Drift Waves in Multicomponent Plasma

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Islamabad, Pakistan
2023

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

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Declaration

I **Fahma Sahreen** hereby declares that this MS thesis titled **Drift Waves in Multicomponent Plasma** has been exclusively composed by myself and is the original report of my research work. The work has not been submitted before for any previous degree. All the details are true to the best of my knowledge and due references and acknowledgments have been made.

At any course of time, if my statement is found to be incorrect, the university has the right to withdraw my MS degree.

Fahma Sahreen

Dedication

Academic Research is the most crucial phase of any degree which surely requires a focused mind, sincere efforts and counselling of our loved ones. This endeavor of mine is wholeheartedly dedicated to my beloved

Father and Mother

who had always been there for me decisively. Their never ending support and love gave me enough strength to get through this challenging part of my degree. Also, I dedicate this thesis to all of my family crew, friends and respected teachers for putting in their generous efforts to help me accomplish my academic research.

Fahma Sahreen

Acknowledgments

My humble gratitude is owed to a number of the people around me. From the head start of this thesis and till final formation and submission, my major thanks go to my Supervisor Dr. Mudassir Ali Shah and Co-Supervisor Dr. Hamid Saleem for their valuable guidance and immense support. Short and extended brainstorming sessions conducted during the entire course of a semester, helped me enough to get a better understanding of research and relevant scholastic concepts. Their expertise and unceasing assistance helped me to boost my questioning ability and solve the admissible research problems.

I would also like to accord my sincere thanks to HOD Physics Dr. Syed Rizwan Hussain for providing enough sources to facilitate the research environment. I would be remiss in not mentioning GEC members Dr. Ayesha Khaliq and Dr. Tajjamul Hussain and other faculty staff with whom I have always been in interaction and have helped me in numerous ways to meet the research formalities.

A very special thanks to all my classmates and friends at the university. Their moral support, informative discussion and kind motivation during all the ups and downs of my academic struggle were quite enough to put me on the right path. A million thanks to my loving parents, and kind siblings for keeping me in high spirits and helping me to execute my academic task. Thank you Everyone!!

Fahma Sahreen

Abstract

Low frequency electrostatic drift waves are studied in multi-component plasmas including usual electron-ion (EI), electron-positron ion (EPI) and dusty plasmas. Cancellation of the contribution of the diamagnetic term in the convective derivative of the ion polarization drift with the collision-less part of the stress tensor is discussed in detail in the case of hot ions. Fourier analysis has been performed to obtain the coupled linear dispersion relation of drift wave and ion acoustic wave (IAW) in electron-ion plasma. Keeping in view this cancellation, the dispersion relation of drift waves in hot ions plasma is derived using the two fluid plasma model. The effect of field-aligned shear flow on the instability of drift waves has also been studied. The propagation characteristics of the drift waves in EPI and dusty plasmas have been investigated and several limiting cases have been discussed. The growth rates of drift wave instability in (EI) plasmas have been estimated numerically using plasma parameters of Joint European Tokamak (JET) and terrestrial ionosphere. It has been shown that the drift wave can become unstable in JET if the parallel shear flow effect is taken into account.

Contents

Nomenclature

λ_{De} Electron Debye Length

Ω_e Electron Cyclotron Frequency

Ω_i Ion Cyclotron Frequency

ω_{pe} Electron Plasma Frequency

ω_{pi} Ion Plasma Frequency

ρ_i Ion Larmor Radius at Ion Temperature

ρ_s Ion Larmor Radius at Electron Temperature

c Speed of Light in Vacuum

c_d Dust Thermal Speed at T_{eff}

c_i Ion Thermal Speed at Ion Temperature

c_s Ion Thermal Speed at Electron Temperature

K_B Boltzmann Constant

n_e Electron Plasma Density

n_i Ion Plasma Density

T_e Electron Temprature

T_i Ion Temperature

v_{en} Electron-Neutral Collision Frequency

Chapter 1

Introduction

1.1 What is Plasma

The universe is defined as the whole of all matter, energy and space. It is mainly composed of dark matter, dark energy and ordinary material consisting of atoms, molecules and other particles in the form of gases, liquids, solids and plasma. Our topic of interest is plasma which is also known as the fourth state of matter. It's an ionized gas having positively charged ions and negatively charged electrons in addition to some concentration of neutral particles. The total number of positive and negative charges is almost equal and hence plasma is quasi-neutral in nature. Plasma is a combination of ions and electrons generating electric and magnetic fields everywhere within the occupied volume. Charged particles interact with each other through long-range electromagnetic forces. Electric field and magnetic field usually enhance the range of motion in plasma therefore, plasma is a quasi-neutral gas of charged and neutral particles exhibiting collective behaviour. Stellar interiors, gaseous nebulae and galaxies are visible because they are in plasma form. Plasma doesn't occur naturally on the earth's surface. Fluorescent tubes and pixels of plasma TV are a few live examples of artificially created plasma on earth [1].

1.2 Plasma Parameters

1.2.1 Debye Length

Plasma has a fundamental characteristic to shield out the potential applied to it or created inside it. For a cold plasma, shielding would be perfect. Thermal motions are dominant in hot plasma. At the edge of a shield, the electric field is weak, so enough thermal energy is available to charges to escape from shielded potential, i.e., $\phi = K_B T_e$. The potential of order $K_B T_e/e$ can leak into plasma and cause a finite electric field to exist at the edge. The potential becomes radius dependent and the thickness of such shielded cloud is given as [2],

$$\lambda_{De} = (\frac{K_B T_e}{4\pi n_{e0} e^2})^{\frac{1}{2}} \tag{1.1}$$

Here λ_{De} is called the electron Debye length and K_B is the Boltzmann constant where T is in degree Kelvin. Temperature is generally expressed in terms of energy units, i.e., $T_j = K_B T_j$ for jth species where $j = e, i$. Electron temperature is used for defining shielding, in general, because electrons being lighter move rapidly as compared to the ions so they form perfect shielding around any accumulated charge. If $T_i \ll T_e$, Debye length λ_{Di} of ions can be ignored.

1.2.2 Quasineutrality

Quasineutrality is a concept of significant importance in plasma. If dimensions of the system are much greater than Debye length then wherever an external potential is introduced in plasma, i.e., a local concentration of charge arises, it is shielded out at a small distance compared to the system's dimension L. This causes the bulk of the plasma free of electric potential. Outside of that shield, the number density of ions is almost equal to the number density of electrons $n_i \approx n_e \approx n$ where n is plasma density. Hence for plasma to exist, λ_{De} must be much smaller than L. Debye shielding is not a valid concept if the number of particles in a Debye sphere are just a few. Plasma must show a behaviour of statistical ensemble such that $N_D = n\frac{4}{3}\pi\lambda_{De}^3 \gg 1$. So far, we have discussed two conditions,

$$\lambda_{De} \ll L \tag{1.2}$$

$$N_{De} \gg 1 \tag{1.3}$$

The third condition deals with collision frequency for which charged particles aren't supposed to collide with neutral particles in a frequent manner because their motion is controlled by electromagnetic forces rather than ordinary hydrodynamic forces. Therefore, the condition:

$$\nu_{en} \ll \omega_{pe} \tag{1.4}$$

must also hold in plasma where $\omega_{pe} = (4\pi n_{e0} e^2/m_e)^{1/2}$ is the electron plasma oscillation frequency and ν_{en} is the electron-neutral collision frequency. The physical meaning of plasma frequency ω_{pe} will be elaborated in the section. (1.2.3). Hence for plasma to exist, the above-mentioned conditions must be satisfied.

1.2.3 Characteristic Frequencies

Plasma has many parameters at different scales of time and space. The characteristic electron and ion plasma frequencies are [3],

$$\omega_{pe} = \sqrt{\frac{4\pi n_{e0} e^2}{m_e}} \tag{1.5}$$

$$\omega_{pi} = \sqrt{\frac{4\pi n_{i0} e^2}{m_i}} \tag{1.6}$$

where n_{e0}, n_{i0} and m_e, m_i are the number density and mass of electron and ion, respectively. Characteristic electron and ion time scale in unmagnetized plasma is obtained as the inverse of their plasma frequency,

$$t_{pe} = \omega_{pe}^{-1} \tag{1.7}$$

$$t_{pi} = \omega_{pi}^{-1} \tag{1.8}$$

Electron and ion plasma periods are obtained as,

$$\tau_{pe} = \frac{2\pi}{\omega_{pe}} \tag{1.9}$$

$$\tau_{pi} = \frac{2\pi}{\omega_{pi}} \tag{1.10}$$

In a magnetized plasma, electron and ion cyclotron frequencies are given by [4],

$$\Omega_e = \frac{e\mathbf{B}_0}{m_e c} \tag{1.11}$$

$$\Omega_i = \frac{e\mathbf{B}_0}{m_i c} \tag{1.12}$$

where $\mathbf{B}_0$ is the constant external magnetic field within the plasma. The spatial scale parameters known as ion and electron Larmor radii are related to circular motion of charged species in the presence of the magnetic field,

$$r_{Le} = \frac{v_{\perp e}}{\Omega_e} \tag{1.13}$$

$$r_{Li} = \frac{v_{\perp i}}{\Omega_i} \tag{1.14}$$

Here $v_{\perp j}$ $(j = e,\ i)$ represents the thermal speed of charged particles in a plane perpendicular to $\mathbf{B}_0$ and r_{Le} and r_{Li} are Larmor radii of electrons and ions, respectively. Since $m_e \ll m_i$, we get $r_{Le} \ll r_{Li}$ and $\Omega_e \gg \Omega_i$. For the existence of plasma, Larmor radii of ions and electrons must be less than the system's dimension L, i.e.,

$$r_{Le} \ll r_{Li} \ll L \tag{1.15}$$

In the limit $r_{Le} \rightarrow 0$, electrons are assumed to move only along field lines.

1.3 Fourier Analysis

Fourier series are quite helpful in modelling and solving partial differential equations. Fourier analysis is relevant to problems of mechanics, electrostatics, heat flow and many other areas. The Fourier series deals with periodic functions. The main idea is to represent complicated periodic functions as a sum of trigonometric functions named sine and cosine. Fourier analysis is an important discovery that has a significant influence on the concept of integration theory, convergence theory and other mathematical functions. For non-periodic functions, the Fourier series becomes the Fourier transform which will be discussed later.

1.3.1 Fourier Series

Fourier series is an infinite series of periodic functions represented in terms of sine and cosine functions. If a function $f(x)$ is defined for real values of x except at some points and repeats itself after a certain period of interval p, such a function is called a periodic function and is written as,

$$f(x+p) = f(x) \tag{1.16}$$

Familiar periodic functions are sine, cosine, tangent and cotangent. Function $f(x) = \tan x$ is not defined for all values of x such as $x = \pm\pi/2, \pm 3\pi/2, ...$ First of all, we represent $f(x)$ as the sum of trigonometric functions which also have period 2π and these are sine and cosine functions. An infinite trigonometric series for a periodic function $f(x)$ of period 2π is of the form [5],

$$\begin{aligned} f(x) &= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + ... \\ &= a_0 + \Sigma_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \end{aligned} \tag{1.17}$$

Here x is a variable and constants a_1, a_0, a_0 and b_n are called the coefficients of the series and $n = 1, 2, ...$ The importance of the Fourier series lies in the fact that it can explain many physical phenomena. It gives solutions of ordinary differential equations (ODEs) and partial differential equations (PDEs) with a clear physical understanding. The fundamental idea is that any periodic function of x can be expressed as a Fourier series. It is obvious that each term has a period of 2π. If the series converges to a certain point, the sum of the series will be a function with period 2π. This kind of function can be expressed in terms of the Fourier series defined as Eq. (1.17). Fourier coefficients are given by Euler formulas,

$$a_0 = 1/2\pi \int_{-\pi}^{\pi} f(x)dx \tag{1.18}$$

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x) \cos nx dx \tag{1.19}$$

$$b_0 = 1/\pi \int_{-\pi}^{\pi} f(x) \sin nx dx \tag{1.20}$$

where n=1, 2, .., . Interesting point is that coefficients a_0, a_n and b_n can be expressed in terms of the functions $f(x)$ because $\sin x$ and $\cos x$ are orthogonal functions. Fourier series can be illustrated by an example of a periodic rectangular wave which is defined as,

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases} \tag{1.21}$$

Fourier series of above function $f(x)$ is obtained as,

$$f(x) = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...) \tag{1.22}$$

1.3.2 Orthogonality of Trigonometric Functions

The orthogonality of trigonometric function is the key to Euler formulas. Trigonometric functions are orthogonal over the interval $\pi \leq x \leq \pi$, $0 \leq x \leq 2\pi$ or period of length 2π. This means that the integral of the product of any two functions over that interval is 0. For any integer m and n

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \qquad (n \neq m) \tag{1.23}$$

$$\int_{-\pi}^{\pi} \sin nx \sin nx dx = 0 \qquad (n \neq m) \tag{1.24}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad (n \neq m \quad or \quad n = m) \tag{1.25}$$

We prove the above equations by integrating both sides of Eq.(1.17) from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x) = \int_{-\pi}^{\pi} [a_0 + \Sigma_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)]dx \tag{1.26}$$

If term-wise integration is permitted, then we get,

$$\int_{-\pi}^{\pi} f(x) = a_0 \int_{-\pi}^{\pi} dx + \Sigma_{n=1}^{\infty}(a_n \int_{\pi}^{\pi} \cos nx + b_n \int_{-\pi}^{\pi} \sin nx dx) \tag{1.27}$$

First term yields $2\pi a_0$. All other integrals are 0. Dividing Eq. (1.27) by 2π gives back Eq. (1.18). Now multiply $\cos mx$ on both sides of Eq. (1.17), we get,

$$\int_{-\pi}^{\pi} f(x) \cos mx = \int_{-\pi}^{\pi} [a_0 + \Sigma_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)] \cos mx dx \tag{1.28}$$

The first term on the right-hand side give an integral of $a_0 \cos mx$, i.e., 0. Integral of $a_n \cos nx \cos mx$ is $a_m \pi$ for $n = m$ and 0 for $n \neq m$ by Eq. (1.23). Integration of $b_n \sin nx \cos mx$ is 0 for all n and m by Eq. (1.25). Thus right side of Eq. (1.27) equals $a_m \pi$. Division by π gives Eq. (1.19), using m instead of n. Now Multiply $\sin mx$ on both sides of Eq. (1.17),

$$\int_{-\pi}^{\pi} f(x) \sin mx = \int_{-\pi}^{\pi} [a_0 + \Sigma_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)] \sin mx dx \tag{1.29}$$

The first term on right side gives integral of $a_0 \sin mx$ which is 0, Integral of $a_n \cos nx \cos mx$ is 0 by Eq. (1.25). Integral of $b_n \sin nx \sin mx$ is $b_m \pi$ if $n = m$ and 0 if $n \neq m$, by Eq. (1.24). This implies Eq. (1.20) where $n = m$. Thus completing our proof of Euler formulas for the Fourier coefficients as stated in Ref. 5.

1.3.3 Fourier Integral

The fourier series is a useful tool for solving problems involving periodic functions. Let us start with a function having period $2L$ and see the change in Fourier series if $L \to \infty$. To illustrate the situation, consider a rectangular wave function $f_L(x)$ having period $2L > 2$ given by (Ref. 5),

$$f_L(x) = \begin{cases} 0, & -L < x \ < -1 \\ 1, & -1 \ < x \ < 1 \\ 0, & -1 \ < x \ < L \end{cases} \tag{1.30}$$

The left part of Fig. 1.1 shows functions for longer periods $2L = 4, 8, 16$, the non-periodic function $f_L(x)$ when L goes to infinity is shown at the end of Fig. 1.1 which can be defined as

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 1 & -1 < x \ < 1 \\ 0 & otherwise \end{cases} \tag{1.31}$$

We will investigate what happens to the Fourier series if L increases. Fourier coefficients for this function are given as,

$$a_0 = \frac{1}{2L} \int_{-1}^{1} dx = \frac{1}{L} \tag{1.32}$$

$$b_n = 0 \tag{1.33}$$

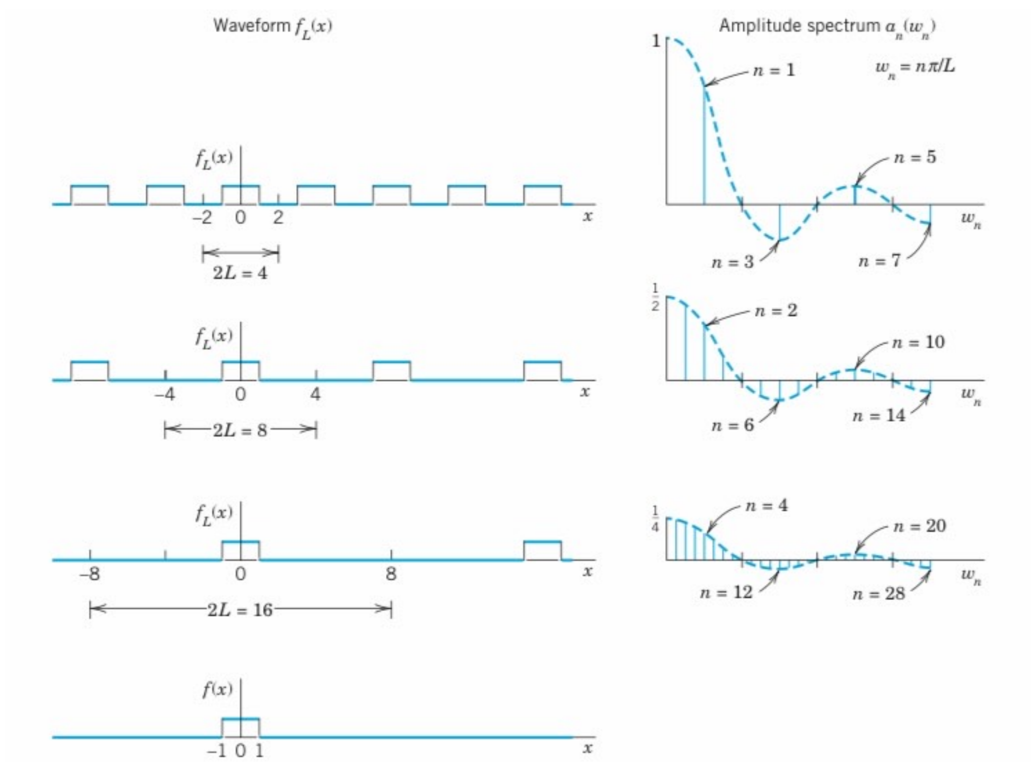

Figure 1.1: Waveforms and Amplitude Spectra

$$a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\frac{\sin\frac{n\pi}{L}}{\frac{n\pi}{L}} \tag{1.34}$$

Note that $b_n = 0$ since f_L is an even function. The sequence of Fourier coefficients is termed as amplitude spectrum of the function f_L. Here $|a_n|$ gives the maximum amplitude of the wave $a_n \cos(n\pi x/L)$ for $n = 1, 2, ..,$. Fig. 1.1 shows the spectrum of amplitude $|a_n|$ for increasing period $2L = 4, 8, 16$. Amplitude becomes more dense towards the positive ω_n-axis. If we define $\omega_n = n\pi/L$ and for an infinite period it will decrease to zero. We draw a conclusion that with increasing periods we should expect an integral which takes all of the values of frequencies ω_n.

1.3.4 Fourier Series to Fourier Integral

A periodic function having period $2L$ can be presented in the form of Fourier series as explained in Sec. 1.3.3,

$$f_L(x) = a_0 + \Sigma_{n=1}^{\infty}(a_n \cos\omega_n x + b_n \sin\omega_n x) \tag{1.35}$$

A question arises, what should we expect if $L \to \infty$? We get an infinite interval so an integral is expected instead of a series involving $\sin\omega x$ and $\cos\omega x$ with $\omega = \omega_n = n\pi/L$ not restricted to multiple integral values. Using coefficients a_n and b_n from Euler formulas, Sec. 1.3.1 and denoting variable of integration as v we can write Fourier series of $f_L(x)$ as,

$$\begin{aligned} f_L(x) =& \frac{1}{2L}\int_{-L}^{L} f_L v dv + \frac{1}{L}\Sigma_{n=1}^{\infty}[\cos\omega_n x \int_{L}^{L} f_L v \cos\omega_n v dv \\ &+ \sin\omega_n x \int_{-L}^{L} f_L v \sin\omega_n v dv] \end{aligned} \tag{1.36}$$

Setting,

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L} \tag{1.37}$$

which gives,

$$\frac{1}{L} = \frac{\Delta\omega}{\pi} \tag{1.38}$$

So we rewrite the Fourier series as,

$$\begin{aligned} f_L(x) =& \frac{1}{2L}\int_{-L}^{L} f_L(v) dv + \frac{1}{L}\Sigma_{n=1}^{\infty}[\cos w_n \Delta\omega \int_{L}^{L} f_L v (\cos\omega_n v dv \\ &+ \sin\omega_n x \Delta\omega \int_{-L}^{L} f_L v \sin\omega_n v dv] \end{aligned} \tag{1.39}$$

In the case when $L \to \infty$, we have noticed in Sec. 1.3.2 that $f(x)$ turns out to be non-periodic. Then $\Delta\omega = \lim_{L\to\infty} \pi/L \to 0$ that is the gap between two adjacent frequencies in the Fourier spectrum becomes infinitesimally small. Therefore, the summation over $\Sigma_{n=1}^{\infty} f(\omega)\Delta\omega = \Sigma_{n=0}^{\infty} = 0$ (because $a_n = 0$) should be expressed as an integral of $\int_0^\infty f(\omega)d\omega$. Furthermore, we denote $\lim_{L\to\infty} f_L(x) = f(x)$ and hence Eq. (1.39) can be written as,

$$f(x) = \frac{1}{\pi}\int_0^\infty [\cos\omega x \int_{-\infty}^{\infty} f(v)\cos\omega v dv + \sin\omega x \int_{-\infty}^{\infty} f(v)\sin\omega v dv] d\omega \tag{1.40}$$

Now introducing the new notations,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv \tag{1.41}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv \tag{1.42}$$

Final representation of of $f(x)$ is given by,

$$f(x) = \int_{0}^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \tag{1.43}$$

It is called the Fourier integral representation of non-periodic function $f(x)$. The fourier series simplify itself if a function is even or odd. Integral representation for even function when $B(\omega) = 0$ is given by,

$$f(x) = \int_{0}^{\infty} A(\omega)\cos\omega x d\omega \quad \text{where} \quad A(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(v)\cos\omega v dv \tag{1.44}$$

Integrand is even for $A(\omega)$ so limits change to twice the integral from 0 to ∞. Similarly, if Fourier integral representation is odd then $A(\omega) = 0$. Hence, Fourier sine integral can be written as,

$$f(x) = \int_{0}^{\infty} B(w)\sin\omega x d\omega \quad \text{where} \quad B(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(v)\sin\omega v dv \tag{1.45}$$

1.3.5 Fourier Transform

Fourier transforms are helpful in the representation of functions from one variable to another. Fourier transform is used for non-periodic functions as well as for spectrum. Integral transforms help in solving all sorts of partial differential and integral equations. Fourier cosine transform deals with even functions using the Fourier cosine integral obtained previously. If we set $A(\omega) = \sqrt{\frac{2}{\pi}}f_c(\omega)$ where c refers to a cosine function, then writing $v = x$ in $A(\omega)$ we get,

$$f_c(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos\omega x dx \tag{1.46}$$

and

$$f(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_c(\omega)\cos\omega x d\omega \tag{1.47}$$

Eq.(1.46) gives a new function: $f_c(\omega)$ from $f(x)$ and is named as Fourier cosine transform of function $f(x)$ where $f(x)$ is called inverse Fourier cosine transform of $f_c(\omega)$. Similarly Fourier sine transform by setting $B(\omega) = \sqrt{\frac{2}{\pi}} f_s(\omega)$ is,

$$f_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx \tag{1.48}$$

where inverse Fourier sine transform is given by,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(\omega) \sin \omega x d\omega \tag{1.49}$$

This whole process of obtaining transform $f_c(\omega)$, $f_s(\omega)$ from a function $f(x)$ is called the Fourier transform method.

1.3.6 Forced Oscillations

Here we discuss a physical application confirming the idea of splitting periodic functions into simpler ones. If a body of mass m attached to the spring having modulus k is subjected to a time-dependent external force $r(t)$ then the differential equation for this problem is,

$$my^{''} + cy^{'} + ky = r(t) \tag{1.50}$$

Eq. (1.50) describes the forced oscillation of a body. Here c and $r(t)$ are the damping constant and driving force for the oscillations of the quantity $y = y(t)$, respectively. If $c = 0$, $r(t) = 0$, Eq. (1.50) reduces to,

$$my^{''}(t) + ky(t) = 0 \tag{1.51}$$

which is the equation of a simple harmonic oscillator and y is the displacement. The oscillation frequency of the oscillator is given as,

$$\omega = (\frac{k}{m})^{\frac{1}{2}} \tag{1.52}$$

Eq.(1.50) is an in-homogeneous ordinary differential equation due to the source term $r(t) \neq 0$, it represents forced oscillations. A general solution of such a differential equation is given as,

$$y(t) = y_c(t) + y_p(t) \tag{1.53}$$

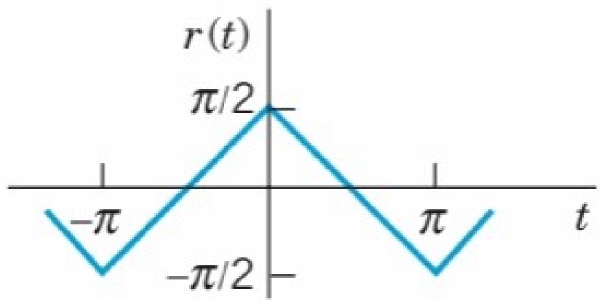

Figure 1.2: Forced oscillations

where $y_c(t)$ is a solution of the homogeneous part of the equation, i.e.,

$$my^{''} + cy^{'} + ky = 0 \tag{1.54}$$

and $y_p(t)$ is the particular solution of nonhomogeneous Eq. (1.50). If $r(t)$ is any periodic function other than a cosine or sine function then a steady state solution can be written as a superposition of oscillations with frequency equal to the frequency of $r(t)$ or the integral multiple of it. Out of these frequencies, if one frequency is close enough to the frequency of $r(t)$ then corresponding oscillations are the dominant part of the system in the presence of applied external force. Let's solve this equation for some random values of the constants by choosing $m = 1$, $c = 0.05$ (g/sec) and $k = 25$ (g/sec^2) then Eq. (1.50) becomes,

$$y^{''} + 0.05y^{'} + 25y = r(t) \tag{1.55}$$

Let us assume that the source term $r(t)$ is given as

$$r(t) = \begin{cases} t + \pi/2, & -\pi < t < 0 \\ -t + \pi/2, & 0 < t < \pi \end{cases} \tag{1.56}$$

Our goal is to work for steady state solution $y(t)$. We will represent $r(t)$ in terms of the Fourier series by working on Fourier coefficients. Calculations yield,

$$a_0 = 0 \tag{1.57}$$

,

$$a_1 = \frac{2 - 2(-1)^n}{\pi n^2} \tag{1.58}$$

$$b_1 = 0 \tag{1.59}$$

Here $b_1 = 0$ clearly shows that $r(t)$ is an even function. Writing $r(t)$ in terms of Fourier series for $b_n = 0$,

$$r(t) = 4/\pi(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + ...) \tag{1.60}$$

Eq. (1.50) can be expressed in following form

$$y^{''} + 0.05y^{'} + 25y = \frac{4}{n^2\pi}\cos nt \tag{1.61}$$

In this example, $r(t)$ turns out to be in sinusoidal form, therefore we use the method of undetermined coefficients [6]. We assume that the steady state solution of Eq. (1.50) has the form

$$y_n = A_n \cos nt + B_n \sin nt \tag{1.62}$$

$$y_n^{'} = -nA_n \sin nt + nB_n \cos nt \tag{1.63}$$

$$y_n^{''} = -n^2 A_n \cos nt - n^2 B_n \sin nt \tag{1.64}$$

Substituting $y_n^{'}$and $y_n^{''}$ into Eq. (1.61) and using the method of equating coefficients, we get,

$$A_n = \frac{4(25 - n^2)}{n^2\pi D_n} \tag{1.65}$$

$$B_n = \frac{0.02}{n\pi D_n} \tag{1.66}$$

where $D_n = (25 - n^2)^2 + (0.05)^2$. Substituting A_n and B_n back into Eq. (1.62), we find,

$$y_n = \frac{4(25 - n^2)}{n^2\pi D_n}\cos nt + \frac{0.02}{n\pi D_n}\sin nt \tag{1.67}$$

The Eq. (1.50) is a linear equation, therefore its solution can be written as a linear combination of several solutions,

$$y = y_1 + y_2 + y_3 + ... \tag{1.68}$$

where y_n is given by Eq. (1.67) for $n = 1, 2, ...$ The amplitude of the oscillations obtained using Eq. (1.62), is given as,

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}} \tag{1.69}$$

which gives the maximum value at $n = 5$. Thus output oscillation in a steady state is five times the frequency of the external driving force which is providing energy to the system. This comprehends the idea of getting resonant frequency in response of external periodic force.

1.4 Acoustic Waves in Dissipative Medium

Acoustic waves are pressure waves generated when a pressure change in fluid causes itself to compress leading to an additional pressure change. Acoustic variables studied under wave propagation are particle velocity, acoustic pressure and density fluctuations in the fluid. The governing equations for an in-viscid and non-conducting fluid medium are [7],

$$\partial_t n + \nabla.(n\vec{u}) = 0 \tag{1.70}$$

$$n(\partial_t + \nabla.\vec{u})\vec{u} = -\nabla p \tag{1.71}$$

$$p = p(\rho) \tag{1.72}$$

Eqs. (1.70-1.72) are known as the mass continuity equation, momentum equation and fluid or gaseous equation of state, respectively. Pressure is expressed in terms of density and entropy. When losses are negligible, entropy remains constant, so pressure can be written in terms of density only. For an adiabatic system

$$\frac{p}{p_0} = (\frac{n}{n_0})^\gamma \tag{1.73}$$

Here γ is the ratio of specific heats at constant pressure and volume, i.e., C_p/C_v. Introducing small perturbations to the system considering acoustic variables,

$$p = p_0 + p_1 \tag{1.74}$$

$$n = n_0 + n_1 \tag{1.75}$$

$$\vec{u} = \vec{u}_1 \tag{1.76}$$

Here p is the total pressure, p_0 is the equilibrium pressure and p_1 refers to the perturbed part. The same scheme holds for density and fluid velocity, respectively. Perturbed acoustic variables are presumed as small quantities of the first order as small signal approximations disturb fluid only in a minor region. Linearized equation of continuity and momentum is written as,

$$\partial_t n_1 + n_0 \nabla \vec{u}_1 = 0 \tag{1.77}$$

$$n_0 \partial_t \vec{u}_1 + \nabla p_1 = 0 \tag{1.78}$$

Equation of state can be presented in the form of a Taylor series by expanding p as a function of n around n_0,

$$p = p_0 + (\partial_n p)_{n_0}(n - n_0) + \frac{1}{2!}(\partial_n^2 p)_{n_0}(n - n_0)^2 + ... \tag{1.79}$$

$$p = p_0 + As + \frac{Bs^2}{2!} + .. \tag{1.80}$$

where

$$A = n_0(\partial_n p)_{n_0} \tag{1.81}$$

$$B = n_0^2(\partial_n p)_{n_0}^2 \tag{1.82}$$

Eq. (1.79) is nonlinear because of the quadratic term $(n - n_0)^2$. In order to make it a linear equation, only the least order terms must retain, i.e., $(n - n_0)$. This gives,

$$p_1 = p - p_0 = As \tag{1.83}$$

If pressure is measured in terms of Pascal where $1Pa = 1g.cm/sec^2$, inquiring the unit of coefficient A which is the adiabatic bulk modulus,

$$\partial_n p \rightarrow g.cm/sec^2/g/cm^3 = cm^2/sec^2 = c^2. \tag{1.84}$$

Here c is the wave speed. Rearranging Eq. (1.74) and Eq. (1.75) as,

$$p_1 = p - p_0 \tag{1.85}$$

$$n_1 = n - n_0 \tag{1.86}$$

Inserting the above equations into linearized Eq. (1.79) gives,

$$p_1 = c_0^2 n_1 \tag{1.87}$$

Taking the time derivative of Eq. (1.77) and subtracting it from the spatial derivative of equation Eq. (1.78) and inserting Eq. (1.87) in it, we get

$$\partial_t^2 p_1 - c_o^2 \nabla^2 p_1 = 0 \tag{1.88}$$

The above equation can be compared with the wave equation, i.e.,

$$\partial_t^2 \vec{u} = c^2 \nabla^2 \vec{u} \tag{1.89}$$

which means that pressure and density perturbation travels through a medium with speed c. For a one-dimensional adiabatic system, the sound speed at equilibrium density and pressure is given by,

$$c = (\partial_n p)^{\frac{1}{2}} = (\frac{\gamma T_e}{m})^{\frac{1}{2}} \tag{1.90}$$

The general solution to Eq. (1.88) is given by,

$$p_{(x,t)} = p e^{\iota(\omega t - kx)} \tag{1.91}$$

Back substitution into Eq. (1.88) gives,

$$\omega^2 = c_0^2 k^2 \tag{1.92}$$

which is called the dispersion relation for acoustic waves in a non-dissipative medium. In order to include the viscous effect in the momentum equation, surface force is presented. An ideal fluid has no viscosity. Using stress tensor and stoke assumptions for a real fluid, we'll calculate the coefficient of viscosity as $4\mu/3$ where μ is the bulk coefficient of viscosity [8]. The Linearized equation of momentum including viscous effect is given as,

$$n_0 \partial_t \vec{u}_1 + \nabla p_1 = \frac{4\mu}{3} \nabla^2 \vec{u}_1 \tag{1.93}$$

Taking the spatial derivative of Eq. (1.77), substituting into momentum Eq. (1.93), we get,

$$n_o \partial_t \vec{u}_1 + \nabla p_1 = -\frac{4\mu}{3} \partial_t \nabla p_1 \tag{1.94}$$

Taking the time derivative of Eq. (1.87) and putting it in the above equation, yields,

$$n_0 \partial_t \vec{u}_1 + \nabla p_1 = -\frac{4\mu}{3 n_0 c_0^2} \partial_t \nabla p_1 \tag{1.95}$$

Inserting the time derivative of Eq. (1.77) into the double time derivative of Eq. (1.87),

$$\partial_t \nabla \vec{u}_1 = -\frac{1}{c_0^2 n_0} \partial_t^2 p \tag{1.96}$$

Inserting Eq. (1.96) into the spatial derivative of Eq. (1.95) gives,

$$\nabla^2 p + \frac{4\nu}{3c_0^2} \partial_t \nabla^2 p - \frac{1}{c_0^2} \partial_t^2 p = 0 \tag{1.97}$$

The above relation is an acoustic wave equation in a dissipative medium. Here $\nu = \mu/n_0$ is known as the kinematic coefficient of viscosity. The propagation vector for this kind of wave is given by $k = \beta - \alpha\iota$ where α is the dissipative coefficient. If the general solution to the above wave equation is given as same as Eq. (1.91). Taking spatial, time and spatial time derivative, respectively,

$$\nabla p_{(x,t)} = -k^2 p_0 e^{\iota(\omega t - kx)} \tag{1.98}$$

$$\partial_t \nabla^2 p_{(x,t)} = -k^2 (\iota\omega) p_0 e^{\iota(\omega t - kx)} \tag{1.99}$$

$$\partial_t p_{(x,t)} = -\omega^2 p_0 e^{\iota(\omega t - kx)} \tag{1.100}$$

Substituting the above equations back into Eq. (1.96) gives,

$$k^2 (1 + \frac{4\nu}{3c_0^2} \iota\omega) - \frac{\omega^2}{c_0^2} = 0 \tag{1.101}$$

$$k = \frac{\pm\frac{\omega}{c}}{\sqrt{1 + \iota\frac{4v\delta_v}{3}}} = \pm(\beta - \iota\alpha) \tag{1.102}$$

Here $\delta_v = w\nu/c_0^2$. If waves travel along the positive direction only, putting k back into the general solution we get,

$$p = p_0 e^{-\alpha x} e^{\iota(kx - \omega t)} \tag{1.103}$$

The above equation is a solution to the viscous wave equation. $e^{-\alpha x}$ shows damping of wave because of viscosity.

1.5 Waves in Plasma

One of the most well-defined properties of plasma is exhibiting numerous kinds of waves. An enormous amount of particles are enclosed in plasma. In order to depict the motion of these particles, an extensive number of modes or waves is required. So a wave is defined as a disturbance that propagates from one medium to another and it is usually ascertained by the mechanical properties of the medium. We restrict ourselves to the situation in which oscillations are small and the amplitude of oscillation is scaled down, then we linearize the fluid equations. We took first-order expansion while ignoring second-order and higher-order terms so whenever oscillation quantities are small if multiplied, they turned out to be small so we consider them as higher-order and oversight them. Different categories of electrostatic waves are discussed in Sec. (1.5.1-1.5.3).

1.5.1 Electron Plasma Waves (EPW)

Thermal motion and energy of the electron generate oscillations in the plasma and make them travel. Assuming magnetic field $\mathbf{B} = 0$ and using fluid model we can write,

$$\partial_t n_e + \nabla.(n_e \mathbf{v}_e) = 0 \tag{1.104}$$

$$m_e n_e(\partial_t + \mathbf{v_e}.\nabla)\mathbf{v}_e = -en_e\mathbf{E} - \nabla p_e \tag{1.105}$$

$$\nabla.\mathbf{E} = 4\pi e n_e \tag{1.106}$$

For an adiabatic system, the equation of state can be modified as,

$$\nabla p_e = \gamma_e T_e \nabla n_e \tag{1.107}$$

Here $\gamma_e = 1 + \frac{2}{f}$ where f is a degree of freedom, so for a one-dimensional case

$$\nabla p = 3T_e \nabla n_e \tag{1.108}$$

For small perturbations

$$n = n_{0e} + n_{e1} \tag{1.109}$$

$$\mathbf{v_e} = \mathbf{v}_{0e} + \mathbf{v}_{e1} \tag{1.110}$$

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 \tag{1.111}$$

Linearised fluid equations are given as,

$$\iota\omega n_{e1} = n_{0e}\iota k v_{e1} \tag{1.112}$$

$$-\iota\omega m_e n_{0e} v_{e1} = -e n_{0e}\mathbf{E}_1 - 3T_e \iota k n_{e1} \tag{1.113}$$

$$\nabla.\mathbf{E}_1 = -4\pi e n_{e1} \tag{1.114}$$

Writing Eq. (1.112) in terms of electron density n_{e1} as,

$$n_{e1} = n_{0e}\frac{k v_{e1}}{\omega} \tag{1.115}$$

Inserting it into Poisson's equation:

$$\mathbf{E}_1 = -\frac{4\pi e n_{0e} k v_{e1}}{\iota k \omega} \tag{1.116}$$

Inserting into Eq. (1.113) in terms of $\mathbf{E}_1$ and n_{e1}, we get,

$$\iota\omega m_e n_{0e} v_{e1} = (e n_{0e}(\frac{-4\pi e}{\iota k}) + 3T_e \iota k) n_{0e}\frac{k v_{e1}}{\omega} \tag{1.117}$$

$$\omega^2 = (\frac{4\pi n_{0e} e^2}{m_e} + \frac{3T_e k^2}{m_e}) v_{e1} \tag{1.118}$$

$$\omega^2 = \omega_{ep}^2 + \frac{3}{2}k^2 v_{th}^2 \tag{1.119}$$

or

$$\omega^2 = \omega_{pe}^2(1 + \frac{3}{2}\lambda_{De}^2 k^2) \tag{1.120}$$

where $\omega_{ep}^2 = 4\pi n_{0e}e^2/m$, $v_{th}^2 = 3T_e/m_e$ and $\lambda_{De}^2 = v_{th}^2/\omega_{ep}^2$. Since the frequency is k dependent, group velocity $\mathbf{v}_g$ is given as,

$$v_g = \frac{d\omega}{dk} = \frac{3}{2}\frac{k}{\omega}v_{th}^2 = \frac{3}{2}\frac{v_{th}^2}{v_\phi} \tag{1.121}$$

where v_ϕ is the phase velocity. At large k, wave information propagate with thermal speed. For a small k (large λ_D), the transmission of information is quite less than v_{th}, owing to the fact that at large λ_D density gradient becomes very small hence thermal motions carry a very small net momentum into close by layers.

1.5.2 Ion Acoustic Waves (IAW)

IAW are longitudinal oscillations that travel through plasma because of the positively charged ions. These waves interact with the electromagnetic field of the ions and through collisions as well. Using plasma approximation, i.e., $n_i = n_e = n$ where magnetic field $B = 0$, equation of continuity and momentum are given as,

$$\partial_t n_i + \nabla.(n_i \mathbf{v}_i) = 0 \tag{1.122}$$

$$m_i n_i(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_i = n_i e\mathbf{E} - \nabla p \tag{1.123}$$

Here n_i is the density of ions based on plasma approximation, $\mathbf{v}$ is the velocity field and m_i is the mass of an ion. If the system is adiabatic such that no heat exchange is possible, we can use the following equation of state for an ideal gas, i.e.,

$$p = Cn_i^{\gamma_i} \tag{1.124}$$

Taking time derivative w.r.t to n_i and dividing by $p = Cn_i^{\gamma_i}$, we get,

$$\frac{\nabla p}{p} = \gamma_i \frac{\nabla n_i}{n_i} \tag{1.125}$$

Inserting ideal gas equation $p_i = n_i T_i$ into the above equation,

$$\nabla p = \gamma_i T_i \nabla n_i \tag{1.126}$$

The final form of the momentum equation is given by,

$$m_i n_i(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_i = n_i e\mathbf{E} - \gamma_i T_i \nabla n_i \tag{1.127}$$

Let's consider perturbations as minimal changes in velocity, density and electric field, and then we write,

$$n_i = n_{0i} + n_{i1} \tag{1.128}$$

$$\mathbf{v_i} = \mathbf{v}_{0i} + \mathbf{v}_{i1} \tag{1.129}$$

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 \tag{1.130}$$

In the absence of oscillations, plasma has equilibrium parameters which bring result such that,

$$\nabla n_{0i} = \partial_t n_{0i} = 0 \tag{1.131}$$

$$\partial_t v_{0i} = v_{0i} = 0 \tag{1.132}$$

$$\partial_t E_0 = E_0 = 0 \tag{1.133}$$

In the presence of oscillations, fluctuations are assumed to be sinusoidal,

$$n_{i1} = |n_{i1}| e^{\iota(kx-\omega t)} \tag{1.134}$$

$$\mathbf{v}_{i1} = |\mathbf{v}_{i1}| e^{\iota(kx-wt)} \mathbf{x} \tag{1.135}$$

$$\mathbf{E} = |\mathbf{E}_1| e^{\iota(kx-wt)} \mathbf{x} \tag{1.136}$$

Using $\partial_t n_i = -\iota\omega n_i$, $\partial_t v_{i1} = -\iota\omega v_{i1}$, $\nabla\phi_1 = \iota k\phi_1$ as $\mathbf{E} = -\nabla\phi$, we can linearize equation of continuity and momentum as,

$$\iota\omega n_{i1} = n_{0i}\iota k v_1 \tag{1.137}$$

$$-\iota\omega m_i n_{0i} v_{i1} = -e n_{0i}\iota k\phi_1 - \gamma_i T_i \iota k n_{i1} \tag{1.138}$$

We can use Boltzmann approximation $n_{e1} = n_{e0}\exp(e\phi_1/T_e)$ for perturbation in the density of electron, similar expression can be written for ions as,

$$n_{i1} = n_{0i}(\frac{e\phi_1}{T_e}) \tag{1.139}$$

Solving for ion density using Eq. (1.137),

$$n_{i1} = n_{0i}\frac{kv_{i1}}{\omega} \tag{1.140}$$

Inserting into Eq. (1.39), we get,

$$\phi_1 = \frac{kv_{i1}T_e}{\omega} \tag{1.141}$$

Inserting in terms of n_{i1} and ϕ_1, Eq.(1.138) takes the form,

$$\omega m_i n_{0i} v_{i1} = (\frac{e\iota k T_e}{e} + \frac{\gamma T_i \iota k}{1})\frac{n_0 \iota k v_{i1}}{i\omega} \tag{1.142}$$

$$\omega^2 = k^2(\frac{T_e}{m_i} + \frac{\gamma_i T_i}{m_i}) \tag{1.143}$$

The above equation is the dispersion relation for IAW in unmagnetized plasma. It basically shows the propagation of unvarying velocity waves and exists only because of thermal motions. In IAW, electrons are dragged along with ions and screen the electric field generated by ions. The IAW configure the region of compression and refraction. The compressed region turning into expanded regions shows that ion thermal motions scatter the ions which brings out the second term in Eq. (1.143). The region of refraction is because of ions being shielded by electrons which can leak the potential of order T_e/e available for ion bunches giving rise to the first term in Eq. (1.143). If quasineutrality doesn't hold then we use Poisson's equation,

$$\nabla.\mathbf{E}_1 = k^2\phi_1 = 4\pi e(n_{i1} - n_{e1}) \tag{1.144}$$

Inserting Boltzmann relation for the electron density in the above equation, we get

$$\phi_1(k^2 + \frac{4\pi n_{0i}e^2}{T_e}) = 4\pi e n_{i1} \tag{1.145}$$

$$\phi_1(k^2\lambda_D^2 + 1) = 4\pi e n_{i1}\lambda_D^2 \tag{1.146}$$

Inserting Eqs. (1.140) and (1.146) into the Eq. (1.138), we get

$$\iota\omega m_i n_{0i} v_{1i} = (e n_{0i}\iota k \frac{4\pi e\lambda_D^2}{1 + k^2\lambda_D^2} + \gamma_i T_i \iota k) n_{0i}\frac{k v_{1i}}{\omega} \tag{1.147}$$

$$\omega^2 = k^2(\frac{T_e}{m_i}\frac{1}{1 + k^2\lambda_D^2} + \frac{\gamma_i T_i}{m_i}) \tag{1.148}$$

This is same as Eq. (1.143) except for the factor $1 + k^2\lambda_D^2$. Our assumption of plasma approximation gives us an error of order $k^2\lambda^2 = (2\pi\lambda_D/\pi)^2$. λ_D is quite small in the majority of the experiments so plasma approximation is justifiable. For $T_i \gg T_e$, dispersion relation reduces to,

$$\omega = k(\frac{\gamma_i T_i}{m_i})^{\frac{1}{2}} \tag{1.149}$$

For $T_e \gg T_i$, dispersion relation takes the form,

$$\omega^2 = k^2\frac{T_e}{m_i(1 + k^2\lambda_D^2)} \tag{1.150}$$

If $k\lambda_D \ll 1$, Eq. (1.150) gives,

$$\frac{\omega}{k} \approx \frac{T_e}{m_i} = c_s \tag{1.151}$$

If, $k\lambda_D \gg 1$, Eq. (1.150) gives,

$$\omega \approx \omega_{pi} \tag{1.152}$$

Ions oscillations are then unshielded by electrons [9].

1.5.3 Ion Acoustic Wave (IAW) in Magnetized Plasma

Consider an electrostatic wave propagating in a magnetized plasma. For a situation where propagation is almost perpendicular to magnetic field but not exactly, we have conditions; $\mathbf{B} = B_0\mathbf{z}, \mathbf{v} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z), \mathbf{v}_\perp \neq \mathbf{v}_\parallel, k_x = 0, \mathbf{E} = -\nabla\phi$, $T_i = 0$. Governing fluid equations are,

$$\partial_t n_i + \nabla.(n_i\mathbf{v}_i) = 0 \tag{1.153}$$

$$m_i n_i(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_i = en_i(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}) - \nabla p_i \tag{1.154}$$

$$-\nabla^2\phi = 4\pi e(n_e - n_i) \tag{1.155}$$

We will ignore the pressure term and linearize the above equation by considering small perturbations. We'll work for the parallel and perpendicular components of velocity. Taking cross product of momentum equation with $\mathbf{z}$, we get

$$m_i n_i(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_i \times \mathbf{z} = en_i(\mathbf{E} \times \mathbf{z} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B} \times \mathbf{z}) \tag{1.156}$$

$$(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_i \times \mathbf{z} = -\frac{e}{m_i}(\nabla\phi \times \mathbf{z} - \frac{1}{c}\mathbf{B}\mathbf{v}_{i\perp}) \tag{1.157}$$

$$\mathbf{v}_{i\perp} = -\frac{e}{m_i\Omega_i}(\nabla\phi \times \mathbf{z}) - \frac{1}{\Omega_i}(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_i \times \mathbf{z} \tag{1.158}$$

where $\Omega_i = e\mathbf{B}/m_i c$. Linearizing the above equation, yields,

$$\mathbf{v}_{i\perp 1} = -\frac{c}{B_0}(\nabla\phi_1 \times \mathbf{z}) - \frac{1}{\Omega_i}\partial_t(\mathbf{v}_{i1} \times \mathbf{z}) \tag{1.159}$$

From the above equation, we can write $\mathbf{v}_{i\perp 1}$ in terms of Cartesian coordinate, i.e., x and y components,

$$v_{ix} = \frac{\iota c}{B_0}k_y\phi_1 + \frac{i\omega}{\Omega_i}v_{iy} \tag{1.160}$$

$$v_{iy} = \frac{\iota\omega}{\Omega_i}v_{ix} \tag{1.161}$$

Putting v_{ix} into v_{iy}

$$v_{iy} = \frac{c\omega}{\Omega_i B_0}k_y\phi_1 - \frac{\omega^2}{\Omega_i^2}v_{iy} \tag{1.162}$$

$$v_{iy}(1+\frac{\omega^2}{\Omega_i^2}) = \frac{c\omega}{\Omega_i B_0}k_y\phi_1 \tag{1.163}$$

For $\omega_i \ll \Omega_i$,

$$v_{iy} = -\frac{c\omega}{\Omega_i B_0}k_y\phi_1 \tag{1.164}$$

For parallel component, the momentum equation modifies as,

$$m_i n_i(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_{iz} = -en_i\partial_z\phi_1 \tag{1.165}$$

After linearization,

$$m_i n_i \iota\omega v_{iz1} = \iota e k_z\phi_1 \tag{1.166}$$

$$v_{iz} = \frac{ek_z}{m_i\omega}\phi_1 \tag{1.167}$$

Linearized equation of continuity in terms of $\mathbf{v}_\perp$ and $\mathbf{v}_\parallel$,

$$\partial_t n_{i1} + n_{0i}(\nabla_\perp.\mathbf{v}_{i\perp} + \partial_\parallel \mathbf{v}_{i\parallel}) = 0 \tag{1.168}$$

Here $v_\parallel = v_z$. Inserting v_x, v_y and v_z into Eq. (1.168), we get,

$$-\iota\omega n_{i1} + \iota n_{0i}(-\frac{c}{\Omega_i B_0}k_y^2 + \frac{e}{m\omega^2}k_z^2)\phi_1 \tag{1.169}$$

$$n_{i1} = n_{0i}(-\frac{c}{\Omega_i B_0}k_y^2 + \frac{e}{m\omega^2}k_z^2)\phi_1 \tag{1.170}$$

Using Boltzmann approximation and Eq. (1.170) into linearized Poisson's equation, we get the result,

$$-\nabla^2\phi_1 = 4\pi e(n_{e1} - n_{i1}) \tag{1.171}$$

$$-\nabla^2\phi_1 = \frac{4\pi e^2 n_{0i}}{T_e}\phi_1 + \frac{4\pi e n_{0i} c}{\Omega_i B_0}k_y^2\phi_1 - \frac{4\pi e^2 n_{0i}}{m_i\omega^2}k_z^2\phi_1 \tag{1.172}$$

$$-1 = \frac{4\pi e^2 n_{0i}}{k^2 T_e} + \frac{4\pi e n_{0i} c}{k^2\Omega_i B_0}k_y^2 - \frac{4\pi e^2 n_{0i}}{k^2 m_i\omega^2}k_z^2 \tag{1.173}$$

Applying quasi neutrality and multiplying with m_i/m_i,

$$\frac{e^2 n_{0i}}{T_e} + \frac{e^2 n_{0i} c}{\Omega_i e B_0}k_y^2 - \frac{e^2 n_0}{m_i\omega^2}k_z^2 = 0 \tag{1.174}$$

$$1 + \frac{m_i c}{\Omega_i e B_0}\frac{T_e}{m_i}k_y^2 - \frac{T_e}{m_i\omega^2}k_z^2 = 0 \tag{1.175}$$

$$1 + \frac{c_s^2}{\Omega_s^2}k_y^2 - \frac{c_s^2}{\omega^2}k_z^2 = 0 \quad (1.176)$$

$$\omega^2(1 + k_y^2\rho_s^2) = c_s^2 k_z^2 \quad (1.177)$$

$$\omega^2 = \frac{c_s^2 k_z^2}{(1 + k_y^2\rho_s^2)} \quad (1.178)$$

The above equation is the dispersion relation for IAW in a magnetized plasma indicating the outcomes of ion parallel motion and ion Larmour radius effect at electron temperature [10]. For a situation where quasineutrality doesn't hold, i.e., $n_e \neq n_i$, Eq. (1.172) modifies as,

$$-\frac{T_e}{4\pi e^2 n_{0i}}k^2 = 1 + \frac{m_i c}{\Omega_i e B_0}\frac{T_e}{m_i}k_y^2 - \frac{T_e}{m_i\omega^2}k_z^2 \quad (1.179)$$

$$-\lambda_D^2 k^2 = 1 + \frac{c_s^2}{\Omega_i^2}k_y^2 - \frac{c_s^2}{\omega^2}k_z^2 \quad (1.180)$$

$$\omega^2(1 + \lambda_D^2 k^2 + \rho_s^2 k_y^2) = c_s^2 k_z^2 \quad (1.181)$$

$$\omega^2 = \frac{c_s^2 k_z^2}{(1 + \lambda_D^2 k^2 + \rho_s^2 k_y^2)} \quad (1.182)$$

The relation has been modified by order of $\lambda_D^2 k^2$, as λ_D is small so plasma approximation holds here as well.

Chapter 2

Parallel Shear Flow Driven Drift Waves

2.1 Introduction

The basic drift wave is a low-frequency electrostatic wave which propagates as a normal mode in inhomogeneous magnetized plasma. This is named as drift wave because it travels along the direction of electron diamagnetic drift. The pressure gradient perpendicular to external magnetic is responsible for the existence of drift waves. The dispersion relation of the fundamental drift mode in the simplest case is obtained by assuming the thermal electrons to be inertia-less ($m_e \to 0$) in the low-frequency limit $\omega \ll \Omega_i = (\frac{eB_0}{m_i c})$ where Ω_i is the ion gyro frequency and B_0 is the ambient magnetic field. If the external magnetic field is given in the z-direction, i.e., $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and the density gradient is along the x-axis, then the wave propagates predominantly along the y-direction with a very small component of wave vector along field lines. The $\mathbf{E} \times \mathbf{B}$ drift of charged particles allows the propagation of drift waves in magnetized plasma. A drift wave appears due to diamagnetic drift which is a fluid drift and is not associated with the single particle motion. The linear and nonlinear drift waves have extensively been investigated in literature [11].

Historically drift waves were discovered as low-frequency fluctuations present at the edge of the plasma column in linear plasma devices [12], [13]. Extensive studies of drift wave stability were conducted by Chen [14]. Later on, a small collisional drift wave was studied by Schllit and Hendal [15].

Whenever is a density gradient in particle distribution, there's always potential for instability of drift waves which is termed universal instability. The basic drift wave turns out to be unstable when its dispersion relation is obtained using the kinematic model. This instability mechanism of drift waves results in the transport of particles energy and momentum in a magnetically confined system [16].

2.2 Dispersion Relation in Cold Ion Plasma

Let us have a picture of a drift wave in a cold ion plasma ($T_i = 0$). Density gradient is along negative x-direction such that $\nabla n_0 = -\mathbf{x}|\frac{dn_0}{dx}|$ while $\mathbf{B} = B_0\mathbf{z}$ acts along z direction. Density perturbations vary sinusoidally along the y axis but remain constant along the x-direction. This variation is slow along the z-axis with a small parallel component of wave vector $k_\parallel$ to justify the Boltzmann relation for electrons. Equation of motion for jth species $(j = i, e)$ is given by [17],

$$m_j n_j(\partial_t + \mathbf{v}_j.\nabla)\mathbf{v}_j = q_j n_j(\mathbf{E} + \frac{1}{c}\mathbf{v}_j \times \mathbf{B}) - \nabla p_j \tag{2.1}$$

Ignoring p_j, then cross product of above equation with $\mathbf{z}$, yields,

$$(\partial_t + \mathbf{v}_j.\nabla)\mathbf{z} \times \mathbf{v}_j = \frac{q_j}{m_j}(\mathbf{z} \times \mathbf{E}) + \frac{q_j}{cm_j}\mathbf{v}_{j\perp}B_0 \tag{2.2}$$

Separating perpendicular velocity $\mathbf{v}_{j\perp}$,

$$\mathbf{v_{j\perp}} = -\frac{c}{B_0}(\mathbf{z} \times \mathbf{E}) + \frac{1}{\Omega_j}(\partial_t + \mathbf{v}_j.\nabla)\mathbf{z} \times \mathbf{v}_j \tag{2.3}$$

$$\mathbf{v}_{j\perp} = \mathbf{v}_E + \mathbf{v}_P \tag{2.4}$$

where,

$$\mathbf{v}_E = -\frac{c}{B_0}(\hat{\mathbf{z}} \times \mathbf{E}) \tag{2.5}$$

$$\mathbf{v}_{Pj} = \frac{1}{\Omega_j}(\partial_t + \mathbf{v}_j.\nabla)\mathbf{z} \times \mathbf{v}_j \tag{2.6}$$

Here $\mathbf{v}_E$ is the $\mathbf{E} \times \mathbf{B}$ drift which is charge independent and $\mathbf{v}_{Pj}$ is the polarization drift. Assuming $\mathbf{v}_E$ to be the dominating part of the perturbed velocity, we obtain,

$$\mathbf{v}_{j\perp 1} \simeq -\frac{c}{B_0}(\mathbf{z} \times \nabla_\perp \phi_1) - \frac{c}{\Omega_j B_0}\partial_t(\nabla_\perp \phi_1) \tag{2.7}$$

where we have approximated,

$$\mathbf{v}_{pj} = -\frac{c}{\Omega_j B_0}\partial_t \nabla_\perp \phi_1 \tag{2.8}$$

The supposition $\mathbf{v}_{Pj} \ll \mathbf{v}_E$ is consistent with $\omega \ll \Omega_j$. The parallel equation of motion for inertialess isothermal electrons is given by,

$$m_e n_e(\partial_t + \mathbf{v}_{ez}.\nabla)\mathbf{v}_{ez} = e n_e \mathbf{E} - \nabla p_e \tag{2.9}$$

Dropping electron inertial terms on left hand side and using $\nabla p_e = -T_e \nabla n_e$, we get,

$$e\nabla\phi = T_e \frac{\nabla n_e}{n_e} \tag{2.10}$$

$$\frac{n_e}{n_0} = \exp(\frac{e\phi}{T_e}) \tag{2.11}$$

After linearization,

$$\frac{n_{e1}}{n_{e0}} = \frac{e\phi_1}{T_e} \tag{2.12}$$

We can write linearized ion continuity equation as,

$$\partial_t n_{i1} + \nabla n_{0i}.\mathbf{v}_{i\perp} + n_{0i}(\nabla_\perp.\mathbf{v}_{i\perp}) = 0 \tag{2.13}$$

Where ion polarization drift and ion's parallel motion have been ignored. Since $\nabla.\mathbf{v}_E = 0$, Eq. (2.8) can be written as,

$$\nabla.\mathbf{v}_{i\perp} = -\frac{c}{\Omega_i B_0}\partial_t \nabla_\perp^2 \phi_1 \tag{2.14}$$

Inserting into Eq. (2.13),

$$\partial_t n_{i1} - \frac{c}{B_0}\nabla n_{0i}.(\mathbf{z}\times\mathbf{E}) - \frac{n_{0i}c}{\Omega_i B_0}\partial_t \nabla_\perp^2 \phi_1 = 0 \tag{2.15}$$

Ignore the third term and divide by n_0,

$$\partial_t \frac{n_{i1}}{n_0} + \frac{c}{B_0}\frac{\nabla n_{0i}}{n_0}.(\mathbf{z}\times\nabla_\perp \phi_1) = 0 \tag{2.16}$$

wehre $n_0 = n_{0i} = n_{0e}$. Applying quasi neutrality condition, i.e., $n_{i1}/n_0 = n_{e1}/n_0 = e\phi/T_e$, we obtain,

$$-\iota\omega\frac{e\phi}{T_e} + \frac{c}{B_0}\frac{\nabla n_{0i}}{n_0}.(\mathbf{z}\times\nabla_\perp \phi_1) = 0 \tag{2.17}$$

Multiplying with T_e/e,

$$\iota\omega\phi_1 = \frac{cT_e}{eB_0}\frac{\nabla n_{0i}}{n_0}.(\mathbf{z}\times\iota k_\perp)\phi_1 \tag{2.18}$$

$$\omega = \frac{cT_e}{eB_0}(-\frac{1}{n_0}|\frac{dn_{0i}}{dx}|\mathbf{x}).(-k_y\mathbf{x}) \tag{2.19}$$

$$\omega = \frac{cT_e}{eB_0}\kappa_n k_y = \omega_e^* \tag{2.20}$$

$$\omega_e^* = v_{de}^* k_y \tag{2.21}$$

where $v_{de}^* = \frac{cT_e}{eB_0}\kappa_n$ and $\kappa_n = |\frac{1}{n_0}\frac{dn_0}{dx}|$. Eq. (2.21) is zeroth order electron diamagnetic drift. It is the simplest dispersion relation of drift waves. Addition of polarization drift will modify Eq. (2.15) as,

$$-\iota\omega\frac{n_{i1}}{n_0} + \frac{c}{B_0}\frac{\nabla n_{0i}}{n_0}.(\mathbf{z}\times\iota k_\perp)\phi_1 - \frac{c}{\Omega_i B_0}\partial_t\nabla_\perp^2\phi_1 = 0 \tag{2.22}$$

Applying quasineutrality,

$$-\iota\omega\frac{e}{T_e}\phi_1 + \frac{c}{B_0}\frac{\nabla n_{0i}}{n_0}.(\mathbf{z}\times\iota k_\perp)\phi_1 - \frac{c}{\Omega_i B_0}(\iota\omega)(k_y^2)\phi_1 = 0 \tag{2.23}$$

Multiplying with T_e/e and m_i/m_i,

$$-\iota\omega\frac{e}{T_e}\phi_1 + \iota\frac{cT_e}{eB_0}\frac{\nabla n_{0i}}{n_0}.(\mathbf{z}\times k_\perp)\phi_1 - \iota\omega\frac{m_i c}{\Omega_i eB_0}\frac{T_e}{m_i}k_y^2\phi_1 = 0 \tag{2.24}$$

$$-\omega + \omega_e^* - \omega\frac{c_s^2}{\Omega_i^2}k_y^2 = 0 \tag{2.25}$$

$$\omega = \frac{\omega_e^*}{1+\rho_s^2 k_y^2} \tag{2.26}$$

Here $c_s^2 = \frac{T_e}{m_i}$ and $\rho_s = \frac{c_s}{\Omega_i}$ are ion thermal speed and Larmor radius at electron temperature, respectively.

2.3 Coupled IAW and Drift Waves

Now we will expand our results for the inertial case of ions by including the effect of their parallel motion $v_{iz} = 0$. If $T_i < T_e$ then using condition, i.e., $v_{ith} \ll \omega/k_z \ll v_{eth}$ [18], we write parallel ion momentum equation while dropping pressure term,

$$m_i n_i(\partial_t + \mathbf{v}_{iz}.\nabla)\mathbf{v}_{iz} = -en_i\nabla\phi_1 \tag{2.27}$$

$$\mathbf{v}_{iz} = \frac{ek_z\phi_1}{m_i\omega} \tag{2.28}$$

Solving Eq. (2.7) in terms of Cartesian coordinates,

$$\mathbf{v}_{i\perp} = -\frac{c\omega k_y\phi_1}{\Omega_i B_0} \tag{2.29}$$

Putting parallel and perpendicular components into a linearized equation of continuity,

$$\partial_t\frac{n_{i1}}{n_0} + \frac{\nabla n_{0i}}{n_0}.\mathbf{v}_{i\perp} + (\nabla_\perp.\mathbf{v}_{i\perp} + \partial_z v_{iz}) = 0 \tag{2.30}$$

$$-\iota\omega\frac{e}{T_e}\phi_1 + \iota\frac{c}{B_0}\frac{\nabla n_0}{n_0}.(\mathbf{z}\times k_\perp)\phi_1 + \iota(-\frac{c\omega k_y^2}{\Omega_i B_0} + \frac{ek_z^2}{m_i\omega})\phi_1 = 0 \tag{2.31}$$

Multiplying by T_e/e and m_i/m_i, respectively,

$$-\omega + \omega_e^* - \omega\frac{m_i c}{\Omega_i e B_0}\frac{T_e}{m_i}k_y^2 + \frac{T_e}{m_i}\frac{k_z^2}{\omega_i} = 0 \tag{2.32}$$

$$-\omega + \omega\omega_e^* - \omega^2\frac{c_s^2}{\Omega_i^2}k_y^2 + c_s^2 k_z^2 = 0 \tag{2.33}$$

$$\omega^2(1 + k_y^2\rho_s^2) - \omega\omega_e^* - c_s^2 k_z^2 = 0 \tag{2.34}$$

The above equation is a linearized dispersion relation for electrostatic drift waves. $k_y^2\rho_s^2$ and $k_z^2 c_s^2$ originate from ion polarization and parallel motion sequentially. Both inertial terms are easily identified as having ion mass m_i. For large $k_\parallel$, drift waves turn into ion acoustic waves as,

$$\omega^2 = \frac{k_z^2 c_s^2}{1 + k_y^2\rho_s^2} \tag{2.35}$$

For $v_{iz} = 0$, the dispersion relation becomes similar to Eq. (2.26) and for $k_y^2\rho_s^2 \ll 1$, the dispersion relation takes the form as,

$$\omega = \omega_e^* = v_{de}^* k_y \tag{2.36}$$

2.4 Drift Wave in Hot Ion Plasma

In the previous section, the temperature of the ions was ignored. Now removing this restriction and writing the equation of motion for $T_i > 0$, we have,

$$m_i n_i(\partial_t + \mathbf{v}_i.\nabla)\mathbf{v}_i = q_i n_i(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}) - \nabla p_i - \nabla.\overleftrightarrow{\pi_i} \tag{2.37}$$

Here p_i and π_i are isotropic pressure and an-isotropic stress tensor due to viscosity, respectively. Separating the perpendicular velocity using the same procedure in the section. (2.2), we get additional velocities as,

$$\mathbf{v}_{Di} = \frac{c}{q_i n_i B_0}(\mathbf{z} \times \nabla p_i) \tag{2.38}$$

$$\mathbf{v}_{\pi i} = \frac{c}{q_i n_i B_0}(\mathbf{z} \times \nabla.\overleftrightarrow{\pi_i}) \tag{2.39}$$

Here $\mathbf{v}_{Di}$ is the ion diamagnetic drift and $\mathbf{v}_{\pi i}$ is the ion viscosity drift. Before we proceed to further calculations, let's have a deeper insight into viscosity tensor and hot ion dynamics in a magnetized plasma.

2.4.1 Pressure Tensor

A stationary fluid is subjected to the forces or internal stresses because of isotropic pressure which is presented by $-\nabla p$, imparting transfer of momentum along one particular direction. In the case of moving fluid, viscous force is also present and responsible for the transfer of momentum in other directions as well, it is represented by viscous stress tensor or shear tensor $\pi_{ij} = m v_i v_j$ which is comprised of 9 components in 3-dimensional space, three diagonal components $(i = j)$ and six off-diagonal elements $(i \neq j)$. Each component of the tensor is defined by the direction in which it is acting but also the orientation of the surface upon which it is acting. Indices i and j specifies the direction and orientation of the surface upon which the stress tensor is acting respectively. Writing stress tensor in matrix from such as [19],

$$\overleftrightarrow{P_{ij}} = \begin{bmatrix} p_{xx} & \pi_{xy} & \pi_{xz} \\ \pi_{yx} & p_{yy} & \pi_{yz} \\ \pi_{zx} & \pi_{zy} & p_{zz} \end{bmatrix} \tag{2.40}$$

Scalar pressure in a fluid is defined as: $p = \frac{1}{3}(\pi_{xx} + \pi_{yy} + \pi_{zz})$. It represents normal stresses where the shear stresses are denoted π_{ij} $(i \neq j)$. Splitting the tensor into two parts, i.e., pressure tensor and shear tensor,

$$\overleftrightarrow{P_{ij}} = \begin{bmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{bmatrix} + \begin{bmatrix} 0 & \pi_{xy} & \pi_{xz} \\ \pi_{yx} & 0 & \pi_{yz} \\ \pi_{zx} & \pi_{zy} & 0 \end{bmatrix} \tag{2.41}$$

$$\overleftrightarrow{P_{ij}} = p_{ii}\overleftrightarrow{I} + \overleftrightarrow{\pi_{ij}} \tag{2.42}$$

Taking divergence of stress tensor according to Einstein summation convention, i.e, $\nabla.\overleftrightarrow{P_{ij}} = \partial_j P_{ij}$

$$\nabla.\overleftrightarrow{P_{ij}} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \begin{bmatrix} p_{xx} & \pi_{xy} & \pi_{xz} \\ \pi_{yx} & p_{yy} & \pi_{yz} \\ \pi_{zx} & \pi_{zy} & p_{zz} \end{bmatrix} \tag{2.43}$$

$$\nabla.\overleftrightarrow{P_{ij}} = \nabla p_{ij} + \nabla.\overleftrightarrow{\pi_{ij}} \tag{2.44}$$

∇p is the divergence of pressure/normal stress where $\nabla.\overleftrightarrow{\pi}$ is the divergence of shear stress. When fluid particles are subjected to collisions, they come up with the average velocity along the direction of fluid velocity on the spot where they made their latest collision. Momentum is transferred to the next particle after every collision. This equalizes velocity at different points. Resulting resistance to this kind of shear flow is comprehended as viscosity presented by off-diagonal components in the stress tensor. A longer mean free path carries more momentum, hence larger is the viscosity. Magnetic plasma shows a similar effect even if there are no collisions. Larmor gyration of fluid particles (particularly ions) sets them into different parts of plasma where they tend to equalize the velocity. This is called the finite Larmor radius effect (FLR) which occurs in response to collisional viscosity [1].

2.4.2 Role of Collision-less Pressure Tensor

So far we have neglected diamagnetic contribution to the polarization and stress tensor drift. This is relevant to the FLR effect (explained in Sec. 2.4.1) which is attained by the inclusion of diamagnetic drift. To avoid complications, temperature gradient and temperature perturbations are kept constant. Utilizing incompressibility condition: $\nabla.\mathbf{v} = 0$ to leading order while substituting drift into $\mathbf{v}_{Pi}$ and $\mathbf{v}_{\pi i}$, we shall assume large mode number, i.e., $k \gg \kappa_n = \frac{\nabla n_0}{n_0}$ and $\nabla \kappa_n = 0$. The stress tensor given by Braginski gives us the effect of viscosity which is relevant to friction between particles and collisionless gyroviscosity (pure FLR effect). Brajniski's gyro viscous components are [20],

$$\pi_{xy} = \pi_{yx} = \frac{n_i T_i}{2\Omega_i}(\partial_x v_x - \partial_y v_y) \tag{2.45}$$

$$\pi_{yy} = -\pi_{xx} = \frac{n_i T_i}{2\Omega_i}(\partial_x v_y + \partial_y v_x) \tag{2.46}$$

Using Einstein summation notation, i.e., $\nabla.\overleftrightarrow{\pi_{ij}} = \pi_{ij}/\partial_j$, Taking derivative w.r.t to velocity and background density, we'll get,

$$\begin{aligned}
(\nabla.\overleftrightarrow{\pi_i})_x = \partial_x \pi_{xx} + \partial_y \pi_{xy} = &- \frac{n_i T_i}{2\Omega_i}(\partial_x^2 v_y + \partial_x \partial_y v_x) - \frac{T_i}{2\Omega_i}(\partial_x v_y + \partial_y v_x)\partial_x n_i \\
&+ \frac{n_i T_i}{2\Omega_i}(\partial_y \partial_x v_x - \partial_y^2 v_y) \\
&= \frac{n_i T_i}{2\Omega_i}(-\partial_x^2 v_y - \partial_x \partial_y v_x + \partial_y \partial_x v_x - \partial_y^2 v_y) \\
&- \frac{T_i}{2\Omega_i}(\partial_x v_y + \partial_y v_x)\partial_x n_i \\
&= -\frac{n_i T_i}{2\Omega_i}(\partial_x^2 v_y + \partial_y^2 v_y) - \frac{T_i}{2\Omega_i}(\partial_x v_y + \partial_y v_x)\partial_x n_i \\
&= -\frac{n_i T_i}{2\Omega_i}\Delta v_y - \frac{T_i}{2\Omega_i}(\partial_x v_y + \partial_y v_x)\partial_x n_i
\end{aligned} \tag{2.47}$$

Now solving for $(\nabla.\overleftrightarrow{\pi_i})_y$

$$
\begin{aligned}
(\nabla.\overleftrightarrow{\pi_i})_y = \partial_x \pi_{yx} + \partial_y \pi_{yy} &= \frac{n_i T_i}{2\Omega_i}(\partial_x^2 v_x - \partial_x \partial_y v_y) + \frac{T_i}{2\Omega_i}(\partial_x v_x - \partial_y v_y)\partial_x n_i \\
&+ \frac{n T_i}{2\Omega_i}(\partial_y \partial_x v_y + \partial_y^2 v_x) \\
&= \frac{n_i T_i}{2\Omega_i}(\partial_x^2 v_y - \partial_x \partial_y v_y + \partial_y \partial_x v_y + \partial_y^2 v_x) \\
&+ \frac{T_i}{2\Omega_i}(\partial_x v_x - \partial_y v_y)\partial_x n_i \\
&= -\frac{n_i T_i}{2\Omega_i}(\partial_x^2 v_x + \partial_y^2 v_x) + \frac{T_i}{2\Omega_i}(\partial_x v_x - \partial_y v_x)\partial_x n_i \\
&= \frac{n_i T_i}{2\Omega_i}\Delta v_x + \frac{T_i}{2\Omega_i}(\partial_x v_x - \partial_y v_y)\partial_x n_i
\end{aligned}
\tag{2.48}
$$

Here $\Delta v_x = \partial_x^2 v_x + \partial_y^2 v_x$ and $\Delta v_y = \partial_x^2 v_y + \partial_y^2 v_y$. Addition of Eq. (2.47) and Eq. (2.48) yields,

$$
\nabla.\overleftrightarrow{\pi_i} = \frac{n_i T_i}{2\Omega_i}(\Delta v_x - \Delta v_y) + \frac{T_i}{2\Omega_i}(\partial_x v_x - \partial_y v_y - \partial_x v_y - \partial_y v_x)\partial_x n_i \tag{2.49}
$$

where

$$
\hat{z} \times \Delta_\perp v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \hat{z} \\ \Delta v_x & \Delta v_y & 0 \end{vmatrix} = \Delta v_x - \Delta v_y \tag{2.50}
$$

$$
\nabla v_y \times \hat{z} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x v_y & \partial_y v_y & 0 \\ 0 & 0 & \hat{z} \end{vmatrix} = -\partial_y v_y - \partial_x v_y \tag{2.51}
$$

$$
\hat{z} \times \nabla v_x = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \hat{z} \\ \partial_x v_x & \partial_y v_x & 0 \end{vmatrix} = -\partial_y v_x + \partial_x v_x \tag{2.52}
$$

Using Eq. (2.50-2.52), Eq. (2.49) can be written in more compact form as,

$$
\nabla.\overleftrightarrow{\pi_i} = \frac{n_i T_i}{2\Omega_i}(\mathbf{z} \times \Delta_\perp v) + \frac{T_i}{2\Omega_i}(\nabla v_y \times \mathbf{z} \times \nabla v_x)\partial_x n_i. \tag{2.53}
$$

Multiplying with n/n and inserting $\kappa = \nabla n/n$, Eq. (2.53) takes the final form as,

$$(\nabla.\overleftrightarrow{\pi_i}) = \frac{n_i T_i}{2\Omega_i}(\mathbf{z} \times \Delta_\perp v + \kappa(\nabla v_y \times \mathbf{z} \times \nabla v_x) \tag{2.54}$$

Taking cross product of above equation with $\mathbf{z}$ and using Eq. (2.39), stress tensor drift can be written as,

$$\begin{aligned}\mathbf{v}_{\pi i} = \frac{c}{en_i B}(\mathbf{z} \times \nabla.\overleftrightarrow{\pi_i}) = &-\frac{cT_i}{2\Omega ieB_0}(\mathbf{z} \times (\mathbf{z} \times \Delta_\perp v) \\ &+ \kappa(\mathbf{z} \times (\nabla v_y \times \mathbf{z} \times \nabla v_x)))\end{aligned} \tag{2.55}$$

Multiplying with m_i/m_i and $2/2$,

$$\mathbf{v}_{\pi i} = \frac{c}{en_i B_0}(\mathbf{z} \times \nabla.\overleftrightarrow{\pi_i}) = \frac{1}{4}\frac{2T_i}{m_i}\frac{m_i c}{\Omega_i e B_0}\Delta_\perp v + \kappa(\mathbf{z} \times \nabla v_y + \Delta v_x) \tag{2.56}$$

$$\mathbf{v}_{\pi i} = \frac{1}{4}\rho_i^2 \Delta_\perp v + \frac{1}{4}\rho_i^2 \kappa(\mathbf{z} \times \nabla v_y + \Delta v_x) \tag{2.57}$$

Here, $\rho_i = 2T_i/m_i\Omega_i^2$ is the ion gyroradius. We are interested in putting stress tensor drift into continuity equation of motion. For that we will calculate an expression having form $\nabla.(n\mathbf{v}_i)$ also including linearized form of κ. Using vector identity, i.e., $\nabla.(n\mathbf{v}_{\pi i}) = \mathbf{v}_{\pi i}.\nabla n_{0i} + n_{0i}\nabla.\mathbf{v}_{\pi i}$, we get,

$$\begin{aligned}\nabla.(n_i\mathbf{v}_{\pi i}) = &\frac{1}{4}\rho_i^2 \nabla n_{0i}.\Delta_\perp v + \frac{1}{4}\rho_i^2 \kappa \nabla n_{0i}(\mathbf{z} \times \nabla v_y + \Delta v_x) + \frac{1}{4}\rho_i^2 n_0 \Delta_\perp(\nabla.\mathbf{v}) \\ &+ \frac{1}{4}\rho_i^2 \kappa n_{0i}(\nabla.(\mathbf{z} \times \nabla v_y)) + \frac{1}{4}\rho_i^2 \kappa n_{0i} \nabla.(\nabla v_x)\end{aligned} \tag{2.58}$$

The second term in Eq. (2.58) becomes zero because we have used the linearized κ. Since $\nabla.(\mathbf{z} \times \nabla v_y)$ is calculated as,

$$\nabla.(\hat{z} \times \nabla v_y) = \nabla.\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \hat{z} \\ \partial_x v_y & \partial_y v_y & 0 \end{vmatrix} = \nabla.[\hat{i}(-\partial_y v_y) + \hat{j}.(\partial_x v_y)] = 0 \tag{2.59}$$

so the $4th$ term also becomes zero, Eq. (2.58) now takes the form as,

$$\nabla.(n\mathbf{v}_{\pi i}) = \frac{1}{4}\rho_i^2 \nabla n_{0i}.\Delta_\perp v + \frac{1}{4}\rho_i^2 n_{0i}\Delta_\perp(\nabla.\mathbf{v}) + \frac{1}{4}\rho_i^2 \kappa n_{0i}\nabla.(\nabla v_x) \tag{2.60}$$

Using $\nabla.\mathbf{v} \approx 0$ and $\nabla n_{0i}.\mathbf{v} = (-dn_{0i}/dx)v_x$, Eq. (2.60) can be written as,

$$\begin{aligned}\nabla.(n\mathbf{v}_{\pi i}) &= \frac{1}{4}\rho_i^2 \nabla n_{0i}.\nabla^2 v_x + \frac{1}{4}\rho_i^2 \nabla n_{0i}.\nabla^2 v_x \\ &= -\frac{1}{2}\rho_i^2 \nabla n_{0i}.\nabla^2 v_x\end{aligned} \tag{2.61}$$

Using Eq. (2.6), ion polarization drift can be written as,

$$\mathbf{v}_{Pi} = \frac{1}{\Omega_i}(\partial_t + \mathbf{v}_i.\nabla)(\mathbf{z} \times \mathbf{v}_i) \tag{2.62}$$

Only perturbed drift will be a part of the last $\mathbf{v_i}$ term because of large mode approximation. Background $\mathbf{v}$ is the only existing term in convective derivative due to linear approximation. We are interested in diamagnetic drift, i.e., background v. Let's check for the contribution of $\mathbf{v}_{Di} = c(\mathbf{z} \times \nabla p_i)/en_i B_0$ into linearized $\nabla.(n\mathbf{v}_{Pi}) = n_{0i}\nabla.\mathbf{v}_{Pi} + \mathbf{v}_{Pi}.\nabla n_{0i}$

$$\nabla.(n_i\mathbf{v}_{Pi}) = \frac{n_{0i}}{\Omega_i}(\mathbf{v}_{Di}.\nabla)\nabla.(\mathbf{z} \times \mathbf{v}_{i1}) + \frac{1}{\Omega_i}(\mathbf{v}_{Di}.\nabla)(\mathbf{z} \times \mathbf{v}_{i1}).\nabla n_{0i} \tag{2.63}$$

The second term in Eq. (2.63) becomes zero because we have used linearized κ. Let us solve the first term,

$$\nabla.(n\mathbf{v}_{Pi}) = \frac{cn_{0i}}{n_{0i}\Omega_i e B_0}(\mathbf{z} \times \nabla p_i.\nabla)\nabla.(\mathbf{z} \times \mathbf{v}_{i1}) \tag{2.64}$$

Since,

$$\nabla.(\hat{z} \times \mathbf{v}_i) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \hat{z} \\ v_x & v_y & 0\end{vmatrix} = -\partial_x v_y + \partial_y v_x \tag{2.65}$$

and

$$(\hat{z} \times \nabla p) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \hat{z} \\ \partial_x p & 0 & 0\end{vmatrix} = \partial_x p \tag{2.66}$$

writing Eq. (2.64) as,

$$\nabla.(n\mathbf{v}_{Pi}) = \frac{cn_{0i}}{n_{0i}\Omega_i e B_0}(\partial_x p_i.\partial_y)(\partial_y v_x - \partial_x v_y) \tag{2.67}$$

Inserting $\partial_x p = -T_i \nabla n_{0i}$,

$$\nabla.(n\mathbf{v}_{Pi}) = \frac{cn_{0i}T_i}{\Omega_i e B_0}(\frac{\nabla n_{0i}}{n_{0i}}.\partial_y)(\partial_y v_x - \partial_x v_y) \tag{2.68}$$

Multiplying with m_i/m_i,

$$\begin{aligned}\nabla.(n\mathbf{v}_{Pi}) &= \frac{m_i c}{\Omega_i e B_0}\frac{T_i}{m_i} n_{0i}(\kappa\partial_y)(\partial_y v_x - \partial_x v_y) \\ &= -\frac{1}{2}\kappa n_{0i}\rho_i^2\partial_y(\partial_y v_x - \partial_x v_y)\end{aligned} \tag{2.69}$$

Adding Eq. (2.61) and (2.69) while inserting $\kappa = -\mathbf{x}\nabla n_0/n_0$,

$$\begin{aligned}\nabla.(n_i\mathbf{v}_{\pi i}) + \nabla.(n_i\mathbf{v}_{Pi}) &= \frac{1}{2}\rho_i^2\kappa n_0\nabla^2 v_x - \frac{1}{2}\rho_i^2\kappa n_{0i}(\partial_y^2 v_x - \partial_y\partial_x y_y) \\ &= \frac{1}{2}\rho_i^2\kappa n_{0i}(\partial_x^2 v_x + \partial_y^2 v_x - \partial_y^2 v_x + \partial_y\partial_x v_y) \\ &= \frac{1}{2}\rho_i^2\kappa n_{0i}\partial_x(\partial_x\hat{i} + \partial_y\hat{j})(v_x + v_y) \\ &= \frac{1}{2}\rho_i^2\kappa n_{0i}\partial_x\nabla.\mathbf{v} \\ &= 0\end{aligned} \tag{2.70}$$

Diamagnetic contributions to $\nabla.(n\mathbf{v}_{Pi})$ are cancelled by stress tensor contribution $\nabla.(n\mathbf{v}_{\pi i})$. Diamagnetic drift not being a particle drift cannot transfer information through convection. Only contribution we get here from time derivative part of polarization drift. So general result as stated as [17],

$$\nabla.[n(\mathbf{v}_{Pi} + \mathbf{v}_{\pi i})] = \nabla.[\frac{n}{\Omega_i}\partial_t(\mathbf{z}\times\mathbf{v}_i)] \tag{2.71}$$

We can compare above result with $\nabla.(n\mathbf{v}_{Di})$. Same physics hold for both equation under condition of having same order FLR parameter, i.e., $k^2\rho^2$. Above result is no longer true in presence of curvature. Same goes for $\nabla.(n\mathbf{v}_{Di})$. Linear contribution from $\mathbf{E}\times\mathbf{B}$ and ion diamagnetic drift is given as,

$$\begin{aligned}\frac{n_{0i}}{\Omega_c}\partial_t\nabla.(\mathbf{z}\times\mathbf{v}_E) &= \frac{cn_{0i}}{\Omega_i B_0}\partial_t\nabla.(\mathbf{z}\times\mathbf{z}\times\mathbf{E}) \\ &= \frac{cn_{0i}}{\Omega_i B_0}\partial_t(\nabla_\perp.\mathbf{E}_\perp)\end{aligned} \tag{2.72}$$

Multiplying with m_i/m_i, e/e, T_i/T_i and using $\mathbf{E} = -\nabla\phi$,

$$\begin{aligned}\frac{n_{0i}}{\Omega_i}\partial_t\nabla.(\mathbf{z}\times\mathbf{v}_E) &= -n_{0i}\frac{m_i c}{\Omega_i e B_0}\frac{T_i}{m_i}\frac{e}{T_i}\partial_t(\nabla_\perp^2\phi) \\ &= -\frac{1}{2}n_{0i}\rho_i^2\partial_t\nabla^2\frac{e\phi}{T_i}\end{aligned} \tag{2.73}$$

For diamagnetic drift,

$$\begin{aligned}\frac{n_{0i}}{\Omega_i}\partial_t\nabla.(\mathbf{z}\times\mathbf{v}_{Di}) &= -\frac{c}{\Omega_i eB_0}\partial_t\nabla.(\mathbf{z}\times\mathbf{z}\times\nabla p_e)\\ &= \frac{cT_i}{\Omega_i eB_0}\partial_t\nabla.(\partial_x p)\end{aligned} \tag{2.74}$$

Using $\partial_x p_i = -T_i\nabla n_{i1}$ and Multiplying by m_i/m_i, e/e

$$\begin{aligned}\frac{n_{0i}}{\Omega_i}\partial_t\nabla.(\mathbf{z}\times\mathbf{v}_{Di}) &= -\frac{m_i c}{\Omega_i eB_0}\frac{T_i}{m_i}\partial_t\nabla^2 n_{i1}\\ &= -\frac{1}{2}\rho_i^2\partial_t\nabla^2 n_{i1}\end{aligned} \tag{2.75}$$

Perturbed density contributes here exclusively. If

$$\nabla.[n(\mathbf{v}_{Pi}+\mathbf{v}_{\pi i})] = \frac{n_{0i}}{\Omega_i}\partial_t\nabla.(\mathbf{z}\times\mathbf{v}_E+\mathbf{v}_D) \tag{2.76}$$

Eq. (2.76) in (ω, k) space takes the form as,

$$\nabla.[n(\mathbf{v}_{Pi}+\mathbf{v}_{\pi i})]_1 = -\frac{1}{2}\rho_i^2 n_{0i}\partial_t\nabla^2\frac{e\phi_1}{T_i} - \frac{1}{2}\rho_i^2\partial_t\nabla^2 n_{i1} \tag{2.77}$$

Using the leading order perturbation, i.e., $n_1/n_0 = \omega_e^* e\phi/\omega T_e$ and multiplying above equation with T_e/T_e,

$$\begin{aligned}\nabla.[n(\mathbf{v}_{Pi}+\mathbf{v}_{\pi i})]_1 &\approx -\frac{1}{2}\rho_i^2\iota k^2(n_{0i}\frac{T_e}{T_i}\omega\frac{e\phi_1}{T_e} + n_{0i}\omega\frac{\omega_e^*}{\omega}\frac{e\phi_1}{T_e})\\ &\approx -\frac{1}{2}n_{0i}\rho_i^2\iota k^2(\frac{T_e}{T_i}\omega+\omega_e^*)\frac{e\phi_1}{T_e}\end{aligned} \tag{2.78}$$

At Electron temperature T_e, Eq. (2.77) takes the form [17],

$$\nabla.[n(\mathbf{v}_{Pi}+\mathbf{v}_{\pi i})]_1 \approx -\iota n_{0i}k^2\rho_s^2(\omega-\omega_i^*)\frac{e\phi_1}{T_e} \tag{2.79}$$

FLR effect comes up with the polarization drift through convection but indeed it is because of time variation of perturbed diamagnetic drift.

2.4.3 Dispersion Relation for Drift Waves in Hot Ion Plasma

Now that we have background familiarity with tensors and their effect along with polarization drift. Let's start with the parallel equation of motion for ions which is written as

$$m_i n_i(\partial_t + \mathbf{v}_{iz}.\nabla)\mathbf{v}_{iz} = -en_i\nabla\phi - \gamma_i T_i \nabla n_i \tag{2.80}$$

Keep $\gamma_i = 1$ for convenience. Linearizing the above equation yields,

$$\mathbf{v}_{iz1} = \frac{ek_z\phi_1}{m\omega} + \frac{T_i}{m_i}\frac{k_z}{\omega}\frac{n_{1i}}{n_{0i}} \tag{2.81}$$

Incorporating into the linearized continuity equation,

$$\partial_t n_{i1} + +\nabla n_{0i}.\mathbf{v}_E + \nabla_\perp.[n_{0i}(\mathbf{v}_E + \mathbf{v}_{Pi} + \mathbf{v}_{\pi i} + \mathbf{v}_{Di})]_1 + n_0\partial_z v_{iz1} = 0 \tag{2.82}$$

where $\nabla.\mathbf{v}_E = 0$ and $\nabla.(n_{0i}\mathbf{v}_{Di}) = \nabla.[n_{0i}(\frac{cT_i}{eB_0}\frac{\mathbf{z}\times\nabla n_{0i}}{n_{0i}})] = 0$. As $\mathbf{v}_{i1} = \mathbf{v}_{E1} + \mathbf{v}_{Di1}$ so using Eq. (2.76) and dividing by n_0,

$$\begin{aligned}&-\partial_t(\frac{n_{i1}}{n_{0i}}) + \frac{\nabla n_{0i}}{n_{0i}}.\frac{c}{B_0}(\mathbf{z}\times\nabla\boldsymbol{\phi_1}) + \frac{c}{\Omega_i B_0}\partial_t\nabla^2\phi_1\\ &+\frac{cT_i}{\Omega_i e B_0}\partial_t\nabla^2\frac{n_{i1}}{n_{0i}} + \partial_z(\frac{ek_z}{m_i\omega} + \frac{T_i k_z}{m_i\omega}\frac{n_{i1}}{n_{0i}}) = 0\end{aligned} \tag{2.83}$$

Using Boltzmann relation and condition $\omega n_{i1}/n_{0i} = \omega_i^* e\phi_1/T_i$

$$\begin{aligned}&-\iota\omega\frac{e}{T_i}\phi_1 + \iota\frac{c}{B_0}\frac{\nabla n_{0i}}{n_{0i}}.(\mathbf{z}\times k_\perp)\phi_1 + \iota\omega\frac{ck_y^2}{\Omega_i B_0}\phi_1\\ &+\iota\omega_i^*\frac{c}{\Omega_i B}\phi_1 + \iota\frac{ek_z^2}{m_i\omega}\phi_1 + \iota\frac{eT_i k_z^2}{m_i\omega}\phi_1 = 0\end{aligned} \tag{2.84}$$

Multiplying by T_e/e and m_i/m_i, respectively,

$$\begin{aligned}&-\omega + \frac{cT_e}{eB}(-\frac{\nabla n_{0i}}{n_{0i}}\mathbf{x}).(-k_y\mathbf{x})) + \omega\frac{m_i c}{\Omega_i e B_0}\frac{T_e}{m_i}k_y^2\\ &+\omega_i^*\frac{m_i c}{\Omega_i e B}\frac{T_e}{m_i} + \frac{T_e}{m_i}\frac{k_z^2}{\omega} + \frac{T_i}{m_i}\frac{k_z^2}{\omega} = 0\end{aligned} \tag{2.85}$$

$$-\omega + \omega_e^* + \omega\frac{c_i^2}{\Omega_i^2}k_y^2 + \omega_i^*\frac{c_s^2}{\Omega_i^2} + \frac{c_s^2}{\omega}k_z^2 + \frac{c_i^2}{\omega}k_z^2 = 0 \tag{2.86}$$

$$\omega^2 - \omega\omega_e^* - k_z^2(c_s^2 + c_i^2) = -k_y^2\rho_s^2(\omega - \omega_i^*)\omega \tag{2.87}$$

The above equation is the linear dispersion relation of coupled IAW and drift waves in hot ion plasma. It must be noted that the term $k_z^2 c_i^2$ is missing in Eq. (11) of Ref. 24.

2.4.4 Contribution of Parallel Shear Flow

Presume an inhomogeneous plasma having shear velocity which is aligned parallel to the magnetic field presented by $\mathbf{v}_{0i} = v_{0i}(x)\mathbf{z}$. Density gradient is given by $\nabla n_{0i} = -\mathbf{x}|\frac{dn_{0i}}{dx}|$. We want to accomplish a steady state for that we have a condition for all species having zero order diamagnetic drift $\mathbf{v}_{0iD} = -\frac{cT_i}{eB_0}(\mathbf{z}\times\nabla n_{0i}/n_{0i})$. Using Eq. (2.6), polarization drift with shear velocity is given as,

$$\mathbf{v}_{Pi} = \frac{1}{\Omega_i}(\partial_t + v_{0iz}(x)\partial_z + \mathbf{v}_i.\nabla)(\mathbf{z}\times\mathbf{v}_i) \tag{2.88}$$

Since Diamagnetic contributions to $\nabla.(n\mathbf{v}_{Pi})$ are absolutely neutralized by stress tensor contribution $\nabla.(n\mathbf{v}_{\pi i})$, as explained previously in Sec. (2.4.2). Diamagnetic drift is unable to transfer information through convection. Hence, the only contribution we get here from the time derivative and shear flow part of polarization drift. So a generic result can be deduced as,

$$\nabla.[n(\mathbf{v}_{Pi} + \mathbf{v}_{\pi i})]_1 = \nabla.[\frac{n_{0i}}{\Omega_i}(\partial_t + v_{0iz}(x)\partial_z)(\mathbf{z}\times\mathbf{v}_{i1})] \tag{2.89}$$

If $\mathbf{v}_{i1} = \mathbf{v}_{E1} + \mathbf{v}_{Di1}$ is considered for drift waves, then Eq. (2.89) becomes

$$\nabla.[n_i(\mathbf{v}_{Pi} + \mathbf{v}_{\pi i})]_1 = \nabla.[\frac{n_{0i}}{\Omega_i}(\partial_t + v_{0iz}(x)\partial_z)(\mathbf{z}\times(\mathbf{v}_{E1} + \mathbf{v}_{Di1}))] \tag{2.90}$$

Inserting values of $\mathbf{v}_E$ and $\mathbf{v}_{Di}$ from Eq. (2.5) and Eq. (2.38) into the above equation,

$$\begin{aligned}\nabla.[n(\mathbf{v}_{Pi}+\mathbf{v}_{\pi i})]_1 &= \nabla.[\frac{n_{0i}}{\Omega_i}(\partial_t + v_{0iz}(x)\partial_z)\frac{c}{B_0}(\mathbf{z}\times\mathbf{z}\times\mathbf{E}_1)\\ &+ \frac{c}{eB_0 n_{0i}}(\mathbf{z}\times\mathbf{z}\times\nabla p_i)]\\ &= \frac{n_{0i}}{\Omega_i}(\partial_t + v_{0iz}(x)\partial_z)(\frac{c}{B_0}(\nabla.\mathbf{E}_{\perp 1})\\ &+ \frac{c}{eB_0 n_{0i}}(\nabla.\nabla_\perp p_i))\end{aligned} \tag{2.91}$$

Separating ∂_t and $v_{0iz}(x)\partial_z$ and inserting $\nabla p_i = -T_i\nabla n_i$,

$$\begin{aligned}\nabla.[n_i(\mathbf{v}_{Pi}+\mathbf{v}_{\pi i})]_1 &= \frac{n_{0i}}{\Omega_i}\partial_t\frac{c}{B_0}(\nabla.\mathbf{E}_{\perp 1}) - \frac{cT_i}{\Omega_i e B_0}\partial_t(\nabla.\nabla_\perp n_{i1})\\ &+ \frac{n_{0i}}{\Omega_i}v_{0z}(x)\partial_z\frac{c}{B_0}(\nabla.\mathbf{E}_{\perp 1})\\ &- \frac{cT_i}{\Omega_i e B_0}v_{0iz}(x)\partial_z(\nabla.\nabla_\perp n_{i1})\\ &= -\frac{n_{0i}c}{\Omega_i B_0}\partial_t\nabla_\perp^2\phi_1 - \frac{cT_i}{\Omega_i e B_0}\partial_t\nabla_\perp^2 n_{i1}\\ &- \frac{n_{0i}c}{\Omega_i B_0}v_{0iz}(x)\partial_z\nabla_\perp^2\phi_1 - \frac{cT_i}{\Omega_i e B_0}v_{0iz}(x)\partial_z\nabla_\perp^2 n_{i1}\end{aligned} \tag{2.92}$$

Inserting the above equation into the linearized equation of continuity,

$$\begin{aligned}(\partial_t + v_{0iz}\partial_z)n_{i1} + \nabla n_{0i}.\mathbf{v}_E - \frac{n_{0i}c}{\Omega_i B_0}(\partial_t + v_{0iz}\partial_z)\nabla_\perp^2\phi_1\\ -\frac{cT_i}{\Omega_i e B_0}(\partial_t + v_{0z}\partial_z)\nabla_\perp^2 n_{i1} + n_{0i}\partial_z v_{iz1} = 0\end{aligned} \tag{2.93}$$

$$\begin{aligned}(-\omega+\omega_{0z})\frac{n_{i1}}{n_{0i}} + \frac{c}{B_0}\frac{\nabla n_{0i}}{n_{0i}}.(\mathbf{z}\times k_y)\phi_1 + \frac{c}{\Omega_i B_0}(-\omega+\omega_{0z})k_y^2\phi_1\\ -\frac{cT_i}{\Omega_i e B_0}(-\omega+\omega_{0z})k_y^2\frac{n_{i1}}{n_{0i}} + k_z v_{iz1} = 0\end{aligned} \tag{2.94}$$

Multiplying with m_i/m_i, T_e/e and e/T_e,

$$\begin{aligned}(\omega-\omega_{0z})\frac{n_{i1}}{n_{0i}} - \frac{cT_e}{eB_0}\kappa_n k_y\frac{e\phi_1}{T_e} + \frac{m_i c}{\Omega_i e B_0}\frac{T_e}{m_i}(\omega-\omega_{0z})k_y^2\frac{e\phi_1}{T_e}\\ +\frac{m_i c}{\Omega_i e B_0}\frac{T_i}{m_i}(\omega-\omega_{0z})k_y^2\frac{n_{i1}}{n_{0i}} - k_z v_{iz1} = 0\end{aligned} \tag{2.95}$$

$$\Omega_\omega \frac{n_{i1}}{n_{0i}} - \omega_e^* \Phi_1 + \Omega_\omega(\rho_s^2 k_y^2 \Phi_1 + \rho_i^2 k_s^2 \frac{n_{i1}}{n_{0i}}) - k_z v_{iz1} = 0 \tag{2.96}$$

The parallel equation of motion in the linear limit is given as,

$$\begin{aligned}(\partial_t + v_{0iz}\partial_z)v_{izl} + v_{ix1}\partial_x v_{0iz}(x) + (\mathbf{v}_{0Di}.\nabla)v_{iz1} =& \frac{e}{m_i}\mathbf{E}_1 - \frac{T_{0i}}{m_i n_{0i}}\nabla n_{i1} \\ &- \frac{1}{m_i n_{0i}}(\nabla.\pi_i)\end{aligned} \tag{2.97}$$

Using relation:

$$-\iota k_z c_i^2 \rho_s^2 k_s^2 (\frac{e\phi}{T_i} + \frac{n_{i1}}{n_{0i}}) + (\mathbf{v}_{0Di}.\nabla)v_{iz1} = -\frac{1}{m_i n_i}(\nabla.\pi_i) \tag{2.98}$$

Inserting into Eq. (2.97),

$$\begin{aligned}(\partial_t + v_{0iz}\partial_z)v_{izl} + v_{ix1}\partial_x v_{0iz}(x) = \frac{e}{m_i}\nabla\phi_1 - \frac{T_{0i}}{m_i}\partial_z \frac{n_{1i}}{n_{0i}} \\ -\iota k_z c_i^2 \rho_s^2 k_y^2 (\frac{e\phi_1}{T_i} + \frac{n_{i1}}{n_{0i}})\end{aligned} \tag{2.99}$$

$$\begin{aligned}(-\omega + \omega_{0iz})v_{iz1} - \frac{c}{B_0}\partial_x v_{0iz} k_y \phi_1 =& - \frac{e}{m_i}k_z\phi_1 - \frac{T_i}{m_i}k_z \frac{n_{i1}}{n_{0i}} \\ &- k_z c_i^2 \rho_i^2 k_y^2 (\frac{e\phi_1}{T_i} + \frac{n_{i1}}{n_{0i}})\end{aligned} \tag{2.100}$$

Multiplying with T_e/e, e/T_e and m_i/m_i

$$\begin{aligned}(-\omega + \omega_{0z})v_{iz1} - \frac{m_i c}{eB_0}\frac{T_e}{m_i}\partial_x v_{0iz} k_y \frac{e\phi_1}{T_e} \\ = -\frac{T_e}{m_i}k_z \frac{e\phi_1}{T_e} - \frac{T_i}{m_i}k_z \frac{n_{i1}}{n_{0i}} - k_z c_i^2 \rho_i^2 k_y^2 (\frac{e\Phi}{T_i} + \frac{n_{i1}}{n_{0i}})\end{aligned} \tag{2.101}$$

$$\begin{aligned}(\omega - \omega_{0z})v_{z1} + \frac{1}{\Omega_i}\partial_x v_{0z} c_s^2 k_y \Phi_1 \\ = c_s^2 k_z \Phi_1 + c_s^2 k_z \frac{n_{i1}}{n_{0i}} + k_z c_i^2 \rho_i^2 k_y^2 \Phi_1 + k_z c_i^2 \rho_i^2 k_y^2 \frac{n_{1i}}{n_{0i}}\end{aligned} \tag{2.102}$$

$$\Omega_\omega v_{zl} + \partial_x v_{0z} c_s^2 k_y \Phi_1 = (1 + \rho_i^2 k_y^2)(c_s^2 k_z \Phi + c_i^2 k_z \frac{n_{1i}}{n_{0i}}) \tag{2.103}$$

Here,

$$\Omega_\omega = \omega - \omega_{0z} \qquad \omega_{0z} = v_{0iz}k_z$$
$$\omega^* = \frac{cT_e}{eB_0}\kappa_n(k_y) = v_{De}ky$$
$$\kappa_n = -\frac{1}{n_0}\frac{dn_0}{dx} \qquad \rho_s^2 = \frac{c_s^2}{\Omega_i^2}$$
$$\rho_i^2 = \frac{c_i^2}{\Omega_i^2} \qquad c_s^2 = \frac{T_e}{m_i}$$
$$c_i^2 = \gamma_i\frac{T_i}{m_i} \qquad \Phi_1 = \frac{e\phi}{T_e}$$

If $v_{0i} = 0$, Eq. (2.92) becomes,

$$\nabla.[n(\mathbf{v}_P + \mathbf{v}_{\pi i})]_1 = -\frac{cn_{0i}}{\Omega_i B_0}(\iota\omega)k_y^2\phi_1 - \frac{cT_i}{\Omega_i eB_0}(\iota\omega)(k_y^2 n_{i1}) \tag{2.104}$$

Multiplying with m_i/m_i, T_e/e and T_e/e

$$\begin{aligned}\nabla.[n(\mathbf{v}_P + \mathbf{v}_{\pi})]_1 &= -n_{0i}\frac{m_i c}{\Omega_i eB_0}\frac{T_i}{m_i}(\iota\omega)k_y^2\frac{e\phi_1}{T_e} - \frac{m_i c}{\Omega_i eB_0}\frac{T_e}{m_i}\frac{T_i}{T_e}(\iota\omega)k_y^2 n_1 \\ &= -\iota n_0\rho_s^2(\iota\omega)k_y^2\frac{e\phi_1}{T_e} - \rho_s^2\frac{T_i}{T_e}(\iota\omega)k_y^2 n_1\end{aligned} \tag{2.105}$$

Following the condition for a (EI) plasma, i.e., $n_{i1}/n_{i0} = n_{e1}/n_{e0} = \omega^* e\phi/\omega T_i$

$$\begin{aligned}\nabla.[n(\mathbf{v}_{Pi} + \mathbf{v}_{\pi i})]_1 &= -\iota\rho_s^2 k_y^2(n_0\omega\frac{e\phi_1}{T_e} - n_0\omega_i^*\frac{e\phi_1}{T_e}) \\ &= -\iota n_0 k_y^2\rho_s^2(\omega - \omega_i^*)\Phi_1\end{aligned} \tag{2.106}$$

This is similar to Eq. (2.79).

2.5 Drift Waves in Electron-Positron-Ion (EPI) Plasma

Shear flow instability has been investigated in pair ion plasma in the linear and non-linear limit [21], [22], [23]. In Ref. 21, cold plasma was considered while in the present work we are dealing with hot plasma. For an (EPI) plasma, Boltzmann approximation for electron and positron is written as,

$$n_{e1} \approx n_0 \exp(\frac{e\phi_1}{T_e}) \approx n_{0e}\frac{e\phi_1}{T_e} \tag{2.107}$$

$$n_{p1} \approx n_{0p} \exp(\frac{-e\phi_1}{T_e}) \approx -n_{0p}\frac{e\phi_1}{T_p} \tag{2.108}$$

Putting n_{e1}, n_{i1} and n_{p1} in linearized Poisson equation as,

$$-\frac{1}{4\pi}\nabla^2\phi_1 = e(n_{e1} + n_{p1} - n_{i1}) \tag{2.109}$$

$$\frac{1}{4\pi}k^2\phi_1 + \frac{e^2 n_{0e}}{T_e}\phi_1 + \frac{e^2 n_{0p}}{T_p}\phi_1 = en_{i1} \tag{2.110}$$

Dividing with $k^2/4\pi$ and n_{0i} on both sides,

$$(1 + \frac{4\pi e^2 n_{0e}}{T_e k^2} + \frac{4\pi e^2 n_{0p}}{T_p k^2})\frac{\phi_1}{n_{0i}} = \frac{4\pi e}{k^2}\frac{n_{i1}}{n_{0i}} \tag{2.111}$$

Multiplying with $\lambda_{De}^2 k^2$

$$(1 + \lambda_{De}^2 k^2 + \frac{n_{0p}}{n_{0e}}\frac{T_e}{T_p})\phi_1\frac{n_{0e}}{n_{i0}} = \frac{T_e}{e}\frac{n_{1i}}{n_{0i}} \tag{2.112}$$

$$(1 + \lambda_{De}^2 + \frac{n_{0p}}{n_{0e}}\frac{T_e}{T_p})\frac{n_{0e}}{n_{0i}}\frac{e\phi_1}{T_e} = \frac{n_{i1}}{n_{0i}} \tag{2.113}$$

$$\eta\Phi_1 = \frac{n_{i1}}{n_{0i}} \tag{2.114}$$

Here, $\eta = (1 + \lambda_{De}^2 + n_{p0}T_e/n_{e0}T_p)n_{e0}/n_{i0}$ and $\lambda_{De}^2 = T_e/4\pi n_{0e}e^2$. Inserting Eq. (2.96) into Eq. (2.103) in terms of v_{z1} and using Eq. (2.114), we'll get,

$$\begin{aligned} &\Omega_\omega^2\frac{n_{1i}}{n_{0i}}\frac{1}{k_z} - \Omega_\omega\omega_e^*\frac{\Phi_1}{k_z} + \Omega_\omega^2(\rho_s^2 k_y^2\Phi_1 + \rho_i^2 k_y^2\frac{n_{i1}}{n_{0i}})\frac{1}{k_z} \\ &- \frac{1}{\Omega_i}\partial_x v_{0iz}c_s^2 k_y\Phi_1 - (1 + \rho_i^2 k_y^2)(c_s^2 k_z\Phi_1 + c_i^2 k_z\frac{n_{1i}}{n_{0i}}) = 0 \end{aligned} \tag{2.115}$$

$$\begin{aligned} &\Omega_\omega^2\eta\Phi_1 - \Omega_\omega\omega_e^*\Phi_1 + \Omega_\omega^2(\rho_s^2 k_y^2 + \rho_i^2 k_y^2\eta)\Phi_1 \\ &\frac{1}{\Omega_i}\partial_x v_{0iz}c_s^2 k_y k_z\Phi_1 - (1 + \rho_i^2 k_y^2)(c_s^2 + c_i^2\eta)k_z^2\Phi_1 = 0 \end{aligned} \tag{2.116}$$

$$\Omega_\omega^2(\eta+\rho_s^2k_y^2+\rho_i^2k_y^2\eta)-\Omega_\omega\omega^*+A_ic_s^2k_yk_z-(1+\rho_i^2k_y^2)(c_s^2+c_i^2\eta)k_z^2=0 \quad (2.117)$$

$$\Omega_\omega^2\lambda-\Omega_\omega\omega^*+A_ic_s^2k_yk_z-(1+\rho_i^2k_y^2)(c_s^2+c_i^2\eta)k_z^2=0 \quad (2.118)$$

Here $A_i=\frac{1}{\Omega_i}\partial_x v_{i0z}$ is normalized shear parameter and $\lambda=\eta+\rho_s^2k_y^2+\rho_i^2k_y^2\eta$. The dispersion relation (2.118) is the same as Eq. (15) of Ref. [24].

Using the quadratic formula, Eq. (2.118) has the solution,

$$\Omega_\omega=\frac{1}{2\lambda}[\omega_e^*\pm\sqrt{\omega_e^{*2}-4\lambda(A_ic_s^2k_yk_z-(1+\rho_i^2k_y^2)(c_s^2+c_i^2\eta)k_z^2)}]=0 \quad (2.119)$$

The instability criteria in this condition read as,

$$A_i>\frac{\omega_e^{*2}}{4\lambda c_s^2k_yk_z}+(1+\rho_i^2k_y^2)(1+\eta\frac{\gamma_iT_i}{T_e})\frac{k_z}{k_y} \quad (2.120)$$

For the case of homogeneous plasma, it is given as,

$$A_i>(1+\rho_i^2k_y^2)(1+\eta\frac{\gamma_iT_i}{T_e})\frac{k_z}{k_y} \quad (2.121)$$

The instability condition emerges through the factor η. Considering its definition, we deduce that positron produces a stabilizing effect.

2.6 Drift Waves in Electron Ion (EI) Plasma

Shear flow instability in (EI) plasma has been suggested earlier in [25]. In the case of (EI) plasma, wavelength exceeds the Debye length for that we have $\eta=1$, so above Eq. (2.118) reduces as,

$$\Omega_\omega^2(1+\rho_s^2k_y^2+\rho_i^2k_y^2)-\Omega_\omega\omega_e^*+A_ic_s^2k_yk_z-(1+\rho_i^2k_y^2)(c_s^2+c_i^2)k_z^2=0 \quad (2.122)$$

The above relation is the same as Eq. (16) of Ref. 24.

For $v_{0z}=0$ and using $n_{i1}/n_{0i}=\omega^*e\phi/\omega T_e$, Eq. (2.122) reduces exactly as Eq. (2.87).

For $T_e=T_i$, Eq. (2.122) sets off as,

$$\Omega_\omega^2(1+2\rho_i^2k_s^2)-\Omega_\omega\omega_e^*+A_ic_i^2k_yk_z-(1+\rho_i^2k_y^2)(2c_i^2)k_z^2=0 \quad (2.123)$$

Stress Tensor's collision-less part modifies the criteria of instability given in Ref. 25 even in the homogeneous limit. Dispersion relation exactly similar to that of Ref 25. is derived as,

$$\Omega^2-2\Omega\kappa_{ne}\rho_ic_ik_y+2A_ic_i^2k_yk_z-2c_i^2k_z^2=0 \quad (2.124)$$

Here $\Omega = \omega - \omega_{0z} - v_{0Di}k_y$. Eq. (48) of Ref. [26] similar to above equation is

$$\Omega^2 - \Omega(\omega_e^* + \omega_i^*) - 2A_i c_i^2 k_y k_z - 2c_i^2 k_z^2 = 0 \tag{2.125}$$

For an isothermal case, $c_s^2 = (\gamma_i T_i + \gamma_e T_e)/m_i = 2T_i/m_i$ has been used. Eqs. (2.124) and (2.125) are not the same as Eq. (2.123). Ω_ω and ω are also defined differently. Eqs. (2.124) and (2.123) can't be reduced into Eq. (2.87) for the limit $v_{0z} = 0$. If we compare the present work with the previous one, Ref. 25 had an extra term $k_y v_{0y}$ in Ω. This term appeared as a consequence of polarization drift. Convective derivative, i.e., $(\partial_t + \mathbf{v}.\nabla)$ was replaced by $(\partial_t + k_y v_{0y} + k_z v_{0z})$. However, the offering of collision-less stress tensor and its annihilation with polarization drift has not been taken into consideration.

For cold ions $T_i \ll T_e$, Eq. (2.122) takes the form,

$$\Omega_\omega^2(1 + \rho_s^2 k_y^2) - \Omega_\omega \omega_e^* + A_i c_s^2 k_y k_z - c_s^2 k_z^2 = 0 \tag{2.126}$$

Solution of Ω_ω is given as,

$$\Omega_\omega = \frac{1}{2(1 + \rho_s^2 k_y^2)}[\omega_e^* \pm \sqrt{\omega_e^{*2} - 4(1 + \rho_s^2 k_y^2)(A_i c_s^2 k_y k_z - c_s^2 k_z^2)}] \tag{2.127}$$

$$\omega = \omega_{0z} + \frac{1}{2(1 + \rho_s^2 k_y^2)}[\omega_e^* \pm \sqrt{\omega_e^{*2} - 4(1 + \rho_s^2 k_y^2)(A_i c_s^2 k_y k_z - c_s^2 k_z^2)}] \tag{2.128}$$

Shear flow instability arises if,

$$4(1 + \rho_s^2 k_y^2)(A_i c_s^2 k_y k_z - c_s^2 k_z^2)) > \omega_e^{*2} \tag{2.129}$$

Real frequency actually has electron diamagnetic drift, i.e.,

$$\omega_r = \omega_{0z} + \frac{\omega_e^*}{2(1 + \rho_s^2 k_y^2)} \tag{2.130}$$

For a homogeneous plasma, we get the same result as of Ref. 25, i.e.,

$$\omega_r = \omega_{0z} \tag{2.131}$$

Note that Eq. (2.128) differs from [27] and [28] in terms of the FLR effect. These equations are derived from kinetic theory where the contribution of stress tensor doesn't occur. Even in the occurrence of homogeneous cases, the condition for shear flow instability becomes a bit different from that found in Ref. 25 because of $\rho_i^2 k_y^2$ factor appearing due to ion stress tensor. For an electron plasma, it is given as,

$$A_i > (1 + \rho_i^2 k_y^2)(1 + \frac{\gamma_i T_i}{T_e})\frac{k_z}{k_y} \tag{2.132}$$

If $T_i \neq 0$, the factor $\rho_i^2 k_y^2$ can't be overlooked in case of short wavelength.

2.6.1 Limiting Cases in Electron Ion (EI) Plasma

In (EI) plasma, Eq. (2.122) can be rewritten as,

$$\Omega_\omega^2(1+\rho_s^2k_y^2+\rho_i^2k_y^2)-\Omega_\omega\omega_e^*-c_s^2k_z^2(1+\rho_i^2k_y^2(1+\frac{c_i^2}{c_s^2})-A_i\frac{k_y}{k_z})=0 \quad (2.133)$$

Uniform Plasma

For the case of homogeneous plasma at $T_i=0$, $\nabla n_{0i}=0$, $\omega_e^*=0$, above equation takes the form as

$$\Omega_\omega^2=\frac{1}{(1+\rho_s^2k_y^2)}(c_s^2k_z^2(1-A_i\frac{k_y}{k_z})) \quad (2.134)$$

If $\partial_x v_{i0z}=0$, above equation appear as,

$$\Omega_\omega^2=\frac{c_s^2k_z^2}{1+\rho_s^2k_y^2} \quad (2.135)$$

If $v_{i0z}=0$, it becomes,

$$\omega^2=\frac{c_s^2k_z^2}{1+\rho_s^2k_y^2} \quad (2.136)$$

It is clear that ω_{0z} here becomes zero. For the limit $1<A_i\frac{k_y}{k_z}$, Eq. (2.134) emerges as,

$$\Omega_\omega=\iota\sqrt{\frac{c_s^2k_z^2}{1+\rho_s^2k_y^2}A_i\frac{k_y}{k_z}} \quad (2.137)$$

$$\Omega_\omega=\iota\gamma \quad (2.138)$$

$$\omega=v_{0z}k_z+\iota\gamma \quad (2.139)$$

Here $\Omega_\omega=\omega-\omega_{oz}$ is termed as Doppler shifted frequency which is purely growing instability and makes IAW disappear. This is also known as D'Angelo mode. The shear flow here is responsible for the wave instability. The real part of the frequency is equivalent to $\omega=v_{0z}k_z$. For the limit $1>A_i\frac{k_y}{K_z}$, instability doesn't appear and Eq. (2.134) is modified IAW dispersion relation along with shear flow lined up to the magnetic field.

Non Uniform Plasma

Let us ignore ion parallel velocity, Eq. (2.133) takes the form,

$$\Omega_\omega^2(1+\rho_s^2k_y^2+\rho_i k_y^2)-\Omega_\omega\omega_e^*=0 \tag{2.140}$$

At $T_i=0$ and $v_{0z}=0$, we get,

$$\omega=\frac{\omega_e^*}{1+\rho_s^2k_y^2} \tag{2.141}$$

which is exactly similar to Eq. (2.26) and is known as the basic drift waves dispersion relation.

2.7 Numerical Analysis

2.7.1 Plasma Scale in JET

Here we will represent numerical solutions by choosing JET parameters where

$$\begin{array}{ll} B=3.4T & T_e=10keV \\ \rho_s=0.29cm & \kappa=3\times10^{-3}cm^{-1} \\ c_s\approx10^8cms^{-1} & v_{De}\approx10^5cms^{-1} \end{array}$$

In order to validate Boltzmann Distribution, we need to attain $k_zv_{the}\gg k_yv_{de}$, that means $k_z\gg0.25\times10^{-4}k_y$ for JET so we chose $k_y=0.3cm^{-1}$ and $k_z=10^{-3}cm^{-1}$. If we want to drop the ion parallel motion then we have to acquire $k_z\ll k_y\times10^{-3}$. We found two real roots by solving Eq. (2.34). A stable region of basic drift waves is presented in Fig. 2.1 which is almost comparable to Fig. 3.7 of Ref. 17. For large k_z, drift waves turn into IAW. Dashed line represents $\omega=\pm c_sk_z$ if $\nabla n_{0i}=0$ and $\rho_s^2k_y^2\ll1$.

We found two imaginary roots of Eq. (2.122) by applying limiting case, i.e. $\nabla n_{0i}=0$ and $\omega_e^*=0$. The parameters are the same as of JET with the addition of shear flow parameter $A_i=0.01$. In Fig. 2.3, we plotted the imaginary frequency of the unstable mode of (EI) plasma against k_z. The result shows growing instability in drift mode because of shear flow.

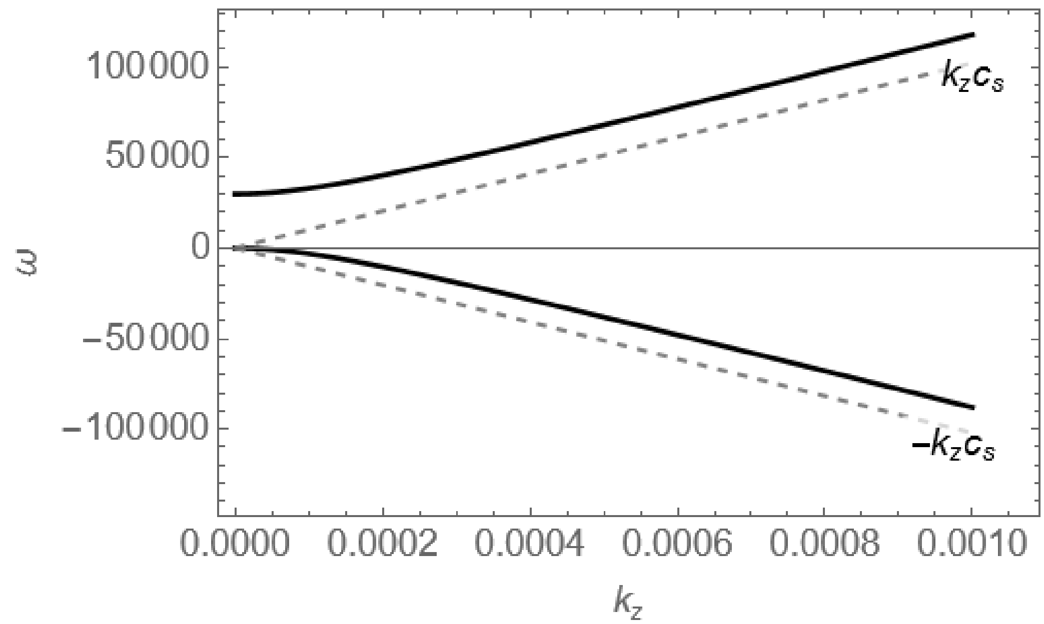

Figure 2.1: Electrostatic Dispersion Relation Eq. (2.34) is plotted against k_z

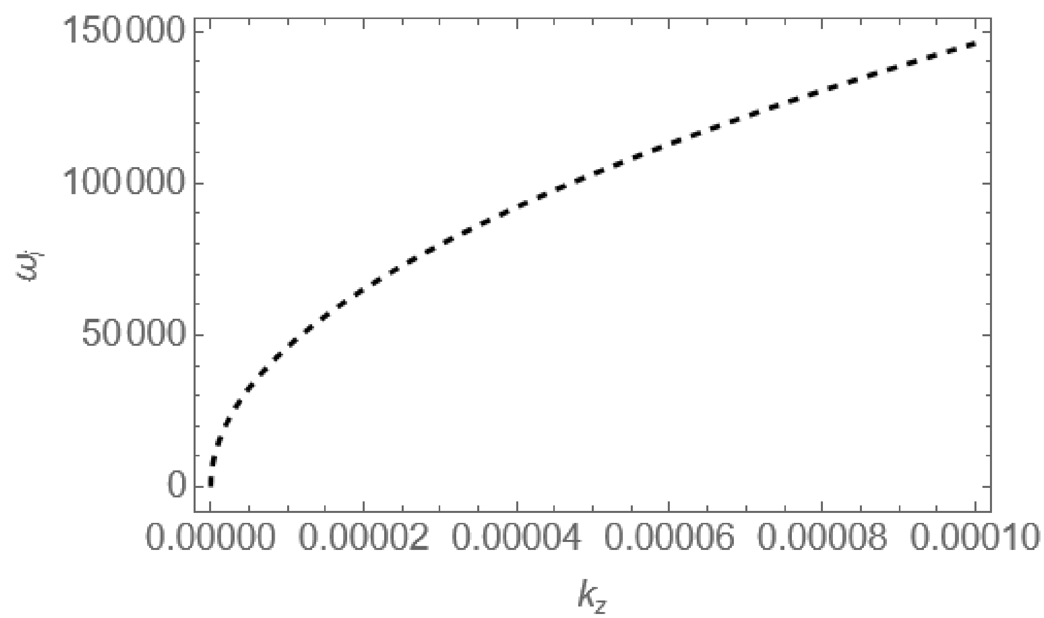

Figure 2.2: Unstable drift mode of Eq. (2.122) is plotted against k_z

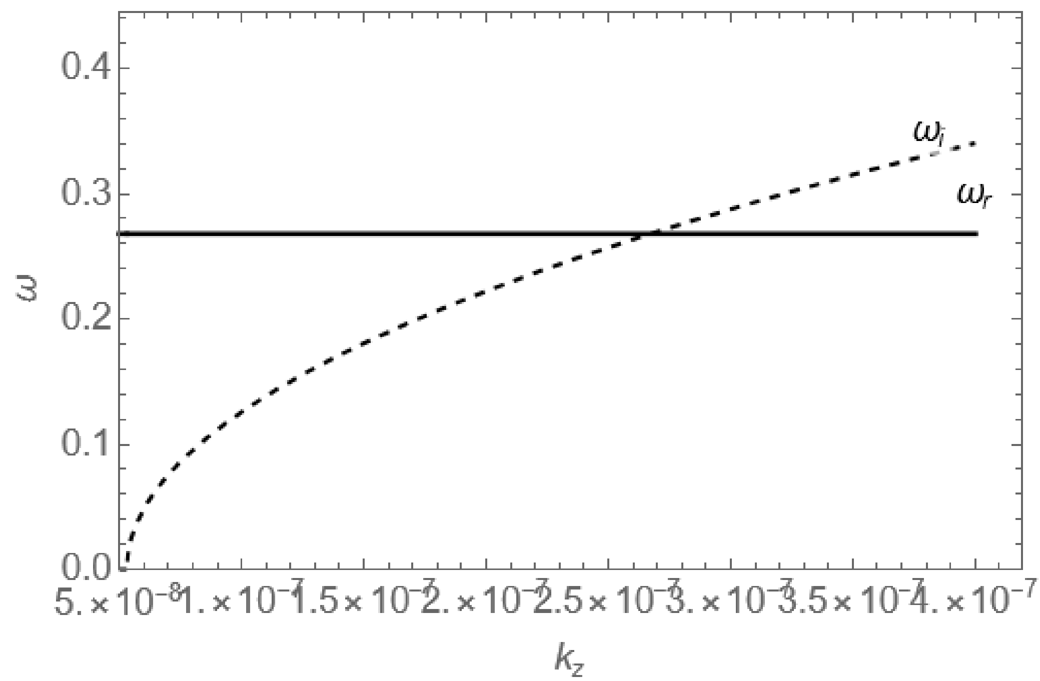

Figure 2.3: Frequencies of the unstable mode of oxygen ion is plotted against k_z

2.7.2 Plasma Scale in Ionosphere

The ionosphere is the earth's uppermost atmosphere comprised of charged and neutral particles. Plasma there has a distinct ratio of oxygen-to-hydrogen densities, electron densities and temperature at different altitudes in the auroral zone of the ionosphere. At an altitude of $800km$, plasma parameters are $B_0 = 0.2G$, $n_{0e} = 10^5 cm^{-3}$, $T_e - 0.13eV$ and $T_i \leq 0.3T_e$. Plasma parameters for oxygen ions are,

$$\Omega_i = 1.21 \times 10^2 rad/s \qquad c_s = 8.905 \times 10^4 cm/s$$
$$\rho_i = 4 \times 10^2 cm \qquad \rho_s = 7.3 \times 10^2 cm$$

All of these values are experimental values [29], [30]. The density gradient scale was chosen to be $\kappa = k_y/50$ while $k_y = 5 \times 10^{-4} cm^{-1}$. The shear flow homogeneity scale for oxygen ions is given as, $A_i = \partial_x v_{0iz}/\Omega_i = 0.1$.

The numerical solution of the growth rate of drift wave instability for oxygen ions is represented in Fig. 2.3. We found two roots of Eq. (2.122) contain both real and imaginary frequency which we have plotted against k_z. It is clear that the imaginary part shows growing instability in drift mode because of shear flow and IAW dissipation. Also, the real part of frequency is approximately larger than the imaginary part. Thus, the linear limit is accurate.

2.8 Drift Wave in Dusty Plasma

Dusty plasma has been an active area of research because of its significance and applicability in space and laboratories [19], [31], [32]. Such a kind of plasma carries a dust grain along with electrons and ions. The flow of electrons and ions on account of their thermal motion, onto the dust grains is the reason behind charge accumulation on them. These dust grains are negatively charged by the reasons of attachment of additional background electrons. Here, ions play the same role as in our previous work. The equilibrium condition states that,

$$n_{0i} = n_{0e} + z_d n_{0d} \tag{2.142}$$

Here $j = i$, e, d and $q_d = -ez_d$. z_d is the charge count carried by the surface of dust rain. Following $m_i \ll m_d$, we assume,

$$n_{i1} \approx n_{0i} \exp(\frac{-e\phi_1}{T_i}) = -n_{0i}\frac{e\phi_1}{T_i} \tag{2.143}$$

$$n_{e1} \approx n_{0e} \exp(\frac{e\phi_1}{T_i}) = n_{0e}\frac{e\phi_1}{T_e} \tag{2.144}$$

Using the linearized Poisson equation,

$$-\frac{1}{4\pi}\nabla^2\phi_1 = (-en_{i1} + en_{e1} - q_d n_{d1}) \tag{2.145}$$

$$-\frac{1}{4\pi}\nabla^2\Phi_1 = \frac{e^2 n_{0i}}{T_i}\phi_1 + \frac{e^2 n_{0e}}{T_e}\phi_1 + ez_d n_{d1} \tag{2.146}$$

$$\frac{n_{1d}}{n_{0d}} = -((\frac{n_{0i}T_e + n_{0e}T_i}{T_e T_i})\frac{e^2\phi_1}{ez_d n_{0d}} + \frac{k^2\phi_1}{4\pi ez_d n_{0d}}) \tag{2.147}$$

$$\frac{n_{1d}}{n_{0d}} = -(e + \frac{k^2}{4\pi ez_d n_{0d}}\frac{T_e T_i z_d n_{0d}}{n_{0i}T_e + n_{0e}T_i})\phi_1 \frac{n_{0i}T_e + n_{0e}T_i}{T_e T_i z_d n_{0d}} \tag{2.148}$$

Dividing by ez_d, we come up with the following result,

$$\frac{n_{d1}}{n_{d0}} = -(1 + \lambda_{De}^2 k^2) ez_d\phi_1/T_{eff} \tag{2.149}$$

$$\frac{n_{d1}}{n_{d0}} = -\mu\Phi_1 \tag{2.150}$$

Here $\lambda_{De}^2 = T_{eff}/4\pi e^2 z_d^2 n_{0d}$, $\Phi_1 = ez_d/\phi_1/T_{eff}$ and $T_{eff} = z_d n_{0d} T_e T_i/(n_{0i}T_e + n_{0e}T_i)$. Writing dust continuity equation as,

$$\begin{aligned}(\partial_t + v_{0dz}\partial_z)n_{1d} + \nabla n_{0d}.\mathbf{v}_E + \frac{n_{0d}}{\Omega_d B_0}(\partial_t + v_{0dz}\partial_z)\nabla_\perp^2\phi_1 \\ -\frac{T_d}{\Omega_d ez_d B_0}(\partial_t + v_{0dz}\partial_z)\nabla_\perp^2 n_{d1} + n_{0d}\partial_z v_{dz1} = 0\end{aligned} \tag{2.151}$$

$$\begin{aligned}(\omega - \omega_{0dz})\frac{n_{d1}}{n_{0d}} + \frac{c}{B_0}\frac{\nabla n_{0d}}{n_{0d}}.(\mathbf{z} \times k_y)\phi_1 + \frac{1}{\Omega_d B_0}(\omega - \omega_{0z})k_y^2\phi_1 \\ -\frac{T_d}{\Omega_d ez_d B_0}(\omega - \omega_{0z})k_y^2\frac{n_{d1}}{n_{0d}} - k_z v_{dz1} = 0\end{aligned} \tag{2.152}$$

Multiplying with T_{eff}/T_{eff}, ez_d/ez_d and n_{0d}, sets out above relation as,

$$\begin{aligned}\Omega_\omega n_{1d} + n_{0d}\omega_d^*\Phi_1 - \frac{n_{0d}}{\Omega_d B_0}\Omega_\omega k_y^2\frac{T_{eff}}{ez_d}\Phi_1 \\ +\frac{T_d k_y^2}{\Omega_d ez_d B_0}(\Omega_\omega)n_{1d} - n_{0d}k_z v_{dz1} = 0\end{aligned} \tag{2.153}$$

Here $\Omega_\omega = \omega - k_z v_{0dz}$, $\omega_d^* = T_{eff}\kappa_{nd}k_y/ez_dB_0$ and $\kappa_{nd} = 1/n_0(dn_{0d}/dx)$. The parallel component of the momentum equation is written as,

$$\begin{aligned}(\partial_t + v_{0dz}\partial_z)v_{z1} + v_{dx1}\partial_x v_{0dz}(x)\\ = -\frac{ez_d}{m_d}\nabla\phi_1 - \frac{T_d}{m_d}\partial_z\frac{n_{d1}}{n_{0d}} - \iota k_z c_d^2\rho_d^2 k_y^2(\Phi_1 + \frac{n_{d1}}{n_{0d}})\end{aligned} \tag{2.154}$$

Solving the above equation yields,

$$\begin{aligned}(-\omega - \omega_{0dz})v_{dz1} + \frac{c}{B_0}\partial_x v_{0dz}(x)k_y\phi_1\\ = -\frac{ez_d}{m_d}k_z\phi_1 - \frac{T_d}{m_d}k_z\frac{n_{d1}}{n_{0d}} - k_z c_d^2\rho_d^2 k_y^2(\Phi_1 + \frac{n_{d1}}{n_{0d}})\end{aligned} \tag{2.155}$$

Multiplying by T_{eff}/T_{eff} and ez_d/ez_d,

$$\begin{aligned}(\omega - \omega_{0dz})v_{dz1} - \frac{T_{eff}k_y}{ez_dB_0}(dxv_{dx0})\frac{ez_d\phi_1}{T_{eff}}\\ = \frac{T_{eff}}{m_d}k_z\frac{ez_d\phi_1}{T_{eff}} + \frac{T_d}{m_d}k_z\frac{n_{0d}}{n_{1d}} - \rho_d^2k_y^2\frac{T_{eff}}{m_d}k_z\Phi_1 - \rho_d^2k_y^2\frac{T_{0d}}{m_d}k_z\frac{n_{d1}}{n_{0d}}\end{aligned} \tag{2.156}$$

$$\Omega_\omega v_{dz1} + \frac{T_{eff}}{ez_dB_0}k_y(dxv_{dx0})\Phi_1 = (1+\rho_d^2k_y^2)(-\frac{T_{eff}}{m_d}\Phi_1 + \frac{T_{0d}}{m_d}\frac{n_{d1}}{n_{0d}})k_z \tag{2.157}$$

Putting Eq. (2.153) into Eq. (2.157) in terms of v_{dz1}, the dispersion relation is obtained as,

$$\begin{aligned}\Omega_\omega^2\frac{n_{d1}}{n_{0d}} + \Omega_\omega\omega_d^*\Phi_1 - \Omega_\omega^2\frac{k_y^2}{\Omega_dB_0}\frac{T_{eff}}{ez_d}\Phi_1 + \Omega_\omega^2\frac{T_dk_y^2}{\Omega_dez_dB_0}\frac{n_{d1}}{n_{0d}}\\ +\frac{T_{eff}}{ez_dB_0}k_yk_z(dxv_{dx0})\Phi_1 - (1+\rho_d^2k_y^2)(-\frac{T_{eff}}{m_d}\Phi_1 + \frac{T_{0d}}{m_d}\frac{n_{d1}}{n_{0d}})k_z^2 = 0\end{aligned} \tag{2.158}$$

Multiplying with T_{eff}/T_{eff}, m_d/m_d and inserting Eq. (2.150),

$$\begin{aligned}\Omega_\omega^2\mu\Phi_1 + \Omega_\omega\omega_d^*\Phi_1 - \Omega_\omega^2k_y^2\frac{m_d}{\Omega_dez_dB_0}\frac{T_{eff}}{m_d}\Phi_1 - \Omega_\omega^2k_y^2\mu\frac{m_d}{\Omega_dez_dB_0}\frac{T_{eff}}{m_d}\frac{T_d}{T_{eff}}\Phi_1\\ +\frac{m_d}{ez_dB_0}\frac{T_{eff}}{m_d}k_yk_z(dxv_{dx0})\Phi_1 - (1+\rho_d^2k_y^2)(-\frac{T_{eff}}{m_d} + \frac{T_{0d}}{T_{eff}}\frac{T_{eff}}{m_d}\mu)k_z^2\Phi_1 = 0\end{aligned} \tag{2.159}$$

$$\chi\Omega_\omega^2 - \omega_d^*\Omega_\omega - c_d^2 k_y k_z A_d - (1+\rho_d^2 k_y^2)(1+\mu\frac{T_{0d}}{T_{eff}})c_d^2 k_z^2 = 0 \tag{2.160}$$

Here $\chi = \mu + \rho_d^2 k_y^2(1 + \mu T_{0d}/T_{d1})$, $c_d^2 = T_{eff}/m_d$, $A_d = 1/\Omega_d(dv_{0d}/dx)$ and $\rho_d = c_d/\Omega_d$. Eq. (2.160) is the coupled linear dispersion relation of dust acoustic wave (DAW) and dust drift wave (DDW) as given in Ref. 24. Using the quadratic formula, the above equation gives us the solution,

$$\Omega_\omega = \frac{1}{2\chi}[\omega_d^* \pm \sqrt{\omega_d^{*2} + 4\chi(c_d^2 k_y k_z A_d + (1+\rho_d^2 k_y^2)(1+\mu\frac{T_{0d}}{T_eff} c_d^2 k_y^2)}] \tag{2.161}$$

The instability indicator is

$$|A_d| > (1+\rho_d^2 k_y^2)(1+\mu\frac{T_{0d}}{T_{eff}})\frac{k_z}{k_y} \tag{2.162}$$

Hence, the negative gradient of shear flow is the main reason behind the instability of drift waves.

Chapter 3

Conclusion

The low-frequency electrostatic waves have been studied in hot ion electron-ion plasma and the cancellation effect of the ion collision-less ion stress tensor with the contribution of the diamagnetic term in the convective derivative of the momentum conservation equation is pointed out using the fluid description using the fluid approach. In addition to it, the linear drift waves in multi-component classical plasmas such as electron-positron ion (EPI) and dusty plasmas are also discussed.

In Chapter 1, we have presented a comprehensive description of plasma and its parameters by describing the plasma's composition, its spatial and temporal scale lengths and characteristic frequencies. The link between the Fourier series and Fourier transformation has been discussed with an example. With increasing the period, say L, of the periodic function, we should expect the summation to take the form of an integral as illustrated in Fig. (1.1). Fourier transforms assist to present any periodic function from one variable into another such that time into frequency and space into wave vector. The forced oscillator has also been studied. The output oscillation is equal to the integral multiple of the frequency of external force which is the source of energy in the system. This validates the idea of enhanced frequency due to the external driving force. The attenuation of forced oscillations in a dissipative medium has been calculated to understand the wave behaviour in collisional plasmas.

The main focus of this thesis is on the linear propagation of drift waves in usual electron-ion plasma taking into account the contribution of the ions temperature T_i. The derivation of the linear dispersion relation in the case of hot ions is not straightforward because of the cancellation of the contribution of diamagnetic drift velocity in the convective derivative of polarization drift with the collision-less part of the ions stress tensor. This point has been highlighted by reproducing the detailed calculations taking help from Weiland's book (Ref. 17). In a cold ion plasma, the dispersion relation of the drift wave is based on the zeroth order diamagnetic drift of electrons. The addition of polarization drift alters the dispersion relations as Eq. (2.26) in which the numerator shows the Larmour radius effect at electron temperature. The coupling of ion acoustic wave (IAW) and drift wave originates when the ion's parallel motion is included in the derivation of linear dispersion relation of low-frequency electrostatic perturbations in the presence of density inhomogeneity.

Shear flow instability has been readdressed in (EI) and (EPI) plasma owing to the effect of elimination of diamagnetic contribution in the polarization drift against collisionless stress tensor part. The limiting cases in (EI) for both uniform and non-uniform plasma have been analyzed. The study revealed that $\partial_x v_{0z} = 0$ and $v_{0z} = 0$ yield dispersion relation for Doppler shifted and primary frequencies of waves. In the limit, $1 \gg A_i k_y / k_z$, the imaginary part of Doppler shifted frequency $\Omega_\omega = \omega - \omega_{0z}$ manifests D Angelo's mode which is purely growing. The real part of the frequency is equivalent to $v_{0z} k_z$.

The dispersion relation of drift waves obtained for hot ion plasmas has been solved using the parameters of JET plasma and the ionospheric plasma. Coupled dispersion relation of drift waves and IAWs has also been solved for the case of JET plasma. Numerical analysis (Fig. 2.1) demonstrates the behaviour of coupled electrostatic drift waves in Joint European Tokamak (JET) for in-homogeneous cases. The results show stable drift waves turning into IAW with increasing k_z. In the case of homogeneous plasma, the imaginary frequency in Fig. 2.2 indicates purely growing instability in JET. We speculate here that drift waves can become unstable in JET if we take into account the shear flow effect. Both the real and the imaginary part of frequencies vs k_z for the case of ionospheric plasma are shown in Fig. 2.3. The imaginary part shows the growing instability of drift mode. The real frequency is approximately larger than the imaginary part of the frequency, therefore, the linear limit is justifiable.

Coupling of dust drift with dust acoustic wave has also been investigated at the end of chapter-2.

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