Some Advancements in Metric Fixed Point Theory

by

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Dedicated to my teacher
Professor Tayyab Kamran, Ph.D

Abstract

Banach contraction principle is one of the most famous results in the literature of fixed point theory and has provided the basis for metric fixed point theory. This result provides us a systematic way to find fixed point of a self mapping. Nadler extended the Banach contraction principle to multi valued mappings using the concept of Hausdorff metric spaces. The purpose of this dissertation is to introduce some more generalized results in the literature of metric fixed point theory. We introduce fixed point theorems for both single and multi valued mappings satisfying the weaker form of contraction conditions on the structure of metric spaces as well as some abstract spaces like, partial metric spaces, uniform spaces, gauge spaces and b-metric spaces.

Preface

This dissertation is based on four chapters. Chapter 1 mainly consist of preliminaries, which we need in the subsequent chapters.

In Chapter 2, we discuss few abstract spaces like uniform spaces, partial metric spaces, b-metric spaces and gauge spaces, together with those properties which we need to prove fixed point theorems at the last chapter.

In Chapter 3, we discuss some new fixed point theorems in metric spaces. This chapter consist of five sections. In the first section of this chapter, we mainly discuss, the error estimates of the iterative sequence for a multi valued mapping satisfying α-ψ-contractive type condition. The purpose of the second section is to show that the notion of modified multi valued α-ψ-contractive type mapping is a real generalization of the multi valued α-ψ-contractive type mapping. In the third section, we have a fixed point theorem for single valued mappings satisfying (α, ψ, ϕ)-contractive condition on a space with two metrics, which is a generalization/extension of α-ψ-contractive type mapping. In the fourth section, we combine the ideas of Semat *et al.* and Wardowski to introduce some new contraction conditions for multi valued mappings and prove corresponding fixed point theorems. The last section of this chapter consists of the existence and stability of best proximity points of nonself multi valued mappings satisfying proximal contraction condition on closed ball of a metric space, known as controlled proximal contraction.

In Chapter 4, we discuss some new fixed point theorems on the abstract spaces which are mentioned in Chapter 2. This chapter also contains five sections. In the first section, we discuss the α-ψ-contractive mappings and corresponding fixed point results in uniform spaces. In the second section, we have some fixed point theorems on partial metric spaces endowed with graph. In the third section, we study new F-contractions for multi valued mappings on b-metric spaces. The fourth section contains the notion of b-gauge spaces and some fixed point theorems on this new structure. In the last section, we discuss some fixed point theorems for Caristi type multi valued mappings on gauge spaces endowed with graph.

Contents

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- Muhammad Usman Ali, Tayyab Kamran and Mihai Postolache: Fixed point theorems for multi valued G-contractions in Hausdorff b-Gauge spaces. Journal of Nonlinear Sciences and Applications 8 (2015), 847-855.

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Chapter 1

Introduction and Preliminaries

The word fixed point of a self mapping $T : X \to X$ is used for a point $x \in X$ which remains unalter under the map, that is $x = Tx$ means $x = T(x)$. Throughout this chapter: (X, d) is metric space and it is assumed to be a complete metric space in all the results of this chapter. Further, throughout the dissertation: we denote the class of all nonempty closed and bounded subsets of X by $CB(X)$ and the class of all nonempty closed subsets of X by $CL(X)$.

In 1922 Banach [38] introduced a contraction principle which is the base of metric fixed point theory and known as Banach contraction principle. He proved that if a self mapping T satisfies the following condition on a complete (X, d), then it has a unique fixed point.

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for each $x, y \in X$, where $k \in [0, 1)$.

It had been observed that if a mapping T satisfies (1.1), then T must be continuous. Thus a question raised whether some contraction condition like (1.1) exists which guarantees the existence of fixed point of mapping, which may be a discontinuous. Kannan [83] and Cheatterja [51] gave positive answer to this question by introducing the following conditions:

$$d(Tx, Ty) \leq k(d(x, Tx) + d(y, Ty)) \quad \forall x, y \in X \tag{1.2}$$

and

$$d(Tx, Ty) \leq k(d(y, Tx) + d(x, Ty)) \quad \forall x, y \in X \tag{1.3}$$

where $k \in [0, \frac{1}{2})$, respectively and proved that if a mapping T satisfies (1.2) or (1.3) on complete (X, d) then it has a unique fixed point. After that we have a lot of extensions/generalizations of (1.1)-(1.3), which can be verified through the bibliography. It is inconvenient to discuss these works briefly. We mention only few but highly cited generalizations, as a tribute to all pioneers of this field.

In 1973 Geraghty [68] introduced the class B of functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0$$

and proved that if a mapping T satisfies the following condition on a complete (X,d), then it has a unique fixed point.

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y) \tag{1.4}$$

for each $x,y \in X$, where $\beta \in B$.

In 2002 Branciari [49] extended the contraction condition [38] as

$$\int_0^{d(Tx,Ty)} \phi(t)\ dt \leq k \int_0^{d(x,y)} \phi(t)\ dt$$

for each $x,y \in X$ and proved that if a mapping T satisfies the above condition on a complete (X,d), then it has a unique fixed point. Where $k \in [0,1)$ and $\phi : [0,\infty) \to [0,\infty)$ is Lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$ and for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)\ dt > 0$.

Jachymski [75] initiated the idea of combining metric fixed point theory and graph theory. He introduced the notion of Banach G-contraction on (X,d) endowed with the directed graph $G = (V,E)$ such that the set of vertex V coincides with X and the set of edges E contains $\{(x,x) : x \in X\}$, but has no parallel edges. Jachymski stated a mapping $T : X \to X$ is Banach G-contraction, if:

(i) $d(Tx,Ty) \leq kd(x,y)$ for each $(x,y) \in E$, where $k \in [0,1)$;

(ii) $(x,y) \in E \Rightarrow (Tx,Ty) \in E$.

Further showed that: if $T : X \to X$ is Banach G-contraction on complete (X,d) endowed with the directed graph $G = (V,E)$ and $(x_0, Tx_0) \in E$, then T has a fixed point, provided T is continuous or G-continuous.

In 2012, Samet *et al.* [128] gave a new contractive condition:

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$$

for each $x,y \in X$, where $\alpha : X \times X \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing mapping with $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for $t > 0$. By using this condition Samet *et al.* proved a fixed point theorem which generalizes Banach contraction principle.

Nadler [111] extended Banach contraction principle to multi valued mapping $T : X \to CB(X)$. He proved that if a mapping $T : X \to CB(X)$ satisfies the following condition on a complete (X,d) then T has a fixed point.

$$H(Tx,Ty) \leq kd(x,y) \tag{1.5}$$

for each $x,y \in X$, where $k \in [0,1)$ and H is Hausdorff metric induced by d, defined as:

$$H(A,B) = \max\left\{\sup_{a\in A}\left(\inf_{b\in B} d(a,b)\right), \sup_{b\in B}\left(\inf_{a\in A} d(b,a)\right)\right\} \text{ for each } A,B \in CB(X).$$

Reich [122] extended the Nadler's result and proved that if a mapping $T : X \to K(X)$, where $K(X)$ is class of all nonempty compact subset X, satisfies the following condition on complete (X, d), then T has a fixed point.

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y)$$

for each $x, y \in X$, where $\varphi : (0, \infty)$ into $[0, 1)$ is such that $\limsup_{r \to t^+} \varphi(r) < 1$, for each $t \in (0, \infty)$.

Reich [122] raised the question: whether the range of T, $K(X)$ can be replaced by $CB(X)$ or $CL(X)$. Mizoguchi and Takahashi [109] gave the positive answer to the conjecture of Reich [122], when the inequality holds also for $t = 0$. In particular, they proved if a mapping $T : X \to CB(X)$ satisfies the above condition on complete (X, d) then T has a fixed point.

Kamran [80] generalized Mizoguchi and Takahashi's theorem by introducing following condition.

$$d(y, Ty) \leq \varphi(d(x, y))d(x, y)$$

for each $x \in X$ and $y \in Tx$, where $\varphi : [0, \infty) \to [0, 1)$ is such that $\limsup_{r \to t^+} \varphi(r) < 1$, for each $t \in [0, \infty)$. With the help of this condition he successfully replaced range of T with $CL(X)$.

As we can see that the Banach contraction principle as well as the other metric fixed point theorems can be generalized by using weaker contraction conditions. This is not the only way to generalize the results of this theory. Another technique to generalize the Banach contraction principle or other metric fixed point theorems is to prove the analogue theorems by using the structure which is more general than metric space.

The preliminaries which we recollect here emphasis only on some weaker contraction conditions. We use/extend these contraction conditions in Chapter 3 and Chapter 4. While to discuss other technique of generalization, first we discuss some abstract spaces in Chapter 2 and then use these structures in Chapter 4.

1.1 α-admissibility and α-ψ-contractive type mappings

Samet *et al.* [128] introduced the notions of α-admissible and α-ψ-contractive type mappings and proved fixed point theorems by using these notions. The simplicity and applicability of these new notions attracted the attention of many researchers working in this area and many interesting fixed point theorems appeared in the literature, see for example [85, 35, 110, 34, 105, 126, 71]. In this section, we list only those generalizations/extensions of these notions which we required in this dissertation. Here, we emphasis on the notions rather than the statements of fixed point theorems unless required. Note that throughout this dissertation:

- α, η are functions from $X \times X$ into $[0, \infty)$;

- Ψ is the family of nondecreasing functions, $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ, [128].

Unless, otherwise stated.

Samet *et al.* [128] introduced the notions of α-admissible and α-ψ-contractive type mappings as follows:

Definition 1.1.1. [128] A mapping $T : X \to X$ is called α-admissible if

$$x, y \in X, \ \alpha(x,y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1. \tag{1.6}$$

Definition 1.1.2. [128] A mapping $T : X \to X$ is called α-ψ-contractive type mapping if for each $x, y \in X$, we have

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)) \tag{1.7}$$

where $\psi \in \Psi$.

By using these notions Samet *et al.* [128] obtained the following theorem.

Theorem 1.1.3. [128] Let $T : X \to X$ is an α-ψ-contractive type mapping satisfying the following conditions:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) a) T is continuous

or

b) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, fixed point of T is unique, if for each $x, y \in X$, there exists $\xi \in X$ such that $\alpha(x,\xi) \geq 1$ and $\alpha(y,\xi) \geq 1$.

One of the earlier generalization on α-ψ-contractive mapping was given by Karapinar *et al.* [85] in following theorem.

Theorem 1.1.4. [85] Let $T : X \to X$ be an α-admissible such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)) \tag{1.8}$$

for all $x, y \in X$, where $\psi \in \Psi$ and

$$M(x,y) = \left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Further, assume that T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Then T has a fixed point.

Salimi *et al.* [126] introduced the notions of α-admissible with respect to η and modified α-ψ-contractive type mappings and proved the corresponding fixed point theorems. Moreover, they claimed that these notions are more general than the notions of [128]. Following are the statements of the notions introduced by Salimi *et al.* [126].

Definition 1.1.5. [126] A mapping $T : X \to X$ is called α-admissible with respect to η, if

$$x, y \in X,\ \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Definition 1.1.6. [126] A mapping $T : X \to X$ is called modified α-ψ-contractive type mapping if

$$x, y \in X,\ \alpha(x, y) \geq \eta(x, y) \Rightarrow d(Tx, Ty) \leq \psi(d(x, y)) \tag{1.9}$$

where $\psi \in \Psi$.

Asl *et al.* [35] extended the notions of α-admissible and α-ψ-contractive mappings to multi valued mappings by introducing the following notions.

Definition 1.1.7. [35] A mapping $T : X \to CL(X)$ is called α_*-admissible if

$$x, y \in X,\ \alpha(x, y) \geq 1 \Rightarrow \alpha_*(Tx, Ty) \geq 1$$

where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$.

Definition 1.1.8. [35] A mapping $T : X \to CL(X)$ is called multi valued α_*-ψ-contractive type mapping if for each $x, y \in X$ we have

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y)) \tag{1.10}$$

where $\psi \in \Psi$.

By using these notions Asl *et al.* [35] proved the following theorem.

Theorem 1.1.9. [35] Let $\psi \in \Psi$ be a strictly increasing map, T be a closed-valued α_*-admissible and α_*-ψ-contractive multi function on X. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$, then $\alpha(x_n, x) \geq 1$ for all n. Then T has a fixed point.

Hussain *et al.* [71] extended the notions of α-admissible with respect to η and modified α-ψ-contractive type mappings to multi valued mappings in the following way.

Definition 1.1.10. [71] A mapping $T : X \to CL(X)$ is called α_*-admissible with respect to η if we have

$$x, y \in X,\quad \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty) \tag{1.11}$$

where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$, $\eta_*(Tx, Ty) = \sup\{\eta(a, b) : a \in Tx, b \in Ty\}$ and η is bounded.

Definition 1.1.11. [71] A mapping $T : X \to CL(X)$ is called modified multi valued α_*-ψ-contractive type mapping (or, multi valued α_*-η-ψ-contractive type mapping) if we have

$$x, y \in X,\ \alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty) \Rightarrow H(Tx, Ty) \leq \psi(d(x, y)) \tag{1.12}$$

where, $\psi \in \Psi$.

Meanwhile, Ali *et al.* [19] introduced the following notion which is more general than the notion given in Definition 1.1.10.

Definition 1.1.12. [19] A mapping $T : X \to CL(X)$ is called generalized α_*-admissible with respect to η, if we have

$$x, y \in X,\quad \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(u, v) \geq \eta(u, v)\ \forall\ u \in Tx \text{ and } v \in Ty. \tag{1.13}$$

1.2 Gauge function

Throughout this dissertation, J denotes an interval on $\mathbb{R}_+ = [0, \infty)$ containing 0, that is an interval of the form $[0, A], [0, A)$ or $[0, \infty)$ and $S_n(t)$ denotes the polynomial $S_n(t) = 1 + t + ... + t^{n-1}$. We use the abbreviation ψ^n for the nth iterate of a function $\psi : J \to J$.

Definition 1.2.1. [118] Let $r \geq 1$. A function $\psi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(i) $\psi(\lambda t) \leq \lambda^r \psi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;

(ii) $\psi(t) < t$ for all $t \in J - \{0\}$.

The first condition of Definition 1.2.1 is equivalent to: $\psi(0) = 0$ and $\psi(t)/t^r$ is nondecreasing on $J - \{0\}$.

Definition 1.2.2. [118] A nondecreasing function $\psi : J \to J$ is said to be a Bianchini-Grandolfi gauge function [44] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \psi^n(t) < \infty,\ \text{ for all } t \in J. \tag{1.14}$$

If a function $\psi : J \to J$ satisfying (1.14), then it also satisfies the following functional equation

$$\sigma(t) = \sigma(\psi(t)) + t. \tag{1.15}$$

Ptak [119] called this function a rate of convergence on J.

Remark 1.2.3. [118] Every gauge function of order $r \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.

Lemma 1.2.4. [118] Let ψ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying

$$\psi(t) = t\phi(t) \quad \text{for all} \quad t \in J \tag{1.16}$$

then the following properties hold:

(i) $0 \leq \phi(t) < 1$ for all $t \in J$;

(ii) $\phi(\lambda t) \leq \lambda^{r-1}\phi(t)$ for all $\lambda \in (0,1)$ and $t \in J$.

Moreover, for each $n \geq 0$ we have

(iii) $\psi^n(t) \leq t\phi(t)^{S_n(r)}$ for all $t \in J$,

(iv) $\phi(\psi^n(t)) \leq \phi(t)^{r^n}$ for all $t \in J$.

1.3 Comparison function

Let $\xi : [0,\infty) \to [0,\infty)$ be a function. First, consider the following conditions:

(i) ξ is increasing function;

(ii) $\xi(t) < t$ for each $t > 0$;

(iii) $\xi(0) = 0$;

(iv) $\{\xi^n(t)\}$ converges to 0 for each $t \geq 0$;

(v) $\sum_{n=0}^{\infty} \xi^n(t)$ converges for each $t > 0$.

The function ξ satisfies (i) and (iv) is said to be a comparison function [42]. The function ξ satisfies (i) and (v) is known as (c)-comparison function [42]. It is easily seen that (i) and (iv) imply (ii); and (i) and (ii) imply (iii) [42].

1.4 F-contractions

Throughout this dissertation, $\mathfrak{F}$ is the class of functions $F : (0,\infty) \to \mathbb{R}$ satisfying the following three assumptions, [134]:

(F_1) F is strictly increasing, that is, for each $a_1, a_2 \in (0,\infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$.

(F_2) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

(F_3) There exists $k \in (0,1)$ such that $\lim_{\mathfrak{d}\to 0^+} \mathfrak{d}^k F(\mathfrak{d}) = 0$.

Following are some examples of such functions.

- $F_a(x) = \ln x$ for each $x \in (0,\infty)$.
- $F_b(x) = x + \ln x$ for each $x \in (0,\infty)$.
- $F_c(x) = -\frac{1}{\sqrt{x}}$ for each $x \in (0,\infty)$.

Wardowski [134] introduced F-contraction and corresponding fixed point theorem in the following way:

Definition 1.4.1. [134] A mapping $T : X \to X$ is an F-contraction if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Remark 1.4.2. [134] Note that if T is F_a-contraction, then it is also Banach contraction. But it is not a case with F_b-contraction.

Theorem 1.4.3. [134] Let $T : X \to X$ be an F-contraction. Then T has a unique fixed point.

Secelean [127] showed that condition (F_2) can be replaced by one of the following condition which is equivalent to (F_2) but easy to handle.

(F_{2a}) $\inf F = -\infty$

or

(F_{2b}) there exists a sequence $\{\mathfrak{d}_n\}$ of positive numbers such that $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

Secelean concluded it on the bases of the following lemma.

Lemma 1.4.4. [127] Let $F : (0,\infty) \to \mathbb{R}$ be an increasing mapping and $\{\mathfrak{d}_n\}$ be a sequence of positive real numbers. Then the following conditions hold.

(i) if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$, then $\lim_{n\to\infty} \mathfrak{d}_n = 0$.

(ii) if $\inf F = -\infty$ and $\lim_{n\to\infty} \mathfrak{d}_n = 0$, then $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.

Minak *et al*. [108] generalized Theorem 1.4.3 in the following way:

Theorem 1.4.5. [108] Let $T : X \to X$ be a mapping for which there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F\Big(\max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}\Big)$$

for each $x, y \in X$ with $d(Tx, Ty) > 0$. If T or F is continuous, then T has a unique fixed point.

Sgroi and Vetro [131] generalized the result of [134] for multi valued mappings in following way:

Theorem 1.4.6. [131] Let $T : X \to CB(X)$, there exists $F \in \mathfrak{F}$ which is continuous from right and $\tau > 0$ such that

$$2\tau + F(H(Tx, Ty)) \leq F(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + Ld(y, Tx)) \quad (1.17)$$

for each $x, y \in X$ with $Tx \neq Ty$, where a_1, a_2, a_3, a_4, satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$, $a_3 \neq 1$ and $L \geq 0$. Then T has a fixed point.

1.5 Multi valued nonself mappings and best proximity points

Let A, B be subsets of (X, d). A point $x \in A$ is called fixed point of a mapping $T : A \to B$, if $x = Tx$. A mapping T has no fixed point if $A \cap B = \emptyset$. In this case $d(x, Tx) > 0$ for all $x \in A$. So, one can explore to find necessary condition so that the minimization problem

$$\min_{x \in A} d(x, Tx)$$

has at least one solution. A point $x^* \in X$ is said to be a best proximity point of mapping $T : A \to B$ if $d(x^*, Tx^*) = dist(A, B)$. When $A = B$, the best proximity point reduces to fixed point of the mapping T. The following well known best approximation theorem is due to Fan [65]. Before stating the result, we first recall that: A subset A of (X, d) is compact, if each sequence in A has a convergent subsequence. A subset A of X is convex if for each $\omega \in (0, 1)$ and $a_1, a_2 \in A$, we have $\omega a_1 + (1 - \omega)a_2 \in A$.

Theorem 1.5.1. [65] Let A be a nonempty compact convex subset of normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that

$$\|x - Tx\| = \inf_{a \in A}\{\|Tx - a\|\}.$$

In literature, we had found that several authors studied best proximity point for nonself mappings on metric spaces including [15, 16, 17, 104, 136, 103, 6, 29, 30, 40, 64, 77].

Now we recollect some notions, definitions and results, for ready references. Throughout this dissertation these notions have same meanings unless otherwise stated: $dist(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$, $d(x, B) = \inf\{d(x, b) : b \in B\}$, $A_0 = \{a \in A : d(a, b) = dist(A, B)$ for some $b \in B\}$, $B_0 = \{b \in B : d(a, b) = dist(A, B)$ for some $a \in A\}$, $CL(B)$ is the set of all nonempty closed subsets of B, $CB(B)$ is the set of all nonempty closed and bounded subsets of B and $B(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$.

Definition 1.5.2. [136] Let (A, B) be a pair of nonempty subsets of (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A$ and

$y_1, y_2 \in B$,

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Abkar and Gbeleh [7] studied the best proximity point of multi valued mappings and gave the following result:

Theorem 1.5.3. [7] Let A and B be two nonempty closed subsets of (X, d) such that A_0 is nonempty. Let $T : A \to CB(B)$ be a mapping satisfying the following conditions:

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) the pair (A, B) satisfies the P-property;

(iii) there exists $\alpha \in (0, 1)$ such that for each $x, y \in A$, we have $H(Tx, Ty) \leq \alpha d(x, y)$.

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = dist(A, B)$.

Basic Lemmas

The following lemmas have vital role in the proof of fixed point theorems for multi valued mappings, we use these lemmas in Chapter 3.

Lemma 1.5.4. [111] Let $B \in CL(X)$ and $x \in X$. Then for each $\epsilon > 0$, there exists $b \in B$ such that $d(x, b) \leq d(x, B) + \epsilon$.

Lemma 1.5.5. [80] Let $B \in CL(X)$ and $x \in X$. Then for each $q > 1$, we have $b \in B$ such that $d(x, b) \leq qd(x, B)$.

Lemma 1.5.6. [18] Let $B \in CL(X)$ and $x \in X$ with $d(x, B) > 0$. Then for each $q > 1$, there exists an element $b \in B$ such that

$$d(x, b) < qd(x, B). \tag{1.18}$$

Chapter 2

Some Abstract Spaces

Metric space is one of the essential and frequently used structure in nonlinear analysis. In this chapter, we discuss some abstract spaces, that is, uniform spaces, partial metric spaces, b-metric spaces and gauge spaces. We discuss these abstract spaces along with the properties that make these spaces special than metric space. This chapter provides the basis for Chapter 4, where we study fixed point theorems by considering these spaces.

2.1 Uniform spaces

Let X be a nonempty set. A nonempty family, ϑ, of subsets of $X \times X$ is called a uniform structure on X if it satisfies the following properties:

(i) if G is in ϑ, then G contains the diagonal $\{(x, x) | x \in X\}$;

(ii) if G is in ϑ and H is a subset of $X \times X$ which contains G, then H is in ϑ;

(iii) if G and H are in ϑ, then $G \cap H$ is in ϑ;

(iv) if G is in ϑ, then there exists H in ϑ, such that, whenever (x, y) and (y, z) are in H, then (x, z) is in G;

(v) if G is in ϑ, then $\{(y, x) : (x, y) \in G\}$ is also in ϑ.

The pair (X, ϑ) is called a uniform space [135] and the element of ϑ is called entourage or neighborhood or surrounding. The pair (X, ϑ) is called a quasi uniform space [48, 135] if property (v) is omitted.

Let $\Delta = \{(x, x) : x \in X\}$ be the diagonal of a nonempty set X. For $V, W \in X \times X$, we shall use the following setting in the sequel

$$V \circ W = \{(x, y) : \text{ there exists } z \in X \text{ with } (x, z) \in W \text{ and } (z, y) \in V\}$$

and

$$V^{-1} = \{(x, y) : (y, x) \in V\}.$$

For a subset $V \in \vartheta$, a pair of points x and y are called V-closed if $(x, y) \in V$ and $(y, x) \in V$. Moreover, a sequence $\{x_n\}$ in X is called a Cauchy sequence for ϑ, if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-closed for $n, m \geq N$. For (X, ϑ), there is a unique topology $\tau(\vartheta)$ on X generated by $V(x) = \{y \in X : (x, y) \in V\}$ where $V \in \vartheta$.

A sequence $\{x_n\}$ in X is convergent to x for ϑ, denoted by $\lim_{n\to\infty} x_n = x$, if for any $V \in \vartheta$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0$. A uniform space (X, ϑ) is called Hausdorff if the intersection of all $V \in \vartheta$ is equal to Δ of X, that is, if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. If $V = V^{-1}$ then we shall say that a subset $V \in \vartheta$ is symmetrical. Throughout the thesis, we shall assume that each $V \in \vartheta$ is symmetrical. For more details, see e.g. [1, 2, 3, 11, 48, 72].

Throughout this section (X, ϑ) is a uniform space. Now, we shall recall the notions of A-distance and E-distance.

Definition 2.1.1. [1, 2] A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an A-distance on (X, ϑ) if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2.1.2. [1, 2] A function $p : X \times X \longrightarrow [0, \infty)$ is said to be an E-distance on (X, ϑ) if

(i) p is an A-distance,

(ii) $p(x, y) \leq p(x, z) + p(z, y), \ \forall x, y, z \in X$.

Example 2.1.3. [1, 2] Let (X, ϑ) be a uniform space and let d be a metric on X. It is evident that (X, ϑ_d) is a uniform space where ϑ_d is a set of all subsets of $X \times X$ containing a "band" $U_\epsilon = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E-distance on (X, ϑ).

Lemma 2.1.4. [1, 2] Let p be an A-distance on (X, ϑ) which is Hausdorff. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. Then, for $x, y, z \in X$, the following results hold:

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

(c) If $p(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is Cauchy sequence in (X, ϑ).

Let p be an A-distance. A sequence in (X, ϑ) with an A-distance is said to be a p-Cauchy if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

Definition 2.1.5. [1, 2] Let p be an A-distance on (X, ϑ). Then:

(i) X is S-complete if for every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} p(x_n, x) = 0$.

(ii) X is p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$, there exists x in X with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$.

(iii) $T : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(T(x_n), T(x)) = 0$.

Remark 2.1.6. Let (X, ϑ) be a Hausdorff and S-complete. If a sequence $\{x_n\}$ be a p-Cauchy sequence, then we have $\lim_{n\to\infty} p(x_n, x) = 0$. Regarding Lemma 2.1.4(b), We derive that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ and hence S-completeness implies p-Cauchy completeness.

2.2 Partial metric spaces

Matthews [102] introduced the notion of partial metric space and extended Banach contraction principle in the setting of partial metric space. The work of Matthews [102] has been extended by many authors, see for example [32, 33, 31, 78, 52, 57, 69, 120, 4, 112, 79, 132, 74, 98, 84, 123, 89, 90]. Using the notion of partial metric on a set X, Aydi *et al.* [37] defined a partial Hausdorff metric on the set of all closed and bounded subsets of X. Moreover, they [37] extended Nadler's fixed point theorem in the setting of a partial Hausdorff metric spaces.

We recollect the following definitions, notions and lemmas for partial(Hausdorff) metric spaces from Matthews [102] and Aydi *et al.* [37].

Definition 2.2.1. [102] A mapping $p : X \times X \to [0, \infty)$ is a partial metric on X, if for all $x, y, z \in X$ the following conditions hold:

(P1) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) = p(y, x)$;

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Remark 2.2.2. [37] If $p(x, y) = 0$ then $(P1)$ and $(P2)$ implies $x = y$ but converse is not true in general.

Example 2.2.3. [102] Let X be the set of all closed intervals of real line $\mathbb{R}$ that is, $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and define a function $p : X \times X \to [0, \infty)$ by, $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$, then (X, p) is a partial metric space.

Lemma 2.2.4. [102] Every metric space is a partial metric space.

Remark 2.2.5. [102] Every partial metric p on X generates a T_0 topology τ_p on X with base as the family of the open balls (p balls) $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}.$$

Definition 2.2.6. [102] Let (X,p) be a partial metric space. Then:

(a) A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ with respect to τ_p if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(b) A sequence $\{x_n\}$ in (X,p) will be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and finite.

(c) (X,p) is a complete partial metric space if each Cauchy sequence $\{x_n\}$ in X converges to some point $x \in X$ with respect to τ_p and $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Throughout this section we assume that (X,p) be a partial metric space.

Remark 2.2.7. [102] The function $d_p : X \times X \to [0,\infty)$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Lemma 2.2.8. [102] A sequence $\{x_n\}$ in (X,d_p) converges to $x \in X$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 2.2.9. [37] Let (X,p) be a partial metric space. Then:

(a) A sequence $\{x_n\}$ in X is Cauchy with respect to p if and only if it is Cauchy with respect to d_p.

(b) (X,p) is complete if and only if the metric space (X,d_p) is complete.

A subset A of (X,p) is a bounded [37], if there exists $x_0 \in A$ such that $p(x_0,a) < p(x_0,x_0) + M$ for each $a \in A$, where $M > 0$. A subset A of (X,p) is closed if it is closed with respect to the topology τ_p on X. Let $CB_p(X)$ denotes the family of all nonempty closed and bounded subsets of (X,p). For $A,B \in CB_p(X)$, $p(x,A) = \inf\{p(x,a) : a \in A\}$, $p(A,B) = \inf\{p(x,y) : x \in A, y \in B\}$. The functions $\delta_p : CB_p(X) \times CB_p(X) \to [0,\infty)$ and $H_p : CB_p(X) \times CB_p(X) \to [0,\infty)$ are defined by $\delta_p(A,B) = \sup\{p(a,B) : a \in A\}$ and $H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$, respectively.

Remark 2.2.10. [37] If $d_p(x,A) = \inf\{d_p(x,a) : a \in A\}$, then it is easy to see that $p(x,A) = 0$ implies that $d_p(x,A) = 0$.

Lemma 2.2.11. [37] Let A is a subset of (X,p), then we have

(i) $a \in \overline{A}$ if and only if $p(a, A) = p(a, a)$.

(ii) If A is closed in (X, p) then A is closed in (X, d_p).

Proposition 2.2.12. [37] For $A, B \in CB_p(X)$, the following properties hold:

(1) $H_p(A, A) \leq H_p(A, B)$;

(2) $H_p(A, B) = H_p(B, A)$;

(3) $H_p(A, C) \leq H_p(A, B) + H_p(B, C) - \inf_{c \in C} p(c, c)$;

(4) $H_p(A, B) = 0$ implies that $A = B$.

Lemma 2.2.13. [37] Let $A, B \in CB_p(X)$ and $h > 1$. Then for any $a \in A$, there exists $b \in B$ such that $p(a, b) \leq h H_p(A, B)$.

2.3 b-metric space

Czerwik [60] introduced the notion of b-metric space. Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric on X, if for each $x, y, z \in X$, there exists $s \geq 1$ satisfying the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The triplet (X, d, s) is said to be a b-metric space. Note that every metric space is a b-metric but converse is not true. Following are some interesting examples of b-metric spaces which are not metric spaces.

Example 2.3.1. [60] Let $p \in (0, 1)$ and $l^p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ endowed with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$ defined by

$$d(\{x_n\}, \{y_n\}) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$$

for each $\{x_n\}, \{y_n\} \in l^p(\mathbb{R})$, is a b-metric space with $s = 2^{1/p}$.

Example 2.3.2. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ is defined by $d(x, y) = |x - y|^2$ for each $x, y \in X$. Clearly $(X, d, 2)$ is a b-metric space, but not a metric space.

Convergence of a sequence in b-metric space is defined in a similar fashion as in metric space. A sequence $\{x_n\} \subseteq X$ is Cauchy sequence in (X, d, s), if for each $\epsilon > 0$ there exists a natural number $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for each $m, n \geq N(\epsilon)$. A b-metric space (X, d, s) is complete if each Cauchy sequence in X converges to some point of X. Note that each convergent sequence in b-metric space has a unique limit point, [60, 14].

The closed and bounded sets in b-metric space, that is (X, d, s), are defined in a similar manner as for a metric space. Thus, throughout the dissertation: For $A, B \in CB(X)$, the function $H : CB(X) \times CB(X) \to [0, \infty)$ defined by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

is said to be a Hausdorff b-metric[61] induced by the b-metric d. For $A, B \in CL(X)$, the function $H : CL(X) \times CL(X) \to [0, \infty)$ defined by

$$H(A, B) = \begin{cases} \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

is said to be a generalized Hausdorff b-metric induced by b-metric d. A Hausdorff b-metric space enjoys the same properties as a Hausdorff metric, expect for the triangular inequality which in Hausdorff b-metric spaces has the form $H(A, B) \leq s[H(A, C) + H(C, B)]$.

Czerwik [61] proved the following results for Hausdorff b-metric spaces.

Lemma 2.3.3. [61] Let (X, d, s) be a b-metric space. For any $A, B \in CL(X)$ and any $x \in X$, we have the following properties:

(a) For $h > 1$ and $a \in A$, there exists $b \in B$ such that $d(a, b) \leq hH(A, B)$.

(b) $d(x, A) = 0 \Leftrightarrow x \in \overline{A} = A$, where $\overline{A}$ denotes the closure of the set A.

(c) $d(x, A) \leq d(x, a)$ for each $a \in A$.

(d) $d(a, B) \leq H(A, B)$ for each $a \in A$.

Czerwik [61] also proved the Nadler's fixed point theorem in the setting of Hausdorff b-metric spaces.

2.4 Gauge spaces

In this section, due to Dugundji [63], we state the spaces which have topology induced by family of pseudo metric spaces. Further, we discuss the condition in which these spaces are Hausdorff. We also discuss the convergence and Cauchyness of sequences in these spaces. First, we recall the pseudo metric space.

Definition 2.4.1. [63] Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called pseudo metric on X, if for each $x, y, z \in X$, we have

(i) $d(x, x) = 0$ for each $x \in X$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, x)$.

Let X be a nonempty set endowed with the pseudo metric d. Then d-ball [63] of radius $\epsilon > 0$ center at $x \in X$ is the set

$$B(x, d, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

Definition 2.4.2. [63] Let X be a nonempty set and $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ be a family of pseudo metrics on X. The topology $\mathfrak{T}(\mathfrak{F})$ having subbases the family

$$\mathfrak{B}(\mathfrak{F}) = \{B(x, d_\nu, \epsilon) : x \in X, d_\nu \in \mathfrak{F} \text{ and } \epsilon > 0\}$$

of balls is called topology induced by the family $\mathfrak{F}$ of pseudo metrics. The pair $(X, \mathfrak{T}(\mathfrak{F}))$ is called a gauge space.

Definition 2.4.3. [63] A family $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ of pseudo metrics is said to be separating if for each pair (x, y) with $x \neq y$, there exists $d_\nu \in \mathfrak{F}$ with $d_\nu(x, y) \neq 0$.

Note that a gauge space is Hausdorff if $\mathfrak{F}$ is separating.

Definition 2.4.4. [63] Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a gauge space with respect to the family $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ of pseudo metrics on X. If $\{x_n\}$ is a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ converges to x if for each $\nu \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n, x) < \epsilon$ for each $n \geq N_0$. We denote it as $x_n \to^{\mathfrak{F}} x$.

(ii) The sequence $\{x_n\}$ is a Cauchy sequence if for each $\nu \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n, x_m) < \epsilon$ for each $n, m \geq N_0$.

(iii) $(X, \mathfrak{T}(\mathfrak{F}))$ is complete if each Cauchy sequence in $(X, \mathfrak{T}(\mathfrak{F}))$ is convergent in X.

(iv) A subset of X is said to be closed if it contains the limit of each convergent sequence of its elements.

Chapter 3

Fixed Point Theorems in Metric Spaces

This chapter consists of five sections. In first four sections, we investigate the existence of fixed points for multi valued/single valued mappings satisfying contraction type conditions on metric spaces. In the last section of this chapter, we investigate the existence of best proximity points for nonself multi valued mappings satisfying proximal contraction type condition on metric spaces. Each section of this chapter contains some examples to elaborate the corresponding results. Here the nonempty set X is endowed with a metric d, that is, (X,d) is a metric space. Further, in results it is considered as a complete metric space. Unless otherwise stated.

3.1 Fixed point theorems for multi valued mappings involving α-function, with error estimates

Proinov [118] investigated the order of convergence of iterative sequences of a mapping satisfying the Hick and Rhoades [70] type contraction condition. By using this technique we investigate the order of convergence of iterative sequences for a multi valued mapping satisfying the contraction condition (3.1) or (3.25). These conditions can be viewed as extensions of (1.7).

Theorem 3.1.1. Let $T : X \to CL(X)$ be an α_*-admissible mapping such that

$$\alpha(x,y)H(Tx,Ty) \leq \psi(d(x,y)), \tag{3.1}$$

for all $x \in X$ and $y \in Tx$ with $d(x,y) \in J$, where ψ is a Bianchini-Grandolfi gauge function on an interval J. Moreover, the strict inequality holds when $d(x,y) \neq 0$. Suppose that there exists $x_0 \in X$ such that $d(x_0,z) \in J$ and $\alpha(x_0,z) \geq 1$ for some $z \in Tx_0$. Then

(i) there exists an orbit $\{x_n\}$ of T in X and $\xi \in X$ such that $\lim_n x_n = \xi$;

(ii) ξ is fixed point of T if and only if the function $f(x) := d(x,Tx)$ is T-orbitally lower semi-continuous at ξ.

Note that, if, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, then $O(T, x_0) = \{x_0, x_1, x_2, \cdots\}$ is said to be an orbit of $T : X \to CL(X)$. A mapping $f : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous at ξ [80], if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \xi$ implies $f(\xi) \leq \liminf_n f(x_n)$.

Proof. Consider $x_1 = z \in Tx_0$. We assume that $d(x_0, x_1) \neq 0$, for otherwise x_0 is a fixed point of T. Define $\rho_0 = \sigma(d(x_0, x_1))$, where σ is defined by (1.14). Since from (1.15), $\sigma(t) \geq t$, we have

$$d(x_0, x_1) \leq \rho_0. \tag{3.2}$$

Notice that $x_1 \in B(x_0, \rho_0)$. It follows from (3.1) that $\alpha(x_0, x_1)H(Tx_0, Tx_1) < \psi(d(x_0, x_1))$. By hypothesis, we have $\alpha(x_0, x_1) \geq 1$. We can choose an $\epsilon_1 > 0$ such that

$$\alpha(x_0, x_1)H(Tx_0, Tx_1) + \epsilon_1 \leq \psi(d(x_0, x_1)).$$

Thus, we have

$$\begin{aligned} d(x_1, Tx_1) + \epsilon_1 &\leq H(Tx_0, Tx_1) + \epsilon_1 \\ &\leq \alpha(x_0, x_1)H(Tx_0, Tx_1) + \epsilon_1 \\ &\leq \psi(d(x_0, x_1)). \end{aligned} \tag{3.3}$$

It follows from Lemma 1.5.4 that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq d(x_1, Tx_1) + \epsilon_1. \tag{3.4}$$

We assume that $d(x_1, x_2) \neq 0$, for otherwise x_1 is a fixed point of T. From inequalities (3.3) and (3.4), we have

$$d(x_1, x_2) \leq \psi(d(x_0, x_1)). \tag{3.5}$$

Note that $d(x_1, x_2) \in J$. Also, we have $x_2 \in B(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \psi(d(x_0, x_1)) \\ &\leq d(x_0, x_1) + \sigma(\psi(d(x_0, x_1))) \\ &= \sigma(d(x_0, x_1)) \quad \text{(by using (1.15))} \\ &= \rho_0. \end{aligned}$$

Since T is an α_*-admissible, then we have $\alpha(x_1, x_2) \geq 1$. Now choose $\epsilon_2 > 0$ such that

$$\alpha(x_1, x_2)H(Tx_1, Tx_2) + \epsilon_2 \leq \psi(d(x_1, x_2)).$$

Thus, we have

$$\begin{aligned} d(x_2, Tx_2) + \epsilon_2 &\leq H(Tx_1, Tx_2) + \epsilon_2 \\ &\leq \alpha(x_1, x_2)H(Tx_1, Tx_2) + \epsilon_2 \\ &\leq \psi(d(x_1, x_2)). \end{aligned} \tag{3.6}$$

It again follows from Lemma 1.5.4 that there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq d(x_2, Tx_2) + \epsilon_2. \tag{3.7}$$

We assume that $d(x_2, x_3) \neq 0$, for otherwise x_2 is a fixed point of T. From (3.5), (3.6) and (3.7), we have

$$d(x_2, x_3) \leq \psi^2(d(x_0, x_1)). \tag{3.8}$$

Note that $d(x_2, x_3) \in J$. Also, we have $x_3 \in B(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq d(x_0, x_1) + \psi(d(x_0, x_1)) + \psi^2(d(x_0, x_1))) \\ &\leq \sum_{j=0}^{\infty} \psi^j(d(x_0, x_1)) \\ &= \sigma(d(x_0, x_1)) = \rho_0. \end{aligned}$$

Repeating the above argument, inductively we obtain the a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n \in Tx_{n-1}, \tag{3.9}$$

$$\alpha(x_{n-1}, x_n) \geq 1, \tag{3.10}$$

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \tag{3.11}$$

$$d(x_{n-1}, x_n) \in J, \text{ and } x_n \in B(x_0, \rho_0). \tag{3.12}$$

We claim that $\{x_n\}$ is Cauchy. For $n, p \in \mathbb{N}$, from (3.11) we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \psi^n(d(x_0, x_1)) + \cdots + \psi^{n+p-1}(d(x_0, x_1)) \\ &\leq \sum_{j=n}^{\infty} \psi^j(d(x_0, x_1)). \end{aligned} \tag{3.13}$$

By using (1.14), it follows from (3.13) that $\{x_n\}$ is Cauchy. Thus, there exists $\xi \in B(x_0, \rho_0)$ with $x_n \to \xi$ as $n \to \infty$. Since $x_n \in Tx_{n-1}$, from (3.1), (3.10) and (3.11), we have

$$\begin{aligned} d(x_n, Tx_n) &\leq \alpha(x_{n-1}, x_n) H(Tx_{n-1}, Tx_n) \\ &\leq \psi(d(x_{n-1}, x_n)) \\ &\leq \psi^n(d(x_0, x_1)). \end{aligned} \tag{3.14}$$

Letting $n \to \infty$, from (3.14), we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.15}$$

Suppose $f(x) = d(x, Tx)$ is T-orbitally lower semi-continuous at ξ, then

$$d(\xi, T\xi) = f(\xi) \leq \liminf_n f(x_n) = \liminf_n d(x_n, Tx_n) = 0.$$

Hence, $\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is fixed point of T then $f(\xi) = 0 \leq \lim_n \inf f(x_n)$. □

Example 3.1.2. Let $X = [-100, \infty)$ be endowed with the usual metric d and let $J = [0, \infty)$. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} [0, \frac{x}{3}] & \text{if } x \geq 0 \\ [x, 0] & \text{otherwise,} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

Take $\psi(t) = \frac{t}{2}$ for each $t \geq 0$. Let $x_0 = 1$, then we have $z = \frac{1}{3} \in Tx_0$ such that $d(x_0, z) \in J$ and $\alpha(x_0, z) = 1$. As we know that $\alpha(x, y) = 1$ for $x, y \in [0, \infty)$. Then, we have $\alpha_*(Tx, Ty) = 1$ whenever $\alpha(x, y) = 1$. Thus, T is an α_*-admissible mapping. For $x \geq 0$ and $y \in Tx$, from (3.1), we have

$$\alpha(x, y)H(Tx, Ty) = \frac{1}{3}|x - y| \leq \frac{1}{2}|x - y| = \psi(d(x, y)),$$

for $x < 0$ and $y \in Tx$, we have

$$\alpha(x, y)H(Tx, Ty) = 0 \leq \frac{1}{2}|x - y| = \psi(d(x, y)).$$

Hence (3.1) holds for each $x \in X$ and $y \in Tx$ with $d(x, y) \in J$. Therefore, all the conditions of Theorem 3.1.1 hold and hence T has a fixed point.

Example 3.1.3. Let $X = [-1, \infty)$ be endowed with the usual metric d and let $J = [0, \infty)$. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} [-1, \frac{x}{3}] & \text{if } x \in [-1, 0) \\ [0, x^2] & \text{if } x \in [0, \frac{3}{5}] \\ [x, e^{x^2}] & \text{if } x \in (\frac{3}{5}, \infty), \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, \frac{3}{5}] \\ 0 & \text{otherwise.} \end{cases}$$

Take $\psi(t) = \frac{24}{25}t$ for each $t \geq 0$. Let $x_0 = \frac{3}{5}$, then we have $z = \frac{9}{25} \in Tx_0$ such that $d(x_0, z) \in J$ and $\alpha(x_0, z) = 1$. As we know that $\alpha(x, y) = 1$ for $x, y \in [0, \frac{3}{5}]$. Then, we have $\alpha_*(Tx, Ty) = 1$ whenever $\alpha(x, y) = 1$. Thus, T is an α_*-admissible mapping. For $x \in [0, \frac{3}{5}]$ and $y \in Tx$, from (3.1), we have

$$\alpha(x, y)H(Tx, Ty) \leq \Big(\frac{3}{5} + \frac{9}{25}\Big)|x - y| = \frac{24}{25}|x - y| = \psi(d(x, y)),$$

for otherwise, we have

$$\alpha(x, y)H(Tx, Ty) = 0 \leq \frac{24}{25}|x - y| = \psi(d(x, y)).$$

Hence (3.1) holds for each $x \in X$ and $y \in Tx$ with $d(x, y) \in J$. Therefore, all the conditions of Theorem 3.1.1 hold and hence T has a fixed point.

Theorem 3.1.4. Let $T : X \to CL(X)$ be an α_*-admissible mapping such that

$$\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y)), \tag{3.16}$$

for all $x \in X$ and $y \in Tx$ with $d(x, y) \in J$, where ψ is a gauge function of order $r \geq 1$ on an interval J and $\phi : J \to \mathbb{R}^+$ is a nondecreasing function defined by (1.16). Moreover, the strict inequality holds when $d(x, y) \neq 0$. Suppose that there exists $x_0 \in X$ such that $d(x_0, z) \in J$ and $\alpha(x, z) \geq 1$ for some $z \in Tx_0$. Then

(i) there exists an orbit $\{x_n\}$ of T in $B(x_0, \rho_0)$ that converges with rate of convergence at least r to a point $\xi \in B(x_0, \rho_0)$, where $\rho_0 = \sigma(d(x_0, x))$ and σ is defined by (1.14);

(ii) for all $n \geq 0$, we have the following a prior estimate

$$d(x_n, \xi) \leq \frac{\lambda^{S_n(r)} d(x_0, x_1)}{1 - \lambda^{r^n}}, \tag{3.17}$$

where $\lambda = \phi(d(x_0, x_1))$;

(iii) for all $n \geq 1$, we have the following a posterior estimate

$$d(x_n, \xi) \quad \leq \quad \frac{\psi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}; \tag{3.18}$$

(iv) for all $n \geq 1$, we have

$$d(x_n, x_{n+1}) \leq \lambda^{S_n(r)} d(x_0, x_1), \tag{3.19}$$

where $\lambda = \phi(d(x_0, x_1))$;

(v) ξ is fixed point of T if and only if the function $f(x) := d(x, Tx)$ is T-orbitally lower semi-continuous at ξ.

Proof. (i) Following the proof of Theorem 3.1.1, we have an orbit $\{x_n\}$ of T at x_0 in $B(x_0, \rho_0)$ such that $\lim_{n\to\infty} x_n = \xi$ and $\xi \in B(x_0, \rho_0)$.
(ii) For $m > n$, by using (3.11) and Lemma 1.2.4-(iii), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq \psi^n(d(x_0, x_1)) + \psi^{n+1}(d(x_0, x_1)) + \cdots + \psi^{m-1}(d(x_0, x_1)) \\ &\leq d(x_0, x_1)[\lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \ldots + \lambda^{S_{m-1}(r)}] \\ &= d(x_0, x_1) \sum_{j=n}^{m-1} \lambda^{S_j(r)}. \end{aligned}$$

Taking n fixed and letting $m \to \infty$, we get

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} \lambda^{S_j(r)}. \tag{3.20}$$

Note that,

$$\sum_{j=n}^{\infty} \lambda^{S_j(r)} = \lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots$$
$$= \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n+r^{n+1}} + \lambda^{r^n+r^{n+1}+r^{n+2}} + \cdots].$$

Since $r \geq 1$, therefore

$$r^n + r^{n+1} \geq 2r^n, \quad r^n + r^{n+1} + r^{n+2} \geq 3r^n \cdots,$$

and

$$\lambda^{r^n+r^{n+1}} \leq \lambda^{2r^n}, \quad \lambda^{r^n+r^{n+1}+r^{n+2}} \leq \lambda^{3r^n} \cdots,$$

since $0 \leq \lambda < 1$. Thus, we have

$$\sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + \cdots] = \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

Substituting this in (3.20), we get

$$d(x_n, \xi) \leq d(x_0, x_1) \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

(iii) For $n \geq 0$, from (3.20), we have

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} [\phi(d(x_0, x_1))]^{S_j(r)}.$$

Putting $n = 0, y_0 = x_n$ and $y_1 = x_1$, we have

$$d(y_0, \xi) \leq d(y_0, y_1) \sum_{j=0}^{\infty} [\phi(d(y_0, y_1))]^{S_j(r)}.$$

Putting $y_0 = x_n$, and $y_1 = x_{n+1}$, we have

$$d(x_n, \xi) \leq d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \tag{3.21}$$
$$\leq \psi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\psi(d(x_n, x_{n-1})))]^{S_j(r)}$$
$$\leq \psi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\psi(d(x_n, x_{n-1})))]^{j}$$
$$= \frac{\psi(d(x_n, x_{n-1}))}{1 - \phi(\psi(d(x_n, x_{n-1})))}, \tag{3.22}$$

since $S_j(r) \geq j$. Now by Lemma 1.2.4-(iv), we have

$$\phi(\psi(d(x_n, x_{n-1}))) \leq [\phi(d(x_n, x_{n-1}))]^r$$

which means that,

$$\frac{1}{1-\phi(\psi(d(x_n,x_{n-1})))} \leq \frac{1}{1-[\phi(d(x_n,x_{n-1}))]^r}. \tag{3.23}$$

For $n \geq 1$, from (3.21), we have

$$\begin{aligned} d(x_n,\xi) &\leq \psi(d(x_n,x_{n-1}))\sum_{j=0}^{\infty}[\phi(\psi(d(x_n,x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\psi(d(x_n,x_{n-1}))}{1-\phi(\psi(d(x_n,x_{n-1})))} \\ &\leq \frac{\psi(d(x_n,x_{n-1}))}{1-[\phi(d(x_n,x_{n-1}))]^r} \quad \text{(by using (3.23))}. \end{aligned}$$

(iv) For $n \geq 1$, by using (3.11) and Lemma 1.2.4-(iii), we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq \psi^n(d(x_1,x_0)) \\ &\leq d(x_0,x_1)\phi(d(x_0,x_1))^{S_n(r)} \\ &= d(x_0,x_1)\lambda^{S_n(r)} \end{aligned}$$

(v) Proof is similar as in the proof of Theorem 3.1.1. □

Corollary 3.1.5. Let $T : X \to CL(X)$ be an α_*-admissible mapping such that

$$\alpha(x,y)H(Tx,Ty) \leq \psi(d(x,y)), \tag{3.24}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x,y) \in J$, where ψ is a gauge function of order $r \geq 1$ on an interval J. Suppose that there exists x_0 in X such that $d(x_0,z) \in J$ and $\alpha(x_0,z) \geq 1$ for some $z \in Tx_0$. Suppose that for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n-1},x_n) \geq 1$ for each $n \in \mathbb{N}$, then $\alpha(x_n,x) \geq 1$ for each $n \in \mathbb{N}$. Then following statements hold:

(i) there exists an orbit $\{x_n\}$ of T in $B(x_0,\rho_0)$ that converges to a fixed point $\xi \in B(x_0,\rho_0)$, where $\rho_0 = \sigma(d(x_0,x))$ and σ is defined by (1.14);

(ii) the estimates *(3.17) - (3.19)* are valid.

Theorem 3.1.6. Let $T : X \to CL(X)$ be a continuous and α_*-admissible mapping such that

$$\alpha(x,y)H(Tx,Ty) \leq \psi(m(x,y)), \quad \text{for all } x \in X \text{ and } y \in Tx, \tag{3.25}$$

with strict inequality holds if $m(x,y) \neq 0$, where ψ is a gauge function of the first order on $J = [0,\infty)$ and

$$m(x,y) = \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\Big\}.$$

Suppose that there exists $x_0 \in X$ such that $d(x_0,z) \in J$ and $\alpha(x_0,z) \geq 1$ for some $z \in Tx_0$. Then following statements hold:

(i) there exists an orbit of T in X that converges to a fixed point of T;

(ii) for all $n \geq 0$, we have the following a prior estimate

$$d(x_n, \xi) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1),$$

where $\lambda = \phi(d(x_0, x_1))$ and $\phi : J \to \mathbb{R}^+$ is a nondecreasing function defined by (1.16);

(iii) for all $n \geq 1$, we have the following a posterior estimate

$$d(x_n, \xi) \leq \frac{\psi(d(x_n, x_{n-1}))}{1 - \phi[\psi(d(x_n, x_{n-1}))]}.$$

Proof. Consider $x_1 = z \in Tx_0$. Define $\rho_0 = \sigma(d(x_0, x_1))$, where σ is defined by (1.14). Since from (1.15), $\sigma(t) \geq t$, we have

$$d(x_0, x_1) \leq \rho_0. \tag{3.26}$$

Assume that $m(x_0, x_1) \neq 0$, for otherwise $d(x_0, Tx_0) \leq m(x_0, x_1) = 0$ and x_0 is a fixed point of T. From (3.25), we have $\alpha(x_0, x_1)H(Tx_0, Tx_1) < \psi(m(x_0, x_1))$. By hypothesis, we have $\alpha(x_0, x_1) \geq 1$. We can choose $\epsilon_1 > 0$ such that

$$\alpha(x_0, x_1)H(Tx_0, Tx_1) + \epsilon_1 \leq \psi(m(x_0, x_1)).$$

Thus, we have

$$\begin{aligned} d(x_1, Tx_1) + \epsilon_1 &\leq H(Tx_0, Tx_1) + \epsilon_1 \\ &\leq \alpha(x_0, x_1)H(Tx_0, Tx_1) + \epsilon_1 \\ &\leq \psi(m(x_0, x_1)). \end{aligned}$$

It follows from Lemma 1.5.4 that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq d(x_1, Tx_1) + \epsilon_1. \tag{3.27}$$

From last two inequalities, we have

$$\begin{aligned} d(x_1, x_2) &\leq \psi(m(x_0, x_1)) \\ &= \psi\Big(\max\Big\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \\ &\qquad \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\Big\}\Big) \\ &= \psi(\max\{d(x_0, x_1), d(x_1, Tx_1)\}), \end{aligned} \tag{3.28}$$

since $\frac{d(x_0,Tx_1)}{2} \leq \max\{d(x_0, x_1), d(x_1, Tx_1)\}$. Assume that $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$. From (3.28), we have

$$d(x_1, Tx_1) \leq d(x_1, x_2) \leq \psi(d(x_1, Tx_1)).$$

Which is not possible. Thus $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1)$. From (3.28), we have

$$d(x_1, x_2) \leq \psi(d(x_0, x_1)).$$

Proceeding inductively in a similar way as in Theorem 3.1.1, we obtain the sequence $\{x_n\}$ in X such that $x_n \to \xi \in X$ as $n \to \infty$. Since T is continuous, by taking limit as $n \to \infty$, we have $\xi \in T\xi$. Estimates (3.17) and (3.18) becomes (3.26) and (3.26) for $r = 1$. □

Remark 3.1.7. Note that our results generalize [35, Theorem 2.1]; [110, Theorem 3.4]; [118, Theorems 4.1 and 4.2; and Corollary 4.5]; [91, Theorem 2.1 and 2.8; and Corollary 2.12]; [62, Theorem 2.1]; [93, Theorems 2.11 and 2.15]; and [10, Theorems 2.1 and 2.2].

3.2 Discussion on modified multi valued α_*-ψ-contractive type mapping

Samet *et al.* [128] introduced the (single valued) α-ψ-contractive mappings via α-admissible self mappings. In this interesting paper [128], the authors examined the existence and uniqueness of a fixed point for such mappings in the frame of complete metric space. This is one of the significant reports in the recent decade, since the announced results of the paper [128] concluded several existing fixed point results, including well-known Banach contraction mapping principle, as corollaries. Following this initial paper, a number of publications appeared on this subject, see e.g. Karapinar and Samet [85], Salimi *et al*. [126, 125], Asl *et al*. [35], Hussain *et al*. [71, 73], Mohammadi *et al*. [110], Amiri *et al*. [34], Minak and Altun [105], Alikhani *et al*. [28]. Among all, we mention the result of Salimi *et al.* [126] in which the authors introduced the notion of modified α-ψ-contractive mappings by the help of another auxiliary function η. As it is expected, the authors [126] established some fixed point theorems for such (single valued) mappings in the setting of complete metric spaces. Later, Mohammadi and Rezapour [107] and independently, Berzig and Karapınar [43], noticed that modified (single valued) α-ψ-contractive type mappings can be considered as a particular case of α-ψ-contractive type mappings. After this observation, it is quite natural to ask that whether the analog of the results of Mohammadi and Rezapour [107], Berzig and Karapınar [43] in the case of multi valued α_*-ψ-contractive type mapping can be obtained.

In this section, we show that the notion of modified multi valued α_*-ψ-contractive type mapping (also called as, multi valued α_*-η-ψ-contractive type mapping) can not be reduced into multi valued α_*-ψ-contractive type mapping. In other words, the notion of multi valued α_*-η-ψ-contractive type mappings is a proper generalization of the concept of multi valued α_*-ψ-contractive type mappings. In addition, we investigate the existence of common fixed points for a sequence of multi valued α_*-η-ψ-contractive type mappings.

As it is mentioned above, in [107, 43], the authors have pointed out the fact that the notion of modified (single valued) α-ψ-contractive type mappings can be considered as a particular case of the concept of (single valued) α-ψ-contractive type mappings. Hence, the announced results in [126] coincide with the related fixed point results of Samet *et al.* [128], and Karapınar *et al.* [85]. More precisely, in [107, 43] the authors showed that if we define

$$\beta(x,y) = \begin{cases} 1, & \text{if } \alpha(x,y) \geq \eta(x,y); \\ 0, & \text{otherwise,} \end{cases}$$

then (1.9) becomes

$$\text{for } x,y \in X \ \beta(x,y)d(Tx,Ty) \leq \psi(d(x,y)). \tag{3.29}$$

Further, T is β-admissible. If we look at (3.29), we see that indeed we have two cases.

(i) when $\beta = 1$ we have

$$d(Tx,Ty) \leq \psi(d(x,y)). \tag{3.30}$$

(ii) when $\beta = 0$ we have

$$0 \cdot d(Tx,Ty) \leq \psi(d(x,y)), \ i.e.\ 0 \leq \psi(d(x,y)). \tag{3.31}$$

Here note that $d(x,y) < \infty$ for all $x,y \in X$. Therefore, from

$$0 \cdot d(Tx,Ty) \leq \psi(d(x,y))$$

we get

$$0 \leq \psi(d(x,y)).$$

By considering the remarks in [107, 43], at the first glance, one would expect that the notion of modified multi valued α_*-η-ψ-contractive type mapping should be a particular case of the concept of multi valued α_*-ψ-contractive type mapping. On the other hand, if we look carefully at the contractive conditions (1.10) and (1.12), we see that it depends upon the metric H. We observe that if we consider a map $T : X \to CB(X)$ then $H(Tx,Ty) < \infty$ for all $x,y \in X$. In this case the fixed point theorems for multi valued α_*-η-ψ-contractive type mappings may be followed from the corresponding theorems for multi valued α_*-ψ-contractive type mappings. Note that if T is a single valued map, then $H(Tx,Ty) = d(x,y) < \infty$, for all $x,y \in X$. This is inconsistent with the observations in [107, 43]. For the case $T : X \to CL(X)$, the value of $H(Tx,Ty)$ may be infinite for some choice of $x,y \in X$. Consequently, a multi valued α_*-η-ψ-contractive type mapping may not imply a multi valued α_*-ψ-contractive type mapping, in general. Indeed, if one would define

$$\beta(x,y) = \begin{cases} 1, & \text{if } \alpha(x,y) \geq \eta(x,y); \\ 0, & \text{otherwise,} \end{cases}$$

then (1.12) appears to reduce into

$$\text{for } x,y \in X \ \beta(x,y)H(Tx,Ty) \leq \psi(d(x,y)). \tag{3.32}$$

Now if we look at (3.32) we again have two cases:

(i) when $\beta(x, y) = 1$ we have

$$H(Tx, Ty) \leq \psi(d(x, y)),$$

(ii) when $\beta(x, y) = 0$ we have

$$0 \cdot H(Tx, Ty) \leq \psi(d(x, y)). \tag{3.33}$$

Now here is the point; when T is bounded then

$$0 \cdot H(Tx, Ty) \leq \psi(d(x, y))$$

implies that $0 \leq \psi(d(x, y))$. Otherwise, it is not true. In other words, when T is not bounded it is not possible to define β in above manner. Following example substantiate our claim.

Example 3.2.1. Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} (-\infty, x] & \text{if } x < 0 \\ [\frac{x}{2}, \infty) & \text{if } x \geq 0 \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\eta : X \times X \to [0, \infty)$ by $\eta(x, y) = \frac{1}{2}$ for each $x, y \in X$. Take $\psi(t) = \frac{t}{2}$ for each $t \geq 0$. If $x, y \geq 0$, then $\alpha_*(Tx, Ty) = 1 > \eta_*(Tx, Ty) = \frac{1}{2}$ which implies

$$H(Tx, Ty) = \frac{1}{2}|x - y| = \psi(d(x, y))$$

for otherwise, we have $\alpha_*(Tx, Ty) = 0 < \eta_*(Tx, Ty) = \frac{1}{2}$. Thus T is modified α_*-η-ψ-contractive type mapping. Observe that $\alpha_*(T(-1), T(1)) = 0$ and $H(T(-1), T(1)) = \infty$. Thus (1.10) doesn't holds when $x = -1, y = 1$ and consequently T is not a multi valued α_*-ψ-contractive type mapping.

Therefore, it is worthwhile to consider fixed point theorems for multi valued α_*-η-ψ-contractive type mappings.

Now, we move towards the second task of this section, which is to investigate the existence of a common fixed point theorem for a sequence of multi valued α_*-η-ψ-contractive type mappings.

Definition 3.2.2. Let $\{T_i : X \to CL(X)\}_{i=1}^{\infty}$ be a sequence of multi valued mappings on (X, d). Let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. We say that the sequence $\{T_i\}$ is α_*-admissible with respect to η, if we have

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(u, v) \geq \eta(u, v) \ \forall \ u \in T_i x \text{ and } v \in T_j y, \tag{3.34}$$

for each $i, j \in \mathbb{N}$. In case when $\alpha(x, y) = 1$ for all $x, y \in X$, the sequence $\{T_i\}$ is a η_*-subadmissible. In case when $\eta(x, y) = 1$ for all $x, y \in X$, the sequence $\{T_i\}$ is α_*-admissible.

Theorem 3.2.3. Let the sequence $\{T_i : X \to CL(X)\}_{i=1}^{\infty}$ be an α_*-admissible with respect to η such that

$$x, y \in X,\ \alpha(x,y) \geq \eta(x,y) \Rightarrow H(T_i x, T_j y) \leq \psi(d(x,y)), \tag{3.35}$$

for each $i, j \in \mathbb{N}$ and ψ be a strictly increasing function in Ψ. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $y_i \in T_i x_0$ for each $i \in \mathbb{N}$ such that $\alpha(x_0, y_i) \geq \eta(x_0, y_i)$;

(ii) if $\{x_i\}$ is a sequence in X with $x_i \to x$ and $\alpha(x_{i-1}, x_i) \geq \eta(x_{i-1}, x_i)$ for each $i \in \mathbb{N}$, then we have $\alpha(x_{i-1}, x) \geq \eta(x_{i-1}, x)$ for each $i \in \mathbb{N}$.

Then, the mappings T_i for $i \in \mathbb{N}$, have a common fixed point.

Proof. By hypothesis, there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. If $x_1 \in T_i x_1$ for each $i \in \mathbb{N}$, then x_1 is a common fixed point of T_i. Let $x_1 \notin T_2 x_1$. Then from (3.35), we have

$$0 < d(x_1, T_2 x_1) \leq H(T_1 x_0, T_2 x_1) \leq \psi(d(x_0, x_1)). \tag{3.36}$$

For $q > 1$ by Lemma 1.5.6, there exists $x_2 \in T_2 x_1$ such that

$$0 < d(x_1, x_2) < q d(x_1, T_2 x_1) \leq q H(T_1 x_0, T_2 x_1) \leq q\psi(d(x_0, x_1)). \tag{3.37}$$

Since, ψ is strictly increasing, from (3.37), we have

$$\psi(d(x_1, x_2)) < \psi(q\psi(d(x_0, x_1))). \tag{3.38}$$

Put $q_1 = \frac{\psi(q\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$. Then $q_1 > 1$. Since the sequence $\{T_i\}_{i=1}^{\infty}$ is α_*-admissible with respect to η, then $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$. If $x_2 \in T_i x_2$ for each $i \in \mathbb{N}$, then x_2 is a common fixed point of T_i. Let $x_2 \notin T_3 x_2$. Then from (3.35), we have

$$0 < d(x_2, T_3 x_2) \leq H(T_2 x_1, T_3 x_2) \leq \psi(d(x_1, x_2)). \tag{3.39}$$

For $q_1 > 1$ by Lemma 1.5.6, there exists $x_3 \in T_3 x_2$ such that

$$\begin{aligned} 0 < d(x_2, x_3) &< q_1 d(x_2, T_3 x_2) \\ &\leq q_1 H(T_2 x_1, T_3 x_2) \\ &\leq q_1 \psi(d(x_1, x_2)) = \psi(q\psi(d(x_0, x_1))). \end{aligned} \tag{3.40}$$

Since, ψ is strictly increasing, from (3.40), we have

$$\psi(d(x_2, x_3)) < \psi^2(q\psi(d(x_0, x_1))). \tag{3.41}$$

Put $q_2 = \frac{\psi^2(q\psi(d(x_0,x_1)))}{\psi(d(x_2,x_3))}$. Then $q_2 > 1$. Continuing in the same way, we get a sequence $\{x_i\}$ in X such that $x_i \in T_i x_{i-1}$, $x_i \neq x_{i-1}$, $\alpha(x_{i-1}, x_i) \geq \eta(x_{i-1}, x_i)$ and

$$d(x_i, x_{i+1}) < \psi^{i-1}(q\psi(d(x_0, x_1))) \text{ for each } i \in \mathbb{N}. \tag{3.42}$$

Let $j > i$, we have

$$d(x_i, x_j) \leq \sum_{n=i}^{n=j-1} d(x_n, x_{n+1}) < \sum_{n=i}^{n=j-1} \psi^{n-1}(q\psi(d(x_0, x_1))).$$

Since $\psi \in \Psi$, then we have

$$\lim_{i,j\to\infty} d(x_i, x_j) = 0. \tag{3.43}$$

Hence $\{x_{i-1}\}$ is Cauchy in (X, d). By completeness of (X, d), there exists $x^* \in X$ such that $x_{i-1} \to x^*$ as $i \to \infty$. By hypothesis (ii), we have $\alpha(x_{i-1}, x^*) \geq \eta(x_{i-1}, x^*)$ for each $i \in \mathbb{N}$. From (3.35), for each $n = 1, 2, \cdots$, we have

$$d(x_i, T_n x^*) \leq H(T_i x_{i-1}, T_n x^*) \leq \psi(d(x_{i-1}, x^*)).$$

Letting $i \to \infty$ in above inequality, we have $d(x^*, T_n x^*) = 0$ for each $n \in \mathbb{N}$. Thus, x^* is a common fixed point of $\{T_i\}$. □

Let us take $T_i = T$ for each $i \in \mathbb{N}$, then Theorem 3.2.3 reduces to following result:

Theorem 3.2.4. Let $T : X \to CL(X)$ be a generalized α_*-admissible mapping with respect to η such that

$$x, y \in X, \ \alpha(x, y) \geq \eta(x, y) \Rightarrow H(Tx, Ty) \leq \psi(d(x, y)), \tag{3.44}$$

where ψ is strictly increasing function in Ψ. Assume that following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;

(ii) if $\{x_i\}$ is a sequence in X with $x_i \to x$ and $\alpha(x_{i-1}, x_i) \geq \eta(x_{i-1}, x_i)$ for each $i \in \mathbb{N}$, then we have $\alpha(x_{i-1}, x) \geq \eta(x_{i-1}, x)$ for each $i \in \mathbb{N}$.

Then, T has a fixed point.

Example 3.2.5. Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} (-\infty, 0] & \text{if } x < 0 \\ \{0, \frac{x}{4}\} & \text{if } 0 \leq x \leq 2 \\ [x^2, \infty) & \text{if } x > 2, \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} \frac{4}{5} & \text{if } x, y \in [0, 2] \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

and $\eta : X \times X \to [0, \infty)$ by $\eta(x, y) = \frac{3}{4}$ for each $x, y \in X$. Take $\psi(t) = \frac{t}{2}$ for each $t \geq 0$. Then, for each $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$, we have

$$H(Tx, Ty) = \frac{1}{4}|x - y| \leq \psi(d(x, y)).$$

Also, T is generalized α_*-admissible mapping with respect to η. For $x_0 = 1$ and $0 \in Tx_0$ we have $\alpha(1,0) > \eta(1,0)$. Moreover, for any sequence $\{x_i\}$ in X with $x_i \to x$ and $\alpha(x_{i-1}, x_i) \geq \eta(x_{i-1}, x_i)$ for each $i \in \mathbb{N}$, we have $\alpha(x_{i-1}, x) \geq \eta(x_{i-1}, x)$ for each $i \in \mathbb{N}$. Therefore, all conditions of Theorem 3.2.4 are satisfied and T has infinitely many fixed points.

Remark 3.2.6. For $T : X \to CL(X)$, contraction condition given in (3.44) is more general than contraction conditions of following form:

$$\alpha(x,y)H(Tx,Ty) \leq \psi(d(x,y)), \tag{3.45}$$

for each $x, y \in X$, where ψ is strictly increasing function in Ψ.

For example, consider T, α, η as defined in Example 3.2.5. Define $\beta : X \times X \to [0, \infty)$ by

$$\beta(x,y) = \begin{cases} 1 & \text{if } \alpha(x,y) \geq \eta(x,y) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$. For $x = 2$ and $y = 2.1$, from (3.45), we have

$$\beta(x,y)H(Tx,Ty) = 0.\infty.$$

As $0.\infty$ is indeterminant form, there is no guaranty, that (3.45) holds for each $x, y \in X$.

3.3 Fixed point theorem for (α, ψ, ϕ)-contractive mappings on spaces with two metrics

The result of this section is due to an inspirational work of Maia [101], Agarwal and O' Regan [12], where we have fixed point theorems on space with two metrics. Here, we discuss a fixed point theorem for (α, ψ, ϕ)-contractive mappings on space with two metrics, which is a generalization of results by Kannan [83], Samet *et al.* [128] and Karapinar *et al.* [85]. Note that here, Φ is a family of functions, $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is continuous and $\phi(0) = 0$.

Definition 3.3.1. A mapping $T : X \to X$ is called (α, ψ, ϕ)-contractive mapping on (X, d), if there exist three functions $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)) + \phi(N(x,y)), \tag{3.46}$$

for each $x, y \in X$, where

$M(x,y) = \max\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$,

and

$N(x,y) = \min\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

Lemma 3.3.2. Let (X, d') be a metric space, d be another metric on X and let $T : (X, d) \to (X, d')$ is uniformly continuous whenever $d' > d$. If $\{x_n\} = \{Tx_{n-1}\}_{n \in \mathbb{N}}$ is Cauchy sequence with respect to d, then $\{x_n\}$ is Cauchy sequence with respect to d'.

Proof. If $d' \leq d$, then trivially $\{x_n\}$ is Cauchy sequence in (X, d'). Next suppose that $d' \nleq d$. Then from uniform continuity of T, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d'(Tx, Ty) < \epsilon \quad \text{whenever } d(x, y) < \delta \text{ for each } x, y \in X. \tag{3.47}$$

As $\{x_n\}$ is Cauchy sequence in (X, d), there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \delta \quad \text{whenever } n, m \geq N. \tag{3.48}$$

From (3.47) and (3.48), we have

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \epsilon \quad \text{whenever } n, m \geq N. \tag{3.49}$$

This shows that $\{x_n\}$ is Cauchy sequence in (X, d'). □

Theorem 3.3.3. Let (X, d') be a complete metric space, d be another metric on X. Suppose that $T : X \to X$ be an (α, ψ, ϕ)-contractive mapping with respect to d. In addition assume that the following conditions hold:

(i) T is an α-admissible map and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(ii) if $d' > d$, then $T : (X, d) \to (X, d')$ is uniformly continuous;

(iii) if $d' \neq d$, then $T : (X, d') \to (X, d')$ is continuous;

(iv) for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_{n-1}, x_n) \geq 1$ for each $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for each $k \in \mathbb{N}$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then from (3.46) by taking $x_1 = Tx_0$ and $x_2 = Tx_1$, we have

$$\begin{aligned}
d(x_1, x_2) &= d(Tx_0, Tx_1) \\
&\leq \alpha(x_0, x_1) d(Tx_0, Tx_1) \\
&\leq \psi(M(x_0, x_1)) + \phi(N(x_0, x_1)) \\
&= \psi\left(\max\left\{d(x_0, x_1), \frac{d(x_0, Tx_0) + d(x_1, Tx_1)}{2}, \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\}\right) \\
&\quad + \phi\left(\min\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), d(x_0, Tx_1), d(x_1, Tx_0)\right\}\right) \\
&= \psi\left(\max\left\{d(x_0, x_1), \frac{d(x_0, x_1) + d(x_1, x_2)}{2}, \frac{d(x_0, x_2)}{2}\right\}\right) + 0 \\
&= \psi(\max\{d(x_0, x_1), d(x_1, x_2)\}),
\end{aligned} \tag{3.50}$$

since $\frac{d(x_0,x_1)+d(x_1,x_2)}{2} \leq \max\{d(x_0, x_1), d(x_1, x_2)\}$ and $\frac{d(x_0,x_2)}{2} \leq \max\{d(x_0, x_1), d(x_1, x_2)\}$. If $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$ then using (3.50), we reach at a contradiction. Thus $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1)$. Hence from (3.50), we have

$$d(x_1, x_2) \leq \psi((x_0, x_1)). \tag{3.51}$$

As T is α-admissible map, by continuing in the same way, we get a sequence $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$ with

$$\alpha(x_{n-1}, x_n) \geq 1, \tag{3.52}$$

and

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^n(d(x_0, x_1)). \tag{3.53}$$

For $n, m \in \mathbb{N}$, we have

$$d(x_n, x_{n+m}) \leq \sum_{i=n}^{n+m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{n+m-1} \psi^i(d(x_0, x_1)). \tag{3.54}$$

Since $\psi \in \Psi$. Hence $\{x_n\}$ is Cauchy in (X, d). From assumption (ii) and Lemma 3.3.2 it follows that $\{x_n\}$ is Cauchy in (X, d'). By the completeness of (X, d'), we have $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We shall show that $x^* = Tx^*$. Assume $x^* \neq Tx^*$. Suppose $d' = d$, then from triangular inequality and condition (iv), we have

$$\begin{aligned}
d(x^*, Tx^*) &\leq d(x^*, x_{n_k+1}) + d(Tx_{n_k}, Tx^*) \\
&\leq d(x^*, x_{n_k+1}) + \alpha(x_{n_k}, x^*) d(Tx_{n_k}, Tx^*) \\
&= d(x^*, x_{n_k+1}) + \psi\Big(\max\Big\{ d(x_{n_k}, x^*), \frac{d(x_{n_k}, Tx_{n_k}) + d(x^*, Tx^*)}{2}, \\
&\quad \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2} \Big\}\Big) + \phi(\min\{d(x_{n_k}, x^*), \\
&\quad d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), d(x_{n_k}, Tx^*), d(x^*, Tx_{n_k})\} \\
&< d(x^*, x_{n_k+1}) + \max\Big\{ d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + d(x^*, Tx^*)}{2}, \\
&\quad \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1})}{2} \Big\} + \phi(\min\{d(x_{n_k}, x^*), \\
&\quad d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), d(x_{n_k}, Tx^*), d(x^*, x_{n_k+1})\}.
\end{aligned} \tag{3.55}$$

Assume that $\beta_k = \min\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), d(x_{n_k}, Tx^*), d(x^*, x_{n_k+1})\}$. Note that as $k \to \infty$, $\beta_k \to 0$ and continuity of ϕ further implies that $\phi(\beta_k) \to \phi(0) = 0$. Now letting $k \to \infty$ in (3.55), we have

$$d(x^*, Tx^*) \leq \frac{d(x^*, Tx^*)}{2}. \tag{3.56}$$

This is a contraction to our assumption. Hence $d(x^*, Tx^*) = 0$. Next consider, $d' \neq d$. Then from triangular inequality, we have

$$d'(x^*, Tx^*) \leq d'(x^*, x_n) + d'(x_n, Tx^*) = d'(x^*, x_n) + d'(Tx_{n-1}, Tx^*). \tag{3.57}$$

Letting $n \to \infty$ in (3.57) and using assumption (iii), we obtain $d'(x^*, Tx^*) = 0$. $\square$

Example 3.3.4. Let $X = (0, \infty)$ be a metric space endowed with metrics

$$d'(x, y) = \begin{cases} |x - y| + 1 & \text{if } x \text{ or } y \text{ or both } x, y \in (0, 1) \\ 0 & \text{if } x = y \in (0, 1) \\ |x - y| & \text{otherwise,} \end{cases}$$

and $d(x,y) = |x-y|$ for each $x,y \in (0,\infty)$. Clearly, X is a complete metric space with respect to metric d'. Define $T : X \to X$ by $Tx = \frac{x+3}{2}$ for each $x \in X$ and $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = 1$ for each $x,y \in X$. Consider $\psi(t) = \frac{t}{2}$ and $\phi(t) = t$ for each $t \geq 0$. Then it is easy to see that T is an (α,ψ,ϕ)-contractive mapping with respect to d and all the other conditions of Theorem 3.3.3 hold. Thus T has a fixed point, which is 3.

Remark 3.3.5. We have the following consequences:

- Theorem 2.2 of Samet *et al.* [128] is a special case of Theorem 3.3.3, when $d = d'$ and $\phi(t) = 0$ for each $t \geq 0$.
- Theorem 2.4 of Karapinar *et al.* [85] is also a special case of Theorem 3.3.3, when $d = d'$ and $\phi(t) = 0$ for each $t \geq 0$.

Thus, we conclude that Theorem 3.3.3 generalizes [128, Theorem2.2] and [85, Theorem 2.4] and those contain therein.

3.4 Multi valued F-contractions and related fixed point theorems with an application

Wardowski [134] introduced a new family of mappings known as F or $\mathfrak{F}$ family. Using the mappings from $\mathfrak{F}$ family he introduced a new contraction condition called F-contraction. This F-contraction nicely generalize the most famous contraction condition, that is, Banach contraction condition. Later on, Acar and Altun [8], Arshad *et al*. [4], Batra and Vashistha [39], Cosentino and Vetro [59], Minak *et al*. [108], Paesano and Vetro [115], Sgroi and Vetro [131], Piri and Kumam [117], Secelean [127], extended the result of Wardowski [134] in different settings. In this section, by combining the ideas of Semat *et al*. [128] and Wardowski [134], we introduce some new contraction conditions for multi valued mappings and prove corresponding fixed point theorems. We also show that many new results in different settings can be obtained from our results. Furthermore, as an application of our result we establish an existence theorem for integral equations. Recall that $\alpha : X \times X \to [0,\infty)$ be a function and $\alpha_*(A,B) = \inf\{\alpha(a,b) : a \in A \text{ and } b \in B\}$.

By slightly modifying the definitions given in [110] and [35], we get the following definitions.

Definition 3.4.1. A mapping $T : X \to CB(X)$ is strictly α-admissible if for each $x \in X$ and $y \in Tx$ such that $\alpha(x,y) > 1$, we have $\alpha(y,z) > 1$ for each $z \in Ty$.

Definition 3.4.2. A mapping $T : X \to CB(X)$ is strictly α_*-admissible mapping if for each $x,y \in X$ with $\alpha(x,y) > 1$, we have $\alpha_*(Tx,Ty) > 1$.

Remark 3.4.3. Note that if a mapping $T : X \to CB(X)$ is strictly α_*-admissible, then it is strictly α-admissible. Converse is not true in general.

Example 3.4.4. Let $X = [-1, 1]$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{0, 1\} & \text{if } x = -1 \\ \{1\} & \text{if } x = 0 \\ \{-x\} & \text{if } x \notin \{-1, 0\} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \neq y. \end{cases}$$

Following the details of [105, Example 1], it is straight forward to see that T is strictly α-admissible but not α_*-admissible.

Now, we state our first F-contraction condition. Then we investigate the existence of fixed points for the mapping satisfying this contraction condition.

Definition 3.4.5. A mapping $T : X \to CB(X)$ is α-F-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(N(x, y)), \tag{3.58}$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)H(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 3.4.6. Let $T : X \to CB(X)$ be an α-F-contraction of Hardy-Rogers-type satisfying the following conditions:

(i) T is strictly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (ii), there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. As $\alpha(x_0, x_1) > 1$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1). \tag{3.59}$$

Since F is strictly increasing, we have

$$F(d(x_1, x_2)) \leq F(\alpha(x_0, x_1)H(Tx_0, Tx_1)). \tag{3.60}$$

From (3.58), we have

$$\begin{aligned}
\tau + F(d(x_1, x_2)) &\leq \tau + F(\alpha(x_0, x_1)H(Tx_0, Tx_1)) \\
&\leq F\Big(a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) + a_3 d(x_1, Tx_1) + \\
&\quad a_4 d(x_0, Tx_1) + L d(x_1, Tx_0)\Big) \\
&\leq F\Big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + \\
&\quad a_4 d(x_0, x_2) + L.0\Big) \\
&\leq F\Big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + \\
&\quad a_4 (d(x_0, x_1) + d(x_1, x_2))\Big) \\
&= F\Big((a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, x_2)\Big). \qquad (3.61)
\end{aligned}$$

Since F is strictly increasing, we get from above that

$$d(x_1, x_2) < (a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, x_2).$$

That is,

$$(1 - a_3 - a_4) d(x_1, x_2) < (a_1 + a_2 + a_4) d(x_0, x_1).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_1, x_2) < d(x_0, x_1).$$

Now, from (3.61), we have

$$\tau + F(d(x_1, x_2)) \leq F(d(x_0, x_1)).$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Let $x_2 \notin Tx_2$. Since, T is strictly α-admissible, we have $\alpha(x_1, x_2) > 1$. There exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \alpha(x_1, x_2) H(Tx_1, Tx_2). \qquad (3.62)$$

Since, F is strictly increasing, we have

$$F(d(x_2, x_3)) \leq F(\alpha(x_1, x_2) H(Tx_1, Tx_2)). \qquad (3.63)$$

From (3.58), we have

$$\begin{aligned}\tau + F(d(x_2, x_3)) &\leq \tau + F(\alpha(x_1, x_2)H(Tx_1, Tx_2)) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2 d(x_1, Tx_1) + a_3 d(x_2, Tx_2) + \\ &\quad a_4 d(x_1, Tx_2) + L d(x_2, Tx_1)\Big) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + \\ &\quad a_4 d(x_1, x_3) + L.0\Big) \\ &\leq F\Big(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + \\ &\quad a_4 (d(x_1, x_2) + d(x_2, x_3))\Big) \\ &= F\Big((a_1 + a_2 + a_4) d(x_1, x_2) + (a_3 + a_4) d(x_2, x_3)\Big). \end{aligned} \tag{3.64}$$

Since F is strictly increasing, we get from above that

$$d(x_2, x_3) < (a_1 + a_2 + a_4) d(x_1, x_2) + (a_3 + a_4) d(x_2, x_3).$$

That is,

$$(1 - a_3 - a_4) d(x_2, x_3) < (a_1 + a_2 + a_4) d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_2, x_3) < d(x_1, x_2).$$

Now from (3.64), we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

So we have

$$F(d(x_2, x_3)) \leq F(d(x_1, x_2)) - \tau \leq F(d(x_0, x_1)) - 2\tau.$$

Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in Tx_{n-1},\ x_{n-1} \neq x_n \text{ and } \alpha(x_{n-1}, x_n) > 1 \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \tag{3.65}$$

Letting $n \to \infty$ in (3.65), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (3.65) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{3.66}$$

Letting $n \to \infty$ in (3.66), we get

$$\lim_{n\to\infty} nd_n^k = 0. \tag{3.67}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $nd_n^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{3.68}$$

To prove that $\{x_n\}$ is Cauchy. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (3.68), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is Cauchy. As (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (iii), we have $\alpha(x_n, x^*) > 1$ for each $n \in \mathbb{N}$. We claim that $d(x^*, Tx^*) = 0$. On contrary suppose that $d(x^*, Tx^*) > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, Tx^*) > 0$ for each $n \geq n_0$. For each $n \geq n_0$, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &< d(x^*, x_{n+1}) + \alpha(x_n, x^*) H(Tx_n, Tx^*) \\ &< d(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) + a_3 d(x^*, Tx^*) + \\ &\quad a_4 d(x_n, Tx^*) + L d(x^*, x_{n+1}). \end{aligned} \tag{3.69}$$

Letting $n \to \infty$ in (3.69), we have

$$d(x^*, Tx^*) \leq (a_3 + a_4) d(x^*, Tx^*) < d(x^*, Tx^*).$$

Which is a contradiction. Thus $d(x^*, Tx^*) = 0$. □

Example 3.4.7. Let $X = \mathbb{N} \cup \{0\}$ be endowed with the usual metric $d(x, y) = |x - y|$ for each $x, y \in X$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \{0, 1\} & \text{if } x = 0, 1 \\ \{x - 1, x\} & \text{if } x > 1 \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in \{0, 1\} \\ \frac{1}{2} & \text{if } x, y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Take $F(x) = x + \ln x$ for each $x \in (0, \infty)$. Under this F, condition (3.58) reduces to

$$\frac{\alpha(x,y)H(Tx,Ty)}{N(x,y)} e^{\alpha(x,y)H(Tx,Ty)-N(x,y)} \leq e^{-\tau} \tag{3.70}$$

for each $x, y \in X$ with $\min\{\alpha(x,y)H(Tx,Ty), N(x,y)\} > 0$. Assume that $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $\tau = \frac{1}{2}$. Clearly, $\min\{\alpha(x,y)H(Tx,Ty), d(x,y)\} > 0$ for each $x, y > 1$ with $x \neq y$. From (3.70) for each $x, y > 1$ with $x \neq y$, we have

$$\frac{1}{2} e^{-\frac{1}{2}|x-y|} < e^{-\frac{1}{2}}.$$

Thus, T is α-F-contraction of Hardy-Rogers-type with $F(x) = x + \ln x$. For $x_0 = 1$, we have $x_1 = 0 \in Tx_0$ such that $\alpha(x_0, x_1) > 1$. Moreover, it is easy to see that T is strictly α-admissible mapping and for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$. Therefore, by Theorem 3.4.6, T has a fixed point in X.

Remark 3.4.8. Note that [131, Theorem 3.4] is not applicable on above example with $F(x) = x + \ln x$. Since for $x = 3$ and $y = 2$, from (1.17), we have $\frac{1}{a_1+a_4} e^{1-a_1-a_4} \leq e^{-2\tau}$, which is impossible.

Definition 3.4.9. A mapping $T : X \to CB(X)$ is α_*-F-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that

$$\tau + F(\alpha_*(Tx,Ty)H(Tx,Ty)) \leq F(N(x,y)), \tag{3.71}$$

for each $x, y \in X$, whenever $\min\{\alpha_*(Tx,Ty)H(Tx,Ty), N(x,y)\} > 0$, where

$$N(x,y) = a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + L d(y,Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 3.4.10. Let $T : X \to CB(X)$ be an α_*-F-contraction of Hardy-Rogers-type satisfying the following conditions:

(i) T is strictly α_*-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. The proof of this theorem is along the same lines as the proof of Theorem 3.4.6 is done. □

Remark 3.4.11. We may replace the condition (iii) of Theorem 3.4.6 and Theorem 3.4.10 by continuity of T.

Definition 3.4.12. A mapping $T : X \to CB(X)$ is α-F-contraction, if there exist a continuous F in $\mathfrak{F}$ and $\tau > 0$ such that

$$\tau + F(\alpha(x, y)H(Tx, Ty)) \leq F(M(x, y)), \tag{3.72}$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y)H(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

with $L \geq 0$.

Theorem 3.4.13. Let $T : X \to CB(X)$ be an α-F-contraction satisfying the following conditions:

(i) T is strictly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (ii), there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. From (3.72), we have

$$\begin{aligned} \tau + F(\alpha(x_0, x_1)H(Tx_0, Tx_1)) &\leq F\Big(\max\Big\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \\ &\qquad \frac{d(x_1, Tx_0) + d(x_0, Tx_1)}{2}\Big\} + Ld(x_1, Tx_0)\Big) \\ &= F\Big(\max\{d(x_0, x_1), d(x_1, Tx_1)\}\Big). \end{aligned} \tag{3.73}$$

As $\alpha(x_0, x_1) > 1$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1). \tag{3.74}$$

Since, F is strictly increasing, we have

$$F(d(x_1, x_2)) \leq F(\alpha(x_0, x_1)H(Tx_0, Tx_1)). \tag{3.75}$$

From (3.73) and (3.75), we have

$$\tau + F(d(x_1, x_2)) \leq F\Big(\max\{d(x_0, x_1), d(x_1, Tx_1)\}\Big). \tag{3.76}$$

If we assume that $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$, then we have a contradiction to (3.76). Thus, $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1)$. From (3.76), we have

$$\tau + F(d(x_1, x_2)) \leq F(d(x_0, x_1)). \tag{3.77}$$

Since T is strictly α-admissible, therefore $\alpha(x_0, x_1) > 1$ implies $\alpha(x_1, x_2) > 1$. If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Let $x_2 \notin Tx_2$. From (3.72), we have

$$\begin{aligned} \tau + F(\alpha(x_1, x_2)H(Tx_1, Tx_2)) &\leq F\Big(\max\Big\{ d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \\ &\quad \frac{d(x_2, Tx_1) + d(x_1, Tx_2)}{2} \Big\} + Ld(x_2, Tx_1) \Big) \\ &= F\Big(\max\{d(x_1, x_2), d(x_2, Tx_2)\} \Big). \end{aligned} \tag{3.78}$$

As $\alpha(x_1, x_2) > 1$, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \alpha(x_1, x_2)H(Tx_1, Tx_2). \tag{3.79}$$

Since F is strictly increasing, we have

$$F(d(x_2, x_3)) \leq F(\alpha(x_1, x_2)H(Tx_1, Tx_2)). \tag{3.80}$$

From (3.78) and (3.80), we have

$$\tau + F(d(x_2, x_3)) \leq F\Big(\max\{d(x_1, x_2), d(x_2, Tx_2)\} \Big). \tag{3.81}$$

If we assume that $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_2, Tx_2)$, then we have a contradiction to (3.81). Thus, $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2)$. From (3.81), we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)). \tag{3.82}$$

From (3.77) and (3.82), we have

$$F(d(x_2, x_3)) \leq F(d(x_0, x_1)) - 2\tau. \tag{3.83}$$

Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in Tx_{n-1},\ x_{n-1} \neq x_n \text{ and } \alpha(x_{n-1}, x_n) > 1 \text{ for each } n \in \mathbb{N}.$$

Moreover,

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \tag{3.84}$$

Letting $n \to \infty$ in (3.84), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (3.84) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{3.85}$$

Letting $n \to \infty$ in (3.85), we get

$$\lim_{n\to\infty} n d_n^k = 0. \tag{3.86}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $nd_n^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{3.87}$$

To prove that $\{x_n\}$ is Cauchy. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (3.87), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is Cauchy. As (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. By condition (iii), we have $\alpha(x_n, x^*) > 1$ for each $n \in \mathbb{N}$. We claim that $d(x^*, Tx^*) = 0$. On contrary suppose that $d(x^*, Tx^*) > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, Tx^*) > 0$ for each $n \geq n_0$. From (3.72), for each $n \geq n_0$, we have

$$\begin{aligned} \tau + F(d(x_{n+1}, Tx^*)) &\leq \tau + F(\alpha(x_n, x^*)H(Tx_n, Tx^*)) \\ &\leq F\Big(\max\Big\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \\ &\qquad \frac{d(x^*, Tx_n) + d(x_n, Tx^*)}{2}\Big\} + Ld(x^*, Tx_n)\Big) \end{aligned}$$

Letting $n \to \infty$ in above inequality and by continuity of F, we get

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)).$$

This implies $\tau \leq 0$. Which is a contradiction. Thus $d(x^*, Tx^*) = 0$. □

Definition 3.4.14. A mapping $T : X \to CB(X)$ is α_*-F-contraction, if there exist a continuous F in $\mathfrak{F}$ and $\tau > 0$ such that

$$\tau + F(\alpha_*(Tx, Ty)H(Tx, Ty)) \leq F(M(x, y)), \tag{3.88}$$

for each $x, y \in X$, whenever $\min\{\alpha_*(Tx, Ty)H(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\Big\} + Ld(y, Tx)$$

with $L \geq 0$.

Theorem 3.4.15. Let $T : X \to CB(X)$ be an α_*-F-contraction satisfying the following conditions:

(i) T is strictly α_*-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. The proof of this theorem is similar to proof of Theorem 3.4.13. □

Remark 3.4.16. If we assume that T is continuous then we can leave condition (iii) and continuity of F from Theorem 3.4.13 and Theorem 3.4.15.

Consequences

In this section, we obtain some fixed point theorems as consequences of our results. It is worth mentioning that these results are also new, according to our knowledge.

Metric space endowed with partial ordering

Here we prove some results for fixed points of multi valued mappings from a partially ordered metric space $(X, d, \preceq)$ into the space of nonempty closed and bounded subsets of the metric space. We begin this subsection by introducing the following definition.

Definition 3.4.17. A mapping $T : X \to CB(X)$ is F_q-contraction of Hardy-Rogers-type on $(X, d, \preceq)$, if there exist $F \in \mathfrak{F}$, $\tau > 0$ and $q > 1$ such that

$$\tau + F(qH(Tx, Ty)) \leq F(N(x, y)), \tag{3.89}$$

for each $x, y \in X$ with $x \preceq y$, whenever $\min\{qH(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + Ld(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 3.4.18. Let $T : X \to CB(X)$ be an F_q-contraction of Hardy-Rogers-type on complete $(X, d, \preceq)$ and satisfying the following conditions:

(i) for each $x \in X$ and $y \in Tx$ such that $x \preceq y$, this implies $y \preceq z$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $x_0 \preceq x_1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, we have $x_n \preceq x$ for each $n \in \mathbb{N}$.

Then, T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 3.4.6 hold. Thus, T has a fixed point. $\square$

Definition 3.4.19. A mapping $T : X \to CB(X)$ is F_q-contraction on $(X, d, \preceq)$, if there exist a continuous F in $\mathfrak{F}$, $\tau > 0$ and $q > 1$ such that

$$\tau + F(qH(Tx, Ty)) \leq F(M(x, y)), \tag{3.90}$$

for each $x, y \in X$ with $x \preceq y$, whenever $\min\{qH(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

with $L \geq 0$.

Theorem 3.4.20. Let $T : X \to CB(X)$ be an F_q-contraction on complete $(X, d, \preceq)$ and satisfying the following conditions:

(i) for each $x \in X$ and $y \in Tx$ such that $x \preceq y$, this implies $y \preceq z$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $x_0 \preceq x_1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, we have $x_n \preceq x$ for each $n \in \mathbb{N}$.

Then, T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 3.4.13 hold. Thus, T has a fixed point. $\square$

Remark 3.4.21. If we replace assumption (i) of above results by
(i'): If $x \preceq y$, then we have $Tx \prec_r Ty$, that is, for each $a \in Tx$ and $b \in Ty$ we have $a \preceq b$.
Then Theorem 3.4.18 and Theorem 3.4.20 follow from Theorem 3.4.10 and Theorem 3.4.15, respectively.

Metric space endowed with graph

In this subsection, we derive some fixed point theorems for multi valued mappings from a metric space (X, d), endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Throughout this subsection, we assume that G is a directed graph such that the set of its vertices V coincides with X (i.e., $V = X$) and the set of its edges E is such that $E \supseteq \triangle$, where $\triangle = \{(x, x) : x \in X\}$. Let us also assume that G has no parallel edges. We can identify G with the pair (V, E).

Definition 3.4.22. A mapping $T : X \to CB(X)$ is graphic F_q-contraction of Hardy-Rogers-type on (X, d) endowed with the graph G, if there exist $F \in \mathfrak{F}$, $\tau > 0$ and $q > 1$ such that

$$\tau + F(qH(Tx, Ty)) \leq F(N(x, y)), \tag{3.91}$$

for each $x, y \in X$ with $(x, y) \in E$, whenever $\min\{qH(Tx, Ty), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + Ld(y, Tx),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 3.4.23. Let $T : X \to CB(X)$ be a graphic F_q-contraction of Hardy-Rogers-type on complete (X, d) endowed with the graph G and satisfying the following conditions:

(i) for each $x \in X$ and $y \in Tx$ with $(x, y) \in E$, this implies $(y, z) \in E$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $(x, y) \in E$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then, T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } (x, y) \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 3.4.6 hold. Thus, T has a fixed point. □

Definition 3.4.24. A mapping $T : X \to CB(X)$ is graphic F_q-contraction on (X, d) endowed with the graph G, if there exist a continuous F in $\mathfrak{F}$, $\tau > 0$ and $q > 1$ such that

$$\tau + F(qH(Tx, Ty)) \leq F(M(x, y)), \tag{3.92}$$

for each $x, y \in X$ with $(x, y) \in E$, whenever $\min\{qH(Tx, Ty), M(x, y)\} > 0$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} + Ld(y, Tx)$$

with $L \geq 0$.

Theorem 3.4.25. Let $T : X \to CB(X)$ be a graphic F_q-contraction on complete (X, d) endowed with the graph G and satisfying the following conditions:

(i) for each $x \in X$ and $y \in Tx$ with $(x, y) \in E$, this implies $(y, z) \in E$ for each $z \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $(x, y) \in E$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$, we have $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then, T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} q & \text{if } (x, y) \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of Theorem 3.4.13 hold. Thus, T has a fixed point. □

Remark 3.4.26. If we replace assumption (i) of above results by
(i'): If $(x, y) \in E$, then we have $(a, b) \in E$ for each $a \in Tx$ and $b \in Ty$.
Then Theorem 3.4.23 and Theorem 3.4.25 follow from Theorem 3.4.10 and Theorem 3.4.15, respectively.

Application

As a consequence of our result, here, we establish an existence theorem for an integral equation. Let $X = (C[a, b], \mathbb{R})$ be the space of all real valued continuous functions defined on $[a, b]$. Note that X is complete [121] with respect to the metric $d_\tau(x, y) = \sup_{t\in[a,b]}\{|x(t) - y(t)|e^{-|\tau t|}\}$. Consider an integral equation of the form

$$x(t) = f(t) + \int_{g(t)}^{h(t)} K(t, s, x(s))ds, \tag{3.93}$$

for $t, s \in [a, b]$. Where $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f, g, h : [a, b] \to \mathbb{R}$ are continuous functions and $g(t) \leq h(t)$ for each $t \in [a, b]$.

Theorem 3.4.27. Let $X = (C[a, b], \mathbb{R})$ and let $T : X \to X$ be the operator defined as

$$Tx(t) = f(t) + \int_{g(t)}^{h(t)} K(t, s, x(s))ds, \tag{3.94}$$

for $t, s \in [a, b]$. Where $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f, g, h : [a, b] \to \mathbb{R}$ are continuous functions and $g(t) \leq h(t)$ for each $t \in [a, b]$. Assume that there exist $\beta : X \to (0, \infty)$ and $\alpha : X \times X \to (0, \infty)$ such that the following conditions hold:

(i) there exists $\tau > 0$ such that

$$|K(t,s,x) - K(t,s,y)| \leq \frac{e^{-\tau}}{\beta(x+y)}|x-y|$$

for each $t, s \in [a,b]$ and $x, y \in X$, moreover,

$$\Big| \int_{g(t)}^{h(t)} \frac{e^{|\tau s|}}{\beta(x(s)+y(s))} ds \Big| \leq \frac{e^{|\tau t|}}{\alpha(x,y)}$$

for each $t \in [a,b]$;

(ii) for $x, y \in X$, $\alpha(x,y) > 1$ implies $\alpha(Tx, Ty) > 1$;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) > 1$;

(iv) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then the integral equation (3.93) has a solution in X.

Proof. First we show that T is an α-F-contraction of Hardy-Rogers-type. For any $x, y \in X$, we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_{g(t)}^{h(t)} |K(t,s,x(s)) - K(t,s,y(s))| ds \\
&\leq \int_{g(t)}^{h(t)} \frac{e^{-\tau}}{\beta(x(s)+y(s))} |x(s) - y(s)| ds \\
&= \int_{g(t)}^{h(t)} \frac{e^{-\tau} e^{|\tau s|}}{\beta(x(s)+y(s))} |x(s) - y(s)| e^{-|\tau s|} ds \\
&\leq e^{-\tau} d_\tau(x,y) \int_{g(t)}^{h(t)} \frac{e^{|\tau s|}}{\beta(x(s)+y(s))} ds \\
&\leq \frac{e^{|\tau t|}}{\alpha(x,y)} e^{-\tau} d_\tau(x,y).
\end{aligned}$$

Thus, we have

$$\alpha(x,y)|Tx(t) - Ty(t)|e^{-|\tau t|} \leq e^{-\tau} d_\tau(x,y) \text{ for each } t \in [a,b].$$

Equivalently

$$\alpha(x,y) d_\tau(Tx, Ty) \leq e^{-\tau} d_\tau(x,y).$$

Clearly natural logarithm belongs to $\mathfrak{F}$. Applying it on above inequality, we get

$$\ln(\alpha(x,y) d_\tau(Tx,Ty)) \leq \ln(e^{-\tau} d_\tau(x,y)),$$

after some simplification, we get

$$\tau + \ln(\alpha(x,y) d_\tau(Tx,Ty)) \leq \ln(d_\tau(x,y)).$$

Thus, T is an α-F-contraction of Hardy-Rogers-type with $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $F(x) = \ln x$. All other conditions of Theorem 3.4.6 are immediately hold. Therefore, the operator (3.94) has a fixed point, that is, the integral equation (3.93) has a solution in X. □

3.5 Existence of best proximity points for controlled proximal contraction

In this section, we discuss sufficient conditions which ensure the existence of best proximity points for multi valued nonself mappings satisfying proximal contraction condition on closed ball of a complete metric space. Moreover, we also study the stability of the best proximity points for such mappings. The following results extend/generalize some results by Lim [100], Abkar and Gbeleh [7]. Note that throughout this section: A and B are nonempty subsets of (X, d). Further, in results, A and B are nonempty closed subsets of complete (X, d).

Definition 3.5.1. Let $x_0 \in A_0$ and $B(x_0, r)$ is a closed ball in (X, d). A mapping $T : A \to CL(B)$ is said to be a proximal contraction on $B(x_0, r)$, if there exists $\alpha \in (0, 1)$ such that

$$H(Tx, Ty) \leq \alpha d(x, y) \text{ for each } x, y \in B(x_0, r) \cap A. \tag{3.95}$$

Now we state and prove the first result of this section.

Theorem 3.5.2. Assume that A_0 is nonempty and $T : A \to CL(B)$ be a mapping satisfying the following conditions:

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) the pair (A, B) satisfies weak P-property;

(iii) there exists $x_0 \in A_0$ such that T is a proximal contraction on the closed ball $B(x_0, r)$ and $d(x_0, Tx_0) + dist(A, B) \leq (1 - \sqrt{\alpha})r$.

Then T has a best proximity point in $B(x_0, r) \cap A_0$.

Proof. By hypothesis (iii), we have $x_0 \in A_0$ such that T is a proximal contraction on closed ball $B(x_0, r)$ and $d(x_0, Tx_0) + dist(A, B) \leq (1 - \sqrt{\alpha})r$. As $x_0 \in A_0$. By (i), we have $y_0 \in Tx_0 \subseteq B_0$. Then there exists $x_1 \in A_0$ such that

$$d(x_1, y_0) = dist(A, B). \tag{3.96}$$

By using triangular inequality, hypothesis (iii) and (3.96), we have

$$d(x_0, x_1) \leq d(x_0, Tx_0) + d(Tx_0, x_1) \leq d(x_0, Tx_0) + d(y_0, x_1) \leq (1 - \sqrt{\alpha})r. \tag{3.97}$$

Since, $x_1 \in A_0 \subseteq A$. Thus, $x_1 \in B(x_0, r) \cap A$. From (3.95), we have

$$d(y_0, Tx_1) \leq H(Tx_0, Tx_1) \leq \alpha d(x_0, x_1). \tag{3.98}$$

As $\alpha > 0$, then by Lemma 1.5.5, we have $y_1 \in Tx_1$ such that

$$d(y_0, y_1) \leq \frac{1}{\sqrt{\alpha}} d(y_0, Tx_1) \leq \sqrt{\alpha} d(x_0, x_1). \tag{3.99}$$

Since $Tx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, y_1) = dist(A, B). \tag{3.100}$$

As (A, B) satisfies the weak P-property. From (3.96) and (3.100), we have

$$d(x_1, x_2) \leq d(y_0, y_1). \tag{3.101}$$

From (3.99) and (3.101), we have

$$d(x_1, x_2) \leq \sqrt{\alpha} d(x_0, x_1). \tag{3.102}$$

Consider the triangular inequality, (3.97) and (3.102), we have

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \sqrt{\alpha} d(x_0, x_1) \\ &\leq (1 - \alpha) r < r. \end{aligned}$$

By construction, we have $x_2 \in A_0 \subseteq A$. Thus $x_2 \in B(x_0, r) \cap A$. Again from (3.95), we have

$$d(y_1, Tx_2) \leq H(Tx_1, Tx_2) \leq \alpha d(x_1, x_2). \tag{3.103}$$

By using Lemma 1.5.5, we have $y_2 \in Tx_2$ such that

$$d(y_1, y_2) \leq \frac{1}{\sqrt{\alpha}} d(y_1, Tx_2) \leq \sqrt{\alpha} d(x_1, x_2). \tag{3.104}$$

Since $Tx_2 \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, y_2) = dist(A, B). \tag{3.105}$$

As (A, B) satisfies the weak P-property. From (3.100) and (3.105), we have

$$d(x_2, x_3) \leq d(y_1, y_2). \tag{3.106}$$

From (3.104) and (3.106), we have

$$d(x_2, x_3) \leq \sqrt{\alpha} d(x_1, x_2) \leq \alpha d(x_0, x_1). \tag{3.107}$$

By considering the triangular inequality, (3.102) and (3.107), we have

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq [1 + \sqrt{\alpha} + (\sqrt{\alpha})^2] d(x_0, x_1) \\ &\leq [1 + \sqrt{\alpha} + (\sqrt{\alpha})^2](1 - \sqrt{\alpha}) r < r. \end{aligned}$$

As $x_3 \in A_0 \subseteq A$. Thus, $x_3 \in B(x_0, r) \cap A$. Continuing in the same way, we get two sequences $\{x_n\} \subseteq A_0$ with $x_n \in B(x_0, r)$ and $\{y_n\} \subseteq B_0$ with $y_n \in Tx_n$ such that

$$d(x_n, y_{n-1}) = dist(A, B) \text{ for each } n \in \mathbb{N}. \tag{3.108}$$

Moreover,

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \leq (\sqrt{\alpha})^n d(x_0, x_1) \text{ for each } n \in \mathbb{N}. \tag{3.109}$$

For $n > m$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} (\sqrt{\alpha})^i d(x_0, x_1) < \sum_{i=n}^{\infty} (\sqrt{\alpha})^i d(x_0, x_1) < \infty. \tag{3.110}$$

Hence $\{x_n\}$ is Cauchy in $B(x_0, r) \cap A \subseteq A$. Similar pattern shows that $\{y_n\}$ is Cauchy in B. Since $B(x_0, r) \cap A$ is closed in A, and A, B are closed subsets of a complete metric space, there exist $x^* \in B(x_0, r) \cap A$ and $y^* \in B$ such that $x_n \to x^*$ and $y_n \to y^*$. By the (3.108), we conclude that $d(x^*, y^*) = dist(A, B)$ as $n \to \infty$. Clearly, $y^* \in Tx^*$, since, $\lim_{n\to\infty} d(y_n, Tx^*) \leq \lim_{n\to\infty} H(Tx_n, Tx^*) = 0$. Hence $dist(A, B) \leq d(x^*, Tx^*) \leq d(x^*, y^*) = dist(A, B)$. Therefore, x^* is a best proximity point of the mapping T. □

Example 3.5.3. Let $X = \mathbb{R}^2$ be endowed with metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Suppose that $A = \{(1, x) : x \in \mathbb{R}\}$ and $B = \{(0, x) : x \in \mathbb{R}\}$. Define $T : A \to CL(B)$ by

$$T(1, x) = \begin{cases} \{(0, 0)\} & \text{if } x \leq 0 \\ \{(0, 0), (0, x/2)\} & \text{if } 0 \leq x \leq 10 \\ \{(0, x)\} & \text{if } x > 10. \end{cases}$$

Let us consider a ball $B(x_0, r)$ with $x_0 = (1, 0.1)$ and $r = 7.5$. Then it is easy to see that T is a proximal contraction on closed ball B((1,0.1),7.5) with $\alpha = \frac{1}{2}$. Also, we have $d(x_0, Tx_0) + dist(A, B) \leq (1 - \sqrt{\alpha})r$. Furthermore, $A_0 = A$, $B_0 = B$; for each $x \in A_0$ we have $Tx \subseteq B_0$ and the pair (A, B) satisfies the weak P-property. Therefore, all the conditions of Theorem 3.5.2 hold and T has a best proximity point.

Corollary 3.5.4. Assume that A_0 is nonempty and $T : A \to B$ be a mapping satisfying the following conditions:

(i) for each $x \in A_0$, we have $Tx \in B_0$;

(ii) the pair (A, B) satisfies the weak P-property;

(iii) there exists $x_0 \in A_0$ such that T is a proximal contraction on the closed ball $B(x_0, r)$, that is,

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for each } x, y \in B(x_0, r) \cap A, \tag{3.111}$$

and $d(x_0, Tx_0) + dist(A, B) \leq (1 - \sqrt{\alpha})r$.

Then T has a best proximity point in $B(x_0, r) \cap A_0$.

If we assume that $X = A = B$, then Theorem 3.5.2 reduces to the following fixed point theorem:

Corollary 3.5.5. Let $T : X \to CL(X)$ be a mapping on complete (X, d). Assume that there exist $x_0 \in X$ and $\alpha \in (0, 1)$ such that

$$H(Tx, Ty) \leq \alpha d(x, y) \text{ for each } x, y \in B(x_0, r)$$

and $d(x_0, Tx_0) \leq (1 - \sqrt{\alpha})r$. Then T has a fixed point.

Stability of Best Proximity points

Stability of fixed points for multi valued mappings was initially investigated by Markin [103] and Nadler [111] with some strong conditions. Lim [100] proved the Stability of fixed points for multi valued contraction mappings by relaxing the condition assumed by Markin [103]. Abkar and Gbeleh [7] discussed the stability of best proximity points for nonself multi valued mappings. In this section, we extend/generalize the stability theorems due to Abkar and Gbeleh [7], and Lim [100].

In this section, B_{T_1} and B_{T_2} denotes the sets of best proximity points of T_1 and T_2 respectively.

Theorem 3.5.6. Assume that A_0 is nonempty and $T_i : A \to CL(B)$, $i = 1, 2$ be mappings satisfying the following conditions:

(i) for each $x \in A_0$, we have $T_i x \subseteq B_0$, $i = 1, 2$;

(ii) the pair (A, B) satisfies the weak P-property;

(iii) for each $i = 1, 2$, there exists $a_i \in A_0$ such that T_i is proximal contraction on closed ball $B(a_i, r_i)$ with same α as a contraction constant, that is,

$$H(T_i x, T_i y) \leq \alpha d(x, y) \text{ for each } x, y \in B(a_i, r_i) \cap A, \tag{3.112}$$

and $d(a_i, T_i a_i) + dist(A, B) \leq (1 - \sqrt{\alpha})r_i$.

Then

$$H(B_{T_1}, B_{T_2}) \leq \frac{1}{1 - \sqrt{\alpha}}[\sup_{x \in A} H(T_1 x, T_2 x) + 2dist(A, B)].$$

Proof. Let $x_0 \in B_{T_1}$, then we have $y_0 \in T_2 x_0$ such that

$$d(x_0, y_0) \leq H(T_1 x_0, T_2 x_0) + dist(A, B).$$

Since $y_0 \in T_2 x_0 \subseteq B_0$, then we have $x_1 \in A_0$ such that

$$d(x_1, y_0) = dist(A, B). \tag{3.113}$$

We know that T_2 is a proximal contraction for closed ball $B(a_2, r_2)$. Without loss of generality, we take $a_2 = x_0$ and $r_2 = r$ such that $d(x_0, T_2 x_0) + dist(A, B) \leq (1 - \sqrt{\alpha})r$. Clearly, $x_1 \in B(x_0, r) \cap A$, since $x_1 \in A_0 \subseteq A$ and

$$d(x_0, x_1) \leq d(x_0, T_2 x_0) + d(T_2 x_0, x_1) \leq d(x_0, T_2 x_0) + d(y_0, x_1) \leq (1 - \sqrt{\alpha})r. \tag{3.114}$$

By hypothesis (iii), we have

$$d(y_0, T_2x_1) \leq H(T_2x_0, T_2x_1) \leq \alpha d(x_0, x_1). \tag{3.115}$$

As $\alpha > 0$, then by Lemma 1.5.5, we have $y_1 \in T_2x_1$ such that

$$d(y_0, y_1) \leq \frac{1}{\sqrt{\alpha}} d(y_0, T_2x_1) \leq \sqrt{\alpha} d(x_0, x_1). \tag{3.116}$$

Since $T_2x_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, y_1) = dist(A, B). \tag{3.117}$$

As (A, B) satisfies the weak P-property. From (3.113) and (3.117), we have

$$d(x_1, x_2) \leq d(y_0, y_1). \tag{3.118}$$

From (3.116) and (3.118), we have

$$d(x_1, x_2) \leq \sqrt{\alpha} d(x_0, x_1). \tag{3.119}$$

Consider the triangular inequality, (3.114) and (3.119), we have

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \sqrt{\alpha} d(x_0, x_1) \\ &\leq (1 - \alpha) r < r. \end{aligned}$$

Also, $x_2 \in A_0 \subseteq A$. Thus, $x_2 \in B(x_0, r) \cap A$. Continuing in the same way, we get two sequences $\{x_n\} \subseteq A_0$ with $x_n \in B(x_0, r) \cap A$ and $\{y_n\} \subseteq B_0$ with $y_n \in T_2x_n$ such that

$$d(x_n, y_{n-1}) = dist(A, B) \text{ for each } n \in \mathbb{N}. \tag{3.120}$$

Moreover,

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \leq (\sqrt{\alpha})^n d(x_0, x_1) \text{ for each } n \in \mathbb{N}. \tag{3.121}$$

For $n > m$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} (\sqrt{\alpha})^i d(x_0, x_1) < \sum_{i=n}^{\infty} (\sqrt{\alpha})^i d(x_0, x_1) < \infty. \tag{3.122}$$

Hence $\{x_n\}$ is Cauchy in $B(x_0, r) \cap A \subseteq A$. Similar pattern shows that $\{y_n\}$ is Cauchy in B. Since $B(x_0, r) \cap A$ is closed in A, and A, B are closed subsets of a complete metric space, there exist $u^* \in B(x_0, r) \cap A$ and $v^* \in B$ such that $x_n \to u^*$ and $y_n \to v^*$. By the (3.120), we conclude that $d(u^*, v^*) = dist(A, B)$ as $n \to \infty$. Clearly, $v^* \in T_2u^*$. Then we have $dist(A, B) \leq$

$d(u^*, T_2u^*) \leq d(u^*, v^*) = dist(A, B)$. Therefore u^* is a best proximity point of T_2. Now, we have

$$\begin{aligned} d(x_0, u^*) &\leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=0}^{\infty} (\sqrt{\alpha})^n d(x_0, x_1) \\ &= \frac{1}{1-\sqrt{\alpha}} d(x_0, x_1) \\ &\leq \frac{1}{1-\sqrt{\alpha}} [d(x_0, y_0) + d(y_0, x_1)] \\ &= \frac{1}{1-\sqrt{\alpha}} [d(x_0, y_0) + dist(A, B)] \\ &\leq \frac{1}{1-\sqrt{\alpha}} [H(T_1x_0, T_2x_0) + 2dist(A, B)]. \end{aligned}$$

Similarly, if $\mathfrak{x}_0 \in B_{T_2}$, then we have $\mathfrak{u}^* \in B_{T_1}$ such that

$$d(\mathfrak{x}_0, \mathfrak{u}^*) \leq \frac{1}{1-\sqrt{\alpha}} [H(T_1\mathfrak{x}_0, T_2\mathfrak{x}_0) + 2dist(A, B)].$$

Thus, we have

$$H(B_{T_1}, B_{T_2}) \leq \frac{1}{1-\sqrt{\alpha}} [\sup_{x \in A} H(T_1x, T_2x) + 2dist(A, B)].$$

□

Example 3.5.7. Let $X = \mathbb{R}^2$ be endowed with metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Suppose that $A = \{(1, x) : x \in \mathbb{R}\}$ and $B = \{(0, x) : x \in \mathbb{R}\}$. Define $T_1, T_2 : A \to CL(B)$ by

$$T_1(1, x) = \begin{cases} \{(0, 0)\} & \text{if } x \leq 0 \\ \{(0, 0), (0, x/2)\} & \text{if } 0 \leq x \leq 10 \\ \{(0, x)\} & \text{if } x > 10, \end{cases}$$

and

$$T_2(1, x) = \begin{cases} \{(0, 1)\} & \text{if } x \leq 1 \\ \{(0, 1), (0, (x+1)/2)\} & \text{if } x > 1. \end{cases}$$

It is easy to see that T_1 is a proximal contraction on closed ball $B(x_0 = (1, 0.1), r = 7.5)$ with $\alpha = \frac{1}{2}$ and $d(x_0, Tx_0) + dist(A, B) \leq (1 - \sqrt{\alpha})r$. Further, T_2 is a proximal contraction on closed ball $B(x_1 = (1, 1.25), r_1 = 8)$ with $\alpha = \frac{1}{2}$ and $d(x_1, Tx_1) + dist(A, B) \leq (1 - \sqrt{\alpha})r_1$. Furthermore, it is easy to see that $A_0 = A$, $B_0 = B$, for each $x \in A_0$ we have $T_ix \subseteq B_0$ for each $i = 1, 2$ and the pair (A, B) satisfies the weak P-property. As all the conditions of Theorem 3.5.6 hold. Thus the conclusion holds. That is,

$$H(B_{T_1}, B_{T_2}) \leq \frac{1}{1-\sqrt{\alpha}} [\sup_{x \in A} H(T_1x, T_2x) + 2dist(A, B)].$$

Note that $B_{T_1} = \{(1, 0)\}$ and $B_{T_2} = \{(1, -1), (1, 1)\}$. Therefore $H(B_{T_1}, B_{T_2}) = 1$.

Let F_{T_1} and F_{T_2} denotes the sets of fixed points of T_1 and T_2 respectively. If we assume that $X = A = B$, then Theorem 3.5.2, reduces to following stability theorem:

Corollary 3.5.8. Let $T_i : X \to CL(X)$, $i = 1, 2$ be mappings on complete (X, d). Assume that there exist $\alpha \in (0, 1)$ and $a_1, a_2 \in X$ such that for each i, we have

$$H(T_i x, T_i y) \leq \alpha d(x, y) \text{ for each } x, y \in B(a_i, r_i) \tag{3.123}$$

and $d(a_i, T_i a_i) \leq (1 - \sqrt{\alpha}) r_i$. Then

$$H(F_{T_1}, F_{T_2}) \leq \frac{1}{1 - \sqrt{\alpha}} \sup_{x \in A} H(T_1 x, T_2 x).$$

Note that in this theorem $B(a_i, r_i)$ are closed balls.

Remark 3.5.9. If r_1, r_2 are sufficiently large such that $B(a_1, r_1)$ and $B(a_2, r_2)$ are equal to X. In this case, from Corollary 3.5.8, we get

Corollary 3.5.10. (Lim [100, Lemma 1]) Let $T_i : X \to CL(X)$, $i = 1, 2$ be α-contractions on complete (X, d), that is,

$$H(T_i x, T_i y) \leq \alpha d(x, y) \text{ for each } x, y \in X$$

where $\alpha \in (0, 1)$. Then

$$H(F_{T_1}, F_{T_2}) \leq \frac{1}{1 - \alpha} \sup_{x \in X} H(T_1 x, T_2 x).$$

Chapter 4

Fixed Point Theorems in Abstract Spaces

In this chapter, we investigate the existence of fixed points for single valued/multi valued mappings on some abstract spaces, like uniform spaces, partial metric spaces, b-metric spaces, gauge spaces and b-gauge spaces. This chapter consists of five sections. In first section, we investigate the existence of fixed points for single valued mappings, while in the remaining four sections we discuss multi valued mappings and existence of their fixed points. This chapter also contains some supporting examples, consequences of the main results and possible applications of few main results.

4.1 Fixed point of α-ψ-contractive type mappings in uniform spaces

In this section, we consider the characterization of the notion for α-ψ-contractive mapping in the context of uniform spaces and prove some fixed point theorems by using this concept. We also use α-admissible pairs to investigate the existence of common fixed points in the setting of uniform spaces. Further, we establish some examples to illustrate the main results of this section. Throughout this section: X is a nonempty set endowed with a uniform structure ϑ, that is, (X,ϑ) is uniform space, and with p as an E-distance. Further, in results, (X,ϑ) is a S-complete Hausdorff uniform space such that p is an E-distance on X.

Definition 4.1.1. A mapping $T : X \to X$ is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)p(Tx,Ty) \leq \psi(p(x,y)), \text{ for all } x,y \in X. \tag{4.1}$$

Theorem 4.1.2. Let $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;

(iii) T is p-continuous.

Then T has a fixed point.

Proof. By hypothesis (ii) of theorem we have $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_{n_0} is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Inductively, we have

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{4.2}$$

From (4.1) and (4.2), it follows that, for all $n \in \mathbb{N}$, we have

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)p(Tx_{n-1}, Tx_n) \leq \psi(p(x_{n-1}, x_n)). \tag{4.3}$$

Iteratively, we derive that

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for all } n \in \mathbb{N}.$$

Since p is an E-distance then for $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m) \\ &\leq \psi^n(p(x_0, x_1)) + \psi^{n+1}(p(x_0, x_1)) + \cdots + \psi^{m-1}(p(x_0, x_1)). \end{aligned} \tag{4.4}$$

To show that $\{x_n\}$ is a p-Cauchy sequence, consider

$$S_n = \sum_{k=0}^{n} \psi^k(p(x_0, x_1)).$$

Thus from (4.4) we have

$$p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \tag{4.5}$$

Since $\psi \in \Psi$, there exists $S \in [0, \infty)$ such that $\lim_{n\to\infty} S_n = S$. Thus by (4.5) we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{4.6}$$

Since p is not symmetric, by repeating the same argument we have

$$\lim_{n,m\to\infty} p(x_m, x_n) = 0. \tag{4.7}$$

Hence the sequence $\{x_n\}$ is a p-Cauchy in the S-complete space X. Thus, there exists $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = 0$ which implies $\lim_{n\to\infty} x_n = u$. Since T is p-continuous, we have $\lim_{n\to\infty} p(Tx_n, Tu) = 0$, which implies that $\lim_{n\to\infty}(x_{n+1}, Tu) = 0$. Hence we have $\lim_{n\to\infty} p(x_n, u) = 0$ and $\lim_{n\to\infty}(x_n, Tu) = 0$. Thus by Lemma 2.1.4-(a) we have $u = Tu$. $\square$

In the following theorem, we omit the p-continuity by replacing a suitable condition on the obtained iterative sequence.

Theorem 4.1.3. Let $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;

(iii) for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Proof. By following the proof of Theorem 4.1.2, we know that $\{x_n\}$ is a p-Cauchy in the S-complete space X. Thus, there exists $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = 0$ which implies $\lim_{n\to\infty} x_n = u$. By using (4.1) and assumption (iii), we get

$$\begin{aligned} p(x_n, Tu) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, Tu) \\ &\leq p(x_n, x_{n+1}) + \alpha(x_n, u)p(Tx_n, Tu) \\ &\leq p(x_n, x_{n+1}) + \psi(p(x_n, u)). \end{aligned}$$

Letting $n \to \infty$ in above inequality, we shall have $\lim_{n\to\infty} p(x_n, Tu) = 0$. Hence we have $\lim_{n\to\infty} p(x_n, u) = 0$ and $\lim_{n\to\infty} p(x_n, Tu) = 0$. Thus by Lemma 2.1.4-(a) we have $u = Tu$. □

Example 4.1.4. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ be endowed with the usual metric d. Define $\vartheta = \{U_\epsilon | \epsilon > 0\}$. It is easy to see that (X, ϑ) is a uniform space. Define $T : X \to X$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{3n+1} & \text{if } x = \frac{1}{n} : n > 1 \\ 1 & \text{if } x = 1, \end{cases} \tag{4.8}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in X - \{1\} \\ 0 & \text{otherwise,} \end{cases} \tag{4.9}$$

and $\psi(t) = \frac{t}{3}$ for all $t \geq 0$. One can easily see that T is α-ψ-contractive and α-admissible mapping. Also for $x_0 = \frac{1}{2}$ we have $\alpha(x_0, Tx_0) = \alpha(Tx_0, x_0) = 1$. Moreover, for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_{n-1}, x_n) = 1$ for each $n \in \mathbb{N}$ we have $\alpha(x_n, x) = 1$ for each $n \in \mathbb{N}$. Therefore by Theorem 4.1.3, T has a fixed point.

In the sequel, we investigate the uniqueness of a fixed point. For this purpose, we introduce the following condition.

(H) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(z, x) \geq 1$ and $\alpha(z, y) \geq 1$.

Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

The following theorem guarantees the uniqueness of a fixed point.

Theorem 4.1.5. Adding the condition (H) in the hypothesis of Theorem 4.1.2 (respectively, Theorem 4.1.3), we obtain the uniqueness of fixed point of T.

Proof. Suppose, on the contrary, that $v \in X$ is another fixed point of T. From (H), there exists $z \in X$ such that

$$\alpha(z, u) \geq 1 \text{ and } \alpha(z, v) \geq 1. \tag{4.10}$$

Owing to the fact that T is α-admissible, from (4.10), we have

$$\alpha(T^n z, u) \geq 1 \text{ and } \alpha(T^n z, v) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{4.11}$$

We define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n = T^n z_0$ for all $n \in \mathbb{N} \cup \{0\}$ and $z_0 = z$. From (4.11) and (4.1), we have

$$p(z_{n+1}, u) = p(Tz_n, Tu) \leq \alpha(z_n, u)p(Tz_n, Tu) \leq \psi(p(z_n, u)), \tag{4.12}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$p(z_n, u) \leq \psi^n(p(z_0, u)), \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} p(z_n, u) = 0. \tag{4.13}$$

Similarly,

$$\lim_{n \to \infty} p(z_n, v) = 0. \tag{4.14}$$

From (4.13) and (4.14) together with Lemma 2.1.4-(a), it follows that $u = v$. Thus we have proved that u is the unique fixed point of T. □

Abdeljawad [5] introduced the following definition.

Definition 4.1.6. [5] A pair of two self mappings $T, Q : X \to X$ is said to be an α-admissible, if for any $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha(Tx, Qy) \geq 1$ and $\alpha(Qx, Ty) \geq 1$.

Definition 4.1.7. A pair of two self mappings $T, Q : X \to X$ is said to be an α-ψ-contractive pair if

$$\alpha(x, y) \max\{p(Tx, Qy), p(Qx, Ty)\} \leq \psi(p(x, y)), \tag{4.15}$$

for each $x, y \in X$, where $\psi \in \Psi$.

Theorem 4.1.8. Suppose that the pair of $T, Q : X \to X$ is an α-ψ-contractive pair satisfying the following conditions:

(i) (T, Q) is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;

(iii) for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.

Then T and Q have a common fixed point.

Proof. By hypothesis (ii) of theorem, we have $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$. Since (T, Q) is an α-admissible pair, then we can construct a sequence such that

$$Tx_{2n} = x_{2n+1}, \quad Qx_{2n+1} = x_{2n+2} \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \; \alpha(x_{n+1}, x_n) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

From (4.15) for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &= p(Tx_{2n}, Qx_{2n+1}) \\ &\leq \alpha(x_{2n}, x_{2n+1}) \max\{p(Tx_{2n}, Qx_{2n+1}), p(Qx_{2n}, Tx_{2n+1})\} \\ &\leq \psi(p(x_{2n}, x_{2n+1})). \end{aligned}$$

Hence, we conclude that

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1})). \tag{4.16}$$

Similarly, we find that

$$\begin{aligned} p(x_{2n+2}, x_{2n+3}) &= p(Qx_{2n+1}, Tx_{2n+2}) \\ &\leq \alpha(x_{2n+1}, x_{2n+2}) \max\{p(Tx_{2n+1}, Qx_{2n+2}), p(Qx_{2n+1}, Tx_{2n+2})\} \\ &\leq \psi(p(x_{2n+1}, x_{2n+2})). \end{aligned}$$

Hence, we derive that

$$p(x_{2n+2}, x_{2n+3}) \leq \psi(p(x_{2n+1}, x_{2n+2})). \tag{4.17}$$

Thus from (4.16) and (4.17), and by induction, we get

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for all } n \in \mathbb{N}. \tag{4.18}$$

We shall show that $\{x_n\}$ is a p-Cauchy sequence. Since p is an E-distance then for $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m) \\ &\leq \psi^n(p(x_0, x_1)) + \psi^{n+1}(p(x_0, x_1)) + \cdots + \psi^{m-1}(p(x_0, x_1)). \end{aligned} \tag{4.19}$$

Now, we shall consider

$$S_n = \sum_{k=0}^{n} \psi^k(p(x_0, x_1)).$$

Thus, from (4.19) we have

$$p(x_n, x_m) \leq S_{m-1} - S_{n-1}. \tag{4.20}$$

Since $\psi \in \Psi$, there exists $S \in [0, \infty)$ such that $\lim_{n\to\infty} S_n = S$. Thus, by (4.20) we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{4.21}$$

Since p is not symmetric, by repeating the same argument we have

$$\lim_{n,m\to\infty} p(x_m, x_n) = 0. \tag{4.22}$$

Hence the sequence $\{x_n\}$ is p-Cauchy in the S-complete space X. Thus, there exists $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = 0$ which implies $\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = u$. By using (4.15) and assumption (iii), we get

$$\begin{aligned} p(x_n, Tu) &\leq p(x_n, x_{2n+2}) + p(x_{2n+2}, Tu) \\ &= p(x_n, x_{2n+2}) + p(Qx_{2n+1}, Tu) \\ &\leq p(x_n, x_{2n+2}) + \alpha(x_{2n+1}, u) \max\{p(Tx_{2n+1}, Qu), p(Qx_{2n+1}, Tu)\} \\ &\leq p(x_n, x_{2n+2}) + \psi(p(x_{2n+1}, u)) \end{aligned} \tag{4.23}$$

Letting $n \to \infty$ in (4.23), we have $p(x_n, Tu) = 0$. Hence we have $\lim_{n\to\infty} p(x_n, u) = 0$ and $\lim_{n\to\infty} p(x_n, Tu) = 0$. Thus by Lemma 2.1.4-(a) we have $u = Tu$. Analogously, one can derive $u = Qu$. Therefore $u = Tu = Qu$. □

Remark 4.1.9. Note that Theorem 4.1.8 is valid if one replace condition (ii) with

(ii)' there exists $x_0 \in X$ such that $\alpha(x_0, Qx_0) \geq 1$ and $\alpha(Qx_0, x_0) \geq 1$.

We shall get the following result by letting $Q = I$ (that is, identity map) in Theorem 4.1.8.

Corollary 4.1.10. Suppose that a mapping $T : X \to X$ is satisfying the condition

$$\alpha(x, y) \max\{p(Tx, y), p(x, Ty)\} \leq \psi(p(x, y)),$$

for each $x, y \in X$, where $\psi \in \Psi$. Also suppose that the following conditions are satisfied:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;

(iii) for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Example 4.1.11. Let (X, d) is a dislocated metric space where $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $d(x, y) = \max\{x, y\}$. Define $\vartheta = \{U_\epsilon | \epsilon > 0\}$, where $U_\epsilon = \{(x, y) \in X^2 : d(x, y) < d(x, x) + \epsilon\}$. It is easy to see that (X, ϑ) is a uniform space. Define $T : X \to X$ by

$$Tx = \begin{cases} 0 \text{ if } x = 0 \\ \frac{1}{2n+1} \text{ if } x = \frac{1}{n} : n > 1 \\ 1 \text{ if } x = 1, \end{cases} \tag{4.24}$$

and $Q : X \to X$ by

$$Qx = \begin{cases} 0 \text{ if } x = 0 \\ \frac{1}{2n} \text{ if } x = \frac{1}{n} : n > 1 \\ 1 \text{ if } x = 1, \end{cases} \tag{4.25}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in X - \{1\} \\ 0 \text{ otherwise}, \end{cases} \tag{4.26}$$

and $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. One can easily see that (T, Q) is an α-ψ-contractive and α-admissible pair. Also for $x_0 = \frac{1}{2}$ we have $\alpha(x_0, Tx_0) = \alpha(Tx_0, x_0) = 1$. Moreover for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$ we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$. Therefore by Theorem 4.1.8, T and Q have a common fixed point.

To investigate the uniqueness of a common fixed point, we introduce the following condition.

(I) For each $x, y \in CFix(T, Q)$, we have $\alpha(x, y) \geq 1$, where $CFix(T, Q)$ is the set of all common fixed points of T and Q.

Theorem 4.1.12. Adding the condition (I) in the hypothesis of Theorem 4.1.8, we obtain the uniqueness of common fixed point of T and Q.

Proof. On the contrary suppose that $u, v \in X$ are two distinct common fixed points of T and Q. From (I) and (4.15) we have

$$p(u, v) \leq \alpha(u, v) \max\{p(Tu, Qv), p(Qu, Tv)\} \leq \psi(p(u, v)) < p(u, v),$$

which is impossible for $p(u, v) > 0$. Consequently, we have $p(u, v) = 0$. Analogously, one can show that $p(v, u) = 0$. Thus we have $u = v$, which is a contradiction to our assumption. Hence T and Q have a unique common fixed point. □

Consequences

Taking in Theorem 4.1.5, $\alpha(x, y) = 1$ for all $x, y \in X$, we immediately obtain the following fixed point theorems.

Corollary 4.1.13. Let $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that

$$p(Tx, Ty) \leq \psi(p(x, y)),$$

for all $x, y \in X$. Then T has a unique fixed point.

By substituting $\psi(t) = kt$, where $k \in [0, 1)$, in Corollary 4.1.13, we get the following

Corollary 4.1.14. Suppose that $T : X \to X$ be a given mapping satisfying

$$p(Tx, Ty) \leq kp(x, y),$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

Taking in Theorem 4.1.12, $\alpha(x, y) = 1$ for all $x, y \in X$, we immediately obtain the following common fixed point theorem.

Corollary 4.1.15. Let $T, Q : X \to X$ be given mappings. Suppose that there exists a function $\psi \in \Psi$ such that

$$\max\{p(Tx, Qy), p(Qx, Ty)\} \leq \psi(p(x, y)),$$

for all $x, y \in X$. Then T and Q have a unique common fixed point.

Taking in Corollary 4.1.10, $\alpha(x, y) = 1$ for all $x, y \in X$, we obtain the following result.

Corollary 4.1.16. Let $T : X \to X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that

$$\max\{p(Tx, y), p(x, Ty)\} \leq \psi(p(x, y)),$$

for all $x, y \in X$. Then T has a unique fixed point.

The notion of cyclic contraction was introduced by Kirk *et al.* [94]. The main advantage of the cyclic contraction is that the mapping satisfying this condition is not needed to be continuous. It has been appreciated by several authors, see e.g. [87, 88, 114, 124] and related reference therein. Now we show that a fixed point theorem for a mapping satisfying the cyclic contraction can also be obtained from our result.

Corollary 4.1.17. Let A_1, A_2 are nonempty closed subsets of X with respect to the topological space $(X, \tau(\vartheta))$. Let $T : Y \to Y$ be a mapping, where $Y = \cup_{i=1}^{2} A_i$. Suppose that the following conditions hold:

(i) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;

(ii) there exists a function $\psi \in \Psi$ such that

$$p(Tx, Ty) \leq \psi(p(x, y)), \text{ for all } (x, y) \in A_1 \times A_2.$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof. Since A_1 and A_2 are closed subsets of X, then (Y, d) is S-complete Hausdorff uniform space. Define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise.} \end{cases}$$

From (ii) and the definition of α, we can write

$$\alpha(x, y)p(Tx, Ty) \leq \psi(p(x, y)),$$

for all $x, y \in Y$. Thus T is an α-ψ-contractive mapping.

Let $(x, y) \in Y \times Y$ such that $\alpha(x, y) \geq 1$. If $(x, y) \in A_1 \times A_2$, from (i), $(Tx, Ty) \in A_2 \times A_1$, which implies that $\alpha(Tx, Ty) \geq 1$. If $(x, y) \in A_2 \times A_1$, from (i), $(Tx, Ty) \in A_1 \times A_2$, which implies that $\alpha(Tx, Ty) \geq 1$. Thus in all cases, we have $\alpha(Tx, Ty) \geq 1$. This implies that T is α-admissible.

Also, from (i), for any $a \in A_1$, we have $(a, Ta) \in A_1 \times A_2$, which implies that $\alpha(a, Ta) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$. This implies from the definition of α that

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \text{ for all } n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed subsets of X with respect to the topological space $(X, \tau(\vartheta))$, we get that

$$(x, x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $x \in A_1 \cap A_2$. Thus we can easily get from the definition of α that $\alpha(x_n, x) \geq 1$ for all n.

Finally, let $x, y \in \text{Fix}(T)$. From (i), this implies that $x, y \in A_1 \cap A_2$. So, for any $z \in Y$, we have $\alpha(z, x) \geq 1$ and $\alpha(z, y) \geq 1$. Thus condition (H) is satisfied.

Now, all the hypotheses of Theorem 4.1.5 are satisfied, and we deduce that T has a unique fixed point that belongs to $A_1 \cap A_2$ (from (i)). □

4.2 Fixed point theorem for a new type of multi valued contraction on partial Hausdorff metric spaces endowed with a graph

Jachymski [75] introduced the notion of Banach G-contraction to extend the notion of Banach contraction, where G is a graph in the metric space whose vertex set coincides with the metric space. He obtained some fixed point theorems for such mappings on complete metric space endowed with graph. Afterwards, many authors extended Banach G-contraction for single as well as for multi valued case, see for examples: Aleomraninejad *et al.* [13], Asl *et al.* [36], Beg *et al.* [41], Bojor [45, 46, 47], Nicolae *et al.* [113], Samreen and Kamran [130, 82, 129], and Tiammee and Suantai [133].

In this section, we first give a generalization of the comparison function by introducing the notion, called K-comparison function. With the help of this notion we introduce a new contractive condition on the structure of partial Hausdorff metric spaces endowed with a graph, called K_G-contractive condition. We further investigate the existence of fixed points for those mappings satisfying K_G-contractive condition.

Following is the definition of K-comparison function.

Definition 4.2.1. A mapping $\zeta : [0, \infty) \to [0, \infty)$ is said to be a K-comparison if the following conditions hold:

(i) for each $t > 0$, we have $\zeta(t) < t$;

(ii) $\zeta(0) = 0$.

Note that any comparison or (c)-comparison function is K-comparison function but converse is not true in general.

Example 4.2.2. Let $\zeta : [0, \infty) \to [0, \infty)$ be a mapping such that

$$\zeta(t) = \begin{cases} \frac{t}{2} & \text{if } 0 \leq t \leq 2 \\ \sqrt{t} & \text{otherwise.} \end{cases}$$

Then, ζ is a K-comparison function which is neither a comparison nor a (c)-comparison function.

We denote the class of K-comparison functions by $\mathfrak{K}$. Throughout this section: X is a nonempty set endowed with a partial metric p, that is, (X, p) is a partial metric space, and with a directed graph G, where the directed graph $G = (V, E)$ is such that the set of its vertices $V = X$ and the set of its edges contains all loops but has no parallel edge. Unless otherwise stated.

Definition 4.2.3. A mapping $T : X \to CB_p(X)$ is said to be a K_G-contractive, if there exists $\zeta \in \mathfrak{K}$ with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$ such that

(i) for each $(x, y) \in E$ with $x \neq y$, we have

$$H_p(Tx, Ty) \leq \zeta\Big(\max\Big\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}\Big\}\Big); \tag{4.27}$$

(ii) if $s \in Tx$ and $t \in Ty$ are such that

$$p(s,t) < p(x,y), \tag{4.28}$$

then we have $(s,t) \in E$, whenever $(x,y) \in E$ with $x \neq y$.

Jachymski [75] imposed the following property on the metric space to replace the continuity of a mapping satisfying the Banach G-contraction. We use this property in our result.

Property (A): ([75], Remark 3.1). For any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E$ for $n \in \mathbb{N}$, then $(x_n, x) \in E$.

In all the results of this section we assume that (X,p) is a complete partial metric space endowed with the graph G and Property (A).

Theorem 4.2.4. Let $T : X \to CB_p(X)$ be a K_G-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

Proof. By hypothesis, we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. If $x_0 = x_1$, then x_0 is a fixed point. Suppose that $x_0 \neq x_1$. Since T is a K_G-contractive mapping, from (4.27), we have

$$\begin{aligned} H_p(Tx_0, Tx_1) &\leq \zeta\Big(\max\Big\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{2}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\Big\}\Big) \\ &< \max\Big\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{2}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\Big\}. \end{aligned} \tag{4.29}$$

Then, there exists $a_1 \in (0, l]$, where $l = \sup_{t>0} \frac{\zeta(t)}{t}$, and obviously a_1 depends on x_0 and x_1, such that

$$H_p(Tx_0, Tx_1) \leq a_1 \max\Big\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{2}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\Big\}. \tag{4.30}$$

Since $a_1 < 1$, then $1/\sqrt{a_1} > 1$. Thus, by using Lemma 2.2.13, we have $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq \frac{1}{\sqrt{a_1}} H_p(Tx_0, Tx_1). \tag{4.31}$$

From (4.30) and (4.31), we get

$$\begin{aligned} p(x_1,x_2) &\leq \sqrt{a_1}\max\Big\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,x_2)}{2}, \frac{p(x_0,x_2)+p(x_1,x_1)}{2}\Big\} \\ &\leq \sqrt{a_1}\max\{p(x_0,x_1), p(x_1,x_2)\}. \end{aligned} \tag{4.32}$$

If we assume that $\max\{p(x_0,x_1), p(x_1,x_2)\} = p(x_1,x_2)$, then we get a contradiction to (4.32). Thus, $\max\{p(x_0,x_1), p(x_1,x_2)\} = p(x_0,x_1)$. From (4.32), we have

$$p(x_1,x_2) \leq \sqrt{a_1} p(x_0,x_1) < p(x_0,x_1). \tag{4.33}$$

From (4.28) and (4.33), we have $(x_1, x_2) \in E$. If $x_1 = x_2$, then x_1 is a fixed point. Suppose that $x_1 \neq x_2$. Again, from (4.27), we have

$$\begin{aligned} H_p(Tx_1, Tx_2) &\leq \zeta\Big(\max\Big\{p(x_1, x_2), \frac{p(x_1, Tx_1) + p(x_2, Tx_2)}{2}, \frac{p(x_1, Tx_2) + p(x_2, Tx_1)}{2}\Big\}\Big) \\ &< \max\Big\{p(x_1, x_2), \frac{p(x_1, Tx_1) + p(x_2, Tx_2)}{2}, \frac{p(x_1, Tx_2) + p(x_2, Tx_1)}{2}\Big\}. \end{aligned}$$

Then, there exists $a_2 \in (0, l]$, and obviously a_2 depends on x_1 and x_2, such that

$$H_p(Tx_1, Tx_2) \leq a_2 \max\Big\{p(x_1, x_2), \frac{p(x_1, Tx_1) + p(x_2, Tx_2)}{2}, \frac{p(x_1, Tx_2) + p(x_2, Tx_1)}{2}\Big\}. \quad (4.34)$$

Since $a_2 < 1$, then $1/\sqrt{a_2} > 1$. Again by using Lemma 2.2.13, we have $x_3 \in Tx_2$ such that

$$p(x_2, x_3) \leq \frac{1}{\sqrt{a_2}} H_p(Tx_1, Tx_2). \quad (4.35)$$

From (4.34) and (4.35), we get

$$\begin{aligned} p(x_2, x_3) &\leq \sqrt{a_1} \max\Big\{p(x_1, x_2), \frac{p(x_1, x_2) + p(x_2, x_3)}{2}, \frac{p(x_1, x_3) + p(x_2, x_2)}{2}\Big\} \\ &\leq \sqrt{a_1} \max\{p(x_1, x_2), p(x_2, x_3)\}. \quad (4.36) \end{aligned}$$

If we assume that $\max\{p(x_1, x_2), p(x_2, x_3)\} = p(x_2, x_3)$, then we get a contradiction to (4.36). Thus, $\max\{p(x_1, x_2), p(x_2, x_3)\} = p(x_1, x_2)$. From (4.36), we have

$$p(x_2, x_3) \leq \sqrt{a_2} p(x_1, x_2) < p(x_1, x_2). \quad (4.37)$$

Also, we have

$$p(x_2, x_3) \leq \sqrt{a_2} p(x_1, x_2) \leq \sqrt{a_2}\sqrt{a_1} p(x_0, x_1).$$

Continuing the same way we get sequences $\{a_n\} \subset (0, l]$ and $\{x_n\} \subset X$ such that $x_n \in Tx_{n-1}$, $x_{n-1} \neq x_n$ and $(x_{n-1}, x_n) \in E$, with

$$p(x_n, x_{n+1}) \leq \sqrt{a_n}\sqrt{a_{n-1}} \cdots \sqrt{a_1} p(x_0, x_1) \text{ for each } n \in \mathbb{N}.$$

Let $n, m \in \mathbb{N}$, by using the triangular inequality, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m}) - \sum_{i=n+1}^{n+m-1} p(x_i, x_i) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m}) \\ &\leq \sqrt{a_n}\sqrt{a_{n-1}} \cdots \sqrt{a_1} p(x_0, x_1) + \sqrt{a_{n+1}}\sqrt{a_n} \cdots \sqrt{a_1} p(x_0, x_1) \\ &\quad + \cdots + \sqrt{a_{n+m-1}}\sqrt{a_{n+m-2}} \cdots \sqrt{a_1} p(x_0, x_1). \quad (4.38) \end{aligned}$$

Let $b = \sup\{\sqrt{a_i} : i \in \mathbb{N}\}$, clearly, $b < 1$. Then from (4.38), we get

$$\begin{aligned} p(x_n, x_{n+m}) &\leq \sqrt{a_n}\sqrt{a_{n-1}} \cdots \sqrt{a_1} p(x_0, x_1) + \sqrt{a_{n+1}}\sqrt{a_n} \cdots \sqrt{a_1} p(x_0, x_1) \\ &\quad + \cdots + \sqrt{a_{n+m-1}}\sqrt{a_{n+m-2}} \cdots \sqrt{a_1} p(x_0, x_1) \\ &\leq [b^n + b^{n+1} + \cdots + b^{n+m-1}] p(x_0, x_1) \\ &< \frac{b^n}{1-b} p(x_0, x_1). \quad (4.39) \end{aligned}$$

Consequently, we have

$$d_p(x_n, x_{n+m}) \leq 2p(x_n, x_{n+m}) < \frac{2b^n}{1-b} p(x_0, x_1).$$

Thus, we conclude that $\{x_n\}$ is Cauchy in (X, d_p). Since (X, p) is a complete partial metric space, by Lemma 2.2.9-(b), (X, d_p) is a complete metric space. Then there exists $x^* \in X$ such that $x_n \to x^* \in X$ with respect to d_p. By Lemma 2.2.8, we have

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_{n+m}) = 0. \tag{4.40}$$

By Property (A), we have $(x_n, x^*) \in E$ for each $n \in \mathbb{N}$. Now, we claim that $p(x^*, Tx^*) = 0$. On contrary suppose that $p(x^*, Tx^*) > 0$. By using the triangular inequality and (4.27), we have

$$\begin{aligned}
p(x^*, Tx^*) &\leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) - p(x_{n+1}, x_{n+1}) \\
&\leq p(x^*, x_{n+1}) + H_p(Tx_n, Tx^*) \\
&\leq p(x^*, x_{n+1}) + \zeta\Big(\max\Big\{p(x_n, x^*), \frac{p(x_n, Tx_n) + p(x^*, Tx^*)}{2}, \\
&\quad \frac{p(x_n, Tx^*) + p(x^*, Tx_n)}{2}\Big\}\Big) \\
&< p(x^*, x_{n+1}) + \max\Big\{p(x_n, x^*), \frac{p(x_n, Tx_n) + p(x^*, Tx^*)}{2}, \\
&\quad \frac{p(x_n, Tx^*) + p(x^*, Tx_n)}{2}\Big\} \\
&\leq p(x^*, x_{n+1}) + \max\Big\{p(x_n, x^*), \frac{p(x_n, x_{n+1}) + p(x^*, Tx^*)}{2}, \\
&\quad \frac{p(x_n, x^*) + p(x^*, Tx^*) - p(x^*, x^*) + p(x^*, x_{n+1})}{2}\Big\}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality,

$$p(x^*, Tx^*) \leq \frac{p(x^*, Tx^*)}{2}.$$

Which is impossible for $p(x^*, Tx^*) > 0$. Thus, $p(x^*, Tx^*) = 0$. Therefore, we have

$$p(x^*, Tx^*) = 0 = p(x^*, x^*).$$

This implies that $x^* \in Tx^*$. □

Example 4.2.5. Let $X = \mathbb{N} \cup \{0\}$ be endowed with a partial metric $p(x, y) = \max\{x, y\}$ and a graph $G = (V, E)$ define as $V = X$ and $E = \{(x, y) : x, y \in \{0, 2, 4, 6, 8, 10, 12, 14, 16\}\} \cup \{(x, x) : x \in \mathbb{N}\}$. Let $T : X \to CB_p(X)$ is defined by

$$Tx = \begin{cases} \{0\} & \text{if } x \in \{0, 2, 6, 10, 12, 14\} \\ \{0, 2\} & \text{if } x = 4 \\ \{0, 4\} & \text{if } x = 8 \\ \{0, 8\} & \text{if } x = 16 \\ \{x+1, x+2\} & \text{otherwise,} \end{cases}$$

and $\zeta : [0,\infty) \to [0,\infty)$ by

$$\zeta(t) = \begin{cases} \frac{t}{2} & \text{if } 0 \le t \le 4 \\ \frac{2t}{3} & \text{if } 4 < t \le 8 \\ \frac{t}{2} & \text{if } t > 8. \end{cases}$$

To see that (4.27) holds, it is sufficient to consider the following cases:

(i) If $(x,y) \in E$ with $x \neq y \in \{0,2,6,10,12,14\}$, then (4.27) trivially holds.

(ii) If $(x,y) \in E$ with $x \in \{0,2,6,10,12,14\}$ and $y = 4$, then

$$H_p(Tx,Ty) = 2 \le \zeta(M_p(x,y)).$$

(iii) If $(x,y) \in E$ with $x \in \{0,2,6,10,12,14\}$ and $y = 8$, then

$$H_p(Tx,Ty) = 4 \le \zeta(M_p(x,y)).$$

(iv) If $(x,y) \in E$ with $x \in \{0,2,6,10,12,14\}$ and $y = 16$, then

$$H_p(Tx,Ty) = 8 \le \zeta(M_p(x,y)).$$

(v) If $(x,y) \in E$ with $x = 4$ and $y = 8$, then

$$H_p(Tx,Ty) = 4 \le \zeta(M_p(x,y)).$$

(vi) If $(x,y) \in E$ with $x = 4$ and $y = 16$, then

$$H_p(Tx,Ty) = 8 \le \zeta(M_p(x,y)).$$

(vii) If $(x,y) \in E$ with $x = 8$ and $y = 16$, then

$$H_p(Tx,Ty) = 8 \le \zeta(M_p(x,y)).$$

where $M_p(x,y) = \max\left\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(y,Tx)}{2}\right\}$. Thus, (4.27) holds. Further it can be observed that for $(x,y) \in E$ with $x \neq y$, if $s \in Tx$ and $t \in Ty$ are such that $p(s,t) < p(x,y)$ then we have $(s,t) \in E$. For $x_0 = 8$, we have $x_1 = 4 \in Tx_0$ such that $(8,4) \in E$. Moreover, Property (A) holds. Therefore, Theorem 4.2.4 guarantees the existence of a fixed point of T.

Example 4.2.6. Let $X = [0,\infty)\times[0,\infty)$ be endowed with a partial metric $p(x,y) = p((x_1,x_2),(y_1,y_2)) = \max\{x_1,y_1\} + \max\{x_2,y_2\}$ and a graph $G = (V,E)$ define as $V = X$ and $E = \{(r,s) : r = (r_1,0), s = (s_1,0) \text{ with } r_1, s_1 \geq 0\} \cup \{(x,x) : x \in X\}$. Let $T : X \to CB_p(X)$ is defined by

$$T(b,a) = \{(0,0),(b/2,a)\} \quad \text{for each } (b,a) \in X,$$

and $\zeta : [0, \infty) \to [0, \infty)$ by

$$\zeta(t) = \begin{cases} \frac{2t}{3} & \text{if } 0 \le t \le 10 \\ \frac{t}{2} & \text{otherwise.} \end{cases}$$

To see that (4.27) holds, we consider the following cases:

(i) If $((u, 0), (v, 0)) \in E$ with $0 \le u < v$, then

$$H_p(T(u, 0), T(v, 0)) = \frac{v}{2} \le \zeta(M_p(x, y)).$$

(ii) If $((u, 0), (v, 0)) \in E$ with $0 \le v < u$, then

$$H_p(T(u, 0), T(v, 0)) = \frac{u}{2} \le \zeta(M_p(x, y)).$$

where $M_p(x, y) = \max\left\{p(x, y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(y,Tx)}{2}\right\}$. Thus, (4.27) holds. Further it can be observed that for $(x, y) \in E$ with $x \neq y$, if $s \in Tx$ and $t \in Ty$ are such that $p(s, t) < p(x, y)$ then we have $(s, t) \in E$. For $x_0 = (1, 0)$, we have $x_1 = (0.5, 0) \in Tx_0$ such that $((1, 0), (0.5, 0)) \in E$. Moreover, Property (A) holds. Therefore, Theorem 4.2.4 guarantees the existence of fixed point of T.

Consequences

Here, we present some other results which can be obtained as a consequence of our result.

By considering $\zeta(t) = \phi(t)t$ for each $t \ge 0$ in Theorem 4.2.4, we get the following result, where ϕ is the same function defined in this corollary.

Corollary 4.2.7. Let $T : X \to CB_p(X)$ be a mapping such that

(i) for each $(x, y) \in E$ with $x \neq y$, we have

$$H_p(Tx, Ty) \le \phi(M_p(x, y))M_p(x, y),$$

where, $M_p(x, y) = \max\left\{p(x, y), \frac{p(x,Tx)+p(y,Ty)}{2}, \frac{p(x,Ty)+p(y,Tx)}{2}\right\}$ and $\phi : [0, \infty) \to [0, 1)$ is such that $\limsup_{t \to r^+} \phi(t) < 1$ for each $r \in [0, \infty)$;

(ii) if $s \in Tx$ and $t \in Ty$ are such that $p(s, t) < p(x, y)$, then we have $(s, t) \in E$, whenever $(x, y) \in E$ with $x \neq y$.

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

It is easy to see that $\zeta(t) = \phi(t)t$ is K-comparison function. Since $\phi : [0, \infty) \to [0, 1)$ is such that $\limsup_{t \to r^+} \phi(t) < 1$ for each $r \in [0, \infty)$, this implies that $\sup_{t>0} \phi(t) < 1$. Thus we have $\zeta(t) = \phi(t)t < t$ and $\zeta(0) = 0$.

Our result is even new if we consider it on the structure of metric space, which is of the following form:

Corollary 4.2.8. Let (X, d) be a complete metric space endowed with a graph G and Property (A). Let $T : X \to CB(X)$ be a mapping such that

(i) for each $(x, y) \in E$ with $x \neq y$, we have

$$H(Tx, Ty) \leq \zeta\Big(\max\Big\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}\Big),$$

where ζ is K-comparison function with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$;

(ii) if $s \in Tx$ and $t \in Ty$ are such that $d(s, t) < d(x, y)$, then we have $(s, t) \in E$, whenever $(x, y) \in E$ with $x \neq y$.

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

Every comparison function or (c)-comparison function is a K-comparison function. Thus the following two results immediately follow from Theorem 4.2.4.

Corollary 4.2.9. Let $T : X \to CB_p(X)$ be a mapping such that

(i) for each $(x, y) \in E$ with $x \neq y$, we have

$$H_p(Tx, Ty) \leq \zeta\Big(\max\Big\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}\Big\}\Big),$$

where ζ is comparison function with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$;

(ii) if $s \in Tx$ and $t \in Ty$ are such that $p(s, t) < p(x, y)$, then we have $(s, t) \in E$, whenever $(x, y) \in E$ with $x \neq y$.

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

Corollary 4.2.10. Let $T : X \to CB_p(X)$ be a mapping such that

(i) for each $(x, y) \in E$ with $x \neq y$, we have

$$H_p(Tx, Ty) \leq \zeta\Big(\max\Big\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}\Big\}\Big),$$

where ζ is (c)-comparison function with $\sup_{t>0} \frac{\zeta(t)}{t} < 1$;

(ii) if $s \in Tx$ and $t \in Ty$ are such that $p(s, t) < p(x, y)$, then we have $(s, t) \in E$, whenever $(x, y) \in E$ with $x \neq y$.

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

We get the following result by taking $\zeta(t) = at$ for each $t \geq 0$, where $a \in [0, 1)$.

Corollary 4.2.11. Let $T : X \to CB_p(X)$ be a mapping such that

(i) for each $(x, y) \in E$ with $x \neq y$, we have

$$H_p(Tx, Ty) \leq a \max\Big\{p(x, y), \frac{p(x, Tx) + p(y, Ty)}{2}, \frac{p(x, Ty) + p(y, Tx)}{2}\Big\},$$

where $a \in [0, 1)$;

(ii) if $s \in Tx$ and $t \in Ty$ are such that $p(s, t) < p(x, y)$, then we have $(s, t) \in E$, whenever $(x, y) \in E$ with $x \neq y$.

Further, assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. Then T has a fixed point.

4.3 Fixed point theorems for F-contraction in b-metric spaces

Cosentino *et al*. [58] introduced a new class of F-contractions in the setting of b-metric spaces. For this purpose they first extended the $\mathfrak{F}$ family to b-metric spaces in the following way:

Let $s \geq 1$ be a real number. Denoted $\mathfrak{F}_s$ by [58] the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is strictly increasing, that is, for each $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$;

(F_2) for each sequence $\{\mathfrak{d}_n\}$ of positive real numbers, we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$;

(F_3) for each sequence $\{\mathfrak{d}_n\}$ of positive real numbers with $\lim_{n\to\infty} \mathfrak{d}_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} {\mathfrak{d}_{\mathfrak{n}}}^k F(\mathfrak{d}_{\mathfrak{n}}) = 0$.

(F_4) for each sequence $\{\mathfrak{d}_n\}$ of positive real numbers such that $\tau + F(s\mathfrak{d}_n) \leq F(\mathfrak{d}_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n\mathfrak{d}_n) \leq F(s^{n-1}\mathfrak{d}_{n-1})$ for each $n \in \mathbb{N}$.

Cosentino *et al*. [58] also showed that the following functions belong to $\mathfrak{F}_s$.

- $F(x) = x + \ln x$, for each $x > 0$.
- $F(x) = \ln x$, for each $x > 0$.

In this section, we first introduce two new Feng and Liu type F-contractions which involve a function α and then establish fixed point theorems for these contractions in the setting of b-metric spaces. We also provide an example to support the result. Finally, we apply the result to obtain existence theorems for Fredholm integral equation in b-metric spaces.

Throughout this section: X is a nonempty set endowed with a b-metric d, that is, (X, d, s) is b-metric space. Further, in all the results, we assume that (X, d, s) is complete b-metric space with $s > 1$. Also note that $\alpha : X \times X \to [0, \infty)$ is a function and $\alpha^*(A, B) = \inf\{\alpha(a, b) : a \in A \text{ and } b \in B\}$.

In following definition, we introduce the notions of α_s-admissible and α_s^*-admissible mappings.

Definition 4.3.1. A mapping $T : X \to CL(X)$ is:

- α_s-admissible if for $x \in X$ and $y \in Tx$ such that $\alpha(x, y) \geq s^2$, we have $\alpha(y, z) \geq s^2$ for each $z \in Ty$.
- α_s^*-admissible if for $x, y \in X$ with $\alpha(x, y) \geq s^2$, we have $\alpha^*(Tx, Ty) \geq s^2$, where $\alpha^*(Tx, Ty) = \inf\{\alpha(u, v) : u \in Tx \text{ and } v \in Ty\}$.

Remark 4.3.2. Note that for $s = 1$ above definition reduces to α-admissible and α_*-admissible, as defined in [110] and [35], respectively.

Example 4.3.3. Let $X = [-1, 1]$ endowed with a b-metric $d(x, y) = |x - y|^2$ with $s = 2$. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} \{0, 1\} & \text{if } x = -1 \\ \{1\} & \text{if } x = 0 \\ \{-x\} & \text{if } x \notin \{-1, 0\} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x = y \\ 5 & \text{if } x \neq y. \end{cases}$$

It is straight forward to see that T is α_s-admissible but not α_s^*-admissible.

Now, we introduce the notion of Feng-Liu-type (F, α)-contraction:

Definition 4.3.4. A mapping $T : X \to CL(X)$ is called Feng-Liu-type (F, α)-contraction, if there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$\tau + F(\alpha(x, y)d(y, Ty)) \leq F(d(x, y)), \tag{4.41}$$

for each $x \in X$ and $y \in Tx$, whenever $\min\{\alpha(x, y)d(y, Ty), d(x, y)\} > 0$.

As we know that, if for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$ for each $n \in \mathbb{N}$, then $O(x_0, T) = \{x_0, x_1, x_2, \cdots\}$ is said to be an orbit of $T : X \to CL(X)$. A mapping $g : X \to \mathbb{R}$ is said to be T-orbitally lower semi continuous at ξ, if $\{x_n\}$ is a sequence in $O(x_0, T)$ and $x_n \to \xi$ implies $g(\xi) \leq \liminf_{n\to\infty} g(x_n)$.

Theorem 4.3.5. Let $T : X \to CL(X)$ be an α_s-admissible Feng-Liu-type (F, α)-contraction with $s > 1$. Assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq s^2$. Then

(i) there exist an orbit $O(x_0, T) = \{x_n : x_n \in Tx_{n-1} \forall n \in \mathbb{N}\}$ and $\xi \in X$ such that $\lim x_n = \xi$;

(ii) ξ is a fixed point of T if and only if $g(x) = d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Proof. By hypothesis, there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq s^2$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. As $\alpha(x_0, x_1) \geq s^2$, there exists $x_2 \in Tx_1$ such that

$$sd(x_1, x_2) \leq \alpha(x_0, x_1) d(x_1, Tx_1). \tag{4.42}$$

Since F is strictly increasing, we have

$$F(sd(x_1, x_2)) \leq F(\alpha(x_0, x_1) d(x_1, Tx_1)). \tag{4.43}$$

From (4.41), we have

$$\tau + F(sd(x_1, x_2)) \leq \tau + F(\alpha(x_0, x_1) d(x_1, Tx_1)) \leq F(d(x_0, x_1)). \tag{4.44}$$

Since T is α_s-admissible, we have $\alpha(x_1, x_2) \geq s^2$. Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in Tx_{n-1},\ x_{n-1} \neq x_n \text{ and } \alpha(x_{n-1}, x_n) \geq s^2 \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$\tau + F(sd(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) \text{ for each } n \in \mathbb{N}. \tag{4.45}$$

In other words, we also have

$$d(x_n, Tx_n) < sd(x_n, Tx_n) \leq sd(x_n, x_{n+1}) < d(x_{n-1}, x_n) \text{ for each } n \in \mathbb{N}. \tag{4.46}$$

Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. Thus by (4.45) and property(F_4), we get

$$\tau + F(s^n d_n) \leq F(s^{n-1} d_{n-1}) \text{ for each } n \in \mathbb{N}.$$

Consequently, we get

$$F(s^n d_n) \leq F(d_0) - n\tau \text{ for each } n \in \mathbb{N}. \tag{4.47}$$

Letting $n \to \infty$ in (4.47), we get $\lim_{n\to\infty} F(s^n d_n) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} s^n d_n = 0$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (s^n d_n)^k F(s^n d_n) = 0.$$

From (4.47) we have

$$(s^n d_n)^k F(s^n d_n) - (s^n d_n)^k d_n^k F(d_0) \leq -(s^n d_n)^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{4.48}$$

Letting $n \to \infty$ in (4.48), we get

$$\lim_{n\to\infty} n(s^n d_n)^k = 0. \tag{4.49}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^n d_n)^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$s^n d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{4.50}$$

To prove that $\{x_n\}$ is Cauchy. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (4.50), we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} s^i d_i \leq \sum_{i=n}^{\infty} s^i d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is a convergent series. Thus, $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is Cauchy. As (X, d, s) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Suppose that $g(x) = d(x, Tx)$ is T-orbitally lower semi continuous at x^*. Then by using (4.46), we have

$$d(x^*, Tx^*) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

Which shows that x^* is a fixed point T. Conversely, if x^* is a fixed point then trivially $g(x) = d(x, Tx)$ is T-orbitally lower semi continuous at x^*. $\square$

Example 4.3.6. Let $X = \mathbb{N} \cup \{0\}$ be endowed with a b-metric $d(x, y) = |x - y|^2$ for each $x, y \in X$ with $s = 2$. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} \{0\} & \text{if } x = 0 \\ \{0, 1, 2, \cdots, x-1\} & \text{if } x \geq 1 \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 5 & \text{if } x, y \in \{0, 1\} \\ 1 & \text{if } x, y > 1 \text{ and } y \neq x - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Take $F(x) = x + \ln x$ for each $x \in (0, \infty)$. Under this F, condition (4.41) reduces to

$$\frac{\alpha(x, y)d(y, Ty)}{d(x, y)} e^{\alpha(x,y)d(y,Ty) - d(x,y)} \leq e^{-\tau} \tag{4.51}$$

for each $x \in X$ and $y \in Tx$ with $\min\{\alpha(x, y)d(y, Ty), d(x, y)\} > 0$. For $\tau = \frac{1}{2}$, it is easy to see that (4.51) satisfies for each $x \in X$ and $y \in Tx$ with $\min\{\alpha(x, y)d(y, Ty), d(x, y)\} > 0$. Thus, T is Feng-Liu-type (F, α)-contraction with $F(x) = x + \ln x$. For $x_0 = 1$, we have $x_1 = 0 \in Tx_0$ such that $\alpha(x_0, x_1) = 5$. Moreover, all other conditions of Theorem 4.3.10 are trivially satisfied. Therefore, T has a fixed point in X.

In following definition we define Feng-Liu-type (F, α^*)-contraction:

Definition 4.3.7. A mapping $T : X \to CL(X)$ is called Feng-Liu-type (F, α^*)-contraction, if there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$\tau + F(\alpha^*(Tx, Ty)d(y, Ty)) \leq F(d(x, y)),$$

for each $x \in X$ and $y \in Tx$, whenever $\min\{\alpha^*(Tx, Ty)d(y, Ty), d(x, y)\} > 0$.

Theorem 4.3.8. Let $T : X \to CL(X)$ be an α_s^*-admissible Feng-Liu-type (F, α^*)-contraction with $s > 1$. Assume that there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq s^2$. Then

(i) there exist an orbit $O(x_0, T) = \{x_n : x_n \in Tx_{n-1} \forall n \in \mathbb{N}\}$ and $\xi \in X$ such that $\lim x_n = \xi$;

(ii) ξ is a fixed point of T if and only if $g(x) = d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Consequences

By using these results we obtain some new fixed point theorems for multi valued mappings in the setting of b-metric space endowed with a partial ordering/graph. If we define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} s^2 & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

then the following result is a direct consequence of the above mentioned results.

Theorem 4.3.9. Let $T : X \to CL(X)$ be a mapping on complete ordered b-metric space $(X, d, s, \preceq)$ with $s > 1$, for which, there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$\tau + F(s^2 d(y, Ty)) \leq F(d(x, y)), \tag{4.52}$$

for each $x \in X$ and $y \in Tx$ with $x \preceq y$, whenever $\min\{s^2 d(y, Ty), d(x, y)\} > 0$. Moreover, the following conditions hold:

(i) for each $x \in X$ and $y \in Tx$ such that $x \preceq y$, we have $y \preceq z$ for each $z \in Ty$;
or
If $x \preceq y$, then we have $Tx \prec_r Ty$, that is, for each $a \in Tx$ and $b \in Ty$, we have $a \preceq b$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $x_0 \preceq x_1$;

Then, there exist an orbit $O(x_0, T) = \{x_n : x_n \in Tx_{n-1} \forall n \in \mathbb{N}\}$ and $\xi \in X$ such that $\lim x_n = \xi$. Furthermore, ξ is a fixed point of T if and only if $g(x) = d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Let $G = (V, E)$ is a directed graph such that the set of its vertices V coincides with X and the set of its edges E is such that $E \supseteq \triangle$, where $\triangle = \{(x, x) : x \in X\}$. Also assume that G has no parallel edges. If we define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} s^2 & \text{if } (x, y) \in E \\ 0 & \text{otherwise} \end{cases}$$

then the following result is direct consequence of the main results.

Theorem 4.3.10. Let $T : X \to CL(X)$ be a mapping on complete b-metric space (X, d, s) endowed with the graph G, having $s > 1$, for which, there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$\tau + F(s^2 d(y, Ty)) \leq F(d(x, y)), \tag{4.53}$$

for each $x \in X$ and $y \in Tx$ with $(x, y) \in E$, whenever $\min\{s^2 d(y, Ty), d(x, y)\} > 0$. Moreover, the following conditions hold:

(i) for each $x \in X$ and $y \in Tx$ such that $(x, y) \in E$, we have $(y, z) \in E$ for each $z \in Ty$;
   or
   If $(x, y) \in E$, then we have $(a, b) \in E$ for each $a \in Tx$ and $b \in Ty$;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $(x_0, x_1) \in E$;

Then, there exist an orbit $O(x_0, T) = \{x_n : x_n \in Tx_{n-1} \forall n \in \mathbb{N}\}$ and $\xi \in X$ such that $\lim x_n = \xi$. Furthermore, ξ is a fixed point of T if and only if $g(x) = d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Applications

In this section, we give existence theorems for Fredholm integral equations. For this purpose, let $X = C([a, b], \mathbb{R})$ be the space of all continuous real valued functions define on $[a, b]$. Note that X is complete b-metric space by considering $d(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2$, with $s = 2$.

Consider the Fredholm integral equation as

$$x(t) = \int_a^b M(t, s, x(s))ds + g(t), \quad t, s \in [a, b] \tag{4.54}$$

where $g \colon [a, b] \to \mathbb{R}$ and $M \colon [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 4.3.11. Let $X = C([a, b], \mathbb{R})$ and let $T \colon X \to X$ be the following operator

$$Tx(t) = \int_a^b M(t, s, x(s))ds + g(t) \text{ for } t, s \in [a, b]$$

where, the function $g \colon [a, b] \to \mathbb{R}$ and $M \colon [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous. Further, assume that the following conditions hold:

(i) there exists a continuous mapping $q\colon X \to [0,\infty)$ such that

$$|M(t,s,x(s)) - M(t,s,Tx(s))| \leq q(s)|x(s) - Tx(s)| \text{ for each } t,s \in [a,b] \text{ and } x \in X.$$

(ii) there exist $\tau > 0$ and $\alpha\colon X \times X \to (0,\infty)$ such that for each $x \in X$, we have

$$\int_a^b q(s)ds \leq \sqrt{\frac{e^{-\tau}}{\alpha(x,Tx)}};$$

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 4$;

(iv) if $x \in X$ is such that $\alpha(x,Tx) \geq 4$, then we have $\alpha(Tx, T^2x) \geq 4$;

(v) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 4$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 4$ for each $n \in \mathbb{N}$.

Then the integral equation (4.54) has a solution.

Proof. We have to show that the operator T satisfies all conditions of Theorem 4.3.5(in single valued case). First, we show that T is (single valued) Feng-Liu-type (F,α)-contraction. For any $x \in X$ we have

$$\begin{aligned}
|Tx(t) - T(Tx(t))|^2 &\leq \left(\int_a^b |M(t,s,x(s)) - M(t,s,Tx(s))|ds\right)^2 \\
&\leq \left(\int_a^b q(s)|x(s) - Tx(s)|ds\right)^2 \\
&\leq \left(\sqrt{\sup_{s\in[a,b]} |x(s) - Tx(s)|^2} \times \int_a^b q(s)ds\right)^2 \\
&= d(x,Tx)\left(\int_a^b q(s)ds\right)^2 \\
&\leq \frac{e^{-\tau}}{\alpha(x,Tx)} d(x,Tx).
\end{aligned}$$

Consequently, we have

$$\alpha(x,Tx)d(Tx,T^2x) \leq e^{-\tau} d(x,Tx) \text{ for each } x \in X.$$

As natural logarithm belongs to $\mathfrak{F}_s$. Applying it on above sides of inequality, and after some simplification, we get

$$\tau + \ln(\alpha(x,Tx)d(Tx,T^2x)) \leq \ln(d(x,Tx)) \text{ for each } x \in X.$$

Thus, $T : X \to X$ is (single valued) Feng-Liu-type (F,α)-contraction with $F(x) = \ln x$. All other conditions of Theorem 4.3.5(in single valued case) are immediately follows by the hypothesis. Therefore, the operator T has a fixed point, that is, the Fredholm integral equation (4.54) has a solution. □

If we define $\alpha(x,y)=4$ for each $x,y\in X$, then above theorem reduces to the following:

Theorem 4.3.12. Let $X=C([a,b],\mathbb{R})$ and let $T\colon X\to X$ be the following operator

$$Tx(t)=\int_a^b M(t,s,x(s))ds+g(t) \text{ for } t,s\in[a,b]$$

where, the function $g\colon[a,b]\to\mathbb{R}$ and $M\colon[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ are continuous. Further, assume that the following conditions hold:

(i) there exists a continuous mapping $q\colon X\to[0,\infty)$ such that

$$|M(t,s,x(s))-M(t,s,Tx(s))|\le q(s)|x(s)-Tx(s)|,$$

for each $t,s\in[a,b]$ and $x\in X$.

(ii) there exists $\tau>0$ such that

$$\int_a^b q(s)ds\le\sqrt{\frac{e^{-\tau}}{4}}.$$

Then the integral equation (4.54) has a solution.

4.4 Fixed point theorems for multi valued G-contractions in Hausdorff b-gauge spaces

Frigon [67] generalized the Banach contraction principle on gauge spaces. Later on many authors continued to work in this direction and obtained several interesting results, see for example, Agarwal *et al.* [9], Cherichi *et al.* [53, 54], Chifu and Petrusel [55], Chis and Precup [56], Lazara and Petrusel [99] and Jleli [76].

In this section, we extend gauge spaces in the setting of b-pseudo metric spaces and prove some fixed point theorems for multi valued mappings in this new setting endowed with a graph. To substantiate the result we construct an example. Moreover, we also discuss a possible application of the result for solving an integral equation.

Now, we begin by introducing the notion of a b_s-pseudo metric space.

Definition 4.4.1. Let X be a nonempty set. A function $d:X\times X\to[0,\infty)$ is called b_s-pseudo metric on X if there exists $s\ge1$ such that for each $x,y,z\in X$, we have

(i) $d(x,x)=0$ for each $x\in X$;

(ii) $d(x,y)=d(y,x)$;

(iii) $d(x,z)\le s[d(x,y)+d(y,z)]$.

Remark 4.4.2. Every b-metric space (X,d,s) is a b_s-pseudo metric space, but the converse is not true.

Example 4.4.3. Let $X = C([0,\infty),\mathbb{R})$. Define a function $d : X \times X \to [0,\infty)$ by $d(x(t), y(t)) = \max_{t\in[0,1]}(x(t) - y(t))^2$. Then:

(i) It is clear that d is not a metric on X.

(ii) d is not a pseudo metric on X. In this respect, consider $x, y, z \in C([0,\infty),\mathbb{R})$ be defined by

$$x(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ t-1 & \text{if } t > 1, \end{cases}$$

$y(t) = 3$ for each $t \ge 0$ and $z(t) = -3$ for each $t \ge 0$. Then, we can see that $d(y, z) = 36 \nleq 18 = d(y, x) + d(x, z)$.

(iii) d is not a b-metric on X. Since, if $u, v \in C([0,\infty),\mathbb{R})$ are defined by

$$u(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ t-1 & \text{if } t > 1, \end{cases}$$

and

$$v(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ 2t-2 & \text{if } t > 1, \end{cases}$$

then $u \ne v$, but $d(u, v) = 0$.

(iv) d is b_2-pseudo metric on X with $s = 2$.

In order to define gauge spaces in the setting of b_s-pseudo metrics we need to define the following.

Definition 4.4.4. Let X be a nonempty set endowed with the b_s-pseudo metric d. The d_s-ball of radius $\epsilon > 0$ centered at $x \in X$ is the set

$$B(x, d, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

Definition 4.4.5. A family $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ of b_s-pseudo metrics is said to be separating if for each pair (x, y) with $x \ne y$, there exists $d_\nu \in \mathfrak{F}$ with $d_\nu(x, y) \ne 0$.

Definition 4.4.6. Let X be a nonempty set and $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ be a family of b_s-pseudo metrics on X. The topology $\mathfrak{T}(\mathfrak{F})$ having subbases the family

$$\mathfrak{B}(\mathfrak{F}) = \{B(x, d_\nu, \epsilon) : x \in X, d_\nu \in \mathfrak{F} \text{ and } \epsilon > 0\}$$

of balls is called topology induced by the family $\mathfrak{F}$ of b_s-pseudo metrics. The pair $(X, \mathfrak{T}(\mathfrak{F}))$ is called a b_s-gauge space. Note that $(X, \mathfrak{T}(\mathfrak{F}))$ is Hausdorff if $\mathfrak{F}$ is separating.

Definition 4.4.7. Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a b_s-gauge space with respect to the family $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ of b_s-pseudo metrics on X and $\{x_n\}$ is a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ converges to x if for each $\nu \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n, x) < \epsilon$ for each $n \geq N_0$. We denote it as $x_n \to^{\mathfrak{F}} x$.

(ii) The sequence $\{x_n\}$ is a Cauchy sequence if for each $\nu \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n, x_m) < \epsilon$ for each $n, m \geq N_0$.

(iii) $(X, \mathfrak{T}(\mathfrak{F}))$ is complete if each Cauchy sequence in $(X, \mathfrak{T}(\mathfrak{F}))$ is convergent in X.

(iv) A subset of X is said to be closed if it contains the limit of each convergent sequence of its elements.

Remark 4.4.8. When $s = 1$, then all above definitions reduce to the corresponding definitions in a gauge space.

Subsequently, in this section, $\mathfrak{A}$ is directed set and X is a nonempty set endowed with a separating complete b_s-gauge structure $\{d_\nu : \nu \in \mathfrak{A}\}$. Further, $G = (V, E)$ is a directed graph in $X \times X$, where the set of its vertices V is equal to X and set of its edges E contains $\{(x, x) : x \in V\}$. Furthermore, G has no parallel edges. For each $d_\nu \in \mathfrak{F}$, $CL_\nu(X)$ denote the set of all nonempty closed subsets of X with respect to d_v. For each $\nu \in \mathfrak{A}$ and $A, B \in CL_\nu(X)$, the function $H_\nu : CL_\nu(X) \times CL_\nu(X) \to [0, \infty)$ defined by

$$H_\nu(A, B) = \begin{cases} \max\left\{ \sup_{x \in A} d_\nu(x, B), \sup_{y \in B} d_\nu(y, A) \right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

is a generalized Hausdorff b_s-pseudo metric on $CL_\nu(X)$. Here, we denote by $CL(X)$ the set of all nonempty closed subsets in the b_s-gauge space $(X, \mathfrak{T}(\mathfrak{F}))$. Now, we move towards our results.

Theorem 4.4.9. Let $T : X \to CL(X)$ be a mapping such that for each $\nu \in \mathfrak{A}$, we have

$$H_\nu(Tx, Ty) \leq a_\nu d_\nu(x, y) + b_\nu d_\nu(x, Tx) + c_\nu d_\nu(y, Ty) + e_\nu d_\nu(x, Ty) + L_\nu d_\nu(y, Tx) \quad \forall\ (x, y) \in E \tag{4.55}$$

where, $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0$, and $s^2 a_\nu + s^2 b_\nu + s^2 c_\nu + 2s^3 e_\nu < 1$.

Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) if $(x, y) \in E$, for $u \in Tx$ and $v \in Ty$ such that $d_\nu(u, v) \leq d_\nu(x, y)$ for each $\nu \in \mathfrak{A}$, then $(u, v) \in E$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

(iv) for each $\{q_\nu : q_\nu > 1\}_{\nu\in\mathfrak{A}}$ and $x \in X$ there exists $y \in Tx$ such that

$$d_\nu(x, y) \leq q_\nu d_\nu(x, Tx) \ \ \forall \ \nu \in \mathfrak{A}.$$

Then T has a fixed point.

Proof. By hypothesis (i), there exist $x_0, x_1 \in X$ such that $x_1 \in Tx_0$ and $(x_0, x_1) \in E$. Now, it follows form (4.55) that

$$H_\nu(Tx_0, Tx_1) \leq a_\nu d_\nu(x_0, x_1) + b_\nu d_\nu(x_0, Tx_0) + c_\nu d_\nu(x_1, Tx_1) + e_\nu d_\nu(x_0, Tx_1) + L_\nu d_\nu(x_1, Tx_0) \ \ \forall \ \nu \in \mathfrak{A}. \tag{4.56}$$

Since $d_\nu(x_1, Tx_1) \leq H_\nu(Tx_0, Tx_1)$ and $d_\nu(x_0, Tx_1) \leq s[d_\nu(x_0, x_1) + d_\nu(x_1, Tx_1)]$, therefore from (4.56), we get

$$d_\nu(x_1, Tx_1) \leq \frac{1}{\xi_\nu} d_\nu(x_0, x_1) \tag{4.57}$$

where, $\xi_\nu = \frac{1-c_\nu-se_\nu}{a_\nu+b_\nu+se_\nu} > 1$. Using hypothesis (iv) there exists $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq \sqrt{\xi_\nu} d_\nu(x_1, Tx_1). \tag{4.58}$$

Combining (4.57) and (4.58), we get

$$d_\nu(x_1, x_2) \leq \frac{1}{\sqrt{\xi_\nu}} d_\nu(x_0, x_1) \ \ \forall \ \nu \in \mathfrak{A}. \tag{4.59}$$

Hypothesis (ii) and (4.59), implies that $(x_1, x_2) \in E$. Continuing in the same way, we get a sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ and

$$d_\nu(x_m, x_{m+1}) \leq \Big(\frac{1}{\sqrt{\xi_\nu}}\Big)^m d_\nu(x_0, x_1) \ \ \forall \ \nu \in \mathfrak{A} \text{ and } m \in \mathbb{N}.$$

For convenience we assume that $\eta_\nu = \frac{1}{\sqrt{\xi_\nu}}$ for each $\nu \in \mathfrak{A}$. Now we show that $\{x_m\}$ is Cauchy. For each $m, p \in \mathbb{N}$ and $\nu \in \mathfrak{A}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} s^i (\eta_\nu)^i d_\nu(x_0, x_1) \\ &\leq \sum_{i=m}^{\infty} (s\eta_\nu)^i d_\nu(x_0, x_1) < \infty \ \ (since\ s\eta_\nu < 1). \end{aligned}$$

This implies that $\{x_m\}$ is Cauchy sequence in X. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$. By using hypothesis (iii), triangular inequality and (4.55), we have

$$\begin{aligned}
d_\nu(x^*, Tx^*) &\leq sd_\nu(x^*, x_{m-1}) + sd_\nu(x_{m-1}, Tx^*) \\
&\leq sd_\nu(x^*, x_{m-1}) + sH_\nu(Tx_m, Tx^*) \\
&\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, Tx_m) + \\
&\quad sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_n(x^*, Tx_m) \\
&\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, x_{m+1}) + \\
&\quad sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_\nu(x^*, x_{m+1}) \\
&\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, x_{m+1}) + sc_\nu d_\nu(x^*, Tx^*) \\
&\quad + se_\nu[sd_\nu(x_m, x^*) + sd_\nu(x^*, Tx^*)] + sL_\nu d_\nu(x^*, x_{m+1}) \ \ \forall\ \nu \in \mathfrak{A}.
\end{aligned}$$

Letting $m \to \infty$, we get

$$d_\nu(x^*, Tx^*) \leq (sc_\nu + s^2 e_\nu) d_\nu(x^*, Tx^*) \ \ \forall\ \nu \in \mathfrak{A}.$$

Which is only possible if $d_\nu(x^*, Tx^*) = 0$. Since the structure $\{d_\nu : \nu \in \mathfrak{A}\}$ on X is separating, we have $x^* \in Tx^*$. □

In case of single valued mapping $T : X \to X$ we have the following result:

Theorem 4.4.10. Let $T : X \to X$ be a mapping such that for each $\in \mathfrak{A}$ we have

$$d_\nu(Tx, Ty) \leq a_\nu d_\nu(x, y) + b_\nu d_\nu(x, Tx) + c_\nu d_\nu(y, Ty) + e_\nu d_\nu(x, Ty) + L_\nu d_\nu(y, Tx) \ \ \forall\ (x, y) \in E \tag{4.60}$$

where, $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0$, and $sa_\nu + sb_\nu + sc_\nu + 2s^2 e_\nu < 1$.

Assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$;

(ii) for $(x, y) \in E$, we have $(Tx, Ty) \in E$, provided $d_\nu(Tx, Ty) \leq d_\nu(x, y)$ for each $\nu \in \mathfrak{A}$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

Then T has a fixed point.

Example 4.4.11. Let $X = C([0, 10], \mathbb{R})$ endowed with the b_2-pseudo metrics $d_n(x(t), y(t)) = \max_{t \in [0,n]} (x(t) - y(t))^2$ for each $n \in \{1, 2, 3, \cdots, 10\}$ and the graph $G = (V, E)$ as $V = X$ and

$$E = \{(x(t), y(t)) : x(t) \leq y(t)\} \cup \{(x(t), x(t)) : x \in X\}.$$

Define $T : X \to X$ by $Tx(t) = \frac{x(t)+1}{5}$, for each $x \in X$. It is easy to see that (4.60) holds with $a_n = 1/5$ and $b_n = c_n = e_n = L_n = 0$ for each $n \in \{1, 2, 3, \cdots, 10\}$. For $x_0 = 0$ and

$x_1 = Tx_0 = 1/5$, we have $(x_0, Tx_0) \in E$. Since T is nondecreasing, for each $(x, y) \in E$, we have $(Tx, Ty) \in E(G)$. For each sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ for each $m \in \mathbb{N}$ and $x_m \to x$, then $(x_m, x) \in E$ for each $m \in \mathbb{N}$. Therefore, all conditions of Theorem 4.4.10 are satisfied and the has a fixed point.

Before going towards our next theorem, we define Ψ_{s^2} family of mappings. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing mapping such that it satisfies the following conditions:

(ψ_1) $\psi(0) = 0$;

(ψ_2) $\psi(\rho t) = \rho\psi(t) < \rho t$ for each $\rho, t > 0$;

(ψ_3) $\sum_{i=1}^{\infty} s^{2i}\psi^i(t) < \infty$ for each $t > 0$, where $s \geq 1$.

Theorem 4.4.12. Let $T : X \to CL(X)$ be a mapping such that for each $\nu \in \mathfrak{A}$ we have

$$H_\nu(Tx, Ty) \leq \psi_\nu(d_\nu(x, y)) \ \ \forall\ (x, y) \in E \tag{4.61}$$

where, $\psi_\nu \in \Psi_{s^2}$. Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) if $(x, y) \in E$, for $u \in Tx$ and $v \in Ty$ such that $\frac{1}{s}d_\nu(u, v) < d_\nu(x, y)$ for each $\nu \in \mathfrak{A}$, then $(u, v) \in E$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$;

(iv) for each $x \in X$, we have $y \in Tx$ such that

$$d_\nu(x, y) \leq s d_\nu(x, Tx) \ \ \forall\ \nu \in \mathfrak{A}.$$

Then T has a fixed point.

Proof. By hypothesis we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (4.61), we get

$$d_\nu(x_1, Tx_1) \leq H_\nu(Tx_0, Tx_1) \leq \psi_\nu(d_\nu(x_0, x_1)) \ \ \forall\ \nu \in \mathfrak{A}. \tag{4.62}$$

By hypothesis (iv), for $x_1 \in X$, we have $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq s d_\nu(x_1, Tx_1) \leq s\psi_\nu(d_\nu(x_0, x_1)) \ \ \forall\ \nu \in \mathfrak{A}. \tag{4.63}$$

Applying ψ_ν, we have

$$\psi_\nu(d_\nu(x_1, x_2)) \leq \psi_\nu(s\psi_\nu(d_\nu(x_0, x_1))) = s\psi_\nu^2(d_\nu(x_0, x_1)) \ \ \forall\ \nu \in \mathfrak{A}.$$

From (4.63), it is clear that $(x_1, x_2) \in E$. Again from (4.61), we have

$$d_\nu(x_2, Tx_2) \leq H_\nu(Tx_1, Tx_2) \leq \psi_\nu(d_\nu(x_1, x_2)) \ \ \forall \ \nu \in \mathfrak{A}. \tag{4.64}$$

By hypothesis (iv), for $x_2 \in X$, we have $x_3 \in Tx_2$ such that

$$d_\nu(x_2, x_3) \leq sd_\nu(x_2, Tx_2) \leq s\psi_\nu(d_\nu(x_1, x_2)) \leq s^2\psi_\nu^2(d_\nu(x_0, x_1)) \ \ \forall \ \nu \in \mathfrak{A}. \tag{4.65}$$

Clearly, $(x_2, x_3) \in E$. Continuing in the same way, we get a sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ and

$$d_\nu(x_m, x_{m+1}) \leq s^m\psi_\nu^m(d_\nu(x_0, x_1)) \ \ \forall \ \nu \in \mathfrak{A}.$$

Now, we show that $\{x_m\}$ is Cauchy sequence. For $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} s^{2i}\psi_\nu^i(d_\nu(x_0, x_1)) < \infty \end{aligned}$$

This implies that $\{x_m\}$ is Cauchy sequence in X. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. Using hypothesis (iv), triangular inequality and (4.61), we have

$$\begin{aligned} d_\nu(x^*, Tx^*) &\leq sd_\nu(x^*, x_{m-1}) + sd_\nu(x_{m-1}, Tx^*) \\ &\leq sd_\nu(x^*, x_{m-1}) + sH_\nu(Tx_m, Tx^*) \\ &\leq sd_\nu(x^*, x_{m-1}) + s\psi_\nu(d_n(x_m, x^*)) \ \ \forall \ \nu \in \mathfrak{A}. \end{aligned}$$

Letting $m \to \infty$, we get $d_\nu(x^*, Tx^*) = 0$ for each $\nu \in \mathfrak{A}$. Since the structure $\{d_\nu : \nu \in \mathfrak{A}\}$ on X is separating, we have $x^* \in Tx^*$. □

By considering $T : X \to X$ in above theorem we get the following one.

Theorem 4.4.13. Let $T : X \to X$ be a mapping such that for each $\nu \in \mathfrak{A}$ we have

$$d_\nu(Tx, Ty) \leq \psi_\nu(d_\nu(x, y)) \ \ \forall \ (x, y) \in E \tag{4.66}$$

where $\psi_\nu \in \Psi_{s^2}$. Assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E$;

(ii) for $(x, y) \in E$, we have $(Tx, Ty) \in E$ provided $\frac{1}{s}d_\nu(Tx, Ty) < d_\nu(x, y)$ for each $\nu \in \mathfrak{A}$;

(iii) if $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Application

Consider the Volterra integral equation of the form:

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds, \ t \in I \tag{4.67}$$

where $f : I \to \mathbb{R}$ and $K : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous functions and K is nondecreasing in its third variable.

Let $X = (C[0, \infty), \mathbb{R})$. Define the family of b_2-pseudo norms by $\|x\|_n = \max_{t \in [0,n]}(x(t))^2$, $n \in \mathbb{N}$. By using this family of b_2-pseudo norms we get a family of b_2-pseudo metrics as $d_n(x, y) = \|x - y\|_n$. Clearly, $\mathfrak{F} = \{d_n : n \in \mathbb{N}\}$ defines b_2-gauge structure on X, which is complete and separating. Define graph $G = (V, E)$ such that $V = X$ and $E = \{(x, y) : x(t) \leq y(t), \forall t \geq 0\}$.

Theorem 4.4.14. Let $X = (C[0, \infty), \mathbb{R})$ and let the operator $T : X \to X$ is define by

$$Tx(t) = f(t) + \int_0^t K(t, s, x(s))ds, t \in I = [0, \infty)$$

where $f : I \to \mathbb{R}$ and $K : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous functions and K is nondecreasing in its third variable. Assume that the following conditions hold:

(i) for each $t, s \in [0, n]$ and $x, y \in X$ with $(x, y) \in E(G)$, there exists a continuous mapping $p : I \times I \to I$ such that

$$|K(t, s, x(s)) - K(t, s, y(s))| \leq \sqrt{p(t, s)d_n(x, y)} \text{ for each } n \in \mathbb{N};$$

(ii) $\sup_{t \geq 0} \int_0^t \sqrt{p(t, s)}ds = a < \frac{1}{\sqrt{2}}$;

(iii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$.

Then the integral equation (4.67) has atleast one solution.

Proof. First we show that for each $(x, y) \in E(G)$, the inequalities (4.60) holds. For any $(x, y) \in E(G)$ and $t \in [0, n]$ for each $n \geq 1$, we have

$$\begin{aligned}
(Tx(t) - Ty(t))^2 &\leq \Big(\int_0^t |K(t, s, x(s)) - K(t, s, y(s))|ds\Big)^2 \\
&\leq \Big(\int_0^t \sqrt{p(t, s)d_n(x, y)}ds\Big)^2 \\
&= \Big(\int_0^t \sqrt{p(t, s)}ds\Big)^2 d_n(x, y) \\
&= a^2 d_n(x, y).
\end{aligned}$$

Thus, we get $d_n(Tx, Ty) \leq a^2 d_n(x, y)$ for each $(x, y) \in E$ and $n \in \mathbb{N}$, with $a^2 < 1/2$. This implies that (4.60) holds with $a_n = a^2$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in \mathbb{N}$. As K is nondecreasing, for each $(x, y) \in E(G)$, we have $(Tx, Ty) \in E(G)$. Therefore, by Theorem 4.4.10, there exists a fixed point of the operator T, that is, integral equation (4.67) has atleast one solution. □

4.5 Fixed point theorems for multi valued Caristi type contractions on gauge spaces

Let (X, d) be a metric space and $T : X \to X$ be a mapping such that there exists a lower semicontinuous function $\phi : X \to [0, \infty)$ satisfying $d(x, Tx) \leq \phi(x) - \phi(Tx)$, then T is called Caristi mapping [50]. Caristi proved in [50], that every Caristi mapping on a complete metric space has a fixed point. Then, Kirk [95] proved that the metric space (X, d) is complete if and only if every Caristi mapping for (X, d) has a fixed point. In this section, we prove some fixed point theorems for Caristi type multi valued mappings on a complete gauge space endowed with a graph. Through out this section we assume that the set X is endowed with directed graph $G = (V, E)$, the set of its vertices V coincides with X and the set of its edges E is such that $E \supseteq \triangle$, where $\triangle = \{(x, x) : x \in X\}$. Also assume that G has no parallel edges.

We use the following definitions in the main results:

Definition 4.5.1. [96] A mapping $T : X \to CL(X)$ is said to be G-continuous if for each sequence $\{x_n\}$ in X such that $(x_n, x_{n+1}) \in E$ and $x_n \to x$, we have $Tx_n \to Tx$.

Definition 4.5.2. [133] A mapping $T : X \to CL(X)$ is said to be edge preserving if

$$(x, y) \in E \Rightarrow \ (a, b) \in E \ \text{ for each } a \in Tx \text{ and } \ b \in Ty.$$

Definition 4.5.3. A mapping $g : X \to [0, \infty)$ is said to be G-lower semi continuous, if for each sequence $\{x_n\}$ in X such that $(x_n, x_{n+1}) \in E$ and $x_n \to x$, we have $g(x) \leq \liminf_{n\to\infty} g(x_n)$.

Subsequently, X is endowed with the graph G and complete gauge structure $\{d_\nu : \nu \in \mathfrak{A}\}$ which is separating.

Theorem 4.5.4. Let $T : X \to CL(X)$ be an edge preserving mapping and let for each $\nu \in \mathfrak{A}$, $\phi_\nu : X \to [0, \infty)$ is a lower semi continuous function such that for each $x \in X$ and $y \in Tx$ with $(x, y) \in E$, we have

$$d_\nu(y, Ty) \leq \phi_\nu(x) - \phi_\nu(y) \text{ for each } \nu \in \mathfrak{A}. \tag{4.68}$$

Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) there exists a sequence $\{q_\nu : q_\nu > 1\}_{\nu\in\mathfrak{A}}$ such that for each $x \in X$, we have $y \in Tx$ satisfying

$$d_\nu(x, y) \leq q_\nu d_\nu(x, Tx) \text{ for each } \nu \in \mathfrak{A}.$$

(iii) for each $\nu \in \mathfrak{A}$, a function $g_\nu : X \to [0, \infty)$ define by $g_\nu(x) = d_\nu(x, Tx)$ is G-lower semi continuous.

Then T has a fixed point.

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (4.68), we have

$$d_\nu(x_1, Tx_1) \le \phi_\nu(x_0) - \phi_\nu(x_1) \text{ for each } \nu \in \mathfrak{A}. \tag{4.69}$$

By using (ii) and (4.69), we have $q_\nu > 1$ for each $\nu \in \mathfrak{A}$ and $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \le q_\nu d_\nu(x_1, Tx_1) \le q_\nu \phi_\nu(x_0) - q_\nu \phi_\nu(x_1) \text{ for each } \nu \in \mathfrak{A}. \tag{4.70}$$

Since T is edge preserving, we have $(x_1, x_2) \in E$. Continuing in the same way we get a sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ and

$$d_\nu(x_m, x_{m+1}) \le q_\nu d_\nu(x_m, Tx_m) \le q_\nu \phi_\nu(x_{m-1}) - q_\nu \phi_\nu(x_m) \text{ for each } m \in \mathbb{N} \text{ and } \nu \in \mathfrak{A}.$$

This implies that for each $\nu \in \mathfrak{A}$ the sequence $\{\phi_\nu(x_m)\}$ is a nonincreasing sequence, there exists $r_\nu \ge 0$ such that $\phi_\nu(x_m) \to r_\nu$ as $m \to \infty$. Now consider $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\le d_\nu(x_m, x_{m+1}) + d_\nu(x_{m+1}, x_{m+2}) + d_\nu(x_{m+2}, x_{m+3}) \\ &\quad + \cdots + d_\nu(x_{m+p-1}, x_{m+p}) \\ &\le q_\nu(\phi_\nu(x_{m-1}) - \phi_\nu(x_m)) + q_\nu(\phi_\nu(x_m) - \phi_\nu(x_{m+1})) + \\ &\quad q_\nu(\phi_\nu(x_{m+1}) - \phi_\nu(x_{m+2})) + \cdots + q_\nu(\phi_\nu(x_{m+p-2}) - \phi_\nu(x_{m+p-1})) \\ &\le q_\nu(\phi_\nu(x_{m-1}) - \phi_\nu(x_{m+p-1})) \text{ for each } \nu \in \mathfrak{A}. \end{aligned} \tag{4.71}$$

This implies that $\{x_m\}$ is Cauchy in X, since $\phi_\nu \to r_\nu$ for each $\nu \in \mathfrak{A}$. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$. Since each $g_\nu(x)$ is G-lower semi continuous then we have $d_\nu(x^*, Tx^*) \le \liminf_m d_\nu(x_m, Tx_m) = 0$ for each $\nu \in \mathfrak{A}$. This implies $x^* \in Tx^*$. □

Theorem 4.5.5. Let $T : X \to CL(X)$ be an edge preserving mapping and let for each $\nu \in \mathfrak{A}$, $\phi_\nu : X \to [0, \infty)$ is a lower semi continuous function such that for each $x \in X$ and $y \in Tx$ with $(x, y) \in E$, we have

$$d_\nu(x, y) \le \phi_\nu(x) - \phi_\nu(y) \text{ for each } \nu \in \mathfrak{A}. \tag{4.72}$$

Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) T is G-continuous.

Then T has a fixed point.

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (4.72), we have

$$d_\nu(x_0, x_1) \le \phi_\nu(x_0) - \phi_\nu(x_1) \text{ for each } \nu \in \mathfrak{A}. \tag{4.73}$$

Since T is edge preserving, we have $(x_1, x_2) \in E$. Thus, continuing in same way we get a sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ and

$$d_\nu(x_m, x_{m+1}) \leq \phi_\nu(x_m) - \phi_\nu(x_{m+1}) \text{ for each } m \in \mathbb{N} \text{ and } \nu \in \mathfrak{A}.$$

This implies that for each $\nu \in \mathfrak{A}$ the sequence $\{\phi_\nu(x_m)\}$ is a nonincreasing sequence, there exists $r_\nu \geq 0$ such that $\phi_\nu(x_m) \to r_\nu$ as $m \to \infty$. Now consider $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\leq d_\nu(x_m, x_{m+1}) + d_\nu(x_{m+1}, x_{m+2}) + d_\nu(x_{m+2}, x_{m+3}) \\ &\quad + \cdots + d_\nu(x_{m+p-1}, x_{m+p}) \\ &\leq \phi_\nu(x_m) - \phi_\nu(x_{m+1}) + \phi_\nu(x_{m+1}) - \phi_\nu(x_{m+2}) + \\ &\quad \phi_\nu(x_{m+2}) - \phi_\nu(x_{m+3}) + \cdots + \phi_\nu(x_{m+p-1}) - \phi_\nu(x_{m+p}) \\ &\leq \phi_\nu(x_m) - \phi_\nu(x_{m+p}) \text{ for each } \nu \in \mathfrak{A}. \end{aligned}$$

This implies that $\{x_m\}$ is Cauchy in X, since $\phi_\nu \to r_\nu$ for each $\nu \in \mathfrak{A}$. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$. As T is G-continuous then we have $x^* \in Tx^*$. □

Theorem 4.5.6. Let $T : X \to CL(X)$ be an edge preserving mapping and let for each $\nu \in \mathfrak{A}$, $\psi_\nu : X \to [0, \infty)$ is an upper semi continuous function such that for each $x, y \in X$ with $(x, y) \in E$, we have

$$d_\nu(y, Ty) \leq \psi_\nu(x) - \psi_\nu(y) \text{ for each } \nu \in \mathfrak{A}. \tag{4.74}$$

Assume that the following conditions hold:

(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$;

(ii) there exists a sequence $\{q_\nu : q_\nu > 1\}_{\nu \in \mathfrak{A}}$ such that for each $x \in X$, we have $y \in Tx$ satisfying

$$d_\nu(x, y) \leq q_\nu d_\nu(x, Tx) \text{ for each } \nu \in \mathfrak{A}.$$

(iii) If $\{x_n\}$ is a sequence in X such that $(x_n, x_{n+1}) \in E$ for each $n \in \mathbb{N}$ and $x_n \to x$, then $(x_n, x) \in E$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

Proof. By hypothesis (i), we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E$. From (4.74), we have

$$d_\nu(x_1, Tx_1) \leq \psi_\nu(x_0) - \psi_\nu(x_1) \text{ for each } \nu \in \mathfrak{A}. \tag{4.75}$$

By using (ii) and (4.75), we have $q_\nu > 1$ for each $\nu \in \mathfrak{A}$ and $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq q_\nu d_\nu(x_1, Tx_1) \leq q_\nu \psi_\nu(x_0) - q_\nu \psi_\nu(x_1) \text{ for each } \nu \in \mathfrak{A}. \tag{4.76}$$

Since T is edge preserving, we have $(x_1, x_2) \in E$. Continuing in the same way we get a sequence $\{x_m\}$ in X such that $(x_m, x_{m+1}) \in E$ and

$$d_\nu(x_m, x_{m+1}) \leq q_\nu d_\nu(x_m, Tx_m) \leq q_\nu \psi_\nu(x_{m-1}) - q_\nu \psi_\nu(x_m) \text{ for each } m \in \mathbb{N} \text{ and } \nu \in \mathfrak{A}.$$

This implies that for each $\nu \in \mathfrak{A}$ the sequence $\{\psi_\nu(x_m)\}$ is a nonincreasing sequence, there exists $r_\nu \geq 0$ such that $\psi_\nu(x_m) \to r_\nu$ as $m \to \infty$. Now consider $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\leq d_\nu(x_m, x_{m+1}) + d_\nu(x_{m+1}, x_{m+2}) + d_\nu(x_{m+2}, x_{m+3}) \\ &\quad + \cdots + d_\nu(x_{m+p-1}, x_{m+p}) \\ &\leq q_\nu(\psi_\nu(x_{m-1}) - \psi_\nu(x_m)) + q_\nu(\psi_\nu(x_m) - \psi_\nu(x_{m+1})) + \\ &\quad q_\nu(\psi_\nu(x_{m+1}) - \psi_\nu(x_{m+2})) + \cdots + q_\nu(\psi_\nu(x_{m+p-2}) - \psi_a(\nu x_{m+p-1})) \\ &\leq q_\nu(\psi_\nu(x_{m-1}) - \psi_\nu(x_{m+p-1})) \text{ for each } \nu \in \mathfrak{A}. \end{aligned} \tag{4.77}$$

This implies that $\{x_m\}$ is Cauchy in X, since $\phi_\nu \to r_\nu$ for each $\nu \in \mathfrak{A}$. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$. By hypothesis (iii), we have $(x_m, x^*) \in E$. From (4.74), we have

$$d_\nu(x^*, Tx^*) \leq \psi_\nu(x_m) - \psi_\nu(x^*) \text{ for each } \nu \in \mathfrak{A}.$$

Letting $m \to \infty$ in above inequality, we have $d_\nu(x^*, Tx^*) = 0$ for each $\nu \in \mathfrak{A}$. Thus, we have $x^* \in Tx^*$. □

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Proforma for PhD Country Directory

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*Subject:	Mathematics
*Title of Dissertation	Some Advancements in Metric Fixed Point Theory
*Date of Enrollment of PhD Scholar in University	09-Sep-2013
Date of Submission of PhD Dissertation to University	July 2016
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Details of Mandatory Research Paper(s) published by the PhD Graduate prior to award of PhD Degree	
*Name of Author(s)	Muhammad Usman Ali, Tayyab Kamran and Liaqat Ali Khan
*Title of Paper	A new type of multivalued contraction in partial Hausdorff metric spaces endowed with a graph
*Name of Journal	Journal of Inequalities and Applications
*Volume; Article ID	2015:205, DOI: 10.1186/s13660-015-0728-y
Page(s)	1-11
Date	2015
*Name of Author(s)	Muhammad Usman Ali, Tayyab Kamran and Mihai Postolache
*Title of Paper	Fixed point theorems for multivalued G-contractions in Hausdorff b-Gauge spaces.
*Name of Journal	Journal of Nonlinear Sciences and Applications
*Volume	8
Page(s)	847-855
Date	2015
*Name of Author(s)	Muhammad Usman Ali, Tayyab Kamran and Erdal Karapinar .
*Title of Paper	Further discussion on modified multivalued α^*-ψ -contractive type mapping

*Name of Journal	Filomat
*Volume	29
Page(s)	1893-1900
Date	2015

* Mandatory fields
__ Proforma with checklist form duly verified by Controller of Examination is to be sent on address
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Check List

Yes **No**

- Minimum CGPA score for admission into a PhD Program of scholar was 3.0 (out of 4.0 in the Semester System) or First Division (Annual System) in M.Phil/M.S/Equivalent.
  ______Yes____ ___________
- Acceptable GAT /GRE Subject Test score prior to admission in PhD Program as per HEC minimum criteria of PhD
  ______Yes____ ___________
- Completed 18 Credit Hours of Course Work ______Yes ________
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- PhD Dissertation has been Evaluated by two PhD experts from technologically/academically advanced foreign countries ______Yes____ ___________
- Less than 20% of similarity index (Plagiarism Test) on the Dissertation before its submission to the two foreign experts, of the discipline from technologically/academically advanced foreign countries.
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