BLACK HOLES IN AN ACCELERATING UNIVERSE

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2010

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A Dissertation Submitted in Partial Fulfillment of the

DEGREE OF DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

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2010

Dedicated to

My parents
and
Prof. Muneer Ahmad Rashid

ACKNOWLEDGEMENT

I express my heart-felt gratitude to Prof. Asghar Qadir, my PhD supervisor, for his guidance and constant support during this research. He encouraged me to work on my independent ideas and groomed my mental research abilities. He always motivated me to visit national and international conferences and symposiums to present my work.

Next I would thank Prof. Francesco De Paolis, for having useful discussions and several suggestions on the improvement of this work. I would like to thank the Salento University of Lecce, Italy for local hospitality during June 2007 and Prof. Remo Ruffini and ICRANet for their hospitality in July, 2009.

I would deeply thank the Abdus Salam International Center for Theoretical Physics (ICTP), Trieste, Italy for inviting and providing me funding to attend various scientific activities during November 2007, June 2008, April 2009 and July 2010.

I am deeply indebted to Prof. Muneer Ahmad Rashid for always welcoming me for any research discussion even if I needed to visit his home to discuss research problems. His smiling face always relieved my tense nerves due to work load.

I would also acknowledge Dr. Khalid Saifullah, Dr. M. Akbar and Dr. Azad Akhtar Siddiqui, Dr. Ibrar Hussain, Dr. Achille Nucita and Mr. M. Umar Farooq for several formal and informal discussions related to my research.

During my PhD, I successfully established collaborations with several scientists from outside Pakistan. To name: (Drs.) Peter K.F. Kuhfittig (USA), Ujjal Debnath, Farook Rahaman, Ranjan Sharma and Mehedi Kalam (India), Hossein Mohseni Sadjadi, Ahmad Sheykhi, Kayoomars Karami, Davood Momeni, Farhad Darabi and Mohammad Reza Setare (Iran) and Emmanuel N. Saridakis (Greece). I really enjoyed a lot working with these scientists and learnt from their experience. I thank all of them for having fruitful collaboration with me.

In a very special way, I would like to thank my parents, for their love and support over all these years, of which, as far as I can remember, have always encouraged me to learn more and to widen my horizons.

Generous financial supports in the form of scholarships, travel grants, research assistantship and research publication awards from NUST are gratefully acknowledged.

Dec, 2010 MUBASHER JAMIL

Abstract

In this thesis, I have considered the scenario of the accretion of phantom energy onto various types of black holes. The phantom energy is a strange kind of energy existing in the universe that drives the cosmic accelerated expansion. Among various black holes, we have chosen the following for our study: Schwarzschild, Riessner-Nordstrom, primordial and BTZ black holes. I also discuss various possibilities of phantom energy as viscous Chaplygin gas and modified variable Chaplygin gas.

The first two chapters are devoted to introductions to cosmology and black holes. In the former, I discuss the standard cosmological model and focus on dark energy theory. In the second chapter, we briefly discuss the astrophysical implications of black holes along with the derivation of two well-known black hole solutions in general relativity. I also discuss a lower space dimensional black hole for our further use.

In the third chapter, we shall discuss the phantom energy accretion onto primordial black holes of various masses. These black holes were formed from the gravitational collapse of primordial soup in the early Universe. They hypothetically radiate energy via Hawking evaporation process. I have found some interesting consequences of the accretion of phantom energy onto a primordial black hole. One of which is that to have the primordial black hole decay now it would have to be more massive initially.

In the fourth chapter, I study the accretion of dark energy on a Reissner-Nordstrom (RN) black hole. Since the RN black hole contains charge, the charge remains unaffected during the accretion of phantom energy. The phantom energy interacts only with the mass of the black hole to decrease it. Due to mass reduction, a stage is naturally reached when the magnitude of charge exceeds the mass of the black hole. At this point, there appears a naked singularity. The appearance of a naked singularity is forbidden according to the Cosmic Censorship Hypothesis (CCH). Hence charged black holes serves the purpose of making naked singularities through my suggested mechanism.

In chapter five, I study the accretion of phantom energy onto a Schwarzschild black hole but taking the former to be represented by extended forms of Chaplygin gas. Our analysis is an extension of the results of [1] who used a linear equation of state for phantom energy. I discuss two cases of polytropic equation of state for phantom energy.

In the sixth chapter, I study the same problem of the accretion of phantom energy on to a Schwarzschild black hole but assuming the former to be viscous. The presence of bulk viscosity in the dark energy can produce the affect of phantom energy (i.e. the equation of state becomes super-negative) and hence cause the accelerated expansion. The accretion mechanism of phantom energy is adapted from the fourth chapter. I will find that the mass of the black hole decreases faster in the viscous phantom case compared to the non-viscous one. Thus bulk viscosity plays a crucial role in the overall evolution of black holes. The origin of bulk viscosity is not clear and is purely ad hoc in the present context.

In seventh chapter, we study the accretion of phantom energy onto a lower space dimensional black hole. An interesting finding is that the rate of change mass of this black hole is independent of its mass and depends only on energy density and pressure of the phantom energy. I also discuss some thermodynamical aspects of the accretion process. In particular, we find that the first and second laws of thermodynamics are violated. I then employ the generalized second law of thermodynamics. Assuming that the later law holds, it yields a condition on the pressure and mass parameters of the model.

Finally in chapter eight, I conclude the thesis and present some open problems related to this work.

Publications From the Thesis

1. M. Jamil, A. Qadir and M.A. Rashid, Eur. Phys. J. C 58 (2008) 325.

2. M. Jamil, Eur. Phys. J. C 62 (2009) 609.

3. F. de Paolis, M. Jamil and A. Qadir, Int. J. Theor. Phys. 49 (2010) 621.

4. M. Jamil and M. Akbar, Gen. Relativ. Gravit. (2010) DOI: 10.1007/s10714-010-1024-2.

5. M. Jamil and A. Qadir, Gen. Relativ. Gravit. (2010) DOI: 10.1007/s10714-010-0928-1.

Contents

Chapter 1

Introduction

Astrophysical observations give mounting evidence that matter and energy in the Universe are distributed neither uniformly nor clumped all over haphazardly. Rather matter-energy is organized into more coherent structures of varying complexity. Astrophysically, on the smallest scale, matter is clumped in the form of dust clouds, planetoids, planets and stars while on the large scale, these stars are grouped together in small clusters (containing a few thousand stars) or galaxies (collections of billions of stars in addition to gas and dust). On a still larger scale, galaxies form clusters and super-clusters and these super-clusters lie on enormous filaments surrounding large voids. Hence it seems that the Universe is very much organized on almost all scales as it contains well-formed structures. This poses a problem of what physical mechanisms brought the Universe to such an organized state. In other words, the formation and evolution of large scale structures is still not well understood. Nevertheless, there is extremely strong evidence that long ago the Universe was remarkably homogeneous. In this thesis I shall model the Universe as if it was perfectly homogeneous and isotropic while noting that the observations leading to my interpretation of an accelerating Universe with dark energy could alternatively be explained by a non-homogeneous cosmology.

In this chapter, I will discuss the standard cosmological model based on the Friedmann-Robertson-Walker (FRW) metric. I also discuss the implications of the model in the present cosmic setting by analyzing the dynamics of dark energy. The FRW model, based on Einstein's general relativity, also has some drawbacks in explaining the several observational features like extreme smoothness of cosmic microwave background, presence of dark matter and recent cosmological acceleration. In the end, I will provide a brief review of well-known black hole solutions for further use in this thesis.

I shall use the following conventions and units: the signature of the underlying metrics will be $(-, +, +, +)$, unless otherwise mentioned. The Planck mass and the reduced Planck mass will be represented by $m_{pl} = (\sqrt{G})^{-1}$ and $M_{pl} = (\sqrt{8\pi G})^{-1}$ respectively, and having taken $c = \hbar\ (= \frac{h}{2\pi}) = 1$. Here G is Newton's gravitational constant, c is the speed of light and h the Planck's constant. The above constants are related as $\kappa^2 = 8\pi G = 8\pi m_{pl}^{-2} = M_{pl}^{-2}$.

1.1 Basics of cosmology

1.1.1 Friedmann-Robertson-Walker Universe

In 1917, Einstein applied his field equations to cosmology [2] and realized to his astonishment, that in general the field equations yield a dynamical Universe. To Einstein this could not be correct so he inserted a term, now called the cosmological constant term, denoted by Λ, into the equations. The equations now yielded a static Universe, in agreement with Einstein's beliefs. However, later it was observationally verified that the Universe was actually expanding, the Universe was indeed dynamical. This was shown by Edwin Hubble in 1929 [3], and Einstein had to withdraw his cosmological constant. Later, Einstein called the inclusion of the cosmological constant 'the biggest blunder of his life'. By including the cosmological constant, he produced what he thought was correct, but in this process failed to be the first to realize that the Universe was expanding. But his 'blunder' was not really as big a blunder as he thought; newer observational facts, have shown that a cosmological constant most probably is present and can be interpreted as representing a Lorentz-invariant vacuum energy with constant density. From the particle physics point of view the cosmological constant naturally arises as an energy density of the vacuum (discussed in detail in section 1.4.2). The cosmological constant turns out to be a measure of the energy density of the vacuum - the state of the lowest energy.

The Einstein field equations (EFE) with cosmological constant term are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1.1}$$

Here $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar obtained by the contraction of the Ricci tensor as $R = g^{\mu\nu}R_{\mu\nu}$, and $g^{\mu\nu}$ is the metric tensor. The cosmological constant is a dimensional parameter with units of $(length)^{-2}$. The aim was to explain the static model of the Universe; the negative pressure induced by the cosmological constant could cancel the equal gravitational pull of the matter keeping the matter distribution in a spherically symmetric form. This idea was in consonance with Mach's principle of inertia which forbade the notion of empty space. But later Friedmann [4] and Lemaitre [5] presented solutions of the EFE, giving expanding Universe scenarios which was later endorsed and observed by E.P. Hubble in the late 1920s [3].

From (1.1.1), one can also write

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}), \tag{1.1.2}$$

where I have used $8\pi GT = -R + 4\Lambda$ which is obtained by taking trace of (1.1.1). The spacetime metric that satisfies the requirements of homogeneity and isotropy is given by the Friedmann-Robertson-Walker (FRW) spacetime

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1.1.3}$$

The above metric is written in 'co-moving coordinates' where $a(t)$ is the scale factor that determines the scale of the expansion of the Universe. Here $k > 0$, $k = 0$ or $k < 0$. All of the homogeneous and isotropic Universe models may be represented by this line-element. For $k > 0$ the spatial surfaces have constant positive curvature and are usually called closed models. For $k = 0$ the spatial surfaces are Euclidean and are called flat models. Lastly, for $k < 0$ the spatial surfaces have constant negative curvature and are called open models.

The geometrical view of the FRW Universe is conveniently obtained by the following substitution

$$\int_0^\chi \frac{dr}{\sqrt{1-kr^2}} = f_k(\chi) = \begin{cases} \sin^{-1}\chi & k = +1 \\ \chi & k = 0 \\ \sinh\chi & k = -1. \end{cases} \tag{1.1.4}$$

Substituting (1.1.4) in (1.1.3) gives

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + f_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{1.1.5}$$

The components of the Ricci tensor are

$$R^0_0 = 3\frac{\ddot{a}}{a}, \tag{1.1.6}$$

$$R^i_j = \Big(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\Big)\delta^i_j, \tag{1.1.7}$$

while the Ricci scalar is

$$R = 6\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\Big). \tag{1.1.8}$$

The stress energy tensor, T^μ_ν determines the matter distribution in spacetime. I consider the perfect fluid approximation for the matter filling the FRW spacetime given by

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu - p\delta^\mu_\nu, \tag{1.1.9}$$

where u^μ is the fluid co-moving four velocity (i.e. the reference frame in which the fluid is at rest) specified by $u^\mu = (1, 0, 0, 0)$. Note that in general, the rest frame is $u^\mu = (\sqrt{|g_{00}|}, 0, 0, 0)$. The fluid approximation holds quite well for the observable Universe where individual galaxies and clusters of galaxies mimic microscopic fluid particles on the grand scale. Therefore the field equations with the inclusion of cosmological constant are

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{1.1.10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.1.11}$$

The effective gravitational energy is given by $\rho+3p$ as the pressure also contributes to gravitation. Note that $p < -\rho/3$ implies repulsive gravitation. Equation (1.1.10) is commonly called the Friedmann equation while (1.1.11) is sometimes termed the Raychaudhri equation [6]. Note that accelerated expansion of the Universe $\ddot{a} > 0$ can be obtained if $\rho+3p < 0$. Here $H = H(t)$ is called the Hubble parameter which determines the expansion rate of the Universe. Its current value is called the Hubble constant, denoted by H_o. According to observations of supernovae of type Ia, it is estimated that [8, 9, 10]

$$H_o = 72 \pm 8 \text{ km/s/Mpc}. \tag{1.1.12}$$

Generally the uncertainty in the determination of its value is customarily defined by introducing another parameter h as

$$H_o = h \times 100 \text{ km/s/Mpc}, \tag{1.1.13}$$

where $h = 0.72 \pm 0.08$ is the uncertainty factor. The Hubble parameter was first introduced by Edwin Hubble to model the recessional velocities of galaxies [3]. He found on the basis of empirical findings, a linear relationship between the recessional velocity v of a galaxy and its corresponding distance r from us. The relation is

$$v = cz = H_o r, \tag{1.1.14}$$

called the 'Hubble Law'. Note that this law is valid for small redshift values. Here z is the cosmological redshift (which is different from the Doppler redshift) parameter described by the ratio of the stretching of the wavelength to the original wavelength λ_o of the incoming light from the distant source i.e.

$$z = \frac{\lambda - \lambda_o}{\lambda_o}, \tag{1.1.15}$$

or

$$1 + z = \frac{\lambda}{\lambda_o} = \frac{a_o}{a(t)}. \tag{1.1.16}$$

This result was observationally obtained in 1929 and was taken as evidence for an expanding Universe. Until then many physicists had believed that the Universe was static. However, after the observational evidence for a dynamic Universe was put forward, they had to admit that this was not the case. The Universe is dynamic and is in a state of expansion.

(1.1.10) and (1.1.11) apparently show that Λ contributes positively to the total pressure and hence it induces a repulsive effect (note that a negative $\Lambda < 0$ produces attractive effect). To obtain *Einstein's static Universe*, I set $H = 0$ and $\ddot{a}/a = 0$ in the above two equations to get

$$\rho = -3p = \frac{3}{8\pi G}\frac{k}{a^2}. \tag{1.1.17}$$

This equation shows that either ρ or p has to be negative to keep the Universe static. Moreover if $\rho + 3p < 0$, then it yields the expanding solution of the Universe, which is why Einstein considered the above solution to be invalid and so added the cosmological constant term in his field equations. For a static

Universe, using $p = 0$, one obtains

$$\rho = \frac{\Lambda}{4\pi G}, \quad \Lambda = \frac{k}{a^2}. \tag{1.1.18}$$

Since $\rho > 0$, one requires $\Lambda > 0$. This further implies $k = +1$, representing a spatially closed Universe of the radius of curvature $a = 1/\sqrt{\Lambda}$. Moreover all the energy density is contained in the parameter ρ. The corresponding metric will be specified as

$$ds^2 = -dt^2 + \frac{dX^2}{1 - \Lambda X^2} + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.1.19}$$

where $X = ar$, a is the scale factor and r is a dimensionless parameter that replaces the radial coordinate. From the consideration of the field equations it was obvious that for any small perturbation different in the static configuration the perturbation would only magnify itself, eventually leading to the Universe to collapse on itself due to over-density of the matter. This Universe model is considered unstable and unphysical.

The Einstein field equations with cosmological constant also yield an exotic solution representing an empty Universe, the so-called the *de Sitter Universe* [11] obtained by taking a positive Λ as the energy-matter source. (A negative Λ gives the *anti-de Sitter Universe.*) The field equation (1.1.10) gives

$$\dot{a}^2 - \varpi^2 a^2 = -k, \tag{1.1.20}$$

where $\varpi^2 \equiv \Lambda/3$. The last equation yields solutions

$$a(t) = \begin{cases} \frac{\sqrt{k}}{\varpi}\cosh\varpi t, & k = +1, \\ e^{\varpi t}, & k = 0, \\ \frac{\sqrt{|k|}}{\varpi}\sinh\varpi t, & k = -1. \end{cases} \tag{1.1.21}$$

This solution has no singularities for any value of k or t i.e. finite $a(t)$, however minimum values of $a(t)$ can be obtained for the three cases: when $k > 0$, the minimum lies at $t = 0$, it also gives a bouncing solution at $t = 0$ i.e. contraction phase followed by expansion; while for $k = 0$, the minimum value of $a(t)$ is obtained only when $t \to -\infty$; and finally for $k < 0$, a coordinate singularity is obtained at $t = 0$.

The de Sitter metric can also be written in terms of spherical coordinates as

$$ds^2 = -(1 - H^2 r^2)dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2(d\theta^2 + \sin^2 d\phi^2). \tag{1.1.22}$$

It is obvious that $r_s \equiv 1/H$ makes the metric singular. Note that in these coordinates, the parameter r has dimensions of length hence the above remark is seen more easily in these coordinates. Contrary to a black hole horizon, a light signal can be sent outside the horizon r_s but no light signal can be received from the outside.

Evolution in the FRW Universe

The most studied cosmological model is based on a widely accepted hypothesis that the observable Universe is spatially homogeneous and isotropic on large scales. Here the scales are of the order of hundreds of megaparsecs (1Mpc$\sim$ 10^6 pc). It is called the weak 'cosmological principle'. This principle is the generalized Copernican principle. Copernicus had said that the Earth is not at the center of the Universe. The generalization says there is no preferred center of the Universe. In other words, there is no special point in the Universe. Isotropy implies that there is no special spatial direction in the Universe, the galaxies are evenly distributed in different angular directions at large scales, hence isotropic. The equations of motion corresponding to FRW spacetime are

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.1.23}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.1.24}$$

An equation relating the energy, the pressure and the scale factor is required to complete the above system i.e. (1.1.23) and (1.1.24). The energy-momentum tensor has to be divergence-free which signals the conservation of energy. The energy conservation equation (or the continuity equation) corresponding to the FRW is obtained by $T^{\mu\nu}_{;\nu} = 0$ to get

$$\dot{\rho} = -3H(\rho + p). \tag{1.1.25}$$

(1.1.23) can alternatively be written as

$$\Omega - 1 = \frac{k}{(aH)^2}, \tag{1.1.26}$$

where $\Omega = \rho/\rho_{cr}$ is the dimensionless density parameter and $\rho_{cr} = 3H^2/8\pi G$ is the critical density and its value approximately is 2×10^{-26} kg/m^3. The matter distribution determines the spatial geometry of the Universe i.e.

$$\Omega > 1 \Leftrightarrow k = +1, \tag{1.1.27}$$

$$\Omega = 1 \Leftrightarrow k = 0, \tag{1.1.28}$$

$$\Omega < 1 \Leftrightarrow k = -1. \tag{1.1.29}$$

A Universe with $\Omega > 1$ (total density is more then the critical density) will expand to a maximum size. After that it will start contracting and will end up in a Big Crunch. A Universe with $\Omega < 1$ (total density is less than the critical density) will expand forever and with ever increasing speed and a $\Omega = 1$ (total density just equal to the critical density), will expand forever with an expansion rate less than that of $\Omega < 1$. See Fig. 1.1

Note that there are two independent equations (1.1.23) and (1.1.24) for three dependent variables p, ρ and $a(t)$. To solve these equations, another equation relating pressure and the density, commonly called the equation of state (EoS) is needed. The simplest choice is

$$p = \omega\rho. \tag{1.1.30}$$

Here ω is a constant called the EoS parameter. There are three widely accepted constituents of the Universe i.e. matter, radiation and the vacuum energy.

Astrophysical and cosmological observations of supernova of type Ia, cosmic microwave background and the distribution properties of the large scale structures [6, 8, 9, 12, 13, 14] show that the observed Universe is spatially flat. Hence I shall assume a flat Universe model ($k = 0$) from here onwards unless otherwise stated.

Thus from the equations (1.1.23), (1.1.25) and (1.1.30), one gets

$$H = \frac{2}{3(1+\omega)(t-t_o)}, \tag{1.1.31}$$

$$a(t) = a_o(t-t_o)^{\frac{2}{3(1+\omega)}}, \tag{1.1.32}$$

$$\rho = \rho_o a^{3(1+\omega)}. \tag{1.1.33}$$

Here ρ_o is the energy density at time t_o. Generally matter is assumed to be pressureless i.e. $p = 0$ or $\omega = 0$, (1.1.32) and (1.1.33) yield

$$\rho \sim a^{-3}, \quad a \propto t^{2/3}. \tag{1.1.34}$$

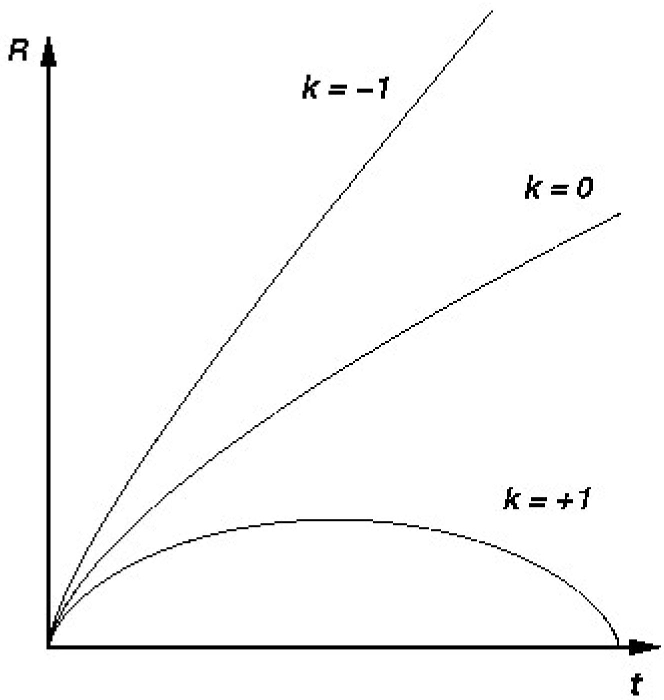

Figure 1.1: Evolution of the Universe (scale factor vs time) for different choices of curvature parameter k. For $k = -1$, the Universe is spatially open and expand forever (hyperbolic spacetime). $k = 0$ corresponds to spatially flat Universe and it also expands forever. However $k = +1$ is spatially closed Universe having finite age and it ends up in a Big Crunch. Recent astrophysical observations indicate that the observable Universe is spatially flat.

Thus the density of matter falls as the volume of the Universe increases. For radiation $\omega = 1/3$, (1.1.32) and (1.1.33) yield

$$\rho \sim a^{-4}, \quad a \propto t^{1/2}. \tag{1.1.35}$$

Thus the density of radiation falls one scale factor faster compared to matter density. Further, for any exotic vacuum energy with $\omega = -1$ (cosmological constant as its density remains constant and pressure is negative), one gets

$$\rho = \rho_o, \quad a \propto e^{Ht}. \tag{1.1.36}$$

Thus the energy density (and hence the pressure) of cosmological constant remains unchanged as the Universe expands.

1.1.2 History of the Universe

The problem regarding the origin of the observable Universe is still an open puzzle in quantum cosmology [15]. The problem arises since almost all cosmological models (with the exception of the steady state model) predict infinitely high temperatures and densities at the moment of the origin. That particular moment is commonly called the 'Big Bang'. The origin of the Universe from a Big Bang singularity requires a deeper understanding of the structure of spacetime near the Planck scale. A theory of quantum gravity, which is as yet not developed, may explain the Big Bang puzzle. For the time being it can be anticipated that the known physical Universe originated from a Big Bang singularity at some time, which can be taken as $t = 0$ (by choosing $t_0 = 0$ in (1.1.32)). At the Planck scale, 10^{-43} sec, quantum effects dominate and the dynamics are to be explained by a theory of quantum gravity. There are several candidates of quantum gravity including string theory, twistors, canonical quantization, loop quantum gravity, causal sets and Hawking's no-boundary proposal, to name a few. However there are several problems with each of these. At this time, the Universe was filled with a plasma of relativistic elementary particles, including mainly quarks, leptons and gauge bosons. It is generally conjectured that the four forces of nature (strong, weak, electromagnetism and gravity) were unified at that time. As quantum effects dominate at this small scale, it might be guessed that spacetime was in a state of chaotic fluctuation. In that case time was not well-defined and the topology of spacetime could not be defined.

If the Universe at $t = 1$s had a temperature of $T = 10^{10}$ K, say, then the temperature would have dropped by a factor of 10 to $T = 10^9$ K at $t = 100$ sec. The initial Universe was a hot Universe dominated by radiation. However, today the Universe is more dominated by matter (dust) than radiation. The transition from a radiation to a dust dominated model, is believed to occur around $t =$44,000 years. Since this time, the dynamics of the Universe has been driven by matter and vacuum energy. As the temperature of the radiation cooled it reached a point where it did not have enough energy to keep the atoms ionized. At around $t =$ 400,000 years matter and radiation decoupled. During the period before this time the radiation was thermalized and in thermal equilibrium with the matter. But at this point, the free electrons could bind to a nucleus and form a neutral atom. Hence, the first atoms in the Universe were created about 400,000 years after the Big Bang. The photons moved freely after this time as there were no free electrons available for Compton-scattering. Effectively, the Universe became transparent. This time in the history of the Universe is called the recombination era. These photons are what make out the cosmic microwave background radiation (CMB). Today the cosmic microwave background radiation has a temperature of about 2.7 K but the radiation was emitted approximately 300,000 years after the Big Bang at a temperature of T=3,000 K. Hence, this radiation is the relic of the Universe, when it was only 300,000 years old.

1.1.3 Some problems in cosmology

Modern cosmology has achieved several successes in explaining various aspects of the Universe. Together with nuclear physics, particle physics and quantum theory, it has successfully explained the formation of light chemical elements, the origin of structures (galaxies and galaxy clusters) and the origin of CMB, to name a few. But there are still some open problems in cosmology which are big challenge for cosmologists. I discuss some of them below.

Horizon problem

The horizon problem originates with the observation of extreme isotropy of the CMB. The CMB data indicates that the radiation is very much like the black body radiation and can be represented by the Planck distribution. It implies that the radiation had been in thermal equilibrium in the past and experienced thermal contact with the matter component. Nevertheless, the CMB data shows that the radiation coming out from the surface of last scattering from two opposite directions could not have thermal contact because the particle horizon of each photon in the last scattering surface covers only a small patch of the sky. The size of the particle horizon at the time of decoupling can be determined as follows. Consider the volume of the particle horizon V_{ph} at the time of decoupling

$$(V_{ph})_d = \left(\frac{t_d}{t_o}\right)^3 V_o, \tag{1.1.37}$$

where t_d and t_o are the time of decoupling and the present time, respectively while V_o is the current horizon volume. Events inside the particle horizon are in causal contact with each other, thus the radiation and everything else inside the $(V_{ph})_d$ had been in thermal contact and thermal equilibrium.

Let us introduce the magnitude of V_o at the time of decoupling $(V_o)_d$, thus

$$(V_o)_d = \frac{a^3(t_d)}{a^3(t_o)} V_o = \left(\frac{t_d}{t_o}\right)^2 V_o. \tag{1.1.38}$$

Comparing the last two equations gives

$$\frac{(V_{ph})_d}{(V_o)_d} = \frac{t_d}{t_o}. \tag{1.1.39}$$

Using the approximated values $t_d = 3 \times 10^5$ years and $t_o = 15 \times 10^9$ years, one obtains the horizon size at the time of decoupling was merely a small patch 2×10^{-5} of the observable part of our Universe, thus representing a horizon problem.

Flatness problem

The flatness problem naively asks why Ω_{tot} is so close to unity or why the spatial curvature k is almost zero. Quantitatively, from the first FRW equation,

$$\Omega_{tot} - 1 = \frac{k}{aH^2} = \frac{k}{\dot{a}^2} = \Omega_k. \tag{1.1.40}$$

If Ω_{tot} is a constant equal to unity, it will remain that forever. However if it is a variable then it will evolve as $\Omega_{tot}(t)$: In matter dominated phase, $|\Omega_{tot} - 1| \propto t^{2/3}$ while in the radiation dominated phase, it will $|\Omega_{tot} - 1| \propto t$. Using this last relation, one can estimate at the time of Big Bang nucleosynthesis, $|\Omega_{tot} - 1| \sim 10^{-16}$ while near the Planck time, the difference was $|\Omega_{tot} - 1| \sim 10^{-60}$. Therefore some rational explanation is required to justify such an exquisite balance. This discussion implies that the Universe started in a state of spatial flatness.

Cosmic coincidence problems

Another puzzle in the standard cosmological model is the coincidence problem or the 'why now?' problem. Briefly put, if Λ is tuned to give $\Omega_\Lambda \sim \Omega_m$ today, then for essentially all of the previous history of the Universe, the cosmological constant was negligible in the dynamics of the expansion, while for the indefinite future, the Universe will undergo a de Sitter-type expansion with Ω_Λ near unity and all other components negligible. However the present Universe is not entirely de Sitter type and has other components like matter and radiation. The present epoch is then very special time in the history of the Universe, the only period when $\Omega_\Lambda \sim \Omega_m$ [16]. In the standard FRW model, both matter and dark energy evolve independently, thus $\rho_m \propto a^{-3}$ and $\rho_\Lambda \propto a^{-3(1+\omega_\Lambda)}$. Therefore the two densities are related as $\rho_\Lambda \propto \rho_m a^{-3}$. In order to solve the coincidence problem, $\rho_\Lambda \propto \rho_m$ is required. Hence the standard cosmological model may require sufficient modification to solve different problems.

One way to attempt to resolve this problem is to proceed with some kind of scaling relation like $\rho_m/\rho_\Lambda = r_m$ and study its dynamics under a constraint equation $r_m(t = t_0) = 1$. The corresponding solutions are called scaling solutions and they have provided a natural solution to the cosmic coincidence problem. The cosmic coincidence problem states: why the cosmic acceleration is so recent or in my presence? Also why the two energy densities of vacuum energy and the matter are so closely comparable at the present time/epoch? In an investigation, Dodelson et al [17] have proposed that the dark energy has periodically dominated in the past so that its preponderance today is natural.

Fine tuning problem

The fine tuning problem asks why is the energy density today so small compared to typical particle physics scale? If $\Omega_m \sim 0.3$ today the missing energy density is of order $10^{-47} GeV^4$, which appears to require the introduction of a new mass scale 14 or so order of magnitude smaller than the electroweak scale. The transition from the earlier matter dominated phase to the current dark energy dominated requires fine tuning of density parameters [18].

The matter energy density is related to the redshift parameter z as

$$\rho_m(z) = (1+z)^3 \rho_{mo}, \tag{1.1.41}$$

where ρ_{mo} is the current matter density. In standard FRW model

$$t(a) = \int_0^a \frac{da'}{a' H(a')} = \int_0^a \frac{da'}{H_o \sqrt{\Omega_m a'^{-1} + \Omega_\Lambda a'^2}}, \tag{1.1.42}$$

where

$$a = \frac{1}{1+z} \tag{1.1.43}$$

for $z = 0$ at current time, $a_o = 1$. Taking $\Omega_m \approx 0.3$, and $\Omega_\Lambda \approx 0.7$ for today, a relationship between the energy density of dark energy to the current matter density is

$$\rho_\Lambda = \frac{\Omega_\Lambda}{\Omega_m} \rho_{mo} \approx 2.33 \rho_{mo}. \tag{1.1.44}$$

Therefore the dark energy occupies more then two times the normal matter density in the observable Universe. One can also obtain the redshift corresponding the transition, using (1.1.16) and (1.1.23)

$$1 + z_{tr} = \left(\frac{2\Omega_\Lambda}{\Omega_m} \right)^{1/3} \approx 1.67 \tag{1.1.45}$$

More precisely from the observational point of view, it is $1 + z_{tr} = 1.46 + 0.13$. To determine the age of the Universe, we substitute the following in (1.1.23),

$$x \equiv a = \frac{1}{1+z}, \tag{1.1.46}$$

to obtain

$$t(z) = \int_0^t dt = H_o^{-1} \int_0^{\frac{1}{1+z}} \frac{dx}{\sqrt{\Omega_m x^{-1} + \Omega_\Lambda x^2}}. \tag{1.1.47}$$

Using $H_o^{-1} \approx t_o$ (current age of the Universe) and $z \approx 0.6$, one obtains $t_{tr} \sim 7.14$ Gyr, showing that the transition time is not that older than the total age of the Universe. However, this estimate for transition time is very crude because the energy density of radiation is ignored.

I would like to mention here that several important problems in cosmology are not discussed here due to their irrelevance like the entropy problem, monopole problem and topological defects, matter- anti-matter asymmetry etc.

1.1.4 Cosmological inflation: a possible solution to cosmic puzzles

A solution to these cosmological problems was suggested by Alan Guth in 1981 [19] by introducing the so-called inflationary model of the Universe. Another motivation to discuss inflation here is that this rapid expansion was driven by the vacuum energy (dark energy) of the 'false vacuum' (i.e. local minimum of the potential function) which interestingly has more energy than the 'true vacuum' (i.e. global minimum of the potential function). Let us first sketch the basic idea behind inflation. The fundamental idea is that the Universe undergoes exponential accelerated expansion, at some time in the past i.e. $\ddot{a} > 0$ and $a(t) \sim e^{Ht}$. It is assumed that the driving force behind inflation was some sort of dark energy, which satisfies $p = -\rho$ (or a cosmological constant). Generically it is thought to occur closer to the GUT scale ($\sim 10^{16}$ GeV). Due to this rapid expansion, small regions of the Universe expanded quickly to much larger sizes, thereby diminishing any spatial curvature and making the Universe flat. The size of the horizon apparently increased indefinitely and topological defects (in particular magnetic monopoles) produced as a result of phase transitions were diluted. Cosmic inflation later on also produced density fluctuations which lead to structure formation.

The vacuum energy density is represented by a scalar field (or an inflaton field) with the Lagrangian density given by

$$L_\Phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi), \tag{1.1.48}$$

where $V(\Phi)$ is the potential energy. The action integral is

$$S = \int d^4x \sqrt{-g} L_\Phi, \tag{1.1.49}$$

where g is the determinant of the metric tensor. The corresponding energy momentum tensor is

$$T^\Phi_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2}(g^{\sigma\gamma} \partial_\sigma \Phi \partial_\gamma \Phi) g_{\mu\nu} - V(\Phi) g_{\mu\nu}. \tag{1.1.50}$$

The lowest energy density of the scalar field is obtained if the kinetic (or the gradient) term vanishes i.e. $\partial_\mu \Phi = 0$ and that the potential is at the minimum

$V(\Phi_{min})$. Hence from (1.1.50) one gets

$$T_{\mu\nu}^{vac} = -V(\Phi_{min})g_{\mu\nu} = -\rho_{vac}g_{\mu\nu}, \tag{1.1.51}$$

which is the Lorentz invariant form of the vacuum energy (the cosmological constant). The dark energy may also be considered as a perfect fluid with the equation of state

$$p_{vac} = -\rho_{vac} \tag{1.1.52}$$

The equation of motion of the scalar field is

$$\ddot{\Phi} + 3H\dot{\phi} + V'(\Phi) = 0. \tag{1.1.53}$$

The scalar field Φ is not itself a physically observable quantity but in the flat Friedmann model, it is related to the pressure and the energy density parameters as

$$\rho = -T_0^0 = \frac{1}{2}\dot{\Phi}^2 + V(\Phi), \quad p = T_i^i = \frac{1}{2}\dot{\Phi}^2 - V(\Phi). \tag{1.1.54}$$

Thus Friedmann equations becomes

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\right], \tag{1.1.55}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{3}\left[\dot{\Phi}^2 - V(\Phi)\right]. \tag{1.1.56}$$

Note that $\kappa^2 = 8\pi G$. Inflation will take place if the *slow roll approximation* is satisfied i.e. neglecting the $\ddot{\Phi}$ in (1.1.53) and neglecting the kinetic energy of Φ compared to the potential energy [20]. Hence the scalar field dynamical equation and the first Friedmann equation becomes

$$\dot{\Phi} \simeq -\frac{V'(\Phi)}{3H}, \tag{1.1.57}$$

$$H^2 \simeq \frac{\kappa^2}{3}V(\Phi). \tag{1.1.58}$$

Here prime denotes differentiation w.r.t. Φ. Differentiating the previous equation and using the value of $\dot{\Phi}$ gives

$$\ddot{\Phi} = -\frac{V''\dot{\Phi}}{3H} + \frac{\dot{H}\dot{\Phi}}{H}. \tag{1.1.59}$$

Now using (1.1.57) and (1.1.58) yields

$$\ddot{\phi} = (-\eta + \epsilon)H\dot{\phi}, \tag{1.1.60}$$

where the *slow roll parameters* are defined as

$$\epsilon \equiv \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2, \tag{1.1.61}$$

$$\eta \equiv M_p^2\frac{V''}{V}. \tag{1.1.62}$$

These parameters will give the slow roll approximation if the *slow roll conditions* $|\epsilon| \ll 1$ and $|\eta| \ll 1$, are satisfied. It can easily be seen that the slow roll conditions indeed yield inflation by $\ddot{a}/a > 0$. One can write

$$\frac{\ddot{a}}{a} = \dot{H} + H^2, \tag{1.1.63}$$

so that inflation will occur when

$$\frac{\dot{H}}{H^2} > -1. \tag{1.1.64}$$

Under slow roll, it becomes

$$\frac{\dot{H}}{H^2} \simeq -\epsilon, \tag{1.1.65}$$

which is small, hence inflation ensured i.e. the Hubble parameter changes very slowly during inflation.

The number of e-foldings during inflation is

$$N = ln\left(\frac{a_f}{a_i}\right), \tag{1.1.66}$$

where a_i and a_f are the initial and final values of the scale factor. In the slow roll approximation, the potential is almost constant in time and therefore the parameter H is also constant. Then the number of e-foldings is given by

$$N = \int_{t_i}^{t_f} H dt. \tag{1.1.67}$$

Since $dt = -(3H/V')d\Phi$, thus the last equation gives

$$N = -\frac{1}{M_p^2} \int\limits_{\Phi_i}^{\Phi_f} \frac{V}{V'} d\Phi. \tag{1.1.68}$$

Numerical value of N turns out to be 60 [15].

The mechanism of cosmological inflation is introduced here since it was driven by the vacuum energy much like that for present accelerated expansion.

1.2 Dark energy

Numerous astrophysical and cosmological observations of supernovae of type Ia [8, 9], Wilkinson Microwave Anisotropy Probe (WMAP) data of the CMB [21], Sloan Digital Sky Survey (SDSS) of the extragalactic Universe [14] and X-ray [22] have shown that the observable Universe is undergoing an accelerated expansion. The type Ia supernova is one of the most powerful tool to probe the expansion rate of the Universe. The type Ia supernova is a burst of a white dwarf star which just reaches the Chandrasekhar limit (1.4 solar mass) due to accretion of matter from its neighboring binary normal (main-sequence) star and then explodes. These supernovae have the same local luminosity since they have roughly the same mass and explosion process. They can serve as 'standard candles' in the Universe. A sample of type Ia supernovae will generate a diagram of Hubble parameter versus distance, through which the information of the expanding velocity in the history of the Universe is obtained. In 1998, two independent teams found that the Universe is accelerating using the observational data of these supernova [8, 9]. I assume that the present evolution is well described by the general relativity theory with the FRW spacetime and the source of gravity is matter in perfect fluid form. Since any normal matter (baryons and radiation) cannot manifest this phenomenon, it can be proposed that some mysterious 'dark energy' is responsible. When dark energy is incorporated in the FRW equations, then the structure of the field equations might be modified. Theoretically, there are many candidates to explain accelerated expansion (to be discussed later in next section), yet there are inherent problems in each of them. One of the problems is to determine the equation of state parameter for dark energy ω_Λ. Observations suggest that $-1.38 < \omega_\Lambda < -0.82$ is not a constant but possesses a parametric form [23]. Another problem is to explain the phantom divide i.e. the transition from the quintessence to the phantom regime. Another unresolved problem is to explain why the energy densities of matter and dark energy are almost of the same order of magnitude, at same time.

Thermodynamic studies of dark energy show that a dark energy dominated Universe will become increasingly hotter [24], while other studies show that phantom dominated Universe possess negative temperature. The same study suggests that dark energy particles can be only massless bosons. There is also an attempt to describe the accelerated expansion without invoking the dark energy i.e. $\Omega_\Lambda = 0$ [25]. In this model, the dark energy phenomenon is explained by some form of particle creation process out of gravitational field.

An explanation of accelerated expansion is to assume a quantum field theoretic vacuum energy ρ_Λ (or ρ_{vac}) or any other dark energy agent/fluid having positive energy density and negative pressure satisfying $p_{vac} < -\rho_{vac}/3$. If dark energy exists then its magnitude is certainly close to the cosmological energy density i.e. $\rho_{cr} \approx 4 \times 10^{-47} GeV^4$, which essentially correspond to 70% to the total cosmological energy density. From theoretical considerations (see section 1.4.2), the value of ρ_{vac} is roughly $10^{50} - 10^{120}$ times higher then is deduced from the empirical data. Thus the dark energy scenario poses three main problems:

1. Why the observed vacuum energy so small or why do all the contributions to the effective cosmological constant term cancel each other up to a very large number of decimal places?

2. Why vacuum energy density is comparable with the matter energy density? In principle, this density should decline so the natural question is to ask why the observed ratio $\rho_m/\rho_{vac} \sim 1$?

3. If the agent behind causing this accelerated expansion is not vacuum energy then what alternative source could be identified at least theoretically? or what is causing the cosmic acceleration? or why is the acceleration happening during the present epoch of the cosmic evolution? Is new physics required to explain this?

1.2.1 What is the vacuum?

Apparently vacuum is a state of emptiness or nothingness. In various branches of theoretical physics, the state of vacuum is interpreted quite differently [26]: *In classical physics*, vacuum corresponds to a state of a system without any particle. *In quantum physics*, the vacuum is not an 'emptiness' rather a more exquisite form of particle-antiparticle production and annihilation. *In relativity*, the vacuum corresponds to a space which is not curved, so-called Minkowskian space. *In cosmology,* the notion of the vacuum and an associated concept of vacuum energy is introduced to describe the mysterious 'dark energy', an energy which induces negative pressure on the cosmological scale (i.e. on the length scale greater than a billion light year) and causes accelerated expansion of the Universe.

Recent astrophysical observations have shown convincingly that the energy density corresponding to vacuum energy Ω_Λ dominates in the Ω_{tot}, the total energy density of the Universe. Before the dark energy domination at a redshift $z \sim 0.5$, the Universe was undergoing decelerated expansion.

$$\Omega_{tot} = \Omega_\Lambda + \Omega_{dm} + \Omega_b + \Omega_\gamma. \tag{1.2.1}$$

Here Ω_{dm} is the density of dark matter; Ω_b corresponds to density of baryons and Ω_γ, the density of photons or radiations. Their approximate values are [27, 28]

$$\Omega_\Lambda \approx 0.73, \quad \Omega_{dm} \approx 0.27, \quad \Omega_b \approx 0.045, \quad \Omega_\gamma \approx 5 \times 10^{-5}. \tag{1.2.2}$$

In literature, several candidates of dark energy are considered including cosmological constant, quintessence, Chaplygin gas and phantom energy, to name a few. I shall discuss these various alternatives in the coming subsections. A contrary viewpoint to dark energy is presented by Wilshire [29, 30, 31] which suggests that cosmic acceleration can be understood as an apparent effect, and dark energy as a misidentification of those aspects of cosmological gravitational energy which by virtue of the strong equivalence principle cannot be localized, namely gradients in the quasi-local gravitational energy associated with spatial curvature gradients, and the kinetic energy of expansion, between bound systems and the volume-average position in freely expanding space. It should be noted that in this thesis, I shall discuss dark energy in the former context.

1.3 Candidates of dark energy

1.3.1 Cosmological constant

The simplest candidate for dark energy is the cosmological constant which is so called because its energy density is constant in time and space. As mentioned earlier, the cosmological constant, was originally introduced by Einstein in 1917 to achieve a static universe (see section 1.1 for details). After Hubble's discovery of the expansion of the universe in 1929, it was dropped by Einstein as it was no longer required. From the point of view of particle physics, however, the cosmological constant naturally arises as an energy density of the vacuum. Moreover, the energy scale of Λ should be much larger than that of the present Hubble constant H_0, if it originates from the vacuum energy density. This is the "cosmological constant problem" and was well known to exist long before the discovery of the accelerated expansion of the universe in 1998.

There have been a number of attempts to solve this problem. An incomplete list includes: adjustment mechanisms, anthropic considerations, changing gravity, quantum gravity, degenerate vacua, causal sets, higher dimensional gravity, supergravity, string theory spacetime foam approach and vacuum fluctuations of the energy density (see [6] for a comprehensive review regarding these attempts).

Special relativity reduces to classical mechanics in the limit of slow speeds $v/c \to 0$ i.e. by setting $c \to \infty$ or $1/c = 0$. General relativity reduces to special relativity if we have sufficiently low masses for the gravitational source. As M appears only in the combination GM/c^2, we can take $G \to 0$ for the 'special relativity limit'. Further, by Dirac quantization procedure we know that the classical limit for quantum theory is $\hbar \to 0$. If in any physical situation none of these can be taken to be zero, we will need a consistent theory of quantum gravity. If we construct dimensional quantities involving the above three constants of nature,we can be sure that at those values quantum gravity cannot be ignored.

$$\begin{aligned} l_p &= \sqrt{G\hbar/c^3} \sim 10^{-33}\text{cm}, \quad \text{Planck length,} \\ t_p &= \sqrt{G\hbar/c^5} \sim 10^{-42}\text{sec}, \quad \text{Planck time,} \\ m_p &= \sqrt{\hbar c/G} \sim 10^{-5}\text{g}, \quad \text{Planck mass.} \end{aligned}$$

Note that the above argument does not say that 'quantum gravity effects *start* at Planck scale', only says 'quantum gravity effects must have started *by* Planck scale'.

Quantum field theory applied naively gives infinite probability for all fields (even for electrodynamics). To get meaningful answers we re-normalize the vectors of the Hilbert space. This re-normalization gives finite answers but modifies the lowest approximation answers. Quantization of the Einstein-Hilbert Lagrangian generically yields a cosmological term related to the onset of quantum gravity. The natural expectation is that it occurs at Planck scale and hence the vacuum cosmological energy density will $\sim mc^2$. Thus the vacuum energy density should be $\sim 10^{16}$erg/cc. The observed accelerated expansion of the Universe corresponds to $\sim 10^{-9}$ erg/cc !

The similarity of the cosmological constant with the vacuum energy density arises from the equivalence of the vacuum energy momentum tensor in (1.1.17) with the cosmological term in (1.1.18). It is due to this fact many authors use vacuum energy and cosmological constant interchangeably. Thus adding these terms one gets the effective cosmological constant term as

$$\frac{\Lambda_{\rm eff}}{8\pi G} = \frac{\Lambda}{8\pi G} + \rho_{vac}. \tag{1.3.1}$$

From the quantum field theory point of view, the vacuum energy density arises by the sum zero point energy density of all the harmonic oscillators as

$$\rho_{vac} = \frac{1}{2}\sum \hbar\omega \tag{1.3.2}$$

A free quantum field can be considered as a collection of an infinite number of harmonic oscillators in the momentum space. Incidently, the zero point vacuum energy of all these oscillators will be infinite collectively. If however the high momentum modes are ignored then

$$\begin{aligned}\rho_{vac} &= \frac{1}{2}\int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{k^2+m^2} \\ &= \frac{1}{4\pi^2}\int_0^\infty dk k^2\sqrt{k^2+m^2}\end{aligned} \tag{1.3.3}$$

where m is the mass of the quantum field and the above equation implies

$$\rho_{vac} \propto k^4. \tag{1.3.4}$$

Assuming a cut-off scale the QFT in which the above integral is finite

$$\rho_{vac} \simeq \frac{k^4_{max}}{16\pi^2} \tag{1.3.5}$$

If one takes $k_{max} = m_{pl} = 1.22 \times 10^{19} GeV$, then

$$\rho_{vac} \approx 10^{74} GeV^4, \tag{1.3.6}$$

which is about 10^{121} order of magnitude larger than the observed value $\rho_\Lambda = \Lambda/4\pi \simeq 10^{-47} GeV^4$. Even if we take an energy scale of quantum chromodynamics (QCD) for k_{max}, we obtain $\rho_{vac} \approx 10^{-3} GeV^4$, which is still much larger than ρ_Λ. This problem was present even before the observational discovery of dark energy in 1998. At that time most people believed that the cosmological constant was exactly zero and tried to explain why it was so. The vanishing of a constant usually implies the existence of some symmetry. In super-symmetric theories, for example, the bosonic degree of freedom as its Fermi counterpart that contributes to the zero-point energy with an opposite sign. If super-symmetry is unbroken, there exists an equal number of bosonic and fermionic degrees of freedom such that the total vacuum energy vanishes [7]. However it is known that super-symmetry is broken at sufficiently high energies (around the scale 10^3 GeV). Hence the vacuum energy is generally non-zero in the world of broken super-symmetry. The origin of this vacuum energy could be a scalar field that comes after the quantum gravity scale.

1.3.2 Quintessence

As discussed earlier, that Λ possesses $\omega_\Lambda = -1$; remains practically constant in time i.e. static; having constant negative pressure and energy density and is also spatially homogeneous. Quintessence is mathematically rich concept to describe dark energy. It is represented by a time dependent and spatially heterogeneous scalar field Φ which is minimally coupled to gravity. Its equation of state is dependent on the redshift parameter $\omega(z)$, in particular, $\omega(z) > -1$ and it also evolves with time. The action for the quintessence is given by (1.1.49). From (1.1.56), $\ddot{a} > 0$ if $\dot{\Phi}^2 < V(\Phi)$. The equation of state for the quintessence field is

$$\omega_\Phi = \frac{p}{\rho} = \frac{\dot{\Phi}^2 - 2V(\Phi)}{\dot{\Phi}^2 + 2V(\Phi)} \tag{1.3.7}$$

Thus the continuity equation becomes

$$\rho = \rho_o \exp\left[-\int 3(1+\omega_\Phi)\frac{da}{a}\right], \tag{1.3.8}$$

where ρ_o is the constant of integration. Note that if $\omega_\Phi = -1$, then it gives $\rho = const$ which corresponds to the slow roll limit i.e. $\dot{\Phi}^2 \ll V(\Phi)$. Conversely if $\dot{\Phi}^2 \gg V(\Phi)$ then it implies $\omega_\Phi = 1$ which yields $\rho \propto a^{-6}$. In general for

$-1 < \omega_\Phi < 1,$

$$\rho \propto a^{-m}, \quad 0 < m < 6. \tag{1.3.9}$$

The range $0 \leq m < 2$ corresponds to accelerated expansion of the Universe as $\omega_\Phi = -1/3$ is the border of acceleration and deceleration. The model of quintessence has also been investigated in the context of 'tracker field', in which the field Q rolls down a potential $V(Q)$ according to an attractor-like solution to the equations of motion [18]. The tracker solution is an attractor in the sense that a very wide range of initial conditions for Q and $\dot{Q}$ rapidly approach a common evolutionary track, so that the cosmology is insensitive to the initial conditions. The tracker field satisfies the similar equation as in (1.2.53). It has also been demonstrated that quintessence field coupled with matter fluid can also drive late cosmic accelerated expansion and simultaneously solve the coincidence problem [32].

The dynamics of ω can be studied as follows: Differentiation of (1.1.23) w.r.t. t and taking $k = 0$ yields

$$\dot{H} = -4\pi G(\rho + p). \tag{1.3.10}$$

Now using (1.1.23) and (1.3.4) yields

$$\omega = -1 - \frac{2\dot{H}}{3H^2}. \tag{1.3.11}$$

For an accelerated Universe, $\omega < -1/3$. When $-1 < \omega < -1/3$, the Universe is in quintessence phase and when $\omega = -1$, it is dominated by the cosmological constant. While for $\omega < -1$, the Universe is in the phantom phase. For an accelerating Universe, $\dot{H} > 0$ while for a decelerating state, $\dot{H} < 0$ is needed. While for a cosmological constant dominated phase, $\dot{H} = 0$. From theoretical point of view, a justification to describe the transition from $\omega > -1$ to $\omega < -1$ or from $\dot{H} < 0$ to $\dot{H} > 0$ is needed.

1.3.3 Chaplygin gas

The dark energy can also be described by an interesting equation of state commonly called the Chaplygin gas given by

$$p = -\frac{X}{\rho} \tag{1.3.12}$$

where X is a non-zero constant parameter which can be either positive or negative. The Chaplygin gas was introduced by Chaplygin in constructing a model to study the lifting force on a plane wing in aerodynamics [33]. It was first used in the cosmological context by Kamenshchik et al [34]. With this equation of state the continuity equation gives

$$\rho = \sqrt{X + \frac{Y}{a^6}}, \tag{1.3.13}$$

where Y is a constant of integration. In the asymptotic limit of the parameters, the behavior of the density is

$$\begin{aligned} \rho &\sim \sqrt{Y} a^{-3}, \quad a \ll (Y/X)^{1/6}, && (1.3.14) \\ \rho &\sim -p \sim \sqrt{A}, \quad a \gg (Y/X)^{1/6}. && (1.3.15) \end{aligned}$$

Therefore at sufficiently early times the Chaplygin gas behaves as pressureless dust when a was small, while for large a the gas behaves like cosmological constant leading to accelerated expansion. One can also calculate the kinetic and the potential parts of the gas using (1.1.54) to get

$$\begin{aligned} \dot{\Phi}^2 &= \frac{Y}{a^6 \sqrt{X + Y/a^6}}, && (1.3.16) \\ V(\Phi) &= \frac{1}{2}\left[\sqrt{X + Y/a^6} + \frac{X}{\sqrt{X + Y/a^6}}\right]. && (1.3.17) \end{aligned}$$

Several generalizations of Chaplygin gas has been proposed in the literature. Each generalization leads to the inclusion of new parameters. It makes the corresponding mathematics of the cosmological model much richer yet the model loses its predictive power. A commonly studied extension of Chaplygin gas is termed generalized Chaplygin gas (GCG) given by [35]

$$p = -\frac{X}{\rho^\alpha}, \tag{1.3.18}$$

where α is a constant to be determined and constrained by the observations. Recently it is proposed that the cosmological model based on Chaplygin gas coupled with dust best fits with the latest supernovae data [36, 37, 38, 39]. Debnath [41] proposed the most general form of the Chaplygin gas EoS by

$$p = X\rho - \frac{Y}{\rho^\alpha}. \tag{1.3.19}$$

$$p = -\frac{B(a)}{\rho^\alpha}, \quad B(a) = B_o a^n. \tag{1.3.20}$$

The last equation was proposed by Yi et al [40]. Merging the last two equations to get

$$p = X\rho - \frac{B(a)}{\rho^\alpha}, \quad B(a) = B_o a^n. \tag{1.3.21}$$

Above α, n and X are constant parameters. Note that for $X = 0$, the generalized Chaplygin gas is obtained while $B(a) = B_o$ yields the modified Chaplygin gas. If $\alpha < 0$, then it represents a polytropic equation of state. Analysis of cosmic microwave background radiation data put constraints on its parameters as $-0.35 \leq X \leq 0.025$ and $-0.021 \leq \alpha \leq 0.54$ [42]. Another such investigation by Bertolami et al [43] suggests that $\alpha > 1$ with 68% confidence level. Thermodynamical evolution of MCG shows that it best fits with other cosmological parameters if $\alpha = 1/4$ and $X = 1/3$ [44]. It is also recently shown that stable attractor solution for MCG exists at $\omega = -1$ i.e. that the EoS of MCG approaches it from the either $\omega > -1$ or $\omega < -1$, independent of the choice of its initial density parameter and the ratio of pressure to critical density [45]. This result suggests that the Universe would not end up in a Big Rip. However another investigation suggests that phantom like Chaplygin gas (which violates the null energy condition $\rho + p > 0$) yields a future singularity, which is different from the Big Rip since it will occur for a finite value of scale factor [46], thus their model supplied a dual singular model. Stability analysis of hybrid Chaplygin gas shows that it can also explain the phantom crossing scenario [47].

Another extension of Chaplygin gas is the 'new generalized Chaplygin gas' and the 'new modified Chaplygin gas' [48], respectively

$$p = \gamma\rho - \frac{\tilde{A}}{\rho^\alpha}, \quad \alpha > 0. \tag{1.3.22}$$

Here $\tilde{A} = -\omega A a^{-3(1+\omega)(1+\alpha)}$, with $A > 0$ constant.

1.3.4 Phantom energy

Phantom-like dark energy (or simply phantom energy) possesses strong negative pressure which can be simulated by a scalar field Φ with negative kinetic energy term in the Lagrangian given by

$$L_\Phi = \frac{l}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi), \tag{1.3.23}$$

here $l = -1$ corresponds to the phantom while $l = +1$ represents the standard scalar field which represent the quintessence field, also $V(\Phi)$ is the potential. Considering the perfect fluid energy momentum tensor, the above Lagrangian gives the following expressions for the energy density and the pressure respectively:

$$\rho_\Phi = \frac{l}{2}\dot{\Phi}^2 + V(\Phi), \quad \rho_\Phi = \frac{l}{2}\dot{\Phi}^2 - V(\Phi). \tag{1.3.24}$$

It results in the violation of the Null energy condition (NEC) $\rho_\Phi + p_\Phi = l\dot{\Phi}^2 > 0$, if $l = -1$. Since the NEC is the basic condition, its violation yields other standard energy conditions to be violated likewise Dominant energy condition ($\rho_\Phi > 0$, $\rho_\Phi \geq |p_\Phi|$) and the Strong energy condition ($\rho_\Phi + p_\Phi > 0$, $\rho_\Phi + 3p_\Phi > 0$). Due to the energy condition violations, it makes the failure of cosmic censorship conjecture and theorems related to black hole thermodynamics. The prime motivation to introduce this weird concept in cosmology does not come from the theory but from the observational data.

The most striking property of phantom energy is that its energy density varies as proportional to the power of the scale factor $a(t)$ and hence grows as the Universe expands (or as $a(t)$ increases) hence

$$\rho \propto a^{3|1+\omega|}, \quad \omega < -1. \tag{1.3.25}$$

which is quite unlike the behavior of normal matter whose density decreases with the growth of the Universe. Thus phantom energy presents a novel scenario of the evolution of the Universe, causing a future singularity commonly called the 'Big Rip'. At the singularity, the energy density of the phantom energy will become infinite and the scale factor shoot to infinity, all in a finite time [49]. Near the imminent singularity, everything is pulled apart and disassociate into the elementary particles. The phantom energy will destroy first the super-gigantic structures like galactic clusters, then galaxies, solar system, atoms and nuclei. Eventually at the Big Rip, even the very fabric of spacetime will be pulled apart.

The novel idea of Big Rip was proposed by Caldwell et al [50]. They asked a question: why restrict our attention exclusively to $\omega \geq -1$? Earlier Caldwell [51] called the matter with $\omega < -1$ as the 'phantom energy'. It was later proposed that phantom energy can support the existence of wormholes (to be discussed later). A Universe starts from the initial Big Bang singularity and ends up with another Big Rip singularity. The prediction of the initial and the final singularity are consistent with the observations.

There is a considerable deal of scientific literature that suggests that Big Rip may not be the necessary final outcome and can be avoided due to quantum processes near the singularity. Such studies are performed with the inclusion of back reaction effects in the calculations. One study by Baushev [52] shows that the phantom field will decay into matter near the Big Rip singularity. As the ripping effect of the phantom energy get stronger, the stretching of quark and anti-quark ($q\bar{q}$) pair increase the potential energy which will be sufficient to produce another such pair. This stretching effect will proceed in every $q\bar{q}$ pair. This pair production will eventually lead to a matter dominated Universe again, thus Big Rip will be avoided with the formation of matter. In this particular model which mimics a cyclic cosmology, there are no cosmological singularities since closer to each such event, matter production takes the lead. In another scenario, it is suggested that phantom fields can decay into one or more phantom particles in addition to ordinary matter particles [53].

In another investigation of phantom energy with the use of phantom scalar field dynamics [55], it is shown that infinite energy density of the phantom field (a consequence of Big Rip) depends on the parameter α in the power-law potential $V(\Phi) \sim \Phi^{\alpha}$. If $\alpha \leq 4$, than $\rho \to \infty$ for $t \to \infty$. Also $\rho \to \infty$ in a finite time if $\alpha > 4$.

In this thesis, I will consider dark energy as a homogeneous and isotropic fluid. In this way, we can characterize the dark energy by an equation of state which parameterizes homogeneous pressure and energy density. However if dark energy is essentially a scalar field, then the strong curvature near the black hole induces inhomogeneity in dark energy. It is expected that in this case the accretion of phantom energy on to the black hole would be somewhat different.

Any viable cosmological model must explain the phantom crossing ($\omega_{de} = -1$) or the phantom divide i.e. the transition of dark energy parameter $\omega_{de} > -1$ to $\omega_{de} < -1$. This problem originates from the observational data which suggests that ω_{de} is not a constant but varies over cosmological times. The two stages of evolution are described separately by quintessence field and the phantom field which correspond to $\omega_{de} > -1$ and $\omega_{de} < -1$ respectively. The phantom transition cannot be described by a single (quintessence or phantom) scalar field alone. However this problem can be avoided by invoking a two-scalar field model and connecting them in a single Lagrangian [56, 57]. This model is termed the quintom model, a merger of two words quintessence and phantom. Alternatively, one may consider single scalar field decomposed into two fields, quintessence and phantom, such that one field dominates at one instant while the other remains negligible [58, 59, 60]. Both of these considerations have gathered much interest in recent years since these explain the phantom crossing scenario.

For the sake of completeness, I would also mention some other possible candidates for dark energy: interacting dark energy-dark matter model [61, 62] agegraphic dark energy [63, 64], holographic dark energy [65, 66], tachyons [67] and K-essence [68] etc. I will not discuss these further due to their irrelevance for the present work.

1.4 Luminosity distance

There are several ways of measuring distances in the expanding universe. For instance one often deals with the co-moving distance which remains unchanged during the evolution and the physical distance which scales proportionally to the scale factor. An observationally relevant way of defining a distance is through the luminosity of a stellar object. The distance d_L known as the luminosity distance, plays a very important role in astronomy including supernova observations.

In Minkowski spacetime, the absolute luminosity L_s of the source and the energy flux F at a distance d is related through $F = L_s/(4\pi d^2)$. By generalizing this to an expanding universe, the luminosity distance, d_L, is defined as

$$d_L^2 \equiv \frac{L_s}{4\pi F}. \tag{1.4.1}$$

Let us consider an object with absolute luminosity L_s located at a coordinate distance χ_s from an observer at $\chi = 0$. The energy of light emitted from the object with time interval Δt_1 is denoted as ΔE_1, whereas the energy which reaches at the sphere with radius χ_s is written as ΔE_0. Note that ΔE_1 and ΔE_0 are proportional to the frequencies of light at $\chi = \chi_s$ and $\chi = 0$, respectively, i.e. $\Delta E_1 \propto \nu_1$ and $\Delta E_0 \propto \nu_0$. The luminosities L_s and L_0 are given by

$$L_s = \frac{\Delta E_1}{\Delta t_1}, \quad L_0 = \frac{\Delta E_0}{\Delta t_0}. \tag{1.4.2}$$

The speed of light is given by $c = \nu_1\lambda_1 = \nu_0\lambda_0$, where λ_1 and λ_0 are the wavelengths at $\chi = \chi_s$ and $\chi = 0$. Then, from (1.1.16) one finds

$$\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0} = \frac{\Delta t_0}{\Delta t_1} = \frac{\Delta E_1}{\Delta E_0} = 1 + z. \tag{1.4.3}$$

Combining the last two equations gives

$$L_s = L_0(1+z)^2. \tag{1.4.4}$$

The light traveling along the χ-direction satisfies the geodesic equation $ds^2 = -dt^2 + a(t)^2 d\chi^2 = 0$. One obtains

$$\chi_s = \int_0^{\chi_s} d\chi = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{h(z')}, \tag{1.4.5}$$

where $h(z) = H(z)/H_0$. From the FRW metric, the area of the sphere at $t = t_0$ is given by $S = 4\pi(a_0 f_k(\chi_s))^2$. Hence the observed energy flux is

$$F = \frac{L_0}{4\pi(a_0 f_k(\chi_s))^2}. \tag{1.4.6}$$

Substituting (1.4.5) and (1.4.6) in (1.4.1), the luminosity distance in an expanding universe:

$$d_L = a_0 f_k(\chi_s)(1+z). \tag{1.4.7}$$

In the flat FRW model with $f_k(\chi) = \chi$,

$$d_L = \frac{1+z}{H_0} \int_0^z \frac{dz'}{h(z')}, \tag{1.4.8}$$

where (1.4.5) is used in obtaining (1.4.8). Then, the Hubble rate $H(z)$ can be expressed in terms of $d_L(z)$:

$$H(z) = \left\{ \frac{d}{dz} \left(\frac{d_L(z)}{1+z} \right) \right\}^{-1}. \tag{1.4.9}$$

Measuring the luminosity distance observationally, one can determine the expansion rate of the universe.

The energy density ρ includes all components present in the Universe i.e.

$$\rho = \sum_i \rho_i^{(0)} (a/a_0)^{-3(1+\omega_i)} = \sum_i \rho_i^{(0)} (1+z)^{3(1+\omega_i)}. \tag{1.4.10}$$

Here ω_i and $\rho_i^{(0)}$ correspond to the equation of state and the present energy density of each component, respectively. Using the first Friedmann equation, the last equation takes the form

$$H^2 = H_0^2 \sum_i \Omega_i^{(0)} (1+z)^{3(1+\omega_i)}, \tag{1.4.11}$$

where $\Omega_i^{(0)} = \rho_i^{(0)}/\rho_{cr}$ is the density parameter for an individual component at the present epoch. Hence, the luminosity distance in a flat geometry is given by

$$d_L = \frac{1+z}{H_0} \int_0^z \frac{dz'}{\sqrt{\sum_i \Omega_i^{(0)} (1+z')^{3(1+\omega_i)}}}. \tag{1.4.12}$$

In Figure 1.2, we provide the luminosity distance versus the redshift for a variety of data sets from the observations of supernovae Ia. The Type Ia supernova (SN Ia) can be observed when white dwarf stars exceed the mass of the Chandrasekhar limit and explode. The belief is that SN Ia are formed in the same way irrespective of where they are in the universe, which means that they have a common absolute magnitude independent of the redshift z. Thus, they can be treated as ideal standard candles. We can measure the apparent magnitude and the redshift observationally, which of course depends upon the objects we observe.

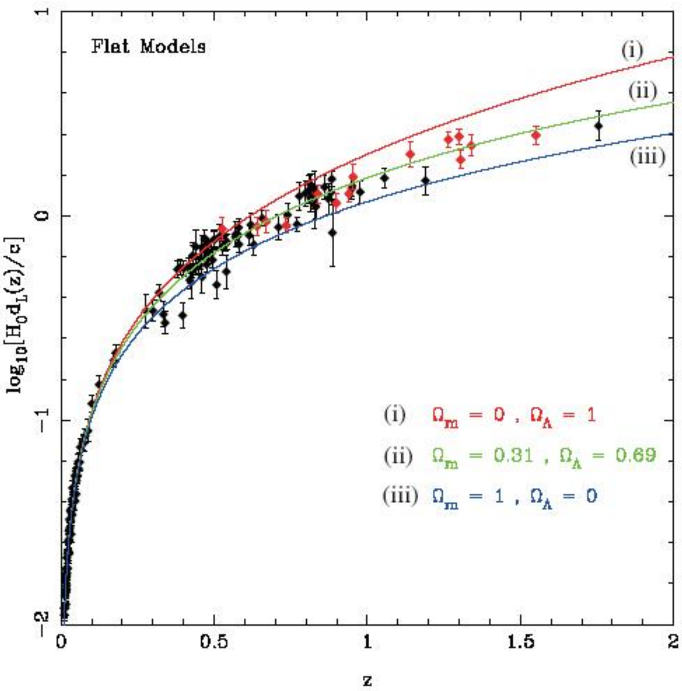

Figure 1.2: The luminosity distance versus the redshift for a flat cosmological model. The black dots come from the 'Gold' data sets by Riess et al [8] whereas the red points show the recent data from the Hubble Space Telescope (HST). The top curve (i) shows data fit with only cosmological constant; the middle curve (ii) shows data fit with 31 percent matter and 69 percent cosmological constant while the bottom curve (iii) provides data fit with the matter only and no cosmological constant. It turns out that the best fit model to the data is the middle curve.

Chapter 2

Black holes in astrophysics, general relativity and beyond

2.1 Black holes in astrophysics

The black hole idea is fairly old. At the end of the eighteenth century, John Michell and Laplace gave heuristic reasoning within Newtonian mechanics showing that light cannot escape from an object more compact than a radius less than $2GM$ (with $c = 1$). This is what is now called the Schwarzschild radius r_s and I shall discuss it in detail in the next section. A particle of mass m cannot escape from an object with mass M only if kinetic energy $mv^2/2$ is larger than the absolute value of the potential energy GMm/R. For $v = c$, one gets the limit on the size of the object M for which nothing can escape. Note that here light to behave as particle-like.

At that time, the necessary conditions for 'dark stars' as these were named by Michel, seemed physically impossible, since these would have extremely high density. In the early 1800's experiments on optical interference led to predominance of the wave theory of light and the end of the corpuscular theory. Since light waves were thought to be unaffected by gravitation, interest in the hypothetical dark stars ceased. But in 1915 Einstein published his General Theory of Relativity, a new theory of gravitation that made fundamental predictions on the effect of gravity on light. A few months after the publication of Einstein's general relativity, Carl Schwarzschild solved the Einstein field equations by assuming a static and spherically symmetric geometry, obtaining what is now called the Schwarzschild solution or metric:

$$ds^2 = -\Big(1 - \frac{r_s}{r}\Big)dt^2 + \Big(1 - \frac{r_s}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2, \tag{2.1.1}$$

where $d\Omega^2$ is the line element of the two-sphere. It is easy to recognize that something strange happens at r_s: Firstly, a clock placed at rest at r shows a proper time $d\tau = (1 - r_s/r)^{1/2}dt$ that shows that at r_s, the clock will run infinitely slow with respect to a clock placed sufficiently far away. An interesting picture of black holes is provided by light cones that allow one to depict the causal structure of any spacetime. The particle trajectories are defined to be timelike, i.e. when $ds^2 < 0$ since physical particles cannot travel faster than light and their trajectories remain confined within the light cones. It is straight forward to draw the light cones in the case of the Schwarzschild spacetime by solving $ds^2 < 0$ for various values of the radial distance r outside and inside r_s. The light cones get squeezed as $r \to r_s$ and the role of r and t reverses at $r < r_s$. Hence every particle, once it crosses the event horizon, is trapped forever.

Black holes may form in a complete gravitational collapse. It can be shown that eventual deformations of the event horizon are quickly dissipated as gravitational radiation: the horizon vibrates according to the quasi-normal modes and finally settles down to an axi-symmetric equilibrium configuration. The most interesting physical property is that black holes are described by only three parameters: mass M, electric charge Q and angular momentum J. Due to its this structural simplicity, John Wheeler coined a famous phrase: *A black hole has no hair*, which was called the 'no hair theorem'. As a consequence, there are now four most famous solutions of Einstein's field equations that represent black hole:

- The Schwarzschild solution [69] is static, spherically symmetric and depends only on M;
- The Reissner-Nordström solution [70, 71] is static, spherically symmetric and depends only on mass M and electric charge Q;
- The Kerr solution [72] is stationary, axis symmetric and depends on M and angular momentum J;
- The Kerr-Newmann solution [73] is stationary, axis-symmetric and depends on all three parameters M, J and Q.

The fact is that the theory of general relativity reliably predicts the existence of black holes but it does not necessarily prove the existence of black holes in the observable Universe because the general theory does not describe the astrophysical processes by which a black hole may form. Thus the astronomical credibility of black holes crucially depends on a good understanding of gravitational collapse of stars and stellar cluster and of direct observational evidence of their existence. Light cannot classically escape black holes but one can hope to detect them indirectly by observing the electromagnetic energy released during accretion processes, typically in the X-ray domain. Search for black holes, therefore, consists in looking for variable X-ray sources, which are binary systems and estimate the mass function that relates observed quantities to unknown masses [74]:

$$f(M_c, M_*) \equiv \frac{pv_*^3}{2\pi G} = \frac{(M_c \mathrm{sin} i)^3}{(M_c + M_*)^2}, \tag{2.1.2}$$

where M_c is the mass of the compact (accreting) object, M_* is the mass of the companion star, v_* is the velocity of the companion star along the line of sight, p is the orbital period of the binary and i is the inclination angle of the binary system. A crucial fact is that M_c cannot be smaller than the value of the mass function - the limit would correspond to a zero mass companion viewed at the maximum angle. Therefore, the best black hole candidates are obtained when the mass function exceeds about three solar masses. Binary systems with compact objects can be divided in two families: the high-mass X-ray binary where the companion star is of high mass and the low mass X-ray binary where the companion star is typically below a solar mass. There are two dozen black hole candidates in binary systems.

In addition to these candidates strongly convincing evidence for the existence of black holes comes from the study of the centers of galaxies (like the Milky Way and many others) and among these are the so-called active galactic nuclei (AGNs). This generic term covers a large family of galaxies including quasars, radio galaxies, Seyfert galaxies, blazers and so on. Recent studies have confirmed that quasars lie at the hearts of galaxies which are themselves too dim to be visible. They are thought to be of the order of the size of our solar system but radiate more than 1000 times as much energy as our entire galaxy. The current explanation is that this enormous power comes from a supermassive black hole which is consuming matter from its surrounding galaxy. It can be considered as established that most galaxies harbor a supermassive black hole (sometimes even a binary system of supermassive black holes) in their centers. The mass of the supermassive black hole roughly linearly correlates with the mass of the galaxy's bulge velocity dispersion σ exhibiting a proportionality $M_{BH} \propto \sigma^4$ [75]. Today the detection of these objects is one of the major goals of extragalactic astronomy. The most convincing method of detection consists in the dynamical analysis of the surrounding matter: gas or stars near the invisible central mass have large dispersion velocities that can be measured spectroscopically.

How do supermassive black holes form? Some theories hold that the first generation of stars included a large proportion of very massive stars all of which formed black holes that somehow merged. Other theories hold that a single seed black hole accreted stars and gas thereby growing more and more with time. The best studied black hole in our galaxy center is very likely the supermassive black hole and is named as Sgr A*. The later is an unusual radio source with nonthermal spectrum, compact size and no detectable motion. Stellar proper motion study [76] found that the motion of the nearby stars is dominated by a compact object with mass about 3.6×10^6 solar mass confined within about 0.05 pc. A point in favor of the Sgr A^* black hole interpretation is that it has been observed to flare in X-rays exhibiting large luminosity variations over short time scales (of the order of minutes). This is consistent with the light crossing and dynamical timescales for the inner few Schwarzschild radii of the accretion flow around a few million solar masses black hole. A problem still open with this black hole is the blackness problem - that is the inferred bolometric luminosity is only a few 10^{36} erg/s that is about eight orders of magnitude below the expected accretion luminosity for a black hole with the same mass. However, the blackness of Sgr A* is not necessarily a problem since several different models to account for it have been proposed (for a review see de Paolis et al [77]).

2.2 Black holes in general relativity

In this section, I provide a derivation of the Schwarzschild solution which represents a static point-like object which is gravitationally isolated lying in a vacuum. The solution will show that this object is not exactly point-like but have a simple structure of a singularity hidden inside a horizon. In general a singularity can be ring-like rather than being point-like. Also the event horizon is more then a surface of infinite redshift: the horizon can expand (due to matter accretion), contract (due to Hawking radiation), radiate energy and rotate thereby producing an effect called the frame-dragging. These are entities that are not only where the coordinates break down but have some geometric significance.

2.2.1 The Schwarzschild solution

Let us start with the line element of flat spacetime in spherical coordinates as [78]:

$$ds^2 = -dt^2 + dr'^2 + r'^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2.1}$$

Introducing at the origin a mass with spherical symmetry. In doing so, the above metric must be modified in a way that it retains spherical symmetry and symmetry with regard to time reversal. This leads to

$$ds^2 = -f_0(r', t)dt^2 + f_1(r', t)dr'^2 + f_2(r', t)r'^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.2}$$

Choosing a new coordinate r such that $f_2(r', t)r'^2 = r^2$. Setting $f_0 = e^\nu$, $f_1 = e^\lambda$, to obtain $g_{00} = -e^\nu$, $g_{11} = e^\lambda$, $g_{22} = r^2$, $g_{33} = r^2\sin^2\theta$; then

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.3}$$

For the empty space surrounding the body, $T_{\mu\nu} = 0$, and the field equations become

$$\mathcal{E}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{2.2.4}$$

Multiplying this by $g^{\mu\nu}$ and contracting gives $R = 0$. The gravitational field equations for vacuum become

$$\begin{aligned}
\mathcal{E}_0^0 &= e^{-\lambda}\Big(\frac{1}{r^2} - \frac{1}{r}\frac{\partial\lambda}{\partial r}\Big) - \frac{1}{r^2} \\
\mathcal{E}_0^1 &= \frac{e^{-\lambda}}{r}\frac{\partial\lambda}{\partial t} \\
\mathcal{E}_2^2 &= \mathcal{E}_3^3 = \frac{1}{2}e^{-\lambda}\Big[\frac{\partial^2\nu}{\partial r^2} + \frac{1}{2}\Big(\frac{\partial\nu}{\partial r}\Big)^2 + \frac{1}{r}\Big(\frac{\partial\nu}{\partial r} - \frac{\partial\lambda}{\partial r}\Big) - \frac{1}{2}\frac{\partial\nu}{\partial r}\frac{\partial\lambda}{\partial r}\Big] \\
&\quad -\frac{1}{2}e^{-\nu}\Big[\frac{\partial^2\lambda}{\partial t^2} + \frac{1}{2}\Big(\frac{\partial\lambda}{\partial t}\Big)^2 - \frac{1}{2}\Big(\frac{\partial\lambda}{\partial t}\Big)\Big(\frac{\partial\nu}{\partial t}\Big)\Big] \\
\mathcal{E}_1^1 &= e^{-\lambda}\Big(\frac{1}{r}\frac{\partial\nu}{\partial r} + \frac{1}{r^2}\Big) - \frac{1}{r^2}.
\end{aligned}$$

Setting the above equations to zero gives the following independent equations

$$\frac{\partial\nu}{\partial r} + \frac{1}{r} - \frac{e^{\lambda}}{r} = 0, \tag{2.2.5}$$

$$\frac{\partial\lambda}{\partial r} - \frac{1}{r} + \frac{e^{\lambda}}{r} = 0, \tag{2.2.6}$$

$$\frac{\partial\lambda}{\partial t} = 0. \tag{2.2.7}$$

The sum of Eqs. (2.2.6) and (2.2.7) gives

$$\frac{\partial}{\partial r}(\nu + \lambda) = 0. \tag{2.2.8}$$

There is no space dependence of the sum of the two functions. It is the fact that $R_{01} = 0$ that gives the time independence. (2.2.7) indicates that such a time dependence can be eliminated anywhere by a coordinate transformation involving only time. This is equivalent to the statement that the assumption of spherical symmetry guarantees the possibility of a time-independent description of the geometry of the space. It is referred as Birchoff theorem. Consequently all time derivatives appearing in the above equations must vanish. The solution of (2.2.6) and (2.2.7) is

$$e^{-\lambda} = e^{\nu} = 1 + \frac{K_0}{r} \tag{2.2.9}$$

The constant K_0 may be determined from the requirement that Newton's law of gravitation be approached, at large distances from the mass. Thus from the geodesic equations, one finds

$$K_0 = -\frac{2GM}{c^2} \tag{2.2.10}$$

Therefore the Schwarzschild metric takes the form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2.11}$$

There are a couple of points worth noting. First, for large r, the metric is approximately that of flat Minkowski spacetime. Second, the metric appears singular when $r = 0$ and when $r = 2GM$. The two values of r have very special physical importance. However their nature is quite different; at $r = 0$ there is a physical singularity where the curvature tensor diverges; at $r = 2GM$, the curvature tensor is well-behaved and finite, but the spacetime has a horizon at $r = 2GM$ in these coordinates.

The physical interpretation of M can be understood by considering a free particle instantaneously at rest outside a spherical body and comparing with the Newtonian limit. In a Newtonian gravitational field the (magnitude of) acceleration of a free particle is

$$g = -\frac{Gm}{r^2} \tag{2.2.12}$$

where m is the mass of the attracting body. According to the theory of general relativity, the acceleration of a test particle is given by the geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{2.2.13}$$

Assuming that the particle is instantaneously at rest in a weak gravitational field, the proper time $d\tau$ can be approximated with dt and set $\frac{dx^\mu}{d\tau} = (1,0,0,0)$ at that particular moment of time. The geodesic equation now simplifies to

$$g = \frac{d^2x^\mu}{dt^2} \approx -\Gamma^r_{tt}. \tag{2.2.14}$$

Since the coordinate basis is used, the connection coefficients are Christoffel symbols and Γ^r_{tt} is given by

$$\Gamma^r_{tt} = \frac{1}{2}g^{r\alpha}\left(\frac{\partial g_{\alpha t}}{\partial t} + \frac{\partial g_{\alpha t}}{\partial t} - \frac{\partial g_{tt}}{\partial x^\alpha}\right) \tag{2.2.15}$$

$$= -\frac{1}{2}(g_{rr})^{-1}\frac{\partial g_{tt}}{\partial r}. \tag{2.2.16}$$

Inserting the found solution into the above equation, to the lowest order

$$g = -\Gamma^r_{tt} = -\frac{M}{r^2}. \tag{2.2.17}$$

Comparing with the classical case one can see that the constant M must be interpreted as the mass of the gravitational body, m times the G i.e. $M = Gm$. If one includes the speed of light c, one gets $g = -Mc^2/r^2$, and hence,

$$M = \frac{Gm}{c^2}. \tag{2.2.18}$$

For mass m the radius

$$R_S = \frac{2Gm}{c^2}, \tag{2.2.19}$$

is called the ‘Schwarzschild radius’. The singularity behavior at the R_S is only a coordinate singularity. Calculating the ‘Kretschmann’s curvature scalar’ defined as the square of the Riemann tensor, one gets

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6}. \tag{2.2.20}$$

This scalar diverges only at the origin.

2.2.2 Reissner-Nordström metric

I now come to gravitational and electromagnetic fields due to a point mass M, with charge Q at rest at the origin. The metric is again spherically symmetric and static. The solution obtained is not only for the Einstein equations but the Maxwell equations as well. When I require a solution to two different sets of field equations, each of which has an effect on the other, one solves the coupled field equations. In this case the coupled Einstein-Maxwell field equations to be solved. The gravitational field enters into the Maxwell equations as I require the covariant divergence of the field tensor to be zero (or more general current density). In the reverse direction, the electromagnetic stress energy tensor given by

$$T^{\nu}_{\mu} = \frac{-1}{4\pi}\Big(- F_{\mu\rho}F^{\nu\rho} + \frac{1}{4}\delta^{\nu}_{\mu}F_{\rho\pi}F^{\rho\pi}\Big), \tag{2.2.21}$$

acts as a source term for the gravitational field. Physically, the energy distribution due to the electromagnetic field has an effective mass, which causes a gravitational field. I will not solve the Maxwell equations but will assume that the usual solution works and verify that my assumption is valid at the end.

Taking the electromagnetic 4-vector potential to be

$$A_\mu = (Q/r, \mathbf{0}). \tag{2.2.22}$$

Then it is easy to see that

$$F_{\mu\nu} = -F_{\nu\mu} = 2\delta^0_{[\mu}\delta^1_{\nu]}Q/r^2. \tag{2.2.23}$$

Inserting this value in the stress-energy tensor gives

$$T^0_0 = T^1_1 = -T^2_2 = -T^3_3 = Q^2 e^{-(\nu+\lambda)}/8\pi r^4. \tag{2.2.24}$$

Since $T^0_0 = T^1_1$, the corresponding field equations yield $\nu'(r) + \lambda'(r) = 0$. Hence once again I can take $\nu(r)+\lambda(r) = 0$. Remembering that here $T = 0$, the Einstein equation for the 22 component is $R^2_2 = 8\pi T^2_2$, which gives

$$-\frac{1}{r^2}\Big[(-re^{-\lambda})+1\Big] = -\frac{Q^2}{r^4}. \tag{2.2.25}$$

Hence solving the above equation yields

$$e^{\nu} = e^{-\lambda} = 1 + \frac{\alpha}{r} + \frac{Q^2}{r^2}. \tag{2.2.26}$$

Again choosing zero charge should yield the Schwarzschild exterior solution. Hence $\alpha = -2M$. Thus

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2, \tag{2.2.27}$$

which is the Reissner-Nordström metric which describes a black hole with an electric charge. By inspecting the above line-element, This spacetime has two horizons

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \tag{2.2.28}$$

These horizons merge into to one in the extremal limit $M = \pm Q$. For $M < |Q|$, there are no horizons, and the singularity at $r = 0$ becomes a so-called naked singularity because it has no surrounding horizons. This is however an nonphysical situation so the bound is $M \geq |Q|$. Also the coordinate singularity at the horizon for this metric can be removed by introducing Kruskal-Szekeres coordinates. The horizons are only coordinate singularities, there are no physical singularities except at $r = 0$.

I must verify that A_μ is a solution of the source-free Maxwell equations when coupled with gravity. Thus

$$F^{\mu\nu}_{;\nu} = \frac{1}{\sqrt{|g|}}(\sqrt{|g|}F^{\mu\nu})_{,\nu} = 0 \tag{2.2.29}$$

Since $\nu + \lambda = 0$, $\sqrt{|g|} = r^2 \sin\theta$. Also from (127),

$$F^{\mu\nu} = -2\delta_0^{[\mu}\delta_1^{\nu]}e^{-(\nu+\lambda)}Q/r^2 = -2\delta_0^{[\mu}\delta_1^{\nu]}Q/r^2. \tag{2.2.30}$$

Hence (2.2.29) is automatically satisfied. It is truly remarkable that the usual solution of Maxwell's equation holds for the coupled Einstein-Maxwell system.

2.2.3 The BTZ (2+1)- dimensional black hole

Let us briefly describe how general relativity is modified in (2+1) dimensions. The Einstein-Hilbert actions becomes

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g}(R - 2\Lambda) + S_m, \tag{2.2.31}$$

where S_m is the matter action. As in (3+1)-dimensions, the resulting Euler-Lagrange equations are the standard Einstein field equations

$$R_{AB} - \frac{1}{2} g_{AB} R + \Lambda g_{AB} = 8\pi G T_{AB}. \quad A, B = 0, 1, 2 \tag{2.2.32}$$

These field equations are written in covariant form and are invariant under the action of the group of diffeomorphisms of the spacetime. The fundamental difference between general relativity in (2+1) and (3+1) dimensions originates in the fact that the curvature tensor in (2+1) dimensions depends linearly on the Ricci tensor [79, 80]. It means that every solution of the vacuum Einstein field equations with $\Lambda = 0$ is flat and that every solution with a non-vanishing cosmological constant has constant curvature. Physically a (2+1) dimensional spacetime has no local degrees of freedom: curvature is concentrated at the location of matter, and there are no gravitational waves. It should also be noted that the Newtonian theory cannot be obtained in (2+1) dimensional gravity (see [79] for more discussion on (2+1) dimensional gravity and its applications).

The unified treatment of space and time is a cornerstone of general relativity. As a practical matter, however, it is sometimes useful to introduce and explicit-although largely arbitrary - division of spacetime into spatial and temporal directions. Such a division is described by the Arnowitt-Deser-Misner (ADM) formalism [79].

Beginning with a spacetime manifold with the topology $[0, 1] \times \Sigma$, where Σ is an open or closed two-surface. Such a spacetime represents a segment of a Universe between an initial surface $\{0\} \times \Sigma$ and a final surface $\{1\} \times \Sigma$, which are assumed to be spacelike. The ADM approach to (2+1)-dimensional general relativity starts with a slicing of the spacetime manifold M into constant time surfaces Σ_t, each provided with a coordinate system $\{x^i\}$ and an induced metric $g_{ij}(t, x^i)$. To obtain the full 3D geometry, I describe the way nearby time slices Σ_t and Σ_{t+dt} fit together. To do so, starting at a point on Σ_t with coordinates x^i, and displace it infinitesimally in the direction normal to Σ_t. The resulting change in proper time can be written as

$$d\tau = Ndt, \tag{2.2.33}$$

where $N(t, x^i)$ is called the lapse function. Such a displacement will not only shift the time coordinate, but will alter the spatial coordinates as well. To allow for this possibility,

$$x^i(t + dt) = x^i(t) - N^i dt, \tag{2.2.34}$$

where $N^i(t, x^i)$ is called the shift vector. By the Lorentzian version of the Pythagoras theorem, the interval between points (t, x^i) and $(t + dt, x^i + dx^i)$ is then

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^i dt). \tag{2.2.35}$$

The last equation is the ADM form of the metric. It is customary in the ADM formalism to establish a new set of conventions that emphasize the role of the surface Σ. For the remainder of this section, spatial indices $i, j, ...$ will be lowered and raised with the spatial metric g_{ij} and its inverse g^{ij}, and not with the full spacetime metric. Note that the components of g^{ij} are not simply the spatial components of the full three-metric $g^{\mu\nu}$; the inverse of the last expression

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & \left(g^{ij} - \frac{N^i N^j}{N^2}\right) \end{pmatrix}. \tag{2.2.36}$$

This convention can cause confusion at first, but it simplifies later notation.

The geometry of the slice Σ_t comprises two elements: the intrinsic geometry of the slice as a two-manifold, and the extrinsic geometry, which describes the embedding of Σ_t in the spacetime M. Just as the intrinsic geometry is determined by the behavior of vectors tangent to Σ_t under parallel transport, the extrinsic geometry is determined by the behavior of vectors normal to Σ_t. In particular, the extrinsic curvature K_{ij} of a surface Σ is defined by

$$K_{\mu\nu} = -\nabla_\mu n_\nu + n_\mu n^\rho \nabla_\rho n_\nu, \tag{2.2.37}$$

where ∇ is the full three-dimensional covariant derivative and n^μ is the unit normal to Σ. In the ADM decomposition (2.2.35), the normal to Σ_t has components $n_\mu = (N, 0, 0)$ and an easy calculation gives

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} -^{(2)} \partial_i N_j -^{(2)} \partial_j N_i). \tag{2.2.38}$$

A general stationary, axially symmetric (2+1)-dimensional metric can be written in the form

$$ds^2 = -N(r)^2dt^2 + f(r)^2dr^2 + r^2(d\phi - N^\phi(r)dt)^2. \tag{2.2.39}$$

The spatial metric g_{ij} takes the form

$$g_{ij} = \begin{pmatrix} f^2 & 0 \\ 0 & r^2 \end{pmatrix} \tag{2.2.40}$$

The only non-zero Ricci scalars are

$$^{(2)}R_{rr} = \frac{f'}{fr}, \tag{2.2.41}$$

$$^{(2)}R_{\phi\phi} = \frac{rf'}{f^3}. \tag{2.2.42}$$

and hence

$$\sqrt{^{(2)}g}\,^{(2)}R = 2\frac{f'}{f^2}. \tag{2.2.43}$$

Since the metric under consideration is stationary, the extrinsic curvature becomes

$$K_{ij} = -\frac{1}{2N}(^{(2)}\nabla_j N_i +^{(2)} \nabla_i N_j). \tag{2.2.44}$$

The only non-vanishing component is

$$K_{r\phi} = -\frac{r^2}{2N}(N^\phi)', \tag{2.2.45}$$

and the corresponding canonical momentum is

$$\pi^r_\phi = -\frac{r^3}{2Nf}(N^\phi)'. \tag{2.2.46}$$

Evaluating the momentum constraint

$$^{(2)}\nabla_j \pi^{ij} = 0 = g^{il}\partial_K \pi^k_l - \frac{1}{2}g^{il}(\partial_l g_{jk})\pi^{jk}. \tag{2.2.47}$$

Since g_{jk} has only diagonal elements and π^{jk} is entirely off-diagonal, the last term of the last equation vanishes. Hence

$$\pi^r_\phi = A, \quad A = constant. \tag{2.2.48}$$

Further, the Hamiltonian constraint equation becomes

$$\frac{2A^2 f}{r^3} - 2\frac{f'}{f} = 0. \tag{2.2.49}$$

Its solution is

$$\frac{1}{f^2} = B^2 + \frac{A^2}{r^2}, \tag{2.2.50}$$

where B^2 is a constant of integration which is required to be positive to ensure that f^2 remains positive for large values of r. To proceed further, one of the dynamical equations of motion coming from varying g_{ij} in the action is needed. The Hamiltonian constraint is

$$\mathfrak{H} = 2fr(\pi^{\phi r})^2 - 2\frac{f'}{f^2}, \tag{2.2.51}$$

the momentum constraint is independent of f and all time derivatives vanish, so the action is

$$I_{\text{eff}} \sim -\int dt \int dr \Big\{ 2Nrf(\pi^{\phi r})^2 - 2N\frac{f'}{f^2} \Big\} + \text{terms independent of } f. \tag{2.2.52}$$

The field equation obtained by varying f is thus

$$\frac{N'}{f^2} + Nr(\pi^{r\phi})^2 = 0. \tag{2.2.53}$$

Combining the last two equations and the constraint $H = 0$, yields

$$\frac{N'}{N} = -\frac{f'}{f}, \tag{2.2.54}$$

or

$$N = f^{-1} \tag{2.2.55}$$

up to a constant factor that can be absorbed by a suitable rescaling of the time coordinate t. To complete the solution, using (2.2.49) to determine N^ϕ:

$$(N^\phi)' = \frac{2Nf}{r^3}\pi^r_\phi = -2\frac{A}{r^3}, \tag{2.2.56}$$

so

$$N^\phi = C + \frac{A}{r^2}. \tag{2.2.57}$$

One has to restrict attention to solutions for which $C = 0$, since otherwise the metric has nonphysical asymptotic behavior.

Substituting (2.2.50), (2.2.55) and (2.2.57) in (2.2.39), one gets

$$\begin{aligned} ds^2 &= -\Big(B^2 + \frac{A^2}{r^2}\Big)dt^2 + \Big(B^2 + \frac{A^2}{r^2}\Big)^{-1}dr^2 + r^2\Big(d\phi + \frac{A}{r^2}dt\Big)^2, && (2.2.58)\\ &= -\Big(Bdt - \frac{A}{B}d\phi\Big)^2 + \Big(B^2 + \frac{A^2}{r^2}\Big)^{-1}dr^2 + \Big(r^2 + \frac{A^2}{B^2}\Big)d\phi^2. && (2.2.59)\end{aligned}$$

This metric can be put in a slightly more standard form by defining

$$\tilde{r} = \frac{1}{B^2}(A^2 + B^2 r)^{1/2} \tag{2.2.60}$$

and hence by the use of above rescaling transformation, one gets

$$ds^2 = -\Big(dt - \frac{A}{B}d\phi\Big)^2 + d\tilde{r}^2 + B^2\tilde{r}^2 d\phi^2. \tag{2.2.61}$$

To find a physical interpretation for the constants of A and B, it is useful to examine the ADM equations of motion in the presence of sources, treating the conical singularity at $r = 0$ as a point particle. In the presence of matter, the field equations obtained from the variation of N and N' become

$$\begin{aligned} H &= -\sqrt{^{(2)}g}\,T_0^0, && (2.2.62)\\ H_i &= -\sqrt{^{(2)}g}\,T_i^0. && (2.2.63)\end{aligned}$$

and the mass of an isolated source is then

$$m = \int d^2x\sqrt{^{(2)}g}\,T_0^0 = -\int d^2x H \tag{2.2.64}$$

The only term in the Hamiltonian constraint that has a chance of behaving peculiarly at $r = 0$ is the spatial curvature $^{(2)}R$. Indeed, recall that curvature can be written in the form

$$\int_\Sigma d^2x\sqrt{^{(2)}g}\,{}^{(2)}R = \int_{\partial\Sigma} d\phi v^\perp = 2\pi v^\perp, \tag{2.2.65}$$

where it is evident from (2.2.46) that

$$v^\perp = -\frac{2}{f} + \mathrm{const} \sim -2B + \mathrm{const}, \tag{2.2.66}$$

as $r \to \infty$. Fixing the constant by noting that when $B = 1$ and $A = 0$, the metric

g_{ij} is that of flat Euclidean two-space for which the last integral vanish. Hence

$$v^{\perp} = 2 - 2B = \frac{\beta}{\pi}, \tag{2.2.67}$$

and the total curvature integral is 2β. Restoring factors of G, the equation for mass becomes

$$m = \frac{1}{16\pi G} \int d^2x \sqrt{^{(2)}}\, ^{(2)}R = \frac{\beta}{8\pi G}. \tag{2.2.68}$$

A similar analysis can be applied to the angular momentum of the source, which in asymptotically Cartesian coordinates is

$$J^{ij} = \int d^2x (x^i T^{0j} - x^j T^{0i}) = 2 \int_{\partial\Sigma} d\phi (x^i \pi^{\perp j} - x^j \pi^{\perp i}). \tag{2.2.69}$$

Setting

$$\pi^{\perp i} = n_j \pi^{j\phi} \partial_\phi x^i = -\frac{A}{Br^2} \epsilon^{ik} x^k \tag{2.2.70}$$

near infinity and see that

$$J^{ij} = \frac{1}{4G} \frac{A}{B} \epsilon^{ij}, \tag{2.2.71}$$

so A/B is a measure of the angular momentum of the source.

Considering the axially symmetric spacetime with negative cosmological constant $\Lambda = -1/l^2$, for which the spacetime is asymptotically anti-de Sitter.

The Hamiltonian constraint is now

$$\frac{2A^2 f}{r^3} - 2\frac{f'}{f^2} - \frac{2r}{l^2} f = 0, \tag{2.2.72}$$

which has its solution

$$\frac{1}{f^2} = B^2 + \frac{A^2}{r^2} + \frac{r^2}{l^2}. \tag{2.2.73}$$

The equation of motion for N now becomes

$$\frac{N'}{f^2} + Nr(\pi^{\phi r})^2 - \frac{Nr}{l^2} = 0. \tag{2.2.74}$$

The solution however is still $N = f^{-1}$.

Renaming some of the constants to get

$$ds^2 = -N^2 dt^2 + r^2 (d\phi^2 + N^\phi dt)^2 + N^{-2} dr^2. \tag{2.2.75}$$

with

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi = -\frac{J}{2r^2}. \tag{2.2.76}$$

This spacetime is the (2+1) dimensional black hole of Banados, Teitelboim and Zanelli (BTZ). It has an event horizon at $r = r_+$ and an inner horizon $r = r_-$, where

$$r_\pm^2 = \frac{l^2}{2}\Big[M \pm \Big(M^2 - \frac{J^2}{l^2}\Big)\Big] \tag{2.2.77}$$

are the zeros of the lapse function N. That $r = r_+$ is a genuine event horizon is most easily seen by changing to Eddington-Finkelstein coordinates

$$dv = dt + \frac{1}{N^2}dr, \quad d\tilde{\phi} = d\phi - \frac{N^\phi}{N^2}dr, \tag{2.2.78}$$

in which the metric becomes

$$ds^2 = -N^2 dv^2 + 2dvdr + r^2(d\tilde{\phi} + N^\phi dv)^2 \tag{2.2.79}$$

It is now evident that the surface $r = r_+$ is a null surface, generated by the geodesics

$$r(\lambda) = r_+, \quad \frac{d\tilde{\phi}}{d\lambda} + N^\phi(r_+)\frac{dv}{d\lambda} = 0. \tag{2.2.80}$$

Note that the lapse function $N^2(r)$ can be factorized as

$$N^2(r) = -\frac{1}{r^2 l^2}(r - r_1)(r - r_2)(r - r_3)(r - r_4), \tag{2.2.81}$$

where $r_1 = r_+$, $r_2 = r_-$, $r_3 = -r_-$ and $r_4 = -r_+$. Calling r_1, r_2 the outer and inner black hole horizons, and r_3, r_4 are the negative horizons. In order for these horizons to be real, the conditions $M > 0$, $-\Lambda J^2 \leq M^2$ must be met.

The rotating BTZ with the incorporation of the charge Q (the charge rotating BTZ) is given by

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2}Q^2 \ln r. \tag{2.2.82}$$

Horizons of the CR-BTZ metric are roots of the lapse function.

1. Usual CR-BTZ black hole when two distinct real roots exist.
2. Extreme CR-BTZ black hole in case of two repeated real roots.
3. Naked CR-BTZ singularity when no real roots exist.

2.3 Energy conditions in general relativity

In the context of general relativity, it is reasonable to expect that the stress-energy tensor will satisfy certain conditions, such as positivity of the energy density and dominance of the energy density over the pressure. Such requirements are embodied in the energy conditions.

To put the energy conditions in concrete form it is useful to assume that the stress energy tensor admits the decomposition

$$T^{\alpha\beta} = \rho \hat{e}_0^\alpha \hat{e}_0^\beta + p_1 \hat{e}_1^\alpha \hat{e}_0^\beta + p_2 \hat{e}_2^\alpha \hat{e}_2^\beta + p_3 \hat{e}_3^\alpha \hat{e}_3^\beta \tag{2.3.1}$$

in which the vectors $\hat{e}_\mu^\alpha$ form an orthonormal basis; they satisfy the relations

$$g_{\alpha\beta} \hat{e}_\mu^\alpha \hat{e}_\nu^\beta = \eta_{\mu\nu}, \tag{2.3.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. The above two equations (2.3.1) and (2.3.2) imply that the quantities ρ and p_i are eigenvalues of the stress-energy tensor and $\hat{e}_\nu^\beta$ are the normalized eigenvectors.

The inverse metric can neatly be expressed in terms of the basis vectors. It is easy to check that the relation

$$g^{\alpha\beta} = \eta^{\mu\nu} \hat{e}_\nu^\alpha \hat{e}_\nu^\beta, \tag{2.3.3}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the inverse of $\eta_{\mu\nu}$.

If the stress-energy tensor is that of a perfect fluid, then $p_1 = p_2 = p_3 \equiv p$. substituting this into (2.3.1) and using (2.3.3) yields

$$\begin{aligned} T^{\alpha\beta} &= \rho \hat{e}_0^\alpha \hat{e}_0^\beta + p(\hat{e}_1^\alpha \hat{e}_0^\beta + \hat{e}_2^\alpha \hat{e}_2^\beta + \hat{e}_3^\alpha \hat{e}_3^\beta) \\ &= \rho \hat{e}_0^\alpha \hat{e}_0^\beta + p(g^{\alpha\beta} + \hat{e}_0^\alpha \hat{e}_0^\beta) \\ &= (\rho + p) \hat{e}_0^\alpha \hat{e}_0^\beta + p g^{\alpha\beta} \end{aligned}$$

The vector $\hat{e}_0^\beta$ is identified with the four-velocity of the perfect fluid.

Some of the energy conditions are formulated in terms of a normalized, future directed, but otherwise arbitrary timelike vector v^α; this represents the four velocity of an arbitrary observer in spacetime. Such a vector can be decomposed as

$$v^\alpha = \gamma(\hat{e}_0^\alpha + a\hat{e}_1^\alpha + b\hat{e}_2^\alpha + c\hat{e}_3^\alpha), \quad \gamma = (1 - a^2 - b^2 - c^2)^{-1/2} \tag{2.3.4}$$

where a, b and c are arbitrary functions of the coordinates, restricted by $a^2 + b^2 + c^2 < 1$. I will also need an arbitrary, future directed null vector k^α which can be expressed as

$$k^\alpha = \hat{e}_0^\alpha + a'\hat{e}_1^\alpha + b'\hat{e}_2^\alpha + c'\hat{e}_3^\alpha, \tag{2.3.5}$$

where a', b', and c' are arbitrary functions of the coordinates, restricted by ${a'}^2 +$

$b'^2 + c'^2 = 1$. Note that the normalization of a null vector is always arbitrary.

Weak energy condition

The weak energy condition states that the energy density of any matter distribution as measured by any observer in spacetime, must be non-negative. Because an observer with four-velocity v^α measures the energy density to be $T_{\alpha\beta}v^\alpha v^\beta$,

$$T_{\alpha\beta}v^\alpha v^\beta \geq 0, \tag{2.3.6}$$

for any future-directed timelike vector v^α. To put this in concrete form, substituting (2.3.1) and (2.3.4)

$$\rho + a^2 p_1 + b^2 p_2 + c^2 p_3 \geq 0. \tag{2.3.7}$$

Because a, b and c are arbitrary, choose $a = b = c = 0$, and this gives $\rho \geq 0$. Alternatively, choose $b = c = 0$, which gives $\rho + a^2 p_1 \geq 0$. Recalling that a^2 must be smaller than unity, obtaining $0 \leq \rho + a^2 p_1 < \rho + p_1$. So $\rho + p_1 > 0$, and similar expressions hold for p_2 and p_3. The weak energy condition therefore implies

$$\rho \geq 0, \quad \rho + p_i > 0. \tag{2.3.8}$$

Null energy condition

The null energy condition makes the same statement as the weak form, except that v^α is replaced by an arbitrary, future directed null vector k^α. Thus,

$$T_{\alpha\beta}k^\alpha k^\beta \geq 0, \tag{2.3.9}$$

is the statement of the null energy condition. Substitution of Eqs. (2.3.1) and (2.3.4) gives

$$\rho + a'^2 p_1 + b'^2 p_2 + c'^2 p_3 \geq 0. \tag{2.3.10}$$

Choosing $b' = c' = 0$ enforces $a' = 1$, and obtaining $\rho + p_i \geq 0$, with similar expressions holding for p_2 and p_3. the null energy condition therefore implies

$$\rho + p_i \geq 0. \tag{2.3.11}$$

Notice that the energy condition implies the null form.

Strong energy condition

The statement of the strong energy condition is

$$\Big(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\Big)v^\alpha v^\beta \geq 0, \tag{2.3.12}$$

or $T_{\alpha\beta}v^\alpha v^\beta \geq -\frac{1}{2}T$, where v^α is any future directed, normalized, timelike vector. Because $T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta} = R_{\alpha\beta}/8\pi$ by virtue of the Einstein field equations, the strong energy condition is really a statement about the Ricci tensor. Substituting Eqs (2.3.1) and (2.3.4) gives

$$\gamma^2(\rho + a^2p_1 + b^2p_2 + c^2p_3) \geq \frac{1}{2}(\rho - p_1 - p_2 - p_3). \tag{2.3.13}$$

Choosing $a = b = c = 0$ enforces $\gamma = 1$, and obtain $\rho + p_1 + p_2 + p_3 \geq 0$. Alternatively, choosing $b = c = 0$ implies $\gamma^2 = 1/(1 - a^2)$ and after some algebra obtain $\rho + p_1 + p_2 + p_3 \geq a^2(p_2 + p_3 - \rho - p_1)$. Because this must hold for any $a^2 < 1$, I have $\rho + p_1 \geq 0$, with similar relations holding for p_2 and p_3. The strong energy condition therefore implies

$$\rho + p_1 + p_2 + p_3 \geq 0, \quad \rho + p_i \geq 0. \tag{2.3.14}$$

It should be noted that the strong energy condition does not imply the weak form.

Dominant energy condition

The dominant energy condition embodies the notion that matter should flow along timelike or null world lines. Its precise statement is that if v^α is an arbitrary, future directed, timelike vector field, then

$$-T^\alpha_\beta v^\beta \ \text{ is a future directed, timelike or null, vector field.} \tag{2.3.15}$$

The quantity $-T^\alpha_\beta v^\beta$ is the matter's momentum density as measured by an observer with four-velocity v^α, and is required to be timelike or null. Substituting (2.3.1) and (2.3.4) and demanding that $-T^\alpha_\beta v^\beta$ not be spacelike gives

$$\rho^2 - a^2p_1^2 - b^2p_2^2 - c^2p_3^2 \geq 0. \tag{2.3.16}$$

Choosing $a = b = c = 0$ gives $\rho^2 \geq 0$, and demanding that $-T^\alpha_\beta v^\beta$ be future directed selects the positive branch: $\rho \geq 0$. Alternatively choosing $b = c = 0$

gives $\rho^2 \geq a^2 p_1^2$. Because this must hold for any $a^2 < 1$, and $\rho \geq |p_1|$, having taken the future direction for $-T^{\alpha}_{\beta} v^{\beta}$. Similar relations hold for p_2 and p_3. The dominant energy condition therefore implies

$$\rho \geq 0, \quad \rho \geq |p_i|. \tag{2.3.17}$$

Chapter 3

Primordial black holes in phantom cosmology

Hawking's discovery that black holes emit thermal radiation due to quantum effects was one of the most important results in 20th century physics [81, 82, 83]. This is because it unified three previously disparate areas of physics - quantum theory, general relativity and thermodynamics - and like all such unifying ideas it has led to deep insights. In practice, only "primordial black holes" (PBH) which formed in the early Universe could be small enough for Hawking radiation to be important. The idea of PBHs did form and their discovery would provide a unique probe of at least four areas of physics: the early Universe; gravitational collapse, high energy physics and quantum gravity. The first topic is relevant because studying PBH formation and evaporation can impose important constraints on primordial inhomogeneities, cosmological phase transitions (including inflation) and varying gravitational constant models [84, 85, 86].

The high density of the early Universe is a necessary but not sufficient condition for PBH formation. One also needs density fluctuations, so that over-dense regions can eventually stop expanding and recollapse. One reason for studying PBH formation and evaporation is that it imposes important constraints on primordial inhomogeneities. PBHs may also form at various phase transitions expected to occur in the early Universe. In some of these one require pre-existing density fluctuations, but in others the PBHs form spontaneously, even if the Universe starts off perfectly smooth.

It was realized many years ago that black holes with a wide range of masses could have formed in the early Universe as a result of the great compression associated with the Big Bang. A comparison of the cosmological density at a time t after the Big Bang with the density associated with a black hole of mass M shows that PBHs would have of order the particle horizon mass at their formation epoch [86]:

$$M_H(t) \sim \frac{c^3 t}{G} \sim 10^{15}\Big(\frac{t}{10^{-23}\mathrm{s}}\Big)\mathrm{g}. \tag{3.0.1}$$

Therefore PBHs that formed in the early history of the Universe must be less massive while those that formed later must be more massive. Black holes formed at Planck time 10^{-43}s would have Planck mass 10^{-8}kg. PBHs could thus span an enormous mass range: those formed at the Planck time (10^{-43}s) would have the Planck mass (10^{-5}g), whereas those formed at 1 s would be as large as $10^5 M_\odot$, comparable to the mass of the holes thought to reside in galactic nuclei. By contrast, black holes forming at the present epoch could never be smaller than about $1M_\odot$. Zeldovich & Novikov [87] first derived (3.0.1) and Hawking [83] was the first person to realize that primordial density perturbations might lead to gravitational collapse on scales above the Planck mass. For a while the existence of PBHs seemed unlikely since Zeldovich & Novikov [87] had pointed out that they might be expected to grow catastrophically. Since a PBH must be of order the horizon size at formation, this suggests that all PBHs could grow to have a mass of order $10^{15} M_\odot$ (the horizon mass at the end of the radiation era). There are strong observational limits on how many such black holes the Universe could contain, so the implication seemed to be that very few PBHs ever existed. Since a PBH must therefore soon become much smaller than the horizon, at which stage cosmological effects become unimportant, it can be concluded that PBHs cannot grow very much at all.

The realization that small PBHs might exist after all prompted Hawking to study their quantum properties. This led to his famous discovery that black holes radiate thermally with a temperature

$$T \sim 10^{-7}\Big(\frac{M}{M_\odot}\Big)^{-1}\mathrm{K}, \tag{3.0.2}$$

so they evaporate on a timescale

$$\tau(M) \sim 10^{64}\Big(\frac{M}{M_\odot}\Big)^{3}\mathrm{yr}. \tag{3.0.3}$$

Only black holes smaller than 10^{15}g would have evaporated by the present epoch, so (3.0.3) implies that this effect could be important only for black holes which formed before 10^{-23}s.

Despite the conceptual importance of this result, it was bad news for PBH enthusiasts. For since PBHs with a mass of 10^{15}g would be producing photons with energy of order 100 MeV at the present epoch, the observational limit on the gamma-ray background intensity at 100 MeV immediately implied that their density is much lower then the critical density [88]. Not only did this render PBHs unlikely dark matter candidates, it also implied that there was little chance of detecting black hole explosions at the present epoch [89]. Nevertheless, it was realized that PBH evaporations could still have interesting cosmological consequences. In particular, they might generate the microwave background [90] or modify the standard cosmological nucleosynthesis scenario [91] or contribute to the cosmic baryon asymmetry [92]. Over the last decade PBHs have been assigned various other cosmological roles. Some people have speculated that PBH evaporation, rather than proceeding indefinitely, could cease when the black hole gets down to the Planck mass. In this case, one could end up with stable Planck mass relics, which would provide dark matter candidates [93].

Physically, black hole region with gravity is so strong that not even light can escape from it. Mathematically one imposes additional conditions of time translation invariance $\mathbf{K}.\mathbf{K}>\mathbf{0}$, where $\mathbf{K}$ is a time-like killing vector and asymptotic flatness $R \to 0$ as $r \to \infty$, with r is the radial spacelike parameter. These conditions are not consistent with cosmological requirements. Since there exists $\mathbf{K}$, therefore energy is conserved. The first law of thermodynamics is energy conservation. This law carries over to black holes.

The area of a black hole can only increase (classically) because the area increases with mass, mass is the same as energy, and energy can only go into a black hole but never come out. It implied that there must be an entropy of the black hole that increases when energy goes into a black hole. Bekenstein [94, 95] proposed that entropy of the black hole is the same as its area. Consequently the area is proportional to square of the mass of the black hole. Later on Hawking [81, 82] gave an explicit constant of proportionality (i.e. 1/4) for the area of the black hole.

In classical thermodynamics if there is an 'extensive' variable for some quantity there will be an 'intensive' variable associated with it. For instance with volume there is pressure and with entropy there is temperature. These quantities are combined in an expression $dE = TdS + pdV$. If the entropy is the black hole area, the temperature will have to be the surface gravity so that the product contributes to the total internal energy. Now surface gravity $\kappa \sim M/r^2$ and $r \sim M$, so the surface gravity is $\sim 1/M$. Thus the larger the black hole the lower the temperature $T \sim 1/M$ since $T \sim \kappa$. It implies that a black hole temperature diverges if mass vanishes or gradually reduces to zero. Thus the 'zeroth law' for the black hole thermodynamics defining temperature, a 'first law' of conservation of energy and a 'second law' defining an entropy that is an increasing quantity with time. However there is no equivalent of the third law "that it takes an infinite number of steps to get to zero temperature".

I am interested in studying the effects of accretion of phantom energy on a static primordial black hole. Carr and Hawking [96] in 1974 considered the formation of black holes of mass 10^2kg and upwards in the early evolution of the Universe. After their attempt, several authors investigated various scenarios of PBH formation [97, 98, 99, 100]. The existence of these small mass black holes was based on the assumption that the early Universe was not entirely spatially smooth but there were density fluctuations or inhomogeneities in the primordial plasma which gravitationally collapsed to form these black holes. Unlike the conventional black holes that are formed by the gravitational collapse of stars or mergers of neutron stars, the primordial black holes (PBH) are formed due to the gravitational collapse of matter without forming any initial stellar object.

Using classical arguments, Penrose and Floyd showed that one can extract rotational energy from a rotating black hole [101]. Penrose went on to argue (see [102] and references therein) that one could take thermal energy from the environs of a black hole and throw it into the black hole to get usable energy out. This would apparently reduce the entropy around the black hole. As such, he had argued that *there must be an entropy of the black hole that increases at least as much as that of its environs decreases.* Hawking had pointed out that in any physical process the area of a black hole always increases [103] just as entropy always increases. This led Bekenstein [94] to propose a linear relationship between the area and entropy of a black hole. Thus Bekenstein [95, 104] generalized the second law of thermodynamics to state that the sum of the entropy of the black hole and its environs never decreases. However, at this stage it seemed that the connection between black holes and thermodynamics was purely formal. At this stage Fulling pointed out that quantization of scalar fields in accelerated frames gives an ambiguous result [105], which seemed to yield radiation seen in the accelerated with a fractional number of particles. Hawking repeated the calculation for an observer near a black hole and obtained the same result by various methods and found that the radiation had a thermal spectrum [83]. This led him to propose that mini-PBHs would evaporate away in a finite time [81, 82].

The corresponding Hawking evaporation process reduces the mass of the black hole by [106]

$$\left.\frac{dM}{dt}\right|_{hr} = -\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2}, \tag{3.0.4}$$

where α is the spin parameter of the emitting particles. Integration of (3.0.4) gives the evolution of PBH mass as

$$M_{hr} = M_i\left(1 - \frac{t}{t_{hr}}\right)^{1/3}, \tag{3.0.5}$$

where the Hawking evaporation time scale t_{hr} is

$$t_{hr} = \frac{G^2}{\hbar c^4}\frac{M_i^3}{3\alpha}. \tag{3.0.6}$$

It is obvious from (3.0.5) that as $t \to t_{hr}$, the mass $M_{hr} \to 0$. Plugging in $t_{hr} = t_o$ (the current age of the Universe) in (3.0.6) gives the mass 10^{12}kg of the PBH that should have been evaporating now. Hence from (3.0.6), it can be estimated that these PBHs were formed before about 10^{-23}sec. For $M_i \gg 10^{14}kg$, $\alpha = 2.011 \times 10^{-4}$, hence (3.0.6) implies $t_{hr} \simeq 2.16 \times 10^{-18}\left(\frac{M}{kg}\right)^3$ sec. While for $5 \times 10^{11}kg \ll M_i \ll 10^{14}$kg, $\alpha = 3.6 \times 10^{-4}$ then (3.0.6) gives $t_{hr} \simeq 4.8 \times 10^{-18}\left(\frac{M}{kg}\right)^3$ sec. Therefore detecting PBHs would be a good tool to probe the very early Universe (closer to the Planck time). The evaporation of PBHs could still have interesting cosmological implications: they might generate the microwave background [87] or modify the standard cosmological nucleosynthesis scenario [108] or contribute to the cosmic baryon asymmetry [93]. Some authors have also considered the possibility of the accretion of matter and dust onto the seed PBH resulting in the formation of super-massive black holes which reside in the centers of giant spiral and elliptical galaxies [109].

3.1 Phantom energy accretion onto a black hole

The FRW equations governing the dynamics of my gravitational system are given by

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_x), \tag{3.1.1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho_m + \rho_x(1+\omega)] \ . \tag{3.1.2}$$

Here ρ_m and ρ_x denote the energy densities of matter and the exotic energy densities respectively. The scale factor $a(t)$ goes like [49]

$$a(t) = \frac{a(t_0)}{[-\omega + (1+\omega)t/t_0]^{-\frac{2}{3(1+\omega)}}} \quad (t > t_0), \tag{3.1.3}$$

where t_0 is the time when the Universe transits from matter to exotic energy domination (which is roughly equal to the age of the Universe). Notice that the scale factor $a(t)$ diverges when the quantity in the square brackets in (3.1.3) vanishes identically i.e.

$$t^* = \frac{\omega}{1+\omega} t_0 \ . \tag{3.1.4}$$

Subtracting t_0 from (3.1.4), I get

$$t^* - t_0 = -\frac{1}{1+\omega} t_0 \ . \tag{3.1.5}$$

The evolution of energy density of the exotic energy is given by

$$\rho_x^{-1} = 6\pi G(1+\omega)^2(t^* - t)^2. \tag{3.1.6}$$

A black hole accreting only the exotic energy has the following rate of change in mass [1]

$$\left.\frac{dM(t)}{dt}\right|_x = \frac{16\pi G^2}{c^5} M^2(\rho_x + p_x) \ . \tag{3.1.7}$$

It is clear that when $\rho_x + p_x < 0$, the mass of the black hole will decrease. I am particularly interested in the evolution of black holes about and after $t = t_0$ since the dark energy is presumably negligible before that time and may not have any noticeable effects on the black hole. Using (3.1.5) and (3.1.6) in (3.1.7), I get

$$\left.\frac{dM(t)}{dt}\right|_x = \frac{8G}{3c^3}\frac{M^2}{t_0^2}(1+\omega) \ . \tag{3.1.8}$$

Therefore the mass change rate for a black hole accreting pure exotic energy is determined by (3.1.8). For the phantom energy accretion, the time scale is obtained by integrating (3.1.8) to get

$$M(t) = M_i \left(1 - \frac{t}{t_x}\right)^{-1}, \tag{3.1.9}$$

where t_x is the characteristic accretion time scale given by

$$t_x^{-1} = \frac{16\pi G^2}{c^5} M_i(\rho_x + p_x). \tag{3.1.10}$$

Using (3.1.5) and (3.1.6) in (3.1.10), I get

$$t_x = \frac{3c^3}{8G} \frac{t_0^2}{M_i(1+\omega)}. \tag{3.1.11}$$

3.2 Evolution of mass due to phantom energy accretion and Hawking evaporation

The expression determining the cumulative evolution of the black hole is obtained by adding (3.0.4) and (3.1.8) i.e.

$$\left.\frac{dM(t)}{dt}\right|_{Total} = \left.\frac{dM}{dt}\right|_{hr} + \left.\frac{dM}{dt}\right|_{x}, \tag{3.2.1}$$

$$= -\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2} + \frac{8G}{3c^3}\frac{M^2}{t_0^2}(1+\omega). \tag{3.2.2}$$

I write the above equation as

$$\frac{dM}{dt} = -aM^2 - \frac{b}{M^2}, \tag{3.2.3}$$

where

$$a = \frac{8G}{3c^3}\frac{\epsilon}{t_0^2}, \quad b = \frac{\hbar c^4 \alpha}{G^2}. \tag{3.2.4}$$

Here $\epsilon = -\omega - 1$. Thus (3.2.3) can be written in the form

$$-\int dt = \frac{1}{b}\int \frac{M^2 dM}{1 + \frac{a}{b}M^4}$$

To integrate above equation, I assume

$$x = \left(\frac{a}{b}\right)^{1/4} M, \tag{3.2.5}$$

which yields

$$-\int dt = \frac{1}{(a^3b)^{1/4}}\int \frac{x^2dx}{1+x^4}, \tag{3.2.6}$$

Note that [110]

$$\begin{aligned}\int \frac{x^{u-1}dx}{1+x^{2n}} &= -\frac{1}{2n}\sum_{k=1}^{n}\cos\left(\frac{u\pi(2k-1)}{2n}\right)\ln\left|1-2x\cos\left(\frac{2k-1}{2n}\right)\pi + x^2\right| \\ &\quad + \frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{u\pi(2k-1)}{2n}\right)\tan^{-1}\left[\frac{x-\cos\left(\frac{2k-1}{2n}\right)\pi}{\sin\left(\frac{2k-1}{2n}\right)\pi}\right], \quad u<2n.\end{aligned} \tag{3.2.7}$$

In my case, $u = 3$ and $n = 2$, hence the above equation yields

$$\int \frac{x^2dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\left|\frac{1-\sqrt{2}x+x^2}{1+\sqrt{2}x+x^2}\right| + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{1+x^2}\right). \tag{3.2.8}$$

On substituting the value of x above gives

$$t = t_0 + \frac{1}{4\sqrt{2}}\ln\left|\frac{1-\sqrt{2}\left(\frac{a}{b}\right)^{1/4}M + \left(\frac{a}{b}\right)^{1/2}M^2}{1+\sqrt{2}\left(\frac{a}{b}\right)^{1/4}M + \left(\frac{a}{b}\right)^{1/2}M^2}\right| + \frac{1}{2\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{2}\left(\frac{a}{b}\right)^{1/4}M}{1+\left(\frac{a}{b}\right)^{1/2}M^2}\right]. \tag{3.2.9}$$

Redefining the values of a and b by assuming $M = mM_i$, where m is a dimensionless parameter and M_i is the initial mass of the black hole. Thus (3.2.3) becomes

$$\frac{dm}{dt} = -a'm^2 - \frac{b'}{m^2}, \tag{3.2.10}$$

where $a' = aM_i$ and $b' = b/M_i^3$. For the terms to be equal strength, I require $a' \approx b'$. Thus

$$M_i \approx \left(\frac{b}{a}\right)^{1/4}. \tag{3.2.11}$$

Now

$$\frac{b}{a} = \frac{3\hbar c^7 t_0^2 \alpha}{8G^3\epsilon}, \quad \text{or,} \quad \epsilon = \frac{3\hbar c^7 t_0^2\alpha}{8G^3M_i^4}. \tag{3.2.12}$$

I can normalize

$$t = t_0 \left[1 - \frac{\frac{1}{4\sqrt{2}} \ln \left| \frac{1-\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4} m + \left(\frac{a'}{b'}\right)^{1/2} m^2}{1+\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4} m + \left(\frac{a'}{b'}\right)^{1/2} m^2} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left[\frac{\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4} m}{1+\left(\frac{a'}{b'}\right)^{1/2} m^2} \right]}{\frac{1}{4\sqrt{2}} \ln \left| \frac{1-\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4} + \left(\frac{a'}{b'}\right)^{1/2}}{1+\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4} + \left(\frac{a'}{b'}\right)^{1/2}} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left[\frac{\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4}}{1+\left(\frac{a'}{b'}\right)^{1/2}} \right]} \right]. \tag{3.2.13}$$

Replacing $p' = a'/b' = \frac{8\epsilon G^3}{3\alpha\hbar c^7 t_0^2} M_i^4 \sim M_i^4$ (the ratio of the phantom component to the Hawking component, in the energy radiated) the above equation becomes

$$t = t_0 \left[1 - \frac{\frac{1}{4\sqrt{2}} \ln \left| \frac{1-\sqrt{2}p'^{1/4}m+p'^{1/2}m^2}{1+\sqrt{2}p'^{1/4}m+p'^{1/2}m^2} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}p'^{1/4}m}{1+p'^{1/2}m^2} \right)}{\frac{1}{4\sqrt{2}} \ln \left| \frac{1-\sqrt{2}p'^{1/4}+p'^{1/2}}{1+\sqrt{2}p'^{1/4}+p'^{1/2}} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}p'^{1/4}}{1+p'^{1/2}} \right)} \right]. \tag{3.2.14}$$

Moveover, the power emission due to Hawking evaporation from the stationary black hole of mass $M \gg 10^{17}$g [106]

$$P = 3.458 \times 10^{46} (M/\mathrm{g})^{-2} \mathrm{ergs}^{-1}, \tag{3.2.15}$$

and for mass $5 \times 10^{14}\mathrm{g} \ll M \ll 10^{17}\mathrm{g}$,

$$P \approx 3.6 \times 10^{16} (M/10^{15}\mathrm{g})^{-2} \mathrm{ergs}^{-1}. \tag{3.2.16}$$

In my analysis, the mass in the above two expressions is replaced by

$$M = \left(\frac{3\hbar c^7 t_0^2 \alpha}{8G^3 \epsilon} \right)^{1/4} \mathrm{g}. \tag{3.2.17}$$

Now choosing $\epsilon = 0.1$, I obtain $M = 8.74029 \times 10^{22}$g which will be evaporating now due to the combined effects of phantom energy and Hawking radiation. Then using (3.2.15), the corresponding power emission will be $P = 4.52661$ erg s^{-1}. Compare this result with that of a black hole of mass $M \simeq 1.05 \times 10^{12}$g evaporating just now due to Hawking radiation only. The corresponding power emission will be $P \simeq 3.144 \times 10^{22}$erg s^{-1}. Note that the power emission from a black hole increases when the effects of phantom energy are incorporated. Similarly, for very large values of $\epsilon \sim 10^{25}$ would give $M = 2.763923 \times 10^{16}$g. Using this mass in (3.2.16), the power emitted is 4.52661×10^{13}erg s^{-1}. However, such large values would lead to a very early Big Rip and hence must be excluded. Thus black holes $\sim 10^{22}$g are of more interest for observational purposes since these are the ones that should be evaporating now.

3.3 Conclusion

In this chapter, I have analyzed the Hawking radiation effects combined with the phantom energy accretion on a stationary black hole. The former process has been thoroughly investigated in the literature. However there is as yet no observational evidence to support it. According to standard theory it is assumed that after the formation of PBHs (of mass $\sim 10^{12}$kg with a Hawking temperature 10^{12}K), they would absorb virtually no radiation or matter whatsoever during their evolution and radiate continuously till they evaporate in a burst of gamma rays at the present time. This scenario assumes that the Hawking temperature for such black holes was always larger than the background temperature of the CMB. Strictly speaking, this cannot be true. Consequently PBHs could have accreted the background radiation (and even some matter) and *grown* in mass. Hence there should be no PBH left to be evaporating right now [111]. However, the above scenario is modified when phantom energy comes into play. When phantom energy and the Hawking process are relevant the total life time scale of the PBH is significantly shortened and the formation of the PBH exploding now is delayed.

From (3.2.14) I obtain the time as a function of mass instead of getting mass as a function of time. To make sense of the results I need to obtain the evolution with time. This is done by inverting the explicit function. I have plotted the normalized time $\tau = t/t_0$ against the dimensionless mass parameter m and m against τ for different choices of the parameter p, in Figures 3.1 - 3.10. It is observed that increasing p' increases the steepness of the curve specifying the mass evolution. Therefore the black hole loses mass faster for larger p' till it vanishes at $\tau = 1$, the present time. In particular, Figures 3.7 and 3.9 show the same evolution of mass for larger values of p'. It appears that the graphs contain a redundant (or nonphysical) part of the mass evolution and the only physically interesting section is above the horizontal curve crossing $t = 0$. Thus in effect, see Figures 3.8 and 3.10, the initial mass of the black hole must be taken as $0.45M_i$ of the value given by for $p' = 5$ and about $0.315M_i$ for $p' = 10$. It is obvious that the results are very insensitive to changes of the parameter ϵ for the phantom energy. As such, they can be regarded as fairly robust.

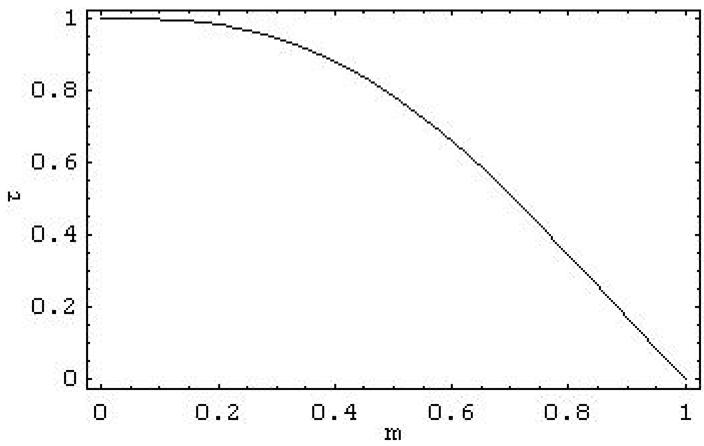

Figure 3.1: The normalized time is plotted against the mass parameter for $p' = 0.1$. The physical picture of decrease in mass of BH is not very clear. Hence in the Figure 3.2, we have re-plotted it with the parameters replaced along the horizontal and vertical axes.

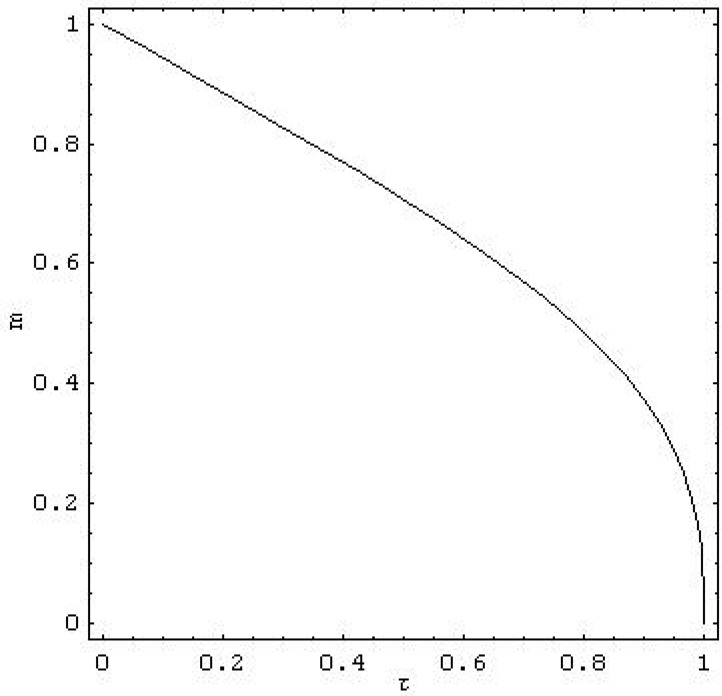

Figure 3.2: The mass parameter is plotted against the normalized time for $p' = 0.1$. Due to the re-definition of mass and time parameters, they don't correspond to the real mass of BH and time. The mass starts to decrease from the initial normalized value (which holds for all masses of BH), and starts decreasing very fast near the Big Rip time (again not real time but it corresponds to the present time) between 0.8 and 1, the later value at which the BH vanishes.

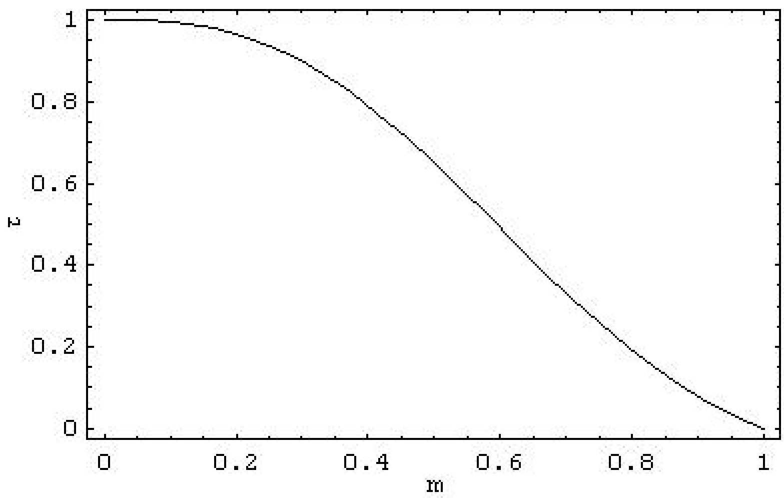

Figure 3.3: The normalized time is plotted against the mass parameter for $p' = 0.5$. Thus the rate of the evaporation of black hole due to phantom energy is 50 percent stronger than that of the Hawking evaporation. It turns out that increasing p' increases the steepness of the evolution curve above.

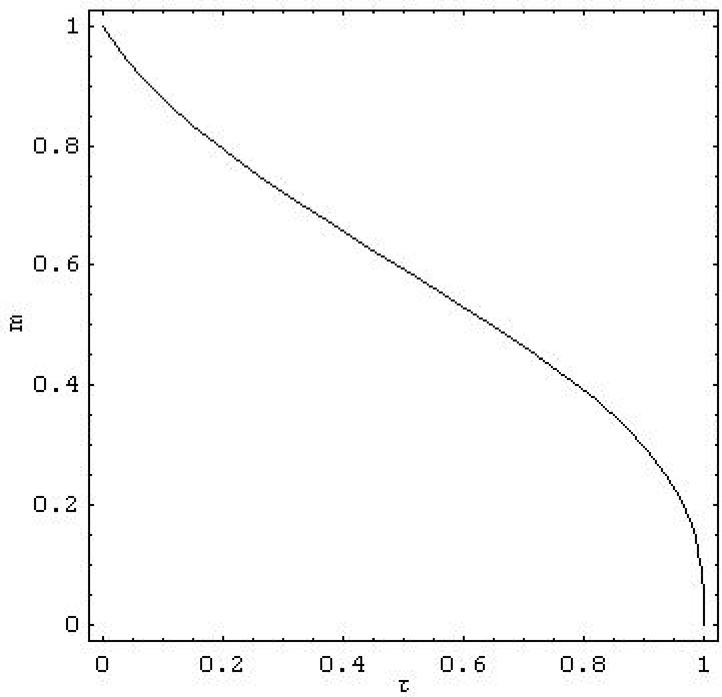

Figure 3.4: The mass parameter is plotted against the normalized time for $p' = 0.5$.

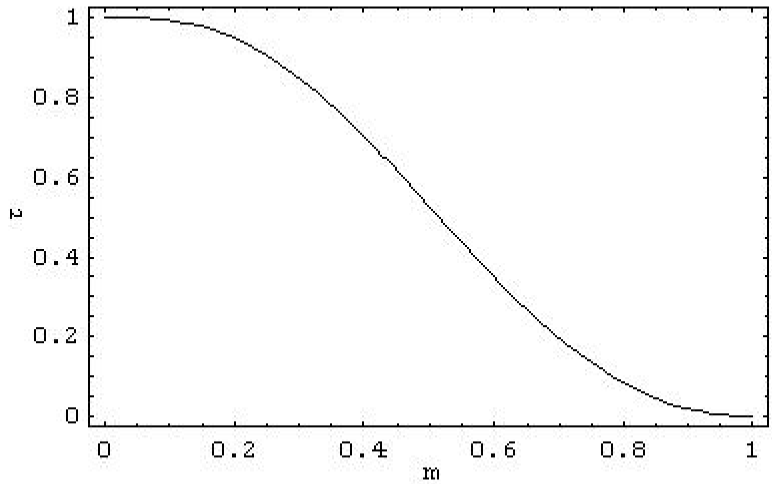

Figure 3.5: The normalized time is plotted against the mass parameter for $p' = 1$. Thus the rate of evaporation of black hole due to phantom energy is equal to that of the Hawking evaporation.

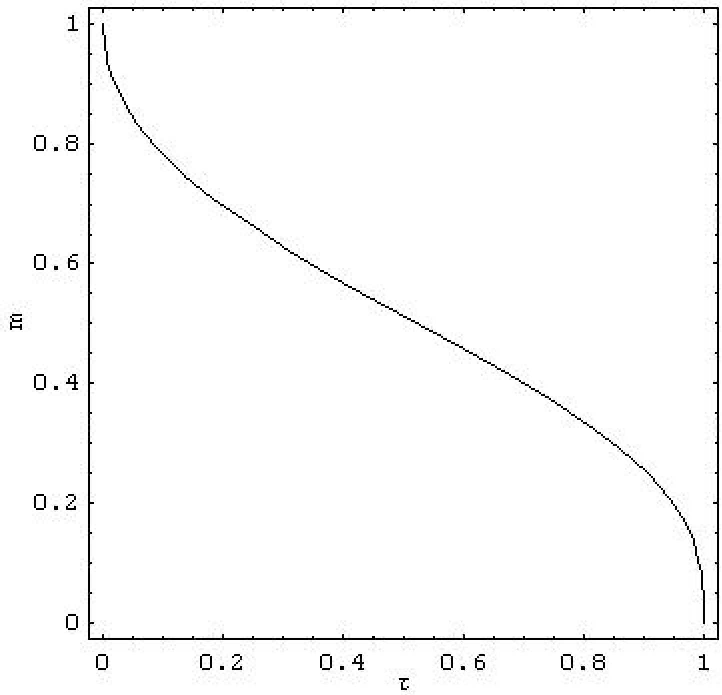

Figure 3.6: The mass parameter is plotted against the normalized time for $p' = 1$. It shows that the evaporation of BH is faster in the beginning and near the end of the normalized time.

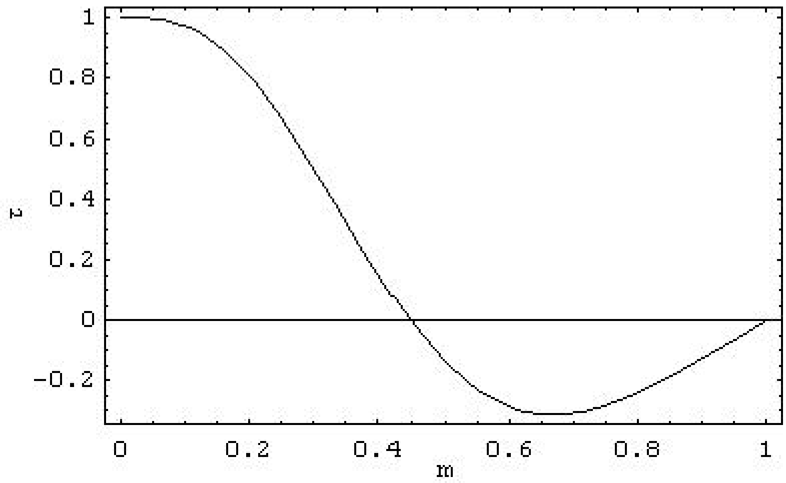

Figure 3.7: The normalized time is plotted against the mass parameter for $p' = 5$. It shows that the rate of the evaporation of black hole due to phantom energy is 5 times stronger(greater) than that of the Hawking evaporation. Notice that the above figure also contains a non-physical region of the evolution curve in which the normalized time acquires negative values. In Figure 3.8 below, we have replotted this figure by removing the non-physical part.

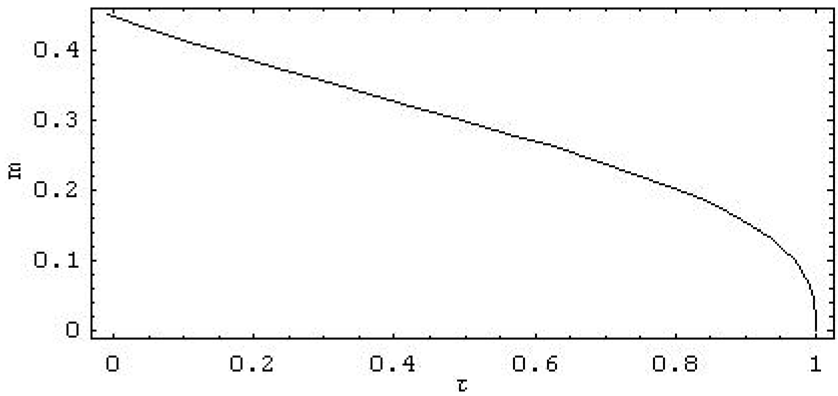

Figure 3.8: The mass parameter is plotted against the normalized time. Here the initial mass of the black hole must be taken as $0.45M_i$ of the value given by for $p' = 5$.

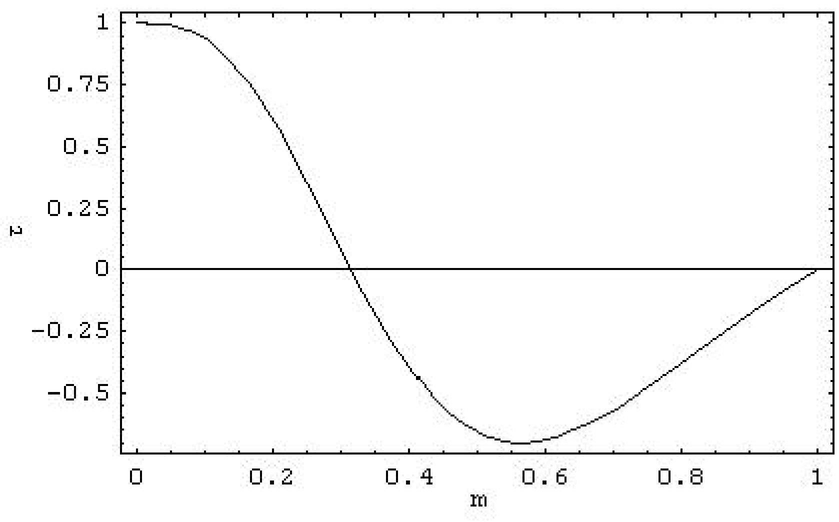

Figure 3.9: The normalized time is plotted against the mass parameter for $p' = 10$. Similar to Figure 3.7 above, this figure contains a huge unphysical portion, which we have removed in Fig. 3.10

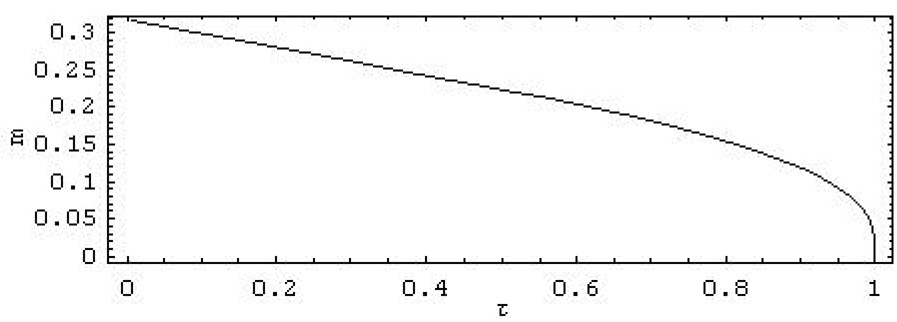

Figure 3.10: The mass parameter is plotted against the normalized time. Here the initial mass of the black hole must be taken as about $0.315 M_i$ for $p' = 10$

Chapter 4

Charged black holes in phantom cosmology

In the last chapter, I discussed the accretion of phantom energy onto a primordial black hole. I now turn my attention to Reissner-Nordström (RN) black hole, which is a spherically symmetrical object with mass M and electric charge Q. I study accretion of phantom energy on a RN black hole (ignoring the case of Hawking radiation by the RN black hole).

In the literature, the RN black hole is discussed in wider contexts. In [112], the author investigated the gravitational lensing by an RN black hole in the weak deflection limit. The author obtained the basic equations for the deflection angle and time delay and found analytical expressions for the positions and amplifications of the primary and secondary images. Due to a net positive charge, the separation between images increases, but no change in the total magnification occurs for an RN black hole. In another study [113], the authors showed that an extremal RN black hole ($Q^2 = M^2$) may be turned into a Kerr-Newmann naked singularity after the capture of a flat and electrically neutral spinning body which is initially gravitationally bound and plunges in radially with its spin aligned to a radial direction. They argued that back-reaction and emission of gravitational radiation would not help to preserve the black hole condition and that conversion to a naked singularity was inevitable. The proposal discussed in [113] was criticized in [114]. The later study showed that no physically reasonable model of such an object can be made. I provide an alternative mechanism of converting a RN black hole into a naked singularity. The mechanism involves accretion of phantom energy onto the black hole.

As discussed in the previous chapter, the accretion of phantom energy reduces the mass of the black hole. In the case of an RN black hole, the charge remains conserved and a stage is reached when the magnitude of the mass becomes smaller than that of the charge. This leads to the formation of a naked singularity.

Penrose proposed the cosmic censorship hypothesis so as to avoid the possibility of unpredictable influences emerging from the singularity, where physical laws break down. As he put it [115], "it is as if there is a cosmic censor board that objects to naked singularities being seen and ensures that they only appear suitably clothed by an event horizon". Due to this conjecture, naked singularities are seldom studied seriously in themselves. A naked singularity is the outcome of a continued gravitational collapse when no event horizon forms hiding the singularity to the asymptotic region [116]. A generic feature of a naked singularity is that of being infinitely red-shifted with respect to any of the non-singular spacetime points. Since no physical influence reaches infinity from the singularity, it is justified to assume the existence of a regular flat (past and future) infinity after the formation of the singularity. In some studies [117, 118], the authors have proposed that a naked singularity, specifically a spinning one, can convert into a black hole as a result of its interaction with the surrounding matter. Moreover if a naked singularity decays into a black hole, then the later will most likely be of a Kerr type. The production of naked singularities as a result of gravitational collapse is still a matter of debate. Naked singularities have far reaching consequences; their spacetime can be causally ill-behaved and they may be sources of cosmic events with anomalously high energy. If they exist it would be legitimate to invoke the validity of a theorem due to Clarke and de Felice [119] which states that a generic strong-curvature naked singularity would give rise to a Cosmic Time Machine (CTM). A CTM is a spacetime which is asymptotically flat and admits closed non-spacelike curves which extend to future infinity.

In the astrophysical context, charged black holes are very unlikely to exist. A charged black hole may form if the initial matter distribution forming it possesses a net electric charge. However, the electrically charged black hole eventually becomes neutral by interacting with surrounding matter (only if that matter carries the opposite electric charge) and converts into a Schwarzschild black hole. One of the motivations to study electrically charged black holes is its role in providing a mechanism to convert it into a naked singularity. A model proposed by Hubeny [120] suggests a mechanism of the conversion of RN black hole into a naked singularity by dropping in a charge. However there some objections to the suggested mechanism [114]. Astrophysically, the only appealing solution is the Kerr-Newmann spacetime, which I shall not deal with in this thesis.

The fate of a stationary uncharged black hole in the phantom energy dominated Universe was investigated by Babichev et al [1]. The phantom energy was assumed to be a perfect fluid. The phantom energy was allowed to fall onto the black hole horizon only in the radial direction. It was concluded that black hole will lose mass steadily due to phantom energy accretion and disappear near the Big Rip. I here adopt their procedure for a static, stationary and charged black hole. Gravitational units ($c = G = 1$) are chosen for this work.

4.1 Accretion onto a charged black hole

In the framework of Newtonian gravity, the accretion of matter onto a compact object was first investigated and formulated by Bondi in 1952 [121]. His work dealt with stationary accretion i.e. model parameters are time-independent. However, this work was not ideally suitable to model accretion onto objects like black holes and compact stars (neutron stars and white dwarfs). A relativistic version of Bondi's work was presented by Michel, almost twenty years later [122]. Using basic principles of conservation laws, he studied the dynamics of accretion and flows near a Schwarzschild black hole. The fluid was taken as a perfect fluid while a polytropic equation of state (relating pressure and density) was chosen for the analysis. He predicted that the accreting gas will get heated up to 10^{12} K while high energy X-rays and gamma rays will be emitted in the process. Michel's work was later on explored in various astrophysical scenarios of accretion onto black holes and quasars etc.

There is strong evidence that the present Universe contains dark energy which is causing accelerated expansion of the Universe. The empirical findings categorically suggests that this dark energy is not due to cosmological constant but a time varying dark energy is responsible. The equation of state parameter of this energy is blow -1, which suggests the presence of phantom energy. This had motivated Babichev et al to study the accretion of phantom energy onto a Schwarzschild black hole. Based on the work of Michel, they concluded that the mass of the black hole decreases with time, finally vanishing near a Big Rip singularity. They mentioned that this disappearance means that even the naked singularities might not get formed and these will also be ripped apart. One can argue against the accretion of phantom energy onto any object by claiming that dark energy is a property of spacetime and that its accretion will be meaningless. Apparently it is true but one can study accretion if dark energy is considered as 'dark energy fluid', and particularly a perfect fluid.

Here I shall assume the steady state accretion of a test perfect fluid with an arbitrary equation of state $p = p(\rho)$, onto a stationary charged black hole, when the event horizon satisfies $M^2 > Q^2$, while the cases of extreme RN black hole $Q^2 = M^2$ and naked singularity $Q^2 > M^2$ are not taken into account. I will show below that when phantom energy is accreted on the black hole, the extreme state of electrically charged black hole is reached during a finite time. It might be asked whether this extreme black hole might convert to a naked singularity or not. Notice that in the case of a Schwarzschild black hole, the naked singularity does not form and it disappears before the Big Rip, however, in the RN case the naked singularity does form. Apparently, this violates the third law of black hole thermodynamics, which says that no physical process can lead a black hole to a naked singularity.

I consider the Reissner-Nordström line element (2.2.27). If $Q^2 > M^2$ then the metric is non-singular everywhere except at the curvature or the irremovable singularity at $r = 0$. Also if $Q^2 \leq M^2$ then the function $f(r)$ has two real roots given by (2.2.28). These roots physically represent the apparent horizons of the RN black hole. The two horizons are termed the inner r_{h-} and the outer r_{h+}. The outer horizon is effectively called the *event horizon* while the inner one is called the *Cauchy horizon* of the black hole. The metric (2.2.27) is then regular in the regions specified by the inequalities: $\infty > r > r_{h+}$, $r_{h+} > r > r_{h-}$ and $r_{h-} > r > 0$. Note that if $Q^2 = M^2$, then it represents an *extreme* RN black hole while if $Q^2 > M^2$, it yields a *naked singularity* at $r = 0$ [123, 124].

The phantom energy is assumed to be a perfect fluid specified by the stress energy tensor (1.2.9). Denoting p as the pressure and ρ as the energy density of the phantom energy. I assume that $p = p(\rho)$ is an arbitrary function of energy density only. Also $u^\mu = (u^t(r), u^r(r), 0, 0)$, is the four velocity of the phantom fluid which satisfies the normalization condition $u^\mu u_\mu = 1$. Assuming that the in-falling phantom fluid does not disturb the global spherical symmetry of the black hole. The energy-momentum conservation $T^{\mu\nu}_{;\nu} = 0$, gives

$$ur^2 M^{-2}(\rho + p)\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2} + u^2} = C_1, \tag{4.1.1}$$

where $u^r = u = dr/ds$ is the radial component of the velocity four vector and $C_1 > 0$ is a constant of integration. While performing the integration in (4.1.1), I chose the positive branch of the logarithmic function since the LHS in (4.1.1) is positive ($u < 0$ for an inward flow and $\rho + p < 0$, the violation of NEC for the phantom energy). Moreover, the second constant of motion is obtained by projecting the energy conservation equation onto the velocity four vector as $u_\mu T^{\mu\nu}_{;\nu} = 0$, which yields

$$ur^2 M^{-2} \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p(\rho')}\right] = -A. \tag{4.1.2}$$

Here $A > 0$ is a constant of integration. Similarly while performing the integration in (4.1.2), I chose the negative branch of the logarithmic function since the LHS in (4.1.2) is negative ($u < 0$). The above ρ_h and ρ_∞ are the energy densities of the phantom energy at the horizon and at infinity respectively. Using (4.1.1) and (4.1.2) gives

$$(\rho + p)\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2} + u^2}\exp\left[-\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p(\rho')}\right] = C_2, \tag{4.1.3}$$

where $C_2 = -C_1/A = \rho_\infty + p(\rho_\infty)$. Integrating the flux of the fluid over the entire cross-section of the event horizon gives the rate of change of mass of black hole as

$$\dot{M} = -4\pi r^2 T^r_t, \tag{4.1.4}$$

where T^r_t determines the momentum density in the radial direction. From (4.1.1) to (4.1.4),

$$\frac{dM}{dt} = 4\pi A M^2 (\rho_\infty(t) + p_\infty(t)), \tag{4.1.5}$$

which clearly demonstrates that mass of black hole decreases if $\rho_\infty + p_\infty < 0$. Note that (4.1.5) can be solved for any equation of state of the form $p = p(\rho)$ or in particular $p = \omega\rho$. In general, (4.1.5) holds for all ρ and p violating the dominant energy condition, thus [125, 126]

$$\frac{dM}{dt} = 4\pi A M^2 (\rho(t) + p(t)). \tag{4.1.6}$$

In the astrophysical context, the mass of black hole is a dynamic quantity. The mass increases by the accretion of matter and can decrease by the accretion of the phantom energy. Since I am not incorporating matter in my model, the mass of black hole will decrease correspondingly.

4.2 Critical accretion

Only in those solutions are relevant that pass through the critical point as these correspond to the material falling into the black hole with monotonically increasing speed. The falling fluid can exhibit variety of behaviors near the critical point of accretion, close to the compact object. For instance, for a given critical point $r = r_c$, the following possibilities arise [127]: (a) $u^2 = c_s^2$ at $r = r_c$, $u^2 \to 0$ as $r \to \infty$, $u^2 < c_s^2$ for $r > r_c$ and $u^2 > c_s^2$ for $r < r_c$. Thus for large distance, the speed of flow becomes negligible (subsonic), at the critical point it is sonic, while the flow becomes supersonic for very small r. Other solutions for the flow near r_c are not of much interest due to their impracticality, like (b) $u^2 < c_s^2$ for all values of r and (c) $u^2 > c_s^2$ for all values of r. Solutions (b) and (c) are not realistic since they describe both subsonic and super-sonic flows for all r. Similarly, (d) $u^2 = c_s^2$ for all values of $r > r_c$ and (e) $u^2 = c_s^2$ for all values of $r < r_c$. Last two solutions are also useless since they give same value of speed at a given r. Hence from this discussion, solution (a) is the only physically motivated, near the critical point. To determine the critical points of accretion I adapt the procedure as specified in Michel [122]. The equation of mass flux $J^r_{;r} = 0$ gives

$$\rho u r^2 = k_1, \tag{4.2.1}$$

where k_1 is constant of integration. Dividing and then squaring (4.1.1) and (4.2.1) gives

$$\left(\frac{\rho + p}{\rho}\right)^2 \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} + u^2\right) = \left(\frac{C_1}{k_1}\right)^2 = C_3. \tag{4.2.2}$$

Here C_3 is a positive constant. Differentiation of (4.2.1) and (4.2.2) and then elimination of $d\rho$ gives

$$\frac{dr}{r}\left[2V^2 - \frac{\frac{M}{r} - \frac{Q^2}{r^2}}{1 - \frac{2M}{r} + \frac{Q^2}{r^2} + u^2}\right] + \frac{du}{u}\left[V^2 - \frac{u^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2} + u^2}\right] = 0, \tag{4.2.3}$$

where

$$V^2 \equiv \frac{d\ln(\rho + p)}{d\ln\rho} - 1. \tag{4.2.4}$$

It is assumed that the flow is smooth at all points of spacetime, however if at any point the denominator D vanishes then the numerator N must also vanish at that point. Mathematically this point is called the *critical point* of the flow [128]. Equating the denominator D and numerator N to zero, so-called *critical point conditions* are given by

$$u_c^2 = \frac{Mr_c - Q^2}{2r_c^2}, \tag{4.2.5}$$

and

$$V_c^2 = \frac{Mr_c - Q^2}{2r_c^2 - 3Mr_c + Q^2}. \tag{4.2.6}$$

Note that by choosing $Q = 0$ in the above equations, one can retrieve the results for the accretion of fluid onto a Schwarzschild black hole [122]. All the quantities with subscript c are defined at the critical point correspondingly. Physically, the critical points represent the *sonic point* of the flow i.e. the point where the speed of flow becomes equal to the speed of sound, $u_c^2 = c_s^2$ or the corresponding Mach number $M_c = 1$. This transition may occur from the initial subsonic to the supersonic or trans-sonic speeds. For any spherically symmetric spacetime, a surface where every point is a sonic point is called a *sound horizon* which itself will be spherical. Any perturbation or disturbance generated in the flow inside the sound horizon ($r < r_c$) is eventually pulled towards the black hole singularity and hence cannot escape to infinity.

It can be seen that the speed of sound (squared) $c_s^2 = \partial p/\partial\rho$ has no physical meaning if the EoS parameter $\omega < 0$ (in $p = \omega\rho$). Thus it will apparently make the exotic cosmic fluids like the cosmological constant, quintessence and the phantom energy unstable that can not be accreted onto the black hole. In order to avoid this problem, Babichev et al [129] introduced a non-homogeneous linear equation of state (nEoS) given by $p = \alpha(\rho - \rho_o)$, where the constants α and ρ_o are free parameters. The nEoS can describe both hydrodynamically stable ($\alpha > 0$) and unstable ($\alpha < 0$) fluids. The parameter ω is related to the nEoS as $\omega = \alpha(\rho - \rho_o)/\rho$. Notice that $\omega < 0$ corresponds to $\alpha > 0$ and $\rho > \rho_o$, thus making the phantom energy as hydrodynamically stable fluid. Therefore the speed of sound c_s is now well-defined with the nEoS for the phantom energy. Hence, the phantom energy can fall onto the RN black hole and can reduce the black hole mass. Since phantom energy reduces only mass and not charge, a stage is reached when the cosmic censorship conjecture becomes violated i.e. $Q^2 > M^2$, the so-called emergence of a naked singularity.

Now physically acceptable solution of (4.2.3) is obtained if $u_c^2 > 0$ and $V_c^2 > 0$, hence

$$2r_c^2 - 3Mr_c + Q^2 \geq 0, \tag{4.2.7}$$

and

$$Mr_c - Q^2 \geq 0. \tag{4.2.8}$$

(4.2.7) can be factorized as

$$2r_c^2 - 3Mr_c + Q^2 = (r_c - r_{c+})(r_c - r_{c-}) \geq 0, \tag{4.2.9}$$

where

$$r_{c\pm} = \frac{1}{4}(3M \pm \sqrt{9M^2 - 8Q^2}), \tag{4.2.10}$$

which are positive satisfying $r_{c+} > r_{c-} > 0$. It is worthwhile to notice that in contrast to the case of accretion onto a Schwarzschild black hole, there are formally two different critical points, with plus and minus signs in the last equation. The limit $Q \to 0$ suggests that the inner critical point r_{c-} is unphysical since it becomes zero if $Q = 0$. In general, for $Q \leq M$, the inner critical point will lie between $r_{h-} \leq r_{c-} \leq r_{h+}$, while the outer one will satisfy $r_{c+} \geq r_{h+}$. Thus only r_{c+} is a physical one since it corresponds to a regular RN spacetime. It is obvious that these roots will be real valued if $9M^2 - 8Q^2 \geq 0$ or

$$\frac{M^2}{Q^2} \geq \frac{8}{9}. \tag{4.2.11}$$

These roots physically represent the locations of the critical or sonic points of the flow near the black hole. Notice that both mass and charge have same dimension of length, therefore all the inequalities here and below represent dimensionless ratios. From (4.2.9), I can see that these critical points specify two regions for the flow: (1) $r_c > r_{c+}$ or (2) $0 < r_c < r_{c-}$. I shall now solve (4.2.7) using (4.2.8) and then deduce a condition for the black hole mass and charge.

To get solutions about the critical points, substitute $r_{c\pm}$ in (4.2.8). For r_{c+}, (4.2.8) gives

$$M\sqrt{9M^2 - 8Q^2} \geq 4Q^2 - 3M^2, \tag{4.2.12}$$

which is satisfied if

$$\frac{M^2}{Q^2} \leq 1, \tag{4.2.13}$$

and

$$\frac{M^2}{Q^2} < \frac{4}{3}. \tag{4.2.14}$$

A comparison of inequalities (4.2.11), (4.2.13) and (4.2.14) imply

$$\frac{8}{9} \leq \frac{M^2}{Q^2} < \frac{4}{3}. \tag{4.2.15}$$

Thus accretion through r_{c+} is possible if the above inequality (4.2.15) is satisfied. It encompasses the two types of black holes in itself: regular and the extreme RN black hole. Interestingly, the naked singularity also falls within the prescribed limits. Thus for all these spacetimes, the accretion is allowed through the critical point r_{c+}. Using $Q = 0$ in the inequality (4.2.15) to retrieve same condition for the Schwarzschild black hole can be misleading. The inequality is deduced using the outer apparent horizon and a critical point. Since Schwarzschild black hole ($Q \to 0$) possesses unique horizon and the critical point, the above inequality cannot be reduced for an uncharged black hole.

Now I consider case (2) when $0 < r_c < r_{c-}$. Substitution of r_{c-} in (4.2.8) gives

$$M\sqrt{9M^2 - 8Q^2} \leq 3M^2 - 4Q^2. \tag{4.2.16}$$

If $3M^2 - 4Q^2 < 0$ then (4.2.12) does not yield any solution. So $3M^2 - 4Q^2 > 0$ which yields

$$\frac{M^2}{Q^2} > \frac{4}{3}, \tag{4.2.17}$$

Further inequality (4.2.12) is satisfied if

$$\frac{M^2}{Q^2} < 1. \tag{4.2.18}$$

Since (4.2.17) and (4.2.18) are mutually inconsistent, there is no solution for r_c in case (2). Thus accretion is not possible through r_{c-}.

Since the mass of black hole is decreasing by the accretion of phantom energy (see 4.1.6), it implies that at least one critical point must exist for the fluid flow, which is specified by r_{c+}. This critical point yields the mass to charge ratio of the black hole in the range specified by (4.2.15) which allows that accretion onto all charged spherically symmetric black holes.

4.3 Conclusion

I have analyzed the effects of accretion of phantom energy onto a charged black hole. When the accreting fluid is phantom, $\rho + p < 0$, the mass of the Reissner-Nordström black hole decreases. This immediately leads us to the question, whether it is possible to transform the Reissner-Nordström black hole into the naked singularity by accretion of phantom. Formally it seems so, since the accreting phantom decreases the black hole mass, while the electric charge of the black hole remains the same. Thus, one can expect that at some finite moment of time a black hole will turn into the naked singularity. Indeed it takes the finite time for the Reissner-Nordström black hole to reach the extreme case. The similar result also holds for the Kerr black hole.

The analysis is performed using two critical points $r_{c\pm}$. It turns out that accretion is possible only through r_{c+} which yields a constraint on the mass to charge ratio given by (4.2.15). This expression incorporates both extremal and non-extremal black holes. Thus all charged black holes will diminish near the Big Rip. Apparently this condition predicts the existence of large charges onto black holes, although astrophysically no such evidence has been successfully deduced from the observations. In theory, the existence of large charges onto black holes is consistently deduced by the general theory of relativity. It needs to be stressed that there is no analogous condition for the Schwarzschild black hole ($Q = 0$). This analysis can be extended for a rotating charged black hole (so-called Kerr-Neumann black hole) to get a deeper insight of the accretion process. This work also serves as the generalization of Michel [122] in terms of the accretion of phantom dark energy onto a charged black hole.

Note that in this study, I ignored the back-reaction effects. Recently Gao et al [130] contested the validity of this result, claiming that the inclusion of the back reaction would result in the opposite process, namely, the mass of a black hole increases in the process of phantom accretion. The particular solution was presented in [130] to support this point. However, the conclusion of [130] is doubtful. It is only valid for imperfect fluids, thus making impossible the application of their arguments to the perfect fluids, for which the effect of black hole mass decreasing was found.

Finally, I stress that the Reissner-Nordström black hole in a phantom energy dominated Universe would, indeed, provide a mechanism to produce a naked singularity.

Chapter 5

Evolution of a Schwarzschild black hole in a Chaplygin gas dominated Universe

I here discuss the accretion of phantom like modified variable Chaplygin gas and the viscous Chaplygin gas severalty onto a black hole. This accretion of the phantom fluid reduces the mass of the black hole. This works serves as the generalization of the earlier work by Babichev et al [1] who initiated the concept of accretion of exotic matter on the black hole. I have built my model on the same pattern by choosing more general EoS for the dark energy.

The outline of the chapter is as follows: In the next section, I discuss the relativistic model of accretion onto a black hole. In third section, I investigate the evolution of the mass of black hole by the accretion of modified variable Chaplygin Gas (MVG) while in the fourth section, I discuss the similar scenario with the viscous generalized Chaplygin gas (VCG).

5.1 Accretion onto a black hole

I consider a Schwarzschild black hole of mass M which is gravitationally isolated and is specified by the line element (2.2.11). The black hole is accreting a Chaplygin gas, which is assumed to be a perfect fluid specified by the stress energy tensor (1.2.9). Represent the pressure and energy density of the Chaplygin gas by p and ρ respectively. Due to static and spherically symmetric nature of the black hole, the velocity four vector is $u^\mu = (u^t(r), u^r(r), 0, 0)$ which satisfies the normalization condition $u^\mu u_\mu = 1$. Thus I am considering only radial in-fall of the Chaplygin gas on the event horizon. Using the energy-momentum conservation for $T^{\mu\nu}_{;\nu} = 0$,

$$ux^2(\rho + p)\sqrt{1 - \frac{2}{x} + u^2} = C_1, \tag{5.1.1}$$

where $x = r/M$ and $u = u^r = dr/ds$, is the radial component of the velocity four vector u^μ and C_1 is a positive constant of integration[1]. The second constant of motion is obtained from $u_\mu T^{\mu\nu}_{;\nu} = 0$, which describes flow of energy in the radial direction and it gives

$$ux^2 \exp\left[\int\limits_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p'(\rho')}\right] = -A, \tag{5.1.2}$$

where A is a positive constant of integration[2] which is determined below for two models of Chaplygin gas. The quantities ρ_∞ and ρ_h are the densities of Chaplygin gas at infinity and at the black hole horizon respectively. Further using (5.1.1) and (5.1.2),

$$(\rho + p)\sqrt{1 - \frac{2}{x} + u^2} \exp\left[-\int\limits_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p'(\rho')}\right] = C_2, \tag{5.1.3}$$

where $C_2 = -C_1/A = \rho_\infty + p(\rho_\infty)$. In order to calculate $\dot{M}$, the rate of change of mass of black hole I integrate the Chaplygin gas flux over the entire horizon as, $\dot{M} = \oint T^r_t dS$ where T^r_t denotes the momentum density of Chaplygin gas in the radial direction and $dS = \sqrt{-g}d\theta d\varphi$ is the surface element of black hole horizon. Using (5.1.1 - 5.1.3), this rate of change is

$$\frac{dM}{dt} = 4\pi AM^2(\rho + p). \tag{5.1.4}$$

Integration of (5.1.4) yields

$$M = M_i\left(1 - \frac{t}{\tau}\right)^{-1}, \tag{5.1.5}$$

which determines the evolution of mass of black hole of initial mass M_i and τ is the characteristic accretion time scale given by

$$\tau^{-1} = 4\pi AM_i(\rho + p). \tag{5.1.6}$$

[1]The reasoning is adapted from (4.1.1).
[2]The reasoning is adapted from (4.1.2).

The number density and energy density of Chaplygin gas are related as

$$\frac{n(\rho_h)}{n(\rho_\infty)} = \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p'(\rho')}\right], \tag{5.1.7}$$

here $n(\rho_h)$ and $n(\rho_\infty)$ are the number densities of the Chaplygin gas at the horizon and at infinity respectively. Further the constant A appearing in (5.1.2) is determined as

$$\frac{n(\rho_h)}{n(\rho_\infty)} ux^2 = -A, \tag{5.1.8}$$

which is an alternative form of energy momentum conservation. Moreover, the critical points of accretion (i.e. the points where the speed of flow achieves the speed of sound $V^2 = c_s^2 = \partial p/\partial\rho$) are determined as follows

$$u_*^2 = \frac{1}{2x_*}, \quad V_*^2 = \frac{u_*^2}{1 - 3u_*^2}, \tag{5.1.9}$$

where $V^2 \equiv \frac{n}{\rho+p}\frac{d(\rho+p)}{dn} - 1$. Finally, the above (5.1.7 - 5.1.9) are combined in a single expression as

$$\frac{\rho_* + p_*(\rho_*)}{n(\rho_*)} = [1 + 3c_s^2(\rho_*)]^{1/2}\frac{\rho_\infty + p(\rho_\infty)}{n(\rho_\infty)}. \tag{5.1.10}$$

5.2 Accretion of modified variable Chaplygin gas

The Chaplygin gas had been proposed to explain the accelerated expansion of the Universe [34]. The observational evidence in favor of cosmological models based on Chaplygin gas is quite encouraging [131, 132, 133, 134]. The Chaplygin gas model favors a spatially flat Universe which agrees with the observational data of Sloan Digital Sky Survey (SDSS) and Supernova Legacy Survey (SNLS) with 95.4 % confidence level [135]. Consequently, various generalizations of Chaplygin gas have been proposed in the literature to incorporate any other dark component in the Universe (see e.g. [136, 137, 138, 139] and references therein).

Consider an equation of state which combines various EoS of Chaplygin gas [41]

$$p = A'\rho - \frac{B(a)}{\rho^\alpha}, \quad B(a) = B_o a^{-\alpha_1}. \tag{5.2.1}$$

Here A', B_o and α_1 are constant parameters with $0 \leq \alpha \leq 1$. For $A' = 0$, (5.2.1) gives generalized Chaplygin gas. Further if $B = B_o$ and $\alpha = 1$, it yields the usual

Chaplygin gas. Also (5.2.1) reduces to modified Chaplygin gas if only $B = B_o$. Moreover, if only $A' = 0$, the same equation represents variable Chaplygin gas.

Considering the background spacetime to be spatially flat ($k = 0$), homogeneous and isotropic represented by Friedmann-Robertson-Walker (FRW) metric. The spacetime is assumed to contain only one component fluid i.e. the phantom energy represented by the Chaplygin gas EoS. The corresponding field equations are

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho, \tag{5.2.2}$$

and

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{\kappa^2}{2}(\rho + 3p), \tag{5.2.3}$$

The energy conservation equation gives the evolution of energy density by

$$\rho = \left[\frac{3B_o(1+\alpha)}{\{3(1+\alpha)(1+A') - \alpha_1\}}\frac{1}{a^{\alpha_1}} + \frac{\Psi}{a^{3(1+\alpha)(1+A')}}\right]^{\frac{1}{1+\alpha}}. \tag{5.2.4}$$

Here Ψ is a constant of integration. Note that to obtain the increasing energy density of phantom energy with respect to scale factor $a(t)$, Requiring the coefficients of $a(t)$ in (5.2.4) to be positive i.e. $\Psi \geq 0$, $B_o(1+\alpha) > 0$ and $3(1+\alpha)(1+A') - \alpha_1 > 0$. Moreover, the exponents of $a(t)$ must be negative i.e. $\alpha_1 < 0$ and $3(1+\alpha)(1+A') < 0$ to obtain increasing ρ. These constraints together imply that $\alpha_1 > 3(1+\alpha)(1+A')$. Another way of getting positive ρ is by setting $\alpha_1 > 0$, $1 + A' > 0$ and $\alpha_1 < 3(1+\alpha)(1+A')$. Further, using (5.1.7) the ratio of the number density of Chaplygin gas near horizon and at infinity is calculated to be

$$\frac{n(\rho_h)}{n(\rho_\infty)} = \left[\frac{\rho_h^{1+\alpha}(1+A') - B(a)}{\rho_\infty^{1+\alpha}(1+A') - B(a)}\right]^{\frac{1}{(1+\alpha)(1+A')}} \equiv \Delta_1. \tag{5.2.5}$$

Notice that the function $B(a)$ can be expressed in terms of ρ implicitly and is determined from (5.2.4). Making use of (5.1.9), the critical points of accretion are given by

$$u_*^2 = \frac{\Delta_2}{1+3\Delta_2}, \quad x_* = \frac{1+3\Delta_2}{2\Delta_2}, \tag{5.2.6}$$

where

$$V_*^2 = A' + \frac{\alpha B(a)}{\rho_*^{\alpha+1}} \equiv \Delta_2 \tag{5.2.7}$$

Thus for the critical points to be finite and positive, either $\Delta_2 > 0$ or $\Delta_2 < 0$ and $\Delta_2 < -1/3$. For the accretion to be critical, the quantity V^2 must become supersonic from the initial subsonic somewhere near the black hole horizon. For the MVG, $\omega = A' - B/\rho^{1+\alpha} < 0$, since $A' < -1$. One can observe that fluids having EoS $\omega < 0$ are hydrodynamically unstable i.e. the speed of sound in that medium can not be defined since $c_s^2 < 0$. In order to overcome this problem Babichev *et al* [129] proposed a redefinition of ω with the help of a generalized linear EoS given by $p = \beta(\rho - \rho_o)$, where β and ρ_o are constant parameters. Here $\beta > 0$ refers to a hydrodynamically stable while $\beta < 0$ corresponds to hydrodynamically unstable fluid. I will not be interested in the later case here. Note that now two parameters ω and β are related by $\omega = \beta(\rho - \rho_o)/\rho$. Further $\omega < 0$ now corresponds to $\beta > 0$ and $\rho_o > \rho$ thereby making the previously unstable fluid, now stable. Also$c_s^2 \equiv \partial p/\partial\rho = \beta$. Since for stability, I require $\beta > 0$ and $0 < c_s^2 < 1$, it leads to $0 < \frac{1}{\rho-\rho_o}(A'\rho - B/\rho^\alpha) < 1$ and $0 < \beta < 1$. Hence the EoS parameter is now well-defined with $A' < -1$ and $\rho_o > \rho$. Thus the stability of the phantom like MVG is guaranteed with the use of generalized linear EoS.

The constant A is determined from (5.1.8) to give

$$-A = \frac{\Delta_1}{4}\left(\frac{1+3\Delta_2}{\Delta_2}\right)^{3/2}. \tag{5.2.8}$$

Using (5.1.6) the characteristic evolution time scale becomes

$$\tau^{-1} = \pi M_i(\rho+p)\frac{\Delta_1}{4}\left(\frac{1+3\Delta_2}{\Delta_2}\right)^{3/2}. \tag{5.2.9}$$

Using (5.2.9) in (5.1.5), the black hole mass is given by

$$M(t) = M_i\left[1 - \pi M_i t(\rho+p)\frac{\Delta_1}{4}\left(\frac{1+3\Delta_2}{\Delta_2}\right)^{3/2}\right]^{-1}, \tag{5.2.10}$$

which determines the evolution of mass of black hole accreting phantom MVG. It can be seen that if the phantom MVG violates the dominant energy condition $\rho+p > 0$ than mass M of the black hole will decrease. Contrary if this condition is satisfied than M will increase. Thus in the classical relativistic regime, this result is in conformity with the result of Babichev et al [1, 129]. I also stress here that although metric (2.2.11) is static, I get a dynamical mass $M(t)$ in (5.2.10). Astrophysically the mass of a black hole is a dynamical quantity: the mass will increase if the black hole accretes classical matter (which satisfies $\rho + p > 0$)

however it will decrease for the exotic phantom energy accretion. The mass can also decrease if the Hawking evaporation process is invoked. Hence the static black holes may not necessarily correspond to the astrophysical black holes. Note that $\omega > 0$ ($\omega < 0$) corresponds to non-phantom (phantom) MVG fluid; although the accretion through the critical point is possible in both the cases, only phantom MVG violating the dominant energy condition will reduce the mass of black hole.

5.3 Accretion of viscous generalized Chaplygin gas

In viscous cosmology, the presence of viscosity corresponds to space isotropy and hence is important in the background of FRW spacetime [140, 141, 142]. The presence of viscous fluid can explain the observed high entropy per baryon ratio in the Universe [143]. It can cause exponential expansion of the Universe and can rule out the initial singularity which mares the standard Big Bang picture. The matter power spectrum in bulk viscous cosmology is also well behaved as there are no instabilities or oscillations on small perturbation scale [144]. Any cosmic fluid having non-zero bulk viscosities has the EoS $p_{\text{eff}} = p + \Pi$, where p is the usual isotropic pressure and Π is the bulk viscous stress given by $\Pi \equiv \xi(\rho)u^{\mu}_{;\mu}$ [145]. The scaling of viscosity coefficient is $\xi = \xi_o \rho^n$ where n is a constant parameter and $\xi(t_o) = \xi_o$. Note that for $0 \leq n \leq 1/2$, I have de Sitter solution and for $n > 1/2$, deflationary solutions. The viscosity coefficient is generally taken to be positive for positive entropy production in conformity with the second law of thermodynamics [146]. Moreover, the entropy corresponding to viscous cosmology is always positive and increasing which is consistent with the thermodynamic arrow of time. Infact the cosmological model with viscosity is consistent with the observational SN Ia data at lower redshifts while it mimics the ΛCDM model in the later cosmic evolution [147]. It is proved in [148, 149] that FRW spacetime filled with perfect fluid and the bulk viscous stresses will violate the dominant energy condition.

Thus the effective pressure is given by

$$p_{\text{eff}} \equiv p + \Pi, \tag{5.3.1}$$

where $\Pi = -3H\xi$ and $p = \chi/\rho^\alpha$ with χ is a constant. Thus in the VCG case, the standard FRW equation becomes [150]

$$\frac{\ddot{a}}{a} = -\frac{1}{3}(\rho + 3p_{\text{eff}}). \tag{5.3.2}$$

Further the energy conservation principle gives

$$\dot{\rho} + 3H(\rho + p_{\text{eff}}) = 0, \tag{5.3.3}$$

which shows that the viscosity term serves as the source term. Using (5.2.2) and (5.3.1) in (5.3.3), gives

$$\frac{a}{3}\frac{d\rho}{da} + \rho + \frac{\chi}{\rho^{\alpha}} - 3\kappa\xi(\rho)\sqrt{\rho} = 0. \tag{5.3.4}$$

Thus solving (5.3.4) yields

$$a(t) = a_o \exp\left[-\int\limits_{\rho_o}^{\rho} \frac{\rho'^{\alpha} d\rho'}{\rho'^{\alpha+1} - 3\kappa\xi(\rho')\rho'^{\alpha+\frac{1}{2}} + \chi}\right]^{\frac{1}{3}}. \tag{5.3.5}$$

For my further analysis I assume ξ to be constant.

The ratio of the number density of VCG near black hole horizon and at infinity is given by

$$\frac{n(\rho_h)}{n(\rho_\infty)} = \exp\left[\int\limits_{\rho_\infty}^{\rho_h} \frac{\rho'^{\alpha} d\rho'}{\rho'^{\alpha+1} - 3\kappa\xi\rho'^{\alpha+\frac{1}{2}} + \chi}\right] \equiv \Delta_3. \tag{5.3.6}$$

The corresponding critical points of accretion are

$$u_*^2 = \frac{\Delta_4}{3\Delta_4 - 1}, \quad x_* = \frac{3\Delta_4 - 1}{2\Delta_4}, \tag{5.3.7}$$

where

$$V_*^2 = -\left(\frac{\alpha\chi}{\rho_*^{\alpha+1}} + \frac{3}{2\sqrt{\rho_*}}\kappa\xi\right) \equiv \Delta_4. \tag{5.3.8}$$

Notice that for the critical points to be finite and positive valued either $\Delta_4 < 0$ or $\Delta_4 > 1/3$. Using (5.1.9) the speed of flow at the critical point is $V^2 = -\Delta_4$. Further, the EoS parameter is $\omega = \chi/\rho^{1+\alpha} - 3\xi H/\rho$ $(= \chi/\rho^{1+\alpha} - \sqrt{3}\kappa\xi/\sqrt{\rho})$. Note that if $\chi < 0$ then $\omega < 0$ and stability of VCG is lost. However, if I here invoke the argument presented in the last section, I can consider accretion with $\omega < 0$. Using the generalized linear EoS $p = \beta(\rho - \rho_o)$ for the phantom energy, I obtain $\beta > 0$ and $\rho_o > \rho$ for $\omega < 0$. Using the definition $c_s^2 \equiv \partial p/\partial\rho = \beta$ and stability requirements $\beta > 0$ and $0 < c_s^2 < 1$ lead to $0 < \frac{1}{\rho-\rho_o}(\chi/\rho^\alpha - \sqrt{3\rho}\kappa\xi) < 1$ and $0 < \beta < 1$. The EoS parameter β is now well-defined with $\chi < 0$ and $\rho_o > \rho$. Therefore the stability of the phantom like VCG is assured with the use of generalized linear EoS.

Using (5.1.8) the constant A is now determined to be

$$-A = \Delta_3 \left(\frac{3\Delta_4 - 1}{2\Delta_4} \right)^{3/2}. \tag{5.3.9}$$

The characteristic evolution time scale is

$$\tau^{-1} = 4\pi M_i(\rho + p)\Delta_3 \left(\frac{3\Delta_4 - 1}{2\Delta_4} \right)^{3/2}. \tag{5.3.10}$$

Using (5.3.9) and (5.3.10) in (5.1.5), the black hole mass evolution as

$$M(t) = M_i \left[1 - 4\pi M_i t(\rho + p)\Delta_3 \left(\frac{3\Delta_4 - 1}{2\Delta_4} \right)^{3/2} \right]^{-1}. \tag{5.3.11}$$

It can be seen that black hole mass will decrease when $\rho + p < 0$ and increase in the opposite case. It is emphasized that this result is valid till the contribution of viscous stress is negligible compared to isotropic stress. For the sake of clarity, it is emphasize that fluid violating the standard energy conditions is termed 'exotic' and hydrodynamically unstable i.e. its existence is not fully guaranteed. But this conclusion is drawn due to the 'bad' choice of the EoS ($p = \omega\rho$) in the analysis. The result is reversed and remedied when the generalized linear EoS in our model is introduced which makes the accretion of exotic fluid much more practical.

5.4 Conclusion

I have investigated the accretion of two different forms of phantom-like Chaplygin gas onto a Schwarzschild black hole. The time scale of accretion and the evolution of mass of black hole are derived in the context of two widely studied Chaplygin gas models namely the modified variable Chaplygin gas and the viscous generalized Chaplygin gas. Although the phantom energy seems to be an unstable fluid as it corresponds to a medium with indeterminate speed of sound and super-luminal speeds. These pathologies arise due to bad choices of the equations of state for the phantom energy and hence can be removed by choosing some suitable transformation from one EoS to another or a totally new EoS for this purpose. This work serves as the generalization of the earlier work by Babichev *et al* [1]. It should be noted that I have ignored matter component in the accretion model. Thus it will be more insightful to incorporate the contributions of matter along with the Chaplygin gas during accretion onto black hole. Moreover my analysis can be extended to the case of rotating black holes as well.

Chapter 6

Black holes in bulk-viscous cosmology

The phenomenon of cosmic-acceleration is among the most compelling problems in cosmology. Our deepest intuition about gravity - that all objects should be attracted to each other - just simply does not holds at cosmological distance scales. Rather than slowing, as Newtonian gravity predicts, the relative velocities of distant galaxies are increasing. The implication is that gravity behaves far differently than it was previously thought or that some mysterious fluid (dark energy) with exotic gravitational properties fills the Universe. Either way, there is new physics beyond the four fundamental forces described by the Standard Particle Physics Model and general relativity [151].

Dark energy with bulk viscosity has a peculiar property to cause accelerated expansion of phantom type in the late evolution of the Universe [140]. It can also alleviate several cosmological puzzles like cosmic age problem [152], coincidence problem [153] and phantom crossing [154]. I will consider phantom energy as an imperfect fluid, implying that the PE could contain non-zero bulk and shear viscosities [142]. The bulk viscosities are negligible for non-relativistic and ultra-relativistic fluids but are important for the intermediate cases. In viscous cosmology, shear viscosities arise in relation to space anisotropy while the bulk viscosity accounts for the space isotropy [140, 141]. Generally, shear viscosities are ignored (as the CMB does not indicate significant anisotropies) and only bulk viscosities are taken into account for the fluids in the cosmological context. Moreover, bulk viscosity related to a grand unified theory phase transition may lead to an explanation of the accelerated cosmic expansion [155].

Babichev et al [1] studied the effects of the accretion of phantom energy onto a Schwarzschild black hole taking PE to be a perfect fluid. As a first approximation, the bulk viscosity can be ignored, but to get a better picture I need to incorporate it into the phantom fluid. I have adapted the procedure of [1, 122] for my calculations.

The plan is as follows: in the next section I review viscous cosmology; in section three I discuss the relativistic model of accretion onto a black hole; in the subsequent section I use results from viscous cosmology for the accretion model; next I give two examples to illustrate the accretion process with a constant and power law viscosity. Later I study black hole evolution in the presence of matter and viscous phantom energy. Finally I conclude with a brief discussion of my results.

6.1 Bulk-viscous cosmology

I assume the background spacetime to be spatially flat, homogeneous, isotropic and spatially flat ($k = 0$) and described by the Friedmann-Robertson-Walker. Introducing p as the effective pressure containing the isotropic pressure p_{pe} and the bulk viscous pressure p_{vis}, given by

$$p = p_{\text{pe}} + p_{\text{vis}}. \tag{6.1.1}$$

Here $\rho = \rho_{\text{pe}} + \rho_{\text{vis}}$ and $p_{\text{vis}} = -\xi u^{\mu}_{;\mu}$, where u^{μ} is the velocity four vector and $\xi = \xi(\rho_{\text{vis}}, t)$ is the bulk viscosity of the fluid [145]. (6.1.1) shows that negative pressure due to viscosity contributes in the effective pressure which cause accelerated expansion. In the FRW model, the expression $u^{\mu}_{;\mu} = 3\dot{a}/a$ holds. Also, ξ is generally taken to be positive in order to avoid the violation of second law of thermodynamics [146].

Assume that the viscous fluid equation of state is

$$p = \omega\rho = (\gamma - 1)\rho. \tag{6.1.2}$$

Note that if $\gamma = 0$ (or $\omega = -1$), (6) represents the EoS for cosmological constant. Furthermore if $\gamma < 0$, it represents phantom energy. In general, for normal matter $1 \leq \gamma < 2$.

The equation governing the evolution of $H(t)$ for a given ξ is

$$2\dot{H} + 3\gamma H^2 - 3\xi H = 0. \tag{6.1.3}$$

On integration, (6.1.3) gives

$$H(t) = \frac{\exp\{\frac{3}{2}\int \xi(t)dt\}}{C + \frac{3}{2}\gamma \exp\{\frac{3}{2}\int \xi(t)dt\}}, \tag{6.1.4}$$

where C is a constant of integration. Note that (6.1.4) can further be solved to get the evolution of $a(t)$ as

$$a(t) = D\Big(C + \frac{3}{2}\gamma \int \exp\Big\{\frac{3}{2}\int \xi(t)dt\Big\}dt\Big)^{\frac{2}{3\gamma}}, \tag{6.1.5}$$

where D is a constant of integration. Thus for a given value of ξ one can obtain expressions of $a(t)$ and hence calculate $\rho(t)$ and $p(t)$.

6.1.1 Accretion onto a black hole

In the background of FRW spacetime, I consider, as an approximation, a gravitationally isolated Schwarzschild black hole (BH) of mass M whose metric is specified by the Schwarzschild line element (2.2.11) with chosen units ($c = 8\pi G = 1$). The background spacetime is assumed to contain one test fluid, namely the phantom energy with non-vanishing bulk viscous stress p_{vis}. The fluid is assumed to fall onto the BH horizon in the radial direction only which is in conformity with the spherical symmetry of the BH. Thus, the velocity four vector of the phantom fluid is $u^\mu = (u^t(r), u^r(r), 0, 0)$ which satisfies the normalization condition $u^\mu u_\mu = -1$. This phantom fluid is specified by the stress energy tensor (1.2.9) [142, 146]. Using the energy momentum conservation for $T^{\mu\nu}$,

$$ur^2M^{-2}(\rho + p)\sqrt{1 - \frac{M}{4\pi r} + u^2} = C_1, \tag{6.1.6}$$

where $u^r = u = dr/ds$ is the radial component of the velocity four vector and $C_1 > 0$ is a constant of integration. The second constant of motion is obtained by contracting the velocity four vector of the phantom fluid with the stress energy tensor $u_\mu T^{\mu\nu}_{;\nu} = 0$, which gives

$$ur^2M^{-2}\exp\Big[\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p(\rho')}\Big] = -A, \tag{6.1.7}$$

where $A > 0$ is a constant of integration. Also ρ_h and ρ_∞ are the energy densities of the phantom fluid at the horizon of the BH, and at infinity respectively. From

(6.1.6) and (6.1.7) I have

$$(\rho+p)\sqrt{1-\frac{M}{4\pi r}+u^2}\exp\Big[-\int_{\rho_\infty}^{\rho_h}\frac{d\rho'}{\rho'+p(\rho')}\Big]=C_2, \qquad (6.1.8)$$

with $C_2=-C_1/A=\rho_\infty+p(\rho_\infty)$. In order to calculate the rate of change of mass of black hole $\dot{M}$, I integrate the flux of the bulk viscous phantom fluid over the entire BH horizon to get

$$\dot{M}=\oint T_t^r dS. \qquad (6.1.9)$$

Here T_t^r determines the energy momentum flux in the radial direction only and $dS=\sqrt{-g}d\theta d\varphi$ is the infinitesimal surface element of the BH horizon. Using (6.1.6) - (6.1.9), gives

$$\frac{dM}{dt}=\frac{AM^2}{16\pi}(\rho+p), \qquad (6.1.10)$$

which clearly demonstrates the vanishing mass of the black hole if $\rho+p<0$. Integration of (6.1.10) leads to

$$M=M_0\Big(1-\frac{t}{\tau}\Big)^{-1}, \qquad (6.1.11)$$

where M_0 is the initial mass of the black hole and modified characteristic accretion time scale $\tau^{-1}=[\frac{AM_0}{16\pi}\{(\rho_{\rm pe}+p_{\rm pe})-\frac{3\xi}{t}\ln(\frac{a}{a_0})\}]$, a_0 being the value of the scale factor at time t_0. Note that during the integration of (6.1.10), I assumed $\rho_{\rm pe}$ and $p_{\rm pe}$ to be constants. In the coming subsections, I shall take these as time dependent entities.

6.1.2 Accretion of viscous phantom fluid

I now study the BH mass evolution in two special cases: (a) constant viscosity; and (b) power law viscosity.

6.1.3 Constant bulk viscosity

For constant viscosity $\xi=\xi_o$, the evolution of $a(t)$ is determined by using (9). It gives

$$a(t)=a_0\Big[1+\frac{\gamma H_o B(t)}{\xi_o}\Big]^{\frac{2}{3\gamma}}, \qquad (6.1.12)$$

where

$$B(t)\equiv\exp\Big(\frac{3t\xi_o}{2}\Big)-1. \qquad (6.1.13)$$

Using (6.1.12) and the Friedmann equation, the density evolution is given by

$$\rho(t) = \frac{\rho_o \exp{(3\xi_o t)}}{\left[1 + \frac{\gamma H_o B(t)}{\xi_o}\right]^2}. \tag{6.1.14}$$

Here $\rho_o = 3H_o^2$. Further, for $\gamma < 0$ the BR singularity occurs in a finite time at

$$\tau = \frac{2}{3\xi_o} \ln\Big(1 - \frac{\xi_o}{H_o\gamma}\Big). \tag{6.1.15}$$

Finally, the BH mass evolution is determined by solving (6.1.10) and (6.1.14) to get

$$M = M_0\Big[1 - \frac{AM_0}{8\pi\gamma}\Big(\frac{\xi_o}{\Delta} - 1\Big)(\xi_o - \gamma H_o)\Big]^{-1}, \tag{6.1.16}$$

where

$$\Delta \equiv \xi_o + (-1 + e^{\frac{3t\xi_o}{2}})\gamma H_o. \tag{6.1.17}$$

This mass is displayed for different values of viscosity at different times in Table 1.

$t \downarrow \xi \rightarrow$	$\xi_1 = 10^{-17}$	$\xi_2 = 10^{-18}$	$\xi_3 = 10^{-19}$	$\xi_4 = 10^{-20}$
$t_1 = 10^7$	3.43427×10^{-4}	2.44662×10^{-3}	6.31285×10^{-3}	7.49184×10^{-3}
$t_2 = 10^{10}$	3.43544×10^{-7}	2.45261×10^{-6}	6.35248×10^{-6}	7.55357×10^{-6}
$t_3 = 10^{13}$	3.43516×10^{-10}	2.45258×10^{-9}	6.35247×10^{-9}	7.55358×10^{-9}
$t_4 = 10^{17}$	1.23994×10^{-14}	2.10182×10^{-13}	5.86096×10^{-13}	7.01997×10^{-13}

Table 1. The mass ratio M/M_0 of black hole for different choices of constant viscosity ξ_o. The initial mass is, throughout, taken to be $50M_\odot$ or 10^{32}kg. The higher the viscosity of the phantom fluid, the sharper the decrease in the BH mass. BHs of all masses, ranging from the solar mass to the intermediate mass to the super-massive, will all meet the same fate

It is apparent from Table.1 that for a fixed viscosity, the mass ratio decreases with time implying that mass of black hole is decreasing for an initial mass. Similarly, at any given time, the mass ratio also decreases with the increase in viscosity. Thus the greater the value of viscosity parameter, the greater would be its effects on the BH mass.

Power law viscosity

If the viscosity has power law dependence upon density i.e. $\xi = \alpha\rho_{\text{vis}}^{s}$, where α and s are constant parameters, it has been shown [148, 156] that it yields cosmologies with a BR if $\sqrt{3\alpha} > \gamma$ and $s = 1/2$. Thus I take $\xi = \alpha\rho^{\frac{1}{2}}$ as a special case. Then the scale factor evolves as

$$a(t) = a_0\Big(1 - \frac{t}{\tau}\Big)^{\frac{2}{3(\gamma-\sqrt{3}\alpha)}}. \tag{6.1.18}$$

The density of phantom fluid evolves as

$$\rho(t) = \frac{4}{3\tau^2(\gamma - \sqrt{3}\alpha)^2}\Big(1 - \frac{t}{\tau}\Big)^{-2}, \tag{6.1.19}$$

or in terms of critical density ρ_{cr} as

$$\rho(t) = \rho_{\text{cr}}\Big(1 - \frac{t}{\tau}\Big)^{-2}. \tag{6.1.20}$$

The corresponding BR time τ is given by

$$\tau = \frac{2}{3(\sqrt{3}\alpha - \gamma)}H_o^{-1}. \tag{6.1.21}$$

Finally, the mass evolution of BH is determined by using (6.1.10) and (6.1.19) is

$$M = M_0\Big[1 + \frac{AM_0}{4\pi(\sqrt{3}\alpha - \gamma)}\frac{t}{\tau(\tau - t)}\Big]^{-1}. \tag{6.1.22}$$

Note that when $\alpha = 0$, this case reduces to that of Babichev et al [1]. The mass in (6.1.22) in displayed for different values of EoS parameter γ at different times in Table 2 and displayed graphically in Figure 6.1. As shown, the mass decreases gradually with the decrease in the EoS parameter γ. Note that I have not graphically displayed the mass for different viscosities given in Table 1 because the variation is not significantly different for most time scales.

$t \downarrow \gamma \rightarrow$	$\gamma_1 = -1 \times 10^{-1}$	$\gamma_2 = -2 \times 10^{-1}$	$\gamma_3 = -3 \times 10^{-1}$	$\gamma_4 = -4 \times 10^{-1}$
$t_1 = 10^{10}$	4.71915×10^{-5}	2.35963×10^{-5}	1.5731×10^{-5}	1.17983×10^{-5}
$t_2 = 10^{13}$	4.79136×10^{-8}	2.35968×10^{-8}	1.57312×10^{-8}	1.17984×10^{-8}
$t_3 = 10^{17}$	4.66492×10^{-12}	2.30523×10^{-12}	1.51867×10^{-12}	1.12539×10^{-12}
$t_4 = 10^{20}$	4.97349×10^{-14}	5.20946×10^{-14}	5.28811×10^{-14}	5.32744×10^{-14}

Table 2. The mass ratio M/M_0 of black hole for different choices of equation of state. The initial mass is $50M_\odot$ or 10^{32}kg. From this table, I can

draw the conclusion that PE containing viscous stresses can play a significant role in the BH mass evolution if the viscosity is sufficiently high for an appropriate EoS.

6.1.4 Examples

I now solve examples to demonstrate the accretion of viscous phantom energy onto a BH. The formalism is adapted from [1].

Viscous linear EoS

I choose the viscous linear EoS, $p = \omega\rho_{\rm pe} - 3H\xi_o$ with $\omega < -1$. The ratio of the number densities of phantom fluid particles at the horizon and at infinity is given by

$$\frac{n(\rho_h^{\rm pe})}{n(\rho_\infty^{\rm pe})} = \Big[\frac{\rho_h^{\rm pe}(1+\omega) - 3\xi_o H}{\rho_\infty^{\rm pe}(1+\omega) - 3\xi_o H}\Big]^{\frac{1}{(1+\omega)}}. \tag{6.1.23}$$

The critical points of accretion (the point where the speed of fluid flow becomes equal to the speed of sound i.e. $u_*^2 = c_s^2$) are given by

$$u_*^2 = \frac{\omega}{1+3\omega}; \quad x_* = \frac{1+3\omega}{2\omega}. \tag{6.1.24}$$

The constant A appearing in (6.1.10) is determined to be

$$A = \frac{|1+3\omega|}{4|\omega|^{3/2}}^{\frac{1+\omega}{2\omega}}. \tag{6.1.25}$$

Notice that the constant A is the same as for the non-viscous case [1]. Also, the density of phantom energy at the horizon is given by

$$\rho_h^{\rm pe} = \frac{3\xi_o H}{1+\omega} + \Big(\frac{4}{A}\Big)^{\frac{\omega-1}{\omega+1}}\Big(\rho_\infty - \frac{3\xi_o H}{1+\omega}\Big). \tag{6.1.26}$$

Moreover, the speed of flow at the horizon is

$$u_h = -\Big(\frac{A}{4}\Big)^{\frac{\omega}{(\omega+1)}}. \tag{6.1.27}$$

The speed is negative as it is directed towards the BH. Also, the characteristic evolution time scale of the BH is given by

$$\tau^{-1} = 4\pi M_0 \frac{(1+3\omega)}{4\omega^{3/2}}^{\frac{1+\omega}{2\omega}} \Big\{\rho_\infty^{\rm pe}(1+\omega) - \frac{3\xi_o}{t}\ln\Big(\frac{a}{a_0}\Big)\Big\}. \tag{6.1.28}$$

Finally, substituting (6.1.28) in (6.1.11) gives the mass evolution of a BH in bulk viscous cosmology

$$M = M_0\Big[1 - 4\pi M_0 t \frac{(1+3\omega)^{\frac{1+\omega}{2\omega}}}{4\omega^{3/2}}\Big\{\rho_\infty^{\rm pe}(1+\omega) - \frac{3\xi_o}{t}\ln\Big(\frac{a}{a_0}\Big)\Big\}\Big]^{-1}. \qquad (6.1.29)$$

Since $\rho_\infty^{\rm pe}$ is unknown for my purpose, I have not evaluated M for different times numerically for tabular and graphical presentation.

Viscous non-linear EoS

I here choose the EoS, $p = \omega\rho_{\rm pe} - 3H\xi(\rho_{\rm vis})$ with $\omega < -1$, where $\xi(\rho_{\rm pe}) = \alpha\rho_{\rm pe}^s$ with α and s are constants. The ratio of number densities is given by

$$\frac{n(\rho_h^{\rm pe})}{n(\rho_\infty^{\rm pe})} = \Big(\frac{\rho_h}{\rho_\infty}\Big)^{\frac{s}{(s-1)(1+\omega)}}\Big(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\Big) \qquad (6.1.30)$$

The constant A is determined to be

$$A = \Big|\Big(\frac{\rho_h}{\rho_\infty}\Big)^{\frac{2s}{(s-1)(1+\omega)}}\Big(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\Big)^3\Big|. \qquad (6.1.31)$$

The speed of flow at the horizon becomes

$$u_h = -\Big(\frac{\rho_h}{\rho_\infty}\Big)^{\frac{2s}{(s-1)(1+\omega)}}\Big(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\Big)^2. \qquad (6.1.32)$$

The critical points of accretion are given by

$$u_*^2 = \frac{\omega - 3s\alpha\rho_h^{s-1}}{1 + 3(\omega - 3s\alpha\rho_h^{s-1})}; \quad x_* = \frac{1 + 3(\omega - 3s\alpha\rho_h^{s-1})}{2(\omega - 3s\alpha\rho_h^{s-1})}. \qquad (6.1.33)$$

The characteristic evolution time scale τ is given by

$$\tau = \Big[4\pi M_0\Big(\frac{\rho_h}{\rho_\infty}\Big)^{\frac{2s}{(s-1)(1+\omega)}}\Big(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\Big)^3\Big\{\rho_\infty(1+\omega) - 3\frac{\alpha\rho_\infty^s}{t}\ln\Big(\frac{a}{a_0}\Big)\Big\}\Big]^{-1}. \qquad (6.1.34)$$

Finally, using (6.1.34) in (6.1.10), the BH mass evolution is given by

$$\begin{aligned} M &= M_0\Big[1 - 4\pi M_0 t\Big(\frac{\rho_h}{\rho_\infty}\Big)^{\frac{2s}{(s-1)(1+\omega)}}\Big(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\Big)^3 \\ &\quad \times\Big\{\rho_\infty(1+\omega) - 3\frac{\alpha\rho_\infty^s}{t}\ln\Big(\frac{a}{a_0}\Big)\Big\}\Big]^{-1}. \end{aligned} \qquad (6.1.35)$$

As before, ρ_∞ is unknown, but further ρ_h is also unknown. As such, I again do not provide a tabular or graphical presentation.

6.1.5 Black holes accreting both matter and viscous phantom fluid

I now consider a two component fluid, the viscous dark energy and matter. The matter part may be composed of both baryonic and non-baryonic matter. It is taken to be a perfect fluid while the PE is taken as a bulk viscous fluid. The corresponding Einstein field equations (EFE) for the two component fluid become:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = T_{\mu\nu} + T^{\mathrm{m}}_{\mu\nu}. \tag{6.1.36}$$

The stress-energy tensor representing the two component fluid is given by

$$T^{\mu\nu} = (\rho + p + \rho_{\mathrm{m}})u^\mu u^\nu + pg^{\mu\nu}. \tag{6.1.37}$$

Here ρ_{m} is the energy density of the pressureless matter. Energy conservation holds independently for both fluids:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{6.1.38}$$

$$\dot{\rho}_{\mathrm{m}} + 3H\rho_{\mathrm{m}} = 0. \tag{6.1.39}$$

Integrating (6.1.39) yields

$$\rho_{\mathrm{m}} = \rho_{\mathrm{m}_0} a^{-3}, \tag{6.1.40}$$

where $\rho_{\mathrm{m}_0} = \rho_{\mathrm{m}}(t_0)$. Similarly, integrating (6.1.38) leads to

$$\rho = \rho_{\mathrm{m}}\left[\left(\Xi + \frac{K}{3}a^{3/2}\right)^2 - 1\right], \tag{6.1.41}$$

where Ξ is a constant and K is given by

$$K = \frac{3\sqrt{3}\xi_o}{\sqrt{\rho_{\mathrm{m}_0}}}, \tag{6.1.42}$$

Thus the total energy density of the two component fluid is given by [144]

$$\rho \equiv \rho + \rho_{\mathrm{m}} = \rho_{\mathrm{m}_0} a^{-3}\left(\Xi + \frac{K}{3}a^{3/2}\right)^2. \tag{6.1.43}$$

Using Eqs. (45) in (16) the evolution of black hole mass is given by

$$\begin{aligned} M &= M_0\Big[1-4\pi A M_0\Big[\frac{\gamma\rho_{\mathrm{m}_0}}{H(t)}\Big\{\frac{K^2}{9}\ln\Big(\frac{a}{a_0}\Big)-\frac{\Xi}{9a^3}(3\Xi+4a^{3/2}K) \\ &\quad +\frac{\Xi}{9a_0^3}(3\Xi+4a_0^{3/2}K)\Big\}\Big]\Big]^{-1}, \end{aligned} \tag{6.1.44}$$

where the scale factor $a(t)$ evolves as

$$a(t)=\Big[\frac{3}{K}(e^{\frac{K}{2}\sqrt{\rho_{\mathrm{m}_0}/3}t+D_1}-\Xi)\Big]^{2/3}, \tag{6.1.45}$$

and D_1 is the constant of integration determined by choosing $t=0$ to get

$$D_1=\frac{2}{K}\ln\Big(\frac{K}{3}a_0^{3/2}+\Xi\Big). \tag{6.1.46}$$

6.2 Conclusion

Up to now have investigated the accretion of two different forms of phantom-like Chaplygin gas onto a Schwarzschild black hole. The time scale of accretion and the evolution of mass of black hole are derived in the context of two widely studied Chaplygin gas models namely the modified variable Chaplygin gas and the viscous generalized Chaplygin gas. Although the phantom energy is an unstable fluid as it corresponds to a medium with indeterminate speed of sound and superluminal speeds. These pathologies arise due to bad choices of the equations of state for the phantom energy and hence can be removed by choosing some suitable transformation from one EoS to another or a totally new EoS for this purpose. This work serves as the generalization of the earlier work by Babichev et al [1, 129]. It should be noted that I have ignored matter component in the accretion model. Thus it will be more insightful to incorporate the contributions of matter along with the Chaplygin gas during accretion onto black hole. Moreover my analysis can be extended to the case of rotating black holes as well.

As pointed out in the next section, I cannot correctly discuss a BR scenario. However I can take a spacetime approximating it sufficiently earlier than the BR. One can than see its asymptotic behavior when the scale factor shoots to infinity, the three terms in (6.1.44) will contribute significantly in the BH mass evolution. The mass will decrease by the accretion of PE ($\gamma<0$) due to its strong negative pressure and is manifested in (6.1.44). Notice that the final expression for BH mass depends only on the initial matter density ρ_{m_0} in addition to constant bulk viscosity ξ_o. The corresponding behavior of BH mass evolution is shown in Figures 6.2 and 6.3 for different values of model parameters. Thus for a shift of parameter γ by 2, yields in the decline of mass ratio by a factor of 2. The decline in the mass of the BH is observed with time showing that phantom energy accretion will be dominant over matter accretion.

I have analyzed the accretion of bulk viscous phantom energy onto a BH. The modeling is based on the relativistic model of accretion for compact objects. The viscosity effects in cosmology are used to give an alternative to cosmic accelerated expansion other then dark energy and quintessence. The evolution of BHs in such a Universe accreting viscous phantom energy would result in a gradual decrease in mass. This gradual decline would be faster than the non-viscous case [1] due to additional terms containing viscosities coupled with mass. Lastly, it is shown that BHs accreting both matter and viscous PE will also meet with the same fate as the viscous forces dominate over the matter component for sufficiently large scale factor $a(t)$.

From this analysis, I can draw the conclusion that PE containing viscous stresses can play a significant role in the BH mass evolution if the viscosity is sufficiently high for an appropriate EoS. Though the viscous stresses are negligibly small $O(10^{-8}Nsm^{-2})$ at the local scale of space and time they can play a significant role in time scales of $\sim$ Gyrs. The higher the viscosity of the phantom fluid, the sharper the decrease in the BH mass. BHs of all masses, ranging from the solar mass to the intermediate mass to the supermassive, will all meet the same fate.

Notice that I have used the Friedmann model which is represented by an asymptotically curved spacetime and at the same time the Schwarzschild black hole, which is asymptotically flat. This may seem contradictory. Schwarzschild black hole has been dealt with in the context of closed Friedmann cosmology [157, 158, 159]. Any global problem in approximating the full situation by a Schwarzschild black hole inserted into Friedmann model arise near the Big Bang or the Big Crunch, defined in terms of the york time [80] as shown elsewhere [160], the effect will be at extremely late times in terms of the usual time parameter. More complete analysis of the asymptotic behavior near a singularity is also available [161], as such near to a singularity in spacetime, the approximation will be extremely good. Consequently our analysis will be satisfactory for black holes formed well after the Big Bang greater then 10^{-40}s and of the Big Rip (presumably much more before 10^{-40}s the rip). Whether there would/would not be a Big Rip as my analysis excludes it.

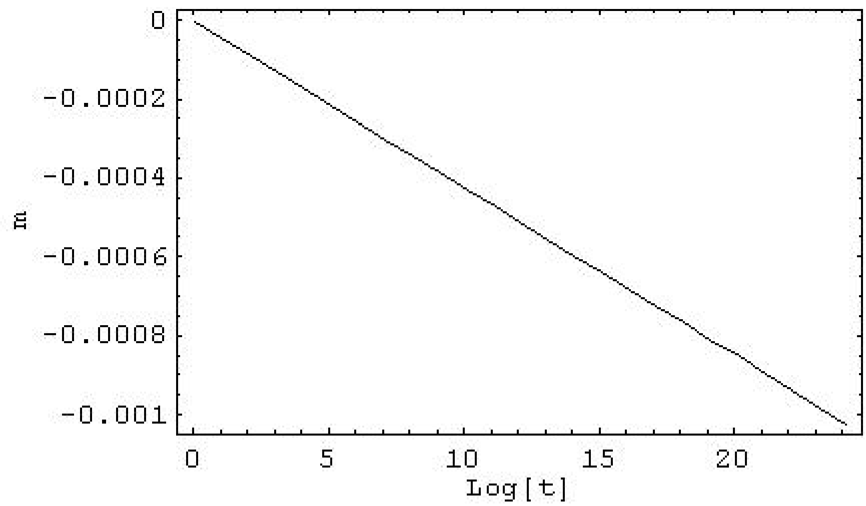

Figure 6.1: For an initial mass of black hole $M_0 = 10^{32}$kg, the evolution of the mass parameter $m = M/M_0 - 1$ is plotted against the logarithmic time with $\alpha = 10^{-5}$ and $t_H = 10^{17}s$. It is shown that the mass of black hole (in suitably normalized units) decreases.

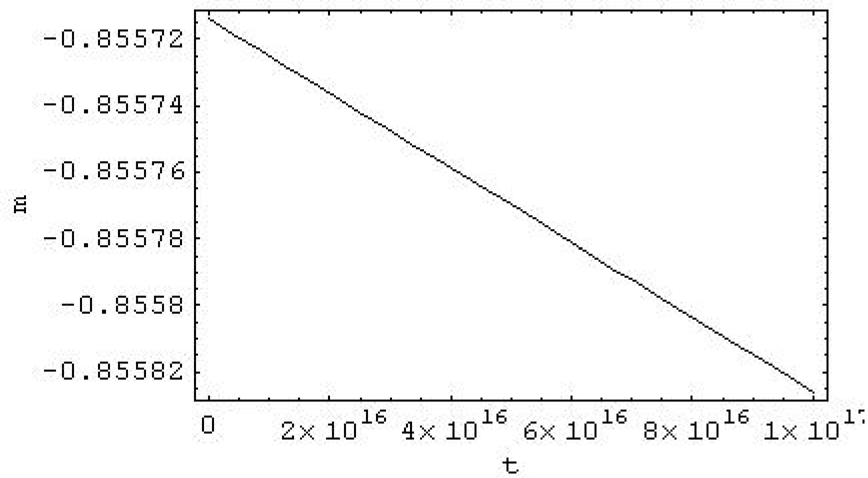

Figure 6.2: For an initial mass of black hole $M_0 = 10^{32}$kg, the evolution of m is plotted against the time parameter t with $A = 1/3$, $\Xi = 3$, $\xi_o = 10^{-16} kgm^{-1}s^{-1}$ and $\gamma = -10^{-1}$ while $H \approx 2.33 \times 10^{-18}$m. It is shown that the mass of black hole (in suitably normalized units) decreases. Here only a small portion of the total evolution profile in Fig. 6.1 is shown.

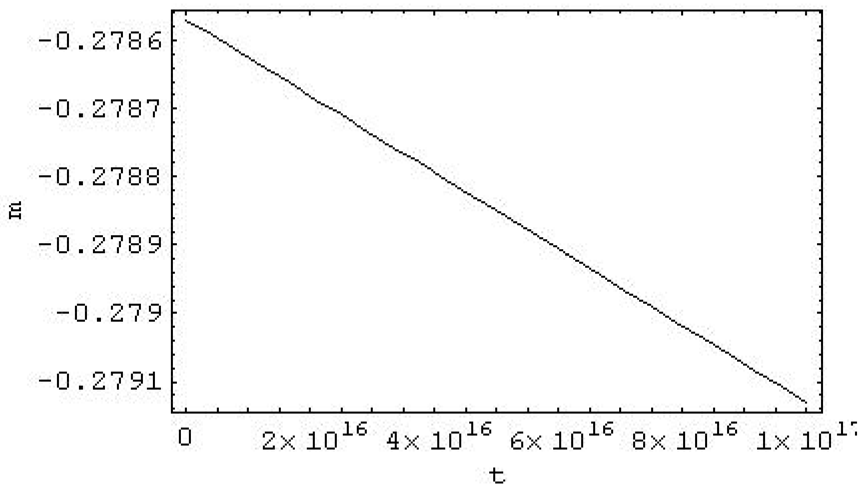

Figure 6.3: For an initial mass of black hole $M_0 = 10^{32}$kg, the evolution of m is plotted against the time parameter t with $A = 1/3$, $\Xi = 3$, $\xi_o = 10^{-16} kgm^{-1}s^{-1}$ and $\gamma = -2 \times 10^{-1}$ while $H \approx 2.33 \times 10^{-18}$m. It is shown that the mass of black hole (in suitably normalized units) decreases. Here only a small portion of the total evolution profile in Fig. 6.1 is shown.

Chapter 7

Phantom energy accretion on a stationary BTZ black hole

7.1 Introduction

Interest in (2+1)-dimensional gravity - general relativity in two spatial dimensions plus time - dates back at least to 1963, when Staruszkiewicz first showed that point particles in a (2+1)-dimensional spacetime could be given a simple and elegant geometrical description [162]. Over the next twenty years occasional papers on classical [163] and quantum mechanical [164] aspects appeared but until recently the subject remains largely a curiosity.

There were some discoveries in (2+1) dimensional gravity that ignited the interests of researchers. In 1984, Deser and Jackiw began a systematic investigation of the behavior of classical and quantum mechanical point sources in (2+1) dimensional gravity [165, 166], showing that such systems exhibit interesting behavior both as toy models for (3+1)-dimensional quantum gravity and as realistic models of cosmic strings. Later on in 1988, Witten showed that (2+1) dimensional general relativity could be rewritten as a Chern-Simons theory, permitting exact computations of topology-changing amplitudes [167, 168]. Over the past decade, (2+1)-dimensional gravity has become an active field of research, drawing insights from general relativity, differential geometry and topology, high energy particle physics, topological field theory and string theory.

In general relativity, there is a serious difficulty concerning the second law of thermodynamics. Although the ordinary second law fails in the presence of black holes and the second law of black hole mechanics fails when quantum effects are taken into account, there is a possibility that the GSL may always hold. If the GSL does hold, it seems clear that one can interpret S_{bh} as representing the physical entropy of a black hole (see Chp. 3 for more details on GSL).

In this chapter I investigate the accretion of exotic phantom energy onto a static uncharged 3-dimensional BTZ black hole. As is obvious that the usual spacetime has three spatial dimensions, so the BTZ black hole is merely a mathematical construct. In the present chapter, I am interested in understanding how the accretion of phantom energy will effect a lower dimensional black hole. I will show that the expression of the evolution of BTZ black hole mass is independent of its mass and dependents only on the energy density and pressure of the phantom energy. Although the mass decreases due to accretion, here the mass is a dimensionless quantity and does not correspond to the physical mass of three dimensional objects. It is well-known that the horizon area of the black hole decreases with the accretion of phantom energy, hence it is essential to study the generalized second law of thermodynamics (GSL) in this case. I show that the validity of GSL in the present model yields an upper bound on the phantom energy pressure. I also demonstrate that the first law of thermodynamics holds in the present construction.

7.2 Model of accretion

Consider the field equations for a (2+1)-dimensional spacetime with a negative cosmological constant Λ

$$G_{ab} + \Lambda g_{ab} = \pi T_{ab}, \quad (a, b = 0, 1, 2) \tag{7.2.1}$$

where G_{ab} is the Einstein tensor in (2+1)-dimension while T_{ab} is the stress energy tensor of the matter field. The units are chosen such that $c = 1$ and $G_3 = 1/8$. Considering the stress-energy tensor to be vacuum, one can obtain the following spherically symmetric metric, a (2+1)-dimensional BTZ black hole [169, 170]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \tag{7.2.2}$$

where $f(r) = -M + r^2/l^2$, M is the dimensionless mass of the black hole and $l^2 = -1/\Lambda$, is a positive constant. The coefficient g_{00} is termed as the lapse function. The event horizon of the BTZ black hole is obtained by setting $f(r) = 0$, which turns out, $r_e = l\sqrt{M}$. Also have $\sqrt{|g|} = r$, where g is the determinant of the metric. To analyze the accretion of phantom energy onto the BTZ black hole, I employ the formalism from the work by Babichev et al [1]. The stress energy momentum tensor representing the phantom energy is the perfect fluid. I denote ρ and p as the energy density and pressure of the phantom energy while $u^a = (u^0, u^1, 0)$ is the velocity three vector of the fluid flow. Also $u^1 = u$ is the radial velocity of the flow while the third component u^2 is zero due to spherical symmetry of the BTZ black hole. There are two important equations of motion in our model: one which controls the conservation of mass flux is $J^a_{;a} = 0$, where J^a is the current density and the other that controls the energy flux $T^a_{0;a} = 0$, across the horizon. Thus the equation of energy conservation $T^{0a}_{;a} = 0$ is

$$ur(\rho + p)\sqrt{f(r) + u^2} = C_1, \tag{7.2.3}$$

where C_1 is an integration constant. Since the flow is inwards the black hole therefore $u < 0$. Also the projection of the energy momentum conservation along the velocity three vector $u_a T^{ab}_{;b} = 0$ (the energy flux equation) is

$$ur \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p}\right] = -A_1. \tag{7.2.4}$$

Here A_1 is a constant and the associated minus sign is taken for convenience. Also ρ_h and ρ_∞ are the energy densities of phantom energy at the BTZ horizon and at infinity respectively. From (7.2.3) and (7.2.4), one gets

$$(\rho + p)\sqrt{f(r) + u^2} \exp\left[-\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p}\right] = C_2, \tag{7.2.5}$$

where $C_2 = -C_1/A_1 = \rho_\infty + p(\rho_\infty)$. The rate of change in the mass of black hole is

$$dM = 2\pi A_1(\rho_\infty + p_\infty)dt. \tag{7.2.6}$$

Note that $\rho_\infty + p_\infty < 0$ (violation of null energy condition) leads to decrease in the mass of the black hole. Moreover, the above expression is also independent of mass contrary to the Schwarzschild black hole and the Reissner-Nordström black hole [171]. Further, the last equation is valid for any general ρ and p violating the null energy condition, thus I can write

$$dM = 2\pi A_1(\rho + p)dt. \tag{7.2.7}$$

7.3 Critical accretion

Only in those solutions that pass through the critical point are relevant as these correspond to the material falling into the black hole with monotonically increasing speed. The falling fluid can exhibit variety of behaviors near the critical point of accretion, close to the compact object. The equation of mass flux or the continuity equation $J^a_{;a} = 0$ is

$$\rho u r = k_1. \tag{7.3.1}$$

Here k_1 is integration constant. From (7.2.4) and (7.3.1), I have

$$\Big(\frac{\rho + p}{\rho}\Big)^2 \Big(f(r) + u^2\Big) = \Big(\frac{C_1}{k_1}\Big)^2 = C_3. \tag{7.3.2}$$

Taking differentials of (7.3.1) and (7.3.2) after simplification, I obtain

$$\frac{du}{u}\Big[-V^2 + \frac{u^2}{f(r) + u^2}\Big] + \frac{dr}{r}\Big[-V^2 + \frac{r^2}{l^2\Big(f(r) + u^2\Big)}\Big] = 0. \tag{7.3.3}$$

Here

$$V^2 \equiv \frac{d\ln(\rho + p)}{d\ln\rho} - 1, \tag{7.3.4}$$

From (7.3.3) if one or the other bracket factor is zero, one gets a turnaround point corresponding double-valued solution in either r or u. The only solution that passes through a critical point is feasible. The feasible solution will correspond to material falling into the object with monotonically increasing velocity. The critical point is obtained by taking the both bracketed factors in (7.3.3) to be zero. This will give us the critical points of accretion. Thus

$$V_c^2 = \frac{r_c^2}{(f(r_c) + u_c^2)l^2}, \tag{7.3.5}$$

$$V_c^2 = \frac{u_c^2}{f(r_c) + u_c^2}. \tag{7.3.6}$$

Above the subscript c refers to the critical quantity. On comparing (7.3.5) and (7.3.6), gives

$$u_c^2 = \frac{r_c^2}{l^2}, \quad V_c^2 = \frac{u_c^2}{-M + 2u_c^2}. \tag{7.3.7}$$

Here u_c is the critical speed of flow at the critical points which I determine below. For physically acceptable solution, $V_c^2 > 0$, hence the restrictions on speeds and the location of the critical points are

$$u_c^2 > \frac{M}{2}, \quad r_c^2 > \frac{r_+^2}{2}. \tag{7.3.8}$$

7.4 Generalized second law of thermodynamics and BTZ black hole

In this section I will discuss the thermodynamic of phantom energy accretion that crosses the event horizon of BTZ black hole. Let us first write the BTZ metric in the form

$$ds^2 = h_{mn}dx^m dx^n + r^2 d\phi^2, \quad m, n = 0, 1 \tag{7.4.1}$$

where $h_{mn} = \text{diag}(-f(r), 1/f(r))$, is a 2-dimensional metric. From the condition of normalized velocities $u^a u_a = -1$, one can obtain the relations

$$u^0 = f(r)^{-1}\sqrt{f(r) + u^2}, \quad u_0 = -\sqrt{f(r) + u^2}. \tag{7.4.2}$$

The components of stress energy tensor are $T^{00} = f(r)^{-1}[(\rho + p)(\frac{f(r)+u^2}{f(r)}) - p]$, and $T^{11} = (\rho + p)u^2 + f(r)p$. These two components help us in calculating the work density which is defined by $W = -\frac{1}{2}T^{mn}h_{mn}$ [172]. In my case it comes out

$$W = \frac{1}{2}(\rho - p). \tag{7.4.3}$$

The energy supply vector is defined by

$$\Psi_n = T_n^m \partial_m r + W\partial_n r. \tag{7.4.4}$$

The components of the energy supply vector are $\Psi_0 = T_0^1 = -u(\rho+p)\sqrt{f(r) + u^2}$, and $\Psi_1 = T_1^1 + W = (\rho+p)\left(\frac{1}{2} + \frac{u^2}{f(r)}\right)$. The change of energy across the apparent horizon is determined through $-dE \equiv -A\Psi$, where $\Psi = \Psi_0 dt + \Psi_1 dr$. The energy crossing the event horizon of the BTZ black hole is given by

$$dE = 4\pi r_e u^2(\rho + p)dt. \tag{7.4.5}$$

Assuming $E = M$ and comparing (7.2.8) and (7.4.5), one can determine the value of constant $A_1 = 2u^2 l\sqrt{M}$.

The entropy of BTZ black hole is

$$S_h = 4\pi r_e. \tag{7.4.6}$$

It can be shown easily that the thermal quantities, change of phantom energy dE, horizon entropy S_h and horizon temperature T_h satisfy the first law $dE = T_h dS_h$, of thermodynamics. After differentiation of (7.4.6) w.r.t. t, and using (7.2.8), I have

$$\dot{S}_h = 8\pi^2 l^2 u^2 (\rho + p). \tag{7.4.7}$$

Since all the parameters are positive in (7.4.7) except that $\rho + p < 0$, it shows that the second law of thermodynamics is violated i.e. $\dot{S}_h < 0$, as a result of accretion of phantom energy on a BTZ black hole.

Now I proceed to the generalized second law of thermodynamics. It is defined by

$$\dot{S}_{tot} = \dot{S}_h + \dot{S}_{ph} \geq 0. \tag{7.4.8}$$

In other words, the sum of the rate of change of entropies of black hole horizon and phantom energy must be positive. I consider event horizon of the BTZ black hole as a boundary of thermal system and the total matter energy within the event horizon is the mass of the BTZ black hole. I also assume that the horizon temperature is in equilibrium with the temperature of the matter-energy enclosed by the event horizon, i.e. $T_h = T_{ph} = T$, where T_{ph} is the temperature of the phantom energy. Similar assumptions for the temperatures T_h and T_{ph} has been studied in [173]. The Einstein field equations satisfy first law of thermodynamics $T_h dS_h = pdA + dE$, at the event horizon [174]. I also assume that the matter-energy enclosed by the event horizon of BTZ black hole also satisfy the first law of thermodynamics given by

$$T_{ph} dS_{ph} = pdA + dE. \tag{7.4.9}$$

Here the horizon temperature is given by

$$T_h = \frac{f'(r)}{4\pi}\bigg|_{r=r_e} = \frac{\sqrt{M}}{2\pi l}. \tag{7.4.10}$$

Assuming that $T_h = T_{ph} = T$, therefore (7.4.9) gives

$$T\dot{S}_{tot} = T(\dot{S}_h + \dot{S}_{ph}) = 4\pi l^2 u(\rho + p)(2\sqrt{M} + \pi l p). \tag{7.4.11}$$

From the above equation, it is clear that the GSL holds provided

$$p \geq -\frac{2\sqrt{M}}{\pi l}. \tag{7.4.12}$$

Since the pressure of the phantom energy is negative, therefore the GSL gives us the lower bound on the pressure of the phantom energy.

7.5 Conclusion

In this chapter, I have investigated the accretion of exotic phantom energy onto a BTZ black hole. The motivation behind this work is to study the accretion dynamics in low dimensional gravity.myanalysis has shown that evolution of mass of a BTZ black hole would be independent of its mass and will be dependent only on the energy density and pressure of the phantom energy in its vicinity. Due to spherical symmetry, the accretion process is simple since the phantom energy falls radially on the black hole. The accretion would be much more interesting when additional parameters like charge and angular momentum are also incorporated in the BTZ spacetime. Similarly, it would be of much interest to perform the above analysis in higher $(n+1)$ dimensional black hole spacetimes.

I also discussed GSL in the BTZ black hole spacetime. I assumed that the event horizon of BTZ black hole acts as a boundary of the thermal system and the phantom energy crossing the event horizon will change the mass of the black hole. I assumed that the horizon temperature is in local equilibrium with the temperature of the matter energy at the event horizon. Under these constraints it is shown that the GSL holds provided the pressure of the phantom energy p has an lower bound $p \geq -\frac{2\sqrt{M}}{\pi l}$, on the black hole parameters (M and l).

Chapter 8

Conclusion

In this thesis I have dealt with the consequences of the observed accelerated expansion of the Universe on the evolution of black holes [171, 183, 184, 185, 186]. I considered only static black holes which evolve with time by accretion (of matter, phantom energy, Chaplygin gas etc) in an expanding Universe. Astrophysically, black holes may not be static objects and can undergo one of the two processes inevitably: absorption of energy-matter from their environs and a relatively slow process of Hawking evaporation which involves emission of massive particles from their horizons. It should be abundantly clear that the former process works efficiently for supermassive black holes (of the order of few million solar masses) while the later one is crucial for small mass black holes (masses of the order of Planck mass). Since I have dealt with astrophysical black holes, with rotations ignored, the evolution of black hole mass is an inevitable phenomenon. Moreover in all cases, the rate of change of BH mass is a decreasing function of time.

Notice that I have used the Friedmann model which is represented by an asymptotically curved spacetime and at the same time black holes, which are asymptotically flat. This may seem contradictory. Any global problem in approximating the full situation by a black hole inserted into Friedmann model arise near the Big Bang or the Big Crunch, the effect will be at extremely late times in terms of the usual time parameter (see conclusion of Chapter 6 for more details).

Dark energy has generally been described by an equation of state. This state parameterization of dark energy assumes that it is homogeneous and isotropic at cosmological scales. However near the black hole horizon, the strong curvature can induce inhomogeneity in the dark energy relative to that of spatial infinity. In such a scenario, the dark energy should be modeled by an inhomogeneous scalar field possessing inhomogeneous pressure and energy density. It is expected that in this case the accretion of phantom energy on to the black hole would be somewhat different. In this connection, a study was performed by Gonzalez and Guzman [177] who deduced that the accretion rate and localization of the scalar field around the black hole depends on the scalar field potential. In the case of the zero potential the scalar field is quickly accreted whereas in the other case the scalar field gets packed near the horizon and some scalar field is radiated away. Moreover the event horizon shrinks with the accretion of the scalar field.

One can ask how dark energy can possibly be absorbed by a black hole. It may seem impossible if dark energy is taken as a property of spacetime. But according to Einstein's field equations which connect geometry with energy-matter, one can treat dark energy as a fluid, and more specifically a perfect fluid. I am not sure about the micro-physical interpretation of dark energy and so I kept my analysis macrophysical. One of the obvious features of dark energy is that it causes cosmic expansion to speed up from the simple expansion to the accelerated one. However, I studied another new feature of dark energy when it interacted with the black holes in the Universe. It was earlier claimed by some people [175] that if dark energy is taken as a quintessence field, then its accretion onto small stellar mass black holes would make them grow to supermassive black holes that reside in the centers of massive elliptical and spiral galaxies. However, drastic features of dark energy emerges as phantom energy.

In the third chapter, I analyzed the Hawking radiation effects combined with the phantom energy accretion on a stationary black hole. The former process has been thoroughly investigated in the literature. However there is as yet no observational support to it. When phantom energy and the Hawking process are relevant the total life time scale of the PBH is significantly shortened and the formation of the PBH that would be exploding now is delayed. In particular, to have the primordial black hole decay now it would have to be more massive initially. I find that the effect of the phantom energy is substantial and the black holes decaying now would be *much* more massive — over 10 orders of magnitude! This effect will be relevant for determining the time of production and hence the number of evaporating black holes expected in a Universe accelerating due to phantom energy.

In the fourth chapter, I analyzed the effects of accretion of phantom energy onto a charged black hole. The analysis is performed using two critical points $r_{c\pm}$.

It turns out that accretion is possible only through r_{c+} which yields a constraint on the mass to charge ratio given by (4.2.16). This expression incorporates both extremal and non-extremal black holes. Thus all charged black holes will diminish near the Big Rip causing the formation of naked singularity. Thus my suggested mechanism leads to the violation of cosmic censorship hypothesis. This work also serves as a generalization of Babichev et al [1] and Michel [122] in terms of the accretion of phantom dark energy onto a charged black hole.

I have analyzed the accretion of bulk viscous phantom energy onto a BH. The modeling is based on the relativistic model of accretion for compact objects. The viscosity effects in cosmology are used to give an alternative to cosmic accelerated expansion other then dark energy and quintessence. The evolution of BHs in such a Universe accreting viscous phantom energy would result in a gradual decrease in mass. This gradual decline would be faster than the non-viscous case [1] due to additional terms containing viscosities coupled with mass. Lastly, it is shown that BHs accreting both matter and viscous PE will also meet with the same fate as the viscous forces dominate over the matter component for sufficiently large scale factor $a(t)$.

In the fifth chapter, I reached the conclusion that phantom energy containing viscous stresses can play a significant role in the BH mass evolution provided the viscosity is sufficiently high for an appropriate EoS. Though the viscous stresses are negligibly small $O(10^{-8}Nsm^{-2})$ at the local scale of space and time they can play a significant role in time scales of $\sim$ Gyrs. The higher the viscosity of the phantom fluid, the sharper the decrease in the BH mass. BHs of all masses, ranging from the solar mass to the intermediate mass to the supermassive, will meet the same fate.

In the sixth chapter, I investigated the accretion of two different forms of phantom-like Chaplygin gas onto a Schwarzschild black hole. The time scale of accretion and the evolution of mass of black hole were derived in the context of two widely studied Chaplygin gas models, namely the modified variable Chaplygin gas and the viscous generalized Chaplygin gas. Although the phantom energy seems to be an unstable fluid as it corresponds to a medium with indeterminate speed of sound and super-luminal speeds, these pathologies arise due to bad choices of the equations of state for the phantom energy and hence can be removed by choosing some suitable transformation from one EoS to another or a totally new EoS for this purpose. This work serves as a generalization of the earlier work by

Babichev [1].

In the seventh chapter, I investigated the accretion of exotic phantom energy onto a BTZ black hole. The motivation behind this work is to study the accretion dynamics in low dimensional gravity. Our analysis showed that the evolution of mass of a BTZ black hole would be independent of its mass and will be dependent only on the energy density and pressure of the phantom energy in its vicinity. I also discussed GSL in the BTZ black hole spacetime. I assumed that the event horizon of a BTZ black hole acts as a boundary of the thermal system and the phantom energy crossing the event horizon will change the mass of the black hole. I assumed that the horizon temperature is in local equilibrium with the temperature of the matter energy at the event horizon. Under these constraints it was shown that the GSL holds provided the pressure of the phantom energy p has an upper bound $p \leq -\frac{2\sqrt{M}}{\pi l}$, on the black hole parameters (M and l).

8.1 Further lines of work

In this thesis, I focused on accretion of dark energy (more specifically phantom energy) onto black holes. The analysis was performed within the framework of Einstein's general relativity. The process of accretion of dark energy onto black holes can be extended in various other ways: it would be of worth checking how the accretion processes and variation of black hole mass changes over time due to rotation. For this purpose, one can take Kerr-Newmann black hole and consider the effects of Hawking radiation and phantom energy accretion on it. Another line of work that needs to be pursued is the investigation of the accretion of inhomogeneous dark energy e.g. scalar field near the EH of black hole. Furthermore, one could consider the evolution of black hole in Friedmann-de Sitter model. This would be worth doing.

In the past few years, several alternative gravity theories, commonly termed 'modified gravity' theories are proposed as an alternative to Einstein's gravity. The most prominent of these are $f(R)$ gravity [176], Gauss-Bonnet gravity [178] (which in fact is a special case of the former theory), Braneworld gravity [179], Divali-Gabadadze-Poratti gravity [180] and scalar-tensor gravity [181]. It would be most interesting to study the accretion of dark energy onto the black hole solutions of these theories and obtain relations which determine how the mass of respective black hole varies over time.

There are numerous black hole solutions in higher dimensional spacetimes including Schwarzschild, RN and Kerr spacetimes. One can study accretion of phantom energy in such spacetimes as well. Moreover, the study of dark energy accretion need not be restricted to black holes only; it can be extended for wormholes (both static and evolving) [182] and also for string theory inspired dilaton-axion black hole.

One can also investigate the thermodynamic features of dark energy accretion onto black holes. This analysis should be similar to that performed in chapter 7 for the BTZ black hole. It is important to verify whether the laws of thermodynamics are respected in the processes under investigation. In particular, it is evident that the second law of thermodynamics is violated, i.e. $\dot{S} < 0$ due to violation of the null energy condition. This suggests that I use the generalized second law of thermodynamics for black holes accreting dark energy. Given the fact that GSL holds in most cases, its utility puts constraints on the model parameters.

Besides phantom energy, one can also consider other types of dark energy candidates including tachyons, quintessence and dilaton fields accreted by various black holes taken in the thesis and calculate the expression for mass variation and critical accretion.

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Eur. Phys. J. C (2008) 58: 325–329
DOI 10.1140/epjc/s10052-008-0761-9

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Charged black holes in phantom cosmology

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Received: 8 May 2008 / Revised: 15 September 2008 / Published online: 22 October 2008

Abstract In the classical relativistic regime, the accretion of phantom-like dark energy onto a stationary black hole reduces the mass of the black hole. We have investigated the accretion of phantom energy onto a stationary charged black hole and have determined the condition under which this accretion is possible. This condition restricts the mass-to-charge ratio in a narrow range. This condition also challenges the validity of the cosmic-censorship conjecture since a naked singularity is eventually produced due to accretion of phantom energy onto black hole.

1 Introduction

Accelerated expansion of the universe has been observed and confirmed by a myriad of sources including the analysis of cosmic microwave background radiation [1], large-scale structure [2] and supernovae SNe Ia data [3, 4]. This expansion is supposedly driven by an exotic vacuum energy having $\rho > 0$ and $p < 0$ (or $\omega < 0$ where $p = \omega\rho$ in the equation of state (EoS) relating pressure to energy density), dominating the observable universe. Inspection of WMAP data suggests that its magnitude is more than 70% of the total energy density of the universe [5]. Among other forms of exotic energies (e.g. quintessence, cosmological constant, k-essence, etc.), the phantom energy with $\omega < -1$ exhibits a similar behavior on a large cosmic scale. The genesis of phantom energy is not clear but it violates the null and weak energy conditions. As these conditions are the weaker one, they ensure that the stronger conditions (i.e. strong and dominant) will be violated automatically [6–8]. These energy conditions guarantee the positive definiteness of the energy densities and pressure densities of all the matter content in the universe. Recent observational data constrain the range of the dark energy by $-1.38 < \omega < -0.82$ at 95% confidence level [9]. Rather the supernova data favor an evolving $\omega(z)$ varying from quintessence ($\omega > -1$) to the phantom regime ($\omega < -1$) [10]. Furthermore, the extrapolation of WMAP data is best fitted with the notion of phantom energy [11].

The energy density and the pressure of the phantom energy can be represented by the minimally coupled spatially homogeneous and time dependent scalar field ϕ having a negative kinetic energy term given by

$$\rho = -\frac{\dot{\phi}^2}{2} + V(\phi), \qquad p = -\frac{\dot{\phi}^2}{2} - V(\phi). \tag{1}$$

Here $V(\phi)$ is the scalar potential and the dot over ϕ represents the derivative with respect to the time parameter t. Note that if the kinetic term in (1) is positive, then it gives the usual dark energy with satisfies all the energy conditions. The above parameters ρ and p are related to the Hubble parameter H by

$$H^2 = \frac{4\pi}{3}\left(-\dot{\phi}^2 + 2V\right), \tag{2}$$

where $H(t) = \dot{a}/a$ and $a(t)$ is the scale factor which arises in the Friedmann–Robertson–Walker spacetime. From (2), we require the potential $V(\phi)$ to be positive. It is argued by using scalar field models of the phantom energy, that it can behave as a long-range repulsive force [12]. The phantom energy possesses some peculiar properties unlike normal matter; e.g. (1) its energy density $\rho(t)$ increases with the expansion of the universe; (2) it ensures the existence and stability of traversable worm holes in the universe [13–17]; (3) also self-gravitating, static and spherically symmetric phantom scalar fields with arbitrary potentials can generate a stable configuration of a regular black hole or apparently non-singular black hole, which inherently possesses an exactly Schwarzschild-like causal structure, but the singularity is replaced by a de Sitter infinity, thereby generating an asymptotically de Sitter expansion beyond the black hole horizon [18, 19]; (4) due to a strong negative pressure the phantom energy can disrupt all gravitationally bound structures, i.e. from galactic clusters to gravitationally

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collapsed objects including black holes [20–25]; (5) it can produce an infinite expansion of the universe in a finite time, thus causing the 'big rip' (i.e. a state when $a(t), \rho(t) \to \infty$ for $t < \infty$) [6, 11].

The big rip is characterized by a future singularity implying a finite age of the universe. It has been proposed that this future singularity can be avoided if the phantom energy is interacting with dark matter [26, 27]. The interaction of phantom energy and dark matter leads to stable attractor solutions at late times and the big rip is avoided in the parameter space [29]. Also, it was argued that there are certain classes of unified dark energy models stable against perturbations, in which Cosmic Doomsday can be avoided [30]. Moreover, in scalar–tensor theories, quantum gravity effects may prevent (or, at least, delay or soften) the cosmic doomsday catastrophe associated with the phantom energy [31, 32]. In Gauss–Bonnet gravity theory and loop quantum cosmology, the big-rip occurrence is avoided [33, 34]. In order to avoid the big rip with phantom matter, it is sufficient to have a phantom scalar field with a potential bounded above by some positive constant [35]. It is also suggested that phantom dark energy with $\omega < -1$ can effectively ameliorate the coincidence problem (i.e. why does the observable universe begin the accelerated expansion so recently and why are we living in an epoch in which the dark energy and the matter energy density are comparable?) [36, 37]. In another model, using vector-like dark energy with a background of a perfect fluid, it is demonstrated that the cosmic-coincidence problem is fairly solved [38].

The fate of a stationary uncharged black hole in a phantom-energy dominated universe was investigated by Babichev et al. [20]. The phantom energy was assumed to be a perfect fluid. The phantom energy was allowed to fall onto the black hole horizon only in the radial direction. It was concluded that the black hole will lose mass steadily due to phantom energy accretion and disappear near the big rip. We here adopt their procedure for a static, stationary and charged black hole. Gravitational units are chosen for this work.

The paper is organized as follows. In the second section, we explain the relativistic model of accretion onto a charged black hole and obtain the black hole mass loss rate. In the third section, we have determined the critical points of the accretion model and have analyzed the dynamics about these points. Finally, we conclude our paper in Sect. 4.

2 Accretion onto a charged black hole

We consider a static and spherically symmetric black hole of mass M having electric charge e, the so-called Reissner–Nordström (RN) case, specified by the line element

$$ds^2 = f(r)\,dt^2 - f(r)^{-1}\,dr^2 - r^2\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big), \quad (3)$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}. \quad (4)$$

If $e^2 > M^2$ then the metric is non-singular everywhere except at the curvature or the irremovable singularity at $r = 0$. Also if $e^2 \le M^2$, then the function $f(r)$ has two real roots given by

$$r_{h\pm} = M \pm \sqrt{M^2 - e^2}. \quad (5)$$

These roots physically represent the apparent horizons of the RN black hole. The two horizons are termed the inner r_{h-} and the outer r_{h+}. The outer horizon is effectively called the *event horizon*, while the inner one is called the *Cauchy horizon* of the black hole. The metric (3) is then regular in the regions specified by the inequalities $\infty > r > r_{h+}$, $r_{h+} > r > r_{h-}$ and $r_{h-} > r > 0$. Note that if $e^2 = M^2$, then it represents an *extreme* RN black hole, while, if $e^2 > M^2$, it yields a *naked singularity* at $r = 0$ [39, 40].

The phantom energy is assumed to be a perfect fluid specified by the stress energy tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}. \quad (6)$$

Here p is the pressure and ρ is the energy density of the phantom energy. Also $u^\mu = (u^t(r), u^r(r), 0, 0)$ is the four-velocity of the phantom fluid which satisfies the normalization condition $u^\mu u_\mu = -1$. We assume that the in-falling phantom fluid does not disturb the global spherical symmetry of the black hole. Further the energy–momentum conservation $T^{\mu\nu}_{;\nu} = 0 = T^{tr}_{;r}$ gives

$$ur^2 M^{-2}(\rho + p)\sqrt{1 - \frac{2M}{r} + \frac{e^2}{r^2} + u^2} = C_1, \quad (7)$$

where $u^r = u = dr/ds$ is the radial component of the velocity four-vector and C_1 is a constant of integration. For inward flow, we shall take $u < 0$. Moreover, the second constant of motion is obtained by projecting the energy conservation equation onto the velocity four-vector as $u_\mu T^{\mu\nu}_{;\nu} = 0$, which yields

$$ur^2 M^{-2} \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p(\rho')}\right] = -A. \quad (8)$$

Here A is a constant of integration. ρ_h and ρ_∞ are the energy densities of the phantom energy at the horizon and at infinity, respectively. From (7) and (8) we have

$$(\rho + p)\sqrt{1 - \frac{2M}{r} + \frac{e^2}{r^2} + u^2} \times \exp\left[-\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p(\rho')}\right] = C_2, \quad (9)$$

where $C_2 = -C_1/A = \rho_\infty + p(\rho_\infty)$. In order to calculate the rate of change of the mass of the black hole we integrate the flux of the fluid over the entire cross-section of the event horizon as follows:

$$\dot{M} = \oint T_t^r \, \mathrm{d}S, \tag{10}$$

where T_t^r determines the momentum density in the radial direction and $\mathrm{d}S = \sqrt{-g}\, \mathrm{d}\theta\, \mathrm{d}\varphi$ is the surface element of the horizon, where g is the determinant of the metric. From (7)–(10), we get

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 4\pi A M^2 \big(\rho_\infty(t) + p_\infty(t)\big), \tag{11}$$

which clearly demonstrates that the mass of the black hole decreases if $\rho_\infty + p_\infty < 0$. Note that (11) can be solved for any equation of state of the form $p = p(\rho)$ or in particular $p = \omega\rho$. In general, (11) holds for all ρ and p violating the dominant energy condition; thus, we can write [43, 44]

$$\frac{dM}{\mathrm{d}t} = 4\pi A M^2 \big(\rho(t) + p(t)\big). \tag{12}$$

In the context of astrophysics, the mass of a black hole is a dynamic quantity. The mass increases by the accretion of matter and can decrease by the accretion of phantom energy. Since we are not incorporating matter in our model, the mass of the black hole will decrease correspondingly.

3 Critical accretion

We are interested only in the solutions that pass through the critical point, as these correspond to the material falling into the black hole with monotonically increasing speed. The falling fluid can exhibit a variety of behaviors near the critical point of accretion, close to the compact object. For instance, for a given critical point $r = r_c$, we have the following possibilities [41]. (a) $u^2 = c_s^2$ at $r = r_c$, $u^2 \to 0$ as $r \to \infty$, $u^2 < c_s^2$ for $r > r_c$ and $u^2 > c_s^2$ for $r < r_c$. Thus for large distance, the speed of flow becomes negligible (subsonic), at the critical point it is sonic, while the flow becomes supersonic for very small r. Other solutions for the flow near r_c are not of much interest due to their impracticality, like the following cases. (b) $u^2 < c_s^2$ for all values of r and (c) $u^2 > c_s^2$ for all values of r. Solutions (b) and (c) are not realistic, since they describe both subsonic and supersonic flows for all r. Similarly, we have (d) $u^2 = c_s^2$ for all values of $r > r_c$ and (e) $u^2 = c_s^2$ for all values of $r < r_c$. The latter two solutions are also useless, since they give the same value of the speed at a given r. Hence, from this discussion, we see that solution (a) is the only one physically motivated, near the critical point. To determine the critical points of accretion we shall adopt the procedure as specified by Michel [42]. The equation of the mass flux $J^\mu_{;\mu} = 0$ gives

$$\rho u r^2 = k_1, \tag{13}$$

where k_1 is a constant of integration. Dividing and then squaring (7) and (13) gives

$$\left(\frac{\rho + p}{\rho}\right)^2 \left(1 - \frac{2M}{r} + \frac{e^2}{r^2} + u^2\right) = \left(\frac{C_1}{k_1}\right)^2 = C_3. \tag{14}$$

Here C_3 is a positive constant. Differentiation of (13) and (14) and then elimination of $\mathrm{d}\rho$ gives

$$\frac{\mathrm{d}u}{u}\left[2V^2 - \frac{\frac{M}{r} - \frac{e^2}{r^2}}{1 - \frac{2M}{r} + \frac{e^2}{r^2} + u^2}\right] + \frac{\mathrm{d}r}{r}\left[V^2 - \frac{u^2}{1 - \frac{2M}{r} + \frac{e^2}{r^2} + u^2}\right] = 0, \tag{15}$$

or

$$\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{u}{r}\frac{\left[V^2 - \frac{u^2}{1 - \frac{2M}{r} + \frac{e^2}{r^2} + u^2}\right]}{\left[2V^2 - \frac{\frac{M}{r} - \frac{e^2}{r^2}}{1 - \frac{2M}{r} + \frac{e^2}{r^2} + u^2}\right]} = \frac{N}{D}, \tag{16}$$

where

$$V^2 \equiv \frac{\mathrm{d}\ln(\rho + p)}{\mathrm{d}\ln\rho} - 1. \tag{17}$$

We have assumed that the flow is smooth at all points of spacetime; however, if at any point the denominator D vanishes, then the numerator N must also vanish at that point. Mathematically this point is called the *critical point* of the flow [45]. Equating the denominator D and numerator N to zero, we can get the so-called *critical point conditions*:

$$u_c^2 = \frac{M r_c - e^2}{2 r_c^2} \tag{18}$$

and

$$V_c^2 = \frac{M r_c - e^2}{2 r_c^2 - 3 M r_c + e^2}. \tag{19}$$

Note that by choosing $e = 0$ in the above equations, we can retrieve the results for the accretion of the fluid onto a Schwarzschild black hole [42]. All the quantities with subscript c are defined at the critical point respectively. Physically, the critical points represent the *sonic point* of the flow, i.e. the point where the speed of flow becomes equal to the speed of sound, $u_c^2 = c_s^2$, or the corresponding Mach number becomes $M_c = 1$. This transition may occur from the initial subsonic to the supersonic or transsonic speeds. For any spherically symmetric spacetime, a surface where every point is a sonic point is called a *sound horizon* which itself will be spherical. Any perturbation or disturbance generated in the flow inside the sound horizon ($r < r_c$) is eventually pulled towards the black hole singularity and hence cannot escape to infinity.

It can be seen that the speed of sound (squared) $c_s^2 = \partial p/\partial \rho$ has no physical meaning if the EoS parameter $\omega < 0$ (in $p = \omega\rho$). Thus, it will apparently make the exotic cosmic fluids like the cosmological constant, quintessence and the phantom energy unstable so that these cannot be accreted onto the black hole. In order to avoid this problem, Babichev et al. [46] introduced a non-homogeneous linear equation of state (nEoS) given by $p = \alpha(\rho - \rho_o)$, where the constants α and ρ_o are free parameters. The nEoS can describe both hydrodynamically stable ($\alpha > 0$) and unstable ($\alpha < 0$) fluids. The parameter ω is related to the nEoS by $\omega = \alpha(\rho - \rho_o)/\rho$. Notice that $\omega < 0$ corresponds to $\alpha > 0$ and $\rho_o > \rho$, thus making the phantom energy a hydrodynamically stable fluid. Therefore the speed of sound c_s is now well defined with the nEoS for the phantom energy. Hence, the phantom energy can fall onto the RN black hole and can reduce the black-hole mass. Since phantom energy reduces only mass and not charge, a stage is reached when the cosmic-censorship conjecture becomes violated, i.e. $e > m$, the emergence of a so-called naked singularity.

Now, a physically acceptable solution of (16) is obtained if $u_c^2 > 0$ and $V_c^2 > 0$; hence we get

$$2r_c^2 - 3Mr_c + e^2 \geq 0, \tag{20}$$

and

$$Mr_c - e^2 \geq 0. \tag{21}$$

Equation (20) can be factorized thus:

$$2r_c^2 - 3Mr_c + e^2 = (r_c - r_{c+})(r_c - r_{c-}) \geq 0, \tag{22}$$

where

$$r_{c\pm} = \frac{1}{4}\left(3M \pm \sqrt{9M^2 - 8e^2}\right), \tag{23}$$

which are positive, satisfying $r_{c+} > r_{c-} > 0$. In general, for $e \leq m$, the inner critical point will lie between $r_{h-} \leq r_{c-} \leq r_{h+}$, while the outer one will satisfy $r_{c+} \geq r_{h+}$. It is obvious that these roots will be real valued if $9M^2 - 8e^2 \geq 0$ or

$$\frac{M^2}{e^2} \geq \frac{8}{9}. \tag{24}$$

These roots physically represent the locations of the critical or sonic points of the flow near the black hole. Notice that both mass and charge have the same dimension of length, therefore all the inequalities here and below represent dimensionless ratios. From (22), we can see that these critical points specify two regions for the flow: (1) $r_c > r_{c+}$ or (2) $0 < r_c < r_{c-}$. We shall now solve (19) using (21) and then deduce a condition for the black-hole mass and charge.

To get solutions about the critical points, we substitute $r_{c\pm}$ in (21). For r_{c+}, (21) gives

$$M\sqrt{9M^2 - 8e^2} \geq 4e^2 - 3M^2, \tag{25}$$

which is satisfied if

$$\frac{M^2}{e^2} \leq 1, \tag{26}$$

and

$$\frac{M^2}{e^2} < \frac{4}{3}. \tag{27}$$

A comparison of inequalities (24), (26) and (27) implies

$$\frac{8}{9} \leq \frac{M^2}{e^2} < \frac{4}{3}. \tag{28}$$

It is interesting to note that these limits on the mass to charge ratio appear in the discussion of pseudo-Newtonian forces [28]. Thus, accretion through r_{c+} is possible if the above inequality (28) is satisfied. It encompasses the two types of black holes in itself: the regular and the extreme RN black hole. Interestingly, the naked singularity also falls within the prescribed limits. Thus, for all these spacetimes, the accretion is allowed through the critical point r_{c+}. We stress here that using $e = 0$ in the inequality (28) to retrieve the same condition for the Schwarzschild black hole may be misleading. The inequality is deduced using the outer apparent horizon and a critical point. Since the Schwarzschild black hole ($e \to 0$) possesses a unique horizon and a critical point, the above inequality cannot be reduced for an uncharged black hole.

Now we consider case (2), $0 < r_c < r_{c-}$. Substitution of r_{c-} in (21) gives

$$M\sqrt{9M^2 - 8e^2} \leq 3M^2 - 4e^2. \tag{29}$$

If $3M^2 - 4e^2 < 0$, then (29) does not yield any solution. So we need $3M^2 - 4e^2 > 0$, which yields

$$\frac{M^2}{e^2} > \frac{4}{3}. \tag{30}$$

Furthermore inequality (29) is satisfied if

$$\frac{M^2}{e^2} < 1. \tag{31}$$

Since (30) and (31) are mutually inconsistent, there is no solution for r_c in case (2). Thus, accretion is not possible through r_{c-}.

Since the mass of the black hole is decreasing by the accretion of the phantom energy (see (12)), it implies that at least one critical point must exist for the fluid flow, which is specified by r_{c+}. This critical point yields the mass-to-charge ratio of the black hole in the range specified by (28), which allows accretion onto all charged spherically symmetric black holes.

4 Conclusion

We have analyzed the effects of accretion of phantom energy onto a charged black hole. The analysis is performed using two critical points $r_{c\pm}$. It turns out that accretion is possible only through r_{c+}, which yields a constraint on the mass-to-charge ratio given by (28). This expression incorporates both extremal and non-extremal black holes. Thus all charged black holes will diminish near the big rip. Apparently this condition predicts the existence of large charges onto black holes, although astrophysically no evidence for such a case has been successfully deduced from the observations. In theory, the existence of large charges on black holes is consistently deduced by the general theory of relativity. It needs to be stressed that there is no analogous condition for the Schwarzschild black hole ($e = 0$). This analysis can be extended for a rotating charged black hole (a so-called Kerr–Neumann black hole) to get deeper insight in the accretion process. This work also serves as the generalization of the case studied by Michel [42] in terms of the accretion of phantom dark energy onto a charged black hole.

Acknowledgement One of us (MJ) would like to thank V. Dokuchaev and E. Babichev for enlightening discussions during this work. We would also thank the anonymous referees for giving useful comments to improve this work.

AQ is grateful to F. De Flice for useful discussions on naked singularities.

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Eur. Phys. J. C (2009) 62: 609–614
DOI 10.1140/epjc/s10052-009-1051-x

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Evolution of a Schwarzschild black hole in phantom-like Chaplygin gas cosmologies

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Received: 2 July 2008 / Revised: 18 November 2008 / Published online: 20 May 2009

Abstract In the classical relativistic regime, the accretion of phantom energy onto a black hole reduces the mass of the black hole. In this context, we have investigated the evolution of a Schwarzschild black hole in the standard model of cosmology using the phantom-like modified variable Chaplygin gas and the viscous generalized Chaplygin gas. The corresponding expressions for accretion time scale and evolution of mass have been derived. Our results indicate that the mass of the black hole will decrease if the accreting phantom Chaplygin gas violates the dominant energy condition and will increase in the opposite case. Thus, our results are in agreement with the results of Babichev et al. who first proposed this scenario.

1 Introduction

The evidence of accelerated expansion in the observable universe is quite compelling and has been confirmed by various astrophysical investigations including observations of supernovae of type Ia [1, 2], anisotropies of the cosmic microwave background radiation [3, 4], large scale structure and galaxy distribution surveys [5]. This expansion of the universe is supposedly driven by an exotic energy commonly called 'dark energy', possessing negative pressure $p < 0$ and positive energy density $\rho > 0$, related by the equation of state (EoS) $p = \omega\rho$. It should be noted that $p = \omega\rho$ is not a true EoS for dark energy, but rather a phenomenological description valid for a certain configuration [6]. Astrophysical data suggest that about two third of the critical energy density is stored in the dark energy component. The corresponding parameter ω is then constrained in the range $-1.38 < \omega < -0.82$ [7]. It shows that the EoS of cosmic fluids is not exactly determined. The genesis of this exotic energy is still unknown. The simplest and the earliest explanation of this phenomenon was provided by the general theory of relativity through the cosmological constant Λ. The observational value of its energy density is 56 to 120 orders of magnitude smaller than that derived from the standard theory [8]. The satisfactory explanation of this phenomenon requires extreme fine tuning of the cosmological parameters. Another problem associated with Λ is the coincidence problem (i.e. why the cosmic accelerated expansion started in the presence of intelligent beings, or alternatively, why the energy densities of matter and dark energy are of the same order at current time) which is as yet explained either through the anthropic principle [9], a variable cosmological constant scenario [10] or by invoking a dark matter–dark energy interaction [11–13]. In this context, several other models have been proposed; among them are models based on homogeneous and time dependent scalar fields termed as quintessence [14], quintom [15, 16] and k-essence [17], to name a few.

The interest in phantom energy arose when Caldwell et al. [18] explored the cosmological consequences of the EoS, $\omega < -1$. The dark energy can achieve this EoS if it is assumed to be a variable quantity i.e. $\omega(z)$, where z is the redshift parameter. Thus ω evolves as follows: for matter dominated universe $\omega = 0$, in quintessence phase $-1 < \omega \leq -1/3$, for the cosmological constant dominated arena $\omega = -1$, while in the phantom regime $\omega < -1$. This scenario appears to be consistent with the observations [19]. In phantom cosmology, the energy density of the phantom energy will become infinite in a finite time leading to the 'big rip', a kind of future singularity. Moreover, due to strong negative pressure of the phantom energy, all stable gravitationally bound objects will be dissociated near the big rip. These findings were later confirmed in [20] by doing numerical analysis for the solar system and the Milky Way galaxy. In this context, the accretion of phantom dark energy onto a black hole was first modeled by Babichev et

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al. [21] who proved that the black hole mass will gradually decrease due to a strong negative pressure of the phantom energy and will tend to zero near the big rip where it will finally disappear. Note that $\omega > -1$ leads to the opposite scenario where the black hole mass increases by accreting dark energy until its event horizon swells up to swallow the whole universe [22]. Later studies [23] showed that quantum effects dominate near the big rip singularity and consequently the mass of the black hole although decreases but stops decreasing at a finite value. In another investigation [24], it was demonstrated that the physical black hole mass will increase due to accretion of phantom energy; consequently the black hole horizon and the cosmological horizon will coincide leading to the black hole singularity becoming naked, all in a finite time. This analysis has been extended for the Riessner–Nördstrom, Kerr–Neumann and Schwarzschild–de Sitter black holes as well [25–30]. This result apparently refutes the cosmic censorship conjecture (or hypothesis) which forbids the occurrence of any naked singularity. However, the formation of naked singularities is not completely ruled out. Numerical simulations of the gravitational collapse of spheroids show that if the collapsing spheroid is sufficiently compact, the singularities are hidden inside the black hole while they become naked (devoid of an apparent horizon) if the spheroid is sufficiently large [31]. The future singularity of the big rip is alternatively avoided by the 'big trip', where the accretion of phantom energy onto a wormhole will increase the size of its throat so much as to engulf the whole universe [32, 33]. Another interesting scenario appears in cyclic cosmology where the masses of black holes first decrease and then increase through the phantom energy accretion and never vanish [34]. The implications of the generalized second law of thermodynamics to the phantom energy accretion onto a black hole imply that accretion will be significant only near the big rip. If this law is violated, then the black hole mass will decrease [35]. The thermodynamical investigations of the phantom energy imply its positive definite entropy, which tends to become constant if the phantom energy largely dominates the universe [36]. This results in the late universe to be hotter compared to the present.

We here discuss the accretion of a phantom-like modified variable Chaplygin gas and the viscous Chaplygin gas separately onto a black hole. This accretion of the phantom fluid reduces the mass of the black hole. This work serves as the generalization of the earlier work by Babichev et al. [21, 37] who initiated the concept of accretion of exotic matter on the black hole. We have built our model on the same pattern by choosing a more general EoS for the dark energy.

The outline of the paper is as follows: in the next section, we discuss the relativistic model of accretion onto a black hole. In the third section, we investigate the evolution of the mass of the black hole by the accretion of a modified variable Chaplygin gas (MVG), while in the fourth section, we discuss the similar scenario with the viscous generalized Chaplygin gas (VCG). Finally, we present the conclusion of our paper. The formalism adopted here is from [21].

2 Accretion onto black hole

We consider a Schwarzschild black hole of mass M which is gravitationally isolated and is specified by the line element (in geometrical units $c = 1 = G$):

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{1}$$

The black hole is accreting a Chaplygin gas, which is assumed to be a perfect fluid specified by the stress energy tensor

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}. \tag{2}$$

Here p and ρ are the pressure and energy density of the Chaplygin gas respectively. Due to the static and spherically symmetric nature of the black hole we assume the velocity four vector $u^\mu = (u^t(r), u^r(r), 0, 0)$, which satisfies the normalization condition $u^\mu u_\mu = -1$. Thus we are considering only radial in-fall of the Chaplygin gas on the event horizon. Using the energy-momentum conservation for $T^{\mu\nu}$, we get

$$ux^2(\rho + p)\sqrt{1 - \frac{2}{x} + u^2} = C_1, \tag{3}$$

where $x = r/M$ and $u = u^r = dr/ds$ is the radial component of the velocity four vector u^μ and C_1 is a constant of integration. The second constant of motion is obtained from $u_\mu T^{\mu\nu}_{;\nu} = 0$, which gives

$$ux^2 \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p'(\rho')}\right] = -A, \tag{4}$$

where A is a constant of integration, which is determined below for two models of the Chaplygin gas. The quantities ρ_∞ and ρ_h are the densities of the Chaplygin gas at infinity and at the black hole horizon respectively. Further, using (3) and (4), we obtain

$$(\rho + p)\sqrt{1 - \frac{2}{x} + u^2} \exp\left[-\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p'(\rho')}\right] = C_2, \tag{5}$$

where $C_2 = -C_1/A = \rho_\infty + p(\rho_\infty)$. In order to calculate $\dot{M}$, the rate of change of the mass of the black hole, we integrate the Chaplygin gas flux over the entire horizon as $\dot{M} = \oint T_t^r dS$ where T_t^r denotes the momentum density of Chaplygin gas in the radial direction and $dS = \sqrt{-g} d\theta\, d\varphi$

is the surface element of black hole horizon. Using (2)–(5), we get this rate of change as

$$\frac{dM}{dt} = 4\pi AM^2(\rho + p). \tag{6}$$

Integration of (6) yields

$$M = M_i\left(1 - \frac{t}{\tau}\right)^{-1}, \tag{7}$$

which determines the evolution of mass of black hole of initial mass M_i and τ is the characteristic accretion time scale given by

$$\tau^{-1} = 4\pi AM_i(\rho + p). \tag{8}$$

The number density and energy density of the Chaplygin gas are related as

$$\frac{n(\rho_h)}{n(\rho_\infty)} = \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho'}{\rho' + p'(\rho')}\right], \tag{9}$$

where $n(\rho_h)$ and $n(\rho_\infty)$ are the number densities of the Chaplygin gas at the horizon and at infinity, respectively. Furthermore, the constant A appearing in (8) is determined as

$$\frac{n(\rho_h)}{n(\rho_\infty)}ux^2 = -A, \tag{10}$$

which is an alternative form of energy-momentum conservation, (4). Moreover, the critical points of accretion (i.e. the points where the speed of flow reaches the speed of sound $V^2 = c_s^2 = \partial p/\partial\rho$) are determined as follows:

$$u_*^2 = \frac{1}{2x_*}, \qquad V_*^2 = \frac{u_*^2}{1 - 3u_*^2}, \tag{11}$$

where $V^2 \equiv \frac{n}{\rho+p}\frac{d(\rho+p)}{dn} - 1$. Finally, the above (9)–(11) are combined in a single expression as

$$\frac{\rho_* + p_*(\rho_*)}{n(\rho_*)} = \left[1 + 3c_s^2(\rho_*)\right]^{1/2}\frac{\rho_\infty + p(\rho_\infty)}{n(\rho_\infty)}. \tag{12}$$

3 Accretion of modified variable Chaplygin gas

The Chaplygin gas had been proposed to explain the accelerated expansion of the universe [38]. It is represented by a simple EoS, $p = -A'/\rho$, where A' is positive constant. The corresponding expression for the evolution of energy density is

$$\rho = \sqrt{A' + \frac{B}{a^6}}, \tag{13}$$

where B is a constant of integration. From (13) we find the following asymptotic behavior for the density [39]:

$$\rho \sim \sqrt{B}a^{-3}, \quad a \ll (B/A')^{1/6}, \tag{14}$$

$$\rho \sim p \sim \sqrt{A'}, \quad a \gg (B/A')^{1/6}. \tag{15}$$

Thus for small a, it gives a matter dominated era at earlier times, while for large a we get a dark energy dominated era at late times. Thus, the Chaplygin gas has the property of giving a unified picture of the evolution of the universe. The observational evidence in favor of cosmological models based on the Chaplygin gas is quite encouraging [40–43]. The Chaplygin gas model favors a spatially flat universe, which agrees with the observational data of the Sloan Digital Sky Survey (SDSS) and Supernova Legacy Survey (SNLS) with 95.4% confidence level [44]. Consequently, various generalizations of Chaplygin gas have been proposed in the literature to incorporate any other dark component in the universe (see e.g. [45–48] and references therein).

We here consider an equation of state which combines various EoS of the Chaplygin gas given by [49]

$$p = A'\rho - \frac{B(a)}{\rho^\alpha}, \quad B(a) = B_o a^{-m}. \tag{16}$$

Here A', B_o and m are constant parameters with $0 \le \alpha \le 1$. For $A' = 0$, (16) gives generalized Chaplygin gas. Further, if $B = B_o$ and $\alpha = 1$, it yields the usual Chaplygin gas. Also (16) reduces to the modified Chaplygin gas if only $B = B_o$. Moreover, if only $A' = 0$, the same equation represents a variable Chaplygin gas.

We consider the background spacetime to be spatially flat ($k = 0$), homogeneous and isotropic, represented by the Friedmann–Robertson–Walker (FRW) metric. The spacetime is assumed to contain only one component fluid i.e. the phantom energy represented by the Chaplygin gas EoS. The corresponding field equations are

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \kappa^2\rho, \tag{17}$$

$$\dot H + H^2 = \frac{\ddot a}{a} = -\frac{\kappa^2}{2}(\rho + 3p), \tag{18}$$

where $\kappa^2 = 8\pi/3$. The conservation of energy is

$$\dot\rho + 3H(\rho + p) = 0. \tag{19}$$

Using (16) and (19), the density evolution is given by

$$\rho = \left[\frac{3B_o(1+\alpha)}{\{3(1+\alpha)(1+A') - m\}}\frac{1}{a^m} + \frac{\Psi}{a^{3(1+\alpha)(1+A')}}\right]^{\frac{1}{1+\alpha}}. \tag{20}$$

Here Ψ is a constant of integration. Note that to obtain the increasing energy density of the phantom energy with respect to the scale factor $a(t)$, we require the coefficients of $a(t)$ in (20) to be positive i.e. $\Psi \geq 0$, $B_o(1+\alpha) > 0$ and $3(1+\alpha)(1+A') - m > 0$. Moreover, the exponents of $a(t)$ must be negative, i.e. $m < 0$ and $3(1+\alpha)(1+A') < 0$ to obtain an increasing ρ. These constraints together imply that $m > 3(1+\alpha)(1+A')$. Another way of getting a positive ρ is by setting $m > 0$, $1+A' > 0$ and $m < 3(1+\alpha)(1+A')$. Further, using (9) the ratio of the number density of the Chaplygin gas near the horizon and at infinity is calculated to be

$$\frac{n(\rho_h)}{n(\rho_\infty)} = \left[\frac{\rho_h^{1+\alpha}(1+A') - B(a)}{\rho_\infty^{1+\alpha}(1+A') - B(a)}\right]^{\frac{1}{(1+\alpha)(1+A')}} \equiv \Delta_1. \qquad (21)$$

Notice that the function $B(a)$ can be expressed in terms of ρ implicitly and is determined from (20). Making use of (11), the critical points of accretion are given by

$$u_*^2 = \frac{\Delta_2}{1+3\Delta_2}, \qquad x_* = \frac{1+3\Delta_2}{2\Delta_2}, \qquad (22)$$

where

$$V_*^2 = A' + \frac{\alpha B(a)}{\rho_*^{\alpha+1}} \equiv \Delta_2. \qquad (23)$$

Thus for the critical points to be finite and positive, we require either $\Delta_2 > 0$ or $\Delta_2 < 0$ and $\Delta_2 < -1/3$. For the accretion to be critical, the fluid velocity V^2 must become supersonic from the initial subsonic somewhere near the black hole horizon. For the MVG, we have $\omega = A' - B/\rho^{1+\alpha} < 0$, since $A' < -1$. One can observe that fluids having EoS $\omega < 0$ are hydrodynamically unstable i.e. the speed of sound in that medium cannot be defined since $c_s^2 < 0$. In order to overcome this problem Babichev et al. [50] proposed a redefinition of ω with the help of a generalized linear EoS given by $p = \beta(\rho - \rho_o)$, where β and ρ_o are constant parameters. Here $\beta > 0$ refers to a hydrodynamically stable fluid, while $\beta < 0$ corresponds to a hydrodynamically unstable fluid. We will not be interested in the latter case here. Note that now the two parameters ω and β are related by $\omega = \beta(\rho - \rho_o)/\rho$. Further, $\omega < 0$ now corresponds to $\beta > 0$ and $\rho_o > \rho$, thereby making the previously unstable fluid, now stable. We also have $c_s^2 \equiv \partial p/\partial \rho = \beta$. Since for stability, we require $\beta > 0$ and $0 < c_s^2 < 1$; it leads to $0 < \frac{1}{\rho - \rho_o}(A'\rho - B/\rho^\alpha) < 1$ and $0 < \beta < 1$. Hence the EoS parameter is now well defined with $A' < -1$ and $\rho_o > \rho$. Thus the stability of the phantom-like MVG is guaranteed with the use of a generalized linear EoS.

The constant A is determined from (10) to give

$$-A = \frac{\Delta_1}{4}\left(\frac{1+3\Delta_2}{\Delta_2}\right)^{3/2}. \qquad (24)$$

Using (8) the characteristic evolution time scale becomes

$$\tau^{-1} = \pi M_i(\rho + p)\frac{\Delta_1}{4}\left(\frac{1+3\Delta_2}{\Delta_2}\right)^{3/2}. \qquad (25)$$

Using (25) in (7), the black hole mass is given by

$$M(t) = M_i\left[1 - \pi M_i t(\rho + p)\frac{\Delta_1}{4}\left(\frac{1+3\Delta_2}{\Delta_2}\right)^{3/2}\right]^{-1}, \qquad (26)$$

which determines the evolution of mass of the black hole accreting phantom MVG. It can be seen that if the phantom MVG violates the dominant energy condition $\rho + p > 0$, then the mass M of the black hole will decrease. Contrary to this condition being satisfied M will increase. Thus in the classical relativistic regime, this result is in conformity with the result of Babichev et al. [21]. We also stress here that although our metric (1) is static, we get a dynamical mass $M(t)$ in (26). Astrophysically the mass of a black hole is a dynamical quantity: the mass will increase if the black hole accretes classical matter (which satisfies $\rho + p > 0$); however, it will decrease for the exotic phantom energy accretion. The mass can also decrease if the Hawking evaporation process is invoked. Hence the static black holes may not necessarily correspond to the astrophysical black holes. We also stress that $\omega > 0$ ($\omega < 0$) corresponds to a non-phantom (phantom) MVG fluid; although the accretion through the critical point is possible in both the cases, only phantom MVG violating the dominant energy condition will reduce the mass of the black hole.

4 Accretion of viscous generalized Chaplygin gas

In viscous cosmology, the presence of viscosity corresponds to space isotropy and hence it is important in the background of FRW spacetime [51–53]. The presence of a viscous fluid can explain the observed high entropy per baryon ratio in the universe [54]. It can cause an exponential expansion of the universe and can rule out the initial singularity which mars the standard big bang picture. The matter power spectrum in bulk viscous cosmology is also well behaved, as there are no instabilities or oscillations on a small perturbation scale [55]. Any cosmic fluid having non-zero bulk viscosities has the EoS $p_{\text{eff}} = p + \Pi$, where p is the usual isotropic pressure and Π is the bulk viscous stress given by $\Pi \equiv \xi(\rho)u^\mu_{;\mu}$ [56]. The scaling of the viscosity coefficient is $\xi = \xi_o \rho^n$ where n is a constant parameter and $\xi(t_o) = \xi_o$. Note that for $0 \leq n \leq 1/2$, we have a de Sitter solution and for $n > 1/2$ we get deflationary solutions. The viscosity coefficient is generally taken to be positive for a positive entropy production in conformity with the second law of thermodynamics [57]. Moreover, the entropy corresponding to viscous cosmology is always positive and increasing, which is consistent with the thermodynamic arrow of time. In fact the cosmological model with viscosity is consistent with the observational SN Ia data at lower redshifts, while it mimics the ΛCDM model in the later cosmic evolution [58]. It is proved in [59, 60] that a FRW spacetime filled with a perfect fluid and the bulk viscous stresses will violate the dominant energy condition.

Thus the effective pressure is given by

$$p_{\rm eff} \equiv p + \Pi, \tag{27}$$

where $\Pi = -3H\xi$ and $p = \chi/\rho^\alpha$ with χ is a constant. Thus in the VCG case, the standard FRW equation becomes [61]

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{2}(\rho + 3p_{\rm eff}). \tag{28}$$

Further the energy conservation principle gives

$$\dot{\rho} + 3H(\rho + p_{\rm eff}) = 0, \tag{29}$$

which shows that the viscosity term serves as the source term. Using (17) and (27) in (29), we get

$$\frac{a}{3}\frac{d\rho}{da} + \rho + \frac{\chi}{\rho^\alpha} - 3\kappa\xi(\rho)\sqrt{\rho} = 0. \tag{30}$$

Thus solving (30) we get

$$a(t) = a_o \exp\left[-\int_{\rho_o}^{\rho} \frac{\rho'^\alpha d\rho'}{\rho'^{\alpha+1} - 3\kappa\xi(\rho')\rho'^{\alpha+\frac{1}{2}} + \chi}\right]^{\frac{1}{3}}. \tag{31}$$

For our further analysis we shall assume ξ to be constant.

The ratio of the number density of VCG near the black hole horizon and at infinity is given by

$$\frac{n(\rho_h)}{n(\rho_\infty)} = \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{\rho'^\alpha d\rho'}{\rho'^{\alpha+1} - 3\kappa\xi\rho'^{\alpha+\frac{1}{2}} + \chi}\right] \equiv \Delta_3. \tag{32}$$

The corresponding critical points of accretion are

$$u_*^2 = \frac{\Delta_4}{3\Delta_4 - 1}, \qquad x_* = \frac{3\Delta_4 - 1}{2\Delta_4}, \tag{33}$$

where

$$V_*^2 = -\left(\frac{\alpha\chi}{\rho_*^{\alpha+1}} + \frac{3}{2\sqrt{\rho_*}}\kappa\xi\right) \equiv \Delta_4. \tag{34}$$

Notice that for the critical points to be finite and positive valued we require either $\Delta_4 < 0$ or $\Delta_4 > 1/3$. Using (11) we see that the speed of flow at the critical point is $V^2 = -\Delta_4$. Further, the EoS parameter is $\omega = \chi/\rho^{1+\alpha} - 3\xi H/\rho$ $(= \chi/\rho^{1+\alpha} - \sqrt{3}\kappa\xi/\sqrt{\rho})$. Note that if $\chi < 0$ then $\omega < 0$ and stability of the VCG is lost. However, if we here invoke the argument presented in the last section, we can consider accretion with $\omega < 0$. Using the generalized linear EoS $p = \beta(\rho - \rho_o)$ for the phantom energy, we obtain $\beta > 0$ and $\rho_o > \rho$ for $\omega < 0$. Using the definition $c_s^2 \equiv \partial p/\partial\rho = \beta$ and the stability requirements $\beta > 0$ and $0 < c_s^2 < 1$ lead to $0 < \frac{1}{\rho - \rho_o}(\chi/\rho^\alpha - \sqrt{3\rho}\kappa\xi) < 1$ and $0 < \beta < 1$. The EoS parameter β is now well defined with $\chi < 0$ and $\rho_o > \rho$. Therefore the stability of the phantom-like VCG is ensured with the use of a generalized linear EoS.

Using (10) the constant A is now determined to be

$$-A = \Delta_3\left(\frac{3\Delta_4 - 1}{2\Delta_4}\right)^{3/2}. \tag{35}$$

The characteristic evolution time scale is

$$\tau^{-1} = 4\pi M_i(\rho + p)\Delta_3\left(\frac{3\Delta_4 - 1}{2\Delta_4}\right)^{3/2}. \tag{36}$$

Using (35) and (36) in (7), we get black hole mass evolution as

$$M(t) = M_i\left[1 - 4\pi M_i t(\rho + p)\Delta_3\left(\frac{3\Delta_4 - 1}{2\Delta_4}\right)^{3/2}\right]^{-1}. \tag{37}$$

It can be seen that the black hole mass will decrease when $\rho + p < 0$ and increase in the opposite case. It is emphasized that this result is valid till the contribution of viscous stress is negligible compared to isotropic stress. For the sake of clarity, we emphasize that the fluid violating the standard energy conditions is termed ‘exotic’ and hydrodynamically unstable, i.e. its existence is not fully guaranteed. But this conclusion is drawn due to the ‘bad’ choice of the EoS ($p = \omega\rho$) in the analysis. The result is reversed and remedied when we introduce the generalized linear EoS in our model, which makes the accretion of exotic fluid much more practical.

5 Conclusion

We have investigated the accretion of two different forms of phantom-like Chaplygin gas onto a Schwarzschild black hole. The time scale of the accretion and the evolution of the mass of the black hole are derived in the context of two widely studied Chaplygin gas models, namely the modified variable Chaplygin gas and the viscous generalized Chaplygin gas. Although the phantom energy is an unstable fluid as it corresponds to a medium with indeterminate speed of sound and super-luminal speeds. These pathologies arise due to bad choices of the equations of state for the phantom energy and hence can be removed by choosing some suitable transformation from one EoS to another or a totally new EoS for this purpose. This work serves as the generalization of the earlier work by Babichev et al. [21]. It should be noted that we have ignored the matter component in the accretion model. Thus it will be more insightful to incorporate the contributions of matter along with the Chaplygin gas during accretion onto the black hole. Moreover our analysis can be extended to the case of rotating black holes as well.

Acknowledgements I would like to thank Muneer Ahmad Rashid for enlightening discussions during this work. Useful criticism on this work from the anonymous referee is also gratefully acknowledged.

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DOI 10.1007/s10714-010-1024-2

RESEARCH ARTICLE

Generalized second law of thermodynamics for a phantom energy accreting BTZ black hole

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Received: 1 December 2009 / Accepted: 19 May 2010

Abstract In this paper, we have studied the accretion of phantom energy on a (2 + 1)-dimensional stationary Banados–Teitelboim–Zanelli (BTZ) black hole. It has already been shown by Babichev et al. that for the accretion of phantom energy onto a Schwarzschild black hole, the mass of black hole would decrease and the rate of change of mass would be dependent on the mass of the black hole. However, in the case of (2 + 1)-dimensional BTZ black hole, the mass evolution due to phantom accretion is independent of the mass of the black hole and is dependent only on the pressure and density of the phantom energy. We also study the generalized second law of thermodynamics at the event horizon and construct a condition that puts an lower bound on the pressure of the phantom energy.

Keywords BTZ black hole · Accretion · Dark energy · Generalized second law of thermodynamics

1 Introduction

It has been found by various astronomical and cosmological observations [1–5] that our universe is currently in the phase of accelerated expansion. In the framework of Einstein's gravity, this accelerated expansion has been explained by the presence of a 'cosmological constant' bearing negative pressure which results in the stretching of the spacetime [6–9]. Many other theoretical models have been presented to explain

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Published online: 09 June 2010

the present accelerated expansion of the universe including based on homogeneous and time dependent scalar field like the quintessence [10–14], Chaplygin gas [15–26] and phantom energy [27–33], to name a few. The phantom energy is characterized by the equation of state $p = \omega\rho$, with $\omega < -1$. It possesses some weird properties: the cosmological parameters like energy density and scale factor become infinite in a finite time; all gravitationally bound objects lose mass with the accretion of phantom energy; the fabric of spacetime is torn apart at the big rip; and that it violates the standard relativistic energy conditions. The astrophysical data coming from the microwave background radiation categorically favors the phantom energy [34]. Motivated from the dark energy models, we model phantom energy by an ideal fluid with negative pressure.

The accretion of dark energy onto a black hole has been studied by many authors [35–40] after the seminal work of Babichev et al. [41,42] who have shown that the mass of the black hole will decrease with time when we consider the accretion of phantom energy. In the Einstein theory of gravity, the accretion of the phantom energy onto Schwarzschild black hole and evaporation of primordial black hole has been discussed [41–43]. It will be interesting to investigate the accretion dynamics in low and higher dimensional gravities. It is also important to investigate accretion dynamics in the extended theories of gravity.

In this paper we investigate the accretion of exotic phantom energy onto a static uncharged 3-dimensional BTZ black hole. We will show that the expression of the evolution of BTZ black hole mass is independent of its mass and dependents only on the energy density and pressure of the phantom energy. It is well-known that the horizon area of the black hole decreases with the accretion of phantom energy, hence it is essential to study the generalized second law of thermodynamics (GSL) in this case [44–46]. We show that the validity of GSL in the present model yields an lower bound on the phantom energy pressure. We also demonstrate that the first law of thermodynamics holds in the present construction.

The plan of the paper is as follows: In second section we model the accretion of phantom energy onto three dimensional BTZ black hole. In third section, we study the GSL for BTZ black hole. Finally we conclude our results.

2 Model of accretion

Consider the field equations for a (2 + 1)-dimensional spacetime with a negative cosmological constant Λ

$$G_{ab} + \Lambda g_{ab} = \pi T_{ab}, \quad (a, b = 0, 1, 2) \tag{1}$$

where G_{ab} is the Einstein tensor in (2 + 1)-dimension while T_{ab} is the stress energy tensor of the matter field. The units are chosen such that $c = 1$ and $G_3 = 1/8$. Considering the stress-energy tensor to be vacuum, one can obtain the following spherically symmetric metric, a (2+1)-dimensional BTZ black hole [47,48]

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\phi^2, \tag{2}$$

where $f(r) = -M + r^2/l^2$, M is the dimensionless mass of the black hole and $l^2 = -1/\Lambda$, is a positive constant. The coefficient g_{00} is termed as the lapse function. The event horizon of the BTZ black hole is obtained by setting $f(r) = 0$, which turns out, $r_e = l\sqrt{M}$. Also we have $\sqrt{|g|} = r$, where g is the determinant of the metric. To analyze the accretion of phantom energy onto the BTZ black hole, we here employ the formalism from the work by Babichev et al. [41,42]. The stress energy momentum tensor representing the phantom energy is the perfect fluid

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}, \tag{3}$$

Here ρ and p are the energy density and pressure of the phantom energy while $u^a = (u^0, u^1, 0)$ is the velocity three vector of the fluid flow. Also $u^1 = u$ is the radial velocity of the flow while the third component u^2 is zero due to spherical symmetry of the BTZ black hole. There are two important equations of motion in our model: one which controls the conservation of mass flux is $J^a_{;a} = 0$, where J^a is the current density and the other that controls the energy flux $T^a_{0;a} = 0$, across the horizon. Since the black hole is stationary, the only component of stress energy tensor of interest is T^{01}. Thus the equation of energy conservation $T^{0a}_{;a} = 0$ is

$$ur(\rho + p)\sqrt{f(r) + u^2} = C_1, \tag{4}$$

where C_1 is an integration constant. Since the flow is inwards the black hole therefore $u < 0$. Also the projection of the energy momentum conservation along the velocity three vector $u_a T^{ab}_{;b} = 0$ (the energy flux equation) is

$$ur \exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p}\right] = -A_1. \tag{5}$$

Here A_1 is a constant and the associated minus sign is taken for convenience. Also ρ_h and ρ_∞ are the energy densities of phantom energy at the BTZ horizon and at infinity respectively. From Eqs. (4) and (5), we obtain

$$(\rho + p)\sqrt{f(r) + u^2} \exp\left[-\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p}\right] = C_2, \tag{6}$$

where $C_2 = -C_1/A_1 = \rho_\infty + p(\rho_\infty)$. The rate of change in the mass of black hole $\dot{M} = -2\pi r T^1_0$, is given by

$$dM = 2\pi A_1(\rho_\infty + p_\infty)\,dt. \tag{7}$$

Note that $\rho_\infty + p_\infty < 0$ (violation of null energy condition) leads to decrease in the mass of the black hole. Moreover, the above expression is also independent of mass contrary to the Schwarzschild black hole and the Reissner–Nordström black hole [35–40]. Further, the last equation is valid for any general ρ and p violating the null energy condition, thus we can write

$$dM = 2\pi A_1(\rho + p)\, dt. \tag{8}$$

3 Critical accretion

We are interested only in those solutions that pass through the critical point as these correspond to the material falling into the black hole with monotonically increasing speed. The falling fluid can exhibit variety of behaviors near the critical point of accretion, close to the compact object. The equation of mass flux or the continuity equation $J^a_{;a} = 0$ is

$$\rho u r = k_1. \tag{9}$$

Here k_1 is integration constant. From Eqs. (4) and (9), we have

$$\left(\frac{\rho + p}{\rho}\right)^2 \left(f(r) + u^2\right) = \left(\frac{C_1}{k_1}\right)^2 = C_3. \tag{10}$$

Taking differentials of (9) and (10) and after simplification, we obtain

$$\frac{du}{u}\left[-V^2 + \frac{u^2}{f(r) + u^2}\right] + \frac{dr}{r}\left[-V^2 + \frac{r^2}{l^2\left(f(r) + u^2\right)}\right] = 0. \tag{11}$$

Here

$$V^2 \equiv \frac{d\ln(\rho + p)}{d\ln\rho} - 1, \tag{12}$$

From (11) if one or the other bracket factor is zero, one gets a turnaround point corresponding double-valued solution in either r or u. The only solution that passes through a critical point is feasible. The feasible solution will correspond to material falling into the object with monotonically increasing velocity. The critical point is obtained by taking the both bracketed factors in Eq. (11) to be zero. This will give us the critical points of accretion. We obtain

$$V_c^2 = \frac{r_c^2}{(f(r_c) + u_c^2)l^2}, \tag{13}$$

$$V_c^2 = \frac{u_c^2}{f(r_c) + u_c^2}. \tag{14}$$

Above the subscript c refers to the critical quantity. On comparing Eqs. (13) and (14), we get

$$u_c^2 = \frac{r_c^2}{l^2}, \quad V_c^2 = \frac{u_c^2}{-M + 2u_c^2}. \tag{15}$$

Here u_c is the critical speed of flow at the critical points which we determine below. For physically acceptable solution, we require $V_c^2 > 0$, hence we get the following restrictions on speeds and the location of the critical points

$$u_c^2 > \frac{M}{2}, \quad r_c^2 > \frac{r_+^2}{2}. \tag{16}$$

4 Generalized second law of thermodynamics and BTZ black hole

In this section we will discuss the thermodynamic of phantom energy accretion that crosses the event horizon of BTZ black hole. Let us first write the BTZ metric in the form

$$ds^2 = h_{mn}\, dx^m\, dx^n + r^2\, d\phi^2, \quad m, n = 0, 1 \tag{17}$$

where $h_{mn} = \mathrm{diag}(-f(r), 1/f(r))$, is a 2-dimensional metric. From the condition of normalized velocities $u^a u_a = -1$, one can obtain the relations

$$u^0 = f(r)^{-1}\sqrt{f(r) + u^2}, \quad u_0 = -\sqrt{f(r) + u^2}. \tag{18}$$

The components of stress energy tensor are $T^{00} = f(r)^{-1}[(\rho + p)(\frac{f(r)+u^2}{f(r)}) - p]$, and $T^{11} = (\rho + p)u^2 + f(r)p$. These two components help us in calculating the work density which is defined by $W = -\frac{1}{2}T^{mn}h_{mn}$ [49]. In our case it comes out

$$W = \frac{1}{2}(\rho - p). \tag{19}$$

The energy supply vector is defined by

$$\Psi_n = T_n^m \partial_m r + W \partial_n r. \tag{20}$$

The components of the energy supply vector are $\Psi_0 = T_0^1 = -u(\rho + p)\sqrt{f(r) + u^2}$, and $\Psi_1 = T_1^1 + W = (\rho + p)(\frac{1}{2} + \frac{u^2}{f(r)})$. The change of energy across the apparent horizon is determined through $-dE \equiv -A\Psi$, where $\Psi = \Psi_0 dt + \Psi_1 dr$. The energy crossing the event horizon of the BTZ black hole is given by

$$dE = 4\pi r_e u^2(\rho + p)\, dt. \tag{21}$$

Assuming $E = M$ and comparing Eqs. (8) and (21), we can determine the value of constant $A_1 = 2u^2 l\sqrt{M}$.

The entropy of BTZ black hole is

$$S_h = 4\pi r_e. \tag{22}$$

It can be shown easily that the thermal quantities, change of phantom energy dE, horizon entropy S_h and horizon temperature T_h satisfy the first law $dE = T_h dS_h$, of thermodynamics. After differentiation of last equation w.r.t. t, and using Eq. (8), we have

$$\dot{S}_h = 8\pi^2 l^2 u^2(\rho + p). \tag{23}$$

Since all the parameters are positive in the above Eq. (23) except that $\rho + p < 0$, it shows that the second law of thermodynamics is violated i.e. $\dot{S}_h < 0$, as a result of accretion of phantom energy on a BTZ black hole.

Now we proceed to the generalized second law of thermodynamics (GSL). It is defined by

$$\dot{S}_{tot} = \dot{S}_h + \dot{S}_{ph} \geq 0. \tag{24}$$

In other words, the sum of the rate of change of entropies of black hole horizon and phantom energy must be positive. We consider event horizon of the BTZ black hole as a boundary of thermal system and the total matter energy within the event horizon is the mass of the BTZ black hole. We also assume that the horizon temperature is in equilibrium with the temperature of the matter-energy enclosed by the event horizon, i.e. $T_h = T_{ph} = T$, where T_{ph} is the temperature of the phantom energy. Similar assumptions for the temperatures T_h and T_{ph} has been studied in [50–54]. We know that the Einstein field equations satisfy first law of thermodynamics $T_h dS_h = pdA + dE$, at the event horizon [55,56]. We also assume that the matter-energy enclosed by the event horizon of BTZ black hole also satisfy the first law of thermodynamics given by

$$T_{ph}\, dS_{ph} = p\, dA + dE. \tag{25}$$

Here the horizon temperature is given by

$$T_h = \left.\frac{f'(r)}{4\pi}\right|_{r=r_e} = \frac{\sqrt{M}}{2\pi l}. \tag{26}$$

In this paper, we are assuming that $T_h = T_{ph} = T$. Therefore Eq. (24) gives

$$T\dot{S}_{tot} = T(\dot{S}_h + \dot{S}_{ph}) = 4\pi l^2 u(\rho + p)(2\sqrt{M} + \pi l p). \tag{27}$$

From the above equation, note that $u < 0$ and $\rho + p < 0$ the GSL holds provided $2\sqrt{M} + \pi l p > 0$ which implies

$$p \geq -\frac{2\sqrt{M}}{\pi l}. \tag{28}$$

Since the pressure of the phantom energy is negative ($p < 0$), therefore the GSL gives us the lower bound on the pressure of the phantom energy.

$$-\frac{2\sqrt{M}}{\pi l} \leq p < 0. \tag{29}$$

The GSL in the phantom energy accretion holds within the inequality (29). Otherwise GSL does not hold which forbid evaporation of BTZ black hole by the phantom accretion [57,58]. In addition, it is not clear whether the GSL should be valid in presence of the phantom fluid not respecting the dominant energy condition [57,58].

5 Conclusion

In this paper, we have investigated the accretion of exotic phantom energy onto a BTZ black hole. The motivation behind this work is to study the accretion dynamics in low dimensional gravity. Our analysis has shown that evolution of mass of a BTZ black hole would be independent of its mass and will be dependent only on the energy density and pressure of the phantom energy in its vicinity. Due to spherical symmetry, the accretion process is simple since the phantom energy falls radially on the black hole. The accretion would be much more interesting when additional parameters like charge and angular momentum are also incorporated in the BTZ spacetime. Similarly, it would be of much interest to perform the above analysis in higher $(n+1)$ dimensional black hole spacetimes.

We also discussed GSL in the BTZ black hole spacetime. We assumed that the event horizon of BTZ black hole acts as a boundary of the thermal system and the phantom energy crossing the event horizon will change the mass of the black hole. We assumed that the horizon temperature is in local equilibrium with the temperature of the matter energy at the event horizon. Under these constraints it is shown that the GSL holds provided the pressure of the phantom energy p has an lower bound $p \geq -\frac{2\sqrt{M}}{\pi l}$, on the black hole parameters (M and l).

Acknowledgments We would like to thank NUST for providing us financial support to visit ICRANet, Pescara, Italy to present this paper at the Second Joint Italian-Pakistani Workshop on Relativistic Astrophysics. We would also thank anonymous referees for their useful comments on this work. Also MJ would thank Emmanuel N. Saridakis, H. Mohseni Sadjadi, Diego Pavon and Ahmad Sheykhi for enlightening discussions during this work.

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Int J Theor Phys (2010) 49: 621–632
DOI 10.1007/s10773-009-0242-4

Black Holes in Bulk Viscous Cosmology

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Received: 27 October 2009 / Accepted: 23 December 2009 / Published online: 31 December 2009

Abstract We investigate the effects of the accretion of phantom energy with non-zero bulk viscosity onto a Schwarzschild black hole and show that black holes accreting viscous phantom energy will lose mass rapidly compared to the non-viscous case. When matter is incorporated along with the phantom energy, the black holes meet with the same fate as bulk viscous forces dominate matter accretion. If the phantom energy has large bulk viscosity, then the mass of the black hole will reduce faster than in the small viscosity case.

Keywords Accretion · Black hole · Bulk viscosity · Phantom energy

1 Introduction

Observations of WMAP [1–3] and supernova of type Ia data [4] have revealed that our Universe is filled with an exotic dark energy apart from dark matter. The nature and composition of this energy is still an open problem but its dynamics is well understood i.e. it causes an approximately exponential expansion of the Universe (see [5–12] for recent reviews on dark energy). Astrophysical data suggest that about two thirds of the critical energy density is stored in the dark energy component. For the equation of state (EoS) parameter $\omega < -1$, the fluid is called phantom energy (PE). Observations show that ω is constrained in the range $-1.38 < \omega < -0.82$ [13], thus providing evidence of phantom energy in the Universe. The PE violates all the energy conditions in all forms (weak, null, strong or dominant). The phantom energy can cause some peculiar phenomena e.g. the existence of wormholes [14,

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15], infinite expansion of the Universe in a finite time causing a Big Rip (BR) and the destruction of all gravitationally bound structures including black holes [16–19]. In particular, black holes will continuously lose mass and disappear near the BR (see [20, 21] for the opposite viewpoint).

Dark energy with bulk viscosity has a peculiar property to cause accelerated expansion of phantom type in the late evolution of the universe [22–24]. It can also alleviate several cosmological puzzles like cosmic age problem [25], coincidence problem [26] and phantom crossing [27]. We will consider phantom energy as an imperfect fluid, implying that the PE could contain non-zero bulk and shear viscosities [28]. The bulk viscosities are negligible for non-relativistic and ultra-relativistic fluids but are important for the intermediate cases. In viscous cosmology, shear viscosities arise in relation to space anisotropy while the bulk viscosity accounts for the space isotropy [22–24, 29]. Generally, shear viscosities are ignored (as the CMB does not indicate significant anisotropies) and only bulk viscosities are taken into account for the fluids in the cosmological context. Moreover, bulk viscosity related to a grand unified theory phase transition may lead to an explanation of the accelerated cosmic expansion [30].

Babichev et al. [17] studied the effects of the accretion of phantom energy onto a Schwarzschild black hole taking PE to be a perfect fluid. As a first order approximation, the bulk viscosity can be ignored, but to get a better picture we need to incorporate it into the phantom fluid. We have adopted the procedure of [17, 31] for our calculations.

The plan of the paper is as follows: in the next section we review viscous cosmology; in section three we discuss the relativistic model of accretion onto a black hole; in the subsequent section we use results from viscous cosmology for the accretion model; next we give two examples to illustrate the accretion process with a constant and power law viscosity. In section six we study black hole evolution in the presence of matter and viscous phantom energy. Finally we conclude the paper with a brief discussion of our results.

2 Bulk-Viscous Cosmology

We assume the background spacetime to be homogeneous, isotropic and spatially flat ($k = 0$) and described by the Friedmann-Robertson-Walker (FRW) metric given by

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{1}$$

where $a(t)$ is the scale factor. We also assume that the spacetime is filled with only one component fluid i.e. the viscous phantom energy of energy density ρ (however, in section six, we shall incorporate matter along with phantom energy). The Einstein field equations for the FRW-metric (in the units $c = 1 = 8\pi G$) are

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho, \tag{2}$$

and

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6}(\rho + 3p), \tag{3}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, p is the effective pressure containing the isotropic pressure p_{pe} and the bulk viscous pressure p_{vis}, given by

$$p = p_{\text{pe}} + p_{\text{vis}}. \tag{4}$$

Here $\rho = \rho_{\mathrm{pe}} + \rho_{\mathrm{vis}}$ and $p_{\mathrm{vis}} = -\xi u^{\mu}_{;\mu}$, where u^{μ} is the velocity four vector and $\xi = \xi(\rho_{\mathrm{vis}}, t)$ is the bulk viscosity of the fluid [32]. Equation (4) shows that negative pressure due to viscosity contributes in the effective pressure which causes accelerated expansion. In the FRW model, the expression $u^{\mu}_{;\mu} = 3\dot{a}/a$ holds. Also, ξ is generally taken to be positive in order to avoid the violation of second law of thermodynamics [33].

The energy conservation equation is

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{5}$$

Assume that the viscous fluid equation of state (EoS) is

$$p = \omega\rho = (\gamma - 1)\rho. \tag{6}$$

Note that if $\gamma = 0$ (or $\omega = -1$), (6) represents the EoS for cosmological constant. Furthermore if $\gamma < 0$, it represents phantom energy. In general, for normal matter $1 \leq \gamma < 2$.

Using (2)–(6), we get the equation governing the evolution of $H(t)$ for a given ξ as

$$2\dot{H} + 3\gamma H^2 - 3\xi H = 0. \tag{7}$$

On integration, (7) gives

$$H(t) = \frac{\exp\{\frac{3}{2}\int \xi(t)dt\}}{C + \frac{3}{2}\gamma \exp\{\frac{3}{2}\int \xi(t)dt\}}, \tag{8}$$

where C is a constant of integration. Note that (8) can further be solved to get the evolution of $a(t)$ as

$$a(t) = D\left(C + \frac{3}{2}\gamma \int \exp\left\{\frac{3}{2}\int \xi(t)dt\right\}dt\right)^{\frac{2}{3\gamma}}, \tag{9}$$

where D is a constant of integration. Thus for a given value of ξ we can obtain expressions of $a(t)$, $\rho(t)$ and $p(t)$ from the system of (5)–(9).

3 Accretion onto Black Hole

In the background of FRW spacetime, we consider, as an approximation, a gravitationally isolated Schwarzschild black hole (BH) of mass M whose metric is specified by the line element:

$$ds^2 = -\left(1 - \frac{M}{4\pi r}\right)dt^2 + \left(1 - \frac{M}{4\pi r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{10}$$

The background spacetime is assumed to contain one test fluid, namely the phantom energy with non-vanishing bulk viscous stress p_{vis}. The fluid is assumed to fall onto the BH horizon in the radial direction only which is in conformity with the spherical symmetry of the BH. Thus, the velocity four vector of the phantom fluid is $u^{\mu} = (u^t(r), u^r(r), 0, 0)$ which satisfies the normalization condition $u^{\mu}u_{\mu} = -1$. This phantom fluid is specified by the stress energy tensor for a viscous fluid [28, 33]:

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}. \tag{11}$$

Using the energy momentum conservation for $T^{\mu\nu}$, we get

$$ur^2M^{-2}(\rho+p)\sqrt{1-\frac{M}{4\pi r}+u^2}=C_1, \tag{12}$$

where $u^r=u=dr/ds$ is the radial component of the velocity four vector and C_1 is a constant of integration. The second constant of motion is obtained by contracting the velocity four vector of the phantom fluid with the stress energy tensor $u_\mu T^{\mu\nu}_{;\nu}=0$, which gives

$$ur^2M^{-2}\exp\left[\int_{\rho_\infty}^{\rho_h}\frac{d\rho'}{\rho'+p(\rho')}\right]=-A, \tag{13}$$

where A is a constant of integration. Also ρ_h and ρ_∞ are the energy densities of the phantom fluid at the horizon of the BH, and at infinity respectively. From (12) and (13) we have

$$(\rho+p)\sqrt{1-\frac{M}{4\pi r}+u^2}\exp\left[-\int_{\rho_\infty}^{\rho_h}\frac{d\rho'}{\rho'+p(\rho')}\right]=C_2, \tag{14}$$

with $C_2=-C_1/A=\rho_\infty+p(\rho_\infty)$. In order to calculate the rate of change of mass of black hole $\dot{M}$, we integrate the flux of the bulk viscous phantom fluid over the entire BH horizon to get

$$\dot{M}=\oint T^r_t\,dS. \tag{15}$$

Here T^r_t determines the energy momentum flux in the radial direction only and $dS=\sqrt{-g}d\theta d\varphi$ is the infinitesimal surface element of the BH horizon. Using (12)–(15), we get

$$\frac{dM}{dt}=\frac{AM^2}{16\pi}(\rho+p), \tag{16}$$

which clearly demonstrates the vanishing mass of the black hole if $\rho+p<0$. Integration of (16) leads to

$$M=M_0\left(1-\frac{t}{\tau}\right)^{-1}, \tag{17}$$

where M_0 is the initial mass of the black hole and modified characteristic accretion time scale $\tau^{-1}=[\frac{AM_0}{16\pi}\{(\rho_{\mathrm{pe}}+p_{\mathrm{pe}})-\frac{3\xi}{t}\ln(\frac{a}{a_0})\}]$, a_0 being the value of the scale factor at time t_0. Note that during the integration of (16), we assumed ρ_{pe} and p_{pe} to be constants. In the coming subsections, we shall take these as time dependent entities.

4 Accretion of Viscous Phantom Fluid

We now study the BH mass evolution in two special cases: (a) constant viscosity; and (b) power law viscosity.

4.1 Constant Bulk Viscosity

For constant viscosity $\xi=\xi_o$, the evolution of $a(t)$ is determined by using (9). It gives

$$a(t)=a_0\left[1+\frac{\gamma H_oB(t)}{\xi_o}\right]^{\frac{2}{3\gamma}}, \tag{18}$$

where

$$B(t) \equiv \exp\left(\frac{3t\xi_o}{2}\right) - 1. \tag{19}$$

Using (5), (6) and (8) the density evolution is given by

$$\rho(t) = \frac{\rho_o \exp(3\xi_o t)}{[1 + \frac{\gamma H_o B(t)}{\xi_o}]^2}. \tag{20}$$

Here $\rho_o = 3H_o^2$. Further, for $\gamma < 0$ the BR singularity occurs in a finite time at

$$\tau = \frac{2}{3\xi_o} \ln\left(1 - \frac{\xi_o}{H_o \gamma}\right). \tag{21}$$

Finally, the BH mass evolution is determined by solving (16) and (20) to get

$$M = M_0 \left[1 - \frac{AM_0}{8\pi\gamma}\left(\frac{\xi_o}{\Delta} - 1\right)(\xi_o - \gamma H_o)\right]^{-1}, \tag{22}$$

where

$$\Delta \equiv \xi_o + (-1 + e^{\frac{3t\xi_o}{2}})\gamma H_o. \tag{23}$$

This mass is displayed for different values of viscosity at different times in Table 1.

It is apparent from Table 1 that for a fixed viscosity, the mass ratio decreases with time implying that mass of black hole is decreasing for an initial mass. Similarly, at any given time, the mass ratio also decreases with the increase in viscosity. Thus the greater the value of viscosity parameter, the greater would be its effects on the BH mass.

4.2 Power Law Viscosity

If the viscosity has power law dependence upon density i.e. $\xi = \alpha\rho_{\text{vis}}^s$, where α and s are constant parameters, it has been shown [35, 36] that it yields cosmologies with a BR if $\sqrt{3\alpha} > \gamma$ and $s = 1/2$. Thus we take $\xi = \alpha\rho^{\frac{1}{2}}$ as a special case. Then the scale factor evolves as

$$a(t) = a_0\left(1 - \frac{t}{\tau}\right)^{\frac{2}{3(\gamma - \sqrt{3}\alpha)}}. \tag{24}$$

Table 1 The mass ratio M/M_0 of black hole for different choices of constant viscosity ξ_o. The initial mass is, throughout, taken to be $50M_\odot$ or 10^{32} kg

$t \downarrow \xi \rightarrow$	$\xi_1 = 10^{-17}$	$\xi_2 = 10^{-18}$	$\xi_3 = 10^{-19}$	$\xi_4 = 10^{-20}$
$t_1 = 10^7$	3.43427×10^{-4}	2.44662×10^{-3}	6.31285×10^{-3}	7.49184×10^{-3}
$t_2 = 10^{10}$	3.43544×10^{-7}	2.45261×10^{-6}	6.35248×10^{-6}	7.55357×10^{-6}
$t_3 = 10^{13}$	3.43516×10^{-10}	2.45258×10^{-9}	6.35247×10^{-9}	7.55358×10^{-9}
$t_4 = 10^{17}$	1.23994×10^{-14}	2.10182×10^{-13}	5.86096×10^{-13}	7.01997×10^{-13}

Table 2 The mass ratio M/M_0 of black hole for different choices of equation of state. The initial mass is $50M_{\odot}$ or 10^{32} kg

$t\downarrow \gamma \rightarrow$	$\gamma_1 = -1\times 10^{-1}$	$\gamma_2 = -2\times 10^{-1}$	$\gamma_3 = -3\times 10^{-1}$	$\gamma_4 = -4\times 10^{-1}$
$t_1 = 10^{10}$	4.71915×10^{-5}	2.35963×10^{-5}	1.5731×10^{-5}	1.17983×10^{-5}
$t_2 = 10^{13}$	4.79136×10^{-8}	2.35968×10^{-8}	1.57312×10^{-8}	1.17984×10^{-8}
$t_3 = 10^{17}$	4.66492×10^{-12}	2.30523×10^{-12}	1.51867×10^{-12}	1.12539×10^{-12}
$t_4 = 10^{20}$	4.97349×10^{-14}	5.20946×10^{-14}	5.28811×10^{-14}	5.32744×10^{-14}

The density of phantom fluid evolves as

$$\rho(t) = \frac{4}{3\tau^2(\gamma - \sqrt{3}\alpha)^2}\left(1 - \frac{t}{\tau}\right)^{-2}, \tag{25}$$

or in terms of critical density ρ_{cr} as

$$\rho(t) = \rho_{\mathrm{cr}}\left(1 - \frac{t}{\tau}\right)^{-2}. \tag{26}$$

The corresponding BR time τ is given by

$$\tau = \frac{2}{3(\sqrt{3}\alpha - \gamma)}H_o^{-1}. \tag{27}$$

Finally, the mass evolution of BH is determined by using (16) and (25) is

$$M = M_0\left[1 + \frac{AM_0}{4\pi(\sqrt{3}\alpha - \gamma)}\frac{t}{\tau(\tau - t)}\right]^{-1}. \tag{28}$$

Note that when $\alpha = 0$, this case reduces to that of Babichev et al. [17]. The mass in (28) in displayed for different values of EoS parameter γ at different times in Table 2 and displayed graphically in Fig. 1. As shown, the mass decreases gradually with the decrease in the EoS parameter γ. Note that we have not graphically displayed the mass for different viscosities given in Table 1 because the variation is not significantly different for most time scales.

5 Examples

We now solve examples to demonstrate the accretion of viscous phantom energy onto a BH. The formalism is adapted from [17].

5.1 Viscous Linear EoS

We choose the viscous linear EoS, $p = \omega\rho_{\mathrm{pe}} - 3H\xi_o$ with $\omega < -1$. The ratio of the number densities of phantom fluid particles at the horizon and at infinity is given by

$$\frac{n(\rho_h^{\mathrm{pe}})}{n(\rho_\infty^{\mathrm{pe}})} = \left[\frac{\rho_h^{\mathrm{pe}}(1+\omega) - 3\xi_o H}{\rho_\infty^{\mathrm{pe}}(1+\omega) - 3\xi_o H}\right]^{\frac{1}{(1+\omega)}}. \tag{29}$$

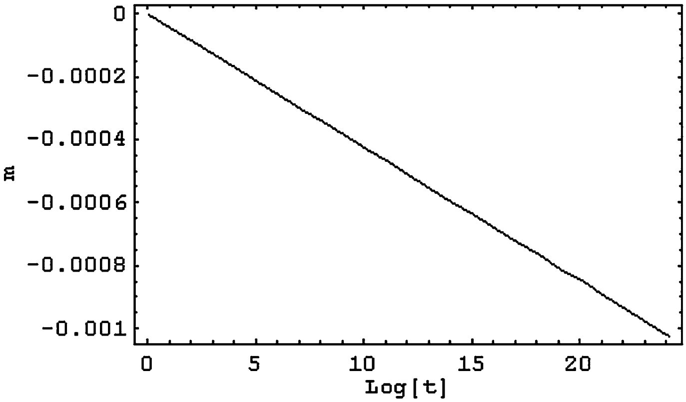

Fig. 1 For an initial mass of black hole $M_0 = 10^{32}$ kg, the evolution of the mass parameter $m = (M/M_0) - 1$ is plotted against the logarithmic time with $\alpha = 10^{-5}$ and $t_H = 10^{17}$ s

The critical points of accretion (the point where the speed of fluid flow becomes equal to the speed of sound i.e. $u_*^2 = c_s^2$) are given by

$$u_*^2 = \frac{\omega}{1+3\omega}; \qquad x_* = \frac{1+3\omega}{2\omega}. \tag{30}$$

The constant A appearing in (16) is determined to be

$$A = \frac{|1+3\omega|}{4|\omega|^{3/2}}^{\frac{1+\omega}{2\omega}}. \tag{31}$$

Notice that the constant A is the same as for the non-viscous case [17]. Also, the density of phantom energy at the horizon is given by

$$\rho_h^{\text{pe}} = \frac{3\xi_o H}{1+\omega} + \left(\frac{4}{A}\right)^{\frac{\omega-1}{\omega+1}} \left(\rho_\infty - \frac{3\xi_o H}{1+\omega}\right). \tag{32}$$

Moreover, the speed of flow at the horizon is

$$u_h = -\left(\frac{A}{4}\right)^{\frac{\omega}{(\omega+1)}}. \tag{33}$$

The speed is negative as it is directed towards the BH. Also, the characteristic evolution time scale of the BH is given by

$$\tau^{-1} = 4\pi M_0 \frac{(1+3\omega)}{4\omega^{3/2}}^{\frac{1+\omega}{2\omega}} \left\{\rho_\infty^{\text{pe}}(1+\omega) - \frac{3\xi_o}{t}\ln\left(\frac{a}{a_0}\right)\right\}. \tag{34}$$

Finally, substituting (34) in (17) we get the mass evolution of a BH in bulk viscous cosmology

$$M = M_0\left[1 - 4\pi M_0 t \frac{(1+3\omega)}{4\omega^{3/2}}^{\frac{1+\omega}{2\omega}} \left\{\rho_\infty^{\text{pe}}(1+\omega) - \frac{3\xi_o}{t}\ln\left(\frac{a}{a_0}\right)\right\}\right]^{-1}. \tag{35}$$

Since $\rho_\infty^{\rm pe}$ is unknown for our purpose, we have not evaluated M for different times numerically for tabular and graphical presentation.

5.2 Viscous Non-linear EoS

We here choose the EoS, $p = \omega\rho_{\rm pe} - 3H\xi(\rho_{\rm vis})$ with $\omega < -1$, where $\xi(\rho_{\rm pe}) = \alpha\rho_{\rm pe}^s$ with α and s are constants. The ratio of number densities is given by

$$\frac{n(\rho_h^{\rm pe})}{n(\rho_\infty^{\rm pe})} = \left(\frac{\rho_h}{\rho_\infty}\right)^{\frac{s}{(s-1)(1+\omega)}} \left(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\right) \tag{36}$$

The constant A appearing in (16) is determined to be

$$A = \left|\left(\frac{\rho_h}{\rho_\infty}\right)^{\frac{2s}{(s-1)(1+\omega)}} \left(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\right)^3\right|. \tag{37}$$

The speed of flow at the horizon becomes

$$u_h = -\left(\frac{\rho_h}{\rho_\infty}\right)^{\frac{2s}{(s-1)(1+\omega)}} \left(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\right)^2. \tag{38}$$

The critical points of accretion are given by

$$u_*^2 = \frac{\omega - 3s\alpha\rho_h^{s-1}}{1 + 3(\omega - 3s\alpha\rho_h^{s-1})}; \qquad x_* = \frac{1 + 3(\omega - 3s\alpha\rho_h^{s-1})}{2(\omega - 3s\alpha\rho_h^{s-1})}. \tag{39}$$

The characteristic evolution time scale τ is given by

$$\tau = \left[4\pi M_0\left(\frac{\rho_h}{\rho_\infty}\right)^{\frac{2s}{(s-1)(1+\omega)}} \left(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\right)^3 \left\{\rho_\infty(1+\omega) - 3\frac{\alpha\rho_\infty^s}{t}\ln\left(\frac{a}{a_0}\right)\right\}\right]^{-1}. \tag{40}$$

Finally, using (40) in (17), the BH mass evolution is given by

$$M = M_0\left[1 - 4\pi M_0 t\left(\frac{\rho_h}{\rho_\infty}\right)^{\frac{2s}{(s-1)(1+\omega)}} \left(\frac{\rho_\infty(1+\omega) - 3H\alpha\rho_\infty^s}{\rho_h(1+\omega) - 3H\alpha\rho_h^s}\right)^3 \right.$$
$$\left. \times\left\{\rho_\infty(1+\omega) - 3\frac{\alpha\rho_\infty^s}{t}\ln\left(\frac{a}{a_0}\right)\right\}\right]^{-1}. \tag{41}$$

As before, ρ_∞ is unknown, but further ρ_h is also unknown. As such, we again do not provide a tabular or graphical presentation.

6 Black Holes Accreting Both Matter and Viscous Phantom Fluid

We now consider a two component fluid, the viscous dark energy and matter. The matter part may be composed of both baryonic and non-baryonic matter. It is taken to be a perfect fluid while the PE is taken as a bulk viscous fluid. The effective pressure is represented by (4). The corresponding Einstein field equations (EFE) for the two component fluid become:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{\rm m}. \tag{42}$$

The stress-energy tensor representing the two component fluid is given by

$$T^{\mu\nu} = (\rho + p + \rho_{\mathrm{m}})u^{\mu}u^{\nu} + pg^{\mu\nu}. \tag{43}$$

Here ρ_{m} is the energy density of the pressureless matter. Energy conservation holds independently for both fluids:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{44}$$

$$\dot{\rho}_{\mathrm{m}} + 3H\rho_{\mathrm{m}} = 0. \tag{45}$$

Integrating (45), we have

$$\rho_{\mathrm{m}} = \rho_{\mathrm{m}_0} a^{-3}, \tag{46}$$

where $\rho_{\mathrm{m}_0} = \rho_{\mathrm{m}}(t_0)$. Similarly, integrating (44) leads to

$$\rho = \rho_{\mathrm{m}}\left[\left(\Xi + \frac{K}{3}a^{3/2}\right)^2 - 1\right], \tag{47}$$

where Ξ is a constant and K is given by

$$K = \frac{3\sqrt{3}\xi_o}{\sqrt{\rho_{\mathrm{m}_0}}}, \tag{48}$$

Thus the total energy density of the two component fluid is given by [34]

$$\rho \equiv \rho + \rho_{\mathrm{m}} = \rho_{\mathrm{m}_0} a^{-3}\left(\Xi + \frac{K}{3}a^{3/2}\right)^2. \tag{49}$$

Using (45) in (16) the evolution of black hole mass is given by

$$\begin{aligned} M = M_0\Bigg[1 - 4\pi A M_0\Bigg[\frac{\gamma\rho_{\mathrm{m}_0}}{H(t)}\Bigg\{\frac{K^2}{9}\ln\left(\frac{a}{a_0}\right) - \frac{\Xi}{9a^3}(3\Xi + 4a^{3/2}K) \\ + \frac{\Xi}{9a_0^3}(3\Xi + 4a_0^{3/2}K)\Bigg\}\Bigg]\Bigg]^{-1}, \end{aligned} \tag{50}$$

where the scale factor $a(t)$ evolves as

$$a(t) = \left[\frac{3}{K}\left(e^{\frac{K}{2}\sqrt{\rho_{\mathrm{m}_0}/3}t + D_1} - \Xi\right)\right]^{2/3}, \tag{51}$$

and D_1 is the constant of integration determined by choosing $t = 0$ to get

$$D_1 = \frac{2}{K}\ln\left(\frac{K}{3}a_0^{3/2} + \Xi\right). \tag{52}$$

As pointed out in the next section, we cannot correctly discuss a BR scenario. However we can take a spacetime approximating it sufficiently earlier than the BR. We can than see its asymptotic behavior. when the scale factor shoots to infinity, the three terms in (50) will contribute significantly in the BH mass evolution. The mass will decrease by the accretion of PE ($\gamma < 0$) due to its strong negative pressure and is manifested in (50). Notice that the

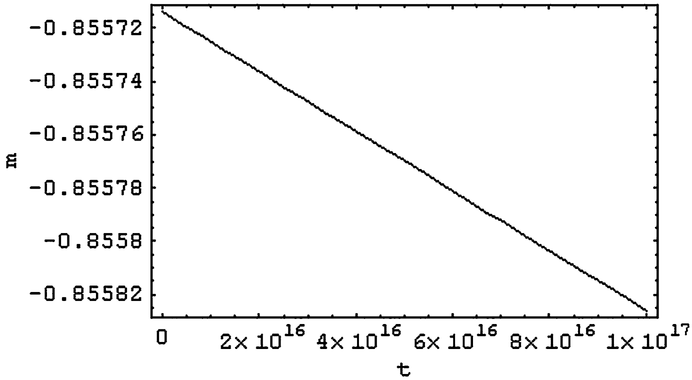

Fig. 2 For an initial mass of black hole $M_0 = 10^{32}$ kg, the evolution of m is plotted against the time parameter t with $A = 1/3$, $\Xi = 3$, $\xi_o = 10^{-16}\ \mathrm{kg\,m^{-1}\,s^{-1}}$ and $\gamma = -10^{-1}$ while $H \approx 2.33 \times 10^{-18}$ m

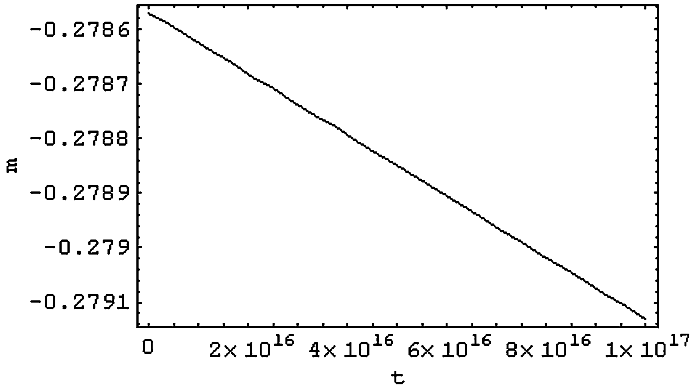

Fig. 3 For an initial mass of black hole $M_0 = 10^{32}$ kg, the evolution of m is plotted against the time parameter t with $A = 1/3$, $\Xi = 3$, $\xi_o = 10^{-16}\ \mathrm{kg\,m^{-1}\,s^{-1}}$ and $\gamma = -2 \times 10^{-1}$ while $H \approx 2.33 \times 10^{-18}$ m

final expression for BH mass depends only on the initial matter density ρ_{m_0} in addition to constant bulk viscosity ξ_o. The corresponding behavior of BH mass evolution is shown in Figs. 2 and 3 for different values of model parameters. Thus for a shift of parameter γ by 2, yields in the decrease of mass ratio by a factor of 2. The decrease in the mass of the BH is observed with time showing that phantom energy accretion will be dominant over matter accretion.

7 Conclusion and Discussion

We have analyzed the accretion of bulk viscous phantom energy onto a BH. The modeling is based on the relativistic model of accretion for compact objects. The viscosity effects in cosmology are used to give an alternative to cosmic accelerated expansion other then dark energy and quintessence. The evolution of BHs in such a Universe accreting viscous phantom energy would result in a gradual decrease in mass. This gradual decrease would be faster than the non-viscous case [17] due to additional terms containing viscosities coupled with mass. Lastly, it is shown that BHs accreting both matter and viscous PE will also meet with the same fate as the viscous forces dominate over the matter component for sufficiently large scale factor $a(t)$.

From this analysis, we can draw the conclusion that PE containing viscous stresses can play a significant role in the BH mass evolution if the viscosity is sufficiently high for an appropriate EoS. Though the viscous stresses are negligibly small $O(10^{-8}\ \mathrm{N\,s\,m^{-2}})$ at the local scale of space and time they can play a significant role in time scales of $\sim$ Gyrs. The higher the viscosity of the phantom fluid, the sharper the decrease in the BH mass. BHs of all masses, ranging from the solar mass to the intermediate mass to the supermassive, will all meet the same fate.

As an extension to this problem, it is interesting to study the accretion of the phantom fluid onto primordial BHs that had formed due to initial density fluctuations in the primordial plasma. The mini-primordial BHs evaporating now via Hawking radiation would have a different initial mass and hence abundance than the standard scenario expects. This work is reported in a separate paper [37].

Notice that we have used the Friedmann model which is represented by an asymptotically curved spacetime and at the same time the Schwarzschild black hole, which is asymptotically flat. This may seem contradictory. Schwarzschild black hole has been dealt with in the context of closed Friedmann cosmology [38–40]. Any global problem in approximating the full situation by a Schwarzschild black hole inserted into Friedmann model arise near the big bang or the big crunch, defined in terms of the York time [41] as shown elsewhere [42], the effect will be at extremely late times in terms of the usual time parameter. More complete analysis of the asymptotic behavior near a singularity is also available [43], as such if we stay near to a singularity in spacetime, the approximation will be extremely good. Consequently our analysis will be satisfactory for black holes formed well after the big bang greater then 10^{-40} s and of the Big Rip (presumably much more before 10^{-40} s the rip). It is clear that we are unable to say whether there would/would not be a Big Rip as our analysis excludes it.

Acknowledgement MJ would like to thank M. Akbar for useful discussions during the work. AQ is grateful to AS-ICTP, Trieste, Italy for travel support and to INFN and the Department of Physics of Salento University at Lecce, for local hospitality.

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DOI 10.1007/s10714-010-0928-1

RESEARCH ARTICLE

Primordial black holes in phantom cosmology

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Received: 9 September 2009 / Accepted: 13 December 2009

Abstract We investigate the effects of accretion of phantom energy onto primordial black holes. Since Hawking radiation and phantom energy accretion contribute to a *decrease* of the mass of the black hole, the primordial black hole that would be expected to decay now due to the Hawking process would decay *earlier* due to the inclusion of the phantom energy. Equivalently, to have the primordial black hole decay now it would have to be more massive initially. We find that the effect of the phantom energy is substantial and the black holes decaying now would be *much* more massive—over ten orders of magnitude! This effect will be relevant for determining the time of production and hence the number of evaporating black holes expected in a universe accelerating due to phantom energy.

Keywords Black hole · Dark energy · Phantom energy · Hawking radiation

1 Introduction

Numerous astrophysical observations are consistent with the standard cold dark matter model with the inclusion of an effective cosmological constant. A classical cosmological constant is generally avoided as quantum gravity attempts lead to a natural expectation of a cosmological term of Planck scale, which is totally at odds with the value required by observation (see for example [1]). As such, it is generally assumed that there is some physical field that comes into play *after the Planck era* and there

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is some principle that excludes the induced Planck energy cosmological term (see for example [2]). This exotic field is often called "dark energy". According to the generally accepted modeling, the latter constitutes more than 70% of the total energy density of the universe while the matter component carries most of the remaining part [3–7]. It is not clear how seriously to take the "quantum gravity" requirements considering that there is no viable theory of quantum gravity to date [1,8]. Further, there is no clear evidence that the onset of quantum gravity will be at Planck scale and not orders of magnitude away from it [9]. There are alternate (classical) explanations of the observations available in the literature (for example by Wiltshire [10–13]) but here we shall follow the generally accepted view of some form of dark energy providing the observed acceleration of the Universe.

The observable universe locally appears to be spatially flat with an equation of state (EoS) parameter (the ratio of pressure to the energy density) $\omega(\equiv p_x/\rho_x) \simeq -1$. The dark energy is an exotic vacuum energy with negative pressure and positive energy density which arises due to quantum vacuum fluctuations in spacetime. Caldwell and co-workers [14,15] considered the possibility of dark energy with the super-negative EoS parameter $\omega < -1$, which they called 'phantom energy'. It also gives negative pressure. The motivation to consider phantom energy as the candidate for dark energy arises from the observational data of the cosmic microwave background power spectrum and supernovae of type Ia. The phantom energy violates the general relativistic energy conditions including the null and dominant ones [16]. Its implications in cosmology give rise to exotic phenomena like an imaginary value of the sound speed, negative temperature, the divergence of the scale factor $a(t)$ and the energy density $\rho_x \sim a^{-3(1+\omega)}$, at a finite time resulting in a 'big rip', an epoch when the spacetime is torn apart [17–21] (see [22] for a review on the big rip singularity). However there are some attempts made recently in which the occurrence of a big rip singularity is avoided by phantom energy decay into matter [23–25]. Another attempt is the 'big trip', in which a wormhole accretes phantom energy and grows so large that it engulfs the whole universe [26]. A similar scenario is also proposed for black holes whereby the black hole event horizon inflates to swallow up the cosmological horizon, resulting in a naked singularity [27]. Moreover, quantum gravitational effects (if they exist) may avoid the big rip singularity. If the big rip cannot be avoided, the smaller the parameter ω, the closer the big rip will be to the present time.

Another weird property of phantom energy is that its accretion onto gravitationally bound structures results in their dissociation and disintegration in a rather slow process. It was first analyzed in [28] for several gravitational systems like the solar system and the Milky Way galaxy. Initially, this possibility was investigated for a Schwarzschild black hole by Babichev et al. [29], who showed that the black hole mass goes to zero near the big rip. Interestingly, in this scenario larger black holes lose mass more rapidly than smaller ones. Later on, their model was extended to the Reissner-Nordstrom [31], Schwarzschild de Sitter [32] and Kerr-Newmann [33,34] black holes. It should be mentioned that it has been argued that the mechanism of accretion followed by Babichev et al. is stationary and does not possess the shift symmetry [30] and hence that the mechanism of dark energy accretion is not realistic and consistent. Nevertheless, we shall follow the Babichev et al. analysis, leaving the detailed analysis for subsequent work, as the effect will be technically difficult to apply and we believe will not make a substantial difference for phantom energy in the neighbourhood of a primordial black hole. It has also been argued that when the back-reaction effects of the accretion process are included in the analysis of Babichev et al. [29], the black hole mass may increase instead of decreasing [35,36], thus avoiding the big rip. Also in cyclic cosmological models, black holes do not tear apart near the turnaround but preserve some nonzero mass [37,38]. We shall ignore the big rip issue here.

2 Hawking evaporation of black holes

We are interested in studying the effects of accretion of phantom energy on a static primordial black hole. Carr and Hawking [39] in 1974 considered the formation of black holes of mass 10^2 kg and upwards in the early evolution of the universe. After their attempt, several authors investigated various scenarios of PBH formation [40–43]. The existence of these small mass black holes was based on the assumption that the early universe was not entirely spatially smooth but there were density fluctuations or inhomogeneities in the primordial plasma which gravitationally collapsed to form these black holes. Unlike the conventional black holes that are formed by the gravitational collapse of stars or mergers of neutron stars, the primordial black holes (PBH) are formed due to the gravitational collapse of matter without forming any initial stellar object. The mass of a PBH can be of the order of the particle horizon mass at the time of its formation [44,45]

$$M_{\mathrm{PBH}} \approx \frac{c^3 t}{G} \approx 10^{12} \left(\frac{t}{10^{-23}\,\mathrm{s}} \right) \mathrm{kg}. \tag{1}$$

Therefore PBHs that formed in the early history of the universe must be less massive while those that formed later must be more massive. Black holes formed at Planck time 10^{-43} s would have Planck mass 10^{-8} kg.

Using classical arguments, Penrose and Floyd showed that one can extract rotational energy from a rotating black hole [46]. Penrose went on to argue (see [8] and references therein) that one could take thermal energy from the environs of a black hole and throw it into the black hole to get usable energy out. This would apparently reduce the entropy around the black hole. As such, he had argued that *there must be an entropy of the black hole that increases at least as much as that of its environs decreases*. Hawking had pointed out that in any physical process the area of a black hole always increases [47] just as entropy always increases. This led Bekenstein [48] to propose a linear relationship between the area and entropy of a black hole. Thus Bekenstein [49,50] generalized the second law of thermodynamics to state that the sum of the entropy of the black hole and its environs never decreases. However, at this stage it seemed that the connection between black holes and thermodynamics was purely formal. At this stage Fulling pointed out that quantization of scalar fields in accelerated frames gives an ambiguous result [51], which seemed to yield radiation seen in the accelerated with a fractional number of particles. Hawking repeated the calculation for an observer near a black hole and obtained the same result by various methods and found that the radiation had a thermal spectrum [52]. This led him to propose that mini-PBHs would evaporate away in a finite time [53].

The corresponding Hawking evaporation process reduces the mass of the black hole by [54]

$$\left.\frac{dM}{dt}\right|_{\rm hr} = -\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2}, \tag{2}$$

where α is the spin parameter of the emitting particles. Integration of Eq. (2) gives the evolution of PBH mass as

$$M_{\rm hr} = M_i\left(1 - \frac{t}{t_{\rm hr}}\right)^{1/3}, \tag{3}$$

where the Hawking evaporation time scale $t_{\rm hr}$ is

$$t_{\rm hr} = \frac{G^2}{\hbar c^4}\frac{M_i^3}{3\alpha}. \tag{4}$$

It is obvious from Eq. (3) that as $t \to t_{\rm hr}$, the mass $M_{\rm hr} \to 0$. Plugging in $t_{\rm hr} = t_o$ (the current age of the universe) in Eq. (4) gives the mass 10^{12} kg of the PBH that should have been evaporating now. Hence from Eq. (1), it can be estimated that these PBHs were formed before about 10^{-23} s. For $M_i \gg 10^{14}$ kg $\alpha = 2.011 \times 10^{-4}$, hence Eq. (4) implies $t_{\rm hr} \simeq 2.16 \times 10^{-18}\left(\frac{M}{\rm kg}\right)^3$ s. While for 5×10^{11} kg $\ll M_i \ll 10^{14}$ kg, $\alpha = 3.6 \times 10^{-4}$ then Eq. (4) gives $t_{\rm hr} \simeq 4.8 \times 10^{-18}\left(\frac{M}{\rm kg}\right)^3$ s. Therefore detecting PBHs would be a good tool to probe the very early universe (closer to the Planck time). The evaporation of PBHs could still have interesting cosmological implications: they might generate the microwave background [55] or modify the standard cosmological nucleosynthesis scenario [56] or contribute to the cosmic baryon asymmetry [57]. Some authors have also considered the possibility of the accretion of matter and dust onto the seed PBH resulting in the formation of super-massive black holes which reside in the centers of giant spiral and elliptical galaxies [58].

3 Phantom energy accretion onto black hole

The FRW equations governing the dynamics of our gravitational system are given by

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_x), \tag{5}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho_m + \rho_x(1+\omega)]. \tag{6}$$

Here ρ_m and ρ_x denote the energy densities of matter and the exotic energy densities respectively. The scale factor $a(t)$ goes like [20]

$$a(t) = \frac{a(t_0)}{[-\omega + (1+\omega)t/t_0]^{-\frac{2}{3(1+\omega)}}} \quad (t > t_0), \tag{7}$$

where t_0 is the time when the universe transits from matter to exotic energy domination (which is roughly equal to the age of the universe). Notice that the scale factor $a(t)$ diverges when the quantity in the square brackets in Eq. (7) vanishes identically i.e.

$$t^* = \frac{\omega}{1+\omega} t_0. \tag{8}$$

Subtracting t_0 from Eq. (8), we get

$$t^* - t_0 = \frac{1}{1+\omega} t_0. \tag{9}$$

The evolution of energy density of the exotic energy is given by

$$\rho_x^{-1} = 6\pi G(1+\omega)^2 (t^* - t)^2. \tag{10}$$

A black hole accreting only the exotic energy has the following rate of change in mass [29]

$$\left.\frac{dM(t)}{dt}\right|_x = \frac{16\pi G^2}{c^5} M^2 (\rho_x + p_x). \tag{11}$$

It is clear that when $\rho_x + p_x < 0$, the mass of the black hole will decrease. We are particularly interested in the evolution of black holes about and after $t = t_0$ since the dark energy is presumably negligible before that time and may not have any noticeable effects on the black hole. Using Eqs. (9) and (10) in (11), we get

$$\left.\frac{dM(t)}{dt}\right|_x = \frac{8G}{3c^3} \frac{M^2}{t_0^2} (1+\omega). \tag{12}$$

Therefore the mass change rate for a black hole accreting pure exotic energy is determined by Eq. (12). For the phantom energy accretion, the time scale is obtained by integrating Eq. (12) to get

$$M(t) = M_i \left(1 - \frac{t}{t_x}\right)^{-1}, \tag{13}$$

where t_x is the characteristic accretion time scale given by

$$t_x^{-1} = \frac{16\pi G^2}{c^5} M_i (\rho_x + p_x). \tag{14}$$

Using Eqs. (9) and (10) in (14), we get

$$t_x = \frac{3c^3}{8G} \frac{t_0^2}{M_i(1+\omega)}. \tag{15}$$

4 Evolution of mass due to phantom energy accretion and Hawking evaporation

The expression determining the cumulative evolution of the black hole is obtained by adding Eqs. (2) and (12) i.e.

$$\left.\frac{dM(t)}{dt}\right|_{\text{Total}} = \left.\frac{dM}{dt}\right|_{\text{hr}} + \left.\frac{dM}{dt}\right|_{x}, \tag{16}$$

$$= -\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2} + \frac{8G}{3c^3}\frac{M^2}{t_0^2}(1+\omega). \tag{17}$$

We write the above equation as

$$\frac{dM}{dt} = -aM^2 - \frac{b}{M^2}, \tag{18}$$

where

$$a = \frac{8G}{3c^3}\frac{\epsilon}{t_0^2}, \quad b = \frac{\hbar c^4 \alpha}{G^2}. \tag{19}$$

Here $\epsilon = -\omega - 1$. Thus (18) can be written in the form

$$-\int dt = \frac{1}{b}\int \frac{M^2 dM}{1 + \frac{a}{b}M^4}$$

To integrate above equation, we assume

$$x = \left(\frac{a}{b}\right)^{1/4} M, \tag{20}$$

which yields

$$-\int dt = \frac{1}{(a^3 b)^{1/4}} \int \frac{x^2 dx}{1 + x^4}, \tag{21}$$

We note that [59]

$$\int \frac{x^{m-1}dx}{1+x^{2n}} = -\frac{1}{2n}\sum_{k=1}^{n}\cos\left(\frac{m\pi(2k-1)}{2n}\right)\ln\left|1-2x\cos\left(\frac{2k-1}{2n}\right)\pi+x^2\right| + \frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{m\pi(2k-1)}{2n}\right)\tan^{-1}\left[\frac{x-\cos\left(\frac{2k-1}{2n}\right)\pi}{\sin\left(\frac{2k-1}{2n}\right)\pi}\right], \quad m<2n. \tag{22}$$

In our case, $m=3$ and $n=2$, hence the above equation yields

$$\int \frac{x^2dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\left|\frac{1-\sqrt{2}x+x^2}{1+\sqrt{2}x+x^2}\right| + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{1+x^2}\right). \tag{23}$$

On substituting the value of x above, we obtain

$$t=t_0+\frac{1}{4\sqrt{2}}\ln\left|\frac{1-\sqrt{2}\left(\frac{a}{b}\right)^{1/4}M+\left(\frac{a}{b}\right)^{1/2}M^2}{1+\sqrt{2}\left(\frac{a}{b}\right)^{1/4}M+\left(\frac{a}{b}\right)^{1/2}M^2}\right|+\frac{1}{2\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{2}\left(\frac{a}{b}\right)^{1/4}M}{1+\left(\frac{a}{b}\right)^{1/2}M^2}\right]. \tag{24}$$

We now redefine the values of a and b by assuming $M=mM_i$, where m is a dimensionless parameter and M_i is the initial mass of the black hole. Thus (18) becomes

$$\frac{dm}{dt} = -a'm^2 - \frac{b'}{m^2}, \tag{25}$$

where $a'=aM_i$ and $b'=b/M_i^3$. For the terms to be equal strength, we require $a'\approx b'$. Thus

$$M_i \approx \left(\frac{b}{a}\right)^{1/4}. \tag{26}$$

Now

$$\frac{b}{a} = \frac{3\hbar c^7 t_0^2\alpha}{8G^3\epsilon}, \quad \text{or} \quad \epsilon = \frac{3\hbar c^7 t_0^2\alpha}{8G^3M_i^4}. \tag{27}$$

We can normalize

$$t=t_0\left[1-\frac{\frac{1}{4\sqrt{2}}\ln\left|\frac{1-\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4}m+\left(\frac{a'}{b'}\right)^{1/2}m^2}{1+\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4}m+\left(\frac{a'}{b'}\right)^{1/2}m^2}\right|+\frac{1}{2\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4}m}{1+\left(\frac{a'}{b'}\right)^{1/2}m^2}\right]}{\frac{1}{4\sqrt{2}}\ln\left|\frac{1-\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4}+\left(\frac{a'}{b'}\right)^{1/2}}{1+\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4}+\left(\frac{a'}{b'}\right)^{1/2}}\right|+\frac{1}{2\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{2}\left(\frac{a'}{b'}\right)^{1/4}}{1+\left(\frac{a'}{b'}\right)^{1/2}}\right]}\right]. \tag{28}$$

Replacing $p = a'/b' = \frac{8\epsilon G^3}{3\alpha \hbar c^7 t_0^2} M_i^4 \sim M_i^4$ (the ratio of the phantom component to the Hawking component, in the energy radiated) the above equation becomes

$$t = t_0 \left[1 - \frac{\frac{1}{4\sqrt{2}} \ln \left| \frac{1-\sqrt{2}p^{1/4}m + p^{1/2}m^2}{1+\sqrt{2}p^{1/4}m + p^{1/2}m^2} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}p^{1/4}m}{1+p^{1/2}m^2} \right)}{\frac{1}{4\sqrt{2}} \ln \left| \frac{1-\sqrt{2}p^{1/4} + p^{1/2}}{1+\sqrt{2}p^{1/4} + p^{1/2}} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}p^{1/4}}{1+p^{1/2}} \right)} \right]. \tag{29}$$

Moveover, the power emission due to Hawking evaporation from the stationary black hole of mass $M \gg 10^{17}$ g [54]

$$P = 3.458 \times 10^{46} (M/\mathrm{g})^{-2} \mathrm{erg\,s}^{-1}, \tag{30}$$

and for mass 5×10^{14} g $\ll M \ll 10^{17}$ g,

$$P \approx 3.6 \times 10^{16} (M/10^{15}\ \mathrm{g})^{-2} \mathrm{erg\,s}^{-1}. \tag{31}$$

In our analysis, the mass in the above two expressions is replaced by

$$M = \left(\frac{3\hbar c^7 t_0^2 \alpha}{8 G^3 \epsilon} \right)^{1/4} \mathrm{g}. \tag{32}$$

Now choosing $\epsilon = 0.1$, we obtain $M = 8.74029 \times 10^{22}$ g which will be evaporating now due to the combined effects of phantom energy and Hawking radiation. Then using Eq. (30), the corresponding power emission will be $P = 4.52661$ erg s^{-1}. We can compare this result with that of a black hole of mass $M \simeq 1.05 \times 10^{12}$ g evaporating just now due to Hawking radiation only. The corresponding power emission will be $P \simeq 3.144 \times 10^{22}$ erg s^{-1}. Note that the power emission from a black hole decreases when the effects of phantom energy are incorporated. Similarly, for very large values of $\epsilon \sim 10^{25}$ would give $M = 2.763923 \times 10^{16}$ g. Using this mass in (31), the power emitted is 4.52661×10^{13} erg s^{-1}. However, such large values would lead to a very early big rip and hence must be excluded. Thus black holes $\sim 10^{22}$ g are of more interest for observational purposes since these are the ones that should be evaporating now.

5 Conclusion

In this paper, we have analyzed the Hawking radiation effects combined with the phantom energy accretion on a stationary black hole. The former process has been thoroughly investigated in the literature. However there is as yet no observational support to it. According to standard theory it is assumed that after the formation of PBHs (of mass $\sim 10^{12}$ kg with a Hawking temperature 10^{12} K), they would absorb virtually no radiation or matter whatsoever during their evolution and radiate continuously till they evaporate in a burst of gamma rays at the present time. This scenario assumes that the Hawking temperature for such black holes was always larger than the background temperature of the CMB. Strictly speaking, this cannot be true. Consequently PBHs

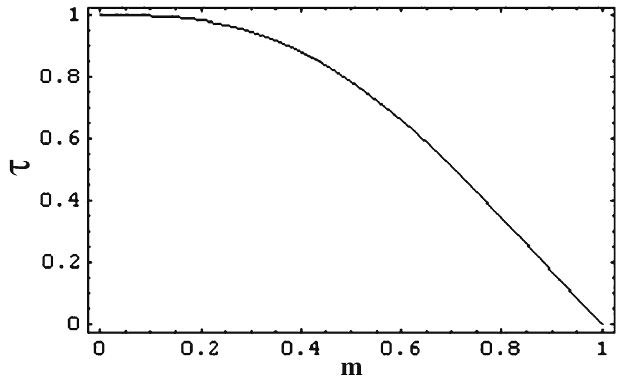

Fig. 1 The normalized time is plotted against the mass parameter for $p = 0.1$

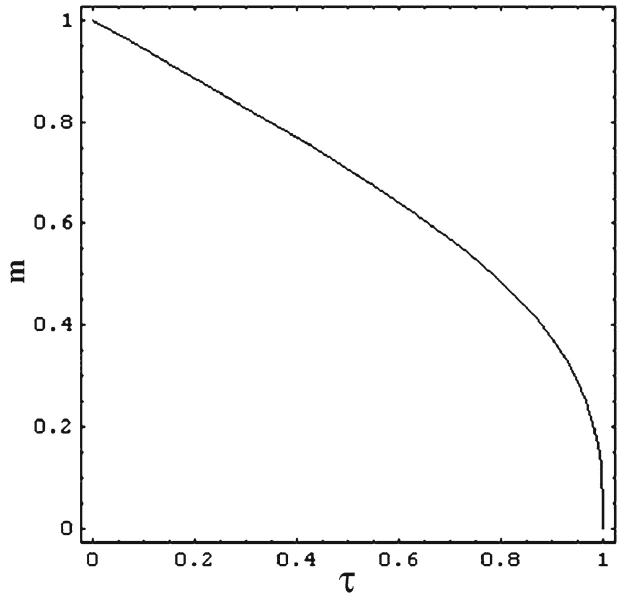

Fig. 2 The mass parameter is plotted against the normalized time for $p = 0.1$

could have accreted the background radiation (and even some matter) and *grown* in mass. Hence there should be no PBH left to be evaporating right now [60]. However, the above scenario is modified when phantom energy comes into play. When phantom energy and the Hawking process are relevant the total life time scale of the PBH is significantly shortened and the formation of the PBH exploding now is delayed.

From Eq. (29) we obtain the time as a function of mass instead of getting mass as a function of time. To make sense of the results we need to obtain the evolution with time. This is done by inverting the explicit function. We have plotted the normalized time $\tau = t/t_0$ against the dimensionless mass parameter m and m against τ for different choices of the parameter p, in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. It is observed

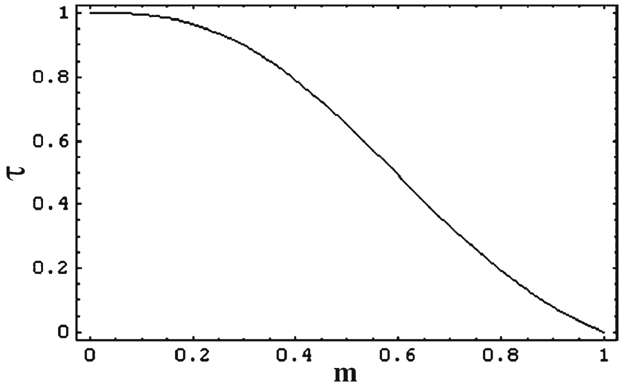

Fig. 3 The normalized time is plotted against the mass parameter for $p = 0.5$

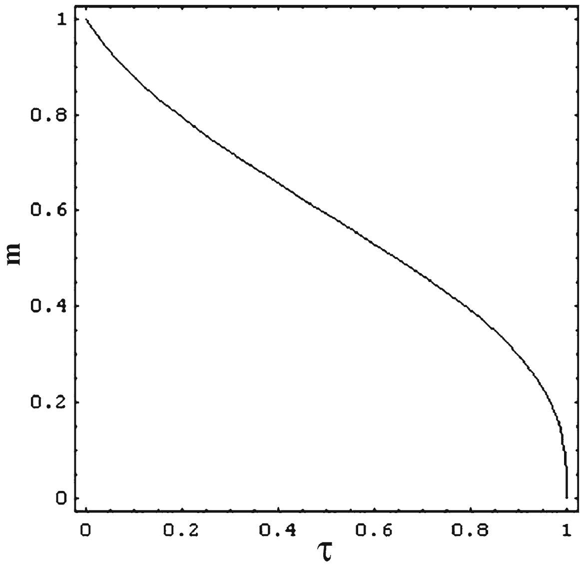

Fig. 4 The mass parameter is plotted against the normalized time for $p = 0.5$

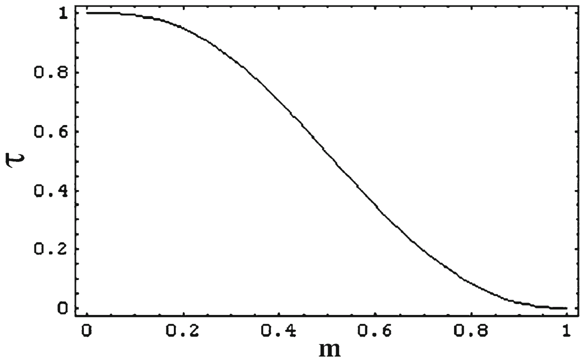

Fig. 5 The normalized time is plotted against the mass parameter for $p = 1$

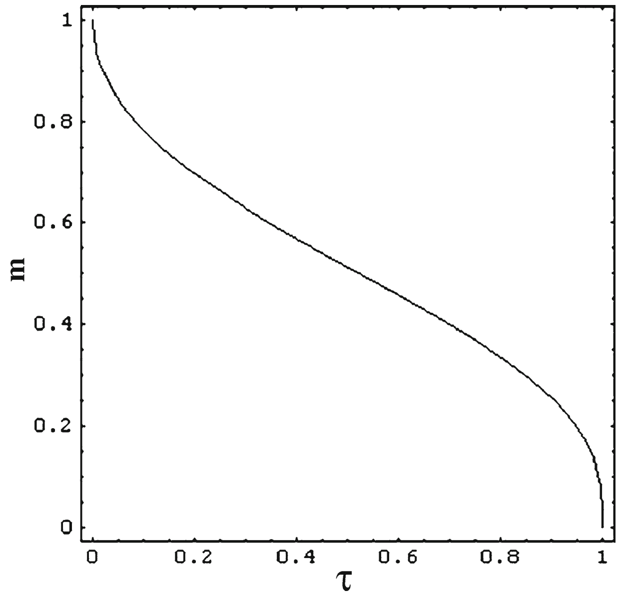

Fig. 6 The mass parameter is plotted against the normalized time for $p = 1$

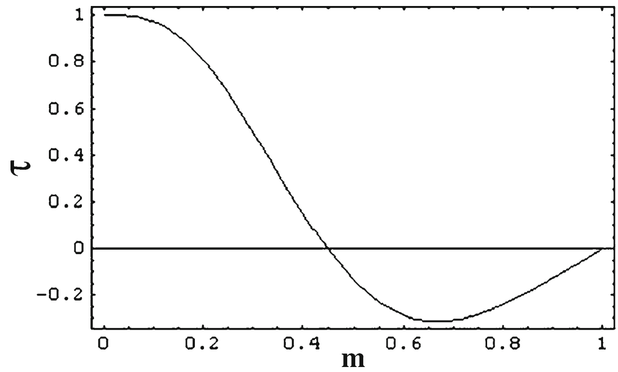

Fig. 7 The normalized time is plotted against the mass parameter for $p = 5$

that increasing p increases the steepness of the curve specifying the mass evolution. Therefore the black hole loses mass faster for larger p till it vanishes at $\tau = 1$, the present time. In particular, Figs. 7 and 9 show the same evolution of mass for larger values of p. It appears that the graphs contain a redundant (or nonphysical) part of the mass evolution and the only physically interesting section is above the horizontal curve crossing $t = 0$. Thus in effect, see Figs. 8 and 10, the initial mass of the black hole must be taken $0.45M_i$ of the value given by for $p = 5$ and about $0.315M_i$ for $p = 10$. It is obvious that the results are very insensitive to changes of the parameter ϵ for the phantom energy. As such, they can be regarded as fairly robust.

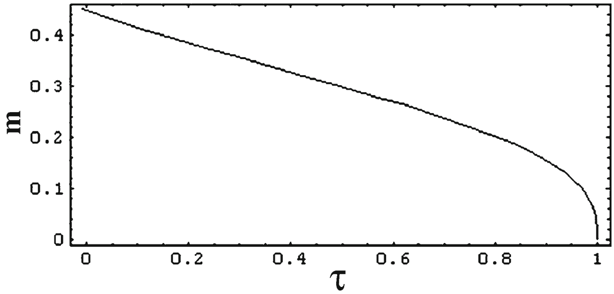

Fig. 8 The mass parameter is plotted against the normalized time for $p = 5$

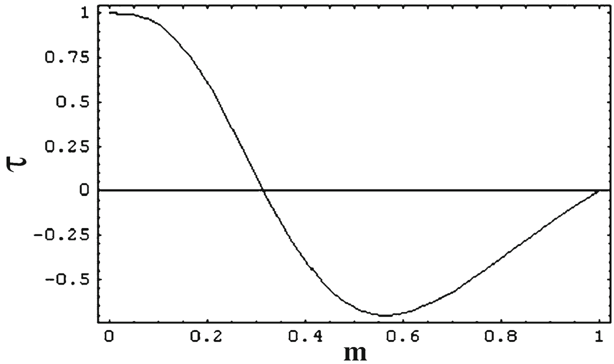

Fig. 9 The normalized time is plotted against the mass parameter for $p = 10$

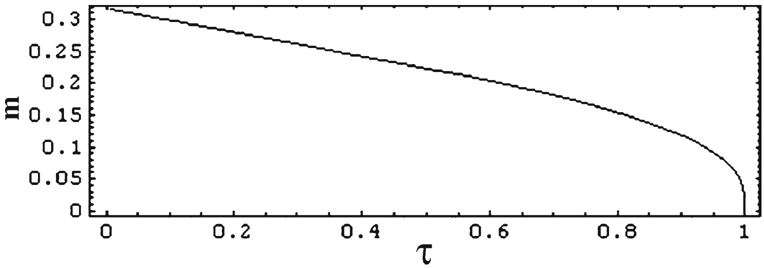

Fig. 10 The mass parameter is plotted against the normalized time for $p = 10$

Acknowledgments One of us (MJ) would like to the thank the National University of Sciences and Technology for full support for participation in the Second Italian-Pakistan Workshop on Relativistic Astrophysics. One of us (AQ) would like to thank ICRANet and Prof. Remo Ruffini for support and hospitality at Pescara, Italy.

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1. M. Jamil and U. Debnath, "FRW cosmology with variable G and Lambda", accepted by Int. J. Theor. Phys.

1. K. Karami, M.S. Khaledian and M. Jamil, "Reconstructing interacting entropy-corrected holographic scalar field models of dark energy in a non-flat universe" accepted by Physica Scripta.

2. H.M. Sadjadi and M. Jamil, "Generalized second law of thermodynamics for FRW cosmology with logarithmic correction", accepted by Europhys. Lett.; arXiv:1002.3588 [gr-qc].

3. M. Jamil and A. Sheykhi, "Interacting entropy corrected agegraphic tachyon dark energy", accepted by Int. J. Theor. Phys.; arXiv:1003.5043 [physics.gen-ph].

4. M. Jamil and I. Hussain, "Accretion of Phantom Energy and Generalized Second Law of Thermodynamics for Einstein-Maxwell-Gauss-Bonnet Black Hole", accepted by Int. J. Theor. Phys.

5. M.R. Setare and M. Jamil, "Cardy-Verlinde formula of Kehagias-Sfetsos black hole", accepted by Int. J. Theor. Phys. arXiv:1001.4716 [physics.gen-ph].

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