KINEMATIC DESIGN OF SERIAL ROBOTIC MECHANISMS USING REVOLUTE AND PRISMATIC JOINTS

By

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ABSTRACT

The Theory of Screws is well known to do kinematic computations. Novel mathematical methods and applications of screw theory were developed gradually over a reasonably long period of time. One of the important usage of the matrix of this theory is the geometric decomposition of the end effector twists into practically implementable joints which are combined together into serial chains. Thereby, calculating and predicting the Single DOF joints orientations, locations and positions for a serial manipulator given the instantaneous end effector twists. Mathematically this design will have multiple solutions, which will be narrowed down considering the structural constraints and further reduced on the basis of user's requirements.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

CHAPTER 2

THE DECOMPOSITION OF INSTANTANEOUS TWISTS; DESIGN AND SELECTION OF JOINTS IN SERIAL ROBOTIC CHAINS

CHAPTER 3

DESIGN OF A SINGLE DOF SERIAL MANIPULATOR

CHAPTER 4

DESIGN OF A TWO DOF SERIAL MANIPULATOR

CHAPTER 5

DESIGN AND SELECTION OF JOINTS IN 3 DOF SERIAL ROBOTIC CHAINS

CHAPTER 6

CONCLUSION

APPENDICES

LIST OF FIGURES

Chapter 1 - INTRODUCTION

1.1 Introduction

Scientists often have the feeling that, through their work, they are learning about some aspect of themselves. Physicists see this connection in their work; so do, for example, psychologists and chemists. In the study of robotics, the connection between the field of study and us is unusually obvious. And, unlike a science that seeks only to analyze, robotics as currently pursued takes the engineering bent toward synthesis. Perhaps it is for these reasons that the field fascinates so many of us.

The study of robotics concerns itself with the desire to synthesize some aspects of human function by the use of mechanisms, sensors, actuators, and computers. Obviously, this is a huge undertaking, which seems certain to require a multitude of ideas from various classical fields [1].

Currently, different aspects of robotics research are carried out by experts in various fields, design of serial manipulators is one of them. It is generally a very well studied topic, however, very little published work is available on the selection of types of joints and their locations in the working space of the mechanisms.

Research on kinematic twists, freedoms, constraints of serial and parallel mechanisms has been done by large number of researchers but geometric decomposition of the end effector twists into practically implementable joints which are combined together into serial chains (include parallel mechanisms but these are not been studied here).

Screw theory, as a theoretical tool, plays an important role in the kinematic analysis of mechanisms and robot manipulators. The principal screws are the basic and important elements of the screw system. In robotics practice, the twist of an end-effector is easily obtained by the linear combination of the joint screws.

The theory of screws is analogous to vector analysis in that both consist of an algebra with a geometric entity as a fundamental element. The vector is a fundamental element for vector analysis, and the screw is the fundamental element for the theory of screws.

1.2 Robot

Robots can be found today in the manufacturing industry, agricultural, military and domestic applications, space exploration, medicine, education, information and communication technologies, entertainment, and many other similar fields.

The word *robot* is mainly used to refer to a wide range of mechanical devices or mechanisms, the common feature of which is that they are all capable of movement and can be used to perform physical tasks. Robots take on many different forms, ranging from humanoid, which mimic the human form and mode of movement, to industrial, whose appearance is dictated by the function they are to perform. Robots can be categorized as robotic manipulators, wheeled robots, legged robots, swimming robots, flying robots, androids and self reconfigurable robots which can apply themselves to a given tasks.

Although the appearance and capabilities of robots vary greatly, all robots share the features of a mechanical, movable structure under some form of control. The structure of a robot is usually mostly mechanical and takes the form of a mechanism having as constituent elements the links connected by joints.

Serial or parallel kinematic chains are concatenated in the robot mechanism. The *serial kinematic chain* is formed by links connected sequentially by joints. Links are connected in series as well as in parallel making one or more closed-loops in a *parallel mechanism*. The mechanical architecture of *parallel robots* is based on parallel mechanisms in which a member called a *moving platform* is connected to a reference member by at least two *limbs* (also called legs or chains) that can be simple or complex. The robot *actuators* are integrated in the limbs usually closed to the fixed member, also called the *base* or the *fixed platform* as shown in Fig 1.2 ahead. The moving platform positions the robot end-effector in space and may have anything between two and six degrees of freedom. Usually, the number of actuators coincides with the degrees of freedom of the mobile platform, exceeding them only in the case of redundantly-actuated parallel robots.

1.3 Historical Remarks on Screws

Since the discovery (attributed to Giulio Mozzi, in the early part of the 19th century) that any three-dimensional rigid body displacement can be accomplished by means of a translation about a unique axis and a rotation about the same axis [2], the concepts of screws and screw displacements have emerged as some of the most convenient means of describing spatial displacement. In 1900, R.S. Ball published his monumental “Theory of Screws” [3]. The theory was developed by Ball as an important tool in the analysis of the kinematic characteristics of mechanisms. After the first decade of the 20^{th} century, the theory of screws received little attention. It was until 1948 that Dimentberg [4] applied the algebra of the theory of screws to the analysis of spatial mechanisms. However, it was Phillips and Hunt [5] in the 1960s who applied the theory of screws to the study of instantaneous kinematics of three bodies in relative motion. Hunt further developed screw theory in aspects of kinematic geometry. Screw theory reveals the nature of the rigid body motion; therefore it plays a very important role in the robotics and kinematics. Hunt proposed various kinematic structures for parallel robots based on the screw theory analysis; he also studied the special configurations of the serial robots using screw theory. Mohamed and Dufy [6] used screw theory to analyze the instantaneous kinematics of the parallel robots. Lipkin and Dufy [7] proposed a new method for hybrid twist and wrench control for a robotic manipulator based on the duality of twist and wrench. Huang and Fang [8] analyzed the kinematic characteristics of three degrees of freedom parallel robot using reciprocal screw theory. Hai-Jun Su, Denis V. Dorozhkin,and Judy M. Vance presented A Screw Theory Approach for the Conceptual Design of Flexible Joints for Compliant Mechanisms [9]

1.4 Definitions and Mathematical Preliminaries

A few definitions and mathematical preliminaries for better understanding of the discussion ahead.

1.4.1 Robot

"A robot is a reprogrammable multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks."(Definition based on Robotics Institute of America (RIA)).

1.4.2 Law Zero

A robot may not injure humanity, or, through inaction, allow humanity to come to harm.

1.4.3 Law One

A robot may not injure a human being, or, through inaction, allow a human being to come to harm, unless this would violate a higher order law.

1.4.4 Law Two

A robot must obey orders given it by human beings, except where such orders would conflict with a higher order law.

1.4.5 Law Three

A robot must protect its own existence as long as such protection does not conflict with a higher order law.

Isaac Asimov proposed these four refined laws of "robotics" to protect us from intelligent generations of robots. Although we are not too far from that time when we really do need to apply Asimov's rules, there is no immediate need, however, it is good to have a plan.

1.4.5 Serial Kinematic Chain

A serial kinematic chain is a single open-loop kinematic chain of rigid bodies (or links) connected by a series of joints. Almost all existing industrial robot manipulators are serial chains, and in most cases simple lower pairs (revolute or prismatic pairs) are used.

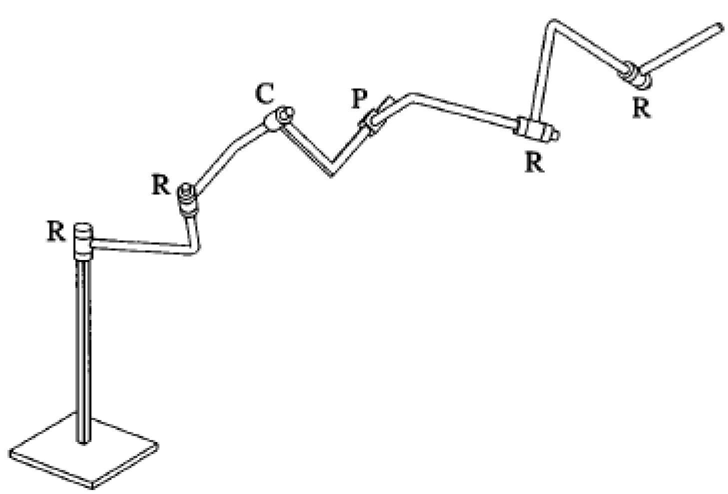


Figure - 1.1 Serial Kinematic Chain

1.4.6 Parallel Kinematic Chain

A parallel kinematic chain is a single or multiple closed-loop kinematic chain of rigid bodies (or links) connected by a series of joints, one end is connected to the base and the other to the moving platform. Examples of parallel actuated kinematic chains are Florida Shoulder and the spatial Stewart Platform. Figure 3 shows an RPUR 5leg parallel manipulator.

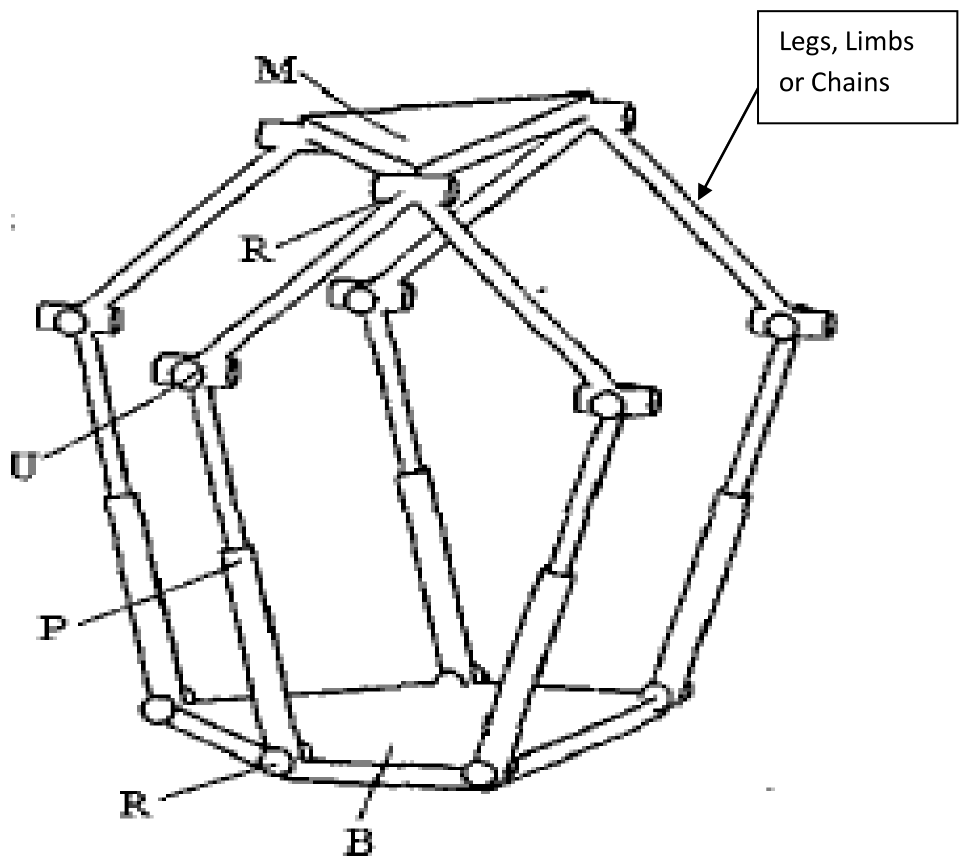


Figure - 1.2 Parallel Kinematic Chain

1.4.7 Difference between Serial and Parallel Devices

A comparison between general serial and parallel devices in terms of some necessary and desirable performance and control characteristics was presented in great detail in [10]. These characteristics are;

- Range of motion or workspace

- Rigidity or stiffness and strength
- Complexity of end-effector positioning formulation
- Complexity of system dynamics
- Precision positioning
- Load carrying distribution through system
- Fabrication (economics)
- Compactness

Chart 1 illustrates a performance chart which indicates, in a relative manner, which characteristics or criteria tend to be favorable for serial and parallel devices. This chart is a modified version of the performance chart presented in [10]

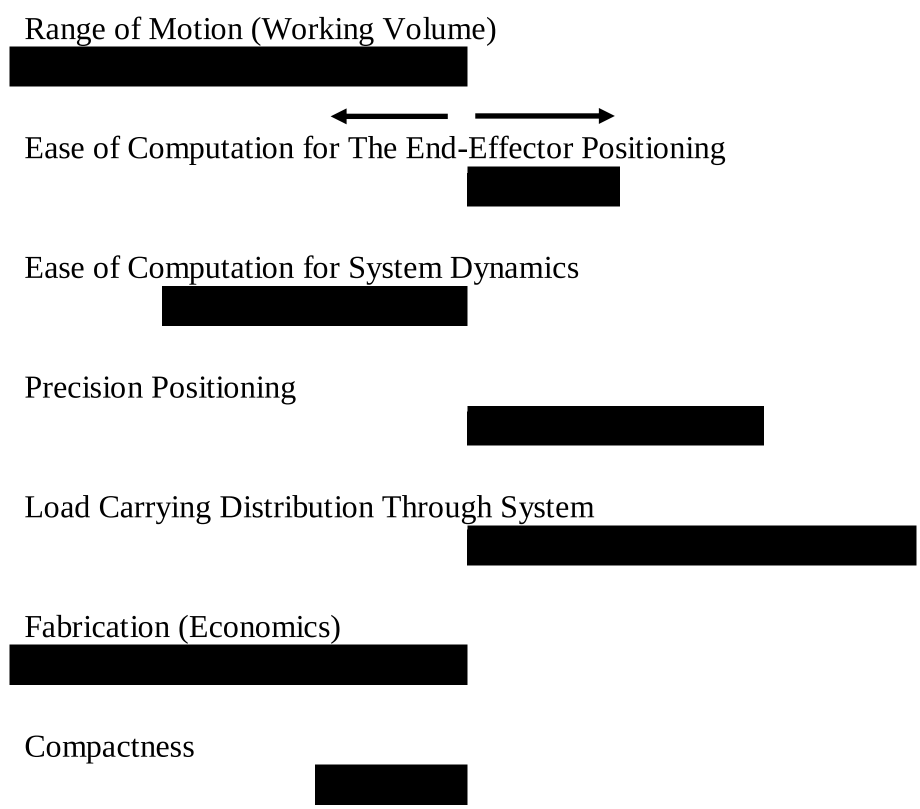


Chart 1.1 Performance Chart Serial and Parallel Kinematic Chains

Chapter 2- THE DECOMPOSITION OF INSTANTANEOUS TWISTS; DESIGN AND SELECTION OF JOINTS IN SERIAL ROBOTIC CHAINS

2.1 Introduction

This study is directed to the problem of identifying the limitations on type, direction and location of kinematic pairs needed to allow specified instantaneous kinematic freedom of the end effector of a serial kinematic chain. This report is restricted to the use of single degree of freedom joints i.e. prismatic (P) and revolute (R) joints.

If a number of end effector twists of a desired serial manipulator are given then this is enough to predict / design the number, type, location and orientation of the joints for this manipulator. The number of such joints is restricted to be the same as the number of given linearly independent twists. In principle, the range of numbers of given twists is between one and six, but the most interesting cases are those where the number of such twists (and hence joints) is two and three. The cases of twist numbers higher than this are simple extension of the arguments derived in the two joints case.

2.2 Screw Theory

While we go into the details of the study it is pertinent here to have a look at the screw theory at a glance.

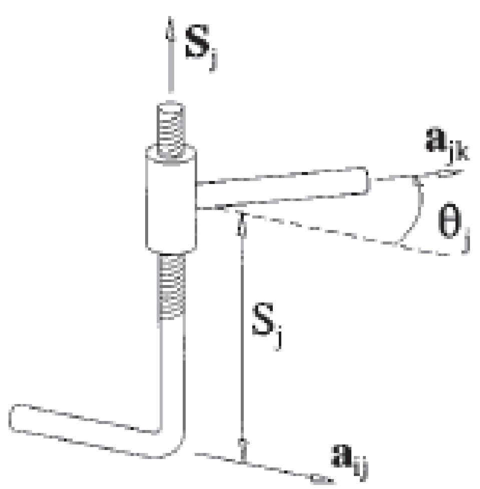


Figure- 2.1 Simple Screw

2.2.1 Introduction

The theory of screws is analogous to vector analysis in that both consists of an algebra with a geometric entity as a fundamental element. The vector is a fundamental element for vector analysis, and the screw is the fundamental element for the theory of screws.

2.2.2 Screw

A classical result in kinematics that will have far reaching implications for us is Charles Theorem. This result, also attributed to Mozi states that "every rigid body displacement can be expressed as a rotation about some fixed axis in space, followed by a pure translation parallel to that axis".

In the sense of rigid body motion, a screw is a way of describing a displacement. It can be thought of as a rotation about an axis and a translation along that same axis. Any general displacement can be described by a screw, and there are methods of converting between screws and other representations of displacements, such as homographic transformations.

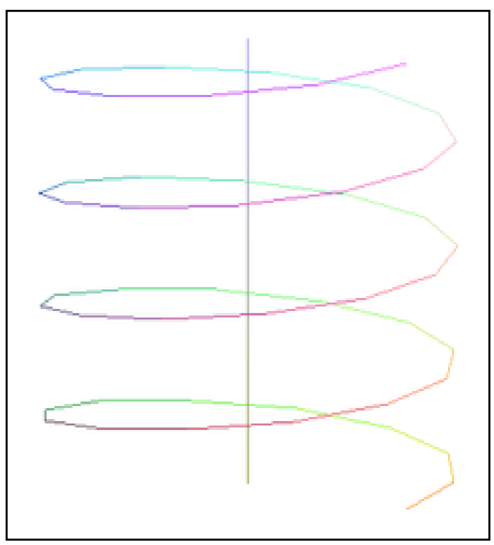


Figure- 2.2 Rotation and Translation about Same Axis

In rigid body dynamics, velocities of a rigid body can be represented by the concept of a screw. This kind of screw is called a twist, and represents the velocity of a body by the direction of its linear velocity, its angular velocity about the axis of translation, and the relationship between the two, called the pitch.

A pure screw is simply a geometric concept which describes a helix. A screw with zero pitch looks like a circle. A screw with infinite pitch looks like a straight line, but is not well defined.

Any motion along a screw can be decomposed into a rotation about an axis followed by a translation along that axis. Any general displacement of a rigid body can therefore be described by a screw.

2.2.2.1 Twist

Twists represent velocity of a body. For example, if you were climbing up a spiral staircase at a constant speed, your velocity would be easily described by a twist. A twist contains six quantities: three linear and three angular.

$$\underbrace{\begin{bmatrix} \omega \\ v \end{bmatrix}}_{twist} = \| \omega \| \underbrace{\begin{bmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{bmatrix}}_{rotation} + \lambda \underbrace{\begin{bmatrix} 0 \\ \hat{\omega} \end{bmatrix}}_{translation} \qquad (12)$$

Where $w = \|w\|$ w^ and λ is called the pitch. Examination of this formula with reference to Figure 2.3 shows that v is the velocity of an imaginary point passing through the origin of the coordinate system in which the twist is expressed and moving together with the object. The twist can be associated with a geometrical line, namely the line passing through r and spanned by w which is left invariant by the rotation.

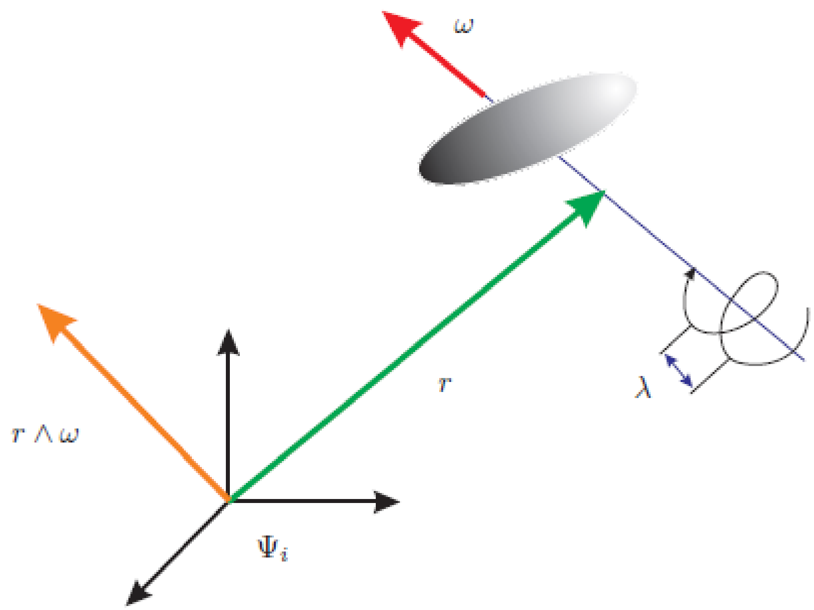


Figure- 2.3 Twist Description

2.2.1.2 Pitch

The ratio between the rotation and the translation is called the pitch λ of the finite motion, sometimes denoted by h. Mathematically this coordinate independent property of motion is written as;

$$\lambda = d/\theta \tag{13}$$

Therefore, a pure rotation is given as

$$\underbrace{\begin{bmatrix} \omega \\ v \end{bmatrix}}_{twist} = \| \omega \| \underbrace{\begin{bmatrix} \hat{\omega} \\ r \wedge \hat{\omega} \end{bmatrix}}_{rotation} \tag{14}$$

For a line at infinity the twist is given as

$$\begin{bmatrix} 0 \\ v \end{bmatrix} \tag{15}$$

2.3 Preliminary Analysis

Given n linearly independent twists t_i for i= 1 to n, it is desired to design a serial manipulator containing j_i joints where these joints whose types, position and orientation have to be designed, can realize any linear combination of the given twists. We are interested in the case where the number of joints is equal to the number of given twists. This can be written as below in mathematical form;-

$$
\begin{aligned}
t_1 &= \sum_{i=1}^{n} \alpha_{1i}\,\$i \\
t_2 &= \sum_{i=1}^{n} \alpha_{2i}\,\$i \\
&\quad \ldots\ldots \\
&\quad \ldots\ldots \\
t_n &= \sum_{i=1}^{n} \alpha_{2i}\,\$i
\end{aligned}
\tag{16}
$$

This can be written in the matrix form:

$$
\underbrace{\begin{pmatrix} t_1 & t_2 & . & . & . & t_n \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}}_{6xn}
=
\underbrace{\begin{pmatrix} \$_1 & \$_2 & . & . & . & \$_n \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}}_{6xn}
\underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{12} & . & . & . & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & . & . & . & \alpha_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \alpha_{n1} & \alpha_{n2} & . & . & . & \alpha_{n1} \end{pmatrix}}_{nxn}
\tag{17}
$$

The essential condition for equivalence is that the coefficient matrix α_{ij} is of full rank, n, such that the mapping between the given twists and the twists on the designed joint screws is one-to-one and onto; in other words, bijective (else multiplication will not be possible)[11].

It should also be noted that the columns of each of the other matrices are linearly independent. The above equation can also be written as;

$$[T] = [J][\alpha] \tag{18}$$

Which can be inverted to give;

$$[T]\,[\alpha]^{-1} = [J] \tag{19}$$

In principle, it is immaterial whether the matrix α that is the coefficient matrix is associated with the matrix of the given twists or that of the joint screws, as in either case it must be of the same dimension and full rank. The design of a serial manipulator is studied for i=1 and 2 in detail and partially for n=3.

Chapter 3- DESIGN OF A SINGLE DOF SERIAL MANIPULATOR

3.1 Introduction

The number of DOF that a manipulator possesses is the number of independent position variables that would have to be specified in order to locate all parts of the mechanism. In other words, it refers to the number of different ways in which a robot arm can move. In the case of typical industrial robots, because a manipulator is usually an open kinematic chain, and because each joint position is usually defined with a single variable, the number of Single DOF joints equals the number of degrees of freedom.

To start with, the design of a single DOF manipulator is not based on the idea that they do exist but on the concept, that we can always isolate any joint of a serial manipulator to see if we can change its location, orientation to achieve the desired end effector twist or not.

3.2 Design of Single DOF Manipulator

A single DOF serial manipulator with linearly independent twists consists of a single joint or as discussed above we can isolate a single joint of any number of joint serial manipulator. But the effect of only that particular single joint will be looked into so the type of the joint depends on the desired end-effector twist caused due to this joint. The end effector twist, for a single DOF manipulator can be written in the screw coordinates in one of the following forms.

$T = [\ \bar{\omega};\ (r \times \bar{\omega})\ ]$ or (20)

$T = [\ \bar{\omega};\ (r \times \bar{\omega}) + h\,\bar{\omega}\ ]$ or (21)

$T = [\ 0;\ \bar{v}\ ]$ (22)

Where

T is the given twist screw

$\bar{\omega}$ is the twist axis or rotational part of the twist

r is the position vector of the axis of twist from the chosen axis system as shown in Fig 3.1

h is the pitch of the screw

$\bar{v}$ is the free vector or the linear velocity of the body

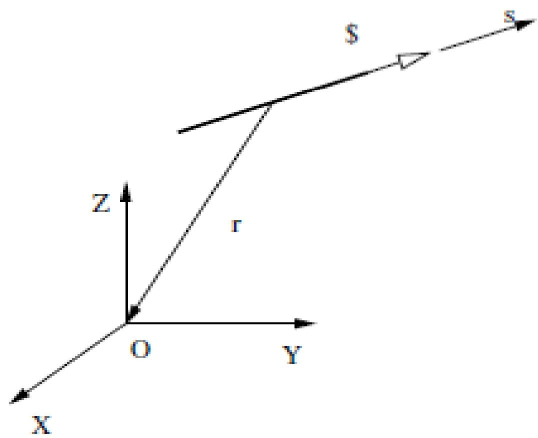


Figure- 3.1 Position Vector r

The twist screw written in the first equation 20 is a zero pitch screw, the equation 21 describes a twist with finite pitch h, and in the equation 22 gives the twist screw of an infinite pitch screw. As we will be discussing only the manipulators with single DOF revolute and prismatic joints so only zero and infinite pitch screws will be taken into account here. For a single DOF serial manipulator the given twist will be written in the following form and the two possible cases for such a twist are discussed below in detail.

$$t_1 = [\, t_{1x} \ t_{1y} \ t_{1z} \ ; \ t_{01x} \ t_{01y} \ t_{01z} \,] \qquad (23)$$

3.2.1 Case 1 – Infinite Pitch Screw Rank of Primary Part =0

In the most general form the given twist will be written in the following form with the primary part equal to zero

$$t_1 = [\, 0 \ 0 \ 0 \ ; \ t_{01x} \ t_{01y} \ t_{01z} \,] \qquad (24)$$

3.2.1.1 Possible Topology

If the twist is given as in the equation 24, and we are required to achieve this twist by using only one joint then it has to be a Prismatic (P) joint i.e. the only option is **(1P)**.

3.2.1.2 Positioning and Orientation

The Prismatic (P) joint has to be oriented in the direction of the vector v, and can in principle be placed anywhere in 3-space.

3.2.2 Case 2 – Zero Pitch Screw Rank of Primary Part =1

In the most general form the given twist will be written in the following form with the primary and secondary part are both non zero.

$t_1 \quad = \quad [\, t_{1x} \;\; t_{1y} \;\; t_{1z} \;\; ; \;\; t_{01x} \;\; t_{01y} \;\; t_{01z} \,]$

For example $t_1 \quad = \quad [\, 0 \;\; 0 \;\; 1 \;\; ; \;\; t_{1x} \;\; t_{1y} \;\; 0 \,]$ (25)

3.2.2.1 Possible Topology

If the twist is given as in the equation 25, and we are required to achieve this twist by using only one joint then it has to be a revolute (R) joint i.e. the only option is **(1R).**

3.2.2.2 Positioning and Orientation

The axis of the (R) joint should be in the direction of the twist, that is, the axis $\bar{\omega}$ and it has the same position vector r. Thus the twist of the end effector is same as that of the joint. This means we cannot place and orient a replacement joint in this case which is different in orientation and placement as of the original joint.

Chapter 4- Design of a Two DOF Serial Manipulator

4.1 Introduction

In the case of a two DOF serial manipulator we need minimum of two joints to obtain the desired twists. The type of joints depends on the desired twists of the end effector. For a two DOF serial manipulator the two linearly independent given twists will be written in the following form. The total end-effector twist in this case is the linear combination of these two twists.

$$\left.\begin{aligned} t_1 &= [\, t_{1x}\ t_{1y}\ t_{1z}\ ;\ t_{01x}\ t_{01y}\ t_{01z} \,] \\ t_2 &= [\, t_{2x}\ t_{2y}\ t_{2z}\ ;\ t_{02x}\ t_{02y}\ t_{02z} \,] \end{aligned}\right\} \quad (26)$$

If the given twists are written in the matrix form denoted by (T) (called as twist matrix from now onward), the type of joint possibilities is checked by finding the rank of the primary part of the matrix given below and is denoted by (P)(called the P matrix from now onward).

$$T = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ t_{2x} & t_{2y} & t_{2z} & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \quad (27)$$

The primary part of the matrix can be written in the matrix form as,

$$P = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{2x} & t_{2y} & t_{2z} \end{pmatrix} \quad (28)$$

The rank of the (P) matrix can be two, one or zero, each case results into a different possibility of alternate joints. The alternate type, location and orientation of the joints depends on the rank of the primary part of the given twists; each case is therefore, discussed below in detail.

4.1.1 Case 1 – Rank of Primary Part =0

In the most general form the twist matrix will have primary part equal to zero so the twist and P matrix for this case can be written as

$$T = \begin{pmatrix} 0 & 0 & 0 & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 0 & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \tag{29}$$

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{30}$$

4.1.1.1 Possible Topology

The manipulator can have only one configuration, that is; (2P) arrangement with the two P joints linearly independent of each other.

4.1.1.2 Positioning and Orientation

When the primary part is possessing rank zero the two prismatic joints can be placed anywhere in three dimensional space as long as they are in the plane formed by the secondary parts of the two twists and are linearly independent of each other.

4.1.2 Case 2 – Rank of Primary Part =1

In the most general form the twist matrix will have two possible twist matrices and (P) matrices.

$$T_1 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ t_{1x} & t_{1y} & t_{1z} & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \tag{31}$$

$$P_1 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{1x} & t_{1y} & t_{1z} \end{pmatrix} \tag{32}$$

$$T_2 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 0 & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \qquad (33)$$

$$P_2 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ 0 & 0 & 0 \end{pmatrix} \qquad (34)$$

4.1.2.1 Case 2 a-Rank of Primary Part =1 (Twist matrix of the form T_1)

If the twist matrix is T_1 with the corresponding primary matrix having rank=1, we can always choose the origin in such a way that the T_1 matrix becomes;

$$T_1 = \begin{pmatrix} 0 & 0 & 1 & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 1 & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \qquad (35)$$

Therefore, the position vectors and the dual parts of the given twists lie in x-y plane. If the position vectors of the two given twists are of the form ($X_n\bar{\imath}$ + $Y_n\,\bar{\jmath}$ + $Z_n\bar{k}$), where n is 1 and 2 in present case, the above twists matrix is then written as;

$$T_1 = \begin{pmatrix} 0 & 0 & 1 & Y_1 & -X_1 & 0 \\ 0 & 0 & 1 & Y_2 & -X_2 & 0 \end{pmatrix} \qquad (36)$$

$$\text{Since } r \times \bar{\omega} = \det \begin{pmatrix} i & j & k \\ X_n & Y_n & Z_n \\ 0 & 0 & 1 \end{pmatrix} \qquad (37)$$

4.1.2.1.1 Possible Topologies

We have the following possibilities, for the position and orientation of the alternate joints.

- 2 R joints with axes in the same direction as of the given twists
- 1R joint and 1P joint

4.1.2.1.2 Positioning and Orientation -2R Case

The 2-R joints will be installed with their axes parallel to the given twists. These joints cannot be placed arbitrarily in space but have to be placed under certain conditions to achieve any linear combination of the given twists, these conditions are discussed below.

Let the two R joints which are oriented in the direction of the given twists are positioned at $(a_1\bar{i} + b_1\bar{j} + c_1\bar{k})$ and $(a_2\bar{i} + b_2\bar{j} + c_2\bar{k})$ respectively. With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_a = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \end{pmatrix} \qquad (38)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ Y_1 & Y_2 \\ -X_1 & -X_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ b_1 & b_2 \\ -a_1 & -a_2 \\ 0 & 0 \end{pmatrix} \qquad (39)$$

The equation can be reduced in the form with each matrix having rank =2.

$$\begin{pmatrix} 1 & 1 \\ Y_1 & Y_2 \\ -X_1 & -X_2 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ b_1 & b_2 \\ -a_1 & -a_2 \end{pmatrix} \qquad (40)$$

For the alpha matrix to have rank equal to two the condition is determinant of alpha matrix should not be equal to zero in mathematical form this condition can be written as

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 \qquad (41)$$

We can write the above matrix equation in the form

$$\begin{pmatrix} \alpha_{11} + \alpha_{21} & \alpha_{12} + \alpha_{22} \\ Y_1\,\alpha_{11} + Y_2\,\alpha_{21} & Y_1\,\alpha_{12} + Y_2\,\alpha_{22} \\ -X_1\,\alpha_{11} - X_2\,\alpha_{21} & -X_1\,\alpha_{12} - X_2\,\alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ b_1 & b_2 \\ -a_1 & -a_2 \end{pmatrix} \qquad (42)$$

This gives the following conditions for equivalence

$$\alpha_{11} + \alpha_{21} = 1 \qquad (43)$$

$$\alpha_{12} + \alpha_{22} = 1 \qquad (44)$$

$$Y_1\,\alpha_{11} + Y_2\,\alpha_{21} = b_1 \qquad (45)$$

$$Y_1\,\alpha_{12} + Y_2\,\alpha_{22} = b_2 \qquad (46)$$

$$X_1\,\alpha_{11} + X_2\,\alpha_{21} = a_1 \qquad (47)$$

$$X_1\,\alpha_{12} + X_2\,\alpha_{22} = a_2 \qquad (48)$$

Putting $\alpha_{11} = 1-\alpha_{21}$ and $\alpha_{22} = 1-\alpha_{12}$ in alpha matrix equivalence condition we get

$$(1-\alpha_{21})(1-\alpha_{12}) - \alpha_{12}\,\alpha_{21} \neq 0$$

or $\quad 1-\alpha_{12} - \alpha_{21} + \alpha_{12}\,\alpha_{21} - \alpha_{12}\,\alpha_{21} \neq 0$

or $\quad 1-\alpha_{12} - \alpha_{21} \neq 0$

so $\quad \alpha_{12} + \alpha_{21} \neq 1 \qquad (49)$

Similarly if we put $\alpha_{21} = 1-\alpha_{11}$ and $\alpha_{12} = 1-\alpha_{22}$ in alpha matrix equivalence condition we get

$$\alpha_{22} + \alpha_{11} \neq 1 \qquad (50)$$

Putting $\alpha_{11} = 1-\alpha_{21}$ and $\alpha_{22} = 1-\alpha_{12}$ the rest of the equivalence conditions become

$$(1-\alpha_{21})\,Y_1 + \alpha_{21}\,Y_2 = b_1 \qquad (51)$$

$$(1-\alpha_{21})\,X_1 + \alpha_{21}\,X_2 = a_1 \qquad (52)$$

$$\alpha_{12}\,Y_1 + (1-\alpha_{12})\,Y_2 = b_2 \qquad (53)$$

$$\alpha_{12}\,X_1 + (1-\alpha_{12})\,X_2 = a_2 \qquad (54)$$

These conditions can be further modified as below:-

$$Y_1 + \alpha_{21}(Y_2 - Y_1) = b_1 \qquad (55)$$

$$X_1 + \alpha_{21}(X_2 - X_1) = a_1 \qquad (56)$$

$$Y_2 - \alpha_{12}(Y_2 - Y_1) = b_2 \qquad (57)$$

$$X_2 - \alpha_{12}(X_2 - X_1) = a_2 \qquad (58)$$

4.1.2.1.3 Solution Set-2R Case

The solution set therefore is the set of following equations with the condition that the given twist axes are aligned with the Z-axis of the co-ordinate system.

$$
\left.
\begin{aligned}
\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} &\neq 0 \\
\alpha_{12} + \alpha_{21} &\neq 1 \\
\alpha_{22} + \alpha_{11} &\neq 1 \\
Y_1 + \alpha_{21}\,(Y_2 - Y_1) &= b_1 \\
X_1 + \alpha_{21}\,(X_2 - X_1) &= a_1 \\
Y_2 - \alpha_{12}\,(Y_2 - Y_1) &= b_2 \\
X_2 - \alpha_{12}\,(X_2 - X_1) &= a_2
\end{aligned}
\right\} \quad (59)
$$

4.1.2.1.4 Numerical Example-2R Case

Let the position vectors of the twists which are directed in the Z-axis of the co-ordinate system are given as;

$r_1 = X_1\bar{\imath} + Y_1\bar{\jmath} + Z_1\bar{k} \quad = \bar{\imath} + 4/3\,\bar{\jmath}$

$r_2 = X_2\bar{\imath} + Y_2\bar{\jmath} + Z_2\bar{k} \quad = 3\bar{\imath} + 2\,\bar{\jmath}$

then;

$$
\begin{aligned}
a_1 &= X_1 + \alpha_{21}\,(X_2 - X_1) \\
&= 1+2\alpha_{21} \\
b_1 &= Y_1 + \alpha_{21}\,(Y_2 - Y_1) \\
&= 4/3 + 2/3\alpha_{21} \\
a_2 &= X_2 + \alpha_{12}\,(X_2 - X_1) \\
&= 3-2\alpha_{12} \\
b_2 &= Y_2 + \alpha_{12}\,(Y_2 - Y_1) \\
&= 2-2/3\alpha_{12}
\end{aligned}
$$

Now we can choose α_{12} and α_{21} with the condition that

$\alpha_{12} + \alpha_{21} \neq 1$

let $\alpha_{12} = 4$ and $\alpha_{21} = 10$ then we have,

$a_1 = 21$

$b_1 = 8$

$a_2 = -5$

$b_2 = -2/3$

and we know that;

$\alpha_{11} + \alpha_{21} = 1$

$\alpha_{12} + \alpha_{22} = 1$

This gives

$\alpha_{11} = -9$

$\alpha_{22} = -3$

The equation of equivalence takes the following form and thus any linear combination of the given twists can be provided by these two joints.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 4/3 & 2 \\ -1 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -9 & 4 \\ 10 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 8 & -2/3 \\ -21 & 5 \\ 0 & 0 \end{pmatrix}$$

4.1.2.1.5 Positioning and Orientation -1R1P Case

The twist matrix will still be the same T_1 matrix, but as the P_1 matrix is of rank one we can install one P and one R joint. The orientation of the prismatic joint and orientation and position of the revolute joint is discussed in detail below.

Let the R joints which is oriented in the direction of the given twists is positioned at $(a_1\bar{i} + b_1\bar{j} + c_1\bar{k})$ and the P joint is oriented in the X-Y plane and its screw is given as $(0 \;\; 0 \;\; 0 \;\; P_x \;\; P_y \;\; 0)$. With these assumptions the equation of the equivalence becomes;

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ Y_1 & Y_2 \\ -X_1 & -X_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ b_1 & Px \\ -a_1 & P_y \\ 0 & 0 \end{pmatrix} \qquad (60)$$

The equation can be reduced in the form with each matrix having rank =2.

$$\begin{pmatrix} 1 & 1 \\ Y_1 & Y_2 \\ -X_1 & -X_2 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_1 & P_x \\ -a_1 & P_y \end{pmatrix} \qquad (61)$$

For the alpha matrix to have rank equal to two the condition is determinant of alpha matrix should not be equal to zero in mathematical form this condition can be written as

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 \qquad (62)$$

We can write the above matrix equation in the form

$$\begin{pmatrix} \alpha_{11} + \alpha_{21} & \alpha_{12} + \alpha_{22} \\ Y_1 \alpha_{11} + Y_2 \alpha_{21} & Y_1 \alpha_{12} + Y_2 \alpha_{22} \\ -X_1 \alpha_{11} - X_2 \alpha_{21} & -X_1 \alpha_{12} - Y_2 \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_1 & P_x \\ -a_1 & P_y \end{pmatrix} \quad (63)$$

This gives the following conditions for equivalence

$$\left.\begin{array}{lcl} \alpha_{11} + \alpha_{21} & = & 1 \\ \alpha_{12} + \alpha_{22} & = & 0 \\ Y_1 \alpha_{11} + Y_2 \alpha_{21} & = & b_1 \\ Y_1 \alpha_{12} + Y_2 \alpha_{22} & = & P_x \\ X_1 \alpha_{11} + X_2 \alpha_{21} & = & a_1 \\ -X_1 \alpha_{12} - X_2 \alpha_{22} & = & Py \end{array}\right\} \quad (64)$$

Putting $\alpha_{11} = 1 - \alpha_{21}$ and $\alpha_{22} = - \alpha_{12}$ in alpha matrix equivalence condition we get

$$(1 - \alpha_{21})(- \alpha_{12}) - \alpha_{12} \alpha_{21} \neq 0$$

or $\quad - \alpha_{12} + \alpha_{12} \alpha_{21} - \alpha_{12} \alpha_{21} \neq 0$

so $\quad \alpha_{12} \neq 0 \qquad (65)$

Similarly if we put $\alpha_{21} = 1 - \alpha_{11}$ and $\alpha_{12} = - \alpha_{22}$ in alpha matrix equivalence condition we get

$$\alpha_{22} \neq 0 \qquad (66)$$

Putting $\alpha_{11} = 1 - \alpha_{21}$ and $\alpha_{22} = -\alpha_{12}$ the rest of the equivalence conditions become

$$(1-\alpha_{21})\, Y_1 + \alpha_{21}\, Y_2 = b_1$$

$$(1-\alpha_{21})\, X_1 + \alpha_{21}\, X_2 = a_1$$

$$\alpha_{12}\, Y_1 - \alpha_{12}\, Y_2 = P_x$$

$$-\alpha_{12}\, X_1 + \alpha_{12}\, X_2 = P_y$$

These conditions can be further modified as below:-

$$Y_1 + \alpha_{21}\,(Y_2 - Y_1) = b_1$$

$$X_1 + \alpha_{21}\,(X_2 - X_1) = a_1$$

$$-\alpha_{12}\,(Y_2 - Y_1) = P_x$$

$$\alpha_{12}\,(X_2 - X_1) = P_y$$

4.1.2.1.6 Solution Set-1R1P

The solution set therefore is the set of following equations with the condition that the given twists axes are aligned with the Z-axis of the co-ordinate system.

$$\left.\begin{array}{lcl} \alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 & & \\ \alpha_{12} \neq 0 & & \\ \alpha_{22} \neq 0 & & \\ Y_1 + \alpha_{21}\,(Y_2 - Y_1) & = & b_1 \\ X_1 + \alpha_{21}\,(X_2 - X_1) & = & a_1 \\ -\alpha_{12}\,(Y_2 - Y_1) & = & P_x \\ \alpha_{12}\,(X_2 - X_1) & = & P_y \end{array}\right\} \qquad (67)$$

4.1.2.1.7 Numerical Example-1R1P

Let the position vectors of the twists which are directed in the Z-axis of the co-ordinate system are given as;

$r_1 = X_1\bar{\imath} + Y_1\bar{\jmath} + Z_1\bar{k} \quad = \bar{\imath} + 4/3\bar{\jmath}$

$r_2 = X_2\bar{\imath} + Y_2\bar{\jmath} + Z_2\bar{k} \quad = 3\bar{\imath} + 2\bar{\jmath}$

then;

$a_1 \quad = X_1 + \alpha_{21} (X_2 - X_1)$

$\quad = 1+2\alpha_{21}$

$b_1 \quad = Y_1 + \alpha_{21} (Y_2 - Y_1)$

$\quad = 4/3 + 2/3\alpha_{21}$

$P_x \quad = - \alpha_{12} (Y_2 - Y_1)$

$\quad = -2/3\ \alpha_{12}$

$P_y \quad = \alpha_{12} (X_2 - X_1)$

$\quad = 2\ \alpha_{12}$

Now we can choose α_{12} and α_{21} with the condition that

$\alpha_{12} \neq 0$

let $\alpha_{12} = 4$ and $\alpha_{21} = 10$ then we have,

$a_1 = 21$

$b_1 = 8$

$Px = -8/3$

$P_y = 8$

and we know that;

$\alpha_{11} + \alpha_{21} = 1$

$\alpha_{12} + \alpha_{22} = 0$ also $\alpha_{22} \neq 0$

This gives

$\alpha_{11} = -9$

$\alpha_{22} = -4$

The equation of equivalence takes the following form and thus any linear combination of the given twists can be provided by these two joints.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 4/3 & 2 \\ -1 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -9 & 4 \\ 10 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 8 & -8/3 \\ -21 & 8 \\ 0 & 0 \end{pmatrix}$$

4.1.2.2 Case 2 b-Rank of Primary Part =1 (Twist matrix of the form T_2)

If the twist matrix is T_2 with the corresponding primary matrix having rank=1, we can always choose the origin in such a way that the T_2 matrix becomes;

$$T_2 = \begin{pmatrix} 0 & 0 & 1 & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 0 & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \qquad (68)$$

Therefore, the dual part of the first twist lies in the x-y plane. If the position vector of the given first twist is of the form $(X_1\bar{i} + Y_1\bar{j} + Z_1\bar{k})$, the above twists matrix is then written as;

$$T_2 = \begin{pmatrix} 0 & 0 & 1 & Y_1 & -X_1 & 0 \\ 0 & 0 & 0 & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \qquad (69)$$

4.1.2.2.1 Possible Topologies

We have the following possibilities, for the position and orientation of the alternate joints.

- If the second twist has only x and y components (or only x, or only y) then the joint possibilities are:-
  - **2 R** joints with axes in the same direction as of the given twists
  - **1R1P** joints with axis of R joint in the same direction as of the given twist and 1P joint oriented in the direction of the second twist in the x-y plane.
- If the second twist has some value in the z axis (no matter we have some x and y component or not) also, then we have only one alternate joint possibility as given below:-
  - 1R joint with axis in the same direction as of the given twist and 1P joint oriented in the direction of the second twist in the x-y-z direction.

4.1.2.2.2 Positioning and Orientation 2R Case (Second Twist oriented in x-y Plane)

The two R joints will be installed with their axis parallel to the given twist with primary part not equal to zero. These joints cannot be placed arbitrarily in space but have to be placed under certain conditions to achieve any linear combination of the given twists, these conditions are discussed below.

Let the two R joints which are oriented in the direction of the given twist (twist with primary part not equal to zero) are positioned at ($a_1\bar{i} + b_1\bar{j} + c_1\bar{k}$) and ($a_2\bar{i} + b_2\bar{j} + c_2\bar{k}$) respectively. With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_a = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \end{pmatrix} \quad (70)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ Y_1 & P_x \\ -X_1 & P_y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ b_1 & b_2 \\ -a_1 & -a_2 \\ 0 & 0 \end{pmatrix} \quad (71)$$

The equation can be reduced in the form with each matrix having rank =2.

$$\begin{pmatrix} 1 & 0 \\ Y_1 & P_x \\ -X_1 & P_y \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ b_1 & b_2 \\ -a_1 & -a_2 \end{pmatrix} \quad (72)$$

For the alpha matrix to have rank equal to two the condition is determinant of alpha matrix should not be equal to zero in mathematical form this condition can be written as

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 \quad (73)$$

we can write the above matrix equation in the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ Y_1 \alpha_{11} + P_x \alpha_{21} & Y_1 \alpha_{12} + P_x \alpha_{22} \\ -X_1 \alpha_{11} + P_y \alpha_{21} & -X_1 \alpha_{12} + P_y \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ b_1 & b_2 \\ -a_1 & -a_2 \end{pmatrix} \quad (74)$$

This gives the following conditions for equivalence

$$\alpha_{11} = 1 \quad (75)$$

$$\alpha_{12} = 1 \quad (76)$$

$$Y_1 \alpha_{11} + P_x \alpha_{21} = b_1 \quad (77)$$

$$Y_1 \alpha_{12} + P_x \alpha_{22} = b_2 \quad (78)$$

$$- X_1 \alpha_{11} + P_y \alpha_{21} = -a_1 \quad (79)$$

$$-X_1 \alpha_{12} + P_y \alpha_{22} = -a_2 \quad (80)$$

Putting $\alpha_{11} = 1$ and $\alpha_{12} = 1$ in alpha matrix equivalence condition we get

$$\alpha_{22} - \alpha_{21} \neq 0$$

or

$$\alpha_{22} \neq \alpha_{21} \quad (81)$$

Putting $\alpha_{11} = 1$ and $\alpha_{12} = 1$ the rest of the equivalence conditions become

$$Y_1 + P_x \alpha_{21} = b_1 \quad (82)$$

$$Y_1 + P_x \alpha_{22} = b_2 \quad (83)$$

$$X_1 - P_y \alpha_{21} = a_1 \quad (84)$$

$$X_1 - P_y \alpha_{22} = a_2 \quad (85)$$

4.1.2.2.3 Solution Set 2R Case (Second Twist oriented in x-y Plane)

The solution set therefore, is the set of following equations with the condition that one of the given twists is aligned with the Z-axis of the co-ordinate system and the other is oriented in the x-y plane (and is due to a prismatic joint).

$$\left.\begin{array}{lcl} \alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 & & \\ \alpha_{11} = 1 & & \\ \text{and} & & \\ \alpha_{12} = 1 & & \\ \alpha_{22} \neq \alpha_{21} & & \\ Y_1 + P_x\,\alpha_{21} & = & b_1 \\ Y_1 + P_x\,\alpha_{22} & = & b_2 \\ X_1 - Py\,\alpha_{21} & = & a_1 \\ X_1 - P_y\,\alpha_{22} & = & a_2 \end{array}\right\} \quad (86)$$

4.1.2.2.4 Numerical Example 2R Case (Second Twist oriented in x-y Plane)

Let the position vectors of the first twist which is directed in the Z-axis of the co-ordinate system is given as;

$$r_1 = X_1\bar{\imath} + Y_1\bar{\jmath} + Z_1\bar{k}$$

$$= 21\bar{\imath} + 8\bar{\jmath}$$

Let the other joints twist be oriented in the direction of the following vector;

v_1=-8/3i + 8j

then;

$$a_1 = X_1 - P_y\,\alpha_{21}$$

$$= 21-8\alpha_{21}$$

$b_1 \quad = Y_1 + P_x\, \alpha_{21}$

$\quad = 8 - 8/3\, \alpha_{21}$

$a_2 \quad = X_1 - P_y\, \alpha_{22}$

$\quad = 21 - 8\alpha_{22}$

$b_2 \quad = Y_1 + P_x\, \alpha_{22}$

$\quad = 8 - 8/3\alpha_{22}$

Now we can choose α_{21} and α_{22} with the condition that

$\alpha_{11} = 1$ and $\alpha_{12} = 1$

$\alpha_{22} \neq \alpha_{21}$

let

$\alpha_{21} = 10/4$ and $\alpha_{22} = 9/4$

then we have,

$a_1 = 1$, $b_1 = 4/3$, $a_2 = 3$, $b_2 = 2$

The equation of equivalence takes the following form and thus any linear combination of the given twists can be provided by these two joints.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 8 & -8/3 \\ -21 & 8 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 10/4 & 9/4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 4/3 & 2 \\ -1 & -3 \\ 0 & 0 \end{pmatrix}$$

4.1.2.2.5 Positioning and Orientation 1R1P Case (Second Twist oriented in x-y Plane)

The twist matrix will still be the same T_2 matrix, but as the P_2 matrix is of rank one we can install one P and one R joint. The orientation of the prismatic joint and orientation and position of the revolute joint is discussed in detail below.

Let the R joint which is oriented in the direction of the given twist is positioned at $(a_1\bar{i} + b_1\bar{j} + c_1\bar{k})$ and the P joint is oriented in the X-Y plane and its screw is given as $(0 \quad 0 \quad 0 \quad P_x \quad P_y \quad 0)$. With these assumptions the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_a = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 0 & P_{x2} & P_{y2} & 0 \end{pmatrix} \qquad (87)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ Y_1 & P_{x1} \\ -X_1 & P_{y1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ b_1 & P_{x2} \\ -a_1 & P_{y2} \\ 0 & 0 \end{pmatrix} \qquad (88)$$

The equation can be reduced in the form with each matrix having rank =2.

$$\begin{pmatrix} 1 & 0 \\ Y_1 & P_{x1} \\ -X_1 & P_{y1} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_1 & P_{x2} \\ -a_1 & P_{y2} \end{pmatrix} \tag{89}$$

For the alpha matrix to have rank equal to two the condition is determinant of alpha matrix should not be equal to zero in mathematical form this condition can be written as

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 \tag{90}$$

We can write the above matrix equation 89 in the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ Y_1\,\alpha_{11} + P_{x1}\,\alpha_{21} & Y_1\,\alpha_{12} + P_{x1}\,\alpha_{22} \\ -X_1\,\alpha_{11} + P_{y1}\,\alpha_{21} & -X_1\,\alpha_{12} + P_{y1}\alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_1 & P_{x2} \\ -a_1 & P_{y2} \end{pmatrix} \tag{91}$$

This gives the following conditions for equivalence

$$\alpha_{11} = 1 \tag{92}$$

$$\alpha_{12} = 0 \tag{93}$$

$$Y_1\,\alpha_{11} + P_{x1}\alpha_{21} = b_1 \tag{94}$$

$$Y_1\,\alpha_{12} + P_{x1}\,\alpha_{22} = P_{x2} \tag{95}$$

$$- X_1\,\alpha_{11} + P_{y1}\,\alpha_{21} = -a_1 \tag{96}$$

$$-X_1\,\alpha_{12} + P_{y1}\,\alpha_{22} = P_{y2} \tag{97}$$

Putting following values of α_{11} and α_{12} in alpha matrix equivalence condition

$\alpha_{11} = 1$

$\alpha_{12} = 0$

we get

$$\alpha_{22} \neq 0 \tag{98}$$

Putting $\alpha_{11} = 1$ and $\alpha_{12} = 1$ the rest of the equivalence conditions become

$$Y_1 + P_{x1}\alpha_{21} = b_1 \tag{99}$$

$$P_{x1}\,\alpha_{22} = P_{x2} \tag{100}$$

$$X_1 - P_{y1}\,\alpha_{21} = a_1 \tag{101}$$

$$P_{y1}\,\alpha_{22} = P_{y2} \tag{102}$$

4.1.2.2.6 Solution Set 1R1P Case (Second Twist oriented in x-y Plane)

The solution set therefore, is the set of following equations with the condition that one of the given twists is aligned with the Z-axis of the co-ordinate system and the other is oriented in the x-y plane (and is due to a prismatic joint).

$$\left.\begin{array}{lcl} \alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 & & \\ \alpha_{11} = 1 \text{ and } \alpha_{12} = 0 & & \\ \alpha_{22} \neq 0 & & \\ Y_1 + P_{x1}\alpha_{21} & = & b_1 \\ P_{x1}\,\alpha_{22} & = & P_{x2} \\ X_1 - P_{y1}\,\alpha_{21} & = & a_1 \\ P_{y1}\,\alpha_{22} & = & P_{y2} \end{array}\right\} \tag{103}$$

4.1.2.2.7 Numerical Example 1R1P Case (Second Twist oriented in x-y Plane)

Let the position vectors of the first twist which is directed in the Z-axis of the co-ordinate system is given as;

$$r_1 = X_1\bar{i} + Y_1\bar{j} + Z_1\bar{k} \quad = 21\bar{i} + 8\bar{j}$$

Let the other joints twist be oriented in the direction of the following vector;

$$v_1 = -8/3i + 8j$$

then;

$$a_1 = X_1 - P_{y1}\,\alpha_{21}$$

$$= 21 - 8\alpha_{21}$$

$$b_1 = Y_1 + P_{x1}\,\alpha_{21}$$

$$= 8 - 8/3\,\alpha_{21}$$

$$P_{x2} = P_{x1}\,\alpha_{22}$$

$$= -8/3\alpha_{22}$$

$$P_{y2} = P_{y1}\,\alpha_{22}$$

$$= 8\alpha_{22}$$

Now we can choose α_{21} and α_{22} with the condition that

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0$$

$$\alpha_{11} = 1 \text{ and } \alpha_{12} = 0$$

$$\alpha_{22} \neq 0$$

let

$$\alpha_{21} = 10/4$$

and

$$\alpha_{22} = 9/4$$

then we have,

$a_1 = 1$

$b_1 = 4/3$

$P_{x2} = -15/4$

$P_{y2} = 18$

The equation of equivalence takes the following form and thus any linear combination of the given twists can be provided by these two joints.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 8 & -8/3 \\ -21 & 8 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 10/4 & 9/4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 4/3 & -15/4 \\ -1 & 18 \\ 0 & 0 \end{pmatrix}$$

4.1.2.2.8 1R1P Case (Second Twist oriented in x-y-z direction)

The twist matrix will still be the same T_2 matrix, but as the P_2 matrix is of rank one we can install one P and one R joint. The orientation of the prismatic joint and orientation and position of the revolute joint is discussed in detail below.

Let the R joint which is oriented in the direction of the given twist is positioned at ($a_1\bar{i} + b_1\bar{j} + c_1\bar{k}$) and the P joint is oriented in the X-Y-Z direction and its screw is given as ($0 \;\; 0 \;\; 0 \;\; P_x \;\; P_y \;\; P_z$). With these assumptions the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_a = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 0 & P_{x2} & P_{y2} & P_{z2} \end{pmatrix} \qquad (104)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ Y_1 & P_{x1} \\ -X_1 & P_{y1} \\ 0 & P_{z1} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ b_1 & P_{x2} \\ -a_1 & P_{y2} \\ 0 & P_{z1} \end{pmatrix} \qquad (105)$$

The equation can be reduced in the form with each matrix having rank =2.

$$\begin{pmatrix} 1 & 0 \\ Y_1 & P_{x1} \\ -X_1 & P_{y1} \\ 0 & P_{z1} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_1 & P_{x2} \\ -a_1 & P_{y2} \\ 0 & P_{z2} \end{pmatrix} \qquad (106)$$

For the alpha matrix to have rank equal to two the condition is determinant of alpha matrix should not be equal to zero in mathematical form this condition can be written as

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 \qquad (107)$$

We can write the above matrix equation in the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ Y_1\alpha_{11} + P_{x1}\alpha_{21} & Y_1\alpha_{12} + P_{x1}\alpha_{22} \\ -X_1\alpha_{11}+P_{y1}\alpha_{21} & -X_1\alpha_{12} + P_{y1}\alpha_{22} \\ P_{z1}\alpha_{21} & P_{z1}\alpha_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_1 & P_{x2} \\ -a_1 & P_{y2} \\ 0 & P_{z2} \end{pmatrix} \quad (108)$$

This gives the following conditions for equivalence

$$\alpha_{11} = 1 \quad (109)$$

$$\alpha_{12} = 0 \quad (110)$$

$$Y_1\alpha_{11} + P_{x1}\alpha_{21} = b_1 \quad (111)$$

$$Y_1\alpha_{12} + P_{x1}\alpha_{22} = P_{x2} \quad (112)$$

$$-X_1\alpha_{11} + P_{y1}\alpha_{21} = -a_1 \quad (113)$$

$$-X_1\alpha_{12} + P_{y1}\alpha_{22} = P_{y2} \quad (114)$$

$$P_{z1}\alpha_{21} = 0$$

P_{z1} cannot be zero initial assumption

so

$$\alpha_{21}=0 \quad (115)$$

$$P_{z1}\alpha_{22} = P_{z2} \quad (116)$$

Putting following values of α_{11} ,α_{12} , and α_{21} in alpha matrix equivalence condition

$\alpha_{11} = 1$

$\alpha_{12} = 0$

$\alpha_{21} = 0$

We get

$$\alpha_{22} \neq 0 \tag{117}$$

Putting $\alpha_{11} = 1$ and $\alpha_{12} = 0$ and $\alpha_{21} = 0$ the rest of the equivalence conditions become

$$Y_1 = b_1 \tag{118}$$

$$P_{x1}\,\alpha_{22} = P_{x2} \tag{119}$$

$$X_1 = a_1 \tag{120}$$

$$P_{y1}\,\alpha_{22} = P_{y2} \tag{121}$$

$$P_{z1}\,\alpha_{22} = P_{z2} \tag{122}$$

4.1.2.2.8 Solution Set 1R1P Case (Second Twist oriented in x-y-z direction)

The solution set therefore, is the set of following equations with the condition that one of the given twists is aligned with the Z-axis of the co-ordinate system and the other is oriented in the x-y-z direction (z component must not be zero).

$$\left.\begin{array}{lcl}
\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 & & \\
\alpha_{11} = 1,\ \alpha_{12} = 0 \text{ and } \alpha_{21} = 0 & & \\
\alpha_{22} \neq 0 & & \\
Y_1 & = & b_1 \\
P_{x1}\,\alpha_{22} & = & P_{x2} \\
X_1 & = & a_1 \\
P_{y1}\,\alpha_{22} & = & P_{y2} \\
P_{z1}\,\alpha_{22} & = & P_{z2}
\end{array}\right\} \tag{123}$$

4.1.2.2.8 Numerical Example 1R1P Case (Second Twist oriented in x-y-z direction)

Let the position vectors of the first twist which is directed in the Z-axis of the co-ordinate system is given as;

$$r_1 = X_1\bar{\imath} + Y_1\bar{\jmath} + Z_1\bar{k} \quad = 21\bar{\imath} + 8\bar{\jmath}$$

Let the other joints twist be oriented in the direction of the following vector;

$$v_1 = 0.5777i + 0.5777j + 0.5777k$$

then;

$$a_1 = X_1$$
$$= 21$$
$$b_1 = Y_1$$
$$= 8$$
$$P_{x2} = P_{x1}\ \alpha_{22}$$
$$= 0.5777\alpha_{22}$$
$$P_{y2} = P_{y1}\ \alpha_{22}$$
$$= 0.5777\alpha_{22}$$
$$P_{z2} = P_{z1}\ \alpha_{22}$$
$$= 0.5777\alpha_{22}$$

Now we can choose α_{22} with the condition that

$$\alpha_{11}\ \alpha_{22} - \alpha_{12}\ \alpha_{21} \neq 0$$

$\alpha_{11} = 1$,

$\alpha_{12} = 0$,

and

$\alpha_{21} = 0$

$\alpha_{22} \neq 0$

α_{22}= 2 then we have,

$a_1 = 21$

$b_1 = 8$

$P_{x2} = 1.154$

$P_{y2} = 1.154$

$P_{z2} = 1.154$

The equation of equivalence takes the following form and thus any linear combination of the given twists can be provided by these two joints.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 8 & 0.5777 \\ -21 & 0.5777 \\ 0 & 0.5777 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 8 & 1.154 \\ -21 & 1.154 \\ 0 & 1.154 \end{pmatrix}$$

4.1.3 Case 3 – Rank of Primary Part =2

In the most general form the twist and (P) matrices will be written as below.

$$T = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ t_{1x} & t_{1y} & t_{1z} & t_{02x} & t_{02y} & t_{02z} \end{pmatrix} \quad (124)$$

$$P = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{2x} & t_{2y} & t_{2z} \end{pmatrix} \quad (125)$$

In this case we have two non parallel skew lines. We, we can choose the origin in such a way that it lies on the first twist screw axis and the Z-axis points along the common normal of the two skew lines as shown in Fig 4.1. If the position vector of the second twist is given as $(X_2\bar{i} + Y_2\bar{j} + Z_2\bar{k})$ with x and y components equal to zero(r only ha z component), and $\bar{\omega}$ is of the form $(l_n\bar{i} + m_n\bar{j} + n_n\bar{k})$ where subscript n=1 and 2 then the twist matrix of the two twists is then written as;

$$T = \begin{pmatrix} l_1 & m_1 & 0 & 0 & 0 & 0 \\ l_2 & m_2 & 0 & -m_2 z_2 & l_2 z_2 & 0 \end{pmatrix} \qquad (126)$$

$$\text{Since } r \times \bar{\omega} = \det \begin{pmatrix} i & j & k \\ X_2 & Y_2 & Z_2 \\ l_n & m_n & n_n \end{pmatrix}$$

where $n_n=0$

And $X_2, Y_2=0$

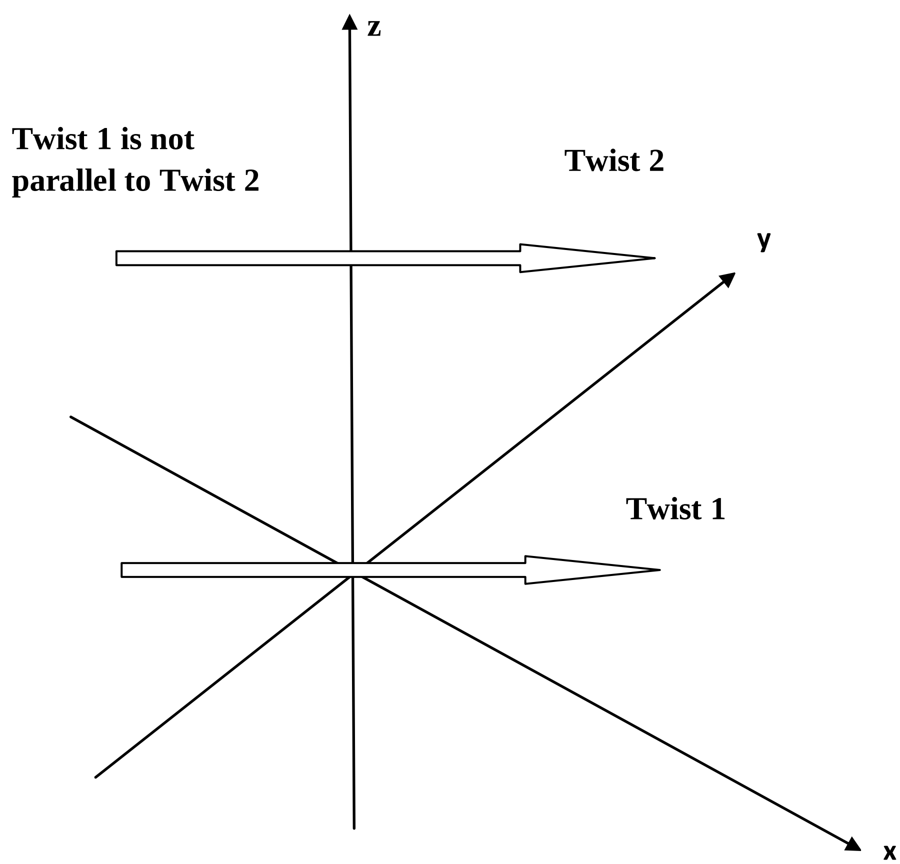


Figure- 4.1 Z-Axis Pointing Normal to Two Skew Lines

4.1.3.1 Possible Topology

The only possibility is to use two R joints with the joint axis parallel to the given twists.

4.1.3.2 Positioning and Orientation

The rank of the primary matrix (P) is two which spans a plane. The primary matrix (P) of the two joints which are needed to achieve this twist must span this plane. If $\bar{\omega}$ of the two joints are given as $(L_n, M_n, 0)$ (as the primary part has to be in the x-y plane) and the position of the joints is given as $(a_n\bar{i} + b_n\bar{j} + c_n\bar{k})$. Then the dual part of the joint screws is given as $(-M_nc_n, L_nc_n, M_na_n-L_nb_n)$.

$$\text{Since } r \times \bar{\omega} = \det \begin{pmatrix} i & j & k \\ a_n & b_n & c_n \\ L_n & M_n & 0 \end{pmatrix}$$

The matrix of the equivalence of the given twist space and the joint space is written as;

$$\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \\ 0 & 0 \\ 0 & -m_2z_2 \\ 0 & l_2z_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \\ M_1 & M_2 \\ 0 & 0 \\ -M_1c_1 & -M_2c_2 \\ L_1c_1 & L_2c_2 \\ M_1a_1 - L_1b_1 & M_2a_2 - L_2b_2 \end{pmatrix} \quad (127)$$

The equivalence condition can be reduced to the following form with each matrix having rank =2.

$$\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \\ 0 & -m_2z_2 \\ 0 & l_2z_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \\ M_1 & M_2 \\ -M_1c_1 & -M_2c_2 \\ L_1c_1 & L_2c_2 \\ M_1a_1 - L_1b_1 & M_2a_2 - L_2b_2 \end{pmatrix} \quad (128)$$

Now by setting $L_1b_1-M_1a_1=0$ and $L_2b_2-M_2a_2 = 0$, as L_n and M_n cannot be equal to zero so both $a_n= b_n = 0$, which is only possible when the joint axis passes through the z-axis. Which means that the position vector of the two joints lie along the z-axis only, $a_n = b_n = 0$. Therefore, for this condition to satisfy the replacement joints can be placed anywhere on the z-axis oriented in the x-y plane linearly independent of each other.

For the alpha matrix to have rank equal to two the condition is determinant of alpha matrix should not be equal to zero in mathematical form this condition can be written as

$$\alpha_{11}\,\alpha_{22} - \alpha_{12}\,\alpha_{21} \neq 0 \qquad (129)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} l_1\alpha_{11} + l_2\,\alpha_{21} & l_1\alpha_{12} + l_2\,\alpha_{22} \\ m_1\alpha_{11} + m_2\alpha_{21} & m_1\alpha_{12} + m_2\alpha_{22} \\ -m_2z_2\alpha_{21} & -m_2z_2\alpha_{22} \\ l_2z_2\alpha_{21} & l_2z_2\alpha_{22} \end{pmatrix} = \begin{pmatrix} L_1 & L_2 \\ M_1 & M_2 \\ -M_1c_1 & -M_2c_2 \\ L_1c_1 & L_2c_2 \end{pmatrix} \quad (130)$$

This gives the following conditions for equivalence of first joint

$$l_1\alpha_{11} + l_2\alpha_{21} = L_1 \quad (131)$$

$$m_1\alpha_{11} + m_2\alpha_{21} = M_1 \quad (132)$$

$$-m_2z_2\alpha_{21} = -M_1c_1 \quad (133)$$

$$l_2z_2\,\alpha_{21} = L_1c_1 \quad (134)$$

From equation 133 we can write

$$c_1 = m_2z_2\alpha_{21}/\,M_1$$

Putting M_1 from equation 132

$$c_1 = m_2z_2\alpha_{21}/\,m_1\alpha_{11} + m_2\alpha_{21} \quad (135)$$

From equation 134 we can write

$$c_1 = l_2z_2\alpha_{21}/\,L_1$$

Putting L_1 from equation 131

$$c_1 = l_2z_2\alpha_{21}/\,l_1\alpha_{11} + l_2\,\alpha_{21} \quad (136)$$

Equating equation 135 and 136

$$m_2z_2\alpha_{21}/\,(m_1\alpha_{11} + m_2\alpha_{21}) = l_2z_2\alpha_{21}/\,(l_1\alpha_{11} + l_2\,\alpha_{21})$$

$$m_2(l_1\alpha_{11} + l_2\,\alpha_{21}) = l_2(m_1\alpha_{11} + m_2\alpha_{21})$$

$$m_1/\,m_2 = l_1/\,l_2 \quad (137)$$

$$m_1/\,l_1 = m_2/\,l_2$$

For the second joint

$$l_1\alpha_{12} + l_2\alpha_{22} = L_2 \quad (138)$$

$$m_1\alpha_{12} + m_2\alpha_{22} = M_2 \quad (139)$$

$$-m_2z_2\alpha_{22} = -M_2c_2 \quad (140)$$

$$l_2z_2\,\alpha_{22} = L_2c_2 \quad (141)$$

From equation 140 we can write

$$c_2 = m_2z_2\alpha_{22}/\,M_2$$

Putting M_2 from equation 139

$$c_2 = m_2 z_2 \alpha_{22} / m_1 \alpha_{12} + m_2 \alpha_{22} \quad (142)$$

From equation141 we can write

$$c_2 = l_2 z_2 \alpha_{22} / L_2 \quad (143)$$

Putting L_2 from equation 138

$$c_2 = l_2 z_2 \alpha_{22} / l_1 \alpha_{12} + l_2 \alpha_{22} \quad (144)$$

Equating equation 142 and 144

$$m_2 z_2 \alpha_{22} / (m_1 \alpha_{12} + m_2 \alpha_{22}) = l_2 z_2 \alpha_{22} / (l_1 \alpha_{12} + l_2 \alpha_{22})$$

$$m_1 / m_2 = l_1 / l_2 \quad (145)$$

Conditions (137) and (144) dictates that the given joints axes should be parallel to each other as it says that the ratio of $m_1/ l_1 = m_2/ l_2$ where m_n, l_n is giving the orientation of the given joints axes. However, we have originally said that the given two joint axis are not parallel to each other for the primary matrix to have rank=2. It brings us to an important deduction as far as this work is concerned that, for spatial cases involving revolute joints taking into account just the primary part rank will not suffice to predict the alternate joints, consideration has to be given to the secondary part of the twist matrix also. Therefore, the present work is applicable to twist matrix with rank =0,1.

Chapter 5- DESIGN AND SELECTION OF JOINTS IN THREE DOF SERIAL ROBOTIC CHAINS

5.1 Introduction

In principle, the range of numbers of given twists is between one and six, but the most interesting cases are those where the number of such twists (and hence joints) is two-three. The cases of twist numbers higher than this are simple extension of the arguments derived in the two-three joints case.

We have already seen in section 4.1.3 that is the case of rank = 2 of the primary matrix of the design of 2 DOF serial manipulator that for spatial cases involving revolute joints taking into account just the primary part rank will not suffice to predict the alternate joints, consideration has to be given to the secondary part of the twist matrix also. Therefore, in the present work only cases of rank =0,1 of the primary matrix will be discussed.

5.2 Design of a Three DOF Manipulator

In the case of three DOF serial manipulator we need minimum of three joints to obtain the desired twists. The type of joints depends on the desired twists of the end effector. If the three linearly independent twists are given as t_1,t_2,t_3 and it is desired to design three joints of a serial manipulator which can provide these three twists, the three given linearly independent end-effector twists can be written as;

$$t_1 = [\, t_{1x}\ t_{1y}\ t_{1z}\ ;\ t_{01x}\ t_{01y}\ t_{01z}\ ] \tag{146}$$

$$t_2 = [\, t_{2x}\ t_{2y}\ t_{2z}\ ;\ t_{02x}\ t_{02y}\ t_{02z}\ ] \tag{147}$$

$$t_3 = [\, t_{3x}\ t_{3y}\ t_{3z}\ ;\ t_{03x}\ t_{03y}\ t_{03z}\ ] \tag{148}$$

The twist matrix T and the P matrix can then be written as below:-

$$T = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ t_{2x} & t_{2y} & t_{2z} & t_{02x} & t_{02y} & t_{02z} \\ t_{3x} & t_{3y} & t_{3z} & t_{03x} & t_{03y} & t_{03z} \end{pmatrix} \tag{149}$$

$$P = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{2x} & t_{2y} & t_{2z} \\ t_{3x} & t_{3y} & t_{3z} \end{pmatrix} \tag{150}$$

The rank of the (P) matrix can be three, two, one or zero, each case results into a different possibility of alternate joints. The alternate type, location and orientation of the joints depends on the rank of the primary part of the given twists. In this thesis only cases of rank zero and one (rank one with parallel revolute joints and prismatic joint if any should be in the plane of the secondary part of the revolute joint twists) are discussed below in detail.

5.2.1 Case 1 –Rank of Primary Part =0

In the most general form the twist matrix will have primary part equal to zero so the twist and P matrix for this case can be written as

$$T = \begin{pmatrix} 0 & 0 & 0 & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 0 & t_{02x} & t_{02y} & t_{02z} \\ 0 & 0 & 0 & t_{03x} & t_{03y} & t_{03z} \end{pmatrix} \tag{151}$$

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{152}$$

5.2.1.1 Possible Topology

The manipulator can have only one configuration, that is; (3P) arrangement with the three P joints linearly independent of each other.

5.2.1.2 Positioning and Orientation

When the primary part is possessing rank zero the three prismatic joints can be placed anywhere in three dimensional space as long as they are linearly independent of each other (when two joints are considered in isolation) and in the plane formed by the secondary part of the twists.

5.2.2 Case 2 – Rank of Primary Part =1

In the most general form the twist matrix will have three possible twist and (P) matrices given below. Each case will be discussed separately in the subsequent paragraphs.

$$T_1 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ t_{2x} & t_{2y} & t_{2z} & t_{02x} & t_{02y} & t_{02z} \\ t_{3x} & t_{3y} & t_{3z} & t_{03x} & t_{03y} & t_{03z} \end{pmatrix} \quad (153)$$

$$P_1 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{2x} & t_{2y} & t_{2z} \\ t_{3x} & t_{3y} & t_{3z} \end{pmatrix} \quad (154)$$

$$T_2 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ t_{2x} & t_{2y} & t_{2z} & t_{02x} & t_{02y} & t_{02z} \\ 0 & 0 & 0 & t_{03x} & t_{03y} & t_{03z} \end{pmatrix} \quad (155)$$

$$P_2 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{2x} & t_{2y} & t_{2z} \\ 0 & 0 & 0 \end{pmatrix} \quad (156)$$

$$T_3 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 0 & t_{02x} & t_{02y} & t_{02z} \\ 0 & 0 & 0 & t_{03x} & t_{03y} & t_{03z} \end{pmatrix} \quad (157)$$

$$P_3 = \begin{pmatrix} t_{1x} & t_{1y} & t_{1z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (158)$$

5.2.2.1 Case 2 a-Rank of Primary Part =1 (Twist matrix of the form T_1)

If the twist matrix is T_1 with the corresponding primary matrix having rank=1, which means the three twists are parallel to each other as shown in Fig 5.1. We can always choose the origin in such a way that the primary part of the twists is directed in the direction of the z-axis, and the first twist axis passes through the origin, the twist matrix then take the following form

$$T_1 = \begin{pmatrix} 0 & 0 & 1 & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 1 & t_{02x} & t_{02y} & t_{02z} \\ 0 & 0 & 1 & t_{03x} & t_{03y} & t_{03z} \end{pmatrix} \qquad (159)$$

The primary part matrix thus becomes;

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad (160)$$

If now the position vectors of the twists are given in the form ($X_n\bar{i} + Y_n \bar{j}+Z_n\bar{k}$), where n= 1 to 3, the above twists matrix is then written as;

$$T_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & Y_2 & -X_2 & 0 \\ 0 & 0 & 1 & Y_3 & -X_3 & 0 \end{pmatrix} \qquad (161)$$

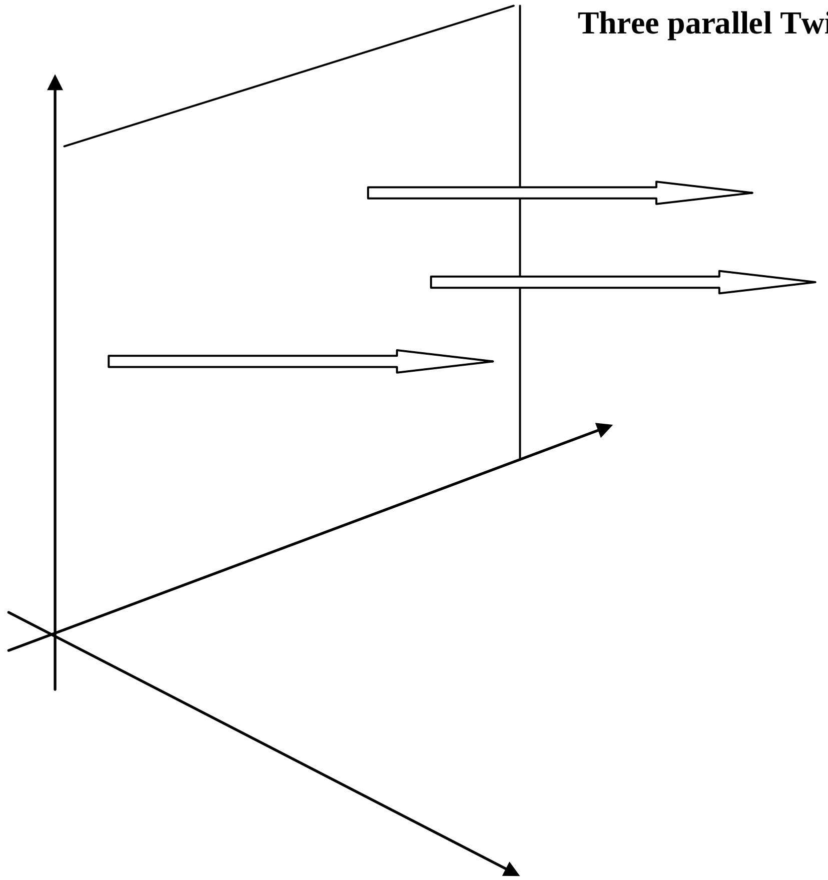


Figure-5.1 Three Parallel Twists

5.2.2.1.1 Possible Topologies

The manipulator can have the following three configurations, with certain conditions on the joints which will be discussed here separately.

- Three parallel R Joints
- Two Parallel R Joints and 1P Joint
- One R joint and 2 P Joints

5.2.2.1.2 Positioning and Orientation -3R Case (Twist matrix of the form T_1)

The three R joints will be installed with their axis parallel to each other and oriented in the direction of the given twists. These joints can be placed anywhere in the plane under certain conditions to achieve any linear combination of the given twists, these conditions are discussed below.

Let the three R joints which are oriented in the direction of the given twists are positioned at $(a_1\bar{i} + b_1\bar{j} + c_1\bar{k})$, $(a_2\bar{i} + b_2\bar{j} + c_2\bar{k})$, and $(a_3\bar{i} + b_3\bar{j} + c_3\bar{k})$respectively. With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a1} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \\ 0 & 0 & 1 & b_3 & -a_3 & 0 \end{pmatrix} \qquad (162)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \\ 0 & 0 & 0 \end{pmatrix} \qquad (163)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \end{pmatrix} \quad (164)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \quad (165)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \end{pmatrix}$$

$$-Y_2X_3+Y_3X_2 \neq 0$$

$$Y_3/ Y_2 \neq X_3/ X_2 \quad (166)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11}+\alpha_{21}+\alpha_{31} & \alpha_{12}+\alpha_{22}+\alpha_{32} & \alpha_{13}+\alpha_{23}+\alpha_{33} \\ Y_2\alpha_{21}+Y_3\alpha_{31} & Y_2\alpha_{22}+Y_3\alpha_{32} & {}_2\alpha_{23}+Y_3\alpha_{33} \\ -X_2\alpha_{21}-X_3\alpha_{31} & -X_2\alpha_{22}-X_3\alpha_{32} & -X_2\alpha_{23}-X_3\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \end{pmatrix} \quad (167)$$

This gives the following conditions for equivalence

$$\alpha_{11}+ \alpha_{21}+ \alpha_{31} =1 \quad (168)$$

$$\alpha_{12}+\alpha_{22}+\alpha_{32} =1 \quad (169)$$

$$\alpha_{13}+\alpha_{23}+\alpha_{33} =1 \quad (170)$$

$$b_1= Y2\ \alpha_{21}+Y_3\alpha_{31} \quad (172)$$

$$b_2= Y_2\ \alpha_{22}+Y_3\alpha_{32} \quad (173)$$

$$b_3= Y_2\alpha_{23}+Y_3\alpha_{33} \quad (174)$$

$$a_1=X_2\alpha_{21}+X_3\alpha_{31} \quad (175)$$

$$a_2=X_2\alpha_{22}+X_3\alpha_{32} \quad (176)$$

$$a_3= X_2\alpha_{23}+X_3\alpha_{33} \quad (177)$$

5.2.2.1.3 Solution Set -3R Case (Twist matrix of the form T_1)

The solution set therefore is the set of following equations with the condition that the given twists axes are aligned with the Z-axis of the co-ordinate system.

$$\left.\begin{aligned}
&\text{Determinant}\begin{pmatrix}\alpha_{11}\ \alpha_{12}\ \ \alpha_{13}\\ \alpha_{21}\ \alpha_{22}\ \ \alpha_{23}\\ \alpha_{31}\ \alpha_{32}\ \ \alpha_{33}\end{pmatrix}\neq 0\\
&Y_3/\ Y_2 \neq X_3/\ X_2\\
&\alpha_{11}+\ \alpha_{21}+\ \alpha_{31}=1\\
&\alpha_{12}+\alpha_{22}+\alpha_{32}\ \ =1\\
&\alpha_{13}+\alpha_{23}+\alpha_{33}\ \ =1\\
&b_1= Y2\ \alpha_{21}+Y_3\alpha_{31}\\
&b_2= Y_2\ \alpha_{22}+Y_3\alpha_{32}\\
&b_3= Y_2\alpha_{23}+Y_3\alpha_{33}\\
&a_1=X_2\alpha_{21+}X_3\alpha_{31}\\
&a_2=X_2\alpha_{22}+X_3\alpha_{32}\\
&a_3= X_2\alpha_{23}+X_3\alpha_{33}
\end{aligned}\right\}\qquad(178)$$

5.2.2.1.4 Positioning and Orientation -2R1P Case (Twist matrix of the form T_1)

The two R one P joints will be installed with the two R joints oriented in the direction of the given twist but different position vectors, the prismatic joint must lie in the plane of the dual parts of the joint twists. Let the two R joints which are oriented in the direction of the given twists are positioned at ($a_1\bar{\imath} + b_1\,\bar{\jmath} + c_1\bar{k}$) and ($a_2\bar{\imath} + b_2\,\bar{\jmath} + c_2\bar{k}$) respectively. Let the P joint is oriented in the X-Y plane and its screw is given as (0 0 0 P_x P_y 0). With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a1} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \\ 0 & 0 & 1 & b_3 & -a_3 & 0 \end{pmatrix} \quad (179)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ b_1 & b_2 & P_x \\ -a_1 & -a_2 & P_y \\ 0 & 0 & 0 \end{pmatrix} \quad (180)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ b_1 & b_2 & P_x \\ -a_1 & -a_2 & P_y \end{pmatrix} \quad (181)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \qquad (182)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \end{pmatrix}$$

$$-Y_2X_3+Y_3X_2 \neq 0$$

$$Y_3/\ Y_2 \neq X_3/\ X_2 \qquad (183)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11}+\alpha_{21}+\alpha_{31} & \alpha_{12}+\alpha_{22}+\alpha_{32} & \alpha_{13}+\alpha_{23}+\alpha_{33} \\ Y_2\alpha_{21}+Y_3\alpha_{31} & Y_2\alpha_{22}+Y_3\alpha_{32} & Y_2\alpha_{23}+Y_3\alpha_{33} \\ -X_2\alpha_{21}-X_3\alpha_{31} & -X_2\alpha_{22}-X_3\alpha_{32} & -X_2\alpha_{23}-X_3\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ b_1 & b_2 & P_x \\ -a_1 & -a_2 & P_y \end{pmatrix} \qquad (184)$$

This gives the following conditions for equivalence

$$\alpha_{11}+\alpha_{21}+\alpha_{31} =1 \qquad (185)$$

$$\alpha_{12}+\alpha_{22}+\alpha_{32} =1 \qquad (186)$$

$$\alpha_{13}+\alpha_{23}+\alpha_{33} =0 \qquad (187)$$

$$b_1= Y2\ \alpha_{21}+Y_3\alpha_{31} \qquad (188)$$

$$b_2= Y_2\ \alpha_{22}+Y_3\alpha_{32} \qquad (189)$$

$$P_x = Y_2\alpha_{23}+Y_3\alpha_{33} \tag{190}$$

$$a_1=X_2\alpha_{21+}X_3\alpha_{31} \tag{191}$$

$$a_2=X_2\alpha_{22}+X_3\alpha_{32} \tag{192}$$

$$P_y = -X_2\alpha_{23}-X_3\alpha_{33} \tag{193}$$

5.2.2.1.5 Solution Set -2R1P Case (Twist matrix of the form T_1)

The solution set therefore is the set of following equations with the condition that the given twists axes are aligned with the Z-axis of the co-ordinate system.

$$\left.\begin{array}{l}
\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \\
Y_3/ Y_2 \neq X_3/ X_2 \\
\alpha_{11}+ \alpha_{21}+ \alpha_{31} =1 \\
\alpha_{12}+\alpha_{22}+\alpha_{32} =1 \\
\alpha_{13}+\alpha_{23}+\alpha_{33} =0 \\
b_1= Y2\ \alpha_{21}+Y_3\alpha_{31} \\
b_2= Y_2\ \alpha_{22}+Y_3\alpha_{32} \\
P_x = Y_2\alpha_{23}+Y_3\alpha_{33} \\
a_1=X_2\alpha_{21+}X_3\alpha_{31} \\
a_2=X_2\alpha_{22}+X_3\alpha_{32} \\
P_y = -X_2\alpha_{23}-X_3\alpha_{33}
\end{array}\right\} \tag{194}$$

5.2.2.1.6 Positioning and Orientation -1R2P Case (Twist matrix of the form T_1)

The one R two P joints will be installed with the R joint oriented in the direction of the given twists, the prismatic joints must lie in the plane of the dual parts of the joint twists, but they have to be linearly independent of each other.

Let the R joint which is oriented in the direction of the given twists is positioned at ($a_1\bar{i} + b_1\,\bar{j} + c_1\bar{k}$). Let the P joints be oriented in the X-Y plane and their screws are given as (0 0 0 P_{x1} P_{y1} 0) and (0 0 0 P_{x2} P_{y2} 0). With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a1} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 0 & P_{x1} & P_{x2} & 0 \\ 0 & 0 & 0 & P_{y1} & P_{y2} & 0 \end{pmatrix} \tag{195}$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ b_1 & P_{x1} & P_{x2} \\ -a_1 & P_{y1} & P_{y2} \\ 0 & 0 & 0 \end{pmatrix} \tag{196}$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & P_{x1} & P_{x2} \\ -a_1 & P_{y1} & P_{y2} \end{pmatrix} \quad (197)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \qquad (198)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & Y_3 \\ 0 & -X_2 & -X_3 \end{pmatrix}$$

$$-Y_2X_3+Y_3X_2 \neq 0$$

$$Y_3/ Y_2 \neq X_3/ X_2 \qquad (199)$$

For the following matrix to have rank = 3 the condition will be determinant not equal to zero which give the following condition;

$$\begin{pmatrix} 1 & 0 & 0 \\ b_1 & P_{x1} & P_{x2} \\ -a_1 & P_{y1} & P_{y2} \end{pmatrix}$$

$P_{x1}\, P_{y2} - P_{x2}\, P_{y1} \neq 0$

$$P_{x1}/ P_{x2} \neq P_{y1}/ P_{y2} \qquad (200)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11}+ \alpha_{21}+ \alpha_{31} & \alpha_{12}+\alpha_{22}+\alpha_{32} & \alpha_{13}+\alpha_{23}+\alpha_{33} \\ Y_2\alpha_{21}+Y_3\alpha_{31} & Y_2\alpha_{22}+Y_3\alpha_{32} & Y_2\alpha_{23}+Y_3\alpha_{33} \\ -X_2\alpha_{21}-X_3\alpha_{31} & -X_2\alpha_{22}-X_3\alpha_{32} & -X_2\alpha_{23}-X_3\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & P_{x1} & P_{x2} \\ -a_1 & P_{y1} & P_{y2} \end{pmatrix} \qquad (201)$$

This gives the following conditions for equivalence

$$\alpha_{11}+ \alpha_{21}+ \alpha_{31} =1 \qquad (202)$$

$$\alpha_{12}+\alpha_{22}+\alpha_{32} =0 \qquad (203)$$

$$\alpha_{13}+\alpha_{23}+\alpha_{33} =0 \qquad (204)$$

$$b_1= Y2\ \alpha_{21}+Y_3\alpha_{31} \qquad (205)$$

$$P_{x1}= Y_2\ \alpha_{22}+Y_3\alpha_{32} \qquad (206)$$

$$P_{x2} = Y_2\alpha_{23}+Y_3\alpha_{33} \qquad (207)$$

$$a_1=X_2\alpha_{21+}X_3\alpha_{31} \qquad (208)$$

$$P_{y1}=-X_2\alpha_{22}-X_3\alpha_{32} \qquad (209)$$

$$P_{y2} = -X_2\alpha_{23}-X_3\alpha_{33} \tag{210}$$

5.2.2.1.7 Solution Set -1R2P Case (Twist matrix of the form T_1)

The solution set therefore is the set of following equations with the condition that the given twists axes are aligned with the Z-axis of the co-ordinate system.

$$\left.\begin{array}{l}
\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \\
Y_3/ Y_2 \neq X_3/ X_2 \\
P_{x1}/ P_{x2} \neq P_{y1}/ P_{y2} \\
\alpha_{11}+ \alpha_{21}+ \alpha_{31} =1 \\
\alpha_{12}+\alpha_{22}+\alpha_{32} =0 \\
\alpha_{13}+\alpha_{23}+\alpha_{33} =0 \\
b_1= Y2\ \alpha_{21}+Y_3\alpha_{31} \\
P_{x1}= Y_2\ \alpha_{22}+Y_3\alpha_{32} \\
P_{x2} = Y_2\alpha_{23}+Y_3\alpha_{33} \\
a_1=X_2\alpha_{21+}X_3\alpha_{31} \\
P_{y1}=-X_2\alpha_{22}-X_3\alpha_{32} \\
P_{y2} = -X_2\alpha_{23}-X_3\alpha_{33}
\end{array}\right\} \tag{211}$$

5.2.2.2 Case 2 b-Rank of Primary Part =1 (Twist matrix of the form T_2)

If the twist matrix is T_2 with the corresponding primary matrix having rank=1, we can always choose the origin in such a way that the primary part of the twists is directed in the direction of the z-axis, and the first twist axis passes through the origin, the twist matrix then take the following form

$$T_2 = \begin{pmatrix} 0 & 0 & 1 & t_{01x} & t_{01y} & t_{01z} \\ 0 & 0 & 1 & t_{02x} & t_{02y} & t_{02z} \\ 0 & 0 & 0 & P_{x1} & P_{y1} & P_{z1} \end{pmatrix} \qquad (212)$$

The primary part matrix thus becomes;

$$P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad (213)$$

If now the position vectors of the twists are given in the form ($X_n\bar{i}+Y_n \bar{j}+Z_n\bar{k}$), where n= 1 to 2, the above twists matrix is then written as (with the assumption that the prismatic joint is in the plane of the dual parts of the twists that is the planer case with the prismatic joint not oriented in the z direction);

$$T_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & Y_2 & -X_2 & 0 \\ 0 & 0 & 0 & P_{x1} & P_{y1} & 0 \end{pmatrix} \qquad (214)$$

5.2.2.2.1 Possible Topologies

The manipulator can have the following three configurations, with certain conditions on the joints which will be discussed later.

- Three parallel R Joints
- Two Parallel R Joints and 1P Joint

- One R joint and 2 P Joints

5.2.2.2.2 Positioning and Orientation -3R Case (Twist matrix of the form T_2)

The three R joints will be installed with their axis parallel to each other and oriented in the direction of the given twists. These joints can be placed anywhere in the three space under certain conditions to achieve any linear combination of the given twists, these conditions are discussed below.

Let the three R joints which are oriented in the direction of the given twists are positioned at $(a_1\bar{i} + b_1\bar{j} + c_1\bar{k})$, $(a_2\bar{i} + b_2\bar{j} + c_2\bar{k})$, and $(a_3\bar{i} + b_3\bar{j} + c_3\bar{k})$respectively. With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a2} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \\ 0 & 0 & 1 & b_3 & -a_3 & 0 \end{pmatrix} \qquad (215)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \\ 0 & 0 & 0 \end{pmatrix} \qquad (216)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \end{pmatrix} \quad (217)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \quad (218)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \end{pmatrix}$$

$$Y_2 P_{y1} + P_{x1} X_2 \neq 0 \qquad (219)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11}+\alpha_{21}+\alpha_{31} & \alpha_{12}+\alpha_{22}+\alpha_{32} & \alpha_{13}+\alpha_{23}+\alpha_{33} \\ Y_2\alpha_{21}+P_{x1}\alpha_{31} & Y_2\alpha_{22}+P_{x1}\alpha_{32} & Y_2\alpha_{23}+P_{x1}\alpha_{33} \\ -X_2\alpha_{21}+P_{y1}\alpha_{31} & -X_2\alpha_{22}+P_{y1}\alpha_{32} & -X_2\alpha_{23+}P_{y1}\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \end{pmatrix} \qquad (220)$$

This gives the following conditions for equivalence

$$\alpha_{11}+ \alpha_{21}+ \alpha_{31} =1 \qquad (221)$$

$$\alpha_{12}+\alpha_{22}+\alpha_{32} =1 \qquad (222)$$

$$\alpha_{13}+\alpha_{23}+\alpha_{33} =1 \qquad (223)$$

$$b_1= Y_2\alpha_{21}+ P_{x1}\alpha_{31} \qquad (224)$$

$$b_2= Y_2\alpha_{22}+ P_{x1}\alpha_{32} \qquad (225)$$

$$b_3= Y_2\alpha_{23}+ P_{x1}\alpha_{33} \qquad (226)$$

$$a_1=X_2\alpha_{21}- P_{y1}\alpha_{31} \qquad (227)$$

$$a_2=X_2\alpha_{22}- P_{y1}\alpha_{32} \qquad (228)$$

$$a_3= X_2\alpha_{23}-P_{y1}\alpha_{33} \qquad (229)$$

5.2.2.2.3 Solution Set -3R Case (Twist matrix of the form T_2)

The solution set therefore is the set of following equations with the condition that the given twist axes are aligned with the Z-axis of the co-ordinate system.

$$\left.\begin{aligned}
&\text{Determinant}\begin{pmatrix}\alpha_{11}\ \alpha_{12}\ \alpha_{13}\\ \alpha_{21}\ \alpha_{22}\ \alpha_{23}\\ \alpha_{31}\ \alpha_{32}\ \alpha_{33}\end{pmatrix}\neq 0\\
&Y_2\,P_{y1} + P_{x1}\,X_2 \neq 0\\
&\alpha_{11}+\alpha_{21}+\alpha_{31}=1\\
&\alpha_{12}+\alpha_{22}+\alpha_{32}=1\\
&\alpha_{13}+\alpha_{23}+\alpha_{33}=1\\
&b_1= Y_2\alpha_{21}+ P_{x1}\alpha_{31}\\
&b_2= Y_2\alpha_{22}+ P_{x1}\alpha_{32}\\
&b_3= Y_2\alpha_{23}+ P_{x1}\alpha_{33}\\
&a_1=X_2\alpha_{21}- P_{y1}\alpha_{31}\\
&a_2=X_2\alpha_{22}- P_{y1}\alpha_{32}\\
&a_3= X_2\alpha_{23}-P_{y1}\alpha_{33}
\end{aligned}\right\}\quad(230)$$

5.2.2.2.4 Positioning and Orientation -2R1P Case (Twist matrix of the form T_2)

The two R one P joints will be installed with the two R joints oriented in the direction of the given twist but with different position vectors, the prismatic joint must lie in the plane of the dual parts of the joint twists oriented in the direction of the third twist.

Let the two R joints which are oriented in the direction of the given twists are positioned at ($a_1\bar{\imath} + b_1\bar{\jmath} + c_1\bar{k}$) and ($a_2\bar{\imath} + b_2\bar{\jmath} + c_2\bar{k}$) respectively. Let the P joint is oriented in the X-Y plane and its screw is given as (0 0 0 P_{x2} P_{y2} 0). With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a2} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \\ 0 & 0 & 0 & P_{x2} & P_{y2} & 0 \end{pmatrix} \qquad (231)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ b_1 & b_2 & P_{x2} \\ -a_1 & -a_2 & P_{y2} \\ 0 & 0 & 0 \end{pmatrix} \qquad (232)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ b_1 & b_2 & P_{x2} \\ -a_1 & -a_2 & P_{y2} \end{pmatrix} \qquad (233)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \qquad (234)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \end{pmatrix}$$

$$Y_2 P_{y1} + P_{x1} X_2 \neq 0 \qquad (235)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11}+\alpha_{21}+\alpha_{31} & \alpha_{12}+\alpha_{22}+\alpha_{32} & \alpha_{13}+\alpha_{23}+\alpha_{33} \\ Y_2\alpha_{21}+P_{x1}\alpha_{31} & Y_2\alpha_{22}+P_{x1}\alpha_{32} & Y_2\alpha_{23}+P_{x1}\alpha_{33} \\ -X_2\alpha_{21}+P_{y1}\alpha_{31} & -X_2\alpha_{22}+P_{y1}\alpha_{32} & -X_2\alpha_{23}+P_{y1}\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ b_1 & b_2 & P_{x2} \\ -a_1 & -a_2 & P_{y2} \end{pmatrix} (236)$$

This gives the following conditions for equivalence

$$\alpha_{11}+\alpha_{21}+\alpha_{31} = 1 \qquad (237)$$

$$\alpha_{12}+\alpha_{22}+\alpha_{32} = 1 \qquad (238)$$

$$\alpha_{13}+\alpha_{23}+\alpha_{33} = 0 \qquad (239)$$

$$b_1 = Y_2\alpha_{21}+P_{x1}\alpha_{31} \qquad (240)$$

$$b_2 = Y_2\alpha_{22}+P_{x1}\alpha_{32} \qquad (241)$$

$$P_{x2} = Y_2\alpha_{23}+P_{x1}\alpha_{33} \qquad (242)$$

$$a_1 = X_2\alpha_{21}-P_{y1}\alpha_{31} \qquad (243)$$

$a_2=X_2\alpha_{22}- P_{y1}\alpha_{32}$ (244)

$P_{y2} = -X_2\alpha_{23}+P_{y1}\alpha_{33}$ (245)

5.2.2.2.5 Solution Set -2R1P Case (Twist matrix of the form T_2)

The solution set therefore is the set of following equations with the condition that the given twist axes are aligned with the Z-axis of the co-ordinate system.

$$\left.\begin{array}{l}
\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \\
\text{Determinant} \begin{pmatrix} 1 & 1 & 0 \\ b_1 & b_2 & P_{x2} \\ -a_1 & -a_2 & P_{y2} \end{pmatrix} \neq 0 \\
Y_2 P_{y1}+ P_{x1} X_2 \neq 0 \\
\alpha_{11}+ \alpha_{21}+ \alpha_{31} =1 \\
\alpha_{12}+\alpha_{22}+\alpha_{32} =1 \\
\alpha_{13}+\alpha_{23}+\alpha_{33} =0 \\
b_1= Y_2\alpha_{21}+ P_{x1}\alpha_{31} \\
b_2=Y_2\alpha_{22}+ P_{x1}\alpha_{32} \\
P_{x2} = Y_2\alpha_{23}+ P_{x1}\alpha_{33} \\
a_1=X_2\alpha_{21}-P_{y1}\alpha_{31} \\
a_2=X_2\alpha_{22}- P_{y1}\alpha_{32} \\
P_{y2} = -X_2\alpha_{23}+P_{y1}\alpha_{33}
\end{array}\right\} \quad (246)$$

5.2.2.2.6 Positioning and Orientation -1R2P Case (Twist matrix of the form T_2)

The one R two P joints will be installed with the R joint oriented in the direction of the given twists, the prismatic joints must lie in the plane of the dual parts of the joint twists, but they have to be linearly independent of each other.

Let the R joint which is oriented in the direction of the given twists is positioned at ($a_1\bar{i} + b_1\bar{j} + c_1\bar{k}$). Let the P joints be oriented in the X-Y plane and their screws are given as (0 0 0 P_{x2} P_{y2} 0) and (0 0 0 P_{x3} P_{y3} 0). With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a2} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 0 & P_{x2} & P_{x3} & 0 \\ 0 & 0 & 0 & P_{y2} & P_{y3} & 0 \end{pmatrix} \qquad (247)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ b_1 & P_{x2} & P_{x3} \\ -a_1 & P_{y2} & P_{y3} \\ 0 & 0 & 0 \end{pmatrix} \qquad (248)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & P_{x2} & P_{x3} \\ -a_1 & P_{y2} & P_{y3} \end{pmatrix} \quad (249)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \qquad (249)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & Y_2 & P_{x1} \\ 0 & -X_2 & P_{y1} \end{pmatrix}$$

$$Y_2\,P_{y1} + P_{x1}\,X_2 \neq 0 \qquad (250)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11}+\alpha_{21}+\alpha_{31} & \alpha_{12}+\alpha_{22}+\alpha_{32} & \alpha_{13}+\alpha_{23}+\alpha_{33} \\ Y_2\alpha_{21}+P_{x1}\alpha_{31} & Y_2\alpha_{22}+P_{x1}\alpha_{32} & Y_2\alpha_{23}+P_{x1}\alpha_{33} \\ -X_2\alpha_{21}+P_{y1}\alpha_{31} & -X_2\alpha_{22}+P_{y1}\alpha_{32} & -_2\alpha_{23}+P_{y1}\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & P_{x2} & P_{x3} \\ -a_1 & P_{y2} & P_{y3} \end{pmatrix} \qquad (251)$$

This gives the following conditions for equivalence

$$\alpha_{11}+\alpha_{21}+\alpha_{31}=1 \qquad (253)$$

$$\alpha_{12}+\alpha_{22}+\alpha_{32}=1 \qquad (254)$$

$$\alpha_{13}+\alpha_{23}+\alpha_{33}=0 \qquad (255)$$

$$b_1=Y_2\alpha_{21}+P_{x1}\alpha_{31} \qquad (256)$$

$$P_{x2}=Y_2\alpha_{22}+P_{x1}\alpha_{32} \qquad (257)$$

$$P_{x3}=Y_2\alpha_{23}+P_{x1}\alpha_{33} \qquad (258)$$

$$a_1=X_2\alpha_{21}-P_{y1}\alpha_{31} \qquad (259)$$

$$P_{y2}=-X_2\alpha_{22}+P_{y1}\alpha_{32} \qquad (260)$$

$$P_{y3}=-X_2\alpha_{23}+P_{y1}\alpha_{33} \qquad (261)$$

5.2.2.2.7 Solution Set -1R2P Case (Twist matrix of the form T_2)

The solution set therefore is the set of following equations with the condition that the given twist axes are aligned with the Z-axis of the co-ordinate system.

$$\left.\begin{array}{l}
\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \\
Y_2 P_{y1} + P_{x1} X_2 \neq 0 \\
\alpha_{11} + \alpha_{21} + \alpha_{31} = 1 \\
\alpha_{12} + \alpha_{22} + \alpha_{32} = 1 \\
\alpha_{13} + \alpha_{23} + \alpha_{33} = 0 \\
b_1 = Y_2 \alpha_{21} + P_{x1} \alpha_{31} \\
P_{x2} = Y_2 \alpha_{22} + P_{x1} \alpha_{32} \\
P_{x3} = Y_2 \alpha_{23} + P_{x1} \alpha_{33} \\
a_1 = X_2 \alpha_{21} - P_{y1} \alpha_{31} \\
P_{y2} = -X_2 \alpha_{22} + P_{y1} \alpha_{32} \\
P_{y3} = -X_2 \alpha_{23} + P_{y1} \alpha_{33}
\end{array}\right\} \quad (262)$$

5.2.2.3 Case 2 a-Rank of Primary Part =1 (Twist matrix of the form T_3)

If the twist matrix is T_3 with the corresponding primary matrix having rank=1, we can always choose the origin in such a way that the primary part of the first twist is directed in the direction of the z-axis, and the first twist axis passes through the origin, the twist matrix then take the following form

$$T_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{x1} & P_{y1} & P_{z1} \\ 0 & 0 & 0 & P_{x2} & P_{y2} & P_{z2} \end{pmatrix} \qquad (263)$$

The primary part matrix thus becomes;

$$P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{264}$$

If now the position vectors of the twists are given in the form ($X_1\bar{i}+Y_1\bar{j}+Z_1\bar{k}$), the above twists matrix is then written as (with the assumption that the prismatic joint is oriented in the x-y plane that is the planer case with the prismatic joint not oriented in the z direction);

.

$$T_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{x1} & P_{y1} & 0 \\ 0 & 0 & 0 & P_{x2} & P_{y2} & 0 \end{pmatrix} \tag{265}$$

5.2.2.3.1 Possible Topologies

The manipulator can have the following three configurations, with certain conditions on the joints which will be discussed later.

- Three parallel R Joints
- Two Parallel R Joints and 1P Joint
- One R joint and 2 P Joints

5.2.2.1.2 Positioning and Orientation -3R Case (Twist matrix of the form T_3)

The three R joints will be installed with their axis parallel to each other and oriented in the direction of the first given twists. These joints can be placed anywhere in the plane under certain conditions to achieve any linear combination of the given twists, these conditions are discussed below.

Let the three R joints which are oriented in the direction of the first given twists are positioned at $(a_1\bar{i} + b_1\bar{j} + c_1\bar{k})$, $(a_2\bar{i} + b_2\bar{j} + c_2\bar{k})$, and $(a_3\bar{i} + b_3\bar{j} + c_3\bar{k})$ respectively. With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a3} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \\ 0 & 0 & 1 & b_3 & -a_3 & 0 \end{pmatrix} \qquad (266)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \\ 0 & 0 & 0 \end{pmatrix} \qquad (267)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \end{pmatrix} \quad (268)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \quad (269)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \end{pmatrix}$$

$$P_{x1}P_{y2} - P_{y1}P_{x2} \neq 0 \quad (270)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ P_{x1}\alpha_{21}+P_{x2}\alpha_{31} & P_{x1}\alpha_{22}+P_{x2}\alpha_{32} & P_{x1}\alpha_{23}+P_{x2}\alpha_{33} \\ P_{y1}\alpha_{21}+P_{y2}\alpha_{31} & P_{y1}\alpha_{22}+P_{y2}\alpha_{32} & P_{y1}\alpha_{23}+P_{y2}\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ -a_1 & -a_2 & -a_3 \end{pmatrix} \quad (271)$$

This gives the following conditions for equivalence

$$\alpha_{11}=1 \quad (272)$$

$$\alpha_{12}=1 \quad (273)$$

$$\alpha_{13}=1 \quad (274)$$

$$b_1= P_{x1}\alpha_{21}+P_{x2}\alpha_{31} \quad (275)$$

$$b_2= P_{x1}\alpha_{22}+P_{x2}\alpha_{32} \quad (276)$$

$$b_3= P_{x1}\alpha_{23}+P_{x2}\alpha_{33} \quad (277)$$

$$a_1= -P_{y1}\alpha_{21}-P_{y2}\alpha_{31} \quad (278)$$

$$a_2= -P_{y1}\alpha_{22}-P_{y2}\alpha_{32} \quad (279)$$

$$a_3= -P_{y1}\alpha_{23}-P_{y2}\alpha_{33} \quad (280)$$

5.2.2.3.3 Solution Set -3R Case (Twist matrix of the form T_3)

The solution set therefore is the set of following equations with the condition that the first given twist axis is aligned with the Z-axis of the co-ordinate system and the other two twists corresponding to prismatic joints are oriented in the x-y plane.

$$\left.\begin{aligned}
&\text{Determinant}\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix} \neq 0\\
&P_{x1}P_{y2} - P_{y1}P_{x2} \neq 0\\
&\alpha_{11}=1\\
&\alpha_{12}=1\\
&\alpha_{13}=1\\
&b_1= P_{x1}\alpha_{21}+P_{x2}\alpha_{31}\\
&b_2= P_{x1}\alpha_{22}+P_{x2}\alpha_{32}\\
&b_3= P_{x1}\alpha_{23}+P_{x2}\alpha_{33}\\
&a_1= -P_{y1}\alpha_{21}-P_{y2}\alpha_{31}\\
&a_2= -P_{y1}\alpha_{22}-P_{y2}\alpha_{32}\\
&a_3= -P_{y1}\alpha_{23}-P_{y2}\alpha_{33}
\end{aligned}\right\} \quad (281)$$

5.2.2.3.4 Positioning and Orientation -2R1P Case (Twist matrix of the form T_3)

The two R one P joints will be installed with the two R joints oriented in the direction of the first given twist but with different position vectors, the prismatic joint must lie in the plane of the dual parts of the joint twists oriented in the direction of the third twist.

Let the two R joints which are oriented in the direction of the given twists are positioned at ($a_1\bar{\imath} + b_1\bar{\jmath} + c_1\bar{k}$) and ($a_2\bar{\imath} + b_2\bar{\jmath} + c_2\bar{k}$) respectively. Let the P joint is oriented in the X-Y plane and its screw is given as (0 0 0 P_{x3} P_{y3} 0). With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$T_{a3} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 1 & b_2 & -a_2 & 0 \\ 0 & 0 & 0 & P_{x3} & P_{y3} & 0 \end{pmatrix} \quad (282)$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ b_1 & b_2 & P_{x3} \\ -a_1 & -a_2 & P_{y3} \\ 0 & 0 & 0 \end{pmatrix} \quad (283)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ b_1 & b_2 & P_{x3} \\ -a_1 & -a_2 & P_{y3} \end{pmatrix} \quad (284)$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \qquad (285)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \end{pmatrix}$$

$$P_{x1}P_{y2} - P_{x2}\, P_{y1} \neq 0 \qquad (286)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ P_{x1}\alpha_{21}+P_{x2}\alpha_{31} & P_{x1}\alpha_{22}+P_{x2}\alpha_{32} & P_{x1}\alpha_{23}+P_{x2}\alpha_{33} \\ P_{y1}\alpha_{21}+P_{y2}\alpha_{31} & P_{y1}\alpha_{22}+P_{y2}\alpha_{32} & P_{y1}\alpha_{23}+P_{y2}\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ b_1 & b_2 & P_{x2} \\ -a_1 & -a_2 & P_{y2} \end{pmatrix} \qquad (287)$$

This gives the following conditions for equivalence

$\alpha_{11}=1$ (288)

$\alpha_{12}=1$ (289)

$\alpha_{13}=0$(287)

$b_1= P_{x1}\alpha_{21}+P_{x2}\alpha_{31}$ (290)

$b_2= P_{x1}\alpha_{22}+P_{x2}\alpha_{32}$ (291)

$P_{x2} = P_{x1}\alpha_{23}+P_{x2}\alpha_{33}$ (292)

$a_1= -P_{y1}\alpha_{21}-P_{y2}\alpha_{31}$ (293)

$a_2= -P_{y1}\alpha_{22}-P_{y2}\alpha_{32}$ (294)

$P_{y2}= P_{y1}\alpha_{23}+P_{y2}\alpha_{33}$ (295)

5.2.2.3.5 Solution Set -2R1P Case (Twist matrix of the form T_3)

The solution set therefore is the set of following equations with the condition that the first given twist axis is aligned with the Z-axis of the co-ordinate system and the other two twists corresponding to prismatic joints are oriented in the x-y plane.

$$\left.\begin{array}{l} \text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \\ P_{x1}P_{y2} - P_{x2}P_{y1} \neq 0 \\ \alpha_{11}=1 \\ \alpha_{12}=1 \\ \alpha_{13}=0 \end{array}\right\}$$

$$
\left.\begin{array}{l}
b_1 = P_{x1}\alpha_{21} + P_{x2}\alpha_{31} \\
b_2 = P_{x1}\alpha_{22} + P_{x2}\alpha_{32} \\
P_{x2} = P_{x1}\alpha_{23} + P_{x2}\alpha_{33} \\
a_1 = -P_{y1}\alpha_{21} - P_{y2}\alpha_{31} \\
a_2 = -P_{y1}\alpha_{22} - P_{y2}\alpha_{32} \\
P_{y2} = P_{y1}\alpha_{23} + P_{y2}\alpha_{33}
\end{array}\right\} \quad (296)
$$

5.2.2.3.6 Positioning and Orientation -1R2P Case (Twist matrix of the form T_3)

The one R two P joints will be installed with the R joint oriented in the direction of the first given twists, the prismatic joints must lie in the x-y plane, but they have to be linearly independent of each other.

Let the R joint which is oriented in the direction of the given twists is positioned at ($a_1\bar{i} + b_1\bar{j} + c_1\bar{k}$). Let the P joints be oriented in the X-Y plane and their screws are given as (0 0 0 P_{x3} P_{y3} 0) and (0 0 0 P_{x4} P_{y4} 0). With these position vectors the dual part of the joint twists can be found as variable. The twist matrix for the alternate joints will become.

$$
T_{a3} = \begin{pmatrix} 0 & 0 & 1 & b_1 & -a_1 & 0 \\ 0 & 0 & 0 & P_{x3} & P_{x4} & 0 \\ 0 & 0 & 0 & P_{y3} & P_{y4} & 0 \end{pmatrix} \quad (297)
$$

The equation of equivalence becomes:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ b_1 & P_{x3} & P_{x4} \\ -a_1 & P_{y3} & P_{y4} \\ 0 & 0 & 0 \end{pmatrix} \qquad (298)$$

The equation can be reduced in the form with each matrix having rank =3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & P_{x3} & P_{x4} \\ -a_1 & P_{y3} & P_{y4} \end{pmatrix}$$

For the alpha matrix to have rank equal to three the condition is determinant of alpha matrix should not be equal to zero, can be written as;

$$\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \qquad (298)$$

For the following matrix to have rank =3 the condition will be determinant not equal to zero which gives the following condition;

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & P_{x1} & P_{x2} \\ 0 & P_{y1} & P_{y2} \end{pmatrix}$$

.

$$P_{x1}P_{y2} - P_{x2}\,P_{y1} \neq 0 \qquad (299)$$

We can write the above matrix equation in the following form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ P_{x1}\alpha_{21}+P_{x2}\alpha_{31} & P_{x1}\alpha_{22}+P_{x2}\alpha_{32} & P_{x1}\alpha_{23}+P_{x2}\alpha_{33} \\ P_{y1}\alpha_{21}+P_{y2}\alpha_{31} & P_{y1}\alpha_{22}+P_{y2}\alpha_{32} & P_{y1}\alpha_{23}+P_{y2}\alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & P_{x3} & P_{x4} \\ -a_1 & P_{y3} & P_{y4} \end{pmatrix} \qquad (298)$$

This gives the following conditions for equivalence

$\alpha_{11}=1$ (298)

$\alpha_{12}=0$ (299)

$\alpha_{13}=0$ (300)

$b_1= P_{x1}\alpha_{21}+P_{x2}\alpha_{31}$ (301)

$P_{x3}= P_{x1}\alpha_{22}+P_{x2}\alpha_{32}$ (302)

$$P_{x4} = P_{x1}\alpha_{23}+P_{x2}\alpha_{33} \qquad (303)$$

$$a_1= -P_{y1}\alpha_{21}-P_{y2}\alpha_{31} \qquad (304)$$

$$P_{y3}= P_{y1}\alpha_{22}+P_{y2}\alpha_{32} \qquad (305)$$

$$P_{y4}= P_{y1}\alpha_{23}+P_{y2}\alpha_{33} \qquad (306)$$

5.2.2.3.7 Solution Set -1R2P Case (Twist matrix of the form T_3)

The solution set therefore is the set of following equations with the condition that the first given twist axis is aligned with the Z-axis of the co-ordinate system and the other two twists corresponding to prismatic joints are oriented in the x-y plane.

$$\left.\begin{array}{l}
\text{Determinant} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \neq 0 \\
P_{x1}P_{y2} - P_{x2}P_{y1} \neq 0 \\
\alpha_{11}=1 \\
\alpha_{12}=1 \\
\alpha_{13}=0 \\
b_1= P_{x1}\alpha_{21}+P_{x2}\alpha_{31} \\
b_2= P_{x1}\alpha_{22}+P_{x2}\alpha_{32} \\
P_{x2} = P_{x1}\alpha_{23}+P_{x2}\alpha_{33} \\
a_1= -P_{y1}\alpha_{21}-P_{y2}\alpha_{31} \\
a_2= -P_{y1}\alpha_{22}-P_{y2}\alpha_{32} \\
P_{y2}= P_{y1}\alpha_{23}+P_{y2}\alpha_{33}
\end{array}\right\} \qquad (307)$$

Chapter 6 – CONCLUSION

6.1 Introduction

This chapter is concerned with the overall outcome of the thesis, and with the suggestions of the further work which can be carried out on the basis of the theory presented in this thesis. First, each chapter is reviewed, and the essential results from each are described; then overall conclusions are proposed; finally, recommendations for further work are made.

6.2 Review

Chapter 1 reviewed the background of the screw theory and the basic definitions of serial and parallel kinematic manipulators.

Chapter 2 presented the more detailed understanding of the screw theory with description of the elements of the screw theory in more detail and the chapter contributed as to how the instantaneous twist of the manipulator can be decomposed to the primitive design of the manipulators. Based upon the linear combination of the specified twists, revolute, prismatic joints can be found to achieve this combination.

Chapter 3 gives the application of the theory presented in chapter 2 to single degree of freedom manipulators and discussed in detail with the help of twist matrices all the possible cases associated with the single DOF manipulators.

The theory presented in chapter 2 is applied to Two DOF serial manipulators in chapter 4 in detail, with detailed examples, all the calculations involved and conditions required. Details of all possible topologies pertaining to two DOF and as to why this theory in the present form is restricted to planer serial manipulators.

Chapter 5 gives a detailed account of the application of the given theory on to 3 DOF serial manipulators (only planer).

6.3 General Conclusion

This thesis is held to be successful in that, a new systematic approach and a theory on instantaneous finite twist has been proposed, which can be used to give alternate designs for a given manipulator. The end effector twist of a given design is used to associate the design to the possible alternate topologies. The design of the revolute and prismatic joints has thus been proposed.

6.4 Further Work

The avenues of further work suggested by the experience of this project are:

- The study be extended to the spatial manipulators working in coordination with eachother.

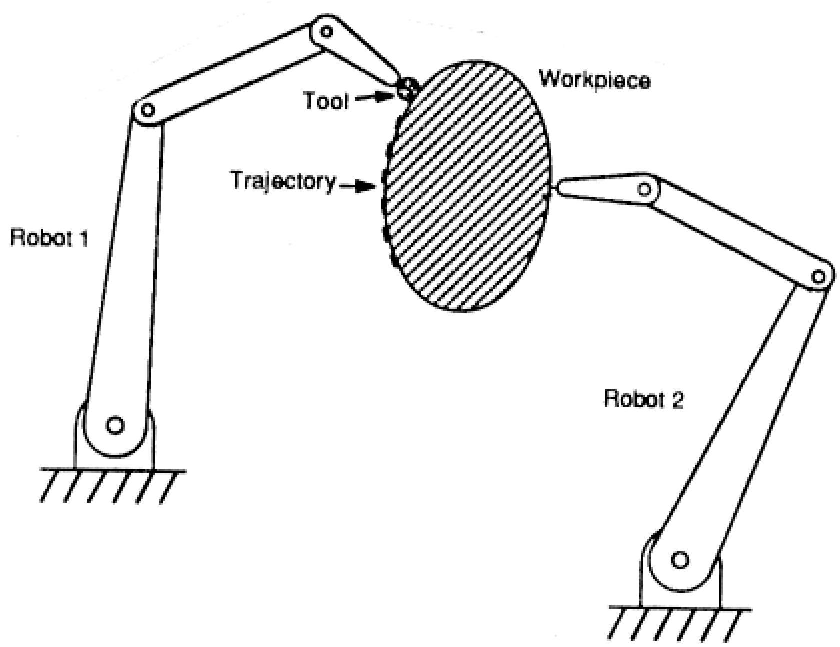


Figure- 6.1 Motion Planning for Multi-Robot Assembly Systems

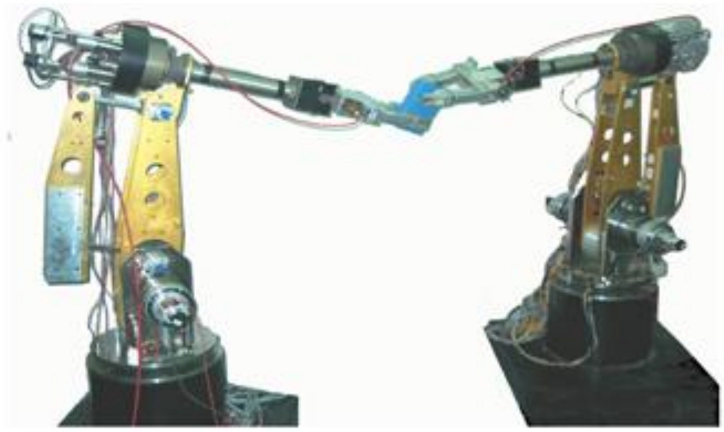


Figure 6.2 A dual-arm cooperative robot

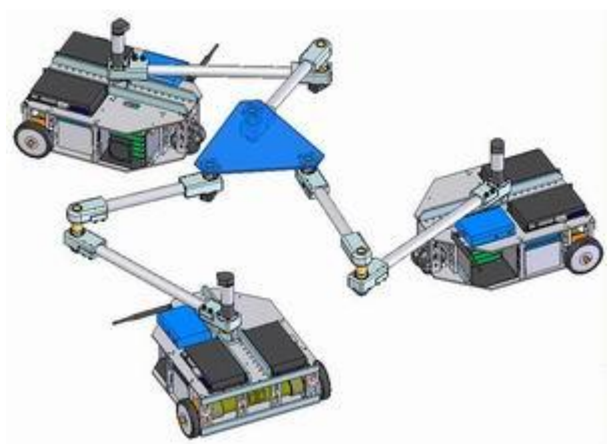


Figure 6.3 Cooperating Robots

- Further investigation can be done on the application of this study to parallel manipulators.

Appendix 'A'

Mathematical Preliminaries

1. Workspace

The workspace of a manipulator is the total volume of space the end-effector can reach. The workspace is constrained by the geometry of the manipulator as well as the mechanical constraints on the joints. The workspace is broken into a reachable workspace and a dexterous workspace. The reachable workspace is the volume of space within which every point is reachable by the end-effector in at least one orientation. The dexterous workspace is the volume of space within which every point can be reached by the end effector in all possible orientations. The dexterous workspace is a subset of the reachable workspace. [12]

2. Vector

A vector is a directed line segment representing a quantity such as force, velocity, etc. which possesses both magnitude and direction. The direction of the quantity is given by the direction of the arrow and the magnitude by the length of the arrow. A vector '**A'** can be represented analytically if we let **i**, **j** and **k** be unit vectors directed along the positive x, y and z axes of a right-handed Cartesian coordinate system. Let the initial point of **A** be located at the origin O and whose terminal point is at coordinates (a_1, a_2, a_3). Then vector A can be represented as

$$\mathbf{A} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}\,. \qquad (1)$$

The vectors $a_1\,\mathbf{i}$, $a_2\,\mathbf{j}$, and $a_3\,\mathbf{k}$ are called the component vectors of A in the x, y and z directions respectively. a_1, a_2, and a_3 are called the x, y and z components of A.

3. Unit Vector

A unit vector is a vector having unit magnitude i.e. a magnitude of 1. Normally denoted by a'¯'or'^'over the name of the vector.

4. Position Vector (or Radius Vector)

A position vector is a vector that extends from the origin of the coordinate system to some point (x, y, z) in space i.e. the vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad (2)$$

or

$$\mathbf{r} = (x, y, z) \qquad (3)$$

5. Sum of Two Vectors

The sum of two vectors expressed in analytical form is obtained by adding corresponding components i.e. if A and B are two vectors such that;

$A = a_1 i + a_2 j + a_3 k$ (4)

$B = b_1 i + b_2 j + b_3 k$ (5)

$A + B = (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k$ (6)

6. Vector (or Cross) Product

The vector (or cross) product A×B of two vectors A and B (defined in 4 and 5) is defined as

$A\times B = |A|\,|B| \sin \theta\, u$ (7)

Where θ is the angle from A to B and u is a unit vector perpendicular to the plane of A and B and so directed that a right-handed screw driven in the direction of u would carry A into B.

$A\times B = (a_2b_3 - b_2a_3)\, i + (a_3b_1 - b_3a_1)\, j + (a_1b_2 - b_1a_2)\, k$ (8)

or A×B = determinant of $\begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$ (9)

7. Rank of a Matrix

The rank of a matrix A is the maximum number of linearly independent row or column vectors of the matrix.

8. Linearly Independence of Vectors

Two vectors $A= (x_1,y_1)$ and $B = (x_1i,y_2)$ are linearly independent if the only simultaneous solution of the system $a(x_1,y_1)+b(x_1,y_2)=0$, $a=0$ and $b=0$, where a and b are both scalars; that is the simultaneous solution of the following two equations;

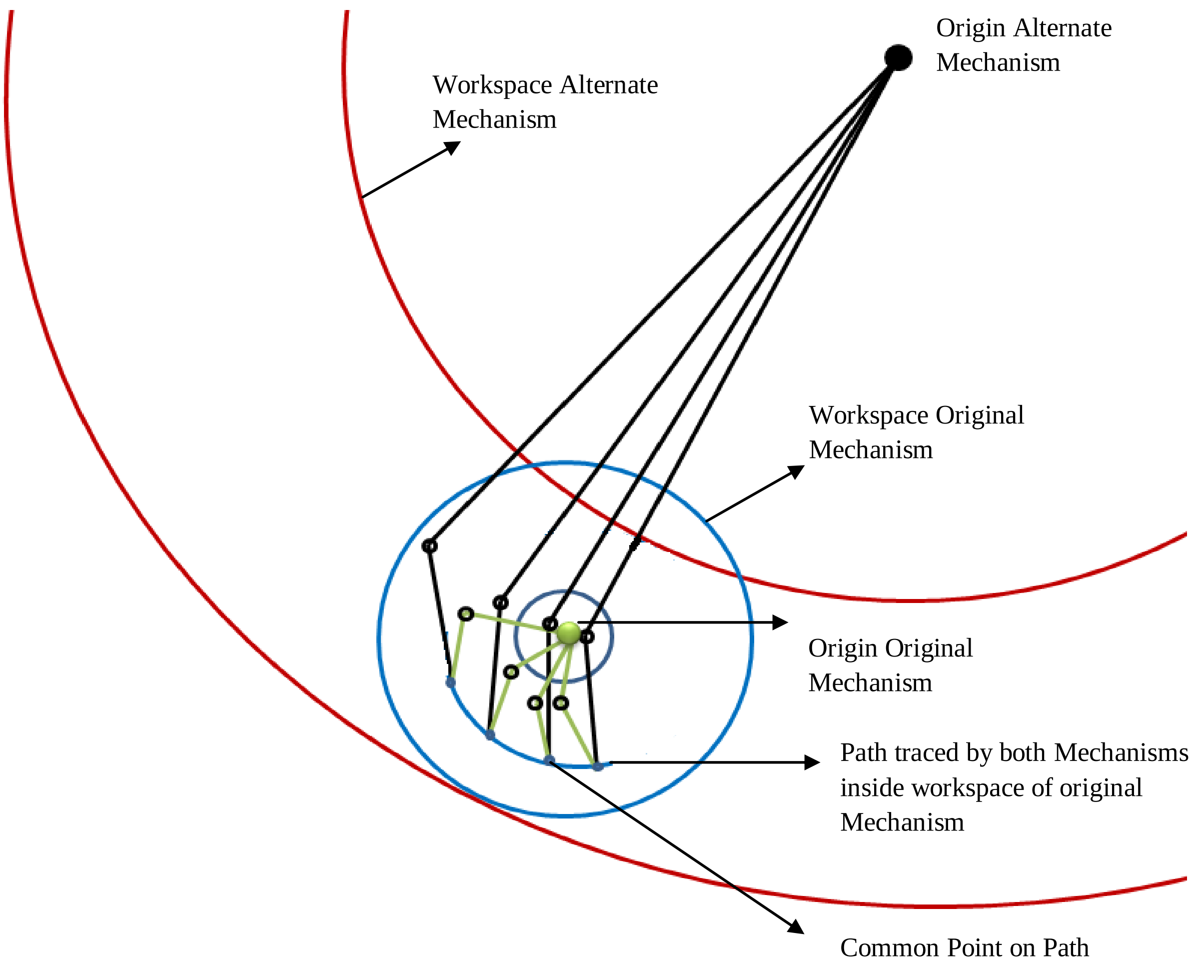
$ax_1+bx_1=0$ (10)

$ay_1+by_1=0$ (11)

Appendix 'B'

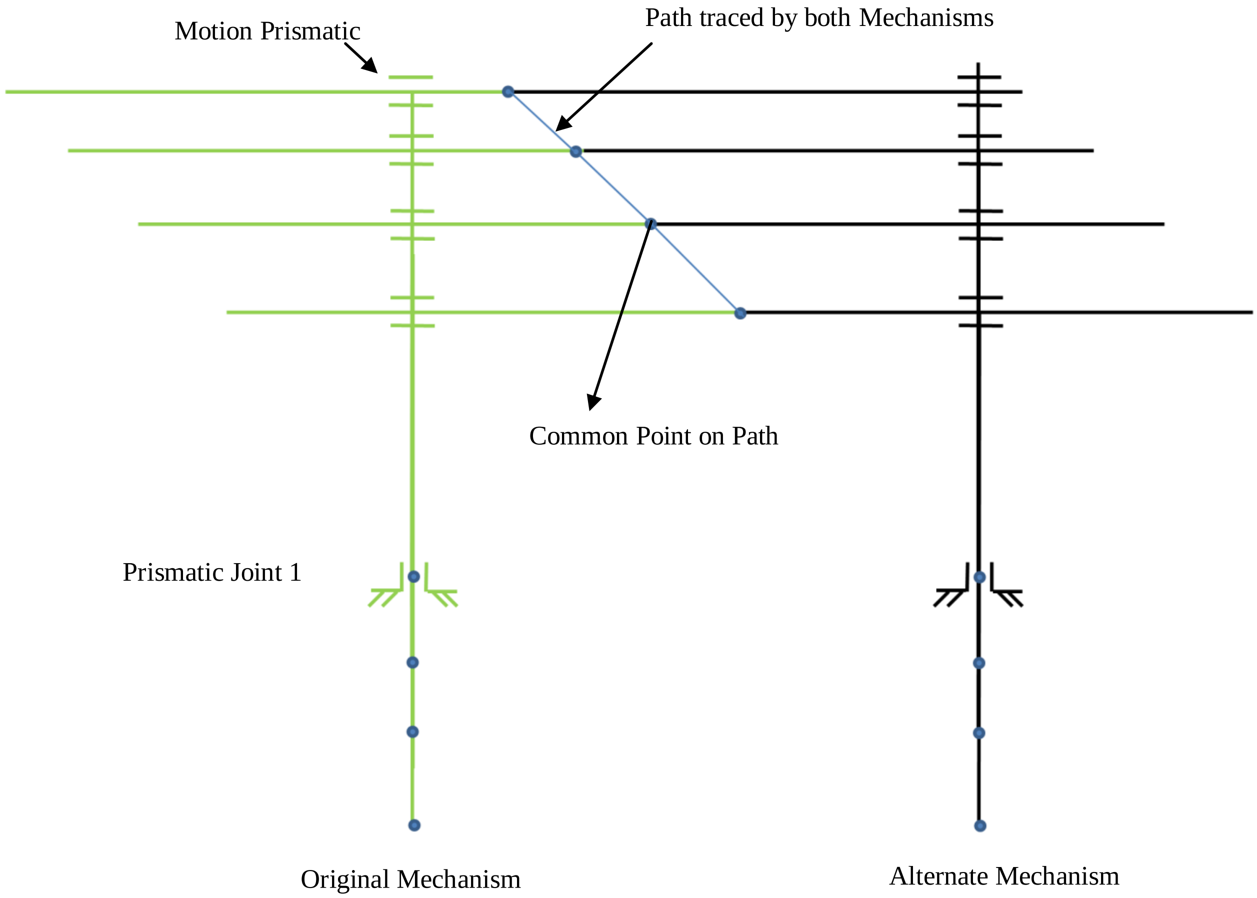
<u>Graphical Representation of Original and Alternate Mechanisms</u>

Two Revolute Joints Mechanism Replaced by Alternate Two Revolute Joints Mechanism (2R to 2R)



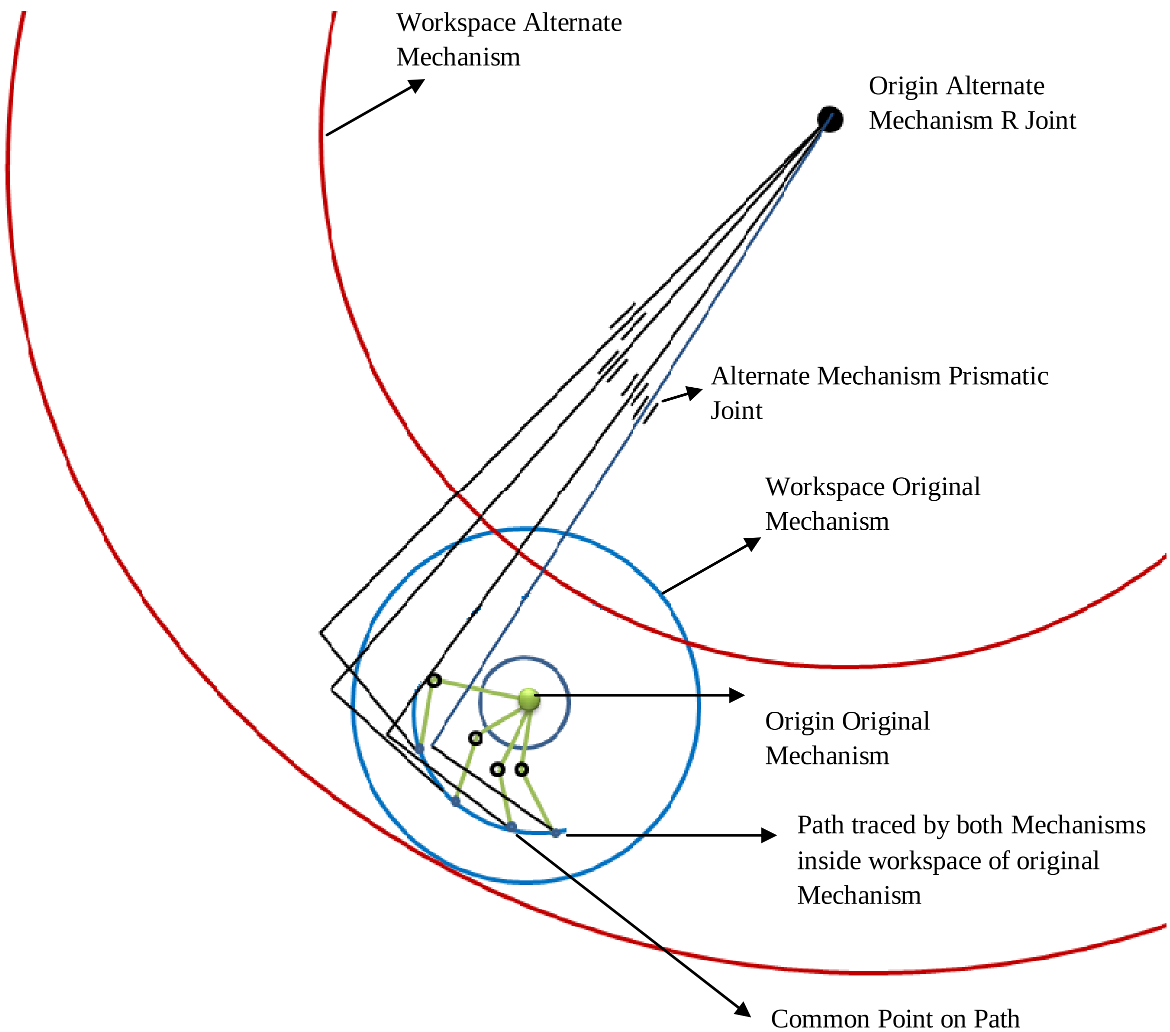
Note:- Graph showing same path traced by end effector of both mechanisms and the path followed by joints

Two Prismatic Joints Mechanism Replaced by Alternate Two Prismatic Joints Mechanism (2P to 2P)



Note: - Both Mechanisms have the same workspace irrespective of the fact that both mechanisms have different joint locations.

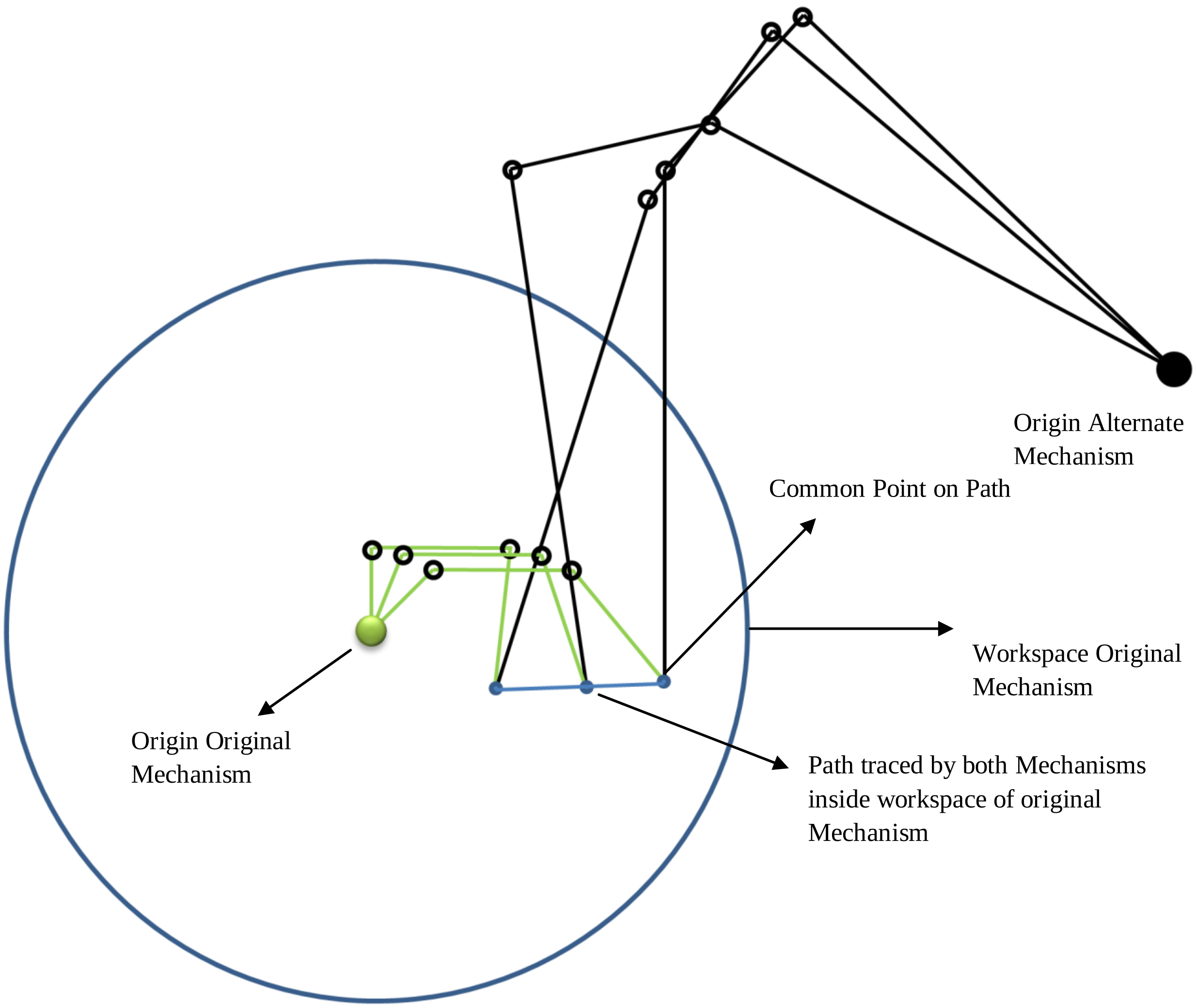
Two Revolute Joints Mechanism Replaced by One Revolute and One Prismatic Joint Mechanism-(2R to 1R1P)



Note:- Graph showing same path traced by end effector of both mechanisms and the path followed by joints

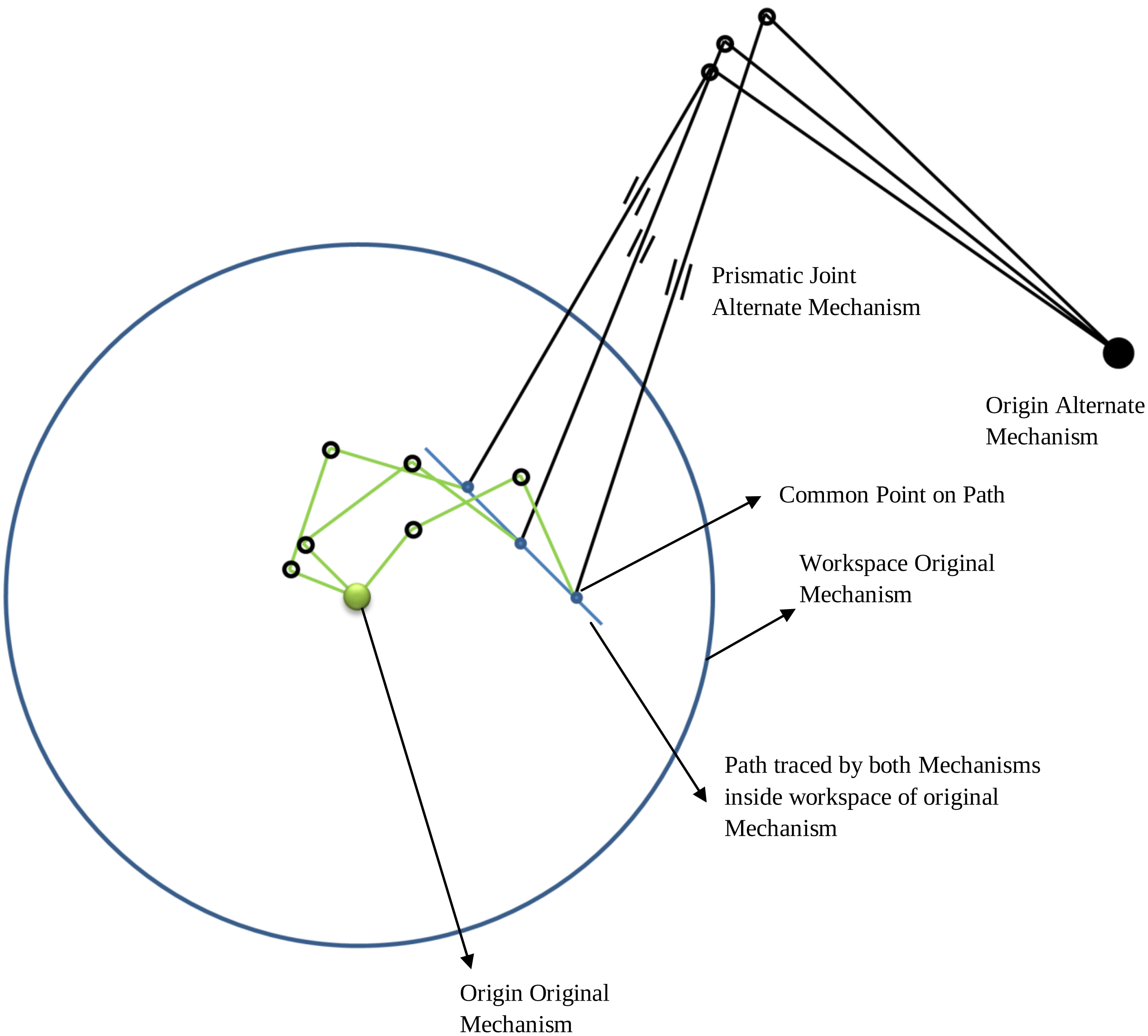
<u>Three Revolute Joints Mechanism Replaced by Three Revolute Joints Mechanism</u>

<u>(3R to 3R)</u>

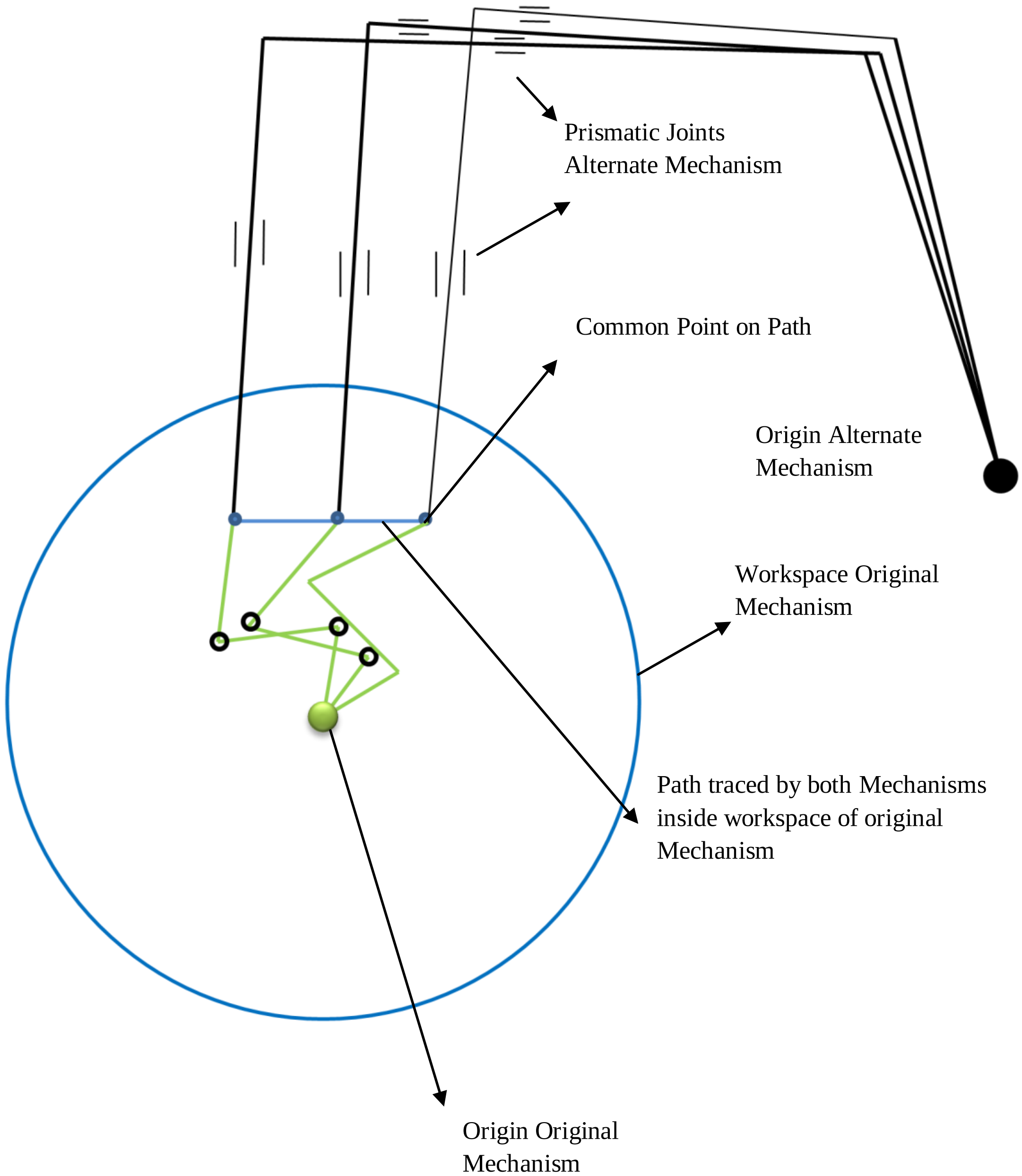


Note:- Graph showing same path traced by end effector of both mechanisms and the path followed by joints

Three Revolute Joints Mechanism Replaced by Two Revolute and One Prismatic Joint Mechanism-(3R to 2R1P)

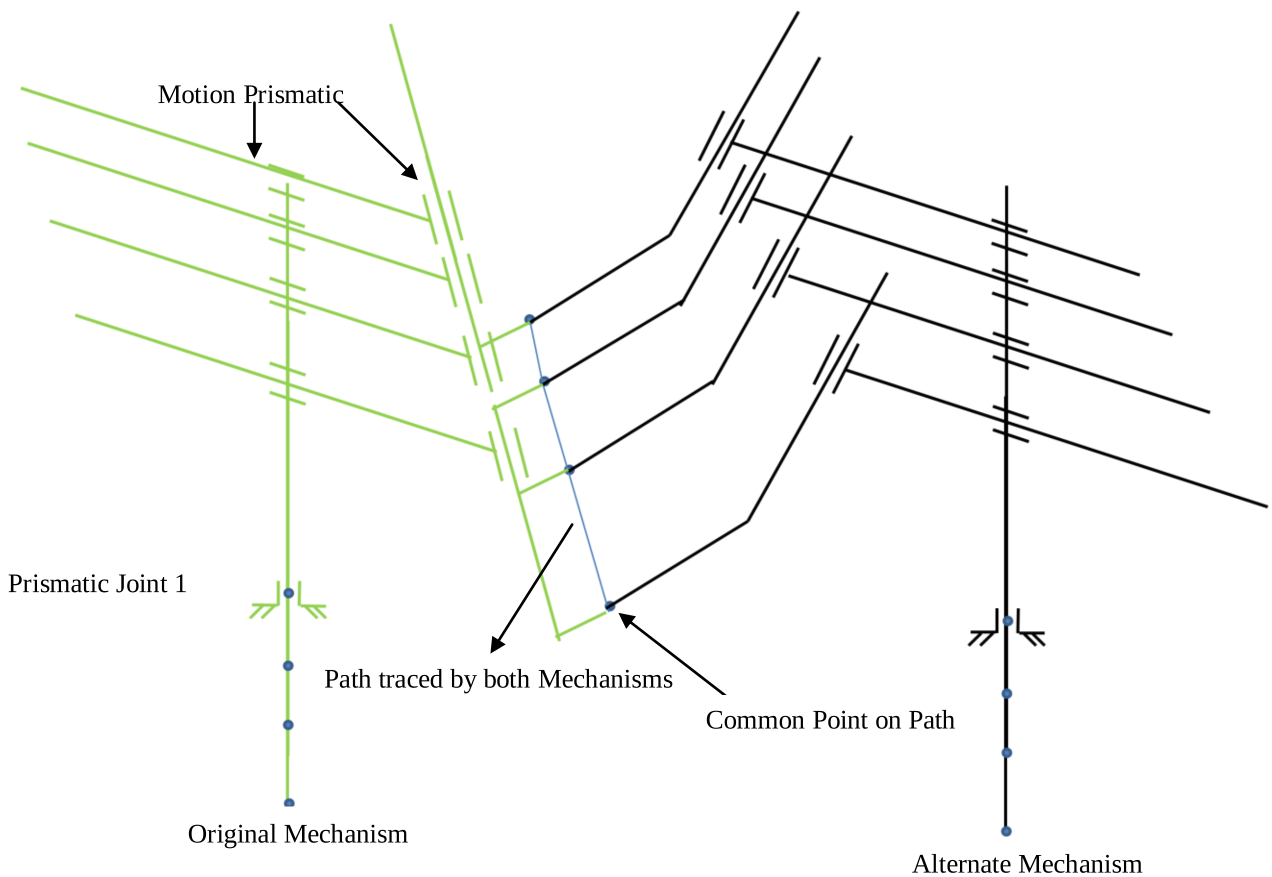


Three Revolute Joints Mechanism Replaced by Two Prismatic and One Revolute Joint Mechanism-(3R to 2P1R)



Three Prismatic Joints Mechanism Replaced by Three Prismatic Joints Mechanism

(3P to 3P)



Note: - Both Mechanisms have the same workspace irrespective of the fact that both mechanisms have different joint locations.

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