DEVELOPMENT OF MODEL ORDER REDUCTION TECHNIQUES

By

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Supervisor Certificate

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(Dr Abdul Ghafoor)

Abstract

Mathematical modeling is an essential feature for the analysis and design of a dynamical system. Generally, large and complex models are obtained from physical systems. Some examples are robotic, telecommunications, network, mechanical and many other complex systems. These systems are governed by the partial differential, Laplace and integro-differential equations etc. For the analysis and design of such systems, reduced order models are desirable that provide a good approximation of the original systems.

In last few decades, notable research work has been done on different aspects of model reduction. Existing techniques of model order reduction mostly suffer from the limitation of absence of original system key properties in reduced order system like passivity, stability, large approximation error and lack of *a priori* error bounds etc.

This thesis investigates frequency weighted balanced model order reduction problem for standard and generalized (singular and non-singular), continuous and discrete linear time invariant systems.

Firstly the frequency weighted model reduction problem is formulated. New frequency weighted model order reduction techniques are proposed for standard continuous and discrete time systems. Frequency interval Gramians based model order reduction techniques (where weights are not explicitly predefined) are also presented for standard continuous and discrete time systems. The proposed techniques guarantee stability even for the case where double sided weightings are employed. *A priori* frequency response error bounds are also derived. The proposed techniques yield mostly low frequency response error when compared to well known existing frequency weighted model reduction techniques.

A generalization of existing frequency interval Gramians based model reduction techniques for Generalized non-singular discrete and continuous time systems is also presented. Moreover, a frequency limited model reduction technique for Generalized descriptor system is also presented. Simple algorithms are also given for preserving the stability of reduced-order models. The work also extends Poor Man's truncated balanced realization for frequency limited case. Numerical examples are also presented for comparison of generalized techniques.

Finally, new techniques to address time interval Gramians based model reduction are also presented for standard continuous time systems. The proposed techniques yield easily computable error bounds and comparable frequency response error.

Dedication

Dedicated to

My Parents, Wife, Children Hanzala, Hafsa and Abdul Hadi

for their love, endless support and encouragement

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All praise for Allah Almighty who has bestowed me with the strength and the passion to accomplish this thesis and I am thankful to Him for His mercy and benevolence. Without His consent I could not have indulged myself in this task. I express my gratitude towards Him with humility and humbleness.

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Publications

The following relevant publications have been produced during PhD period.

1. M. Imran, A. Ghafoor, and V. Sreeram, "Frequency weighted model order reduction technique and error bound", Automatica, In Press, 2014. Impact factor 3.132.
2. M. Imran and A. Ghafoor, "Model reduction of Descriptor systems using frequency limited Gramians ", Journal of The Franklin Institute, In Press, 2014. Impact factor 2.26.
3. M. Imran and A. Ghafoor, "Stability preserving model order reduction technique using frequency limited Gramians for discrete time systems ", IEEE Transactions on Circuits and Systems II: Express Briefs, vol. 61, no. 9, pp. 716-720, 2014. Impact factor 1.187.
4. M. Imran, A. Ghafoor, and V. Sreeram, "Frequency weighted model order reduction technique and error bounds for discrete time systems", Mathematical Problems in Engineering, vol. 2014, pp. 1-8. Impact factor 1.082.
5. M. Imran and A. Ghafoor, "Limited frequency Gramians based model reduction for Generalized non-singular systems ", IMA Journal of Mathematics, Control and Information, In Press, 2014. Impact factor 0.967.
6. M. Imran, A. Ghafoor, and V. Sreeram, "Limited frequency interval Gramians based model reduction for Generalized non-singular discrete time systems", IET Control Theory and Applications, Accepted with revisions. Impact factor 1.844.
7. M. Imran and A. Ghafoor, "Limited frequency Gramians based model reduction technique and error bound ", Circuits, Systems and Signal Processing, Submitted, 2014. Impact factor 1.264.
8. M. Imran, A. Ghafoor, and V. Sreeram, "Time limited Gramians-based model reduction technique and error bound", Journal of The Franklin Institute, Submitted, 2014. Impact factor 2.26.
9. M. Imran and A. Ghafoor,"Model reduction of discrete time systems using time limited Gramians", IEEE Transactions on Circuits and Systems II: Express Briefs, Submitted, 2014. Impact factor 1.187.
10. M. Imran, A. Ghafoor, S. Akram, and V. Sreeram, "Limited frequency interval Gramians based model reduction for nonsingular generalized systems", Australian Control Conference, Perth, pp. 441-444, 2013.

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Acronyms

MOR	Model Order Reduction
ROMs	Reduced Order Models
ROC	Reduced Order Controller
BT	Balanced Truncation
BSPA	Balanced Singular Perturbation Approximation
FWMR	Frequency Weighted Model Reduction
FLMR	Frequency Limited Model Reduction
EVD	Eigenvalue Decomposition
SVD	Singular Value Decomposition
HSV	Hankel Singular Values
PMTBR	PoorMan's Balanced Truncated Realization
EPMTBR	Extended PoorMan's Balanced Truncated Realization
TLMR	Time Limited Model Reduction
PCALE	Projected Continuous Algebraic Lyapunov Equation
dB	Decibel
diag	Diagonal

Notations

Transfer function- Standard	$G(s) = C(sI - A)^{-1}B + D$
	$G(z) = C(zI - A)^{-1}B + D$
	$\Leftrightarrow$ State-space realization $\{A, B, C, D\}$
Transfer function- Generalized	$G(s) = C(sE - A)^{-1}B + D$
	$G(z) = C(zE - A)^{-1}B + D$
	$\Leftrightarrow$ State-space realization $\{E, A, B, C, D\}$
$\|G(j\omega)\|_\infty$	sup_ω, if $G(j\omega)$ is a transfer function (matrix)
	where $\bar{\sigma}(jw)$ is a maximum singular vale of $G(j\omega)$
$P > 0$	Positive definite matrix P
	Symmetric matrix P with positive eigenvalues
$P \geq 0$	Positive semi-definite matrix
	Symmetric matrix P with non-negative eigenvalues
X^T	Transpose of matrix or a vector X
X^H	Hermitian of matrix or a vector X
X^{-1}	Inverse of matrix X
$\lambda_j[X]$	Eigenvalues of X
$\|X\|$	Modulus of X
$\sigma(X)$	Singular value of X

Chapter 1

Introduction

1.1 Overview

The derivation of a reasonable mathematical model is fundamental to obtain a good understanding of the dynamical behavior of a physical system in question or to control its behavior in order to achieve desired performance specifications. In practice, modelling of complex systems (such as chip design, fluid flow, mechanical systems simulation) yields very large scale systems. Despite the advancement of technology and the ever increasing computational speed, the analysis, control and optimization of large scale systems is challenging (if not impossible) due to expensive computations and storage requirements. Therefore, process of generating a low-dimensional or reduced-order models (ROMs) that provides a good estimate of the original full order system is known as model order reduction (MOR). In general, the aim of MOR is to find ROMs which approximate the input output behaviour of the original systems. This is achieved with a lower storage requirement as well as evaluation time. MOR has played a significant role in modern control system design and caught lot of attention in the last few decades [1, 2, 5, 19, 24, 26, 28, 32, 33, 50, 56, 57, 60, 71, 76, 77, 89, 92, 93, 94, 95, 96, 97].

One of the significant factors in MOR is the reduction error which is obtained from the difference between the original and ROMs frequency response. In addition system properties like stability, input output behaviour are also equally important to be preserved in MOR. Numerical properties such as computational speed and accuracy, storage requirements etc. play a vital role in computational efficiency of the MOR techniques. The error bound formula for MOR technique gives some idea of the approximation error. It will assist the designer to choose MOR technique for the concerned application.

1.1.1 Balanced truncation

Given a full order continuous time original stable system be $G(s) = C(sI - A)^{-1}B + D$ where $\{A, B, C, D\}$ is its n^{th} order minimal realization. The objective of MOR is to find a ROM $G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$, where $\{A_{11}, B_1, C_1, D\}$ is an r^{th} order $(r < n)$ minimal realization such that the approximation error $\|G(s) - G_r(s)\|_\infty$ is minimized.

Balanced truncation (BT) [1] (please see Appendix A) is most commonly used MOR technique which preserves stability in ROMs and yields *a priori* frequency response error bounds [5]. In BT, the controllability and observability Gramians are transformed into an internally balanced system. An internally balanced realization has equal controllable and observable states. The ROMs are obtained by truncating the least controllable and least observable states. Hence, using BT [1] technique, the error obtained is considerably smaller, which indicates good performance of ROMs. Besides BT, other schemes such as Hankel optimal approximation [2], Pade approximation [3], Krylov technique [4] etc. are also useful for solving MOR problem.

BT performs well at higher frequencies, therefore for better performance at lower frequencies, balanced singular perturbation approximation (BSPA) is used (please see Appendix B). The ROMs obtained via the BSPA [33, 34] are also stable and balanced. Moreover, the error bound for the BT also holds for BSPA [33, 34]. However, the ROMs obtained via the BSPA [33, 34] may be proper even for the strictly proper original systems.

1.1.2 Frequency weighted model reduction

In MOR, it is important to have small reduction error between the original system and ROM for all the frequencies. However, sometimes the error is more critical over a certain band of frequency rather than other frequencies. This is true for the case, when ROMs are used in feedback control design [5, 73]. This leads the concept of using frequency weights in MOR procedure, also known as frequency weighted model reduction (FWMR) problem. Note that, controller order reduction problem can be reformulated as FWMR (please see Appendix C).

Given the original full order stable system $G(s) = C(sI - A)^{-1}B + D$, the stable input weighting system $V_i(s) = C_V(sI - A_V)^{-1}B_V + D_V$ and a stable output weighting system $W_o(s) = C_W(sI - A_W)^{-1}B_W + D_W$, where $\{A, B, C, D\}$, A_V, B_V, C_V, D_V and A_W, B_W, C_W, D_W are n^{th}, p^{th} and q^{th} order minimal realization respectively, the main objective is to find a stable ROM $G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r$ where $\{A_r, B_r, C_r, D_r\}$ is an r^{th} order $(r < n)$ minimal realization, such that $\|W_o(s)(G(s) - G_r(s))V_i(s)\|_\infty$ is made as small as possible. This is known as two sided FWMR problem as shown in Fig. 1.1. If one of the weights is identity, the problem is known as one sided FWMR, where the objective is to find a ROM $G_r(s)$, such that $\|(G(s) - G_r(s))V_i(s)\|_\infty$ (in case of input weighting) and $\|W_o(s)(G(s) - G_r(s))\|_\infty$ (in case of output weighting) is made as small as possible. Fig. 1.2 and Fig. 1.3 represent the input and output FWMR error systems respectively. Enns [5] was the first to formulate this problem by introducing frequency weightings to the BT [1] technique.

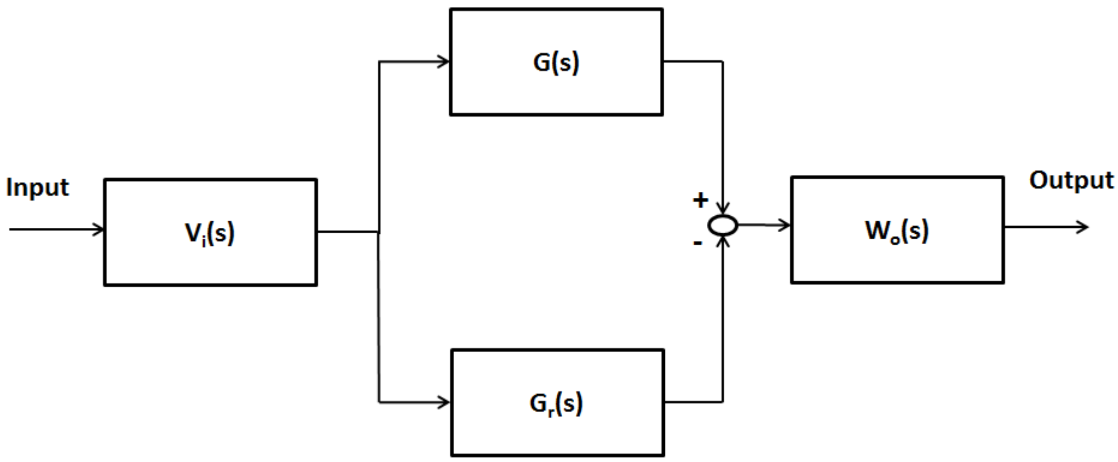

Figure 1.1: Input-output FWMR error system

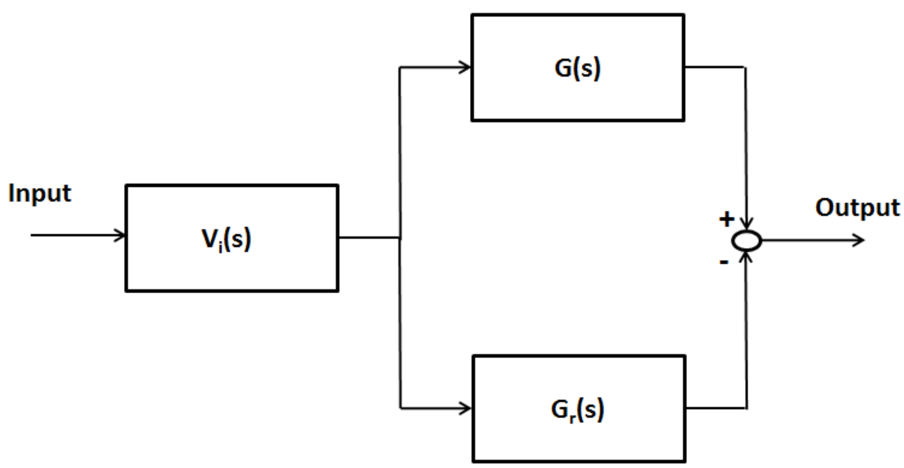

Figure 1.2: Input FWMR error system

Enns [5] technique was an extension of BT [1] to incorporate frequency weights. These weights are useful for the frequency shaping of the MOR error. Enns technique [5] may use input weighting, output weighting or both. However, for one sided weighting, stability of ROMs is guaranteed but for double sided weighting case, stability is not guaranteed. To overcome this instability problem of double sided weighting, several modifications to Enns technique have been proposed [7, 19, 22, 23, 25, 26, 72, 73, 75, 80, 81, 83, 84].

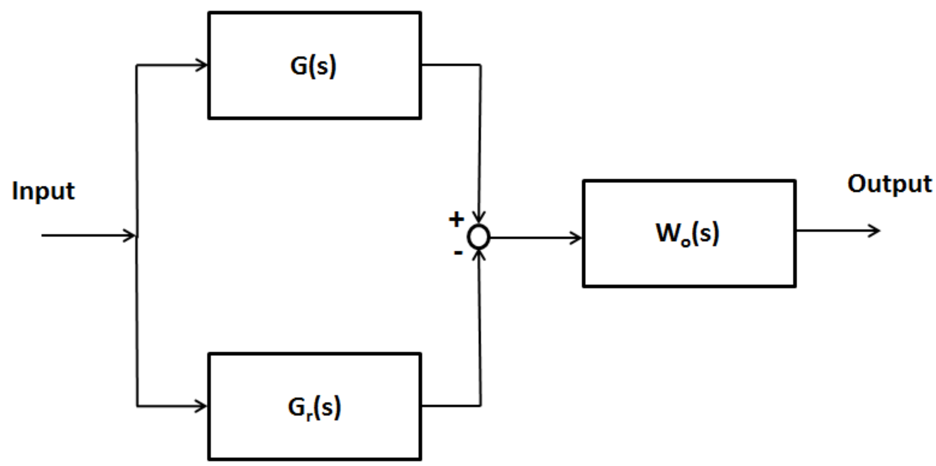

Figure 1.3: Output FWMR error system

To overcome Enns drawbacks, Lin and Chiu [19] has proposed a different technique that guarantees stability when double sided weightings are present. However, their technique can work only when the weighing function used is strictly proper and no pole zero cancellation occurs when forming the augmented system. These limitations of Lin and Chiu technique were later modified by Sreeram et al [23] and Varga and Anderson [25], where [23] generalized [19] to include proper weights, while [25] retains the stability of the system even when pole zero cancellation occur. However, Varga and Anderson technique [25] produces the same results as Enns [5] especially in controller reduction applications. So far controller reduction problem, if Enns technique [5] yields unstable ROMs, so does by Varga and Anderson [25] technique.

Wang et al's technique [26] has also addressed instability problem of Enns [5], which not only provide stable ROMs in the presence of double sided weightings but also derived error bounds. The approximation error of Wang et al technique [26] was later improved by Varga and Anderson [25]. As pointed out by Sreeram [22], this technique ([26] and its modification by Varga and Anderson [25]) are realization dependent. This means that for same original system, different models can be obtained from different realizations.

Another group of techniques based on partial fraction was originally proposed by Latham and Anderson [83]. A number of FWMR techniques based on partial fraction expansion idea have followed [7,23,73,75,80,81,84]. Error bounds exist for some special type of weighting functions [7, 23, 80, 81]. Sreeram and Ghafoor [7] technique provides lower approximation error, but this technique is adhoc with no theocratical justification [82]. Sahlan and Sreeram

[81], although provides lower error as compared to Enns technique [5], and other well-known FWMR techniques [19,25,26], but this technique is realization dependent. Moreover, finding general transformation matrices for the weights to reduce the weighted approximation error remains a challenging open problem [81].

1.1.3 Frequency limited model reduction problem

The objective of FWMR technique is to make the weighted error $\|W_o(s)(G(s) - G_r(s))V_i(s)\|_\infty$ as small as possible, where $V_i(s)$ is some input weighting and $W_o(s)$ is some output weighting given to the system. These input and output weightings are often fictitious (until unless specified by the user) and results may vary by changing these weightings. In many cases, the problem is to approximate original system $G(s)$ over a certain frequency band $[\omega_1, \omega_2]$ and no input and output weightings are given. Using FWMR for such cases, the designer need to construct weights to reflect this frequency band. Choosing weightings is also itself a problem [30].

Gawronski and Juang [29] proposed frequency limited model reduction (FLMR) technique where the frequency weights are not explicitly predefined, but approximation is considered in certain frequency intervals $[\omega_1, \omega_2]$ without construction of input and output weightings by frequency domain representation of Gramians. In this technique, Gramians are defined for a desired frequency intervals. However, it can also yield unstable ROMs for stable original system (like Enns technique [5]). Moreover, there are no error bounds. Motivated by [26], Gugercin and Antoulas [30] has modified Gawronski and Juang technique to provide ROMs. Motivated from (Varga and Anderson [25] modification to Wang et al's technique [26]), Ghafoor and Sreeram [8] proposed another modification to Gawronski and Juang [29] technique to provide stable ROMs. Both techniques [30, 87] carry frequency response error bounds subject to fulfilment of certain rank conditions. However, like [25, 26], these techniques are also realization dependant.

The problem of FWMR with given weightings and FLMR without predefined weightings becomes equivalent as shown in [30] (please see Appendix D).

Gawronski and Juang has also introduced a concept of time limited Gramians based model reduction (TLMR). Unfortunately, TLMR also lacks the stability of ROMs and also does not carry frequency response error bounds.

Generalized descriptor systems are also useful and find their presence in a number of applications which include semidiscretization of partial differential equations, multi-body dynamics with constraints, electrical circuit simulation and micro-electro-mechanical system [44,45,46,47,48,49]. Many techniques for MOR of such systems appear in [3,55,56,57,58]. However, there is no work in literature that is linked with limited frequency Gramians for generalized systems.

1.1.4 Problem summary

Existing FWMR, FLMR and TLMR techniques may yield unstable ROMs, have no or poor error bounds and yield more approximation error.

1.1.5 Summary of contributions

Various FWMR, FLMR and TLMR techniques [10]- [18] for standard and generalized state space systems for both continuous and discrete time are proposed which always yield stable ROMs, have easily computable error bounds and mostly yield less approximation error.

1.2 Thesis outline

The thesis is divided into seven chapters. A brief description of each chapter is outlined here.

Chapter 2 (partially published in [10,12]) presents a new FWMR technique. The proposed technique guarantees stability of the ROMs in the presence of double sided weightings for continuous and discrete time systems. The proposed technique also yields frequency response *a priori* error bounds. Computational aspects of existing and proposed techniques are also given. Several numerical examples are given to show the effectiveness of the proposed technique in reducing the approximation error in the selected frequency band.

Chapter 3 (partially published in [11, 16]) presents a new FLMR technique (without explicitly pre-defined weights) for continuous and discrete time systems. ROMs are guaranteed to be stable and error bounds are also available. The proposed technique mostly produces better approximation error as compared to Gugercin and Antoulas [30] and Ghafoor and Sreeram [8, 87] techniques, in the desired frequency interval.

Chapter 4 (partially published in [13, 14, 18]) presents FLMR technique for generalized non-singular continuous and discrete time systems. The numerical simulations and comparison of frequency response error for the proposed generalized techniques are also included.

Chapter 5 (partially published in [15]) presents FLMR technique for Generalized descriptor (singular) system. The proposed technique generalizes the results of Gawronski and Juang [29] technique for large-scale Generalized descriptor systems using frequency interval Gramians. Simple algorithms are also given for preserving the stability of ROMs. The work also extends Poor Man's truncated balanced realization (PMTBR) technique [61] to include frequency limited Gramians for Generalized descriptor systems. A numerical comparison of proposed techniques is also included.

Chapter 6 (partially appear in [17]) proposes TLMR techniques. The proposed techniques extend the results of time limited Gawronski and Juang [29] and Gugercin and Antoulas [69] techniques. The proposed techniques also yield error bounds. Numerical examples are given to show the effectiveness of the proposed techniques in reducing the approximation error in the selected time interval.

Finally, chapter 7 concludes the thesis with some suggestions for future research.

Chapter 2

FWMR: A New Technique

2.1 Introduction

MOR is an essential feature for the analysis and design of modern control systems. The ROM should possess the properties of original system. Moreover, it is desired that the approximation error between the original system and ROM is small for whole frequency range. BT [1] (for standard state space system) is a popular scheme for this purpose, since it preserves stability and has frequency response *a priori* error bounds. It is desired that reduction algorithm minimizes the error for whole frequency range. However, for certain scenarios the approximation error is more critical in certain frequency intervals rather than whole frequency range.

Enns [5] FWMR has extended the BT method to include frequency weightings in the reduction procedure. This technique may include input, output and double sided weightings. Enns technique ensures stability of ROMs for single sided case only. For double sided weighting case, this technique may yield unstable ROMs [23]. To rectify the instability issue of Enns technique, various modifications exist in literature including [7,19,25,26] etc.

Lin and Chiu [19] technique is only applicable when weightings are strictly proper. It was improved to include more general proper weightings in [23]. Varga and Anderson [25] technique also yields proper models for strictly proper original systems. Ghafoor and Sreeram [7] technique although provides alternate solution to Enns instability problem but it is a parameterized technique. Wang et al's [26] technique is relatively useful, since other techniques [19,25] are not applicable for controller reduction problem. Moreover, besides stability solution to Enns [5] technique, Wang et al's [26] technique also yields easily computable expression for *a priori* error bounds. However, Wang et al's technique involves square root of eigenvalue decomposition (EVD) of some matrices, therefore may yield large error.

In this chapter (partially published in [10,12]), a new technique is proposed for FWMR for both continuous and discrete time systems. The stability is guaranteed even for double sided weighting and a large change in some of eigenvalues is circumvented by pursuing similar effect on all eigenvalues. The proposed technique provides comparable frequency response error and yields easily computable *a priori* error bounds. Numerical examples are given to show the usefulness of the proposed technique. The results are compared with the existing FWMR techniques.

2.2 Preliminaries

Here we review some of the existing FWMR techniques, namely Enns [5], Wang et al [26] and Varga and Anderson [25] etc.

2.2.1 Enns technique

Let $G(s)$ and $G(z)$ be the transfer function of a stable original continuous and discrete time systems respectively with the following minimal realization, $\{A, B, C, D\}$. Similarly, let $V_i(s)$, $W_o(s)$, $V_i(z)$, $W_o(z)$ be the transfer functions of stable input and output weights of continuous and discrete time with the following minimal realizations: $\{A_i, B_i, C_i, D_i\}$ and $\{A_o, B_o, C_o, D_o\}$ respectively. The augmented systems are given by

$$G(s)V_i(s) = C_i(sI - A_i)^{-1}B_i + D_i \quad (2.1)$$

$$W_o(s)G(s) = C_o(sI - A_o)^{-1}B_o + D_o \quad (2.2)$$

$$G(z)V_i(z) = C_i(zI - A_i)^{-1}B_i + D_i \quad (2.3)$$

$$W_o(z)G(z) = C_o(zI - A_o)^{-1}B_o + D_o \quad (2.4)$$

where

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] = \left[\begin{array}{cc|c} A & BC_V & BD_V \\ 0 & A_V & B_V \\ \hline C & DC_V & DD_V \end{array}\right] \quad (2.5)$$

$$\left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right] = \left[\begin{array}{cc|c} A_W & B_WC & B_WD \\ 0 & A & B \\ \hline C_W & D_WC & D_WD \end{array}\right] \quad (2.6)$$

Let the Gramians

$$P_i = \begin{bmatrix} P_E & P_{12} \\ P_{12}^T & P_V \end{bmatrix}, \quad Q_o = \begin{bmatrix} Q_W & Q_{12}^T \\ Q_{12} & Q_E \end{bmatrix}$$

satisfy the following Lyapunov equations:

$$A_i P_i + P_i A_i^T + B_i B_i^T = 0 \tag{2.7}$$

$$A_o^T Q_o + Q_o A_o + C_o^T C_o = 0 \tag{2.8}$$

For discrete time systems above Lyapunov equations are:

$$A_i P_i A_i^T - P_i + B_i B_i^T = 0 \tag{2.9}$$

$$A_o^T Q_o A_o - Q_o + C_o^T C_o = 0 \tag{2.10}$$

By expanding the (1,1) and (2,2) blocks of equations (2.7) and (2.8) respectively, we have:

$$AP_E + P_E A^T + X_E = 0 \tag{2.11}$$

$$A^T Q_E + Q_E A + Y_E = 0 \tag{2.12}$$

where

$$X_E = BC_V P_{12}^T + P_{12} C_V^T B^T + BD_V D_V^T B^T \tag{2.13}$$

$$Y_E = C^T B_W^T Q_{12}^T + Q_{12} B_o C + C^T D_W^T D_W C \tag{2.14}$$

Similarly, expansion of (1,1) and (2,2) blocks of equations (2.9) and (2.10) respectively, yield

$$AP_E A^T - P_E + X_E = 0 \tag{2.15}$$

$$A^T Q_E A - Q_E + Y_E = 0 \tag{2.16}$$

where

$$X_E = AP_{12} C_V^T B^T + BC_V P_{12}^T A^T + BC_V P_V {C_V}^T B^T + BD_V D_V^T B^T \tag{2.17}$$

$$Y_E = C^T B_W^T Q_{12}^T A + A^T Q_{12} B_W C + C^T B_W^T Q_W B_W C + C^T D_W^T D_W C \tag{2.18}$$

Let T be contragredient matrix (used to transform the original system realization) obtained as

$$T^T Q_E T = T^{-1} P_E T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix} \tag{2.19}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$. By partitioning the transformed realization, we have

$$\begin{aligned} \hat{A} &= T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ \hat{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ \hat{C} &= CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\ \hat{D} = D,\ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \end{aligned}$$

where $A_{11} \in R^{r\times r}$. The ROM is obtained as:

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$$
$$G_r(z) = C_1(zI - A_{11})^{-1}B_1 + D$$

Remark 2.2.1 *For input weighting only, P_E and Q are used for calculating the balancing (contragredient) transformation T in (2.19). Likewise for the case of output weighting only, the matrices P and Q_E are used for calculating the balancing (contragredient) transformation T in (2.19), where P and Q are unweighted Gramians computed as:*

$$\begin{aligned} AP + PA^T + BB^T &= 0 \\ A^TQ + QA + C^TC &= 0 \\ APA^T - P + BB^T &= 0 \\ A^TQA - Q + C^TC &= 0 \end{aligned}$$

Remark 2.2.2 *Since in Enns technique, it is not guaranteed to ensure $X_E \geq 0$ and $Y_E \geq 0$, the ROMs may not remain stable in the case of double-sided weightings.*

2.2.2 Wang et al's technique

The limitation of Enns technique that it may provide unstable ROMs in the presence of both input and output weightings was overcome by [26] technique. This technique [26] guarantees the positive semi-definiteness of symmetric matrices X_E (2.13) and Y_E (2.14). A discrete time version of [26]) technique is proposed by Campbell et al's [85]. Let the new controllability P_{WS} and observability Q_{WS} Gramians (calculated by solving the following Lyapunov equations) respectively

$$AP_{WS} + P_{WS}A^T + B_{WS}B_{WS}^T = 0 \tag{2.20}$$

$$A^TQ_{WS} + Q_{WS}A + C_{WS}^TC_{WS} = 0 \tag{2.21}$$

Similarly, for discrete time case, the Lyapunov equations are

$$AP_{WS}A^T - P_{WS} + B_{WS}B_{WS}^T = 0 \tag{2.22}$$

$$A^TQ_{WS}A - Q_{WS} + C_{WS}^TC_{WS} = 0 \tag{2.23}$$

are used to obtain contragredient matrix T as

$$T^TQ_{WS}T = T^{-1}P_{WS}T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$.

The fictitious input B_{WS} and output C_{WS} matrices shown in the above Lyapunov equations are defined as, $B_{WS} = U_{WS}|S_{WS}|^{1/2}$ and $C_{WS} = |R_{WS}|^{1/2}V_{WS}^T$, respectively. Since the expressions $U_{WS},\ S_{WS},\ V_{WS},$ and R_{WS} are calculated by orthogonal EVD $X_E = U_{WS}S_{WS}U_{WS}^T$ and $Y_E = V_{WS}R_{WS}V_{WS}^T$, where $S_{WS} = diag(s_1, s_2, \cdots, s_n)$, $R_{WS} = diag(r_1, r_2, \cdots, r_n)$, $|s_1| \geq |s_2| \geq \cdots \geq |s_n| \geq 0$ and $|r_1| \geq |r_2| \geq \cdots \geq |r_n| \geq 0$. The ROMs are calculated by partitioning the transformed realization.

Remark 2.2.3 *Since $X_E \leq B_{WS}B_{WS}^T \geq 0$, $Y_E \leq C_{WS}^TC_{WS} \geq 0$, $P_{WS} > 0$ and $Q_{WS} > 0$, the minimality of $\{A, B_{WS}, C_{WS}\}$ is guaranteed. Thus the stability of ROMs in the case of double-sided weighting follows immediately from the stability of unweighted BT.*

Theorem 2.2.1 *[26] The following expression for error bounds holds (subject to fulfillment of* $rank\begin{bmatrix} B_{WS} & B \end{bmatrix} = rank\begin{bmatrix} B_{WS} \end{bmatrix}$ *and* $rank\begin{bmatrix} C_{WS} \\ C \end{bmatrix} = rank\begin{bmatrix} C_{WS} \end{bmatrix}$*):*

$$\|W_o(s)(G(s) - G_r(s))V_i(s)\|_\infty \leq 2\|W_o(s)L|_\infty \|KV_i(s)\|_\infty \sum_{j=r+1}^{n} \sigma_j$$

$$\|W_o(s)(G(z) - G_r(z))V_i(s)\|_\infty \leq 2\|W_o(z)L|_\infty \|KV_i(z)\|_\infty \sum_{j=r+1}^{n} \sigma_j$$

where

$$L = CV_{WS} diag(|r_1|^{-\frac{1}{2}}, |r_2|^{-\frac{1}{2}}, \cdots, |r_{li}|^{-\frac{1}{2}}, 0, \cdots, 0)$$

$$K = diag(|s_1|^{-\frac{1}{2}}, |s_2|^{-\frac{1}{2}}, \cdots, |s_{ko}|^{-\frac{1}{2}}, 0, \cdots, 0)U_{WS}^T B$$

$li = rank\,[X_E]$ *and* $ko = rank\,[Y_E]$.

2.2.3 Varga and Anderson modification to Wang et al's technique

Varga and Anderson [25] proposed a modification to Wang et al's technique [26] by reducing the Gramians distance to Enns choice (i.e, the size of $P_{WS} - P_E$ and $Q_{WS} - Q_E$). The proposed transformation simultaneously diagonalized the Gramians $\acute{P}_{VA}$ and $\acute{Q}_{VA}$

$$T^{-1}\acute{P}_{VA}T^{-T} = T^T\acute{Q}_{VA}T = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$ and the Gramians $\acute{P}_{VA}$ and $\acute{Q}_{VA}$ satisfy the following Lyapunov equations:

$$A\acute{P}_{VA} + \acute{P}_{VA}A^T + B_{VA}B_{VA}^T = 0 \quad (2.24)$$

$$A^T\acute{Q}_{VA} + \acute{Q}_{VA}A + C_{VA}^T C_{VA} = 0 \quad (2.25)$$

Similarly, for discrete time case, the Lyapunov equations become

$$A\acute{P}_{VA}A^T - \acute{P}_{VA} + B_{VA}B_{VA}^T = 0 \quad (2.26)$$

$$A^T\acute{Q}_{VA}A - \acute{Q}_{VA} + C_{VA}^T C_{VA} = 0 \quad (2.27)$$

The new fictitious matrices B_{VA} and C_{VA} are defined as $B_{VA} = U_{VA_1} S_{VA_1}^{1/2}$ and $C_{VA} = R_{VA_1}^{1/2} V_{VA_1}^T$, respectively. The terms $U_{VA_1}, S_{VA_1}, R_{VA_1}$ and V_{VA_1} are obtained from the orthogonal EVD of symmetric matrices

$$X_E = \begin{bmatrix} U_{VA_1} & U_{VA_2} \end{bmatrix} \begin{bmatrix} S_{VA_1} & 0 \\ 0 & S_{VA_2} \end{bmatrix} \begin{bmatrix} {U_{VA_1}}^T \\ {U_{VA_2}}^T \end{bmatrix} \tag{2.28}$$

$$Y_E = \begin{bmatrix} V_{VA_1} & V_{VA_2} \end{bmatrix} \begin{bmatrix} R_{VA_1} & 0 \\ 0 & R_{VA_2} \end{bmatrix} \begin{bmatrix} {V_{VA_1}}^T \\ {V_{VA_2}}^T \end{bmatrix} \tag{2.29}$$

where $\begin{bmatrix} S_{VA_1} & 0 \\ 0 & S_{VA_2} \end{bmatrix} = diag\{s_1, s_2, \ldots, s_n\}$, $\begin{bmatrix} R_{VA_1} & 0 \\ 0 & R_{VA_2} \end{bmatrix} = diag\{r_1, r_2, \ldots, r_n\}$, $S_{VA_1} > 0, S_{VA_2} \leq 0$, $R_{VA_1} > 0$ and $R_{VA_2} \leq 0$. The ROMs are calculated by partitioning the transformed realization. Since

$$X_E \leq B_{VA} B_{VA}^T \leq B_{WS} B_{WS}^T \geq 0$$

$$Y_E \leq C_{VA}^T C_{VA} \leq C_{WS}^T C_{WS} \geq 0$$

and the realization $\{A, B_{VA}, C_{VA}\}$ is minimal, the stability of ROM in the case of double sided weightings is guaranteed.

Remark 2.2.4 *An error bound for this technique [25] also exists, similar to one in [26], subject to fulfillment of* $rank \begin{bmatrix} B_{VA} & B \end{bmatrix} = rank \begin{bmatrix} B_{VA} \end{bmatrix}$ *and* $rank \begin{bmatrix} C_{VA} \\ C \end{bmatrix} = rank \begin{bmatrix} C_{VA} \end{bmatrix}$.

2.3 Proposed technique

In Wang et al's [26] technique, the symmetric (generally indefinite) matrices X_E and Y_E are made positive definite (or positive semidefinite) by taking the absolute values of the eigenvalues obtained by EVD of X_E and Y_E. This may lead to a large change in some eigenvalues and may not effect other eigenvalues. To pursue a similar effect on all eigenvalues, a new technique (partially published in [10, 12]) is proposed which not only provides stability but also yields comparable frequency response error and error bound. Let a new controllability P_{IG} and observability Q_{IG} Gramians respectively, are calculated by solving the following

Lyapunov equations:

$$AP_{IG} + P_{IG}A^T + B_{IG}B_{IG}^T = 0 \tag{2.30}$$

$$A^TQ_{IG} + Q_{IG}A + C_{IG}^TC_{IG} = 0 \tag{2.31}$$

Similarly, for discrete time case, Lyapunov equations become

$$AP_{IG}A^T - P_{IG} + B_{IG}B_{IG}^T = 0 \tag{2.32}$$

$$A^TQ_{IG}A - Q_{IG} + C_{IG}^TC_{IG} = 0 \tag{2.33}$$

The matrices B_{IG} and C_{IG} are the new fictitious input and output matrices respectively and are defined as:

$$B_{IG} = \begin{cases} U(S - s_nI)^{1/2} & \text{for } s_n < 0 \\ US^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{2.34}$$

$$C_{IG} = \begin{cases} (R - r_nI)^{1/2}V^T & \text{for } r_n < 0 \\ R^{1/2}V^T & \text{for } r_n \geq 0. \end{cases} \tag{2.35}$$

The terms $U,\ S,\ V$, and R are calculated by the orthogonal EVD of symmetric matrices $X_E = USU^T$ and $Y_E = VRV^T$, where $S = diag(s_1, s_2, \cdots, s_n)$, $R = diag(r_1, r_2, \cdots, r_n)$, $s_1 \geq s_2 \geq \cdots \geq s_n$, and $r_1 \geq r_2 \geq \cdots \geq r_n$. Note that, the matrices B_{IG} and C_{IG} are constructed by pursuing similar effect on all eigenvalues of symmetric matrices X_E and Y_E.

A contragredient transformation matrix T (used to transform the original system realization) is obtained as

$$T^TQ_{IG}T = T^{-1}P_{IG}T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$. The ROMs are calculated by partitioning the transformed realization as

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$$

$$G_r(z) = C_1(zI - A_{11})^{-1}B_1 + D$$

Remark 2.3.1 *Since* $X_E \leq B_{IG}B_{IG}^T \geq 0$, $Y_E \leq C_{IG}^T C_{IG} \geq 0$, $P_{IG} > 0$ *and* $Q_{IG} > 0$. *Therefore, the realization* (A, B_{IG}, C_{IG}, D) *is minimal. Moreover, the ROMs are stable. The stability of ROM follows from the stability of unweighted BT [20].*

Remark 2.3.2 *Let*

$$L_{IG} = \begin{cases} CV(R - r_n I)^{-1/2} & \textit{for } r_n < 0 \\ CVR^{-1/2} & \textit{for } r_n \geq 0 \end{cases} \tag{2.36}$$

$$K_{IG} = \begin{cases} (S - s_n I)^{-1/2} U^T B & \textit{for } s_n < 0 \\ S^{-1/2} U^T B & \textit{for } s_n \geq 0. \end{cases} \tag{2.37}$$

a. *If* $rank\,[B_{IG} \;\; B] = rank\,[B_{IG}]$, *then* $B = B_{IG}K_{IG}$.

b. *If* $rank \begin{bmatrix} C_{IG} \\ C \end{bmatrix} = rank\,[C_{IG}]$, *then* $C = L_{IG}C_{IG}$.

*The existence of these rank conditions can be shown in a similar way as in [26]. It is also shown in [26] that conditions given in **a.** and **b.** are almost always true.*

Theorem 2.3.1 *The following error bound for the proposed technique holds*

(i) $\|W_o(s)(G(s) - G_r(s))V_i(s)\|_\infty \leq 2\|W_o(s)L_{IG}\|_\infty \|K_{IG}V_i(s)\|_\infty \sum_{j=r+1}^{n} \sigma_j$

(ii) $\|(G(s) - G_r(s))V_i(s)\|_\infty \leq 2\|K_{IG}V_i(s)\|_\infty \sum_{j=r+1}^{n} \sigma_j$

(iii) $\|W_o(s)(G(s) - G_r(s))\|_\infty \leq 2\|W_o(s)L_{IG}\|_\infty \sum_{j=r+1}^{n} \sigma_j$

Proof: For brevity, we show only the proof of (i) here, proofs of (ii) and (iii) can be obtained similarly. By partitioning $B_{IG} = \begin{bmatrix} B_{IG_1} \\ B_{IG_2} \end{bmatrix}$, $C_{IG} = \begin{bmatrix} C_{IG_1} & C_{IG_2} \end{bmatrix}$ and substituting $B_1 = B_{IG_1}K_{IG}$, $C_1 = L_{IG}C_{IG_1}$ respectively yields

$$\begin{aligned}
&\|W_o(s)(G(s) - G_r(s))V_i(s)\|_\infty \\
&= \|W_o(s)(C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1)V_i(s)\|_\infty \\
&= \|W_o(s)(L_{IG}C_{IG}(sI - A)^{-1}B_{IG}K_{IG} - L_{IG}C_{IG_1}(sI - A_{11})^{-1}B_{IG_1}K_{IG})V_i(s)\|_\infty \\
&= \|W_o(s)L_{IG}(C_{IG}(sI - A)^{-1}B_{IG} - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1})K_{IG}V_i(s)\|_\infty \\
&\leq \|W_o(s)L_{IG}\|_\infty \|C_{IG}(sI - A)^{-1}B_{IG} - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1}\|_\infty \|K_{IG}V_i(s)\|_\infty
\end{aligned}$$

If $\{A_{11}, B_{IG_1}, C_{IG_1}, D\}$ is a ROM obtained by partitioning the balanced realization $\{A, B_{IG}, C_{IG}, D\}$, we have from [5]

$$\|C_{IG}(sI - A)^{-1}B_{IG} - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1}\|_\infty \le 2 \sum_{j=r+1}^{n} \sigma_j$$

the results follows.

Remark 2.3.3 *A similar error bound holds for the discrete time case also as:*

(i) $\|W_o(z)(G(z) - G_r(z))V_i(z)\|_\infty \le 2\|W_o(z)L_{IG}\|_\infty \|K_{IG}V_i(z)\|_\infty \sum_{j=r+1}^{n} \sigma_j$

(ii) $\|(G(z) - G_r(z))V_i(z)\|_\infty \le 2\|K_{IG}V_i(z)\|_\infty \sum_{j=r+1}^{n} \sigma_j$

(iii) $\|W_o(z)(G(z) - G_r(z))\|_\infty \le 2\|W_o(z)L_{IG}\|_\infty \sum_{j=r+1}^{n} \sigma_j$

Following example provides insight into the construction of B_{IG} and C_{IG} for the case when $s_n < 0$ and $r_n < 0$.

Example 2.3.1 *Consider the* 3^{rd} *order system [6]* $G(s) = (s + 2)/(s^3 + 2s^2 + 3s + 1)$ *with the weights* $V_i(s) = W_o(s) = 1/(s^2 + 4s + 2)$*, respectively. Eigenvalues of matrices* X_E *and* Y_E *are* $\{0.0292, 0.0000, -0.0190\}$ *and* $\{0.9296, -0.0000, -0.0274\}$ *respectively, which yields* $s_n = -0.0190$*,* $r_n = -0.0274$*,* $|S_{WS}| = diag\{0.0292, 0.0000, 0.0190\}$*,* $|R_{WS}| = diag\{0.09296, 0.0000, 0.0274\}$*,* $(S - s_nI) = diag\{0.0482, 0.0190, 0.0000\}$*, and* $(R - r_nI) = diag\{0.1204, 0.0274, 0.0000\}$*. Equations (2.34) and (2.35) are used to construct* B_{IG} *and* C_{IG}*. The Hankel singular values (HSV) obtained by [26] and proposed techniques are* $\{0.2082, 0.204, 0.0129\}$ *and* $\{0.2443, 0.0323, 0.0227\}$ *respectively. The norm of* L, K, L_{IG} *and* K_{IG} *are* 2.2859*,* 4.5564*,* $2.2529, 3.5450$ *respectively.*

Example 2.3.2 *Let* $G(s) = (4s^2+6s+2)/(s^3+5s^2+4s+2)$*, with weights* $V_i(s) = 1/(s+1)$ *and* $W_o(s) = 1/(s + 4)$*, respectively. HSV obtained by [26] and proposed techniques are* $\{0.1474, 0.1176, 0.0136\}$ *and* $\{0.1389, 0.1312, 0.0153\}$ *respectively.*

Remark 2.3.4 *For the case when symmetric matrices* $X_E \ge 0$ *and* $Y_E \ge 0$*, then* $P_E = P_{WS} = P_{IG}$ *and* $Q_E = Q_{WS} = Q_{IG}$*. Consequently, the ROMs obtained using Enns [5], Wang et al's [26] and proposed technique are the same. Otherwise* $P_E < P_{IG}$*,* $Q_E < Q_{IG}$

and $(\lambda_j[P_E Q_E])^{1/2} \leq (\lambda_j[P_{IG}Q_{IG}])^{1/2}$. *However (as can be seen from above examples), a direct relationship among Gramians and HSV of proposed and Wang et al's [26] techniques cannot be given (when* $X_E \ngeq 0$ *and* $Y_E \ngeq 0$*).*

Remark 2.3.5 *For the case when input* $V_i(s)$, $V_i(z)$ *weights are co-inner and output* $W_o(s)$, $W_o(z)$ *weights are inner [21], then* $P = P_E = P_{WS} = P_{IG}$ *and* $Q = Q_E = Q_{WS} = Q_{IG}$.

2.3.1 Computational aspects

The balancing procedure involves computation of transformation matrix using controllability and observability Gramians. Sometimes these matrices become numerically low rank especially in large scale systems (possibly) due to rapid decay of their eigenvalues [27]. Due to this reason, the balancing procedure becomes inefficient. Accuracy enhancing techniques for different FWMR techniques appear in [6, 25].

For the unweighted case, Hammarling [9] technique is used to obtain Cholesky factors of Gramian matrices from original system realization without actually computing controllability and observability Gramian matrices respectively.

In the FWMR techniques, the Cholesky factors of the Gramian matrices are obtained from the augmented system realizations. Let $\bar{S}$ and $\bar{R}$ be the Cholesky factors of the augmented system Gramians matrices P_i and Q_o of equations (2.7) and (2.8) respectively,

$$P_i = \bar{S}\bar{S}^T = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} S_{11}^T & 0 \\ S_{12}^T & S_{22}^T \end{bmatrix}$$
$$= \begin{bmatrix} S_{11}S_{11}^T + S_{12}S_{12}^T & S_{12}S_{22}^T \\ S_{22}S_{12}^T & S_{22}S_{22}^T \end{bmatrix} = \begin{bmatrix} P_E & P_{12} \\ P_{12}^T & P_V \end{bmatrix}$$

and

$$Q_o = \bar{R}^T\bar{R} = \begin{bmatrix} R_{11}^T & 0 \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$
$$= \begin{bmatrix} R_{11}^T R_{11} & R_{11}^T R_{12} \\ R_{12}^T R_{11} & R_{22}^T R_{22} + R_{12}^T R_{12} \end{bmatrix} = \begin{bmatrix} Q_W & Q_{12}^T \\ Q_{12} & Q_E \end{bmatrix}$$

By making use of the Cholesky factors $\bar{S}$ and $\bar{R}$ calculated above, the Cholesky factors corresponding to Gramians in FWMR techniques namely [5, 26] and proposed technique can be obtained as follows:

1. **Enns technique [5]:** The Cholesky factors
$S_E = [S_{11} \;\; S_{12}]$ and $R_E = \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}$ satisfy [25]

$$P_E = S_E S_E^T = S_{11}S_{11}^T + S_{12}S_{12}^T = \begin{bmatrix} S_{11} & S_{12} \end{bmatrix} \begin{bmatrix} S_{11}^T \\ S_{22}^T \end{bmatrix}$$

$$Q_E = R_E^T R_E = R_{22}^T R_{22} + R_{12}^T R_{12} = \begin{bmatrix} R_{22}^T & R_{12}^T \end{bmatrix} \begin{bmatrix} R_{22} \\ R_{12} \end{bmatrix}$$

2. **Wang et al's technique [26]:** The Cholesky factors $\bar{S}_{WS}$ and $\bar{R}_{WS}$ satisfy $P_{WS} = \bar{S}_{WS}\bar{S}_{WS}^T$ and $Q_{WS} = \bar{R}_{WS}^T \bar{R}_{WS}$, where P_{WS} (2.20) and Q_{WS} (2.21) [6].

3. **Proposed technique:** The Cholesky factors $\bar{S}_{IG}$ and $\bar{R}_{IG}$ satisfy $P_{IG} = \bar{S}_{IG}\bar{S}_{IG}^T$ and $Q_{IG} = \bar{R}_{IG}^T \bar{R}_{IG}$, where P_{IG} (2.30) and Q_{IG} (2.31).
Next we establish a relationship between Cholesky factors and Gramian matrices of Enns and proposed technique. Equation (2.30) and (2.31) can be expressed as:

$$\begin{aligned}
&A(P_E + P_{ad}) + (P_E + P_{ad})A^T + (X_E - s_n I) = 0, && \text{for } s_n < 0 \\
&AP_E + P_E A^T + X_E = 0, && \text{for } s_n \geq 0 \\
&A^T(Q_E + Q_{ad}) + (Q_E + Q_{ad})A + (Y_E - r_n I) = 0, && \text{for } r_n < 0 \\
&A^T Q_E + Q_E A + Y_E = 0, && \text{for } r_n \geq 0 \\
&AP_{ad} + P_{ad}A^T - s_n I = 0, && \text{for } s_n < 0 \\
&A^T Q_{ad} + Q_{ad}A - r_n I = 0, && \text{for } r_n < 0
\end{aligned}$$

Similarly, for discrete time case equations (2.32) and (2.33) can be expressed as:

$$\begin{aligned}
&A(P_E + P_{ad})A^T - (P_E + P_{ad}) + (X_E - s_n I) = 0, && \text{for } s_n < 0 \\
&AP_E A^T - P_E + X_E = 0, && \text{for } s_n \geq 0 \\
&A^T(Q_E + Q_{ad})A - (Q_E + Q_{ad}) + (Y_E - r_n I) = 0, && \text{for } r_n < 0 \\
&A^T Q_E A - Q_E + Y_E = 0, && \text{for } r_n \geq 0 \\
&AP_{ad}A^T - P_{ad} - s_n I = 0, && \text{for } s_n < 0 \\
&A^T Q_{ad} A - Q_{ad} - r_n I = 0, && \text{for } r_n < 0
\end{aligned}$$

Since

$$X_{IG} = U(S - s_n I)^{1/2}(S - s_n I)^{1/2}U^T = X_E - s_n I, \qquad \text{for } s_n < 0$$
$$X_{IG} = U(S)^{1/2}(S)^{1/2}U^T = X_E, \qquad \text{for } s_n \geq 0$$
$$Y_{IG} = V^T(R - r_n I)^{1/2}(R - r_n I)^{1/2}V = Y_E - r_n I, \qquad \text{for } r_n < 0$$
$$Y_{IG} = V^T(R)^{1/2}(R)^{1/2}V = Y_E, \qquad \text{for } r_n \geq 0$$

By using Hammarling technique to calculate the Cholesky factors of the Gramians P_{ad} and Q_{ad} from the realization $\{A, \sqrt{-s_n}I, \sqrt{-r_n}I, D\}$, we can write $P_{ad} = \bar{S}_{ad}\bar{S}_{ad}^T$ and $Q_{ad} = \bar{R}_{ad}^T\bar{R}_{ad}$. Therefore, P_{IG} (2.30) and Q_{IG} (2.31) can be expressed as:

$$P_{IG} = \bar{S}_{IG}\bar{S}_{IG}^T = P_E + P_{ad} = S_{11}S_{11}^T + S_{12}S_{12}^T + S_{ad}S_{ad}^T = \begin{bmatrix} S_{11} & S_{12} & S_{ad} \end{bmatrix} \begin{bmatrix} S_{11}^T \\ S_{12}^T \\ S_{ad}^T \end{bmatrix}$$

$$Q_{IG} = \bar{R}_{IG}^T\bar{R}_{IG} = Q_E + Q_{ad} = R_{22}^T R_{22} + R_{12}^T R_{12} + R_{ad}^T R_{ad} = \begin{bmatrix} R_{22}^T & R_{12}^T & R_{ad}^T \end{bmatrix} \begin{bmatrix} R_{22} \\ R_{12} \\ R_{ad} \end{bmatrix}$$

Remark 2.3.6 *Note that, the Cholesky factors for the Enns and the proposed technique respectively, are computed directly from the augmented system realization using Hammarling technique without calculating the augmented system realization Gramian matrices P_i and Q_o. However, Cholesky factorization for Wang et al's [26] technique is computed directly from the corresponding frequency weighted realization using Hammarling technique without calculating the associated frequency weighted Gramians [8].*

2.4 Numerical examples

Here numerical results of both continuous and discrete time linear state space systems are presented.

2.4.1 Continuous time case

Example 2.4.1 *Consider the fourth order system used in [6, 19, 23, 25, 26]*

Table 2.1: Frequency weighted errors and error bounds comparison for ROMs

Weighting	Order	Enns [5] Technique	Wang et al's [26] Technique		Proposed Technique	
		Error	Error	Error Bound	Error	Error Bound
Input	1	1.1310	1.1270	2.4488	1.1270	1.7861
	2	0.1342	0.1367	0.4573	0.1240	0.4502
	3	0.0654	0.0658	0.1155	0.0678	0.0900
Output	1	1.1244	1.1182	2.0463	1.1193	1.9866
	2	0.1553	0.1552	0.3616	0.1552	0.3540
	3	0.0593	0.0593	0.0921	0.0592	0.0901
Two Sided	1	2.1291	2.1213	7.2898	2.1234	4.9323
	2	0.2260	0.2720	1.4895	0.2424	1.2789
	3	0.1131	0.1151	0.3228	0.1075	0.2446

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 \\ 1/2 & -3/2 \\ 1 & -5 \\ -1/2 & 1/6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4/15 & 1 & 0 & 1 \end{bmatrix}$$

with following 2^{nd} order input and output weightings

$$A_i = A_o = 4.5I_2, B_i = B_o = 3I_2, C_i = C_o = 1.5I_2, D_i = D_o = I_2$$

respectively. Table 2.1 gives the corresponding results.

Example 2.4.2 *Consider a stable 6^{th} order system (Example 3.1 of [6])*

$$A = \begin{bmatrix} -13 & 1 & 0 & 0 & 0 & 0 \\ -75 & 0 & 1 & 0 & 0 & 0 \\ -210 & 0 & 0 & 1 & 0 & 0 \\ -320 & 0 & 0 & 0 & 1 & 0 \\ -600 & 0 & 0 & 0 & 0 & 1 \\ -80 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0.6 \\ 2 & 0.9 \\ 5 & 0.3 \\ 1 & 0.6 \\ 0.5 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Table 2.2: Frequency weighted errors and error bounds comparison for ROMs

Weighting	Order	Enns [5] Technique	Wang et al's [26] Technique		Proposed Technique	
		Error	Error	Error Bound	Error	Error Bound
Input	1	139.6132	136.5005	361.3248	134.4076	411.4761
	2	18.3129	18.3316	92.4481	18.7287	95.7778
	3	21.1996	21.2479	44.8325	20.2999	48.7198
	4	9.1666	9.1205	19.8312	8.9602	21.7747
	5	2.6430	2.4827	5.5316	2.3935	6.6900
Output	1	131.4251	108.5365	356.2655	108.2467	171.4624
	2	16.5652	15.4991	86.6817	15.6764	40.3546
	3	16.6552	14.7750	40.9833	15.2463	20.8452
	4	8.0065	8.2056	17.7719	7.2529	9.8677
	5	2.4785	4.4455	5.7643	1.8129	2.6186
Two Sided	1	131.5077	99.4405	667.2325	99.2950	365.7043
	2	16.3252	15.2951	168.6104	15.7326	80.9650
	3	16.5315	14.6251	78.0990	14.5335	42.8066
	4	7.7922	7.8043	32.8717	7.0779	20.3537
	5	2.1083	3.5243	10.1790	2.4644	5.6365

with following 2^{nd} *order input and output weights*

$$A_i = \begin{bmatrix} -10 & 0 \\ 0 & -4.5 \end{bmatrix}, B_i = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, C_i = \begin{bmatrix} -2.7333 & 0 \\ 0 & -3 \end{bmatrix},$$
$$A_o = \begin{bmatrix} -4.1 & 0 \\ 0 & -4.5 \end{bmatrix}, B_o = \begin{bmatrix} -5.4667 & 0 \\ 0 & -6 \end{bmatrix}, C_o = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix},$$
$$D_o = D_i = I_2$$

respectively. Table 2.2 gives the corresponding results.

Table (2.1 and 2.2) shows the frequency weighted errors obtained by Enns [5], Wang et al's [26] and the proposed technique for input, output and double sided weighting cases. It is observed that the proposed technique compares well (mostly produces lower error and tighter error bounds) with existing FWMR techniques. Moreover, Enns technique may yield unstable model and has no *a priori* error bound.

2.4.2 Discrete time case

Example 2.4.3 *Consider (example C appeared in [23]) a* 4^{th} *order stable discrete time system*

$$G(z) = \frac{z^3}{z^4 + 1.1z^3 - 0.01z^2 - 0.275z - 0.06}$$

with the following weightings

$$V_i(z) = W_o(z) = \frac{z + 0.9}{z + 0.1}$$

The first order ROM obtained by Enns [5] technique is unstable while ROM obtained by Varga and Anderson's, Campbell et al's and proposed techniques are stable yielding frequency response errors $112.9338, 100.8739$ and 94.116 respectively. Note that, proposed technique provides stability with relatively lower error when compared to other techniques.

Example 2.4.4 *Consider a* 6^{th} *order stable low pass digital elliptic filter with* 0.2 *dB of peak-to-peak ripple and a minimum stopband attenuation of* 20 *dB represented by*

$$G(z) = \frac{0.1054z^6 - 0.1944z^5 + 0.1187z^4 - 0.1187z^2 + 0.1944z - 0.1054}{z^6 - 2.9621z^5 + 4.8325z^4 - 4.9819z^3 + 3.5245z^2 - 1.5262z + 0.3657}$$

with the following input and output weightings respectively,

$$V_i(z) = \frac{z^3 + 3.0081z^2 + 1.9944z + 1.0325}{z^3 + 0.2z^2 + 0.75z + 0.2}$$

$$W_o(z) = \frac{z^3 + 2.97z^2 + 2.9403z + 0.9703}{z^3 + 1.1619z^2 + 0.6959z + 0.1378}$$

Table 2.3 gives the comparison of error and error bounds for ROMs obtained by Enns, Varga and Anderson's, Campbell et al's and proposed techniques for the input and two sided weighting cases. Note that, the proposed technique mostly yields lower error as compared to other techniques.

Example 2.4.5 *Consider a* 4^{th} *order stable discrete time system [87]*

$$G(z) = \frac{10^{-3}(3.315z^3 - 4.9695z^2 + 2.1668z - 0.24002)}{z^4 - 3.7035z^3 + 5.1957z^2 - 3.2718z + 0.77986}$$

Table 2.3: Frequency weighted errors and error bounds comparison for ROMs

Weighting	Order	Enns Technique [5]	Campbell et al's Technique [85]		Varga and Anderson's Technique [86]		Proposed Technique	
			Error	Error Bound	Error	Error Bound	Error	Error Bound
Two Sided	1	21.1254	20.5953	1634.2	21.6491	725.7718	15.6065	2549.0
	2	31.9647	32.8319	978.34	32.8863	433.8123	18.4571	1623.9
	3	35.0441	32.3860	590.48	33.9063	255.7017	26.1274	998.92
	4	28.7611	31.4710	236.41	30.4331	102.7518	30.7929	427.13
	5	12.7538	29.5760	117.81	12.7660	50.4647	25.6547	203.36
Input	1	7.0257	7.1275	145.811	7.2356	92.6748	7.0140	242.5983
	2	10.4643	10.7354	87.9789	10.7694	55.8603	10.6714	149.7017
	3	11.2055	10.3852	53.0816	10.8477	32.9346	9.9857	90.8656
	4	8.9654	10.0342	21.8067	9.6079	13.5877	8.4277	41.1182
	5	2.4435	3.1720	10.4718	2.8446	6.3761	3.2061	16.5445

Table 2.4: Frequency weighted errors and error bounds comparison for ROMs

Weighting	Order	Enns Technique [5]	Campbell et al's Technique [85]		Varga and Anderson's Technique [86]		Proposed Technique	
			Error	Error Bound	Error	Error Bound	Error	Error Bound
Input	1	0.0216	0.0245	0.6024	0.0241	0.3321	0.0240	0.6783
	2	0.0021	0.0027	0.2582	0.0026	0.1403	0.0025	0.2853
	3	0.0015	0.0025	0.0453	0.0023	0.0247	0.0021	0.0508

with the following input weighting

$$V_i(z) = \frac{z^2 - 0.1z - 0.05}{z^2 - 0.9z + 0.75}$$

Table 2.4 gives the comparison of error and error bounds for ROMs obtained by Enns, Varga and Anderson's, Campbell et al's and proposed techniques for the input weighting case. Note that, the proposed technique compares well and yields relatively lower error as compared to other techniques.

Example 2.4.6 *Consider a 4^{th} order stable low pass digital Chebyshev type 1 filter with*

Table 2.5: Frequency weighted errors and error bounds comparison for ROMs

Weighting	Order	Enns Technique [5]	Campbell et al's Technique [85]		Varga and Anderson's Technique [86]		Proposed Technique	
			Error	Error Bound	Error	Error Bound	Error	Error Bound
Output	1	1.7905	1.7898	8.2727	1.7900	5.6369	1.7867	6.2509
	2	0.8967	0.8126	2.9832	0.8520	1.9243	0.7296	2.5083
	3	0.5098	0.4868	1.4488	0.4979	0.9219	0.4740	0.9543

$0.1dB$ *of peak-to-peak ripples in the passband represented by:*

$$G(z) = \frac{0.49z^4 - 0.9799z^2 + 0.49}{z^4 - 0.2893z^3 - 0.6629z^2 + 0.0246z + 0.2904}$$

with the following output weighting

$$W_o(z) = \frac{z - 0.2}{z^2 - 0.4z + 0.5}$$

Table 2.5 gives the comparison of error and error bounds for ROMs obtained by Enns, Varga and Anderson's, Campbell et al's and proposed techniques for output weighting. Note that, the proposed technique compares well and yields relatively lower error as compared to other techniques.

2.5 Conclusion

In this chapter, a new FWMR technique for continuous and discrete time systems is proposed based on pursuing similar effect on the eigenvalues of weighted augmented input and output realization matrices. Unlike the Enns, the proposed technique yields not only stable ROMs for the case of double-sided weighting but also gives easily computable *a priori* frequency response error bounds. Computational aspects of existing and proposed FWMR techniques are also given. The numerical examples show that the proposed technique compares well with other well-known techniques generally yielding lower error and tighter error bounds for ROMs of different order.

The next chapter considers FLMR problem.

Chapter 3

FLMR: A New Technique

3.1 Introduction

Previous chapter considers FWMR, here we consider FLMR for continuous and discrete time systems.

Gawronski and Juang [29] proposed a FLMR technique, where weights are not explicitly predefined, but approximation is considered in certain frequency intervals for linear continuous and discrete time systems. The Gramians are defined for a desired frequency intervals. However, it can also yield unstable ROMs for stable original system and there are no error bounds. To overcome instability problem of Gawronski and Juang, some FLMR techniques have been proposed in literature including Wang and Zilouchian [90], Gugercin and Antoulas [30], Ghafoor and Sreeram [8].

Gawronski and Juang technique [29] was modified by Gugercin and Antoulas [30] to provide stable models and error bounds. This technique ensures stability by taking absolute values of eigenvalues. This causes a large change in some of the eigenvalues and little effect on rest of eigenvalues. Ghafoor and Sreeram [8] technique tends to minimize the approximation error and yields stable ROMs and has error bound expression also. This technique ensures stability by retaining only positive eigenvalues and truncating negative eigenvalues. Like Gugercin and Antoulas [30], this technique does not have a similar effect on all eigenvalues.

In this chapter (partially published in [11, 16]), a FLMR technique is proposed which provides stable ROMs by pursuing similar effect on all eigenvalues for both continuous and discrete time systems. The proposed technique provides comparable frequency response error and yields easily computable *a priori* error bounds. Numerical examples are given to show the usefulness and comparison of proposed technique with the existing FLMR techniques.

3.2 Preliminaries

Consider a linear time invariant continuous and discrete time systems respectively

$$G(s) = C(sI - A)^{-1}B + D \tag{3.1}$$

$$G(z) = C(zI - A)^{-1}B + D \tag{3.2}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$ and $\{A, B, C, D\}$ is its n^{th} order minimal realization with m inputs and p outputs. The problem of MOR is to to find a ROM with

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D \tag{3.3}$$

$$G_r(z) = C_1(zI - A_{11})^{-1}B_1 + D \tag{3.4}$$

which approximates the original system (in the desired frequency band $[\omega_1, \omega_2]$ where $\omega_2 > \omega_1$), where $A_{11} \in R^{r\times r}$, $B_1 \in R^{r\times m}$, $C_1 \in R^{p\times r}$, $D \in R^{p\times m}$ with $r < n$.

Let P and Q be the frequency domain controllability and observability Gramians

$$\text{Continuous time}\begin{cases} P = \frac{1}{2\pi}\int_{-\infty}^{\infty}(j\omega I - A)^{-1}BB^T(-j\omega I - A^T)^{-1}d\omega \\ Q = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\omega I - A^T)^{-1}C^TC(j\omega I - A)^{-1}d\omega \end{cases}$$

$$\text{Discrete time}\begin{cases} P = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\omega} I - A)^{-1}BB^T(e^{-j\omega} I - A^T)^{-1}d\omega \\ Q = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\omega} I - A^T)^{-1}C^TC(e^{j\omega} I - A)^{-1}d\omega \end{cases}$$

are the solution of following Lyapunov equations:

$$\text{Continuous time}\begin{cases} AP + PA^T + BB^T = 0 \\ A^TQ + QA + C^TC = 0 \end{cases}$$

$$\text{Discrete time}\begin{cases} APA^T - P + BB^T = 0 \\ A^TQA - Q + C^TC = 0 \end{cases}$$

3.2.1 Gawronski and Juang's technique

The discrete time version of Gawronski and Juang's technique [29] also appears in Wang and Zilouchian [90]. The controllability and observability Gramians for limited frequency domain are defined as

$$\text{Continuous time} \begin{cases} AP_{GJ} + P_{GJ}A^T + X_{GJ} = 0 \\ A^T Q_{GJ} + Q_{GJ}A + Y_{GJ} = 0 \end{cases} \tag{3.5}$$

$$\text{Discrete time} \begin{cases} AP_{GJ}A^T - P_{GJ} + X_{GJ} = 0 \\ A^T Q_{GJ}A - Q_{GJ} + Y_{GJ} = 0 \end{cases} \tag{3.6}$$

where

$$\text{Continuous time} \begin{cases} X_{GJ} = \left(S(\omega_2) - S(\omega_1)\right) BB^T + BB^T \left(S^H(\omega_2) - S^H(\omega_1)\right) \\ Y_{GJ} = \left(S^H(\omega_2) - S^H(\omega_1)\right) C^T C + C^T C \left(S(\omega_2) - S(\omega_1)\right) \\ S(\omega) = \frac{j}{2\pi} \ln\left((j\omega I + A)(-j\omega I + A)^{-1}\right) \end{cases}$$

$$\text{Discrete time} \begin{cases} X_{GJ} = BB^T F^H + FBB^T \\ Y_{GJ} = C^T C F + F^H C^T C \\ F = -\frac{\omega_2 - \omega_1}{4\pi} I + \frac{1}{2\pi} \int_{\delta\omega} (e^{j\omega} I - A)^{-1} d\omega \end{cases}$$

$S^H(\omega)$ and F^H are Hermitian of $S(\omega)$ and F respectively. Let

$$T^T Q_{GJ} T = T^{-1} P_{GJ} T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$ and T is a contragredient matrix used to transform the original system realization. ROMs are obtained by partitioning the transformed realization.

Remark 3.2.1 *One can consider the multiple frequency intervals for approximation. For example, for two intervals* $[\omega_1, \omega_2]$ *and* $[\omega_3, \omega_4]$, $\omega_1 < \omega_2$, $\omega_3 < \omega_4$.

Remark 3.2.2 *Since the symmetric matrices* X_{GJ} *and* Y_{GJ} *may not be positive semidefinite, the ROMs obtained by Gawronski and Juang [29] and Wang and Zilouchian's [90] techniques may not be stable. This issue was solved by [30] and [87] and error bounds were also derived.*

3.2.2 Gugercin and Antoulas's technique

The instability problem of Gawronski and Juang [29] was solved by Gugercin and Antoulas [30]. The discrete time version of [30] was proposed by Ghafoor and Sreeram [87] Algorithm 1. Let the controllability P_{GA} and observability Q_{GA} Gramians respectively, obtained as the solutions to Lyapunov equations

$$AP_{GA} + P_{GA}A^T + B_{GA}B_{GA}^T = 0 \tag{3.7}$$

$$A^TQ_{GA} + Q_{GA}A + C_{GA}^TC_{GA} = 0 \tag{3.8}$$

Similarly, for discrete time systems above Lyapunov equations become

$$AP_{GA}A^T - P_{GA} + B_{GA}B_{GA}^T = 0 \tag{3.9}$$

$$A^TQ_{GA}A - Q_{GA} + C_{GA}^TC_{GA} = 0 \tag{3.10}$$

Equations (3.7) and (3.8) are used to find a contragredient matrix T as:

$$T^TQ_{GA}T = T^{-1}P_{GA}T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_i \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1,\ \sigma_r > \sigma_{r+1},\ B_{GA} = U_{GA}|S_{GA}|^{1/2},\ C_{GA} = |R_{GA}|^{1/2}V_{GA}^T$. The terms $U_{GA},\ S_{GA},\ V_{GA}$, and R_{GA} are obtained as $X_{GJ} = U_{GA}S_{GA}U_{GA}^T$ and $Y_{GJ} = V_{GA}R_{GA}V_{GA}^T$, where

$$S_{GA} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & s_n \end{bmatrix}, \quad R_{GA} = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}$$

$|s_1| \geq |s_2| \geq \cdots \geq |s_n| \geq 0$ and $|r_1| \geq |r_2| \geq \cdots \geq |r_n| \geq 0$. The ROMs are calculated by partitioning the transformed realization.

Remark 3.2.3 *Since $X_{GJ} \leq B_{GA}B_{GA}^T$, $Y_{GJ} \leq C_{GA}^TC_{GA}$ and the realization (A, B_{GA}, C_{GA}) is minimal and the stability of ROMs is also guaranteed. This technique also yields frequency* *response error bounds.*

3.2.3 Ghafoor and Sreeram's technique

Ghafoor and Sreeram [8] also addresses the instability problem of Gawronski and Juang [29] technique. The discrete time version of this technique appears in [87] Algorithm 2. Ghafoor and Sreeram [8] defined the controllability P_{GS} and observability Q_{GS} Gramians respectively, as the solutions to Lyapunov equations

$$AP_{GS} + P_{GS}A^T + B_{GS}B_{GS}^T = 0 \tag{3.11}$$

$$A^TQ_{GS} + Q_{GS}A + C_{GS}^TC_{GS} = 0 \tag{3.12}$$

Similarly, for discrete time systems above Lyapunov equations become

$$AP_{GS}A^T - P_{GS} + B_{GS}B_{GS}^T = 0 \tag{3.13}$$

$$A^TQ_{GS}A - Q_{GS} + C_{GS}^TC_{GS} = 0 \tag{3.14}$$

The equations (3.11) and (3.12)to find a contragredient matrix T as:

$$T^TQ_{GS}T = T^{-1}P_{GS}T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_i \geq \sigma_{i+1},\ i = 1, 2, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$, $B_{GS} = U_{GS_1}S_{GS_1}^{1/2}$, $C_{GS} = R_{GS_1}^{1/2}V_{GS_1}^T$. The terms U_{GS_1}, S_{GS_1}, V_{GS_1}, and R_{GS_1} are obtained as

$$X_{GJ} = \begin{bmatrix} U_{GS_1} & U_{GS_2} \end{bmatrix} \begin{bmatrix} S_{GS_1} & 0 \\ 0 & S_{GS_2} \end{bmatrix} \begin{bmatrix} U_{GS_1}^T \\ U_{GS_2}^T \end{bmatrix}$$

$$Y_{GJ} = \begin{bmatrix} V_{GS_1} & V_{GS_2} \end{bmatrix} \begin{bmatrix} R_{GS_1} & 0 \\ 0 & R_{GS_2} \end{bmatrix} \begin{bmatrix} V_{GS_1}^T \\ V_{GS_2}^T \end{bmatrix}$$

where $\begin{bmatrix} S_{GS_1} & 0 \\ 0 & S_{GS_2} \end{bmatrix} = diag(s_1, s_2, \cdots, s_n)$, $\begin{bmatrix} R_{GS_1} & 0 \\ 0 & R_{GS_2} \end{bmatrix} = diag(r_1, r_2, \cdots, r_n)$, $s_1 \geq s_2 \geq \cdots \geq s_n$, $r_1 \geq r_2 \geq \cdots \geq r_n$, $S_{GS_1} = diag(s_1, s_2, \cdots, s_l)$, $R_{GS_1} = diag(r_1, r_2, \cdots, r_l)$, $s_1 \geq s_2 \geq \cdots \geq s_l > 0$, $r_1 \geq r_2 \geq \cdots \geq r_l > 0$. The ROMs are calculated by partitioning the transformed realization.

Remark 3.2.4 *Since* $X_{GJ} \leq B_{GS}B_{GS}^T \leq B_{GA}B_{GA}^T$*,* $Y_{GJ} \leq C_{GS}^T C_{GS} \leq C_{GA}^T C_{GA}$ *and the realization* $\{A, B_{GS}, C_{GS}\}$ *is minimal, and the stability of the ROMs is also guaranteed. Moreover, an error bound (similar to [30]) also exists.*

3.3 Proposed technique

In Gugercin and Antoulas [30] technique (and its discrete time version [87]), the symmetric matrices X_{GJ} and Y_{GJ} are ensured positive /semipositive definite by taking the square root of absolute values of the eigenvalues obtained by EVD of symmetric matrices X_{GJ} and Y_{GJ}. This sometimes leads to a large change in some eigenvalues and may not effect other eigenvalues. On the other end, Ghafoor and Sreeram [8] ensures positive definiteness of the matrices X_{GJ} and Y_{GJ} by taking only positive eigenvalues and truncating negative eigenvalues. This technique also does not have similar effect on all eigenvalues.

In the following, a technique (partially published in [11, 16]) is proposed where effort is to have a similar effect on all eigenvalues of indefinite matrices X_{GJ} and Y_{GJ}. The ROMs obtained are guaranteed to be stable. Moreover, it yields frequency response error bound and improved frequency response error. Let new controllability P_{IG} and observability Q_{IG} Gramians, respectively, be calculated by solving the following Lyapunov equations:

$$AP_{IG} + P_{IG}A^T + B_{IG}B_{IG}^T = 0 \tag{3.15}$$

$$A^T Q_{IG} + Q_{IG}A + C_{IG}^T C_{IG} = 0 \tag{3.16}$$

Similarly, for discrete time case, Lyapunov equation becomes

$$AP_{IG}A^T - P_{IG} + B_{IG}B_{IG}^T = 0 \tag{3.17}$$

$$A^T Q_{IG}A - Q_{IG} + C_{IG}^T C_{IG} = 0 \tag{3.18}$$

The matrices B_{IG} and C_{IG} are the new fictitious input and output matrices respectively defined as:

$$B_{IG} = \begin{cases} U_{IG}(S_{IG} - s_n I)^{1/2} & \text{for } s_n < 0 \\ U_{IG} S_{IG}^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{3.19}$$

$$C_{IG} = \begin{cases} (R_{IG} - r_n I)^{1/2} V_{IG}^T & \text{for } r_n < 0 \\ R_{IG}^{1/2} V_{IG}^T & \text{for } r_n \geq 0. \end{cases} \tag{3.20}$$

The terms U_{IG}, S_{IG}, V_{IG}, and R_{IG} are calculated as $X_{GJ} = U_{IG} S_{IG} U_{IG}^T$ and $Y_{GJ} = V_{IG} R_{IG} V_{IG}^T$, where $S_{IG} = diag(s_1, s_2, \cdots, s_n)$, $R_{IG} = diag(r_1, r_2, \cdots, r_n)$, $s_1 \geq s_2 \geq \cdots \geq s_n$, and $r_1 \geq r_2 \geq \cdots \geq r_n$.

Consider a contragredient transformation matrix T (used to transform the original system) is obtained as

$$T^T Q_{IG} T = T^{-1} P_{IG} T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$. The ROMs are obtained by partitioning the transformed realization.

Remark 3.3.1 *Since $X_{GJ} \leq B_{IG} B_{IG}^T$, $Y_{GJ} \leq C_{IG}^T C_{IG}$, $B_{IG} B_{IG}^T \geq 0$, $C_{IG}^T C_{IG} \geq 0$, $P_{IG} > 0$ and $Q_{IG} > 0$. Therefore, the realization (A, B_{IG}, C_{IG}) is minimal. Moreover, the ROMs are guaranteed to be stable.*

Theorem 3.3.1 *The following error bound for the proposed technique hold if the rank conditions $rank\,[B_{IG}\ \ B] = rank\,[B_{IG}]$ and $rank \begin{bmatrix} C_{IG} \\ C \end{bmatrix} = rank\,[C_{IG}]$ (which follows from [8, 26, 30]) are satisfied*

$$\|G(s) - G_r(s)\|_\infty \leq 2 \|L_{IG}\| \|K_{IG}\| \sum_{j=r+1}^{n} \sigma_j$$

$$\|G(z) - G_r(z)\|_\infty \leq 2 \|L_{IG}\| \|K_{IG}\| \sum_{j=r+1}^{n} \sigma_j$$

where

$$L_{IG} = \begin{cases} CV_{IG}(R_{IG} - r_n I)^{-1/2} & \textit{for } r_n < 0 \\ CV_{IG}R_{IG}^{-1/2} & \textit{for } r_n \geq 0. \end{cases} \tag{3.21}$$

$$K_{IG} = \begin{cases} (S_{IG} - s_n I)^{-1/2}U_{IG}^T B & \textit{for } s_n < 0 \\ S_{IG}^{-1/2}U_{IG}^T B & \textit{for } s_n \geq 0 \end{cases} \tag{3.22}$$

Proof: Since $rank\,[B_{IG} \;\; B] = rank\,[B_{IG}]$ and $rank \begin{bmatrix} C_{IG} \\ C \end{bmatrix} = rank\,[C_{IG}]$, the relationships $B = B_{IG}K_{IG}$ and $C = L_{IG}C_{IG}$ hold. By partitioning $B_{IG} = \begin{bmatrix} B_{IG_1} \\ B_{IG_2} \end{bmatrix}$, $C_{IG} = \begin{bmatrix} C_{IG_1} & C_{IG_2} \end{bmatrix}$ and substituting $B_1 = B_{IG_1}K_{IG}$, $C_1 = L_{IG}C_{IG_1}$ respectively yields

$$\begin{aligned} &\|G(s) - G_r(s)\|_\infty = \|C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1\|_\infty \\ &= \|L_{IG}C_{IG}(sI - A)^{-1}B_{IG}K_{IG} - L_{IG}C_{IG_1}(sI - A_{11})^{-1}B_{IG_1}K_{IG}\|_\infty \\ &= \|L_{IG}(C_{IG}(sI - A)^{-1}B_{IG} - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1})K_{IG}\|_\infty \\ &\leq \|L_{IG}\|\|C_{IG}(sI - A)^{-1}B_{IG} - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1}\|_\infty\|K_{IG}\| \end{aligned}$$

If $\{A_{11}, B_{IG_1}, C_{IG_1}\}$ is ROM obtained by partitioning a balanced realization $\{A, B_{IG}, C_{IG}\}$, we have [5]

$$\|C_{IG}(sI - A)^{-1}B_{IG} - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1}\|_\infty \leq 2\sum_{j=r+1}^{n} \sigma_j.$$

Therefore,

$$\|G(s) - G_r(s)\|_\infty \leq 2\|L_{IG}\|\|K_{IG}\|\sum_{j=r+1}^{n} \sigma_j$$

The proof of discrete time version follows similarly.

Remark 3.3.2 *When $X_{GJ} \ngeq 0$ and $Y_{GJ} \ngeq 0$, then*

$$X_{IG} = B_{IG}B_{IG}^T = X_{GJ} - s_n I, \qquad Y_{IG} = C_{IG}^T C_{IG} = Y_{GJ} - r_n I$$

$$P_{IG} = P_{GJ} + P_{ad}, \qquad Q_{IG} = Q_{GJ} + Q_{ad},$$

where

$$\text{Continuous time}\begin{cases} A(P_{GJ}+P_{ad})+(P_{GJ}+P_{ad})A^T+(X_{GJ}-s_nI)=0, & \textit{for} \quad s_n<0\\ A^T(Q_{GJ}+Q_{ad})+(Q_{GJ}+Q_{ad})A+(Y_{GJ}-r_nI)=0, & \textit{for} \quad r_n<0\\ AP_{ad}+P_{ad}A^T-s_nI=0, & \textit{for} \quad s_n<0\\ A^TQ_{ad}+Q_{ad}A-r_nI=0, & \textit{for} \quad r_n<0 \end{cases}$$

$$\text{Discrete time}\begin{cases} A(P_{GJ}+P_{ad})A^T-(P_{GJ}+P_{ad})+(X_{GJ}-s_nI)=0, & \textit{for} \quad s_n<0\\ A^T(Q_{GJ}+Q_{ad})A-(Q_{GJ}+Q_{ad})+(Y_{GJ}-r_nI)=0, & \textit{for} \quad r_n<0\\ AP_{ad}A^T-P_{ad}-s_nI=0, & \textit{for} \quad s_n<0\\ A^TQ_{ad}A-Q_{ad}-r_nI=0, & \textit{for} \quad r_n<0 \end{cases}$$

Remark 3.3.3 *For the case when symmetric matrices $X_{GJ}\geq 0$ and $Y_{GJ}\geq 0$, then $P_{GJ}=P_{GA}=P_{GS}=P_{IG}$ and $Q_{GJ}=Q_{GA}=Q_{GS}=Q_{IG}$. Otherwise $P_{GJ}<P_{IG}$ and $Q_{GJ}<Q_{IG}$. Moreover, frequency limited HSV satisfies: $(\lambda_j[P_{GJ}Q_{GJ}])^{1/2}\leq(\lambda_j[P_{IG}Q_{IG}])^{1/2}$.*

3.4 Numerical examples

Numerical results of both continuous and discrete time standard systems are presented.

3.4.1 Continuous time case

Example 3.4.1 *Consider a linear time invariant stable 6^{th} order system with the following state space representations*

$$\begin{aligned} A &= \begin{bmatrix} -9 & -29 & -100 & -82 & -19 & -2\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\\ B &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T\\ C &= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}\\ D &= 0 \end{aligned}$$

The ROMs are obtained using Gawronski and Juang [29], Gugercin and Antoulas [30], Ghafoor and Sreeram [8] and proposed techniques for limited frequency intervals $[\omega_1,\omega_2]$. It is observed that 4^{th} order ROM obtained by Gawronski and Juang technique [29] is unstable with poles $-1.2229 \pm 3.4602i$, $0.1322 \pm 2.7913i$ respectively, while ROMs calculated by proposed and Gugercin and Antoulas [30], Ghafoor and Sreeram [8] techniques are stable for the frequency interval $[\omega_1, \omega_2] = [5, 8]$ rad/s.

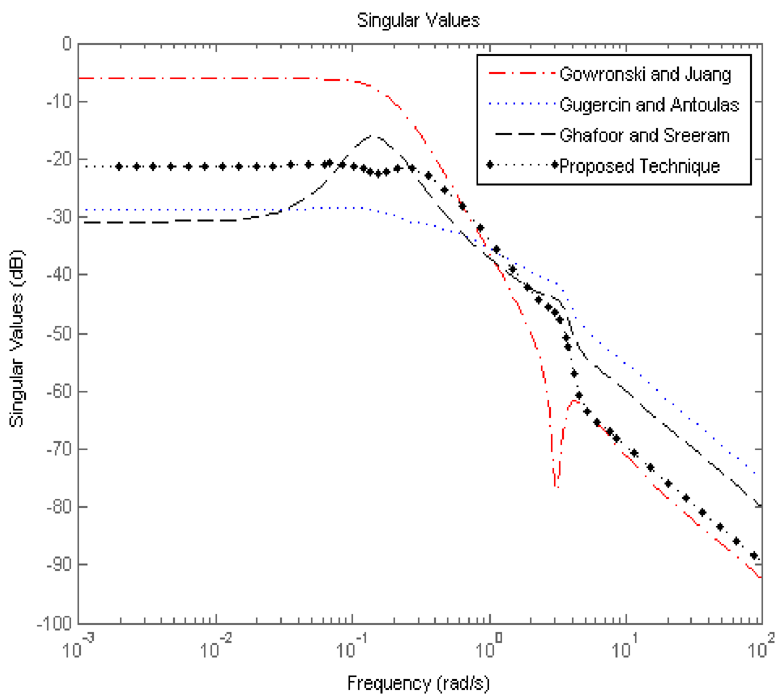

Figure 3.1: $\sigma\left[G(s) - G_r(s)\right]$ in the desired frequency range $[\omega_1, \omega_2] = [2, 7]$ rad/sec.

Figure 3.1 compares the singular value plots for the error function, $\sigma\left[G(s) - G_r(s)\right]$, where $G_r(s)$ are the 2^{nd} order ROMs obtained using Gawronski and Juang [29], Gugercin and Antoulas [30], Ghafoor and Sreeram [8] and proposed techniques for the frequency range $[\omega_1, \omega_2] = [2, 7]$ rad/sec. Figure 3.2 represents close up view of the error plot in the desired frequency range.

Example 3.4.2 *Consider the 6^{th} order stable three mass mechanical system shown in Fig. 3.3, also studied in [8, 29]. The masses are $m_1 = 11$, $m_2 = 5$ and $m_3 = 10$. The stiffnesses $k_1 = k_4 = 10$, $k_2 = 50$ $k_3 = 55$ and dampings $d_i = 0.01k_i$, $i = 1, 2, 3, 4$. The single input u is applied giving $f_1 = u$, $f_2 = 2u$, $f_3 = 5u$; the output is $y = 2q_1 - 2q_2 + 3q_3$, where q_i is displacement of the i^{th} mass, and f_i is the force applied to that mass. The state space representation of the system is*

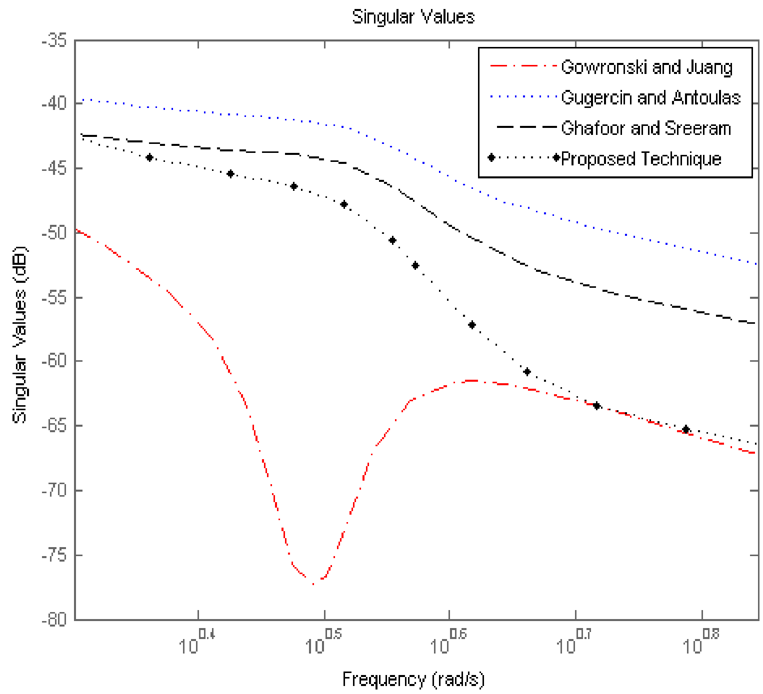

Figure 3.2: Close up view of $\sigma\left[G(s) - G_r(s)\right]$ in the desired frequency range $[\omega_1, \omega_2] = [2, 7]$ rad/sec.

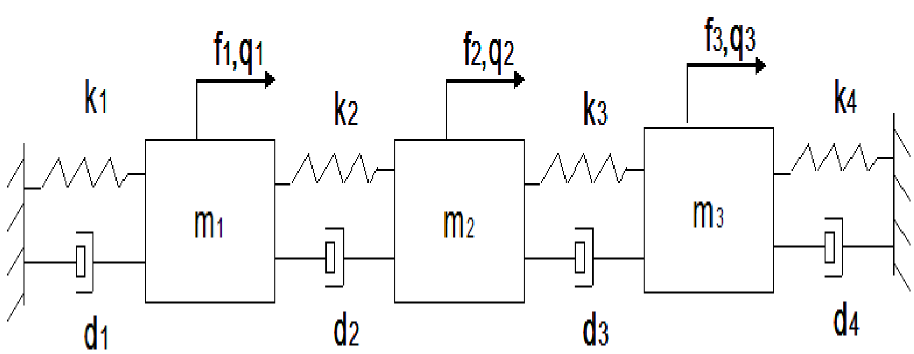

Figure 3.3: Simple three mass mechanical system

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -5.4545 & 4.5455 & 0 & -0.0545 & 0.0455 & 0 \\ 10 & -21 & 11 & 0.1000 & -0.2100 & 0.1100 \\ 0 & 5.5000 & -6.5000 & 0 & 0.0550 & -0.0650 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0.0909 & 0.4000 & -0.5000 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 2 & -2 & 3 & 0 & 0 & 0 \end{bmatrix}$$

$$D = 0$$

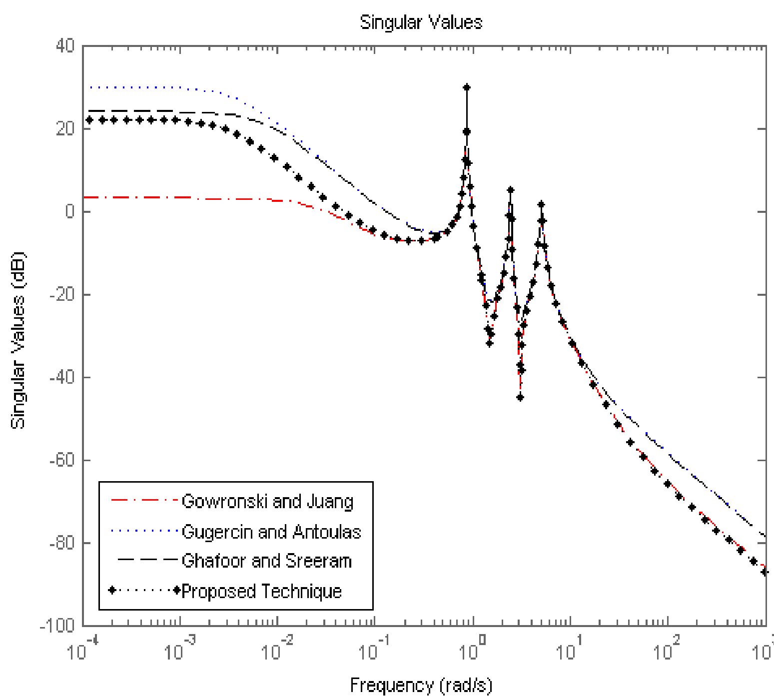

Figure 3.4: $\sigma\left[G(s) - G_r(s)\right]$ in the desired frequency range $[\omega_1, \omega_2] = [1, 5]$ rad/sec.

Figure 3.4 compares the $\sigma\left[G(s) - G_r(s)\right]$, where $G_r(s)$ are the 1^{st} order ROMs obtained using Gawronski and Juang [29], Gugercin and Antoulas [30], Ghafoor and Sreeram [8] and proposed techniques for the frequency range $[\omega_1, \omega_2] = [1, 5]$ rad/sec. Figure 3.5 represents close up view of the error plot in the desired frequency range.

Example 3.4.3 *Consider an* 8^{th} *order stable analog Chebyshev type 1 bandpass filter with* 20 *dB of peak-to-peak ripple in the passband with following state space representation*

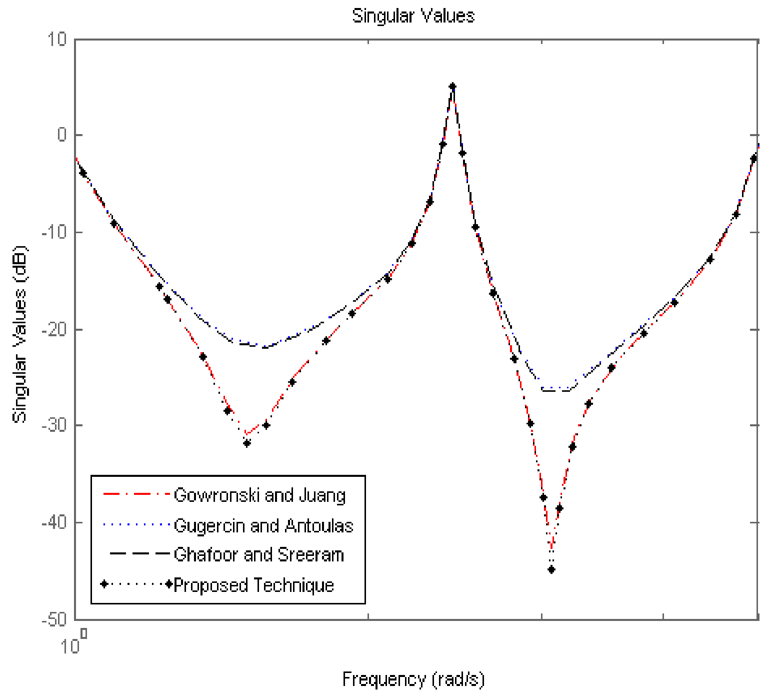

Figure 3.5: Close up view of $\sigma\left[G(s)-G_r(s)\right]$ in the desired frequency range $[\omega_1,\omega_2]=[1,5]$ rad/sec.

$$A = \begin{bmatrix} -0.6490 & -5.3691 & 0 & 0 & 3.8730 & 0 & 0 & 0 \\ 5.3691 & 0 & 0 & 0 & 0 & 3.8730 & 0 & 0 \\ 0 & 36.5054 & -0.2688 & -12.9391 & 0 & 0 & 3.8730 & 0 \\ 0 & 0 & 12.9391 & 0 & 0 & 0 & 0 & 3.8730 \\ -3.8730 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.8730 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.8730 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.8730 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0.0136 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = 0$$

Figure 3.6 compares the singular value plots for the error function, $\sigma\left[G(s)-G_r(s)\right]$, where $G_r(s)$ are the 4^{th} order ROMs obtained using Gawronski and Juang [29], Gugercin

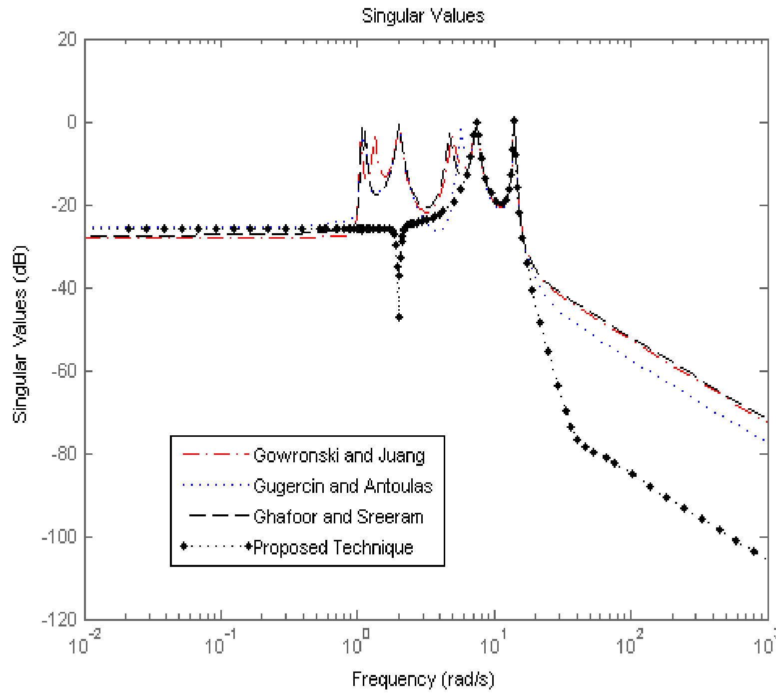

Figure 3.6: $\sigma\left[G(s) - G_r(s)\right]$ in the desired frequency range $[\omega_1, \omega_2] = [1, 10]$ rad/sec.

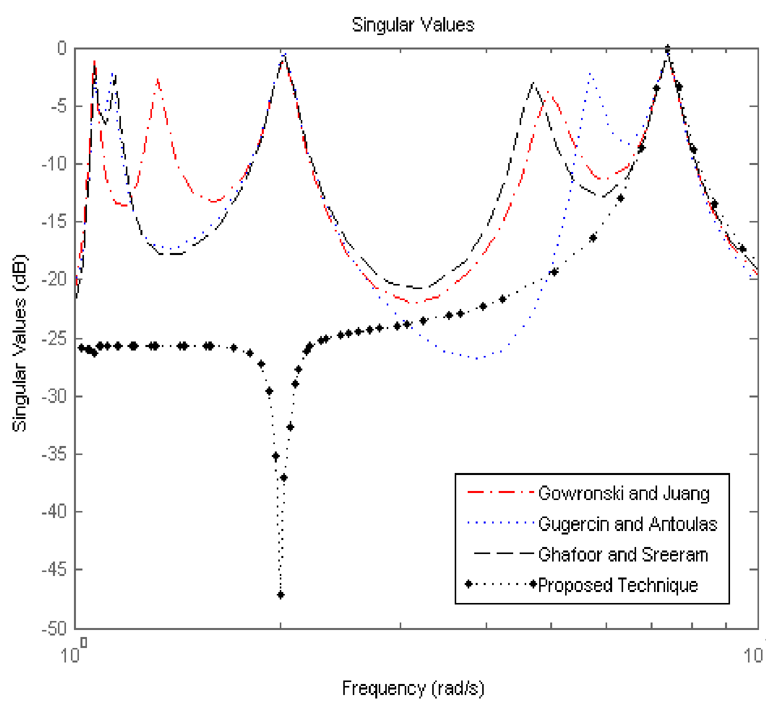

Figure 3.7: Close up view of $\sigma\left[G(s) - G_r(s)\right]$ in the desired frequency range $[\omega_1, \omega_2] = [1, 10]$ rad/sec.

and Antoulas [30], Ghafoor and Sreeram [8] and proposed techniques for the frequency range $[\omega_1, \omega_2] = [1, 10]$ rad/sec. Figure 3.7 represents close up view of the error plot in the desired frequency range.

Example 3.4.4 *Consider a* 6^{th} *order stable analog Chebyshev type 2 highpass filter with stopband ripple of* 11 *dB and stopband edge frequency of 11.3 Hz with following state space representation*

$$A = \begin{bmatrix} 0 & 7.2396 & 0 & 0 & 0 & 0 \\ -7.2396 & -10.9901 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13.4910 & 0 & 0 \\ -21.7779 & -53.5399 & -13.4910 & -8.0453 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 17.6523 \\ -34.6578 & -85.2046 & -3.8176 & -12.8035 & -17.6523 & -2.9448 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 7.2396 & 0 & 35.2689 & 0 & 56.1278 \end{bmatrix}^T$$

$$C = \begin{bmatrix} -0.6175 & -1.5180 & -0.0680 & -0.2281 & -0.0327 & -0.0525 \end{bmatrix}$$

$$D = 0$$

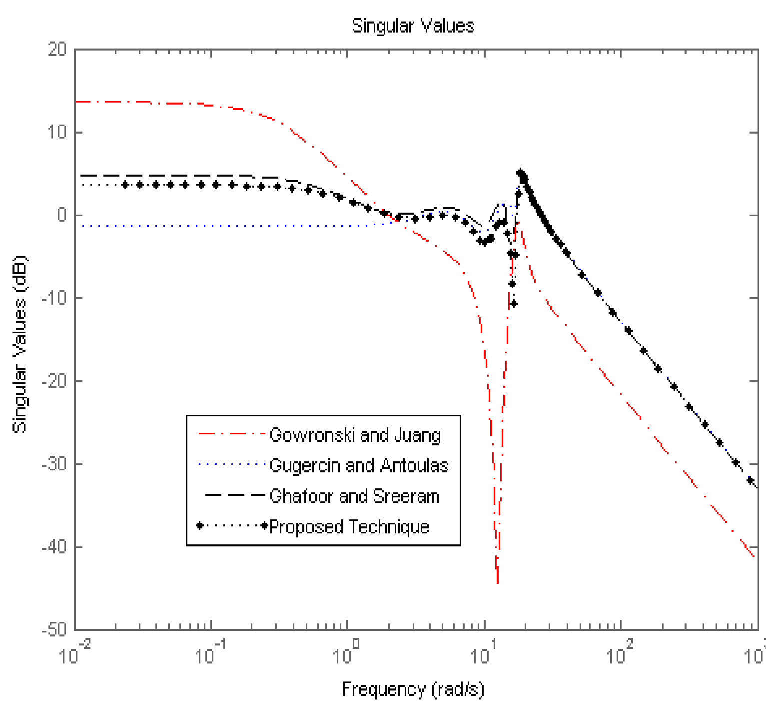

Figure 3.8: $\sigma\left[G(s) - G_r(s)\right]$ in the desired frequency range $[\omega_1, \omega_2] = [10, 15]$ rad/sec.

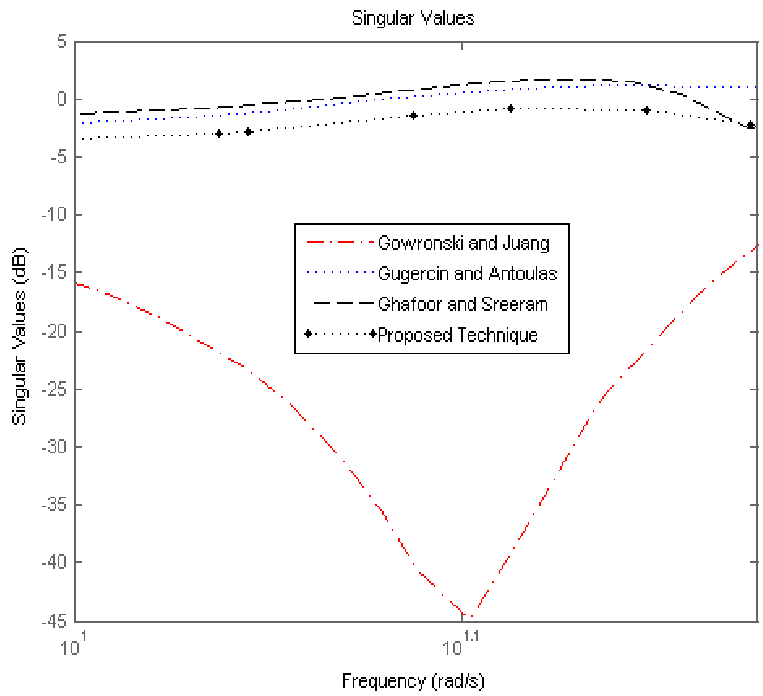

Figure 3.9: Close up view of $\sigma\left[G(s)-G_r(s)\right]$ in the desired frequency range $[\omega_1, \omega_2] = [10, 15]$ rad/sec.

Figure 3.8 compares the singular value plots for the error function, $\sigma\left[G(s)-G_r(s)\right]$, where $G_r(s)$ are the 3^{rd} order ROMs obtained using Gawronski and Juang [29], Gugercin and Antoulas [30], Ghafoor and Sreeram [8] and proposed techniques for the frequency range $[\omega_1, \omega_2] = [10, 15]$ rad/sec. Figure 3.9 represents close up view of the error plot in the desired frequency range.

Discussion: It is observed from examples (3.4.1 - 3.4.4) that in the desired frequency interval Gawronski and Juang technique mostly gives better approximation error, but it sometimes yields unstable ROMs as given in example 3.4.1. The proposed technique provides comparatively good approximation as compared to Gugercin and Antoulas and Ghafoor and Sreeram technqiues in the desired frequency interval. Gugercin and Antoulas and Ghafoor and Sreeram and proposed techniques have the advantage of providing stable ROMs and carry error bounds also.

3.4.2 Discrete time case

Example 3.4.5 *Consider a 4^{th} order stable discrete time system [23] represented by*

$$G(z) = \frac{z^3}{z^4 + 1.1z^3 - 0.01z^2 - 0.275z - 0.06}$$

Table 3.1: The ROMs in the frequency range 0.3π to 0.5π

Techniques	Order 1	Order 2
Wang and Zilouchian's [90]	$\frac{(0.9324)}{(z+1.5654)}$	$\frac{(1.2281z-1.2082)}{(z+1.7747)(z-1.1679)}$
Algorithm 1 [87]	$\frac{(1.2887)}{(z+0.8797)}$	$\frac{(0.9793z+0.2416)}{(z+0.7531)(z+0.6818)}$
Algorithm 2 [87]	$\frac{(1.1276)}{(z+0.9238)}$	$\frac{(0.9785z+0.2614)}{(z+0.7264+0.0502i)(z+0.7264-0.0502i)}$
Proposed	$\frac{(1.0677)}{(z+0.9315)}$	$\frac{(0.9820z+0.4569)}{(z+0.7962+0.1282i)(z+0.7962-0.1282i)}$

Table 3.1 shows the ROMs obtained by using Wang and Zilouchian's [90], Ghafoor and Sreeram's algorithms (1 and 2) [87] and proposed techniques in the frequency range 0.3π to 0.5π respectively. Note that, the ROMs obtained for the first and second order are unstable for Wang and Zilouchian's [90] technique, whereas the Ghafoor and Sreeram's algorithms (1 and 2) [87] and proposed technique yields stable ROMs. The third order ROM (not shown in table 3.1) obtained by these techniques is stable.

Example 3.4.6 *Consider a 6^{th} order stable discrete time system represented by*

$$G(z) = \frac{\begin{array}{r}0.011z^6 + 0.0635z^5 + 0.1653z^4 + 0.2169z^3 \\ +0.1565z^2 + 0.7124z + 0.0882\end{array}}{z^6 - 1.5z^5 + 2.3z^4 - 2.1z^3 + 1.5z^2 - 0.69z + 0.2}$$

Fig. 3.10 represents the frequency response errors, $\sigma[G(z) - G_r(z)]$, where $H_r(z)$ is the fourth order ROM produced by BT [1], Wang and Zilouchian's [90], Ghafoor and Sreeram's algorithms (1 and 2) [87] and proposed techniques respectively. The frequency interval for computing error response is 0.35π to 0.45π. Fig. 3.11 represents closeup view of frequency response error.

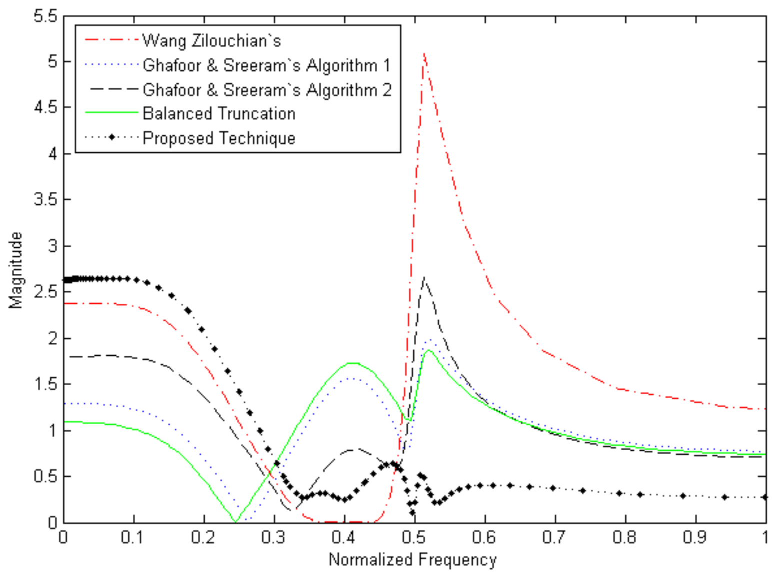

Figure 3.10: Frequency response error comparison of the ROMs

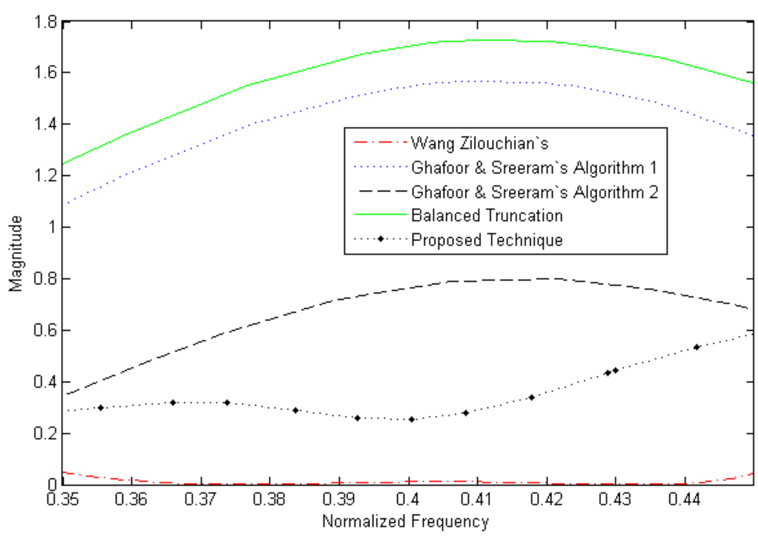

Figure 3.11: Frequency response error comparison - a closeup view

Example 3.4.7 *Consider a* 6^{th} *order stable discrete time system represented by*

$$G(z) = \frac{z^5 + 5z^4 + 7z^3 + 8z^2 + 10z + 1}{4z^6 + 3.5z^5 + 3z^4 + 2z^3 + 1.2z^2 + z + 0.4}$$

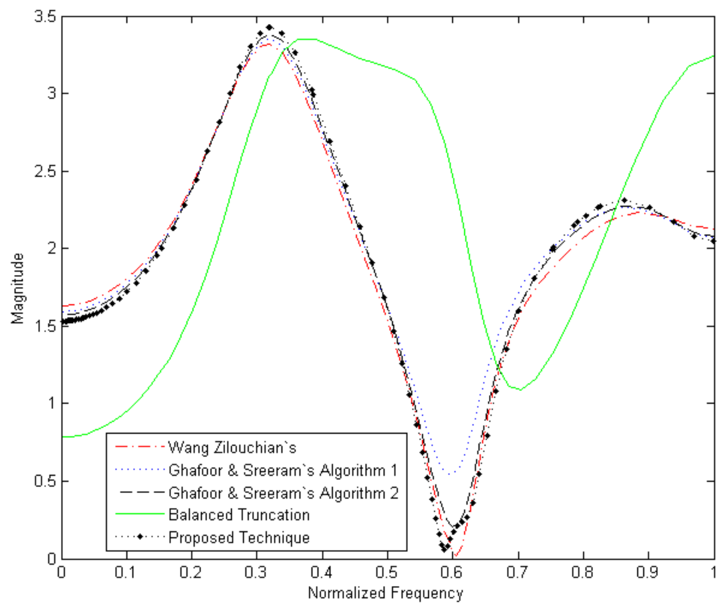

Figure 3.12: Frequency response error comparison of the ROMs

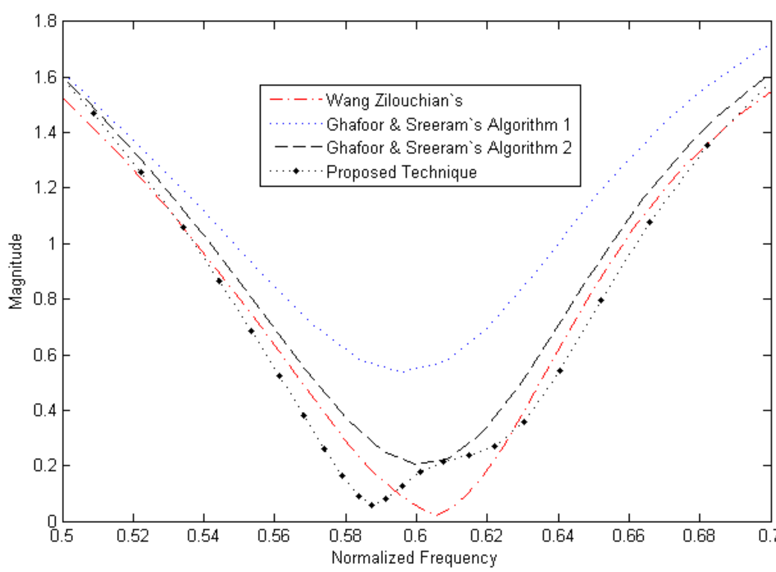

Figure 3.13: Frequency response error comparison - a closeup view

Fig. 3.12 represents the frequency response errors, $\sigma[G(z) - G_r(z)]$, where $H_r(z)$ is the second order ROM produced by BT [1], Wang and Zilouchian's [90], Ghafoor and Sreeram's algorithms (1 and 2) [87] and proposed techniques respectively. The frequency interval for computing error response is 0.5π to 0.7π. Fig. 3.13 represents closeup view of frequency response error, where the unweighted BT plot is not shown because of its large value.

Example 3.4.8 *Consider a* 6^{th} *order stable discrete time system represented by*

$$G(z) = \frac{\begin{array}{r} 0.0107z^6 + 0.0642z^5 + 0.1595z^4 + 0.2168z^3 \\ +0.1525z^2 + 0.0704z + 0.0009 \end{array}}{\begin{array}{r} z^6 - 1.4637z^5 + 2.2838z^4 - 2.0587z^3 \\ +1.4467z^2 - 0.6746 + 0.1825 \end{array}}$$

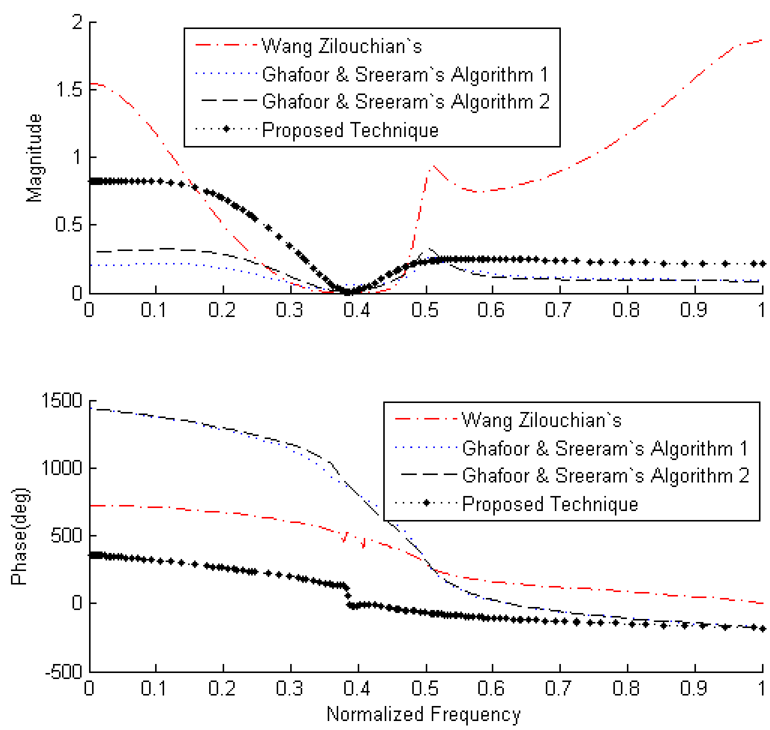

Figure 3.14: Frequency response with magnitude and phase error comparison of the ROMs

Fig. 3.14 represents the magnitude and phase of the frequency response errors, $\sigma[G(z) - G_r(z)]$, where $H_r(z)$ is the fourth order ROM produced by using Wang and Zilouchian's [90], Ghafoor and Sreeram's algorithms (1 and 2) [87] and proposed techniques respectively. The frequency interval for computing error response is 0.37π to 0.42π. Fig. 3.15 represents closeup view of the magnitude and phase of frequency response errors. It is observed that the proposed technique compares well with other existing techniques when both magnitude and phase of frequency response error are considered in the desired frequency interval. Note

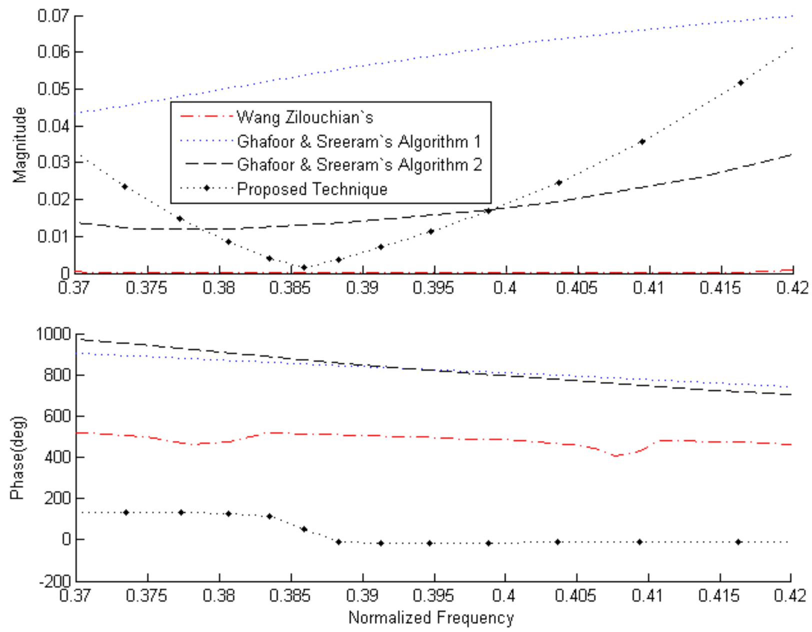

Figure 3.15: Frequency response with magnitude and phase error comparison - a closeup view

that, the ROM obtained using Wang and Zilouchian's technique is unstable having a pole at $z = 1.5699$.

Discussion: It is observed (from examples 3.4.5 - 3.4.8) that in the desired frequency interval Wang and Zilouchian's technique gives better approximation error, but it sometimes yields unstable ROMs as given in example 3.4.5. The proposed technique provides comparatively good approximation as compared to Ghafoor and Sreeram's algorithms (1 and 2) in the desired frequency interval. Ghafoor and Sreeram's technique algorithms (1 and 2) and proposed techniques have the advantage of providing stable ROMs and carry error bounds also.

3.5 Conclusion

In this chapter, a FLMR technique for continuous and discrete time systems is proposed. The ROMs obtained are guaranteed to be stable and error bounds are also available. The proposed technique compares well with other existing FLMR techniques (including Gawronski and Juang [29], Gugercin and Antoulas [30] and Ghafoor and Sreeram [8] for continuous time case and Wang and Zilouchian's [90], Ghafoor and Sreeram's algorithms (1 and 2) [87] for discrete time case) in the desired frequency interval.

In the next chapter, we extend FLMR results for generalized non-singular systems.

Chapter 4

FLMR: Generalized Non-Singular Systems

4.1 Introduction

In this chapter we propose FLMR for generalized non-singular continuous and discrete time systems.

Generalized descriptor systems find application in a variety of disciplines including electrical circuits, power systems, multibody systems etc. [41, 56, 57]. Such systems are expressed in generalized state space representation with non-identity E matrix (please see eq. 4.1) instead of standard state space system where matrix E is identity. A generalized descriptor systems may have singular or nonsingular matrix E.

Different techniques for the model reduction of generalized descriptor systems (with singular matrix E) exist in literature including BT technique [56, 57, 64], Krylov projection techniques [58] etc. Likewise, techniques for different versions of generalized descriptor systems for periodic, time varying systems, etc. also exist in literature [35, 91] and references therein. However, there is no work in literature to the best of author's knowledge linked with limited frequency Gramians for generalized systems with nonsingular matrix E.

In this chapter (partially published in [13,14,18]), generalization of (Gawronski and Juang [29], Gugercin and Antoulas [30], Ghafoor and Sreeram [8] limited frequency techniques for continuous time and Wang and Zilouchian [90] and Ghafoor and Sreeram [87] limited frequency techniques for discrete time) systems for generalized nonsingular systems are proposed. Error bounds and numerical examples are also given.

4.2 Preliminaries

Consider a linear time invariant generalized stable continuous and discrete time system respectively with following state space representation

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{4.1}$$

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{4.2}$$

$$G(s) = C(sE - A)^{-1}B + D \tag{4.3}$$

$$G(z) = C(zE - A)^{-1}B + D \tag{4.4}$$

where $E, A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$ and $\{E, A, B, C, D\}$ is its n^{th} order minimal realization. The system (4.1) is called the standard state space system if $E = I$ (where I is identity matrix). Otherwise, the system (4.1) is in generalized form and is known as generalized state space system. A descriptor system is a special form of the generalized state space system if E is singular. In this chapter, we consider a special case of the generalized form of the system (4.1) with nonsingular matrix E.

MOR problem for the generalized system (with nonsingular E matrix) (4.1) is to find a ROM

$$\text{Continuous time} \begin{cases} E_r \dot{x}_r(t) = A_r x_r(t) + B_r u(t) \\ y_r(t) = C_r x_r(t) + D_r u(t) \end{cases}$$

$$\text{Discrete time} \begin{cases} E_r x_r(k+1) = A_r x_r(k) + B_r u(k) \\ y_r(k) = C_r x_r(k) + D_r u(k) \end{cases}$$

where E_r, $A_r \in R^{r,r}$, $B_r \in R^{r,m}$, $C_r \in R^{p,r}$ $D_r \in R^{p,m}$ and $r < n$. It is assumed that the pencil $\lambda E - A$ is regular, i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in C$.

One can easily convert system (4.1) into the standard state space system as following

$$\text{Continuous time} \begin{cases} \dot{x}(t) = E^{-1}Ax(t) + E^{-1}Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$\text{Discrete time} \begin{cases} x(k+1) = E^{-1}Ax(k) + E^{-1}Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

However, in the MOR scenario, this transformation is not feasible due to: (i) in transformation based model reduction, transformation is applied to the original system state space realization, to balance the system, and for truncation. (ii) for practical system models, guaranteeing $E = I$ is not easy unless we transform. In case of transformation, the structure of the original system realization is not maintainable. (iii) converting generalized form of system (4.1) with nonsingular matrix to a standard state space $E = I$, may cause large errors at an early stage of computation possibly due to ill-conditioning of E [31] (iv) for large scale systems computation of E^{-1} is expensive, hence not feasible.

Following examples somehow elaborate few of above limitations.

Example 4.2.1 *Consider a third order stable generalized system satisfying eq. (4.1)*

$$E=\begin{bmatrix}1 & 0 & 5\\ 2 & 4 & 1\\ 1 & 0 & 6\end{bmatrix},\ A=\begin{bmatrix}-20 & 4 & 2\\ 3 & -4 & 2\\ 2 & 2 & -7\end{bmatrix},\ B=\begin{bmatrix}2\\ 0\\ 1\end{bmatrix},\ C=\begin{bmatrix}1 & 0 & 0\end{bmatrix},\ D=0$$

with regular pencil and magnitude of eigenvalues are $-144.4727, -0.3668$ *and* -1.6605 *respectively.*

Since with matrix E is non-singular, we can convert it into standard state space system like

$$\hat{A}=E^{-1}A=\begin{bmatrix}-130 & 14 & 47\\ 60.25 & -7.50 & -20.75\\ 22 & -2 & -9\end{bmatrix},\ \hat{B}=E^{-1}B=\begin{bmatrix}7\\ -3.25\\ -1\end{bmatrix},\ C=\begin{bmatrix}1 & 0 & 0\end{bmatrix},\ D=0$$

The transformation matrix obtained for original realization is

$$T=\begin{bmatrix}-0.3291 & -0.2381 & -2.8789\\ 0.0236 & -0.5737 & 1.1638\\ -0.3425 & -0.4831 & 0.5374\end{bmatrix}$$

while transformation matrix obtained by standard state space system is

$$\hat{T}=\begin{bmatrix}-1.9623 & 1.7712 & -0.1096\\ 0.9189 & -0.9049 & -1.4244\\ -0.2273 & -0.7878 & 0.4631\end{bmatrix}.$$

When these two different transformations obtained are applied to the original realization, we will get two different ROMs. Moreover, transformation matrix obtained by standard state space system in present scenario when applied to original realization will not give us ROM in generalized form. Hence, structure of original system realization is not maintainable in ROMs.

Example 4.2.2 *Consider the example 4.2.1 with following non-singular matrix* E

$$E = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 4 & 1 \\ 1 & 0 & 4.98 \end{bmatrix}$$

The condition number for matrix E *is 2909. Although, matrix* E *is non-singular but it is ill conditioned and may cause large round off errors.*

4.3 Gramians of generalized non-singular system

For a system of eq. (4.1), let P_G and Q_G be the generalized Gramians for continuous time systems are as follows:

$$\begin{aligned} P_G &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (j\omega E - A)^{-1} B B^T (-j\omega E^T - A^T)^{-1} d\omega && (4.5) \\ Q_G &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} E^T (-j\omega E^T - A^T)^{-1} C^T C (j\omega E - A)^{-1} E d\omega \\ &= E^T \hat{Q}_G E && (4.6) \end{aligned}$$

where

$$\hat{Q}_G = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-j\omega E^T - A^T)^{-1} C^T C (j\omega E - A)^{-1} d\omega$$

which satisfy the following generalized Lyapunov equations:

$$A P_G E^T + E P_G A^T = -BB^T \quad (4.7)$$

$$A^T E^{-T} Q_G + Q_G E^{-1} A = -C^T C \quad (4.8)$$

$$A^T \underbrace{E^{-T} Q_G E^{-1}}_{\hat{Q}_G} E + E^T \underbrace{E^{-T} Q_G E^{-1}}_{\hat{Q}_G} A = -C^T C \quad (4.9)$$

where $Q_G = E^T \hat{Q}_G E$.

Similarly, let P_G and Q_G be the generalized Gramians for discrete time systems, are given by

$$
\begin{aligned}
P_G &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\omega}E-A)^{-1}BB^T(e^{-j\omega}E^T-A^T)^{-1}d\omega && (4.10)\\
Q_G &= \frac{1}{2\pi}\int_{-\pi}^{\pi}E^T(e^{-j\omega}E^T-A^T)^{-1}C^TC(e^{j\omega}I-A)^{-1}Ed\omega && (4.11)\\
&= E^T\hat{Q}_GE
\end{aligned}
$$

where

$$
\hat{Q}_G = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\omega}E^T-A^T)^{-1}C^TC(e^{j\omega}I-A)^{-1}d\omega
$$

The Gramians P_G and Q_G satisfies the following Lyapunov equations

$$
\begin{aligned}
AP_GA^T-EP_GE^T &= -BB^T && (4.12)\\
A^TE^{-T}Q_GE^{-1}A-E^TE^{-T}Q_GE^{-1}E &= -C^TC\\
A^T\underbrace{E^{-T}Q_GE^{-1}}_{\hat{Q}_G}A-E^T\underbrace{E^{-T}Q_GE^{-1}}_{\hat{Q}_G}E &= -C^TC && (4.13)
\end{aligned}
$$

Note that, we do not require to compute E^{-1}. It has been shown in [36, 37, 38, 39], that for a stable generalized system with regular pencil, the Gramians P_G and Q_G have unique Hermitian, positive semidefinite solutions of generalized Lyapunov equations (4.7) and (4.9).

4.4 Generalized balanced truncation technique

The following section presents the generalized BT technique [31]. BT for the generalized nonsingular system is related to generalized controllability and observability Gramians (4.5) and (4.6). These Gramians are the solution of generalized Lyapunov equations (4.7) and (4.8). As the pencil $\lambda E-A$ is assumed to be stable, P_G and Q_G are positive semi definite and there exists factorization $P_G = M^TM$ and $Q_G = N^TN$. Matrices M and N are cholesky factors of the Gramians.

By considering the singular value decomposition (SVD) of the product MN^T as

$$
MN^T = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}
$$

where the matrices U, Σ and V^T are partitioned at a given reduced dimension r such that $\Sigma_1 = diag(\sigma_1, ..., \sigma_r)$, $\Sigma_2 = diag(\sigma_{r+1}, ..., \sigma_n), \sigma_i \geq 0$ for all i and $\sigma_r > \sigma_{r+1}$. Notice that

$\sigma_1, \sigma_2, ..., \sigma_n$ are the HSV of the system.

Now in order to find the ROM of a given system, we use balancing free square root BT algorithm because it provides more accurate ROMs in the presence of rounding errors [42,43]. After SVD calculation of MN^T, QR factorization are computed.

$$M^T U_1 = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}, N^T V_1 = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} \bar{R} \\ 0 \end{bmatrix}$$

where $K_1, L_1 \in \Re^{n \times r}$ have orthonormal columns and $\tilde{R}, \bar{R} \in \Re^{r \times r}$ are upper triangular. The ROM in generalized state space form is obtained as

$$(E_r, A_r, B_r, C_r, D_r) = (L_1^T K_1, L_1^T E^{-1} A K_1, L_1^T E^{-1} B, C K_1, D)$$

The realization obtained by above algorithm satisfies [31,40]

$$\|G(s) - G_r(s)\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j$$

4.5 A generalization of Gawronski and Juang's FLMR technique

Gawronski and Juang [29] (its discrete time version Wang and Zilouchian [90]) proposed FLMR technique for the standard state space system. Here, we generalize the Gawronski and Juang [29] FLMR technique.

4.5.1 Continuous time systems

Let us define the generalized frequency domain controllability interval Gramian P_{GGJ} and observability interval Gramian Q_{GGJ} respectively as

$$P_{GGJ} = \frac{1}{2\pi} \int_{\delta\omega} (j\omega E - A)^{-1} B B^T (-j\omega E^T - A^T)^{-1} d\omega \tag{4.14}$$

$$Q_{GGJ} = \frac{1}{2\pi} \int_{\delta\omega} E^T (-j\omega E^T - A^T)^{-1} C^T C (j\omega E - A)^{-1} E d\omega \tag{4.15}$$

where $\delta\omega$ is the integration interval $[-\omega_2 \longrightarrow -\omega_1]$ and $[\omega_1 \longrightarrow \omega_2]$ and $\omega_2 > \omega_1$. These Gramians satisfy the following generalized Lyapunov equations

$$AP_{GGJ}E^T + EP_{GGJ}A^T = -BB^TF^*E^T - EFBB^T = -X_{GGJ} \tag{4.16}$$

$$A^T \underbrace{E^{-T}Q_{GGJ}E^{-1}}_{Q\hat{GGJ}} E + E^T \underbrace{E^{-T}Q_{GGJ}E^{-1}}_{Q\hat{GGJ}} A = -C^TCFE - E^TF^*C^TC = -Y_{GGJ} \tag{4.17}$$

where $Q_{GGJ} = E^TQ\hat{_{GGJ}}E$ and

$$F = \frac{1}{2\pi}\int_{\delta\omega}(j\omega E - A)^{-1}d\omega \tag{4.18}$$

The Gramians P_{GGJ} and Q_{GGJ} obtained by above Lyapunov equations are used in calculation of matrices M and N, which are further used to obtain ROM as elaborated in section 4.4.

Next, we elaborate the process of computing Gramians P_{GGJ} and Q_{GGJ} of equations (4.16), (4.17) respectively.

For controllability Gramian eq. (4.7), can be written as

$$(j\omega E - A)P_GE^T + EP_G(-j\omega E^T - A^T) = BB^T$$

Pre-multiplying by $(j\omega E - A)^{-1}$, post-multiplying by $(-j\omega E^T - A^T)^{-1}$ and integrating both sides yields

$$\frac{1}{2\pi}\int_{\delta\omega}P_GE^T(-j\omega E^T - A^T)^{-1}d\omega + \frac{1}{2\pi}\int_{\delta\omega}(j\omega E - A)^{-1}EP_Gd\omega = \frac{1}{2\pi}\int_{\delta\omega}(j\omega E - A)^{-1}BB^T(-j\omega E^T - A^T)^{-1}d\omega \tag{4.19}$$

Substituting eq. (4.18) and eq. (4.14) in eq. (4.19) yields

$$P_GE^TF^* + FEP_G = P_{GGJ} \tag{4.20}$$

where F^* is conjugate transpose of F. Note that, $AFE = EFA$, which is shown as following:

$$\begin{aligned}
AFE &= \frac{1}{2\pi}\int_{\delta\omega} A(j\omega E - A)^{-1} E d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} \left(E^{-1}\left(j\omega E - A\right)A^{-1}\right)^{-1} d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} \left(j\omega A^{-1} - E^{-1}\right)^{-1} d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} \left(A^{-1}\left(j\omega E - A\right)E^{-1}\right)^{-1} d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} E(j\omega E - A)^{-1} A d\omega \\
&= EFA
\end{aligned}$$

Now using eq. (4.12) and substituting eq. (4.32) in left hand side of eq. (4.24) yields right hand side of eq. (4.24) as follows:

$$\begin{aligned}
AP_{GGJ}E^T + EP_{GGJ}A^T &= A\left(P_G E^T F^* + FEP_G\right)E^T + E\left(P_G E^T F^* + FEP_G\right)A^T \\
&= AP_G E^T F^* E^T + EP_G E^T F^* A^T + AFEP_G E^T + EFEP_G A^T \\
&= AP_G E^T F^* E^T + EP_G A^T F^* E^T + EFAP_G E^T + EFEP_G A^T \\
&= (AP_G E^T + EP_G A^T)F^* E^T + EF(AP_G E^T + EP_G A^T) \\
&= -BB^T F^* E^T - EFBB^T
\end{aligned}$$

Similarly for observability Gramian, from eq. (4.9) we have,

$$\left(-j\omega E^T - A^T\right)E^{-T}Q_G E^{-1}E + E^T E^{-T} Q_G E^{-1}\left(j\omega E - A\right) = C^T C$$

Pre-multiplying by $E^T(-j\omega E^T - A^T)^{-1}$, post-multiplying by $(j\omega E - A)^{-1}E$ and integrating both sides yields

$$\frac{1}{2\pi}\int_{\delta\omega} Q_G E^{-1} E(j\omega E - A)^{-1} E d\omega + \frac{1}{2\pi}\int_{\delta\omega}(-j\omega E^T - A^T)^{-1} E^T E^{-T} Q_M d\omega = \\ \frac{1}{2\pi}\int_{\delta\omega} E^T(-j\omega E^T - A^T)^{-1} C^T C (j\omega E - A)^{-1} E d\omega$$

$$Q_G FE + E^T F^* Q_G = Q_{GGJ} \tag{4.21}$$

Now by substituting value of eq. (4.21) in left hand side of eq. (4.17) yields right hand side of eq. (4.17).

Remark 4.5.1 *Multiple frequency intervals can be considered for approximation. For example, for two intervals* $[\omega_a, \omega_b]$ *and* $[\omega_c, \omega_d]$, $\omega_a < \omega_b < \omega_c < \omega_d$*, the integration range* $\delta\omega$ *involve integration from* $[-\omega_d \longrightarrow -\omega_c]$, $[-\omega_b \longrightarrow -\omega_a]$, $[\omega_a \longrightarrow \omega_b]$ *and* $[\omega_c \longrightarrow \omega_d]$.

Remark 4.5.2 *The symmetric matrices* X_{GGJ} *and* Y_{GGJ} *are not guaranteed to be positive semidefinite, therefore the ROMs obtained by generalized Gawronski & Juang technique are not guaranteed to be stable.*

4.5.2 Discrete time systems

In order to derive generalized Wang and Zilouchian's technique (a discrete time counter part of Gawronski and Junag [29]) for nonsingular discrete time system, let us define the generalized frequency domain controllability interval Gramian P_{GWZ} and observability interval Gramian Q_{GWZ} respectively as

$$\begin{aligned} P_{GWZ} &= \frac{1}{2\pi}\int_{\delta\omega}(e^{j\omega}E - A)^{-1}BB^T(e^{-j\omega}E^T - A^T)^{-1}d\omega \\ &= \frac{1}{2\pi}\int_{\delta\omega}(E- Ae^{-j\omega})^{-1}BB^T(E^T - A^Te^{j\omega})^{-1}d\omega \end{aligned} \quad (4.22)$$

$$\begin{aligned} Q_{GWZ} &= \frac{1}{2\pi}\int_{\delta\omega} E^T(e^{-j\omega}E^T - A^T)^{-1}C^TC(e^{j\omega}E - A)^{-1}Ed\omega \\ &= \frac{1}{2\pi}\int_{\delta\omega} E^T(E^T - A^Te^{j\omega})^{-1}C^TC(E - Ae^{-j\omega})^{-1}Ed\omega \end{aligned} \quad (4.23)$$

where $\delta\omega$ is the integration interval $[\omega_1, \omega_2]$ and $0 \leq \omega_1 \leq \omega_2 \leq \pi$. These Gramians P_{GWZ} and Q_{GWZ} are the solution of following Lyapunov equations

$$AP_{GWZ}A^T - EP_{GWZ}E^T = -X_{GWZ} \quad (4.24)$$

$$A^T \underbrace{E^{-T}Q_{GWZ}E^{-1}}_{\hat{Q}_{GWZ}} A - E^T \underbrace{E^{-T}Q_{GWZ}E^{-1}}_{\hat{Q}_{GWZ}} E = -Y_{GWZ} \quad (4.25)$$

where $Q_{GWZ} = E^T\hat{Q}_{GWZ}E$ and

$$X_{GWZ} = BB^TF^* + FBB^T \quad (4.26)$$

$$Y_{GWZ} = C^TCF + F^*C^TC \quad (4.27)$$

$$F = -\frac{\Delta\omega}{4\pi}I + EF_1 \quad (4.28)$$

and

$$F_1 = \frac{1}{2\pi}\int_{\delta\omega}(E - Ae^{-j\omega})^{-1}d\omega \tag{4.29}$$

Let

$$T^T Q_{GWZ} T = T^{-1} P_{GWZ} T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$ and T is a contragredient matrix. The ROMs are calculated by partitioning the transformed original system realization. Next, we elaborate the process of computing Gramians (4.24) and (4.25) respectively. From eq. (4.12), we can write

$$AP_G(e^{-j\omega}E^T - A^T) + (e^{j\omega}E - A)P_G E^T e^{-j\omega} = -(e^{j\omega}E - A)P_G(e^{-j\omega}E^T - A^T) +$$
$$EP_G(e^{-j\omega}E^T - A^T)e^{j\omega} + (e^{j\omega}E - A)P_G e^{-j\omega}E^T = BB^T \tag{4.30}$$

Pre-multiplying by $(e^{-j\omega}E - A)^{-1}$, post-multiplying by $(e^{-j\omega}E^T - A^T)^{-1}$ and integrating both sides of equation (4.30) yields

$$-\frac{1}{2\pi}\int_{\delta\omega} P_G d\omega + \frac{1}{2\pi}\int_{\delta\omega}(e^{j\omega}E - A)^{-1}EP_G e^{j\omega}d\omega + \frac{1}{2\pi}\int_{\delta\omega} P_G E^T(e^{-j\omega}E^T - A^T)^{-1}e^{-j\omega}d\omega$$
$$= \frac{1}{2\pi}\int_{\delta\omega}(e^{j\omega}E - A)^{-1}BB^T(e^{-j\omega}E^T - A^T)^{-1}d\omega$$
$$-\frac{1}{2\pi}\int_{\delta\omega} P_G d\omega + \frac{1}{2\pi}\int_{\delta\omega}(E - Ae^{-j\omega})^{-1}EP_G d\omega + \frac{1}{2\pi}\int_{\delta\omega} P_G E^T(E^T - A^T e^{j\omega})^{-1}d\omega$$
$$= \frac{1}{2\pi}\int_{\delta\omega}(e^{j\omega}E - A)^{-1}BB^T(e^{-j\omega}E^T - A^T)^{-1}d\omega \tag{4.31}$$

Substituting eq. (4.29) and eq. (4.22) in eq. (4.31)

$$-\frac{\Delta\omega}{2\pi}P_G + F_1 E P_G + P_G E^T F_1{}^* = P_{GWZ} \tag{4.32}$$

where ${F_1}^*$ is conjugate transpose of F_1. Now substituting the value of eq. (4.32) in eq. (4.24) yields

$$\begin{aligned}
AP_{GWZ}A^T - EP_{GWZ}E^T &= A\left(-\frac{\Delta\omega}{2\pi}P_G + F_1EP_G + P_GE^T{F_1}^*\right)A^T - \\
&\quad E\left(-\frac{\Delta\omega}{2\pi}P_G + F_1EP_G + P_GE^T{F_1}^*\right)E^T \\
&= -\frac{\Delta\omega}{2\pi}\{AP_GA^T - EP_GE^T\} + AF_1EP_GA^T - \\
&\quad E(F_1EP_G)E^T + AP_GE^T{F_1}^*A^T - E(P_GE^T{F_1}^*)E^T \\
&= -\frac{\Delta\omega}{2\pi}\{AP_GA^T - EP_GE^T\} + EF_1AP_GA^T - \\
&\quad E(F_1EP_G)E^T + AP_GA^T{F_1}^*E^T - E(P_GE^T{F_1}^*)E^T \\
&= -\frac{\Delta\omega}{2\pi}\{AP_GA^T - EP_GE^T\} + EF_1\{AP_GA^T - \\
&\quad EP_GE^T\} + \{AP_GA^T - EP_GE^T\}{F_1}^*E^T \\
&= \frac{\Delta\omega}{2\pi}BB^T - EF_1BB^T - BB^T{F_1}^*E^T \\
&= \frac{\Delta\omega}{4\pi}BB^T + BB^T\frac{\Delta\omega}{4\pi} - EF_1BB^T - BB^T{F_1}^*E^T \\
&= -(-\frac{\Delta\omega}{4\pi}I + EF_1)BB^T - BB^T(-\frac{\Delta\omega}{4\pi}I + {F_1}^*E^T) \\
&= -FBB^T - BB^TF^*
\end{aligned}$$

Note that, $AF_1E = EF_1A$, which can be shown as follows:

$$\begin{aligned}
AF_1E &= \frac{1}{2\pi}\int_{\delta\omega} A(E - Ae^{-j\omega})^{-1}Ed\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} \left(E^{-1}\left(E - Ae^{-j\omega}\right)A^{-1}\right)^{-1}d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} \left(A^{-1} - E^{-1}e^{-j\omega}\right)^{-1}d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} \left(A^{-1}\left(E - Ae^{-j\omega}\right)E^{-1}\right)^{-1}d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} E(E - Ae^{-j\omega})^{-1}Ad\omega \\
&= EF_1A
\end{aligned}$$

Similar process can be followed for computing observability Gramian of eq. (4.25).

Remark 4.5.3 *The symmetric matrices X_{GWZ} and Y_{GWZ} obtained by the above procedure may not gurantee positive semidefiniteness, hence the ROMs calculated by generalized Wang and Zilouchian method may not be stable. This limitation of X_{GWZ} and Y_{GWZ} sometimes* *causes hinderence in the balancing process. This is due to the fact that the controllability P_{GWZ} and observability Q_{GWZ} Gramians are not guaranteed to be positive definite (when $X_{GWZ} \ngeq 0$ and $Y_{GWZ} \ngeq 0$). Some more discussion on this topic is included in numerical examples section.*

4.6 A generalization of Gugercin and Antoulas's FLMR technique

The stability problem of Gawronski and Juang [29] technique (that the ROMs stability is not always guaranteed) was solved by Gugercin and Antoulas [30] for standard systems. The discrete time version of [30] was proposed by Ghafoor and Sreeram [87] Algorithm 1. In the following, we generalize Gugercin and Antoulas [30] FLMR technique (and its discrete time version [87]) to address stability problem of generalized Gawronski and Juang [29] technique. Let the new controllability Gramian P_{GGA} and observability Gramian Q_{GGA} be obtained as solutions to the generalized Lyapunov equations

$$AP_{GGA}E^T + EP_{GGA}A^T + B_{GGA}B_{GGA}^T = 0 \tag{4.33}$$

$$E^T \underbrace{E^{-T}Q_{GGA}E^{-1}}_{\hat{Q}_{GGA}} A + A^T \underbrace{E^{-T}Q_{GGA}E^{-1}}_{\hat{Q}_{GGA}} E + C_{GGA}^T C_{GGA} = 0 \tag{4.34}$$

are used to obtain matrices M and N, which are further used to obtain ROM as elaborated in section 4.4.

Similarly, for generalized discrete time case, the Lyapunov equations become

$$AP_{GGA}A^T - EP_{GGA}E^T + B_{GGA}B_{GGA}^T = 0 \tag{4.35}$$

$$A^T \underbrace{E^{-T}Q_{GGA}E^{-1}}_{Q_{GGA}} A - E^T \underbrace{E^{-T}Q_{GGA}E^{-1}}_{Q_{GGA}} E + C_{GGA}^T C_{GGA} = 0 \tag{4.36}$$

The matrices B_{GGA} and C_{GGA} in the above generalized Lyapunov equations are input and output matrices obtained as $B_{GGA} = U_{GGA}|S_{GGA}|^{1/2}$ and $C_{GGA} = |R_{GGA}|^{1/2}V_{GA}^T$, respectively. The terms U_{GGA}, S_{GGA}, V_{GGA}, and R_{GGA} are acquired from the EVD of matrices $X_{GGJ} = U_{GGA}S_{GGA}U_{GGA}^T$ and $Y_{GGJ} = V_{GGA}R_{GGA}V_{GGA}^T$, where $S_{GGA} = diag(s_1, s_2, \ldots, s_n)$, $R_{GGA} = diag(r_1, r_2, \ldots, r_n)$, $|s_1| \geq |s_2| \geq \cdots \geq |s_{n-1}| \geq |s_n| \geq 0$ and $|r_1| \geq |r_2| \geq \ldots |r_{n-1}| \geq |r_n| \geq 0$. The ROMs are calculated by partitioning the transformed realization. Since $X_{GGJ} \leq B_{GGA}B_{GGA}^T \geq 0$, $Y_{GGJ} \leq C_{GGA}^T C_{GGA} \geq 0$ and the minimal realization (E, A, B_{GGA}, C_{GGA}) is obtained, hence the ROM is guaranteed to be stable.

Remark 4.6.1 *Let*

$$L_{GGA} = CV_{GGA}diag(|r_1|^{-\frac{1}{2}}, |r_2|^{-\frac{1}{2}}, \cdots, |r_{ni}|^{-\frac{1}{2}}, 0, \cdots, 0) \quad (4.37)$$

$$K_{GGA} = diag(|s_1|^{-\frac{1}{2}}, |s_2|^{-\frac{1}{2}}, \cdots, |s_{no}|^{-\frac{1}{2}}, 0, \cdots, 0)U_{GGA}^T B \quad (4.38)$$

where $ni = rank\,[X_{GGJ}]$ *and* $no = rank\,[Y_{GGJ}]$.

a. *If* $rank\,[B_{GGA}\ \ B] = rank\,[B_{GGA}]$*, then* $B = B_{GGA}K_{GGA}$.

b. *If* $rank\begin{bmatrix} C_{GGA} \\ C \end{bmatrix} = rank\,[C_{GGA}]$*, then* $C = L_{GGA}C_{GGA}$.

*The existence of these rank conditions can be shown in a similar way as in [26]. It is also shown in [26] that conditions given in **a.** and **b.** are almost always true.*

Theorem 4.6.1 *The following error bounds holds*

$$\|G(s) - G_r(s)\|_\infty \leq 2\|L_{GGA}\|\|K_{GGA}\| \sum_{j=r+1}^{n} \sigma_i$$

$$\|G(z) - G_r(z)\|_\infty \leq 2\|L_{GGA}\|\|K_{GGA}\| \sum_{j=r+1}^{n} \sigma_i$$

Proof: By partitioning $B_{GGA} = \begin{bmatrix} B_{GGA_1} \\ B_{GGA_2} \end{bmatrix}$, $C_{GGA} = \begin{bmatrix} C_{GGA_1} & C_{GGA_2} \end{bmatrix}$ and substituting $B_r = B_{GGA_1}K_{GGA}$, $C_r = L_{GGA}C_{GGA_1}$ respectively yields

$$\begin{aligned}
&\|G(s) - G_r(s)\|_\infty \\
&\quad = \|C(sE-A)^{-1}B - C_r(sE-A_r)^{-1}B_r\|_\infty \\
&\quad = \|L_{GGA}C_{GGA}(sE-A)^{-1}B_{GGA}K_{GGA} - L_{GGA}C_{GGA_1}(sE-A_r)^{-1}B_{GGA_1}K_{GGA}\|_\infty \\
&\quad = \|L_{GGA}(C_{GGA}(sE-A)^{-1}B_{GGA} - C_{GGA_1}(sE-A_r)^{-1}B_{GGA_1})K_{GGA}\|_\infty \\
&\quad \leq \|L_{GGA}\|\|C_{GGA}(sE-A)^{-1}B_{GGA} - C_{GGA_1}(sE-A_r)^{-1}B_{GGA_1}\|_\infty\|K_{GGA}\|
\end{aligned}$$

If $\{A_r, B_{GGA_1}, C_{GGA_1}\}$ is ROM obtained by partitioning a balanced realization $\{A, B_{GGA}, C_{GGA}\}$, we have from [31, 40]

$$\|C_{GGA}(sE-A)^{-1}B_{GGA} - C_{GGA_1}(sE-A_r)^{-1}B_{GGA_1}\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i.$$

The result follows. The proof of discrete time version follows similarly.

4.7 A generalization of Ghafoor and Sreeram's FLMR technique

Inspired by Ghafoor and Sreeram [8] modification to Gugercin and Antoulas [30] technique (a discrete time version of [8] appears in [87] Algorithm 2), we propose a modification to the generalized Gugercin and Antoulas technique by reducing the distance between the Gramians, $P_{GGA} - P_{GGJ}$ and $Q_{GGA} - Q_{GGJ}$. Let P_{GGS} be the new controllability and and Q_{GGS} be the observability Gramians respectively, are calculated as the solutions to the following Lyapunov equations:

$$AP_{GGS}E^T + EP_{GGS}A^T + B_{GGS}B_{GGS}^T = 0 \tag{4.39}$$

$$E^T \underbrace{E^{-T}Q_{GGS}E^{-1}}_{\hat{Q}_{GGS}} A + A^T \underbrace{E^{-T}Q_{GGS}E^{-1}}_{\hat{Q}_{GGS}} E + C_{GGS}^T C_{GGS} = 0 \tag{4.40}$$

are used to obtain matrices M and N, which are further used to obtain ROM as elaborated in section 4.4. Similarly, for generalized discrete time case, the Lyapunov equation becomes

$$AP_{GGS}A^T - EP_{GGS}E^T + B_{GGS}B_{GGS}^T = 0 \tag{4.41}$$

$$A^T \underbrace{E^{-T}Q_{GGS}E^{-1}}_{Q_{GGS}} A - E^T \underbrace{E^{-T}Q_{GGS}E^{-1}}_{Q_{GGS}} E + C_{GGS}^T C_{GGS} = 0 \tag{4.42}$$

The new fictitious matrices B_{GGS} and C_{GGS} shown in eq. (4.39) and (4.40) are defined as $B_{GGS} = U_{GGS_1}S_{GGS_1}^{1/2}$ and $C_{GGS} = R_{GGS_1}^{1/2}V_{GGS_1}^T$, respectively. The terms U_{GGS_1}, S_{GGS_1}, V_{GGS_1}, and R_{GGS_1} are calculated by the orthogonal EVD of matrices

$$X_{GGJ} = \begin{bmatrix} U_{GGS_1} & U_{GGS_2} \end{bmatrix} \begin{bmatrix} S_{GGS_1} & 0 \\ 0 & S_{GGS_2} \end{bmatrix} \begin{bmatrix} U_{GGS_1}^T \\ U_{GGS_2}^T \end{bmatrix}$$

$$Y_{GGJ} = \begin{bmatrix} V_{GGS_1} & V_{GGS_2} \end{bmatrix} \begin{bmatrix} R_{GGS_1} & 0 \\ 0 & R_{GGS_2} \end{bmatrix} \begin{bmatrix} V_{GGS_1}^T \\ V_{GGS_2}^T \end{bmatrix}$$

where $\begin{bmatrix} S_{GGS_1} & 0 \\ 0 & S_{GGS_2} \end{bmatrix} = diag(s_1, s_2, \cdots, s_n)$,

$\begin{bmatrix} R_{GGS_1} & 0 \\ 0 & R_{GGS_2} \end{bmatrix} = diag(r_1, r_2, \cdots, r_n)$, $s_1 \geq s_2 \geq \cdots \geq s_n$, $r_1 \geq r_2 \geq \cdots \geq r_n$, $S_{GGS_1} = diag(s_1, s_2, \cdots, s_l)$, $R_{GGS_1} = diag(r_1, r_2, \cdots, r_l)$, $s_1 \geq s_2 \geq \cdots \geq s_l > 0$, $r_1 \geq$

$r_2 \geq \cdots \geq r_l > 0$. The ROMs are calculated by partitioning the transformed realization. Since $X_{GGJ} \leq B_{GGS}B_{GGS}^T \leq B_{GGA}B_{GGA}^T \geq 0$, $Y_{GGJ} \leq C_{GGS}^T C_{GGS} \leq C_{GGA}^T C_{GGA} \geq 0$ and the realization $\{E, A, B_{GGS}, C_{GGS}\}$ is minimal, the stability of the ROM is guaranteed.

Remark 4.7.1 *Let*

$$\begin{aligned} L_{GGS} &= CV_{GGS_1}R_{GGS_1}^{-\frac{1}{2}} &(4.43)\\ K_{GGS} &= S_{GGS_1}^{-\frac{1}{2}}U_{GGS_1}^T B. &(4.44)\end{aligned}$$

a. *If* $rank\,[B_{GGS}\;\; B] = rank\,[B_{GGS}]$, *then* $B = B_{GGS}K_{GGS}$.

b. *If* $rank \begin{bmatrix} C_{GGS} \\ C \end{bmatrix} = rank\,[C_{GGS}]$, *then* $C = L_{GGS}C_{GGS}$.

The existence of these rank conditions can be shown in a similar way as in [26].

Theorem 4.7.1 *The following error bound holds*

$$\|G(s) - G_r(s)\|_\infty \leq 2\|L_{GGS}\|\|K_{GGS}\| \sum_{j=r+1}^{n} \sigma_i$$

$$\|G(z) - G_r(z)\|_\infty \leq 2\|L_{GGS}\|\|K_{GGS}\| \sum_{j=r+1}^{n} \sigma_i$$

Proof: By partitioning $B_{GGS} = \begin{bmatrix} B_{GGS_1} \\ B_{GGS_2} \end{bmatrix}$, $C_{GGS} = \begin{bmatrix} C_{GGS_1} & C_{GGS_2} \end{bmatrix}$ and substituting $B_r = B_{GGS_1}K_{GGS}$, $C_r = L_{GGS}C_{GGS_1}$ respectively yields

$$\begin{aligned} &\|G(s) - G_r(s)\|_\infty \\ &\quad = \|C(sE-A)^{-1}B - C_r(sE-A_r)^{-1}B_r\|_\infty \\ &\quad = \|L_{GGS}C_{GGS}(sE - A)^{-1}B_{GGS}K_{GGS} - L_{GGS}C_{GGS_1}(sE - A_r)^{-1}B_{GGS_1}K_{GGS}\|_\infty \\ &\quad = \|L_{GGS}(C_{GGS}(sE - A)^{-1}B_{GGS} - C_{GGS_1}(sE - A_r)^{-1}B_{GGS_1})K_{GGS}\|_\infty \\ &\quad \leq \|L_{GGS}\|\|C_{GGS}(sE - A)^{-1}B_{GGS} - C_{GGS_1}(sE - A_r)^{-1}B_{GGS_1}\|_\infty\|K_{GGS}\| \end{aligned}$$

If $\{A_r, B_{GGS_1}, C_{GGS_1}\}$ is ROM obtained by partitioning a balanced realization $\{A, B_{GGS}, C_{GGS}\}$, we have from [31,40]

$$\|C_{GGS}(sE - A)^{-1}B_{GGS} - C_{GGS_1}(sE - A_r)^{-1}B_{GGS_1}\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i.$$

Table 4.1: The errors and error bounds for the ROMs.

Techniques					
r	Generalized Gawronski & Juang	Generalized Gugercin & Antoulas		Generalized Ghafoor & Sreeram	
	Error	Error	Bound	Error	Bound
1	0.0038	0.0037	0.0105	0.0038	0.0045
2	$Unstable$	0.0011	0.0032	0.0010	0.0012

The result follows. The proof of discrete time version follows similarly.

4.8 Numerical examples

Numerical results for both continuous and discrete time generalized non-singular systems are presented.

4.8.1 Continuous time case

Example 4.8.1 *Consider a third order stable generalized system satisfying eq. (4.1)*

$$E = \begin{bmatrix} 4 & 0 & 0 \\ 0.5 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -4 & 1 & 2 \\ 1 & -8 & 1 \\ 2 & 1 & -20 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D = 0$$

with regular pencil and magnitude of eigenvalues are $-20.0741, -0.8788$ *and* -4.1096 *respectively.*

Table 4.1 shows error and error bounds for ROMs obtained by using generalized Gugercin and Antoulas and Ghafoor and Sreeram techniques for the frequency range $[\omega_1, \omega_2] = [22, 25]$ rad/s. Moreover, table 4.1 also shows errors obtained by using generalized Gawronski and Juang technique.

Table 4.2 shows the ROMs obtained by using generalized Gawronski and Juang, generalized Gugercin and Antoulas and generalized Ghafoor and Sreeram techniques for the frequency range $[\omega_1, \omega_2] = [22, 25]$ rad/s. Note that the ROM obtained for second order is unstable for generalized Gawronski and Juang technique whereas the generalized Gugercin and Antoulas and generalized Ghafoor & Sreeram techniques yield stable ROMs.

Fig. 4.1 shows the comparison of the singular values plot for the error function $\sigma[G(s) - G_r(s)]$, where $G_r(s)$ is the second order ROM obtained using different general-

Table 4.2: The ROMs obtained for generalized Gawronski and Juang , generalized Gugercin and Antoulas and generalized Ghafoor and Sreeram techniques in the desired frequency range $[\omega_1, \omega_2] = [22, 25]$ rad/sec

Techniques	Order 1 (reduced model)	Order 2 (reduced model)
Generalized Gawronski & Juang	$\frac{0.268}{(s+0.8943)}$	$\frac{0.2681s-0.015}{(s-0.057)(s+0.8828)}$
Generalized Gugercin & Antoulas	$\frac{0.2681}{(s+0.8947)}$	$\frac{0.2684s+0.2846}{(s+0.8551)(s+1.0933)}$
Generalized Ghafoor & Sreeram	$\frac{0.2861}{(s+1.018)}$	$\frac{0.2861s+0.1989}{(s+0.7271)(s+1.0212)}$

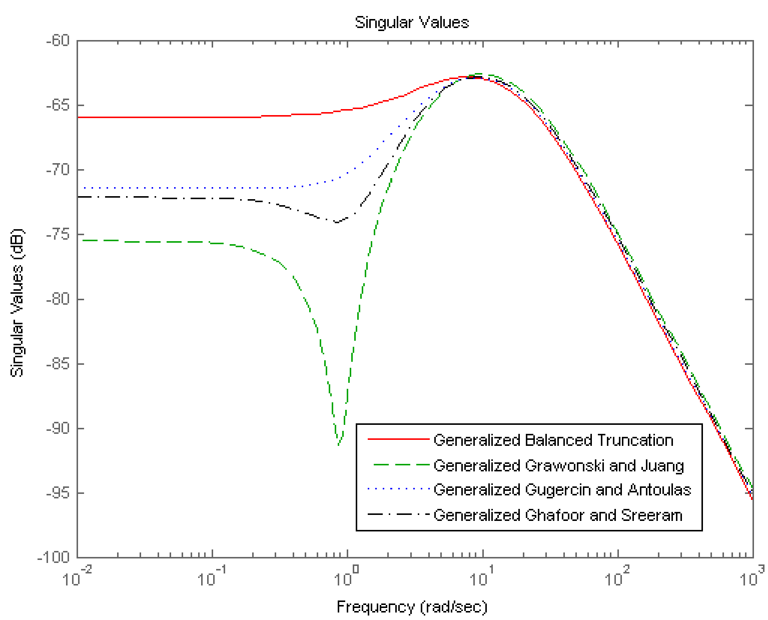

Figure 4.1: Singular values plot for the error function $\sigma[G(s) - G_r(s)]$ in the desired frequency interval $[\omega_1, \omega_2] = [0.1, 10] rad/s$.

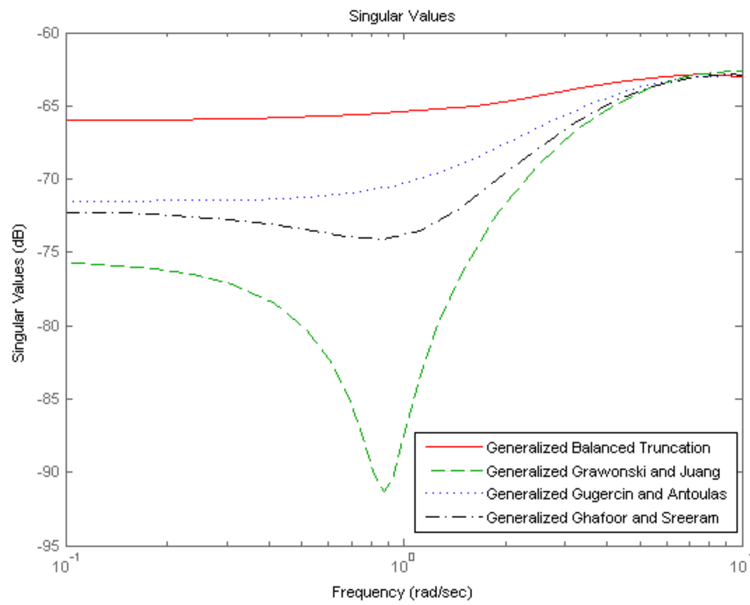

Figure 4.2: Closeup view of singular values plot for the error function $\sigma[G(s) - G_r(s)]$ in the desired frequency interval $[\omega_1, \omega_2] = [0.1, 10] rad/s$.

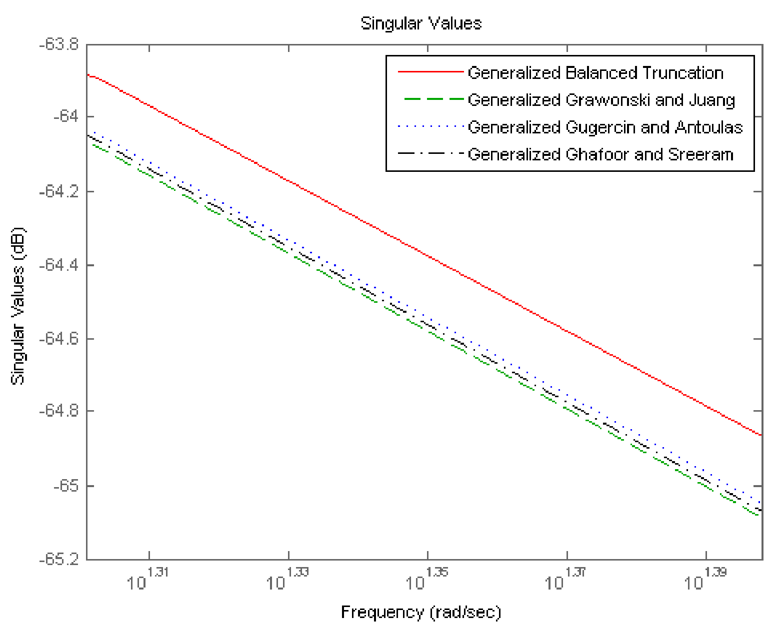

Figure 4.3: Closeup view of singular values plot for the error function $\sigma[G(s) - G_r(s)]$ in the desired frequency interval $[\omega_1, \omega_2] = [20, 25] rad/s$.

ized schemes e.g. generalized BT, generalized Gawronski and Juang, generalized Gugercin and Antoulas and generalized Ghafoor and Sreeram techniques for the frequency interval $[\omega_1, \omega_2] = [0.1, 10]$ rad/s. Fig. 4.2 and Fig. 4.3 show the closeup view of error plots in the desired frequency range for the second and first order ROM respectively. Note that in the desired frequency range generalized Gawronski and Juang technique yields better approximation as compared to other techniques, however it can yield unstable ROMs.

Example 4.8.2 *For error bound calculation, we have considered a sixth order stable generalized system satisfying eq. (4.1)*

$$E = \begin{bmatrix} 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -5.4545 & 4.5455 & 0 & -0.0545 & 0.0455 & 0 \\ 10 & -21 & 11 & 0.1000 & -0.2100 & 0.1100 \\ 0 & 5.5000 & -6.5000 & 0 & 0.0550 & -0.0650 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & 0 & 0 & 0.0909 & 0.4000 & -0.5000 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -2 & 3 & 0 & 0 & 0 \end{bmatrix}, D = 0$$

with regular pencil and magnitude of eigenvalues are $-0.0632 + 3.7631i, -0.0632 - 3.7631i, -0.0405 + 1.6629i, -0.0405 - 1.6629i, -0.0012 + 0.2372i$ *and* $-0.0012 - 0.2372i$ *respectively.*

Table 4.3 shows errors and error bounds for the ROMs obtained using generalized Gugercin and Antoulas and Ghafoor and Sreeram techniques for the frequency range $[\omega_1, \omega_2] = [1, 5]$ rad/s. Moreover, table 4.3 also shows errors obtained by using the generalized Gawronski and Juang technique.

Table 4.3: The errors and error bounds for the ROMs.

	Generalized				
r	Gawronski & Juang	Gugercin & Antoulas		Ghafoor & Sreeram	
	Error	Error	Bound	Error	Bound
1	21.3363	21.4377	235.551	21.4718	47.080
2	5.3301	5.3305	28.2643	5.3306	8.9963
3	5.2505	5.2758	20.8997	5.2444	6.3184
4	0.2008	5.1884	13.6711	3.1392	3.6980
5	0.1963	4.9643	6.7526	1.5995	1.8107

Discussion: Note that in the above examples (4.8.1 - 4.8.2) ROMs obtained by generalized Gawronski and Juang technique provide better approximation error in the desired frequency band, but it may not guarantee stability of ROMs. Although, generalized Gugercin and Antoulas technique may provide large approximation error in the desired frequency range, but it has the advantage of providing stable ROMs. Generalized Ghafoor and Sreeram technique provides stable ROMs that gives better approximation error as compared to generalized Gugercin and Antoulas technique.

4.8.2 Discrete time case

Example 4.8.3 *Consider a* 4^{th} *order original stable generalized nonsingular discrete time system with state space representation* $[E, A, B, C, D]$ *where*

$$E = \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0.6 & 1 & 0 & 0 \\ 0.2 & 0 & 2 & 0.2 \\ 0.2 & 0 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.2650 & -.6974 & 0.2011 & -0.2819 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2.1463 & -0.3652 & 0.1734 & -0.2591 \end{bmatrix}, D = 1$$

Table 4.4: The ROMs obtained for generalized Wang and Zilouchian's, generalized Ghafoor and Sreeram's methods Algorithm (1 and 2) in the desired frequency range 0.2π to 0.8π

Methods	Reduced system of order 1	Reduced system of order 2
Generalized Wang & Zilouchian's	$\frac{(z+1.9465)}{(z+1.4042)}$	$\frac{(z+7.9251)(z+0.5523)}{(z-0.1766+0.6620i)(z-0.1766-0.6620i)}$
Generalized Ghafoor & Sreeram's Algorithm 1	$\frac{(z+1.6898)}{(z+0.9265)}$	$\frac{(z+2.9635)(z+0.4140)}{(z-0.3080+0.7434i)(z-0.3080-0.7434i)}$
Generalized Ghafoor & Sreeram's Algorithm 2	$\frac{(z+0.7480)}{(z+0.7380)}$	$\frac{(z+10.2877)(z+0.6855)}{(z-0.0989+0.3859i)(z-0.0989-0.3859i)}$

Table 4.4 shows ROMs obtained by using generalized Wang and Zilouchian's method, generalized Ghafoor and Sreeram's method Algorithms (1 and 2) in the frequency range 0.2 π to 0.8 π respectively. Note that ROM obtained for first order is unstable for generalized Wang and Zilouchian's method whereas the generalized Ghafoor and Sreeram's method Algorithms (1 and 2) yield stable ROMs. The reduced 3^{rd} order ROM (not shown in table 4.4) obtained by each method is stable.

Example 4.8.4 *Consider the same system used for example 4.8.3 with following matrix E with following state space representation $[E, A, B, C, D]$ where*

$$E = \begin{bmatrix} 2 & 0.25 & 0 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Fig. 4.4 and Fig. 4.6 represent the frequency response $G(z)$ and Fig. 4.5 and Fig. 4.7 represents frequency response errors ($\sigma[G(z)-G_r(z)]$), for the second and third order ROMs obtained by proposed generalized methods in the frequency range 0.5 π to 0.9 π respectively.

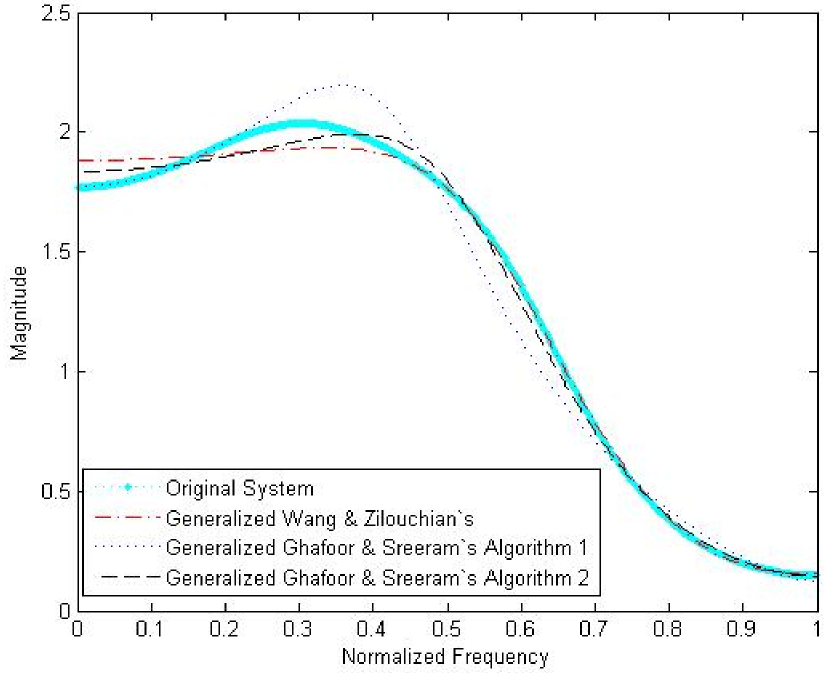

Figure 4.4: Frequency response comparison of original and the ROMs

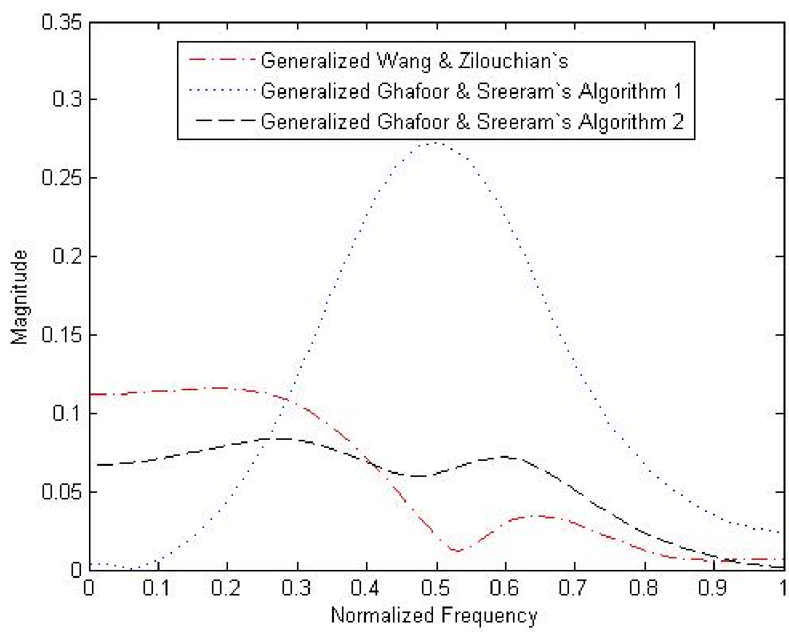

Figure 4.5: Frequency response error comparison of the ROMs

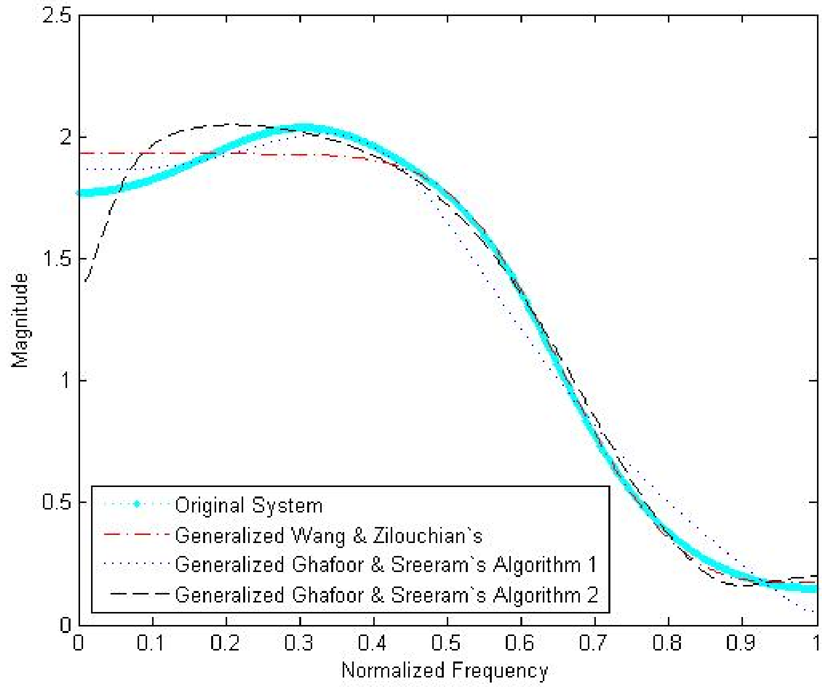

Figure 4.6: Frequency response comparison of original and the ROMs

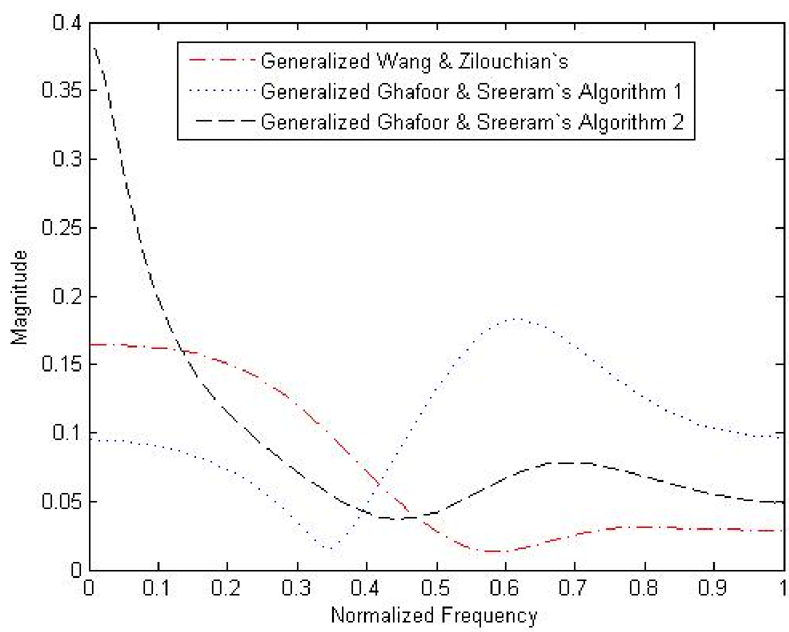

Figure 4.7: Frequency response error comparison of the ROMs

Moreover, original system frequency response in the frequency range 0 to π is also included in Fig. 4.4 and Fig. 4.6.

Example 4.8.5 *Consider an 8^{th} order original stable generalized nonsingular discrete time system with state space representation $[E, A, B, C, D]$ where*

$$E = \begin{bmatrix} 6 & 1 & 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$A = \begin{bmatrix} 1.0025 & -0.9479 & 0.8402 & -1.2326 & 0.7154 & -0.5542 & 0.2343 & -0.1642 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0.0675 & -0.3329 & 0.0565 & 0.3208 & 0.0481 & -0.3064 & 0.0158 & 0.0562 \end{bmatrix},$$

$$D = 0.0673$$

Fig. 4.8 represents the frequency responses for second order ROMs obtained by generalized Wang and Zilouchian method and generalized Ghafoor and Sreeram's method Algorithms (1 and 2) in the frequency range $0.3\ \pi$ to $0.6\ \pi$. Moreover, Fig. 4.8 also includes original system frequency response in the frequency range 0 to π. Fig. 4.9 and Fig. 4.10 represent the frequency response errors for the second and fourth order ROMs produced

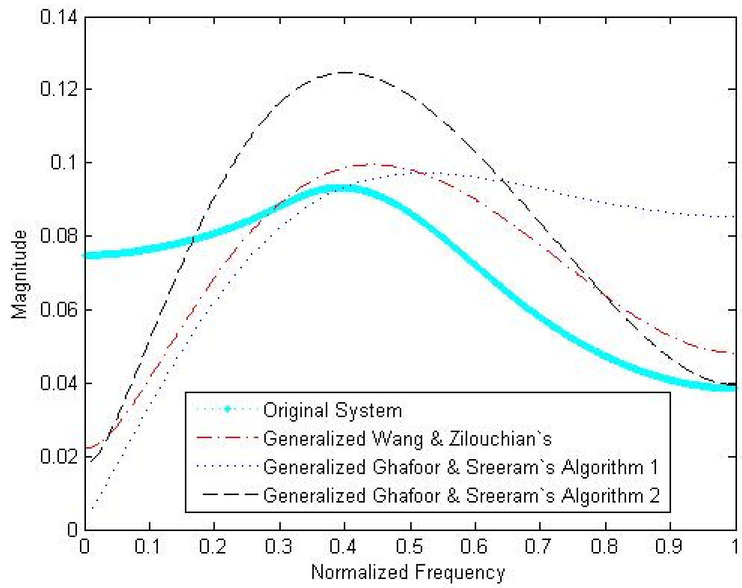

Figure 4.8: Frequency response comparison of original and the ROMs

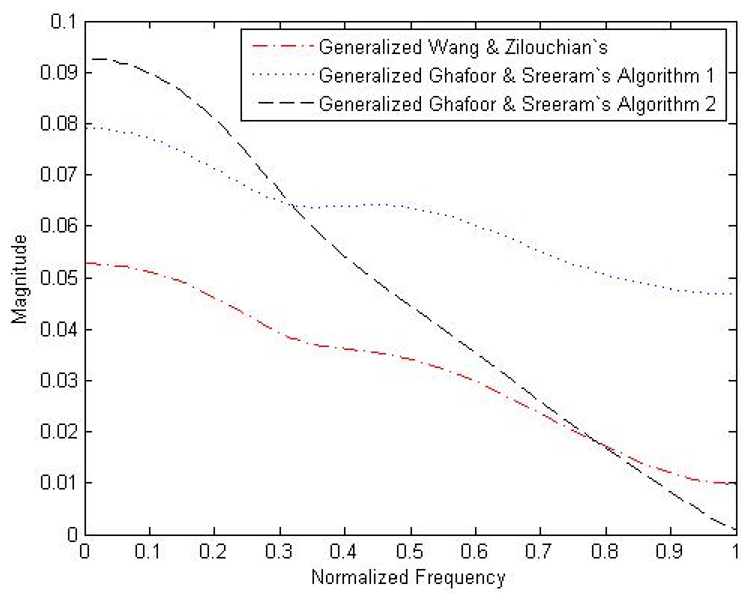

Figure 4.9: Frequency response error comparison of the ROMs

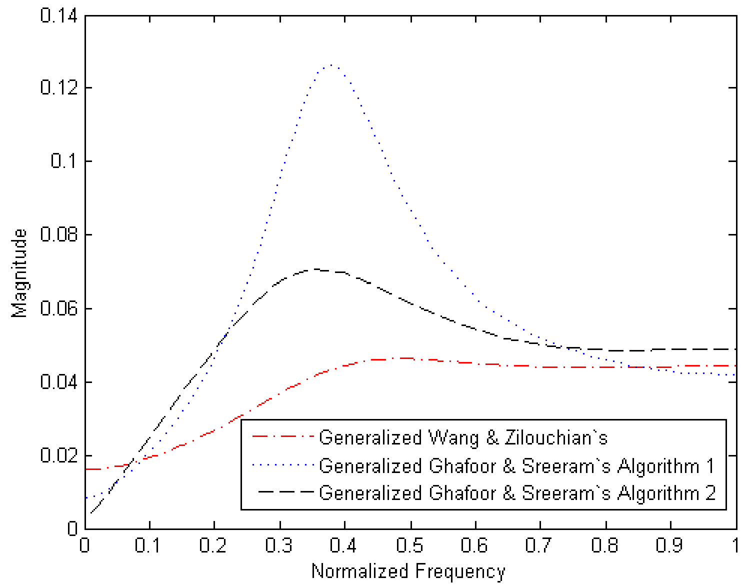

Figure 4.10: Frequency response error comparison of the ROMs

by generalized Wang and Zilouchian's method, generalized Ghafoor and Sreeram method Algorithms (1 and 2) respectively, in the frequency range $0.3\ \pi$ to $0.6\ \pi$.

Example 4.8.6 *Consider a 20^{th} order original stable generalized nonsingular discrete time system with state space representation $[E, A, B, C, D]$ where*

$$E = \begin{bmatrix} E_1 & E_2 \\ E_2^T & E_3 \end{bmatrix}, \text{where } E_1 = \begin{bmatrix} 10 & 7 & 4 & 2 & 1 \\ 0 & 8 & 5 & 1 & 1 \\ 1 & 5 & 7 & 0 & 2 \\ 1 & 2 & 3 & 6 & 1 \\ 1 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad E_2 = 0_{5\times 15}, \quad E_3 = I_{15\times 15}$$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \text{where}$$

$$A_1 = [-2.8156\ -1.9564\ 0.1688\ -2.8071\ -6.7556\ -4.3113\ -1.7602\ -4.4958$$
$$-6.0439 - 4.0588\ -3.0452\ -3.3386\ -2.8658\ -2.0864\ -1.5842\ -1.0681$$
$$-0.6456\ -0.4648 - 0.2843\ -0.0829]$$

$$A_2 = \begin{bmatrix} I_{19\times 19} & 0_{19\times 1} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0_{1\times 19} \end{bmatrix}^T$$

$$C = [-0.0346\ -0.1470\ \ 0.0021\ \ 0.5189\ -0.0831\ -1.5288\ -0.0216\ \ 2.5273$$
$$-0.0743\ -3.149 - 0.0374\ \ 2.5415\ -0.0352\ -1.5014\ -0.0195\ 0.5403$$
$$-0.0079\ -0.1287\ -0.0035\ \ 0.0113]$$

$$D = 0.0123$$

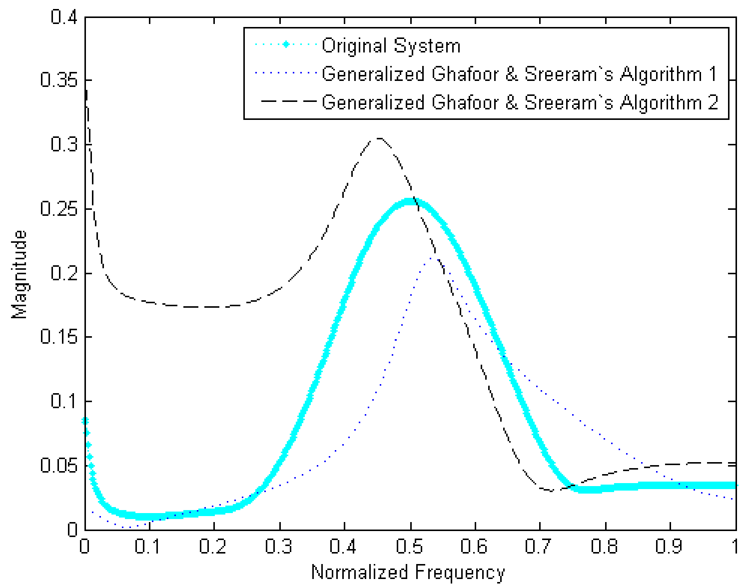

Figure 4.11: Frequency response comparison of original and the ROMs

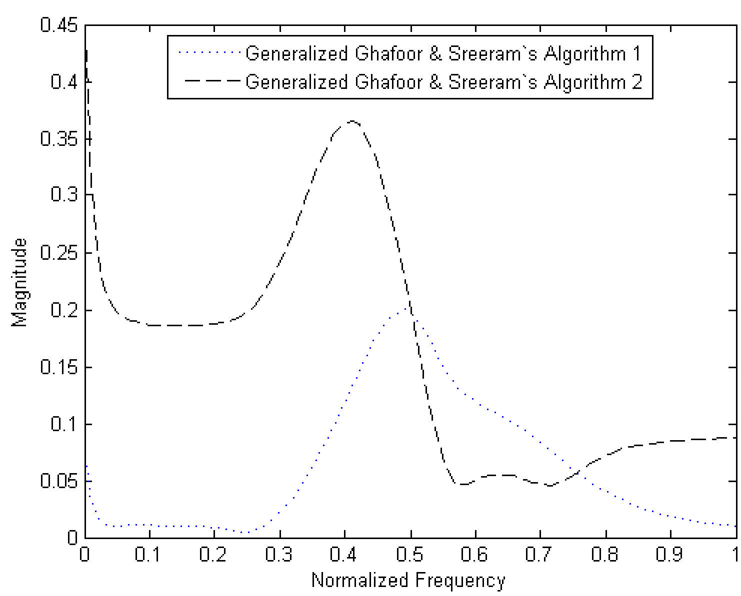

Figure 4.12: Frequency response comparison of original and the ROMs

Fig. 4.11 and Fig. 4.12 represent frequency response and frequency response errors for the seventh order ROMs obtained by generalized Ghafoor and Sreeram's method Algorithms (1 and 2) in the frequency interval $0.5\,\pi$ to $0.7\,\pi$ respectively. Moreover, Fig. 4.12 also includes original system frequency response in the frequency range 0 to π.

Discussion: It is observed that in the desired frequency interval generalized Wang and Zilouchian's method yields lower approximation error but it may give unstable ROMs. Generalized Ghafoor and Sreeram's method Algorithms (1 and 2) provide comparable results and have the advantage of providing stable ROMs. The numerical comparison of generalized Wang and Zilouchian method is not shown in example 4.8.6 because of limitation of symmetric matrices X_{GWZ} and Y_{GWZ} which become indefinite and thereby cause hinderance in the balancing procedure. Due to this limitation, the controllability Gramian P_{GWZ} is not positive definite e.g. some eigenvalues of P_{GWZ} in example 4.8.6 are -0.0404, -0.0038, -0.0016, -0.0013, -0.0003, -0.0001. Likewise observability Gramian Q_{GWZ} is also not positive definite e.g. some eigenvalues of Q_{GWZ} in example 4.8.6 are -0.0384, -0.0030, -0.0012, -0.0010, -0.0004, -0.0004, -0.0001. So balancing using generalized Wang and Zilouchian method in example 4.8.6 is not possible.

4.9 Conclusion

In this chapter, three existing FLMR techniques including Gawronski and Juang [29], Gugercin and Antoulas [30] and Ghafoor and Sreeram [8] for continuous time systems and the following three discrete time (Wang and Zilouchian's [90], Ghafoor and Sreeram's algorithms (1 and 2) [87]) FLMR techniques (previously proposed for standard state space systems) are modified for generalized nonsingular systems, are presented. The comparison of results shows that although generalized Gawronski and Juang technique may sometimes yield unstable ROMs, but it gives a better approximation as compared to other two techniques. Generalized Gugercin and Antoulas and generalized Ghafoor and Sreeram techniques provide stable ROMs and have error bounds. Moreover, the generalized Wang and Zilouchian method may produce unstable ROMs but generalized Ghafoor and Sreeram's method Algorithms (1 and 2) always yield stable ROMs for discrete time systems.

In the next chapter, we further generalize the results for descriptor (singular) systems.

Chapter 5

FLMR: Generalized Descriptor Systems

5.1 Introduction

Descriptor systems find their presence in a number of applications which include semidiscretization of partial differential equations, multi-body dynamics with constraints, electrical circuit simulation and micro-electro-mechanical system [44, 45, 46, 47, 48, 49]. The derivation of a reasonable mathematical model is fundamental to obtain a good understanding of the dynamical behavior of a physical system in question or to control its behavior in order to achieve desired performance specifications. In practice, modelling of complex systems (such as chip design, fluid flow, mechanical systems simulation) yield very large scale descriptor systems. Despite the advancement of technology and the ever increasing computational speed, the analysis, control and optimization of large scale systems is challenging (if not impossible) due to expensive computations and storing requirements. Therefore, generating low-dimensional or ROMs that provide a good estimate of the original full order system has caught a lot of attention in both mathematical and engineering communities [10, 51, 52, 53, 54, 88].

Currently, there exist many methods for model reduction of descriptor systems [56,57,58]. Some of these methods are based on Krylov subspaces [3,55]. Although, these methods have been applied successfully for several different applications of microsystem design and circuit simulations, however, these methods yield good local estimates, but do not provide global error bounds.

BT (a popular technique which has been previously proposed for standard state space systems [1]) has also been extended to descriptor systems [56]. This extension is based on the analysis of improper and proper Gramians defined as solutions of the projected generalized Lyapunov equations [56, 57]. Some important properties of the BT model reduction technique include preservation of stability and existence of an *a priori* error bound for the ROM. The frequency response error tend towards zero at very high frequencies, but it is generally non-zero at very low frequencies. A reverse statement is true in the BSPA case [59].

Ideally, one would like to have the small reduction error for all frequencies. Since, in many practical scenarios, it is desirable that in certain frequency intervals the ROMs have small frequency response errors. Gawronski and Juang [29] introduced a scheme for standard state space systems, where the frequency weights are explicitly not predefined and approximation is considered in a given frequency interval.

To the best of authors knowledge, FLMR for descriptor systems has never been considered in literature. In this chapter, we present FLMR scheme (partially published in [15]) for general descriptor systems. The method generalizes the results by [29] for large-scale descriptor systems using frequency interval Gramians. Simple algorithms are also given for preserving the stability of ROMs. The work also extends Poor Man's truncated balanced realization (PMTBR) scheme [61] to incorporate FLMR (proposed EPMTBR) for descriptor systems. Note that (as also pointed out in [62]), PMTBR method uses only controllability Gramian, therefore it works well for symmetric systems where controllability and observability matrices are same. But it may not work well for general (unsymmetrical systems like RLC interconnect systems) descriptor systems. However, since EPMTBR considers both controllability and observability Gramians, therefore, it works even for non-symmetrical systems. Practical numerical examples are also incorporated to show the successful application of the proposed method in the desired frequency range.

5.2 Preliminaries

Let n^{th} order continuous-time linear time-invariant system with transfer function

$$G(s) = C(sE - A)^{-1}B + D \tag{5.1}$$

where $E, A \in \Re^{n,n}$, $B \in \Re^{n,m}$, $C \in \Re^{p,n}$, $D \in \Re^{p,m}$. Here, n is the order of system (5.1), m is the number of inputs and p is the number of outputs. If $E = I$, then (5.1) is a standard state space system, else, system (5.1) is called a generalized state space system or a descriptor system. This may include a singular or non-singular matrix E. Some work linked with model reduction of small scale systems having non-singular E matrix appears in [18]. However, this paper deals with the model reduction of generalized large scale descriptor systems, irrespective of E is singular or non-singular.

A MOR problem for the system (5.1) consists in finding a ROM

$$G_r(s) = \widetilde{C}_r(s\widetilde{E}_r - \widetilde{A}_r)^{-1}\widetilde{B}_r + \widetilde{D} \tag{5.2}$$

where r is the dimension of ROM such that $r \ll n$. Assume that the pencil $\lambda E - A$ is regular, i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in C$. Then there exist nonsingular matrices T_l and T_r such that

$$E = T_l \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T_r, \quad A = T_l \begin{bmatrix} A_f & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T_r, \tag{5.3}$$

where N and A_f are in Jordan canonical form matrices, where N is a nilpotent matrix of nilpotency index ν, see [63]. A_f contains finite eigenvalues of $\lambda E - A$ and N contains eigenvalue at infinity. Since n_∞ and n_f are the dimensions of infinite and finite eigenvalues associated with the deflating subspaces of pencil $\lambda E - A$ respectively, and ν be the index of the pencil $\lambda E - A$. The matrices

$$P_r = T_r^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_r, \quad P_l = T_l \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1} \tag{5.4}$$

are the spectral projectors against the left and right deflating subspaces of the pencil $\lambda E - A$ associated with the finite eigenvalues along the left and right deflating subspaces corresponding to the eigenvalues at infinity [57]. Furthermore, $Q_r = I - P_r$ and $Q_l = I - P_l$ are complementary projectors.

Consider the descriptor system (5.1) with the regular matrix pencil $\lambda E - A$. A transfer function shown in (5.1) maps the input-output relation in the frequency domain. The transfer function $\mathbf{G}(s)$ is said to be proper if $\lim_{s\to\infty} \mathbf{G}(s) < \infty$, and it is strictly proper if $\lim_{s\to\infty} \mathbf{G}(s) = 0$. Let the matrices

$$B = T_l \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}, \qquad C = [\, C_f, \; C_\infty \,]\, T_r \tag{5.5}$$

be partitioned accordingly to E and A in (5.3). Then the transfer function $\mathbf{G}(s)$ can be separated as $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$, where $\mathbf{G}_{sp}(s) = C_f(sI - A_f)^{-1}B_f$ and

$$\mathbf{P}(s) = C_\infty(sN - I)^{-1}B_\infty + D = -\sum_{i=0}^{\nu-1} C_\infty N^i B_\infty s^i + D$$

are, correspondingly, the strictly proper part and the polynomial part of $\mathbf{G}(s)$. Clearly, $\mathbf{G}(s)$ is said to be proper if and only if $C_\infty N^i B_\infty = 0$ for $i \geq 1$. It is strictly proper, if, additionally, $-C_\infty B_\infty + D = 0$.

Consider the proper observability $\mathcal{G}_{po}$ and controllability $\mathcal{G}_{pc}$ Gramians (of the system (5.1)

$$\mathcal{G}_{po} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-\imath\omega E - A)^{-T}P_r^T C^T C P_r(\imath\omega E - A)^{-1}d\omega, \tag{5.6}$$

$$\mathcal{G}_{pc} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\imath\omega E - A)^{-1}P_l B B^T P_l^T(-\imath\omega E - A)^{-T}d\omega, \tag{5.7}$$

are the symmetric, unique, positive semidefinite solutions of the projected continuous-time algebraic Lyapunov equations (PCALEs) [56]

$$E^T\mathcal{G}_{po}A + A^T\mathcal{G}_{po}E = -P_r^T C^T C P_r, \quad \mathcal{G}_{po} = P_l^T\mathcal{G}_{po}P_l, \tag{5.8}$$

$$E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T = -P_l B B^T P_l^T, \quad \mathcal{G}_{pc} = P_r\mathcal{G}_{pc}P_r^T, \tag{5.9}$$

Moreover, the improper observability $\mathcal{G}_{io}$ and controllability $\mathcal{G}_{ic}$ Gramians

$$\mathcal{G}_{io} = \frac{1}{2\pi}\int_{0}^{2\pi}(e^{-\imath\omega}E - A)^{-T}Q_r^T C^T C Q_r(e^{\imath\omega}E - A)^{-1}d\omega. \tag{5.10}$$

$$\mathcal{G}_{ic} = \frac{1}{2\pi}\int_{0}^{2\pi}(e^{\imath\omega}E - A)^{-1}Q_l B B^T Q_l^T(e^{-\imath\omega}E - A)^{-T}d\omega, \tag{5.11}$$

are symmetric, unique, positive semidefinite solutions of the projected discrete-time algebraic Lyapunov equations

$$A^T\mathcal{G}_{io}A - E^T\mathcal{G}_{io}E = Q_r^T C^T C Q_r, \qquad \mathcal{G}_{io} = Q_l^T\mathcal{G}_{io}Q_l. \tag{5.12}$$

$$A\,\mathcal{G}_{ic}A^T - E\,\mathcal{G}_{ic}E^T = Q_l B B^T Q_l^T, \quad \mathcal{G}_{ic} = Q_r\mathcal{G}_{ic}Q_r^T, \tag{5.13}$$

From the improper and proper Gramians, we can define the improper and proper HSV of the descriptor system (5.1) that are of great interest in BT model reduction. The improper HSV θ_j are defined as the square roots of the largest n_∞ eigenvalues of $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$, while the proper HSV σ_j the square roots of the largest n_f eigenvalues of the matrix $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$. The BT approach for the model reduction of the descriptor system (5.1) is to transform given system into a balanced form such that

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = diag(\sigma_1, \ldots, \sigma_{n_f}, 0, \ldots, 0),$$
$$\mathcal{G}_{ic} = \mathcal{G}_{io} = diag(0, \ldots, 0, \theta_1, \ldots, \theta_{n_\infty})$$

and to truncate the states associated with zero improper and small proper HSV. Note that the equations that represent the improper HSV define a manifold in which the solution dynamics take place. Thus, a truncation of the states associated with the small nonzero improper HSV may cause an inaccurate approximation [64].

5.3 Proposed technique

5.3.1 Frequency limited proper Gramians

Let $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ be the frequency limited proper controllability and observability Gramians of the system (5.1) defined via

$$\mathcal{G}_{pcf} = \frac{1}{2\pi}\int_{\delta\omega}(\iota\omega E - A)^{-1}P_l BB^T P_l^T(-\iota\omega E - A)^{-T}d\omega, \quad (5.14)$$

$$\mathcal{G}_{pof} = \frac{1}{2\pi}\int_{\delta\omega}(-\iota\omega E - A)^{-T}P_r^T C^T C P_r(\iota\omega E - A)^{-1}d\omega, \quad (5.15)$$

where

$$\delta\omega = \bigcup_{j=1}^{q}([-\beta_j, -\alpha_j] \cup [\alpha_j, \beta_j])$$

is the frequency range of operation with $0 \le \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_q < \beta_q$. For simplicity, we will consider the frequency interval $\delta\omega = [-\beta, -\alpha] \cup [\alpha, \beta]$ only. Note that the integration interval should be symmetric with respect to zero. This will guarantee that the Gramians $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ are real, symmetric and positive semidefinite. In the special case, when $\alpha_1 = 0$ and $\beta_1 = \infty$, the Gramians $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ coincide with the proper Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$, respectively. For $E = I$, the Gramians in (5.14) and (5.15) are the frequency limited Gramians for standard state space systems considered in [29].

The following proposition shows that the frequency limited Gramians are the solutions of certain projected Lyapunov equations.

Proposition 5.3.1 *The frequency limited proper Gramians $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ defined as in* (5.14) *and* (5.15)*, respectively, satisfy the PCALEs*

$$E\,\mathcal{G}_{pcf}A^T + A\mathcal{G}_{pcf}E^T = -E\,FP_lBB^TP_l^T - P_lBB^TP_l^TF^TE^T, \qquad \mathcal{G}_{pcf} = P_r\mathcal{G}_{pcf}P_r^T, \quad (5.16)$$

$$E^T\,\mathcal{G}_{pof}A + A^T\mathcal{G}_{pof}E = -E^TF^TP_r^TC^TCP_r - P_r^TC^TCP_rFE, \qquad \mathcal{G}_{pof} = P_l^T\mathcal{G}_{pof}P_l, \quad (5.17)$$

where

$$F = \frac{1}{2\pi}\int_{\delta\omega}(\imath\omega E - A)^{-1}\,d\omega. \tag{5.18}$$

Proof *First of all note that the matrix F in (5.18) is real. Indeed,*

$$\begin{aligned}\bar{F} &= \frac{1}{2\pi}\int_{-\beta}^{-\alpha}(-\imath\omega E - A)^{-1}\,d\omega + \frac{1}{2\pi}\int_{\alpha}^{\beta}(-\imath\omega E - A)^{-1}\,d\omega\\ &= \frac{1}{2\pi}\int_{\alpha}^{\beta}(\imath\omega E - A)^{-1}\,d\omega + \frac{1}{2\pi}\int_{-\beta}^{-\alpha}(\imath\omega E - A)^{-1}\,d\omega = F.\end{aligned}$$

We now show the following relations

$$\mathcal{G}_{pcf} = FE\,\mathcal{G}_{pc} + \mathcal{G}_{pc}E^TF^T, \tag{5.19}$$

$$\mathcal{G}_{pof} = F^TE^T\,\mathcal{G}_{po} + \mathcal{G}_{po}EF, \tag{5.20}$$

between the frequency limited proper Gramians $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ and the proper Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$. From the PCALE (5.9) we have

$$\begin{aligned}P_lBB^TP_l^T &= -E\,\mathcal{G}_{pc}A^T - A\,\mathcal{G}_{pc}E^T\\ &= E\,\mathcal{G}_{pc}(-\imath\omega E^T - A^T) + (\imath\omega E - A)\mathcal{G}_{pc}E^T.\end{aligned}$$

Pre-multiplying and post-multiplying the both sides of this equation by $(\imath\omega E - A)^{-1}$ and $(-\imath\omega E^T - A^T)^{-1}$, respectively, and integrating on $\delta\omega$, we obtain (5.19). The equation (5.20) can be derived analogously from the PCALE (5.8).

Further, we obtain from (5.19) that

$$E\,\mathcal{G}_{pcf}A^T + A\mathcal{G}_{pcf}E^T = EFE\,\mathcal{G}_{pc}A^T + E\mathcal{G}_{pc}E^TF^TA^T + AFE\,\mathcal{G}_{pc}E^T + A\mathcal{G}_{pc}E^TF^TE^T. \tag{5.21}$$

Next we show that $AFE = EFA$. Indeed, using Weierstrass canonical form (5.3) we get

$$\begin{aligned}
EFA &= \frac{1}{2\pi}\int_{\delta\omega} E(\iota\omega E - A)^{-1}A d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} T_l \begin{bmatrix} (\iota\omega I - A_f)^{-1}A_f & 0 \\ 0 & N(\iota\omega N - I)^{-1} \end{bmatrix} T_r\, d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} T_l \begin{bmatrix} A_f(\iota\omega I - A_f)^{-1} & 0 \\ 0 & (\iota\omega N - I)^{-1}N \end{bmatrix} T_r\, d\omega \\
&= \frac{1}{2\pi}\int_{\delta\omega} A(\iota\omega E - A)^{-1}E d\omega = AFE.
\end{aligned}$$

Hence, we can continue (5.21) as

$$\begin{aligned}
E\,\mathcal{G}_{pcf}A^T + A\mathcal{G}_{pcf}E^T &= EF\left(E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T\right) + \left(E\mathcal{G}_{pc}A^T + A\mathcal{G}_{pc}E^T\right)F^TE^T \\
&= -EFP_lBB^TP_l^T - P_lBB^TP_l^TF^TE^T.
\end{aligned}$$

Equation $\mathcal{G}_{pcf} = P_r\mathcal{G}_{pcf}P_r^T$ can be verified using the Weierstrass canonical form (5.3), (5.4) and (5.14). Thus, $\mathcal{G}_{pcf}$ satisfies the PCALE (5.16). Equation (5.17) follows similarly from (5.20).

An energy interpretation of the frequency limited Gramians of standard state space systems appears in [60]. The result can be extended for descriptor systems in a straightforward way.

5.3.2 Proposed algorithms

Similar to [56], the proper frequency limited singular values ξ_j of the descriptor system (5.1) are calculated by taking square roots of the largest n_f eigenvalues of the matrix $\mathcal{G}_{pcf}E^T\mathcal{G}_{pof}E$. The FLMR method for (5.1) is to transform given system into a balancing form such that

$$\begin{aligned}
\mathcal{G}_{pcf} = \mathcal{G}_{pof} &= diag(\xi_1, \ldots, \xi_{n_f}, 0, \ldots, 0), \\
\mathcal{G}_{ic} = \mathcal{G}_{io} &= diag(0, \ldots, 0, \theta_1, \ldots, \theta_{n_\infty})
\end{aligned}$$

and truncation the states associated with the small ξ_j and zero θ_j. Consider the following algorithm.

Algorithm 1 *FLMR.*

Given the original full order stable system $[\,E,\, A,\, B,\, C,\, D\,]$ such that $\lambda E - A$ is regular, i.e., $\det(\lambda E - A) \neq 0$ *for some $\lambda \in C$ and a frequency range $\delta\omega$*

1. *Calculate* R_p *and* L_p *by the Cholesky factorization of the frequency limited proper Gramians*
$\mathcal{G}_{pcf} = R_p R_p^T$ *and* $\mathcal{G}_{pof} = L_p L_p^T$.

2. *Calculate* R_i *and* L_i *by the Cholesky factorization of improper Gramians* $\mathcal{G}_{ic} = R_i R_i^T$ *and* $\mathcal{G}_{io} = L_i L_i^T$ *which satisfies* (5.13) *and* (5.12)*, respectively.*

3. *Calculate the SVD for proper part*

$$L_p^T E R_p = [\,U_{p1},\, U_{p2}\,] \begin{bmatrix} \Sigma_{p1} & 0 \\ 0 & \Sigma_{p2} \end{bmatrix} [\,V_{p1},\, V_{p2}\,]^T, \tag{5.22}$$

where the matrices $[\,U_{p1},\, U_{p2}\,]$ *and* $[\,V_{p1},\, V_{p2}\,]$ *consist of orthonormal columns,* $\Sigma_{p1} = diag(\xi_1, \ldots, \xi_{\ell_f})$ *and* $\Sigma_{p2} = diag(\xi_{\ell_f+1}, \ldots, \xi_{r_p})$ *with* $r_p = rank(L_p^T E R_p)$.

4. *Calculate SVD for improper part*

$$L_i^T A R_i = U_{i3} \Theta_{i3} V_{i3}^T,$$

where V_{i3} *and* U_{i3} *consists of orthonormal columns, also* $\Theta_{i3} = diag(\theta_1, \ldots, \theta_{\ell_\infty})$ *with* $\ell_\infty = rank(L_i^T A R_i)$.

5. *Calculate the ROM*

$$[\,\widetilde{E}_r,\, \widetilde{A}_r,\, \widetilde{B}_r,\, \widetilde{C}_r,\, \widetilde{D}\,] = [\,W_1^T E T,\, W_1^T A T,\, W_1^T B,\, C T,\, D\,]$$

with the projection matrices

$$\begin{aligned} W_1 &= [\,L_p U_{p1} \Sigma_{p1}^{-1/2},\, L_i U_{i3} \Theta_{i3}^{-1/2}\,], \\ T &= [\,R_p V_{p1} \Sigma_{p1}^{-1/2},\, R_i V_{i3} \Theta_{i3}^{-1/2}\,]. \end{aligned}$$

Since the ROM has the system matrices of the form

$$\begin{aligned} \tilde{E} &= \begin{bmatrix} I_{\ell_f} & 0 \\ 0 & \tilde{E}_\infty \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} \tilde{A}_f & 0 \\ 0 & I_{\ell_\infty} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} \Sigma_{p1}^{-1/2} U_{p1}^T L_p^T B \\ \Theta_{i3}^{-1/2} U_{i3}^T L_i^T B \end{bmatrix} = \begin{bmatrix} \tilde{B}_f \\ \tilde{B}_\infty \end{bmatrix}, \\ \tilde{C} &= \left[C R_p V_{p1} \Sigma_{p1}^{-1/2},\, C R_i V_{i3} \Theta_{i3}^{-1/2} \right] = \left[\tilde{C}_f,\, \tilde{C}_\infty \right], \end{aligned}$$

where

$$\tilde{A}_f = \Sigma_{p1}^{-1/2} U_{p1}^T L_p^T A R_p V_{p1} \Sigma_{p1}^{-1/2},$$
$$\tilde{E}_\infty = \Theta_{i3}^{-1/2} U_{i3}^T L_i^T E R_i V_{i3} \Theta_{i3}^{-1/2}.$$

By making use of Weierstrass canonical form (5.3) and (5.4), (5.5), the Cholesky factorization of the improper Gramians is determined as

$$\begin{aligned} R_i &= T_r^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -B_\infty & -NB_\infty & \cdots & -N^{\nu-1}B_\infty \end{bmatrix}, \\ L_i &= T_l^{-T} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -C_\infty^T & -N^T C_\infty^T & \cdots & -(N^{\nu-1})^T C_\infty^T \end{bmatrix}. \end{aligned}$$

Then the matrix $L_i^T E R_i$ can be written as

$$L_i^T E R_i = \begin{bmatrix} C_\infty N B_\infty & \cdots & C_\infty N^{\nu-1} B_\infty & 0 \\ \vdots & & 0 & \vdots \\ C_\infty N^{\nu-1} B_\infty & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Thus, if the system (5.1) is proper, then $L_i^T E R_i = 0$ and, hence, $\tilde{E}_\infty = 0$. Here, the ROM transfer function takes the form

$$\tilde{\mathbf{G}}(s) = \tilde{C}_f (sI - \tilde{A}_f)^{-1} \tilde{B}_f - \tilde{C}_\infty \tilde{B}_\infty + D.$$

Since the matrix $EFP_lBB^TP_l^T + P_lBB^TP_l^TF^TE^T$ in (5.16) and the matrix $E^TF^TP_r^TC^TCP_r + P_r^TC^TCP_rFE$ in (5.17) may not fulfill the condition of positive semidefiniteness, hence the stability of the ROM calculated via Algorithm 1 is not guaranteed. In order to guarantee the stability of the ROM, we combine the proper Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ with the frequency limited Gramians $\mathcal{G}_{pof}$ and $\mathcal{G}_{pcf}$, respectively, as presented in the following algorithms.

Algorithm 2 *Stability-preserving FLMR.*

Given the original stable full order system $[\,E,\, A,\, B,\, C,\, D\,]$ *such that* $\lambda E - A$ *is regular, i.e.,* $\det(\lambda E - A) \neq 0$ *for some* $\lambda \in C$ *and a frequency range* $\delta\omega$

1. *Calculate* R_p *and* L_p *by the Cholesky factorization of the Gramians* $\mathcal{G}_{po} = L_pL_p^T$*. and* $\mathcal{G}_{pcf} = R_pR_p^T$ *respectively.*

2)-5) The same as in Algorithm 1.

Algorithm 3 *Stability-preserving FLMR.*

Given the original stable full order system $[\,E,\,A,\,B,\,C,\,D\,]$ *such that* $\lambda E - A$ *is regular, i.e.,* $\det(\lambda E - A) \neq 0$ *for some* $\lambda \in C$ *and a frequency range* $\delta\omega$

1. *Calculate* R_p *and* L_p *by the Cholesky factorization of the Gramians* $\mathcal{G}_{pof} = L_p L_p^T$ *and* $\mathcal{G}_{pc} = R_p R_p^T$ *respectively.*

2)-5) The same as in Algorithm 1.

Proposition 5.3.2 *The ROMs calculated by using Algorithms 2 and 3 are asymptotically stable.*

proof *The proof follows similarly to one given in [5] for one-sided FWMR case.*

In case of a symmetric descriptor system (5.1) with $E = E^T$, $A = A^T$ and $B = C^T$, Algorithms 2 and 3 yield the same result. Moreover, for any general descriptor system (5.1), when $\delta\omega = (-\infty, \infty)$, the proposed algorithms yield the same results as in [57].

5.3.3 Computation of frequency limited Gramians

In the following section we elaborate the process for calculation of the Cholesky factors of the frequency limited Gramians $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$.

Since the right-hand sides in the projected Lyapunov equations (5.16) and (5.17) are possibly indefinite, we cannot use these equations to compute the required Cholesky factors directly without actually computing the frequency limited Gramians as it is done in classical BT [57, 65]. Moreover, the representations (5.19) and (5.20) are also useless since it involves computation of Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ which are not preferred in large-scale system case. Moreover, the matrix F involves the computation of the integral over the desired frequency range. Further research/investigation is necessary for computing Cholesky factors of frequency limited Gramians $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ obtained using equations (5.16) and (5.17) or (5.19) and (5.20).

Therefore, we determine the Cholesky factors of $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ by evaluating the integrals (5.14) and (5.15) using a quadrature formula with nodes ω_j and weights γ_j.

So for application of quadrature rules, the Gramian $\mathcal{G}_{pcf}$ in (5.14) can be approximated as

$$\begin{aligned}\mathcal{G}_{pcf} &\approx \frac{1}{2\pi}\sum_{j=1}^{k}\gamma_j\left\{(\iota\omega_j E - A)^{-1}P_l BB^T P_l^T(-\iota\omega_j E - A)^{-T}\right.\\ &\left.+(-\iota\omega_j E - A)^{-1}P_l BB^T P_l^T(\iota\omega_j E - A)^{-T}\right\}. \end{aligned} \tag{5.23}$$

Assuming that all γ_j are positive, we rewrite (5.23) as

$$\mathcal{G}_{pcf} \approx \tfrac{1}{2}\left[\,B_1,\,\bar{B}_1,\,\ldots,\,B_i,\,\bar{B}_i\,\right]\left[\,B_1,\,\bar{B}_1,\,\ldots,\,B_i,\,\bar{B}_i\,\right]^*, \tag{5.24}$$

where $B_j = \sqrt{\gamma_j/\pi}(\iota\omega_j E - A)^{-1}P_l B$. Since the number of columns of B and the number of nodes ω_j are typically less as compared to the state-space order n [65] the matrix in the right-hand side in (5.24) is referred as a low-rank approximation of $\mathcal{G}_{pcf}$ and $1/\sqrt{2}\left[\,B_1,\,\bar{B}_1,\,\ldots,\,B_i,\,\bar{B}_i\,\right]$ is its low-rank Cholesky factor. Note that this factor is complex. However, taking into account

$$\left[\,B_j,\,\bar{B}_j\,\right]\left[\,B_j,\,\bar{B}_j\,\right]^* = 2\left[\,Re(B_j),\,Im(B_j)\,\right]\left[\,Re(B_j),\,Im(B_j)\,\right]^T,$$

we obtain $\mathcal{G}_{pcf} \approx \tilde{R}_p\tilde{R}_p^T$ with a real low-rank Cholesky factor

$$\tilde{R}_p = \left[Re(B_1),\,\ldots,\,Re(B_i),\,Im(B_1),\,\ldots,\,Im(B_i)\right]. \tag{5.25}$$

Similarly, the Gramian $\mathcal{G}_{pof}$ can be approximated as $\mathcal{G}_{pof} \approx \tilde{L}_p\tilde{L}_p^T$, where

$$\tilde{L}_p = \left[\,Re(C_1),\,\ldots,\,Re(C_i),\,Im(C_1),\,\cdots,\,Im(C_i)\,\right] \tag{5.26}$$

with $C_j = \sqrt{\gamma_j/\pi}(-\iota\omega_j E - A)^{-T}P_r^T C^T$.

A major difficulty in the computation of the frequency limited proper Gramians is that the spectral projectors P_l and P_r are needed. However, in several applications including, multibody systems with constraints, electrical circuits and computational fluid dynamics, due to the special form of the matrices E and A, this form can be exploited to obtain the projectors P_l and P_r in explicit form, see [65]. Hence, in the following section it is assumed that these projectors are given.

5.4 Proposed extended version of PMTBR method (EPMTBR)

In this section we propose extended version of PMTBR method [61] and compare it with the FLMR technique.

The proposed EPMTBR method is based on balancing the symmetric, positive semidefinite matrices

$$\begin{aligned}\mathcal{G}_c^{PM} &= \frac{1}{2\pi}\int_{\delta\omega}(\imath\omega E - A)^{-1}BB^T(-\imath\omega E - A)^{-T}d\omega,\\ \mathcal{G}_o^{PM} &= \frac{1}{2\pi}\int_{\delta\omega}(-\imath\omega E - A)^{-T}C^TC(\imath\omega E - A)^{-1}d\omega.\end{aligned}$$

Note that these matrices are related to the frequency limited Gramians as

$$\mathcal{G}_{pcf} = P_r\mathcal{G}_c^{PM}P_r^T, \qquad \mathcal{G}_{pof} = P_l^T\mathcal{G}_o^{PM}P_l.$$

Using the same approach as in Section 5.3.3, $\mathcal{G}_c^{PM}$ and $\mathcal{G}_o^{PM}$ can be approximated by low-rank matrices $\mathcal{G}_c^{PM} \approx RR^T$ and $\mathcal{G}_o^{PM} \approx LL^T$, where

$$\begin{aligned}R &= \left[Re(\hat{B}_1),\, \dots,\, Re(\hat{B}_i),\, Im(\hat{B}_1),\, \dots,\, Im(\hat{B}_i)\right],\\ L &= \left[Re(\hat{C}_1),\, \dots,\, Re(\hat{C}_i),\, Im(\hat{C}_1),\, \cdots,\, Im(\hat{C}_i)\right]\end{aligned}$$

with the matrices $\hat{B}_j = \sqrt{\gamma_j/\pi}(\imath\omega E - A)^{-1}B$ and $\hat{C}_j = \sqrt{\gamma_j/\pi}(-\imath\omega E - A)^{-T}C^T$. Therefore,

$$\begin{aligned}L^TER &= L^T(P_l + Q_l)E(P_r + Q_r)R\\ &= L^TP_lEP_rR + L^TQ_lEQ_rR\\ &= L_p^TER_p + L^TQ_lEQ_rR.\end{aligned}$$

Using (5.3), (5.4) and (5.5), we obtain that the matrix $L^TQ_lEQ_rR$ has the form

$$L^TQ_lEQ_rR = \frac{1}{\pi}\left[\sqrt{\gamma_j\gamma_q}\sum_{i=0}^{\nu-1}\alpha_{jq,i}C_\infty N^{i+1}B_\infty\right]_{j,q=1}^{2i,2i},$$

where $\alpha_{jq,i}$ are real scalars depending on ω_j and ω_q.

If the system (5.1) transfer function $\mathbf{G}(s)$ is proper, then $L^TQ_lEQ_rR = 0$ and L^TER has the same SVD as $L_p^TER_p$. Here the proposed EPMTBR provides the ROM

$$\tilde{E}_{\mathrm{PM}} = W_{\mathrm{PM}}^TET_{\mathrm{PM}}, \;\; \tilde{A}_{\mathrm{PM}} = W_{\mathrm{PM}}^TAT_{\mathrm{PM}},$$

$$\tilde{B}_{\mathrm{PM}} = W_{\mathrm{PM}}^TB, \;\; \tilde{C}_{\mathrm{PM}} = CT_{\mathrm{PM}}, \;\; \tilde{D}_{\mathrm{PM}} = D,$$

where $W_{\mathrm{PM}} = LU_{p1}\Sigma_{p1}^{-1/2}$ and $T_{\mathrm{PM}} = RV_{p1}\Sigma_{p1}^{-1/2}$. We obtain

$$\tilde{E}_{\mathrm{PM}} = I, \;\; \tilde{A}_{\mathrm{PM}} = \tilde{A}_f + \hat{A}_f, \;\; \tilde{B}_{\mathrm{PM}} = \tilde{B}_f + \hat{B}_f, \;\; \tilde{C}_{\mathrm{PM}} = \tilde{C}_f + \hat{C}_f,$$

where

$$\begin{aligned}\hat{A}_f &= \frac{1}{\pi}\Sigma_{p1}^{-1/2}U_{p1}^T\left[\sqrt{\gamma_j\gamma_q}\,C_\infty B_\infty\right]_{j,q=1}^{2i,2i}V_{p1}\Sigma_{p1}^{-1/2},\\ \hat{B}_f &= \frac{1}{\pi}\Sigma_{p1}^{-1/2}U_{p1}^T\left[\sqrt{\gamma_1}\,I,\ldots,\sqrt{\gamma_i}\,I,\ 0,\ldots,0\right]^T C_\infty B_\infty,\\ \hat{C}_f &= \frac{1}{\pi}C_\infty B_\infty\left[\sqrt{\gamma_1}\,I,\ldots,\sqrt{\gamma_i}\,I,\ 0,\ldots,0\right]V_{p1}\Sigma_{p1}^{-1/2}.\end{aligned}$$

Thus, if $\mathbf{G}(s)$ is strictly proper then the FLMR and EPMTBR methods provide the same ROM. However, if $\mathbf{G}(s)$ is the proper system but not strictly proper then the result may be different.

5.5 Numerical examples

In the following section we will demonstrate some numerical illustrative examples to show the usefulness of the suggested model reduction methods for descriptor systems. In order to apply these methods, we incorporate two separate models: a semidiscretized Stoke's equation and inlet flow system.

As elaborated in section 5.3.3, we require to determine the cholesky factors of $\mathcal{G}_{pcf}$ and $\mathcal{G}_{pof}$ by evaluating the integrals (5.14) and (5.15) using a quadrature formula with nodes ω_j and weights γ_j. For this purpose, different quadrature rules e.g. Trapezoidal, Boole and Gauss are explored. Trapezoidal and Boole quadrature rules make use of equal spacing between the nodes. Gauss quadrature rule has the flexibility of using (optimal) spacing between nodes [66], and hence gives relatively good results as compared to Trapezoidal and Boole. The difference between Boole and Gauss is in the process of calculation of weights and nodes. We have included results using Boole and Gauss quadrature rules. The procedure for the calculation of weights and nodes for quadrature formulas are shown in [67]. Similarly, the procedure for calculation of spectral projectors P_l and P_r for the large scale systems used in following examples has been discussed in [65].

Example 5.5.1 ***Semidiscretized Stoke's equation***

Consider the system that represents the flow mechanism of an incompressible liquid described by semidiscritized Stoke's equation. The model description for the above system is given in [57]. By using the finite element method, Stoke's equations lead to the descriptor *system (5.1)*

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1^T, & B_2^T \end{bmatrix}^T, \quad C = [\, C_1, \; C_2 \,], \quad D = 0,$$

where E_{11} is nonsingular and A_{12} has full column rank. By making use of given form, the projectors P_r and P_l for the right and left deflating subspaces of the pencil $\lambda E - A$ are calculated in [65]. In our experiments, the full order descriptor system is of order $n = 17175$. The dimensions of infinite and finite eigenvalues of deflating subspaces are $n_\infty = 11550$ and $n_f = 5625$, respectively.

For the model reduction of the semidiscretized Stokes equation, we will utilize the FLMR method described in Algorithm 1, where the desired frequency interval is

$$\delta\omega = \left[-10^4, -10^2\right] \cup \left[10^2, 10^4\right]$$

and R_p and L_p obtained by Cholesky factorization of the frequency limited proper Gramians are replaced by the low-rank Cholesky factors, $\tilde{R}_p$ and $\tilde{L}_p$ as in (5.25) and (5.26), respectively. For computing these factors we use the composite Boole quadrature with 9, 25 and 49 equidistant nodes. Moreover, results are also obtained using Gauss quadrature rule.

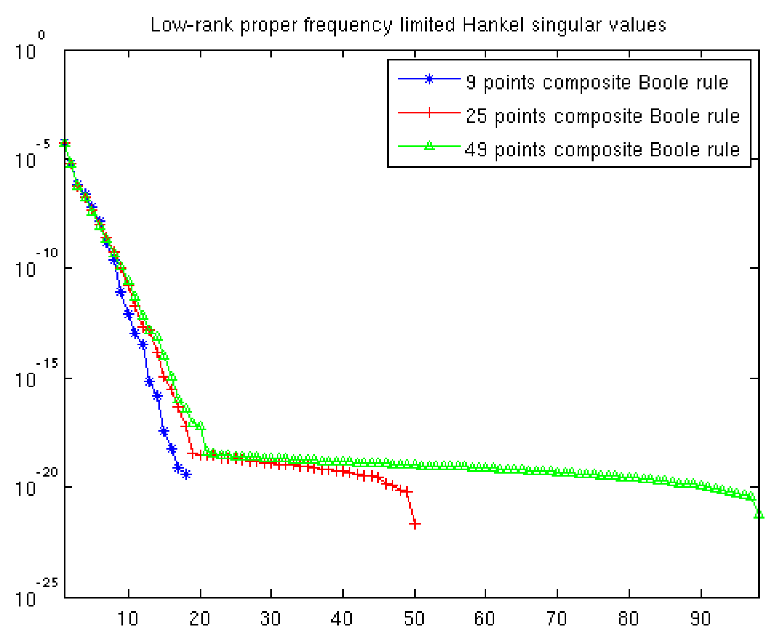

Figure 5.1: Approximate low-rank frequency limited HSV of Stoke's equation.

Fig. 5.1 shows low-rank proper frequency limited HSV ξ_j using 9, 25 and 49 equidistant

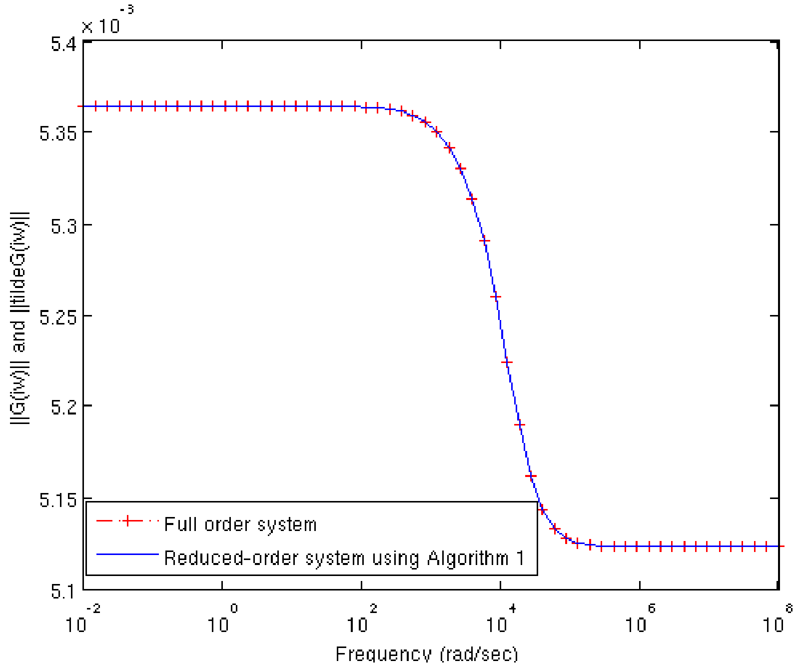

Figure 5.2: Frequency response plot for the full order and the ROM.

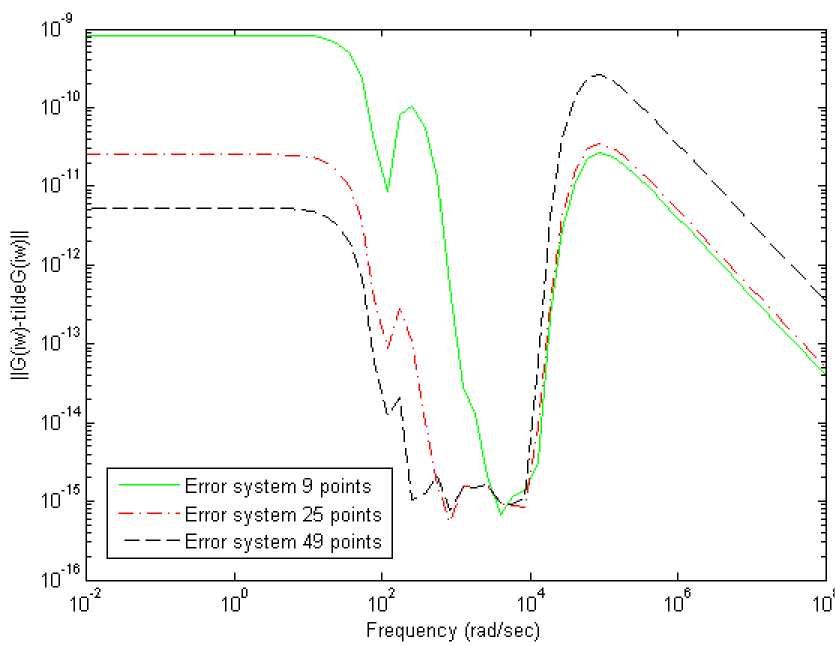

Figure 5.3: Frequency response absolute errors between the full order and the ROM using Boole quadrature rule.

nodes composite Boole quadrature. Note that Boole quadrature rule approximates integral function slightly better than the trapezoidal quadrature rule. Moreover, the integral function approximation slightly improves with more samples as compared to less samples for given frequency intervals.

Similar results are also obtained using Gauss quadrature rule. Note that, the non-zero

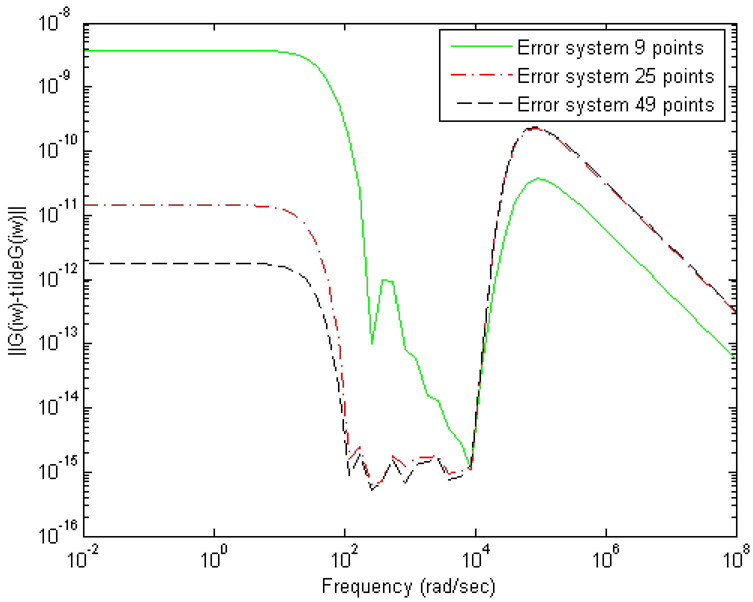

Figure 5.4: Frequency response absolute errors between the full order and the ROM using Gauss quadrature rule.

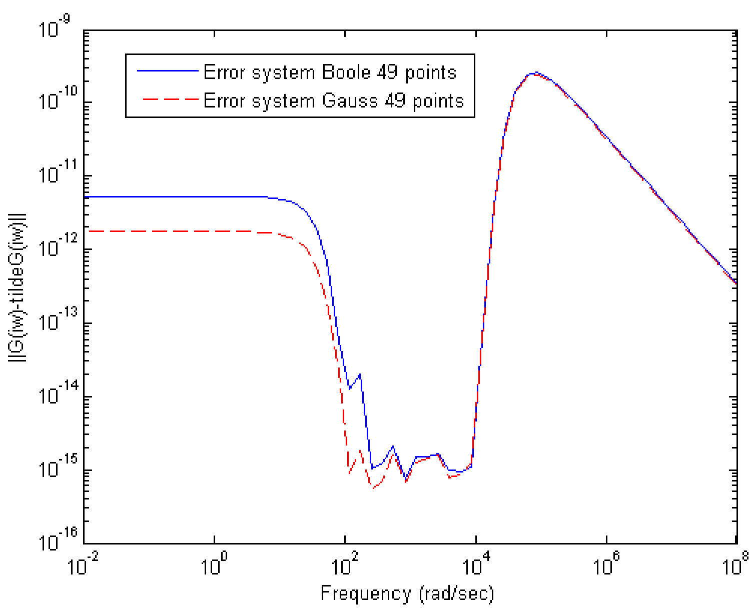

Figure 5.5: Comparison between Boole and Gauss Rule using 49 points $[\omega_1, \omega_2] = [10^2, 10^4] rad/s$.

singular values for a given quadrature are not more than twice the points due to contributions of positive and negative frequency intervals.

An approximation of the Stoke's equation by a model of order $r = 18$ ($\ell_f = 17$, $\ell_\infty = 1$) is calculated by using the frequency limited balance truncation method. The frequency responses of the absolute values of the full order and the ROMs are presented in Fig. 5.2.

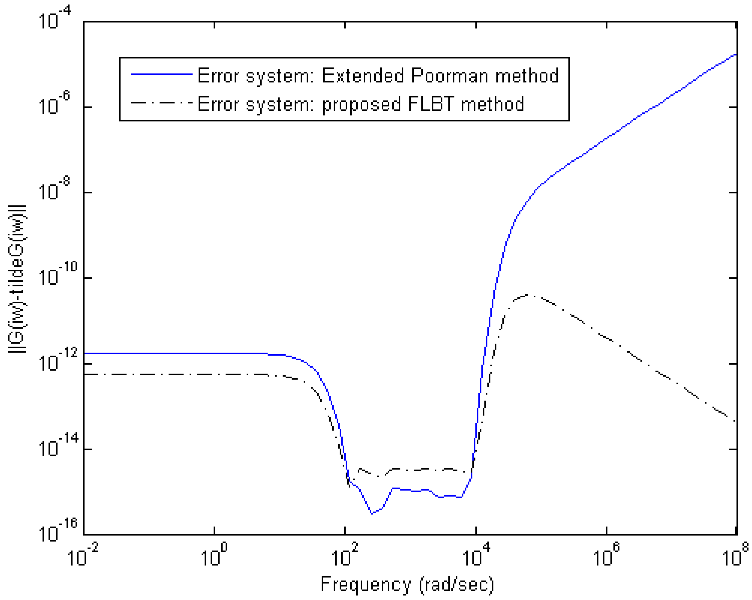

Figure 5.6: Frequency response error between the full order and ROM using the proposed EPMTBR and proposed FLMR techniques

The results in Fig. 5.2 are difficult to differentiate. Fig. 5.3, represents the absolute errors $\|\mathbf{G}(\iota\omega) - \widetilde{\mathbf{G}}(\iota\omega)\|_2$ for a frequency interval of $\delta\omega \in [\,10^{-2},\ 10^{8}\,]$, where the calculation of ROM $\widetilde{\mathbf{G}}(\iota\omega)$ involves 9, 25 and 49 equidistant nodes composite Boole quadrature rule. Note that, there is a significant improvement in approximation error in the case of 49 (as compared to 9 and 25) equidistant nodes composite Boole quadrature rule.

Fig. 5.4 shows the absolute error plot for the reduced system using 9, 25 and 49 points Gauss rule. In this case there is a slight improvement of approximation error in the case of 49 (as compared to 9 and 25) points Gauss rule.

Fig. 5.5 shows the comparison of the error plot obtained by using 49 points Boole and Gauss rule. Note that, 49 points Gauss quadrature rule shows improved error performance in comparison to 49 points Boole quadrature rule. Note that, absolute errors are smaller in the desired frequency range $\delta\omega \in [\,10^{2},\ 10^{4}\,]$.

A numerical comparison between proposed EPMTBR and proposed FLMR techniques using Gauss quadrature rule is shown in Fig. 5.6.

Example 5.5.2 ***Supersonic inlet flow system***

We consider unsteady flow through a supersonic diffuser system as described in [46]. The model is of order $n = 11730$ and has 1 input and 2 outputs. The dimensions of the infinite and finite eigenvalues of the deflating subspaces are $n_\infty = 407$ and $n_f = 11323$, respectively.

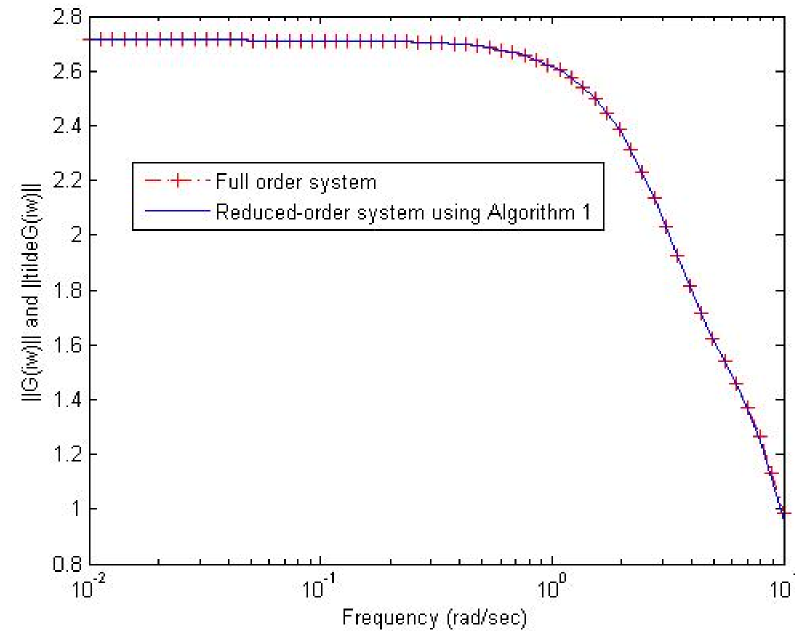

Figure 5.7: Frequency response plot for the full order and the ROM.

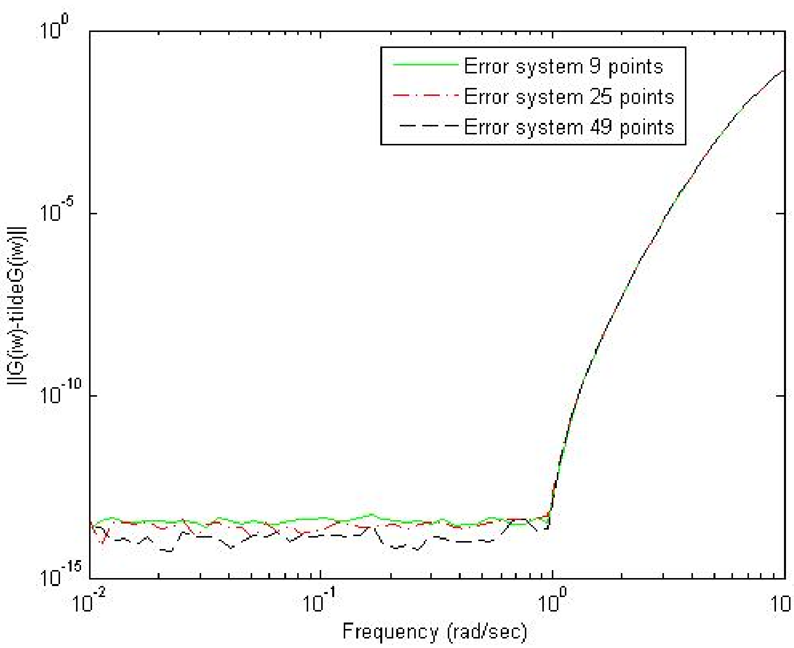

Figure 5.8: Frequency response absolute errors between full order and ROM using Boole quadrature rule.

We approximate the supersonic inlet flow system by a system of order $r = 9$ ($\ell_f = 8, \ell_\infty = 1$) calculated by the frequency limited balance truncation method. Frequency responses for absolute values of the full order and the ROMs are presented in Fig. 5.7. The results in Fig. 5.7 are difficult to differentiate. Fig. 5.8 represents the absolute error plot $\|\mathbf{G}(\iota\omega)-\widetilde{\mathbf{G}}(\iota\omega)\|_2$

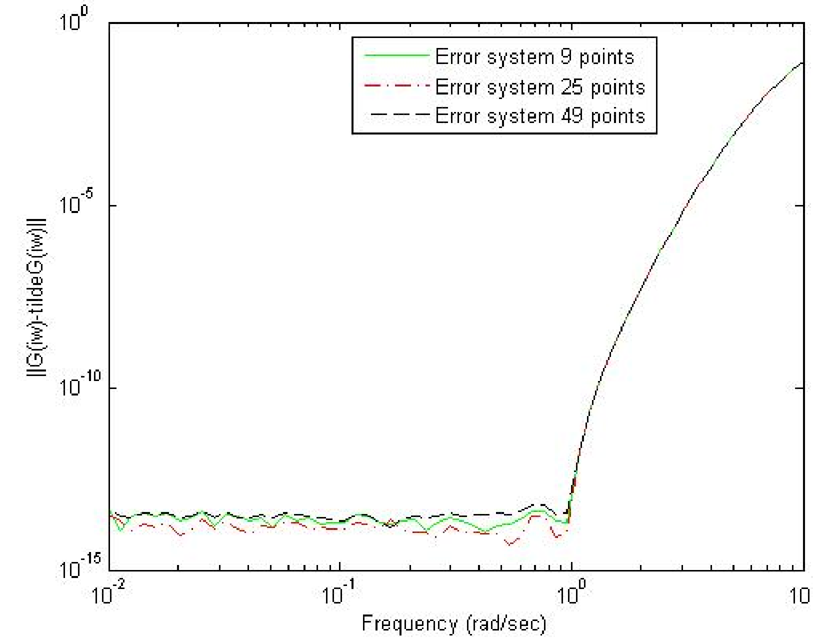

Figure 5.9: Frequency response absolute errors between full order and ROM using Gauss quadrature rule.

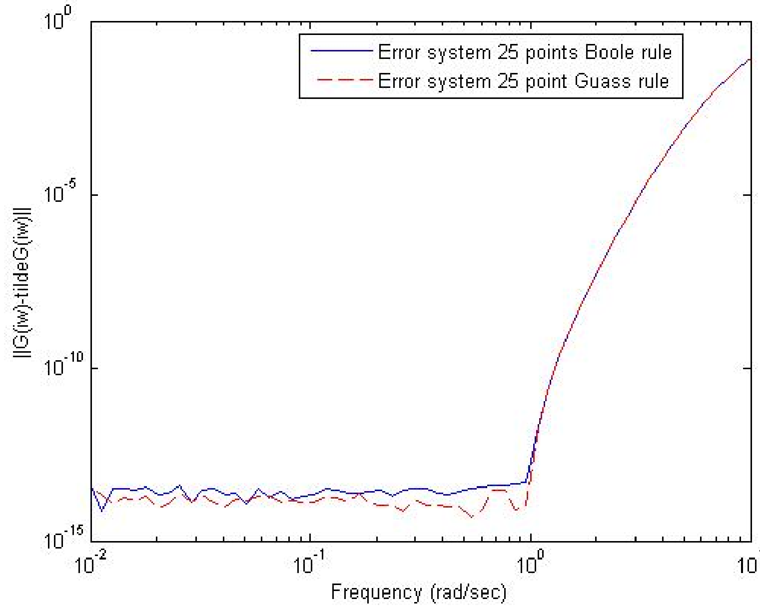

Figure 5.10: Comparison between Boole and Gauss rule using 25 points $[\omega_1, \omega_2] = [0.1, 1] rad/s$.

for a frequency interval $\delta\omega \in [\,10^{-2},\ 10^{1}\,]$, where the calculation of ROM $\widetilde{\mathrm{G}}(\iota\omega)$ involves 9, 25 and 49 equidistant nodes composite Boole quadrature rule. Note that, in this case there is a significant improvement in approximation error 49 equidistant nodes composite Boole quadrature rule (as compared to 9 and 25) equidistant nodes composite Boole quadrature rule.

Fig. 5.9 shows the absolute error plot for the ROM using 9, 25 and 49 points Gauss rule. Note that in this case 25 points gives better results as compared to (9 and 49) points Gauss rule. Fig. 5.10 shows the comparison of the error plot obtained by using 25 points Boole and Gauss rule.

Note that absolute errors are smaller in the desired frequency range $\delta\omega \in [0.1,\ 1]$. It is observed in [68] that by increasing the order of the system, Gauss quadrature weights show fluctuations and therefore affect the overall results, this can also be inferred from Fig. 5.9.

5.6 Conclusion

In this chapter, for a large-scale descriptor systems, we have presented FLMR method using generalized results of [29] for frequency interval Gramians. Stability is preserved using algorithm 2 and 3. Existing PMTBR method is also extended to include frequency limited Gramians. The effectiveness of the proposed method in the desired frequency range is shown using practical numerical examples. The frequency response error in the desired frequency range is reduced.

In the next chapter we propose TLMR techniques.

Chapter 6

TLMR: New Techniques

6.1 Introduction

In this chapter we propose TLMR techniques for continuous time systems.

Many practical MOR problems are inherently dependent upon time intervals [70, 71]. Sometimes, it is required to approximate the original high-order system better in specified time interval than over the entire time range. In order to address this problem, Gawronski and Juang [29] proposed a time limited balanced model order reduction technique, based on time-limited controllability and observability Gramians. However, the disadvantage of this technique is it may yield unstable models for stable original systems. Furthermore, the frequency response error bounds are also not available for this technique. To overcome the instability problem, Gugercin and Antoulas [69] proposed a modified procedure based on [26] from which the frequency response error bounds can also be obtained easily.

In this chapter, modifications to Gawronski and Juang [29] TLMR are proposed. These modified techniques (partially appear in [17]) not only ensure stability of ROMs but also provide comparable error and frequency response error bounds. Numerical examples are also given to show the usefulness of the proposed techniques. The results are compared with the existing TLMR techniques.

6.2 Preliminaries

Consider a linear time invariant continuous time system

$$G(s) = C(sI - A)^{-1}B + D, \tag{6.1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$ and $\{A, B, C, D\}$ is its n^{th} order minimal realization with m inputs and p outputs. The problem of MOR is to to find

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D, \tag{6.2}$$

which approximates the original system (in the desired time interval $[t_1, t_2]$ where $t_2 > t_1$), where $A_{11} \in R^{r\times r}$, $B_1 \in R^{r\times m}$, $C_1 \in R^{p\times r}$, $D \in R^{p\times m}$ with $r < n$.

Let P and Q be the controllability and observability Gramians

$$P = \int_0^\infty e^{A\tau} B B^T e^{A^T\tau} d\tau \tag{6.3}$$

$$Q = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau \tag{6.4}$$

are the solution of following Lyapunov equations:

$$AP + PA^T + BB^T = 0 \tag{6.5}$$

$$A^T Q + QA + C^T C = 0 \tag{6.6}$$

6.2.1 Gawronski and Juang's TLMR technique

Gawronski and Juang [29] proposed a TLMR technique, where approximation is given in a finite time interval. The controllability and observability Gramians for a finite time interval are defined as

$$P_{GJ} = \int_{t_1}^{t_2} e^{A\tau} B B^T e^{A^T\tau} d\tau = \int_0^\infty e^{A\tau} X_c(t) e^{A^T\tau} d\tau \tag{6.7}$$

$$Q_{GJ} = \int_{t_1}^{t_2} e^{A^T\tau} C^T C e^{A\tau} d\tau = \int_0^\infty e^{A^T\tau} Y_o(t) e^{A\tau} d\tau \tag{6.8}$$

where $X_c = e^{At_1} B B^T e^{A^T t_1} - e^{At_2} B B^T e^{A^T t_2}$ and $Y_o = e^{A^T t_1} C^T C e^{At_1} - e^{A^T t_2} C^T C e^{At_2}$. These Gramians are the solution of following Lyapunov equations

$$AP_{GJ} + P_{GJ} A^T + X_c = 0 \tag{6.9}$$

$$A^T Q_{GJ} + Q_{GJ} A + Y_o = 0 \tag{6.10}$$

Let

$$T^T Q_{GJ} T = T^{-1} P_{GJ} T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$ and T is a contragredient matrix used to transform the original system realization. ROMs are obtained by partitioning the transformed realization.

Remark 6.2.1 *The matrices X_c and Y_o are not guaranteed to be positive semidefinite, the models obtained by Gawronski and Juang technique may not be stable [69, 71].*

6.2.2 Gugercin and Antoulas's TLMR technique

Let the controllability P_{GA} and observability Q_{GA} Gramians respectively, obtained as the solutions to Lyapunov equations

$$AP_{GA} + P_{GA}GA^T + B_{GA}B_{GA}^T = 0 \tag{6.11}$$

$$GA^TQ_{GA} + Q_{GA}A + C_{GA}^TC_{GA} = 0 \tag{6.12}$$

are used to find a contragredient matrix T as:

$$T^TQ_{GA}T = T^{-1}P_{GA}T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_i \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1,\ \sigma_r > \sigma_{r+1},\ B_{GA} = U_{GA}|S_{GA}|^{1/2},\ C_{GA} = |R_{GA}|^{1/2}V_{GA}^T$. The terms $U_{GA},\ S_{GA},\ V_{GA}$, and R_{GA} are obtained as $X_c = U_{GA}S_{GA}U_{GA}^T$ and $Y_o = V_{GA}R_{GA}V_{GA}^T$, where $S_{GA} = diag(s_1, s_2, \ldots, s_n),\ R_{GA} = diag(r_1, r_2, \ldots, r_n)$. $|s_1| \geq |s_2| \geq \cdots \geq |s_n| \geq 0$ and $|r_1| \geq |r_2| \geq \cdots \geq |r_n| \geq 0$. ROMs are calculated by partitioning the transformed realization.

Remark 6.2.2 *Since $X_c \leq B_{GA}B_{GA}^T$, $Y_o \leq C_{GA}^TC_{GA}$ and the realization (A, B_{GA}, C_{GA}) is minimal and the stability of ROMs is also guaranteed this technique also yields frequency response error bounds.*

6.3 Proposed techniques

In this section, we proposed two techniques (partially appear in [17]) to address instability problem of Gawronski and Juang [29] technique. Moreover, the proposed techniques also carry frequency response error bounds.

In Gugercin and Antoulas [69] technique, the symmetric matrices X_c and Y_o are ensured positive /semipositive definite by taking the square root of absolute values of the eigenvalues obtained by EVD of symmetric matrices X_c and Y_o. This sometimes leads to a large change in some eigenvalues and may not effect other eigenvalues.

6.3.1 Proposed technique 1

Motivated from [8], we modify Gawronski and Juang technique to yield stable ROMs and frequency response error bounds. This technique guarantees stability of ROMs by performing EVD of symmetric matrices $X_c(t)$ and $Y_o(t)$ and retaining the positive eigenvalues and truncating the negative eigenvalues. Let the new controllability P_G and observability Q_G Gramians respectively, be solutions of the following Lyapunov equations

$$AP_G + P_G A^T + B_G B_G^T = 0 \tag{6.13}$$

$$A^T Q_G + Q_G A + C_G^T C_G = 0 \tag{6.14}$$

to find a contragredient matrix T (used to transform the original system realization) as:

$$T^T Q_G T = T^{-1} P_G T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_i \geq \sigma_{i+1},\ i = 1, 2, \ldots, n-1, \sigma_r > \sigma_{r+1}, B_G = U_{G_1} S_{G_1}^{1/2}, C_G = R_{G_1}^{1/2} V_{G_1}^T$. The terms $U_{G_1},\ S_{G_1},\ V_{G_1}$, and R_{G_1} are obtained as

$$X_c = \begin{bmatrix} U_{G_1} & U_{G_2} \end{bmatrix} \begin{bmatrix} S_{G_1} & 0 \\ 0 & S_{G_2} \end{bmatrix} \begin{bmatrix} U_{G_1}^T \\ U_{G_2}^T \end{bmatrix}$$

$$Y_o = \begin{bmatrix} V_{G_1} & V_{G_2} \end{bmatrix} \begin{bmatrix} R_{G_1} & 0 \\ 0 & R_{G_2} \end{bmatrix} \begin{bmatrix} V_{G_1}^T \\ V_{G_2}^T \end{bmatrix}$$

where $\begin{bmatrix} S_{G_1} & 0 \\ 0 & S_{G_2} \end{bmatrix} = diag(s_1, s_2, \cdots, s_n)$, $\begin{bmatrix} R_{G_1} & 0 \\ 0 & R_{G_2} \end{bmatrix} = diag(r_1, r_2, \cdots, r_n)$, $s_1 \geq s_2 \geq \cdots \geq s_n, r_1 \geq r_2 \geq \cdots \geq r_n, S_{G_1} = diag(s_1, s_2, \cdots, s_l), R_{G_1} = diag(r_1, r_2, \cdots, r_l)$, $s_1 \geq s_2 \geq \cdots \geq s_l > 0, r_1 \geq r_2 \geq \cdots \geq r_l > 0$. ROMs are calculated by partitioning the transformed realization.

Remark 6.3.1 *Since $X_c \leq B_G B_G^T \leq B_{GA} B_{GA}^T$, $Y_o \leq C_G^T C_G \leq C_{GA}^T C_{GA}$ and the realization $\{A, B_G, C_G\}$ is minimal, the stability of the ROMs is therefore guaranteed.*

Theorem 6.3.1 *The following error bound holds (subject to existence of $rank\,[B_G \;\; B] = rank\,[B_G]$ and $\begin{bmatrix} C_G \\ C \end{bmatrix} = rank\,[C_G]$).*

$$\|G(s) - G_r(s)\|_\infty \leq 2\|L_G\|\|K_G\| \sum_{j=r+1}^{n} \sigma_i$$

where $L_G = C V_{G_1} R_{G_1}^{-\frac{1}{2}}$ and $K_G = S_{G_1}^{-\frac{1}{2}} U_{G_1}^T B$.

Proof: Since $rank\,[B_G \;\; B] = rank\,[B_G]$ and $rank \begin{bmatrix} C_G \\ C \end{bmatrix} = rank\,[C_G]$, the relationships $B = B_G K_G$ and $C = L_G C_G$ hold. By partitioning $B_G = \begin{bmatrix} B_{G_1} \\ B_{G_2} \end{bmatrix}, C_G = \begin{bmatrix} C_{G_1} & C_{G_2} \end{bmatrix}$ and substituting $B_1 = B_{G_1} K_G$, $C_1 = L_G C_{G_1}$ respectively yields

$$\begin{aligned}
\|G(s) - G_r(s)\|_\infty &= \|C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1\|_\infty \\
&= \|L_G C_G (sI - A)^{-1} B_G K_G - L_G C_{G_1}(sI - A_{11})^{-1} B_{G_1} K_G\|_\infty \\
&= \|L_G (C_G (sI - A)^{-1} B_G - C_{G_1}(sI - A_{11})^{-1} B_{G_1}) K_G\|_\infty \\
&\leq \|L_G\| \|C_G (sI - A)^{-1} B_G - C_{G_1}(sI - A_{11})^{-1} B_{G_1}\|_\infty \|K_G\|
\end{aligned}$$

If $\{A_{11}, B_{G_1}, C_{G_1}\}$ is ROM obtained by partitioning a balanced realization $\{A, B_G, C_G\}$, we have from Enns [5]

$$\|C_G (sI - A)^{-1} B_G - C_{G_1}(sI - A_{11})^{-1} B_{G_1}\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j.$$

The result follows.

Remark 6.3.2 *The $rank\,[B_G \;\; B] = rank\,[B_G]$ and $rank \begin{bmatrix} C_G \\ C \end{bmatrix} = rank\,[C_G]$ is almost always true [26].*

6.3.2 Proposed technique 2

The proposed technique 1 ensures positive definiteness of the matrices X_c and Y_o by taking only positive eigenvalues and truncating negative eigenvalues. This technique also does not have similar effect on all eigenvalues.

In the following, (motivated from [10]) another TLMR technique is proposed where effort is to have a similar effect on all eigenvalues of indefinite matrices X_c and Y_o. The ROM obtained are guaranteed to be stable. Moreover, it yields frequency response error bound and improved error. Let new controllability P_I and observability Q_I Gramians, respectively, are calculated by solving the following Lyapunov equations:

$$AP_I + P_I A^T + B_I B_I^T = 0 \tag{6.15}$$

$$A^T Q_I + Q_I A + C_I^T C_I = 0 \tag{6.16}$$

where B_I and C_I are the new fictitious input and output matrices respectively defined as:

$$B_I = \begin{cases} U_I(S_I - s_n I)^{1/2} & \text{for } s_n < 0 \\ U_I S_I^{1/2} & \text{for } s_n \geq 0 \end{cases} \tag{6.17}$$

$$C_I = \begin{cases} (R_I - r_n I)^{1/2} V_I^T & \text{for } r_n < 0 \\ R_I^{1/2} V_I^T & \text{for } r_n \geq 0. \end{cases} \tag{6.18}$$

The terms $U_I,\ S_I,\ V_I$, and R_I are calculated as $X_c = U_I S_I U_I^T$ and $Y_o = V_I R_I V_I^T$, where $S_I = diag(s_1, s_2, \cdots, s_n)$, $R_I = diag(r_1, r_2, \cdots, r_n)$, $s_1 \geq s_2 \geq \cdots \geq s_n$, and $r_1 \geq r_2 \geq \cdots \geq r_n$.

Let a contragredient transformation matrix T (used to transform the original system) is obtained as

$$T^T Q_I T = T^{-1} P_I T^{-T} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, 3, \ldots, n-1$, $\sigma_r > \sigma_{r+1}$. ROMs are calculated by partitioning the transformed realization.

Remark 6.3.3 *Since* $X_c \leq B_I B_I^T$, $Y_o \leq C_I^T C_I$, $B_I B_I^T \geq 0$, $C_I^T C_I \geq 0$, $P_I > 0$ *and* $Q_I > 0$.

Therefore, the realization (A, B_I, C_I) is minimal. Moreover, the ROMs are guaranteed to be stable.

Theorem 6.3.2 *The following error bound for the proposed technique holds if the rank conditions $rank\,[B_I \;\; B] = rank\,[B_I]$ and $rank \begin{bmatrix} C_I \\ C \end{bmatrix} = rank\,[C_I]$ are satisfied*

$$\|G(s) - G_r(s)\|_\infty \leq 2\|L_I\|\|K_I\| \sum_{j=r+1}^{n} \sigma_j$$

where

$$L_I = \begin{cases} CV_I(R_I - r_n I)^{-1/2} & \text{for } r_n < 0 \\ CV_I R_I^{-1/2} & \text{for } r_n \geq 0. \end{cases} \tag{6.19}$$

$$K_I = \begin{cases} (S_I - s_n I)^{-1/2} U_I^T B & \text{for } s_n < 0 \\ S_I^{-1/2} U_I^T B & \text{for } s_n \geq 0. \end{cases} \tag{6.20}$$

Proof: Since $rank\,[B_I \;\; B] = rank\,[B_I]$ and $rank \begin{bmatrix} C_I \\ C \end{bmatrix} = rank\,[C_I]$, the relationships $B = B_I K_I$ and $C = L_I C_I$ hold. By partitioning $B_I = \begin{bmatrix} B_{I_1} \\ B_{I_2} \end{bmatrix}, C_I = \begin{bmatrix} C_{I_1} & C_{I_2} \end{bmatrix}$ and substituting $B_1 = B_{I_1} K_I$, $C_1 = L_I C_{I_1}$ respectively yield

$$\begin{aligned} \|G(s) - G_r(s)\|_\infty &= \|C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1\|_\infty \\ &= \|L_I C_I(sI - A)^{-1} B_I K_I - L_I C_{I_1}(sI - A_{11})^{-1} B_{I_1} K_I\|_\infty \\ &= \|L_I(C_I(sI - A)^{-1} B_I - C_{I_1}(sI - A_{11})^{-1} B_{I_1}) K_I\|_\infty \\ &\leq \|L_I\|\|C_I(sI - A)^{-1} B_I - C_{I_1}(sI - A_{11})^{-1} B_{I_1}\|_\infty \|K_I\| \end{aligned}$$

If $\{A_{11}, B_{I_1}, C_{I_1}\}$ is ROM obtained by partitioning a balanced realization $\{A, B_I, C_I\}$, we have from [5]

$$\|C_I(sI - A)^{-1} B_I - C_{I_1}(sI - A_{11})^{-1} B_{I_1}\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j.$$

The result follows.

Remark 6.3.4 *The* $rank\,[B_I \;\; B] = rank\,[B_I]$ *and* $rank \begin{bmatrix} C_I \\ C \end{bmatrix} = rank\,[C_I]$ *is almost always true ([26]).*

Remark 6.3.5 *For the case when symmetric matrices* $X_c(t) \geq 0$ *and* $Y_o(t) \geq 0$*, then* $P_{GJ} = P_G = P_S = P_I$ *and* $Q_{GJ} = Q_G = Q_S = Q_I$*. Otherwise* $P_{GJ} < P_I$ *and* $Q_{GJ} < Q_I$*. Moreover, time limited HSV satisfies:* $(\lambda_j[P_{GJ}Q_{GJ}])^{1/2} \leq (\lambda_j[P_I Q_I])^{1/2}$.

Remark 6.3.6 *When* $X_c \ngeq 0$ *and* $Y_o \ngeq 0$*, then*

$$X_I = B_I B_I^T = X_c(t) - s_n I, \quad Y_I = C_I^T C_I = Y_o(t) - r_n I$$

$$P_I = P_{GJ} + P_{ad}, \qquad Q_I = Q_{GJ} + Q_{ad}$$

where

$$\begin{aligned} AP_{ad} + P_{ad}A^T - s_n I &= 0, \quad \textit{for } s_n < 0 \\ A^T Q_{ad} + Q_{ad}A - r_n I &= 0, \quad \textit{for } r_n < 0 \end{aligned}$$

6.4 Numerical examples

Example 6.4.1 *Consider a linear time invariant stable* 3^{rd} *order system with the following state space representation.*

$$\begin{aligned} A &= \begin{bmatrix} -8.6040 & -4.8344 & -0.0192 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\ C &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ D &= 0 \end{aligned}$$

The first order ROM obtained by Gawronski and Junag [29] is unstable with pole $s = 0.0000313$ while Gugercin and Antoulas [69] and proposed techniques (1 and 2) provide stable ROM within the time interval $[t_1, t_2] = [0, 8]$ sec.

Example 6.4.2 *Consider a linear time invariant stable* 6^{th} *order system with the following* *state space representations*

$$A = \begin{bmatrix} -9 & -29 & -100 & -82 & -19 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0 & 0 & -44 & 44 & 38 & 14 \end{bmatrix}$$

$$D = 0$$

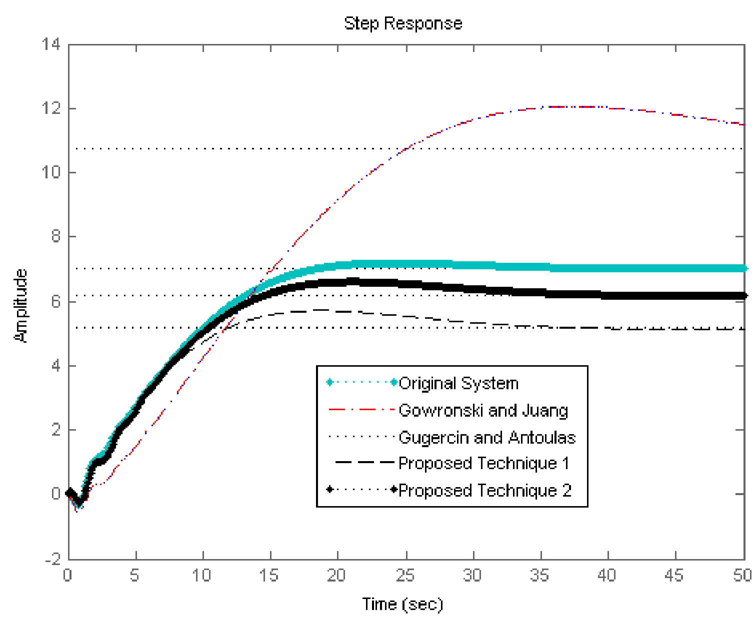

Figure 6.1: Step response in the desired time interval [0,10] sec.

Fig. 6.1 and Fig. 6.2 represent the step response and step response error for the fourth order ROMs obtained using Gawronski and Juang [29], Gugercin and Antoulas [69] and proposed techniques (1 and 2) for the time interval $[t_1, t_2] = [0, 10]$ sec. Fig. 6.3 represents the close up view of the error plot for the step response. Fig. 6.4 and Fig. 6.5 represent the impulse response and impulse response error for the fourth order ROMs obtained using Gawronski and Juang [29], Gugercin and Antoulas [69] and proposed techniques (1 and 2)

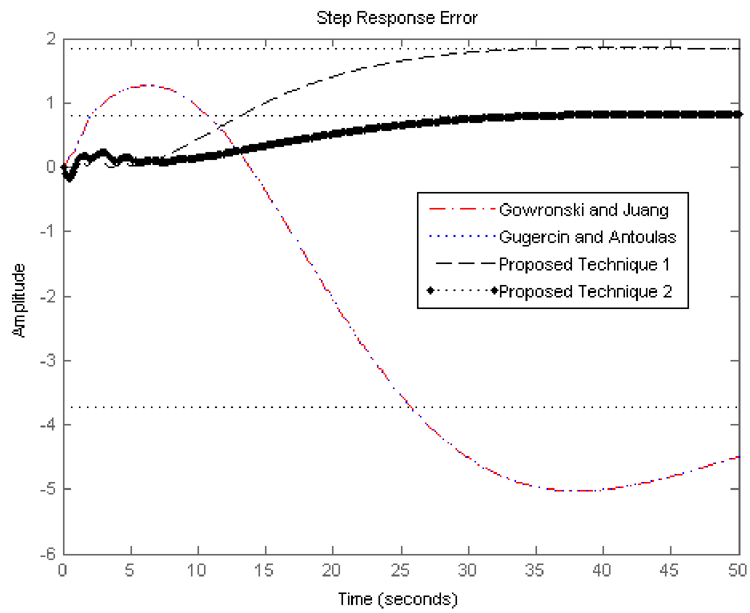

Figure 6.2: Close up view of step response error in the desired time interval [0,10] sec.

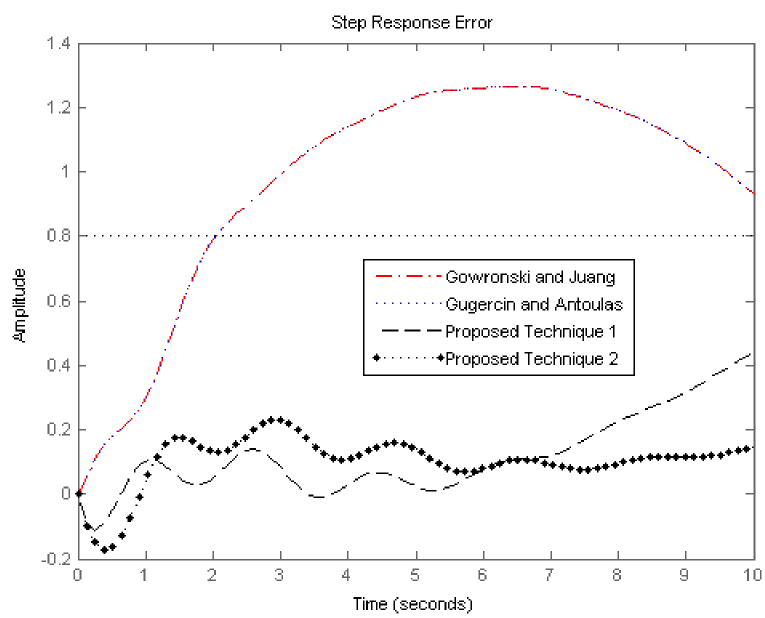

Figure 6.3: Close up view of step response error in the desired time interval [0,10] sec.

for the time interval $[t_1, t_2] = [0, 10]$ sec. Moreover, Fig. 6.1 and Fig. 6.4 also represent step and impulse response of original system. It is observed in Fig. 6.3, the proposed techniques provide comparatively less impulse response error as compared to other techniques.

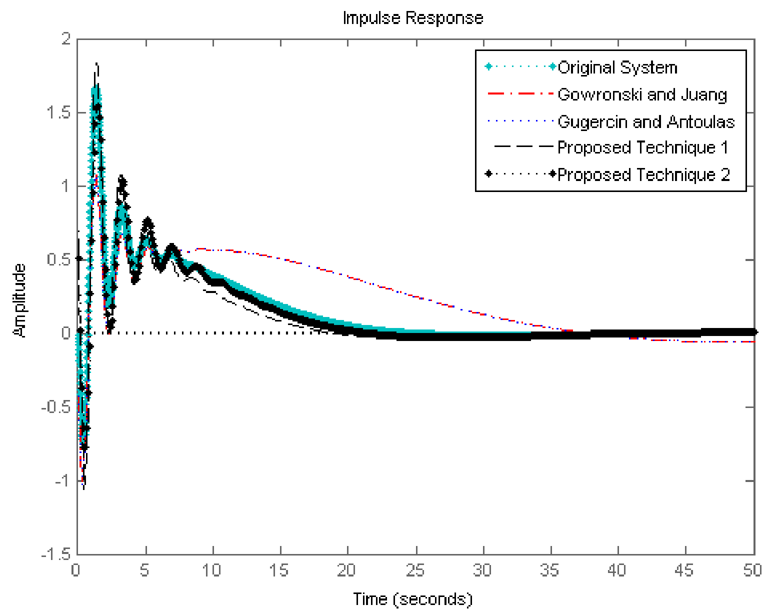

Figure 6.4: Impulse response in the desired time interval [0,10] sec.

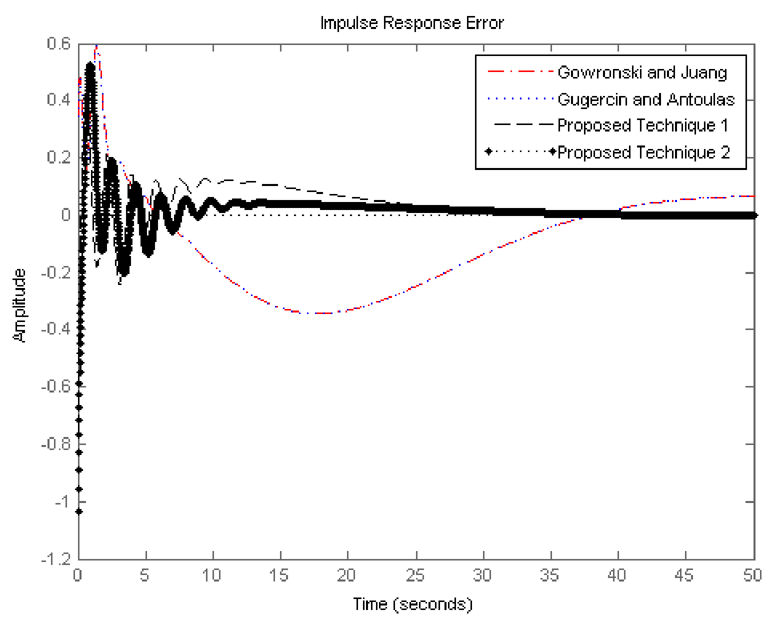

Figure 6.5: Close up view of impulse response error in the desired time interval [0,10] sec.

Example 6.4.3 *Consider a 5^{th} order stable system with following state space representation*

$$A = \begin{bmatrix} -3 & -4.3 & -3 & -1.2 & -0.2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$
$$C = \begin{bmatrix} 0.1 & 0.1 & 0.2 & 0.1 & 0 \end{bmatrix}$$
$$D = 0$$

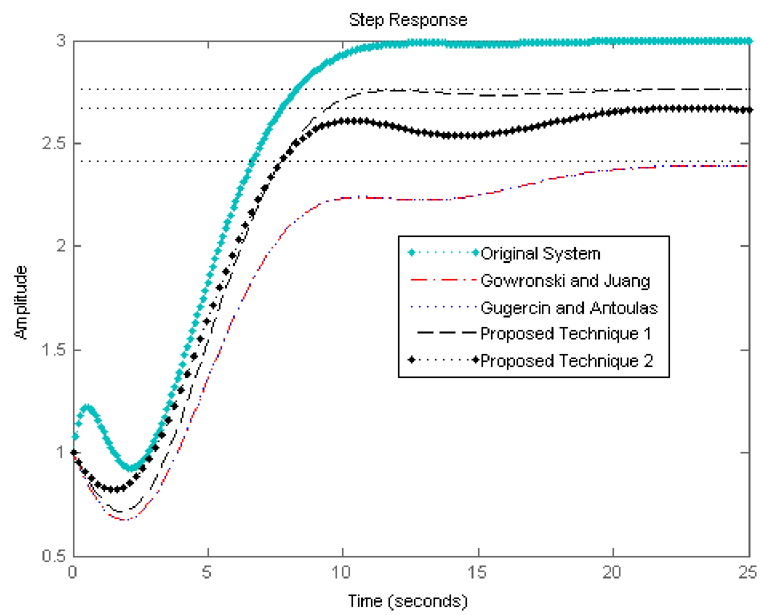

Figure 6.6: Step response in the desired time interval [0,7] sec.

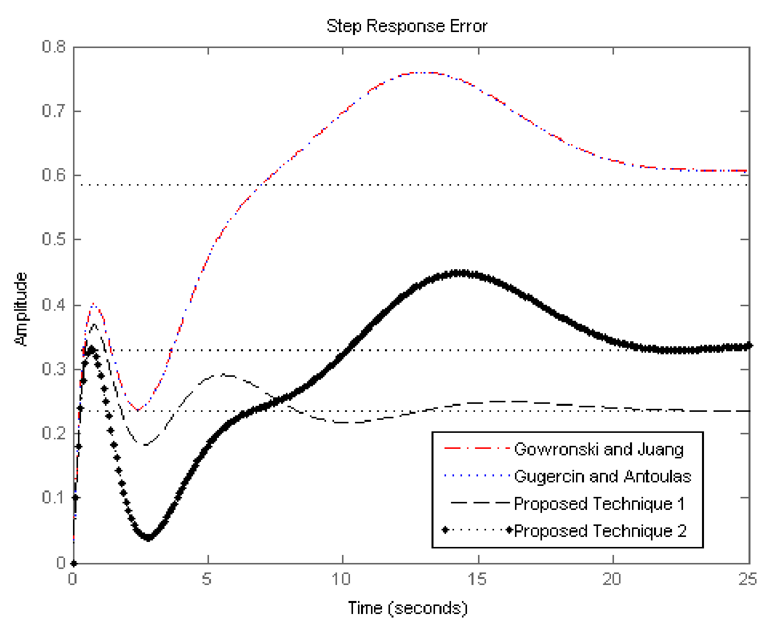

Figure 6.7: Close up view of step response error in the desired time interval [0,7] sec.

Fig. 6.6 and Fig. 6.7 represent the step response and step response error for the third

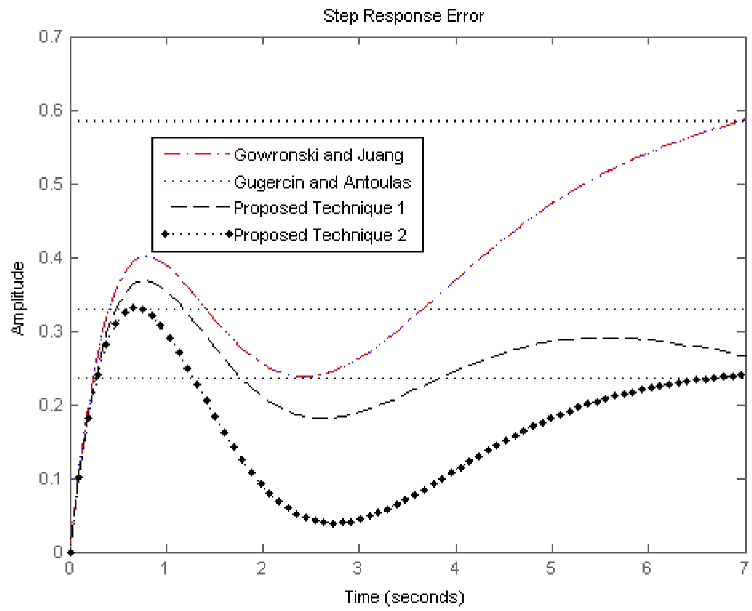

Figure 6.8: Close up view of step response error in the desired time interval [0,7] sec.

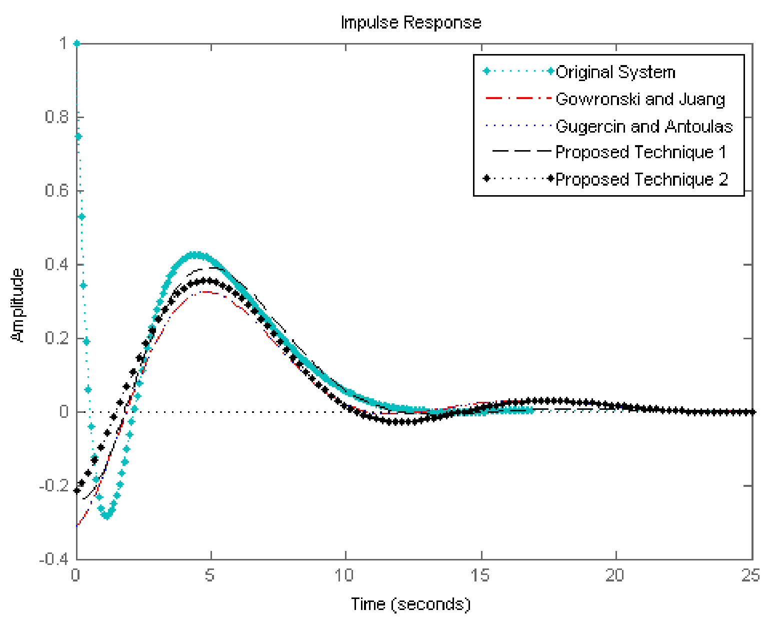

Figure 6.9: Impulse response in the desired time interval [0,7] sec.

order ROMs obtained using Gawronski and Juang [29], Gugercin and Antoulas [69] and proposed techniques (1 and 2) for the time interval $[t_1, t_2] = [0, 7]$ sec. Fig. 6.8 represents the close up view of the error plot for the step response. Fig. 6.9 and Fig. 6.10 represent the impulse response and impulse response error for the third order ROMs obtained using Gawronski and Juang [29], Gugercin and Antoulas [69] and proposed techniques (1 and 2)

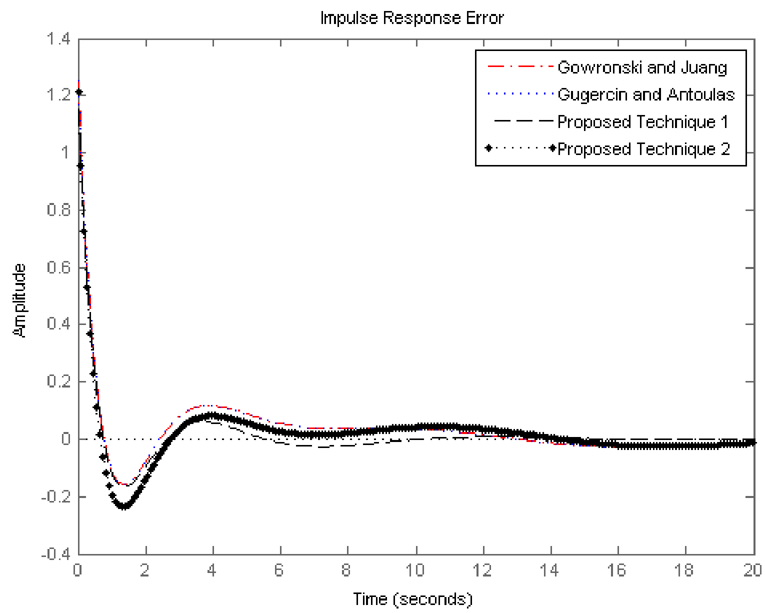

Figure 6.10: Close up view of impulse response error in the desired time interval [0,7] sec.

for the time interval $[t_1, t_2] = [0, 7]$ sec. Moreover, Fig. 6.6 and Fig. 6.9 also represent step and impulse response of original system. It is observed in Fig. 6.7, proposed techniques provide comparable impulse response error as compared to other techniques.

Discussion: It is observed (from examples 6.4.1 - 6.4.3) that in the desired time interval Gawronski and Juang [29], Gugercin and Antoulas [69] and proposed techniques (1 and 2) provide good approximation in the desired time interval. However, Gawronski and Juang [29] sometimes give unstable ROMs and Gugercin and Antoulas [69] and proposed techniques yield stable ROMs.

6.5 Conclusion

In this chapter, two new TLMR techniques were proposed. The proposed techniques extended the results of time limited Gawronski and Juang [29], Gugercin and Antoulas [69]. The ROMs are guaranteed to be stable and error bounds are also available. The proposed techniques mostly yield better approximation as compared to Gugercin and Antoulas [69] technique in the desired time interval.

Next chapter concludes the thesis and identifies some future research directions.

Chapter 7

Conclusions and Future Work

7.1 Overview of the thesis

This thesis proposes the FWMR, FLMR and TLMR techniques for linear continuous and discrete time for standard and generalized (singular and nonsingular) systems.

In chapter 2, a FWMR technique for both continuous and discrete time systems is proposed which provides stable ROMs for double sided weighting case and also yields *a priori* frequency response error bound. The proposed technique provides solution to Enns [5] instability problem for double sided weighting case. The simulation results show that the proposed technique mostly yields lower frequency response error and stable ROM in the presence of double sided weighting.

Chapter 3 proposes FLMR technique for linear continuous and discrete time systems. The proposed technique extends the results of Gawronski and Juang [29], Gugercin and Antoulas [30] and Ghafoor and Sreeram [8] (for continuous time) and Wang and Zilouchian's [90], Ghafoor and Sreeram's algorithms (1 and 2) [87] (for discrete time) systems. The proposed technique mostly yields better approximation as compared to Gugercin and Antoulas [30] and Ghafoor and Sreeram [8,87] in the desired frequency interval. The ROMs are guaranteed to be stable and error bounds are also available.

Chapter 4 generalizes the FLMR techniques for general (non-singular) linear continuous and discrete time systems. The error bounds are also given. The numerical simulations represent the comparison of frequency response error for the proposed generalized techniques.

Chapter 5 further improves FLMR technique for Generalized descriptor (singular) systems. The technique extends the results of Gawronski and Juang [29] (initially proposed for standard state space systems) for generalized descriptor system. Moreover, the work also extends Poor Man's truncated balanced realization (PMTBR) technique [61] to include frequency limited Gramians for generalized descriptor systems. The numerical comparisons of proposed techniques are also presented.

Chapter 6 proposes two new TLMR techniques. The proposed techniques extended the results of time limited Gawronski and Juang [29], Gugercin and Antoulas [69]. The ROMs are guaranteed to be stable and error bounds are also available. The proposed techniques mostly yield better approximation as compared to Gugercin and Antoulas [69] technique in the desired time interval.

7.2 Future research

Some open research areas from the thesis are summarized below:

- Wang et al's [26], Varga and Anderson [25], Gugercin and Antoulas [30], Ghafoor and Sreeram [8] and proposed FWMR techniques (chapter 2) are realization dependent. Which realization of the original system can produce lower approximation error and tighter error bounds remains an open problem.

- FLMR techniques (chapter 3) use BT. It is interesting to see using different model reduction techniques like Hankel norm, Krylov, Pade approximation techniques instead of BT yield better results or not.

- There are number of formulas for the fictitious input and output matrices used in reduction procedure of different techniques. It is not clear which among these yields the best result in terms of lower weighted approximation error and needs further investigation.

- Stability of ROMs is not guaranteed in case of Enns [5] double sided weighting case due to indefiniteness of matrices X_E and Y_E. Similarly, Gawronski and Juang [29] and Wang and Zilouchian [90] may yield unstable ROMs. This will remain an open problem for future research work.

- Proposed techniques are only applicable for stable linear time invariant original system. It is interesting to see whether these results remain valid for non-linear and time varying systems.

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Appendix A

A.1 Balanced truncation

Consider the full order continuous time original stable system be $G(s) = C(sI-A)^{-1}B+D$ where $\{A, B, C, D\}$ is its n^{th} order minimal realization. Let $P > 0$ and $Q > 0$ be the controllability and observability Gramians satisfying the following Lyapunov equations:

$$AP + PA^T + BB^T = 0 \tag{A.1}$$

$$QA + A^TQ + C^TC = 0 \tag{A.2}$$

For discrete time case, let the original stable system be $G(z) = C(zI - A)^{-1}B + D$, the Lyapunov equations are:

$$APA^T - P + BB^T = 0 \tag{A.3}$$

$$A^TQA - Q + C^TC = 0 \tag{A.4}$$

Let T be the transformation obtained via simultaneously diagonalizing the Gramians P and Q

$$T^{-1}PT^{-T} = T^TQT = \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

where $\Sigma_1 = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}, \Sigma_2 = diag\{\sigma_{r+1}, \ldots, \sigma_n\}, \sigma_j = \sigma_{j+1}, j = 1, 2, \ldots, n-1,$

$\sigma_r > \sigma_{r+1}$ and σ_j are the HSV. Transforming and partitioning the original system:

$$\hat{A} = T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \hat{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$\hat{C} = CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \hat{D} = D \tag{A.5}$$

where $A_{11} \in \Re^{r \times r}(r < n)$. The ROM is given as $G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$ and $G_r(z) = C_1(zI - A_{11})^{-1}B_1 + D$ for continuous and discrete time systems respectively.

The error bounds of the technique can be obtained from [2, 5]

$$\|G(s) - G_r(s)\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j.$$

for continuous time systems.

$$\|G(z) - G_r(z)\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j.$$

for discrete time systems.

Remark A.1.1 *A given realization* $\{A, B, C, D\}$ *can be transformed to a balanced realization* $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ *if and only if it is asymptotically stable and minimal.*

Remark A.1.2 *The balanced realization is unique up to the ordering of HSV* σ_j *and an orthogonal transformation that commutes with* Σ*.*

Remark A.1.3 *The subsystem* A_{ii} *is asymptotically stable if* Σ_1 *and* Σ_2 *have no common diagonal elements. Moreover, the subsystem* $\{A_{ii}, B_i, C_i, D\} for (i = 1, 2, ...)$ *is controllable and observable.*

Appendix B

B.1 Balanced singular perturbation approximation

Let the stable original system have the balanced realization (A.5) and the transfer function $G(s)$ be written in the form:

$$G(s) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sI_k - A_{11} & -A_{12} \\ -A_{21} & sI_{n-k}A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Decomposing the transfer function $G(s) = G_1(s) + G_2(s)$ gives

$$\begin{aligned} G_1(s) &= C_{spa}(s)(sI_k - A_{spa}(s))^{-1}B_{spa}(s) + D \\ G_2(s) &= C_2(sI_{n-k} - A_{22})^{-1}B_2 \end{aligned}$$

where

$$\begin{aligned} A_{spa}(s) &= A_{11} + A_{12}(sI_{n-k} - A_{22})^{-1}A_{21} \\ B_{spa}(s) &= B_1 + A_{12}(sI_{n-k} - A_{22})^{-1}B_2 \\ C_{spa}(s) &= C_1 + C_2(sI_{n-k} - A_{22})^{-1}A_{21} \end{aligned} \tag{B.1}$$

If the subsystem $G_2(s)$ is stable and its states have very fast transient dynamics in the neighbourhood of $s = \sigma_0$, then by ignoring $G_2(s)$, the ROM of $G(s)$ can be approximated by $G_{spa}(\sigma_0) = C_{spa}(\sigma_0)(sI - A_{spa}(\sigma_0))^{-1}B_{spa}(\sigma_0) + D_{spa}(\sigma_0)$ where $D_{spa}(\sigma_0) = D + C_2(\sigma_0 I - A_{22})^{-1}B_2$ and $A_{spa}(\sigma_0), B_{spa}(\sigma_0), C_{spa}(\sigma_0)$ are defined as in B.1 by substituting s with σ_0.

The two extreme cases of the generalized BSPA are:

1. at $\sigma_0 = 0$, the ROM is

$$G_{spa}(0) = C_{spa}(0)(sI - A_{spa}(0))^{-1}B_{spa}(0) + D_{spa}(0)$$

where

$$\begin{aligned} A_{spa}(0) &= A_{11} + A_{12}A_{22}^{-1}A_{21} \\ B_{spa}(0) &= B_1 + A_{12}A_{22}^{-1}B_2 \\ C_{spa}(0) &= C_1 + C_2A_{22}^{-1}A_{21} \\ D_{spa}(0) &= D + C_2A_{22}^{-1}B_2 \end{aligned}$$

which is the BSPA [33, 34].

2. at $\sigma_0 = \infty$, the ROM corresponds to the BT [1], as $A_{spa}(\infty) \to A_{11}, B_{spa}(\infty) \to B_1, C_{spa}(\infty) \to C_1$ and $D_{spa}(\infty) \to D$.

Remark B.1.1 *The BSPA [33, 34] and BT [1], are related via a frequency inversion $s \to 1/s$, as follows:*

1. *Given $G(s)$ in the balanced realization form, define $H(s) = G(1/s)$.*

2. *Let $H_r(s)$ be a ROM obtained via BT of $H(s)$.*

3. *Set $G_r(s) = H_r(1/s)$, where $G_r(s)$ is the ROM obtained via BT of $G(s)$.*

Remark B.1.2 *The BSPA technique [33, 34] yields better approximation at low frequencies in contrast to BT technique [1].*

Remark B.1.3 *Almost all the properties of the continuous time system case BSPA [33, 34] and BT [1] are preserved in discrete time case. However, in the discrete time system case, the ROM realization obtained by directly truncating the original balanced realization may not be balanced [6].*

C.1 Controller reduction: FWMR

C.1.1 Stability consideration for FWMR

Let the transfer function matrix of a linear time invariant plant be $G(s)$ with $K(s)$ is a high order stabilizing controller as shown in Fig. C.1. Defining $K_r(s)$ as the reduced order controller (ROC) as shown in Fig. C.2, where Fig. C.2(a) and Fig. C.2(b) are equivalent. The ROC $K_r(s)$ is stabilizing controller under the following sufficient conditions [78, 79]:

1. $K(s)$ and $K_r(s)$ have the same number of poles in the open right half plane and and no poles on the imaginary axis.

2. Either

$$\|(K(s) - K_r(s))G(s)(I + K(s)G(s))^{-1}\|_\infty < 1$$

or

$$\|(I + G(s)K(s))^{-1}G(s)(K(s) - K_r(s))\|_\infty < 1$$

where $G(s)(I + K(s)G(s))^{-1} = (I + G(s)K(s))^{-1}G(s)$

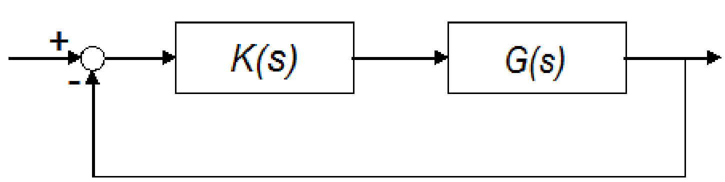

Figure C.1: Closed loop system diagram [6]

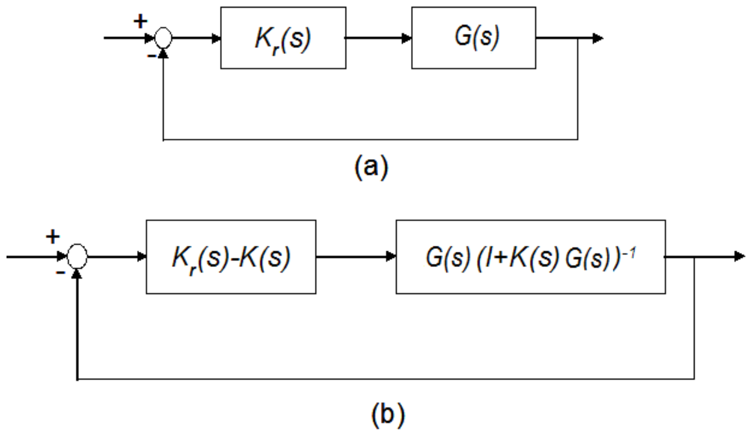

Figure C.2: Closed loop system diagram [6]

C.1.2 Closed-loop transfer function consideration for FWMR

The closed loop transfer function matrices with the high order controller $K(s)$ and ROC $K_r(s)$ for Fig. C.1 be

$$C(s) = G(s)K(s)(I + G(s)K(s))^{-1}$$

and closed loop transfer function for Fig. C.2(a) be

$$C_r(s) = G(s)K_r(s)(I + G(s)K_r(s))^{-1}$$

then

$$\begin{aligned} C(s) - C_r(s) &= G(s)K(s)(I + G(s)K(s))^{-1} - G(s)K_r(s)(I + G(s)K_r(s))^{-1} \\ &\approx (I + G(s)K(s))^{-1}G(s)(K(s) - K_r(s))(I + G(s)K(s))^{-1} \end{aligned}$$

which suggests the following approximation problem [79]. Find the ROC such that

1. $K(s)$ and $K_r(s)$ have the same number of poles in the open right half plane, and no poles on the imaginary axis.

2. The index

$$\|(I + G(s)K(s))^{-1}G(s)(K(s) - K_r(s))(I + G(s)K(s))^{-1}\|_\infty$$

is minimized.

Relative error of the difference i.e. $C(s)^{-1}(C(s)-C_r(s))$ can also be used to measure the closeness of the closed loop transfer function. By assuming $G(s)$ and $K(s)$ are square and invertible, then the following equation is given:

$$C(s)^{-1}(C(s)-C_r(s)) \approx K(s)^{-1}(K(s)-K_r(s))(I+G(s)K(s))^{-1}$$

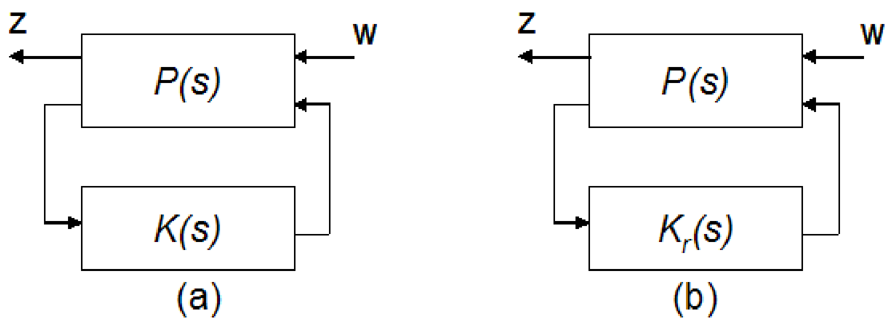

Figure C.3: Closed loop system diagram [6]

C.2 H_∞ / Modern controller reduction

Consider a feedback control system as shown in Fig. C.3(a) where $P(s)$ the plant with input w and output z, controlled by full order controller $K(s)$, the ROC $K_r(s)$ can be obtained as in Fig. C.3(b).

Partitioning $P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$, $K(s)$ and $K_r(s)$ can then be expresses in linear fractional transformation form as in [74]:

$$\begin{aligned} T_{zw}(s) &= P_{11}(s) + P_{12}(s)K(s)(I - P_{22}K(s))^{-1}P_{21}(s) \\ \bar{T}_{zw}(s) &= P_{11}(s) + P_{12}(s)K_r(s)(I - P_{22}K_r(s))^{-1}P_{21}(s) \end{aligned}$$

C.2.1 Stability consideration for FWMR

Suppose $K(s)$ and $K_r(s)$ have same number of right half plane poles, then the closed loop system $\bar{T}_{wz}(s)$ is stable if either of the following sufficient conditions is satisfied

$$\|(I - P_{22}(s)K(s))^{-1}P_{22}(s)(K - K_r(s))\|_\infty < 1$$

$$\|(K - K_r(s))(I - P_{22}(s)K(s))^{-1}P_{22}(s)\|_\infty < 1$$

C.2.2 Closed-loop transfer function consideration for FWMR

$$\begin{aligned} T_{zw} - \bar{T}_{zw} = P_{12}(s)K(s)(I - P_{22}(s)K(s))^{-1}P_{21}(s) - \\ P_{12}K_r(s)(I - P_{22}(s)K_r(s))^{-1}P_{21}(s) \qquad \text{(C.1)} \\ \approx P_{12}(s)(I - K(s)P_{22}(s))^{-1}(K(s) - K_r(s))(I - P_{22}(s)K(s))^{-1}P_{21}(s) \end{aligned}$$

Equation (C.1) suggests the following approximation problem. Find ROC $K_r(s)$ such that full order controller $K(s)$ and the ROC $K_r(s)$ have same number of poles in the right half plane, and the index $\|P_{12}(s)(I - K(s)P_{22})^{-1}(K(s) - K_r(s))(I - p_{22}(s)K(s))^{-1}P_{21}(s)\|_\infty$ is minimized.

Minimizing $\|P_{12}(s)(I - K(s)P_{22})^{-1}(K(s) - K_r(s))(I - P_{22}(s)K(s))^{-1}P_{21}(s)\|_\infty$ is the optimal solution which sometimes may not be found, so we seek a stabilizing ROC $K_r(s)$ such that

$$\|P_{12}(s)(I - K(s)P_{22})^{-1}(K(s) - K_r(s))(I - p_{22}(s)K(s))^{-1}P_{21}(s)\|_\infty < \gamma$$

where γ is a positive constant.

Note that, in a special case when $P(s) = \begin{bmatrix} 0 & P_{12}(s) \\ P_{21}(s) & 0 \end{bmatrix} = \begin{bmatrix} 0 & W(s) \\ V(s) & 0 \end{bmatrix}$ then $T_{zw} - \bar{T}_{zw} = P_{12}(s)(K(s) - K_r(s))P_{21}(s)$.

Appendix D

D.1 Equivalence between FWMR and FLMR techniques

Consider

$$G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D$$

be a given system with state equation $\dot{x}(t) = Ax(t) + Bu(t)$. Let $X(s) \triangleq (sI - A)^{-1}B$ be the transfer function from input $u(t)$ to the state $x(t)$. The controllability Gramian $\bar{P}$ in frequency domain can be defined as:

$$\begin{aligned} \bar{P} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)X(j\omega)^T d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega I - A)^{-1}BB^T(-j\omega I - A^T)^{-1}d\omega \end{aligned}$$

Suppose there is an input weighting with transfer function $V_i(s)$ with impulse response $v_i(t)$. The new state equation now becomes $\dot{\tilde{x}}(t) = A\tilde{x}(t) + B(v_i * u)(t)$, where $*$ is convolution operator. Hence the transformed input weighted transfer function $\tilde{X}(s)$ is shown as:

$$\tilde{X}(s) \triangleq (sI - A)^{-1}BV_i(s)$$

Now the controllability Gramian $\tilde{P}$ for weighted system can be shown as:

$$\begin{aligned} \bar{P} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{X}(j\omega)\tilde{X}(j\omega)^T d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega I - A)^{-1}BV_i(j\omega)V_i(-j\omega)^T B^T(-j\omega I - A^T)^{-1}d\omega \qquad \text{(D.1)} \end{aligned}$$

Now by assuming $V_i(s)$ be a bandpass filter over the frequency band $[\omega_1, \omega_2]$ with unit amplitude. Eq. (D.1) can now becomes

$$\bar{P} = \bar{P}_\Omega = \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1}(j\omega I - A)^{-1}BB^T(-j\omega I - A^T)^{-1}d\omega + \frac{1}{2\pi}\int_{-\omega_2}^{\omega_1}(j\omega I - A)^{-1}BB^T(-j\omega I - A^T)^{-1}d\omega$$

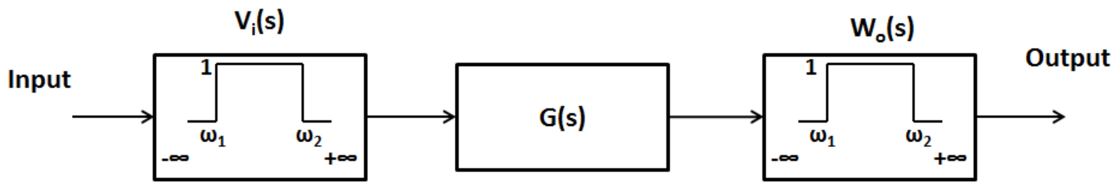

Figure D.1: Input/Output augmented systems with bandpass weightings

So FWMR problem becomes FLMR problem [29] by choosing the weights $V_i(s)$ and $W_o(s)$ as perfect bandpass filter over the desired frequency band as shown in D.1. The modified weighted error expression becomes $\|(G(s) - G_r(s)\|_\infty$ over the frequency band $[\omega_1, \omega_2]$. A similar expression can also be shown for TLMR technique.

Remark D.1.1 *As pointed out in [30] infinite dimensional realization are needed to obtain perfect bandpass filters. However, for present scenario, the required bandpass filters are approximated by low-order bandpass filter. Also as the order of the weighting increases, we get closer to perfect bandpass filters [30].*