Solvability of Systems of Differential Equations by Complex Symmetry Analysis

by

Muhammad Safdar

Supervised by: **Asghar Qadir**

A thesis submitted to the
Centre for Advanced Mathematics & Physics,
National University of Sciences & Technology,
H–12, Kashmir Highway, Islamabad, Pakistan
for the degree of **Doctor of Philosophy**

2013

© Copyright by NUST 2013
All Rights Reserved

Dedicated To

My Father

Malik Hassan Akhter

and

My Teacher

(***Late***) ***Muhammad Arif***

Abstract

Complex symmetry analysis (CSA) establishes a connection between a class of n and $2n$, dimensional complex and real systems of ordinary/partial differential equations (ODEs/PDEs), respectively. Similarly, another class of systems of PDEs is extractable from the base complex systems of ODEs by CSA. The equivalence of the base complex systems under an invertible transformation of the dependent and independent variables has been exploited to study the corresponding real systems. Of particular interest is the extension of Lie groups of transformations developed to solve scalar and systems of non-linear second order ODEs, to two and four dimensional systems of the same order, respectively. For scalar equations, they may be used to either reduce them to the free particle equation (*linearize*) or integrate them. The former requires an 8-dimensional algebra while a 2-dimensional solvable algebra is sufficient to apply the latter. In this thesis, those two and four dimensional systems are characterized that arise from linearizable or integrable scalar and two dimensional systems of second order ODEs, respectively. Further, invariance of systems of PDEs under equivalence transformations is studied with CSA by employing the invariance properties of the base complex PDEs.

Three linearizable classes of two dimensional systems of cubically semi-linear (in the first derivative) second order ODEs has appeared so far in the literature. A comparison of two of the classes, obtainable from geometric methods and CSA, is presented. Both these classes are transformable to the system of free particle equations subject to certain linearization conditions, even though their general (cubic) semi-linear forms are proved to be inequivalent under point transformations.

There are five equivalence classes of two dimensional linearizable systems of second order ODEs namely, those with 5, 6, 7, 8 and 15-dimensional Lie algebras. For those systems that arise from a scalar complex linearizable second order ODE, treated as a pair of real ODEs, a reduced optimal canonical form is established. Of the five only three equivalence classes with 6, 7 or 15-dimensional algebras are recovered by this procedure. Both the equations of these systems are found to satisfy Cauchy-Riemann (CR) equations with respect to the dependent variables. Therefore, here as elsewhere in this thesis, such systems are called CR-structured systems.

A class of non-linearizable two dimensional CR-structured systems of second order ODEs is presented to show that the linearizability of the scalar complex equations is not sufficient to map the emerging systems to linear forms. A general system of n second order ODEs with $2n$ symmetry generators may not be amenable to quadratures by real symmetry analysis. However, it is shown that the CR-structured systems may be solvable by a procedure called *complex-linearization* even if they have fewer symmetries than required to linearize or integrate them.

A symmetry generator of the base complex ODE associates a pair of *Lie-like* operators with the CR-structured systems. It is proved that all such operators are not necessarily real symmetries of the emerging system. A criterion has been developed which shows when and how the real symmetry generators of the CR-structured systems of two second order ODEs are extractable from the associated complex Lie symmetries of the base ODEs.

The most general complex-linearizable form and the complex-linearization criteria for four dimensional systems of second order ODEs are derived by extending the geometric linearization criteria presented for two dimensional systems of cubically semi-linear second order ODEs. Two canonical forms of such systems have been derived by employing CSA on a system of dimension two once and a scalar equation twice. A specific form of the complex linearizing transformations associated with the base two dimensional systems is shown to furnish the reduction of the corresponding four dimensional complex-linearizable systems to the free particle Newtonian systems.

Semi-invariants for a class of systems of two linear parabolic type PDEs in two independent variables under equivalence transformations of the dependent variables have been deduced. This class of systems of two linear parabolic type PDEs and the real transformations that map such systems into themselves with different coefficients in general, are shown to correspond to complex scalar linear parabolic equations and associated complex transformations, respectively. Moreover, the semi-invariants for such systems of PDEs also correspond to complex Ibragimov invariants of the complex scalar linear parabolic PDEs. Particular cases of systems of parabolic type equations, i.e., when they are uncoupled or coupled in a special manner, have been studied with CSA.

Contents

Publications from the Thesis

- M. Safdar, A. Qadir and S. Ali, Inequivalence of classes of linearizable systems of cubically semi-linear ordinary differential equations obtained by real and complex symmetry analysis, *Math. Comp. Appl.* **16** (4) (2011) 923-34.
- M. Safdar, A. Qadir and S. Ali, Linearizability of systems of two second order ODEs using complex symmetry analysis, *Math. Prob. Engg.* **2011** (2011) 171834.
- M. Safdar, S. Ali and F.M. Mahomed, Linearization of systems of four second order ordinary differential equations, *Pramana - journal of physics.* **77**(3) (2011) 581-94.
- F.M. Mahomed, M. Safdar and J. Zama, Ibragimov-type Invariants for a System of Two Linear Parabolic Equations, *Commun. Nonlinear Sci. Numer. Simulat.* **17** (2012) 3140-3147.
- S. Ali, A. Qadir and M. Safdar, Symmetry solutions of two-dimensional systems not solvable by symmetry analysis, arXiv:1104.3837.
- F.M. Mahomed, A. Qadir, M. Safdar, Lie-like operators and Lie point symmetries for two dimensional systems of second order ODEs, submitted.
- F.M. Mahomed and M. Safdar, Complete classification for systems of two second order ordinary differential equations by using complex symmetry analysis, in progress.

Chapter 1

Introduction

A well-known problem of mathematics is solving equations whether they are algebraic, ODEs or PDEs. The attempts Abel, Galois, Lagrange and Ruffini made to construct procedures to solve (algebraic) polynomial equations laid the foundations of modern algebra [12, 100]. The problem of solving differential equations (DEs) encountered in the second half of the 19th century when the descriptions of numerous physical phenomena were found to involve the notions of derivatives and integrals introduced by Leibniz and Newton. The determination of the functional dependence between the variables involved in a DE took a similar course to that adopted earlier by Abel and Galois (see for instance [91]) in solving the polynomial equations. In Abel and Galois procedures, the concept of admitted group played a crucial role.

In the course of development of the theory of DEs several *ad hoc* integration methods were adopted to study a few special classes of such equations. Derivation of solution procedures for most of the DEs was mainly considered due to their appearance in the description of the physical problems. Indeed, the equivalence of DEs under invertible point transformations had been exploited to solve those equations which were reducible to integrable ones. The entirety of such equations that are transformable to each other by point transformations is an *equivalence class* of DEs. The seminal works of Brioschi [13], Cockle [20], Halphen [34] and Laguerre [45, 46], motivated Forsyth [26] to use the similarity between the algebraic properties of differential and algebraic equations to calculate the *invariants* of linear DEs. Though there may exist a few or infinitely many mappings to relate the concerned DEs, his attempts did provide practical algorithms to find them. Further, Sophus Lie explored the invariance and equivalence properties of DEs, under the groups of continuous transformations to construct possible classes of solvable DEs [59]. His efforts started by introducing the concepts of *infinitesimal transformations* and *one parameter group of transformations* in the realm of DEs. The most significant realization that led him to create his theory was the non-trivial transformations which leave an ODE form invariant (called a *symmetry* of that ODE). The device he constructed to find exact solutions of ODEs, is known as *Lie groups* and their associated *Lie algebras* due to Hermann Weyl. Lie observed that to work out the canonical forms and solutions of ODEs by utilizing their symmetries, the infinitesimal generators of the symmetry group must explicitly be determined [11, 77, 94]. For about ten years Lie studied the subject of linear PDEs which helped him to reformulate the integration problems in terms of the associated linear PDEs. He took suitable manipulations of the determining equations apart from solving them explicitly. In this way a working algorithm was given for determining the vector fields associated with the symmetry transformations [51].

After Lie's death, the theory of continuous groups of transformations and related symmetries [48]-[59] to solve DEs had been almost forgotten. Most of the work on solving DEs, which relied on the database-like tools containing collections of solvable equations [44, 81, 85], remained unchanged and hardly any concepts of symmetry were considered to solve the DEs. The most common questions which Lie's theory of continuous groups encountered were the following: to what extent is Lie's theory of continuous groups constructive; and how much effort is needed to go through the huge amount of analytic calculations in adopting a solution scheme based on symmetries of the DEs? Drach [21] put together some evidence to argue that Lie partially achieved his goal in generalizing the Galois theory of groups to DEs. Such arguments were based on a few observations which highlight the limitations of Lie's theory of continuous groups, for example, a single symmetry of a scalar ODE can integrate it by quadrature, i.e., reduces the order of equation by one but the determining equation for finding symmetries of first order scalar ODEs is under-determined. Also an equation may have a solution in closed form without having Lie symmetries [29]. However, the applications governed by DEs very often have a special structure expressed by its symmetries. The recognition of such structures leads to the development of a better understanding of the underlying problem which often helps to construct the solution algorithms for DEs.

Another major discovery based on the analogies of differential and algebraic equations was Loewy's theory of linear ODEs and its generalizations to systems of linear PDEs. He gave an algorithm to reduce linear ODEs [47, 60] and their representations to irreducible lowest degree equations. An altogether different attempt was Janet's theory of linear PDEs [42] and the description of the canonical forms for systems of such equations which are known as Janet's bases. Both these results are closely related to Lie's theory, for instance, the latter describes the symmetry type of the DE by identifying the determining equations as a Janet basis. Picard and Vessiot [80] have also generalized Galois theory to linear ODEs apart from Lie's efforts in Leipzig. However, their *differential Galois theory* have no connection with Lie's though the points of departure as well as the underlying group theories were the same. The Picard-Vessiot theory was claimed to be a proper extension of Galois theory due to a quite satisfactory theoretical base but is of limited use as compared to Lie'e theory established for the non-linear DEs.

The solution schemes for DEs assembled on the basis of their symmetries essentially involve the construction of transformations to map them to simple solvable forms. The equivalence of non-linear and linear second order ODEs under certain classes of point transformations, i.e., the linearization [51] of the second order ODEs was a major breakthrough of the group theory applied to DEs. Lie derived the most general (cubically semi-linear in the first derivative) form of a linearizable second order ODE by targeting the free particle equation with general point transformations [35]. Invariant linearizability criteria were presented in terms of a set of four linear PDEs involving two auxiliary functions apart from the coefficients appearing in the linearizable form. All the scalar second order linear ODEs were also recognized equivalent via point transformations which obviates the reduction of any linearizable second order ODE to the free particle equation. Consequently, Lie proved the existence of only one equivalence class which includes all the linear and linearizable second order ODEs, i.e., all such equations have an 8-dimensional Lie symmetry algebra. Lie also observed that all scalar second order non-linear ODEs are equivalent under *contact transformations* which do not only transform the dependent and/or independent variables but map the first order derivatives of the former. These transformations do not play any significant role in linearizing the second order ODEs but appear non-trivial for higher order cases as these are used to transform non-linear third and fourth order scalar ODEs to simplest linear forms.

Tresse [95, 96] made use of the relative invariants of the equivalence group of transformations to reformulate the existing linearizability criteria for second order ODEs by eliminating the auxiliary functions. Cartan [14] uniquely associated geometric structures with the second order ODEs to study their equivalence and linearization [33]. Chern [19] utilized the geometric invariants associated with more general transformations to linearize third order ODEs [18]. However, the problem was not completely solved, as the practical criteria to construct such transformations were not described and the targeted canonical forms of third order ODEs were restricted to constant coefficients. Further, Grebot [32] considered a subclass of the general point transformations [31] to extend the linearizable class of higher order ODEs. Later, Neut and Petitot [76] generalized this scheme to target the linear third order ODEs with arbitrary coefficients. However, the equivalence conditions were not explicitly determined. Linearization of these ODEs has also been studied by means of non-point transformations [22].

Meleshko [73] developed an algorithm to linearize third order ODEs, based on Lie linearization criteria for second order ODEs. Finding the integrating factors [6, 17, 36, 75] or Lie point symmetries of the underlying equations are the key features of his linearization scheme. This algorithm was also implemented to solve fourth order ODEs by linearizing the corresponding reduced second order ODEs. Ibragimov and Meleshko used point [39] as well as contact transformations [40] to study the linearization problem for the third order ODEs. They derived criteria to find the linearizing transformations whether they are point or contact, apart from just giving the coefficients of the resulting canonical forms of the third order ODEs. Separately, Ibragimov *et al* [41] also addressed the linearization problem for the fourth order ODEs and described a procedure to find the linearizing point transformations and the corresponding linear forms. A systematic approach has been developed for constructing the linearizing transformations for non-linear ODEs in [15, 90].

Work has been done on reducing two dimensional systems of second order non-linear ODEs to linear, solvable forms (see, e.g., [16, 70, 71, 98]). Several physical problems involve systems of second order ODEs, for instance, systems of free particle equations and the coupled or uncoupled, damped or undamped harmonic oscillators. Linearization furnishes the key to recognize such systems when encountered in some practical application or physical phenomena, where they might appear in a disguised form. Recently, Merker [74] characterized systems of m free particle equations. In particular, for two dimensional systems of second order ODEs he established the most general linearizable form which is cubically semi-linear in the first derivatives of both the dependent variables, by employing a set of general point transformations. Following Lie's classical method he derived the necessary and sufficient conditions to map systems to the Newtonian free particle system.

Wafo Soh and Mahomed presented linearization criteria for systems of second order ODEs [98] in terms of the associated Lie algebras. The underlying idea that played an essential role in constructing this *algebraic* linearization algorithm was Lie's criteria for the scalar ODEs, which requires a connected 2-dimensional algebra to reduce them to the free particle equation. They proved that a 4-dimensional connected or Abelian Lie algebra is the requirement to map non-linear systems of two quadratically semi-linear second order ODEs to some linear forms or systems of free particle equations, respectively. Moreover, the procedure to construct the linearizing point transformations was given explicitly which was also generalized to higher dimensional systems.

Mahomed and Leach [69] investigated the equivalence problem for equations of order three or higher. They broke down the class of nth order scalar equations into three equivalence subclasses, namely the subclasses of equations with $n+1$, $n+2$ or $n+4$-dimensional Lie point symmetry algebras for $n \geq 3$. Recall that for $n = 2$ there is only the 8-dimensional Lie algebra available. Gorringe and Leach addressed the problem of group classification of the linear systems of two second order ODEs with constant coefficients. They proved such systems may have a 7, 8 or 15-dimensional Lie algebra [30]. A system of n non-homogeneous linear second order ODEs involves $2n^2 + n$ arbitrary coefficients which was the major difficulty to deal with them for the classification purpose. Wafo Soh and Mahomed made a remarkable attempt to reduce the number of coefficients [97], by invertible point transformations which ensure the same symmetry structure of systems mapped into each other. They obtained an optimal canonical form [97], to resolve the issue of complete group classification of linear two dimensional systems of second order ODEs. They obtained five equivalence classes by proving that the allowable dimensions of the symmetry Lie algebras for such linear systems are $5, 6, 7, 8$ or 15.

A connection between a two dimensional system of quadratically semi-linear second order ODEs and scalar cubically semi-linear second order ODEs was used to re-derive Lie's linearization conditions [38] by geometric method. Further, a relationship between Killing vectors of the manifold (isometries) and symmetries of systems of the geodesic equations on the manifold [25] has been exploited to construct the linearizability conditions for maximally symmetric systems of two quadratically semi-linear ODEs [71, 83]. Treating the coefficients of systems as Christoffel symbols and requiring the Riemann curvature tensor zero results in a $sl(n+2,\mathbb{R})$ algebra. Mahomed and Qadir presented the most general class of linearizable systems which can be transformed to a system of free particle equations by *geometric* linearization [70]. For the construction of their linearization criteria, their departure point was a system of n geodesic equations which is transformable to a system of the free particle equations if the underlying space is flat. They re-derived the Lie linearization criteria for scalar second order ODEs by projecting the system of two geodesic equations down by one dimension [5] and using the geometric linearization developed for the quadratically semi-linear systems [71]. They described a procedure to construct the linearizing point transformations and extended this scheme to linearize systems of n cubically semi-linear second order ODEs.

CSA deals with those systems of DEs that arise from systems of complex ODEs. The base ODEs are said to be complex because their dependent variables are considered as complex functions of the complex or real independent variables. Such complex equations with the complex and real independent variables yield systems of PDEs and ODEs on splitting into the real and imaginary parts, respectively. Likewise, systems of PDEs can be studied in terms of the base systems of complex PDEs with CSA. For definiteness, consider a second order ODE written in semi-linear form where both the dependent and independent variables are complex. Then the right side of the equation is an arbitrary function of the complex first derivative of the dependent variable as well as of the dependent and independent ones (which are also complex). Writing these complex variables in terms of the real and imaginary parts and breaking the equation into the two parts, as a consequence a system of two second order PDEs emerges with a set of two CR-equations which enforces the existence of complex derivatives for the base complex equations. Moreover, both the equations of such systems satisfy another set of CR-equations, i.e., when both the equations are differentiated with respect to real dependent variables and their derivatives they constitute CR-type equations. Likewise, if the dependent variable in the semi-linear form of the complex base scalar ODE is considered as a complex function of a real independent variable then this procedure yields a system of two second order ODEs. In contrast with the earlier case, here both the emerging real ODEs are found to satisfy only the later set of CR-equations, i.e., the CR-type equations which involve derivatives of both the equations of the system with respect to the dependent variables and their derivatives. Therefore, these systems are called CR-structured systems.

The CR-structured two dimensional systems can be solved: by extending the linearization or integration algorithms developed for the base scalar ODEs to these systems; and by employing complex symmetry generators associated with the base complex ODEs to deal with the variational problem for such systems [2, 3]. The connection between the scalar and systems of two ODEs allows us to study the equivalence and linearization of systems by means of the equivalence of the base scalar ODEs [4]. An obvious question would be whether every CR-structured system inherits all the symmetry properties (integrability, linearizability or solvability) of the base complex equation? In other words, whether all the systems in this class form an equivalence class if they are obtained from one equivalence class of the corresponding complex base equation. If the answer is affirmative, then this class of systems can be solved trivially, but no conclusion has been arrived at yet. Therefore, in this thesis, we study two and four dimensional CR-structured systems of second order ODEs which emerge from the linear or linearizable complex scalar or systems of second order ODEs. In this work, we address in detail the following issues associated with CR-structured systems of second order ODEs: (i) equivalence of the most general complex-linearizable systems and the most general form of the geometric linearizable systems; (ii) the symmetry structure of linearizable CR-structured systems; (iii) solvability of non-linearizable CR-structured systems with less than four Lie point symmetries; (iv) the role of complex vector fields of the base complex ODEs in providing Lie symmetries for CR-structured systems; (v) linearization of four dimensional systems. Semi-invariants for a class of systems of two parabolic type PDEs are also deduced by exploiting the invariance of scalar complex parabolic PDEs.

In the second chapter of this thesis, a comparison of the complex-linearizable and linearizable classes of two dimensional systems of cubically semi-linear (in the first derivative) second order ODEs is presented. CSA was employed to obtain the complex-linearizable class of systems of two second order ODEs [4]. It was obtained by treating the dependent variable of the most general form of a linearizable scalar ODE as a complex function and splitting it into real and imaginary parts. Separately, by projecting a system of three quadratically semi-linear ODEs, namely the geodesic equations, down to two dimensional systems of cubically semi-linear second order ODEs led to the geometric linearizable class [70]. However, there exist another class of such systems which is obtained by mapping an arbitrary two dimensional system into a system of free particle equations via general point transformations [74]. This form is the most general linearizable form of systems as compared to the geometric linearizable form of two dimensional systems of second order ODEs. The difference was apparent in the former two classes due to the number of coefficients appearing in the associated systems, which leads to a question: are all these classes distinct or do they have some overlap or are identical under point transformations? The point transformations are considered to argue their existence to map one class [4] into the other [70]. This investigation [86] leads to the result that there are no point transformations to put both of them into a single equivalence class and they are distinct from each other.

The third chapter deals with the symmetry structure of the linearizable class of two dimensional CR-structured systems of second order ODEs [87], obtainable from complex *linearizable* base ODEs. The first issue raised concerns the conditions which must be satisfied by an arbitrary general system of two second order ODEs to correspond to a complex second order ODE. It is shown that the coefficients appearing in the general form of systems must satisfy the CR-equations to correspond to complex scalar ODEs. These CR-equations are different from those associated with systems of PDEs as they involve the derivatives of the coefficients of the systems with respect to both the dependent variables and their first derivatives. Further, an *optimal canonical form* of linear CR-structured systems has been established by using linear forms of the base second order equations, other than the free particle case. The linear form obtained by CSA is the reduced optimal canonical form because it involves one arbitrary coefficient. This form of two dimensional linear CR-structured systems has been investigated to obtain their symmetry structure, which gave three of the five equivalence classes [97]. These three equivalence classes of the *linearizable* CR-structured systems of two second order ODEs are found to have a 6, 7 or 15-dimensional Lie point symmetry algebra. Moreover, it is proved that there does not exist a 5 or 8-dimensional Lie point symmetry algebra for the CR-structured linear systems of two second order ODEs.

The complex linearizing transformations of the base equations play an essential role in providing the real linearizing transformations for the corresponding CR-structured systems. However, the complex transformations do not necessarily transform the corresponding systems to linear forms. In fact, if they are of the form

$$\tau_1 : (real, complex) \rightarrow (real, complex),$$

i.e., the point transformations which map the real independent and complex dependent variables to real independent and complex dependent variables, then they can be used to map the corresponding systems to their simplest forms. A class of two dimensional CR-structured *non-linearizable* systems of cubically semi-linear second order ODEs arising from a linearizable scalar ODE [89], is presented in the fourth chapter. The complex solutions of the complex base ODEs are employed to deduce solutions of the associated CR-structured two dimensional systems of this class. The CR-structured two dimensional systems are shown to be solvable regardless of the paucity in the least number of symmetries [94] required to integrate or linearize them. Moreover, the role of the CR-equations in establishing a correspondence of solutions of the CR-structured non-linearizable systems with the base scalar second order ODEs is presented diagrammatically. The complex linearizing transformations τ_1 described above and

$$\tau_2 : (real, complex) \rightarrow (complex, complex),$$

which map the real independent and complex dependent variables to complex independent and dependent ones, may reduce a complex scalar ODE to the free particle equation with a real and complex independent variable, respectively. Therefore, the former equation with real independent variable, splits into a system of two ODEs while the later yields a system of PDEs along with the usual CR-equations. This system of PDEs and the associated CR-equations are shown to provide a solution algorithm which only requires complex linearizability of the complex base equation to solve the emerging systems with less than four symmetry generators.

The subsequent chapter is on the complex symmetry generators [72] associated with the CR-structured two dimensional systems of second order ODEs [66]. The components of these complex symmetries are found to split into components of the real symmetry generators of the CR-structured systems earlier. Here it is shown that this statement is not true in general, though the complex symmetry generators of the associated complex ODEs may yield the complete symmetry algebras of the emerging systems, they often result in a few extra operators. In fact, the complex generators either split into two operators called *Lie-like operators* which are different from Lie symmetries or provide components of two Lie point symmetries for the CR-structured systems. Moreover, decomposition of a complex vector field may result in a symmetry generator and a "Lie-like" operator for the associated systems. The conditions which relates such operators with the systems of ODEs are known as Lie-like conditions and are presented here. Furthermore, these conditions are refined to reduce them to Lie symmetry conditions for systems by using the CR-structure of the components of the Lie-like operators. Exploiting these refinements it is proved that in general such operators and symmetry generators associated with the CR-structured systems are different from one another. The refined Lie-like conditions provide criteria which show when and how the Lie point symmetries of the CR-structured systems are extractable from complex symmetry generators of the base complex second order ODEs.

In the sixth chapter, linearization of a class of systems of four second order ODEs is addressed [88] which emerge from complex two dimensional systems of second order. Two canonical forms of such systems have been derived with fewer arbitrary coefficients than their real analogues. The concept of *nested complexification* is introduced, i.e., complexification of a scalar ODE twice. The procedure of nested complexification yields the simplest possible linear class of four dimensional systems. The geometric linearization criteria for systems of two cubically semi-linear second order ODEs [70] are extended to a class of systems of four ODEs by CSA. In fact, this is an extension of complex-linearization as it only ensure the linearizability of the base complex two dimensional systems, i.e., linearization of the corresponding real four dimensional systems is not guaranteed by the criteria presented. It is shown that a complex-linearizable four dimensional system can be linearized if complex transformations of the form

$$\tau_3 : (real, (complex, complex)) \rightarrow (real, (complex, complex)),$$

are associated with the complex base system to transform it into a maximally symmetric two dimensional system.

Invariance of a class of systems of two linear parabolic type PDEs under complex equivalence transformations of the dependent variables has been studied in the seventh chapter. If a linear parabolic PDE involves a complex dependent function of a complex independent variable then splitting it into the real and imaginary parts provides a coupled system of two real linear PDEs. These systems of parabolic type PDEs are transformable to the same family by utilizing the equivalence mappings of the corresponding complex PDE if it can be mapped to itself via such transforms [67]. Moreover, these systems have fewer arbitrary coefficients than their classical analogues and all these coefficients also satisfy the CR-equations. The algebraic properties of the systems may not necessarily correspond to those of the base complex PDEs, but the equivalence transformations for such systems can be obtained from the complex split of the equivalence transformations of the corresponding scalar PDEs. It is due to this subclass of linear equivalence transformations of the dependent variables of systems of linear PDEs, that the reductions of CR-structured systems to their simple forms are achieved. Semi-invariants under changes of dependent variables obtained for this class of systems of PDEs are found to correspond to complex Laplace-type invariants [38] of the scalar complex linear parabolic PDE. Moreover, the real transformations which yield such semi-invariants, also emerge from the complex transformation of the dependent variable that produced the complex semi-invariants of the complex parabolic PDEs. It is shown that the real and imaginary parts of the complex Ibragimov invariants of the complex scalar parabolic PDE are actually the Ibragimov-type invariants for the emerging class of systems of PDEs.

Further, invariance of special subclasses (uncoupled and coupled) of systems of two parabolic type PDEs has been investigated. These subclasses turned out to be same as one may have from the general real class of systems of two parabolic PDEs by restricting their coefficients in a special manner. Therefore, these uncoupled and coupled cases are shown to establish a fundamental connection between real and complex symmetry analysis for such systems of parabolic type PDEs. Indeed, this work extends the invariance criteria deduced for the scalar linear second order PDEs, to invariance criteria for systems of second order linear PDEs. This correspondence is not apparent in the work pursued in this thesis, for systems of two second order ODEs, i.e., Lie linearization theorem for the scalar second order ODEs is not extendable to linearization criteria for the corresponding CR-structured systems of ODEs. Therefore, invariance of systems of PDEs is investigated here in order to characterize CR-structured systems of DEs and their associated invariants. These results show that invariance of the CR-structured systems of DEs may follow from invariance of the base complex DEs but under certain subclasses of the invertible equivalence point transformations of the variables.

1.1 Lie Symmetry Analysis for Scalar Second Order ODEs

The basic definitions and necessary tools to deal with ODEs by symmetry methods are provided in this section. It is devoted to the development of the theory of continuous groups and its translation to workable algorithms to solve scalar ODEs. For instance, Lie groups, symmetry transformations, the corresponding infinitesimal generators and the algorithm to find them is discussed (see, [94]). Moreover, multi-parameter Lie groups and Lie algebras are also given in detail. In the literature one often encounters two methods to solve second order ODEs by using their symmetries; the first is the successive reduction of order and the second is the reduction to Lie's canonical forms [35, 94], once such equations posses two symmetry generators. Further, for the second order ODEs a linearization theorem [35] based on invertible point transformations, is presented which requires an 8-dimensional algebra. The classification of second order ODEs with respect to 2-dimensional algebras is provided to reveal their integrability. The subsequent subsection summarizes the equivalence and linearization of two dimensional systems of second order ODEs [62]. Symmetry analysis for systems or higher order ODEs gets complicated, e.g., the extension of Lie's linearization algorithm to two dimensional systems of second order or scalar third order ODEs is difficult to manage without computer algorithms. Despite these obstacles, a few linearization algorithms for systems are developed [4, 70] which are given in the third section. Then, in the last section CSA is proposed as a tool for studying symmetry properties of systems by means of the invariance properties of the base complex ODEs.

1.1.1 Point Transformations and Infinitesimal Generators

One uses an appropriate change of the dependent and independent variables, known as point transformations

$$\tilde{x} = \tilde{x}(x, u), \quad \tilde{u} = \tilde{u}(x, u), \tag{1.1}$$

which maps points (x, u) into points $(\tilde{x}, \tilde{u})$, when dealing with a scalar nth order ODE

$$u^{(n)}(x) = \omega(x, u, u', \ldots, u^{(n-1)}), \tag{1.2}$$

to reduce it into a simplest possible form (see, e.g., [35, 94]). Finding all such invertible transformations of a given ODE (especially the discrete ones) is very difficult and certainly hopeless task. Therefore, the context of symmetries requires dependence of the point transformations (1.1) on at least one arbitrary parameter ϵ, which then become

$$\tilde{x} = \tilde{x}(x, u; \epsilon), \quad \tilde{u} = \tilde{u}(x, u; \epsilon). \tag{1.3}$$

Let these transformations (1.3) lie in a region $D \subset \mathbb{R}^2$ that are defined for each $\tilde{x}$, $\tilde{u}$ in D, depending on the parameter $\epsilon \in S \subset \mathbb{R}$, with a law of composition $\varphi(\epsilon, \delta)$, then they form a one parameter group of transformations if: for each $\epsilon \in S$, the transformations (1.3) are one-to-one onto D, which implies $(\tilde{x}, \tilde{u}) \in D$; S with the law of composition φ forms a group; $\tilde{x}(x, u; e) = x$, and $\tilde{u}(x, u; e) = u$, at $\epsilon = e$ that are called identity transformations; if $\tilde{x} = \tilde{x}(x, u, \epsilon)$, $\tilde{u} = \tilde{u}(x, u, \epsilon)$ and $\tilde{\tilde{x}} = \tilde{\tilde{x}}(\tilde{x}, \tilde{u}, \delta)$, $\tilde{\tilde{u}} = \tilde{\tilde{u}}(\tilde{x}, \tilde{u}, \delta)$, then $\tilde{\tilde{x}} = \tilde{\tilde{x}}(x, u, \varphi(\epsilon, \delta))$, and $\tilde{\tilde{u}} = \tilde{\tilde{u}}(x, u, \varphi(\epsilon, \delta))$. Moreover, a group of transformations is called a one parameter Lie group if it satisfies the following conditions (including above axioms, given for groups of transformations): ϵ is a continuous parameter and $\epsilon = 0$ yields the identity transformations; the defining functions $\tilde{x}$ and $\tilde{u}$ are infinitely differentiable with respect to x, u and analytic functions of ϵ in S; $\varphi(\epsilon, \delta)$ is an analytic function of ϵ and δ.

A group of transformations is admitted by an ODE if it leaves the family of its solution curves invariant, i.e., maps solutions of the underlying equation into solutions of the same equation. In other words, it re-parametrizes the solution curves. The one parameter group moves an arbitrary point (x_0, u_0) in the (x, u)-plane when the parameter varies. The images of the arbitrary point move along some line or curve. Repeating this procedure by considering different initial points yields representative curves in the (x, u)-plane, of those points which are transformable to each other under the action of the group. The set of curves (called *orbits* of the group) representing the one parameter group of transformations (1.3) is completely characterized by the field of its tangent vectors $\mathbf{Z}$, in the plane. To describe this idea concisely one needs to expand (1.3) by taking (x, u) an arbitrary point, in the form

$$\begin{aligned} \tilde{x}(x, u; \epsilon) &= x + \epsilon\xi(x, u) + \cdots = x + \epsilon\mathbf{Z}x + \cdots, \\ \tilde{u}(x, u; \epsilon) &= u + \epsilon\eta(x, u) + \cdots = u + \epsilon\mathbf{Z}u + \cdots, \end{aligned} \tag{1.4}$$

where $\xi(x, u)$ and $\eta(x, u)$ are defined as

$$\xi(x, u) = \frac{\partial\tilde{x}}{\partial\epsilon}, \quad \eta(x, u) = \frac{\partial\tilde{u}}{\partial\epsilon}, \quad \text{at } \epsilon = 0, \tag{1.5}$$

which are components of the operator

$$\mathbf{Z} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}, \tag{1.6}$$

which can also be written as

$$\mathbf{Z} = \xi\partial_x + \eta\partial_u. \tag{1.7}$$

The operator $\mathbf{Z}$, is called an infinitesimal generator of (1.1) as its repeated application generates these finite transformations when

$$\frac{\partial\tilde{x}}{\partial\epsilon} = \xi(\tilde{x}, \tilde{u}), \quad \frac{\partial\tilde{u}}{\partial\epsilon} = \eta(\tilde{x}, \tilde{u}), \tag{1.8}$$

are integrated with the initial conditions $\tilde{x} = x$, $\tilde{u} = u$ at $\epsilon = 0$. It is apparent from the above relations that the integral curves of $\mathbf{Z}$, are the representatives of the group orbits, which are uniquely determined by the infinitesimal operators. It is not necessary to use (1.3) in the context of symmetry analysis for DEs instead of (1.1), which clearly are not the members of the one parameter group of transformations. The one parameter group of transformations (1.3) is mainly considered to find and use the corresponding symmetries, though the symmetry generators are linear operators, the underlying transformations might be complicated to deal with. Furthermore, the symmetry generator (1.6) is reducible to a simplest (canonical) form $\mathbf{Z} = \partial_{\tilde{x}}$, when its components are worked out in terms of the new variables $(\tilde{x}, \tilde{u})$, by solving the system of equations

$$\mathbf{Z}\tilde{x} = 1, \quad \mathbf{Z}\tilde{u} = 0. \tag{1.9}$$

The canonical representation of (1.6) is guaranteed by the theory of PDEs which always yields a non-trivial solution of (1.9).

1.1.2 Extended Transformations and Prolongations of Symmetry Generators

A symmetry generator or transformation of the form (1.1) and (1.3) requires an extension or prolongation to include all the derivatives (see, e.g., [94]), i.e., these extensions of (1.6) must be up to nth derivative to act on (1.2). To this aim, the derivatives can trivially be transformed by defining

$$\begin{aligned}
\tilde{u}' &= \frac{d\tilde{u}(x,u;\epsilon)}{d\tilde{x}(x,u;\epsilon)} = \frac{u'(\partial\tilde{u}/\partial u) + (\partial\tilde{u}/\partial x)}{u'(\partial\tilde{x}/\partial u) + (\partial\tilde{x}/\partial x)} = \tilde{u}'(x,u,u';\epsilon),\\
\tilde{u}'' &= \frac{d\tilde{u}'(x,u,u';\epsilon)}{d\tilde{x}(x,u;\epsilon)} = \tilde{u}''(x,u,u',u'';\epsilon),\\
&\vdots \qquad\qquad\qquad \vdots\\
\tilde{u}^{(n)} &= \frac{d\tilde{u}^{(n-1)}(x,u,u',\dots,u^{(n-2)};\epsilon)}{d\tilde{x}(x,u;\epsilon)} = \tilde{u}'(x,u,u',\dots,u^{(n-1)};\epsilon).
\end{aligned} \tag{1.10}$$

These are the derivatives of, and with respect to, the transformed dependent and independent variables, respectively. Expanding the above relations in the same manner as it was done before in (1.4) leads to

$$\begin{aligned}
\tilde{u}' &= u' + \epsilon\eta'(x,u,u') + \cdots,\\
\tilde{u}'' &= u'' + \epsilon\eta''(x,u,u',u'') + \cdots,\\
&\vdots \qquad\qquad \vdots\\
\tilde{u}^{(n)} &= u^{(n)} + \epsilon\eta^{(n)}(x,u,u',\dots,u^{(n-1)}) + \cdots,
\end{aligned} \tag{1.11}$$

where $\eta, \eta', \dots, \eta^{(n)}$ are defined by

$$\eta' = \frac{\partial\tilde{u}'}{\partial\epsilon}, \dots, \eta^{(n)} = \frac{\partial\tilde{u}^{(n)}}{\partial\epsilon}, \quad \text{at } \epsilon = 0. \tag{1.12}$$

On the other hand, invoking (1.4) the set of equations (1.10) becomes

$$\begin{aligned}
\tilde{u}' &= \frac{du + \epsilon d\eta + \cdots}{dx + \epsilon d\xi + \cdots} = u' + \epsilon\left(\frac{d\eta}{dx} - u'\frac{d\xi}{dx}\right) + \cdots,\\
&\vdots \qquad\qquad\qquad \vdots\\
\tilde{u}^{(n)} &= \frac{du^{(n-1)} + \epsilon d\eta^{(n-1)} + \cdots}{dx + \epsilon d\xi + \cdots} = u^{(n)} + \epsilon\left(\frac{d\eta^{(n-1)}}{dx} - u^{(n)}\frac{d\xi}{dx}\right) + \cdots,
\end{aligned} \tag{1.13}$$

where d/dx is the total derivative operator given by

$$\frac{d}{dx} = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + \cdots. \tag{1.14}$$

Comparison of equations (1.11) and (1.13) result in the following formula

$$\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - u^{(n)}\frac{d\xi}{dx}, \tag{1.15}$$

which determines the components of the extended generator $\mathbf{Z}$ to any order n recursively. Notice that $\eta^{(n)}$ is not the n^{th} derivative of η as is clear from (1.15). All these observations are summarized below.

Definition 1.1.1. The nth order extension or prolongation of the symmetry generator denoted by $\mathbf{Z}^{[n]}$, is represented by

$$\mathbf{Z}^{[n]} = \xi\partial_x + \eta\partial_u + \eta'\partial_{u'} + \cdots + \eta^{(n)}\partial_{u^{(n)}}, \tag{1.16}$$

where $\eta^{(j)}$'s for $j = 1, \ldots, n$, are obtainable from (1.13) or (1.15).

Since geometrically a family of curves represents the general solution of an ODE, therefore, such an ODE admits a one parameter Lie group of point transformations if solution curves are mapped into solution curves under the action of the Lie group. It leads to the following definition and theorem stated in [94].

Definition 1.1.2. (**Invariance of an ODE:**) The one-parameter Lie group of transformations (1.3) is admitted by the ODE (1.2) or leaves it invariant if and only if its nth extension leaves the solution curves invariant.
That is to say, under a transformation (1.1) the ODE of the form (1.2) does not change; both of them imply

$$\tilde{u}^{(n)}(\tilde{x}) = \omega(\tilde{x}, \tilde{u}, \tilde{u}', \ldots, \tilde{u}^{(n-1)}). \tag{1.17}$$

Theorem 1.1.3. *An nth extended infinitesimal symmetry generator* (*1.16*) *is admitted by an nth order ODE* (*1.2*) *if and only if*

$$\boldsymbol{Z}^{[n]}[u^{(n)}(x) - \omega(x, u, u', \ldots, u^{(n-1)})] = 0, \tag{1.18}$$

i.e.,

$$\eta^{(n)}(x, u, u', \ldots, u^{(n)}) = \boldsymbol{Z}^{[n-1]}\omega(x, u, u', \ldots, u^{(n-1)}), \tag{1.19}$$

when $u^{(n)}(x) - \omega(x, u, u', \ldots, u^{(n-1)}) = 0$.

Definition 1.1.4. (**An alternate formulation of symmetry:**) An nth order ODE (1.2) can be identified as a PDE in $n+1$, variables (see, e.g., [94])

$$\mathbf{A}f = (\partial_x + u'\partial_u + u''\partial_{u'} + \cdots + \omega\partial_{u^{(n-1)}})f = 0, \tag{1.20}$$

which can be given a concise form

$$\mathbf{A}f = a^j(x^k)\frac{\partial}{\partial x^i}f(x^k) = 0, \quad j, k = 1, \ldots, n+1. \tag{1.21}$$

The first integrals, i.e., a function $z = z(x, u, u', \ldots, u^{(n-1)})$, which is constant along the solutions of (1.2)

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + u'\frac{\partial z}{\partial u} + u''\frac{\partial z}{\partial u'} + \cdots + u^{(n)}\frac{\partial z}{\partial u^{(n-1)}} = 0, \tag{1.22}$$

when $u^{(n)} - \omega = 0$, provide such a connection between ODEs and PDEs. Consequently, the comparison of (1.20) and (1.22) obviates that every solution φ^α, of the former is a first integral of the equation (1.2) and conversely. Further, a set of n functionally independent solutions of (1.20) corresponds to the general solution of (1.2) that is obtainable from

$$\varphi^\alpha(x, u, u', \ldots, u^{(n-1)}) = \varphi_0^\alpha, \quad \alpha = 1, \ldots, n, \tag{1.23}$$

after eliminating all the derivatives of u from it. In order to describe the conditions which relate a symmetry generator $\mathbf{Z}$, with the PDE (1.20) consider its solution set φ^α. Since a symmetry maps solutions into solutions thus

$$\mathbf{Z}\varphi^\alpha = \Omega^\alpha(\varphi^\beta), \tag{1.24}$$

which leads to

$$\mathbf{A}\varphi^\alpha = 0 = \mathbf{A}\Omega^\alpha, \tag{1.25}$$

as φ^α being a solution implies that $\Omega^\alpha(\varphi^\beta)$ is also a solution. A linear operator like the operators $\mathbf{Z}$ and $\mathbf{A}$ is their commutator

$$[\mathbf{Z}, \mathbf{A}] = \mathbf{ZA} - \mathbf{AZ} = -(\mathbf{A}\xi)\partial_x + [(\mathbf{Z}u') - (\mathbf{A}\eta)]\partial_u + \cdots + [(\mathbf{Z}\omega) - (\mathbf{A}\eta^{(n-1)})]\partial_{u^{(n-1)}}. \tag{1.26}$$

The relations presented in (1.25) yields

$$[\mathbf{Z}, \mathbf{A}]\varphi^\alpha = \mathbf{Z}(\mathbf{A}\varphi^\alpha) - \mathbf{A}(\mathbf{Z}\varphi^\alpha) = 0. \tag{1.27}$$

Since this is true for all φ^α, therefore, $[\mathbf{Z}, \mathbf{A}]f = 0$, has the same solution as (1.20). It implies that

$$[\mathbf{Z}, \mathbf{A}] = \lambda(x, u, \ldots, u^{(n-1)})\mathbf{A}, \tag{1.28}$$

must be satisfied for $\mathbf{Z}$, to be a symmetry of (1.20). Now comparing the coefficients of $\partial_x, \partial_u, \partial_{u'}, \ldots$ in the above equation after inserting (1.26), yields

$$-\mathbf{A}\xi = -(\frac{\partial\xi}{\partial x} + u'\frac{\partial\xi}{\partial u}) = \lambda, \tag{1.29}$$

which defines λ. Writing $\mathbf{A}$ as

$$\mathbf{A} = \frac{d}{dx} \quad (\text{mod } u^{(n)} = \omega), \tag{1.30}$$

and using (1.29), equation (1.28) expands to

$$\begin{aligned}
&(\eta' - \frac{d\eta}{dx})\partial_u + (\eta'' - \frac{d\eta'}{dx})\partial_{u'} + \cdots + (\mathbf{Z}\omega - \frac{d\eta^{(n-1)}}{dx})\partial_{u^{(n-1)}} \\
&= -\frac{d\xi}{dx}(u'\partial_u + u''\partial_{u'} + \cdots + \omega\partial_{u^{(n-1)}}) \quad (\text{mod } u^{(n)} = \omega).
\end{aligned} \tag{1.31}$$

Notice that the above equation determines the components of an extended generator $\mathbf{Z}$ of order n, which are given in (1.15). Equation (1.31) leads us to

$$\mathbf{Z}\omega = A\eta^{(n-1)} - \omega\frac{d\xi}{dt}, \tag{1.32}$$

which derives the following conditions.

1.1.3 Lie Point Symmetry Conditions

The necessary tools are now in place to work out Lie point symmetry conditions [94] for an ODE of the form (1.2) which yield a practicable algorithm to find the components of the vector field **Z**. Likewise, the most general form of an nth order ODE corresponding to an operator (1.16) is obtainable from this algorithm. The present work mainly concerns those systems which emerge from a scalar second order ODE, therefore, the explicit derivation of the symmetry conditions for these equations is presented. To this end, apply the second extension of **Z**, on both sides of a scalar second order ODE

$$\mathbf{Z}^{[2]}u'' = \mathbf{Z}^{[1]}\omega(x, u, u'), \tag{1.33}$$

when

$$u'' = \omega(x, u, u'). \tag{1.34}$$

It implies

$$\eta''(x, u, u', u'') = \xi(x, u)\omega_{,x} + \eta(x, u)\omega_{,u} + \eta'(x, u, u')\omega_{,u'}, \tag{1.35}$$

where

$$\begin{aligned} \eta' &= \eta_{,x} + (\eta_{,u} - \xi_{,x})u' - \xi_{,u}u'^2, \\ \eta'' &= \eta_{,xx} + (2\eta_{,xu} - \xi_{,xx})u' + (\eta_{,uu} - 2\xi_{,xu})u'^2 - \xi_{,uu}u'^3 + (\eta_{,u} - 2\xi_{,x} - 3\xi_{,u}u')u'', \end{aligned} \tag{1.36}$$

where $,x$ and $,u$ in the subscripts denote the partial derivatives of the coefficients with respect to the independent and dependent variables, respectively, are easily obtainable from (1.15). Inserting (1.36) into (1.35) constitutes the Lie point symmetry conditions for scalar second order ODEs.

1.1.4 Multiple Parameter Lie Groups and Lie Algebras

The transformations (1.3) may depend on more than one parameter such as

$$\tilde{x} = \tilde{x}(x, u; \epsilon_N), \quad \tilde{u} = \tilde{u}(x, u; \epsilon_N), \quad \text{where } N = 1, \ldots, r, \tag{1.37}$$

to make an r-parameter group by satisfying all the group axioms with distinct ϵ_N (see, e.g., [11]). An infinitesimal generator $\mathbf{Z}_N$, can be associated with each parameter ϵ_N, by the following relation

$$\begin{aligned} \mathbf{Z}_N &= \xi_N \partial_x + \eta_N \partial_u, \\ \xi_N(x, u) &= \frac{\partial \tilde{x}}{\partial \epsilon_N}, \quad \eta_N(x, u) = \frac{\partial \tilde{u}}{\partial \epsilon_N}, \quad \text{at } \epsilon_N = 0. \end{aligned} \tag{1.38}$$

Definition 1.1.5. Consider an r-parameter Lie group of transformations (1.37) with infinitesimal generator (1.38). The commutator of $\mathbf{Z}_M$ and $\mathbf{Z}_N$ is another linear operator

$$[\mathbf{Z}_M, \mathbf{Z}_N] = \mathbf{Z}_M\mathbf{Z}_N - \mathbf{Z}_N\mathbf{Z}_M = (\mathbf{Z}_M\xi_N - \mathbf{Z}_N\xi_M)\,\partial_x + \cdots . \tag{1.39}$$

From which it immediately follows that

$$[\mathbf{Z}_M, \mathbf{Z}_N] = -[\mathbf{Z}_N, \mathbf{Z}_M]. \tag{1.40}$$

Theorem 1.1.6. *The commutator of any two infinitesimal generators of an r-parameter Lie group of point transformations is also an infinitesimal generator, in particular*

$$[\boldsymbol{Z}_M, \boldsymbol{Z}_N] = C^P_{MN}\boldsymbol{Z}_P, \tag{1.41}$$

where the coefficients C^P_{MN}, for $M, N, P = 1, \ldots, r$, are called the structure constants.

Definition 1.1.7. Equations (1.41) are called the commutation relations of the r-parameter Lie group of transformations (1.37) associated with (1.38).

The Jacobi identity

$$[\mathbf{Z}_M, [\mathbf{Z}_N, \mathbf{Z}_P]] + [\mathbf{Z}_N, [\mathbf{Z}_P, \mathbf{Z}_M]] + [\mathbf{Z}_P, [\mathbf{Z}_M, \mathbf{Z}_N]] = 0, \tag{1.42}$$

holds for any three generators $\mathbf{Z}_M, \mathbf{Z}_N$ and $\mathbf{Z}_P$.

Theorem 1.1.8. *The structure constants given by the commutation relations (1.41), satisfy the following relations*

$$\begin{aligned} &C^P_{MN} + C^P_{NM} = 0, \\ &C^P_{MN}C^Q_{ST} + C^P_{NT}C^Q_{SM} + C^P_{TM}C^Q_{SN} = 0. \end{aligned} \tag{1.43}$$

Definition 1.1.9. A Lie algebra $\mathcal{L}$, is a vector space over some field with an additional law of combination of elements in $\mathcal{L}$, satisfying (1.40) and (1.42) with, most importantly, closure with respect to commutation. In particular, the symmetry generators $\mathbf{Z}_M$, for $M = 1, \ldots, r$, of an r-parameter Lie group of transformations form an r-dimensional Lie algebra $\mathcal{L}^r$, over $\mathbb{R}$, if for any $\mathbf{Z}_M, \mathbf{Z}_N, \mathbf{Z}_P \in \mathcal{L}^r$, and $a, b \in \mathbb{R}$:

1. $a\mathbf{Z}_M + b\mathbf{Z}_N \in \mathcal{L}^r$;
2. $\mathbf{Z}_M + \mathbf{Z}_N = \mathbf{Z}_N + \mathbf{Z}_M$;
3. $\mathbf{Z}_M + (\mathbf{Z}_N + \mathbf{Z}_P) = (\mathbf{Z}_M + \mathbf{Z}_N) + \mathbf{Z}_P$;
4. $[\mathbf{Z}_M, \mathbf{Z}_N] \in \mathcal{L}^r$;
5. $[\mathbf{Z}_M, \mathbf{Z}_N] = -[\mathbf{Z}_N, \mathbf{Z}_M]$;

6. $[\mathbf{Z}_M,[\mathbf{Z}_N,\mathbf{Z}_P]]+[\mathbf{Z}_N,[\mathbf{Z}_P,\mathbf{Z}_M]]+[\mathbf{Z}_P,[\mathbf{Z}_M,\mathbf{Z}_N]]=0$;

7. $[a\mathbf{Z}_M+b\mathbf{Z}_N,\mathbf{Z}_P]=a[\mathbf{Z}_M,\mathbf{Z}_P]+b[\mathbf{Z}_N,\mathbf{Z}_P]$.

Definition 1.1.10. A subspace $\mathcal{S}\subset\mathcal{L}$ is called a subalgebra of the Lie algebra $\mathcal{L}$, if for any $\mathbf{Z}_M,\mathbf{Z}_N\in\mathcal{S},\ [\mathbf{Z}_M,\mathbf{Z}_N]\in\mathcal{S}$.

Definition 1.1.11. A subalgebra $\mathcal{S}\subset\mathcal{L}$, is called an ideal or normal subalgebra of $\mathcal{L}$, if for any $\mathbf{Y}\in\mathcal{S}$, $\mathbf{Z}\in\mathcal{L}$, $[\mathbf{Y},\mathbf{Z}]\in\mathcal{S}$.

Definition 1.1.12. $\mathcal{L}^q$, is a q-dimensional solvable Lie algebra if there exists a chain of subalgebras

$$\mathcal{L}^1\subset\mathcal{L}^2\subset\cdots\subset\mathcal{L}^{(q-1)}\subset\mathcal{L}^q=\mathcal{L}^q, \tag{1.44}$$

such that $\mathcal{L}^{(k)}$, is a k-dimensional Lie algebra and $\mathcal{L}^{(k-1)}$, is an ideal of $\mathcal{L}^{(k)}\ for\ k=1,2,\ldots,q$.

Definition 1.1.13. $\mathcal{L}$, is called an Abelian Lie algebra if for any $\mathbf{Z}_M,\mathbf{Z}_N\in\mathcal{L},\ [\mathbf{Z}_M,\mathbf{Z}_N]=0$.

Theorem 1.1.14. *Every Abelian Lie algebra is a solvable Lie algebra.*

Theorem 1.1.15. *Every* 2*-dimensional Lie algebra is solvable.*

The main focus so far was to demonstrate the procedures which connect infinitesimal transformations with their generators and ODEs. Apart from being devoted to basic manipulations of symmetry generators and associated Lie point symmetry algebras this review concerns two solution schemes described for the scalar second order ODEs: the first one is their integration; and the equivalence and linearization problems comprise the second.

1.1.5 Integration of Second Order ODEs

A Lie point symmetry algebra of dimension one ensures the reduction of order of a second order equation by one. Therefore, a 2-dimensional algebra is expected to integrate a second order equation completely. A classification of integrable scalar second order ODEs with 2-dimensional algebras is illustrated [35, 62, 94] on the basis of the structural properties of the commutators $[\mathbf{Z}_1,\mathbf{Z}_2]$ and $\delta=\xi_1\eta_2-\eta_1\xi_2$, where $\mathbf{Z}_1=\xi_1\partial_x+\eta_1\partial_u$ and $\mathbf{Z}_2=\xi_2\partial_x+\eta_2\partial_u$. The structural properties here represent the invariance of both the above quantities under the change of bases in $\mathcal{L}^2$ and the variables x, u. The following theorem is stated for the classification of such algebras [35].

Theorem 1.1.16. *Any* 2*-dimensional algebra can be reduced to one of the following four inequivalent classes*

1. $[\boldsymbol{Z}_1,\boldsymbol{Z}_2]=0,\ \delta\neq0,$

2. $[\boldsymbol{Z}_1,\boldsymbol{Z}_2]=0,\ \delta=0,$

3. $[\boldsymbol{Z}_1, \boldsymbol{Z}_2] \neq 0,\ \delta \neq 0,$

4. $[\boldsymbol{Z}_1, \boldsymbol{Z}_2] \neq 0,\ \delta = 0.$

by a suitable change of basis.

Invariance of the above algebras under the dependent and independent variables reduces them to those given in Table 1.1. The representative scalar second order ODEs admitting these algebras, where the basis are given in the canonical variables, are also mentioned. The Lie algebras $\mathcal{L}^s$ for

Type	symmetry generators	Representative equations
I	$\mathbf{Z}_1 = \partial_x,\ \mathbf{Z}_2 = \partial_u,$	$u'' = \omega(u'),$
II	$\mathbf{Z}_1 = \partial_u,\ \mathbf{Z}_2 = x\partial_u,$	$u'' = \omega(x),$
III	$\mathbf{Z}_1 = \partial_u,\ \mathbf{Z}_2 = x\partial_x + u\partial_u,$	$u'' = \frac{1}{x}\omega(u'),$
IV	$\mathbf{Z}_1 = \partial_u,\ \mathbf{Z}_2 = u\partial_u,$	$u'' = u'\omega(x),$

Table 1.1: Lie canonical forms of scalar equations.

$2 < s \leq 8$ associated with the second order equations led to the problem of complete classification of such equations. Lie proved that the algebras associated with the second order equations have the maximum $0, 1, 2, 3$ or 8 dimensions. Therefore, an equation with the s-dimensional algebra for $s \geq 4$ necessarily admits an 8-dimensional algebra. The equations with such an algebra of vector fields are said to be maximally symmetric, e.g., the free particle equation. Lie presented a linearization test to uncover maximally symmetric ODEs which may arise in a disguised (non-linear) form, via invertible point transformations.

1.1.6 Invariants of Algebraic and Differential Equations

The *invariants* and *differential invariants* of groups of equivalence transformations of algebraic and families of DEs are abbreviated as invariants of algebraic and DEs, respectively. The invariants of subgroups of the equivalence groups of transformations are termed as *semi-invariants*. Differential invariants can be used to reduce the DEs to the equivalent integrable ones. Before going to them invariants for the algebraic equations are discussed below in detail (see, e.g., [35]).

Definition 1.1.17. Equivalence transformations of an nth degree algebraic equation

$$P_n(x) \equiv \alpha_0 x^n + n\alpha_1 x^{n-1} + \frac{n(n-1)}{2\cdot 1}\alpha_2 x^{n-2} + \cdots + n\alpha_{n-1}x + \alpha_n = 0, \tag{1.45}$$

are the invertible mappings of the form $\tilde{x} = \varrho(x)$ which convert it to an equation of the same form and degree but in general with different coefficients, i.e., to an equation

$$\tilde{P}_n(\tilde{x}) \equiv \tilde{\alpha}_0 \tilde{x}^n + n\tilde{\alpha}_1 \tilde{x}^{n-1} + \frac{n(n-1)}{2\cdot 1}\tilde{\alpha}_2 \tilde{x}^{n-2} + \cdots + n\tilde{\alpha}_{n-1}\tilde{x} + \tilde{\alpha}_n = 0. \tag{1.46}$$

Both the equations here are written in the standard form with the Binomial coefficients which are often used for successive calculation of the related invariants.

Proposition 1.1.18. *The linear fractional transformations*

$$\tilde{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \tag{1.47}$$

comprise the most general group of equivalence transformations of (1.45) with a constraint

$$\alpha\delta - \beta\gamma \neq 0. \tag{1.48}$$

Definition 1.1.19. A function $\varpi(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is said to be an invariant of the equation (1.45) if it is not altered by the transformations (1.47), i.e., under these equivalence mappings

$$\varpi(\alpha_0, \alpha_1, \ldots, \alpha_n) = \varpi(\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n). \tag{1.49}$$

Rewriting the equation (1.45) by taking $x = \frac{u}{v}$

$$Q_n(u, v) \equiv \alpha_0 u^n + n\alpha_1 u^{n-1} v + \frac{n(n-1)}{2 \cdot 1}\alpha_2 u^{n-2} v^2 + \cdots + n\alpha_{n-1} u v^{n-1} + \alpha_n v^n = 0, \tag{1.50}$$

then the group of linear fractional transformations (1.47) becomes

$$\tilde{u} = \alpha u + \beta v, \quad \tilde{v} = \gamma u + \delta v. \tag{1.51}$$

Further, setting $\alpha = \exp(a)$ and $\delta = \exp(b)$, and considering the infinitesimal transformations

$$\tilde{u} \approx u + (au + \beta v), \quad \tilde{v} \approx v + (\gamma u + bv), \tag{1.52}$$

for which the inverse can be written in the first order of precision relative to the small parameters $a,\ b,\ \beta,\ \gamma$ as

$$u \approx \tilde{u} - (a\tilde{u} + \beta\tilde{v}), \quad v \approx \tilde{v} - (\gamma\tilde{u} + b\tilde{v}), \tag{1.53}$$

leads to *infinitesimal method* to deal with the algebraic invariants.

Definition 1.1.20. An equivalence transformation associated with an nth order general linear homogeneous ODE with variable coefficients

$$u^{(n)} + n\beta_1(x)u^{(n-1)} + \frac{n!\beta_2(x)}{(n-2)!2!}u^{(n-2)} + \cdots + n\beta_{n-1}(x)u' + \beta_n(x)u = 0, \tag{1.54}$$

is an invertible change of the dependent and independent variables (x, u) that preserves the linearity, homogeneity and order of the ODE.

Proposition 1.1.21. *An infinite group of linear transformations of the dependent variables*

$$u = \varsigma(x)v, \quad \varsigma(x) \neq 0, \tag{1.55}$$

and invertible change of the independent variable

$$\tilde{x} = \sigma(x), \quad \sigma'(x) \neq 0, \tag{1.56}$$

where $\varsigma(x)$ and $\sigma(x)$ are arbitrary n times continuously differentiable functions, represents the most general form of the equivalence mappings for equation (1.54).

To illustrate the infinitesimal approach for finding the semi-invariants of an ODE consider the equation (1.54) when $n = 3$ which reads as

$$u''' + 3\beta_1(x)u'' + 3\beta_2(x)u' + \beta_3(x)u = 0. \tag{1.57}$$

Implementing a subgroup of the transformations (1.55), i.e., let $\varsigma(x) = 1 - \epsilon\tilde{\varsigma}(x)$ where ϵ is a small parameter, the above equation becomes

$$v''' + 3\tilde{\beta}_1(x)v'' + 3\tilde{\beta}_2(x)v' + \tilde{\beta}_3(x)v = 0, \tag{1.58}$$

where

$$\begin{aligned}
\tilde{\beta}_1 &\approx \beta_1 - \epsilon\tilde{\varsigma}', \\
\tilde{\beta}_2 &\approx \beta_2 - \epsilon(\tilde{\varsigma}'' + 2\beta_1\tilde{\varsigma}'), \\
\tilde{\beta}_3 &\approx \beta_3 - \epsilon(\tilde{\varsigma}''' + 3\beta_1\tilde{\varsigma}'' + 3\beta_2\tilde{\varsigma}').
\end{aligned} \tag{1.59}$$

The group generator readable from the above formula (1.59) when prolonged to include the derivatives of $\beta_j,\ j = 1, 2, 3$ has the form

$$\begin{aligned}
\mathbf{Z}_\varsigma = {} & \tilde{\varsigma}'\partial_{\beta_1} + (\tilde{\varsigma}'' + 2\beta_1\tilde{\varsigma}')\partial_{\beta_2} + (\tilde{\varsigma}''' + 3\beta_1\tilde{\varsigma}'' + 3\beta_2\tilde{\varsigma}')\partial_{\beta_3} + \tilde{\varsigma}''\partial_{\beta_1'} \\
& + (\tilde{\varsigma}''' + 2\beta_1\tilde{\varsigma}'' + 2\beta_1'\tilde{\varsigma}')\partial_{\beta_2'} + (\tilde{\varsigma}^{(iv)} + 3\beta_1\tilde{\varsigma}''' + 3\beta_2\tilde{\varsigma}'' + 3\beta_1'\tilde{\varsigma}'' + 3\beta_2'\tilde{\varsigma}')\partial_{\beta_3'} + \cdots .
\end{aligned} \tag{1.60}$$

Definition 1.1.22. The differential invariants of (1.59) are the semi-invariants associated with the equation (1.58). These semi-invariants $\hat{h}$ are functions of the coefficients β_j and their derivatives $\beta_j', \beta_j'', \ldots$ of a finite order that are compatible with $\mathbf{Z}_\varsigma(\hat{h}) = 0$.

There are two independent semi-invariants of the third order ODE (1.58) given below

$$\hat{h}_1 = \beta_2 - \beta_1^2 - \beta_1', \quad \hat{h}_2 = \beta_3 - 3\beta_1\beta_2 + 2\beta_1^3 + 2\beta_1\beta_1' - \beta_2'. \tag{1.61}$$

1.1.7 Equivalence Classes of Scalar ODEs

Two ODEs or in general DEs are said to be equivalent under point transformations if one can be mapped into the other [62] and both are in agreement from the point of view of their associated symmetry algebras.

(a) First order ODEs. All the first order ODEs are equivalent to each other, therefore, an equation of the form $u'(x) = \tilde{\omega}(x, u)$, is always transformable to $u' = 0$ by point transformations.

(b) Linear second order ODEs. The simplest linear form of a second order ODE, namely the free particle equation has the 8-dimensional algebra $sl(3, \mathbb{R})$. All the linear scalar second order ODEs are transformable to each other under point transformations. Therefore, any linear second order equation

$$u'' + \alpha(x)u' + \beta(x)u = 0, \tag{1.62}$$

where prime denotes the differentiation with respect to x, is reducible to the free particle equation

$$\tilde{u}'' = 0, \tag{1.63}$$

here prime represents differentiation with respect to $\tilde{x}$. This implies that there exist only one equivalence class of such ODEs.

(c) Linear ODEs of order n, $n \geq 3$. Such linear equations are not necessarily transformable to their simplest form. The maximal Lie algebra of an nth order ODE where $n \geq 3$ is at most r-dimensional where $r \leq n + 4$.

1.1.8 Lie Linearization Theorem

Lie's linearization theorem (see, e.g., [35]) assumes that the equation (1.34) linearizes to the simplest equation (1.63), by a change of variables

$$\tilde{x} = \varphi(x, u), \quad \tilde{u} = \psi(x, u). \tag{1.64}$$

Accordingly, the derivatives $\tilde{u}'$ and $\tilde{u}''$ transform as

$$\tilde{u}' = \frac{D_x(\psi)}{D_x(\varphi)}, \quad \tilde{u}'' = \frac{D_x(D_x(\psi))}{D_x(\varphi)}. \tag{1.65}$$

The free particle equation (1.63), by inserting the above derivative transformations takes the form

$$D_x(\varphi) \cdot D_x^2(\psi) - D_x(\psi) \cdot D_x^2(\varphi) = 0, \tag{1.66}$$

where

$$D_x(\varphi) = \varphi_{,x} + u'\varphi_{,u},$$
$$D_x^2(\varphi) = \varphi_{,xx} + 2u'\varphi_{,xu} + u'^2\varphi_{,uu} + u''\varphi_{,u}. \tag{1.67}$$

Similarly, the expressions for $D_x(\psi)$ and $D_x^2(\psi)$ when inserted into (1.66) yield

$$(\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x})u'' + (\varphi_{,u}\psi_{,uu} - \psi_{,u}\varphi_{,uu})u'^3 + (\varphi_{,x}\psi_{,uu} + 2\varphi_{,u}\psi_{,xu} - \psi_{,x}\varphi_{,uu} - 2\psi_{,u}\varphi_{,xu})u'^2$$
$$+(\varphi_{,u}\psi_{,xx} + 2\varphi_{,x}\psi_{,xu} - \psi_{,u}\varphi_{,xx} - 2\psi_{,x}\varphi_{,xu})u' + \varphi_{,x}\psi_{,xx} - \psi_{,x}\varphi_{,xx} = 0. \tag{1.68}$$

Hence, dividing by the non-zero Jacobian of (1.64) $\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x}$, and writing

$$\varphi_{,u}\psi_{,uu} - \psi_{,u}\varphi_{,uu} = F_3(x,u)(\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x}),$$
$$\varphi_{,x}\psi_{,uu} - \psi_{,x}\varphi_{,uu} + 2(\varphi_{,u}\psi_{,xu} - \psi_{,u}\varphi_{,xu}) = F_2(x,u)(\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x}),$$
$$\varphi_{,u}\psi_{,xx} - \psi_{,u}\varphi_{,xx} + 2(\varphi_{,x}\psi_{,xu} - \psi_{,x}\varphi_{,xu}) = F_1(x,u)(\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x}),$$
$$\varphi_{,x}\psi_{,xx} - \psi_{,x}\varphi_{,xx} = F_0(x,u)(\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x}), \tag{1.69}$$

the general equation (1.34) reduces to

$$u'' + F_3(x,u)u'^3 + F_2(x,u)u'^2 + F_1(x,u)u' + F_0 = 0, \tag{1.70}$$

which is the most general candidate for linearization of a second order ODE.

Theorem 1.1.23. *The following statements are equivalent:*
(I) *A second order ODE (1.34) is transformable to the free particle equation (1.63) by point transformations (1.64).*
(II) *Equation (1.34) has an 8-dimensional Lie point symmetry algebra.*
(III) *The coefficients F_3, F_2, F_1 and F_0 satisfy the following integrability conditions*

$$\frac{\partial z}{\partial x} = z^2 - F_0 w - F_1 z + \frac{\partial F_0}{\partial u} + F_0 F_2,$$
$$\frac{\partial z}{\partial u} = -zw + F_0 F_3 - \frac{1}{3}\frac{\partial F_2}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial u},$$
$$\frac{\partial w}{\partial x} = zw - F_0 F_3 - \frac{1}{3}\frac{\partial F_1}{\partial u} + \frac{2}{3}\frac{\partial F_2}{\partial x},$$
$$\frac{\partial w}{\partial u} = -w^2 + wF_2 + zF_3 + \frac{\partial F_3}{\partial x} - F_1 F_3, \tag{1.71}$$

with two auxiliary functions w and z.
(IV) *The coefficients of the equation (1.70) satisfy the following set of constraints*

$$3F_{3,xx} - 2F_{2,xu} + F_{1,uu} - 3(F_1F_3)_{,x} + 3(F_0F_3)_{,u} + (F_2^2)_{,x} + 3F_3F_{0,u} - F_2F_{1,u} = 0,$$
$$3F_{0,uu} - 2F_{1,xu} + F_{2,xx} - 3(F_0F_3)_{,x} + 3(F_0F_2)_{,u} - (F_1^2)_{,u} - 3F_0F_{3,x} + F_1F_{2,x} = 0. \tag{1.72}$$

(V) *Equation (1.34) admits a 2-dimensional algebra spanned by connected operators of the form* $\boldsymbol{Z}_1 = \xi_1\partial_x + \eta_1\partial_u$, $\boldsymbol{Z}_2 = \xi_2\partial_x + \eta_2\partial_u$, *such that*

$$\xi_1\eta_2 - \eta_1\xi_2 = 0. \tag{1.73}$$

To summarize Lie Linearization test for scalar second order ODEs: equation (1.70) can be linearized if and only if the coefficients appearing in it result in the compatibility of over-determined system (1.69). Integration of (1.69) furnishes the linearizing point transformations (1.64) to map (1.70) to the free particle equation (1.63).

1.1.9 Noether Symmetries and First Integrals

The procedures to determine the Lie point symmetries and use them to solve DEs have been demonstrated in the preceding subsections. Another important application of these infinitesimal transformations to physical problems is the construction of conservation laws (see, e.g., [11]). An equation in the divergence free form

$$div\ \upsilon = D_j\upsilon^j, \tag{1.74}$$

reveals a conservation law associated with a physical system, where the vector function υ may depend on the independent variables, dependent variables (state functions) of the system and their derivatives to some order κ and D_j are the total derivative operators. The conservation law $D_t\upsilon^1 = 0$ arises for systems in classical mechanics where time t is the only independent variable, implies that υ^1 is a constant of motion. These constants of motion can reduce the number of degrees of freedom of the system by restricting its motion. Moreover, finding conservation laws for a system is often the first step towards its solution, i.e., finding more conservation laws for the system means one getting closer to the complete solution.

Emmy Noether proved that for every Lie point symmetry admitted by the action integral of the Lagrangian system one can constructively find a conservation law. *Hamilton's principle of least action* states that a Lagrangian $L = L(t, \mathbf{q}, \mathbf{v})$ where t is time, $\mathbf{q} = q^1, q^2, \ldots, q^m$ are coordinates and $\mathbf{v} = d\mathbf{q}/dt$ are velocities of particles of the system, characterizes a mechanical system. The trajectories $\mathbf{q} = \mathbf{q}(t)$ of the particles that provide an extremum of the action $S = \int_{t_0}^{t_1} L(t, \mathbf{q}, \mathbf{v})dt$ determine the motion of the system. Considering an increment $\delta q = \delta q(t)$ with the assumption that it is small everywhere in $t_0 \leq t \leq t_1$ and $\delta q(t_0) = \delta q(t_1) = 0$ causes the variation of the action integral. By employing these assumptions and necessary conditions for S to have an extremum lead to the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial v^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{1.75}$$

Noether presented a fundamental formula to construct conservation laws for Euler-Lagrange equations once their symmetries are determined. For example, in the view of Noether's description of conservation laws the conservation of angular momentum and energy are related to rotational and translational invariance in time, respectively. The determination of the invariance properties of Euler-Lagrange equations which come from a variational problem of the action integral leads us to conserved quantities.

Definition 1.1.24. The scalar second order ODE (1.34) has a Lagrangian $L(x,u,u')$, if it is equivalent to the Euler-Lagrange equation

$$\frac{d}{dx}(\frac{\partial L}{\partial u'}) - \frac{\partial L}{\partial u} = 0. \tag{1.76}$$

Definition 1.1.25. The operator (1.16) is called a Noether point symmetry generator corresponding to a Lagrangian $L(x,u,u')$ of the equation (1.34) if there exists a gauge function $B(x,u)$ such that

$$\mathbf{Z}^{[1]}(L) + D(\xi)L = D(B), \tag{1.77}$$

where D is the total derivative operator of the form (1.14).

The utility of an available Noether point symmetry generator lies in the following three theorems.

Theorem 1.1.26. *If $\boldsymbol{Z}$ is a Noether point symmetry generator corresponding to a lagrangian $L(x,u,u')$ of (1.34), then*

$$I = \xi L + (\eta - u'\xi)\frac{\partial L}{\partial u'} - B, \tag{1.78}$$

is a first integral of (1.34) or a conserved quantity associated with (1.76), with respect to the operator $\boldsymbol{Z}$.

Theorem 1.1.27. *The first integral I, associated with the Noether point symmetry $\boldsymbol{Z}$, satisfies*

$$\boldsymbol{Z}^{[1]}I = 0, \tag{1.79}$$

i.e., $\boldsymbol{Z}$ is a point symmetry generator of the first integral I of the scalar ODE (1.34).

Theorem 1.1.28. *If there exists a Noether point symmetry generator for a given Lagrangian $L(x,u,u')$ of (1.34) then the scalar ODE has solution in terms of quadrature.*

1.1.10 Lie Point Symmetry Group Classification in the Plane

For scalar second order ODEs Lie presented a complete classification. He proved that the number of symmetries $\mathbf{Z}_\vartheta$ for such equations is bounded by $0 \leq \vartheta \leq 8$. Further, he recognized that the algebra of vector fields acting in the plane associated with these ODEs have dimensions $0, 1, 2, 3$ or 8. All the other equations that have an r-dimensional algebras where $4 \leq r \leq 7$ are transformable to ODEs with an 8-dimensional algebra. Apart from Lie's contributions in this classification there also appeared a few extensions [29, 68, 78], all these algebra realizations are presented here. All these

Algebra	Realizations in the (x,u) plane
C_1	$Z_1 = \partial_x$
$C^I_{2;1}$	$Z_1 = \partial_x,\, Z_2 = \partial_u$
$C^{II}_{2;1}$	$Z_1 = \partial_u,\, Z_2 = x\partial_u$
$C^I_{2;2}$	$Z_1 = \partial_u,\, Z_2 = x\partial_x + u\partial_u$
$C^{II}_{2;2}$	$Z_1 = \partial_u,\, Z_2 = u\partial_u$
$C_{3;2}$	$Z_1 = \partial_x,\, Z_2 = \partial_u,\, Z_3 = x\partial_u$
$C^I_{3;3}$	$Z_1 = \partial_x,\, Z_2 = \partial_u,\, Z_3 = x\partial_x + (x+u)\partial_u$
$C^{II}_{3;3}$	$Z_1 = \partial_u,\, Z_2 = x\partial_u,\, Z_3 = \partial_x + u\partial_u$
$C^I_{3;4}$	$Z_1 = \partial_x,\, Z_2 = \partial_u,\, Z_3 = x\partial_x$
$C^{II}_{3;4}$	$Z_1 = \partial_u,\, Z_2 = x\partial_u,\, Z_3 = x\partial_x + u\partial_u$
$C^I_{3;5}$	$Z_1 = \partial_x,\, Z_2 = \partial_u,\, Z_3 = x\partial_x + u\partial_u$
$C^{II}_{3;5}$	$Z_1 = \partial_u,\, Z_2 = x\partial_u,\, Z_3 = u\partial_u$
$C^I_{3;6}$	$Z_1 = \partial_x,\, Z_2 = \partial_u,\, Z_3 = x\partial_x + au\partial_u,\, a \neq 0, 1$
$C^{II}_{3;6}$	$Z_1 = \partial_u,\, Z_2 = x\partial_u,\, Z_3 = (1-a)x\partial_x + u\partial_u,\, a \neq 0, 1$
$C^I_{3;7}$	$Z_1 = \partial_x,\, Z_2 = \partial_u,\, Z_3 = (bx+u)\partial_x + (bu-x)\partial_u$
$C^{II}_{3;7}$	$Z_1 = x\partial_u,\, Z_2 = \partial_u,\, Z_3 = (1+x^2)\partial_x + (x+b)u\partial_u$
$C^I_{3;8}$	$Z_1 = \partial_u,\, Z_2 = x\partial_x + u\partial_u,\, Z_3 = 2xu\partial_x + u^2\partial_u$
$C^{II}_{3;8}$	$Z_1 = \partial_u,\, Z_2 = x\partial_x + u\partial_u,\, Z_3 = 2xu\partial_x + (u^2 - x^2)\partial_u$
$C^{III}_{3;8}$	$Z_1 = \partial_u,\, Z_2 = x\partial_x + u\partial_u,\, Z_3 = 2xu\partial_x + (u^2 + x^2)\partial_u$
$C^{IV}_{3;8}$	$Z_1 = \partial_u,\, Z_2 = u\partial_u,\, Z_3 = u^2\partial_u$
$C_{3;9}$	$Z_1 = (1+x^2)\partial_x + xu\partial_u,\, Z_2 = xu\partial_x + (1+u^2)\partial_u,$ $Z_3 = u\partial_x - x\partial_u$

Table 1.2: Realizations of 1-, 2-, and 3-dimensional algebras in the real plane.

algebra realizations do not necessarily yield the corresponding second order ODEs. Indeed, there exist a few realizations in the Table 1.2 which do not represent a maximal Lie point algebra, in fact, some of the 3-dimensional algebras lead to those equations which have 8-dimensional algebras. Therefore, leaving these two cases Table 1.2 can be refined to include only those algebras which yield the corresponding ODEs and the non-linear ones, given in Table 1.3.

1.2 Lie Symmetry Analysis for Systems of Second Order ODEs

In this section Lie point symmetry conditions for the scalar second order ODEs have been extended to systems of dimension n, [94]. Two linearizable classes of such systems are also given with the linearizability criteria developed for both the classes. A complex-linearizable class of two dimensional

Algebra	Realizations	Representative equations
1-dimensional	$Z_1 = \partial_x$	$u'' = \omega(u, u')$
2-dimensional	$Z_1 = \partial_x,\ Z_2 = \partial_u$	$u'' = \omega(u')$
2-dimensional	$Z_1 = \partial_u$ $Z_2 = x\partial_x + u\partial_u$	$xu'' = \omega(u')$
3-dimensional	$Z_1 = \partial_x,\ Z_2 = \partial_u$ $Z_3 = x\partial_x + (x+u)\partial_u$	$u'' = A\exp(-u')$
3-dimensional	$Z_1 = \partial_x,\ Z_2 = \partial_u$ $Z_3 = x\partial_x + au\partial_u$	$u'' = Au'^{\frac{a-2}{a-1}}, a \neq 0, \frac{1}{2}, 2$
3-dimensional	$Z_1 = \partial_x,\ Z_2 = \partial_u$ $Z_3 = (bx+u)\partial_x + (bu-x)\partial_u$	$u'' = A(1+u'^2)^{\frac{3}{2}}\exp(b\arctan u')$
3-dimensional	$Z_1 = \partial_u,\ Z_2 = x\partial_x + u\partial_u$ $Z_3 = 2xu\partial_x + u^2\partial_u$	$xu'' = Au'^3 - \frac{1}{2}u'$
3-dimensional	$Z_1 = \partial_u,\ Z_2 = x\partial_x + u\partial_u$ $Z_3 = 2xu\partial_x + (u^2 - x^2)\partial_u$	$xu'' = u' + u'^3 + A(1+u'^2)^{3/2}$
3-dimensional	$Z_1 = \partial_u,\ Z_2 = x\partial_x + u\partial_u$ $Z_3 = 2xu\partial_x + (u^2 + x^2)\partial_u$	$xu'' = u' - u'^3 + A(1-u'^2)^{3/2}$
3-dimensional	$Z_1 = (1+x^2)\partial_x + xu\partial_u,$ $Z_2 = xu\partial_x + (1+u^2)\partial_u$ $Z_3 = u\partial_x - x\partial_u$	$u'' = A[\frac{1+u'^2+(u-xu')^2}{1+x^2+u^2}]^{3/2}$
8-dimensional	$Z_1 = \partial_x,\ Z_2 = \partial_u,\ Z_3 = x\partial_x$ $Z_4 = x\partial_u,\ Z_5 = u\partial_x,\ Z_6 = u\partial_u$ $Z_7 = x^2\partial_x + xu\partial_u,\ Z_8 = xu\partial_x + u^2\partial_u$	$u'' = 0$

Table 1.3: Lie group classification of the scalar second order ODEs.

CR-structured systems concludes this chapter.

1.2.1 Lie Point Symmetry Conditions for Systems

The derivation of the Lie point symmetry conditions for systems of n second order ODEs takes the same route (see, e.g., [94]) as has already been illustrated for a scalar second order ODE. Hence, extending that algorithm to the case of n, dependent variables S^a, where $a = 1, 2, \ldots, n$ and one independent variable t, one arrives at the Lie symmetry conditions for systems. Consider a system of n second order equations

$$\ddot{S}^a = \omega^a(t, S^j, \dot{S}^j), \quad a, j = 1, \ldots, n, \tag{1.80}$$

where dot represents differentiation with respect to t. The first integrals of the system (1.80) are the $2n$ functionally independent solutions $\phi^a = \phi^a(t, S^a, \dot{S}^a)$, of the PDE

$$\mathbf{A}f = (\partial_t + \dot{S}^a\partial_{S^a} + \omega^a(t, S^j, \dot{S}^j)\partial_{\dot{S}^a})f = 0, \tag{1.81}$$

where summation is over repeated index a. The equivalence of (1.80) and (1.81) is proved by following the same reasoning presented earlier for the scalar second order equations. The only difference is the number of variables and the structure of A. Every solution $S^a = S^a(t, \phi^a)$ of (1.80) depends on $2n$ constants of integration ϕ^a. It follows from $\phi^a_{,t}$ and the equation (1.80) that $A\phi^a = 0$. Conversely, if the solution of (1.81) is available to compute $S^a = S^a(t, \phi^a)$, then

$$\begin{aligned} \mathbf{A}\phi^a &= (\partial_t + \dot{S}^a\partial_{S^a} + \omega^a\partial_{\dot{S}^a})\phi^a = 0, \\ \phi^a_{,t} &= (\partial_t + \dot{S}^a\partial_{S^a} + \ddot{S}^a\partial_{\dot{S}^a})\phi^a = 0. \end{aligned} \tag{1.82}$$

These equations yield $(\ddot{S}^a - \omega^a)\phi^a_{\dot{S}^a} = 0$, which is possible for the functionally independent solutions only if $\ddot{S}^a = \omega^a$. The point transformations in this case are extended to n dependent variables S^a, and one independent variable t, which yield the following generator of group transformations

$$\mathbf{X}^{[1]} = \xi(t, S^j)\partial_t + \eta^a(t, S^j)\partial_{S^a} + \dot{\eta}^a(t, S^j, \dot{S}^j)\partial_{\dot{S}^a}. \tag{1.83}$$

It is related with (1.80) if

$$[\mathbf{X}^{[1]}, \mathbf{A}] = \lambda\mathbf{A}. \tag{1.84}$$

The extensions of the finite group of transformations can easily be read from the components of (1.84) which exactly are

$$-\mathbf{A}\xi = -\frac{d\xi}{dt} = \lambda, \quad \dot{\eta}^a = \frac{d\eta^a}{dt} - \dot{S}^a\frac{d\xi}{dt}. \tag{1.85}$$

The symmetry conditions for systems of n ODEs are

$$\mathbf{X}\omega^a = \mathbf{A}\dot{\eta}^a - \omega^a\frac{d\xi}{dt}, \tag{1.86}$$

which are extendable to the following expression

$$\begin{aligned} &\xi\omega^a_{,t} + \eta^b\omega^a_{,b} + (\eta^b_{,t} + \dot{S}^c\eta^b_{,c} - \dot{S}^b\xi_{,t} - \dot{S}^b\dot{S}^c\xi_{,c})\omega^a_{,\dot{S}^b} \\ &+2\omega^a(\xi_{,t} + \dot{S}^b\xi_{,b}) + \omega^b(\dot{S}^a\xi_{,b} - \eta^a_{,b}) + \dot{S}^a\dot{S}^b\dot{S}^c\xi_{,bc} \\ &+2\dot{S}^a\dot{S}^c\xi_{,tc} - \dot{S}^b\dot{S}^c\eta^a_{,bc} + \dot{S}^a\xi_{,tt} - 2\dot{S}^b\eta^a_{,tb} - \eta^a_{,tt} = 0. \end{aligned} \tag{1.87}$$

after using the prolongations of the components of the vector field $\mathbf{X}$. For a concrete comparison of the symmetry conditions for a general two dimensional system of second order ODEs

$$\begin{aligned} y'' &= \omega_1(x, y, z, y', z'), \\ z'' &= \omega_2(x, y, z, y', z'), \end{aligned} \tag{1.88}$$

where prime denotes differentiation with respect to the independent variable x, in the later part of this work, their derivation is given below. They are also readable from (1.87) when there are two dependent y, z, and one independent variable x, the symmetry conditions consist of the following set of equations

$$\begin{aligned}
&\xi\omega_{1,x}+\eta_1\omega_{1,y}+\eta_2\omega_{1,z}+[\eta_{1,x}+(\eta_{1,y}-\xi_{,x})y^{'}+(\eta_{1,z}-\xi_{,z}y^{'})z^{'}-\xi_{,y}y^{'2}]\omega_{1,y^{'}}\\
&+[\eta_{2,x}+(\eta_{2,z}-\xi_{,x})z^{'}+(\eta_{2,y}-\xi_{,y}z^{'})y^{'}-\xi_{,z}z^{'2}]\omega_{1,z^{'}}-\eta_{1,xx}-(2\eta_{1,xy}-\xi_{,xx})y^{'}\\
&-2\eta_{1,xz}z^{'}-(\eta_{1,y}-2\xi_{,x}-3\xi_{,y}y^{'}-2\xi_{,z}z^{'})y^{''}-(\eta_{1,z}-\xi_{,z}y^{'})z^{''}\\
&-(\eta_{1,yy}-2\xi_{,xy}-2\xi_{,yz}z^{'})y^{'2}-(\eta_{1,zz}-\xi_{,zz}y^{'})z^{'2}-2(\eta_{1,yz}-\xi_{,xz})y^{'}z^{'}+\xi_{,yy}y^{'3}=0,
\end{aligned} \tag{1.89}$$

and

$$\begin{aligned}
&\xi\omega_{2,x}+\eta_1\omega_{2,y}+\eta_2\omega_{2,z}+[\eta_{1,x}+(\eta_{1,y}-\xi_{,x})y^{'}+(\eta_{1,z}-\xi_{,z}y^{'})z^{'}-\xi_{,y}y^{'2}]\omega_{2,y^{'}}\\
&+[\eta_{2,x}+(\eta_{2,z}-\xi_{,x})z^{'}+(\eta_{2,y}-\xi_{,y}z^{'})y^{'}-\xi_{,z}z^{'2}]\omega_{2,z^{'}}-\eta_{2,xx}-2\eta_{2,xy}y^{'}\\
&-(2\eta_{2,xz}-\xi_{,xx})z^{'}-(\eta_{2,y}-\xi_{,y}z^{'})y^{''}-(\eta_{2,z}-2\xi_{,x}-2\xi_{,y}y^{'}-3\xi_{,z}z^{'})z^{''}\\
&-(\eta_{2,yy}-\xi_{,yy}z^{'})y^{'2}-(\eta_{2,zz}-2\xi_{,xz}-2\xi_{,yz}y^{'})z^{'2}-2(\eta_{2,yz}-\xi_{,xy})y^{'}z^{'}+\xi_{,zz}z^{'3}=0.
\end{aligned} \tag{1.90}$$

Alternatively, these can be derived if the system (1.88) admits the following symmetry

$$\begin{aligned}
\mathbf{X}^{[2]} &= \xi(x,y,z)\partial_x+\eta_1(x,y,z)\partial_y+\eta_2(x,y,z)\partial_z+\eta_1'(x,y,z,y',z')\partial_{y'}+\eta_2'(x,y,z,y',z')\partial_{z'}\\
&+\eta_1''(x,y,z,y',z',y'',z'')\partial_{y''}+\eta_2''(x,y,z,y',z',y'',z'')\partial_{z''},
\end{aligned} \tag{1.91}$$

then the symmetry conditions are

$$\eta_1'' = \xi\omega_{1,x}+\eta_1\omega_{1,y}+\eta_2\omega_{1,z}+\eta_1'\omega_{1,y'}+\eta_2'\omega_{1,z'}, \tag{1.92}$$

$$\eta_2'' = \xi\omega_{2,x}+\eta_1\omega_{2,y}+\eta_2\omega_{2,z}+\eta_1'\omega_{2,y'}+\eta_2'\omega_{2,z'}, \tag{1.93}$$

where

$$\eta_1^{'} = \eta_{1,x}+(\eta_{1,y}-\xi_{,x})y^{'}+(\eta_{1,z}-\xi_{,z}y^{'})z^{'}-\xi_{,y}y^{'2}, \tag{1.94}$$

$$\eta_2^{'} = \eta_{2,x}+(\eta_{2,z}-\xi_{,x})z^{'}+(\eta_{2,y}-\xi_{,y}z^{'})y^{'}-\xi_{,z}z^{'2}, \tag{1.95}$$

and

$$\begin{aligned}
\eta_1^{''} &= \eta_{1,xx}+(2\eta_{1,xy}-\xi_{,xx})y^{'}+2\eta_{1,xz}z^{'}+(\eta_{1,y}-2\xi_{,x}-3\xi_{,y}y^{'}-2\xi_{,z}z^{'})y^{''}\\
&+(\eta_{1,z}-\xi_{,z}y^{'})z^{''}+(\eta_{1,yy}-2\xi_{,xy}-2\xi_{,yz}z^{'})y^{'2}+2(\eta_{1,yz}-\xi_{,xz})y^{'}z^{'}\\
&+(\eta_{1,zz}-\xi_{,zz}y^{'})z^{'2}-\xi_{,yy}y^{'3},
\end{aligned} \tag{1.96}$$

$$\begin{aligned}
\eta_2^{''} &= \eta_{2,xx}+(2\eta_{2,xz}-\xi_{,xx})z^{'}+2\eta_{2,xy}y^{'}+(\eta_{2,z}-2\xi_{,x}-2\xi_{,y}y^{'}-3\xi_{,z}z^{'})z^{''}\\
&+(\eta_{2,y}-\xi_{,y}z^{'})y^{''}+(\eta_{2,zz}-2\xi_{,xz}-2\xi_{,yz}y^{'})z^{'2}+2(\eta_{2,yz}-\xi_{,xy})y^{'}z^{'}\\
&+(\eta_{2,yy}-\xi_{,yy}z^{'})y^{'2}-\xi_{,zz}z^{'3}.
\end{aligned} \tag{1.97}$$

1.2.2 Symmetry Structure of Linearizable Two Dimensional Systems of Second Order ODEs

The general system of n linear homogeneous second order ODEs involve $2n^2+n$, arbitrary coefficients. These coefficients make it difficult to address the equivalence problem for such systems. In other words, one needs to map this general linear form to an appropriate linear form with fewer arbitrary coefficients. A remarkable reduction in the number of coefficients was presented in [97], which is stated in the following theorem.

Theorem 1.2.1. *Any system of n second order non-homogeneous linear ODEs*

$$\ddot{\boldsymbol{u}} = \boldsymbol{A}\dot{\boldsymbol{u}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{c}, \tag{1.98}$$

can be mapped invertibly to one of the following forms

$$\ddot{\boldsymbol{v}} = \boldsymbol{C}\boldsymbol{v}, \tag{1.99}$$

$$\ddot{\boldsymbol{w}} = \boldsymbol{D}\boldsymbol{w}, \tag{1.100}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are $n \times n$ matrix functions, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{c}$, are vector functions and dot represents differentiation relative to the independent variable t.

In particular, for a system of two second order ODEs ($n = 2$) there are total 10 coefficients for the system (1.98). It is reducible to the first and second canonical forms, (1.99) and (1.100), respectively. Thus a system with 4 arbitrary coefficients

$$\begin{aligned} \ddot{w}_1 &= d_{11}(t)w_1 + d_{12}(t)w_2, \\ \ddot{w}_2 &= d_{21}(t)w_1 + d_{22}(t)w_2, \end{aligned} \tag{1.101}$$

can be obtained by using the equivalence of (1.98) and the counterpart of the Laguerre-Forsyth second canonical form (1.100) under point transformations. The number of arbitrary coefficients can be further reduced to three [97] by the change of variables

$$\tilde{y} = w_1/\rho(t), \;\; \tilde{z} = w_2/\rho(t), \;\; x = \int^t \rho^{-2}(s)ds, \tag{1.102}$$

where ρ satisfies

$$\rho'' - \frac{d_{11} + d_{22}}{2}\rho = 0, \tag{1.103}$$

to the linear system

$$\begin{aligned} \tilde{y}'' &= \tilde{d}_{11}(x)\tilde{y} + \tilde{d}_{12}(x)\tilde{z}, \\ \tilde{z}'' &= \tilde{d}_{21}(x)\tilde{y} - \tilde{d}_{11}(x)\tilde{z}, \end{aligned} \tag{1.104}$$

with

$$\tilde{d}_{11} = \frac{\rho^3(d_{11} - d_{22})}{2}, \ \ \tilde{d}_{12} = \rho^3 d_{12}, \ \ \tilde{d}_{21} = \rho^3 d_{21}. \tag{1.105}$$

This procedure of reduction of arbitrary coefficients for linearizable systems simplifies the classification problem enormously. System (1.104) is called the *optimal canonical form* for linear systems of two second order ODEs, as it has the fewest arbitrary coefficients, namely three. This optimal canonical form yields five equivalence classes of linearizable systems of two second order ODEs, namely with $5, 6, 7, 8$ or 15-dimensional Lie point symmetry algebras. The following theorem was stated for the maximally symmetric two dimensional system of second order ODEs.

Theorem 1.2.2. *Any linear or linearizable system is transformable to a two dimensional system of the free particle equations*

$$y'' = 0, \quad z'' = 0, \tag{1.106}$$

if and only if it has a 15*-dimensional algebra of the Lie point symmetries.*

Moreover, it is proved that a linear system with a 5 or 6-dimensional algebra is only reducible, by point transformations to those linear systems which have arbitrary coefficients. Furthermore, linear systems with constant as well as variable coefficients can have a 7 or 8-dimensional algebras.

1.2.3 Noether Symmetries and First Integrals for Systems

Consider the symmetry generator

$$\mathbf{X} = \xi(x, y, z)\partial_x + \eta_1(x, y, z)\partial_y + \eta_2(x, y, z)\partial_z, \tag{1.107}$$

which has first extension

$$\mathbf{X}^{[1]} = \mathbf{X} + (\frac{d}{dx}\eta_1 - y'\frac{d}{dx}\xi)\partial_{y'} + (\frac{d}{dx}\eta_2 - z'\frac{d}{dx}\xi)\partial_{z'}, \tag{1.108}$$

where

$$\frac{d}{dx} = \partial_x + y'\partial_y + z'\partial_z, \tag{1.109}$$

is the total derivative. A system of two second order ODEs of the form (1.88) admits a Lagrangian $L(x, y, z, y', z')$, if the system is equivalent to the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dx}(\frac{\partial L}{\partial y'}) - \frac{\partial L}{\partial y} &= 0, \\ \frac{d}{dx}(\frac{\partial L}{\partial z'}) - \frac{\partial L}{\partial z} &= 0, \end{aligned} \tag{1.110}$$

Definition 1.2.3. The vector field $\mathbf{X}$ is called a Noether point symmetry generator corresponding to a Lagrangian $L(x, y, z, y', z')$ of the system (1.88) if there exist a gauge function $B(x, y, z)$ such that

$$\mathbf{X}^{[1]}(L) + D(\xi)L = D(B), \tag{1.111}$$

here D is the total differentiation operator given by (1.109).

Theorem 1.2.4. *If $\boldsymbol{X}$ is a Noether point symmetry generator corresponding to a Lagrangian of the system (1.88), then*

$$I = \xi L + (\eta_1 - \xi y')\frac{\partial L}{\partial y'} + (\eta_2 - \xi z')\frac{\partial L}{\partial z'} - B \tag{1.112}$$

is a Noether first integral of (1.88) associated with $\boldsymbol{X}$.

1.3 Geometric Linearization

A connection between a linearizable cubically semi-linear ODE (1.70) and systems of two geodesic equations has been established [38, 83]. This connection extends Lie's linearization criteria for scalar ODEs to systems of two quadratically semi-linear ODEs. To understand the correspondence, consider $x = x(t)$, $y = y(t)$ and set $y(t) = u(x(t))$. If dot denotes differentiation with respect to t then

$$\dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}, \tag{1.113}$$

and prime with respect to x then $u' = du/dx$. Applying the chain rule yields

$$\dot{y} = u'\dot{x}, \quad \ddot{y} = u''\dot{x}^2 + u'\ddot{x}, \quad \dot{x}^3 u'' = \dot{x}\ddot{y} - \dot{y}\ddot{x}. \tag{1.114}$$

Using these relations in equation (1.70) leads to

$$\begin{aligned} &\dot{x}^3(u'' + F_3 u'^3 + F_2 u'^2 + F_1 u' + F_0) \\ &= \dot{x}(\ddot{y} + \alpha_1\dot{y}^2 + \alpha_2\dot{x}\dot{y} + F_0\dot{x}^2) - \dot{y}(\ddot{x} - F_3\dot{y}^2 - \alpha_3\dot{x}\dot{y} - \alpha_4\dot{x}^2), \end{aligned} \tag{1.115}$$

where

$$\alpha_1 + \alpha_3 = F_2, \quad \alpha_2 + \alpha_4 = F_1. \tag{1.116}$$

In this way equation (1.70) can be projected onto the (x, y) plane to obtain the geodesic equations

$$\ddot{S}^i + \Gamma^i_{jk}\dot{S}^j\dot{S}^k = 0, \quad i, j, k = 1, 2, \tag{1.117}$$

where $S^1 = x$, $S^2 = y$ and Γ^i_{jk} are Christoffel symbols, which depend on S^i and in terms of the metric tensor read as

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}). \tag{1.118}$$

These are symmetric in the lower pair of indices and have $n^2(n+1)/2$ components. Comparing (1.115) and (1.117) these symbols in the present case are

$$\Gamma^1_{11} = -\alpha_4, \quad \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{2}\alpha_3, \quad \Gamma^1_{22} = -F_3,$$
$$\Gamma^2_{11} = F_0, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\alpha_2, \quad \Gamma^2_{22} = \alpha_1. \tag{1.119}$$

A necessary and sufficient condition for the system of two quadratically semi-linear ODEs (1.117) to be reducible to the fee particle equations is that the Riemann curvature tensor constructed from (1.119) vanishes, i.e.,

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml} = 0. \tag{1.120}$$

R^i_{jkl} is skew symmetric in the lower last two indices and satisfies

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0. \tag{1.121}$$

Computing the components of Riemann curvature tensor leads to

$$R^1_{112} = -\frac{1}{2}\alpha_{3,x} + \alpha_{4,y} + F_0F_3 - \frac{1}{4}\alpha_2\alpha_3,$$
$$R^1_{212} = \frac{1}{2}\alpha_{3,y} - F_{3,x} - \frac{1}{4}\alpha_3^2 + \alpha_4F_3 + \frac{1}{2}\alpha_2F_3 - \frac{1}{2}\alpha_1\alpha_3,$$
$$R^2_{112} = \frac{1}{2}\alpha_{2,x} - F_{0,y} + \frac{1}{4}\alpha_2^2 - \alpha_1F_0 + \frac{1}{2}\alpha_2\alpha_4 - \frac{1}{2}\alpha_3F_0,$$
$$R^2_{212} = -\frac{1}{2}\alpha_{2,y} + \alpha_{1,x} - F_0F_3 + \frac{1}{4}\alpha_2\alpha_3. \tag{1.122}$$

Thus equations (1.120) provide the above system of four first order PDEs in $\alpha_j,\ F_j$ for $j = 1, 2, 3, 4$. There are two conditions (1.116) which relate these quantities, substituting $\alpha_1 = F_2 - \alpha_3$ and $\alpha_4 = F_1 - \alpha_2$ in (1.122) and solving them with respect to the derivatives of $\alpha_2,\ \alpha_3$ leads to (1.71) when α_2 and α_3 are denoted by $2z$ and $2w$, respectively.

The re-derivation of the Lie's compatibility conditions (1.71) for the scalar second order ODEs by geometric methods lead to linearizability criteria for a system of quadratically semi-linear second order ODEs [83]. Such systems are assumed to contain only the quadratic (in first derivatives of the dependent variables) terms, i.e., a system of the form (1.117) for $i, j, k = 1, 2, \ldots, n$. In this general case there is no need to treat Γ^i_{jk} as Christoffel symbols, indeed, these can be thought of as coefficients of the relevant terms appearing in the concerned system. This consideration helps to write the Linearization criteria in an explicit form. To this end, a system of three geodesic equations was considered and again the corresponding space was assumed to be flat which in terms of Riemann curvature tensor is given by the following set of equations

$$(\Gamma^i_{j2})_x - (\Gamma^i_{j1})_y + \Gamma^i_{m1}\Gamma^m_{j2} - \Gamma^i_{m2}\Gamma^m_{j1} = 0,$$
$$(\Gamma^i_{j3})_x - (\Gamma^i_{j1})_z + \Gamma^i_{m1}\Gamma^m_{j3} - \Gamma^i_{m3}\Gamma^m_{j1} = 0,$$
$$(\Gamma^i_{j3})_y - (\Gamma^i_{j2})_z + \Gamma^i_{m2}\Gamma^m_{j3} - \Gamma^i_{m3}\Gamma^m_{j2} = 0. \tag{1.123}$$

This set of equations end up with 27 conditions as a whole but not all of them are linearly independent, therefore, in this case 9 equations comprising linearization criteria in terms of the coefficients of the system of three geodesic equations. In general, for a system of n such equations

$$\ddot{S}^i + \Gamma^i_{jk}\dot{S}^j\dot{S}^k = 0, \quad i, j, k = 1, ..., n, \tag{1.124}$$

there are total n^2 independent equations which represent the linearization criteria for an n dimensional system. For a system of the form (1.124) with constant coefficients the linearizability criteria reduce to

$$\Gamma^i_{mk}\Gamma^m_{jl} = \Gamma^i_{ml}\Gamma^m_{jk}, \tag{1.125}$$

which are trivially satisfied for $k = l$ but are non-trivial otherwise.

Theorem 1.3.1. *The necessary and sufficient condition for a system of n second order quadratically semi-linear ODEs for n dependent variables of the form (1.124) to be linearizable via point transformations and admit $sl(n+2, \mathbb{R})$ symmetry algebra is that the Riemann tensor vanishes.*

After recognizing the correspondence between linearizability criteria for systems of two geodesic equations and scalar second order ODEs, geometric procedure is then extended to derive linearizability conditions for a general system of second order cubically semi-linear ODEs [71]. The most general form of a linearizable system of cubically semi-linear second order ODEs

$$S^{a''} + A_{bc}S^{a'}S^{b'}S^{c'} + B^a_{bc}S^{b'}S^{c'} + C^a_bS^{b'} + D^a = 0, \qquad a = 1, \ldots, n-1, \tag{1.126}$$

was established by projecting the system (1.124) down [5] by one dimension. This is achieved by inserting

$$\dot{S}^a = \frac{dS^a}{dx^1}\dot{x}^1, \quad \ddot{S}^a = \frac{d^2S^a}{dx^{1^2}} + \frac{dx^a}{dx^1}\ddot{x}^1, \quad a = 2, \ldots, n, \tag{1.127}$$

into (1.124) and canceling $\dot{x}^{1^2}$ throughout. The coefficients of (1.126) in terms of the Γ^a_{bc}'s are

$$A_{bc} = -\Gamma^1_{bc}, \; B^a_{bc} = \Gamma^a_{bc} - 2\delta^a_{(c}\Gamma^1_{b)1}, \; C^a_b = 2\Gamma^a_{1b} - \delta^a_b\Gamma^1_{11}, \; D^a = \Gamma^a_{11}. \tag{1.128}$$

The simplest non-trivial case of (1.126), namely a system of two second order cubically semi-linear ODEs

$$\begin{aligned} y'' + \alpha_1 y'^3 + 2\alpha_2 y'^2 z' + \alpha_3 y' z'^2 + \beta_1 y'^2 + 2\beta_2 y' z' + \beta_3 z'^2 + \gamma_1 y' + \gamma_2 z' + \delta_1 = 0, \\ z'' + \alpha_1 y'^2 z' + 2\alpha_2 y' z'^2 + \alpha_3 z'^3 + \beta_4 y'^2 + 2\beta_5 y' z' + \beta_6 z'^2 + \gamma_3 y' + \gamma_4 z' + \delta_2 = 0, \end{aligned} \tag{1.129}$$

emerges from a system of three ODEs of the form (1.124) by projection. Here the coefficients are in general functions of x, y, z. Notice that there are fifteen coefficients in the above system while the base system have eighteen. Therefore, three of the Christoffel symbols are lost, selecting Γ^1_{12}, Γ^2_{12}

and Γ^3_{33} as arbitrary, and solving fifteen Γ^a_{bc}'s of (1.128) in terms of A_{bc}, B^a_{bc}, C^a_b and D^a as well as Γ^1_{12}, Γ^2_{12} and Γ^3_{33} leads to the required constraint equations. For example, Γ^a_{bc}'s in which arbitrary elements appear can be read from (1.128) as

$$\begin{aligned}
&\Gamma^1_{11} = 2\Gamma^2_{12} - C^2_2, \quad \Gamma^1_{13} = \frac{1}{2}(\Gamma^3_{33} - B^3_{33}),\\
&\Gamma^2_{22} = 2\Gamma^1_{12} + B^2_{22}, \quad \Gamma^2_{23} = \frac{1}{2}(\Gamma^3_{33} + 2B^2_{23} - B^3_{33}),\\
&\Gamma^3_{13} = \Gamma^2_{12} + \frac{1}{2}C^3_3 - \frac{1}{2}C^2_2, \quad \Gamma^3_{23} = \Gamma^1_{12} + B^3_{23},
\end{aligned} \tag{1.130}$$

and similarly all the others. There are total 27 conditions which come from the flat space requirement imposed by equations (1.123), out of these 24 are linearly independent due to (1.121). Further, these can be reduced to fifteen conditions on the coefficients which represent the linearizability criteria for the corresponding system. All these observations can be stated in the following theorem.

Theorem 1.3.2. *A system of the form* (*1.129*) *is linearizable if the coefficients satisfy the following fifteen conditions*

$$\begin{aligned}
&\frac{1}{2}\gamma_{3,x} - \delta_{2,y} + \frac{1}{4}\gamma_4\gamma_3 + \frac{1}{4}\gamma_1\gamma_3 - \delta_1\beta_4 - \delta_2\beta_5 = 0,\\
&\frac{1}{2}\gamma_{2,x} - \delta_{1,z} + \frac{1}{4}\gamma_2\gamma_4 + \frac{1}{4}\gamma_2\gamma_1 - \delta_1\beta_2 - \delta_2\beta_3 = 0,
\end{aligned}$$

$$\begin{aligned}
&\beta_{4,x} - \frac{1}{2}\gamma_{3,y} - \alpha_1\delta_2 + \frac{1}{2}\gamma_3\beta_1 + \frac{1}{2}\gamma_4\beta_4 - \frac{1}{2}\gamma_1\beta_4 - \frac{1}{2}\gamma_3\beta_5 = 0,\\
&\beta_{3,x} - \frac{1}{2}\gamma_{2,z} - \alpha_3\delta_1 + \frac{1}{2}\gamma_2\beta_6 - \frac{1}{2}\gamma_2\beta_2 - \frac{1}{2}\gamma_4\beta_3 + \frac{1}{2}\gamma_1\beta_3 = 0,\\
&\beta_{5,x} - \frac{1}{3}\beta 1, x + \frac{1}{6}\gamma_{1,y} - \frac{4}{3}\delta_2\alpha_2 - \frac{2}{3}\gamma_2\beta_4 + \frac{2}{3}\gamma_3\beta_2 - \frac{1}{2}\gamma_{4,y} = 0,
\end{aligned}$$

$$\begin{aligned}
&-\alpha_{2,y} + \alpha_{1,z} - \alpha_1\beta_2 - \alpha_2\beta_5 + \alpha_2\beta_1 + \alpha_3\beta_4 = 0,\\
&-\alpha_{3,y} + \alpha_{2,z} - \alpha_1\beta_3 - \alpha_2\beta_6 + \alpha_2\beta_2 + \alpha_3\beta_5 = 0,
\end{aligned}$$

$$\begin{aligned}
&-\alpha_{2,x} + \frac{5}{6}\alpha_2\gamma_1 + \frac{1}{3}\alpha_3\gamma_3 - \frac{1}{3}\beta_{5,z} + \beta_3\beta_4 + \frac{1}{6}\alpha_2\gamma_4 - \beta_2\beta_2 - \frac{2}{3}\beta_{2,y} + \frac{1}{3}\beta_6 + \frac{2}{3}\beta_{1,z} - \frac{1}{3}\alpha_1\gamma_2 = 0,\\
&-\alpha_{2,x} + \frac{1}{6}\alpha_2\gamma_1 + \frac{2}{3}\beta_{6,y} + \frac{5}{6}\alpha 2\gamma_4 + \frac{1}{3}\alpha_1\gamma_2 - \frac{2}{3}\beta_{5,z} - \frac{1}{3}\alpha_3\gamma_3 - \frac{1}{3}\beta_{2,y} + \frac{1}{3}\beta_{1,z} + \beta_3\beta_4 - \beta_2\beta_5 = 0,
\end{aligned}$$

$$\begin{aligned}
&-\alpha_{3,x} + \frac{1}{2}\alpha_3\gamma_1 + \frac{1}{2}\alpha_3\gamma_4 - \beta_{3,y} + \beta_{2,z} - \beta_1\beta_3 + \beta_2\beta_2 - \beta_2\beta_6 + \beta_3\beta_5 = 0,\\
&-\alpha_{1,x} + \frac{1}{2}\alpha_1\gamma_1 + \frac{1}{2}\alpha_1\gamma_4 + \beta_{5,y} - \beta_{4,z} - \beta_4\beta_6 + \beta_2\beta_4 + \beta_5\beta_5 + \beta_1\beta_5 = 0,
\end{aligned}$$

$$\begin{aligned}
&-\frac{2}{3}\beta_{1,x} + \frac{1}{3}\gamma_{1,y} - \frac{1}{2}\beta_6\gamma_3 + \alpha_1\delta_1 - \frac{2}{3}\alpha_2\delta_2 - \frac{1}{3}\beta_4\gamma_2 + \frac{5}{6}\beta_2\gamma_3 + \beta_{5,x} - \frac{1}{2}\gamma_{3,z} + \frac{1}{2}\beta_5\gamma_4 - \frac{1}{2}\beta_5\gamma_1 = 0,\\
&\delta_{1,y} + \delta_1\beta_1 + \delta_2\beta_2 - \delta_2\beta_6 + \frac{1}{2}\gamma_{4,z} - \frac{1}{2}\gamma_{1,x} - \delta_{2,z} + \frac{1}{4}\gamma_4\gamma_4 - \frac{1}{4}\gamma_1\gamma_1 - \delta_1\beta_5 = 0,
\end{aligned}$$

$$\beta_{2,x} + \frac{1}{2}\gamma_{2,y} - 2\alpha_2\delta_1 + \frac{1}{2}\beta_5\gamma_2 + \frac{1}{2}\beta_1\gamma_2 + \frac{1}{2}\beta_2\gamma_4 - \frac{1}{2}\beta_2\gamma_1 - \beta_3\gamma_3 - \gamma_{1,z} - \alpha_3\delta_2 = 0,$$
$$-\beta_{2,x} + \beta_{6,x} + \gamma_{2,y} - \beta_5\gamma_2 + \beta_1\gamma_2 + \beta_2\gamma_4 - \beta_2\gamma_1 - \frac{1}{2}\gamma_{4,z} - \frac{1}{2}\gamma_{1,z} - 2\alpha_3\delta_2 = 0. \quad (1.131)$$

The geometric approach of projection gives a linearizable class of systems which can be mapped only to a system of the free particle equations. Symmetry classification of linearizable two dimensional systems of second order ODEs made it clear that there are five equivalence classes for such systems. Thus geometric approach covers only one of them.

1.4 Systems Obtainable from Complex ODEs and PDEs

CSA extends Lie's theory of continuous groups developed for the scalar second order ODEs to systems of two PDEs and ODEs of the same order [1]. The invariance and algebraic properties of the base complex equations have been employed to solve the corresponding systems of PDEs and ODEs. But so far no conclusions have been drawn which can solely determine the relationship between algebraic symmetry properties of the base equations and emerging systems. In this thesis these issues are studied to investigate this relationship. To demonstrate the solution procedures developed earlier with the help of CSA consider a second order ODE in general semi-linear form

$$u^{''}(z) = \omega(z, u, u^{'}), \quad (1.132)$$

where prime denoted differentiation with respect to z. Treating $u(z)$ as a complex dependent function of a complex independent variable z. With these assumptions equation (1.132) becomes a complex equation. Just as z splits into real and imaginary parts, i.e., $z = x + iy$, the real and imaginary parts of $u''(z)$ are each real valued functions of z, or equivalently of x and y. A similar argument can be given for ω on the right side of (1.132), therefore, if $f_1,\ f_2$ and $\omega_1,\ \omega_2$ denote the real and imaginary parts of u and ω then (1.132) decomposes to a system of two PDEs

$$f_{1,xx} - f_{1,yy} + 2f_{2,xy} = 4\omega_1(x, y, f_1, f_2, f_3, f_4),$$
$$f_{2,xx} - f_{2,yy} - 2f_{1,xy} = 4\omega_2(x, y, f_1, f_2, f_3, f_4), \quad (1.133)$$

where

$$f_3 = \frac{1}{2}(f_{1,x} + f_{2,y}), \quad f_4 = \frac{1}{2}(f_{2,x} - f_{1,y}). \quad (1.134)$$

The existence of complex derivatives is assumed for the equation (1.132) which becomes a PDE instead of an ODE due to complex function $u(z)$, to extract the system (1.133), hence the functions f_1 and f_2 satisfy CR-equations

$$f_{1,x} = f_{2,y}, \quad f_{1,y} = -f_{2,x}. \quad (1.135)$$

Likewise, considering $u(x)$ in the equation (1.34) as a complex function of a real variable x, then it can be broken into real and imaginary parts

$$u(x) = f_1(x) + if_2(x), \tag{1.136}$$

to obtain a system of two second order ODEs for the two parts

$$\begin{aligned} f_1'' &= \omega_1(x, f_1, f_2, f_1', f_2'), \\ f_2'' &= \omega_2(x, f_1, f_2, f_1', f_2'), \end{aligned} \tag{1.137}$$

here prime denotes differentiation with respect to x. The semi-linearity of (1.132) leads to a non-trivial system of elegantly coupled equations (1.137), in particular, when the base equation is in a linearizable form (1.70). One may also extract a system of two PDEs if the same procedure is repeated when the departure point is a complex PDE with 1 dependent and m independent variables

$$H(x^m, u, u_m, u_{nm}, \ldots) = 0, \tag{1.138}$$

instead of the base ODE of the form

$$\omega(x, u, u^{'}, u^{''}, \ldots) = 0, \tag{1.139}$$

considered earlier. Decomposing PDE (1.138) yields the following system of two PDEs

$$\begin{aligned} H_1(x^m, f_1, f_2, f_{1,m}, f_{2,m}, f_{1,nm}, f_{2,nm}, \ldots) &= 0, \\ H_2(x^m, f_1, f_2, f_{1,m}, f_{2,m}, f_{1,nm}, f_{2,nm}, \ldots) &= 0. \end{aligned} \tag{1.140}$$

Notice that not every system of two PDEs (1.133) and (1.140) or ODEs (1.137) can be obtained from a scalar ODE or PDE by projecting the dependent variables of these systems to complex functions of the complex or real (independent) variables. The analyticity of the complex functions of the complex or real independent variables leads to constraints, i.e., the CR-equations which must be satisfied by both the equations of systems of PDEs or ODEs to be extractable from a base complex DE. The characterization for the correspondence is given in the subsequent chapters.

1.5 Complex-Linearization

The linearizing transformations of the base ODEs (1.132) have been employed [2] to map the corresponding systems of PDEs and ODEs to linear forms. Indeed, the most general linearizable forms of systems (1.133) and (1.137) have been derived. As a linearizable scalar second order ODE is cubically semi-linear in the first derivative of the dependent variable, therefore, the emerging linearizable systems are also at most cubic in the first order derivatives of the dependent variables, i.e., systems of the form

$$f_{1,xx} - f_{1,yy} + 2f_{2,xy} - 4A_1 f_3^3 + 12A_1 f_3 f_4^2 + 12A_2 f_3^2 f_4 - 4A_2 f_4^3 - 4B_1 f_3^2$$
$$+8B_2 f_3 f_4 + 4B_1 f_4^2 - 4C_1 f_3 + 4C_2 f_4 - 4D_1 = 0,$$
$$f_{2,xx} - f_{2,yy} - 2f_{1,xy} - 12A_1 f_3^2 f_4 + 4A_1 f_4^3 - 4A_2 f_3^3 f_4 + 12A_2 f_3 f_4^2 - 8B_1 f_3 f_4$$
$$-4B_2 f_3^2 + 4B_2 f_4^2 - 4C_2 f_3 - 4C_1 f_4 - 4D_2 = 0, \tag{1.141}$$

where $f_3,\ f_4$ are the same as given in (1.134), and

$$f_1'' + A_1 f_1'^3 - 3A_2 f_1'^2 f_2' - 3A_1 f_1' f_2'^2 + A_2 f_2'^3 + B_1 f_1'^2 - 2B_2 f_1' f_2' - B_1 f_2'^2 + C_1 f_1' - C_2 f_2' + D_1 = 0,$$
$$f_2'' + A_2 f_1'^3 + 3A_1 f_1'^2 f_2' - 3A_2 f_1' f_2'^2 - A_1 f_2'^3 + B_2 f_1'^2 + 2B_1 f_1' f_2' - B_2 f_2'^2 + C_2 f_1' + C_1 f_2' + D_2 = 0. \tag{1.142}$$

Here the coefficients in (1.141) and (1.142) A_j, B_j, C_j, and D_j are functions of x, y, f_1, f_2 and x, f_1, f_2, respectively. These are the most general forms of systems of two PDEs and ODEs that can be linearized by complex methods. The next issue is the construction of the invariant linearizability criteria for these systems emerging from a complex base equation. To this end, obviously the first choice is to break the complex functions of the complex and real independent variables in the associated Lie's linearization conditions (1.72) for the scalars. For a system of ODEs of the form (1.142) these linearizability conditions (1.72) can be decomposed with the help of (1.136) to the following set of four equations

$$12A_{1,xx} + 12C_1 A_{1,x} - 12C_2 A_{2,x} - 6D_1 A_{1,f_1} - 6D_1 A_{2,f_2} + 6D_2 A_{2,f_1} -$$
$$6D_2 A_{1,f_2} + 12A_1 C_{1,x} - 12A_2 C_{2,x} + C_{1,f_1 f_1} - C_{1,f_2 f_2} + 2C_{2,f_1 f_2} -$$
$$12A_1 D_{1,f_1} - 12A_1 D_{2,f_2} + 12A_2 D_{2,f_1} - 12A_2 D_{1,f_2} + 2B_1 C_{1,f_1} + 2B_1 C_{2,f_2} -$$
$$2B_2 C_{2,f_1} + 2B_2 C_{1,f_2} - 8B_1 B_{1,x} + 8B_2 B_{2,x} - 4B_{1,xf_1} - 4B_{2,xf_2} = 0,$$

$$12A_{2,xx} + 12C_2 A_{1,x} + 12C_1 A_{2,x} - 6D_2 A_{1,f_1} - 6D_2 A_{2,f_2} - 6D_1 A_{2,f_1} +$$
$$6D_1 A_{1,f_2} + 12A_2 C_{1,x} + 12A_1 C_{2,x} + C_{2,f_1 f_1} - C_{2,f_2 f_2} - 2C_{1,f_1 f_2} -$$
$$12A_2 D_{1,f_1} - 12A_2 D_{2,f_2} - 12A_1 D_{2,f_1} + 12A_1 D_{1,f_2} + 2B_2 C_{1,f_1} + 2B_2 C_{2,f_2} +$$
$$2B_1 C_{2,f_1} - 2B_1 C_{1,f_2} - 8B_2 B_{1,x} - 8B_1 B_{2,x} - 4B_{2,xf_1} + 4B_{1,xf_2} = 0,$$

$$24D_1 A_{1,x} - 24D_2 A_{2,x} - 6D_1 B_{1,f_1} - 6D_1 B_{2,f_2} + 6D_2 B_{2,f_1} - 6D_2 B_{1,f_2} +$$
$$12A_1 D_{1,x} - 12A_2 D_{2,x} + 4B_{1,xx} - 4C_{1,xf_1} - 4C_{2,xf_2} - 6B_1 D_{1,f_1} -$$
$$6B_1 D_{2,f_2} + 6B_2 D_{2,f_2} - 6B_2 D_{1,f_2} + 3D_{1,f_1 f_1} - 3D_{1,f_2 f_2} + 6D_{2,f_1 f_2} + 4C_1 C_{1,f_1}$$
$$+4C_1 C_{2,f_2} - 4C_2 C_{2,f_1} + 4C_2 C_{1,f_2} - 4C_1 B_{1,x} + 4C_2 B_{2,x} = 0,$$

$$24D_2A_{1,x}+24D_1A_{2,x}-6D_2B_{1,f_1}-6D_2B_{2,f_2}-6D_1B_{2,f_1}+6D_1B_{1,f_2}+$$
$$12A_2D_{1,x}+12A_1D_{2,x}+4B_{2,xx}-4C_{2,xf_1}+4C_{1,xf_2}-6B_2D_{1,f_1}-$$
$$6B_2D_{2,f_2}-6B_1D_{2,f_1}+6B_1D_{1,f_2}+3D_{2,f_1f_1}-3D_{2,f_2f_2}-6D_{1,f_1f_2}+4C_2C_{1,f_1}-$$
$$4C_2C_{2,f_2}+4C_1C_{2,f_1}-4C_1C_{1,f_2}-4C_2B_{1,x}-4C_1B_{2,x}=0. \tag{1.143}$$

In the same way, one can obtain similar conditions for systems of PDEs of the form (1.133) by decomposing $u(z)$ into real and imaginary parts in (1.72). Notice that the decomposed conditions corresponding to (1.72), both for the systems of PDEs and ODEs do not ensure the linearizability of the emerging systems. The only claim which came from CSA is solvability of those systems which emerge from complex equations if their coefficients are compatible with the conditions (1.72). These systems may not be linearizable even if they are derived from a linearizable base ODE. Therefore, such systems are said to be *complex-linearizable*. The following result [4] was stated for such two dimensional systems (1.141) and (1.142).

Theorem 1.5.1. *Systems of two PDEs* (*1.133*) *and ODEs* (*1.137*) *are complex-linearizable if they are of the form* (*1.141*) *and* (*1.142*), *i.e., they are at most cubically semi-linear in the first order derivatives of the dependent variables and their coefficients* (*being real and imaginary parts of the coefficients of* (*1.70*)) *satisfy* (*1.72*).

Chapter 2

Inequivalent Geometric and Complex-Linearizable Two Dimensional Systems of Second Order ODEs

Equivalence of the scalar as well as of two dimensional systems of second order ODEs under point transformations is well documented in literature. Here a similar problem has been addressed for the most general linearizable class (1.129) and the complex-linearizable class (1.142) of two dimensional systems of second order ODEs. These linearizable forms of systems have been derived by adopting different methods. The first one appeared by extending the geometric linearization for systems of $2n$ quadratically semi-linear ODEs to systems of $2n-1$ cubically semi-linear second order ODEs. Invariant linearizability criteria have been presented explicitly to map such cubic semi-linear systems of two second order ODEs to the maximally symmetric systems. Therefore, the targeted linear form can only be a two dimensional system of free particle equations, which reveals the existence of one equivalence class of these systems (1.129) if they are linearizable. On the other hand, (1.142) was obtained by extending the Lie's linearization scheme for scalar second order ODEs to systems. The symmetry structure of this linearizable class has not been investigated, however, given in the subsequent chapter. The linearization criteria and the most general linearizable form of the CR-structured two dimensional systems of second order ODEs have not been derived. However, in the later part of the thesis both these issues are also addressed.

The system (1.129) has *fifteen* arbitrary coefficients whereas (1.142) has *only eight*. The obvious question would be whether both these classes are distinct or one is transformable into the other by point transformations. The reason to investigate the equivalence of (1.129) and (1.142) under point transformations is simply the absence of knowledge about the equivalence classes of the CR-structured linearizable systems. In case, if there exist invertible point transformations to map (1.129) and (1.142) into each other then being members of the same equivalence class, i.e., of maximally symmetric two dimensional systems of second order ODEs, they must have 15-dimensional symmetry algebras. In the first section this issue is addressed by considering point transformations of specific form, i.e., linear transformations of the dependent and independent variables with constant coefficients. It is found that both these classes of linearizable systems are disjoint. In the next two sections linear point transformations with variable coefficients and arbitrary point transformations are considered to study the equivalence problem of the classes (1.129) and (1.142). This quest arrived at the same result, i.e., there do not exist point transformations to put both these classes into one equivalence class.

2.1 Inequivalence of Two Classes of Linearizable Systems under Constant Linear Transformations

In the following theorem constant linear point transformations are considered to study the equivalence of the two most general linearizable classes of two dimensional systems of second order ODEs (1.129) and (1.142).

Theorem 2.1.1. *The linearizable and complex-linearizable classes of two dimensional systems of second order ODEs (1.129) and (1.142) are not related by constant linear transformations of the form*

$$f_1 = a_1 y + a_2 z, \quad f_2 = a_3 y + a_4 z, \tag{2.1}$$

where $a_j, \ j = 1, 2, 3, 4$ *are constants.*

Proof. It is clear that the transformations (2.1) must be invertible and hence $a_1a_4 - a_2a_3 \neq 0$. Inserting these transformations and their derivatives

$$f_1' = a_1 y' + a_2 z', \quad f_2' = a_3 y' + a_4 z', \tag{2.2}$$

and

$$f_1'' = a_1 y'' + a_2 z'', \quad f_2'' = a_3 y'' + a_4 z'', \tag{2.3}$$

in the system (1.142) yields a system which involves y, z and x as the dependent and independent variables, respectively. To compare the resulting system of ODEs with the system (1.129) it can be simplified to the following form

$$\begin{aligned}
&y'' + \tfrac{1}{a_1a_4-a_2a_3}[\{(A_1a_4 - A_2a_2)(a_1^3 - 3a_1a_3^2) + (A_2a_4 + A_1a_2)(a_3^3 - 3a_1^2a_3)\}y'^3 + 3\{(A_1a_4\\
&-A_2a_2)(a_1^2a_2 - a_2a_3^2 + 2a_1a_3a_4) + (A_2a_4 + A_1a_2)(a_3^2a_4 - a_1^2a_4 - 2a_1a_2a_3)\}y'^2z' + 3\{(A_1a_4\\
&-A_2a_2)(a_1a_2^2 - a_1a_4^2 - 2a_2a_3a_4) + (A_2a_4 + A_1a_2)(a_3a_4^2 - a_2^2a_3 - 2a_1a_2a_4)\}y'z'^2 + \{(A_1a_4\\
&-A_2a_2)(a_2^3 - 3a_2a_4^2) + (A_2a_4 + A_1a_2)(a_4^3 - 3a_2^2a_4)\}z'^3 + \{(B_1a_4 - B_2a_2)(a_1^2 - a_3^2) - 2(B_2a_4\\
&+B_1a_2)a_1a_3\}y'^2 + 2\{(B_1a_4 - B_2a_2)(a_1a_2 - a_3a_4) - (B_2a_4 + B_1a_2)(a_1a_4 - a_2a_3)\}y'z'\\
&+\{(B_1a_4 - B_2a_2)(a_2^2 - a_4^2) - 2(B_2a_4 + B_1a_2)a_2a_4\}z'^2 + \{(C_1a_4 - C_2a_2)a_1 - (C_2a_4\\
&+C_1a_2)a_3\}y' + \{(C_1a_4 - C_2a_2)a_2 - (C_2a_4 + C_1a_2)a_4\}z' + D_1a_4 - D_2a_2] = 0,
\end{aligned}$$

$$\begin{aligned}
&z'' - \tfrac{1}{a_1a_4-a_2a_3}[\{(A_1a_3 - A_2a_1)(a_1^3 - 3a_1a_3^2) + (A_2a_3 + A_1a_1)(a_3^3 - 3a_1^2a_3)\}y'^3 + 3\{(A_1a_3\\
&-A_2a_1)(a_1^2a_2 - a_2a_3^2 + 2a_1a_3a_4) + (A_2a_3 + A_1a_1)(a_3^2a_4 - a_1^2a_4 - 2a_1a_2a_3)\}y'^2z' + 3\{(A_1a_3\\
&-A_2a_1)(a_1a_2^2 - a_1a_4^2 - 2a_2a_3a_4) + (A_2a_3 + A_1a_1)(a_3a_4^2 - a_2^2a_3 - 2a_1a_2a_4)\}y'z'^2 + \{(A_1a_3\\
&-A_2a_1)(a_2^3 - 3a_2a_4^2) + (A_2a_3 + A_1a_1)(a_4^3 - 3a_2^2a_4)\}z'^3 + \{(B_1a_3 - B_2a_1)(a_1^2 - a_3^2) - 2(B_2a_3\\
&+B_1a_1)a_1a_3\}y'^2 + 2\{(B_1a_3 - B_2a_1)(a_1a_2 - a_3a_4) - (B_2a_3 + B_1a_1)(a_1a_4 - a_2a_3)\}y'z'\\
&+\{(B_1a_3 - B_2a_1)(a_2^2 - a_4^2) - 2(B_2a_3 + B_1a_1)a_2a_4\}z'^2 + \{(C_1a_3 - C_2a_1)a_1 - (C_2a_3\\
&+C_1a_1)a_3\}y' + \{(C_1a_3 - C_2a_1)a_2 - (C_2a_3 + C_1a_1)a_4\}z' + D_1a_3 - D_2a_1] = 0.
\end{aligned} \tag{2.4}$$

Equating coefficients of the cubic (in the first derivative) terms in systems (2.4) and (1.129) leads to

$$\begin{aligned}
&A_1(a_1^3a_4 + a_2a_3^3 - 3a_1a_3(a_3a_4 + a_1a_2)) + A_2(a_3^3a_4 - a_2a_1^3 - 3a_1a_3(a_1a_4 - a_2a_3)) = \delta\alpha_1,\\
&A_1(-6a_1a_3(a_2^2 + a_4^2)) + A_2(-3a_1^2 + 3a_3^2)(a_2^2 + a_4^2) = 2\delta\alpha_2,\\
&A_1(-3a_1a_4 - 3a_2a_3)(a_2^2 + a_4^2) + A_1(3a_3a_4 - 3a_1a_2)(a_2^2 + a_4^2) = \delta\alpha_3,\\
&A_1(-2a_2a_4(a_2^2 + a_4^2)) + A_2(a_4^2 - a_2^2)(a_2^2 + a_4^2) = 0,\\
&A_1(a_1^4 + 6a_1^2a_3^2 - a_3^4) + A_2(a_1^4 + 3a_1^3a_3 - 3a_1^2a_3^2 - a_1a_3^3) = 0,\\
&A_1(-3a_1^3a_2 + 9a_1^2a_3a_4 + 3a_1a_2(2a_3^2 + a_4^2) - 3a_3^3a_4) + A_2(3a_1^3a_4 + 9a_1^2a_2a_3 - 9a_1a_3^2a_4 - 3a_2a_3^3) = \delta\alpha_1,\\
&A_1(3a_1^2(a_4^2 - a_1^2) + 6a_2a_3a_4(a_1 + a_2) + 3a_3^2(a_2^2 - a_4^2)) + 6A_2(a_1^2a_2a_4 + a_1a_3(a_2^2 - a_4^2) - a_2a_3^2a_4) = 2\delta\alpha_2,\\
&A_1(a_1a_4^3 - 3a_1a_2^2a_4 + a_2^3a_3 - 3a_2a_3a_4^2) + A_2(-a_1a_2^3 + 3a_1a_2a_4^2 - 3a_2^2a_3a_4 + a_3a_4^3) = \delta\alpha_3,
\end{aligned} \tag{2.5}$$

where $\delta = a_1a_4 - a_2a_3 \neq 0$ and $A_1,\ A_2$ are linearly independent coefficients. The solution of the set of equations (2.5) is $a_1 = a_2 = a_3 = a_4 = 0$, which is clearly incompatible with invertibility condition.

Thus there do not exist constant linear point transformations (2.1) that map the cubically semi-linear terms of the linearizable system of two ODEs obtained by a scalar second order complex ODE to cubically semi-linear terms of the system of two ODEs provided by the geometric approach of projections.

2.2 Inequivalence of Two Classes of Linearizable Systems under Linear Point Transformations with Variable Coefficients

Here the previous analysis is generalized to those point transformations which have variable coefficients, i.e., coefficients are functions of the independent variable. The following theorem shows inequivalence of the two linearizable classes (1.129) and (1.142) under linear point transformations with variable coefficients.

Theorem 2.2.1. *The complex-linearizable and linearizable classes of systems of ODEs provided by CSA and geometric methods are not transformable to each other by linear point transformations with variable coefficients*

$$f_1 = a_1(x)y + a_2(x)z, \quad f_2 = a_3(x)y + a_4(x)z. \tag{2.6}$$

Proof. To establish this result, one needs to proceed on the same lines adopted in Theorem 1, which at the first step require the following derivatives

$$\begin{aligned} f_1' &= a_1'y + a_2'z + a_1y' + a_2z', \\ f_2' &= a_3'y + a_4'z + a_3y' + a_4z', \end{aligned} \tag{2.7}$$

$$\begin{aligned} f_1'' &= a_1''y + a_2''z + 2(a_1'y' + a_2'z') + a_1y'' + a_2z'', \\ f_2'' &= a_3''y + a_4''z + 2(a_3'y' + a_4'z') + a_3y'' + a_4z'', \end{aligned} \tag{2.8}$$

where prime denotes differentiation with respect to x. Notice that the equations (2.7) and (2.8) contain a few extra terms as compared to (2.2) and (2.3). However, these do not appear with the cubic (in first derivatives) terms in the resulting system which reads as

$$\begin{aligned}
&y'' + \tfrac{1}{a_1(x)a_4(x)-a_2(x)a_3(x)}[\{(A_1a_4(x) - A_2a_2(x))(a_1^3(x) - 3a_1(x)a_3^2(x)) + (A_2a_4(x) + A_1a_2(x)) \\
&(a_3^3(x) - 3a_1^2(x)a_3(x))\}y'^3 + 3\{(A_1a_4(x) - A_2a_2(x))(a_1^2(x)a_2(x) - a_2(x)a_3^2(x) + 2a_1(x)a_3(x)a_4(x)) \\
&+(A_2a_4(x) + A_1a_2(x))(a_3^2(x)a_4(x) - a_1^2(x)a_4(x) - 2a_1(x)a_2(x)a_3(x))\}y'^2z' + 3\{(A_1a_4(x) - A_2a_2(x)) \\
&(a_1(x)a_2^2(x) - a_1(x)a_4^2(x) - 2a_2(x)a_3(x)a_4(x)) + (A_2a_4(x) + A_1a_2(x))(a_3(x)a_4^2(x) - a_2^2(x)a_3(x) \\
&-2a_1(x)a_2(x)a_4(x))\}y'z'^2 + \{(A_1a_4(x) - A_2a_2(x))(a_2^3(x) - 3a_2(x)a_4^2(x)) + (A_2a_4(x) + A_1a_2(x)) \\
&(a_4^3(x) - 3a_2^2(x)a_4(x))\}z'^3] = 0, \\
&z'' - \tfrac{1}{a_1(x)a_4(x)-a_2(x)a_3(x)}[\{(A_1a_3(x) - A_2a_1(x))(a_1^3(x) - 3a_1(x)a_3^2(x)) + (A_2a_3(x) + A_1a_1(x)) \\
&(a_3^3(x) - 3a_1^2(x)a_3(x))\}y'^3 + 3\{(A_1a_3(x) - A_2a_1(x))(a_1^2(x)a_2(x) - a_2(x)a_3^2(x) + 2a_1(x)a_3(x)a_4(x)) \\
&+(A_2a_3(x) + A_1a_1(x))(a_3^2(x)a_4(x) - a_1^2(x)a_4(x) - 2a_1(x)a_2(x)a_3(x))\}y'^2z' + 3\{(A_1a_3(x) - A_2a_1(x)) \\
&(a_1(x)a_2^2(x) - a_1(x)a_4^2(x) - 2a_2(x)a_3(x)a_4(x)) + (A_2a_3(x) + A_1a_1(x))(a_3(x)a_4^2(x) - a_2^2(x)a_3(x) \\
&-2a_1(x)a_2(x)a_4(x))\}y'z'^2 + \{(A_1a_3(x) - A_2a_1(x))(a_2^3(x) - 3a_2(x)a_4^2(x)) + (A_2a_3(x) + A_1a_1(x)) \\
&(a_4^3(x) - 3a_2^2(x)a_4(x))\}z'^3] = 0.
\end{aligned} \tag{2.9}$$

Comparing the coefficients of $y'^3,\ y'^2z',\ y'z'^2,\ z'^3$ in the above system and (1.129) leads to a system of equations of the form (2.5). The solution of this system for the present case with variable coefficients is again $a_j(x) = 0,\ j = 1, 2, 3, 4$.

Thus, once again, it is proved that the most general forms of the complex-linearizable systems obtained by CSA and geometric linearizable systems can not be mapped into each other by linear point transformations with variable coefficients.

2.3 Inequivalence of Two Classes of Linearizable Systems under Arbitrary Point Transformations

For completeness arbitrary point transformations are considered in this section to relate the most general complex-linearizable and linearizable forms of the two dimensional systems of second order ODEs. The following theorem establishes the general inequivalence of systems (1.129) and (1.142) under arbitrary point transformations.

Theorem 2.3.1. *The most general complex-linearizable and linearizable forms* (*1.129*) *and* (*1.142*) *are not transformable to each other by arbitrary point transformations*

$$f_1 = \alpha(x, y, z),\ \ f_2 = \beta(x, y, z). \tag{2.10}$$

Proof. The transformations (2.10) are invertible, therefore, these satisfy the condition

$$\alpha_{,y}\beta_{,z} - \alpha_{,z}\beta_{,y} \neq 0. \tag{2.11}$$

First and second derivatives of these transformations are

$$\begin{aligned} f_1' &= \alpha_{,x} + \alpha_{,y}y' + \alpha_{,z}z', \\ f_2' &= \beta_{,x} + \beta_{,y}y' + \beta_{,z}z', \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} f_1'' &= \alpha_{,xx} + 2\alpha_{,xy}y' + 2\alpha_{,xz}z' + \alpha_{,yy}y'^2 + 2\alpha_{,yz}y'z' + \alpha_{,zz}z'^2 + \alpha_{,y}y'' + \alpha_{,z}z'', \\ f_2'' &= \beta_{,xx} + 2\beta_{,xy}y' + 2\beta_{,xz}z' + \beta_{,yy}y'^2 + 2\beta_{,yz}y'z' + \beta_{,zz}z'^2 + \beta_{,y}y'' + \beta_{,z}z''. \end{aligned} \tag{2.13}$$

Inserting them in system (1.142) and simplifying results in following system

$$\begin{aligned} &y'' + \tfrac{1}{\alpha_{,y}\beta_{,z}-\alpha_{,z}\beta_{,y}}[\{(A_1\beta_{,z} - A_2\alpha_{,z})(\alpha_{,y}^3 - 3\alpha_{,y}\beta_{,y}^2) + (A_2\beta_{,z} + A_1\alpha_{,z})(\beta_{,y}^3 - 3\alpha_{,y}^2\beta_{,y})\}y'^3 \\ &+3\{(A_1\beta_{,z} - A_2\alpha_{,z})(\alpha_{,y}^2\alpha_{,z} - \beta_{,y}^2\alpha_{,z} - 2\alpha_{,y}\beta_{,y}\beta_{,z}) + 3(A_2\beta_{,z} + A_1\alpha_{,z})(\beta_{,y}^2\beta_{,z} - \alpha_{,y}^2\beta_{,z} \\ &-2\alpha_{,y}\alpha_{,z}\beta_{,y})\}y'^2z' + 3\{(A_1\beta_{,z} - A_2\alpha_{,z})(\alpha_{,y}\alpha_{,z}^2 - \alpha_{,y}\beta_{,z}^2 - 2\alpha_{,z}\beta_{,y}\beta_{,z}) + 3(A_2\beta_{,z} + A_1\alpha_{,z}) \\ &(\beta_{,y}\beta_{,z}^2 - \alpha_{,z}^2\beta_{,y} - 2\alpha_{,y}\alpha_{,z}\beta_{,z})\}y'z'^2 + \{(A_1\beta_{,z} - A_2\alpha_{,z})(\alpha_{,z}^3 - 3\alpha_{,z}\beta_{,z}^2) + (A_2\beta_{,z} + A_1\alpha_{,z}) \\ &(\beta_{,z}^3 - 3\alpha_{,z}^2\beta_{,z})\}z'^3] = 0, \end{aligned}$$

$$\begin{aligned}&z'' - \tfrac{1}{\alpha_{,y}\beta_{,z}-\alpha_{,z}\beta_{,y}}[\{(A_1\beta_{,y}-A_2\alpha_{,y})(\alpha_{,y}^3-3\alpha_{,y}\beta_{,y}^2)+(A_2\beta_{,y}+A_1\alpha_{,y})(\beta_{,y}^3-3\alpha_{,y}^2\beta_{,y})\}y'^3\\&+3\{(A_1\beta_{,y}-A_2\alpha_{,y})(\alpha_{,y}^2\alpha_{,z}-\beta_{,y}^2\alpha_{,z}-2\alpha_{,y}\beta_{,y}\beta_{,z})+3(A_2\beta_{,y}+A_1\alpha_{,y})(\beta_{,y}^2\beta_{,z}-\alpha_{,y}^2\beta_{,z}\\&-2\alpha_{,y}\alpha_{,z}\beta_{,y})\}y'^2z'+3\{(A_1\beta_{,y}-A_2\alpha_{,y})(\alpha_{,y}\alpha_{,z}^2-\alpha_{,y}\beta_{,z}^2-2\alpha_{,z}\beta_{,y}\beta_{,z})+3(A_2\beta_{,y}+A_1\alpha_{,y})\\&(\beta_{,y}\beta_{,z}^2-\alpha_{,z}^2\beta_{,y}-2\alpha_{,y}\alpha_{,z}\beta_{,z})\}y'z'^2+\{(A_1\beta_{,y}-A_2\alpha_{,y})(\alpha_{,z}^3-3\alpha_{,z}\beta_{,z}^2)+(A_2\beta_{,y}+A_1\alpha_{,y})\\&(\beta_{,z}^3-3\alpha_{,z}^2\beta_{,z})\}z'^3]=0.\end{aligned} \tag{2.14}$$

Now comparing the coefficients of like powers of the first order derivatives of the dependent variables appearing in the above system and (1.129), the following system of PDEs is obtained

$$\begin{aligned}&(A_1\beta_{,z}-A_2\alpha_{,z})(\alpha_{,y}^3-3\alpha_{,y}\beta_{,y}^2)+(A_2\beta_{,z}+A_1\alpha_{,z})(\beta_{,y}^3-3\alpha_{,y}^2\beta_{,y})+3(A_1\beta_{,y}-A_2\alpha_{,y})\\&(\alpha_{,y}^2\alpha_{,z}-\beta_{,y}^2\alpha_{,z}-2\alpha_{,y}\beta_{,y}\beta_{,z})+3(A_2\beta_{,y}+A_1\alpha_{,y})(\beta_{,y}^2\beta_{,z}-\alpha_{,y}^2\beta_{,z}-2\alpha_{,y}\alpha_{,z}\beta_{,y})=0,\end{aligned}$$

$$\begin{aligned}&(A_1\beta_{,z}-A_2\alpha_{,z})(\alpha_{,y}^2\alpha_{,z}-\beta_{,y}^2\alpha_{,z}-2\alpha_{,y}\beta_{,y}\beta_{,z})+(A_2\beta_{,z}+A_1\alpha_{,z})(\beta_{,y}^2\beta_{,z}-\alpha_{,y}^2\beta_{,z}\\&-2\alpha_{,y}\alpha_{,z}\beta_{,y})+(A_1\beta_{,y}-A_2\alpha_{,y})(\alpha_{,y}\alpha_{,z}^2-\alpha_{,y}\beta_{,z}^2-2\alpha_{,z}\beta_{,y}\beta_{,z})+(A_2\beta_{,y}+A_1\alpha_{,y})(\beta_{,y}\beta_{,z}^2\\&-\alpha_{,z}^2\beta_{,y}-2\alpha_{,y}\alpha_{,z}\beta_{,z})=0,\end{aligned}$$

$$\begin{aligned}&3(A_1\beta_{,z}-A_2\alpha_{,z})(\alpha_{,y}\alpha_{,z}^2-\alpha_{,y}\beta_{,z}^2-2\alpha_{,z}\beta_{,y}\beta_{,z})+3(A_2\beta_{,z}+A_1\alpha_{,z})(\beta_{,y}\beta_{,z}^2-\alpha_{,z}^2\beta_{,y}\\&-2\alpha_{,y}\alpha_{,z}\beta_{,z})+(A_1\beta_{,y}-A_2\alpha_{,y})(\alpha_{,z}^3-3\alpha_{,z}\beta_{,z}^2)+(A_2\beta_{,y}+A_1\alpha_{,y})(\beta_{,z}^3-3\alpha_{,z}^2\beta_{,z})=0,\end{aligned}$$

$$(A_1\beta_{,z}-A_2\alpha_{,z})(\alpha_{,z}^3-3\alpha_{,z}\beta_{,z}^2)+(A_2\beta_{,z}+A_1\alpha_{,z})(\beta_{,z}^3-3\alpha_{,z}^2\beta_{,z})=0,$$

$$(A_1\beta_{,y}-A_2\alpha_{,y})(\alpha_{,y}^3-3\alpha_{,y}\beta_{,y}^2)+(A_2\beta_{,y}+A_1\alpha_{,y})(\beta_{,y}^3-3\alpha_{,y}^2\beta_{,y})=0. \tag{2.15}$$

This system of PDEs has no solution, i.e., solving the above system of PDEs with the condition (2.11) and incorporating the CR-structure of the coefficients $A_j,\ j=1,2$, one arrives at an inconsistency. Thus there exist no point transformations to relate the cubically semi-linear terms of system (1.129) and (1.142).

The inequivalence results proved in this chapter for the complex-linearizable and linearizable classes (1.129) and (1.142) imply that both the systems do not form an equivalence class. This fact indicates that either the linearizable two dimensional CR-structured systems of second order ODEs may not be reduced to a system of free particle equations or the most general linearizable form of the complex-linearizable systems is not cubically semi-linear in the first derivatives of the dependent variables.

Chapter 3

Linearizable Two Dimensional CR-Structured Systems of ODEs

A scalar second order ODE requires at least an 8-dimensional algebra to be linearizable, thereby possessing *only one equivalence class*. Whereas linearizable systems of two second order ODEs have five equivalence classes [97], i.e., the allowable number of symmetries for such systems is 5, 6, 7, 8 or 15. The procedure of constructing a pair of real functions of two variables from a single complex function of a complex variable, leading to a system of PDEs, entails the use of the CR-equations in a transparent way. However, the role of these equations for a system of ODEs is far from clear as the CR-equations require two independent variables. Here it is shown that the CR-equations with respect to both the dependent variables comprise the conditions for the correspondence of a scalar and system of two second order ODEs. Separately, the criteria to identify these CR-structured systems with a complex linearizable base equation have been shown earlier [1, 4] to solely rest on the set of equations (1.143) apart from just satisfying CR-equations. Such identification leads to the *complex-linearizable* CR-structured systems.

Those systems which correspond to a complex linearizable ODE were shown to be linearizable earlier [4], where complex invertible point transformations of the base equations were used to extract the real transformations to linearize emerging systems. However, it was not proved that linearizability of the complex scalar second order ODEs is sufficient to linearize such systems of second order ODEs of dimension two. In this chapter, full symmetry structure of linearizable CR-structured systems of second order ODEs is presented. For this purpose a *reduced optimal canonical form* for the targeted linear CR-structured systems obtainable from a complex linear equation is derived. Of the five equivalence classes appeared due to real symmetry analysis [97], only three depart as a result of this analysis. This difference arises due to the fact that in CSA equivalence of *scalar* second order ODEs is invoked to obtain the reduced optimal form, while in real symmetry analysis equivalence of linear *systems* of two ODEs was used to derive their optimal form. Indeed, former have fewer arbitrary coefficients than the latter. It is proved that complex-linearizable systems are transformable to one of the three equivalence classes if a 6, 7 or 15-dimensional algebra is associated with them. Interesting fact is that only one equivalence class of the linearizable base ODE gives rise to three classes for the corresponding CR-structured linearizable systems.

3.1 Conditions for the Correspondence of Systems and Base Scalar ODEs

Following the classical Lie procedure [74], one uses invertible point transformations

$$X = X(x,y,z), \;\; Y = Y(x,y,z), \;\; Z = Z(x,y,z), \tag{3.1}$$

to map the general system of two second order ODEs in semi-linear form to the simplest form

$$Y'' = 0, \;\; Z'' = 0, \tag{3.2}$$

where the prime denotes differentiation with respect to X. Under (3.1) the derivatives transform as

$$\begin{aligned} Y' &= \frac{D_x(Y)}{D_x(X)} = F_1(x,y,z,y',z'), \\ Z' &= \frac{D_x(Z)}{D_x(X)} = F_2(x,y,z,y',z'), \end{aligned} \tag{3.3}$$

and

$$Y'' = \frac{D_x(F_1)}{D_x(X)}, \;\; Z'' = \frac{D_x(F_2)}{D_x(X)}, \tag{3.4}$$

where D_x is the total derivative operator. This yields

$$\begin{aligned} & y'' + \alpha_{11}y'^3 + \alpha_{12}y'^2z' + \alpha_{13}y'z'^2 + \alpha_{14}z'^3 + \beta_{11}y'^2 + \beta_{12}y'z' + \beta_{13}z'^2 \\ & + \gamma_{11}y' + \gamma_{12}z' + \delta_1 = 0, \\ & z'' + \alpha_{21}y'^3 + \alpha_{22}y'^2z' + \alpha_{23}y'z'^2 + \alpha_{24}z'^3 + \beta_{21}y'^2 + \beta_{22}y'z' + \beta_{23}z'^2 \\ & + \gamma_{21}y' + \gamma_{22}z' + \delta_2 = 0, \end{aligned} \tag{3.5}$$

the coefficients being functions of the dependent and independent variables which are given in the Appendix A-1. Notice that system (3.5) represents the most general form of a system of two second order ODEs that can be a candidate for linearization. Because it has 20 arbitrary coefficients while the linearizable and complex-linearizable forms of systems (1.129) and (1.142) derived earlier have 15 and 8, respectively. The following theorem establishes the correspondence of systems of second order ODEs and the complex base ODEs, when both are given in a general semi-linear form.

Theorem 3.1.1. *A general two dimensional system of second order ODEs (1.137) corresponds to a complex equation (1.132) if and only if* ω_1 *and* ω_2 *satisfy the CR-equations, i.e.,*

$$\begin{aligned} \omega_{1,f_1} &= \omega_{2,f_2}, \quad \omega_{1,f_2} = -\omega_{2,f_1}, \\ \omega_{1,f_1'} &= \omega_{2,f_2'}, \quad \omega_{1,f_2'} = -\omega_{2,f_1'}, \end{aligned} \tag{3.6}$$

where $\omega(x,u,u') = \omega_1(x,f_1,f_2,f_1',f_2') + i\omega_2(x,f_1,f_2,f_1',f_2')$.

Two dimensional complex-linearizable system (1.142) obtainable from the most general form of a complex linearizable equation is also cubically semi-linear. Clearly, the coefficients $A_j,\ B_j,\ C_j$ and D_j, for $j = 1,2$, satisfy the CR-equations, i.e., $A_{1,f_1} = A_{2,f_2},\ A_{1,f_2} = -A_{2,f_1}$ and vice versa. It is obvious that (1.70) generates a system by decomposing the complex coefficients into real and imaginary parts as

$$\begin{aligned} F_3(x,u) &= A_1(x,f_1,f_2) + iA_2(x,f_1,f_2), \quad F_2(x,u) = B_1(x,f_1,f_2) + iB_2(x,f_1,f_2), \\ F_1(x,u) &= C_1(x,f_1,f_2) + iC_2(x,f_1,f_2), \quad F_0(x,u) = D_1(x,f_1,f_2) + iD_2(x,f_1,f_2), \end{aligned} \tag{3.7}$$

where all the coefficients are analytic. The system of the form (1.142) is called complex-linearizable as it comes from a complex linearizable ODE (1.70). In order to establish correspondence between cubically semi-linear forms (1.142) and (3.5), the following theorem is stated.

Theorem 3.1.2. *A system of the form (1.142) corresponds to (3.5) if and only if the coefficients of both the systems satisfy the following set of equations:*

$$\begin{aligned} &\alpha_{11} = -\tfrac{1}{3}\alpha_{13} = \tfrac{1}{3}\alpha_{22} = -\alpha_{24} = A_1, \quad \beta_{11} = \tfrac{1}{2}\beta_{22} = -\beta_{13} = B_1, \\ &\alpha_{14} = -\tfrac{1}{3}\alpha_{12} = -\tfrac{1}{3}\alpha_{23} = \alpha_{21} = A_2, \quad \beta_{21} = -\tfrac{1}{2}\beta_{12} = -\beta_{23} = B_2, \\ &\gamma_{11} = \gamma_{22} = C_1, \quad \gamma_{21} = -\gamma_{12} = C_2, \quad \delta_1 = D_1, \quad \delta_2 = D_2. \end{aligned} \tag{3.8}$$

The above theorem identifies those two dimensional systems which can be candidates for complex-linearization. It may be pointed out that the coefficients of (3.5) also satisfy the CR-equations as a result of (3.8) to be complex-linearizable.

3.2 Reduced Optimal Canonical Form of Linear CR-Structured Systems

The simplest forms for linear systems of two second order ODEs corresponding to complex scalar ODEs can be established by invoking the equivalence of scalar second order linear ODEs under point transformations. Consider a general linear scalar complex second order ODE

$$u'' = \zeta_1(x)u' + \zeta_2(x)u + \zeta_3(x), \tag{3.9}$$

where prime denotes differentiation relative to x and $u(x) = f_1(x) + if_2(x)$ is a complex function of the real independent variable x. As all the linear scalar second order ODEs are equivalent under point transformations, so equation (3.9) is equivalent to the following scalar second order complex ODEs

$$u'' = \zeta_4(x)u', \tag{3.10}$$

$$u'' = \zeta_5(x)u, \tag{3.11}$$

whereas (3.10) and (3.11) are also transformable to each other. Indeed these three forms are reducible to the free particle equation. These three complex scalar linear ODEs belong to the same equivalence class, that is, all have 8-dimensional Lie symmetry algebras. The subsequent sections are devoted to prove that the systems obtainable by these forms using CSA have more than one equivalence class. To extract systems of two linear ODEs from (3.10) and (3.11) consider $\zeta_4(x) = \alpha_1(x) + i\alpha_2(x)$ and $\zeta_5(x) = \alpha_3(x) + i\alpha_4(x)$ to obtain two linear forms of systems of two linear ODEs

$$\begin{aligned} f_1'' &= \alpha_1(x)f_1' - \alpha_2(x)f_2', \\ f_2'' &= \alpha_2(x)f_1' + \alpha_1(x)f_2', \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} f_1'' &= \alpha_3(x)f_1 - \alpha_4(x)f_2, \\ f_2'' &= \alpha_4(x)f_1 + \alpha_3(x)f_2. \end{aligned} \tag{3.13}$$

This discussion leads to the following theorem.

Theorem 3.2.1. *If a system of two second order ODEs is linearizable via invertible complex point transformations then it can be mapped to one of the two forms (3.12) or (3.13).*

Notice that here *only two arbitrary coefficients* arise in both the linear forms whereas the minimum number obtained before was three, that is, a system of the form (1.104). Such a reduction in the number of coefficients is possible only in the case of linear forms of systems that are extractable by CSA. In fact (3.13) can be reduced further by the change of variables

$$F_1 = f_1/\rho(x), \quad F_2 = f_2/\rho(x), \quad X = \int^x \rho^{-2}(s)ds, \tag{3.14}$$

where ρ satisfies

$$\rho'' - \alpha_3\rho = 0, \tag{3.15}$$

to

$$\begin{aligned} F_1'' &= -\beta(X)F_2, \\ F_2'' &= \beta(X)F_1, \end{aligned} \tag{3.16}$$

where $\beta = \rho^4\alpha_4$, and prime denotes differentiation with respect to X. This result can be stated in the form of a theorem.

Theorem 3.2.2. *Any linear system of two second order ODEs of the form* (*3.13*) *with two arbitrary coefficients is transformable to a simplest system of two linear ODEs* (*3.16*) *with one arbitrary coefficient via real point transformations* (*3.14*) *where* (*3.15*) *holds.*

The system (3.16) provides the reduced optimal canonical form associated with linearizable CR-structured systems that contains a single coefficient $\beta(X)$, which is an arbitrary function of X. The equivalence of systems (3.12) and (3.13) can be established via invertible point transformations, which leads to the following theorem.

Theorem 3.2.3. *Two linear forms of the systems of two second order ODEs* (*3.12*) *and* (*3.13*) *are equivalent via invertible point transformations*

$$\begin{aligned} f_1 &= M_1(x)\tilde{f}_1 - M_2(x)\tilde{f}_2 + f_1^*, \\ f_2 &= M_1(x)\tilde{f}_2 + M_2(x)\tilde{f}_1 + f_2^*, \end{aligned} \tag{3.17}$$

of the dependent variables only, where $M_1(x)$, $M_2(x)$ *are two linearly independent solutions of*

$$\begin{aligned} \alpha_1 M_1 - \alpha_2 M_2 &= 2M_1', \\ \alpha_2 M_1 + \alpha_1 M_2 &= 2M_2', \end{aligned} \tag{3.18}$$

and f_1^*, f_2^* *are the particular solutions of* (*3.12*).

Proof. Differentiating the set of equations (3.17) and inserting the result in the linear form (3.12), routine calculations show that under the conditions (3.18) system (3.12) can be mapped to (3.13) where

$$\begin{aligned} \alpha_3(x) &= \frac{1}{M_1^2 + M_2^2}\left(M_1(\alpha_1 M_1' - \alpha_2 M_2' - M_1'') + M_2(\alpha_1 M_2' + \alpha_2 M_1' - M_2'')\right), \\ \alpha_4(x) &= \frac{1}{M_1^2 + M_2^2}\left(M_1(\alpha_1 M_2' + \alpha_2 M_1' - M_2'') - M_2(\alpha_1 M_1' - \alpha_2 M_2' - M_1'')\right). \end{aligned} \tag{3.19}$$

Thus the linear form (3.12) is reducible to (3.16).

Remark 3.2.4. Any non-linear system of two second order ODEs that is linearizable by complex methods can be mapped invertibly to a system of the form (3.16) with one coefficient which is an arbitrary function of the independent variable.

3.3 Symmetry Structure of Linearizable Two Dimensional Systems of Second Order ODEs

In this section the symmetry structure of linearizable CR-structured two dimensional systems associated with the complex scalar linearizable second order ODEs is obtained by subjecting the reduced optimal canonical form into symmetry conditions. A system of PDEs arises as a result whose solution provides the symmetry generators for the corresponding linearizable systems of two second order ODEs. The following theorem states the main result about equivalence classes of CR-structured systems which can be linearized.

Theorem 3.3.1. *Linearizable systems of two second order ODEs reducible to the linear form (3.16) via invertible complex point transformations, have* 6, 7 *or* 15*-dimensional Lie point symmetry algebras.*

Proof. The symmetry conditions provide the following set of PDEs for the system (3.16)

$$\xi_{,xx} = \xi_{,xf_1} = \xi_{,f_1f_1} = \eta_{1,f_2f_2} = \eta_{2,f_1f_1} = 0, \tag{3.20}$$

$$\eta_{1,f_1f_1} - 2\xi_{,xf_1} = \eta_{1,f_1f_2} - \xi_{,xf_2} = \eta_{2,f_1f_2} - \xi_{,xf_1} = \eta_{2,f_2f_2} - 2\xi_{,xf_2} = 0, \tag{3.21}$$

$$\xi_{,xx} - 2\eta_{1,xf_1} - 3\beta(x)\xi_{,f_1}f_2 + \beta(x)\xi_{,f_2}f_1 = \eta_{1,xf_2} + \beta(x)\xi_{,f_2}f_2 = 0, \tag{3.22}$$

$$\xi_{,xx} - 2\eta_{2,xf_2} + 3\beta(x)\xi_{,f_2}f_1 - \beta(x)\xi_{,f_1}f_2 = \eta_{2,xf_1} - \beta(x)\xi_{,f_1}f_1 = 0, \tag{3.23}$$

$$\eta_{1,xx} + \beta(x)(\eta_{1,f_2}f_1 + 2\xi_{,x}f_2 - \eta_{1,f_1}f_2 + \eta_2) + \beta'(x)f_2\xi = 0, \tag{3.24}$$

$$\eta_{2,xx} + \beta(x)(\eta_{2,f_2}f_1 - 2\xi_{,x}f_1 - \eta_{2,f_1}f_2 - \eta_1) - \beta'(x)f_1\xi = 0. \tag{3.25}$$

Equations (3.22)-(3.25) involve an arbitrary function of the independent variable and its first derivatives. Equations (3.20) and (3.21) yield the following solution set

$$\begin{aligned} \xi &= \gamma_1(x)f_1 + \gamma_2(x)f_2 + \gamma_3(x), \\ \eta_1 &= \gamma_1'(x)f_1^2 + \gamma_2'(x)f_1f_2 + \gamma_4(x)f_1 + \gamma_5(x)f_2 + \gamma_6(x), \\ \eta_2 &= \gamma_1'(x)f_1f_2 + \gamma_2'(x)f_2^2 + \gamma_7(x)f_1 + \gamma_8(x)f_2 + \gamma_9(x). \end{aligned} \tag{3.26}$$

Equations (3.22) and (3.23), imply

$$\beta(x)\gamma_1(x) = 0 = \beta(x)\gamma_2(x). \tag{3.27}$$

Now assuming $\beta(x)$ to be zero, non-zero constant and arbitrary function of x, generates the following cases.

Case 1.1. $\beta(x) = 0$.
The set of determining equations (3.20)-(3.25) will reduce to a trivial system of PDEs

$$\begin{aligned} &\eta_{1,xx} = \eta_{1,xf_2} = \eta_{1,f_2f_2} = 0, \\ &\eta_{2,xx} = \eta_{2,xf_1} = \eta_{2,f_1f_1} = 0, \\ &2\xi_{,xf_1} - \eta_{1,f_1f_1} = 2\xi_{,xf_2} - \eta_{2,f_2f_2} = 0, \\ &\xi_{,xf_2} - \eta_{1,f_1f_2} = \xi_{,xf_1} - \eta_{2,f_1f_2} = 0, \\ &\xi_{,xx} - 2\eta_{1,xf_1} = \xi_{,xx} - 2\eta_{2,xf_2} = 0, \end{aligned} \tag{3.28}$$

which can be extracted classically for the system of free particle equations. Solving it a 15-dimensional Lie point symmetry algebra is derived.

Case 1.2. $\beta(x) \neq 0$.
Then (3.27) implies $\gamma_1(x) = \gamma_2(x) = 0$ and (3.26) reduces to

$$\begin{aligned} \xi &= \gamma_3(x), \\ \eta_1 &= (\tfrac{\gamma_3'(x)}{2} + c_3)f_1 + c_1 f_2 + \gamma_6(x), \\ \eta_2 &= c_2 f_1 + (\tfrac{\gamma_3'(x)}{2} + c_4)f_2 + \gamma_9(x). \end{aligned} \tag{3.29}$$

Here two subcases arise, i.e., either $\beta(x)$ is a non-zero constant or an arbitrary function of x.

Case 1.2.1. $\beta(x)$ *is a non-zero constant.*
As equations (3.24) and (3.25) involve the derivatives of $\beta(x)$, which will now be zero, equations (3.22)-(3.25) and (3.29) yield a 7-dimensional Lie algebra. The explicit expressions of the symmetry generators involve trigonometric functions. For a simple demonstration of the algorithm consider $\beta(x) = 1$, then the solution of the set of the determining equations is

$$\xi = c_1,$$

and

$$\begin{aligned} \eta_1 &= c_2 f_1 - [c_4 e^{x/\sqrt{2}} + c_3 e^{-x/\sqrt{2}}]\sin(x/\sqrt{2}) + c_6 e^{x/\sqrt{2}}\cos(x/\sqrt{2}) \\ &+ c_5 e^{-x/\sqrt{2}}\cos(x/\sqrt{2}) + c_7 f_2, \\ \eta_2 &= [-c_6 e^{x/\sqrt{2}} + c_5 e^{-x/\sqrt{2}}]\sin(x/\sqrt{2}) - c_4 e^{x/\sqrt{2}}\cos(x/\sqrt{2}) - c_2 f_2 \\ &+ c_3 e^{-x/\sqrt{2}}\cos(x/\sqrt{2}) + c_7 f_1. \end{aligned} \tag{3.30}$$

This yields a 7-dimensional symmetry algebra.

Case 1.2.2.1. $\beta(x) = x^{-2}, x^{-4}$ or $(x+1)^{-4}$.
Equations (3.22)-(3.25) and (3.29) yield a 7-dimensional Lie algebra. Thus the 7-dimensional algebras can be related with systems which have variable coefficients in their linear forms, apart from the linear forms with constant coefficients.

Case 1.2.2.2. $\beta(x) = x^{-1}, x^2$, $x^2 \pm c_0$ or e^x.
Here equations (3.22)-(3.25) and (3.29), result in a 6-dimensional Lie point symmetry algebra. The explicit expressions involve special functions, e.g., for $\beta(x) = x^{-1}$, x^2, $x^2 \pm c_0$ one extracts Bessel functions. Likewise, for $\beta(x) = e^x$ there are six symmetries, including the generators $f_1\partial_{f_1} - e^x f_2\partial_{f_2}$, $f_2\partial_{f_2} + e^x f_1\partial_{f_1}$. The remaining four generators come from the solution of an ODE of order four.

Thus there is only a 6, 7 or 15-dimensional algebra for those linearizable systems of two second order ODEs that are transformable to (3.16) via invertible complex point transformations. There is no need to investigate the remaining two linear forms (3.12) and (3.13), because these are transformable to system (3.16), that is, all these forms have the same symmetry structures. The linear forms providing 6 or 7-dimensional algebras here are obtainable by linear forms extractable from (1.104), with a 6 or 7-dimensional algebra, respectively. Consider (1.104) with all the coefficients to be non-zero constants, i.e., $\tilde{d}_{11}(x) = a_0$, $\tilde{d}_{12}(x) = b_0$ and $\tilde{d}_{21}(x) = c_0$, where

$$a_0^2 + b_0 c_0 \neq 0. \tag{3.31}$$

This system provides seven symmetry generators. The linear form (3.16) also provides a 7-dimensional algebra with constant coefficients satisfying (3.31), while the 8-dimensional symmetry algebra was extracted [97] by assuming

$$a_0^2 + b_0 c_0 = 0. \tag{3.32}$$

Such linear forms cannot be obtained from (3.16). These two examples explain why a 7-dimensional algebra can be obtained from (3.16), but a linear form with an 8-dimensional algebra is not obtainable from it.

To prove these observations consider arbitrary point transformations of the form

$$\tilde{f}_1 = a(x) f_1 + b(x) f_2, \quad \tilde{f}_2 = c(x) f_1 + d(x) f_2. \tag{3.33}$$

Case a. If $a(x) = a_0$, $b(x) = b_0$, $c(x) = c_0$ and $d(x) = d_0$ are constants then (3.33) implies

$$\begin{aligned} \tilde{f}_1'' &= a_0 f_1'' + b_0 f_2'', \\ \tilde{f}_2'' &= c_0 f_1'' + d_0 f_2''. \end{aligned} \tag{3.34}$$

Using (1.104) and (3.33) in the above equation gives

$$\begin{aligned} &(a_0 d_0 - b_0 c_0) f_1'' = ((a_0 d_0 + b_0 c_0)\tilde{d}_{11}(x) + c_0 d_0 \tilde{d}_{12}(x) - a_0 b_0 \tilde{d}_{21}(x)) f_1 \\ &+ (2 b_0 d_0 \tilde{d}_{11}(x) + d_0^2 \tilde{d}_{12}(x) - b_0^2 \tilde{d}_{21}(x)) f_2, \\ &(a_0 d_0 - b_0 c_0) f_2'' = ((a_0 d_0 + b_0 c_0)\tilde{d}_{11}(x) + c_0 d_0 \tilde{d}_{12}(x) - a_0 b_0 \tilde{d}_{21}(x)) f_2 \\ &+ (2 a_0 c_0 \tilde{d}_{11}(x) + c_0^2 \tilde{d}_{12}(x) - a_0^2 \tilde{d}_{21}(x)) f_1, \end{aligned} \tag{3.35}$$

where $a_0 d_0 - b_0 c_0 \neq 0$. Using (3.13), (3.35) and the linear independence of the $\tilde{d}$'s, gives

$$\begin{aligned} &a_0 b_0 = c_0 d_0 = 0, \\ &a_0^2 - b_0^2 = c_0^2 - d_0^2 = 0, \\ &a_0 d_0 + b_0 c_0 = a_0 c_0 - b_0 d_0 = 0, \end{aligned} \tag{3.36}$$

which has a solution $a_0 = b_0 = c_0 = d_0 = 0$, which is inconsistent with the requirement $a_0 d_0 - b_0 c_0 \neq 0$.

Case b. If $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are arbitrary functions of x then

$$\tilde{f}_1'' = a(x)f_1'' + b(x)f_2'' + a''(x)f_1 + b''(x)f_2 + 2a'(x)f_1' + 2b'(x)f_2',$$
$$\tilde{f}_2'' = c(x)f_1'' + d(x)f_2'' + c''(x)f_1 + d''(x)f_2 + 2c'(x)f_1' + 2d'(x)f_2'. \tag{3.37}$$

Thus it yields

$$\begin{aligned}(ad-bc)f_1'' = {} & [(ad+bc)\tilde{d}_{11} + cd\tilde{d}_{12} - ab\tilde{d}_{21} - a''d + c''b]f_1 + (2bd\tilde{d}_{11} \\ & + d^2\tilde{d}_{12} - b^2\tilde{d}_{21} - b''d + d''b)f_2 - 2d(a'f_1' + b'f_2') + 2b(c'f_1' + d'f_2'),\end{aligned} \tag{3.38}$$

$$\begin{aligned}(ad-bc)f_2'' = {} & (2ac\tilde{d}_{11} + c^2\tilde{d}_{12} - a^2\tilde{d}_{21} - a''c + c''a)f_1 + [(ad+bc)\tilde{d}_{11} \\ & + cd\tilde{d}_{12} - ab\tilde{d}_{21} - b''c + d''a]f_2 - 2c(a'f_1' + b'f_2') + 2a(c'f_1' + d'f_2').\end{aligned} \tag{3.39}$$

Comparison of the coefficients and use of the linear independence of $\tilde{d}$'s results in

$$a'(x) = b'(x) = c'(x) = d'(x) = 0, \tag{3.40}$$

which implies that it reduces to a system of the form (3.35), it leads again to the same result stated in the following theorem.

Theorem 3.3.2. *The linear forms for systems of two second order ODEs obtainable by real symmetry analysis with* 5 *or* 8*-dimensional algebras are not transformable to* (*3.13*) *by invertible point transformations.*

Before presenting some illustrative applications of the theory developed a refinement of Theorem (3.3.1) due to Theorem (3.3.2) is given in the following remark.

Remark 3.3.3. There are *only* 6, 7 or 15-dimensional algebras for the CR-structured linearizable two dimensional systems of second order ODEs, i.e., there are no 5 or 8-dimensional Lie point symmetry algebras for such systems.

3.4 Applications

Consider a system of non-homogeneous geodesic-type ODEs

$$f_1'' + f_1'^2 - f_2'^2 = \Omega_1(x, f_1, f_2, f_1', f_2'),$$
$$f_2'' + 2f_1'f_2' = \Omega_2(x, f_1, f_2, f_1', f_2'). \tag{3.41}$$

where Ω_1 and Ω_1 are linear functions of the dependent variables and their derivatives. This system corresponds to a complex scalar equation

$$u'' + u'^2 = \Omega(x, u, u'), \tag{3.42}$$

which is either transformable to the free particle equation or one of the linear forms (3.9)-(3.11), by means of the complex transformations

$$\chi = \chi(x),\ U(\chi) = e^{u}. \tag{3.43}$$

Which are further transformable to the free particle equation by utilizing another set of invertible complex point transformations. Generally, the system (3.41) is transformable to a system of the free particle equations or a linear system of the form

$$\begin{aligned} F_1'' &= \widetilde{\Omega}_1(\chi, F_1, F_2, F_1', F_2') - \widetilde{\Omega}_2(\chi, F_1, F_2, F_1', F_2'), \\ F_2'' &= \widetilde{\Omega}_2(\chi, F_1, F_2, F_1', F_2') + \widetilde{\Omega}_1(\chi, F_1, F_2, F_1', F_2'), \end{aligned} \tag{3.44}$$

where $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ can only be linear functions of the dependent variables and their derivatives, via an invertible change of variables obtainable from (3.43). The linear form (3.44) can be mapped to a maximally symmetric system if and only if there exist some invertible complex transformations of the form τ_1, otherwise these forms can not be reduced further. This is the reason why only three equivalence classes namely with 6, 7 and 15-dimensional algebras for systems corresponding to linearizable complex equations with only one equivalence class, are obtained. An example of a non-linear system that admits a 15-dimensional algebra which can be mapped to the free particle system using (3.43) is presented below. Furthermore, four systems of quadratically semi-linear ODEs transformable to (3.44) via (3.43) that are not further reducible to the free particle system are also considered.

1. Consider (3.41) with

$$\begin{aligned} \Omega_1 &= -\frac{2}{x} f_1', \\ \Omega_2 &= -\frac{2}{x} f_2', \end{aligned} \tag{3.45}$$

it admits a 15-dimensional algebra. The real linearizing transformations

$$\chi(x) = \frac{1}{x},\ F_1 = e^{f_1}\cos(f_2),\ F_2 = e^{f_1}\sin(f_2), \tag{3.46}$$

obtainable from the complex transformations (3.43) with $U(\chi) = F_1(\chi) + iF_2(\chi)$, map the above non-linear system to $F_1'' = 0$, $F_2'' = 0$. Moreover, the solution of (3.45) corresponds to the solution of the corresponding complex equation

$$u'' + u'^2 + \frac{2}{x} u' = 0. \tag{3.47}$$

2. Now consider Ω_1 and Ω_2 to be linear functions of the first derivatives f_1', f_2', i.e., system (3.41) with

$$\Omega_1 = c_1 f_1' - c_2 f_2',$$
$$\Omega_2 = c_2 f_1' + c_1 f_2', \tag{3.48}$$

which admits a 7-dimensional algebra, provided both c_1 and c_2, are not simultaneously zero. It is associated with the complex equation

$$u'' + u'^2 - cu' = 0. \tag{3.49}$$

Using the transformations (3.43) to generate the real transformations

$$\chi(x) = x,\ F_1 = e^{f_1}\cos(f_2),\ F_2 = e^{f_1}\sin(f_2), \tag{3.50}$$

which map the non-linear system to a linear system of the form (3.12), i.e.,

$$F_1'' = c_1 F_1' - c_2 F_2',$$
$$F_2'' = c_2 F_1' + c_1 F_2', \tag{3.51}$$

which also has a 7-dimensional symmetry algebra and corresponds to

$$U'' - cU' = 0. \tag{3.52}$$

All the linear second order ODEs are transformable to the free particle equation thus the above equation can be transformed to $\widetilde{U}'' = 0$, using

$$(\chi(x), U) \to (\widetilde{\chi} = \alpha + \beta e^{c\chi(x)}, \widetilde{U} = U), \tag{3.53}$$

where c must be complex in order to yield a linear coupled system (3.51). But these complex transformations can not generate real transformations to reduce the corresponding system (3.51) to a maximally symmetric system.

3. A system with a 6-dimensional Lie algebra is obtainable from (3.41) by introducing a linear function of x in the above coefficients, i.e.,

$$\Omega_1 = (1+x)(c_1 f_1' - c_2 f_2'),$$
$$\Omega_2 = (1+x)(c_2 f_1' + c_1 f_2'), \tag{3.54}$$

in (3.41), then the same transformations (3.50) convert the above system into a linear system

$$F_1'' = (1+\chi)\left(c_1 F_1' - c_2 F_2'\right),$$
$$F_2'' = (1+\chi)\left(c_2 F_1' + c_1 F_2'\right), \tag{3.55}$$

where both systems (3.54) and (3.55) are in agreement on the dimensions (i.e., six) of their symmetry algebras. Again, the above system is a special case of the linear system (3.12).

4. Considering $\Omega_1 = c_1,\ \Omega_2 = c_2$, where $c_j\ (j = 1, 2)$ are non-zero constants, then under the same real transformations (3.50), the non-linear system considered here takes the form

$$\begin{aligned} F_1'' &= c_1 F_1 - c_2 F_2, \\ F_2'' &= c_2 F_1 + c_1 F_2. \end{aligned} \tag{3.56}$$

Notice that the transformations used to carry out linearization of systems emerge from the complex point transformations

$$\tau_1 : (x, u(x)) \to (\chi(x), U(x, u)), \tag{3.57}$$

where $u(x)$, is an analytic complex function. In fact, these transformations reveal the linearizability of the complex-linearizable two dimensional CR-structured systems of second order ODEs. As it is observed that a complex base equation can be linearized by transformations other than the above, i.e., the transformations of the form

$$\tau_2 : (x, u(x)) \to (\chi(x, u), U(x, u)). \tag{3.58}$$

These can not be decomposed into real linearizing transformations in order to map the corresponding systems to linear forms. These observations make it obvious to expect that a complex-linearizable two dimensional CR-structured system of second order ODEs may not be linearized. The transformations $\tau_j,\ j = 1, 2$, sufficiently characterize the associated CR-structured complex-linearizable systems. For completeness a class of non-linearizable systems of two second order ODEs has been investigated in the subsequent chapter which also emerge from the complex scalar linearizable ODEs.

Chapter 4

Symmetry Solutions of Two Dimensional Systems Not Solvable by Symmetry Analysis

A scalar second order ODE can be integrated through quadratures if it has a specific (solvable) 2-dimensional algebra [35], while integration of two dimensional systems of second order ODEs may not be accomplished with four symmetry generators [94]. This chapter is a continuation of the previous one as it also deals with those two dimensional *non-linearizable* CR-structured systems which arise from a linearizable complex scalar second order ODE. Though the complex transformations of the form (3.57) and (3.58) may reduce the base complex equation to the free particle equations, the latter can not be employed to transform the emerging systems to systems of free particle equations. The reason for this has been discussed in the first section where the association of CR-equations with complex-linearizable systems is presented. The CR-equations do not only relate systems with the base equations, indeed, provide an algorithm to solve such CR-structured systems even if they are not linearizable. The core of this chapter is to solve those CR-structured systems which can not be solved by linearization as was done in the previous chapter by making use of the transformations of the form (3.57).

The complex transformations τ_2 are different from τ_1 because they can not be used to obtain the real linearizing transformations to map the associated systems to linear forms. The systems generated by a base linearizable complex ODE where the associated transformations are of type τ_2 are proved to be non-linearizable here by showing that such systems have at most *four* Lie point symmetries [97]. However, in the case of four symmetry generators the integration of associated systems is not guaranteed by symmetry methods. Hence for less than four generators *they are not solvable*. Nevertheless, they may correspond to a scalar linearizable complex ODE which solves corresponding systems by employing the CR-equations.

4.1 Classification

A classification of two-dimensional systems of second order ODEs has been presented here which correspond to those base ODEs that are solvable by standard symmetry methods. The CR-structured systems which emerge from a linearizable or integrable complex second order ODE are called *complex-solvable* systems. Let Υ_1 be a class of systems that includes both the complex-linearizable and complex-solvable systems. Further, dividing the complex-linearizable systems into two disjoint classes Υ_2 and Υ_3 with respect to their *real* symmetry algebras of dimension $d(\Upsilon_2) > 4$ and $d(\Upsilon_3) \leq 4$, respectively. Since the dimensions of the algebras associated with the systems in the classes Υ_2 and Υ_3 are not the same, therefore there do not exist complex transformations that map systems in Υ_2 to systems in Υ_3 and vice versa. These three classes are related by the following relations

$$\Upsilon_1 \cap \Upsilon_2 = \Upsilon_2, \quad \Upsilon_1 \cap \Upsilon_3 = \Upsilon_3, \quad \Upsilon_2 \cap \Upsilon_3 = \emptyset. \tag{4.1}$$

Since there exists five linearizable classes of two-dimensional systems with symmetry algebra $d \geq 5$, therefore each candidate of Υ_3 is non-linearizable in standard Lie theory. The class Υ_3 is the main subject of discussion in this chapter.

Linearization problem for the class Υ_2 is addressed in the previous chapter [87]. In the subsequent sections an algorithm is constructed to achieve solvability of complex-linearizable two dimensional systems of second order ODEs belonging to Υ_3. Though this algorithm is not based on integration of such systems, yet efficiently solves them by exploiting complex solutions of the base ODEs.

4.2 Role of the CR-Equations for Systems of Two Second Order ODEs

An important implication of the CR-equations has been observed to split the systems obtainable from complex linearizable equations, into two classes namely linearizable and non-linearizable. The complex transformations τ_j, $j = 1, 2$, reduce a non-linear complex equation to the free particle equation. However, the former reduce systems to a linear form while the latter can not be invoked to linearize systems. The reason for this is difference in the complex free particle equations one obtains when transforming a complex ODE by point transformations (3.57) and (3.58). The transformations τ_1, yield a complex free particle equation $U'' = 0$, where prime denotes the differentiation with respect to the real independent variable. Therefore, it decomposes into a two dimensional system of the free particle equations. The complex transformations of the independent variable in τ_2, involve two real functions

$$\chi(x, u) = \chi_1(x, f_1, f_2) + i\chi_2(x, f_1, f_2). \tag{4.2}$$

The dependent function $U(\chi)$ is also complex which gives two real functions F_1 and F_2. Both of them are not only functions of χ_1 but also of χ_2. The linearized scalar equation $U''=0$, fails to produce the free particle system because the prime denotes differentiation with respect to χ, which upon using the chain rule yields

$$\frac{\partial}{\partial\chi}=\frac{1}{2}\left(\frac{\partial}{\partial\chi_1}-i\frac{\partial}{\partial\chi_2}\right). \tag{4.3}$$

Hence the complex free particle equation provides a system of two PDEs

$$\begin{aligned} F_{1_{\chi_1\chi_1}}-F_{1_{\chi_2\chi_2}}+2F_{2_{\chi_1\chi_2}}&=0,\\ F_{2_{\chi_1\chi_1}}-F_{2_{\chi_2\chi_2}}-2F_{1_{\chi_1\chi_2}}&=0. \end{aligned} \tag{4.4}$$

Now, by definition, a complex Lie point transformation is analytic. Thus τ_2 is analytic. Since the derivative u' transforms into a complex derivative U', which exists if and only if $U(\chi)$ is complex analytic and is preserved under τ_2. Therefore

$$F_{1_{\chi_1}}=F_{2_{\chi_2}},\quad F_{1_{\chi_2}}=-F_{2_{\chi_1}}, \tag{4.5}$$

which are the CR-equations. It is the solution of system (4.4) subject to condition (4.5), which upon invoking invertible transformations τ_2, provide solutions of the original system. Hence the following result is stated to encompass these observations.

Theorem 4.2.1. *All the complex-linearizable two dimensional systems can be transformed into system (4.4), (4.5) under the transformations of the form* τ_2.

The role of the CR-equations can diagrammatically be explained by employing the original idea of Riemann that a complex function may be regarded as the dependence of one plane on another plane unlike the dependence of a real function on a line. It requires the solution of system (4.4) and (4.5) that is

$$\begin{aligned} F_1(\chi_1,\chi_2)&=c_1\chi_1+c_2\chi_2+c_3,\\ F_2(\chi_1,\chi_2)&=c_1\chi_2-c_2\chi_1+c_4, \end{aligned} \tag{4.6}$$

where c_j $(j=1,2,3,4)$ are constants. These are two complex planes determined by χ_1 and χ_2 with normals

$$\begin{aligned} \mathbf{n}_1&=[c_1,\ c_2],\\ \mathbf{n}_2&=[c_2,-c_1]. \end{aligned} \tag{4.7}$$

Thus they intersect at right angles, i.e.,

$$\mathbf{n}_1\cdot\mathbf{n}_2=0, \tag{4.8}$$

giving a straight line in intersection. The linear or liearizable scalar second order ODEs lie on the straight line, which can be regarded as an intersection of two planes at right angles in the complex-linearization of two dimensional systems of second order ODEs. The role of χ_2 is to slice the three dimensional space $\mathbb{R}^3 = \{(x, f_1, f_2)\}$ into two coordinate planes. The solution f_1 and f_2 of the system under consideration can be found by solving (4.6) with the use of F_1 and F_2 obtainable from $U(\chi)$, which can be determined by τ_2. Figure 1 gives the digrammatic explanation of complex-linearization and the role of the CR-equations for systems of two second order ODEs. It proves the following result.

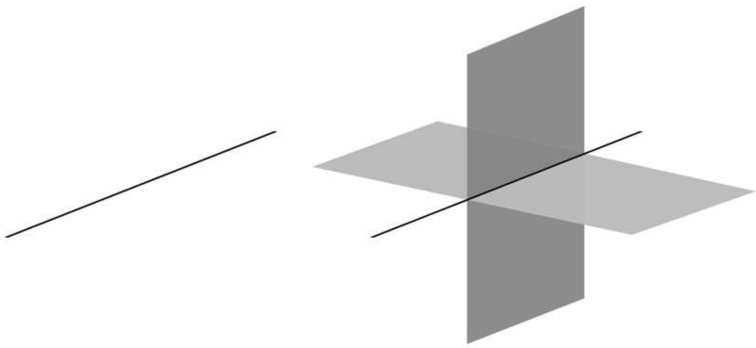

Figure 4.1: Diagrammatic explanation of complex-linearization: The straight line embedded in a three dimensional space $\mathbb{R}^3$ as the intersection of two mutually perpendicular planes.

Theorem 4.2.2. *The necessary and sufficient condition for a two dimensional system* (*1.142*) *to be complex-linearizable is that the two planes determined by* (*4.6*) *intersect at right angle resulting in a straight line which corresponds to scalar linear equations.*

The subsequent sections are presented to illustrate the complex-solvability of two dimensional systems with the aid of examples on Υ_1, Υ_2 and Υ_3. The complex-integrability is demonstrated first and then the complex-linearizability of systems with algebras of dimensions 4, 3, 2, 1.

4.3 Υ_1-class: Complex-Integrable Systems

This class contains both the complex-linearizable and complex-solvable systems. An example of the latter is given here, i.e., a system with two symmetries which emerges from an integrable base complex ODE is solved. Consider a non-linear coupled system

$$f_1'' = \frac{(f_1^2 - f_2^2)f_1' + 2f_1f_2f_2'}{(f_1^2 + f_2^2)^2},$$
$$f_2'' = \frac{(f_1^2 - f_2^2)f_2' - 2f_1f_2f_1'}{(f_1^2 + f_2^2)^2}, \tag{4.9}$$

it has a 2-dimensional algebra $[\mathbf{X}_1, \mathbf{X}_2] = 2\mathbf{X}_1$, where

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}. \tag{4.10}$$

Using standard Lie analysis it is not straight-forward to carry out integration of this system. The solution procedure adopted here highlights the crucial steps involved in using the complex transformations in the form of invariants and differential invariants of symmetries. As both the equations of the above system of ODEs satisfy CR-equations therefore such a system can be mapped to a scalar complex ODE

$$u'' = \frac{u'}{u^2}. \tag{4.11}$$

The symmetry generators $\mathbf{X}_1 = \partial_x$ and $\mathbf{X}_2 = 2x\partial_x + u\partial_u$ comprise a 2-dimensional solvable Lie algebra $\mathbf{G}_2$ admitted by (4.11). Hence, the above equation can be integrated to find a symmetry solution by using both approaches; canonical coordinates and differential invariants. Since scaling is inherited under $\mathbf{X}_1$ therefore canonical coordinates relative to this symmetry have been employed. The canonical transformations

$$\chi = u, \quad \psi = x, \quad U(\chi) = \frac{d\psi}{d\chi} = \frac{1}{u'}, \tag{4.12}$$

convert (4.11) into a first order equation

$$U' = -\frac{U^2}{\chi^2}. \tag{4.13}$$

Integrating above equation leads to the following solution

$$U(\chi) = -\frac{\chi}{1 + c\chi}, \tag{4.14}$$

which upon using invertible complex transformations leads to an implicit solution

$$cu(x) - ln(1 + cu(x)) - c^2(b - x) = 0, \tag{4.15}$$

of the equation (4.11). To workout the solution of the system (4.9) realification of (4.13) is the first step that provides a system of PDEs

$$\begin{aligned} F_{1,\chi_1} + F_{2,\chi_2} &= \frac{-2(\chi_1^2 - \chi_2^2)(F_1^2 - F_2^2) - 8\chi_1\chi_2F_1F_2}{(\chi_1^2 + \chi_2^2)^2}, \\ F_{2,\chi_1} - F_{1,\chi_2} &= 4\frac{\chi_1\chi_2(F_1^2 - F_2^2) - (\chi_1^2 - \chi_2^2)F_1F_2}{(\chi_1^2 + \chi_2^2)^2}, \end{aligned} \tag{4.16}$$

as χ is a complex independent variable. That is why the system (4.9) is reducible to a pair of first-order PDEs rather ODEs. This is a similar situation that arises in complex-linearization except the difference that here the target equation is a reduced solvable ODE not a linear one. The solution of the above system of PDEs is

$$F_1 = -\frac{\chi_1 + c_1(\chi_1^2 + \chi_2^2)}{1 + (c_1^2 + c_2^2)(\chi_1^2 + \chi_2^2) + 2(c_1\chi_1 - c_2\chi_2)},$$
$$F_2 = -\frac{\chi_2 - c_2(\chi_1^2 + \chi_2^2)}{1 + (c_1^2 + c_2^2)(\chi_1^2 + \chi_2^2) + 2(c_1\chi_1 - c_2\chi_2)}, \tag{4.17}$$

which can easily be read from (4.14) when $c = c_1 + ic_2$, hence F_1 and F_2 satisfy the CR-equations with respect to χ_1 and χ_2. The above solution can be inverted by invoking the real transformations

$$\chi_1 = f_1, \quad \chi_2 = f_2, \quad F_1 = \frac{f_1'}{f_1'^2 + f_2'^2}, \quad F_2 = \frac{-f_2'}{f_1'^2 + f_2'^2}, \tag{4.18}$$

to obtain solution of system (4.9). Therefore, (4.15) decomposes into a solution of the system (4.9) due to its correspondence with the complex base ODE (4.11).

4.4 Υ_2-class: Linearizable Systems (Revisited)

For completeness a linearizable system with a 7−dimensional Lie algebra from the previous chapter

$$f_1'' + f_1'^2 - f_2'^2 = c_1 f_1' - c_2 f_2',$$
$$f_2'' + 2f_1' f_2' = c_2 f_1' + c_1 f_2', \tag{4.19}$$

is given here to highlight the use of complex transformations at the two steps involved in the complex-linearization of the above system of ODEs. It admits a 7-dimensional algebra, provided both c_1 and c_2 are not simultaneously zero. Using the transformations

$$\chi(x) = x, \; F_1 = e^{f_1}\cos(f_2), \; F_2 = e^{f_1}\sin(f_2), \tag{4.20}$$

relative to τ_1, it can be mapped to a linear system

$$F_1'' = c_1 F_1' - c_2 F_2',$$
$$F_2'' = c_2 F_1' + c_1 F_2', \tag{4.21}$$

which also has a 7-dimensional symmetry algebra. Since the number of symmetries are the same for both systems it ensures that there exist invertible point transformations to map them into each other. To find them put on *complex glasses* and observe that the base non-linear equation $u'' + u'^2 = cu'$, can be mapped invertibly to $U'' = cU'$, using transformations

$$\chi(x) = x, \; U = e^{u(x)}. \tag{4.22}$$

As all the linear scalar equations are equivalent therefore the transformed complex ODE can be further mapped to a complexified free particle equation $\widetilde{U}'' = 0$, via

$$\widetilde{\chi} = \sigma + \varsigma e^{c\chi(x)}, \quad \widetilde{U} = U, \tag{4.23}$$

where σ and ς are complex constants. It is vital to identify the distinction between the two crucial steps here. In the first step, the process of realification of (4.22) to find the real transformations that linearized system (4.19) to (4.21). In the second step, though complex transformations $(\chi, U) \rightarrow (\widetilde{\chi}, \widetilde{U})$ map the base ODEs to the free particle equation, these can not be employed to transform system (4.21) to maximally symmetric system. Because, the independent variable $\widetilde{\chi}$ is complex so long as c is a complex number, i.e., $c = c_1 + ic_2$, it jumps off the real line by adding an extra dimension superficially and results in a set of linear PDEs. Elegantly, this larger 4−dimensional space $(\chi_1, \chi_2, F_1(\chi_1, \chi_2), F_2(\chi_1, \chi_2))$, where χ_2 adds the superficial dimension, is also equipped with an analytic structure contains the general solution curve of the system (4.21). It is noteworthy an *independent* dimension due to complex c, is

$$e^{c\chi(x)} = e^{c_1\chi} \cos(c_2\chi) + ie^{c_1\chi} \sin(c_2\chi). \tag{4.24}$$

Hence, $\chi_2(x) = e^{c_1\chi} \sin(c_2\chi(x))$. Thus, the solution of system (4.21) can be found from the complex solution of the linearized equation by applying the inverse transformations. Clearly, this hidden connection can only be uncovered by investigating the underlying complex structure.

4.5 Υ_3-class: Complex-Linearizable Systems

The non-linearizable class of systems is associated with those complex non-linear equations that are transformable to the complex free particle equation via complex transformations of the form $(real, complex) \rightarrow (complex, complex)$, namely τ_2. Both the classes (linearizable/non-linearizable) of systems associated with complex scalar linearizable equations are disjoint relative to dimensions of their symmetry algebras. Hence they are inequivalent under invertible point transformations. In order to demonstrate the solvability of the non-linearizable class of systems consider the following two dimensional system

$$\begin{aligned} f_1'' - \beta f_1'^3 + 3\gamma f_1'^2 f_2' + 3\beta f_1' f_2'^2 - \gamma f_2'^3 = 0, \\ f_2'' - \gamma f_1'^3 - 3\beta f_1'^2 f_2' + 3\gamma f_1' f_2'^2 + \beta f_2'^3 = 0, \end{aligned} \tag{4.25}$$

where $\beta = \beta(x, f_1, f_2)$ and $\gamma = \gamma(x, f_1, f_2)$. It corresponds to a linearizable scalar second order ODE, i.e., it is complex-linearizable if and only if

$$\beta_{xx} = 0, \quad \gamma_{xx} = 0, \tag{4.26}$$

to satisfy the set of equations (1.143). Note that the system (4.25) is not linearizable as it has fewer than five symmetry generators but the base complex equation is transformable to the free particle equation via complex transformations of the form τ_2 if

$$F_3(x, u) = \alpha_1(x)\alpha_2(u), \tag{4.27}$$

where $\alpha_1(x)$ must be a linear function. In particular, the invertible complex linearizing transformations in this case are

$$x = U, \quad u = \chi, \tag{4.28}$$

which reduce the complex non-linear equation $u''(x) - \alpha_1(x)\alpha_2(u)u'^3 = 0$, to

$$U'' + \alpha_2(\chi)(c_1 U + c_2) = 0. \tag{4.29}$$

The complex solution obtained by complex linearization provides the solution of the associated system (4.25) on its split into real and imaginary parts. It is due to the elegance of complex variables that systems which can not be dealt with standard Lie symmetry approach are nevertheless solvable by the complex symmetry approach!

4.5.1 Solvable Systems with 4-Dimensional Algebra

Considering $\beta(x, f_1, f_2) = 1$ and $\gamma(x, f_1, f_2) = 0$ in (4.25) yields a coupled system

$$\begin{aligned} f_1'' - f_1'^3 + 3f_1' f_2'^2 &= 0, \\ f_2'' - 3f_1'^2 f_2' + f_2'^3 &= 0, \end{aligned} \tag{4.30}$$

which is complex-linearizable and has only four symmetries

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = \partial_{f_1}, \quad \mathbf{X}_3 = \partial_{f_2}, \quad \mathbf{X}_4 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}, \tag{4.31}$$

with Lie algebra

$$\begin{aligned} [\mathbf{X}_1, \mathbf{X}_2] = 0, \quad [\mathbf{X}_1, \mathbf{X}_3] = 0, \quad [\mathbf{X}_2, \mathbf{X}_3] = 0, \\ [\mathbf{X}_1, \mathbf{X}_4] = 2\mathbf{X}_1, \quad [\mathbf{X}_2, \mathbf{X}_4] = \mathbf{X}_2, \quad [\mathbf{X}_3, \mathbf{X}_4] = \mathbf{X}_3. \end{aligned} \tag{4.32}$$

Therefore, it is not one of the linearizable classes of two dimensional systems. Now in order to solve system (4.30) consider the corresponding complex equation

$$u'' - u'^3 = 0. \tag{4.33}$$

which can be linearized as it has an 8-dimensional Lie algebra. This is transformable to the linear equation

$$U'' + 1 = 0, \tag{4.34}$$

by inverting the role of the dependent and independent variables $\chi = u,\ U = x$. It has the solution $2U = -\chi^2 + a\chi + b$, where a and b are complex constants. This complex solution decomposes to provide

$$\begin{aligned} f_1(x) &= ((2x + a_1)^2 + a_2^2)^{1/4} \cos \frac{1}{2}\left(\arctan\left(\frac{a_2}{2x + a_1}\right)\right) + b_1, \\ f_2(x) &= ((2x + a_1)^2 + a_2^2)^{1/4} \sin \frac{1}{2}\left(\arctan\left(\frac{a_2}{2x + a_1}\right)\right) + b_2, \end{aligned} \tag{4.35}$$

which is a solution of (4.30). Likewise, if the system of PDEs

$$F_{1_{\chi_1\chi_1}} - F_{1_{\chi_2\chi_2}} + 2F_{2_{\chi_1\chi_2}} + 4 = 0,$$
$$F_{2_{\chi_1\chi_1}} - F_{2_{\chi_2\chi_2}} - 2F_{1_{\chi_1\chi_2}} = 0, \tag{4.36}$$

which corresponds to the transformed linear complex ODE (4.34) is solved along with the CR-equations (4.5), then it leads to the solution of system (4.30). Such a system could be solved by real symmetry analysis because it has four symmetry generators. However, in the subsequent subsections those systems are given that are *not* solvable by real symmetry methods as they have *fewer symmetry generators than required to integrate them.*

4.5.2 Solvable Systems with 3-Dimensional Algebra

It is easy to construct a system from (4.25) which has only three symmetries. For example observe that in (4.25) the functions β and γ can be at most linear functions of independent variable x. Hence the following complex-linearizable system

$$f_1'' - x f_1'^3 + 3x f_1' f_2'^2 = 0,$$
$$f_2'' - 3x f_1'^2 f_2' + x f_2'^3 = 0, \tag{4.37}$$

is obtained by involving x linearly in the coefficients to remove the x-translation. It has the following 3-dimensional Abelian Lie algebra

$$\mathbf{X}_1 = x\partial_x, \quad \mathbf{X}_2 = \partial_{f_1}, \quad \mathbf{X}_3 = \partial_{f_2}. \tag{4.38}$$

Following the same procedure as developed in the previous case, i.e., transforming the base complex non-linear equation

$$u'' - x u'^3 = 0, \tag{4.39}$$

to a linear form $U'' = -U(\chi)$ which has the solution

$$U(\chi) = c_1 \cos(\chi) + c_2 \sin(\chi). \tag{4.40}$$

Inverting it the solution

$$u(x) = \arctan\left(\frac{x}{\sqrt{c - x^2}}\right) + b, \tag{4.41}$$

for the non-linear ODE (4.39) arises. The system of PDEs associated with the transformed linear equation reads as

$$F_{1_{\chi_1\chi_1}} - F_{1_{\chi_2\chi_2}} + 2F_{2_{\chi_1\chi_2}} + 4F_1(\chi_1, \chi_2) = 0,$$
$$F_{2_{\chi_1\chi_1}} - F_{2_{\chi_2\chi_2}} - 2F_{1_{\chi_1\chi_2}} + 4F_2(\chi_1, \chi_2) = 0. \tag{4.42}$$

Considering the set of equations (4.5) and (4.42) as a system the following solution is obtained

$$F_1 = \frac{1}{2}(-\chi_1^2 + \chi_2^2) + a_1\chi_1 + a_2\chi_2 + a_3,$$
$$F_2 = -\chi_1\chi_2 + a_1\chi_2 - a_2\chi_1 + a_4. \tag{4.43}$$

Inverting this solution by complex linearizing transformations (4.28) yields the solution of system (4.37) which is *not* solvable by real symmetry methods but by complex-linearization.

4.5.3 Solvable Systems with 2-Dimensional Algebra

Consider the system

$$f_1'' - f_1 f_1'^3 + 3f_2 f_1'^2 f_2' + 3f_1 f_1' f_2'^2 - f_2 f_2'^3 = 0,$$
$$f_2'' - f_2 f_1'^3 - 3f_1 f_1'^2 f_2' + 3f_2 f_1' f_2'^2 + f_1 f_2'^3 = 0, \tag{4.44}$$

which has only two Lie symmetries

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x\partial_x + \frac{1}{3}f_1\partial_{f_1} + \frac{1}{3}f_2\partial_{f_2}. \tag{4.45}$$

The system (4.44) is solvable due to its correspondence with the complex scalar second order ODE

$$u'' - uu'^3 = 0, \tag{4.46}$$

which *is* linearizable, despite having only a 2-dimensional algebra. Implementing the change of dependent and independent variables (4.28) the above equation leads to a system of PDEs

$$F_{1_{\chi_1\chi_1}} - F_{1_{\chi_2\chi_2}} + 2F_{2_{\chi_1\chi_2}} + 4\chi_1 = 0,$$
$$F_{2_{\chi_1\chi_1}} - F_{2_{\chi_2\chi_2}} - 2F_{1_{\chi_1\chi_2}} + 4\chi_2 = 0, \tag{4.47}$$

which determines the solution of the system (4.44) by inverting the solution

$$F_1 = \exp(\chi_2)[c_1\sin(\chi_1) + c_2\cos(\chi_1)] + \exp(-\chi_2)[c_3\sin(\chi_1) + c_4\cos(\chi_1)],$$
$$F_2 = \exp(\chi_2)[c_1\cos(\chi_1) - c_2\sin(\chi_1)] - \exp(-\chi_2)[c_3\cos(\chi_1) - c_4\sin(\chi_1)], \tag{4.48}$$

of the system of PDEs comprised of (4.5) and (4.47).

Notice that even a scalar second order ODE requires at least two symmetries to be solvable, while here a system of ODEs has been solved. In the following section a system with *only one* symmetry generator is solved by CSA, *one symmetry generator is insufficient to solve even a scalar second order ODE by standard Lie approach.* Nevertheless, *one can go further* !

4.5.4 Solvable Systems with 1-Dimensional Algebra

Consider $\beta(x, f_1, f_2) = xf_1$ and $\gamma(x, f_1, f_2) = xf_2$, in (4.25) which gives

$$
\begin{aligned}
&f_1'' - xf_1f_1'^3 + 3xf_2f_1'^2f_2' + 3xf_1f_1'f_2'^2 - xf_2f_2'^3 = 0,\\
&f_2'' - xf_2f_1'^3 - 3xf_1f_1'^2f_2' + 3xf_2f_1'f_2'^2 + xf_1f_2'^3 = 0.
\end{aligned} \tag{4.49}
$$

This system is non-linearizable as it has *only* a scaling symmetry $\mathbf{X}_1 = x\partial_x$. The corresponding scalar second order complex ODE is

$$u'' - xuu'^3 = 0, \tag{4.50}$$

which has an 8-dimensional algebra and linearizes to

$$U'' + \chi U = 0, \tag{4.51}$$

which is the Airy equation whose solutions are Airy functions extended to the complex plane. The solution of the complex linearized equation for $U(\chi)$ is given by

$$U(\chi) = c_1\mathrm{Ai}(-\chi) + c_2\mathrm{Bi}(-\chi), \tag{4.52}$$

where $\mathrm{Ai}(-\chi)$ and $\mathrm{Bi}(-\chi)$ are the two Airy functions. Inverting (4.52) yields a solution of the associated non-linear equation which implicitly provides a solution

$$
\begin{aligned}
&Re(c_1\mathrm{Ai}(-f_1 - if_2) + c_2\mathrm{Bi}(-f_1 - if_2) = x,\\
&im(c_1\mathrm{Ai}(-f_1 - if_2) + c_2\mathrm{Bi}(-f_1 - if_2) = 0,
\end{aligned} \tag{4.53}
$$

for the system (4.49). The system of PDEs which establishes the correspondence between the solutions of the complex base ODE and emerging system in this case is of the form

$$
\begin{aligned}
&F_{1_{\chi_1\chi_1}} - F_{1_{\chi_2\chi_2}} + 2F_{2_{\chi_1\chi_2}} + 4(\chi_1F_1 - \chi_2F_2) = 0,\\
&F_{2_{\chi_1\chi_1}} - F_{2_{\chi_2\chi_2}} - 2F_{1_{\chi_1\chi_2}} + 4(\chi_2F_1 + \chi_1F_2) = 0.
\end{aligned} \tag{4.54}
$$

4.5.5 Coupled-Modified-Emden System (Revisited)

Now consider an example of a physical system studied in [3]

$$
\begin{aligned}
&f_1'' + 3f_1f_1' - 3f_2f_2' + f_1^3 - 3f_1f_2^2 = 0,\\
&f_2'' + 3f_2f_1' + 3f_1f_2' - f_2^3 + 3f_1^2f_2 = 0.
\end{aligned} \tag{4.55}
$$

This system has three symmetries $\mathbf{X}_1,\ \mathbf{X}_2,\ \mathbf{X}_3$, where

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x\partial_x - f_1\partial_{f_1} - f_2\partial_{f_2}, \quad \mathbf{X}_3 = x^2\partial_x - 2xf_1\partial_{f_1} - 2xf_2\partial_{f_2}, \tag{4.56}$$

with Lie algebra $[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1$, $[\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2$ and $[\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3$. The system (4.55) is solvable *only* by complex-linearization. The complex magic is played by the complex transformations

$$\chi = x - \frac{1}{u}, \quad U = \frac{x^2}{2} - \frac{x}{u}, \tag{4.57}$$

which map the non-linear complex equation into the complex free particle equation, whose solution after using the above transformations directly yield the solution of the system (4.55) [3]. In this case (4.4) and (4.5) yield

$$\begin{aligned} F_1 &= a_1\chi_1 - a_2\chi_2 + b_1, \\ F_2 &= a_2\chi_1 + a_1\chi_2 + b_2, \end{aligned} \tag{4.58}$$

where

$$\chi_1 = x - \frac{f_1}{f_1^2 + f_2^2}, \quad \chi_2 = \frac{f_1}{f_1^2 + f_2^2}, \tag{4.59}$$

and

$$F_1 = \frac{x^2}{2} - \frac{f_1}{f_1^2 + f_2^2}, \quad F_2 = \frac{f_2}{f_1^2 + f_2^2}. \tag{4.60}$$

Now by solving (4.58) for f_1 and f_2, with the invocation of equations (4.59) and (4.60), provides the same solution

$$\begin{aligned} f_1(x) &= \frac{2x^3 - 6x^2a_1 + 4(a_2^2 + a_1^2 - b_1)x + 4a_1b_1 + 4a_2b_2}{x^4 - 4x^3a_1 + 4((a_2^2 + a_1^2 - b_1)x^2 + 2(a_2b_2 + a_1b_1)x + b_1^2 + b_2^2)}, \\ f_2(x) &= \frac{(2x^2 + 4b_1)a_2 + 4b_2(x - a_1)}{x^4 - 4x^3a_1 + 4((a_2^2 + a_1^2 - b_1)x^2 + 2(a_2b_2 + a_1b_1)x + b_1^2 + b_2^2)}, \end{aligned} \tag{4.61}$$

of system (4.55), as obtained earlier by CSA [3] up to a redefinition of constants.

Both the previous chapters dealt with the CR-structured complex-linearizable systems of two second order ODEs in order to characterize them. The symmetry structure of such linearizable systems had been investigated to provide their equivalence classes. Moreover, a systematic procedure was demonstrated to solve these systems if they can not be linearized by complex methods. This characterization of two dimensional CR-structured systems of second order ODEs is incomplete as *the most general linearizable form* and *invariant linearization criteria* have not been provided so far. However, in the last chapter a few evidence are presented to address these issues in detail, which may separate the linearizable systems from the complex-linearizable ones.

Chapter 5

Lie-Like Operators and Lie Point Symmetries for Two Dimensional Systems

After exploring the invariance and equivalence properties of the DEs their classifications with respect to the associated symmetry algebras have been stressed in the literature. Lie presented a complete classification of scalar second order ODEs (see, e.g., [62]), where each such class represents an entirety of those ODEs that are transformable into one another by point transformations. He considered all possible continuous groups acting in the plane to present complete complex classification. Lie proved that for a second order ODE the dimensions of the Lie algebras of vector fields are $0, 1, 2, 3$ or 8, whereas a maximal $4, 5, 6$ or 7-dimensional algebra cannot be associated with second order ODEs. Therefore, if an equation admits such an algebra realization then it is always mappable to the simplest linear one by an invertible change of the variables.

In this chapter, the role of the complex vector fields associated with the CR-structured two dimensional systems of second order ODEs due to their correspondence with the complex scalar second order ODEs has been explored. A discussion on obtaining the most general forms of systems of two ODEs by making use of the admitted complex symmetry algebras of the base equations is also given. Lie-like conditions have been derived in the first section which relate the real and imaginary parts of the complex symmetry generators of the base complex equation called *Lie-like* operators with the CR-structured systems. The algebra realizations of the complex maximally symmetric equations are described in terms of the emerging systems. Then those cases are considered where the base complex equation has non-maximal algebra realizations. For such realizations it is shown that the Lie algebras admitted by the CR-structured systems are subsets of the sets of the Lie-like operators. However, for the former cases it may not be true and this chapter mainly addresses this issue. The complex symmetry generators are earlier found to split into real symmetry generators of the corresponding systems of ODEs and PDEs [2], in particular, the complex equations that admit at most 2-dimensional algebras. Here it shown that the situation is not the same for n-dimensional algebras (for $2 < n < 8$) and their representative scalar and systems of ODEs.

In the third section, it is shown that the complex symmetry generators of the scalar base second order ODEs do not necessarily split into the symmetry generators of the associated systems of ODEs. These complex generators may yield a few operators other than the symmetry generators, these are the Lie-like operators which satisfy the Lie-like conditions but they are not compatible with the Lie symmetry conditions for systems. It is observed that a complex symmetry may either split into two Lie-like operators which are different from Lie symmetries [66] or provide components of two Lie point symmetries for the corresponding system. Moreover, decomposition of the components of a complex symmetry may result into a symmetry generator and a Lie-like operator of the associated system. A refinement of the Lie-like conditions by using the underlying CR-structure of the components of the operators is presented here. This refinement show that these conditions contain the Lie symmetry conditions and a few extra terms which play a decisive role in obtaining Lie symmetries for systems from the complex ones of the base ODEs. The criteria presented here show the difference between Lie-like and Lie point symmetry conditions for the CR-structured two dimensional systems of second order ODEs. Indeed, it show when and how the Lie point symmetries of the CR-structured systems are extractable from complex symmetry generators of the base complex ODEs.

In the last section a discussion on the *Noether-like* operators and associated first integrals is given. The complex-linearizable class of two dimensional non-linearizable systems of second order ODEs (4.25) is considered to compare the first integrals provided by Noether symmetries and Noether-like operators associated with this class of systems. As the base complex ODEs are linearizable so there exist 5-dimensional complex Noether algebras which yield five first integrals in each case. Decomposing them provide Noether-like operators and associated first integrals. It is shown that the Noether symmetries and first integrals admitted by this class of systems due to real symmetry analysis are less than those obtained by CSA. This comparison shows the usefulness of the Lie and Noether-like operators for systems.

5.1 Lie-Like Conditions for Systems

A Lie point symmetry generator (1.16) when prolonged to include the first and second derivatives, is admitted by a scalar second order ODE (1.34) if both the generator and ODE satisfy the symmetry conditions (1.35). In this case the first and second prolongation coefficients are

$$\eta' = \eta_{,x} + (\eta_{,u} - \xi_{,x})u' - \xi_{,u}u'^2, \tag{5.1}$$

and

$$\eta'' = \eta_{,xx} + (2\eta_{,xu} - \xi_{,xx})u' + (\eta_{,uu} - 2\xi_{,xu})u'^2 - \xi_{,uu}u'^3 + (\eta_{,u} - 2\xi_{,x} - 3\xi_{,u}u')u'', \tag{5.2}$$

respectively. The system (1.137) emerges from a scalar second order equation if the dependent variable $u(x)$ of the scalar ODE is considered as a complex function of a real independent variable x, i.e., if $u(x) = f_1(x) + if_2(x)$. In other words, such systems are obtainable from scalar equations if both the equations of the system (1.137) satisfy the CR-equations

$$\omega_{1,f_1} = \omega_{2,f_2}, \quad \omega_{2,f_1} = -\omega_{1,f_2}, \quad \omega_{1,f_1'} = \omega_{2,f_2'}, \quad \omega_{2,f_1'} = -\omega_{1,f_2'}. \tag{5.3}$$

The symmetry generator (1.16) represents a complex vector field due to the presence of the complex function $u(x)$. Considering $\xi = \xi_1 + i\xi_2$, $\eta = \eta_1 + i\eta_2$, $\eta' = \zeta_1' + i\zeta_2'$ and $\eta'' = \zeta_1'' + i\zeta_2''$, in (1.16), for $n = 2$, yields a pair of operators of the form

$$\mathbf{X}_1^{[2]} = \xi_1\partial_x + \frac{1}{2}(\eta_1\partial_{f_1} + \eta_2\partial_{f_2} + \zeta_1'\partial_{f_1'} + \zeta_2'\partial_{f_2'} + \zeta_1''\partial_{f_1''} + \zeta_2''\partial_{f_2''}), \tag{5.4}$$

$$\mathbf{X}_2^{[2]} = \xi_2\partial_x + \frac{1}{2}(\eta_2\partial_{f_1} - \eta_1\partial_{f_2} + \zeta_2'\partial_{f_1'} - \zeta_1'\partial_{f_2'} + \zeta_2''\partial_{f_1''} - \zeta_1''\partial_{f_2''}), \tag{5.5}$$

when decomposed into the real and imaginary parts. The coefficients ζ_j' and ζ_j'' for $j = 1, 2$, are obtained from (5.1) and (5.2), therefore, they are different from the set of equations (1.94)-(1.97). In order to derive the Lie-like conditions consider the action of a complex symmetry on the complex scalar ODE (1.34)

$$\mathbf{Z}^{[2]}u'' = \mathbf{Z}^{[1]}\omega(x, u, u'), \tag{5.6}$$

which leads to the complex symmetry conditions (1.35). This equation can be decomposed into real and imaginary parts to give

$$\begin{aligned} \mathbf{X}_1^{[2]}f_1'' - \mathbf{X}_2^{[2]}f_2'' &= \mathbf{X}_1^{[1]}\omega_1(x, f_1, f_2, f_1', f_2') - \mathbf{X}_2^{[1]}\omega_2(x, f_1, f_2, f_1', f_2'), \\ \mathbf{X}_2^{[2]}f_1'' + \mathbf{X}_1^{[2]}f_2'' &= \mathbf{X}_1^{[1]}\omega_2(x, f_1, f_2, f_1', f_2') + \mathbf{X}_2^{[1]}\omega_1(x, f_1, f_2, f_1', f_2'), \end{aligned} \tag{5.7}$$

because $\mathbf{Z}^{[2]} = \mathbf{X}_1^{[2]} + i\mathbf{X}_2^{[2]} = \xi\partial_x + \eta\partial_u + \eta'\partial_{u'} + \eta''\partial_{u''}$, where all ξ, η, η' and η'' are complex due to presence of the complex function u of real variable x and its derivatives. Inserting the operators (5.4) and (5.5) in (5.7) one arrives at the following conditions

$$\begin{aligned} \zeta_1'' = {} & \xi_1\omega_{1,x} - \xi_2\omega_{2,x} + \tfrac{1}{2}(\eta_1\omega_{1,f_1} + \eta_2\omega_{1,f_2} + \zeta_1'\omega_{1,f_1'} + \zeta_2'\omega_{1,f_2'} \\ & -\eta_2\omega_{2,f_1} + \eta_1\omega_{2,f_2} - \zeta_2'\omega_{2,f_1'} + \zeta_1'\omega_{2,f_2'}), \end{aligned} \tag{5.8}$$

$$\begin{aligned} \zeta_2'' = {} & \xi_1\omega_{2,x} + \xi_2\omega_{1,x} + \tfrac{1}{2}(\eta_1\omega_{2,f_1} + \eta_2\omega_{2,f_2} + \zeta_1'\omega_{2,f_1'} + \zeta_2'\omega_{2,f_2'} \\ & +\eta_2\omega_{1,f_1} - \eta_1\omega_{1,f_2} + \zeta_2'\omega_{1,f_1'} - \zeta_1'\omega_{1,f_2'}). \end{aligned} \tag{5.9}$$

The first extension coefficients ζ_j', can be obtained by splitting (1.15) for $n = 1$, into the real and imaginary parts which leads to the following equations

$$\begin{aligned} \zeta_1' &= D_1\eta_1 - D_2\eta_2 - f_1'(D_1\xi_1 - D_2\xi_2) + f_2'(D_2\xi_1 + D_1\xi_2), \\ \zeta_2' &= D_2\eta_1 + D_1\eta_2 - f_1'(D_2\xi_1 + D_1\xi_2) - f_2'(D_1\xi_1 - D_2\xi_2). \end{aligned} \tag{5.10}$$

Here the total derivative operator (1.14) appears in two parts

$$\begin{aligned} D_1 &= \tfrac{\partial}{\partial x} + \tfrac{1}{2}(f_1'\tfrac{\partial}{\partial f_1} + f_2'\tfrac{\partial}{\partial f_2} + f_1''\tfrac{\partial}{\partial f_1'} + f_2''\tfrac{\partial}{\partial f_2'}),\\ D_2 &= \tfrac{1}{2}(f_2'\tfrac{\partial}{\partial f_1} - f_1'\tfrac{\partial}{\partial f_2} + f_2''\tfrac{\partial}{\partial f_1'} - f_1''\tfrac{\partial}{\partial f_2'}). \end{aligned} \tag{5.11}$$

Inserting these D_j's in (5.10) yields

$$\begin{aligned} \zeta_1' = \eta_{1,x} + \tfrac{1}{2}[(\eta_{1,f_1} + \eta_{2,f_2} - 2\xi_{1,x})f_1' - (\eta_{2,f_1} - \eta_{1,f_2} - 2\xi_{2,x})f_2'\\ -(\xi_{1,f_1} + \xi_{2,f_2})(f_1'^2 - f_2'^2) + 2(\xi_{2,f_1} - \xi_{1,f_2})f_1'f_2'], \end{aligned} \tag{5.12}$$

$$\begin{aligned} \zeta_2' = \eta_{2,x} + \tfrac{1}{2}[(\eta_{2,f_1} - \eta_{1,f_2} - 2\xi_{2,x})f_1' + (\eta_{1,f_1} + \eta_{2,f_2} - 2\xi_{1,x})f_2'\\ -2(\xi_{1,f_1} + \xi_{2,f_2})f_1'f_2' - (\xi_{2,f_1} - \xi_{1,f_2})(f_1'^2 - f_2'^2)]. \end{aligned} \tag{5.13}$$

Similarly, for the second prolongation (i.e., $n = 2$) of these Lie-like operators the following relation

$$\zeta_1'' + i\zeta_2'' = (D_1 + iD_2)(\zeta_1' + i\zeta_2') - (f_1'' + if_2'')[(D_1 + iD_2)(\xi_1 + i\xi_2)], \tag{5.14}$$

leads to

$$\begin{aligned} \zeta_1'' &= (\xi_{1,f_2f_2} - 2\xi_{2,f_1f_2} - \xi_{1,f_1f_1})(f_1'^3 - 3f_1'f_2'^2) + (\xi_{2,f_1f_1} - 2\xi_{1,f_1f_2} - \xi_{2,f_2f_2})(3f_1'^2f_2' - f_2'^3)\\ &-(4\xi_{1,xf_1} + 4\xi_{2,xf_2} - \eta_{1,f_1f_1} - 2\eta_{2,f_1f_2} + \eta_{1,f_2f_2})(f_1'^2 - f_2'^2) + (4\eta_{1,f_1f_2} + 2\eta_{2,f_2f_2} - 2\eta_{2,f_1f_1}\\ &-8\xi_{1,xf_2} + 8\xi_{2,xf_1})f_1'f_2' + [4(\eta_{1,xf_1} + \eta_{2,xf_2} - \xi_{1,xx}) - 6(\xi_{1,f_1} + \xi_{2,f_2})f_1'' - 6(\xi_{1,f_2} - \xi_{2,f_1})f_2'']f_1'\\ &+[4(\eta_{1,xf_2} - \eta_{2,xf_1} + \xi_{2,xx}) - 6(\xi_{1,f_2} - \xi_{2,f_1})f_1'' + 6(\xi_{1,f_1} + \xi_{2,f_2})f_2'']f_2' + 4\eta_{1,xx}\\ &+(2\eta_{1,f_1} + 2\eta_{2,f_2} - 8\xi_{1,x})f_1'' + (2\eta_{1,f_2} - 2\eta_{2,f_1} + 8\xi_{2,x})f_2'', \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} \zeta_2'' &= (\xi_{2,f_2f_2} + 2\xi_{1,f_1f_2} - \xi_{2,f_1f_1})(f_1'^3 - 3f_1'f_2'^2) + (\xi_{1,f_2f_2} - 2\xi_{2,f_1f_2} - \xi_{1,f_1f_1})(3f_1'^2f_2' - f_2'^3)\\ &+(4\xi_{1,xf_2} - 4\xi_{2,xf_1} + \eta_{2,f_1f_1} - 2\eta_{1,f_1f_2} - \eta_{2,f_2f_2})(f_1'^2 - f_2'^2) + (4\eta_{2,f_1f_2} - 2\eta_{1,f_2f_2} + 2\eta_{1,f_1f_1}\\ &-8\xi_{2,xf_2} - 8\xi_{1,xf_1})f_1'f_2' + [4(\eta_{2,xf_1} - \eta_{1,xf_2} - \xi_{2,xx}) + 6(\xi_{1,f_2} - \xi_{2,f_1})f_1'' - 6(\xi_{1,f_1} + \xi_{2,f_2})f_2'']f_1'\\ &+[4(\eta_{1,xf_1} + \eta_{2,xf_2} - \xi_{1,xx}) - 6(\xi_{1,f_1} + \xi_{2,f_2})f_1'' - 6(\xi_{1,f_2} - \xi_{2,f_1})f_2'']f_2' + 4\eta_{2,xx}\\ &+(2\eta_{2,f_1} - 2\eta_{1,f_2} - 8\xi_{2,x})f_1'' + (2\eta_{1,f_1} + 2\eta_{2,f_2} - 8\xi_{1,x})f_2''. \end{aligned} \tag{5.16}$$

Apparently, the extensions (5.12)-(5.16) are different from those given in (1.92)-(1.97) because different procedures have been adopted to derive them. Notice that the Lie-like conditions are associated with the system (1.137) where (5.3) holds, while the Lie symmetry conditions are derived for general form of such system, i.e., both the equations of system (1.88) may not satisfy the CR-equations. In general, the Lie point symmetry operators associated with systems form an algebra while the Lie-like ones do not [66].

5.2 Most General Forms of Systems that Admit Lie-Like Operators

Consider the 8-dimensional maximal complex Lie algebra

$$\begin{aligned} &\mathbf{Z}_1 = \partial_x,\quad \mathbf{Z}_2 = \partial_u,\quad \mathbf{Z}_3 = x\partial_x,\quad \mathbf{Z}_4 = u\partial_u,\quad \mathbf{Z}_5 = x\partial_u,\\ &\mathbf{Z}_6 = u\partial_x,\quad \mathbf{Z}_7 = x^2\partial_x + xu\partial_u,\quad \mathbf{Z}_8 = xu\partial_x + u^2\partial_u, \end{aligned} \tag{5.17}$$

where the representative equation is the free particle complex equation

$$u'' = 0. \tag{5.18}$$

The complex symmetry generators $\mathbf{Z}_1 - \mathbf{Z}_8$, yield fourteen Lie-like operators

$$\begin{aligned}
&\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_{f_1},\ \mathbf{X}_3 = \partial_{f_2},\ \mathbf{X}_4 = x\partial_x,\ \mathbf{X}_5 = x\partial_{f_1},\ \mathbf{X}_6 = x\partial_{f_2},\ \mathbf{X}_7 = f_1\partial_x,\ \mathbf{X}_8 = f_2\partial_x,\\
&\mathbf{X}_9 = f_1\partial_{f_1} + f_2\partial_{f_2},\ \mathbf{X}_{10} = f_2\partial_{f_1} - f_1\partial_{f_2},\ \mathbf{X}_{11} = x^2\partial_x + \tfrac{x}{2}(f_1\partial_{f_1} + f_2\partial_{f_2}),\\
&\mathbf{X}_{12} = x(f_2\partial_{f_1} - f_1\partial_{f_2}),\ \mathbf{X}_{13} = xf_1\partial_x + \tfrac{1}{2}(f_1^2 - f_2^2)\partial_{f_1} + f_1f_2\partial_{f_2},\\
&\mathbf{X}_{14} = xf_2\partial_x + f_1f_2\partial_{f_1} + \tfrac{1}{2}(f_2^2 - f_1^2)\partial_{f_2},
\end{aligned} \tag{5.19}$$

for the associated system of free particle equations

$$f_1'' = 0, \quad f_2'' = 0. \tag{5.20}$$

The above system is maximally symmetric, therefore, it admits a 15-dimensional Lie point symmetry algebra

$$\begin{aligned}
&\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_{f_1},\ \mathbf{X}_3 = \partial_{f_2},\ \mathbf{X}_4 = x\partial_x,\ \mathbf{X}_5 = x\partial_{f_1},\ \mathbf{X}_6 = x\partial_{f_2},\ \mathbf{X}_7 = f_1\partial_x,\ \mathbf{X}_8 = f_2\partial_x,\\
&\mathbf{X}_9 = f_1\partial_{f_1},\ \mathbf{X}_{10} = f_2\partial_{f_2},\ \mathbf{X}_{11} = f_2\partial_{f_1},\ \mathbf{X}_{12} = f_1\partial_{f_2},\ \mathbf{X}_{13} = x^2\partial_x + xf_1\partial_{f_1} + xf_2\partial_{f_2},\\
&\mathbf{X}_{14} = xf_1\partial_x + f_1^2\partial_{f_1} + f_1f_2\partial_{f_2},\ \mathbf{X}_{15} = xf_2\partial_x + f_1f_2\partial_{f_1} + f_2^2\partial_{f_2}.
\end{aligned} \tag{5.21}$$

From (5.19) and (5.21) it is clear that all the Lie-like operators are not the Lie point symmetries but most of the operators yield symmetries for the system.

Now consider another example where a non-linear system of two second order ODEs

$$\begin{aligned}
&f_1'' - \frac{1}{f_1^2 + f_2^2}\left(A_1f_1 + A_2f_2 + \frac{f_1}{2}(f_1'^2 - f_2'^2) + f_2f_1'f_2'\right) = 0,\\
&f_2'' - \frac{1}{f_1^2 + f_2^2}\left(A_2f_1 - A_1f_2 + f_1f_1'f_2' + \frac{f_2}{2}(f_2'^2 - f_1'^2)\right) = 0,
\end{aligned} \tag{5.22}$$

emerge from a scalar base equation

$$uu'' - \frac{1}{2}u'^2 - A = 0. \tag{5.23}$$

The complex symmetries associated with the above complex equation form a 3-dimensional algebra

$$\mathbf{Z}_1 = \partial_x,\ \mathbf{Z}_2 = x\partial_x + u\partial_u,\ \mathbf{Z}_3 = x^2\partial_x + 2xu\partial_u, \tag{5.24}$$

whereas the system (5.22) have three symmetry generators

$$\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2},\ \mathbf{X}_3 = x^2\partial_x + 2xf_1\partial_{f_1} + 2xf_2\partial_{f_2}, \tag{5.25}$$

and five Lie-like operators

$$\begin{aligned}
&\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2},\ \mathbf{X}_3 = f_2\partial_{f_1} - f_1\partial_{f_2},\\
&\mathbf{X}_4 = x^2\partial_x + xf_1\partial_{f_1} + xf_2\partial_{f_2},\ \mathbf{X}_5 = xf_2\partial_{f_1} - xf_1\partial_{f_2}.
\end{aligned} \tag{5.26}$$

Notice that here also all the Lie-like operators are not the Lie point symmetries and when the commutators of these Lie-like operators are computed it leads to a new operator, i.e.,

$$\mathbf{X}_6 = x\mathbf{X}_5, \tag{5.27}$$

repeating this procedure by including the new operators one arrives at the following

$$\begin{aligned} \mathbf{X}_7 &= x\mathbf{X}_6 = x^2\mathbf{X}_5, \\ \mathbf{X}_8 &= x\mathbf{X}_7 = x^2\mathbf{X}_6 = x^3\mathbf{X}_5, \\ \mathbf{X}_9 &= \cdots = x^4\mathbf{X}_5. \\ &\vdots \qquad \vdots \end{aligned} \tag{5.28}$$

Hence these observations help in stating the following theorem.

Theorem 5.2.1. *The Lie-like operators associated with a two dimensional CR-structured system of second order ODEs do not form an algebra.*

The most general forms of systems of two second order ODEs associated with 1, 2 and 3 Lie-like operators (which are also Lie point symmetries for systems) are given in Table 5.1. In Table 5.2

Complex Equation	Lie-like Operators	Associated Systems
$u'' = \omega(u, u')$	$\mathbf{X}_1 = \partial_x$	$f_1'' = \omega_1(f_1, f_2, f_1', f_2')$
		$f_2'' = \omega_2(f_1, f_2, f_1', f_2')$
$u'' = \omega(x, u')$	$\mathbf{X}_1 = \partial_{f_1},\ \mathbf{X}_2 = \partial_{f_2}$	$f_1'' = \omega_1(x, f_1', f_2')$
		$f_2'' = \omega_2(x, f_1', f_2')$
$u'' = \omega(u')$	$\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_{f_1}$	$f_1'' = \omega_1(f_1', f_2')$
	$\mathbf{X}_3 = \partial_{f_2}$	$f_2'' = \omega_2(f_1', f_2')$

Table 5.1: All the Lie-like operators are the Lie point symmetries for the associated systems.

those systems are presented which are spanned by 4, 5 and 6 Lie-like operators out of which a few are the Lie point symmetries.

5.3 Comparison of the Lie-Like Operators and Lie Point Symmetries for Systems

This section is devoted to the comparison of the criteria to obtain Lie-like operators with the Lie point symmetry conditions. The allowable number of the Lie point symmetry generators associated with the scalar complex and CR-structured systems of ODEs is $\mathbf{Z}_\vartheta$ and $\mathbf{X}_\kappa$, where $\vartheta = 0 - 3$ or 8,

Complex Equation	Lie-like Operators	Associated Systems
$uu'' = \omega(u')$	$\mathbf{X}_1 = \partial_x$ $\mathbf{X}_2 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}$ $\mathbf{X}_3 = f_2\partial_{f_1} - f_1\partial_{f_2}$	$f_1'' = (f_1^2+f_2^2)^{-1}(f_1\omega_1 + f_2\omega_2)$ $f_2'' = (f_1^2+f_2^2)^{-1}(f_1\omega_2 - f_2\omega_1)$ where $\omega_i = \omega_i(f_1', f_2')$
$xu'' = \omega(u')$	$\mathbf{X}_1 = \partial_{f_1}, \mathbf{X}_2 = \partial_{f_2}$ $\mathbf{X}_3 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}$ $\mathbf{X}_4 = f_2\partial_{f_1} - f_1\partial_{f_2}$	$xf_1'' = \omega_1(f_1', f_2')$ $xf_2'' = \omega_2(f_1', f_2')$
$u'' = A(u')^{\frac{a-2}{a-1}}$ $a \neq 0, \frac{1}{2}, 2$	$\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_{f_1},\ \mathbf{X}_3 = \partial_{f_2}$ $\mathbf{X}_4 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}$ $\mathbf{X}_5 = f_2\partial_{f_1} - f_1\partial_{f_2}$	$f_1'' = f_1'^4 + f_2'^4 - 6f_1'^2 f_2'^2$ $f_2'' = 4f_1'f_2'(f_1'^2 - f_2'^2)$
$u'' = Ae^{-u'}$	$\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_{f_1},\ \mathbf{X}_3 = \partial_{f_2}$ $\mathbf{X}_4 = 2x\partial_x + (x + f_1)\partial_{f_1} + f_2\partial_{f_2}$ $\mathbf{X}_5 = f_2\partial_{f_1} - f_1\partial_{f_2}$	$f_1'' = Ae^{-f_1'}\cos(f_2')$ $f_2'' = -Ae^{-f_1'}\sin(f_2')$
$xu'' = Au'^3 - \frac{1}{2}u'$	$\mathbf{X}_1 = \partial_{f_1},\ \mathbf{X}_2 = \partial_{f_2}$ $\mathbf{X}_3 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}$ $\mathbf{X}_4 = f_2\partial_{f_1} - f_1\partial_{f_2}$ $\mathbf{X}_5 = 4xf_1\partial_x + (f_1^2 - f_2^2)\partial_{f_1} + 2f_1f_2\partial_{f_2}$ $\mathbf{X}_6 = 4xf_2\partial_x + 2f_1f_2\partial_{f_1} - (f_1^2 - f_2^2)\partial_{f_2}$	$xf_1'' = Af_1'^3 - 3Af_1'f_2'^2 - \frac{1}{2}f_1'$ $xf_2'' = 3Af_1'^2f_2' - Af_2'^3 - \frac{1}{2}f_2'$

Table 5.2: Systems for which all the associated Lie-like operators are not the Lie point symmetries.

and $\kappa = 0 - 8$ or 15. Theorem 1 shows that the number of symmetries and associated operators is not the same. As for the simplest maximally symmetric system that arises due to a complex free particle equation with $\vartheta = 8$, there exist fourteen Lie-like operators which clearly is not in the limit set above for κ. Furthermore, this investigation leads us to establish a criterion which replaces the Lie point symmetry conditions for two dimensional CR-structured systems of second order ODEs. For comparison the dependent variables f_1 and f_2 which appeared in the CR-structured systems in the previous discussion are replaced by y and z in the subsequent sections of this chapter.

Theorem 5.3.1. *The real and imaginary parts of the components* $(\xi(x,u), \eta(x,u))$ *of the complex symmetry generator* (*1.16*), *for* $n = 2$, *arise in the following combinations*

$$\boldsymbol{X}_1 = \xi_1\partial_x + \eta_1\partial_y + \eta_2\partial_z, \tag{5.29}$$

and

$$\boldsymbol{X}_2 = \xi_2\partial_x + \eta_2\partial_y - \eta_1\partial_z, \tag{5.30}$$

to give the symmetry generators of the emerging CR-structured systems subject to constraints

$$2\xi_{1,xz}y'z' - 2\xi_{1,xy}z'^2 + 2\xi_{1,zz}y'z'^2 + \xi_{1,yz}y'^2z' - \xi_{1,yz}z'^3 + \xi_{1,z}z'y'' + (2\xi_{1,z}y' - 3\xi_{1,y}z' - 2\xi_{2,x})z''$$
$$-\xi_{2,xx}z' + (\xi_{1,y}z'^2 - \xi_{1,z}y'z' + \xi_{2,x}z')w_{1,y'} + (\xi_{1,z}y'^2 - \xi_{1,y}y'z' - \xi_{2,x}y')w_{1,z'} - \xi_2\omega_{2,x} = 0, \tag{5.31}$$

$$2\xi_{1,xy}y'z' - 2\xi_{1,xz}y'^2 + 2\xi_{1,yy}y'^2z' + \xi_{1,yz}y'z'^2 - \xi_{1,yz}y'^3 + \xi_{1,y}y'z'' + (2\xi_{1,y}z' - 3\xi_{1,z}y' + 2\xi_{2,x})y''$$
$$+\xi_{2,xx}y' + (\xi_{1,y}z'^2 - \xi_{1,z}y'z' + \xi_{2,x}z')w_{2,y'} + (\xi_{1,z}y'^2 - \xi_{1,y}y'z' - \xi_{2,x}y')w_{2,z'} + \xi_2\omega_{1,x} = 0, \tag{5.32}$$

and

$$\xi_1\omega_{1,x} - \xi_2\omega_{2,x} - \xi_2\omega_{1,x} - (\xi_{2,xx} - \xi_{1,xx})y' - \xi_{2,xx}z' - [2(\xi_{2,x} - \xi_{1,x}) + 3(\xi_{2,y} - \xi_{1,y})y' + (2\xi_{2,z}$$
$$-3\xi_{1,z})z']y'' - [2\xi_{2,x} + (\xi_{2,z} - 3\xi_{1,z})y' + 3\xi_{1,y}z']z'' - [2(\xi_{2,xy} - \xi_{1,xy}) + (2\xi_{2,yz} - 3\xi_{1,yz})z']y'^2$$
$$-2(\xi_{2,xz} - \xi_{1,xz})y'z' - [2\xi_{1,xy} + (\xi_{2,zz} - 3\xi_{1,zz})y']z'^2 - (\xi_{2,yy} - \xi_{1,yy})y'^3 - \xi_{1,yz}z'^3 + [(\xi_{2,x}$$
$$-\xi_{1,x})y' + \xi_{2,x}z' + (\xi_{2,y} - \xi_{1,y})y'^2 + (\xi_{2,z} - 2\xi_{1,z})y'z' + \xi_{1,y}z'^2]\omega_{1,y'} + [-\xi_{2,x}y' + (\xi_{2,x} - \xi_{1,x})z'$$
$$+\xi_{1,z}y'^2 + (\xi_{2,y} - 2\xi_{1,y})y'z' + (\xi_{2,z} - \xi_{1,z})z'^2]\omega_{1,z'} = 0, \tag{5.33}$$

$$\xi_1\omega_{2,x} + \xi_2\omega_{1,x} - \xi_2\omega_{2,x} - (\xi_{2,xx} - \xi_{1,xx})z' + \xi_{2,xx}y' - [2(\xi_{2,x} - \xi_{1,x}) + (2\xi_{2,y} - 3\xi_{1,y})y' + 3(\xi_{2,z}$$
$$-\xi_{1,z})z']z'' - [-2\xi_{2,x} + (\xi_{2,y} - 3\xi_{1,y})z' + 3\xi_{1,z}y']y'' - [2(\xi_{2,xz} - \xi_{1,xz}) + (2\xi_{2,yz} - 3\xi_{1,yz})y']z'^2$$
$$-2(\xi_{2,xy} - \xi_{1,xy})y'z' - [2\xi_{1,xz} + (\xi_{2,yy} - 3\xi_{1,yy})z']y'^2 - (\xi_{2,zz} - \xi_{1,zz})z'^3 - \xi_{1,yz}y'^3 + [(\xi_{2,x}$$
$$-\xi_{1,x})y' + \xi_{2,x}z' + (\xi_{2,y} - \xi_{1,y})y'^2 + (\xi_{2,z} - 2\xi_{1,z})y'z' + \xi_{1,y}z'^2]\omega_{2,y'} + [-\xi_{2,x}y' + (\xi_{2,x} - \xi_{1,x})z'$$
$$+\xi_{1,z}y'^2 + (\xi_{2,y} - 2\xi_{1,y})y'z' + (\xi_{2,z} - \xi_{1,z})z'^2]\omega_{2,z'} = 0, \tag{5.34}$$

respectively.

Proof. The derivation of the Lie point symmetry conditions (1.92)-(1.97) involves a single symmetry generator of the form (1.91) whereas Lie-like conditions involve a pair of operators (5.4) and (5.5). To reduce the former set of equations to compare them with the latter the CR-equations which both the equations ω_j of the associated system satisfy, are used. Indeed, the real and imaginary parts of the components $\xi(x, u)$ and $\eta(x, u)$ of the complex vector fields also satisfy the CR-equations

$$\xi_{1,y} = \xi_{2,z}, \quad \xi_{1,z} = -\xi_{2,y}, \quad \eta_{1,y} = \eta_{2,z}, \quad \eta_{1,z} = -\eta_{2,y}. \tag{5.35}$$

Moreover, utilizing the CR-equations for η_j it is found that ζ'_j, satisfy the CR-equations with respect to both the dependent variables as well as their first derivatives. These results help in reducing the Lie-like conditions and corresponding extensions given in (5.8)-(5.16) to the following equations

$$\zeta''_1 = \xi_1\omega_{1,x} - \xi_2\omega_{2,x} + \eta_1\omega_{1,y} + \eta_2\omega_{1,z} + \zeta'_1\omega_{1,y'} + \zeta'_2\omega_{1,z'}, \tag{5.36}$$

$$\zeta''_2 = \xi_1\omega_{2,x} + \xi_2\omega_{1,x} + \eta_1\omega_{2,y} + \eta_2\omega_{2,z} + \zeta'_1\omega_{2,y'} + \zeta'_2\omega_{2,z'}, \tag{5.37}$$

$$\zeta'_1 = \eta_{1,x} + (\eta_{1,y} - \xi_{1,x})y' + (\eta_{1,z} + \xi_{2,x})z' - \xi_{1,y}(y'^2 - z'^2) - 2\xi_{1,z}y'z', \tag{5.38}$$

$$\zeta'_2 = \eta_{2,x} + (\eta_{2,y} - \xi_{2,x})y' + (\eta_{2,z} - \xi_{1,x})z' + \xi_{1,z}(y'^2 - z'^2) - 2\xi_{1,y}y'z', \tag{5.39}$$

and

$$\zeta_1'' = \eta_{1,xx} + (2\eta_{1,xy} - \xi_{1,xx})y' + (2\eta_{1,xz} + \xi_{2,xx})z' + (\eta_{1,yy} - 2\xi_{1,xy})y'^2 + 2(\eta_{1,yz} - 2\xi_{1,xz})y'z'$$
$$+(\eta_{1,zz} + 2\xi_{1,xy})z'^2 - \xi_{1,yy}y'^3 - 3\xi_{1,yz}y'^2z' - 3\xi_{1,zz}y'z'^2 + \xi_{1,yz}z'^3 + (\eta_{1,y} - 2\xi_{1,x} - 3\xi_{1,y}y'$$
$$-3\xi_{1,z}z')y'' + (\eta_{1,z} + 2\xi_{2,x} - 3\xi_{1,z}y' + 3\xi_{1,y}z')z'', \tag{5.40}$$

$$\zeta_2'' = \eta_{2,xx} + (2\eta_{2,xy} - \xi_{2,xx})y' + (2\eta_{2,xz} - \xi_{1,xx})z' + (\eta_{2,yy} + 2\xi_{1,xz})y'^2 + 2(\eta_{2,yz} - 2\xi_{1,xy})y'z'$$
$$+(\eta_{2,zz} - 2\xi_{1,xz})z'^2 + \xi_{1,yz}y'^3 - 3\xi_{1,yy}y'^2z' - 3\xi_{1,yz}y'z'^2 - \xi_{1,zz}z'^3 + (\eta_{2,y} - 2\xi_{2,x} + 3\xi_{1,z}y'$$
$$-3\xi_{1,y}z')y'' + (\eta_{2,z} - 2\xi_{1,x} - 3\xi_{1,y}y' - 3\xi_{1,z}z')z'', \tag{5.41}$$

respectively. Now comparing the Lie-like conditions and the Lie symmetry conditions, i.e., considering firstly the equations (1.94), (1.95), (5.38) and (5.39) imply

$$\zeta_1' = \eta_1' - \xi_{1,z}y'z' + \xi_{1,y}z'^2 + \xi_{2,x}z',$$
$$\zeta_2' = \eta_2' - \xi_{1,y}y'z' + \xi_{1,z}y'^2 - \xi_{2,x}y'. \tag{5.42}$$

Likewise, equations (1.96), (1.97), (5.40) and (5.41) lead to the following set of equations

$$\zeta_1'' = \eta_1'' - 2\xi_{1,xz}y'z' + 2\xi_{1,xy}z'^2 - 2\xi_{1,zz}y'z'^2 - \xi_{1,yz}y'^2z' + \xi_{1,yz}z'^3 - \xi_{1,z}z'y''$$
$$-(2\xi_{1,z}y' - 3\xi_{1,y}z' - 2\xi_{2,x})z'' + \xi_{2,xx}z',$$
$$\zeta_2'' = \eta_2'' - 2\xi_{1,xy}y'z' + 2\xi_{1,xz}y'^2 - \xi_{1,yz}y'z'^2 - 2\xi_{1,yy}y'^2z' + \xi_{1,yz}y'^3 - \xi_{1,y}y'z''$$
$$-(2\xi_{1,y}z' - 3\xi_{1,z}y' + 2\xi_{2,x})y'' - \xi_{2,xx}y'. \tag{5.43}$$

Inserting these ζ_j' and ζ_j'' in equations (5.8) and (5.9) one arrives at the reduced Lie-like conditions

$$\eta_1'' - \xi_1\omega_{1,x} - \eta_1\omega_{1,y} - \eta_2\omega_{1,z} - \eta_1'\omega_{1,y'} - \eta_2'\omega_{1,z'} = 2\xi_{1,xz}y'z' - 2\xi_{1,xy}z'^2 + 2\xi_{1,zz}y'z'^2 + \xi_{1,yz}y'^2z'$$
$$-\xi_{1,yz}z'^3 + \xi_{1,z}z'y'' + (2\xi_{1,z}y' - 3\xi_{1,y}z' - 2\xi_{2,x})z'' - \xi_{2,xx}z' + (\xi_{1,y}z'^2 - \xi_{1,z}y'z' + \xi_{2,x}z')\omega_{1,y'}$$
$$+(\xi_{1,z}y'^2 - \xi_{1,y}y'z' - \xi_{2,x}y')\omega_{1,z'} - \xi_2\omega_{2,x}, \tag{5.44}$$

$$\eta_2'' - \xi_1\omega_{2,x} - \eta_1\omega_{2,y} - \eta_2\omega_{2,z} - \eta_1'\omega_{2,y'} - \eta_2'\omega_{2,z'} = 2\xi_{1,xy}y'z' - 2\xi_{1,xz}y'^2 + 2\xi_{1,yy}y'^2z' + \xi_{1,yz}y'z'^2$$
$$-\xi_{1,yz}y'^3 + \xi_{1,y}y'z'' + (2\xi_{1,y}z' - 3\xi_{1,z}y' + 2\xi_{2,x})y'' + \xi_{2,xx}y' + (\xi_{1,y}z'^2 - \xi_{1,z}y'z' + \xi_{2,x}z')\omega_{2,y'}$$
$$+(\xi_{1,z}y'^2 - \xi_{1,y}y'z' - \xi_{2,x}y')\omega_{2,z'} + \xi_2\omega_{1,x}. \tag{5.45}$$

Notice that these equations include the Lie symmetry conditions. Therefore, it is clear from (5.44) and (5.45) that the components ξ_1, η_1 and η_2 of the operator (5.4) constitute a real symmetry generator of the form $\xi_1\partial_x + \eta_1\partial_y + \eta_2\partial_z$ for the corresponding system if ξ_1 and ξ_2 are compatible with (5.31) and (5.32). Although, this set of equations plays a decisive role in obtaining Lie symmetries from the associated operators but only for an operator of the form (5.4), a similar effort is needed to establish the criteria for the second operator (5.5) to be a Lie symmetry generator, with the components ξ_2, η_2 and $-\eta_1$,i.e., a lie symmetry of the form $\xi_2\partial_x + \eta_2\partial_y - \eta_1\partial_z$. Merely replacing ξ_1 by ξ_2 in the reduced conditions will not serve the purpose as Lie point symmetry conditions involve derivatives of ξ with respect to x. Though the terms which contain the derivatives of ξ_j's with respect to y, z can be replaced using the CR-equations, those with only x derivatives do not allow the replacements. Therefore, it is needed to get rid of such derivative terms involving ξ_1 in the symmetry conditions by adding similar terms with ξ_2. As a result another set of the reduced Lie-like conditions relative to ξ_2 is obtained

$$\begin{aligned}&\eta_1'' - \xi_2\omega_{1,x} - \eta_1\omega_{1,y} - \eta_2\omega_{1,z} - \eta_1'\omega_{1,y'} - \eta_2'\omega_{1,z'} = \xi_1\omega_{1,x} - \xi_2\omega_{2,x} - \xi_2\omega_{1,x} - (\xi_{2,xx} - \xi_{1,xx})y' \\ &-\xi_{2,xx}z' - [2(\xi_{2,x} - \xi_{1,x}) + 3(\xi_{2,y} - \xi_{1,y})y' + (2\xi_{2,z} - 3\xi_{1,z})z']y'' - [2\xi_{2,x} + (\xi_{2,z} - 3\xi_{1,z})y' \\ &+3\xi_{1,y}z']z'' - [2(\xi_{2,xy} - \xi_{1,xy}) + (2\xi_{2,yz} - 3\xi_{1,yz})z']y'^2 - 2(\xi_{2,xz} - \xi_{1,xz})y'z' - [2\xi_{1,xy} + (\xi_{2,zz} \\ &-3\xi_{1,zz})y']z'^2 - (\xi_{2,yy} - \xi_{1,yy})y'^3 - \xi_{1,yz}z'^3 + [(\xi_{2,x} - \xi_{1,x})y' + \xi_{2,x}z' + (\xi_{2,y} - \xi_{1,y})y'^2 + (\xi_{2,z} \\ &-2\xi_{1,z})y'z' + \xi_{1,y}z'^2]\omega_{1,y'} + [-\xi_{2,x}y' + (\xi_{2,x} - \xi_{1,x})z' + \xi_{1,z}y'^2 + (\xi_{2,y} - 2\xi_{1,y})y'z' + (\xi_{2,z} \\ &-\xi_{1,z})z'^2]\omega_{1,z'},\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}&\eta_2'' - \xi_2\omega_{2,x} - \eta_1\omega_{2,y} - \eta_2\omega_{2,z} - \eta_1'\omega_{2,y'} - \eta_2'\omega_{2,z'} = \xi_1\omega_{2,x} + \xi_2\omega_{1,x} - \xi_2\omega_{2,x} - (\xi_{2,xx} - \xi_{1,xx})z' \\ &+\xi_{2,xx}y' - [2(\xi_{2,x} - \xi_{1,x}) + (2\xi_{2,y} - 3\xi_{1,y})y' + 3(\xi_{2,z} - \xi_{1,z})z']z'' - [-2\xi_{2,x} + (\xi_{2,y} - 3\xi_{1,y})z' \\ &+3\xi_{1,z}y']y'' - [2(\xi_{2,xz} - \xi_{1,xz}) + (2\xi_{2,yz} - 3\xi_{1,yz})y']z'^2 - 2(\xi_{2,xy} - \xi_{1,xy})y'z' - [2\xi_{1,xz} + (\xi_{2,yy} \\ &-3\xi_{1,yy})z']y'^2 - (\xi_{2,zz} - \xi_{1,zz})z'^3 - \xi_{1,yz}y'^3 + [(\xi_{2,x} - \xi_{1,x})y' + \xi_{2,x}z' + (\xi_{2,y} - \xi_{1,y})y'^2 + (\xi_{2,z} \\ &-2\xi_{1,z})y'z' + \xi_{1,y}z'^2]\omega_{2,y'} + [-\xi_{2,x}y' + (\xi_{2,x} - \xi_{1,x})z' + \xi_{1,z}y'^2 + (\xi_{2,y} - 2\xi_{1,y})y'z' + (\xi_{2,z} \\ &-\xi_{1,z})z'^2]\omega_{2,z'}.\end{aligned} \tag{5.47}$$

Therefore, a Lie point symmetry of a two dimensional system of second order ODEs of the form $\mathbf{X}_2$, i.e., the generator (5.30) emerges from an operator (5.5) if ξ_1 and ξ_2 satisfy the set of equations (5.33) and (5.34).

The above theorem is sufficient to distinguish between the Lie and Lie-like operators associated with the CR-structured two dimensional systems, as well as it provides conditions to obtain the former from the later, which makes the existence of Lie algebras obvious. It is observed that most of the Lie point symmetries for such systems come from the components of the complex vector fields of the base ODEs. Though all the complex symmetries may not split to yield all the Lie symmetries of the corresponding systems, they contain the information about the Lie point symmetry algebras associated with the CR-structured systems.

In the subsequent section four applications of the theory developed are presented by considering those CR-structured systems which correspond to non-linear integrable or linearizable and linear scalar ODEs of second order.s

5.4 Examples

1. The most general form of a two dimensional system of second order ODEs associated with the following 3-dimensional algebra of vector fields

$$\mathbf{X}_1 = \partial_y,\ \mathbf{X}_2 = \partial_z,\ \mathbf{X}_3 = x\partial_x + y\partial_y + z\partial_z. \tag{5.48}$$

is

$$\begin{aligned} xy'' &= \omega_1(y', z'), \\ xz'' &= \omega_2(y', z'). \end{aligned} \tag{5.49}$$

The symmetry generators given in (5.48) arise from the following complex 2-dimensional algebra

$$\mathbf{Z}_1 = \partial_u,\ \mathbf{Z}_2 = x\partial_x + u\partial_u. \tag{5.50}$$

Four operators are associated with the system (5.49) due to the above complex symmetry algebra and all of them satisfy (5.44)-(5.47). It can trivially be seen that $\mathbf{Z}_1$ satisfies the set of equations (5.31)-(5.34). Thus it generates two symmetry generators of the system. However, only one of the operators obtainable from $\mathbf{Z}_2$ yields a Lie point symmetry as (5.33) and (5.34) are not satisfied by the operator $z\partial_y - y\partial_z$. Consider the following three operators

$$\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = x\partial_x + \frac{1}{2}(y\partial_y + z\partial_z),\ \mathbf{X}_3 = z\partial_y - y\partial_z, \tag{5.51}$$

which yield the two dimensional system

$$\begin{aligned} y'' &= \frac{1}{y^2 + z^2}(y\omega_1(y', z') + z\omega_2(y', z')), \\ z'' &= \frac{1}{y^2 + z^2}(y\omega_2(y', z') - z\omega_1(y', z')), \end{aligned} \tag{5.52}$$

by using the reduced conditions (5.44)-(5.47). This system has only a 2-dimensional Lie point symmetry algebra which consists of ∂_x and $x\partial_x + y\partial_y + z\partial_z$ whereas $\mathbf{X}_3$ is inconsistent with the Lie point symmetry conditions for systems. The complex base equations in both the above cases are $xu'' = \omega(u')$ and $uu'' = \omega(u')$, respectively. The complex symmetries which give rise to these complex equations are (5.50) and $\mathbf{Z}_1 = \partial_x,\ \mathbf{Z}_2 = x\partial_x + u\partial_u$, respectively.

2. Consider a maximally symmetric system of two second order ODEs

$$\begin{aligned} y'' + \alpha y' &= 0, \\ z'' + \alpha z' &= 0, \end{aligned} \tag{5.53}$$

where α is a constant. This is a CR-structured system which has a 15-dimensional algebra

$$\begin{aligned} &\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_y,\ \mathbf{X}_3 = \partial_z,\ \mathbf{X}_4 = y\partial_y,\ \mathbf{X}_5 = y\partial_z,\ \mathbf{X}_6 = z\partial_y,\ \mathbf{X}_7 = z\partial_z,\ \mathbf{X}_8 = e^{-\alpha x}\partial_z, \\ &\mathbf{X}_9 = e^{-\alpha x}\partial_y,\ \mathbf{X}_{10} = e^{\alpha x}\partial_x,\ \mathbf{X}_{11} = e^{\alpha x}y\partial_x,\ \mathbf{X}_{12} = e^{\alpha x}z\partial_x,\ \mathbf{X}_{13} = \tfrac{-z}{\alpha}\partial_x + yz\partial_y + z^2\partial_z, \\ &\mathbf{X}_{14} = \tfrac{-y}{\alpha}\partial_x + y^2\partial_y + yz\partial_z,\ \mathbf{X}_{15} = \tfrac{-e^{-\alpha x}}{\alpha}\partial_x + e^{-\alpha x}y\partial_y + e^{-\alpha x}z\partial_z. \end{aligned} \tag{5.54}$$

The complex Lie point symmetry algebra associated with the complex base equation is

$$\begin{aligned}&\mathbf{Z}_1=\partial_x,\ \mathbf{Z}_2=\partial_u,\ \mathbf{Z}_3=u\partial_u,\ \mathbf{Z}_4=e^{\alpha x}\partial_x,\ \mathbf{Z}_5=e^{-\alpha x}\partial_u,\\&\mathbf{Z}_6=ue^{\alpha x}\partial_x,\ \mathbf{Z}_7=\tfrac{-u}{\alpha}\partial_x+u^2\partial_u,\ \mathbf{Z}_8=-\tfrac{e^{-\alpha x}}{\alpha}\partial_x+ue^{-\alpha x}\partial_u.\end{aligned}\tag{5.55}$$

These complex symmetries yield fourteen operators for the system (5.53) that are compatible with (5.8) and (5.9). It is apparent from the symmetry algebra given for system (5.53) in (5.54) that the first twelve symmetry generators $\mathbf{X}_1-\mathbf{X}_{12}$ are composed of the real and imaginary parts of the first six complex symmetries $\mathbf{Z}_1-\mathbf{Z}_6$ of the complex equation. However, the split of the complex symmetry generator $\mathbf{Z}_7$ do not come out as Lie point symmetries because the set of equations (5.31)-(5.34) for ξ_j's in this case are non-zero. One Lie point symmetry namely, $\mathbf{X}_{15}$ of the system corresponds to $\mathbf{Z}_8$, i.e., only the components of $\Re(\mathbf{Z}_8)$ is the Lie point symmetry. Notice that $\mathbf{X}_{13}$ and $\mathbf{X}_{14}$ are not obtainable from the procedure developed here. It implies that all the complex symmetries (or their parts) do not necessarily decompose into all the symmetry generators of the corresponding systems.

3. There are fifteen Lie-like operators for the following system

$$\begin{aligned}y''+x(y'^3-3y'z'^2)+\frac{2}{x}y'=0,\\z''+x(3y'^2z'-z'^3)+\frac{2}{x}z'=0.\end{aligned}\tag{5.56}$$

The complex analogue of the operators associated with the above system is the following 8-dimensional complex Lie point symmetry algebra

$$\begin{aligned}&\mathbf{Z}_1=x\partial_x,\ \mathbf{Z}_2=\partial_u,\ \mathbf{Z}_3=x^2\sin u\partial_x,\ \mathbf{Z}_4=-x^2\cos u\partial_x,\ \mathbf{Z}_5=\sin u\partial_x+\tfrac{1}{x}\cos u\partial_u,\\&\mathbf{Z}_6=-\cos u\partial_x+\tfrac{1}{x}\sin u\partial_u,\ \mathbf{Z}_7=x\sin 2u\partial_x+\cos 2u\partial_u,\ \mathbf{Z}_8=-x\cos 2u\partial_x+\sin 2u\partial_u.\end{aligned}\tag{5.57}$$

Despite the fact that the system (5.56) has fifteen operators it is non-linearizable by point transformations. This implies that this system must have a symmetry algebra of dimension less than five which leads us to the same result that all the operators in this case are not Lie symmetries of the system (5.56). Therefore, the conditions given in Theorem 1 leave us with the 3-dimensional Lie algebra

$$\mathbf{X}_1=x\partial_x,\ \mathbf{X}_2=\partial_y,\ \mathbf{X}_3=\partial_z,\tag{5.58}$$

out of fifteen operators for the system (5.56). As both the equations of this system satisfy the CR-equations with respect to the dependent variables and their first derivatives; it corresponds to a complex non-linear base equation $xu''+x^2u'^3+u'=0$, which is linearizable because it has an 8-dimensional complex symmetry algebra (5.57).

4. Consider a quadratically semi-linear CR-structured system of second order ODEs

$$\begin{aligned}y''+y'^2-z'^2=\beta_1y'-\beta_2z',\\z''+2y'z'=\beta_2y'+\beta_1z',\end{aligned}\tag{5.59}$$

where β_1 and β_2 are constants. It is linearizable as it has the following 7-dimensional algebra

$$\begin{aligned}
&\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_y,\ \mathbf{X}_3 = \partial_z,\\
&\mathbf{X}_4 = e^{-y}\sin z\partial_y + e^{-y}\cos z\partial_z,\ \mathbf{X}_5 = -e^{-y}\cos z\partial_y + e^{-y}\sin z\partial_z,\\
&\mathbf{X}_6 = -e^{(\beta_1 x-y)}\sin(\beta_2 x - z)\partial_y + e^{(\beta_1 x-y)}(\cos\beta_2 x\cos z + \sin\beta_2 x\sin z)\partial_z,\\
&\mathbf{X}_7 = -e^{(\beta_1 x-y)}\cos(\beta_2 x - z)\partial_y + e^{(\beta_1 x-y)}(\cos\beta_2 x\sin z - \sin\beta_2 x\cos z)\partial_z.
\end{aligned} \tag{5.60}$$

The system (5.59) is associated with a linearizable complex equation of the form

$$u'' + u'^2 - \beta u' = 0, \tag{5.61}$$

as the CR-equations hold for the equations of the system. The complex base equation is also linearizable as it has an 8-dimensional complex algebra

$$\begin{aligned}
&\mathbf{Z}_1 = \partial_x,\ \mathbf{Z}_2 = \partial_u,\ \mathbf{Z}_3 = e^{-u}\partial_u,\ \mathbf{Z}_4 = e^{\beta x-u}\partial_u,\ \mathbf{Z}_5 = \tfrac{-1}{\beta}e^{\beta x}\partial_x,\\
&\mathbf{Z}_6 = e^{-\beta x+u}\partial_x,\ \mathbf{Z}_7 = \tfrac{e^u}{\beta}\partial_x + e^u\partial_u,\ \mathbf{Z}_8 = \tfrac{e^{\beta x}}{\beta}\partial_x + e^{\beta x}\partial_u.
\end{aligned} \tag{5.62}$$

As it is clear from (5.60) that all these generators are obtainable from first four complex symmetry generators given in (5.62), i.e., the components of these complex vector fields yield all the symmetries for system (5.59). All the components of the remaining complex vector fields do not satisfy Lie point symmetry conditions for systems as well as these choices of ξ_j's fail to satisfy (5.31)-(5.34).

5.5 Noether-Like Operators, Noether Symmetries and Associated First Integrals

The remaining sections of this chapter present a comprehensive comparison of the Lie symmetries and Lie-like operators. This section considers the applications of the Lie-like operators admitted by the CR-structured systems of two second order ODEs. This discussion relies on the Noether-like operators [23, 24] and their role in providing the associated first integrals. Therefore, one needs complete description of the Noether-like operators of the CR-structured systems to deal with them.

If a CR-structured system of the form (1.137) have Lagrangians $L_j(x, f_1, f_2, f_1', f_2')$, where $j = 1, 2$, then it is equivalent to Euler-Lagrange equations of the form

$$\frac{\partial L_1}{\partial f_1} + \frac{\partial L_2}{\partial f_2} - \frac{d}{dx}(\frac{\partial L_1}{\partial f_1'} + \frac{\partial L_2}{\partial f_2'}) = 0, \tag{5.63}$$

$$\frac{\partial L_1}{\partial f_2} - \frac{\partial L_2}{\partial f_1} + \frac{d}{dx}(\frac{\partial L_2}{\partial f_1'} - \frac{\partial L_1}{\partial f_2'}) = 0. \tag{5.64}$$

Definition 5.5.1. The operators of the form (5.4) and (5.5) are said to be Noether-like operators corresponding to Lagrangians $L_1(x, f_1, f_2, f_1', f_2')$ and $L_2(x, f_1, f_2, f_1', f_2')$ of (1.137) if there exist gauge functions $B_1(x, f_1, f_2)$ and $B_2(x, f_1, f_2)$ such that

$$\begin{aligned}
&\mathbf{X}_1^{[1]}L_1 - \mathbf{X}_2^{[1]}L_2 + (d_x\xi_1)L_1 - (d_x\xi_2)L_2 = d_xB_1,\\
&\mathbf{X}_1^{[1]}L_2 + \mathbf{X}_2^{[1]}L_1 + (d_x\xi_1)L_2 + (d_x\xi_2)L_1 = d_xB_2.
\end{aligned} \tag{5.65}$$

Theorem 5.5.2. *If $\boldsymbol{X}_j$, are the Noether-like operators corresponding to $L_j(x, f_1, f_2, f_1', f_2')$, for $j = 1, 2$, then*

$$I_1 = \xi_1 L_1 - \xi_2 L_2 + \frac{1}{2}[(\eta_1 - y'\xi_1 + z'\xi_2)(\frac{\partial L_1}{\partial y'} + \frac{\partial L_2}{\partial z'}) - (\eta_2 - y'\xi_2 - z'\xi_1)(\frac{\partial L_2}{\partial y'} - \frac{\partial L_1}{\partial z'})] - B_1,$$
$$I_2 = \xi_1 L_2 + \xi_2 L_1 + \frac{1}{2}[(\eta_1 - y'\xi_1 + z'\xi_2)(\frac{\partial L_2}{\partial y'} - \frac{\partial L_1}{\partial z'}) + (\eta_2 - y'\xi_2 - z'\xi_1)(\frac{\partial L_1}{\partial y'} + \frac{\partial L_2}{\partial z'})] - B_2, \tag{5.66}$$

are the first integrals of the system (1.137) associated with the Noether-like operators $\boldsymbol{X}_j$.

Theorem 5.5.3. *The first integrals I_j, associated with the Noether-like operators $\boldsymbol{X}_j$, are compatible with the following set of equations*

$$\boldsymbol{X}_1^{[1]} I_1 - \boldsymbol{X}_2^{[1]} I_2 = 0,$$
$$\boldsymbol{X}_1^{[1]} I_2 + \boldsymbol{X}_2^{[1]} I_1 = 0, \tag{5.67}$$

and

$$DI_1 = DI_2 = 0, \tag{5.68}$$

where D is the total derivative operator given by (1.109).

5.6 First Integrals Associated with the Complex-Linearizable Systems

The following four cases deal with the Noether-like operators and the corresponding first integrals associated with the complex-linearizable class of systems considered in the previous chapter. These cases describe the difference between real and complex symmetry analysis.

Case 1. The following two Lagrangians

$$L_1 = 2f_1 + \frac{f_1'}{f_1'^2 + f_2'^2}, \quad L_2 = 2f_2 - \frac{f_2'}{f_1'^2 + f_2'^2}, \tag{5.69}$$

are associated with the system (4.30) which admits the following 4-dimensional algebra

$$\mathbf{X}_1 = \partial_x, \ \mathbf{X}_2 = \partial_{f_1}, \ \mathbf{X}_3 = \partial_{f_2}, \ \mathbf{X}_4 = 2x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}. \tag{5.70}$$

It is found that three gauge functions

$$B_1 = c_1, \quad B_2 = 2x, \quad B_3 = c_2, \tag{5.71}$$

exist for $\mathbf{X}_j$ where $j = 1, 2, 3$, respectively, relative to L_1. Similarly, L_2 generates

$$B_1 = c_3, \quad B_2 = c_4, \quad B_3 = 2x, \tag{5.72}$$

with the same point symmetries as mentioned above. Thus a 3-dimensional Noether algebra is found to exist for the system (4.30) and three first integrals are as given below

$$\begin{aligned} I_1 &= 2f_1 + \tfrac{2f_1'}{f_1'^2+f_2'^2} - c_1, \\ I_2 &= \tfrac{1}{f_1'^2+f_2'^2} - \tfrac{2f_1'^2}{(f_1'^2+f_2'^2)^2} - 2x, \\ I_3 &= \tfrac{-2f_1'f_2'}{(f_1'^2+f_2'^2)^2} - c_2, \end{aligned} \tag{5.73}$$

associated with $\mathbf{X}_j$, relative to L_1. Similarly for the same Noether symmetries relative to L_2 the following first integrals are obtained

$$\begin{aligned} I_1 &= 2f_2 - \tfrac{2f_2'}{f_1'^2+f_2'^2} - c_3, \\ I_2 &= \tfrac{2f_1'f_2'}{(f_1'^2+f_2'^2)^2} - c_4, \\ I_3 &= \tfrac{-1}{f_1'^2+f_2'^2} + \tfrac{2f_2'^2}{(f_1'^2+f_2'^2)^2} - 2x. \end{aligned} \tag{5.74}$$

Notice that the first integrals given in (5.73) and (5.74) are related by the CR-equations due to the Lagrangians L_j which also satisfy the CR-equations. Now consider the base complex equation (4.33) where the associated complex Lagrangian is

$$L = 2u + \frac{1}{u'}. \tag{5.75}$$

The following five Lie point symmetries

$$\mathbf{Z}_1 = \partial_x,\ \mathbf{Z}_2 = \partial_u,\ \mathbf{Z}_3 = u\partial_x,\ \mathbf{Z}_4 = (u^3 - 2xu)\partial_x - 2u^2\partial_u,\ \mathbf{Z}_5 = (3u^2 - 2x)\partial_x - 4u\partial_u, \tag{5.76}$$

yield five gauge functions

$$B_1 = c,\ B_2 = 2x,\ B_3 = 2x + u^2,\ B_4 = \frac{3u^4}{2} - 2xu^2 - 2x^2,\ B_5 = 4u^3, \tag{5.77}$$

respectively. The corresponding complex first integrals are

$$\begin{aligned} &I_1 = 2u + \tfrac{2}{u'} - c,\ I_2 = \tfrac{-1}{u'^2} - 2x,\ I_3 = u^2 - 2x + \tfrac{2u}{u'}, \\ &I_4 = \tfrac{1}{2u'^2}((u^2 - 2x)u' + 2u)^2,\ I_5 = 2u^3 - 4xu + \tfrac{(6u^2-4x)}{u'} + \tfrac{4u}{u'^2}, \end{aligned} \tag{5.78}$$

which are compatible with

$$\mathbf{Z}_j^{[1]} I_j = 0, \quad j = 1,..,5. \tag{5.79}$$

Therefore, 10 Noether-like operators emerge from the complex Noether symmetry generators of the base equation. These Noether-like operators yield the following ten first integrals

$$\begin{aligned} &I_1 = 2f_1 + \tfrac{2f_1'}{f_1'^2+f_2'^2} - c_1,\quad I_2 = 2f_2 - \tfrac{2f_2'}{f_1'^2+f_2'^2} - c_2, \\ &I_3 = -2x - \tfrac{f_1'^2-f_2'^2}{(f_1'^2+f_2'^2)^2},\quad I_4 = \tfrac{2f_1'f_2'}{(f_1'^2+f_2'^2)^2}, \end{aligned}$$

$$I_5 = f_1^2 - f_2^2 - 2\left(x - \tfrac{f_1 f_1' + f_2 f_2'}{f_1'^2 + f_2'^2}\right), \quad I_6 = 2\left(f_1 f_2 + \tfrac{f_1' f_2 - f_1 f_2'}{f_1'^2 + f_2'^2}\right),$$

$$I_7 = f_1^4 + f_2^4 - 6f_1^2 f_2^2 + 4x^2 - 4x(f_1^2 - f_2^2) + 4\tfrac{(f_1^3 - 3f_1 f_2^2 - 2x f_1)f_1' + (3f_1^2 f_2 - f_2^3 - 2x f_2)f_2'}{f_1'^2 + f_2'^2}$$
$$+4\tfrac{(f_1^2 - f_2^2)(f_1'^2 - f_2'^2) + 4f_1 f_2 f_1' f_2'}{(f_1'^2 + f_2'^2)^2},$$
$$I_8 = f_1 f_2(f_1^2 - f_2^2) - 2x f_1 f_2 + \tfrac{(3f_1^2 f_2 - f_2^3 - 2x f_2)f_1' - (f_1^3 - 3f_1 f_2^2 - 2x f_1)f_2'}{f_1'^2 + f_2'^2}$$
$$+\tfrac{2f_1 f_2(f_1'^2 - f_2'^2) - 2(f_1^2 - f_2^2)f_1' f_2'}{(f_1'^2 + f_2'^2)^2},$$

$$\begin{aligned} I_9 &= f_1^3 - 3f_1 f_2^2 - 2x f_1 + \tfrac{(3f_1^2 - 3f_2^2 - 2x)f_1' + 6f_1 f_2 f_2'}{f_1'^2 + f_2'^2} + 2\tfrac{f_1(f_1'^2 - f_2'^2) + 2f_2 f_1' f_2'}{(f_1'^2 + f_2'^2)^2}, \\ I_{10} &= 3f_1^2 f_2 - f_2^3 - 2x f_2 + \tfrac{6f_1 f_2 f_1' + (3f_2^2 - 3f_1^2 + 2x)f_2'}{f_1'^2 + f_2'^2} + 2\tfrac{f_2(f_1'^2 - f_2'^2) - 2f_1 f_1' f_2'}{(f_1'^2 + f_2'^2)^2}. \end{aligned} \tag{5.80}$$

Case 2. A system of two second order cubically semi-linear ODEs (4.37) emerges from a linearizable scalar second order complex ODE (4.39). The Lagrangian associated with this base equation is $L = 2xu + \frac{1}{u'}$. It yields five gauge functions

$$\begin{aligned} &B_1 = x^2, \quad B_2 = 2xu\sin u + 2x\cos u, \quad B_3 = 2xu\cos u - 2x\sin u, \\ &B_4 = x^2(2u\cos 2u - \sin 2u), \quad B_5 = x^2(2u\sin 2u + \cos 2u), \end{aligned} \tag{5.81}$$

relative to five Lie point symmetries

$$\begin{aligned} &\mathbf{Z}_1 = \partial_u, \quad \mathbf{Z}_2 = \sin u\partial_x, \quad \mathbf{Z}_3 = \cos u\partial_x, \\ &\mathbf{Z}_4 = x\cos 2u\partial_x + \sin 2u\partial_u, \quad \mathbf{Z}_5 = x\sin 2u\partial_x - \cos 2u\partial_u, \end{aligned} \tag{5.82}$$

that are the Noether symmetries. The first integrals of (4.39) associated with (5.82) are

$$\begin{aligned} &I_1 = \tfrac{-1}{u'^2} - x^2, \quad I_2 = \tfrac{\sin u}{u'} - x\cos u, \quad I_3 = \tfrac{\cos u}{u'} + x\sin u, \\ &I_4 = x^2\sin 2u - \tfrac{\sin 2u}{u'^2} + 2\tfrac{x\cos 2u}{u'}, \quad I_5 = -x^2\cos 2u + \tfrac{\cos 2u}{u'^2} + 2\tfrac{x\sin 2u}{u'}. \end{aligned} \tag{5.83}$$

Breaking the complex Lagrangian into real and imaginary parts yields

$$L_1 = 2xf_1 + \frac{f_1'}{f_1'^2 + f_2'^2}, \quad L_2 = 2xf_1 - \frac{f_2'}{f_1'^2 + f_2'^2}. \tag{5.84}$$

Further the gauge functions for the system (4.37) due to the associated Lagrangians are found to correspond to the complex gauge functions generated by the complex Noether algebra admitted by the base ODE. Therefore, the corresponding first integrals are

$$I_1 = -x^2 - \frac{f_1'^2 - f_2'^2}{(f_1'^2 + f_2'^2)^2}, \quad I_2 = \frac{2f_1' f_2'}{(f_1'^2 + f_2'^2)^2}$$

$$I_3 = \frac{\sin(f_1)\cosh(f_2)f_1' + \cos(f_1)\sinh(f_2)f_2'}{f_1'^2 + f_2'^2} - x\cos(f_1)\cosh(f_2),$$
$$I_4 = \frac{\cos(f_1)\sinh(f_2)f_1' - \sin(f_1)\cosh(f_2)f_2'}{f_1'^2 + f_2'^2} + x\sin(f_1)\sinh(f_2),$$

$$I_5 = \frac{\cos(f_1)\cosh(f_2)f_1' - \sin(f_1)\sinh(f_2)f_2'}{f_1'^2 + f_2'^2} + x\sin(f_1)\cosh(f_2),$$
$$I_6 = -\frac{\sin(f_1)\sinh(f_2)f_1' + \cos(f_1)\cosh(f_2)f_2'}{f_1'^2 + f_2'^2} + x\cos(f_1)\sinh(f_2),$$

$$I_7 = -x^2\cos(f_1)\cosh(2f_2) + 2\left(\frac{x\sin(2f_1)\cosh(2f_2)f_1' + x\cos(2f_1)\sinh(2f_2)f_2'}{f_1'^2 + f_2'^2}\right)$$
$$+\frac{\cos(2f_1)\cosh(2f_2)(f_1'^2 - f_2'^2) - 2\sin(2f_1)\sinh(2f_2)f_1'f_2'}{(f_1'^2 + f_2'^2)^2},$$
$$I_8 = x^2\sin(2f_1)\sinh(2f_2) + 2\left(\frac{x\cos(2f_1)\sinh(2f_2)f_1' - x\sin(2f_1)\cosh(2f_2)f_2'}{f_1'^2 + f_2'^2}\right)$$
$$-\frac{\sin(2f_1)\sinh(2f_2)(f_1'^2 - f_2'^2) + 2\cos(2f_1)\cosh(2f_2)f_1'f_2'}{(f_1'^2 + f_2'^2)^2},$$

$$I_9 = x^2\sin(2f_1)\cosh(2f_2) + 2\left(\frac{x\cos(2f_1)\cosh(2f_2)f_1' - x\sin(2f_1)\sinh(2f_2)f_2'}{f_1'^2 + f_2'^2}\right)$$
$$-\frac{\sin(2f_1)\cosh(2f_2)(f_1'^2 - f_2'^2) + 2\cos(2f_1)\sinh(2f_2)f_1'f_2'}{(f_1'^2 + f_2'^2)^2},$$
$$I_{10} = x^2\cos(2f_1)\sinh(2f_2) - 2\left(\frac{x\sin(2f_1)\sinh(2f_2)f_1' + x\cos(2f_1)\cosh(2f_2)f_2'}{f_1'^2 + f_2'^2}\right)$$
$$-\frac{\cos(2f_1)\sinh(2f_2)(f_1'^2 - f_2'^2) - 2\sin(2f_1)\cosh(2f_2)f_1'z'}{(f_1'^2 + f_2'^2)^2}. \quad (5.85)$$

Now consider the real symmetry algebra of the system (4.37)

$$\mathbf{X}_1 = \partial_{f_1},\ \mathbf{X}_2 = \partial_{f_2},\ \mathbf{X}_3 = x\partial_x, \quad (5.86)$$

which provides two gauge functions

$$B_1 = x^2, \quad B_2 = C_1, \quad (5.87)$$

for $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, relative to L_1. Similarly with L_2

$$B_1 = C_2, \quad B_2 = x^2, \quad (5.88)$$

and same symmetries. Thus a 2-dimensional Noether algebra is found to exist for the system (4.37) and the first integrals are as given below

$$I_1 = \frac{1}{f_1'^2 + f_2'^2} - \frac{2f_1'^2}{(f_1'^2 + f_2'^2)^2} - x^2, \quad I_2 = \frac{-2f_1'f_2'}{(f_1'^2 + f_2'^2)^2} - C_1, \quad (5.89)$$

and

$$I_1 = \frac{2f_1'f_2'}{(f_1'^2 + f_2'^2)^2} - C_2, \quad I_2 = \frac{-1}{f_1'^2 + f_2'^2} + \frac{2z'^2}{(f_1'^2 + f_2'^2)^2} - x^2. \quad (5.90)$$

relative to L_1 and L_2. Hence, CSA is far ahead from the real symmetry analysis as it yields 10 first integrals for the same system with the help of the Noether-like operators.

Case 3. The Lagrangians of a complex-linearizable system of two second order cubically semi-linear ODEs (4.44) emerge from the complex Lagrangian $L = u^2 + \frac{1}{u'}$ of the base linearizable equation (4.46). Following five gauge functions

$$B_1 = C, \quad B_2 = 2x + \frac{u^3}{3}, \quad B_3 = \frac{u^4}{2}, \quad B_4 = \frac{2}{3}u^6 - xu^3 - 3x^2, \quad B_5 = -3u^5, \tag{5.91}$$

are found to be related with following Lie symmetry generators

$$\begin{aligned} &\mathbf{Z}_1 = \partial_x,\ \mathbf{Z}_2 = u\partial_x,\ \mathbf{Z}_3 = u^2\partial_x - 2\partial_u, \\ &\mathbf{Z}_4 = (u^4 - 3xu)\partial_x - 3u^2\partial_u,\ \mathbf{Z}_5 = (-5u^3 + 6x)\partial_x + 12u\partial_u. \end{aligned} \tag{5.92}$$

The complex first integrals are therefore

$$\begin{aligned} &I_1 = u^2 + 2/u' - C, \quad I_2 = \tfrac{1}{3}\left((2u^3 - 6x) + 6u/u'\right), \quad I_3 = \tfrac{1}{2}\left(\tfrac{(2+u^2u')^2}{u'^2}\right), \\ &I_4 = \tfrac{1}{3}\left(\tfrac{(u^3-3x)u'+3u}{u'}\right)^2, \quad I_5 = -2u^5 + 6xu^2 - 2\tfrac{(5u^3-6x)}{u'} - 12\tfrac{u}{u'^2}. \end{aligned} \tag{5.93}$$

Lagrangians associated with system (4.44) are

$$L_1 = (f_1^2 - f_2^2) + \frac{f_1'}{f_1'^2 + f_2'^2}, \quad L_2 = 2f_1f_2 - \frac{f_2'}{f_1'^2 + f_2'^2}, \tag{5.94}$$

which yields 10 first integrals corresponding to the complex first integrals due to Noether-like operators emerging from $\mathbf{Z}_j$, for $j = 1, 2, 3, 4, 5$. For the comparison of above results obtained by CSA with those which real symmetry analysis provide, one needs to consider the Lagrangians (5.94) of the system (4.44) with the following 2-dimensional algebra

$$\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = 3x\partial_x + f_1\partial_{f_1} + f_2\partial_{f_2}. \tag{5.95}$$

Only one gauge function is found

$$B = C_1, \tag{5.96}$$

that exist for $\mathbf{X}_1$, relative to L_1. Similarly L_2 generates

$$B = C_2. \tag{5.97}$$

The first integral that arise due to a 1-dimensional Noether algebra relative to L_1 is

$$I = f_1^2 - f_2^2 + \frac{2f_1'}{f_1'^2 + f_2'^2} - C_1. \tag{5.98}$$

Similarly for the same Noether symmetry with L_2 the following first integral is found

$$I = 2f_1f_2 - \frac{2f_2'}{f_1'^2 + f_2'^2} - C_2. \tag{5.99}$$

Case 4. Another system of two second order cubically semi-linear ODEs with variable coefficients, i.e., the coefficients are functions of both the dependent and independent variables (4.49) is complex linearizable. The corresponding Lagrangians are

$$\begin{aligned} L_1 &= xf_1^2 - xf_2^2 + \tfrac{f_1'}{f_1'^2+f_2'^2}, \\ L_2 &= 2xf_1f_2 - \tfrac{f_2'}{f_1'^2+f_2'^2}. \end{aligned} \tag{5.100}$$

The system (4.49) admits one Lie point symmetry generator

$$\mathbf{X}_1 = x\partial_x, \tag{5.101}$$

and there does not exist any gauge function corresponding to it, which implies that there is 0-dimensional Noether algebra. As far as CSA is concerned there are 10 Noether-like operators as the two dimensional system is complex-linearizable. Therefore, there are ten first integrals corresponding to these operators.

All these cases demonstrate that CSA is a powerful tool to deal with the CR-structured two dimensional systems, indeed, for higher dimensional systems. The only requirement is the linearizability or integrability of the base systems or scalar equations. For instance, if one move from two to four dimensions for systems of second order ODEs where the base system is solvable by symmetry methods then solvability of the emerging four dimensional system is achievable due to CSA.

Chapter 6

Linearization of Systems of Four Second Order ODEs

In this chapter, complex-linearization has been extended to four dimensional systems of second order ODEs. The canonical forms established to obtain the symmetry structure of two dimensional systems are used in providing the corresponding canonical representatives of systems of four ODEs. These representatives arise by considering the dependent variables of the optimal [97] and reduced optimal [87] canonical forms, as complex functions of the real independent variables [88]. These are the '*special classes*' of linear four dimensional systems as the number of coefficients is *six* and *two*, respectively. These canonical representatives may give rise to a few subclasses from the entirety of the equivalence classes associated with the linearizable systems of second order ODEs of dimension four.

The second issue raised here concerns the linearization of systems of four second order ODEs. As in case of the complex-linearizable two dimensional systems the base complex linearizable maximally symmetric equations (1.70), were exploited to give the most general complex-linearizable form and complex-linearization conditions. Likewise, to obtain the complex-linearizable systems of four second order ODEs the linearizable class of systems of the form (1.129) has been considered which contains only maximally symmetric systems with a 15-dimensional Lie point symmetry algebra. The linearizability criteria associated with this class of systems of cubically semi-linear ODEs encompass fifteen constraint equations written in terms of the coefficients of the systems. Hence the class of emerging *linearizable* four dimensional systems is proved to be transformable to a system of the Newtonian free particle equations with maximally symmetry algebra identical to $sl(6,\mathbb{R})$. It is also shown that the most general form of such complex-linearizable systems of four ODEs is cubically semi-linear in the first derivatives of the dependent variables and coefficients satisfy a set of thirty complex-linearization conditions.

The conditions derived here do not solve the linearization problem for four dimensional systems of second order ODEs solely as these only ensure complex-linearizability of such systems. In fact, the complex linearizing transformations of the base complex two dimensional systems determine linearizability of the emerging four dimensional systems. In the last section an analogue of the transformations τ_1, for two dimensional systems has been used to obtain the real linearizing transformations to map the emerging four dimensional complex-linearizable systems of second order ODEs to their simplest possible forms.

6.1 Canonical Forms for Systems of Four Second Order ODEs

The general four dimensional system of linear ODEs is given by

$$f_j'' - \kappa_{jk} f_k' - \lambda_{jk} f_k - c_j = 0, \quad (j, k = 1, \ldots, 4) \tag{6.1}$$

where summation over repeated indices is assumed and the functions κ_{jk}, λ_{jk} and c_j are arbitrary functions of x. The above system has 36 coefficients. The algebraic classification of different symmetry algebras and number of linearizable classes of such systems is not known except for the maximally symmetric case $sl(6, \mathbb{R})$ of the free particle Newtonian system. Here the interest lies in extracting the canonical forms of four dimensional systems of second order ODEs by CSA.

There are two possible ways to extract canonical forms for systems of four linear second order ODEs. Firstly, using the equivalence of the complex linear scalar ODEs. In [87], it was shown that the optimal canonical form for a special class of systems of two second order ODEs with Lie algebras of dimensions 6 and 7 contains only one coefficient. This special class of two dimensional system corresponding to a linear scalar complex equation gives rise to a class of those four dimensional systems that involve only two arbitrary coefficients. This procedure is known as *nested complexification*, namely, to complexify the two dimensional system obtained from a scalar complex equation, i.e.,

$$\begin{aligned} f_1'' + \alpha_1(x) f_3 - \alpha_2(x) f_4 = 0, \\ f_2'' + \alpha_1(x) f_4 + \alpha_2(x) f_3 = 0, \\ f_3'' - \alpha_1(x) f_1 + \alpha_2(x) f_2 = 0, \\ f_4'' - \alpha_1(x) f_2 - \alpha_2(x) f_1 = 0. \end{aligned} \tag{6.2}$$

It corresponds to a linear two dimensional complex system (3.16) when $F_1 = f_1 + if_2$, $F_2 = f_3 + if_4$ and $\beta = \alpha_1 + i\alpha_2$. Furthermore, (3.16) can be mapped to a scalar complex equation via point transformations

$$\nu_1 = \rho u_1, \quad \nu_2 = \rho u_2, \quad x = \int^t \rho^{-2}(s) ds, \tag{6.3}$$

where $\rho'' = \alpha_3 \rho$ and $\beta = \rho^4 \alpha_4$. These are stated in the theorem below.

Theorem 6.1.1. *There exists an optimal canonical form of four dimensional systems of linear second order ODEs arising from the nested complexification that contains only two arbitrary coefficients.*

Secondly a canonical form of linearizable systems can also be derived by utilizing the equivalence of systems of two complex ODEs. For example the following system

$$\begin{aligned}
&f_1'' - \alpha_1(x)f_1 + \alpha_2(x)f_2 - \alpha_3(x)f_3 + \alpha_4(x)f_4 = 0,\\
&f_2'' - \alpha_2(x)f_1 - \alpha_1(x)f_2 - \alpha_4(x)f_3 - \alpha_3(x)f_4 = 0,\\
&f_3'' - \alpha_5(x)f_1 + \alpha_6(x)f_2 + \alpha_1(x)f_3 - \alpha_2(x)f_4 = 0,\\
&f_4'' - \alpha_6(x)f_1 - \alpha_5(x)f_2 + \alpha_2(x)f_3 + \alpha_1(x)f_4 = 0,
\end{aligned} \tag{6.4}$$

with six arbitrary coefficients can be reduced to (1.104), using $\tilde{y} = f_1 + if_2,\ \tilde{z} = f_3 + if_4$ and when all three parameters in (1.104) are considered complex functions of the real independent variable. The set of equations (6.4) represents another canonical form which contains six arbitrary coefficients. These are the two linear forms for systems of four second order ODEs with fewer arbitrary coefficients obtainable in the manner described. Notice that these forms can not be obtained using real symmetry analysis because the canonical representative of a four dimensional system requires at least $n^2 = 16$, arbitrary coefficients. These results are stated in the form of the following theorem.

Theorem 6.1.2. *Any linearizable system of four second order non-linear ODEs obtainable from a scalar complex ODE or a system of two second order complex ODEs is transformable to one of the forms (6.2) or (6.4), respectively.*

6.2 Complex-Linearization Criteria for Systems of Four Second Order ODEs

The derivation of the complex-linearization criteria for a class of systems of four second order ODEs require a base complex linearizable system. To this end, the linearizable class (1.129) determines the explicit complex-linearization criteria for those four dimensional systems that may be maximally symmetric, i.e., has $sl(6, \Re)$ Lie point symmetry algebra. This provides a set of thirty equations to be satisfied by the coefficients that arise in this class of systems. In this regard the following theorem is stated without proof but can be proved easily following the same procedure adopted for two dimensional systems in [87].

Theorem 6.2.1. *A system of four second order cubically semi-linear ODEs*

$$\begin{aligned}
&f_1'' + (\alpha_{11}f_1' - 3\alpha_{12}f_2' + 2\alpha_{21}f_3' - 2\alpha_{22}f_4' + \beta_{11}^1)f_1'^2 - (3\alpha_{11}f_1' - \alpha_{12}f_2' + 2\alpha_{21}f_3'\\
&-2\alpha_{22}f_4' - \beta_{11}^1)f_2'^2 + (\alpha_{31}f_1' - \alpha_{32}f_2' + \beta_{31}^1)f_3'^2 + (\alpha_{31}f_1' + \alpha_{32}f_2' - \beta_{31}^1)f_4'^2\\
&+2(2\alpha_{22}f_1'f_2'f_3' + 2\alpha_{21}f_1'f_2'f_4' + \alpha_{32}f_1'f_3'f_4' - \alpha_{31}f_2'f_3'f_4') - 2(\beta_{12}^1f_1'f_2' - \beta_{21}^1f_1'f_3'\\
&+\beta_{22}^1f_1'f_4' + \beta_{22}^1f_2'f_3' + \beta_{21}^1f_2'f_4' + \beta_{32}^1f_3'f_4') + \gamma_{11}^1f_1' - \gamma_{12}^1f_2' + \gamma_{21}^1f_3' - \gamma_{22}^1f_4' + \delta_{11} = 0,
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
&f_2'' + (\alpha_{12}f_1' + 3\alpha_{11}f_2' + 2\alpha_{22}f_3' + 2\alpha_{21}f_4' + \beta_{12}^1)f_1'^2 - (3\alpha_{12}f_1' + \alpha_{11}f_2' + 2\alpha_{22}f_3'\\
&+2\alpha_{21}f_4' - \beta_{12}^1)f_2'^2 + (\alpha_{32}f_1' + \alpha_{31}f_2' + \beta_{32}^1)f_3'^2 - (\alpha_{32}f_1' + \alpha_{31}f_2' - \beta_{32}^1)f_4'^2\\
&+2(2\alpha_{21}f_1'f_2'f_3' + 2\alpha_{22}f_1'f_2'f_4' - \alpha_{31}f_1'f_3'f_4' - \alpha_{32}f_2'f_3'f_4') + 2(\beta_{11}^1f_1'f_2' + \beta_{22}^1f_1'f_3'\\
&+\beta_{21}^1f_1'f_4' + \beta_{21}^1f_2'f_3' - \beta_{22}^1f_2'f_4' + \beta_{31}^1f_3'f_4') + \gamma_{12}^1f_1' + \gamma_{11}^1f_2' + \gamma_{22}^1f_3' + \gamma_{21}^1f_4' + \delta_{12} = 0,
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
&f_3'' + (\alpha_{11}f_3' - \alpha_{12}f_4' + \beta_{11}^2)f_1'^2 - (\alpha_{11}f_3' - \alpha_{12}f_4' - \beta_{11}^2)f_2'^2 + (2\alpha_{21}f_1' - 2\alpha_{22}f_2'\\
&+\alpha_{31}f_3' - 3\alpha_{32}f_4' + \beta_{31}^2)f_3'^2 - (2\alpha_{21}f_1' - 2\alpha_{22}f_2' + 3\alpha_{31}f_3' - \alpha_{32}f_4' - \beta_{31}^2)f_4'^2\\
&-2(\alpha_{12}f_1'f_2'f_3' + \alpha_{11}f_1'f_2'f_4' + 2\alpha_{22}f_1'f_3'f_4' + 2\alpha_{21}f_2'f_3'f_4') - 2(\beta_{12}^2f_1'f_2' - \beta_{21}^2f_1'f_3'\\
&+\beta_{22}^2f_1'f_4' + \beta_{22}^2f_2'f_3' + \beta_{21}^2f_2'f_4' + \beta_{32}^2f_3'f_4') + \gamma_{11}^2f_1' - \gamma_{12}^2f_2' + \gamma_{21}^2f_3' - \gamma_{22}^2f_4' + \delta_{21} = 0,
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
&f_4'' + (\alpha_{12}f_3' + \alpha_{11}f_4' + \beta_{21}^2)f_1'^2 - (\alpha_{12}f_3' + \alpha_{11}f_4' - \beta_{12}^2)f_2'^2 + (2\alpha_{22}f_1' + 2\alpha_{21}f_2'\\
&+\alpha_{32}f_3' + 3\alpha_{31}f_4' + \beta_{32}^2)f_3'^2 - (2\alpha_{22}f_1' + 2\alpha_{21}f_2' + 3\alpha_{32}f_3' + \alpha_{31}f_4' - \beta_{32}^2)f_4'^2\\
&+2(\alpha_{11}f_1'f_2'f_3' - \alpha_{12}f_1'f_2'f_4' + 2\alpha_{21}f_1'f_3'f_4' - 2\alpha_{22}f_2'f_3'f_4') + 2(\beta_{11}^2f_1'f_2' - \beta_{22}^2f_1'f_3'\\
&+\beta_{21}^2f_1'f_4' + \beta_{21}^2f_2'f_3' - \beta_{22}^2f_2'f_4' + \beta_{31}^2f_3'f_4') + \gamma_{12}^2f_1' + \gamma_{11}^2f_2' + \gamma_{22}^2f_3' + \gamma_{21}^2f_4' + \delta_{22} = 0,
\end{aligned} \tag{6.8}$$

where

$$\alpha_{jk} \neq \alpha_{kj},\ \beta_{jk}^l \neq \beta_{kj}^l,\ \gamma_{jk}^l \neq \gamma_{kj}^l,\ \delta_{jk} \neq \delta_{kj}, \tag{6.9}$$

is complex-linearizable if and only if the coefficients satisfy 30 *constraint equations given in the Appendix A-2.*

Once the linearizability criteria are satisfied by a system the next step involves the construction of the invertible linearizing transformations which can often be difficult. In the subsequent section it is illustrated that how a complex linearizing transformation of the form

$$X = X(x), \quad U = U(x, u, v), \quad V = V(x, u, v). \tag{6.10}$$

plays the magic of generating invertible transformations to linearize the corresponding non-linear systems of four ODEs. Notice that the independent variable X depends *only* on x. The above transformation is a special case of the general invertible complex point transformations

$$X = X(x, u, v), \quad U = U(x, u, v), \quad V = V(x, u, v), \tag{6.11}$$

for two dimensional systems. The dependence of X on the complex variables (x, u, v) makes it a complex independent variable. This may seem strange as the decomposition of linearized system would yield a system of PDEs instead of ODEs. This is true. However, the fact of the matter is that the resultant system is equipped with the CR-equations which makes the system integrable and the solution is obtained in the original variables upon using invertible transformations. This line of approach is followed in [87]- [89] for two dimensional non-linear systems corresponding to non-linear complex scalar equations and it was found that there exists two classes of such systems, linearizable and non-linearizable systems. It was shown that there exists non-linearizable systems with Lie algebras of dimensions $4, 3, 2, 1$ which can be complex-linearized. In this chapter, the investigation is restricted to transformations of the form (6.10) that yield a set of real invertible transformations for systems of four ODEs. Furthermore the solution of a non-linear system can be obtained from the linearized system by utilizing such real transformations. The complete characterization of all four dimensional non-linear systems and their Lie algebras subject to the complex transformations (6.11) is a separate issue which has not been dealt with here.

6.3 Computer Algorithm

The complex-linearizability conditions include thirty PDEs of the first order in the coefficients of a non-linear system of four ODEs, relative to the dependent and independent variables. In order to avoid cumbersome calculations a Computer Algebra System is used, e.g., MAPLE or CRACK to check the complex-linearizability of a system. For this purpose one needs to translate the thirty constraint equations in the form of a computer code in MAPLE to test complex-linearizability for a non-linear system of the form (6.5)−(6.8) whose coefficients are used as an input. The code also determines the symmetry algebra of a system.

6.4 Applications

A variety of examples is considered to verify the theory developed for the linearization of systems of four second order ODEs. It is important to identify that non-linearity depends on the coupling of both derivative and non-derivative terms in such systems for which there could be many choices. Furthermore there can be $3! = 6$ types of coupling, e.g., the first equation can be coupled with the other three in the system. The first example illustrates how the complex variable approach reveal the complete integrability of a system that has the maximal algebra $sl(6, \Re)$. The example 3 is of great interest for two main reasons. Firstly it is a geodesic type system of ODEs in which the non-linearity arises due to the quadratic terms in the first order derivatives. Such equations appear frequently in Relativity in the study of shortest paths on curved manifold. Secondly this system can be extended to higher dimensions to yield the linearization of a system of $4n$ ODEs which has the Lie algebra $sl(2(2n + 1), \Re)$, $n \geq 1$.

1. Consider a quadratically semi-linear system of four second order ODEs

$$
\begin{aligned}
&f_1'' - f_1' + f_1'^2 - f_2'^2 = 0, \\
&f_2'' - f_2' + 2f_1'f_2' = 0, \\
&f_3'' - f_3' + f_3'^2 - f_4'^2 = 0, \\
&f_4'' - f_4' + 2f_3'f_4' = 0,
\end{aligned}
\tag{6.12}
$$

where the first and last two equations are coupled in the first derivatives of the dependent functions f_1, f_2 and f_3, f_4, respectively. Substituting all the coefficients $\beta^1_{11} = \beta^2_{31} = 1$ and $\gamma^1_{11} = \gamma^2_{21} = -1$ in the linearizability conditions then it can be verified that above system is complex-linearizable. In order to find the linearizing transformations to map system (6.12) into a linear target system $F''_j = 0,\ j = 1, \ldots, 4$, the following transformations

$$\begin{aligned} &X = \exp(x), \\ &F_1 = \exp(f_1)\cos(f_2), \quad F_2 = \exp(f_1)\sin(f_2), \\ &F_3 = \exp(f_3)\cos(f_4), \quad F_4 = \exp(f_3)\sin(f_4). \end{aligned} \tag{6.13}$$

are obtained by complex transformations of the form (6.10). These are the complex transformations which reveal linearizability of 4-dimensional complex-linearizable systems of second order ODEs.

2. An example of a quadratically semi-linear system with the same coupling type as above but the coefficients involve the dependent and independent variables is

$$\begin{aligned} &f''_1 - \tfrac{f_1(f'^2_1 - f'^2_2) + 2f_2 f'_1 f'_2}{f_1^2 + f_2^2} - x f_1 = 0, \\ &f''_2 - \tfrac{2f_1 f'_1 f'_2 - f_2(f'^2_1 - f'^2_2)}{f_1^2 + f_2^2} - x f_2 = 0, \\ &x f''_3 + x f'^2_3 - x f'^2_4 + 2 f'_3 = 0, \\ &x f''_4 + 2x f'_3 f'_4 + 2 f'_4 = 0. \end{aligned} \tag{6.14}$$

Complex-linearizability of this system is ensured by Theorem (6.2.1), i.e., the coefficients

$$\begin{aligned} &\beta^1_{11} = \tfrac{-f_1}{f_1^2 + f_2^2}, \quad \beta^1_{12} = \tfrac{f_2}{f_1^2 + f_2^2}, \quad \beta^2_{31} = 1, \\ &\gamma^2_{21} = \tfrac{2}{x}, \quad \delta_{11} = -x f_1, \quad \delta_{12} = -x f_2, \end{aligned} \tag{6.15}$$

satisfy thirty constraint equations given for linearizability. The linear target system in this case is

$$F''_1 = X^{-4}, \quad F''_2 = F''_3 = F''_4 = 0, \tag{6.16}$$

which upon utilizing the transformations

$$\begin{aligned} &X = \tfrac{1}{x}, \\ &F_1 = \tfrac{1}{2x}\ln(f_1^2 + f_2^2), \quad F_2 = \tfrac{1}{x}\arctan\left(\tfrac{f_2}{f_1}\right), \\ &F_3 = \exp(f_3)\cos(f_4), \quad F_4 = \exp(f_3)\sin(f_4), \end{aligned} \tag{6.17}$$

arises. Notice that the above linearizing transformations come from invertible complex transformations

$$X = \frac{1}{x}, \quad U = \frac{1}{x}\log u, \quad V = \exp v. \tag{6.18}$$

CRACK is used to verify that the number of symmetries of the system (6.14) is 35 whereas in all other examples MAPLE-code is considered to verify the number of symmetries.

3. Now consider a coupled system of four real ODEs of geodesic type given by

$$
\begin{aligned}
&f_1'' + f_1'^2 - f_2'^2 = 0,\\
&f_2'' + 2f_1'f_2' = 0,\\
&f_3'' + f_3'^2 - f_4'^2 + 2(f_1'f_3' - f_2'f_4') = 0,\\
&f_4'' + 2(f_3'f_4' + f_1'f_4' + f_3'f_2') = 0,
\end{aligned}
\tag{6.19}
$$

where the first two equations are coupled in the first derivatives of f_1 and f_2 while the other two equations contain the coupling relative to the first derivatives of all the dependent functions. This can be easily checked that the above system is complex-linearizable by subjecting the coefficients $\beta^1_{11} = \beta^2_{21} = \beta^2_{31} = 1$ into the corresponding conditions and it can be mapped to $F_j'' = 0$, by the real linearizing transformations

$$
\begin{aligned}
&F_1 = \exp(f_1)\cos(f_2), \quad F_2 = \exp(f_1)\sin(f_2),\\
&F_3 = \exp(f_1 + f_3)\cos(f_2 + f_4), \quad F_4 = \exp(f_1 + f_3)\sin(f_2 + f_4).
\end{aligned}
\tag{6.20}
$$

The generalization of system (6.19) in a $(4n + 1)$-dimensional space can trivially be done as the first two equations are coupled with each other while the other two equations contains full coupling. Take a system of $2n$ such equations of the form

$$
\begin{aligned}
&f_j'' + f_j'^2 - g_j'^2 = 0,\\
&g_j'' + 2f_j'g_j' = 0,
\end{aligned}
\tag{6.21}
$$

where $j = 1, ..., n$, and another system of $2n$ ODEs that contains full coupling, i.e.,

$$
\begin{aligned}
&h_j'' + h_j'^2 - k_j'^2 + 2(f_j'h_j' - g_j'k_j') = 0,\\
&k_j'' + 2(h_j'k_j' + f_j'k_j' + h_j'g_j') = 0,
\end{aligned}
\tag{6.22}
$$

where $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{k}$, are n, dimensional vectors with $4n$, dependent functions of x. The above system (6.21)-(6.22) can also be regarded as a surface in a $(12n+1)$-dimensional space $(x, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{f}', \mathbf{g}', \mathbf{h}'$, $\mathbf{k}', \mathbf{f}'', \mathbf{g}'', \mathbf{h}'', \mathbf{k}'')$. Introduce a complex structure on the $12n$-dimensional space with the identification

$$
u_j = u_j(x) := f_j(x) + ig_j(x), \quad v_j = v_j(x) := h_j(x) + ik_j(x),
\tag{6.23}
$$

which map the systems (6.21)-(6.22) to systems of $2n$ ODEs in a $2n$-complex dimensional space

$$
u_j'' + u_j'^2 = 0,
\tag{6.24}
$$

and

$$
v_j'' + v_j'^2 + 2u_j'v_j' = 0, \qquad j = 1, 2, \ldots, n,
\tag{6.25}
$$

respectively. Both these systems can be linearized to the complexified free particle system

$$
U_j'' = 0, \quad V_j'' = 0,
\tag{6.26}
$$

using the complex transformations

$$U_j = \exp u_j, \quad V_j = \exp(u_j + v_j). \tag{6.27}$$

Consequently the system (6.21)-(6.22) is linearized to the maximally symmetric system, $sl(2(2n+1),\Re)$, i.e.,

$$\begin{aligned} F_j^{''} = 0, \quad G_j^{''} = 0, \\ H_j^{''} = 0, \quad K_j^{''} = 0, \end{aligned} \tag{6.28}$$

using the real transformations obtained from (6.27). The next example is also of geodesic type with all equations coupled together.

4. This example presents a case where the non-linear system is a fully coupled four dimensional system

$$\begin{aligned} &f_1^{''} - f_1^{'2} + f_2^{'2} - f_3^{'2} + f_4^{'2} = 0, \\ &f_2^{''} - 2(f_1^{'}f_2^{'} + f_3^{'}f_4^{'}) = 0, \\ &f_3^{''} - 2(f_1^{'}f_3^{'} - f_2^{'}f_4^{'}) = 0, \\ &f_4^{''} - 2(f_1^{'}f_4^{'} + f_2^{'}f_3^{'}) = 0. \end{aligned} \tag{6.29}$$

This can be obtained from a system of two complex ODEs, i.e., from the complexified Newtonian system

$$u^{''} - u^{'2} - v^{'2} = 0, \quad v^{''} - 2u^{'}v^{'} = 0, \tag{6.30}$$

which has four dimensional Lie algebra with commutation relations

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta] = 0, \quad [\mathbf{X}_\alpha, \mathbf{X}_4] = \mathbf{X}_\alpha, \quad \alpha, \beta = 1, 2, 3 \tag{6.31}$$

identified as $L_{4,2}$ of symmetries. This is reducible to the free particle system as the coefficients $\beta_{11}^1 = \beta_{31}^1 = \beta_{21}^2 = -1$ satisfy the complex-linearizability conditions. The following transformations of the dependent variables

$$\begin{aligned} F_1 &= c_1 \exp(f_3 - f_1)\cos(f_4 - f_2) + c_2 \exp(-f_3 - f_1)\cos(f_4 + f_2), \\ F_2 &= c_1 \exp(f_3 - f_1)\sin(f_4 - f_2) + c_2 \exp(-f_3 - f_1)\sin(f_4 + f_2), \\ F_3 &= c_2 \exp(-f_3 - f_1)\cos(f_4 + f_2) - c_1 \exp(f_3 - f_1)\cos(f_4 - f_2), \\ F_4 &= -c_2 \exp(-f_3 - f_1)\sin(f_4 + f_2) - c_1 \exp(f_3 - f_1)\sin(f_4 - f_2), \end{aligned} \tag{6.32}$$

maps the non-linear system to $F_j^{''} = 0, \ j = 1, \ldots, 4.$

5. Finally, consider a maximally symmetric system with variable coefficients

$$\begin{aligned}
&xf_1'' + f_1' + xf_1'^2 - xf_2'^2 = 0,\\
&xf_2'' + f_2' + 2xf_1'f_2' = 0,\\
&xf_3'' + f_3' + xf_3'^2 - xf_4'^2 + 2xf_1'f_3' - 2xf_2'f_4' = 0,\\
&xf_4'' + f_4' + 2xf_3'f_4' + 2xf_1'f_4' + 2xf_2'f_3' = 0.
\end{aligned} \tag{6.33}$$

This system is linearizable and can be mapped to a system of the free particle equations via the following change of the dependent and independent variables

$$\begin{aligned}
&X = \ln x,\\
&F_1 = \exp(f_1)\cos(f_2), \quad F_2 = \exp(f_1)\sin(f_2),\\
&F_3 = \exp(f_1 + f_3)\cos(f_2 + f_4), \quad F_4 = \exp(f_1 + f_3)\sin(f_2 + f_4).
\end{aligned} \tag{6.34}$$

These are obtainable by the complex linearizing transformations of the corresponding complex-linearizable system. Linearizability is also ensured by the coefficients $\beta_{11}^1 = \beta_{21}^2 = \beta_{31}^2 = 1$ and $\gamma_{11}^1 = \gamma_{21}^2 = x^{-1}$ of the non-linear system, i.e., as they satisfy the complex-linearization conditions and transformations are of the form (6.10).

Notice that all the systems which can be linearized by complex methods require a subclass of arbitrary point transformations where the transformed independent variable is only a function of the old independent real variable. Therefore, the complex-linearization procedure can be extended to systems of $2n$ second order ODEs and a subclass of the complex transformations of the base systems of n ODEs singles out the linearizable systems from the general complex-linearizables.

Chapter 7

Ibragimov-Type Invariants for a System of Two Linear Parabolic Equations

In this chapter, invariance of systems of two second order linear PDEs is studied by employing CSA on the scalar linear parabolic PDE

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u, \tag{7.1}$$

with two independent variables, where $a \neq 0$, b and c are arbitrary functions of t and x. Equation (7.1) arises in many important applications, e.g., the Fokker-Planck PDE [84] belongs to this class which models many phenomena [7,8]. The Black-Sholes [9] as well as the bond-pricing equations [99] are also in the family of equations (7.1). Lie's theory of groups is used to analyze the Black-Scholes [27] and bond pricing equations [28]. Algebraic properties of one and two dimensional Fokker-Planck equations and their exact solutions have been investigated in [92] and [93], respectively. Moreover, the reduction of one-factor bond pricing parabolic equations to the classical linear heat equation is achieved by symmetry method [82]. Furthermore, fundamental solutions have been derived for two zero-coupon bond-pricing PDEs. Lie [53] provided a complete group classification of (7.1), he extracted four canonical forms of this equation which admit 1, 2, 4 and 6 dimensional non-trivial point symmetry algebras (apart from the infinite dimensional algebra of trivial symmetries). The family of parabolic equations (7.1) with $a = 1$ and $b = 0$ reduces to fourth Lie canonical form which was utilized [79] to study this class further. A transformation procedure is also developed [10] which relies on symmetries of such parabolic equation.

Equivalence transformations that map equation (7.1) into itself have been employed to deduce the associated semi-invariants and invariants. Laplace type semi-invariants of this equation are derived [37] under linear transformation of the dependent variable. These semi-invariants are referred here as Ibragimov invariants. Separately, transformations of the independent variables are also utilized [43] to obtain associated semi-invariants. In the reference cited, joint invariant equation for (7.1) is also found by considering transformations of both the dependent and independent variables.

These joint invariants provided necessary and sufficient conditions to reduce it to the classical heat equation. The criteria for such a reduction are given in terms of the coefficients of the equation and their partial derivatives. Furthermore, the work on joint invariant equations for the family of equations (7.1) and reduction to the heat equation is also addressed [63] along with refinement of the invariant conditions. The reducibility of (7.1) to second Lie canonical form is achieved and a complete invariant characterization is provided for all the four Lie canonical forms in [61].

Semi-invariants for a class of systems of two parabolic type PDEs are derived [67] in this chapter. It is shown that the invariance of the base complex parabolic equation reveals the invariance of the emerging systems of parabolic type PDEs. The semi-invariants of systems are deduced by making use of their correspondence with the scalar parabolic PDE and it is shown that these Ibragimov-type semi-invariants also correspond to those of the base scalar parabolic PDE, i.e., Laplace-type semi-invariants. In obtaining these semi-invariants a subclass of the equivalence transformations of the dependent variables for systems is considered. These systems, their associated equivalence transformations and corresponding Ibragimov-type semi-invariants are found to have CR-structure with respect to both the dependent variables. Further, such systems of two parabolic type PDEs are shown to be reducible to their simple solvable forms via complex linear equivalence transformations of the complex dependent variable associated with the base parabolic PDE.

The invariance correspondence of the complex base and real systems of DEs presented here has not been reported so far in this thesis. On splitting the invariant quantities associated with the complex scalar parabolic equation (7.1), into the real and imaginary parts constitute invariants of the corresponding systems, which ensure reduction of systems of PDEs to simple linear forms if the base complex equations are also reducible to simplest ones. Therefore, this chapter on invariance of systems of PDEs is included to develop a better understanding of the complex symmetry method for DEs. The results presented here for systems of two parabolic type PDEs reveal invariant characterization of the CR-structured systems of DEs.

7.1 Parabolic Type Systems and Associated Equivalence Transformations

Linear transformation of (only) the dependent variable of the parabolic PDE (7.1) is of the form

$$u = \sigma(t,x)\bar{u}, \quad \sigma(t,x) \neq 0, \tag{7.2}$$

that maps the family (7.1) into itself. That is under (7.2) the parabolic equation (7.1) remains linear and homogeneous but the transformed PDE

$$\bar{u}_t = \bar{a}\bar{u}_{xx} + \bar{b}\bar{u}_x + \bar{c}\bar{u}, \tag{7.3}$$

has new coefficients $\bar{a}$, $\bar{b}$ and $\bar{c}$ which are

$$\bar{a} = a, \quad \bar{b} = b + 2\frac{\sigma_x}{\sigma}a, \quad \bar{c} = \frac{L(\sigma)}{\sigma}, \tag{7.4}$$

in which the operator L is defined as

$$L = a\frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x} - \frac{\partial}{\partial t} + c. \tag{7.5}$$

Now focus on the linear parabolic type system of two PDEs in two independent variables t and x

$$\begin{aligned} v_t &= \alpha_1 v_{xx} + \alpha_2 w_{xx} + \beta_1 v_x + \beta_2 w_x + \gamma_1 v + \gamma_2 w, \\ w_t &= \alpha_3 v_{xx} + \alpha_4 w_{xx} + \beta_3 v_x + \beta_4 w_x + \gamma_3 v + \gamma_4 w, \end{aligned} \tag{7.6}$$

where the coefficients α_j, β_j and γ_j for $j = 1, .., 4$ are functions of t and x. The equivalence transformations of the dependent variables, as can be verified, of this family (7.6) of linear parabolic type equations are as follows

$$\begin{aligned} v &= \tau_1(t,x)\bar{v} + \tau_2(t,x)\bar{w}, \\ w &= \tau_3(t,x)\bar{v} + \tau_4(t,x)\bar{w}. \end{aligned} \tag{7.7}$$

The coefficients of the transformed system of PDEs under the linear change (7.7) of the dependent variables of system (7.6), can be written in terms of τ_i's and coefficients of (7.6).

Here a special class of systems (7.6) which corresponds to a complex scalar linear parabolic equation of the form (7.1) is considered. Such complex base equation has complex dependent variable of the real independent variables, i.e., $u(t,x) = v(t,x) + iw(t,x)$. Moreover, all the coefficients of complex base PDE (7.1) are also considered complex functions of the independent variables t and x, to extract a *coupled* system of two linear parabolic type PDEs

$$\begin{aligned} v_t &= a_1 v_{xx} - a_2 w_{xx} + b_1 v_x - b_2 w_x + c_1 v - c_2 w, \\ w_t &= a_1 w_{xx} + a_2 v_{xx} + b_1 w_x + b_2 v_x + c_1 w + c_2 v. \end{aligned} \tag{7.8}$$

The subset of equivalence transformations of the dependent variables (7.7) of the form

$$\begin{aligned} v &= \sigma_1(t,x)\bar{v} - \sigma_2(t,x)\bar{w}, \\ w &= \sigma_2(t,x)\bar{v} + \sigma_1(t,x)\bar{w}, \end{aligned} \tag{7.9}$$

are arrived at by the complex transformations (7.2) if $\sigma(t,x) = \sigma_1(t,x) + i\sigma_2(t,x)$. This subset of the equivalence transformations (7.9) maps the system of linear PDEs (7.8) to

$$\begin{aligned} \bar{v}_t &= \bar{a}_1 \bar{v}_{xx} - \bar{a}_2 \bar{w}_{xx} + \bar{b}_1 \bar{v}_x - \bar{b}_2 \bar{w}_x + \bar{c}_1 \bar{v} - \bar{c}_2 \bar{w}, \\ \bar{w}_t &= \bar{a}_1 \bar{w}_{xx} + \bar{a}_2 \bar{v}_{xx} + \bar{b}_1 \bar{w}_x + \bar{b}_2 \bar{v}_x + \bar{c}_1 \bar{w} + \bar{c}_2 \bar{v}, \end{aligned} \tag{7.10}$$

where

$$
\begin{aligned}
&\bar{a}_1 = a_1, \quad \bar{a}_2 = a_2,\\
&\bar{b}_1 = b_1 + \tfrac{2[\sigma_1(a_1\sigma_{1x}-a_2\sigma_{2x})+\sigma_2(a_1\sigma_{2x}+a_2\sigma_{1x})]}{\sigma_1^2+\sigma_2^2},\\
&\bar{b}_2 = b_2 + \tfrac{2[-\sigma_2(a_1\sigma_{1x}-a_2\sigma_{2x})+\sigma_1(a_1\sigma_{2x}+a_2\sigma_{1x})]}{\sigma_1^2+\sigma_2^2},\\
&\bar{c}_1 = \tfrac{\sigma_1(L_1\sigma_1-L_2\sigma_2)+\sigma_2(L_1\sigma_2+L_2\sigma_1)}{\sigma_1^2+\sigma_2^2},\\
&\bar{c}_2 = \tfrac{\sigma_1(L_1\sigma_2+L_2\sigma_1)-\sigma_2(L_1\sigma_1-L_2\sigma_2)}{\sigma_1^2+\sigma_2^2},
\end{aligned}
\tag{7.11}
$$

and the operators L_1 and L_2 are the real and imaginary parts of the complex operator (7.5), i.e.,

$$
\begin{aligned}
L_1 &= a_1\tfrac{\partial^2}{\partial x^2} + b_1\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial t} + c_1,\\
L_2 &= a_2\tfrac{\partial^2}{\partial x^2} + b_2\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial t} + c_2.
\end{aligned}
\tag{7.12}
$$

Remark 7.1.1. The linear parabolic type system (7.8) is a special class of the family of the linear parabolic systems (7.6). Equations (7.8) have six arbitrary coefficients whereas (7.6) has twelve.

Now consider the case for which system (7.8) reduces to an uncoupled

$$
\begin{aligned}
v_t &= a_1 v_{xx} + b_1 v_x + c_1 v,\\
w_t &= a_1 w_{xx} + b_1 w_x + c_1 w,
\end{aligned}
\tag{7.13}
$$

and a coupled system of two PDEs

$$
\begin{aligned}
v_t &= -a_2 w_{xx} - b_2 w_x - c_2 w,\\
w_t &= a_2 v_{xx} + b_2 v_x + c_2 v.
\end{aligned}
\tag{7.14}
$$

These systems are extractable from (7.8) by considering $a_2 = b_2 = c_2 = 0$ and $a_1 = b_1 = c_1 = 0$, respectively or alternatively assuming the coefficients a, b and c, real or pure imaginary.

These systems (7.13) and (7.14) will be investigated later. The essence of this section is that the system (7.8) has equivalence transformations, that is, a linear change of the dependent variables (7.9) which maps it into the same family with (in general) new coefficients (7.11).

7.2 Semi-Invariants under Transformation of the Dependent Variables

In this section Ibragimov-type invariants of the linear parabolic type system (7.8) under the dependent variables transformations (7.9) are derived by reverting to the base scalar complex linear parabolic equation (7.1) with complex coefficients.

Consider the following two non-trivial semi-invariants a and K [37]

$$
K = \frac{1}{2}b^2 a_x + (a_t + a a_{xx} - a_x^2)b + (a a_x - ab)b_x - a b_t - a^2 b_{xx} + 2a^2 c_x,
\tag{7.15}
$$

referred to as a Laplace-type or Ibragimov invariants in [37]. The Ibragimov semi-invariant K can be written compactly [61] as

$$k = \frac{\partial}{\partial x}\left(c - a\left(\frac{b}{2a}\right)_x - \frac{b^2}{4a}\right) + \frac{\partial}{\partial t}\left(\frac{b}{2a}\right), \tag{7.16}$$

where $K = -2a^2 k$.

The following results are based on the semi-invariants a and k. The scalar linear parabolic PDE (7.1) is equivalent to the parabolic equation (7.3), i.e., $\bar{L}\bar{u} = 0$, via the linear change (7.2) if and only if

$$\bar{a} = a, \quad \bar{k} = k, \tag{7.17}$$

where $\bar{k}$ is

$$\bar{k} = \frac{\partial}{\partial x}\left(\bar{c} - \bar{a}\left(\frac{\bar{b}}{2\bar{a}}\right)_x - \frac{\bar{b}^2}{4\bar{a}}\right) + \frac{\partial}{\partial t}\left(\frac{\bar{b}}{2\bar{a}}\right). \tag{7.18}$$

The construction of σ in the transformation (7.2) is achieved by solving the system [61]

$$\frac{\sigma_x}{\sigma} = \frac{\bar{b}}{2\bar{a}} - \frac{b}{2a}, \quad \frac{\sigma_t}{\sigma} = c - a\left(\frac{b}{2a}\right)_x - \frac{b^2}{4a} - \bar{c} + \bar{a}\left(\frac{\bar{b}}{2\bar{a}}\right)_x + \frac{\bar{b}^2}{4\bar{a}}. \tag{7.19}$$

As an example to be used later, let (7.1) has constant coefficients, then it can be reduced to the classical heat equation

$$\bar{u}_t = \bar{a}\bar{u}_{xx}, \quad \bar{a} = constant, \tag{7.20}$$

under a change of the dependent variable

$$u = \exp\left(ct - \frac{b^2}{4a}t - \frac{b}{2a}x\right)\bar{u}, \tag{7.21}$$

since $\bar{a} = a$ and $\bar{k} = k$. This transformation is a consequence of the following relations

$$\frac{\sigma_x}{\sigma} = -\frac{b}{2a}, \quad \frac{\sigma_t}{\sigma} = c - \frac{b^2}{4a}, \tag{7.22}$$

which appear due to (7.19).

Ibragimov-type invariants of the linear parabolic type system (7.8) under linear changes of the dependent variables (7.9) can be derived by the complex split of a, k, $\bar{a}$ and $\bar{k}$, i.e.,

$$\bar{a}_1 = a_1, \quad \bar{a}_2 = a_2, \quad \bar{k}_1 = k_1, \quad \bar{k}_2 = k_2, \tag{7.23}$$

where

$$\begin{aligned} k_1 = &\frac{\partial}{\partial x}\left(c_1 - \frac{1}{2}\left[a_1\left(\frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}\right)_x - a_2\left(\frac{a_1b_2 - a_2b_1}{a_1^2 + a_2^2}\right)_x\right] - \frac{1}{4}\left[\frac{a_1b_1^2 - a_1b_2^2 + 2a_2b_1b_2}{a_1^2 + a_2^2}\right]\right) \\ &+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}\right), \\ k_2 = &\frac{\partial}{\partial x}\left(c_2 - \frac{1}{2}\left[a_1\left(\frac{a_1b_2 - a_2b_1}{a_1^2 + a_2^2}\right)_x + a_2\left(\frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}\right)_x\right] - \frac{1}{4}\left[\frac{a_2b_2^2 - a_2b_1^2 + 2a_1b_1b_2}{a_1^2 + a_2^2}\right]\right) \\ &+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{a_1b_2 - a_2b_1}{a_1^2 + a_2^2}\right). \end{aligned} \tag{7.24}$$

Note that the transformed k's are with the bars over coefficients a_j, b_j and c_j where $j = 1, 2$. All these observations lead to the following result.

Proposition 7.2.1. *The linear parabolic system*

$$\begin{aligned} L_1 v - L_2 w &\equiv -v_t + a_1 v_{xx} - a_2 w_{xx} + b_1 v_x - b_2 w_x + c_1 v - c_2 w = 0, \\ L_1 w + L_2 v &\equiv -w_t + a_1 w_{xx} + a_2 v_{xx} + b_1 w_x + b_2 v_x + c_1 w + c_2 v = 0, \end{aligned} \tag{7.25}$$

is equivalent to

$$\begin{aligned} \bar{L}_1 \bar{v} - \bar{L}_2 \bar{w} &\equiv -\bar{v}_t + \bar{a}_1 \bar{v}_{xx} - \bar{a}_2 \bar{w}_{xx} + \bar{b}_1 \bar{v}_x - \bar{b}_2 \bar{w}_x + \bar{c}_1 \bar{v} - \bar{c}_2 \bar{w} = 0, \\ \bar{L}_1 \bar{w} + \bar{L}_2 \bar{v} &\equiv -\bar{w}_t + \bar{a}_1 \bar{w}_{xx} + \bar{a}_2 \bar{v}_{xx} + \bar{b}_1 \bar{w}_x + \bar{b}_2 \bar{v}_x + c\bar{c}_1 \bar{w} + \bar{c}_2 \bar{v} = 0, \end{aligned} \tag{7.26}$$

under linear transformations (*7.9*) *if and only if* (*7.23*) *holds, i.e.,*

$$\bar{a}_1 = a_1, \quad \bar{a}_2 = a_2, \quad \bar{k}_1 = k_1, \quad \bar{k}_2 = k_2, \tag{7.27}$$

where

$$\begin{aligned} \bar{k}_1 = {} & \frac{\partial}{\partial x}\left(\bar{c}_1 - \frac{1}{2}\left[\bar{a}_1\left(\frac{\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2}{\bar{a}_1^2 + \bar{a}_2^2}\right)_x - \bar{a}_2\left(\frac{\bar{a}_1\bar{b}_2 - \bar{a}_2\bar{b}_1}{\bar{a}_1^2 + \bar{a}_2^2}\right)_x\right] - \frac{1}{4}\left[\frac{\bar{a}_1\bar{b}_1^2 - \bar{a}_1\bar{b}_2^2 + 2\bar{a}_2\bar{b}_1\bar{b}_2}{\bar{a}_1^2 + \bar{a}_2^2}\right]\right) \\ & + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2}{\bar{a}_1^2 + \bar{a}_2^2}\right), \\ \bar{k}_2 = {} & \frac{\partial}{\partial x}\left(\bar{c}_2 - \frac{1}{2}\left[\bar{a}_1\left(\frac{\bar{a}_1\bar{b}_2 - \bar{a}_2\bar{b}_1}{\bar{a}_1^2 + \bar{a}_2^2}\right)_x + \bar{a}_2\left(\frac{\bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2}{\bar{a}_1^2 + \bar{a}_2^2}\right)_x\right] - \frac{1}{4}\left[\frac{\bar{a}_2\bar{b}_1^2 - \bar{a}_2\bar{b}_1^2 + 2\bar{a}_1\bar{b}_1\bar{b}_2}{\bar{a}_1^2 + \bar{a}_2^2}\right]\right) \\ & + \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\bar{a}_1\bar{b}_2 - \bar{a}_2\bar{b}_1}{\bar{a}_1^2 + \bar{a}_2^2}\right), \end{aligned} \tag{7.28}$$

and σ_1, σ_2 *in the linear transformations* (*7.9*) *can be deduced from* (*7.19*) *via a complex split.*

The proof of this proposition follows immediately from the preceding discussion including the construction of the linear transformation encapsulated in σ.

7.2.1 Uncoupled Parabolic Type Systems

Here an investigation of the Ibragimov-type invariants of the system (7.8) when it maps to the system of the form (7.13) are pursued. Notice that these systems are special cases of the system of PDEs (7.8). If the targeted parabolic type system is uncoupled, i.e., of the form

$$\begin{aligned} \bar{v}_t &= \bar{a}_1 \bar{v}_{xx} + \bar{b}_1 \bar{v}_x + \bar{c}_1 \bar{v}, \\ \bar{w}_t &= \bar{a}_1 \bar{w}_{xx} + \bar{b}_1 \bar{w}_x + \bar{c}_1 \bar{w}. \end{aligned} \tag{7.29}$$

Then the transformed semi-invariants satisfy $\bar{a}_1 = a_1$, $\bar{a}_2 = 0$, $\bar{k}_2 = 0$ and

$$\bar{k}_1 = \frac{\partial}{\partial x}\left(\bar{c}_1 - \bar{a}_1\left(\frac{\bar{b}_1}{2\bar{a}_1}\right)_x - \frac{\bar{b}_1^2}{4\bar{a}_1}\right) + \frac{\partial}{\partial t}\left(\frac{\bar{b}_1}{2\bar{a}_1}\right). \tag{7.30}$$

7.2.2 Coupled Parabolic Type Systems

For a coupled system of the form

$$\begin{aligned}\bar{v}_t &= -\bar{a}_2\bar{w}_{xx} - \bar{b}_2\bar{w}_x - \bar{c}_2\bar{w},\\ \bar{w}_t &= \bar{a}_2\bar{v}_{xx} + \bar{b}_2\bar{v}_x + \bar{c}_2\bar{v},\end{aligned} \tag{7.31}$$

the transformed semi-invariants satisfy $\bar{a}_1 = 0$, $\bar{a}_2 = a_2$, $\bar{k}_1 = 0$ and

$$\bar{k}_2 = \frac{\partial}{\partial x}\left(\bar{c}_2 - \bar{a}_2\left(\frac{\bar{b}_2}{2\bar{a}_2}\right)_x - \frac{\bar{b}_2^2}{4\bar{a}_2}\right) + \frac{\partial}{\partial t}\left(\frac{\bar{b}_2}{2\bar{a}_2}\right). \tag{7.32}$$

These sub-sections can be summarized in the following result.

Proposition 7.2.2. *The linear parabolic type system (7.8) is reducible to the uncoupled system (7.29) and the special coupled system (7.31) via equivalence transformations (7.9) if and only if $\bar{a}_1 = a_1$, $\bar{a}_2 = 0$, $\bar{k}_2 = 0$ and $\bar{k}_1$ as in (7.30) and $\bar{a}_2 = a_2$, $\bar{a}_1 = 0$, $\bar{k}_1 = 0$ and $\bar{k}_2$ as in (7.32), respectively.*

The proof follows at once from the discussion contained above in sub-sections (7.2.1) and (7.2.2). Consequently one arrives at the following.

Corollary 7.2.3. *The parabolic type system (7.8) is reducible to the simpler uncoupled and coupled systems*

$$\begin{aligned}\bar{v}_t &= \bar{a}_1\bar{v}_{xx},\\ \bar{w}_t &= \bar{a}_1\bar{w}_{xx},\end{aligned} \tag{7.33}$$

and

$$\begin{aligned}\bar{v}_t &= -\bar{a}_2\bar{w}_{xx},\\ \bar{w}_t &= \bar{a}_2\bar{v}_{xx},\end{aligned} \tag{7.34}$$

by means of the linear dependent variables change (7.9) if and only if $\bar{a}_1 = a_1$, $\bar{a}_2 = 0$ and $\bar{a}_1 = 0$, $\bar{a}_2 = a_2$, respectively while $\bar{k}_1 = k_1 = 0 = k_2 = \bar{k}_2$ for both the cases, viz. uncoupled and special coupled systems of PDEs.

Note that if a_1 is constant in the Corollary then the criteria for reduction to simplest system, that is, the classical heat system of PDEs follows immediately.

The linear parabolic type uncoupled and coupled systems of PDEs (7.13) and (7.14) are not only the special cases of (7.8) but these also appear as subcases of the general class of systems of two linear parabolic type PDEs (7.6). Thus the Proposition (7.2.2) implies that the sub-class (7.9) of the equivalence transformations (7.7) is found to map special subcases (uncoupled and coupled linear parabolic type systems of two PDEs) of the general class (7.6) into one of its simplest forms or the same family of equations with different coefficients.

In the next section some applications of the results obtained are pursued.

7.3 Applications

Here examples of certain parabolic type systems and their reductions are considered.

1. A constant coefficient parabolic type system

$$\begin{aligned} v_t &= a_1 v_{xx} - a_2 w_{xx} + b_1 v_x - b_2 w_x + c_1 v - c_2 w, \\ w_t &= a_1 w_{xx} + a_2 v_{xx} + b_1 w_x + b_2 v_x + c_1 w + c_2 v, \end{aligned} \tag{7.35}$$

is reducible to the classical heat system

$$\begin{aligned} \bar{v}_t &= \bar{a}_1 \bar{v}_{xx} - \bar{a}_2 \bar{w}_{xx}, \\ \bar{w}_t &= \bar{a}_2 \bar{v}_{xx} + \bar{a}_1 \bar{w}_{xx}, \end{aligned} \tag{7.36}$$

since

$$\bar{a}_1 = a_1, \quad \bar{a}_2 = a_2, \quad \bar{k}_1 = k_1 = 0, \quad \bar{k}_2 = k_2 = 0. \tag{7.37}$$

The transformation is given by the complex split of (7.21), i.e.,

$$\begin{aligned} v &= e^{\alpha_1}(\bar{v}\cos(\alpha_2) - \bar{w}\sin(\alpha_2)), \\ w &= e^{\alpha_1}(\bar{w}\cos(\alpha_2) + \bar{v}\sin(\alpha_2)), \end{aligned} \tag{7.38}$$

where

$$\begin{aligned} \alpha_1 &= \left(c_1 - \frac{a_1(b_1^2 - b_2^2) + 2a_2 b_1 b_2}{4(a_1^2 + a_2^2)}\right) t - \left(\frac{a_1 b_1 + a_2 b_2}{2(a_1^2 + a_2^2)}\right) x, \\ \alpha_2 &= \left(c_2 - \frac{2a_1 b_1 b_2 - a_2 b_1^2 + a_2 b_2^2}{4(a_1^2 + a_2^2)}\right) t - \left(\frac{a_1 b_2 - a_2 b_1}{2(a_1^2 + a_2^2)}\right) x. \end{aligned} \tag{7.39}$$

Notice that this transformation is much more complicated than the corresponding transformation (7.21), but elegantly furnishes the equivalence of (7.35) and (7.36).

2. The time-dependent linear parabolic type system

$$\begin{aligned} v_t &= x v_{xx} + c_1(t) v - c_2(t) w, \\ w_t &= x w_{xx} + c_1(t) w + c_2(t) v, \end{aligned} \tag{7.40}$$

has $a_1 = x$, $k_1 = 0 = k_2$ and by is reducible to

$$\begin{aligned} \bar{v}_t &= x \bar{v}_{xx}, \\ \bar{w}_t &= x \bar{w}_{xx}. \end{aligned} \tag{7.41}$$

The transformation is

$$\begin{aligned} v &= e^{\int c_1(t)dt}\left(\bar{v}\cos\left(\int c_2(t)dt\right) - \bar{w}\sin\left(\int c_2(t)dt\right)\right), \\ w &= e^{\int c_1(t)dt}\left(\bar{w}\cos\left(\int c_2(t)dt\right) + \bar{v}\sin\left(\int c_2(t)dt\right)\right). \end{aligned} \tag{7.42}$$

This is quite a messy transformation compared to the scalar case of the PDE

$$u_t = xu_{xx} + c(t)u, \tag{7.43}$$

which reduces to $\bar{u}_t = x\bar{u}_{xx}$ via $u = e^{\int c(t)dt}\bar{u}$.

3. Consider the variable coefficient linear diffusive system

$$\begin{aligned} v_t &= -\tfrac{1}{2}A^2x^2v_{xx} - Bxv_x + Cv, \\ w_t &= -\tfrac{1}{2}A^2x^2w_{xx} - Bxw_x + Cw, \end{aligned} \tag{7.44}$$

which actually corresponds to the complex Black-Scholes equation (set $u = v + iw$)

$$u_t = -\frac{1}{2}A^2x^2u_{xx} - Bxu_x + Cu. \tag{7.45}$$

Here $\bar{a}_1 = a_1, \bar{a}_2 = a_2 = 0$, $\bar{k}_1 = k_1 = 0$ and $\bar{k}_2 = k_2 = 0$. Thus the system (7.44) can be reduced to

$$\begin{aligned} \bar{v}_t &= -\tfrac{1}{2}A^2x^2\bar{v}_{xx}, \\ \bar{w}_t &= -\tfrac{1}{2}A^2x^2\bar{w}_{xx}, \end{aligned} \tag{7.46}$$

via

$$\begin{aligned} v &= \left(C - \tfrac{1}{2}B - \tfrac{1}{2}\tfrac{B^2}{A^2}\right)tx^{-B/A^2}\bar{v}, \\ w &= \left(C - \tfrac{1}{2}B - \tfrac{1}{2}\tfrac{B^2}{A^2}\right)tx^{-B/A^2}\bar{w}. \end{aligned} \tag{7.47}$$

4. Finally consider some special cases, i.e., uncoupled and coupled parabolic type systems of PDEs. If $a_2 = b_2 = c_2 = 0$ or $a_1 = b_1 = c_1 = 0$ in (7.39) then as a result the following equivalence transformations

$$\begin{aligned} v &= e^{\alpha_3}\overline{v}, \\ w &= e^{\alpha_3}\overline{w}, \end{aligned} \tag{7.48}$$

and

$$\begin{aligned} v &= e^{\alpha_4}(\overline{v}\cos(\alpha_5) - \overline{w}\sin(\alpha_5)), \\ w &= e^{\alpha_4}(\overline{w}\cos(\alpha_5) + \overline{v}\sin(\alpha_5)), \end{aligned} \tag{7.49}$$

are found, respectively. Here $\alpha_3 = \left(c_1 - \frac{b_1^2}{4a_1}\right)t - \frac{b_1}{2a_1}x$, $\alpha_4 = -\frac{b_2}{2a_2}x$ and $\alpha_5 = \left(c_2 - \frac{b_2^2}{4a_2}\right)t$. These transformations (7.48) and (7.49) reduce the following systems

$$\begin{aligned} v_t &= a_1v_{xx} + b_1v_x + c_1v, \\ w_t &= a_1w_{xx} + b_1w_x + c_1w, \end{aligned} \tag{7.50}$$

and

$$\begin{aligned} v_t &= -a_2w_{xx} - b_2w_x - c_2w, \\ w_t &= a_2v_{xx} + b_2v_x + c_2v, \end{aligned} \tag{7.51}$$

to an uncoupled and a coupled system of PDEs

$$\begin{aligned} \bar{v}_t &= \bar{a}_1\bar{v}_{xx}, \\ \bar{w}_t &= \bar{a}_1\bar{w}_{xx}, \end{aligned} \tag{7.52}$$

and

$$\begin{aligned} \bar{v}_t &= -\bar{a}_2 \bar{w}_{xx}, \\ \bar{w}_t &= \bar{a}_2 \bar{v}_{xx}, \end{aligned} \tag{7.53}$$

respectively, which satisfy the invariant conditions accordingly. Another uncoupled system with variable coefficients

$$\begin{aligned} v_t &= v_{xx} + x v_x + \tfrac{x^2}{4} v, \\ w_t &= w_{xx} + x w_x + \tfrac{x^2}{4} w, \end{aligned} \tag{7.54}$$

is equivalent to the simpler system of the form

$$\begin{aligned} \bar{v}_t &= \bar{v}_{xx} + \bar{v}, \\ \bar{w}_t &= \bar{w}_{xx} + \bar{w}, \end{aligned} \tag{7.55}$$

via the equivalence transformations

$$\begin{aligned} \bar{v} &= v e^{(\frac{3}{2}t + \frac{1}{4}x^2)}, \\ \bar{w} &= w e^{(\frac{3}{2}t + \frac{1}{4}x^2)}. \end{aligned} \tag{7.56}$$

In this case $\bar{a} = a = 1$ and $k_1 = k_2 = \bar{k}_1 = \bar{k}_2 = 0$.

Chapter 8

Summary and Conclusions

Lie proved that all linear scalar second order ODEs are transformable to each other, which implies that all these equations belong to one equivalence class. His earliest attempts were to exploit invariance of linear DEs under invertible transformations of variables to map linear DEs to equations with known solutions. The equivalence problem for linear DEs prompts the question of existence of invertible point transformations to connect them with non-linear ones. With this aim, Lie presented the linearization theorem for scalar second order ODEs. He proved that a non-linear ODE is transformable to the free particle equation if and only if it is at most cubically semi-linear in the first derivative of the dependent variable and satisfies a set of constraint equations given in terms of its coefficients. Though simple in principle, the linearization problem is complicated by the requirement of eight Lie point symmetry generators which restricts its applicability to just maximally symmetric second order ODEs. However, an integration technique was developed to find the exact solutions of such ODEs which requires only a 2-dimensional solvable algebra of symmetry generators.

A two dimensional system of second order ODEs can be linearized if it has a 5, 6, 7, 8 or 15-dimensional Lie point symmetry algebra [97]. Such systems are integrable if they have at least a 4-dimensional algebra. Despite these classifications for linearizable and integrable two dimensional systems the work was incomplete. One needs practical algorithms to construct the linearization criteria and transformations to map systems to linear ones. Among a few, geometric [70, 71] and algebraic linearization [98] schemes were developed to accomplish these tasks. CSA provides us means to study systems in terms of a base equation, such that, their linearization and solvability may follow from the base ODEs [1–4]. The only constraints on such systems of second order ODEs are the CR-equations (with respect to the dependent variables and their first order derivatives), to correspond to a base scalar equation [87]. Therefore, CSA singles out a special class of systems from the general one which may be solved by employing the usual symmetry methods but can be solved solely by CSA [89] if the base equation is solvable. The CR-equations do not only establish a connection between systems and base equations, they play an essential role in connecting their solutions and symmetry generators.

Two classes of systems of cubically semi-linear ODEs were provided by geometric and complex-linearization procedures. The class which the geometric approach provides is transformable to a system of free particle equations subject to a set of constraints on the coefficients, hence is maximally symmetric. The equivalence or inequivalence of these classes was investigated under point transformations. (The symmetry structure of the class derived with CSA has not been investigated earlier and was worked out here.) It was shown that they are in general distinct for systems of genuinely cubically semi-linear second order ODEs. The equivalence of the two classes of systems of second order ODEs that arise due to CSA and geometric approach has been investigated here but there exist the most general class of such linearizable systems [74]. It has twenty arbitrary coefficients and, in principle, can be linearized to the five equivalence classes [97]. The CR-structured linearizable systems have three out of these five equivalence classes, it is worth while to investigate the existence of transformations between this class and the most general one. The equivalence problem for the linear CR-structured systems with one arbitrary coefficient and the general linear form with three has been undertaken in chapter 3. The results obtained propose that these two classes are transformable into each other because it is proved that both the optimal and reduce optimal linear forms generate 6, 7 and 15-dimensional symmetry algebras. These observations imply the existence of certain relationship between these linearizable classes via invertible maps, which needs to be investigated.

Group classification of two dimensional linearizable systems of second order ODEs [97] has uncovered five equivalence classes with a $5, 6, 7, 8$, or 15-dimensional symmetry algebras. It was not clear that CSA is limited to these five classes or that it provides all the remaining four classes with the same algebras mentioned above or it may go beyond this number. Therefore, the issue of the classification of linearizable two dimensional CR-structured systems of second order ODEs provided by CSA, was being investigated here. The classification of such general linearizable systems was obtained by using the equivalence of corresponding linear systems under invertible transforms of the variables. An "optimal canonical form" played a decisive role in unraveling the symmetry structure. This canonical form was achieved by invertible transformations where the invertibility of these mappings insures that the symmetry structure is preserved. That optimal canonical form of the linear systems of two second order ODEs led to five linearizable classes with respect to Lie point symmetry algebras with dimensions mentioned above. The existence of a "reduced optimal canonical form" for CR-structured two dimensional linear systems of ODEs was proved. This reduced canonical form provided three equivalence classes, namely with 6, 7 or $15-$dimensional point symmetry algebras. Two cases were not amenable to the use of complex symmetries: those of 5 and $8-$dimensional algebras. The systems corresponding to a complex linearized scalar ODE involve one parameter which can only cover *three* possibilities: (a) it is zero; (b) it is a non-zero constant; and (c) it is a non-constant function.

The non-existence of 5 and 8-dimensional algebras for the CR-structured linear forms appearing due to CSA has been proved by showing that these forms are not equivalent to those provided by the real symmetry approach for systems [97] with 5 and 8 generators. It remains to recover these two linearizable classes by complex methods as the most general complex-linearizable form derived here and the corresponding linear systems are not found to generate them. The CR-structured systems studied here are complex-linearizable as they emerge from those equations that can be linearized by means of the transformations of the form $\tau_j,\ j=1,2$. These transformations were found to play a comprehensive role in deciding whether the emerging systems could be linearized or not.

The next issue raised in this thesis concerns those two dimensional CR-structured systems that are non-linearizable but correspond to a scalar base linearizable equation which can be linearized by invertible complex transformations τ_2. The real transformations corresponding to the complex transformations above cannot be used to linearize the real systems [89]. A straight line representing the solution of the (complex) free particle equation must be the intersection of two planes that are the complex-linearizable system of ODEs, at right angles to incorporate the CR-equations. Thus the complex extensions are very important and they generalize our understanding in a natural way. It was shown that complex-linearization adopts such a procedure to solve systems with 1, 2, 3, and 4-dimensional algebras. Clearly, from symmetry algebra the linearizability of such systems is not achievable by real symmetry analysis, indeed, systems cannot be integrated with less than four symmetry generators.

Lie presented a complete classification for the scalar second order ODEs with the explicit forms of these equations and associated algebra realizations (see [62]). He proved that the allowable number of symmetries for such ODEs is 0, 1, 2, 3, or 8. By considering the dependent variable in these realizations as a complex function of the real independent variable, one obtains the CR-structured systems as well as the Lie-like operators which are not the Lie point symmetry generators. Lie-like conditions are derived here to associate the complex symmetry generators of the base equations with the emerging systems. It was shown that these Lie-like operators do not form an algebra [66] as do the Lie symmetries associated with the CR-structured systems. Further, the real and imaginary parts of the complex symmetry generators are found to satisfy the CR-equations with respect to the dependent variables as well as their first order derivatives. By making use of these observations the Lie-like conditions are reduced for a comparison with the Lie point symmetry conditions [72]. It was proved that the Lie point symmetry conditions for two dimensional CR-structured systems of second order ODEs are a part of the Lie-like conditions. These results have shown the difference between Lie-like operators and Lie symmetries. Indeed, a set of four equations is presented which recognizes Lie point symmetries out of Lie-like operators associated with a CR-structured system.

Two canonical forms for linear systems of four second order ODEs to which such a non-linear system can be mapped by complex methods [88], are presented in this work. The equivalence of linear scalar complex ODEs as well as the equivalence of systems of two complex ODEs of second order were used as tools to establish these forms for systems of four ODEs. The next step was to construct complex-linearizability criteria for such non-linear systems. The most general forms of these systems have been established which are transformable to a system of free particle equations, i.e., it is at most cubically semi-linear in the first derivatives of the dependent variables of the system. For this purpose the geometric linearization is extended to four dimensional CR-structured systems. These base systems are, in general, algebraically inequivalent to those obtainable from CSA. This complex-linearizable class of four dimensional systems can be linearized if the complex linearizing transformations of the base equations have a specific form, which was a critical observation made earlier for two dimensional CR-structured systems. These complex linearizing transformations were shown to single out the linearizable systems from the complex-linearizable four dimensional systems of second order ODEs.

Complex-linearization criteria for systems of four second order ODEs can also be established by implementing the complexification procedure on a scalar second order ODE twice. This class of four dimensional complex-linearizable systems of second order ODEs

$$f_j'' = \omega_j(x, f_1, f_2, f_3, f_4, f_1', f_2', f_3', f_4'), \quad j = 1, 2, 3, 4, \tag{8.1}$$

where prime denotes differentiation with respect to x, would be different from the one given here. One obvious difference may appear due to number of arbitrary coefficients, as this would have fewer than the one studied in this thesis. The number of arbitrary coefficients reduces due to the constraints on such systems to correspond to a complex linearizable ODE. These constraints are obviously the CR-equations which in this case comprised of the following eight equations

$$\begin{aligned} \omega_{1,f_1} &= \omega_{2,f_2} = \omega_{3,f_3} = \omega_{4,f_4}, \\ -\omega_{1,f_2} &= \omega_{2,f_1} = -\omega_{3,f_1} = \omega_{4,f_3}, \\ -\omega_{1,f_3} &= -\omega_{2,f_2} = \omega_{3,f_1} = \omega_{4,f_2}, \\ \omega_{1,f_4} &= -\omega_{2,f_3} = -\omega_{3,f_2} = \omega_{4,f_1}, \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} \omega_{1,f_1'} &= \omega_{2,f_2'} = \omega_{3,f_3'} = \omega_{4,f_4'}, \\ -\omega_{1,f_2'} &= \omega_{2,f_1'} = -\omega_{3,f_1'} = \omega_{4,f_3'}, \\ -\omega_{1,f_3'} &= -\omega_{2,f_2'} = \omega_{3,f_1'} = \omega_{4,f_2'}, \\ \omega_{1,f_4'} &= -\omega_{2,f_3'} = -\omega_{3,f_2'} = \omega_{4,f_1'}. \end{aligned} \tag{8.3}$$

It seems that the complex transformations of the form τ_1 may yield the real linearizing transformations for the emerging complex-linearizable systems when complexified twice as τ_3 did when two dimensional systems are complexified once.

Linear parabolic type equations have been extensively studied in the literature and such scalar equations were mainly focussed. Here complex methods are used to study a special class of systems of two linear parabolic type equations in order to deduce new semi-invariants called Ibragimov-type semi-invariants [67]. It is shown that the semi-invariants under equivalence transformations of the dependent variables, restricted to a special class, can be achieved for a special class of two linear parabolic type equations in two independent variables that can be obtained from the analytic continuation of the scalar complex linear (1+1) parabolic equation. The Ibragimov-type semi-invariants as well as maps for this class of systems of two linear parabolic type equations are found to correspond to the complex Laplace type invariants [37] and maps associated with the base complex scalar (1+1) parabolic PDE, respectively. It is now clear that this special class of systems of two linear parabolic type equations have different algebraic properties when its symmetries are deduced from its complex counterpart compared to those of the system itself (see [65]). However, the consistency of the equivalence criteria had been proved in this thesis, i.e., semi-invariants of the system of two linear parabolic type PDEs correspond to the complex Laplace-type semi-invariants of the base complex PDE. Moreover, an answer to the inverse problem, i.e., when a system of two linear parabolic type PDEs arise from a scalar complex linear (1+1) parabolic equation was also provided.

This thesis mainly dealt with those CR-structured systems that are complex-linearizable. The complex base equations generating the complex-linearizable class of two dimensional systems that consist of both the linearizable and non-linearizable systems, are necessarily maximally symmetric. A complete classification of the scalar second order ODEs was presented that includes such equations with only 3 or lower dimensional algebra realizations. These 3-dimensional algebras are of particular interest as they may yield a linearizable two dimensional system with five or six symmetry generators. For this purpose one needs to investigate the Lie-like operators associated with systems due to a 3-dimensional complex algebra of the base equations. For instance, if all the Lie-like operators of such systems are Lie point symmetries than it must generate a linearizable class of systems. However, the complex Lie algebras of scalar second order ODEs with dimensions less than three would obviously yield non-linearizable CR-structured systems.

The most general form of a complex-linearizable system that can be linearized to the reduced optimal canonical form has not been achieved so far. Targeting the free particle equation with the transformations of the form τ_1 provides a quadratically semi-linear second order ODE, whereas τ_2 leads to a cubic (in first derivative) ODE. Therefore, a complex-linearizable system is (at most) cubically semi-linear but the linearizable CR-structured systems belonging to this class would be at most quadratic in the first derivatives of the dependent variables. To present this idea concisely, one needs to look into the procedure Lie adopted in deriving the most general linearizable form of scalar second order ODEs via invertible transformations. For this purpose consider

$$\tilde{x} = \varphi(x, u), \quad \tilde{u} = \psi(x, u). \tag{8.4}$$

These invertible maps connect $\tilde{u}'' = 0$ with

$$u'' + \frac{1}{(\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x})}[(\varphi_{,u}\psi_{,uu} - \psi_{,u}\varphi_{,uu})u'^3 + (\varphi_{,x}\psi_{,uu} + 2\varphi_{,u}\psi_{,xu} - \psi_{,x}\varphi_{,uu} - 2\psi_{,u}\varphi_{,xu})u'^2$$
$$+(\varphi_{,u}\psi_{,xx} + 2\varphi_{,x}\psi_{,xu} - \psi_{,u}\varphi_{,xx} - 2\psi_{,x}\varphi_{,xu})u' + \varphi_{,x}\psi_{,xx} - \psi_{,x}\varphi_{,xx}] = 0, \tag{8.5}$$

subject to invertibility constraint $\varphi_{,x}\psi_{,u} - \varphi_{,u}\psi_{,x} \neq 0$. The arbitrary functions $\varphi(x,u)$ and $\psi(x,u)$ involved in these transformations can be restricted to generate the corresponding scalar second order ODEs. The following cases arise, showing that a quadratically semi-linear second order ODE can be generated with the restricted form of the arbitrary point transformations, i.e., τ_1.

Case (i). $\mathcal{T}_1 : \tilde{x} = \varphi(x),\ \tilde{u} = \psi(x,u)$,

$$u'' + \frac{1}{(\varphi_{,x}\psi_{,u})}[(\varphi_{,x}\psi_{,uu})u'^2 + (2\varphi_{,x}\psi_{,xu} - \psi_{,u}\varphi_{,xx})u' + \varphi_{,x}\psi_{,xx} - \psi_{,x}\varphi_{,xx}] = 0, \tag{8.6}$$

Case (ii). $\mathcal{T}_2 : \tilde{x} = \varphi(u),\ \tilde{u} = \psi(x,u)$,

$$u'' - \frac{1}{(\varphi_{,u}\psi_{,x})}[(\varphi_{,u}\psi_{,uu} - \psi_{,u}\varphi_{,uu})u'^3 + (2\varphi_{,u}\psi_{,xu} - \psi_{,x}\varphi_{,uu})u'^2 + (\varphi_{,u}\psi_{,xx})u'] = 0, \tag{8.7}$$

Case (iii). $\mathcal{T}_3 : \tilde{x} = \varphi(x,u),\ \tilde{u} = \psi(x)$,

$$u'' - \frac{1}{(\varphi_{,u}\psi_{,x})}[(-\psi_{,x}\varphi_{,uu})u'^2 + (\varphi_{,u}\psi_{,xx} - 2\psi_{,x}\varphi_{,xu})u' + \varphi_{,x}\psi_{,xx} - \psi_{,x}\varphi_{,xx}] = 0, \tag{8.8}$$

Case (iv). $\mathcal{T}_4 : \tilde{x} = \varphi(x,u),\ \tilde{u} = \psi(u)$,

$$u'' + \frac{1}{(\varphi_{,x}\psi_{,u})}[(\varphi_{,u}\psi_{,uu} - \psi_{,u}\varphi_{,uu})u'^3 + (\varphi_{,x}\psi_{,uu} - 2\psi_{,u}\varphi_{,xu})u'^2 - (\psi_{,u}\varphi_{,xx})u'] = 0, \tag{8.9}$$

Case (v). $\mathcal{T}_5 : \tilde{x} = \varphi(x),\ \tilde{u} = \psi(u)$,

$$u'' + \frac{1}{(\varphi_{,x}\psi_{,u})}[(\varphi_{,x}\psi_{,uu})u'^2 - (\psi_{,u}\varphi_{,xx})u'] = 0, \tag{8.10}$$

Case (vi). $\mathcal{T}_6 : \tilde{x} = \varphi(u),\ \tilde{u} = \psi(x)$,

$$u'' + \frac{1}{(\varphi_{,u}\psi_{,x})}[(\psi_{,x}\varphi_{,uu})u'^2 - (\varphi_{,u}\psi_{,xx})u'] = 0. \tag{8.11}$$

There does not exist a linearizable form of the scalar second order ODE corresponding to $\mathcal{T}_7 : \tilde{x} = \varphi(x),\ \tilde{u} = \psi(x)$ and $\mathcal{T}_8 : \tilde{x} = \varphi(u),\ \tilde{u} = \psi(u)$. By looking at the above cases it is apparent that the most general linearizable form of a CR-structured complex-linearizable two dimensional system of second order ODEs would be quadratic non-linear in the first derivatives of the dependent variables. This remains as an open problem to characterize such linearizable systems with respect to the corresponding complex linearizing transformations. By fixing the problem of the most general linearizable form of the CR-structured two dimensional systems of second order ODEs one may look for the invariant linearization criteria for these systems instead of the complex-linearization criteria.

The CR-equations were shown to help in developing solution algorithms to solve those systems that are complex-linearizable. Indeed, these can be linearized with certain complex transformations. The question of complex-integrability, which concerns those CR-structured systems that arise from complex integrable base equations and can be integrated, remains unanswered. The integration of two dimensional systems of second order ODEs relies on atleast a 4-dimensional algebra but it is not necessary that all the Lie-like operators obtained from a 2-dimensional complex algebra are the symmetries of the corresponding systems. Furthermore, it is clear from the 2-dimensional solvable algebras associated with the integrable scalar ODEs that all the complex-integrable systems could not be integrated. For example, consider the symmetries of the equation $u'' - \omega(u') = 0$, which admits $\mathbf{Z}_1 = \partial_x$ and $\mathbf{Z}_2 = \partial_u$. Though all the Lie-like operators emerging from these complex symmetries are symmetry generators of the complex-integrable CR-structured system, they are, in general, insufficient to integrate it completely. The complex-integrable systems may provide a smaller class as comparative to the class of complex-linearizable ones.

For four dimensional complex-linearizable systems of second order ODEs only the linearizable systems were presented in this thesis, one needs to discover the symmetry structure of such systems. It has already been observed that for two dimensional linearizable systems emerging from complex linearizable ODEs have three equivalence classes, though the base equations have only one equivalence class. It implies that the four dimensional systems obtainable from complex systems that can be linearized, may have more than five equivalence classes because the base systems have five. By extending these observations to four dimensional complex-linearizable systems one may extract more than one equivalence class even if the base systems are *only* maximally symmetric. Indeed, there will be four dimensional systems that are complex-linearizable but could not be linearized via complex linearizing transformations of the base systems. The CR-equations for higher dimensional systems could be incorporated to give solution algorithms as was done for two dimensional complex-linearizable systems. Then the complex-integrability of these systems needs attention to integrate or solve them by complex methods. There are no such workable algorithms established with symmetry analysis for even three dimensional systems. Therefore, CSA might be exploited to extend and translate Lie's theory of continuous group into practicable algorithms, for solving the linearization and integrability problems for higher than four dimensional systems of second order ODEs.

One might wonder how the procedures developed can be extended to odd dimensional systems of equations. To obtain a system of $2n$ ODEs one can take a system of n ODEs, by regarding it as complex and splitting. This method will not work for odd dimensions. An extension of the procedure has been developed [64] by splitting iteratively starting with a scalar base equation. Among others, this gave a three dimensional system of ODEs. The procedure could be used by increasing the number of iterations or starting with a higher dimensional system and using a second iteration, to obtain any dimensional system — even or odd. An interesting characterization of the Lie-like operators has been found contrary to those associated with the two dimensional systems of second order ODEs. Though for systems of dimension two it is shown that all the Lie symmetry generators are obtainable from the Lie-like operators, it is not necessarily the case when one moves even one dimension further by employing CSA.

The systems of PDEs addressed in chapter 7 emerge from a linear parabolic PDE by considering the dependent variable as a complex function of the two real independent variables, i.e., $u(x) = f_1(x) + if_2(x)$. Another class of systems of PDEs is obtainable from a scalar ODE $u''(x) = \omega(x, u(x), u'(x))$ when the dependent variable is a complex function of a complex independent variable, i.e., $u(z) = f_1(z) + if_2(z)$ where $z = x + iy$. As a result one finds a system of two PDEs

$$f_{1,xx} - f_{1,yy} + 2f_{2,xy} = 4\Omega_1(x, y, f_1, f_2, \frac{1}{2}(f_{1,x} + f_{2,y}), \frac{1}{2}(f_{2,x} - f_{1,y})),$$
$$f_{2,xx} - f_{2,yy} - 2f_{2,xy} = 4\Omega_2(x, y, f_1, f_2, \frac{1}{2}(f_{1,x} + f_{2,y}), \frac{1}{2}(f_{2,x} - f_{1,y})). \quad (8.12)$$

The complex-linearization criteria for such systems of two PDEs have been derived [4] and the complex linearizing transformations associated with the base scalar ODEs have been employed to linearize the emerging systems of PDEs. Systems of PDEs which emerge from scalar ODEs are said to be complex-linearizable under certain conditions obtained by splitting the complex variables in Lie's linearization conditions for the scalars. The work on complex-linearization presented in this thesis has shown that the complex linearizing transformations of the associated base ODEs determine whether the emerging systems of ODEs are linearizable or not. However, linearization of a system of two PDEs obtainable from a complex scalar ODE may follow from those complex transformations which can not be used to map two dimensional systems of ODEs to the simplest linear ones. In order to clarify this point consider the higher-dimensional coupled system of modified Lane-Emden type presented in [4]

$$f_{1,xx} - f_{1,yy} + 2f_{2,xy} = -6f_1(f_{1,x} + f_{2,y}) + 6f_2(f_{2,x} - f_{1,y}) - 4f_1^3 + 12f_1f_2^2,$$
$$f_{2,xx} - f_{2,yy} - 2f_{2,xy} = -6f_2(f_{1,x} + f_{2,y}) - 6f_1(f_{2,x} - f_{1,y}) - 12f_1^2f_2 + 4f_2^3, \quad (8.13)$$

where $f_{1,x} = f_{2,y}$ and $f_{1,y} = -f_{2,x}$. This system of PDEs is reducible to

$$F_{1,xx} - F_{1,yy} + 2F_{2,xy} = 0,$$
$$F_{2,xx} - F_{2,yy} - 2F_{2,xy} = 0, \quad (8.14)$$

under the following transformations

$$X = x - \frac{f_1}{f_1^2 + f_2^2}, \quad Y = y + \frac{f_2}{f_1^2 + f_2^2},$$
$$F_1 = \frac{x^2 - y^2}{2} - \frac{f_1^2 + f_2^2}{xf_1 + yf_2}, \quad F_2 = xy - \frac{f_1^2 + f_2^2}{yf_1 - xf_2}. \quad (8.15)$$

Notice that these types of transformations do not generate the real linearizing transformations for a system of ODEs. Therefore, in the case of PDEs, when dealt with by complex symmetry methods, one may expect to go beyond the limits observed for the systems of ODEs in this thesis. The essence of the above discussion is that a system of PDEs, if obtained from a complex base scalar or systems of ODEs may inherit almost all the symmetry properties of the base. For example a system of two

PDEs, if obtained from a linearizable base ODE, can always be linearized. If this is true then the linearization problem for a system consisting of two or more PDEs can be dealt with by complex transformations of the base system of ODEs. The characterization of such systems of PDEs, or those obtainable by double splitting of the base ODEs would be interesting.

Another major issue which can be dealt with in CSA is linearization of systems of third order ODEs. To this end, linearizability criteria and linearizing transformations associated with the scalar third order ODEs [39, 40, 73] can be used to obtain the complex-linearizability conditions and corresponding transformations to map the emerging systems of two third order ODEs to simplest possible linear forms, respectively. Such linearizing point transformations must be of the form τ_1 in order to reduce the associated non-linear systems to their corresponding linear forms. For instance, consider the following systems of two third order non-linear ODEs

$$\begin{aligned} &f_1''' + \frac{3}{x}(1 + xf_1')f_1'' - 3f_2''f_2' + f_1'^3 - 3f_1'f_2'^2 + \frac{3}{x}(f_1'^2 - f_2'^2) = 0, \\ &f_2''' + \frac{3}{x}(1 + xf_1')f_2'' + 3f_1''f_2' - f_2'^3 + 3f_1'^2f_2' + \frac{6}{x}(f_1'f_2') = 0, \end{aligned} \tag{8.16}$$

and

$$\begin{aligned} &f_1''' - \left(\frac{6(f_1f_1' + f_2f_2')}{f_1^2 + f_2^2} + \frac{3}{x}\right)f_1'' - \left(\frac{6(f_2f_1' - f_1f_2')}{f_1^2 + f_2^2}\right)f_2'' + \frac{6(f_1^2 - f_2^2)(f_1'^3 - 3f_1'f_2'^2)}{(f_1^2 + f_2^2)^2} \\ &\quad + \frac{12f_1f_2(3f_1'^2 - f_2'^3)}{(f_1^2 + f_2^2)^2} + \frac{6f_1(f_1'^2 - f_2'^2)}{x(f_1^2 + f_2^2)} + \frac{12f_2f_1'f_2'}{x(f_1^2 + f_2^2)} + \frac{6f_1'}{x^2} + \frac{6f_1}{x^3} = 0, \\ &f_2''' + \left(\frac{6(f_2f_1' - f_1f_2')}{f_1^2 + f_2^2}\right)f_1'' - \left(\frac{6(f_1f_1' + f_2f_2')}{f_1^2 + f_2^2} + \frac{3}{x}\right)f_2'' - \frac{12f_1f_2(f_1'^3 - 3f_1'f_2'^2)}{(f_1^2 + f_2^2)^2} \\ &\quad + \frac{6(f_1^2 - f_2^2)(3f_1'^2f_2' - f_2'^3)}{(f_1^2 + f_2^2)^2} - \frac{6f_2(f_1'^2 - f_2'^2)}{x(f_1^2 + f_2^2)} + \frac{12f_1f_1'f_2'}{x(f_1^2 + f_2^2)} + \frac{6f_2'}{x^2} + \frac{6f_2}{x^3} = 0, \end{aligned} \tag{8.17}$$

where prime denotes differentiation with respect to x. These systems are transformable to the simplest linear form

$$F_1''' = 0, \quad F_2''' = 0, \tag{8.18}$$

here prime denotes differentiation with respect to χ. The complex transformations used to linearize the above systems are of the form

$$\chi = x, \quad U = x\exp(u), \tag{8.19}$$

and

$$\chi = x, \quad U = 1/xu, \tag{8.20}$$

respectively. Notice that the above complex transformations are analogues of τ_1 and they linearize the following complex third order ODEs

$$u''' + \frac{3}{x}(1 + xu')u'' + u'^3 + \frac{3}{x}u'^2 = 0, \tag{8.21}$$

and

$$u''' - \left(\frac{6u'}{u} + \frac{3}{x}\right)u'' + \frac{6u'^3}{u^2} + \frac{6u'^2}{xu} + \frac{6u'}{x^2} + \frac{6u}{x^3} = 0, \tag{8.22}$$

to $U''' = 0$. The following set of CR-equations

$$\begin{aligned} &\omega_{1,f_1} = \omega_{2,f_2}, \quad \omega_{1,f_2} = -\omega_{2,f_1}, \quad \omega_{1,f_1'} = \omega_{2,f_2'}, \\ &\omega_{1,f_2'} = -\omega_{2,f_1'}, \quad \omega_{1,f_1''} = \omega_{2,f_2''}, \quad \omega_{1,f_2''} = -\omega_{2,f_1''}, \end{aligned} \tag{8.23}$$

establishes a correspondence between the complex-linearizable two dimensional systems of third order ODEs and the complex base linearizable scalar ODEs. Though complex methods deal with the special CR-structured classes of systems of DEs, these can be incorporated to solve (linearize/integrate) higher dimensional systems of higher order ODEs. Invariance of the parabolic, hyperbolic and elliptic PDEs under equivalence transformations of the dependent and independent variables can also be extended to CR-structured systems of PDEs. CSA would be helpful in uncovering the algebraic symmetry properties of those systems of DEs which are difficult to deal with the usual symmetry methods by adopting Lie's theory in full generality.

Chapter 9

Appendix

9.1 Appendix A-1

Inserting $F_1(x, y, z, y', z')$ and $F_2(x, y, z, y', z')$ from (3.3) into (3.4) leads to

$$\begin{aligned} \frac{D_x(X).D_x^2(Y) - D_x(Y).D_x^2(X)}{(D_x(X))^3} &= 0, \\ \frac{D_x(X).D_x^2(Z) - D_x(Z).D_x^2(X)}{(D_x(X))^3} &= 0. \end{aligned} \quad (9.1)$$

Substituting

$$\begin{aligned} &D_x(X) = X_x + y^{'}X_y + z^{'}X_z, \\ &D_x^2(X) = X_{xx} + 2y^{'}X_{xy} + 2z^{'}X_{xz} + y^{'2}X_{yy} + 2y^{'}z^{'}X_{yz} + z^{'2}X_{zz} + y^{''}X_y + z^{''}X_z, \end{aligned} \quad (9.2)$$

and similar expressions for $D_x(Y),\ D_x(Z),\ D_x^2(Y)$ and $D_x^2(Z)$ in (9.1), yield

$$\begin{aligned} &\bar{\alpha}_1 y^{''} + \bar{\alpha}_2 z^{''} + \bar{\beta}_1 y^{'3} + \bar{\beta}_2 y^{'2}z^{'} + \bar{\beta}_3 y^{'}z^{'2} + \bar{\beta}_4 z^{'3} + \bar{\gamma}_1 y^{'2} + \bar{\gamma}_2 y^{'}z^{'} + \bar{\gamma}_3 z^{'2} \\ &+\bar{\delta}_1 y^{'} + \bar{\delta}_2 z^{'} + \bar{\epsilon}_1 = 0, \\ &\bar{\alpha}_3 y^{''} + \bar{\alpha}_4 z^{''} + \bar{\beta}_5 y^{'3} + \bar{\beta}_6 y^{'2}z^{'} + \bar{\beta}_7 y^{'}z^{'2} + \bar{\beta}_8 z^{'3} + \bar{\gamma}_4 y^{'2} + \bar{\gamma}_5 y^{'}z^{'} + \bar{\gamma}_6 z^{'2} \\ &+\bar{\delta}_3 y^{'} + \bar{\delta}_4 z^{'} + \bar{\epsilon}_2 = 0. \end{aligned} \quad (9.3)$$

The coefficients of the above system of ODEs are

$$\begin{aligned} &\bar{\alpha}_1 = X_xY_y - Y_xX_y + z^{'}(X_zY_y - Y_zX_y), \quad \bar{\alpha}_2 = X_xY_z - Y_xX_z + y^{'}(X_yY_z - Y_yX_z), \\ &\bar{\alpha}_3 = X_xZ_y - Z_xX_y + z^{'}(X_zZ_y - Z_zX_y), \quad \bar{\alpha}_4 = X_xZ_z - Z_xX_z + y^{'}(X_yZ_z - Z_yX_z), \\ &\bar{\beta}_1 = X_yY_{yy} - Y_yX_{yy}, \quad \bar{\beta}_2 = X_zY_{yy} - Y_zX_{yy} + 2(X_yY_{yz} - Y_yX_{yz}), \\ &\bar{\beta}_3 = X_yY_{zz} - Y_yX_{zz} + 2(X_zY_{yz} - Y_zX_{yz}), \quad \bar{\beta}_4 = X_zY_{zz} - Y_zX_{zz}, \\ &\bar{\beta}_5 = X_yZ_{yy} - Z_yX_{yy}, \quad \bar{\beta}_6 = X_zZ_{yy} - Z_zX_{yy} + 2(X_yZ_{yz} - Z_yX_{yz}), \\ &\bar{\beta}_7 = X_yZ_{zz} - Z_yX_{zz} + 2(X_zZ_{yz} - Z_zX_{yz}), \quad \bar{\beta}_8 = X_zZ_{zz} - Z_zX_{zz}, \end{aligned} \quad (9.4)$$

$$
\begin{aligned}
\bar{\gamma}_1 &= X_x Y_{yy} - Y_x X_{yy} + 2(X_y Y_{xy} - Y_y X_{xy}),\\
\bar{\gamma}_2 &= 2[X_x Y_{yz} + X_y Y_{xz} + X_z Y_{xy} - (Y_x X_{yz} + Y_y X_{xz} + Y_z X_{xy})],\\
\bar{\gamma}_3 &= X_x Y_{zz} - Y_x X_{zz} + 2(X_z Y_{xz} - Y_z X_{xz}),\\
\bar{\gamma}_4 &= X_x Z_{yy} - Z_x X_{yy} + 2(X_y Z_{xy} - Z_y X_{xy}),\\
\bar{\gamma}_5 &= 2[X_x Z_{yz} + X_y Z_{xz} + X_z Z_{xy} - (Z_x X_{yz} + Z_y X_{xz} + Z_z X_{xy})],\\
\bar{\gamma}_6 &= X_x Z_{zz} - Z_x X_{zz} + 2(X_z Z_{xz} - Z_z X_{xz}),\\
\bar{\delta}_1 &= X_y Y_{xx} - Y_y X_{xx} + 2(X_x Y_{xy} - Y_x X_{xy}),\\
\bar{\delta}_2 &= X_z Y_{xx} - Y_z X_{xx} + 2(X_x Y_{xz} - Y_x X_{xz}),\\
\bar{\delta}_3 &= X_y Z_{xx} - Z_y X_{xx} + 2(X_x Z_{xy} - Z_x X_{xy}),\\
\bar{\delta}_4 &= X_z Z_{xx} - Z_z X_{xx} + 2(X_x Z_{xz} - Z_x X_{xz}),\\
\bar{\epsilon}_1 &= X_x Y_{xx} - Y_x X_{xx}, \quad \bar{\epsilon}_2 = X_x Z_{xx} - Z_x X_{xx}.
\end{aligned} \tag{9.5}
$$

System (9.3) yields a system of the form (3.5) with the following coefficients

$$
\begin{aligned}
&\alpha_{11} = \tau_1(\bar{\alpha}_4\bar{\beta}_1 - \bar{\alpha}_2\bar{\beta}_5),\ \alpha_{12} = \tau_1(\bar{\alpha}_4\bar{\beta}_2 - \bar{\alpha}_2\bar{\beta}_6),\ \alpha_{13} = \tau_1(\bar{\alpha}_4\bar{\beta}_3 - \bar{\alpha}_2\bar{\beta}_7),\\
&\alpha_{14} = \tau_1(\bar{\alpha}_4\bar{\beta}_4 - \bar{\alpha}_2\bar{\beta}_8),\ \alpha_{21} = \tau_2(\bar{\alpha}_3\bar{\beta}_1 - \bar{\alpha}_1\bar{\beta}_5),\ \alpha_{22} = \tau_2(\bar{\alpha}_3\bar{\beta}_2 - \bar{\alpha}_1\bar{\beta}_6),\\
&\alpha_{23} = \tau_2(\bar{\alpha}_3\bar{\beta}_3 - \bar{\alpha}_1\bar{\beta}_7),\ \alpha_{24} = \tau_2(\bar{\alpha}_3\bar{\beta}_4 - \bar{\alpha}_1\bar{\beta}_8),\\
&\beta_{11} = \tau_1(\bar{\alpha}_4\bar{\gamma}_1 - \bar{\alpha}_2\bar{\gamma}_4),\ \beta_{12} = \tau_1(\bar{\alpha}_4\bar{\gamma}_2 - \bar{\alpha}_2\bar{\gamma}_5),\ \beta_{13} = \tau_1(\bar{\alpha}_4\bar{\gamma}_3 - \bar{\alpha}_2\bar{\gamma}_6),\\
&\beta_{21} = \tau_2(\bar{\alpha}_3\bar{\gamma}_1 - \bar{\alpha}_1\bar{\gamma}_4),\ \beta_{22} = \tau_2(\bar{\alpha}_3\bar{\gamma}_2 - \bar{\alpha}_1\bar{\gamma}_5),\ \beta_{23} = \tau_2(\bar{\alpha}_2\bar{\gamma}_3 - \bar{\alpha}_1\bar{\gamma}_6),\\
&\gamma_{11} = \tau_1(\bar{\alpha}_4\bar{\delta}_1 - \bar{\alpha}_2\bar{\delta}_3),\ \gamma_{12} = \tau_1(\bar{\alpha}_4\bar{\delta}_2 - \bar{\alpha}_2\bar{\delta}_4),\ \gamma_{21} = \tau_2(\bar{\alpha}_3\bar{\delta}_1 - \bar{\alpha}_1\bar{\delta}_3),\\
&\gamma_{22} = \tau_2(\bar{\alpha}_3\bar{\delta}_2 - \bar{\alpha}_1\bar{\delta}_4),\ \delta_1 = \tau_1(\bar{\alpha}_4\bar{\epsilon}_1 - \bar{\alpha}_2\bar{\epsilon}_2),\ \delta_2 = \tau_2(\bar{\alpha}_3\bar{\epsilon}_1 - \bar{\alpha}_1\bar{\epsilon}_2),
\end{aligned} \tag{9.6}
$$

where $\tau_1 = -\tau_2 = \bar{\alpha}_1\bar{\alpha}_4 - \bar{\alpha}_2\bar{\alpha}_3$.

9.2 Appendix A-2

$$2(\gamma^2_{11,x} - \delta_{21,f_1} - \delta_{22,f_2}) + (\gamma^2_{21}\gamma^2_{11} - \gamma^2_{22}\gamma^2_{12} + \gamma^1_{11}\gamma^2_{11} - \gamma^1_{12}\gamma^2_{12}) - 4(\beta^2_{11}\delta_{11} - \beta^2_{12}\delta_{12} + \beta^2_{21}\delta_{21} - \beta^2_{22}\delta_{22}) = 0,$$
$$2(\gamma^2_{12,x} + \delta_{21,f_2} - \delta_{22,f_1}) + (\gamma^2_{21}\gamma^2_{12} + \gamma^2_{11}\gamma^2_{22} + \gamma^1_{11}\gamma^2_{12} + \gamma^1_{12}\gamma^2_{11}) - 4(\beta^2_{12}\delta_{11} + \beta^2_{11}\delta_{12} + \beta^2_{22}\delta_{21} + \beta^2_{21}\delta_{22}) = 0,$$

$$4\beta^2_{11,x} - \gamma^2_{11,f_1} - \gamma^2_{12,f_2} - 4(\alpha_{11}\delta_{21} - \alpha_{12}\delta_{22}) + 2(\beta^1_{11}\gamma^2_{11} - \beta^1_{12}\gamma^2_{12}) + 2(\beta^2_{11}\gamma^2_{21} - \beta^2_{12}\gamma^2_{22}) - 2(\beta^2_{11}\gamma^1_{11} - \beta^2_{12}\gamma^1_{12}) - 2(\beta^2_{21}\gamma^2_{11} - \beta^2_{22}\gamma^2_{12}) = 0,$$
$$4\beta^2_{12,x} + \gamma^2_{11,f_2} - \gamma^2_{12,f_1} - 4(\alpha_{11}\delta_{22} + \alpha_{12}\delta_{21}) + 2(\beta^1_{12}\gamma^2_{11} + \beta^1_{11}\gamma^2_{12}) + 2(\beta^2_{12}\gamma^2_{21} + \beta^2_{11}\gamma^2_{22}) - 2(\beta^2_{12}\gamma^1_{11} + \beta^2_{11}\gamma^1_{12}) - 2(\beta^2_{21}\gamma^2_{12} + \beta^2_{22}\gamma^2_{11}) = 0,$$

$$12\beta^2_{21,x} - 4\beta^1_{11,x} + \gamma^1_{11,f_1} + \gamma^1_{12,f_2} - 16(\alpha_{21}\delta_{21} - \alpha_{22}\delta_{22}) - 8(\beta^2_{11}\gamma^1_{21} - \beta^2_{12}\gamma^1_{22}) + 8(\beta^1_{21}\gamma^2_{11} - \beta^1_{22}\gamma^2_{12}) - 3(\gamma^2_{21,f_1} + \gamma^2_{22,f_2}) = 0,$$

$$12\beta^2_{22,x} - 4\beta^1_{12,x} + \gamma^1_{12,f_1} - \gamma^1_{11,f_2} - 16(\alpha_{22}\delta_{21} + \alpha_{21}\delta_{22}) - 8(\beta^2_{11}\gamma^1_{22} + \beta^2_{12}\gamma^1_{21}) + 8(\beta^1_{21}\gamma^2_{12} + \beta^1_{22}\gamma^2_{11}) - 3(\gamma^2_{22,f_1} + \gamma^2_{21,f_2}) = 0,$$

$$2(\gamma^1_{21,x} - \delta_{11,f_3} - \delta_{12,f_4}) + \gamma^1_{21}\gamma^2_{21} - \gamma^1_{22}\gamma^2_{22} + \gamma^1_{21}\gamma^1_{11} - \gamma^1_{22}\gamma^1_{12} - 4(\beta^1_{21}\delta_{11} - \beta^1_{22}\delta_{12} + \beta^1_{31}\delta_{21} - \beta^1_{32}\delta_{22}) = 0,$$
$$2(\gamma^1_{22,x} - \delta_{12,f_3} + \delta_{11,f_4}) + \gamma^1_{21}\gamma^2_{22} + \gamma^1_{22}\gamma^2_{21} + \gamma^1_{21}\gamma^1_{12} + \gamma^1_{22}\gamma^1_{11} - 4(\beta^1_{21}\delta_{12} + \beta^1_{22}\delta_{11} + \beta^1_{31}\delta_{22} + \beta^1_{32}\delta_{21}) = 0,$$

$$4\beta^1_{31,x} - \gamma^1_{21,f_3} - \gamma^1_{22,f_4} - 4(\alpha_{31}\delta_{11} - \alpha_{32}\delta_{12}) + 2(\beta^2_{31}\gamma^1_{21} - \beta^2_{32}\gamma^1_{22} - \beta^1_{21}\gamma^1_{21} + \beta^1_{22}\gamma^1_{22} - \beta^1_{31}\gamma^2_{21} + \beta^1_{32}\gamma^2_{22} + \beta^1_{31}\gamma^1_{11} - \beta^1_{32}\gamma^1_{12}) = 0,$$
$$4\beta^1_{32,x} - \gamma^1_{22,f_3} + \gamma^1_{21,f_4} - 4(\alpha_{32}\delta_{11} + \alpha_{31}\delta_{12}) + 2(\beta^2_{32}\gamma^1_{21} + \beta^2_{31}\gamma^1_{22} - \beta^1_{21}\gamma^1_{22} - \beta^1_{22}\gamma^1_{21} - \beta^1_{31}\gamma^2_{22} - \beta^1_{32}\gamma^2_{21} + \beta^1_{31}\gamma^1_{12} + \beta^1_{32}\gamma^1_{11}) = 0,$$

$$(\alpha_{21,f_1} + \alpha_{22,f_2} - \alpha_{11,f_3} - \alpha_{12,f_4}) + 2(\alpha_{11}\beta^1_{21} - \alpha_{12}\beta^1_{22} + \alpha_{21}\beta^2_{21} - \alpha_{22}\beta^2_{22} - \alpha_{21}\beta^1_{11} + \alpha_{22}\beta^1_{12} - \alpha_{31}\beta^2_{11} + \alpha_{32}\beta^2_{12}) = 0,$$
$$(\alpha_{22,f_1} - \alpha_{21,f_2} - \alpha_{12,f_3} + \alpha_{11,f_4}) + 2(\alpha_{11}\beta^1_{22} + \alpha_{12}\beta^1_{21} + \alpha_{21}\beta^2_{22} + \alpha_{22}\beta^2_{21} - \alpha_{21}\beta^1_{12} - \alpha_{22}\beta^1_{11} - \alpha_{31}\beta^2_{12} - \alpha_{32}\beta^2_{11}) = 0,$$

$$(\alpha_{31,f_1} + \alpha_{32,f_2} - \alpha_{21,f_3} - \alpha_{22,f_4}) + 2(\alpha_{11}\beta^1_{31} - \alpha_{12}\beta^1_{32} + \alpha_{21}\beta^2_{31} - \alpha_{22}\beta^2_{32} - \alpha_{21}\beta^1_{21} + \alpha_{22}\beta^1_{22} - \alpha_{31}\beta^2_{21} + \alpha_{32}\beta^2_{22}) = 0,$$
$$(\alpha_{32,f_1} - \alpha_{31,f_2} - \alpha_{22,f_3} + \alpha_{21,f_4}) + 2(\alpha_{11}\beta^1_{32} + \alpha_{12}\beta^1_{31} + \alpha_{21}\beta^2_{32} + \alpha_{22}\beta^2_{31} - \alpha_{21}\beta^1_{22} - \alpha_{22}\beta^1_{21} - \alpha_{31}\beta^2_{22} - \alpha_{32}\beta^2_{21}) = 0,$$

$$-6\alpha_{21,x} + 5(\alpha_{21}\gamma^1_{11} - \alpha_{22}\gamma^1_{12}) + 2(\alpha_{31}\gamma^2_{11} - \alpha_{32}\gamma^2_{12}) - 2(\beta^2_{21,f_3} + \beta^2_{22,f_4}) + 6(\beta^1_{31}\beta^2_{11} - \beta^1_{32}\beta^2_{12}) + (\alpha_{21}\gamma^2_{21} - \alpha_{22}\gamma^2_{22}) - 6(\beta^1_{21}\beta^2_{21} - \beta^1_{22}\beta^2_{22}) - 4(\beta^1_{21,f_1} + \beta^1_{22,f_2}) + 2(\beta^2_{31,f_1} + \beta^2_{32,f_2}) + 4(\beta^1_{11,f_3} + \beta^1_{12,f_4}) - 2(\alpha_{11}\gamma^1_{21} - \alpha_{12}\gamma^1_{22}) = 0,$$
$$-6\alpha_{22,x} + 5(\alpha_{21}\gamma^1_{12} + \alpha_{22}\gamma^1_{11}) + 2(\alpha_{31}\gamma^2_{12} + \alpha_{32}\gamma^2_{11}) - 2(\beta^2_{22,f_3} + \beta^2_{21,f_4}) + 6(\beta^1_{31}\beta^2_{12} + \beta^1_{32}\beta^2_{11}) + (\alpha_{21}\gamma^2_{22} + \alpha_{22}\gamma^2_{21}) - 6(\beta^1_{21}\beta^2_{22} + \beta^1_{22}\beta^2_{21}) - 4(\beta^1_{22,f_1} - \beta^1_{21,f_2}) + 2(\beta^2_{32,f_1} - \beta^2_{31,f_2}) + 4(\beta^1_{12,f_3} - \beta^1_{11,f_4}) - 2(\alpha_{12}\gamma^1_{21} + \alpha_{11}\gamma^1_{22}) = 0,$$

$$-2\alpha_{11,x} + (\alpha_{11}\gamma^1_{11} - \alpha_{12}\gamma^1_{12}) - 2(\beta^2_{11}\beta^2_{31} - \beta^2_{12}\beta^2_{32}) + \beta^2_{21,f_1} + \beta^2_{22,f_2} - \beta^2_{11,f_3} - \beta^2_{12,f_4} + 2(\beta^2_{11}\beta^1_{21} - \beta^2_{12}\beta^1_{22} + \beta^2_{21}\beta^2_{21} - \beta^2_{22}\beta^2_{22}) + (\alpha_{11}\gamma^2_{21} - \alpha_{12}\gamma^2_{22}) - 2(\beta^2_{21}\beta^1_{11} - \beta^2_{22}\beta^1_{12}) = 0,$$
$$-2\alpha_{12,x} + (\alpha_{11}\gamma^1_{12} + \alpha_{12}\gamma^1_{11}) - 2(\beta^2_{31}\beta^2_{12} + \beta^2_{11}\beta^2_{32}) + (\beta^2_{22,f_1} - \beta^2_{21,f_2} - \beta^2_{12,f_3} + \beta^2_{11,f_4}) + 2(\beta^2_{11}\beta^1_{22} + \beta^1_{21}\beta^2_{12} + 2\beta^2_{21}\beta^2_{22}) + (\alpha_{11}\gamma^2_{22} + \alpha_{12}\gamma^2_{21}) - 2(\beta^2_{21}\beta^1_{12} + \beta^2_{22}\beta^1_{11}) = 0,$$

$$-2\alpha_{31,x} + (\alpha_{31}\gamma^1_{11} - \alpha_{32}\gamma^1_{12}) + (\alpha_{31}\gamma^2_{21} - \alpha_{32}\gamma^2_{22}) - (\beta^1_{31,f_1} + \beta^1_{32,f_2} - \beta^1_{21,f_3} - \beta^1_{22,f_4}) - 2(\beta^1_{11}\beta^1_{31} - \beta^1_{12}\beta^1_{32} - \beta^1_{21}\beta^1_{21} + \beta^1_{22}\beta^1_{22} + \beta^1_{21}\beta^2_{31} - \beta^1_{22}\beta^2_{32} - \beta^1_{31}\beta^2_{21} + \beta^1_{32}\beta^2_{22}) = 0,$$
$$-2\alpha_{32,x} + (\alpha_{32}\gamma^1_{11} + \alpha_{31}\gamma^1_{12}) + (\alpha_{31}\gamma^2_{22} + \alpha_{32}\gamma^2_{21}) - (\beta^1_{32,f_1} - \beta^1_{31,f_2} - \beta^1_{22,f_3} + \beta^1_{21,f_4}) - 2(\beta^1_{11}\beta^1_{32} + \beta^1_{12}\beta^1_{31} - 2\beta^1_{21}\beta^1_{22} + \beta^1_{21}\beta^2_{32} + \beta^1_{22}\beta^2_{31} - \beta^1_{31}\beta^2_{22} - \beta^1_{32}\beta^2_{21}) = 0,$$

$-8\beta^1_{11,x}+2(\gamma^1_{11,f_1}+\gamma^1_{12,f_2})-6(\beta^2_{31}\gamma^2_{11}-\beta^2_{32}\gamma^2_{12})+12(\alpha_{11}\delta_{11}-\alpha_{12}\delta_{12})-8(\alpha_{21}\delta_{21}-\alpha_{22}\delta_{22})-4(\beta^2_{11}\gamma^1_{21}-\beta^2_{12}\gamma^1_{22})+10(\beta^1_{21}\gamma^2_{11}-\beta^1_{22}\gamma^2_{12})+12\beta^2_{21,x}-3(\gamma^2_{11,f_3}+\gamma^2_{12,f_4})+6(\beta^2_{21}\gamma^2_{21}-\beta^2_{22}\gamma^2_{22}-\beta^2_{21}\gamma^1_{11}+\beta^2_{22}\gamma^1_{12})=0,$
$-8\beta^1_{12,x}+2(\gamma^1_{12,f_1}-\gamma^1_{11,f_2})-6(\beta^2_{31}\gamma^2_{12}+\beta^2_{32}\gamma^2_{11})+12(\alpha_{12}\delta_{11}+\alpha_{11}\delta_{12})-8(\alpha_{21}\delta_{22}+\alpha_{22}\delta_{21})-4(\beta^2_{12}\gamma^1_{21}+\beta^2_{11}\gamma^1_{22})+10(\beta^1_{21}\gamma^2_{12}+\beta^1_{22}\gamma^2_{11})+12\beta^2_{22,x}-3(\gamma^2_{12,f_3}-\gamma^2_{11,f_4})+6(\beta^2_{22}\gamma^2_{21}+\beta^2_{21}\gamma^2_{22}-\beta^2_{22}\gamma^1_{11}-\beta^2_{21}\gamma^1_{12})=0,$

$-6\alpha_{21,x}+2(\beta^2_{31,f_1}+\beta^2_{32,f_2})+(\alpha_{21}\gamma^1_{11}-\alpha_{22}\gamma^1_{12})+5(\alpha_{21}\gamma^2_{21}-\alpha_{22}\gamma^2_{22})+2(\alpha_{11}\gamma^1_{21}-\alpha_{12}\gamma^1_{22}-\beta^2_{21,f_3}-\beta^2_{22,f_4}-\alpha_{31}\gamma^2_{11}+\alpha_{32}\gamma^2_{12})+6(\beta^2_{11}\beta^1_{31}-\beta^2_{12}\beta^1_{32}-\beta^2_{21}\beta^1_{21}+\beta^2_{22}\beta^1_{22})-(\beta^1_{21,f_1}+\beta^1_{22,f_2}-\beta^1_{11,f_3}-\beta^1_{12,f_4})=0,$
$-6\alpha_{22,x}+2(\beta^2_{32,f_1}-\beta^2_{31,f_2})+(\alpha_{21}\gamma^1_{12}+\alpha_{22}\gamma^1_{11})+5(\alpha_{21}\gamma^2_{22}+\alpha_{22}\gamma^2_{21})+2(\alpha_{11}\gamma^1_{22}+\alpha_{12}\gamma^1_{21})-2(\beta^2_{22,f_3}-\beta^2_{21,f_4}+\alpha_{31}\gamma^2_{12}+\alpha_{32}\gamma^2_{11})+6(\beta^2_{12}\beta^1_{31}+\beta^2_{11}\beta^1_{32}-\beta^2_{21}\beta^1_{22}-\beta^2_{22}\beta^1_{21})-\beta^1_{22,f_1}+\beta^1_{21,f_2}+\beta^1_{12,f_3}-\beta^1_{11,f_4}=0,$

$4\beta^1_{21,x}+\gamma^1_{21,f_1}+\gamma^1_{22,f_2}-8(\alpha_{21}\delta_{11}-\alpha_{22}\delta_{12})+2(\beta^2_{21}\gamma^1_{21}-\beta^2_{22}\gamma^1_{22}+\beta^1_{11}\gamma^1_{21}-\beta^1_{12}\gamma^1_{22}+\beta^1_{21}\gamma^2_{21}-\beta^1_{22}\gamma^2_{22}-\beta^1_{12}\gamma^1_{11}+\beta^1_{22}\gamma^1_{12})-4(\beta^1_{31}\gamma^2_{11}-\beta^1_{32}\gamma^2_{12})-2(\gamma^1_{11,f_3}+\gamma^1_{12,f_4})-4(\alpha_{31}\delta_{21}-\alpha_{32}\delta_{22})=0,$
$4\beta^1_{22,x}+\gamma^1_{22,f_1}-\gamma^1_{21,f_2}-8(\alpha_{22}\delta_{11}+\alpha_{21}\delta_{12})+2(\beta^2_{22}\gamma^1_{21}+\beta^2_{21}\gamma^1_{22}+\beta^1_{12}\gamma^1_{21}+\beta^1_{11}\gamma^1_{22}+\beta^1_{21}\gamma^2_{22}+\beta^1_{22}\gamma^2_{21}-\beta^1_{21}\gamma^1_{12}-\beta^1_{22}\gamma^1_{11})-4(\beta^1_{31}\gamma^2_{12}+\beta^1_{32}\gamma^2_{11})-2(\gamma^1_{12,f_3}-\gamma^1_{11,f_4})-4(\alpha_{32}\delta_{21}+\alpha_{31}\delta_{22})=0,$

$-4\beta^1_{21,x}+4\beta^2_{31,x}+2(\gamma^1_{21,f_1}+\gamma^1_{22,f_2})-4(\gamma^1_{21}\beta^2_{21}-\gamma^1_{22}\beta^2_{22}-\beta^1_{11}\gamma^1_{21}+\beta^1_{12}\gamma^1_{22}-\beta^1_{21}\gamma^2_{21}+\beta^1_{22}\gamma^2_{22}+\beta^1_{21}\gamma^1_{11}-\beta^1_{22}\gamma^1_{12})-\gamma^2_{21,f_3}-\gamma^2_{22,f_4}-\gamma^1_{11,f_3}-\gamma^1_{12,f_4}-8(\alpha_{31}\delta_{21}-\alpha_{32}\delta_{22})=0,$
$-4\beta^1_{22,x}+4\beta^2_{32,x}+2(\gamma^1_{22,f_1}-\gamma^1_{21,f_2})-4(\gamma^1_{22}\beta^2_{21}+\gamma^1_{21}\beta^2_{22}-\beta^1_{11}\gamma^1_{22}-\beta^1_{12}\gamma^1_{21}-\beta^1_{21}\gamma^2_{22}-\beta^1_{22}\gamma^2_{21}+\beta^1_{21}\gamma^1_{12}+\beta^1_{22}\gamma^1_{11})-\gamma^2_{22,f_3}+\gamma^2_{21,f_4}-\gamma^1_{12,f_3}+\gamma^1_{11,f_4}-8(\alpha_{32}\delta_{21}-\alpha_{31}\delta_{22})=0,$

$2(\delta_{11,f_1}+\delta_{12,f_2})+4(\beta^1_{11}\delta_{11}-\beta^1_{12}\delta_{12}+\beta^1_{21}\delta_{21}-\beta^1_{22}\delta_{22}-\beta^2_{31}\delta_{21}+\beta^2_{32}\delta_{22})+2(\gamma^2_{21,x}-\gamma^1_{11,x}-\delta_{21,f_3}-\delta_{22,f_4})+\gamma^2_{21}\gamma^2_{21}-\gamma^2_{22}\gamma^2_{22}-\gamma^1_{11}\gamma^1_{11}+\gamma^1_{12}\gamma^1_{12}-4(\beta^2_{21}\delta_{11}-\beta^2_{22}\delta_{12})=0,$
$2(\delta_{12,f_1}-\delta_{11,f_2})+4(\beta^1_{11}\delta_{12}+\beta^1_{12}\delta_{11}+\beta^1_{21}\delta_{22}+\beta^1_{22}\delta_{21}-\beta^2_{31}\delta_{22}-\beta^2_{32}\delta_{21})+2(\gamma^2_{22,x}-\gamma^1_{12,x}-\delta_{22,f_3}+\delta_{21,f_4}+\gamma^2_{21}\gamma^2_{22}-\gamma^1_{11}\gamma^1_{12})-4(\beta^2_{21}\delta_{12}+\beta^2_{22}\delta_{11})=0,$

Bibliography

[1] S. Ali, *Complex Lie symmetries for differential equations*, PhD thesis, National University of Sciences and Technology, 2009.

[2] S. Ali, F.M. Mahomed and A. Qadir, Complex Lie symmetries for Scalar Second order ordinary differential equations, *Nonlinear Analysis: Real World Applications.* **10** (2009) 3335.

[3] S. Ali, F.M. Mahomed and A. Qadir, Complex Lie symmetries for variational problems, *J.N.M.P.* **15** (2008) 25.

[4] S. Ali, F.M. Mahomed and A. Qadir, Linearization criteria for systems of two second order differential equations by complex methods, *Nonlinear Dyn.* **66** (2011) 77.

[5] A.V. Aminova, N.A.M. Aminov, Projective geometry of systems of differential equations: general conceptions, *Tensor N S.* **62** (2000) 65.

[6] S.C. Anco and G. Bluman, Integrating factors and first integrals of ordinary differential equations, *Eur. J. Appl. Math.* **9** (1998) 245.

[7] F.T. Arecchi, G.S. Rodari and A. Sona, Statistics of the laser radiation at threshold, *Phys. Lett.* **25A**(1) (1967) 59.

[8] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect*, John Wiley, New York, 1982.

[9] J. Black and M. Scholes, The pricing of options and corporate liabilities, *J. of Political Economy.* **81** (1973) 637.

[10] G.W. Bluman, On the transformation of diffusion processes into the Wiener process, *SIAM J. Applied Math.* **39**(2) (1980) 238.

[11] G.W. Bluman and S. Kumei, *Symmetries of Differential Equations*, Springer, Berlin, 1990.

[12] N. Bourbaki, *Eléments d'histoire des mathématiques*, Masson Editeur, Paris, 1984 [English translation, *Elements of the History of Mathematics*, Springer, Berlin, 1994].

[13] F. Brioschi, Les invariants des équations différentielles linéaires, *Acta Mathematica.* **14** (1890) 233.

[14] E. Cartan, Sur les variétés de courbure constante d'un espace euclidien ou non euclidien, *Bull. Soc. Math. France.* **52** (1924) 205.

[15] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, A systematic method of finding linearizing transformations for nonlinear ordinary differential equations, arXiv:1008.3233.

[16] V.K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the complete integrability and linearization of nonlinear ordinary differential equations - Part V: Linearization of coupled second order equations, *Proc. R. Soc. London A.* **465** (2009) 2369.

[17] E.S. Cheb-Terrab and A.D. Roche, *J.N.M.P.* **3** (1996) 341.

[18] S.S. Chern, Sur la géométrie d'une équation différentielle du troisème orde, *C.R. Acad. Sci. Paris.* **204** (1937) 1227.

[19] S.S. Chern, The geometry of the differential equation $y^{'''} = F(x, y, y^{'}, y^{''})$, *Sci. Rep. Nat. Tsing Hua Univ.* **4** (1940) 97.

[20] J. Cockle, Correlations of analysis, *Philosophical Magzine.* **24**(IV) (1862) 531.

[21] J. Drach, *Essai sur une théorie général de l'intégration et sur la classification des transcendentes*, Gauthier-Villars et C^{ie}, Paris, 1898.

[22] L.G.S. Duarte, C. Moreira and F.C. Santos, Linearization under non-point transformation, *J. Phys. A: Math.Gen.* **27** (1994) L739.

[23] M.U. Farooq, S. Ali and F.M. Mahomed, Two dimensional systems that arise from the Noether classification of Lagrangians on the line, *Appl. Math. Comput.* **217** (2011) 6959.

[24] M.U. Farooq, S. Ali and A. Qadir, Invariants of two dimensional systems via complex Lagrangians with applications, *Commun. Nonlinear Sci. Numer. Simulat.* **16** (2011) 1804.

[25] T. Feroze, F.M. Mahomed and A. Qadir, The connection between isometries and symmetries of geodesic equations of the underlying spaces, *Nonlinear Dyn.* **45** (2006) 65.

[26] A.R. Forsyth, Invariants, covariants and quotient-derivatives associated with linear differential equations, *Philosophical Transactions.* **179** (1888) 377.

[27] R.K. Gazizov and N.H. Ibragimov, Lie symmetry analysis of differential equations in finance, *Nonlinear Dyn.* **17** (1998) 387.

[28] J. Goard, New solutions to the bond-pricing equation via Lie's classical method, *Mathematical and Computer Modelling.* **32** (2000) 299.

[29] A. Gonzàles-Lòpez, N. Kamran and P. Olver, Lie algebras of vector fields in the real plane, *Proceedings of the London Mathematical Society.* **64**(3) (1992) 339.

[30] V.M. Gorringe and P.G.L. Leach, Lie point symmetries for systems of second order linear ordinary differential equations, *Quaestiones Mathematicae.* **11**(1) (1988) 95.

[31] G. Grebot, The characterization of third order ordinary differential equations admitting a transitive fibre preserving point symmetry group, *J. Math. Anal. Appl.* **206** (1997) 364.

[32] G. Grebot, The linearization of third order ODEs, Preprint, 1996.

[33] C. Grissom, G. Thompson and G. Wilkens, Linearization of second order ordinary differential equations via Cartan's equivalence method, *J. Differential Equations.* **77**(1) (1989) 1.

[34] G.H. Halphen, *Sur les invariants différentielles*, Thèse présentée *à* la Faculté de Sciences de Paris, 1878.

[35] N.H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, Chichester, Wiley, 1999.

[36] N.H. Ibragimov, Invariant Lagrangians and a new method of integration of nonlinear equations, *J. Math. Anal. Appl.* **304** (2005) 212.

[37] N.H. Ibragimov, Laplace type invariants for parabolic equations, *Nonlinear Dyn.* **28** (2002) 125.

[38] N.H. Ibragimov and Franco Magri, Geometric proof of Lie's linearization theorem, *Nonlinear Dyn.* **36** (2004) 41.

[39] N.H. Ibragimov, S.V. Meleshko, Linearization of third order ordinary differential equations by point transformations, *Archives of ALGA.* **1** (2004) 71.

[40] N.H. Ibragimov and S.V. Meleshko, Linearization of third order ordinary differential equations by point and contact transformations, *J. Math. Analy. Appl.* **308** (2005) 266.

[41] N.H. Ibragimov, S.V. Meleshko and S. Suksern, Linearization of fourth order ordinary differential equations by point transformations, *Archives of ALGA.* **4** (2007) 113.

[42] M. Janet, Les systèmes d'*é*quations aux dérivées partielles, *J. mathématiques.* **83** (1920) 65.

[43] I.K. Johnpillai and F.M. Mahomed, Singular invariant equation fro the (1+1) Fokker-Planck equation, *J. Phys. A: Math. Gen.* **34** (2001) 11033.

[44] E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen, I. Gewöhnliche Differentialgleichungen*, Akademische Verlagsgesellschaft, Leipzig, 1961.

[45] E. Laguerre, Sur les *é*quations diff*é*rentielles lin*é*aires du troisi*è*me ordre, *Comptes Rendus.* **88** (1879) 116.

[46] E. Laguerre, Sur quelques invariants des èquations diffèrentielles, *Comptes Rendus.* **88** (1879) 224.

[47] Z. Li, F. Schwarz and S. Tsarev, Factoring zero-dimensional ideals of linear partial differential operators, *Proceedings of the ISSAC, ACM Press.* **2** (2002) 168.

[48] S. Lie, Allgemeine theorie der partiellen differentialgleichungen erster ordnung, *Mathematische Annalen.* **11** (1877) 464 [*Gesammelte Abhandlungen.* **IV** 163].

[49] S. Lie, *Differential Equations*, Chelsea, New York, 1967.

[50] S. Lie, *Geometrie der Berührungstransformationen*, Teubner, Leipzig, 1896 [Reprinted by Chelsea Publishing Company, New York, 1977].

[51] S. Lie, Klassifikation und integration von gewöhnlichen differentialgleichungen zwischen x, y, die eine gruppe von transformationen gestatten I, II, II and IV, *Archiv for Mathematik.* **8** and **9** (1883) 187, 249, 371, 431. [*Gesammelte Abhandlungen.* **V** 240, 282, 362 and 432].

[52] S. Lie, *Lectures on differential equations with known infinitesimal transformations*, Teubner, Leipzig [In German, Lie's lectures by G. Sheffers, 1891].

[53] S. Lie, On integration of a class of linear partial differential equations by means of definite integrals, *Archiv for Mathematik og Naturvidenskab.* **VI**(3) (1881) 328.

[54] S. Lie, *Theorie der Transformationsgruppen I, II and III*, Teubner, Leipzig, 1888 [Reprinted by Chelsea Publishing Company, New York, 1970].

[55] S. Lie, *Ü*ber differentialinvarianten, *Mathematische Annalen.* **24** (1884) 537 [*Gesammelte Abhandlungen.* **VI** 95].

[56] S. Lie, *Ü*ber unendliche kontinuierliche gruppen, *Christ. Forh. Aar, Nr.* **12** (1883) [*Gesammelte Abhandlungen.* **V** 314].

[57] S. Lie, Verallgemeinerung und neue verwertung des jacobischen multiplikators, *Christ. Forb.* (1874) 255 [*Gesammelte Abhandlungen.* **III** 188].

[58] S. Lie, *Vorlesungen Über Continuierliche Gruppen*, Teubner, Leipzig, 1893 [Reprinted by Chelsea Publishing Company, New York, 1971].

[59] S. Lie, *Vorlesungen Über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen*, Teubner, Leipzig, 1891 [Reprinted by Chelsea Publishing Company, New York, 1967].

[60] A. Loewy, Über vollständig reduzible lineare homogene differentialgleichungen, *Mathematische Annalen.* **56** (1906) 89.

[61] F.M. Mahomed, Complete invariant characterization of scalar linear (1+1) parabolic equations, *J.N.M.P.* **15** (2008) 112.

[62] F.M. Mahomed, Symmetry group classification of ordinary differential equations: Survey of some results, *Math. Meth. Appl. Sci.* **30** (2007) 1995.

[63] F.M. Mahomed and C.A. Pooe, Invariant equations for (1+1) linear parabolic equations, *Proceedings of the International Conference Modern Group Analysis 9, 16th ISNA, Moscow State University, Moscow.* (2003) 525.

[64] F.M. Mahomed and A. Qadir, Higher dimensional systems of differential equations obtained by iterative use of complex methods, arXiv:1108.5251v1.

[65] F.M. Mahomed and Rehana Naz, A note on the Lie symmetries of complex partial differential equations and their split real systems, *Pramana - journal of physics.* **77**(3) (2011) 483.

[66] F.M. Mahomed and M. Safdar, Complete classification for systems of two second order ordinary differential equations by using complex symmetry analysis (work in progress).

[67] F.M. Mahomed, M. Safdar and J. Zama, Ibragimov-type Invariants for a System of Two Linear Parabolic Equations, *Commun. Nonlinear Sci. Numer. Simulat.* **17** (2012) 3140.

[68] F.M. Mahomed and P.G.L. Leach, Lie algebras associated with second order ordinary differential equations, *J. Math. Phy.* **30** (1989) 2770.

[69] F.M. Mahomed and P.G.L. Leach, Symmetry Lie algebra of nth order ordinary differential equations, *J. Math. Anal. Appl.* **151** (1990) 80.

[70] F.M. Mahomed and A. Qadir, Invariant linearization criteria for systems of cubically semi-linear second order ordinary differential equations, *J.N.M.P.* **16** (2009) 1.

[71] F.M. Mahomed and A. Qadir, Linearization for a system of two second order quadratically semi-linear ordinary differential equations, *NonLinear Dyn.* **48** (2007) 417.

[72] F.M. Mahomed, A. Qadir, M. Safdar, Lie-like operators and Lie point symmetries for two dimensional systems of second order ODEs, in progress.

[73] S.V. Meleshko, On linearization of third order ordinary differential equations, *J. Phys. A: Math. Gen. Math.* **39** (2006) 15135.

[74] J. Merker, Characterization of the Newtonian free particle system in $m \geq 2$ unknown dependent variables, *Acta Applicandae Mathematicae.* **92** (2006) 125.

[75] S. Moyo and P.G.L. Leach, Symmetry properties of autonomous integrating factors, *SIGMA.* **1** (2005) 024.

[76] S. Neut and M. Petitot, La géométrie de l'équation $y''' = f(x, y, y', y'')$, *C. R. Acad. Sci. Paris. I.* **335** (2002) 515.

[77] P. Olver, *Application of Lie Groups to Differential Equations*, Springer, Berlin, 1986.

[78] P. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, 1995.

[79] L.V. Ovsiannikov, *Group properties of differential equations*, Academic Press, New York, 1982.

[80] E. Picard, Sur les groupes de transformation des équations diffèrentielles linéaires, *Comptes Rendus.* **86** (1883) 1131.

[81] A.D. Polyanin and V.V. Zaitsev, *Handbuch der Linearen Differentialgleichungen*, Exakte Lösungen, Spektrum Akademischer Verlag, Heidelberg, 1996.

[82] C.A. Pooe, F.M. Mahomed and C. Wafo Soh, Fundamental solutions for zero-coupon bond pricing models, *Nonlinear Dyn.* **36** (2004) 69.

[83] A. Qadir, Linearization of ordinary differential equations by using geometry, *SIGMA.* **3** (2007) 1.

[84] H. Risken, *The Fokker-Planck Equation, Methods of Solution and Applications 2nd Edition*, Springer-Verlag, Berlin, 1989.

[85] P.L. Sachdev, *A Compendium on Nonlinear Ordinary Differential Equations*, Wiley-Interscience, New York, 1997.

[86] M. Safdar, A. Qadir and S. Ali, Inequivalence of classes of linearizable systems of cubically semi-linear ordinary differential equations obtained by real and complex symmetry analysis, *Math. Comp. Appl.* **16**(4) (2011) 923.

[87] M. Safdar, A. Qadir and S. Ali, Linearizability of systems of two second order ODEs using complex symmetry analysis, *Math. Prob. Engg.* **2011** (2011) 171834.

[88] M. Safdar, S. Ali and F.M. Mahomed, Linearization of systems of four second order ordinary differential equations, *Pramana - journal of physics.* **77**(3) (2011) 581.

[89] S. Ali, A. Qadir and M. Safdar, Symmetry solutions of two-dimensional systems not solvable by symmetry analysis, arXiv:1104.3837.

[90] W. Sarlet, F.M. Mahomed and P.G.L. Leach, Symmetries of nonlinear differential equations and linearization, *J Phys. A: Math. Gen.* **20** (1987) 277.

[91] F. Schwarz, *Algorithmic Lie Theory for Solving Ordinary Differential Equations*, Taylor and Francis Group, LLC, 2008.

[92] W.M. Shtelen and V.I. Stognii, Symmetry properties of one- and two-dimensional Fokker-Planck equations, *J. Phy. A: Math. Gen.* **22** (1989) L539.

[93] S. Spichak and V. Stognii, Symmetry classification and exact solutions of the one-dimensional Fokker-Planck equation with arbitrary coefficients of drifts and diffusion, *J. Phys. A: Math. Gen.* **32** (1999) 8341.

[94] H. Stephani, *Differential Equations: Their Solution Using Symmetries*, Cambridge University Press, Cambridge, UK, 1989.

[95] A. Tresse, *Détermination des Invariants Ponctuels de Léquation Différentielle Ordinaire du Second Ordre* $y'' = \omega(x, y, y')$ *Preisschriften der fürstlichen Jablonowski'fschen Geselschaft* ***XXXII***, Leipzig, Herzel, 1896.

[96] A. Tressé, Sur les Invariants differentiels des groupes continus de transformations, *Acta. Math.* **18** (1894) 1.

[97] C. Wafo Soh and F.M. Mahomed, Symmetry breaking for a system of two linear second order ordinary differential equations, *Nonlinear Dyn.* **22** (2000) 121.

[98] C. Wafo Soh and F.M. Mahomed, Linearization criteria for a system of second order ordinary differential equations, *Int. J. of NonLinear Mech.* **36** (2001) 671.

[99] P. Wilmott, *Derivatives: The theory and practice of financial engineering*, John Wiley, West Sussex England, 1998.

[100] H. Wussing, *Die Genesis des Abstrakten Gruppenbegriffes*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1969.