Convex Functions and Exponential Convexity

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Dedication

- This thesis is dedicated to my family. For their endless love, prayer, support and encouragement

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First and foremost my thanks goes to Almighty Allah, the most gracious and the most merciful.
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May Allah bless them all (Ameen!).

Ubaid ur Rehman Atif

Abstract

One of the impressive application of the theory of convex functions is to the study of classical inequalities. Here, we show that how the theory provides an elementary, elegant, and unifed treatment of some of the best known inequalities in mathematics. The goal of this study is to give a short summary of the main results of the Hermite-Hadamard inequality for AG-convex functions, which are a special type of convex functions that the author has been studying for the past year. Also we present some interesting nontrivial examples to support our inequalities. Furthermore, we focus on exponentially convex functions and establish some important results equipped with integration, we also present some examples to support our main results.

Contents

Chapter 1

Introduction of Convex Functions

Convex and exponential convex functions are fundamental concepts in the field of pure mathematics, particularly in the study of optimization and convex analysis. Convex functions have been extensively studied due to their interesting and useful properties, such as global optimality and uniqueness of solutions. On the other hand, exponential convex functions have recently gained attention due to their ability to model complex nonlinear systems in a tractable way. If a function is called convex, it means that its graph lies above its chords. In other words, any two points belonging to the graph of the function, the line segment joining these two points must be positioned either above or on the graph. Convex functions are characterized by their non-negative second derivative and are known to have several interesting properties, such as being Lipschitz continuous and having subgradients at every point. Exponential convex functions, on the other hand, are functions that can be articulated as the the exponential of a convex function. These functions have been extensively studied in recent years due to their ability to model complex nonlinear systems in a tractable way. In particular, they are often used in the study of optimization, where they have been shown to have nice theoretical properties and can be efficiently optimized. In recent years, exponential convex functions have also found applications in various areas of pure mathematics, including geometry, analysis, and number theory. The study of exponential convex functions continues to be an active area of research, with many open problems and exciting developments.[1]

1.1 Convex Sets

For any pair of points $x_{1,}$ x_2, a set Q in R^n is considered convex within the set, the line connecting them is also part of the set. Mathematically, this can be expressed as: for all x_1, x_2 in Q and for any scalar value μ in the interval $[0, 1]$, the expression $\mu \mathrm{x}_1 + (1 - \mu)\mathrm{x}_2$ must also be a part of Q [2].

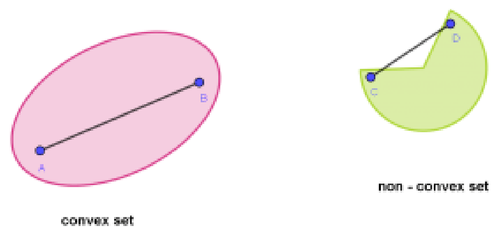


Figure 1.1: Convex and Non-Convex Set

Example 1. *A line, triangular region, etc.*

1.2 Properties of convex sets

(a) Any arbitrary collection of convex sets $\{Q_i | i \in I\}$, the intersection denoted as $\cap_{i\in I}\mathrm{Q}_i$ is a convex set.

(b) The vector sum of two convex sets Q_1 and Q_2, denoted as Q_1+ Q_2, is also convex set.

(c) For any scalar value α and convex set Q, the set obtained by scaling Q by α, denoted as αQ is also convex [2].

1.3 Convex Functions

Definition 1.3.1. *A function J mapping convex set Q in R^n to R is considered convex if, for all pairs of points x_1,x_2 in Q and for all scalar values μ with $0 \leq \mu \leq 1$, the following inequality holds*

$$J\left(\mu\left(x_1\right) + \left(1-\mu\right)x_2\right) \leq \mu\ J\left(x_1\right) + \ \left(1-\mu\right)\ J\left(x_2\right). \tag{1.1}$$

If this inequality is satisfied with strict inequality, then J is considered a strictly convex function over Q. On the other hand, In the case of the inequality being reversed, then J is considered concave. If the reversed inequality is strict, then J is referred to as a strictly concave function.

Example 1. $J : R \to R, \;\; J(x) = |x|$

Solution: For all x_1,x_2 in R we have

$$\begin{aligned} J\left(\mu x_1 + (1-\mu)\, x_2\right) &= |\mu x_1 + (1-\mu)\, x_2|, \quad \mu \in [0,1] \\ &\leq |\mu x_1| + |(1-\mu)\, x_2| \\ &= \mu\, |x_1| + (1-\mu)\, |x_2| \\ &= \mu \; J\left(x_1\right) + (1-\mu) \;\; J\;\left(x_2\right) \end{aligned}$$

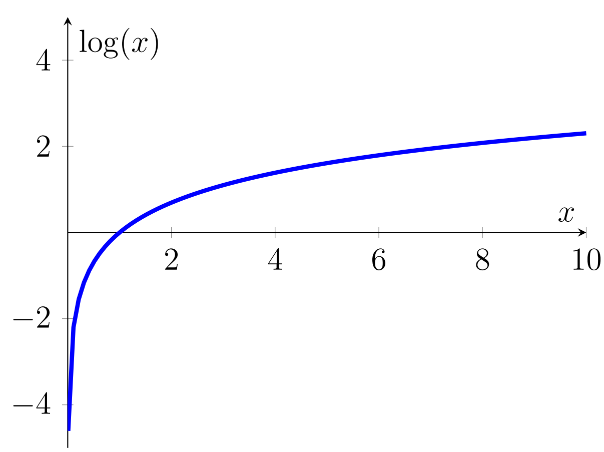
hence, $J(x) = |x|$ satisfy (1.1) so it is convex.

Definition 1.3.2. *A function $J : R \to R$ that has two continuous derivatives is convex if $J''(x) \geq 0$ for all real x.*

Example 2. *Show $-\log(x)$ is convex for $x > 0$.*

Solution: Let $J(x) = -\log(x)$,

$J'(x) = -\frac{1}{x}$, $J''(x) = \frac{1}{x^2}$, when x is greater than 0 (with the assumption that $-\log(x)$ is defined), $J''(x)$ always remains positive, so $-\log(x)$ is convex. To verify this, it is evident from a plotted graph that $\log(x)$ exhibits concavity, thereby confirming the convex nature of $-\log(x)$.



1.4 Properties of convex functions

(a) If Q is a convex set, and J and K are both convex functions defined on Q in R^n, then $J+K$, αJ(where α is a nonnegative scalar) and and the maximum of $J(x)$ and $K(x)$, represented as Max $\{\ J(x), K(x)\}$, are also convex on Q.

(b) A convex function is not required to be continuous. for example:

$$J(x) = \begin{cases} x^2 & if\ 0 \leq x < 1 \\ 2 & if\ x = 1 \end{cases}$$

(c) A Convex function is not necessarily differentiable.

An example of this is the function $J(x) = |x|$, where $x \in R$ is convex but lacks differentiability at x = 0.

(d) Let $J(x)$ is a convex function on a subset Q of R^n that is also convex. In such cases every local minimum of J within Q is also a global minimum.

(e) A function $J(x)$ is convex if and only if for every point in the domain of the function, the Hessian matrix of J is positive semi-definite.

For a function$J$. The Hessian matrix is defined as

$$\mathrm{H}_J = \begin{bmatrix} \frac{\partial^2 J}{\partial x_1' \partial x_1'} & \frac{\partial^2 J}{\partial x_1' \partial x_2'} & \cdot & \cdot & \cdot & \frac{\partial^2 J}{\partial x_1' \partial x_n'} \\ \frac{\partial^2 J}{\partial x_2' \partial x_1'} & \frac{\partial^2 J}{\partial x_2' \partial x_2'} & \cdot & \cdot & \cdot & \frac{\partial^2 J}{\partial x_2' \partial x_n'} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^2 J}{\partial x_n' \partial x_1'} & \frac{\partial^2 J}{\partial x_n' \partial x_2'} & \cdot & \cdot & \cdot & \frac{\partial^2 J}{\partial x_n' \partial x_n'} \end{bmatrix}$$

A square matrix consisting partial derivatives of the second-order of a function is known as the Hessian matrix, it gives information about the curvature of a function, if it's positive semi-definite, it means the function is convex.

1.5 Continuity and differentiability

On the interval $[g, h]$, a finite and convex function is bounded above by M, where M is the maximum value between $\{J(g), J(h)\}$, since for any $w = \mu g + (1-\mu)h$ within the interval,

$$J(w) \leq \mu J(g) + (1-\mu)J(h) \leq \mu M + (1-\mu)M = M. \tag{1.2}$$

Formulating an arbitrary point in a prescribed form $(g+h)/2+t$. We can observe that it is bounded from below. Then,
$J(\frac{g+h}{2}) \leq \frac{1}{2}J(\frac{g+h}{2}+t) + \frac{1}{2}J(\frac{g+h}{2}-t)$ or $J(\frac{g+h}{2}+t) \geq 2J(\frac{g+h}{2}) - J(\frac{g+h}{2}-t)$.
With M serving as the upper bound, $-J[\frac{(g+h)}{2}-t] \geq -M$, so
$J(\frac{g+h}{2}+t) \geq 2J(\frac{g+h}{2}) - M = m$.
On the boundary points one can readily observe that the convex function might not exhibit continuity . In that area, there is a possibility of upward jumps. However, the function is not just continuous on interior, but it is also satisfies the stronger condition. For any closed sub interval $[g,h]$ of the interior of the domain we will prove that, there is a constant C such that for any pair of points $x_1, x_2 \in [g,h]$, the following holds

$$|J(x_1) - J(x_2)| \leq C|x_1 - x_2|. \tag{1.3}$$

When a function meets the requirement of equation (1.3) for some C and all x_1 and x_2 in the given interval. It is referred to as satisfying a Lipschitz condition or being Lipschitz.

Definition 1.5.1. *The function J is absolutely continuous throughout the interval $[g,h]$ if, for any positive value ϵ, we can always identify a $\delta > 0$, such that for any collection of disjoint open subintervals $\{(g_i,h_i)\}_{i=1}^n$, of $[g,h]$ with $\sum_{i=1}^n (h_i - g_i)^n < \delta$ and*

$$\textstyle\sum_{i=1}^n |f(h_i) - f(g_i)| < \epsilon$$

Theorem 1.5.1. [3]*If a function J maps a convex set Q to R is convex function, then for any closed interval $[g,h]$ within the int(Q) the function J satisfies a Lipschitz condition. As a result, the function J is absolutely continuous on $[g,h]$ and continuous on int(Q)*

Proof. To satisfy the condition, select a value greater than zero, represented by ϵ, ensuring that both g minus ϵ and h plus ϵ belong to the set Q and Assume that m

and M denote the lower and upper bounds for J on $[g-\epsilon, h+\epsilon]$. If g and x_2 are two different points of $[g, h]$, set

$w = x_2 + \frac{\epsilon}{|x_2-x_1|}(x_2 - x_1), \quad \lambda = \frac{|x_2-x_1|}{\epsilon+|x_2-x_1|}$,

then $w \in [g - \epsilon, h + \epsilon]$, $x_2 = \lambda w + (1 - \lambda)x_1$, and we have

$J(x_2) \leq \lambda J(w) + (1 - \lambda)J(x_1) = \lambda[J(w) - J(x_1)] + J(x_1)$

$J(x_2) - J(x_1) \leq \lambda(M - m) < \frac{|x_2-x_1|}{\epsilon}(M - m) = C|x_2 - x_1|$,

where $C = (M - m)/\epsilon$. Since this holds true for all values of x_1 and x_2 within the range of $[g, h]$, we observe that $|J(x_2) - J(x_1)| \leq C|x_2 - x_1|$ as desired.

Next, Let us remember that $J(s)$ exhibits absolute continuity throughout the interval $[g, h]$ if, for any positive value ϵ, we can always identify a $\delta > 0$ such that for any collection of disjoint open subintervals $\{(g_i, h_i)\}_{i=1}^n$, of $[g, h]$ with $\sum_{i=1}^n (h_i - g_i)^n < \delta$ and $\sum_{i=1}^n |f(h_i) - f(g_i)| < \epsilon$. Clearly the choice δ=ϵ/C meets this requirement.

Finally, the fact that continuity of J on int(Q) can be attributed to the arbitrary nature of $[g, h]$. □

1.6 Jensen's Inequality

The Jensen inequality was first introduced by the Danish mathematician, Johan Jensen[4] in 1906. It is an important concept in mathematics and is widely used in many fields such as economics, physics, and statistics. The study of inequalities has been a crucial part of the classical literature of mathematics, and the critical analysis of inequalities has helped to shape the field into what it is today.

One of the most influential works on inequalities was published in 1934 by the mathematicians Hardy, Littlewood, and Polya[5]. This book provided a comprehensive and sophisticated analysis of inequalities and helped to popularize the subject among mathematicians.

Jensen's inequality has several different forms, but in essence, it states that when a convex transformation is applied to a mean, the result is always less than or equal to the mean obtained after the transformation. This concept has many practical applications, such as in the calculation of expected values in statistics or in the study of optimization problems in economics.

Overall, the study of inequalities has been instrumental in advancing the field of mathematics, and Jensen's inequality is just one example of how these concepts are used to solve complex problems and make important discoveries.

Definition 1.6.1. *Consider J be a convex function defined over the interval Q. If $x_1, x_2, x_3, ..., x_n \in Q$ and $\mu_1, \mu_2, \mu_3, ..., \mu_n$ are nonnegative real numbers such that $\mu_1 + \mu_2 + \mu_3 + ... + \mu_n = 1$, then*

$$J\left(\sum \mu_i x_i\right) \leq \sum \mu_i J(x_i). \tag{1.4}$$

where i from 1 to n.

Definition 1.6.2. *Assume J be a convex function defined on the entire real number line, and the function K is an integrable function over the interval $[0, 1]$ and it follows that the composition of J with K, denoted as $J \circ K$ also integrable over the interval $[0, 1]$. Then*

$$J\left(\int_0^1 K(x)dx\right) \leq \int_0^1 (J \circ K)(x)dx. \tag{1.5}$$

1.7 Hermite-Hadamard inequality

The inequality offers an approximation for the average value of a continuous convex function over an interval. More specifically,

if the convex function J that maps the interval $[g, h]$ to the set of real numbers is such a function with $g<h$, then

$$J\left(\frac{g+h}{2}\right) \leq \frac{1}{h-g}\int_g^h J(x)\,dx \leq \frac{J(g) + J(h)}{2} \tag{1.6}$$

Ch. Hermite discovered this inequality in 1881. E. F. Beckenbach, the expert of history and convex functions theory told about the proof of the inequality (1.6) was originally formulated by J. Hadamard in the year 1893. Now Eq (1.6) is read as Hermite-Hadamard inequality, while in the past it was read as Hadamard inequality.

Consider Q denote a vector space over the field U, where U is either R or C and x_1, x_2 in Q, $x_1 \neq x_2$

Consider the following

$[x_1, x_2] := (1-\mu)\mathrm{x}_1 + (\mu)\mathrm{x}_2, \;\; \mu \;\in\; [0,1].$

The Hermite-Hadamard integral inequality holds for every convex function defined on the interval $[x_1,\, x_2]$ $\in$Q, (see[6])

$$J\left(\frac{x_1+x_2}{2}\right) \leq \int_0^1 J[(1-\mu)x_1 + (\mu)\, x_2]d\mu \;\leq \frac{J\,(x_1) \;+\; J\,(x_2)}{2} \tag{1.7}$$

This can be deduced from (1.6) for the convex function $K(x_1, x_2)$:[0, 1]$\rightarrow R$

Definition 1.7.1. *Consider a function $J(x)$:*

(a) *J has a left derivative at p if*

(1) For δ>0, J is defined on (p-δ, p) in R and

(2) $\lim_{u\rightarrow 0_-} \frac{f(p+u)-f(p)}{u}$ *exists*

If such a limit exists, it is denoted by $J'_-(p)$

(b) J has a right derivative at p if

(1) For δ>0, J is defined on (p, p+δ) in R and

(2)$\lim_{u\rightarrow 0_+} \frac{f(p+u)-f(p)}{u}$ *exists*

If such a limit exists, it is denoted by $J'_+(p)$

Chapter 2

Log convex function

2.1 Log convex function

A function J defined over a convex set Q, with range in the positive real numbers is referred to as multiplicatively-convex,AG-convex, or log-convex if the logarithm of J is a convex function, or equivalently, if the following inequality is satisfied for every pair of x_1, x_2 in Q and μ in $[0, 1]$:

$$J\left(\mu\left(x_1\right)+(1-\mu)\,x_2\right)\leq\left[\ J\left(x_1\right)\right]^{\mu}\ \left[\ J\left(x_2\right)\right]^{1-\mu} \tag{2.1}$$

A log convex function is convex because $J = \exp\left(\ln J\right)$. If J and K are convex and K is increasing, then the composition of the two functions, $K \circ J$, is also convex. However, it is not necessarily true that every convex function is log-convex. This follows directly from above inequality because, by employing the arithmetic-geometric mean inequality. We have

$$\left[\ J\left(x_1\right)\right]^{\mu}\ \left[\ J\left(x_2\right)\right]^{1-\mu}\ \leq\ \mu J\left(x_1\right)+(1-\mu)\,J\left(x_2\right) \tag{2.2}$$

for all $x_1, x_2 \in Q$ and $\mu \in [0, 1]$

Example 1. *The function $J(x) = \exp|x|^p$, for $p \geq 1$ is a log convex.*

2.2 Properties of log convex functions

(a) If $J(x)$ is log convex, then $J(px)$ is also log convex for any positive constant p.

(b) If $J(x)$ and $K(x)$ are both log convex, then $J(x)K(x)$ is also log-convex.

(c) If $J(x)$ is a log convex function & $K(x)$ is an increasing function, then $K(J(\mathrm{x}))$ is also log convex.

(d) If $J(\mathrm{x})$ is log convex & non-decreasing, then its integral is also log convex.

Theorem 2.2.1. [7]*(i)Consider a function $J : [g, h] \to [0, +\infty)$ that is both differentiable and strictly decreasing, for any x' in the interval $(g, h]$ the function $J(x') > 0$. Define $J(x') = \int_g^{x'} J(t)dt$ for all x' in the interval $(g, h]$. Then the function J is log concave.*

(ii) Consider J be a function from $[g, h]$ to $[0, +\infty)$ is twice differentiable log concave, $J(x')$ and $J'(x')$ is greater than zero for x' in the interval $(g, h]$. Define $J(x') = \int_g^{x'} J(t)dt$ with x' in the interval $(g, h]$. Then J is log concave.

(iii) Let $J : [g, h] \to [0, +\infty)$ be a twice differentiable log convex function, $J(x')$ and $J'(x')$ both are greater than zero for x' in the interval $(g, h]$, $\lim_{x' \to a+} J^2(x')/J'(x')$ is equal to zero. Define $J(x') = \int_g^{x'} J(t)dt$ with x' in the interval $(g, h]$. Then J is a log convex function.

Proof. Let

$$Z(x') := \frac{J''(x')J(x') - (J'(x'))^2}{J'(x')}, \quad x' \in (g, h].$$

Then

$$Z(x') := \frac{\left(\int_g^{x'} J(t)dt\right)'' \cdot \left(\int_g^{x'} J(t)dt\right) - J^2(x')}{J'(x')}$$
$$= \int_g^{x'} J(t)dt - \left(\frac{J^2(x')}{J'(x')}\right)$$

and

$$
\begin{aligned}
Z'(x') = &\ J(x') - \left[\frac{2J(x')\,(J'(x'))^2 - J^2(x')J''(x')}{(J'(x'))^2}\right] \\
&= J(x') \cdot \left[\frac{J(x')J''(x') - (J'(x'))^2}{(J'(x'))^2}\right] \leq 0.
\end{aligned}
$$

Then Z is decreasing. We have

$$
Z(x') \leq \lim_{x' \to g+} Z(x') = -\lim_{x' \to g+} \frac{J^2(x')}{J'(x')} \leq 0,
$$

and

$$
J''(x')J(x') - (J'(x'))^2 \leq 0
$$

This complete the proof.

Theorem 2.2.2. [7]*Consider $J : [g, h] \subseteq \mathbb{R} \to \mathbb{R}_+$ be a log convex(concave), p in the interval (g, h), $J'_-(p)$ and $J'_+(p)$ both are not equal to zero. Then*

$$
\int_g^h J(t)dt \geq (\leq) \left(\frac{(J(p))^2}{J'_-(p)}\right)\left[1 - \exp\left(-(p-g)\frac{J'_-(p)}{J(p)}\right)\right] + \frac{(J(p))^2}{J'_+(p)}\left[\exp\left((h-p)\frac{J'_+(p)}{J(p)}\right) - 1\right] \tag{2.3}
$$

holds $\iff$ J is a qe^{rx}-type function, $q > 0$, and r in $\mathbb{R}$.

Proof. Since $J'_+(p)$ is not equal to zero, we can choose v in the interval (p, h) such that $J(v)$ is not equal to $J(p)$. For any (t) belongs to (v, h) and $\mu = \frac{v-p}{t-p}$, $v = (1-\mu)p + \mu t$ hold. Then

$$
\begin{aligned}
J(v) = J((1-\mu)p + \mu t) \leq (\geq)(J(p))^{1-\mu}(J(t))^{\mu}, \qquad (2.4) \\
J(t)(\geq) \leq (J(v))^{\frac{t-p}{v-p}}(J(p))^{\frac{-(t-v)}{(v-p)}}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\int_v^h J(t)dt(\geq) \leq \int_v^h (J(v))^{\frac{t-p}{v-p}} \cdot (J(p))^{\frac{-(t-v)}{(v-p)}} dt \\
= \frac{(J(p))^{\frac{v}{v-p}}}{(J(v))^{\frac{p}{v-p}}} \int_v^h \left(\frac{J(v)}{J(p)}\right)^{\frac{t}{v-p}} dt \\
= \frac{(v-p)(J(p))^{\frac{v}{v-p}}}{(J(v))^{\frac{p}{v-p}}(\ln J(v) - \ln J(p))} \left[\left(\frac{J(v)}{J(p)}\right)^{\frac{h}{v-p}} - \left(\frac{J(v)}{J(p)}\right)^{\frac{v}{v-p}}\right] \\
= \frac{(v-p)J(v)\left[\left(\frac{J(v)}{J(p)}\right)^{\frac{(h-v)}{(v-p)}} - 1\right]}{\ln J(v) - \ln J(p)}.
\end{aligned}$$

Let $v \to p+$, we have

$$\begin{aligned}
\int_p^h J(t)dt \geq (\leq) \lim_{v\to p+} \frac{J(v)[\exp\{(h-v)(\ln J(v) - \ln J(p))/(v-p)\} - 1]}{(\ln J(v) - \ln J(p))/(v-p)} \\
= J(p) \lim_{v\to p+} \frac{\left[\exp\left\{(h-p)\frac{\ln J(v)-\ln J(p)}{J(v)-J(p)} \cdot \frac{J(v)-J(p)}{v-p}\right\} - 1\right]}{\frac{\ln J(v)-\ln J(p)}{J(v)-J(p)} \cdot \frac{J(v)-J(p)}{v-p}} \\
= \frac{(J(p))^2}{J'_+(p)} \left\{\exp\left((h-p)\frac{J'_+(p)}{J(p)}\right) - 1\right\}.
\end{aligned}$$

Similarly,

$$\int_g^p J(t)dt \geq (\leq) \frac{(J(p))^2}{J'_-(p)} \left(1 - \exp\left\{-(p-g)\frac{J'(p)}{J(p)}\right\}\right)$$

Hence

$$\begin{aligned}
\int_g^h J(t)dt = \int_g^p J(t)dt + \int_p^h J(t)dt \\
\geq (\leq) \frac{(J(p))^2}{J'_-(p)} \left[1 - \exp\left(-(p-g)\frac{J'(p)}{J(p)}\right)\right] + \frac{(J(p))^2}{J'_+(p)} \left[\exp\left((h-p)\frac{J'_+(p)}{J(p)}\right) - 1\right].
\end{aligned}$$

the above equality holds because of (2.4), if and only if

$$J(\mu(x_1) + (1-\mu)x_2) = [J(x_1)]^\mu [J(x_2)]^{1-\mu}$$

holds for any $x_1, x_2 \in [g,h], \mu \in (0,1)$. *Then* $\exists\ r, m$ *in* $\mathbb{R}$*, such that*

$$\ln J(x_1) = rx_1 + m, \quad J(x_1) = e^m (e^r)^{x_1}.$$

This complete the proof of Theorem 2.2.2.

Corollary 2.2.1. [7]*Consider* $J : [g,h] \subseteq \mathbb{R} \to \mathbb{R}_+$*be a log convex(concave) function,* p *in the interval* $[g,h]$ *and* $J'(p)$ *is not equal to zero. Then*

$$\int_g^h J(t)dt \geq (\leq) \frac{(J(p))^2}{J'(p)} \left[\exp\left((h-p)\frac{J'(p)}{J(p)} \right) - \exp\left(-(p-g)\frac{J'(p)}{J(p)} \right) \right].$$

Specifically, if $J'(g), J'(h)$ *are not equal to zero or* $J'((g+h)/2)$ *is not equal to zero, we have*

$$\begin{aligned} &\int_g^h J(t)dt \geq (\leq) \frac{(J(g))^2}{J'(g)} \left[\exp\left((h-g)\frac{J'(g)}{J(g)} \right) - 1 \right], \\ &\int_g^h J(t)dt \geq (\leq) \frac{(J(h))^2}{J'(h)} \left[1 - \exp\left(-(h-g)\frac{J'(h)}{J(h)} \right) \right] \end{aligned} \tag{2.5}$$

or

$$\int_g^h J(t)\ dt \geq (\leq) \frac{\left(J\left(\frac{g+h}{2}\right)\right)^2}{J'\left(\frac{g+h}{2}\right)} \left[\exp\left(\frac{h-g}{2} \cdot \frac{J'\left(\frac{g+h}{2}\right)}{J\left(\frac{g+h}{2}\right)} \right) - \exp\left(-\frac{h-g}{2} \cdot \frac{J'\left(\frac{g+h}{2}\right)}{J\left(\frac{g+h}{2}\right)} \right) \right],$$

holds $\iff$ J *is a* qe^{rx}*-type function,* q *is greater than zero,*
and r *in* $\mathbb{R}$.

2.3 Hermite-Hadamard inequality for log-convex

We know that Hermite-Hadamard inequality

$$J\left(\frac{g+h}{2}\right) \leq \frac{1}{h-g} \int_g^h J(x)\, dx \leq \frac{J(g) + J(h)}{2} \tag{2.6}$$

Apply the inequality mentioned above for log convex functions $J:$ [g,h] $\rightarrow$ (0,∞) , we

have

$$\ln\left[J\left(\frac{g+h}{2}\right)\right] \leq \frac{1}{h-g}\int_g^h \ln J\left(x\right)dx \leq \frac{\ln J\left(g\right) \;+\; \ln J\left(h\right)}{2} \tag{2.7}$$

from above inequality we get

$$J\left(\frac{g+h}{2}\right) \leq \exp\left[\frac{1}{h-g}\int_g^h \ln J\left(x\right)dx\right] \leq \sqrt{J\left(g\right)\; J\left(h\right)} \tag{2.8}$$

This is Hermite Hadamard inequality for log convex functions.

Theorem 2.3.1. *(Dragomir-Mond, 1998[8]).Consider $J : Q \rightarrow [0, \infty)$ be a log convex mapping on Q and g,h belongs to Q with g is less than h. Then one has inequality:*

$$J\left(\frac{g+h}{2}\right) \leq \frac{1}{(h-g)}\int_g^h \sqrt{J\left(x\right)\; J\left(g+h-x\right)dx} \leq \sqrt{J\left(g\right)\; J\left(h\right)} \tag{2.9}$$

Theorem 2.3.2. *(Dragomir-Mond, 1998[8]).Consider $J : Q \rightarrow (0, \infty)$ be a log convex mapping and g,h belongs to Q with g is less than h.Then*

$$J\left(\tfrac{g+h}{2}\right) \leq \exp\left[\tfrac{1}{h-g}\int_g^h \ln J\left(x\right)dx\right]$$

$$\leq \tfrac{1}{h-g}\int_g^h \sqrt{J\left(x\right)J\left(g+h-x\right)dx} \leq \sqrt{J\left(g\right)\; J\left(h\right)}$$

$$\leq \tfrac{1}{h-g}\int_g^h J\left(x\right)dx \leq L\left(J\left(g\right)\;,\; J\left(h\right)\right)$$

Where for strictly positive real number q,r, the L(q,r) is logrithmic mean, i.e., L(q,r) = $\frac{q-r}{\ln q - \ln r}$ if $q \neq r$ and L(q,q) = q. The last above inequality was obtained in a different context in [9].

Theorem 2.3.3. [10].*Consider $J : Q \rightarrow [0, \infty)$ be a log convex defined on set Q, g,h $\in Q$ under the condition g<h. Then*

$$J\left(\frac{g+h}{2}\right) \leq \left(\frac{1}{h-g}\int_g^h \sqrt{J\left(x\right)}dx\right)^2 \leq \frac{1}{h-g}\int_g^h J\left(x\right)dx \tag{2.10}$$

Theorem 2.3.4. *(Dragomir, 2001 [11]). Consider the function $J: Q \to (0,\infty)$ be a diffeentiable log convex on the interior of Q (Q°) and g,h belongs to the Q° with g is less than h. Then*

$$\frac{\frac{1}{h-g}\int_g^h J(x)dx}{J\left(\frac{g+h}{2}\right)} \geq L\left(\exp\left[\frac{J'\left(\frac{g+h}{2}\right)}{J\left(\frac{g+h}{2}\right)}\left(\frac{h-g}{2}\right)\right], \exp\left[-\frac{J'\left(\frac{g+h}{2}\right)}{J\left(\frac{g+h}{2}\right)}\left(\frac{h-g}{2}\right)\right]\right) \geq 1$$

Theorem 2.3.5. *(Dragomir, 2001[11]).Consider $J: Q \to (0,\infty)$ be a diffentiable log convex function on the interior of Q (Q°) and g,h in Q° with g<h.Then*

$$\frac{\frac{J(g)\ +\ J(h)}{2}}{\frac{1}{h-g}\int_g^h J(x)dx} \geq 1+\ln\left[\frac{\int_g^h J(x)dx}{\int_g^h J(x)\exp\left(\frac{J'(x)}{J(x)}\left(\frac{g+h}{2}-x\right)\right)dx}\right]$$

$$\geq 1+\ln\left[\frac{\frac{1}{h-g}\int_g^h J(x)dx}{J\left(\frac{g+h}{2}\right)}\right] \geq 1$$

The Hadamard refinement of the Hermite-Hadamard inequality can be stated as follows:

Lemma 2.3.1. *(Dragomir, 1994 [12]).Consider $J: [g,h] \to R$ be a convex and an arbitrary division of $[g,h]$ $g = x_{\circ} < x_1 < \ldots\ldots < x_n = h$ with $n \geq 2$. Then*

$$J\left(\frac{g+h}{2}\right) \leq \frac{1}{h-g}\sum J\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i) \tag{2.11}$$

$$\leq \frac{1}{h-g}\int_g^h J(x)\,dx \leq \frac{1}{h-g}\sum\left(\frac{J(x_i)+J(x_{i+1})}{2}\right)(x_{i+1}-x_i) \leq \frac{J(g)+J(h)}{2} \tag{2.12}$$

where i from 0 to n-1.

Theorem 2.3.6. *(Dragomir, 2015 [13]).Consider $J: [g,h] \to R$ be a log convex function on $[g,h]$ and an arbitrary division of $[g,h]$ $g = x_{\circ} < x_1 < \ldots\ldots < x_n = h$ with $n \geq 1$ and i from 1 to n-1. Then*

$$J\left(\frac{g+h}{2}\right) \leq \prod\left[J\left(\frac{x_i+x_{i+1}}{2}\right)\right]^{\frac{x_{i+1}-x_i}{h-g}}$$

$$\leq \exp\left(\frac{1}{h-g}\int_g^h \ln J(x)\ dx\right)$$

$$\leq \prod\left[\sqrt{J(x_i)\,J(x_{i+1})\,dx}\right]^{\frac{x_{i+1}-x_i}{h-g}} \leq (J(g)\ J(h))^{\frac{1}{2}}$$

Proof: If we write the first inequality in above lemma for the given function $J = \ln J$

then we obtain

$$\ln J\left(\tfrac{g+h}{2}\right) \leq \tfrac{1}{h-g}\sum \ln J\left(\tfrac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$$
$$\leq \tfrac{1}{h-g}\int_g^h \ln J(x)\,dx \quad \leq \tfrac{1}{h-g}\sum\left(\tfrac{\ln J(x_i)+\ln J(x_{i+1})}{2}\right)(x_{i+1}-x_i)$$
$$\leq \tfrac{\ln J(g)+\ln J(h)}{2}.$$

Where i from 1 to n-1. This complete the proof.

Chapter 3

Exponential convex function

3.1 Exponential Convex Functions

The concept of exponentially convex functions was first developed by S. N. Bernstein in [14], approximately eighty years ago. Subsequently, D. V. Widder introduced these functions as a subset of convex functions on a specified interval (g, h) in his publication [15]. Conveniently, Widder's [16] book provides an excellent account of the theory that ultimately leads to the integral representation of exponentially convex functions.

Definition 3.1.1. *A positive function J is called exponentially convex if*

$$\exp J\ \left(\mu\left(x_1\right)+(1-\mu)\left(x_2\right)\right) \leq\ (\mu) \exp J\left(x_1\right)\ +\ (1-\mu) \exp J\left(x_2\right)$$

for all x_1, x_2 *in* Q *and* μ *belongs to* $[0,1]$,

Example 1. $\ln(x)$ *is exponential-convex for* $x >$ *0,*

3.2 Properties of exponential convex functions

(a)If $J(x)$ is exponential-convex, then $\exp(J(x))$ is always greater than zero.
(b) If $J(x)$ is exponential-convex, then $\exp(J(px))$ is also exponential-convex for any positive constant p.

(c) If $J(x)$ and $K(x)$ are exponential-convex, then $\exp(J(x)+K(x))$ is also exponential-convex.
(d) If $J(x)$ is exponential-convex and $K(x)$ is an increasing function, then $\exp(K(J(x)))$ is also exponential-convex.

3.3 Jensen Inequality for exponential convex functions

Let us recall Jensen inequality, let J be a convex function over the interval Q. If $x_1, x_2, x_3, ..., x_n \in Q$ and $\mu_1, \mu_2, \mu_3, ..., \mu_n$ are non negative real numbers such that $\mu_1 + \mu_2 + \mu_3 + ... + \mu_n = 1$, then

$$J\left(\sum_{i=1}^{n} \mu_i x_i\right) \leq \sum_{i=1}^{n} \mu_i J(x_i)$$

applying the above inequality to exponential convex functions we obtain

$$\exp J\left(\sum_{i=1}^{n} \mu_i x_i\right) \leq \sum_{i=1}^{n} \mu_i \exp J(x_i)$$

3.4 Hermite Hadamard inequality adapted for the case of exponential convex functions

Recall Hermite Hadamard inequality

$$J\left(\frac{g+h}{2}\right) \leq \frac{1}{h-g}\int_g^h J(x)\,dx \leq \frac{J(g) + J(h)}{2}$$

applying the above inequality to exponential convex functions we obtain

$$\exp J\left(\frac{g+h}{2}\right) \leq \frac{1}{h-g}\int_g^h \exp J(x)dx \leq \frac{\exp J(g) + \exp J(h)}{2}$$

Theorem 3.4.1. *(Dragomir, 2015 [13]).Consider the function $J : [g, h] \to (0,\infty)$ be a log convex. Then*

$$1 \leq \exp\left(\frac{1}{8}\left(\frac{J_+\ \left(\frac{g+h}{2}\right)-\ J_-\ \left(\frac{g+h}{2}\right)}{J\left(\frac{g+h}{2}\right)}\right)(h-g)\right)$$
$$\leq \frac{\sqrt{J(g)\ .\ J(h)}}{\exp\left(\frac{1}{h-g}\int_g^h \ln J(x)dx\right)}$$
$$\leq \exp\left(\frac{1}{8}\left(\frac{J_-\ (h)}{J(h)} - \frac{J+\ (g)}{J(g)}\right)(h-g)\right)$$

and

$$1 \leq \exp\left(\frac{1}{8}\left(\frac{J_+\ \left(\frac{g+h}{2}\right)-\ J_-\ \left(\frac{g+h}{2}\right)}{J\left(\frac{g+h}{2}\right)}\right)(h-g)\right)$$
$$\leq \frac{\exp\left(\frac{1}{h-g}\int_g^h \ln\ \ J\ \ (x)dx\right)}{J\left(\frac{g+h}{2}\right)}$$
$$\leq \exp\left(\frac{1}{8}\left(\frac{J_-\ (h)}{J(h)} - \frac{J+\ (g)}{J(g)}\right)(h-g)\right)$$

Theorem 3.4.2. *(Dragomir, 2015 [13]). Consider $J : [g,h] \to$ (0,∞) be a log convex function on $[g,h]$ and an arbitrary division of $[g,h]$, $g = x_\circ < x_1 <$ ...... $< x_n = h$ with n $\geq$ 1. Then*

$$\exp\left(\frac{1}{h-g}\int_g^h \ln J\left(x\right)dx\right) \leq \frac{1}{h-g}\sum\int_{x_i}^{x_{i+1}}\sqrt{J\left(x\right).\ J\left(x_i+x_{i+1}-x\right)}\ dx$$
$$\leq \frac{1}{h-g}\int_g^h J\left(x\right)dx$$

where i from 0 to n-1

Proof: The fact that we have

$$\exp\left[\frac{1}{h-g}\int_g^h \ln J\left(x\right)dx\right] = \exp\left[\frac{1}{h-g}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\ln J\left(x\right)dx\right]$$
$$= \exp\left[\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{h-g}\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln J\left(x\right)dx\right)\right]$$

since $\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{h-g} = 1$, *then by using Jensen inequality for the exp convex function we have*

$$\exp\left[\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{h-g}\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln J\left(x\right)dx\right)\right]$$
$$\leq \sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{h-g}\exp\left(\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln J\left(x\right)dx\right)$$

Using the first inequality (of Theorem 3.4.2) on each the intervals $[x_i, x_{i+1}]$, we have

$$\exp\left[\frac{1}{(x_{i+1}-x_i)}\int_{x_i}^{x_{i+1}}\ln J\left(x\right)dx\right]$$
$$\leq \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\sqrt{J\left(x\right).\ J\left(x_i+x_{i+1}-x\right)}\ dx$$
$$\leq \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}J\left(x\right)dx$$

for every i belongs to $\{0, 1, \ldots.., n-1\}$

When the inequality mentioned above is multiplied by $\frac{x_{i+1}-x_i}{h-g}$ *and we take the summation from i equals 0 to n-1, we can obtain* $\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{h-g}\exp\left[\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\ln J\left(x\right)dx\right]$

$\leq \frac{1}{h-g}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\sqrt{J\ (x)\,J(x_i+x_{i+1}-x)dx}$

$\leq \frac{1}{h-g}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} J\,(x)\,dx$

$= \frac{1}{h-g}\int_g^h J\,(x)\,dx$

making use of above first 3 inequalities we get the desire result.

Corollary 3.4.1. *Let* $J : [g,h] \to (0,\infty)$ *be a log convex function on the interval* $[g,h]$, x_1 *and* x_2 *in* $[g,h]$, *then*

$$\exp\left[\frac{1}{h-g}\int_g^h \ln J\,(x_1)\,dx\right]$$
$$\leq \frac{1}{h-g}\left[\int_g^{x_2}\sqrt{J\,(x_1)\,.\ J\,(g+x_2-x_1)}dx+\int_{x_2}^h\sqrt{J\,(x_1)\,.\ J\,(h+x_2-x_1)}dx\right]$$
$$\leq \frac{1}{h-g}\int_g^h J\,(x_1)\,dx$$

Theorem 3.4.3. *[13]Let* $J : [g,h] \to (0,\infty)$ *be a log convex function within the specified interval* $[g,h]$. *Then for every* p *greater than zero we obtain the following inequality*

$$J\left(\frac{g+h}{2}\right) \leq \exp\left[\frac{1}{h-g}\int_g^h \ln J\,(x)\,dx\right]$$
$$\leq \left(\frac{1}{h-g}\int_g^h J^p\,(x)\,J^p\,(g+h-x)\,dx\right)^{\frac{1}{2p}}$$
$$\leq \left(\frac{1}{h-g}\int_g^h J^{2p}\,(x)\,dx\right)^{\frac{1}{2p}}$$
$$\leq \begin{cases} [L_{2p-1}\,(J\,(g)\ ,\ J\,(h))]^{1-\frac{1}{2p}}\,[L\ (J\,(g)\ ,\ J\,(h))]^{\frac{1}{2p}}, & p\neq\frac{1}{2}; \\ L\ (J\,(g)\ ,\ J\,(h))\,, & p=\frac{1}{2} \end{cases}$$

if $p \in \left(0,\frac{1}{2}\right)$, *then we have*

$$J\ \left(\frac{g+h}{2}\right) \leq \exp\left[\frac{1}{h-g}\int_g^h \ln\ J\ (x)\,dx\right]$$
$$\leq \left(\frac{1}{h-g}\int_g^h\ J^{\ p}\,(x)\ \ J^{\ p}\,(g+h-x)\,dx\right)^{\frac{1}{2p}}$$
$$\leq \left(\frac{1}{h-g}\int_g^h J^{2p}\,(x)\,dx\right)^{\frac{1}{2p}}$$
$$\leq \frac{1}{h-g}\int_g^h J\,(x)\,dx$$

Proposition 1. *[17]Let the function* $J : Q \to \mathbb{R}$. *The both statements are the equivalent.*

(i) The function J *shows exponential convexity on* Q

(ii) J *is a continuous and*

$$\sum \xi_i\xi_j J\left(\frac{x_i+x_j}{2}\right) \geq 0 \tag{3.1}$$

for any ξ_i belongs to $\mathbb{R}$ and any $x_i \in Q, 1 \leq i \leq n$ $1 \leq j \leq n$.
For exponentially convex on $\mathbb{R}$ we will just say exponentially convex function.

Remark 1. *From (3.1) we have the following statementss:*
(i) if the function J is exponentially convex on the interval Q then $J(x)$ is greater than and equal to zero, for all x in Q; for every $p \geq 0$, pJ is also exponentially convex;
(ii) if J_1 and J_2 both are exponentially convex on the interval Q, then $J_1 + J_2$ is a exponentially convex on the interval Q
(iii) if the function J is exponentially cnvex. Then for every d, t in $\mathbb{R}, x \mapsto J(dx)$ and $x \mapsto J(x-t)$ both the functions are exponentially convex.

Corollary 3.4.2. *If the function J is convex exponentially on the interval Q. Then the matrix*

$$\left[J\left(\frac{x_i + x_j}{2}\right)\right]$$

is a positively semi-definite, and

$$\det\left[J\left(\frac{x_i + x_j}{2}\right)\right] \geq 0$$

for any n in $\mathbb{N}$, x_i belongs to $Q, i, j = 1, \ldots, n$.

Corollary 3.4.3. *If the function $J : Q \to R_+$ is exponentially convex, then J is a log convex i.e ln J is convex.*
Proof.For n=2 and from(3.1), we obtain

$$\xi_1^2 J(x_1) + 2\xi_1\xi_2 J(\tfrac{x_1+x_2}{2}) + \xi_2^2 J(x_2) \geq 0$$

for all ξ_1, ξ_2 belongs to set of real numbers and all x_1, x_2 in Q, .hence

$$J^2(\frac{x_1 + x_2}{2}) \leq J(x_1)J(x_2) \quad for\ all\ x_1, x_2 \in Q \tag{3.2}$$

since the function J is continuous. We have

$$J\left(\mu\left(x_1\right) + \left(1-\mu\right)\left(x_2\right)\right) \leq \left[J\left(x_1\right)\right]^{\mu}\left[J\left(x_2\right)\right]^{(1-\mu)} \tag{3.3}$$

for every x_1, x_2 *in* Q *and* μ *belongs to* $[0,1]$.

Proposition 2. [17]*Suppose that the function* J *is nonnegative continuous defined on the interval* Q. *s.t*

$$J^2(\frac{x_1+x_2}{2}) \leq J(x_1)J(x_2), \; x_1, x_2 \in Q \tag{3.4}$$

if $J(x'_\circ)$ *is zero for some* $x'_\circ$ *in* Q,*then* $J=0$ *on the interval* Q

proof.For every x_2 *in* Q. *Then from* (3.4)

$$J\left(\frac{1}{2^n}x'_\circ + \left(1-\frac{1}{2^n}\right)x_2\right) \leq J\left(x'_\circ\right)^{\frac{1}{2^n}} J\left(x_2\right)^{1-\frac{1}{2n}} \tag{3.5}$$

So that $J\left(\frac{1}{2^n}x'_\circ + \left(1-\frac{1}{2^n}\right)x_2\right)$ *is zero and* $J\left(x_2\right) = \lim_n \left(\frac{1}{2^n}x'_\circ + \left(1-\frac{1}{2^n}\right)x_2\right) = 0$

Remark 2. *From the previous proposition, we have that if an exponentially convex function at few points is zero, then the function is also zero on the entire domain.*

Remark 3. *A function* $J : Q \to (0,\infty)$ *is called log convex in J-sense if it satisfies (3.4)*

Theorem 3.4.4. [17]*Let the function* $J : Q \to (0,\infty)$, *is log convex and derivable. Consider* $K : Q \times Q \to (0,\infty)$ *is defined by*

$$K\left(x', y'\right) = \left\{ \begin{array}{ll} \left(\frac{J(x')}{J(y')}\right)^{\frac{1}{x'-y'}}, & x' \neq y' \\ \exp\left(\frac{J'(x')}{J(y')}\right), & x' = y' \end{array} \right\} \tag{3.6}$$

if $(x'_1 \leq x_2{\prime})$ *and* $(y'_1 \leq y'_2)$. *then*

$$K(x'_1, y'_1) \leq K(x'_2, y'_2) \tag{3.7}$$

Proof.Given that the function $\ln J$ *is a convex, we hold (see[18],p. 2)*

$$\frac{\ln J(x'_1) - \ln J(y'_1)}{x'_1 - y'_1} \leq \frac{\ln J(x'_2) - \ln J(y'_2)}{x'_2 - y'_2}, \tag{3.8}$$

for $x'_1 \leq x'_2$ *and* $y'_1 \leq y'_2$; x'_1 *is not equal to* y'_1 *and* x'_2 *is also not equal to* y'_2 *concluding* $K(x'_1, y'_1) \leq K(x'_2, y'_2)$.

if $x'_1 = y'_1 \leq x'_2$ *we apply the* $\lim_{y'_1 - x'_1}$ *to (3.8) to conclude*

$$K(x_1', y_1') \leq K(x_2', y_2')$$

Theorem 3.4.5. *The function $J : Q \to R$ is an exponentially convex on the interval Q if and only if*

$$J(x) = \int_{-\infty}^{\infty} \exp(tx) d\sigma(t), x \in Q \tag{3.9}$$

for few non-decreasing function σ $from$ R to R

proof.See [19], p. 211. A first application of theorem 3.4.5 is that exponential convexity is closed under multiplication.

Corollary 3.4.4. *If J_1, J_2 are exponentially convex function o Q. Then $J_1 J_2$ is also exponentially convex on the interval Q.*

proof.For any $n \in N$ and all $\xi_i \in R, x_{i \in Q},\ i = 1, \ldots.., n$ we have

$$\sum \xi_i \xi_j J_1 \left(\tfrac{x_i + x_j}{2}\right) J_2 \left(\tfrac{x_i + x_j}{2}\right)$$
$$= \int_{-\infty}^{\infty} \sum \xi_i \exp \tfrac{t x_i}{2} \xi_j \exp \tfrac{t x_j}{2} J_1 \left(\tfrac{x_i + x_j}{2}\right) \sigma_2(dt) \geq 0$$

where i,j=1,...,n and we use the integral representation (3.9) for the function J_2.

Example 1. *For every $\alpha > 0$, the function $J : (0, \infty) \to \mathbb{R}$ defined by*

$$J(x) = x^{-\alpha}$$

is exponentially convex on the interval $(0, \infty)$, since $x^{-\alpha} = \int_0^{\infty} \exp -xt \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt$ (see[20] , p. 210).

Example 2. *For every $\alpha > 0$, the function*

$$J(x) = \exp -\alpha\sqrt{x}$$

is exponentially convex on $(0, \infty)$, since $\exp -\alpha\sqrt{x} = \int_0^{\infty} \exp -xt \exp -\alpha^2/4t \frac{\alpha}{2\sqrt{\pi t^3}} dt, x > 0$ (see[20], p. 214).

Further analytical properties and more examples of exponentially convex functions are contained in the following theorem from [21].

Theorem 3.4.6. *[17]If $J(s), s \in \mathbb{R}^d$, is exponentially convex, then $K(t) = J(\mathrm{i}t), t \in \mathbb{R}^d$ and the whole function, is a positively definite. Conversely, Assuming $K(t), t \in \mathbb{R}^d$, is whole positively definite, the function $J(s) = K(-\mathrm{i}s), s \in \mathbb{R}^d$ is an exponentially convex.*

Using the previous theorem, the following examples were constructed in [21].

Example 3. *A characteristic function of the uniform distribution on $[0, 1]$ is $K(t) = \frac{\exp it-1}{it}$. Applying Theorem 3.4.6, we get exponentially convex function*

$$J(x) = \frac{\exp x - 1}{x}.$$

Example 4. *A characteristic function of the normal distribution $N(\mu, \sigma^2)\,; \mu$ in $\mathbb{R}, \sigma > 0$, where is $K(t) = \exp it\mu - \frac{1}{2}\sigma^2 t^2$. Applying Theorem 3.4.6, we get the exponentially convex function*

$$J(x) = \exp -\mu x + \sigma^2 x^2.$$

We now proceed with other properties of exponential convexity.

Theorem 3.4.7. *Assume that $J : Q \to \mathbb{R}$ is an exponentially convex function on the interval Q. Then*

(i) For any $m \in \mathbb{N}$ we have

$$J^{(m)}(x) = \int_{-\infty}^{\infty} t^m \exp(tx) d\sigma(t)$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is some non-decreasing function.
(ii) For any $m \in \mathbb{N}$ the function $x \mapsto J^{(2m)}(x)$ is exponentially convex.
(iii) For any $x \in Q$ and all $m \in \mathbb{N}$ we have

$$\left(J^{(2m+2)}(x)\right)^2 \leq J^{(2m)}(x) J^{(2m+4)}(x) \tag{3.10}$$

(iv) for any n in $\mathbb{N}$ *and for all* x_i *belongs to* Q *and* i *from* 1 *to* n*, the matrix*

$$\left[J^{(i+j)}\left(\frac{x_i+x_j}{2}\right)\right] \tag{3.11}$$

is positive-semidefinite. where i, j=0,...,n

Proof. (i) and (ii)-part follow by using integral representation (3.9). Proofs of (iii) and (iv) can be found in [22] and [23] respectively.

Remark 4. *From Theorem 3.4.7 it follows that if* J *is a non-constant exponentially convex function then* $J^{(2m)}$ *is also a non-constant exponentially convex function on* Q*, for any* $m \in \mathbb{N}$*. Now, it is obvious that no polynomial can be exponentially convex.*

Theorem 3.4.8. *Let* $J : Q \to \mathbb{R}$ *be an exponentially convex function.*
(i) If $0 \in Q, x_i \in Q$*,* i *from* 1 *to* n*, and* $\sum x_i$ *belongs to* Q*, then for* $m \geq 2$*, where* m *is any natural number.*

$$J^{m-1}(0)J\left(\sum_{i=1}^{m} x_i\right) \geq \prod J\left(x_i\right),$$

for all $i = 1, \ldots m, x_i \geq 0$ *or* $x_i \leq 0$
(ii) If $0 \in Q$*, and* $x_1, x_2 \in Q$ *such that* $x_1 \cdot x_2 < 0, x_1 + x_2 \in Q$*, then*

$$J(0)J(x_1+x_2) \leq J(x_1)J(x_2).$$

(iii) If $x \in Q$*, and* i, j *are odd natural numbers, then*

$$J(x)J^{(i+j)}(x) \geq J^{(i)}(x)J^{(j)}(x).$$

Proof. (i)

$$J^{m-1}(0)J\left(\sum x_i\right) = \left(\int_{-\infty}^{\infty}\sigma(dt)\right)^{m-1}\int_{-\infty}^{\infty}\prod \exp(tx_i)\sigma(dt)$$

We now apply Chebyshev's inequality (see [18], p. 197) for $p(t) = 1, f_i(t) = \exp tx_i, i = 1, \dots, m.$

(ii) We apply Chebyshev's inequality for $p(t) = 1, f(t) = \exp(tx), f(t) = \exp ty.$

(iii)

$$J(x)J^{(i+j)}(x) = \int_{-\infty}^{\infty} \exp tx\sigma(dt) \int_{-\infty}^{\infty} t^i t^j \exp(tx)\sigma(dt)$$

We now apply Chebyshev's inequality for $p(t) = \exp tx, f(t) = t^i, g(t) = t^j.$

Chapter 4

Conclusions and Recommendations

4.1 Conclusions

In this chapter we are going to write the summary and conclusion of our work presented in this thesis. This thesis mainly focuses to describe and unify three concepts of mathematics namely, convex function, log convex function and exponential convexity. Motivated by the ideas of convex functions and its applications in different fields of mathematics and other Sciences. Furthermore, keeping in mind that convex functions has a pivotal role to solve variety of problems of engineering, economics and weather sciences and so on. We begin with Chapter1, define the convex set, convex function and Hermite Hadamard inequality. In Chapter2, we review the results of Hermite Hadamard inequality for log convex function(see,[4]). Moreover by studying the techniques of refinement of inequalities in [4] for finding different result for log convex function. In Chapter3, the concept of exponential convexity is introduced. Additionally, the chapter examines Jensen's inequality adapted for exponential convex functions and also extends the Hermite-Hadamard inequality to this particular class of functions. In addition, exponential convexity also plays a key role in regularization, loss function design, and model fitting, providing support for the successful application of machine learning.

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