X$ with $f^n x \in [y]_{\widetilde{G}}$ for all $n \geq 1$ then $r(f^n x, y) \to 0$.

Subsequently, we call a graph G satisfying property ($\mathcal{C}_f$) or ($\mathcal{H}_f$) as ($\mathcal{C}_f$)-graph or ($\mathcal{H}_f$)-graph respectively. It is easily seen that in a graph G the property ($\mathcal{C}$) subsumes property ($\mathcal{C}_f$) for any self-mapping f on X but converse may not hold as shown below.

Example 3.0.9. [102] Let $X = [0, 1]$ endowed with usual metric $d(x, y) = |x - y|$. Consider a graph G consisting of $V(G) := X$ and $E(G) := \{(\frac{n}{n+1}, \frac{n+1}{n+2}) : n \in \mathbb{N}\} \cup \{(\frac{x}{2^n}, \frac{x}{2^{n+1}}) : n \in \mathbb{N}, x \in [0, 1]\} \cup \{(\frac{x}{2^{2n}}, 0) : n \in \mathbb{N}, x \in [0, 1]\}$. Note that G does not satisfy property ($\mathcal{C}$) as $\frac{n}{n+1} \to 1$. Whereas by defining $f : X \to X$ as $fx = \frac{x}{2}$, G satisfies property ($\mathcal{C}_f$). Since, $f^n x = \frac{x}{2^n} \to 0$ as $n \to \infty$.

Similarly, any graph G satisfying the property ($\mathcal{H}$) also satisfies ($\mathcal{H}_f$) whereas next example shows that the converse may not hold.

Example 3.0.10. [102]Let $X = [0, 1]$ endowed with usual metric $d(x, y) = |x - y|$. Consider a graph G consisting of $V(G) := X$ and $E(G) := \{(\frac{n}{n+1}, 0) : n \in \mathbb{N}\} \cup \{(\frac{x}{2^n}, 0) : n \in \mathbb{N}, x \in [0, 1]\} \cup \{(0, 1)\}$. Since, $\frac{n}{n+1} \to 1$ but $r(\frac{n}{n+1}, 1) = |\frac{n}{n+1} - 0| + |0 - 1| \nrightarrow 0$. Thus property ($\mathcal{H}$) does not hold in G. On the other hand by defining $f : X \to X$ as $fx = \frac{x}{2}$ then G satisfies ($\mathcal{H}_f$).

3.1 Weakly G-contractive mappings

Motivated by Jachymski [52] using the language of graph theory, we obtain some fixed point results that unify and extend main results by Harjani and Sadarangani [46]. In particular, we show that Theorems 2.5.5−2.5.9 are special cases of our results. Consequently, as an application of our results we obtain a fixed point result for weakly contractive cyclic mappings and for α-type weakly contractive mappings. An example has been established to demonstrate the degree of generality of our result over some pre-existing results. Inspired form [52] we introduce the following intuitive notion.

Definition 3.1.1. Let (X,d) be a metric space endowed with the a graph G. A mapping $f : X \to X$ is called weakly G-contractive if it satisfies the following two conditions. For $x, y \in X$:

$$(fx, fy) \in E(G) \text{ whenever } (x,y) \in E(G), \tag{3.1}$$

$$d(fx, fy) \leq d(x,y) - \psi(d(x,y)) \text{ whenever } (x,y) \in E(G), \tag{3.2}$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous nondecreasing such that ψ is positive on $(0,\infty)$ and $\psi(0) = 0$.

Example 3.1.2. Let (X,d) be a metric space. Consider the graph G_0 defined by $G_0 = (X, X \times X)$. Then any weakly contractive map is weakly G-contractive.

3.1.1 Fixed point theorems for weakly G-contractive mappings

Let us denote by $r(x,y) = \sum_{i=1}^{m} d(z_{i-1}, z_i)$, where $\{z_i\}_{i=0}^{m}$ is a path from x to y in G. We start with the following lemma.

Lemma 3.1.3. Let (X,d) be a metric space and $f : X \to X$ be a weakly G-contractive map. Then for any $x \in X$ and $y \in [x]_{\tilde{G}}$ we have

$$\lim_{n\to\infty} d(f^n x, f^n y) = \lim_{n\to\infty} r(f^n x, f^n y) = 0. \tag{3.3}$$

Proof. Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there exists a path $x = z_0, z_1, \cdots, z_l = y$ in the graph G. As f is weakly G-contractive, from (3.1) and (3.2) we get

$$(f^n z_{i-1}, f^n z_i) \in E(G) \ \forall i = 1, 2, \cdots, l, \quad \forall n \in \mathbb{N} \tag{3.4}$$

and

$$
\begin{aligned}
d(f^n z_{i-1}, f^n z_i) &\leq d(f^{n-1} z_{i-1}, f^{n-1} z_i) - \psi(d(f^{n-1} z_{i-1}, f^{n-1} z_i)) \quad (3.5)\\
&\leq d(f^{n-1} z_{i-1}, f^{n-1} z_i) \quad \forall i = 1, 2, \cdots, l, \quad \forall n \in \mathbb{N}. \quad (3.6)
\end{aligned}
$$

This shows that $\{d(f^n z_{i-1}, f^n z_i)\}$ is a nonincreasing sequence of nonnegative real numbers, bounded below by 0, thus convergent. Let $d(f^n z_{i-1}, f^n z_i) \to \gamma$. Taking limit as $n \to \infty$ in (3.5) we get $\gamma \leq \gamma - \psi(\gamma) \leq \gamma$. Therefore, $\psi(\gamma) = 0$ and by using the properties of ψ we get $\gamma = 0$. Thus

$$
\lim_{n\to\infty} d(f^n z_{i-1}, f^n z_i) = 0 \; \forall i = 1, 2, \cdots, l, \forall n \in \mathbb{N}. \quad (3.7)
$$

By triangular inequality, we have

$$
\begin{aligned}
d(f^n x, f^n y) &= d(f^n z_0, f^n z_k)\\
&\leq d(f^n z_0, f^n z_1) + d(f^n z_1, f^n z_2) + \cdots + d(f^n z_{k-1}, f^n z_l).
\end{aligned}
$$

Taking limit as $n \to \infty$ and using (3.7), we get

$$
\lim_{n\to\infty} d(f^n x, f^n y) \leq \lim_{n\to\infty} r(f^n x, f^n y) = 0.
$$

□

Assume that $x \in X$ and a is any positive real number. For convenience we define the following:

$$
\mathfrak{B}_a(x) = \{y \in [x]_{\widetilde{G}} : r(x, y) < a, \text{for at least one path between } x \text{ and } y \text{ in } \widetilde{G}\}
$$

and

$$
\overline{\mathfrak{B}_a(x)} = \{y \in [x]_{\widetilde{G}} : r(x, y) \leq a, \text{for at least one path between } x \text{ and } y \text{ in } \widetilde{G}\}.
$$

Proposition 3.1.4. Let (X, d) be a metric space and f be a weakly G-contractive mapping from X into X. Let there exists $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$ then the sequence $\{f^n x_0\}$ is Cauchy.

Proof. Since $fx_0 \in [x_0]_{\tilde{G}}$ then from Lemma 3.1.3 we obtain

$$\lim_{n\to\infty} r(f^{n+1}x_0, f^n x_0) = 0. \tag{3.8}$$

Now, we will show that $\{f^n x_0\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} r(f^{n+1}x_0, f^n x_0) = 0$, for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$r(f^{n_0+1}x_0, f^{n_0}x_0) \leq \inf_j \Big\{\frac{\epsilon}{2}, \psi\big(d(x_j, f^{n_0}x_0)\big)\Big\}, \tag{3.9}$$

where the vertex $x_j \in X$ is adjacent to $f^{n_0}x_0$ with a single edge. Since, $fx_0 \in [x_0]_{\tilde{G}}$ then by induction there exists a path between $f^{n_0}x_0$ and $f^{n_0+1}x_0$ in $\widetilde{G}$. This evokes the existence of at least one vertex $x_j \in X$ adjacent to $f^{n_0}x_0$. We claim that $f\big(\overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}\big) \subset \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$. Let $z \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$. Let $\{y_i\}_{i=0}^{l}$ be a path between z and $f^{n_0}x_0$ such that $y_0 = z$ and $y_l = f^{n_0}x_0$ then $\{fy_i\}_{i=0}^{l}$ is a path between fz and $ff^{n_0}x_0$. So that $fz \in [f^{n_0+1}x_0]_{\widetilde{G}} = [f^{n_0}x_0]_{\widetilde{G}}$.

Then two cases arise:

Case 1. If $0 < r(z, f^{n_0}x_0) \leq \frac{\epsilon}{2}$.

Since $z \in [f^{n_0}x_0]$, using (3.2) & (3.9) along the path $\{fz, \cdots, f^{n_0+1}x_0, \cdots, f^{n_0}x_0\}$, we have

$$\begin{aligned} r(fz, f^{n_0}x_0) &= r(fz, f^{n_0+1}x_0) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &= \sum_{i=1}^{l} d(fy_{i-1}, fy_i) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq \sum_{i=1}^{l} d(y_{i-1}, y_i) - \sum_{i=1}^{l} \psi(d(y_{i-1}, y_i)) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq r(z, f^{n_0}x_0) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Case 2. If $\frac{\epsilon}{2} < r(z, x_{n_0}) \leq \epsilon$.

In this case, again using (3.2) & (3.9) along the path $\{fz, \cdots, f^{n_0+1}x_0, \cdots, f^{n_0}x_0\}$, we have

$$\begin{aligned} r(fz, f^{n_0}x_0) &= r(fz, f^{n_0+1}x_0) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &= \sum_{i=1}^{l} d(fy_{i-1}, fy_i) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq \sum_{i=1}^{l} d(y_{i-1}, y_i) - \sum_{i=1}^{l} \psi(d(y_{i-1}, y_i)) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq r(z, f^{n_0}x_0) - \psi(d(y_{l-1}, y_l)) + r(f^{n_0+1}x_0, f^{n_0}x_0) \\ &\leq r(z, f^{n_0}x_0) - \psi(d(y_{l-1}, f^{n_0}x_0)) + \psi(d(y_{l-1}, f^{n_0}x_0)) \leq \epsilon. \end{aligned}$$

This proves our claim. As $f^{n_0+1}x_0 \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$ then $f(f^{n_0+1}x_0) \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$. Thus, $f^{n_0+k}x_0 \in [f^{n_0}x_0]_{\widetilde{G}}$ for $k = 1, 2, \cdots$. Repeating the same procedure it follows that $f^n x_0 \in \overline{\mathfrak{B}_\epsilon(f^{n_0}x_0)}$, for $n \geq n_0$. It infers that for each $n \geq n_0$

$$d(f^n x_0, f^{n_0}x_0) \leq r(f^n x_0, f^{n_0}x_0) \leq \epsilon.$$

It simply yields $f^n x_0 \in \overline{B}(f^{n_0}x_0; \epsilon)$ for all $n \geq n_0$. Finally, it vindicates that $\{f^n x_0\}$ is a Cauchy sequence in X. □

Theorem 3.1.5. Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a weakly G-contractive mapping. Suppose that the following conditions holds

(i) G satisfies property $(\mathcal{C}_f)$,

(ii) there exists some $x_0 \in X_f := \{x \in X : (x, fx) \in E(G)\}$.

Then $f|_{[x_0]_{\widetilde{G}}}$ has a unique fixed point $\xi \in [x_0]_{\widetilde{G}}$ and $f^n y \to \xi$ for any $y \in [x_0]_{\widetilde{G}}$.

Proof. Let $x_0 \in X_f$ i.e, $(fx_0, x_0) \in E(G)$ then $fx_0 \in [x_0]_{\widetilde{G}}$. Thus Proposition 3.1.4 yields $\{f^n x_0\}$ is Cauchy. Since X is complete there exists $\xi \in X$ such that $f^n x_0 \to \xi$.
Suppose condition (i) holds. Then there exists a subsequence $\{f^{n_k}x_0\}$ of $\{f^n x_0\}$ and $p \in \mathbb{N}$ such that $(\xi, f^{n_k}x_0) \in E(G)$ for all $k \in \mathbb{N}$ and $k \geq p$. Now, using (3.2) we have for all $k \geq n_0$

$$\begin{aligned} d(f\xi, \xi) &\leq d(f\xi, f^{n_k+1}x_0) + d(f^{n_k+1}x_0, \xi) \\ &\leq d(\xi, f^{n_k}x_0) - \psi(d(\xi, f^{n_k}x_0)) + d(f^{n_k+1}x_0, \xi) \\ &\leq d(\xi, f^{n_k}x_0) + d(f^{n_k+1}x_0, \xi). \end{aligned}$$

Letting $k \to \infty$ we get $d(f\xi, \xi) = 0$. Thus ξ is fixed point of f. We observe that $\{x_0, fx_0, \cdots, f^{n_1}x_0, \cdots, f^p x_0, \xi\}$ is path from x_0 to ξ in $\widetilde{G}$. It vindicates $\xi \in [x_0]_{\widetilde{G}}$. Now let $y \in [x_0]_{\widetilde{G}}$ be arbitrary. Then by Lemma 3.1.3 we have

$$\lim_{n\to\infty} d(f^n y, f^n x_0) = 0.$$

Thus $\lim_{n\to\infty} f^n(y) = \xi$.

Suppose f has two fixed points ξ and η. Then it follows from Lemma 3.1.3 that

$$d(\xi, \eta) = d(f^n \xi, f^n \eta).$$

Taking limit as $n \to \infty$ we get $\xi = \eta$. □

Remark 3.1.6. Indeed f is a Picard operator on X if G is a weakly connected graph because $X := [x_0]_{\widetilde{G}}$.

Theorem 3.1.7. Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a weakly G-contractive mapping. Assume that G is weakly connected satisfying property $(\mathcal{H}_f)$. Then f is Picard operator.

Proof. Let G is weakly connected and $x_0 \in X$ then there exists a path between x_0 and fx_0 in $\widetilde{G}$ or equivalently $fx_0 \in [x_0]_{\widetilde{G}}$. By Proposition 3.1.4, $\{f^n x_0\}$ is Cauchy. Since, X is complete then $f^n x_0 \to \xi \in X$. Since, G is weakly connected so that for each n there exists a path of finite length from $f^n x_0$ to ξ. Let $\{z_i^n\}_{i=0}^m$ be a path between $f^n x_0$ to ξ with $z_0^n = f^n x_0$ and $z_m^n = \xi$ then

$$\begin{aligned} d(\xi, f\xi) &\leq d(\xi, f^{n+1}x_0) + d(f^{n+1}x_0, f\xi) \\ &\leq d(\xi, f^{n+1}x_0) + \sum_{i=1}^{m} d(fz_{i-1}^n, fz_i^n) \\ &\leq d(\xi, f^{n+1}x_0) + \sum_{i=1}^{m} d(z_{i-1}^n, z_i^n) - \sum_{i=1}^{m} \psi(d(z_{i-1}^n, z_i^n)) \\ &\leq d(\xi, f^{n+1}x_0) + r(f^n x_0, \xi). \end{aligned}$$

Since G is weakly connected then $f^n x_0 \in [\xi]_{\widetilde{G}}$ for all n so that the right hand side of above inequality converges to 0 as $n \to \infty$. Thus, we conclude $f\xi = \xi$. Let $y \in X := [x_0]$ be arbitrary then by Lemma 3.1.3, $f^n y \to \xi$. Uniqueness of fixed point can be proved similarly as in Theorem 3.1.5. □

Theorem 3.1.8. Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a weakly G-contractive mapping. Suppose that the following conditions holds:

(i) f is orbital G-continuous;

(ii) there exists some $x_0 \in X_f := \{x \in X : (x, fx) \in E(G)\}$.

Then f has a fixed point $\xi \in X$ and $f^n y \to \xi$ for each $y \in [x_0]_{\widetilde{G}}$. Moreover, if G is weakly connected then f is Picard operator.

Proof. Since, $(x_0, fx_0) \in E(G)$ imply $fx_0 \in [x_0]_{\widetilde{G}}$ then from Proposition 3.1.4 $\{f^n x_0\}$ is Cauchy. Since, X is complete there exists $\xi \in X$ such that $f^n x_0 \to \xi$. By orbital G-continuity as $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \geq 1$ we obtain $f f^n x_0 \to f\xi$. Hence, $f\xi = \xi$. Let $y \in [x_0]_{\widetilde{G}}$ be arbitrary then from Lemma 3.1.3, $f^n y \to \xi$. Uniqueness can be easily followed. □

Theorem 3.1.9. Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a weakly G-contractive mapping. Suppose that the following conditions holds

(i) f is orbitally continuous,

(ii) there exists some $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$.

Then f has a fixed point $\xi \in X$ and $f^n y \to \xi$ for any $y \in [x_0]_{\widetilde{G}}$. Moreover, if G is weakly connected then f is Picard operator.

Proof. Since, $fx_0 \in [x_0]_{\widetilde{G}}$ then from Proposition 3.1.4 $\{f^n x_0\}$ is Cauchy so that the Completeness propert of X yields $f^n x_0 \to \xi \in X$. Since, f is orbitally continuous then $f f^n x_0 \to f\xi$. Hence, $f\xi = \xi$. Let $y \in [x_0]_{\widetilde{G}}$ be arbitrary then from Lemma 3.1.3, $f^n y \to \xi$. □

Remark 3.1.10. Let (X, d) be a metric space and $\succeq$ be a partial order in X. Define the graph G_1 by

$$E(G_1) = \{(x, y) \in X \times X : x \succeq y\}.$$

Note that for this graph, condition (3.1) means f is nondecreasing with respect to this order. Furthermore for the graph G_1 property $(\mathcal{C})$ is equivalent to the statement; that any nondecreasing sequences $\{x_n\}$, with $x_n \to x$ has a subsequence $\{x_{n_k}\}$ such that $x \succeq x_{n_k}$ for all $k \in \mathbb{N}$, which is certainly weaker than condition (2.19). Furthermore, the weak connectivity of the graph gives condition (2.22). Therefore, Theorems 2.5.5, 2.5.6 and 2.5.7 are special cases of Theorem 3.1.5 when $G = G_1$ which satisfies property $(\mathcal{C})$.

Proposition 3.1.11. Let (X, d) be a metric space endowed with a graph G and $f : X \to X$ be a weakly G-contractive mapping. Then f is weakly G^{-1}-contractive as well as weakly $\tilde{G}$-contractive.

Proof. Let $(x, y) \in E(G^{-1})$, then $(y, x) \in E(G)$. Since f is weakly G-contractive, $(fy, fx) \in E(G)$. Thus $(fx, fy) \in E(G^{-1})$. Therefore, condition (3.1) is satisfied for the graph G^{-1}. As $(y, x) \in E(G)$ and f is weakly G-contractive, from (3.2) we get $d(fy, fx) \leq d(y, x) - \psi(d(y, x))$ and symmetry of d implies that $d(fx, fy) \leq d(x, y) - \psi(d(x, y))$. Thus condition (3.2) also holds for the graph G^{-1}. Hence f is weakly G^{-1}-contractive mapping. Similar argument shows that f is weakly $\tilde{G}$-contractive mapping. □

Remark 3.1.12. Consider the following graph in the metric space (X, d)

$$E(G_2) = \{(x, y) \in X \times X : x \succeq y \vee y \succeq x\}.$$

For this graph (3.1) holds if f is monotone with respect to the order. Moreover, $G_2 = \tilde{G}_1$ and it follows from above proposition that if f is weakly G_1-contractive it is weakly G_2-contractive. Therefore Theorems 2.5.8 and 2.5.9 are special cases of Theorem 3.1.5 when $G = G_2$ which satisfies property $(\mathcal{C})$.

Lemma 3.1.13. Let (X, d) be a metric space endowed with a graph and $f : X \to X$ be a G-contractive map. Suppose for some $x_0 \in X, fx_0 \in [x_0]_{\tilde{G}}$. Then

(i) $f([x_0]_{\tilde{G}}) \subseteq [x_0]_{\tilde{G}}$,

(ii) $f|_{[x_0]_{\tilde{G}}}$ is a $\tilde{G}_{x_0}$-contractive.

Proof. Let $x \in [x_0]_{\tilde{G}}$. Then there is a path $x = z_0, z_1, \cdots, z_l = x_0$ between x and x_0. Since f is G-contractive, $(fz_{i-1}, fz_i) \in E(G) \forall i = 1, 2, \cdots, l$. Thus $fx \in [fx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$.
Suppose $(x, y) \in E(\tilde{G}_{x_0})$. Then $(fx, fy) \in G$, since f is G-contractive. But $[x_0]_{\tilde{G}}$ is f invariant, so we conclude that $(fx, fy) \in E(\tilde{G}_{x_0})$. Condition (3.2) is satisfied automatically, since $\tilde{G}_{x_0}$ is a subgraph of G. □

3.1.2 Applications and an illustrative example

The following example substantiates the validity of our results over some pre-existing results in literature.

Example 3.1.14. Let $X := [0, \infty)$ equipped with the usual metric d. Let $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = \frac{t^2}{2}$. Define a mapping $f : X \to X$ as

$$fx = \begin{cases} x - \frac{x^2}{2}, & \text{if } x \in [0,1] \setminus \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\} \\ 2x, & \text{otherwise.} \end{cases}$$

Consider the graph G such that $V(G) := X$ and $E(G) := \Omega \cup \big\{(0, x) : x \in [0,1] \setminus \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}\big\}$.

It can be easily seen that G satisfies (3.1) and (3.2). Theorem 3.1.5 yields 0 is a fixed point of f. On the other hand one cannot invoke Theorem 2.2.1 (specifically by taking $x = 0, y = 1$, (2.4) does not hold). Also we note that f is not a Banach G-contraction. Since, $\{(0, \frac{1}{n})\}_{n \geq 3} \subset E(G)$, then

$$d(f0, f\frac{1}{n}) = |\frac{1}{n} - \frac{1}{2n^2}| \leq \kappa \, |\frac{1}{n}| = \kappa \, d(0, \frac{1}{n}). \tag{3.10}$$

Letting $n \to \infty$, (3.10) yields $\kappa \geq 1$.

In [63] authors introduced the notions of cyclic representations and cyclic contractions. From Theorem 3.1.5 one can easily invoke [58, Theorem 6] as follows.

Corollary 3.1.15. ([58, Theorem 6]) Let (X, d) be a complete metric space. Let m be a positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that the following conditions hold:

1.$\cup_{i=1}^m A_i$ is cyclic representation of Y with respect to f and

2. there exists $\psi : [0, \infty) \to [0, \infty)$ where ψ is continuous, nondecreasing, positive on $(0, \infty)$, $\psi(0) = 0$ and the following holds:

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \text{ for } x \in A_i, y \in A_{i+1};\ A_{m+1} = A_1.$$

Then f has a unique fixed point $\xi \in \cap_{i=1}^{m} A_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^{m} A_i$.

Proof. Since, A_i, $i \in \{1, \cdots, m\}$ are closed then (Y, d) is a complete metric space. Consider a graph G consisting of $V(G) := Y$ and $E(G) := \Omega \cup \{(x, y) \in Y \times Y : x \in A_i, y \in A_{i+1}; i = 1, \cdots, m\}$. For such a graph the first condition in view of Definition 2.7.1 infers that f preserves edges. Thus from second condition it follows that f is a weakly G-contractive mapping. Now let $f^n x \to x^* \in Y$ such that $(f^n x, f^{n+1} x) \in E(G)$ for all $n \geq 1$ then in view of Definition (2.7.1), sequence $\{f^n x\}$ has infinitely many terms in each A_i so that one can easily extract a subsequence of $\{f^n x\}$ converging to x^* in each A_i, since A_i's are closed then $x^* \in \cap_{i=1}^{m} A_i$. Now it is easy to form a subsequence $\{f^{n_k} x\}$ in some A_j, $j \in \{1, \cdots, m\}$ such that $(f^{n_k} x, x^*) \in E(G)$ for $k \geq 1$. Thus G is weakly connected and satisfies property $(\mathcal{C})$. Hence, conclusion follows from Theorem 3.1.5. □

Definition 3.1.16. ([36, 37]) A metric space (X, d) is said to be ϵ-chainable, for some $\epsilon > 0$, if for $x, y \in X$ there exist $x_i \in X$; $i = 0, 1, 2, \cdots, l$ with $x_0 = x, x_l = y$ such that $d(x_{i-1}, x_i) < \epsilon$ for $i = 1, 2, \cdots, l$.

Now we state and prove another consequence of Theorem 3.1.9.

Theorem 3.1.17. Let (X, d) be a complete ϵ-chainable metric space. Let $\psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing, positive on $(0, \infty)$ and $\psi(0) = 0$. Assume that $f : X \to X$ satisfies,

$$d(x, y) < \epsilon \quad \Longrightarrow \quad d(fx, fy) \leq d(x, y) - \psi(d(x, y)), \tag{3.11}$$

for all $x, y \in X$. Then f is a Picard operator.

Proof. Let $G := (V(G), E(G))$ such that $V(G) := X$ and $E(G) := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. Since X is ϵ-chainable which shows G is weakly connected. Let $(x, y) \in E(G)$, from (3.11) we have,

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \leq d(x, y) < \epsilon.$$

Thus $(fx, fy) \in E(G)$ so that f is weakly G-contractive. Further, (3.11) implies that f is continuous. The conclusion follows from Theorem 3.1.9. $\square$

Now we define the following.

Definition 3.1.18. Let (X, d) be a metric space and $f : X \to X$. The mapping f is said to be an α-type weakly contractive mapping if

$$\alpha(x, y)d(fx, fy) \leq (x, y) - \psi(d(x, y)), \quad \text{for all } x, y \in X, \tag{3.12}$$

where $\alpha : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ is continuous nondecreasing such that ψ is positive on $(0, \infty)$ and $\psi(0) = 0$.

Theorem 3.1.19. Let (X, d) be a complete metric space. Suppose that $f : X \to X$ be an α-type weakly contractive mapping and satisfies the following conditions:

(i) f is α-admissible, i.e., $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$ for every $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.

Then f has a fixed point.

Proof. Consider a graph $G := \big(V(G), E(G)\big)$ consisting of

$$V(G) := X \quad \text{and} \quad E(G) := \{(x, y) \in X \times X : \alpha(x, y) \geq 1\}.$$

Then for such a graph condition (i) implies that f preserves edges and condition (ii) invokes the existence of some $x_0 \in X$ with $(x_0, fx_0) \in E(G)$. Let $(x, y) \in E(G)$ then $\alpha(x, y) \geq 1$ and inequality (3.12) yields

$$d(fx, fy) \leq \alpha(x, y)d(fx, fy) \leq (x, y) - \psi(d(x, y)). \tag{3.13}$$

Thus f is weakly G-contractive mapping. Moreover it is easy to observe that condition (iii) is equivalent to property $(\mathcal{C})$. Hence all the conditions of Theorem 3.1.7 are satisfied and the conclusion follows. $\square$

3.2 Integral G-contractions

Branciari [23] generalized the Banach contraction principle by proving the existence of unique fixed point of a mapping on a complete metric space satisfying a general contractive condition of integral type. In this section motivated by the work of Jachymski [52] and Branciari [23], we introduce two new contraction conditions for mappings on complete metric spaces. Moreover, using these contractive conditions, we obtain some fixed point theorems for mappings on complete metric spaces. Our results generalize and unify some results by above mentioned authors. In the following we introduce the notion of integral G-contraction which basically unifies the two different concepts of contractions in (Jachymski [52], Branciari [23]).

Definition 3.2.1. A mapping $f : X \to X$ is called an integral G-contraction if for all $x, y \in X$:

$$(fx, fy) \in E(G) \quad \text{whenever} \quad (x, y) \in E(G) \tag{3.14}$$

and

$$(x, y) \in E(G) \quad \text{implies} \quad \int_0^{d(fx,fy)} \phi(s)ds \leq \kappa \int_0^{d(x,y)} \phi(s)ds \tag{3.15}$$

for some $\kappa \in (0, 1)$ and $\phi \in \Phi$. Here Φ denote the class of all mappings $\phi : [0, +\infty) \to [0, +\infty)$ which are Lebesgue integrable, summable on each compact subset of $[0, +\infty)$, nonnegative and for each $\epsilon > 0$, $\int_0^\epsilon \phi(s)ds > 0$.

Remark 3.2.2. Note that if $f : X \to X$ satisfies (2.9) then f is an integral G_1-contraction where $G_1 = (X, X \times X)$. Moreover, every Banach G-contraction is an integral G-contraction (take $\phi(x) = 1$), but the converse may not hold.

3.2.1 Fixed point theorems for integral G-contractions

We start by proving the following trivial proposition.

Proposition 3.2.3. Let $f : X \to X$ be an integral G-contraction with contraction constant $\kappa \in (0, 1)$ and $\phi \in \Phi$, then

(i) f is both an integral G^{-1}-contraction and an integral $\widetilde{G}$-contraction with the same contraction constant $\kappa \in (0, 1)$ and ϕ.

(ii) $[x]_{\widetilde{G}}$ is f-invariant and $f|_{[x_0]_{\widetilde{G}}}$ is an integral $\widetilde{G}_{x_0}$-contraction provided that there exists some $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$.

Proof. (i) It is a direct consequence of symmetry of d.

(ii) Let $x \in [x_0]_{\widetilde{G}}$. Then there is a path $x = z_0, z_1, \cdots, z_m = x_0$ between x and x_0 in $\widetilde{G}$. Since f is an integral G-contraction then $(fz_{i-1}, fz_i) \in E(\widetilde{G}) \forall i = 1, 2, \cdots, l$. Thus $fx \in [fx_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$.
Suppose that $(x, y) \in E(\widetilde{G}_{x_0})$ then $(fx, fy) \in E(G)$ as f is an integral G-contraction. But $[x_0]_{\widetilde{G}}$ is f invariant, so we conclude that $(fx, fy) \in E(\widetilde{G}_{x_0})$. Furthermore, (3.15) is satisfied automatically because $\widetilde{G}_{x_0}$ is a subgraph of G. □

Lemma 3.2.4. Let $f : X \to X$ be an integral G-contraction and $y \in [x]_{\widetilde{G}}$, then

$$\lim_{n\to\infty} d(f^n x, f^n y) = 0. \tag{3.16}$$

Proof. Let $x \in X$ and $y \in [x]_{\widetilde{G}}$ then there exists $N \in \mathbb{N}$ such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for all $i = 1, 2, \cdots, N$. By Proposition 3.2.3 it follows that $(f^n x_{i-1}, f^n x_i) \in E(\widetilde{G})$ and

$$\int_0^{d(f^n x_{i-1}, f^n x_i)} \phi(s)ds \leq \kappa \int_0^{d(f^{n-1} x_{i-1}, f^{n-1} x_i)} \phi(s)ds, \tag{3.17}$$

holds for all $n \in \mathbb{N}$ and $i = 0, 1, 2, \cdots, N$. Denoting, $d_n = d(f^n x_{i-1}, f^n x_i)$ for all $n \in \mathbb{N}$. If $d_m = 0$ for some $m \in \mathbb{N}$ then it follows from (3.17) that $d_n = 0$ for all $n \in \mathbb{N}$ with $n > m$. Therefore, in this case, $\lim_{n\to\infty} d_n = 0$. Now, assume that $d_n > 0$. We claim that $\{d_n\}$ is non-increasing sequence. Otherwise, there exists $n_\circ \in \mathbb{N}$ such that $d_{n_\circ} > d_{n_\circ - 1}$. Now, using properties of ϕ, it follows from (3.17) that

$$0 < \int_0^{d_{n_\circ - 1}} \phi(s)ds \leq \int_0^{d_{n_\circ}} \phi(s)ds \leq \kappa \int_0^{d_{n_\circ - 1}} \phi(s)ds.$$

Since, $0 < c < 1$, this yields a contradiction. Therefore, $\lim_{n\to\infty} d_n = r \geq 0$. Let $r > 0$, then it follows from (3.17) that

$$\int_0^r \phi(s)ds = \lim_{n\to\infty} \int_0^{d_n} \phi(s)ds \leq \kappa \lim_{n\to\infty} \int_0^{d_{n-1}} \phi(s)ds = \kappa \int_0^r \phi(s)ds. \tag{3.18}$$

It implies $(1 - \kappa) \int_0^r \phi(s)ds \leq 0$ and it further implies $\int_0^r \phi(s)ds = 0$ which is a contradiction. Thus, $\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(f^n x_{i-1}, f^n x_i) = 0, \forall i = 1, 2, \cdots, N$, in both cases. From triangular inequality we have $d(f^n x, f^n y) \leq \sum_{i=1}^{N} d(f^n x_{i-1}, f^n x_i)$ and letting $n \to \infty$ gives $\lim_{n\to\infty} d(f^n x, f^n y) = 0$. $\square$

Now we define a subclass of integral G-contractions. We call this a class of sub-integral G-contractions. Let us denote with Ψ the class of all mappings $\phi \in \Phi$ satisfying the following condition:

$$\int_0^{\alpha+\beta} \phi(s)ds \leq \int_0^{\alpha} \phi(s)ds + \int_0^{\beta} \phi(s)ds \tag{3.19}$$

for every $\alpha, \beta \geq 0$. Note that every constant function $\phi(x) = k > 0$ belongs to the class Ψ.

Example 3.2.5. Define $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ by

$$\phi_1(x) = \frac{1}{x+1}, \qquad \phi_2(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that ϕ_1, ϕ_2 satisfy (3.19) and thus belong to the class Ψ.

Definition 3.2.6. We say that an integral G-contraction is a sub-integral G-contraction if $\phi \in \Psi$.

Lemma 3.2.7. Let $f : X \to X$ be a sub-integral G-contraction and $y \in [x]_{\widetilde{G}}$. Then there exists $p(x, y) \geq 0$ such that

$$\int_0^{d(f^n x, f^n y)} \phi(s)ds \leq \kappa^n p(x, y), \quad \forall \quad n \in \mathbb{N}. \tag{3.20}$$

Proof. Let $x \in X$ and $y \in [x]_{\widetilde{G}}$ then there exists $N \in \mathbb{N}$ such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for all $i = 1, 2, \cdots, N$. Since $\phi \in \Psi$ using triangular inequality, it follows that

$$\int_0^{d(f^n x, f^n y)} \phi(s)ds \leq \sum_{i=1}^{N} \int_0^{d(f^n x_{i-1}, f^n x_i)} \phi(s)ds, \tag{3.21}$$

and

$$\int_0^{d(f^n x_{i-1}, f^n x_i)} \phi(s)ds \leq \kappa^n \int_0^{d(x_{i-1}, x_i)} \phi(s)ds. \tag{3.22}$$

Since, $(f^n x_{i-1}, f^n x_i) \in E(G)$ for all $i = 1, 2, 3, ..., N$ and $n \in \mathbb{N}$ so that from (3.21) and (3.22) we get

$$\begin{aligned} \int_0^{d(f^n x, f^n y)} \phi(s)ds &\leq \kappa^n \sum_{i=0}^{N} \int_0^{d(x_{i-1}, x_i)} \phi(s)ds \\ &= \kappa^n p(x, y), \end{aligned} \tag{3.23}$$

where $p(x,y) = \sum_{i=0}^{N} \int_0^{d(x_{i-1},x_i)} \phi(s)ds$. □

Theorem 3.2.8. Let $f : X \to X$ be a sub-integral G-contraction. Assume that:

(i) $X_f := \{x \in X : (x, fx) \in E(G)\} \neq \emptyset$;

(ii) G is a $(\mathcal{C}_f)$-graph.

Then, for any $x_0 \in X_f$, $f|_{[x_0]_{\widetilde{G}}}$ is a Picard operator. Further, if G is weakly connected then f is a Picard operator.

Proof. Let $x_0 \in X_f$ then $fx_0 \in [x_0]_{\widetilde{G}}$. Suppose $m > n \geq 1$ so that using Lemma (3.2.7), we obtain

$$\begin{aligned}
\int_0^{d(f^m x_0, f^n x_0)} \phi(s)ds &\leq \int_0^{d(f^n x_0, f^{n+1} x_0)} \phi(s)ds + \cdots + \int_0^{d(f^{m-1} x_0, f^m x_0)} \phi(s)ds \\
&\leq (\kappa^n + \kappa^{n+1} + \kappa^{n+2} + \ldots + \kappa^{m-1}) p(x_0, fx_0) \\
&\leq [\frac{\kappa^n}{1-\kappa}] p(x_0, fx_0) \ \to 0 \ \ as \ \ n \to \infty.
\end{aligned}$$

It follows that $\{f^n x_0\}$ is a Cauchy sequence in X. Therefore, $f^n x_0 \to \xi \in X$. Let y is another element in $[x_0]_{\widetilde{G}}$ then it follows from Lemma 3.2.4 that $f^n y \to \xi$, too. Next, we show that ξ is a fixed point of f. Since, $f^n x_0 \to \xi \in X$ and $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$ and G is $(\mathcal{C}_f)$-graph then there exists a subsequence $\{f^{n_k} x_0\}$ of $\{f^n x_0\}$ and $p \in \mathbb{N}$ such that $(f^{n_k} x_0, \xi) \in E(G)$ for all $k \geq p$. Therefore, $(x_0, fx_0, f^2 x_0, ..., f^{n_p} x_0, \xi)$ is a path in G and so in $\widetilde{G}$ from x_0 to ξ, thus $\xi \in [x_0]_{\widetilde{G}}$. From (3.15), we get

$$\int_0^{d(f^{n_k+1} x_0, f\xi)} \phi(s)ds \leq \kappa \int_0^{d(f^{n_k} x_0, \xi)} \phi(s)ds \quad \forall \ k \in \mathbb{N},$$

letting $k \to \infty$, we have $\int_0^{d(\xi, f\xi)} \phi(s)ds = 0$, which implies that $d(\xi, f\xi) = 0$. This shows that $f|_{[x_0]_{\widetilde{G}}}$ is a Picard operator. Moreover, if G is weakly connected then f is a Picard operator, since $[x_0]_{\widetilde{G}} = X$. □

Remark 3.2.9. Theorem 3.2.8 generalizes claims 4^0 & 5^0 of [52, Theorem 3.2].

Corollary 3.2.10. Let (X, d) be a complete metric space endowed with a graph G such that G is a $(\mathcal{C}_f)$-graph. Then the following statements are equivalent:

(1) G is weakly connected.

(2) Every sub-integral G-contraction f on X is a Picard operator provided that $X_f \neq \emptyset$.

Proof. (1) $\implies$ (2) : It is immediate from Theorem 3.2.8.
(2) $\implies$ (1) : On the contrary suppose that G is not weakly connected then $\widetilde{G}$ is disconnected, i.e., there exists $x_\circ \in X$ such that $[x_\circ]_{\widetilde{G}} \neq \emptyset$ and $X \backslash [x_\circ]_{\widetilde{G}} \neq \emptyset$. Let $y_\circ \in X \backslash [x_\circ]_{\widetilde{G}}$, we construct a self-mapping f as:

$$fx = \begin{cases} x_\circ & \text{if} \quad x \in [x_\circ]_{\widetilde{G}} \\ y_\circ & \text{if} \quad x \in X \setminus [x_\circ]_{\widetilde{G}}. \end{cases}$$

Let $(x, y) \in E(G)$ then $[x]_{\widetilde{G}} := [y]_{\widetilde{G}}$ which implies $fx = fy$. Hence $(fx, fy) \in E(G)$ as G contains all loops and further (3.15) is trivially satisfied. But $x_\circ$ and $y_\circ$ are two fixed points of f contradicting the fact that f has a unique fixed point. □

Theorem 3.2.11. Let $f : X \to X$ be a sub-integral G-contraction. Assume that f is orbitally G-continuous and $X_f := \{x \in X : (x, fx) \in E(G)\} \neq \emptyset$. Then for any $x_0 \in X_f$ and $y \in [x_0]_{\widetilde{G}}$, $\lim_{n\to\infty} f^n y = \xi \in X$ where ξ is a fixed point of f. Further, if G is weakly connected then f is a Picard operator.

Proof. Let $x_0 \in X_f$ then the arguments used in the proof of Theorem 3.2.8 imply that $\{f^n x_0\}$ is a Cauchy sequence. Therefore, $f^n x_0 \to \xi \in X$. Since $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$ and f is orbitally G-continuous therefore $\xi = \lim_{n\to\infty} f f^n x_0 = ft$. Note that if y is another element from $[x_0]_{\widetilde{G}}$ then it follows from Lemma 3.2.4 that $\lim_{n\to\infty} f^n y = \xi$. Finally, if G is weakly connected then $[x_0]_{\widetilde{G}} := X$ which yields that f is a Picard operator. □

Remark 3.2.12. Theorem 3.2.11 generalizes claims 2^0 & 3^0 of [52, Theorem 3.3].

Theorem 3.2.13. Let $f : X \to X$ be a sub-integral G-contraction. Assume that f is orbitally continuous and if there exists some $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$ then, for $y \in [x_0]_{\widetilde{G}}$, $\lim_{n\to\infty} f^n y = \xi \in X$ where ξ is a fixed point of f. Further, if G is weakly connected then f is a Picard operator.

Proof. Let $x_0 \in X$ be such that $fx_0 \in [x_0]_{\widetilde{G}}$ then using the same arguments as in the proof of Theorem 3.2.8, $\{f^n x_0\}$ is Cauchy and thus $\lim_{n\to\infty} f^n x_0 = \xi \in X$. Moreover,

$\xi = \lim_{n\to\infty} ff^n x_0 = f\xi$, as f is orbitally continuous. Note that if y is another element from $[x_0]_{\widetilde{G}}$ then it follows from Lemma 3.2.4 that $\lim_{n\to\infty} f^n y = \xi$. If G is weakly connected then $[x_0]_{\widetilde{G}} := X$ this yields that f is a Picard operator. $\square$

Remark 3.2.14. Theorem 3.2.13 generalizes claims 2^0 & 3^0 of [52, Theorem 3.4] and thus generalizes and extends the results of Nieto and Rodrt́ýguez-Lt́opez [72, Theorems 2.1 and 2.3], Petrusel and Rus [84, Theorem 4.3] and Ran and Reurings [90, Theorem 2.1].

Corollary 3.2.15. Let (X,d) be a complete metric space endowed with a graph G. Then the following statements are equivalent:

(1) G is weakly connected.

(2) Every sub-integral G-contraction f on X is a Picard operator provided f is orbitally continuous.

Proof. (1) $\Longrightarrow$ (2): It is obvious from Theorem 3.2.13.
(2) $\Longrightarrow$ (1): Note that the example constructed in Corollary 3.2.10 is orbitally continuous. $\square$

Remark 3.2.16. Corollary 3.2.15 generalizes claims 2^0 & 3^0 of [52, Corollary 3.3].

Theorem 3.2.17. Let $f : X \to X$ be an integral G-contraction. Assume that the following assertions hold:

(i) there exists $x_0 \in X_f$ such that

$$\int_0^{d(fx,fy)} \phi(s)ds \leq \kappa \int_0^{d(x,y)} \phi(s)ds \;\forall x,y \in \mathcal{O}(x_0) \subset [x_0]_{\widetilde{G}} \tag{3.24}$$

where, $\mathcal{O}(x_0) = \{x_0, fx_0, f^2x_0, \cdots\}$;

(ii) G is a $(\mathcal{C}_f)$-graph.

Then, for any $x_0 \in X_f$, $f\,|_{[x_0]_{\widetilde{G}}}$ is a Picard operator. Furthermore, if G is weakly connected then f is a Picard operator.

Proof. Let $x_0 \in X_f$, then $fx_0 \in [x_0]_{\widetilde{G}}$. Now, it follows from Proposition 3.2.3(ii) that $\mathcal{O}(x_0) \subset [x_0]_{\widetilde{G}}$. Moreover, from Lemma 3.2.4 we get

$$\lim_{n\to\infty} d(f^n x_0, f^{n+1}x_0) = 0. \tag{3.25}$$

We claim that $\{f^n x_0\}$ is Cauchy sequence. Otherwise, there exists some $\epsilon > 0$ in such a way that for each $k \in \mathbb{N}$ there are $m_k, n_k \in \mathbb{N}$ with $n_k > m_k > k$, satisfying

$$d(f^{m_k}x_0, f^{n_k}x_0) \geq \epsilon. \tag{3.26}$$

We may choose sequences $\{m_k\}, \{n_k\}$ such that corresponding to m_k the natural number n_k is the smallest satisfying (3.26). Therefore,

$$\epsilon \leq d(f^{n_k}x_0, f^{m_k}x_0) \leq d(f^{n_k}x_0, f^{n_k-1}x_0) + d(f^{n_k-1}x_0, f^{m_k}x_0) < d(f^{n_k}x_0, f^{n_k-1}x_0) + \epsilon.$$

On letting $k \to \infty$ and using (3.25) we get

$$\lim_{k\to\infty} d(f^{n_k}x_0, f^{m_k}x_0) = \epsilon. \tag{3.27}$$

Moreover, using (3.25) and(3.27), it follows from

$$d(f^{n_k-1}x_0, f^{m_k-1}x_0) \leq d(f^{n_k-1}x_0, f^{n_k}x_0) + d(f^{n_k}x_0, f^{m_k}x_0) + d(f^{m_k}x_0, f^{m_k-1}x_0),$$

and

$$d(f^{n_k}x_0, f^{m_k}x_0) \leq d(f^{n_k}x_0, f^{n_k-1}x_0) + d(f^{n_k-1}x_0, f^{m_k-1}x_0) + d(f^{m_k-1}x_0, f^{m_k}x_0),$$

that

$$\lim_{n\to\infty} d(f^{n_k-1}x_0, f^{m_k-1}x_0) = \epsilon. \tag{3.28}$$

Since $f^{n_k-1}x_0, f^{m_k-1}x_0 \in \mathcal{O}(x_0)$ it follows from assertion (ii) that

$$\int_0^{d(f^{n_k}x_0, f^{m_k}x_0)} \phi(s)ds \leq \kappa \int_0^{d(f^{n_k-1}x_0, f^{m_k-1}x_0)} \phi(s)ds. \tag{3.29}$$

Letting $k \to \infty$ and using (3.27) and (3.28) we get $(1-\kappa)\int_0^\epsilon \phi(s)ds \leq 0$. As $0 < \kappa < 1$ this implies that $\epsilon = 0$. Therefore, $\{f^n x_0\}$ is Cauchy sequence in X. The rest of the proof runs on same lines as the proof of Theorem 3.2.8. □

Remark 3.2.18. Theorem 3.2.17 generalizes [23, Theorem 2.1].

Remark 3.2.19. The conclusion of Theorem 3.2.17 that f is a Picard operator provided that G is weakly connected remains valid if we replace assertion (ii) by $(ii)'$ f is orbital G-continuous or $(ii)''$ f is orbitally continuous.

3.2.2 Applications and illustrative example

Following example elucidates the degree of generality of Theorem 3.2.17 over main results of Branciari [23] and Jachymski [52].

Example 3.2.20. Let $X := [0,1]$ be equipped with the usual metric d. Define $f : X \to X$, $\phi : [0,+\infty) \to [0,+\infty)$ by

$$fx = \begin{cases} \frac{x}{1+px} & \text{if } x = \frac{1}{n}, \\ 0 & \text{if } x \neq \frac{1}{n}, \end{cases} \qquad \text{and} \quad \phi(s) = \begin{cases} s^{\frac{1}{s}-2}(1-\log s) & \text{if } s > 0, \\ 0 & \text{if } s = 0, \end{cases}$$

for all $n \in \mathbb{N}$ and $p \geq 1$ is any fixed positive integer. Consider the graph G such that $V(G) := X$ and $E(G) := \Omega \cup \{(0,x) : x \in X\} \cup \{(\frac{1}{n+1}, \frac{1}{n}) : n \in \mathbb{N}\}$. We observe that (3.14) holds. Moreover, $\int_0^\tau \phi(s)ds = \tau^{\frac{1}{\tau}}$, so that (3.15) is equivalent to

$$d(fx,fy)^{\frac{1}{d(fx,fy)}} \leq \kappa\, d(x,y)^{\frac{1}{d(x,y)}} \quad \text{for } (x,y) \in E(G). \tag{3.30}$$

Next we show that (3.30) is satisfied for $\kappa = \frac{1}{1+p} < 1$.
Case i. Let $(x,x) \in E(G)$ then (3.15) is trivially satisfied.
Case ii. Let $(0,x) \in E(G); x \neq \frac{1}{n}$ for $n \in \mathbb{N}$ the condition (3.15) is trivially satisfied. Let $(0,x) \in E(G); x = \frac{1}{n}$ for $n \in \mathbb{N}$ then

$$d(fx,fy)^{\frac{1}{d(fx,fy)}} = \left|\frac{1}{n+p} - 0\right|^{\frac{1}{\left|\frac{1}{n+p}-0\right|}} = \frac{1}{(n+p)^{(n+p)}}, \tag{3.31}$$

and

$$d(x,y)^{\frac{1}{d(x,y)}} = \left|\frac{1}{n} - 0\right|^{\frac{1}{\left|\frac{1}{n}-0\right|}} = \frac{1}{(n)^n}. \tag{3.32}$$

From inequality (3.30) we need to show that

$$\frac{1}{(n+p)^{(n+p)}} \leq \frac{1}{(1+p)(n)^n} \tag{3.33}$$

or equivalently,

$$\Big[\frac{n}{n+p}\Big]^{n}\frac{1}{(n+p)^{p}} \leq \frac{1}{1+p},$$

Since, $\frac{1}{(n+p)^p} \leq \frac{1}{(n+p)} < \frac{1}{(1+p)}$ for all $n \in \mathbb{N}$ and $\frac{n}{n+p} < 1$. Thus inequality (3.33) is satisfied.

Case iii. Let $(\frac{1}{n+1}, \frac{1}{n}) \in E(G)$ for $n \in \mathbb{N}$ then we have

$$d(fx, fy)^{\frac{1}{d(fx,fy)}} = \Big|\frac{1}{n+1+p} - \frac{1}{n+p}\Big|^{\frac{1}{\left|\frac{1}{n+1+p}-\frac{1}{n+p}\right|}} = \Big[\frac{1}{(n+1+p)(n+p)}\Big]^{(n+1+p)(n+p)} \tag{3.34}$$

and

$$d(x, y)^{\frac{1}{d(x,y)}} = \Big|\frac{1}{n} - \frac{1}{n+1}\Big|^{\frac{1}{\left|\frac{1}{n}-\frac{1}{n+1}\right|}} = \Big[\frac{1}{n(n+1)}\Big]^{n(n+1)}. \tag{3.35}$$

We need to show that

$$\Big[\frac{1}{(n+1+p)(n+p)}\Big]^{(n+1+p)(n+p)} \leq \frac{1}{(1+p)}\Big[\frac{1}{n(n+1)}\Big]^{n(n+1)}, \tag{3.36}$$

on rearranging we have

$$\frac{[n(n+1)]^{n(n+1)}}{[(n+1+p)(n+p)]^{(n+1+p)(n+p)}} \leq \frac{1}{(p+1)},$$

or,

$$\Big[\frac{n}{n+p}\Big]^{n(n+1)}\Big[\frac{n+1}{n+1+p}\Big]^{n(n+1)}\frac{1}{[(n+1+p)(n+p)]^{(p^2+2np+p)}} \leq \frac{1}{1+p}.$$

By analyzing L.H.S, we see that $\frac{n}{n+p} < 1$ and $\frac{n+1}{n+1+p} < 1$ for all $n \in \mathbb{N}$ and $\frac{1}{[(n+1+p)(n+p)]^{(p^2+2np+p)}} < \frac{1}{[(n+1+p)(n+p)]} < \frac{1}{(n+p)} \leq \frac{1}{(1+p)}$ for all $n \in \mathbb{N}$. Which infers that inequality (3.36) is indeed true. Therefore, f is an integral G-contraction with contraction constant $\kappa = \frac{1}{1+p}$. Note that G is weakly connected $(\mathcal{C})$-graph and (3.24) also holds for $\mathcal{O}(0)$. Thus all the conditions of Theorem 3.2.17 are satisfied and f is a Picard operator with fixed point 0. Note that f is not a Banach G-contraction, since, for $(\frac{1}{n+1}, \frac{1}{n}) \in E(G)$,

$$\frac{d(f\frac{1}{n+1}, f\frac{1}{n})}{d(\frac{1}{n+1}, \frac{1}{n})} = \frac{\left|\frac{1}{n+1+p} - \frac{1}{n+p}\right|}{\left|\frac{1}{n} - \frac{1}{n+1}\right|} \to 1, \text{ as } n \to \infty.$$

By setting $p = 2$ in above example we have

$$d(f\frac{1}{2}, f\frac{3}{4})^{\frac{1}{d(f\frac{1}{2},f\frac{3}{4})}} = (\frac{1}{4})^4 = d(\frac{1}{2}, \frac{3}{4})^{\frac{1}{d(\frac{1}{2},\frac{3}{4})}}.$$

Therefore one can not apply Theorem 2.2.6 [23].

Following result is an important consequence of Theorem 3.2.8.

Theorem 3.2.21. Let (X, d) be a complete metric space. Let m be a positive integer, $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume:

(i) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;

(ii) there exists $\phi \in \Psi$ such that $\int_0^{d(fx,fy)} \phi(s)ds \leq \kappa \int_0^{(d(x,y))} \phi(s)ds$ whenever, $x \in A_i, y \in A_{i+1}$, where $A_{m+1} = A_1$.

Then f has a unique fixed point $\xi \in \cap_{i=1}^m A_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^m A_i$.

Proof. Since, A_i, $i \in \{1, \cdots, m\}$ are closed then (Y, d) is complete metric space. Consider a graph G consisting of $V(G) := Y$ and $E(G) := \Omega \cup \{(x, y) \in Y \times Y : x \in A_i, y \in A_{i+1}; i = 1, \cdots, m\}$. By (i) & (ii) it follows that f is sub-integral G-contraction. Now let $f^n x \to x^* \in Y$ such that $(f^n x, f^{n+1}x) \in E(G)$ for all $n \geq 1$ then in view of (2.29) the sequence $\{f^n x\}$ has infinitely many terms in each A_i so that one can easily extract a subsequence of $\{f^n x\}$ converging to x^* in each A_i, since A_i's are closed then $x^* \in \cap_{i=1}^m A_i$. Now it is easy to form a subsequence $\{f^{n_k} x\}$ in some A_j, $j \in \{1, \cdots, m\}$ such that $(f^{n_k} x, x^*) \in E(G)$ for $k \geq 1$, it vindicates G is weakly connected $(\mathcal{C}_f)$-graph. Hence, conclusion follows from Theorem 3.2.8. □

Remark 3.2.22. Taking $\phi(s) = 1$, Theorem 3.2.21 subsumes the main result of [63].

Definition 3.2.23. Let (X, d) be a metric space and $f : X \to X$. We call the mapping f an α-type sub-integral contraction if

$$\alpha(x, y) \int_0^{d(fx,fy)} \phi(s)ds \leq \kappa \int_0^{d(x,y)} \phi(s)ds, \quad \text{for all } x, y \in X, \tag{3.37}$$

where $\alpha : X \times X \to [0, \infty)$, $\kappa \in (0, 1)$ and $\phi \in \Psi$.

Theorem 3.2.24. Let (X, d) be a complete metric space. Suppose that $f : X \to X$ be an α-type sub-integral contraction and satisfies the following conditions:

(i) f is α-admissible, i.e., $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$ for every $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.

Then f has a fixed point.

Proof. Consider a graph $G := \big(V(G), E(G)\big)$ consisting of

$$V(G) := X \quad \text{and} \quad E(G) := \{(x, y) \in X \times X : \alpha(x, y) \geq 1\}.$$

Then for such a graph condition (i) implies that f preserves edges i.e., $(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$. Let $(x, y) \in E(G)$ then $\alpha(x, y) \geq 1$ and inequality (3.37) yields

$$\int_0^{d(fx,fy)} \phi(s)ds \leq \alpha(x, y) \int_0^{d(fx,fy)} \phi(s)ds \leq \kappa \int_0^{d(x,y)} \phi(s)ds. \tag{3.38}$$

Thus f is sub-integral G-contraction. The condition (ii) invokes the existence of some $x_0 \in X$ with $(x_0, fx_0) \in E(G)$ so that $X_f \neq \emptyset$. Furthermore it is easy to observe that condition (iii) yields that G is a $(\mathcal{C})$-graph. Hence all the conditions of Theorem 3.2.8 are satisfied and the conclusion follows. □

3.3 Conclusion

1. In Section 3.1 we introduced the notion of weakly G-contractive maps and established some fixed point theorems for such mappings. The notion of weakly G-contractive maps generalizes and unifies the notion of Banach G-contractions and weakly contractive maps. Therefore Theorems 3.1.5, 3.1.7 subsume not only the class of weakly contractive maps but also class of all Banach G-contractions. This elicits the novelty of the results. Moreover, Example 3.1.14 illuminates the degree of generality of our result over some pre-existing results.

2. The purpose of Section 3.2 is stipulated with the concept of integral G-contractions. The notion of an integral G-contraction not only generalizes/extends the notion of a Banach G-contraction but it also improves the integral inequality (2.9). Whereas, the notion of a sub-integral G-contraction generalizes the notion of a Banach G-contraction and partially generalizes the integral inequality (2.9). Therefore, Theorem 3.2.8 generalizes/extends some results of Jachymski [52] and provides partial improvement to

the main result of Branciari [23]. At this point, a very natural question is bound to be posed: Are the conclusions of Theorem 3.2.8, 3.2.11, 3.2.13 still valid for integral G-contractions? We have provided a partial answer to this question in Theorem 3.2.17 on the expense of inequality (3.24). But it remains open to investigate an affirmative answer without the crucial condition of (3.24). Moreover, Example 3.2.20 invokes and elucidates the generality of Theorem 3.2.17.

Chapter 4

Fixed point theorems in b-metric spaces

The motivation behind the present chapter is stipulated with the notion of b-metric space which propelled us to undertake some investigations by flavoring the underlying ambient structure with a graph G [103]. The idea intrigued us to present two different classes of contraction mappings which are discussed in Section 4.2. Consequently, we apply our results to obtain fixed point theorems for cyclic contractions. Moreover, we also obtain b-metric version of Theorem 2.4.3 due to Samet et al.[100].

In Section 4.3 two convergence theorems are presented for the class of φ-contractions in b-metric space where φ is a gauge function of order $r \geq 1$. Furthermore, the error estimates for the convergence of proposed iterative process are also calculated. To illuminate the novelty of obtain result an example is furnished. As an application we obtain an existence theorem for the solution of first order differential equation wherein the iterative scheme converges to the solution with the higher order as compared to the Picard method where the convergence to the solution is linear.

4.1 Preliminaries

Subsequently, throughout this chapter let (X, d) be a b-metric space (unless specified otherwise) with a coefficient $s \geq 1$. We recall some auxiliary notions and results in a b-metric space [8, 22, 31, 49] which are needed subsequently.

Definition 4.1.1. Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is:

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$;

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Definition 4.1.2. A b-metric space (X, d) is complete if every Cauchy sequence in X converges.

Remark 4.1.3. Let (X, d) be a b-metric space then:

(i) every convergent sequence has a unique limit;

(ii) every convergent sequence is Cauchy;

(iii) in general the b-metric d is not a continuous functional [32].

Definition 4.1.4. Let (X, d) be a b-metric space and A be a nonempty subset of X then closure $\overline{A}$ of A is the set consisting of all points of A and its limit points. Moreover, A is closed if and only if $A = \overline{A}$.

In the following the b-metric version of Cantor's intersection theorem is given which can be easily established running on the same lines as in the proof of its metric version.

Theorem 4.1.5. [18] Let (X, d) be a complete b-metric space then every nested sequence of closed balls has a non-empty intersection.

Definition 4.1.6. [48] Let $f : D \subset X \to X$ and there exist some $x \in D$ such that the set $\mathcal{O}(x) = \{x, fx, f^2x, \cdots\} \subset D$. The set $\mathcal{O}(x)$ is known as an orbit of $x \in D$. A function G from D into the set of real numbers is said to be f-orbitally lower semi-continuous at $t \in X$ if $\{x_n\} \subset \mathcal{O}(x)$ and $x_n \to t$ implies $G(t) \leq \liminf G(x_n)$.

Matkowski [66] introduced the class of φ-contractions in metric fixed point theory to generalize Banach contraction principle and subsequently further study was developed in this setting by different authors when underlying space was taken to be a partially ordered set (see, e.g., [2, 77]). For details on φ contractions we refer the readers to [11, 97].

Berinde [12] took a further step to investigate φ contractions when the framework was taken to be a b-metric space and for some technical reasons he had to introduce the notion of b-comparison function in particular he obtained some estimations for rate of convergence [12]. See also [12, 20, 78, 83].

Definition 4.1.7. (Berinde, [12]) Let $s \geq 1$ be a fixed real number. A non-decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is known as a b-comparison function if the following holds;
$(vi)_\varphi$ $\sum_{n=0}^{\infty} s^n \varphi^n(t)$ converges for all $t \in \mathbb{R}^+$.

The concept of b-comparison function coincides with the comparison function when $s = 1$. Let (X, d) be a b-metric space with coefficient $s \geq 1$, then $\varphi(t) = at$; $t \in \mathbb{R}^+$ with $0 < a < \frac{1}{s}$ is a b-comparison function.

4.2 b-metric endowed with a graph G

Throughout this section let (X, d) be a b-metric space with coefficient $s \geq 1$ and Ω is the diagonal of the cartesian product $X \times X$. G is a directed graph such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Omega$. Assume that G has no parallel edges. We assign to each edge having vertices x and y a unique element $d(x, y)$.

4.2.1 Fixed point theorems for mappings in b-metric endowed a graph

We introduce the following definition.

Definition 4.2.1. We say that a mapping $f : X \to X$ is a b-(φ, G) contraction if for all $x, y \in X$:

$$(fx, fy) \in E(G) \quad \text{whenever} \quad (x, y) \in E(G); \tag{4.1}$$

$$d(fx, fy) \leq \varphi(d(x, y)) \quad \text{whenever} \quad (x, y) \in E(G), \tag{4.2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function.

Remark 4.2.2. Note that a Banach G-contraction is a b-(φ, G) contraction.

Example 4.2.3. Any constant mapping $f : X \to X$ is a b-(φ, G) contraction for any graph G with $V(G) = X$.

Example 4.2.4. Any self mapping f on X is trivially a b-(φ, G_1) contraction, where $G_1 = (V(G), E(G)) = (X, \Omega)$.

Example 4.2.5. Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = |x - y|^2$. Then d is a b-metric on X with $s = 2$. Assume the self mapping $fx = \frac{x}{2}$, for all $x \in X$. Then f is a b-(φ, G_0) contraction with $\varphi(t) = \frac{t}{4}$ and $G_0 = (X, X \times X)$. Note that d is not a metric on X.

Definition 4.2.6. Two sequences $\{x_n\}$ and $\{y_n\}$ in X are said to be equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$ and if each of them is a Cauchy sequence then they are called Cauchy equivalent.

As a direct consequence of Definition 4.2.6, we get the following remark.

Remark 4.2.7. Let $\{x_n\}$ and $\{y_n\}$ be equivalent sequences in X. (i) If $\{x_n\}$ converges to x then $\{y_n\}$ also converges to x and vice versa. (ii) $\{y_n\}$ is a Cauchy sequence whenever $\{x_n\}$ is a Cauchy sequence and vice versa.

Proposition 4.2.8. Let $f : X \to X$ be a b-(φ, G) contraction where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function then

(i). f is a b-$(\varphi, \widetilde{G})$ contraction and a b-(φ, G^{-1}) contraction as well.

(ii). $[x_0]_{\widetilde{G}}$ is f-invariant and $f|_{[x_0]_{\widetilde{G}}}$ is a b-$(\varphi, \widetilde{G}_x)$ contraction provided that $x_0 \in X$ is such that $fx_0 \in [x_0]_{\widetilde{G}}$.

Proof. (i). It follows from (d2) (Definition 1.4.1).
(ii). Let $x \in [x_0]_{\widetilde{G}}$. Then there is a path $x = x_0, z_1, \cdots, z_l = x_0$ between x and x_0. Since f is a b-(φ, G) contraction then $(fz_{i-1}, fz_i) \in E(G) \forall i = 1, 2, \cdots, l$. Thus $fx \in [fx_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$. Suppose $(x, y) \in E(\tilde{G}_{x_0})$. Then $(fx, fy) \in E(G)$ as f is a b-(φ, G) contraction. But $[x_0]_{\widetilde{G}}$ is f invariant, so we conclude that $(fx, fy) \in E(\tilde{G}_{x_0})$. the condition (4.2) is satisfied automatically as $\tilde{G}_{x_0}$ is a subgraph of G.

□

From now on we assume that coefficient of b-comparison function φ is at least as large as the coefficient of b-metric s.

Lemma 4.2.9. Let $f : X \to X$ be a b-(φ, G) contraction where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a b-comparison function. Then given any $x \in X$ and $y \in [x]_{\widetilde{G}}$, two sequences $\{f^n x\}$ and $\{f^n y\}$ are equivalent.

Proof. Let $x \in X$ and $y \in [x]_{\widetilde{G}}$ then there exists a path $\{x_i\}_{i=0}^{l}$ in $\widetilde{G}$ from x to y with $x_0 = x$, $x_l = y$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$. From Proposition 4.2.8 as f is a b-$(\varphi, \widetilde{G})$ contraction. So that

$$(f^n x_{i-1}, f^n x_i) \in E(\widetilde{G}) \text{ implies } d(f^n x_{i-1}, f^n x_i) \leq \varphi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)) \tag{4.3}$$

for all $n \in \mathbb{N}$ and $i = 0, 1, 2, \cdots, l$. Hence,

$$d(f^n x_{i-1}, f^n x_i) \leq \varphi^n(d(x_{i-1}, x_i)) \;\; \forall n \in \mathbb{N} \text{ and } i = 0, 1, 2, \cdots, l. \tag{4.4}$$

We observe that $\{f^n x_i\}_{i=0}^{l}$ is a path in $\widetilde{G}$ from $f^n x$ to $f^n y$. From (d3) Definition 1.4.1 and (4.4) we have,

$$d(f^n x, f^n y) \leq \sum_{i=1}^{l} s^i d(f^n x_{i-1}, f^n x_i) \leq \sum_{i=1}^{l} s^i \varphi^n(d(x_{i-1}, x_i)). \tag{4.5}$$

Letting $n \to \infty$ we obtain $d(f^n x, f^n y) \to 0$. □

Proposition 4.2.10. Let f be a b-(φ, G) contraction where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a b-comparison function. Suppose that there is x_0 in X such that $fx_0 \in [x_0]_{\tilde{G}}$. Then $\{f^n x_0\}$ is a Cauchy sequence in X.

Proof. Since $fx_0 \in [x_0]_{\tilde{G}}$. Let $\{y_i\}_{i=0}^{r}$ be a path from x_0 to fx_0 then using the same arguments as in Lemma 4.2.9, we arrive at

$$d(f^n x_0, f^{n+1} x_0) \leq \sum_{i=1}^{r} s^i \varphi^n(d(y_{i-1}, y_i)), \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

Let $m > n \geq 1$, then from above inequality it follows for $p \geq 1$

$$\begin{aligned} d(f^n x_0, f^{n+p} x_0) &\leq sd(f^n x_0, f^{n+1} x_0) + s^2 d(f^{n+1} x_0, f^{n+2} x_0) + \cdots + s^p d(f^{n+p-1} x_0, f^{n+p} x_0) \\ &\leq \frac{1}{s^{n-1}} \big[\sum_{j=n}^{n+p-1} s^j d(f^j x_0, f^{j+1} x_0) \big] \\ &\leq \frac{1}{s^{n-1}} \big[\sum_{i=1}^{r} s^i \sum_{j=n}^{n+p-1} s^j \varphi^j(d(y_{i-1}, y_i)) \big]. \end{aligned} \tag{4.7}$$

Denoting for each $i = 1, 2, \cdots, r$

$$S_n^i = \sum_{k=0}^{n} s^k \varphi^k(d(y_{i-1}, y_i)), \quad n \geq 1$$

relation (4.7) becomes

$$d(f^n x_0, f^{n+p} x_0) \leq \frac{1}{s^{n-1}} \Big[\sum_{i=1}^{r} s^i [S_{n+p-1}^i - S_{n-1}^i] \Big], \tag{4.8}$$

Since, φ is a b-comparison function, so that for each $i = 1, 2, \cdots, r$ we obtain

$$\sum_{k=0}^{\infty} s^k \varphi^k(d(y_{i-1}, y_i)) < \infty.$$

Then corresponding to each i there is a real number S^i such that

$$\lim_{n \to \infty} S_n^i = S^i. \tag{4.9}$$

In view of (4.9) relation (4.8) gives $d(f^n x_0, f^{n+p} x_0) \to 0$ as $n \to \infty$. It infers that $\{f^n x_0\}$ is a Cauchy sequence in X. □

In the following the notions of $(\mathcal{C}_f)$ and $(\mathcal{H}_f)$ graphs are refurnished in the settings of b-metric space.

Definition 4.2.11. Let $f : X \to X$, $y \in X$ and the sequence $\{f^n y\}$ in X is such that $f^n y \to x^*$ with $(f^n y, f^{n+1} y) \in E(G)$ for $n \in \mathbb{N}$.

1. We say that a graph G is a $(\mathcal{C}_f)$-graph if there exists a subsequence $\{f^{n_k} y\}$ and a natural number p such that $(f^{n_k} y, x^*) \in E(G)$ for all $k \geq p$ [101].

2. We say that a graph G is an $(\mathcal{H}_f)$-graph if $f^n y \in [x^*]_{\widetilde{G}}$ for $n \geq 1$ then $r(f^n y, x^*) \to 0$ (as $n \to \infty$). Where $r(f^n y, x^*) = \sum_{i=1}^{M} s^i d(z_{i-1}, z_i)$; $\{z_i\}_{i=0}^{M}$ is a path from $f^n y$ to x^* in $\widetilde{G}$.

Obviously every $(\mathcal{C})$-graph is a $(\mathcal{C}_f)$-graph for any self mapping f on X but the converse may not hold as shown in Chapter 2 (see, Example 3.0.9).

Example 4.2.12. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}$ with respect to the b-metric $d(x, y) = |x-y|^2$ and I is an identity map on X. Consider a graph G_2 consisting of $V(G_2) = X$ and

$E(G_2) = \{(\frac{1}{n}, \frac{1}{n+1}), (\frac{1}{n+1}, n), (n, 0), (\frac{1}{5n}, 0); n \in \mathbb{N}\}$.

Since, $x_n = \frac{1}{n} \to 0$ as $n \to \infty$. We note that G_2 is a $(\mathcal{C}_I)$-graph but $r(x_n, 0) = 2|\frac{1}{n} - \frac{1}{n+1}|^2 + 2^2|\frac{1}{n+1} - n|^2 + 2^2 n^2 \nrightarrow 0$ as $n \to \infty$. Thus G_2 is not an $(\mathcal{H}_I)$-graph.

Example 4.2.13. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{\frac{\sqrt{5}}{n} : n \in \mathbb{N}\} \cup \{0\}$ with respect to b-metric $d(x, y) = |x - y|^2$ and I is identity map on X. Consider a graph G_3 consisting of $V(G_3) = X$ and

$E(G_3) = \{(\frac{1}{n}, \frac{1}{n+1}), (\frac{1}{n+1}, \frac{\sqrt{5}}{n}), (\frac{\sqrt{5}}{n}, 0); n \in \mathbb{N}\}$.

Since, $x_n = \frac{1}{n} \to 0$ as $n \to \infty$. Clearly G_3 is not a $(\mathcal{C}_I)$-graph but it is easy to verify that G_3 is an $(\mathcal{H}_I)$-graph.

Above examples show that for a given f the notions of $(\mathcal{C}_f)$-graph and $(\mathcal{H}_f)$-graph remain independent when the underlying space is a nontrivial b-metric space even if f is an identity map.

Theorem 4.2.14. Let (X, d) be a complete b-metric space such that d is continuous. Let $f : X \to X$ be a b-(φ, G) contraction where φ is a b-comparison function. Assume that the following conditions hold:

(i) there is x_0 in X for which (x_0, fx_0) is an edge in $\widetilde{G}$;

(ii) G is a $(\mathcal{C}_f)$-graph.

then f has a unique fixed point $\xi \in [x_0]_{\widetilde{G}}$ and for any $y \in [x_0]_{\widetilde{G}}$, $f^n y \to \xi$. Further if G is weakly connected then f is Picard operator.

Proof. It follows from Proposition 4.2.10 that $\{f^n x_0\}$ is a Cauchy sequence in X. Since X is complete there exists $\xi \in X$ such that $f^n x_0 \to t$. Since, $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$ and G is a $(\mathcal{C}_f)$ graph there exists a subsequence $\{f^{n_k} x_0\}$ of $\{f^n x_0\}$ and $p \in \mathbb{N}$ such that $(f^{n_k} x_0, \xi) \in E(G)$ for all $k \geq p$. Observe that $(x_0, fx_0, f^2 x_0, \cdots, f^{n_1} x_0, \cdots, f^{n_p} x_0, \xi)$ is a path in $\widetilde{G}$. Therefore, $\xi \in [x_0]_{\widetilde{G}}$. From (4.2), we get

$$d(f^{n_k+1} x_0, f\xi) \leq \varphi(d(f^{n_k} x_0, \xi)) < d(f^{n_k} x_0, \xi) \quad \forall k \geq n_0, \tag{4.10}$$

letting $k \to \infty$ we obtain $\lim_{k\to\infty} f^{n_k+1}x_0 = f\xi$, as d is continuous. Since $\{f^{n_k}x_0\}$ is a subsequence of $\{f^n x_0\}$, we conclude that $f\xi = \xi$. Finally, if $y \in [x_0]_{\widetilde{G}}$, it follows from Lemma 4.2.7 that $f^n y \to \xi$.

$\square$

Theorem 4.2.15. Let (X, d) be a complete b-metric space such that d is continuous. Let $f : X \to X$ be a b-(φ, G) contraction where φ is a b-comparison function. If G is weakly connected $(\mathcal{H}_f)$-graph then f is Picard operator.

Proof. Let G is weakly connected $(\mathcal{H}_f)$-graph. From the Proposition 4.2.10 $f^n x_0 \to \xi \in X$, then $r(f^n x_0, \xi) \to 0$ as $n \to \infty$. Now for each $n \in \mathbb{N}$ let $\{y_i^n\}; i = 0, 1, \cdots, M_n$ be a path from $f^n x_0$ to ξ with $y_0 = \xi$ and $y_{M_n}^n = f^n x_0$ in $\widetilde{G}$ then

$$\begin{aligned} d(\xi, f\xi) &\leq s[d(\xi, f^{n+1}x_0) + d(f^{n+1}x_0, f\xi)] \\ &\leq s[d(\xi, f^{n+1}x_0) + \sum_{i=1}^{M_n} s^i d(fy_{i-1}^n, fy_i^n)] \\ &\leq s[d(\xi, f^{n+1}x_0) + \sum_{i=1}^{M_n} s^i \varphi(d(y_{i-1}^n, y_i^n))] \\ &< s[d(\xi, f^{n+1}x_0) + \sum_{i=1}^{M_n} s^i d(y_{i-1}^n, y_i^n)] = s[d(\xi, f^{n+1}x_0) + r(f^n x_0, \xi)], \end{aligned}$$

letting $n \to \infty$ above inequality yields $f\xi = \xi$. Let $y \in [x_0]_{\widetilde{G}} := X$ be arbitrary then from Lemma 4.2.9 and Remark 4.2.7 it is easily seen that $f^n y \to \xi$. $\square$

Following example shows that the condition of $(\mathcal{C}_f)$-gragh or $(\mathcal{H}_f)$-graph in the hypothesis of Theorem 4.2.14 & 4.2.15 can't be dropped.

Example 4.2.16. Let $X = [0, 1]$, $d(x, y) = |x - y|^2$ and $fx = x/2$ for all $x \in (0, 1]$ and $f0 = 1/2$. Then (X, d) is a complete b-metric space with $s = 2$. Further, d is continuous and f is a b-(φ, G_1) contraction (with $\varphi(t) = t/4$) where $V(G_1) = X$ and $E(G_1) = \{(x, y) \in (0, 1]\times(0, 1]; x \geq y\}\cup\{(0, 0), (0, 1)\}$. Note that G_1 is weakly connected but f has no fixed point in $[x_0]_{\widetilde{G_1}} = X$. Observe that G_1 is not a $(\mathcal{C}_f)$-graph because the sequence $f^n x = \frac{x}{2^n} \to 0$ for $x \in (0, 1]$ and $(f^n x, f^{n+1}x) \in E(G_1); n \in \mathbb{N}$ but it does not contain any subsequence such that

$(x_{n_k}, 0) \in E(G_1)$. Also we note that for any fixed $x \in (0,1]$, $r(f^n x, 0) = 2[|\frac{x}{2^n} - 1|^2 + |1-0|^2] \nrightarrow 0$ as $n \to \infty$.

The notions of orbital and orbital G-continuity for a self-mapping f can be induced in b-metric space intuitively as follows.

Definition 4.2.17. Let (X, d) be a b-metric space. A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n\in\mathbb{N}}$ of positive integers, $f^{k_n}x \to y$ implies $f(f^{k_n}x) \to fy$ as $n \to \infty$. A mapping $f : X \to X$ is called orbitally G-continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n\in\mathbb{N}}$ of positive integers, $f^{k_n}x \to y$ and $(f^{k_n}x, f^{k_n+1}x) \in E(G)$ $\forall n \in \mathbb{N}$ imply $f(f^{k_n}x) \to fy$.

Theorem 4.2.18. Let (X, d) be a complete b-metric space, f be a b-(φ, G) contraction where φ is a b-comparison function. Assume that d is continuous, f is orbitally G-continuous and there is x_0 in X for which (x_0, fx_0) is an edge in G. Then f has a fixed point $\xi \in X$. Moreover, for any $y \in [x_0]_{\widetilde{G}}$, $f^n y \to \xi$.

Proof. It follows from Proposition 4.2.10 that $\{f^n x_0\}$ is a Cauchy sequence in (X, d). Since X is complete there exists $\xi \in X$ such that $\lim_{n\to\infty} f^n x_0 = \xi$. Since $(f^n x_0, f^{n+1}x_0) \in E(G)$ for all $n \in \mathbb{N}$ and f is orbitally G-continuous. Therefore, continuity of d implies that $f\xi = \xi$. Let $y \in [x_0]_{\widetilde{G}}$ be arbitrary then it follows from Lemma 4.2.9 that $\lim_{n\to\infty} f^n y = \xi$. □

Slightly strengthening the continuity condition on f our next theorem deals with the graph G which may fail to have the property that there is x_0 in X for which (x_0, fx_0) is an edge in G.

Theorem 4.2.19. Let (X, d) be a complete b-metric space, f be a b-(φ, G) contraction where φ is a b-comparison function. Assume that d is continuous, f is orbitally continuous and there is x_0 in X for which $fx_0 \in [x_0]_{\widetilde{G}}$. Then for any $y \in [x_0]_{\widetilde{G}}$, $f^n y \to \xi \in X$ where ξ is a fixed point of f.

Proof. It follows from Proposition 4.2.10 that $\{f^n x_0\}$ is a Cauchy sequence in X. Since X is complete there exists $\xi \in X$ such that $f^n x_0 \to \xi$. Since, f is orbitally continuous then $\lim_{n\to\infty} ff^n x_0 = f\xi$ which yields $f\xi = \xi$. Let $y \in [x_0]_{\widetilde{G}}$ be arbitrary then from Lemma 4.2.9, $\lim_{n\to\infty} f^n y = \xi$. □

Remark 4.2.20. In addition to the hypothesis of Theorem 4.2.18 and Theorem 4.2.19 if we assume that G is weakly connected then f will become Picard operator [84] on X.

Remark 4.2.21. Theorem 4.2.14 generalizes/extends claims 4^0 & 5^0 of [52, Theorem 3.2] and [78, Theorem 4(1)]. Theorem 4.2.18 generalizes claims 2^0 & 3^0 of [52, Theorem 3.3]. Theorem 4.2.19 generalizes claims 2^0 & 3^0 of [52, Theorem 3.4] and thus generalizes extends results of Nieto and Rodríguez-López [72, Theorems 2.1 and 2.3], Petrusel and Rus [84, Theorem 4.3] and Ran and Reurings [90, Theorem 2.1]. We mention here that Theorem 4.2.14 can not be improved using comparison function instead of b-comparison function (see, Łukawska and Jachymski [42, Example 2])

We observe that Theorem 4.2.19 can be used to extend famous fixed point theorem of Edelstein to the case of $b-$metric space. We need to define the notion of ϵ-chainable property for the b-metric space.

Definition 4.2.22. A b-metric space (X, d) is said to be ϵ-chainable for some $\epsilon > 0$ if for each $x, y \in X$ there exist $x_i \in X;\ i = 0, 1, 2, \cdots, l$ with $x_0 = x, x_l = y$ such that $d(x_{i-1}, x_i) < \epsilon$ for $i = 1, 2, \cdots, l$.

Corollary 4.2.23. Let (X, d) be a complete ϵ-chainable b-metric space. Assume that d is continuous. Let there exists a b-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f : X \to X$ satisfies,

$$d(x, y) < \epsilon \quad \text{implies} \quad d(fx, fy) \leq \varphi(d(x, y)), \tag{4.11}$$

for all $x, y \in X$. Then f is a Picard operator.

Proof. Consider a graph G consisting of $V(G) := X$ and $E(G) := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. Since X is ϵ-chainable so G is weakly connected. Let $(x, y) \in E(G)$, from (4.11) we have,

$$d(fx, fy) \leq \varphi(d(x, y)) < d(x, y) < \epsilon.$$

Then $d(fx, fy) \in E(G)$. Therefore, in view of (4.11) f is b-(φ, G) contraction. Further, (4.11) implies that f is continuous. Now the conclusion follows by using Theorem 4.2.19. □

Now we establish a fixed point theorem using a general contractive condition.

Theorem 4.2.24. Let (X, d) be a complete b-metric space, G be a $(\mathcal{C}_f)$-graph in $X \times X$ such that $V(G) = X$ and $f : X \to X$ be an edge preserving mapping. Assume that d is continuous and there exist $\delta, \beta, \gamma \geq 0$ with $s\delta + (s+1)\beta + s(s+1)\gamma < 1$ and for all $(x, y) \in E(G)$

$$d(fx, fy) \leq \delta d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)]. \tag{4.12}$$

If there is x_0 in X for which (x_0, fx_0) is an edge in G then f has a fixed point in $[x_0]_{\widetilde{G}}$.

Proof. Since f is edge preserving then $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$. From (4.12) and using (d3) Definition 1.4.1 it follows that

$$\begin{aligned} d(f^n x_0, f^{n+1} x_0) \quad &\leq \quad \delta d(f^{n-1} x_0, f^n x_0) + \beta[d(f^{n-1} x_0, f^n x_0) + d(f^n x_0, f^{n+1} x_0)] \\ &\quad + \gamma s[d(f^{n-1} x_0, f^n x_0) + d(f^n x_0, f^{n+1} x_0)]. \end{aligned}$$

On rearranging,

$$d(f^n x_0, f^{n+1} x_0) \leq \left[\frac{\delta + \beta + \gamma s}{1 - \beta - \gamma s}\right] d(f^{n-1} x_0, f^n x_0).$$

Repeating iteratively we have,

$$d(f^n x_0, f^{n+1} x_0) \leq \left[\frac{\delta + \beta + \gamma s}{1 - \beta - \gamma s}\right]^n d(x_0, fx_0). \tag{4.13}$$

For $m > n \geq 1$ and using (d3) Definition 1.4.1, we have

$$\begin{aligned} d(f^n x_0, f^m x_0) \quad &\leq \quad sd(f^n x_0, f^{n+1} x_0) + s^2 d(f^{n+1} x_0, f^{n+2} x_0) + ... + s^{m-n} d(f^{m-1} x_0, f^m x_0) \\ &\leq \quad \frac{1}{s^{n-1}} d(x_0, fx_0) \sum_{j=n}^{m-1} s^j \left[\frac{\delta + \beta + \gamma s}{1 - \beta - \gamma s}\right]^j \qquad \text{(using 4.13)} \\ &< \quad \frac{1}{s^{n-1}} d(x_0, fx_0) \sum_{j=n}^{\infty} s^j \left[\frac{\delta + \beta + \gamma s}{1 - \beta - \gamma s}\right]^j. \end{aligned} \tag{4.14}$$

Since, $s\left[\frac{\delta+\beta+\gamma s}{1-\beta-\gamma s}\right] < 1$ then $\{f^n x_0\}$ is a Cauchy sequence in X. By completeness of X the sequence $\{f^n x_0\}$ converges to some $\xi \in X$. Since, G is a $(\mathcal{C}_f)$-graph, there exists a subsequence $\{f^{n_k} x_0\}$ and a natural number p such that $(f^{n_k} x_0, \xi) \in E(G)$ for all $k \geq p$. From (4.12) for all $k \geq p$ we have

$$\begin{aligned} d(f^{n_k+1} x_0, f\xi) \quad &\leq \quad \delta d(f^{n_k} x_0, \xi) + \beta[d(f^{n_k} x_0, f^{n_k+1} x_0) + d(\xi, f\xi)] \\ &+ \quad \gamma[d(f^{n_k} x_0, f\xi) + d(\xi, f^{n_k+1} x_0)]. \end{aligned} \tag{4.15}$$

Since the b-metric d is continuous and $\beta + \gamma < 1$ so letting $k \to \infty$ inequality (4.15) yields $f\xi = \xi$. Also note that $(x_0, fx_0, f^2x_0, \cdots, f^{n_1}x_0, \cdots, f^{n_p}x_0, \xi)$ is a path in G and hence in $\widetilde{G}$, therefore $\xi \in [x_0]_{\widetilde{G}}$. □

We note that Theorem 4.2.24 does not guarantee the uniqueness of fixed point but this can be accomplished under some assumptions as in the following theorem.

Theorem 4.2.25. In addition to the hypothesis of Theorem 4.2.24 we further assume that if $\delta + 2s\beta + 2\gamma < 1$ for the same set of $\delta, \beta, \gamma \geq 0$ and for any two fixed points ξ_1, ξ_2 there exists $z \in X$ such that (ξ_1, z) and $(\xi_2, z) \in E(G)$. Then f has a unique fixed point.

Proof. Let ξ_1 and ξ_2 are two fixed points of f then there exists $z \in X$ such that $(\xi_1, z), (\xi_2, z) \in E(G)$. By induction we have $(\xi_1, f^n z), (\xi_2, f^n z) \in E(G)$ for all $n = 0, 1, \cdots$. From (4.12) we have,

$$\begin{aligned} d(\xi_1, f^n z) &\leq \delta d(\xi_1, f^{n-1}z) + \beta d(f^{n-1}z, f^n z) + \gamma[d(\xi_1, f^n z) + d(f^{n-1}z, \xi_1)] \\ &\leq \delta d(\xi_1, f^{n-1}z) + \beta s[d(f^{n-1}z, \xi_1) + d(\xi_1, f^n z)] + \gamma[d(\xi_1, f^n z) + d(f^{n-1}z, \xi_1)]. \end{aligned}$$

On rearranging,

$$d(\xi_1, f^n z) \leq [\frac{\delta + s\beta + \gamma}{1 - s\beta - \gamma}] d(\xi_1, f^{n-1}z), \quad \text{for all } n = 1, 2, \cdots. \tag{4.16}$$

Continuing recursively, (4.16) gives

$$d(\xi_1, f^n z) \leq [\frac{\delta + s\beta + \gamma}{1 - s\beta - \gamma}]^n d(\xi_1, z). \tag{4.17}$$

Since, $[\frac{\delta+s\beta+\gamma}{1-s\beta-\gamma}] < 1$ then $\lim_{n\to\infty} d(\xi_1, f^n z) = 0$. Similarly one can show that $\lim_{n\to\infty} d(\xi_2, f^n z) = 0$. Thus by using (d3) Definition 1.4.1 it infers that $d(\xi_1, \xi_2) = 0$. □

Suppose that $(X, \preceq)$ is a partially ordered set. Consider graph G_2 consisting of $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$ and $V(G_2)$ coincides with X. We note that if a self mapping f is monotone with respect to the order $\preceq$ then for the graph G_2, it is obvious that f is edge preserving or equivalently we can say that f maps comparable elements onto comparable elements.

Following corollaries are the direct consequences of Theorem 4.2.25.

Corollary 4.2.26. Let (X, d) be a complete metric space where X is a partially ordered set with respect to $\preceq$. Let $f : X \to X$ be nondecreasing (or nonincreasing) with respect to $\preceq$. Assume that there exists $\delta, \beta, \gamma \geq 0$ with $\delta + 2\beta + 2\gamma < 1$ such that,

$$d(fx, fy) \leq \delta d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)],$$

for all comparable $x, y \in X$. If the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(ii) for nondecreasing (or nonincreasing) sequence $\{x_n\} \to x \in X$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \preceq x$, for all k.

Then f has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$ then the fixed point is unique.

Corollary 4.2.27. Let (X, d) be a complete metric space where X is a partially ordered set with respect to $\preceq$. Let $f : X \to X$ be nondecreasing (or nonincreasing) with respect to $\preceq$. Assume that there exists a constant $0 < c < 1/2$ such that,

$$d(fx, fy) \leq c[d(x, fx) + d(y, fy)],$$

for all comparable $x, y \in X$. If the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(ii) for nondecreasing (or nonincreasing) sequence $\{x_n\} \to x \in X$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \preceq x$, for all k.

Then f has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$ then the fixed point is unique.

Corollary 4.2.28. Let (X, d) be a complete metric space where X is a partially ordered set with respect to $\preceq$. Let $f : X \to X$ be nondecreasing (or nonincreasing) with respect to $\preceq$. Assume that there exists a constant $0 < c < 1/2$ such that,

$$d(fx, fy) \leq c[d(x, fy) + d(y, fx)],$$

for all comparable $x, y \in X$. If the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(ii) for nondecreasing (or nonincreasing) sequence $\{x_n\} \to x \in X$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \preceq x$, for all k.

Then f has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$ then the fixed point is unique.

4.2.2 Some consequences and applications

We note that in Theorem 4.2.24 the condition "there is x_0 in X for which (x_0, fx_0) is an edge in G" yields $f^n x_0 \to \xi$, where $\xi \in X$ is a fixed point of f. Consider a graph $G := (X, X \times X)$. For such graph under the assumptions of Theorems 4.2.24, 4.2.25 it infers that f is a Picard operator. Thus many standard fixed point theorems can be easily deduced from Theorem 4.2.25.

Corollary 4.2.29. (Hardy and Rogers [45]). Let (X, d) be a complete metric space and let $f : X \to X$. Suppose that there exists constants $\delta, \beta, \gamma \geq 0$ such that

$$d(fx, fy) \leq \delta d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)],$$

for all $x, y \in X$, where $\delta + 2\beta + 2\gamma < 1$ then f has a unique fixed point in X.

Corollary 4.2.30. (Kannan [57]). Let (X, d) be a complete metric space and let $f : X \to X$. Suppose that there exists a constants c such that

$$d(fx, fy) \leq c[d(x, fx) + d(y, fy)],$$

for all $x, y \in X$, where $0 < c < 1/2$ then f is Picard operator.

Corollary 4.2.31. (Chatterjea [26]). Let (X, d) be a complete metric space and let $f : X \to X$. Suppose that there exists a constants c such that

$$d(fx, fy) \leq c[d(x, fy) + d(y, fx)],$$

for all $x, y \in X$, where $0 < c < 1/2$ then f is Picard operator.

The concept of cyclic representations and cyclic contractions can also be invoked intuitively when the underlying space is a b-metric. Let X be a nonempty set endowed with a b-metric d. Let m be a positive integer and $\{X_i\}_{i=1}^m$ be nonempty closed subsets of X and $f : \cup_{i=1}^m X_i \to \cup_{i=1}^m X_i$ be an operator. Then $X := \cup_{i=1}^m X_i$ is known as a cyclic representation of X with respect to f if

$$f(X_1) \subset X_2, \cdots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1 \tag{4.18}$$

and operator f is known as cyclic operator.

Theorem 4.2.32. Let (X, d) be a complete b-metric space such that d is a continuous functional on $X \times X$. Let m be a positive integer, $\{X_i\}_{i=1}^m$ nonempty closed subsets of X, $Y := \cup_{i=1}^m X_i$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a b-comparison function and $f : Y \to Y$. Further suppose that

(*i*) $\cup_{i=1}^m X_i$ is cyclic representation of Y with respect to f;

(*ii*) $d(fx, fy) \leq \varphi(d(x, y))$ whenever, $x \in X_i, y \in X_{i+1}$, where $X_{m+1} = X_1$.

Then f has a unique fixed point $\xi \in \cap_{i=1}^m X_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^m X_i$.

Proof. We note that (Y, d) is a complete b-metric space. Let us consider a graph G consisting of $V(G) := Y$ and $E(G) := \Omega \cup \{(x, y) \in Y \times Y : x \in X_i, y \in X_{i+1}; i = 1, \cdots, m\}$. By (i) it follows that f preserves edges. Thus for this graph G in view of condition (*ii*) the mapping f is $b-(\varphi, G)$ contraction. Now let $f^n x \to x^*$ in Y such that $(f^n x, f^{n+1} x) \in E(G)$ for all $n \geq 1$ then in view of (4.18) the sequence $\{f^n x\}$ has infinitely many terms in each X_i so that one can easily extract a subsequence of $\{f^n x\}$ converging to x^* in each X_i. Since X_i's are closed then $x^* \in \cap_{i=1}^m X_i$. Now it is easy to form a subsequence $\{f^{n_k} x\}$ in some X_j, $j \in \{1, \cdots, m\}$ such that $(f^{n_k} x, x^*) \in E(G)$ for $k \geq 1$, it implies that G is a weakly connected $(\mathcal{C}_f)$-graph and thus conclusion follows from Theorem 4.2.14. □

Remark 4.2.33. [79, Theorem 2.1(1)] can be deduced from Theorem 4.2.32 if (X, d) is a metric space.

On the same lines as in proof of Theorem 4.2.32 we obtain the following consequence of Theorem 4.2.25.

Theorem 4.2.34. Let (X,d) be a complete b-metric space such that d is continuous functional on $X \times X$. Let m be positive integer, $\{X_i\}_{i=1}^m$ nonempty closed subsets of X, $Y := \cup_{i=1}^m X_i$ and $f : Y \to Y$. Further suppose that

(i) $\cup_{i=1}^m X_i$ is cyclic representation of Y with respect to f;

(ii) there exist $\delta, \beta, \gamma \geq 0$ with $s\delta + s(s+1)\beta + s(s+1)\gamma < 1$ such that
$d(fx, fy) \leq \delta d(x,y) + \beta[d(x,fx) + d(y,fy)] + \gamma[d(x,fy) + d(y,fx)]$,
whenever, $x \in X_i, y \in X_{i+1}$, where $X_{m+1} = X_1$.

Then f has a fixed point $\xi \in \cap_{i=1}^m X_i$.

Remark 4.2.35. [15, Theorem 7] and [82, Theorem 3.1] can be deduced from Theorem 4.2.34 but it does not ensure f to be a Picard operator.

Remark 4.2.36. We note that in proof [15, Theorem 7] author's argument to prove G (as assumed in proof of Theorem 4.2.32) a $(\mathcal{C})$-graph is valid only if the terms of sequence $\{x_n\}$ are Picard iterations otherwise it is void. For example let $Y = \cup_{i=1}^3 X_i$ where $X_1 = \{\frac{1}{n} : n \text{ is odd}\} \cup \{0\}$, $X_2 = \{\frac{1}{n} : n \text{ is even}\} \cup \{0\}$, $X_3 = \{0\}$ and define $f : Y \to Y$ as $fx = \frac{x}{2}; x \in Y \setminus X_2$ and $fx = 0; x \in X_2$. We see that (4.18) is satisfied and $\frac{1}{n} \to 0$ but X_3 does not contain infinitely many terms of $\{x_n\}$. In the following we give the corrected argument to prove G a $(\mathcal{C})$-graph.
Let $x_n \to x$ in X such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$. Keeping in mind construction of G there exists at least one pair of closed sets $\{X_j, X_{j+1}\}$ for some $j \in \{1, 2, \cdots, m\}$ such that both sets contain infinitely many terms of sequence $\{x_n\}$. Since X_i's are closed so that $x \in X_j \cap X_{j+1}$ for some $j \in \{1, \cdots, m\}$ and thus one can easily extract a subsequence such that $(x_{n_k}, x) \in E(G)$ holds for $k \geq 1$.

Definition 4.2.37. Let (X,d) be a b-metric space and $f : X \to X$. We call the mapping f a $(b, \alpha - \varphi)$ contraction if

$$\alpha(x,y)\varphi(fx, fy) \leq d(x,y), \qquad \text{for all } x, y \in X, \tag{4.19}$$

where $\alpha : X \times X \to [0, \infty)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a b-comparison functions.

Remark 4.2.38. We note that if (X, d) is a simple metric space i.e., $s = 1$ then Definition 4.2.37 coincides with Definition 2.4.1 due to Samet et al. [100].

Theorem 4.2.39. Let (X, d) be a complete b-metric space such that d is a continuous functional. Suppose that $f : X \to X$ be a $(b, \alpha - \varphi)$ contraction and satisfies the following conditions:

(i) f is α-admissible, i.e., $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$ for every $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.

Then f has a fixed point.

Proof. Consider a graph $G := \big(V(G), E(G)\big)$ consisting of

$$V(G) := X \quad \text{and} \quad E(G) := \{(x, y) \in X \times X : \alpha(x, y) \geq 1\}.$$

For the graph G the condition (i) implies that f preserves edges i.e., $(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$. Let $(x, y) \in E(G)$ then $\alpha(x, y) \geq 1$ and from inequality (4.19) we obtain

$$\varphi(fx, fy) \leq \alpha(x, y)\varphi(fx, fy) \leq d(x, y) \tag{4.20}$$

Thus f is b-(φ, G) contraction. The condition (ii) vindicates the existence of some $x_0 \in X$ with $(x_0, fx_0) \in E(G)$. Furthermore it is easy to observe that the condition (iii) implies that G is a $(\mathcal{C})$-graph. Hence all the conditions of Theorem 3.2.8 (1) are satisfied and the conclusion follows. □

Remark 4.2.40. Theorem 4.2.39 extends Theorems 2.4.2 and 2.4.3 due to Samet et al. [100] to the case of b-metric spaces. Thus for $(s = 1)$ when d is a simple metric, Theorem 4.2.39 subsumes Theorems 2.4.2 and 2.4.3 as special cases.

Theorem 4.2.41. Let (X, d) be a complete b-metric space such that d is a continuous functional. Suppose that there exist $\delta, \beta, \gamma \geq 0$ with $s\delta + s(s+1)\beta + s(s+1)\gamma < 1$ and

$\alpha : X \times X \to [0, \infty)$ such that $f : X \to X$ satisfies the following contractive condition:

$$\alpha(x, y)d(fx, fy) \leq \delta d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)], \quad \text{for all } x, y \in X. \tag{4.21}$$

Assume the following conditions hold:

(i) f is α-admissible, i.e., $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$ for every $x, y \in X$;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$ for all n.

Then f has a fixed point.

Proof. The proof follows the same lines as in Theorem 4.2.39. Consider a graph $G := \big(V(G), E(G)\big)$ consisting of

$$V(G) := X \quad \text{and} \quad E(G) := \{(x, y) \in X \times X : \alpha(x, y) \geq 1\}.$$

For the graph G the condition (i) implies that f preserves edges i.e., $(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$. Let $(x, y) \in E(G)$ then $\alpha(x, y) \geq 1$ and from inequality (4.21) we obtain

$$d(fx, fy) \leq \alpha(x, y)d(fx, fy) \leq \delta d(x, y) + \beta[d(x, fx) + d(y, fy)] + \gamma[d(x, fy) + d(y, fx)]. \tag{4.22}$$

Thus f satisfies (4.12). The condition (ii) infers the existence of some $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. Furthermore, the condition (iii) implies that G is a $(\mathcal{C})$-graph. Hence all the conditions of Theorem 4.2.24 are satisfied and the conclusion follows. □

Now we establish an existence theorem for the solution of an integral equation as a consequence of our Theorem 4.2.14.

Theorem 4.2.42. Consider the integral equation

$$x(t) = \int_a^b k(t, s, x(s))ds + g(t), \quad t \in [a, b], \tag{4.23}$$

where $k : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}^n$ is continuous. Assume that

(i) $k(t, s, .) : \mathbb{R}^n \to \mathbb{R}^n$ is nondecreasing for each $t, s \in [a, b]$;

(ii) there exists a $(c)-$comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and a continuous function $p : [a, b] \times [a, b] \to \mathbb{R}^+$ such that
$|k(t, s, x(s)) - k(t, s, y(s))| \le p(t, s)\varphi(|x(s) - y(s)|)$ for each $t, s \in [a, b]$ and $x \le y$ (i.e., $x(t) \le y(t);\ \forall t \in [a, b]$);

(iii) $\sup_{t\in[a,b]} \int_a^b p(t, s)ds \le 1$;

(iv) there exists $x_0 \in C([a, b], \mathbb{R}^n)$ such that $x_0(t) \le \int_a^b k(t, s, x_0(s))ds + g(t)$ for all $t \in [a, b]$.

Then the integral equation (4.23) has a unique solution in the set $\{x \in C([a, b], \mathbb{R}^n) : x(t) \le x_0(t)$ or $x(t) \ge x_0(t); \forall t \in [a, b]\}$.

Proof. Let $(X, ||.||_\infty)$ where $X = C([a, b], \mathbb{R}^n)$ and define a mapping $T : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$ by

$$Tx(t) = \int_a^b k(t, s, x(s))ds + g(t), \quad t \in [a, b]. \tag{4.24}$$

Consider a graph G consisting of $V(G) := X$ and $E(G) = \{(x, y) \in X \times X : x(t) \le y(t) \forall t \in [a, b]\}$. From property (i) we observe that the mapping T is nondecreasing, thus T preserves edges. Furthermore G is a $(\mathcal{C})$-graph that is for every nondecreasing sequence $\{x_n\} \subset X$ which converges to some $z \in X$ then $x_n(t) \le z(t)$ for all $t \in [a, b]$. Now for every $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\begin{aligned}
|Tx(t) - Ty(t)| &\le \int_a^b |k(t, s, x(s)) - k(t, s, y(s))|ds \\
&\le \int_a^b p(t, s)\varphi(|x(s) - y(s)|)ds \\
&\le \varphi(||x - y||_\infty) \int_a^b p(t, s)ds.
\end{aligned}$$

Hence, $d(Tx, Ty) \le \varphi(d(x, y))$. From (iv) we have, $(x_0, Tx_0) \in E(G)$, so that $[x_0]_{\widetilde{G}} = \{x \in C([a, b], \mathbb{R}^n) : x(t) \le x_0(t)$ or $x(t) \ge x_0(t); \forall t \in [a, b]\}$. The conclusion follows from Theorem 4.2.14. □

Note that Theorem 4.2.42 specifies the region of solution which invokes the novelty of our result.

4.3 Contractions using gauge function φ

Recently, Proinov [85] extended Banach contraction principle with higher order of convergence. He proposed an iterative scheme for a mapping satisfying a contractive condition which involves gauge function of order $r \geq 1$ and obtained error estimates as well. His results include as special cases some results of Mysovskih [69], Rheinboldt [92], Gel'man [38] and Huang [50] and others. In [64] authors extended the results of Proinov to the case of multi-valued mappings.

Inspired by the work of Proinov [85] in this section we investigate whether the consequences of his results hold when the underlying structure is replaced with a b-metric space. We give an affirmative answer to this question. Our results generalize main results of Proinov [85] and thus subsume many results of authors [38, 50, 69, 92]. We establish an example to substantiate the validity of our results. Consequently, in Subsection 3.3.2 we also obtain an existence theorem for the solution of an initial value problem.
A gauge function $\varphi : J \to J$ satisfies: $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0,1)$ and $t \in J$; and $\varphi(t) < t$ for all $t \in J \setminus \{0\}$.

Definition 4.3.1. ([14]) A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini Grandolfi gauge function if $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t \in J$.

Subsequently, Let (X, d) be a b-metric space with coefficient $s \geq 1$. We assume that $f : D \subset X \to X$ be an operator and there exist some $x_0 \in D$ such that $\mathcal{O}(x_0) \subset D$. Let the operator f satisfies the following iterated contractive condition:

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \quad \text{for all } x \in \mathcal{O}(x_0) \text{ such that } d(x, fx) \in J, \tag{4.25}$$

where φ is a gauge function of order $r \geq 1$ on an interval J. We establish two convergence theorems for iterative processes of the type

$$x_{n+1} = fx_n, \quad n = 0, 1, 2, \cdots, \tag{4.26}$$

where f satisfies (4.25).

4.3.1 b-Bianchini Grandolfi gauge functions

In [85] Proinov proved his main results by assuming Bianchini Grandolfi gauge functions and the mapping f satisfying contractive condition (4.36) when the underlying set is endowed with a metric (see, Corollary 4.3.16). But in the setting of b-metric space for some technical reasons we have to restrict ourselves to the gauge functions satisfying $\sum_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ for all $t \in J$ where s is the coefficient of b-metric space. Furthermore, taking into account such crucial condition in order to calculate a priori and a posteriori estimates we consider the gauge functions of the form:

$$\varphi(t) = t\frac{\phi(t)}{s} \quad \text{for all } t \in J, \tag{4.27}$$

where $s \geq 1$ is the coefficient of b-metric d and ϕ is nonnegative nondecreasing function on J such that

$$0 \leq \phi(t) < 1 \quad \text{for all } t \in J. \tag{4.28}$$

Remark 4.3.2. One can always define a non-negative non-decreasing function ϕ on J satisfying (4.27) and (4.28) as follows:

$$\phi(t) = \begin{cases} \frac{s\varphi(t)}{t}, & \text{if } t \in J \setminus \{0\} \\ 0, & \text{if } t = 0, \end{cases} \tag{4.29}$$

where s is the coefficient of b-metric d.

For a fixed $s \geq 1$, let us consider the following simple examples of gauge functions of order r:

(i) $\varphi(t) = \frac{ct}{s},\ 0 < c < 1$ is a gauge function of order 1 on $J = [0, \infty)$;

(ii) $\varphi(t) = \frac{ct^r}{s},\ (c > 0, r > 1)$ is a gauge function of order r on $J = [0, h)$ where $h = (\frac{1}{c})^{\frac{1}{(r-1)}}$.

It is essential to mention here that to establish the fixed point theorem (see, Theorem 4.3.14) we do not necessarily require the gauge functions φ satisfying (4.27),(4.28). But we consider the gauge function such that $\sum_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ for all $t \in J$ where s is a coefficient of b-metric space.

Lemma 4.3.3. Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying (4.27) and (4.28). Then,

(1). $0 \leq \frac{\phi(t)}{s} < 1$ for all $t \in J$,

(2). $\phi(\mu t) \leq \mu^{r-1}\phi(t)$ for all $\mu \in (0, 1)$ and $t \in J$.

Remark 4.3.4. When d is a simple metric then $s = 1$. In such case every gauge function satisfying $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ is of the form $\varphi(t) = t\phi(t)$ where ϕ is nonnegative nondecreasing function on J [85]. Thus the condition $0 \leq \phi(t) < 1$ for all $t \in J$ becomes superfluous and is directly followed from Lemma 4.3.3.

Lemma 4.3.5. Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying (4.27) and (4.28). Then for every $n \geq 0$ we have

(1). $\varphi^n(t) \leq t\big[\frac{\phi(t)}{s}\big]^{P_n(r)}$ for all $t \in J$,

(2). $\phi(\varphi^n(t)) \leq s\big[\frac{\phi(t)}{s}\big]^{r^n}$ for all $t \in J$.

Proof. (1). Set $\mu = \frac{\phi(t)}{s}$ and let $t \in J$. Then from Lemma 4.3.3 we obtain $0 \leq \mu < 1$. For $\mu = 0$ the case is trivial. We shall prove (1) by using mathematical induction. For $n = 0, 1$ the property (1) is trivially satisfied as it reduces to an equality. Let it also holds for any integer $n \geq 1$, i.e.,

$$\varphi^n(t) \leq t\mu^{P_n(r)}.$$

Since φ is nondecreasing on J so we obtain (as $t\mu^{P_n(r)} \in J$ because $t \in J$ and $\mu < 1$)

$$\varphi^{n+1}(t) \leq \varphi[t\mu^{P_n(r)}] \leq \mu^{rP_n(r)}\varphi(t) \leq \mu^{rP_n(r)}t\frac{\phi(t)}{s} = t\mu^{rP_n(r)+1} = t\mu^{P_{n+1}(r)}.$$

(2). By making use of Lemma 4.3.3 and monotonicity of ϕ then (1) leads to the following

$$\phi\big(\varphi^n(t)\big) \leq \phi\big(t\big[\frac{\phi(t)}{s}\big]^{P_n(r)}\big) \leq \big[\frac{\phi(t)}{s}\big]^{(r-1)P_n(r)}\phi(t) = s\big[\frac{\phi(t)}{s}\big]^{1+(r-1)P_n(r)} = s\big[\frac{\phi(t)}{s}\big]^{r^n}$$

which completes the proof. □

Definition 4.3.6. Let $q \geq 1$ be a fixed real number. A nondecreasing function $\varphi : J \to J$ is said to be a b-Bianchini Grandolfi gauge function with a coefficient q on J if:

$$\sigma(t) = \sum_{n=0}^{\infty} q^n\varphi^n(t) < \infty \quad \text{for all } t \in J. \tag{4.30}$$

We note that a b-Bianchini Grandolfi gauge function also satisfies following functional equation:

$$\sigma(t) = q\sigma(\varphi(t)) + t. \tag{4.31}$$

It is easy to see that every b-Bianchini Grandolfi gauge function is also Bianchini Grandolfi [14] gauge function but converse may not hold. A b-Bianchini Grandolfi gauge function having coefficient $q_1 \geq 1$ is also a b-Bianchini Grandolfi gauge function having coefficient $q_2 \geq 1$ for every $q_2 \leq q_1$.

From now on, we always assume that the coefficient of b-Bianchini Grandolfi gauge function is at least as large as the coefficient of b-metric space.

Lemma 4.3.7. Every gauge function of order $r \geq 1$ defined by (4.27) and (4.28) is a b-Bianchini Grandolfi gauge function with coefficient $s \geq 1$.

Proof. It is immediately followed from first part of Lemma 4.3.5 and using the fact that $P_n(r) \geq n$ for $r \geq 1$ and $n \geq 0$. □

4.3.2 Fixed point theorems using gauge function φ

For convenience we define a function $E : D \to \mathbb{R}^+$ by $E(x) = d(x, fx)$ and assume that there exist some $x_0 \in D$ such that $\mathcal{O}(x_0) \subset D$ so that the condition (4.25) can be put in the form:

$$E(fx) \leq \varphi(E(x)) \quad \text{for all } x \in \mathcal{O}(x_0) \text{ such that } E(x) \in J. \tag{4.32}$$

Lemma 4.3.8. Suppose $x_0 \in X$ be such that $\mathcal{O}(x_0) \subset D$. Assume that $E(x_0) \in J$ then $E(x_n) \in J$ for all $n \geq 0$.

Proof. Since, $x_0, x_1, x_2, \cdots, x_n$ are well defined and belong to D. From (4.32) we have

$$E(x_1) = d(x_1, x_2) \leq \varphi(d(x_0, x_1)) = \varphi(E(x_0)) \in J \quad (\text{as } E(x_0) \in J).$$

Hence, $E(x_1) \in J$. Similarly, iterating successively we get $E(x_n) \in J$ for all $n \geq 0$. □

Definition 4.3.9. Suppose $x_0 \in D$ be such that $\mathcal{O}(x_0) \subset D$ and $E(x_0) \in J$. Then for every iterate $x_n \in D, n \geq 0$ we define the closed ball $\overline{B}(x_n, \rho_n)$ with center at x_n and radius $\rho_n = s\sigma(E(x_n))$, where $\sigma : J \to \mathbb{R}^+$ is defined by (4.30).

Lemma 4.3.10. Suppose $x_0 \in D$ be such that $\mathcal{O}(x_0) \subset D$ and $E(x_0) \in J$. Assume that $\overline{B}(x_n, \rho_n) \subset D$ for some $n \geq 0$ then $x_{n+1} \in D$ and $\overline{B}(x_{n+1}, \rho_{n+1}) \subset \overline{B}(x_n, \rho_n)$.

Proof. Since, $E(x_0) \in J$ then Lemma 4.3.8 invokes $E(x_n) \in J$ for all $n \geq 0$. The condition (4.31) implies $\sigma(t) \geq t$ for all $t \in J$. We have

$$d(x_n, x_{n+1}) \leq \sigma(d(x_n, x_{n+1})) \leq s\sigma(d(x_n, x_{n+1})) = \rho_n.$$

Thus $x_{n+1} \in \overline{B}(x_n, \rho_n) \subset D$. Now let $x \in \overline{B}(x_{n+1}, \rho_{n+1})$. As $E(x_n) \in J$ so that from (4.32) we have $E(x_{n+1}) \leq \varphi(E(x_n))$. By making use of (4.31) we get

$$\begin{aligned} d(x, x_n) &\leq s[d(x, x_{n+1}) + E(x_n)] \\ &\leq s[\rho_{n+1} + E(x_n)] = s[s\sigma(E(x_{n+1})) + E(x_n)] \\ &\leq s[s\sigma(\varphi(E(x_n))) + E(x_n)] = s\sigma(E(x_n)) = \rho_n. \end{aligned}$$

Hence, $x \in \overline{B}(x_n, \rho_n)$. □

Definition 4.3.11. (Initial Orbital Point). We say that a point $x_0 \in D$ is an initial orbital point of f if $E(x_0) \in J$ and $\mathcal{O}(x_0) \subset D$.

Following lemma is obvious.

Lemma 4.3.12. For every initial orbital point $x_0 \in D$ of f and every $n \geq 0$ we have

$$E(x_{n+1}) \leq \varphi(E(x_n)) \quad \text{and} \quad E(x_n) \leq \varphi^n(E(x_0)).$$

Furthermore, if φ is a gauge function of order $r \geq 1$ defined by (4.27) and (4.28) then

$$E(x_n) \leq E(x_0)\mu^{P_n(r)} \quad \text{and} \quad \phi(E(x_n)) \leq s\mu^{r^n} = \phi(x_0)\mu^{r^n - 1},$$

where $\mu = \frac{\phi(E(x_0))}{s}$ and ϕ is nonnegative nondecreasing on J satisfying (4.27) and (4.28).

Proof. By making use of Lemma 4.3.8 we obtain $E(x_{n+1}) \leq \varphi(E(x_n))$. Since φ is nondecreasing then it is easily followed that $E(x_n) \leq \varphi^n(E(x_0))$. Now from Lemma 4.3.5 (1) we have

$$E(x_n) \leq \varphi^n(E(x_0)) \leq E(x_0)\Big[\frac{\phi(E(x_0))}{s}\Big]^{P_n(r)} = E(x_0)\mu^{P_n(r)}.$$

By using Lemma 4.3.5 (2) we obtain

$$\phi(E(x_n)) \le \phi(\varphi^n(E(x_0))) \le s\Big[\frac{\phi(E(x_0))}{s}\Big]^{r^n} = s\mu^{r^n}.$$

□

Following lemma gives bounds for inclusion radii and throughout its proof we will make use of the following facts,

$$0 \le \phi(t) < 1, \quad P_j(r) \ge j, \quad 0 \le \mu^{r^n} < 1,$$

where $r \ge 1, \mu = \frac{\phi(E(x_0))}{s}$ and $j = 0, 1, 2, \cdots$.

Lemma 4.3.13. Suppose $x_0 \in D$ is an initial orbital point of f and φ is a gauge function of order $r \ge 1$. Let ϕ be a nonnegative and nondecreasing on J defined by (4.27) and (4.28). Then for radii $\rho_n = s\sigma(E(x_n)), n = 0, 1, 2, \cdots$, the following estimates hold:

1. $\rho_n \le sE(x_n)\sum_{j=0}^{\infty}\big[\phi(E(x_n))\big]^{P_j(r)} \le \frac{sE(x_n)}{1-\phi(E(x_n))}$,
2. $\rho_n \le sE(x_n)\sum_{j=0}^{\infty}\big[\phi(E(x_0))\mu^{r^n-1}\big]^{P_j(r)} \le \frac{sE(x_n)}{1-\phi(E(x_0))\mu^{r^n-1}}$,
3. $\rho_n \le sE(x_0)\mu^{P_n(r)}\sum_{j=0}^{\infty}\big[\phi(E(x_0))\mu^{r^n-1}\big]^{P_j(r)} \le sE(x_0)\frac{\mu^{P_n(r)}}{1-\phi(E(x_0))\mu^{r^n-1}}$,
4. $\rho_{n+1} \le s\varphi(E(x_n))\sum_{j=0}^{\infty}\big[\phi(\varphi(E(x_n)))\big]^{P_j(r)} \le \frac{s\varphi(E(x_n))}{1-\phi(\varphi(E(x_n)))}$,
5. $\rho_{n+1} \le s\varphi(E(x_n))\sum_{j=0}^{\infty}\big[\phi(E(x_0))\mu^{r^{n+1}-1}\big]^{P_j(r)} \le \frac{s\varphi(E(x_n))}{1-\phi(E(x_0))\mu^{r^{n+1}-1}}$,

where $\mu = \frac{\phi(E(x_0))}{s}$.

Proof. 1. From definition of ρ_n we have

$$\begin{aligned}
\rho_n = s\sigma(E(x_n)) &= s\sum_{j=0}^{\infty} s^j\varphi^j(E(x_n)) \\
&\le s\sum_{j=0}^{\infty} s^j E(x_n)\big[\frac{\phi(E(x_n))}{s}\big]^{P_j(r)} \qquad \text{(using Lemma 4.3.5)} \\
&= sE(x_n)\sum_{j=0}^{\infty} s^j\big[\frac{\phi(E(x_n))}{s}\big]^{P_j(r)} \qquad (4.33) \\
&\le sE(x_n)\sum_{j=0}^{\infty}[\phi(E(x_n))]^j = \frac{sE(x_n)}{1-\phi(E(x_n))}.
\end{aligned}$$

2. From (4.33) we have

$$\begin{aligned}
\rho_n &\leq sE(x_n)\sum_{j=0}^{\infty}\big[\phi(E(x_n))\big]^{P_j(r)} \\
&\leq sE(x_n)\sum_{j=0}^{\infty}\big[s\mu^{r^n}\big]^{P_j(r)} \quad \text{(using second part of Lemma 4.3.12)} \\
&= sE(x_n)\sum_{j=0}^{\infty}\big[\phi(E(x_0))\mu^{r^n-1}\big]^{P_j(r)} \\
&\leq sE(x_n)\sum_{j=0}^{\infty}\big[\phi(E(x_0))\mu^{r^n-1}\big]^{j} = \frac{sE(x_n)}{1-\phi(E(x_0))\mu^{r^n-1}}.
\end{aligned}$$

3. By making use of first part of Lemma 4.3.12 in above we have

$$\begin{aligned}
\rho_n &\leq sE(x_n)\sum_{j=0}^{\infty}\big[\phi(E(x_0))\mu^{r^n-1}\big]^{P_j(r)} \\
&\leq sE(x_0)\mu^{P_n(r)}\sum_{j=0}^{\infty}\big[\phi(E(x_0))\mu^{r^n-1}\big]^{j} \\
&\leq \frac{sE(x_0)\mu^{P_n(r)}}{1-\phi(E(x_0))\mu^{r^n-1}}.
\end{aligned}$$

4. Now by making use of Lemma 4.3.5 we have

$$\begin{aligned}
\rho_{n+1} = s\sigma(E(x_{n+1})) &= \sum_{j=0}^{\infty} s^j\varphi(E(x_{n+1})) \\
&\leq sE(x_{n+1}))\sum_{j=0}^{\infty} s^j\big[\frac{\phi(E(x_{n+1}))}{s}\big]^{P_j(r)} \\
&\quad \text{(as } E(x_{n+1}) \leq \varphi(E(x_n) \text{ and } \phi \text{ is nondecreasing)} \\
&\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}\big[\phi(\varphi(E(x_n)))\big]^{P_j(r)} \\
&\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}\big[\phi(\varphi(E(x_n)))\big]^{j} = \frac{s\varphi(E(x_n))}{1-\phi(\varphi(E(x_n)))}.
\end{aligned}$$

5. From (4) we have,

$$\begin{aligned}\rho_{n+1} &\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}\left[\phi(E(x_{n+1}))\right]^{P_j(r)} \\ &\leq s\varphi(E(x_n))\sum_{j=0}^{\infty}\left[\phi(E(x_0))\mu^{r^{n+1}-1}\right]^{P_j(r)} \qquad \text{(using Lemma 4.3.12)} \\ &\leq \frac{s\varphi(E(x_n))}{1-\phi(E(x_0))\mu^{r^{n+1}-1}}.\end{aligned}$$

□

Now we proceed to formulate the following fixed point theorems.

Theorem 4.3.14. Let $f : D \subset X \to X$ be an operator on a complete b-metric space (X, d) such that the b-metric is continuous and f satisfies (4.25) with a b-Bianchini Grandolfi gauge function φ of order $r \geq 1$ on an interval J with coefficient $s \geq 1$. Then starting from an initial orbital point x_0 of f the iterative sequence (4.26) remains in $\overline{B}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n), n = 0, 1, 2, \cdots$, where $\rho_n = s\sigma(d(x_n, x_{n+1})), \sigma$ defined in (4.31) and $s \geq 1$ is a coefficient of b-metric space. Furthermore, for each $n \geq 1$ we have

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})).$$

If $\xi \in D$ and the function $E(x) = d(x, fx)$ on D is f-orbitally lower semi-continuous at ξ then ξ is a fixed point of f.

Proof. Since $x_0 \in D$ is an initial orbital point of f then from Lemma 4.3.10 we have

$$\overline{B}(x_{n+1}, \rho_{n+1}) \subset \overline{B}(x_n, \rho_n) \quad \text{for all } n \geq 0.$$

So that $x_n \in \overline{B}(x_0, \rho_0)$ for all $n \geq 0$. According to the definition of ρ and using Lemma 4.3.12 we have

$$\begin{aligned}\rho_n = s\sigma(E(x_n)) &\leq s\sigma(\varphi^n(E(x_0))) \\ &= s\sum_{j=0}^{\infty} s^j\varphi^j(\varphi^n(E(x_0))) \\ &= \frac{1}{s^{n-1}}\sum_{j=n}^{\infty} s^j\varphi^j(\varphi^n(E(x_0))) \quad \text{for all } n \geq 0. \qquad (4.34)\end{aligned}$$

Since φ is a b-Bianchini Grandolfi gauge function then from (4.34) we obtain

$$\rho_n \to 0 \quad \text{as} \quad n \to \infty, \tag{4.35}$$

which implies that $\{\overline{B}(x_n, \rho_n)\}$ is a nested sequence of closed balls. By Cantor's theorem (for complete b-metric spaces), we deduce that there exists a unique point ξ such that $\xi \in \overline{B}(x_n, \rho_n)$ for all $n \geq 0$ and $x_n \to \xi$ or equivalently, $\lim_{n\to\infty} d(x_n, \xi) = 0$. From (d3) Definition 1.4.1 we have

$$d(\xi, fx_n) \leq s[d(\xi, x_n) + d(x_n, fx_n)] = s[d(\xi, x_n) + d(x_n, x_{n+1})].$$

Letting $n \to \infty$ and since the b-metric is continuous we obtain

$$\lim_{n\to\infty} d(\xi, fx_n) = 0.$$

If $\xi \in D$ and $E(x) = d(x, fx)$ is f-lower semi continuous at ξ then

$$d(\xi, f\xi) = E(\xi) \leq \lim_{n\to\infty} \inf E(x_n) = \lim_{n\to\infty} \inf d(x_n, x_{n+1}) = 0,$$

which infers $\xi = f\xi$. Furthermore, from Lemma 4.3.12 we obtain the following

$$\begin{aligned} d(x_n, x_{n+1}) = E(x_n) &\leq \varphi(E(x_{n-1})) \\ &= \varphi(d(x_{n-1}, x_n)). \end{aligned}$$

□

Remark 4.3.15. Theorem 4.3.14 gives a generalization of [85, Theorem 4.1] and extends it to the case of b-metric spaces. It reduces to [85, Theorem 4.1] when $s = 1$. Hence Theorem 4.3.14 not only extends the result of Proinov [85] but in turn it also includes results of Bianchini and Grandolfi [14] and Hicks [47] as special cases.

Corollary 4.3.16. ([85, Theorem 4.1]) Let (X, d) be a complete metric space and $f : D \subset X \to X$ be an operator satisfying

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \quad \text{for all } x \in D \text{ and } fx \in D \text{ with } d(x, fx) \in J, \tag{4.36}$$

where φ is a Bianchini Grandolfi gauge function on an interval J. Then starting from an initial point x_0 of f the iterative sequence $\{x_n\}$ remains in $\overline{B}(x_0, \rho_0)$ and converges to a point

ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n) : n = 0, 1, \cdots$ where $\rho_n = \sigma(d(x_n, fx_n))$ and $\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t)$. Moreover, if $\xi \in D$ and f is continuous at ξ then ξ is a fixed point of f.

Proof. Let $x_0 \in D$ be an initial point then it is an initial orbital point. It follows from Lemma 4.3.8 that $E(x_n) \in J$ for $n = 0, 1, 2, \cdots$. Thus from (4.36) we have

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \quad \text{for all } x \in \mathcal{O}(x_0) \text{ with } d(x, fx) \in J. \tag{4.37}$$

Thus Theorem 4.3.14 yields $x_n \to \xi \in X$. Also since the iterative sequence $\{x_n\} \in \mathcal{O}(x_0)$ and the mapping f is continuous at point ξ then $fx_n \to f\xi$. Thus

$$E(\xi) = d(\xi, f\xi) \leq \lim_{n\to\infty} s[d(\xi, x_{n+1}) + d(x_{n+1}, f\xi)] = 0 \leq \lim_{n\to\infty} \inf E(x_n).$$

This yiels f-orbital lower semi-continuity of $E(x) = d(x, fx)$ at point ξ. Hence the conclusion follows from Theorem 4.3.14. □

Theorem 4.3.17. Let $f : D \subset X \to X$ be an operator on a complete b-metric space (X, d) such that the b-metric is continuous and let f satisfies (4.25) with a b-Bianchini Grandolfi gauge function φ of order $r \geq 1$ and coefficient s on an interval J. Further, suppose that $x_0 \in D$ is an initial orbital point of f then following statements hold true.

1. The iterative sequence (4.26) remains in $\overline{B}(x_0, \rho_0)$ and converges with rate of convergence at least $r \geq 1$ to a point ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n), n = 0, 1, \cdots$, and

$$\begin{aligned}\rho_n &= sd(x_n, x_{n+1}) \sum_{j=0}^{\infty} \left[\phi(d(x_n, x_{n+1}))\right] \\ &\leq \frac{sd(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))}\end{aligned} \tag{4.38}$$

   where ϕ is nonnegative and nondecreasing function on J satisfying (4.27) and (4.28).

2. For all $n \geq 0$ the following priori estimate holds:

$$\begin{aligned}d(x_n, \xi) &\leq \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{\infty} \phi(E(x_0))^{P_j(r)} \\ &= d(x_0, fx_0) \frac{\phi(E(x_0))^{P_n(r)}}{s^{n-1}[1 - \phi(E(x_0))^{r^n}]}.\end{aligned} \tag{4.39}$$

3. For all $n \geq 1$ the following posteriori estimate holds:

$$\begin{aligned} d(x_n, \xi) &\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} \big[\phi(\varphi(d(x_n, x_{n-1})))\big]^{P_j(r)} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(d(x_n, x_{n-1}))\big[\frac{\phi(d(x_n, x_{n-1}))}{s}\big]^{r-1}}. \end{aligned} \tag{4.40}$$

4. For all $n \geq 1$, we have

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1}) \leq \mu^{P_n(r)} d(x_0, fx_0). \tag{4.41}$$

5. If $\xi \in D$ and the function $G(x) = d(x, fx)$ on D is f-orbitally lower semi-continuous at ξ then ξ is a fixed point of f.

Proof. 1. From Theorem 4.3.14 it follows that iterative sequence (4.26) converges to $\xi \in X$ and further $\xi \in \overline{B}(x_n, \rho_n)$ for all $n = 0, 1, 2, \cdots$. Moreover, estimate (1) of Lemma 4.3.13 implies

$$\rho_n \leq sd(x_n, x_{n+1}) \sum_{j=0}^{\infty} \big[\phi(d(x_n, x_{n+1}))\big]^{P_j(r)} \leq \frac{sd(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))}$$

2. For $m > n$

$$\begin{aligned} d(x_n, x_m) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n-1} d(x_{m-2}, x_{m-1}) + s^{m-n} d(x_{m-1}, x_m) \\ &= \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j E(x_j) \\ &\leq \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j \varphi^j (E(x_0)) \quad \text{(from Lemma 4.3.12 } E(x_n) \leq \varphi^n(E(x_0))) \\ &\leq \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j E(x_0) \Big[\frac{\phi(E(x_0))}{s}\Big]^{P_j(r)} \quad \text{(using Lemma 4.3.5)} \\ &\leq \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{m-1} \lambda^{P_j(r)}, \end{aligned}$$

where $\lambda = \phi(E(x_0))$. Keeping n fixed and letting $m \to \infty$ we get

$$d(x_n, \xi) \leq \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)} = \frac{d(x_0, fx_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)}. \tag{4.42}$$

Since,

$$r^n + r^{n+1} \geq 2r^n,\ r^n + r^{n+1} + r^{n+2} \geq 3r^n, \cdots,$$

then,

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n},\ \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n}, \cdots,$$

which gives

$$\begin{aligned}
\sum_{j=n}^{\infty} \lambda^{P_j(r)} &= \lambda^{P_j(r)} + \lambda^{P_{j+1}(r)} + \cdots \\
&= \lambda^{P_n(r)} \big[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + \cdots\big] \\
&\leq \lambda^{P_j(r)} \big[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + \cdots\big] = \frac{\lambda^{P_n(r)}}{1 - \lambda^{r^n}}. \qquad (4.43)
\end{aligned}$$

Hence from (4.42) we obtain

$$d(x_n, \xi) \leq \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{\infty} \phi(E(x_0))^{P_j(r)} = d(x_0, fx_0) \frac{\phi(E(x_0))^{P_n(r)}}{s^{n-1}[1 - \phi(E(x_0))^{r^n}]}$$

3. From (4.42) we have for $n \geq 0$

$$d(x_n, \xi) \leq \frac{d(x_0, x_1)}{s^{n-1}} \sum_{j=n}^{\infty} \big[\phi(d(x_0, x_1))\big]^{P_j(r)}.$$

Setting $n = 0$, $y_0 = x_0$ and $y_1 = x_1$ we have

$$d(y_0, \xi) \leq sd(y_0, y_1) \sum_{j=0}^{\infty} \big[\phi(d(y_0, y_1))\big]^{P_j(r)}.$$

Setting again $y_0 = x_n$ and $y_1 = x_{n+1}$ gives

$$\begin{aligned}
d(x_n, \xi) &\leq sd(x_n, x_{n+1}) \sum_{j=0}^{\infty} \big[\phi(d(x_n, x_{n+1}))\big]^{P_j(r)} \\
&\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} \big[\phi(\varphi(d(x_n, x_{n-1})))\big]^{P_j(r)} \\
&\leq s\varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} \big[\phi(\varphi(d(x_n, x_{n-1})))\big]^{j} \\
&= \frac{s\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))}. \qquad (4.44)
\end{aligned}$$

From Lemma 4.3.5(2) we obtain

$$\phi(\varphi(d(x_n, x_{n-1}))) \leq s\big[\frac{\phi(d(x_n, x_{n-1}))}{s}\big]^r = \phi(d(x_n, x_{n-1}))\big[\frac{\phi(d(x_n, x_{n-1}))}{s}\big]^{r-1} \tag{4.45}$$

which implies

$$\frac{1}{1-\phi(\varphi(d(x_n, x_{n-1})))} \leq \frac{1}{1-\phi(d(x_n, x_{n-1}))\big[\frac{\phi(d(x_n,x_{n-1}))}{s}\big]^{r-1}}. \tag{4.46}$$

Thus from (4.44) and (4.46) we deduce for $n \geq 1$,

$$\begin{aligned} d(x_n, \xi) &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1-\phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{s\varphi(d(x_n, x_{n-1}))}{1-\phi(d(x_n, x_{n-1}))\big[\frac{\phi(d(x_n,x_{n-1}))}{s}\big]^{r-1}}. \end{aligned}$$

4.

$$\begin{aligned} d(x_{n+1}, x_n) = E(x_n) &\leq \varphi(E(x_{n-1})) \\ &= E(x_{n-1})\frac{\phi(E(x_{n-1}))}{s} \\ &\leq E(x_0)\mu^{P_{n-1}(r)}\mu^{r^{n-1}} \quad \text{using Lemma (4.3.3)} \\ &= E(x_0)\mu^{P_{n-1}(r)+r^{n-1}} \\ &= E(x_0)\mu^{P_n(r)} = \mu^{P_n(r)}d(x_0, fx_0). \end{aligned}$$

5. Its proof runs on the same lines as the proof of Theorem 4.3.14. □

Remark 4.3.18. For $s = 1$ Theorem 4.3.17 reduces to [85, Theorem 4.2]. It also generalizes (taking $s = 1$ and $\varphi(t) = \lambda t,\ 0 < \lambda < 1$) results of Rheinboldt [92, 12.3.2], Kornstaedt [65, Satz 4.1], Hicks and Rhoades [48] and Park [80, Theorem 2]. First two conclusions of Theorem 4.3.17 are due to Gel'man [38, Theorem3] (taking $s = 1$ and $\varphi(t) = ct^r, c \geq 0, r \geq 1$). It also yields some results of Hicks [47, Theorem 3].

Corollary 4.3.19. Let $f : X \to X$ be an operator on a complete b-metric space (X, d) such that the b-metric is continuous. Further, assume that f satisfies:

$$d(fx, fy) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in X \text{ with } d(x, y) \in J, \tag{4.47}$$

where φ is a b-Bianchini Grandolfi gauge function of order $r \geq 1$ on an interval J and with coefficient $s \geq 1$. Assume that $x_0 \in X$ is such that $d(x_0, fx_0) \in J$. Then the following statements hold.

(i) The iterative sequence (4.26) converges to a fixed point ξ of f.

(ii) The operator f has a unique fixed point in $S = \{x \in X : d(x, \xi) \in J\}$.

(iii) The estimates (4.38)-(4.41) are valid.

Proof. From (4.47) we have

$$d(fx, fy) \leq \varphi(d(x, y)) < d(x, y),$$

which gives the continuity of f in b-metric space (X, d). Thus conclusions (i) and (iii) follow immediately from Theorem 4.3.17. Let ξ' is another fixed point of f in S then $d(\xi, \xi') \in J$. It follows from (4.47) that $d(\xi, \xi') \leq \varphi(d(\xi, \xi'))$ which yields $\xi' = \xi$. □

Remark 4.3.20. For $s = 1$ when b-metric space under consideration is a simple metric space then above corollary coincides with [85, Corollary 4.4]. Thus conclusions of Corollary 4.3.19 are consequences of results of Matkowski [66].

4.3.3 Application and illustrative example

The following example elucidates the degree of generality of our results.

Example 4.3.21. Let $X := \{x_1, x_2, x_3\}$. Define a function $d : X \times X \to \mathbb{R}^+$ as

$$d(x_1, x_2) = \frac{1}{k^2}, \quad d(x_2, x_3) = \frac{1}{k-1}, \quad d(x_1, x_3) = \frac{1}{k}, \quad d(x_i, x_j) = d(x_j, x_i)$$

$$\text{and } d(x_i, x_i) = 0 \text{ for all } i, j = 1, 2, 3,$$

where $k \geq 3$ is any positive integer. It is an easy exercise to see that d is a b-metric with coefficient $s \geq \frac{k^2}{k^2-1} > 1$. Define $f : X \to X$ as

$$fx_1 = x_1, \qquad fx_2 = x_1, \qquad fx_3 = x_2.$$

Setting $\varphi(t) = t^2$ on $J = [0, \frac{1}{k-1}]$ then φ is a b-Bianchini Grandolfi gauge function with coefficient $\frac{k^2}{(k^2-1)}$ having order 2. Moreover, it is easily seen that all conditions of Theorem 4.3.14 are satisfied.

On the other hand assume that x_1, x_2, x_3 are real numbers and the set $\{x_1, x_2, x_3\}$ is endowed with the Euclidean metric d_e. For each gauge function φ defined on some interval $[0,h)$ one can find $\frac{1}{n_0} \in [0,h)$ for some $n_0 \in \mathbb{N}$. In such case, identifying as $x_1 = \frac{1}{n_0}, x_2 = \frac{2}{n_0}, x_3 = \frac{3}{n_0}$. Assume f as defined above then with respect to Euclidean metric d_e we have,

$$d_e(f\frac{2}{n_0}, f\frac{3}{n_0}) = d_e(\frac{1}{n_0}, \frac{2}{n_0}) = \frac{1}{n_0} \leq \varphi(d_e(\frac{2}{n_0}, \frac{3}{n_0})) = \varphi(\frac{1}{n_0})$$

which contradicts the definition of φ. Hence, one can not invoke main results of Proinov [85, Theorem 4.1, 4.2, Cororraly 4.4].

Theorem 4.3.22. Consider the following initial value problem,

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{4.48}$$

Assume that:

(*i*) f is continuous;

(*ii*) f satisfies the condition

$$|f(t,x) - f(t,y)| \leq k|x(t) - y(t)|^r \text{ for } (t,x),(t,y) \in R; \tag{4.49}$$

(*iii*) f is bounded on R i,e.,

$$|f(t,x)| \leq \frac{k^r}{2}; \tag{4.50}$$

where $R = \{(t,x) : |t - t_0| \leq (\frac{1}{k})^{r-1},\ |x - x_0| \leq \frac{k}{2}\}$, $r \geq 2$ and $0 < k < 1$.
Then the initial value problem (4.48) has a unique solution on the interval $I = [t_0 - (\frac{1}{k})^{r-1}, t_0 + (\frac{1}{k})^{r-1}]$.

Proof. Let $C(I)$ be the space of all continuous real valued functions on I where $I = [t_0 - (\frac{1}{k})^{r-1}, t_0 + (\frac{1}{k})^{r-1}]$ with the usual supremum metric, i.e.,

$$d(x,y) = \max_{t \in I} |x(t) - y(t)|.$$

Integrating 4.48 gives

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) d\tau. \tag{4.51}$$

Indeed finding the solution of initial value problem (4.48) is equivalent of finding the fixed point of self mapping $T : X \to X$ defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau, \tag{4.52}$$

where $X = \{x \in C(I) : |x(t) - x_0| \leq \frac{k}{2}; k > 0\}$ then X is closed subspace of $C(I)$. We see that if $\tau \in I$ then $|\tau - t_0| \leq (\frac{1}{k})^{r-1}$ and $x \in X$ gives $|x(\tau) - x_0| \leq \frac{k}{2}$. Thus $(\tau, x(\tau)) \in R$ and since f is continuous on R therefore the integral in (4.52) exists and T is defined for each $x \in X$. To see that T maps X to itself. We use (4.52) to write

$$\begin{aligned}
|Tx(t) - x_0| &= |\int_{t_0}^{t} f(\tau, x(\tau))d\tau| \\
&\leq \int_{t_0}^{t} |f(\tau, x(\tau))|d\tau \\
&\leq \frac{k^r}{2}|t - t_0| \quad \text{(using (4.50))} \\
&\leq \frac{k^r}{2}(\frac{1}{k})^{r-1} = \frac{k}{2}.
\end{aligned}$$

Now by using (4.49) we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_{t_0}^{t} |f(\tau, x(\tau)) - f(\tau, y(\tau))| \\
&\leq k\int_{t_0}^{t} |x(\tau) - y(\tau)|^r d\tau \\
&\leq k\big(\max_{\tau\in I}|x(\tau) - y(\tau)|\big)^r|t - t_0| \\
&\leq k\big(\max_{\tau\in I}|x(\tau) - y(\tau)|\big)^r(\frac{1}{k})^{r-1} \\
&= (\frac{1}{k})^{r-2}\big(\max_{\tau\in I}|x(\tau) - y(\tau)|\big)^r,
\end{aligned} \tag{4.53}$$

Thus (4.53) implies

$$d(Tx, Ty) \leq (\frac{1}{k})^{r-2}(d(x, y))^r. \tag{4.54}$$

We take $J = [0, k]$. So that it suffices to take $\varphi(u) = (\frac{1}{k})^{r-2}u^r$ for $u \in [0, k]; k < 1$ then φ is a gauge function of order $r \geq 2$. Also, for $u \in J - \{0\}$ we have

$$\varphi(u) = (\frac{1}{k})^{r-2}u^r = (\frac{1}{k})^{r-2}u^2u^{r-2} \leq (\frac{1}{k})^{r-2}u^2k^{r-2} = u^2 < u. \tag{4.55}$$

Thus from (4.54) we obtain $d(Tx, Ty) \leq \varphi(d(x, y))$ for all $x, y \in X$ and $d(x, y) \in J$. Further, for any $x \in X$ it is easily seen that $d(x, x_0) \leq \frac{k}{2}$ which yields $d(x, y) \leq k$ for $x, y \in X$.

Therefore, all the conditions of Corollary 4.3.19 are satisfied. Hence the iterative sequence $x_n = Tx_{n-1};\ n = 1, 2, \cdots$ converges to the unique fixed point of T at a rate of convergence $r \geq 2$. On the other hand Picard iterations converges to the solution linearly. □

4.4 Conclusion

1. Section 4.2 is devoted to some fixed point theorems in b-metric space endowed with a graph G. In this context we defined the notions of b-(φ, G) contractions and Hardy and Rogers type G-contractions and established fixed point theorems for such classed (see, Theorem 4.2.14 & 4.2.25). Consequently, our results evoke the notions of cyclic contractions in b-metric space which substantiates the validity of results (see, Theorem 4.2.32 & 4.2.34). Moreover, b-(φ, G) contractions subsumes the so called notion of $\alpha - \varphi$ contractions due to Samet et al. [100] (see, Theorem 4.2.39).

2. In Section 4.3 we have established two convergence theorems in the setting of a b-metric such that the self-mapping satisfies a contraction condition involving a gauge function of order $r \geq 1$. The gauge function φ has to satisfy the condition $\sum_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ where $s \geq 1$ is the coefficient of underlying b-metric space. An example (Example 4.3.21) has been furnished to assess the degree of generality of our results. As an application we established an existence theorem for the solution of an initial value problem which not only gives the unique solution but also locates the domain for solution.

Chapter 5

Fixed point theorems in probabilistic metric spaces

Because of its comprehensive and panoptic aspect the Banach contraction principle has been extended in many different directions for single and multi-valued mappings not only in metric space but in probabilistic metric space as well. As we have mentioned earlier that Nieto and Rodríguez-López [73], Ran and Reurings [90], Petruşel and Rus [84] established some elegant results for contractions in partially ordered metric spaces. Afterwards Jachymski [52] established a nice generalization by utilizing graph theoretic approach.

Motivated by the work of Jachymski one can pose a very natural question: Is it possible to establish a probabilistic version of the main result of Jachymski [52] (see, Corollary 5.2.12)? In this chapter we give an affirmative answer to this question [56]. Our results are substantial generalizations and improvements of corresponding the results of Jachymski [52] and Sehgal [104] and others (see, e.g., [72, 73, 90]). Subsequently, we apply our main results to the setting of cyclical contractions and to that of (ϵ, δ)-contractions as well.

5.1 Preliminaries

In the following we recall some basic notions which can be found in [67, 104, 107].

Definition 5.1.1. A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (briefly, t-norm) if the following conditions hold:

(i) Δ is associative and commutative,

(ii) $\Delta(a,1) = a$ for all $a \in [0,1]$,

(iii) $\Delta(a,b) \leq \Delta(c,d)$ for all $a,b,c,d \in [0,1]$ with $a \leq c$ and $b \leq d$.

Typical examples of t-norms are $\Delta_M(a,b) = \min\{a,b\}$ and $\Delta_P(a,b) = ab$.

Definition 5.1.2. (Hadzić [43], Hadzić and Pap [44]) A t-norm Δ is said to be of $\mathscr{H}$-type if the family of functions $\{\Delta^n(t)\}_{n\in\mathbb{N}}$ is equicontinuous at $t = 1$, where $\Delta^n : [0,1] \to [0,1]$ is recursively defined by:

$$\Delta^1(t) = \Delta(t,t), \text{ and } \Delta^n(t) = \Delta(\Delta^{n-1}(t), t); \quad t \in [0,1], n = 2, 3, \cdots .$$

A trivial example of a t-norm of $\mathscr{H}$-type is $\Delta_M := \min$, but there exists t-norms of $\mathscr{H}$-type with $\Delta \neq \Delta_M$ (see, e.g.,[43]).

Definition 5.1.3. A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X, \mathscr{F}, \Delta)$ where $(X, \mathscr{F})$ is a PM-space, Δ is a t-norm and instead of (PM3) in Definition 1.5.2 it satisfies the following triangle inequality:

(PM3)$'$ $F_{x,z}(t+s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$,

for all $x, y, z \in X$ and $t, s \geq 0$.

Remark 5.1.4. (Sehgal [104]) Let (X, d) be a metric space. Define $F_{xy}(t) = \epsilon_0(t - d(x,y))$ for all $x, y \in X$ and $t > 0$. Then the triple $(X, \mathscr{F}, \Delta_M)$ is a Menger PM-space induced by the metric d. Furthermore, $(X, \mathscr{F}, \Delta_M)$ is complete if and only if d is complete.

Schweizer et al. [108] introduced the concept of neighborhood in PM-spaces. For $\varepsilon > 0$ and $\delta \in (0,1]$ the (ε, δ)-neighborhood of $x \in X$ is denoted by $\mathcal{N}_x(\varepsilon, \delta)$ and is defined by

$$\mathcal{N}_x(\varepsilon, \delta) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \delta\}.$$

Furthermore, if $(X, \mathscr{F}, \Delta)$ is a Menger PM-space with $\sup_{0<a<1} \Delta(a,a) = 1$ then the family of neighborhoods $\{\mathcal{N}_x(\varepsilon, \delta) : x \in X, \varepsilon > 0, \delta \in (0,1]\}$ determines a Hausdorff topology for X.

Definition 5.1.5. Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space.

(1) A sequence $\{x_n\}$ in X converges to an element x in X (we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) if for every $\varepsilon > 0$ and $\delta > 0$ there exists a natural number $N(\varepsilon, \delta)$ such that $F_{x_n,x}(\varepsilon) > 1 - \delta$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is Cauchy sequence if, for every $\varepsilon > 0$ and $\delta > 0$ there exists a natural number $N(\varepsilon, \delta)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \delta$, whenever $n, m \geq N$.

(3) A Menger PM-space is complete if and only if every Cauchy sequence in X converges to a point in X.

Theorem 5.1.6. (Sehgal [104]) Let $(X, \mathscr{F}, \Delta)$ be a complete Menger PM-space where Δ is a continuous t-norm satisfying $\Delta(x, x) > x$ for each $x \in [0, 1]$. Let $f : X \to X$ and there exists $\kappa; 0 < \kappa < 1$ such that f satisfies the following contraction condition

$$F_{fx,fy}(\kappa t) \geq F_{x,y}(t), \quad (t > 0) \text{ for all } x, y \in X. \tag{5.1}$$

Then there exists a unique fixed point $\xi \in X$. Furthermore, $f^n y \to \xi$ for every $y \in X$.

Sherwood [109] established the following fixed point theorem for the class of t-norms of $\mathscr{H}$-type.

Theorem 5.1.7. Let $(X, \mathscr{F}, \Delta)$ be a complete Menger PM-space where Δ is a t-norm of $\mathscr{H}$-type. Let $f : X \to X$ and there exists $\kappa; 0 < \kappa < 1$ such that f satisfies the following contraction condition

$$F_{fx,fy}(\kappa t) \geq F_{x,y}(t), \quad (t > 0) \text{ for all } x, y \in X. \tag{5.2}$$

Then there exists a unique fixed point $\xi \in X$. Furthermore, $f^n y \to \xi$ for every $y \in X$.

Now we attribute to some basic notations from graph theory which are needed subsequently. Let (X, d) be a metric space, Ω be the diagonal of the Cartesian product $X \times X$, G be a directed graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Omega$. Assume that G has no parallel edges. For each $(x, y) \in E(G)$ we attribute a unique distance distribution function $F_{x,y}$.

5.2 Probabilistic G-contractions

We start with the following instinctive definition.

Definition 5.2.1. Let $(X, \mathscr{F}, \Delta)$ be a PM-space. A mapping $f : X \to X$ is said to be a probabilistic G-contraction if there exists $\kappa \in (0, 1)$ such that the following condition holds for all $x, y \in X$:

$$(fx, fy) \in E(G) \quad \text{whenever} \quad (x, y) \in E(G), \tag{5.3}$$

$$(x, y) \in E(G) \text{ implies } F_{fx,fy}(\kappa t) \geq F_{x,y}(t) \ (t > 0). \tag{5.4}$$

Example 5.2.2. Let (X, d) be a metric space endowed with a graph G and the mapping $f : X \to X$ be a Banach G-contraction. Then the induced Menger PM space $(X, \mathscr{F}, \Delta_M)$ is a probabilistic G-contraction.

To see this, let $(x, y) \in E(G)$ then $(fx, fy) \in E(G)$ as f is a Banach G-contraction. Further, there exists $\kappa \in (0, 1)$ such that for $x, y \in X$ with $(x, y) \in E(G)$ we have

$$d(fx, fy) \leq \kappa d(x, y). \tag{5.5}$$

Now for $t > 0$ we have

$$\begin{aligned} F_{fx,fy}(\kappa t) &= \epsilon_0(\kappa t - d(fx, fy)) \\ &\geq \epsilon_0(\kappa t - \kappa d(x, y)) = F_{x,y}(t). \end{aligned}$$

Thus f satisfies (5.4).

From Example 5.2.2 it infers that every Banach G-contraction is a probabilistic G-contraction with the same contraction constant.

5.2.1 Fixed point theorems for probabilistic G-contractions

We start with the following proposition.

Proposition 5.2.3. Let $f : X \to X$ be a probabilistic G-contraction with a contraction constant $\kappa \in (0, 1)$. Then

(i) f is a probabilistic $\widetilde{G}-$contraction and a probabilistic $G^{-1}-$contraction with the same contraction constant κ.

(ii) $[x_0]_{\widetilde{G}}$ is $f-$invariant and $f|_{[x_0]_{\widetilde{G}}}$ is probabilistic $\widetilde{G}_{x_0}-$contraction provided that $x_0 \in X$ is such that $fx_0 \in [x_0]_{\widetilde{G}}$.

Proof. (i) It follows from symmetry of $F_{x,y}$.
(ii) Let $x \in [x_0]_{\tilde{G}}$. Then there is a path $x = z_0, z_1, \cdots, z_l = x_0$ between x and x_0. Since f is a probabilistic $G-$contraction then $(fz_{i-1}, fz_i) \in E(G) \forall i = 1, 2, \cdots, l$. Thus $fx \in [fx_0]_{\tilde{G}} = [x_0]_{\tilde{G}}$.
Suppose $(x, y) \in E(\tilde{G}_{x_0})$. Then $(fx, fy) \in E(G)$ as f is a probabilistic $G-$contraction. But $[x_0]_{\tilde{G}}$ is f invariant, so we conclude that $(fx, fy) \in E(\tilde{G}_{x_0})$. The condition (5.4) is satisfied automatically, since $\tilde{G}_{x_0}$ is a subgraph of G.

□

Lemma 5.2.4. Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space under a $t-$norm Δ satisfying $\sup_{a<1} \Delta(a, a) = 1$. Assume that the mapping $f : X \to X$ is a probabilistic G-contraction. Let $y \in [x]_{\widetilde{G}}$ then $F_{f^n x, f^n y}(t) \to 1$ as $n \to \infty$ $(t > 0)$. Moreover, $f^n x \to z \in X$ $(n \to \infty)$ if and only if $f^n y \to z$ $(n \to \infty)$.

Proof. Let $x \in X$ and $y \in [x]_{\widetilde{G}}$ then there exists a path $(x_i); i = 0, 1, 2, \cdots, l$ in $\widetilde{G}$ from x to y with $x_0 = x$, $x_l = y$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$. From Proposition 5.2.3 it infers that f is a probabilistic $\widetilde{G}-$contraction. By induction for $t > 0$ we have, $(f^n x_{i-1}, f^n x_i) \in E(\widetilde{G})$ and $F_{f^n x_{i-1}, f^n x_i}(\kappa t) \geq F_{f^{n-1} x_{i-1}, f^{n-1} y_i}(t)$ for all $n \in \mathbb{N}$ and $i = 1, \cdots, l$. Thus we obtain

$$\begin{aligned} F_{f^n x_{i-1}, f^n x_i}(t) &\geq F_{f^{n-1} x_{i-1}, f^{n-1} x_i}(\frac{t}{\kappa}) \\ &\geq F_{f^{n-2} x_{i-1}, f^{n-2} x_i}(\frac{t}{\kappa^2}) \\ &\cdots \\ &\geq F_{x_{i-1}, x_i}(\frac{t}{\kappa^n}) \to 1 \quad (\text{as } n \to \infty). \end{aligned}$$

Let $t > 0$ and $\delta > 0$ be given and since $\sup_{a<1} \Delta(a, a) = 1$ then there exists $\lambda(\delta) \in (0, 1)$ such that $\Delta(1 - \lambda, 1 - \lambda) > 1 - \delta$. Choose a natural number n' such that for all $n \geq n'$ we have,

$F_{f^nx_0,f^nx_1}(\frac{t}{2}) > 1-\lambda$ and $F_{f^nx_1,f^nx_2}(\frac{t}{2}) > 1-\lambda$. We get for all $n \geq n'$

$$\begin{aligned} F_{f^nx_0,f^nx_2}(t) &\geq \Delta(F_{f^nx_0,f^nx_1}(\frac{t}{2}), F_{f^nx_1,f^nx_2}(\frac{t}{2})) \\ &\geq \Delta(1-\lambda, 1-\lambda) > 1-\delta, \end{aligned}$$

so that $F_{f^nx_0,f^nx_2}(t) \to 1$ as $n \to \infty, (t > 0)$. Continuing recursively one can easily show that

$$F_{f^nx_0,f^nx_l}(t) \to 1 \ \text{ as } \ n \to \infty, \quad (t > 0).$$

Let $f^nx \to z \in X$. Let $t > 0$ and $\delta > 0$ be given, since $\sup_{a<1} \Delta(a,a) = 1$ then there exists $\lambda(\delta) \in (0,1)$ such that $\Delta(1-\lambda, 1-\lambda) > 1-\delta$. Choose a natural number n_0 such that for all $n \geq n_0$ we have, $F_{f^nx,f^ny}(\frac{t}{2}) > 1-\lambda$ and $F_{z,f^nx}(\frac{t}{2}) > 1-\lambda$. So that for all $n \geq n_0$, we have,

$$\begin{aligned} F_{z,f^ny}(t) &\geq \Delta(F_{z,f^nx}(\frac{t}{2}), F_{f^nx,f^ny}(\frac{t}{2})) \\ &\geq \Delta(1-\lambda, 1-\lambda) > 1-\delta. \end{aligned}$$

Hence, $f^ny \to z$ as $n \to \infty$. □

Every t-norm can be extended in a unique way to an n-array as: $\Delta_{i=1}^0 x_i = 0$, $\Delta_{i=1}^n x_i = \Delta(\Delta_{i=1}^{n-1} x_i, x_n)$ for $n = 1, 2, \cdots$. Let $(x_i)_{i=1}^l$ be a path between two vertices x and y in a graph G. Let us denote with $L_{x,y}(t) = \Delta_{i=1}^l F_{x_{i-1},x_i}(t)$ for all t. Clearly the function $L_{x,y}$ is monotone nondecreasing.

In the following we extend the notions of $(\mathcal{C}_f)$ and $(\mathcal{H}_f)$ graphs to probabilistic metric spaces.

Definition 5.2.5. Let $(X, \mathscr{F}, \Delta)$ be a PM-space and $f : X \to X$. Suppose there exists a sequence $\{f^nx\}$ in X such that $f^nx \longrightarrow x^*$ and $(f^nx, f^{n+1}x) \in E(G)$ for $n \in \mathbb{N}$. We say that:

(*i*) G is a $(\mathcal{C}_f)$-graph in X if there exists a subsequence $\{f^{n_k}x\}$ of $\{f^nx\}$ and a natural number N such that $(f^{n_k}x, x^*) \in E(G)$ for $k \geq N$;

(*ii*) G is an $(\mathcal{H}_f)$-graph in X if $f^nx \in [x^*]_{\widetilde{G}}$ for $n \geq 1$ and the sequence of functions $\{L_{f^nx,x^*}(t)\}$ converges to $\epsilon_0(t)$ uniformly as $n \to \infty$ $(t > 0)$.

Example 5.2.6. Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space induced by the metric $d(x,y) = |x-y|$ on $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}$ and I be an identity map on X.

Consider the graph G_2 consisting of $V(G_2) = X$ and

$E(G_2) = \{(\frac{1}{n}, \frac{1}{n+1}), (\frac{1}{n+1}, n), (n, 0), (\frac{1}{5n}, 0); n \in \mathbb{N}\}$.

We note that $x_n = \frac{1}{n} \to 0$ as $n \to \infty$. Also it is easy to see that G_2 is a $(\mathcal{C}_I)$-graph. But since, $\Delta(a, b) = \min\{a, b\}$ then

$$\begin{aligned} L_{x_n,0}(t) &= \Delta(\Delta(\epsilon_0(t - |\frac{1}{n} - \frac{1}{n+1}|), \epsilon_0(t - |\frac{1}{n+1} - n|)), \epsilon_0(t - n)) \\ &= \epsilon_0(t - n) \nrightarrow \epsilon_0(t) \qquad \text{as } n \to \infty. \end{aligned}$$

Thus G_2 is not an $(\mathcal{H}_I)$-graph.

Example 5.2.7. Let $(X, \mathscr{F}, \Delta)$ be a Menger PM space induced by the metric $d(x, y) = |x-y|$ on $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{\frac{\sqrt{5}}{n+1} : n \in \mathbb{N}\} \cup \{0\}$ and I be an identity map on X. Consider the graph G_3 consisting of $V(G_3) = X$ and

$E(G_3) = \{(\frac{1}{n}, \frac{1}{n+1}), (\frac{1}{n}, \frac{\sqrt{5}}{n+1}), (\frac{\sqrt{5}}{n+1}, 0); n \in \mathbb{N}\}$.

Since, $x_n = \frac{1}{n} \to 0$ as $n \to \infty$. Clearly G_4 is not a $(\mathcal{C}_I)$-graph. But,

$$L_{x_n,0}(t) = \epsilon_0(t - \frac{\sqrt{5}}{n+1}) \to \epsilon_0(t) \text{ as } n \to \infty (t > 0).$$

Thus G_3 is an $(\mathcal{H}_I)$-graph.

From above examples we note that the notions of $(\mathcal{C}_f)$-graph and $(\mathcal{H}_f)$-graph are independent even if f is an identity map.

Following lemma is essential to prove our fixed point results.

Lemma 5.2.8. (Miheţ [68]) Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space under a t-norm Δ of $\mathscr{H}$-type. Let $\{x_n\}$ be a sequence in X and there exists $\kappa \in (0, 1)$ such that:

$$F_{x_n,x_{n+1}}(\kappa t) \geq F_{x_{n-1},x_n}(t) \quad \text{for all } n \in \mathbb{N}, t > 0.$$

Then $\{x_n\}$ is a Cauchy sequence.

Theorem 5.2.9. Let $(X, \mathscr{F}, \Delta)$ be a complete Menger PM-space under a t-norm Δ of $\mathscr{H}$-type. Assume that the mapping $f : X \to X$ is a probabilistic G-contraction and there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$ then the following assertions hold.

(*i*). If G is a $(\mathcal{C}_f)$-graph then f has a unique fixed point $\xi \in [x_0]_{\widetilde{G}}$ and for any $y \in [x_0]_{\widetilde{G}}$, $f^n y \to \xi$. Moreover, if G is weakly connected then f is a Picard operator .

(*ii*). If G is weakly connected $(\mathcal{H}_f)$-graph then f is a Picard operator .

Proof. Since f is a probabilistic G-contraction and there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. By induction $(f^n x_0, f^{n+1} x_0) \in E(G)$ for all $n \geq 1$ and for $t > 0$,

$$F_{f^n x_0, f^{n+1} x_0}(\kappa t) \geq F_{f^{n-1} x_0, f^n x_0}(t) \quad \text{for all } n \geq 1. \tag{5.6}$$

(*i*). Since, t-norm Δ is $\mathscr{H}$-type then from Lemma 5.2.8 it infers that $\{f^n x_0\}$ is a Cauchy sequence in X. From completeness of Menger PM-space X there exists $\xi \in X$ such that

$$\lim_{n \to \infty} f^n x_0 = \xi. \tag{5.7}$$

Now we prove that ξ is a fixed point of f. Let G be a $(\mathcal{C}_f)$-graph. Then there exists a subsequence $\{f^{n_k} x_0\}$ of $\{f^n x_0\}$ and $N \in \mathbb{N}$ such that $(f^{n_k} x_0, \xi) \in E(G)$ for all $k \geq N$. Note that $(x_0, fx_0, f^2 x_0, ..., f^{n_1} x_0, \cdots, f^{n_N} x_0, \xi)$ is a path in G and so in $\widetilde{G}$ from x_0 to ξ, thus $\xi \in [x_0]_{\widetilde{G}}$. Since f is probabilistic G-contraction and $(f^{n_k} x_0, \xi) \in E(G)$ for all $k \geq N$. For $t > 0$ and $k \geq N$, we get

$$\begin{aligned} F_{f^{n_k+1} x_0, f\xi}(t) &\geq F_{f^{n_k+1} x_0, f\xi}(\kappa t) \\ &\geq F_{f^{n_k} x_0, \xi}(t) \to 1 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Thus we obtain,

$$\lim_{k \to \infty} f^{n_k+1} x_0 = f\xi. \tag{5.8}$$

Hence, we conclude that $f\xi = \xi$. Now let $y \in [x_0]_{\widetilde{G}}$ then from Lemma 5.2.4 we get

$$\lim_{n \to \infty} f^n y = \xi. \tag{5.9}$$

Next we prove the uniqueness of fixed point. Suppose $\xi^* \in [x_0]_{\widetilde{G}} = [\xi]_{\widetilde{G}}$ such that $f\xi^* = \xi^*$, then from Lemma 5.2.4 for $t > 0$, we have,

$$F_{\xi, \xi^*}(t) = F_{f^n \xi, f^n \xi^*}(t) \to 1 \quad n \to \infty. \tag{5.10}$$

Hence, $\xi^* = \xi$. Moreover, if G is weakly connected then f is Picard operator as $[x_0]_{\widetilde{G}} := X$.

(ii). Let G be a weakly connected $(\mathcal{H}_f)$-graph. By using the same arguments as in the first part of proof we obtain $\lim_{n\to\infty} f^n x_0 = \xi$. Since G is weakly connected for each $n \in \mathbb{N}$ there exist a finite $M_n \in \mathbb{N}$ such that $(z_i^n); i = 0, \cdots, M_n$ be a path in $\widetilde{G}$ from $f^n x_0$ to ξ with $z_0^n = f^n x_0$, $z_{M_n}^n = \xi$ and $(z_{i-1}^n, z_i^n) \in E(\widetilde{G})$.

$$\begin{aligned}
F_{\xi,f\xi}(t) &\geq F_{\xi,f\xi}(\kappa t)\\
&\geq \Delta(F_{\xi,f^{n+1}x_0}(\frac{\kappa t}{2}), F_{f^{n+1}x_0,f\xi}(\frac{\kappa t}{2}))\\
&\geq \Delta(F_{\xi,f^{n+1}x_0}(\frac{\kappa t}{2}), \Delta_{i=1}^{M_n} F_{fz_{i-1}^n,fz_i^n}(\frac{\kappa t}{2M_n}))\\
&\geq \Delta(F_{\xi,f^{n+1}x_0}(\frac{\kappa t}{2}), \Delta_{i=1}^{M_n} F_{z_{i-1}^n,z_i^n}(\frac{t}{2M_n}))\\
&= \Delta(F_{\xi,f^{n+1}x_0}(\frac{\kappa t}{2}), L_{f^n x_0,\xi}(\frac{t}{2M_n}))\\
&\geq \Delta(F_{\xi,f^{n+1}x_0}(\frac{\kappa t}{2}), L_{f^n x_0,\xi}(\frac{t}{2M})), \qquad (5.11)
\end{aligned}$$

where $M = \max\{M_n : n \in \mathbb{N}\}$. Since, G is an $(\mathcal{H}_f)$-graph and $(f^n x_0, f^{n+1}x_0) \in E(G)$ for $n \in \mathbb{N}$ with $\lim_{n\to\infty} f^n x_0 = \xi \in X$. Then sequence of functions $\{L_{f^n x_0,\xi}(t)\}$ converges to $\epsilon_0(t), (t > 0)$ uniformly. Let $t > 0$ and $\delta > 0$ be given, as the family $\{\Delta^p(t)\}_p \in \mathbb{N}$ is equicontinuous at point $t = 1$ there exists $\lambda(\delta) \in (0, 1)$ such that $\Delta^p(1 - \lambda) > 1 - \delta$ for every $p \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have, $F_{\xi,f^{n+1}x_0}(\frac{\kappa t}{2}) > 1 - \lambda$ and $L_{f^n x_0,\xi}(\frac{t}{2M}) > 1 - \lambda$. So that in view of (5.11) for all $n \geq n_0$ we have

$$\begin{aligned}
F_{\xi,f\xi}(t) &\geq \Delta(1 - \lambda, 1 - \lambda)\\
&= \Delta^1(1 - \lambda) > 1 - \delta. \qquad (5.12)
\end{aligned}$$

Hence, we deduce $f\xi = \xi$. Finally, let $y \in X := [x_0]_{\widetilde{G}}$ be arbitrary then from Lemma 5.2.4, $\lim_{n\to\infty} f^n y = \xi$. □

Corollary 5.2.10. Let $(X, \mathscr{F}, \Delta)$ be a complete Menger PM-space under a t-norm Δ of $\mathscr{H}$-type. Assume that X is endowed with a graph G which is either $(\mathcal{C}_f)$-graph or $(\mathcal{H}_f)$-graph. Then following statements are equivalent:

(i) G is weakly connected.

(ii) For every probabilistic G-contraction f on X if there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$ then f is a Picard operator .

Proof. $(i) \Rightarrow (ii)$: It is immediate from Theorem 5.2.9.

$(ii) \Rightarrow (i)$: Suppose that G is not weakly connected. Then $\widetilde{G}$ is disconnected, i.e., there exists $x^* \in X$ such that $[x^*]_{\widetilde{G}} \neq \emptyset$ and $X \setminus [x^*]_{\widetilde{G}} \neq \emptyset$. Let $y^* \in X \setminus [x^*]_{\widetilde{G}}$, we construct a self mapping f by:

$$fx = \begin{cases} x^* & \text{if} \quad x \in [x^*]_{\widetilde{G}} \\ y^* & \text{if} \quad x \in X \setminus [x^*]_{\widetilde{G}}. \end{cases}$$

Let $(x, y) \in E(G)$ then $[x]_{\widetilde{G}} := [y]_{\widetilde{G}}$ which implies $fx = fy$ hence $(fx, fy) \in E(G)$ as G contains all loops. Thus the mapping f preserves edges. Also, for $t > 0$ and $\kappa \in (0, 1)$ we have, $F_{fx,fy}(\kappa t) = 1 \geq F_{x,y}(t)$ thus (5.4) is trivially satisfied. But x^* and y^* are two fixed points of f contradicting the fact that f is a Picard operator. □

Remark 5.2.11. Taking $G = (X, X \times X)$, Theorem 5.2.9 improves and extends result of Sehgal [104, Theorem 3] to all Menger PM-spaces with t-norms of $\mathscr{H}$-type. Theorem 5.2.9 generalizes claim 4^0 of [52, Theorem 3.2] and thus we have the following consequence.

Corollary 5.2.12. (Jachymski [52, Theorem 3.2]) Let (X, d) be a complete metric space endowed with a graph G. Assume that the mapping $f : X \to X$ is a Banach G-contraction and the following property is satisfied:

($\mathcal{P}$) for any sequence $\{x_n\}$ in X, if $x_n \to x$ in X and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$ then there exists a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x) \in E(G)$ for all $k \geq 1$.

If there exists $x_0 \in X$ with $(x_0, fx_0) \in E(G)$. Then $f|_{[x_0]_{\widetilde{G}}}$ is a Picard operator . Furthermore, if G is weakly connected then f is a Picard operator.

Proof. Let $(X, \mathscr{F}, \Delta_M)$ be the Menger PM-space induced by the metric d. Since, the mapping f is a Banach G-contraction then it is a probabilistic G-contraction (see, Example 5.2.2) and property ($\mathcal{P}$) invokes G is a $(\mathcal{C}_f)$-graph. Hence the conclusion follows from Theorem 5.2.9(i). □

Example 5.2.13. Let $(X, \mathscr{F}, \Delta_M)$ be a Menger PM-space where $X = [0, \infty)$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for $t > 0$. Then $(X, \mathscr{F}, \Delta_M)$ is complete. Define a selfmapping f on X by:

$$fx = \begin{cases} \frac{x^2}{p} & \text{if } \ x = \frac{1}{n} \text{ and } p \geq 3 \text{ is fixed integer,} \\ 0, & \text{otherwise.} \end{cases} \tag{5.13}$$

Further assume that X is endowed with a graph G consisting of $V(G) := X$ and $E(G) := \Omega \cup \{(\frac{1}{n}, \frac{1}{m}) : n, m \in \mathbb{N} \text{ and } n|m\} \cup \{(x, 0) : x \neq \frac{1}{n}\}$. It can be seen that f is a probabilistic G-contraction with $\kappa = \frac{2}{p}$ and satisfies all conditions of Theorem 5.2.9 (i).
Note that for $x = 1$ and $y = \frac{5}{6}$ and for each $\kappa \in (0, 1)$ we can easily set $0 < t < \frac{1}{6(1-\kappa)}$ such that,

$$\frac{\kappa t}{\kappa t + |\frac{1}{3} - 0|} < \frac{t}{t + |1 - \frac{5}{6}|},$$

or,

$$F_{f1, f\frac{5}{6}}(\kappa t) < F_{1, \frac{5}{6}}(t) \ \text{ for } 0 < t < \frac{1}{6(1-\kappa)}.$$

Hence, one can not invoke [104, Theorem 3].

In the following we extend the notions of orbital continuity and orbital G-continuity of a self-mapping to the case of probabilistic metric spaces.

Definition 5.2.14. Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space under a t-norm Δ of $\mathscr{H}$-type. A mapping $f : X \to X$ is said to be: (i) continuous at point $x \in X$ if whenever $x_n \to x$ in X implies $fx_n \to fx$ as $n \to \infty$; (ii) orbitally continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n \in \mathbb{N}}$ of positive integers, $f^{k_n}x \to y$ implies $f(f^{k_n}x) \to fy$ as $n \to \infty$; (iii) orbitally G-continuous if for all $x, y \in X$ and any sequence $\{k_n\}_{n \in \mathbb{N}}$ of positive integers, $f^{k_n}x \to y$ and $(f^{k_n}x, f^{k_n+1}x) \in E(G)$ $\forall n \in \mathbb{N}$ imply $f(f^{k_n}x) \to fy$ (see, [52]).

Theorem 5.2.15. Let $(X, \mathscr{F}, \Delta)$ be a complete Menger PM-space under a $t-$norm Δ of $\mathscr{H}$-type. Assume that the mapping $f : X \to X$ is a probabilistic G-contraction such that f is orbitally G-continuous and let there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. Then f has a unique fixed point $\xi \in X$ and for every $y \in [x_0]_{\widetilde{G}}$, $f^n y \to \xi$. Moreover, if G is weakly connected then f is a Picard operator .

Proof. Let $(x_0, fx_0) \in E(G)$ then by induction $(f^n x_0, f^{n+1}x_0) \in E(G)$ for all $n \in \mathbb{N}$. By using Lemma 5.2.8 it follows that $f^n x_0 \to \xi \in X$. Since, f orbitally G-continuous then $f(f^n x_0) \to f\xi$. This gives $\xi = f\xi$. From Lemma 5.2.4 for any $y \in [x_0]_{\widetilde{G}}$, $f^n y \to \xi$.

□

Remark 5.2.16. We note that in Theorem 5.2.15 the assumption that f is orbitally G-continuous can be replaced by orbital continuity or continuity of f.

Remark 5.2.17. Theorem 5.2.15 generalizes and extends claims 2^0 & 3^0 [52, Theorem 3.3] and claim 3^0 [52, Theorem 3.4].

As a consequence of Theorems 5.2.9, 5.2.15 we obtain following corollary which is actually a probabilistic version of Theorem 2.5.3 and thus generalizes and extends results of Nieto and Rodríguez-López [72, Theorems 2.1 and 2.3], Petruşel and Rus [84, Theorem 4.3] and Ran and Reurings [90, Theorem 2.1].

Corollary 5.2.18. Let $(X, \preceq)$ be a partially ordered set and $(X, \mathscr{F}, \Delta)$ be a complete Menger PM-space under a t-norm Δ of $\mathscr{H}$-type. Assume that the mapping $f : X \to X$ is nondecreasing (nonincreasing) with respect to the order "$\preceq$" on X and there exists $\kappa \in (0, 1)$ such that:

$$F_{fx,fy}(\kappa t) \geq F_{x,y}(t) \quad \text{for all } x, y \in X, \text{ with } x \preceq y, \ (t > 0). \tag{5.14}$$

Also suppose that either

(*i*) f is continuous, or

(*ii*) for every nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x$ in X we have $x_n \preceq x$ for all $n \geq 1$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point. Furthermore, if $(X, \preceq)$ is such that: every pair of elements of X has an upper or lower bound, then f is a Picard operator .

Proof. Consider a graph G_1 consisting of $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \preceq y\}$. If f is nondecreasing then it preserves edges with respect to graph G_1 and condition (5.14) becomes equivalent to (5.4). Thus f is a probabilistic G_1-contraction. In case f is nonincreasing consider G_2 with $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } x \succeq y\}$ and vertex set which coincides with X. Actually, $G_2 := \widetilde{G_1}$ and from Proposition 5.2.3 if f is a probabilistic G_1-contraction then it is a probabilistic G_2 contraction. Now if f is continuous then conclusion follows from Theorem 5.2.15. On the other hand if (*ii*) holds then G_1 and G_2 are $(\mathcal{C}_f)$-graphs and conclusions follow from first part of Theorem 5.2.9. □

By relaxing $\mathscr{H}$-type condition on t-norm, our next result deals with compact Menger PM-space using the following class of graphs as fixed point property is closely related to connectivity of graph.

Definition 5.2.19. Let $(X, \mathscr{F})$ be a PM-space endowed with a graph G and $f : X \to X$. Assume the the sequence $\{f^n x\}$ in X with $(f^n x, f^{n+1}x) \in E(G)$ for $n \in \mathbb{N}$ and $F_{f^n x, f^{n+1}x}(t) \to 1 (t > 0)$. We say that the graph G is an (E_f)-graph if for any subsequence $f^{n_k}x \to z \in X$ there exists a natural number N such that $(f^{n_k}x, z) \in E(G)$ for all $k \geq N$.

Theorem 5.2.20. Let $(X, \mathscr{F}, \Delta)$ be a compact Menger PM-space under a $t-$norm Δ satisfying $\sup_{a<1} \Delta(a, a) = 1$. Assume that the mapping $f : X \to X$ is a probabilistic G-contraction and let there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$. If G is an (E_f)-graph then f has a unique fixed point $\xi \in [x_0]_{\widetilde{G}}$.

Proof. Since, $(x_0, fx_0) \in E(G)$ then $(f^n x_0, f^{n+1}x_0) \in E(G)$ for $n \in \mathbb{N}$ and

$$\begin{aligned} F_{f^n x_0, f^{n+1}x_0}(t) &\geq F_{f^{n-1}x_0, f^n x_0}(\frac{t}{\kappa}) \\ &\vdots \\ &\geq F_{x_0, fx_0}(\frac{t}{\kappa^n}) \to 1 \quad \text{as} \quad n \to \infty \ (t > 0). \end{aligned}$$

From compactness let $\{f^{n_k}x_0\}$ be a subsequence such that $f^{n_k}x_0 \to \xi \in X$. Let $t > 0$ and $\delta > 0$ be given, since $\sup_{a<1} \Delta(a, a) = 1$ then there exists $\lambda(\delta) \in (0, 1)$ such that $\Delta(1 - \lambda, 1 - \lambda) > 1 - \delta$ choose $n' \in \mathbb{N}$ such that for all $k \geq n'$ we have, $F_{f^{n_k}x_0, \xi}(\frac{t}{2}) > 1 - \lambda$ and $F_{f^{n_k}x_0, f^{n_k+1}x_0}(\frac{t}{2}) > 1 - \lambda$. Then we obtain

$$\begin{aligned} F_{f^{n_k+1}x_0, \xi}(t) &\geq \Delta(F_{f^{n_k+1}x_0, f^{n_k}x_0}(\frac{t}{2}), F_{f^{n_k}x_0, \xi}(\frac{t}{2})) \\ &\geq \Delta(1 - \lambda, 1 - \lambda) > 1 - \delta. \end{aligned}$$

Thus, $f^{n_k+1}x_0 \to \xi$.

Choose $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ we have, $F_{f^{n_k+1}x_0, \xi}(\frac{t}{2}) > 1-\lambda$ and $F_{f^{n_k}x_0, \xi}(\frac{t}{2\kappa}) > 1-\lambda$. Since, G is an (E_f)-graph there exists $N \in \mathbb{N}$ such that $(f^{n_k}x_0, \xi) \in E(G)$ for all $k \geq N$. Let $n_0 = \max\{n_1, N\}$ then for $k \geq n_0$ we get

$$\begin{aligned} F_{f\xi, \xi}(t) &\geq \Delta(F_{f^{n_k+1}x_0, f\xi}(\frac{t}{2}), F_{f^{n_k+1}x_0, \xi}(\frac{t}{2}) \\ &\geq \Delta(F_{f^{n_k}x_0, \xi}(\frac{t}{2\kappa}), F_{f^{n_k+1}x_0, \xi}(\frac{t}{2}) \\ &\geq \Delta(1 - \lambda, 1 - \lambda) > 1 - \delta. \end{aligned}$$

Hence, $f\xi = \xi$. Note that $\{x_0, fx_0, \cdots, f^{n_1}x_0, \cdots, f^{n_N}x_0, \xi\}$ is a path in $\widetilde{G}$, so that $\xi \in [x_0]_{\widetilde{G}}$. □

So far it remains to be investigated. Is it possible to extend Theorem 5.2.20 to all complete Menger PM-spaces?

Definition 5.2.21. [104] Let $(X, \mathscr{F})$ be a PM-space. Let $\varepsilon > 0$ and $0 < \delta < 1$ be a fixed real numbers. A mapping $f : X \to X$ is said to be an (ε, δ)-contraction if there exists a constant $\kappa \in (0, 1)$ such that for $x \in X$ and $y \in \mathcal{N}_x(\varepsilon, \delta)$ we have,

$$F_{fx,fy}(\kappa t) \geq F_{x,y}(t) \quad \text{for all } t > 0 \tag{5.15}$$

The PM space $(X, \mathscr{F})$ is said to be (ε, δ)-chainable if for each $x, y \in X$ there exists a finite sequence $(x_n)_{n=0}^{N}$ of elements in X with $x_0 = x$ and $x_N = y$ such that $x_{i+1} \in \mathcal{N}_{x_i}(\varepsilon, \delta)$ for $i = 0, 1, \cdots, N-1$.

It is important to note that every (ε, δ)-contraction mapping is continuous. Let $x_n \to x$ in X then there exists a natural number $N(\varepsilon, \delta)$ such that $x_n \in \mathcal{N}_x(\varepsilon, \delta)$ for all $n \geq N$. Thus for $t > 0$ and for all $n \geq N$ we obtain,

$$\begin{aligned} F_{fx_n,fx}(t) &\geq F_{fx_n,fx}(\kappa t) \\ &\geq F_{x_n,x}(t) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Hence, $fx_n \to fx$.

Theorem 5.2.22. Let $(X, \mathscr{F}, \Delta)$ be a complete (ε, δ)-chainable Menger PM-space under a t-norm Δ of $\mathscr{H}$-type. Let the mapping $f : X \to X$ is an (ε, δ)-contraction. Then f is a Picard operator .

Proof. Consider the graph G consisting of $E(G) = \{(x, y) \in X \times X : F_{x,y}(\varepsilon) > 1 - \delta\}$ and $V(G)$ coincides with X. Let $x, y \in X$. Since the PM-space $(X, \mathscr{F})$ is (ε, δ)-chainable there exists a finite sequence $(x_i)_{i=0}^{N}$ in X with $x_0 = x$ and $x_N = y$ such that $F_{x_i,x_{i+1}}(\varepsilon) > 1 - \delta$ for $i = 0, 1, \cdots N-1$. Hence, $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \cdots, N-1$. This implies G is connected. Let $(x, y) \in E(G)$ then $y \in \mathcal{N}_x(\varepsilon, \delta)$. Since, the mapping f is an (ε, δ)-contraction thus (5.4) is satisfied. Finally we have,

$$\begin{aligned} F_{fx,fy}(\varepsilon) &\geq F_{fx,fy}(\kappa\varepsilon) \\ &\geq F_{x,y}(\varepsilon) > 1 - \delta. \end{aligned}$$

Thus, $(fx, fy) \in E(G)$. Hence, f is a probabilistic G-contraction and the conclusion follows from Theorem 5.2.15. □

Remark 5.2.23. Theorem 5.2.22 has an advantage over Theorem 7 of Sehgal and Bharucha-Reid [104] which is only restricted to continuous t-norms satisfying $\Delta(t, t) \geq t$. Moreover, the proof of our result is rather simple and easy which evokes novelty of Theorem 5.2.22.

Definition 5.2.24. (Edelstein [36, 37]) The metric space (X, d) is ε-chainbale for some $\varepsilon > 0$ if for every $x, y \in X$ there exists finite sequence $(x_i)_{n=0}^N$ of elements in X with $x_0 = x$, $x_N = y$ and $d(x_i, x_{i+1}) < \varepsilon$ for $i = 0, 1, \cdots, N - 1$.

Remark 5.2.25. [104] If (X, d) is an ε-chainable metric space then induced Menger PM-space $(X, \mathscr{F}, \Delta_M)$ is an (ε, δ)-chainable space.

Corollary 5.2.26. (Edelstein [36, 37]) Let (X, d) be a complete ε-chainable metric space. Let $f : X \to X$ and there exists $\kappa \in (0, 1)$ such that

$$\forall_{x,y \in X}\{d(x, y) < \varepsilon \implies d(fx, fy) \leq \kappa\, d(x, y)\}. \tag{5.16}$$

Then f is a Picard operator.

Proof. Since the metric space (X, d) is ε-chainable then the induced Menger PM-space $(X, \mathscr{F}, \Delta)$ is (ε, δ)-chainable for each $0 < \delta < 1$. We only need to show that the self mapping f on X is (ε, δ)-contraction. Let $x, y \in X$ be such that $y \in \mathcal{N}_x(\varepsilon, \delta)$, i.e, $F_{x,y}(\varepsilon) > 1 - \delta$ or $\epsilon_0(\varepsilon - d(x, y)) > 1 - \delta$. By definition of ϵ_0 it implies $d(x, y) < \varepsilon$ and thus $d(fx, fy) \leq \kappa\, d(x, y)$. Now for $t > 0$ we get

$$\begin{aligned} F_{fx,fy}(\kappa\, t) &= \epsilon_0(\kappa\, t - d(fx, fy)) \\ &\geq \epsilon_0(t - d(x, y)) \\ &= F_{x,y}(t). \end{aligned}$$

Hence the conclusion follows from Theorem 5.2.22. □

5.2.2 Application

The notion of cyclic contractions by Kirk et al. [63] can be extended in probabilistic metric space as well.

Let X be a nonempty set and $(X, \mathscr{F}, \Delta)$ be a PM-space. Suppose m be a positive integer and $\{A_i\}_{i=1}^m$ be nonempty closed subsets of X and $f : \cup_{i=1}^m A_i \to \cup_{i=1}^m A_i$ be an operator. Then $X := \cup_{i=1}^m A_i$ is known as cyclic representation of X with respect to f if;

$$f(A_1) \subset A_2, \cdots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1 \tag{5.17}$$

and the operator f is known as cyclic operator [63].

In the following we present the probabilistic version of main result of [63], as a last consequence of Theorem 5.2.9.

Theorem 5.2.27. Let $(X, \mathscr{F}, \Delta)$ be a complete Menger PM-space under a t-norm Δ of $\mathscr{H}$-type. Let m be a positive integer, $\{A_i\}_{i=1}^m$ nonempty closed subsets of X, $Y := \cup_{i=1}^m A_i$ and $f : Y \to Y$. Assume that:

- (i) $\cup_{i=1}^m A_i$ is cyclic representation of Y with respect to f;
- (ii) $\exists\, \kappa \in (0, 1)$ such that $F_{fx,fy}(\kappa t) \geq F_{x,y}(t),\ (t > 0)$ whenever, $x \in A_i, y \in A_{i+1}$, where $A_{m+1} = A_1$.

Then f has a unique fixed point $\xi \in \cap_{i=1}^m A_i$ and $f^n y \to \xi$ for any $y \in \cup_{i=1}^m A_i$.

Proof. Since, $\cup_{i=1}^m A_i$ is closed then $(Y, \mathscr{F}, \Delta)$ is complete. Let us consider a graph G consisting of $V(G) := Y$ and $E(G) := \Omega \cup \{(x, y) \in Y \times Y : x \in A_i, y \in A_{i+1}; i = 1, \cdots, m\}$. By (i) it follows that f preserves edges. Now let $f^n x \to x^*$ in Y such that $(f^n x, f^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$. Then by (5.17) it infers that the sequence $\{f^n x\}$ has infinitely many terms in each $A_i; i \in \{1, 2, \cdots, m\}$. So that one can easily identify a subsequence of $\{f^n x\}$ converging to x^* in each A_i and since A_i's are closed then $x^* \in \cap_{i=1}^m A_i$. Thus, we can easily form a subsequence $\{f^{n_k} x\}$ in some A_j, $j \in \{1, \cdots, m\}$ such that $(f^{n_k} x, x^*) \in E(G)$ for $k \geq 1$. It elicits G is weakly connected $(\mathcal{C}_f)$-graph. Hence, by Theorem 5.2.9 conclusion follows. □

5.3 Conclusion

The notion of probabilistic G-contractions extends/generalizes the notion of Banach G-contractions and the contractions on partially order sets. The obtained results based on this intuitive approach are of great agrement to that presented in [104]. Probabilistic cyclic contractions elucidate the novelty of our results. Moreover Example 5.2.2 substantiates the degree of generality of our main results. Our further goal is to extend this notion by weakening contractiveness assumption.

Bibliography

[1] M. Abbas, T. Nazir, S. Radenović, Common fixed points for four maps in partially ordered metric spaces, *Applied mathematics letters* (2011), 1520–1526.

[2] R. P. Agarwal, M. A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, **87** (2008), 109–116.

[3] Ya. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, new results in operator theory, *Advances and Appl.*, *Birkhauser verlag, Basel*, (ed. by I. Gohberg and Yu Lyubich), **98** (1997), 7–22.

[4] S. M. A. Aleomraninejad, Sh. Rezapour, N. Shahzad, Some fixed point results on a metric space with a graph, *Topology and its Applications*, **159** (2012), 659–663.

[5] M. A. Alghamdi, A. Petrusel, N. Shahzad, A fixed point theorem for cyclic generalized contractions in metric spaces, *Fixed point theorey and applications*, (2012): 122.

[6] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, *J. Math. Anal. Appl.*, **322** (2006), 796–802.

[7] H. Aydi, M. F. Bota, S. Mitrovic, E. Karapinar, A fixed point theorem for set-valued quasi-contractions in b-metric spaces, *Fixed Point Theory Appl.*, (2012), 2012:88.

[8] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal., Gos. Ped. Inst. Unianowsk,* **30**, (1989), 26–37.

[9] S. Banach, Sur les opťerations dans les ensembles abstraits et leur application aux ťequations intťegrales. *Fund. Math.,* **3** (1922), 133–181.

[10] I. Beg, A. R. Butt, S. Radojević, The contraction principle for set valued mappings on a metric space with a graph, *Comput. Math. Appl.*, **60** (2010), 1214–1219.

[11] V. Berinde, Contractii Generalizate si Aplicatii, *Editura Cub Press, Baia Mare*, **22**, (1997).

[12] V. Berinde, Generalized contractions in quasimetric spaces, Seminar on Fixed Point Theory, *Babes-Bolyai University of Cluj-Napoca*, **3** (1993), 3–9.

[13] V. Berinde, Sequences of operators and fixed points in quasimetric spaces, *Studia Univ. Babes-Bolyai, Math.,* **16** (1996), 23–27.

[14] R. M. Bianchini, M. Grandolfi, Transformaziani di tipo contracttivo generalizzato in uno spazio metrico, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fiz. Mat. Natur.*, **45** (1968), 212–216.

[15] F. Bojor, Fixed point theorems for Reich type contractions on metric spaces with a graph, *Nonlinear Anal.*, **75** (2012), 3895–3901.

[16] F. Bojor, Fixed points of Kannan mappings in metric spaces endowed with a graph, *An. Şt. Univ. Ovidius Constanţa*, **20**(1) (2012), 31–40.

[17] M. Boriceanu, M. Bota, A. Petruşel, Multivalued fractals in b-metric spaces, *Cent. Eur. J. Math.* **8**(2) (2010), 367–377.

[18] M. Bota, A. Molnar, Cs. Varga, On Ekland's variantional principle in b-metric spaces, *Fixed Point Theory*, **12**(1) (2011), 21–28.

[19] M. F. Bota-Boriceanu, A. Petruşel, Ulam-Hyers stability for operatorial equations. *An. Stiint. Univ. Al. I. Cuza Iasi. Mat.* (N.S.) **57**(1) (2011), 65–74.

[20] M. Bota, A. Molnár, C. Varga, On Ekeland's variational principle in b-metric spaces. *Fixed Point Theory*, **12** (2011), 21–28.

[21] M. F. Bota, E. Karapinar, A note on ŞSome results on multi-valued weakly Jungck mappings in b-metric spaceŤ, *Cent. Eur. J. Math.* 11, (2013), **9**, 1711–1712.

[22] N. Bourbaki, Topologie generale, *Herman, Paris,* (1974).

[23] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, **29** (2002), 531–536.

[24] F. E. Browder, On the convergence of successive approximations for nonlinear functional equations, *Nedrel. Akad. Wetensch. Math.*, **30** (1968), 27–35.

[25] Dz. Burgic, S. Kalabusic, M. R. S. Kulenovic, Global attractivity results for mixed monotone mappings in partially ordered complete metric spaces, *Fixed point theory and applications*, (2009): 762478.

[26] S. K. Chatterjea, Fixed point theorems, *Comptes Rendus de l' Academie Bulgare des Sciences,* **25** (1972), 727–730.

[27] Y. Cho, K. S. Park, S. S. Chang, Fixed point theorems in metric spaces and probabilistic metric spaces, *Int. J. Math. Math. Sci.*, **19** (1996), 243–252.

[28] L. B. Ćirić, A generalization of Banach contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267–273.

[29] L. B. Ćirić, N. Cakid, M. Rajocic, J. S. Uma, Monotone generalized nonlinear contractions in partially ordered metric spaces, *Fixed point theory and applications*, (2008): 131294.

[30] E. A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, *New York, McGraw-Hill*, (1955).

[31] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Sem. Mat. Univ. Modena*, **46** (1998), 263–276.

[32] S. Czerwik, Functional equations and inequalities in several variables, *World Scientific*, (2002).

[33] T. Dinevari, M. Frigon, Fixed point results for multivalued contractions on a metric space with a graph, *J. Math. Anal. Appl.*, (2013), (in press).

[34] H. S. Ding, L. Li, S. Radenoviċ, Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces, *Fixed point theory and applications*, (2012): 96.

[35] A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatiblemappings satisfying contractive conditions of integral type, *J. Math. Anal. Appl.*, **329** (2007), 31–45.

[36] M. Edelstein, On fixed and periodic points under contractive mappings, *J. London Math. Soc.*, **37** (1962), 74–79.

[37] M. Edelstein, An extension of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **12** (1961), 7–10.

[38] A. Gel'man, A certain fixed-point principle, *Dokl. Akad. Nauk* SSSR, **198** (1950), 506–508 (in Russian); *Sovit Math. Dokl.*, **12** (1971), 813–816 (Engl. transl.).

[39] T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applictions, *Nonlinear Anal.*, **65** (2006), 1379–1393.

[40] Z. Golubović, Z. Kadelburg, S. Radenović, Coupled coincidence points of mappings in ordered partial metric spaces, *Abstract and applied analysis*, (2012).

[41] A. Granas and J. Dugundji, Fixed point theory, *Springer-Verlag, New York*, (2003), MR1987179 (2004d:58012)

[42] G. Gwóźdź-Łukawska, J. Jachymski, IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem, *J. Math. Anal. Appl.*, **356** (2009), 453–463.

[43] O. Hadzić, Fixed point theorems for multivalued mapping in probabilistic metric spaces, *Fuzzy Sets and Systems* **88** (1997), 219–226.

[44] O. Hadzić, E. Pap, Fixed point theory in probabilistic metric spaces, *Kluwer Academic Publishers, Dordrecht*, (2001).

[45] G. E. Hardy and T. D. Rogers, A generalization of a fixed point theorem of Reich, *Canadian Mathematical Bulletin,* **16** (1973), 201–206.

[46] J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mapping in patially ordered sets, *Nonlinear Analysis*, **71** (2009), 3403–3410.

[47] T. L. Hicks, Another view of fixed point theory, *Math. Japonica*, **35** (1990), 231–234.

[48] T. L. Hicks, B. E. Rhoades, A Banach type fixed point theorem, *Math. Japonica*, **24** (1979), 327–330.

[49] J. Heinonen, Lectures on Analysis on Metric Spaces, *Springer Berlin*, (2001).

[50] Z. D. Huang, On Newton's method under Holder continuous derivative, *J. Math. Nal. Appl.*, **270** (2002), 332–339.

[51] N. Hussain, E. Karapinar, P. Salimi, P. Vetro, Fixed point results for G^m Meir-Keeler contractive and $G-(\alpha,\psi)$-Meir-Keeler contractive mappings, *Fixed point theory and applications*, (2013): 34.

[52] J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.*, **136** (2007), 1359–1373.

[53] J. Jachymaski, Order-theoretic aspects of ametric fixed point theory, Handbook of metric fixed point theory (eds., W. A. Kirk and B.Sims), *Kluwer Acad. Publ., Dordrechet*, (2001), MR1904289 (2003f:54094), 613–641.

[54] M. Jleli, E. Karapinar, B. Samet, Best proximity points for generalized $\alpha-\psi$-proximal contractive type mappings, *Journal of applied mathematics*, (2013):534127.

[55] R. Johnsonbaugh, Discrete Mathmatics, *Prentice-Hall, Inc., New Jersey*, (1997).

[56] T. Kamran, M. Samreen, N. Shahzad, Probabilistic G-contractions, *Fixed Point Theory and Applications,* (2013): 223.

[57] R. Kannan, Some results on fixed points-ii, *Amer. Math. Monthly*, **76** (1969), 405–408.

[58] E. Karapinar, Fixed point theory for cyclic weak ϕ-contraction. *Appl. Math. Lett.* **24**(6) (2011), 822–825.

[59] E. Karapinar, Best Proximity Points Of Cyclic Mappings, *Appl. Math. Lett.,* 25 (2012) **11**, 1761–1766.

[60] E. Karapinar, B. Samet, Generalized $\alpha - \psi$ contractive type mappings and related fixed point theorems with applications, *Abstract and applied analysis*, (2012): 793486.

[61] E. Karapinar, K. Sadarangani, Fixed point theory for cyclic $(\varphi - \psi)$-contractions, *Fixed point theory and applications*, (2011): 69.

[62] E. Karapinar, I. M. Erhan, Cyclic Contractions and Fixed Point Theorems, *Filomat* **26**:4 (2012), 777–782.

[63] W. A. Kirk, P. S. Srinivasan, P. Veeranmani, Fixed points for mappings satisfying cyclical contractive condition, *Fixed point theory*, **4**(1) (2003), 79–89.

[64] Q. Kiran, T. Kamran, Nadler's type principle with higher order of convergence, *Nonlinear Analysis*, **69** (2008), 4106–4120.

[65] H. J. Kornstaedt, Funktionalungleichungen und iterationsverfahren, *Aequationes Math.*, **13** (1975), 21–45.

[66] J. Matkowski, Integrable solutions of functional equations, *Dissertations Math. (Rozprawy Mat.)*, (1975), 127.

[67] K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci. U. S. A.*, **28** (1942), 535–537.

[68] D. Miheţ, A Generalization of a contraction principle in probabilistic metric spaces. Part II, *International Journal of Mathematics and Mathematical Sciences*, **5** (2005), 729–736.

[69] I. P. Mysovskih, On the convergence of L. V. Kantorovic's method of solution of functionsl equations and its applications, *Dokl. Acal. Nauk* SSSR, **70** (1950), 565–568 (in Russian).

[70] V. V. Nemytzki, The fixed point method in analysis (Russian), *Usp. Mat. Nauk,* **1** (1936), 141–174.

[71] J. J. Nieto, R. L. Pouso and R. Rodríguez-López, Fixed point theorems in ordered abstract spaces, *Proc. Amer. Math. Soc.*, **135** (2007), 2505–2517.

[72] J. J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, **22** (2005), 223–239.

[73] J. J. Nieto, R. Rodríguez-López, Existance and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin.(Engl. Ser.)* **23** (2007), 2205–2212.

[74] J. J. Nieto, R. Rodríguez-López, Existence of extremal solutions for quadratic fuzzy equations, *Fixed point theory and applications*, (2005), 321-342.

[75] O. Onicescu, Nombres et Systdmes Aléatoires, *Éditions de l'Académie de la R. P. Roumaine, Bucarest,* (1964).

[76] D. O'Regan, A. Petrusel, Fixed point theorems in ordered metric spaces, *J. Math. Anal. Appl.* **341** (2008), 1241–1252.

[77] D. O'Regan, A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, *J. Math. Anal. Appl.*, **341** (2008), 1241–1252.

[78] M. Pacurar, A fixed point result for φ-contractions on b-metric spaces without the boundedness assumption, preprint.

[79] M. Pacurar, I. A. Rus, Fixed point theory for cyclic φ-contractions, *Nonlinear Analysis*, **72** (2010), 1181–1187.

[80] S. Park, A unified approach to fixed points of contractive maps, *J. Korean Math. Soc.*, **16** (1980), 95–105.

[81] V. Parvaneh, J. R. Roshan, S. Radenović, Existence of tripled coincedence points in ordered b-metric spaces and an application to a system of integral equations, *Fixed point theory and applications*, **1**(2013), 1–19.

[82] M. A. Petric, Some remarks concerning Ċiriċ-Reich-Rus operators, *Creat. Math. Inform.,* **18**(2) (2009), 188-193.

[83] T. P. Petru, M. Boriceanu, Fixed point results for generalized φ-contraction on a set with two metrics. *Topol. Methods Nonlinear Anal.*, **33**(2), (2009), 315–326.

[84] A. Petrusel and I. A. Rus, Fixed point theorems in ordered L-spaces, *Proc. Amer. Math. Soc.* **134** (2006), 411–418.

[85] P. D. Proinov, A generalization of the Banach contraction principle with high order of convergence of successive approximations, *Nonlinear Anal.*, **67** (2007), 2361–2369.

[86] P. D. Proinov, New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *Journal of Complexity,* **26** 1, (2010), 3–42.

[87] S. Radenoviĉ, Z. Kadelburg, Generalized weak contractions in partially ordered metric spaces, *Comput. Math. Appl.*, **60** (2010), 1776–1783.

[88] S. Raju., E. Karapinar, K. Tai, Reich Type Cyclic Orbital and Reich Type Meir-Keeler Contraction Maps, *Journal of Nonlinear Analysis and Optimization: Theory and Applications,* **4**, 1(2013).

[89] E. Rakotch, A note on contractive mappings, *Proc. A. M. S.*, **13** (1962), 459–465.

[90] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, **132** (2004), 1435–1443.

[91] S. Reich, Fixed points of contractive functions, *Boll. Unione Mat. Ital.*, **5** (1972), 26–42.

[92] W. C. Rheinboldt, A unified convergence theory for a class of iterative processes, *SIAM J. Numer. Anal.*, **5** (1968), 42–63.

[93] B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Analysis*, **47** (2001), 2683–2693.

[94] B. E. Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, **63** (2003), 4007–4013.

[95] I.A. Rus, Picard operators and applications, *Scientiae Mathematicae Japonicae*, **58** (2003)1, 191–219.

[96] I. A. Rus, Cyclic representations and fixed points, *Ann. T. Popoviciu Seminar Funct. Eq. Approx. Convexity*, **3** (2005), 171–178.

[97] I.A. Rus, Generalized Contractions and Applications, *Cluj Univ. Press*, (2001).

[98] I. A. Rus, A. Petruşel, G. Petruşel, Fixed point Theory, *Cluj University Press, Cluj-Napoca*, (2008).

[99] I. A. Rus, Stability of attractor of a φ-contractions system. Seminar on Fixed point theory, *Babes-Bolyai Univ.,* **3** (1998), 31–34.

[100] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha-\psi$ contractive type mappings, *Nonlinear Analysis*, **75** (2012), 2154–2165.

[101] M. Samreen, T. Kamran, Fixed point theorems for integral G-contractions, *Fixed Point Theory Appl.*, (2013):149.

[102] M. Samreen, T. Kamran, Fixed point theorems for weakly G-contractive mappings on a metric space endowed with a graph, *Filomat* (In press).

[103] M. Samreen, T. Kamran, N. Shahzad, Some fixed point theorems in b-metric space endowed with graph, *Abstract and Applied Analysis,* (2013): 967132.

[104] V. M. Sehgal, A. T. Bharucha-Reid, Fixed points of contraction mappings on Probabilistic metric spaces, *Math. System Theory*, **6** (1972), 97–102.

[105] B. Schweizer, Probabilistic metric spaces–the first 25 years, *The N.Y. Statistician*, **19** (1967), 3–6.

[106] B. Schweizer, Probabilistic metric spaces, Probabilistic Methods in Applied Mathematics (A. T. Bharucha-Reid, ed.), *Academic Press, New York* **4**.

[107] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, *Elsevier/ North-Holland, New York*, (1983).

[108] B. Schweizer, A. Sklar, E. Thorp, The metrization of statistical metric spaces, *Pacific J. Math.*, **10** (1960), 673–675.

[109] H. Sherwood, Complete probabilistic metric spaces, *Z. Wahr. verw. Geb.*, **20** (1971), 117–128.

[110] T. Suzuki, Meir-Keeler contractions of integral type are still Meir-Keeler contractions, *Int. J. Math. Math. Sci.*, (2007):39281, 6 pages.

[111] M. Turinici, Abstract comparison principle and multivariables Gronwall-Bellman Inequalities, *J. Math. Ana. and Appl.*, (1986) 117, 100–127.

[112] P. Vijayaraju, B. E. Rhoades, and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, (2005), 2359–2364.

[113] T. Zamfirescu, Fixed point theorems in metric spaces, *Arch. Math.* (Basel), (1972), 292–298.