Generalized convex functions with some integral properties

Naila Mehreen

Regn.# NUST201690369PSNS7116S

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Supervised by Dr. Matloob Anwar

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan

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Examination Committee:

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	Name	Dr. Sabir Hussain	
	Designation	Associate Professor	
	Official Address	University of Engineering & Technology, Lahore	

b)	**External Examiner:**		Signature
	Name	Dr. Khuram Ali Khan	
	Designation	Assistant Professor	
	Official Address	Department of Mathematics University of Sargodha, Sargodha.	

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	Designation	Principal	
	Official Address	NUST-SNS	

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Dedicated to my parents

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Abstract

Convex function, its generalizations and inequalities involving convex functions have many applications in various fields of science. Aim of this dissertation is to study the several generalized convex functions in one dimension as well as in two dimensions and the inequalities via classical and several generalized fractional integrals with some applications. In this thesis, we established several inequalities via conformable and new conformable fractional integrals for p-convex functions. Inequalities involving Katugampola fractional integrals are also proved for s-convex functions in second sense and m-convex functions and some application to special mean are also given. Some mean value theorems are given for p-convex functions and s-convex functions in first sense. Classical integral inequalities are obtained for some new class of convex functions called exponentially p-convex functions and exponentially s-convex functions in second sense with some applications. Riemann−Liouville fractional integrals inequalities are proved for (s, p)-convex functions and some classical integral inequalities are obtained for co-ordinated (s, p)-convex functions. Furthermore, we also obtained Hermite-Hadamard and Fejér type inequalities for co-ordinated harmonically convex functions in Riemann−Liouville fractional integrals and Katugampola fractional integrals.

Contents

List of Publications from the thesis

1. N. Mehreen, M. Anwar, Integral inequalities for some convex functions via generalized fractional integrals, *J. Inequal. Appl.*, **2018**:208 2018.
2. N. Mehreen, M. Anwar, Hermite-Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications, *J. Inequal. Appl.*, **2019**:92 2019.
3. N. Mehreen, M. Anwar, Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for p-convex functions via new fractional conformable integral operators, *J. Math. Computer Sci.*, **19**:230–240, 2019.
4. N. Mehreen, M. Anwar, Hermite-Hadamard type inequalities for co-ordinated harmonically convex functions via Katugampola fractional integrals. *American Journal Mathematics and Sciences*, **9(1)**, 2020.
5. N. Mehreen, M. Anwar, Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for p-convex functions via conformable fractional integrals , *J. Inequal. Appl.*, **2020**:107, 2020.

Chapter 1

Introduction

This chapter includes the background knowledge and significance of the main constitute of this thesis. It reveals that the core idea of this thesis is based on the useful areas of research that have many applications. It also includes some basic definitions and results of previous research.

1.1 Convex functions and some generalizations

In optimization theory, convexity is one of the most common and useful hypothesis. Convexity usually worked as a global sense to proposition, some how it is introduced to give local validity. Furthermore, in classical Fermat theorem or in Kuhn-Tucker conditions for nonlinear programming, convexity is used to obtain sufficient conditions. Convexity, in microeconomics, also plays key role in duality theory and in equilibrium theory. For further historical background and applications see [12, 36, 49]. During the past few decades, convexity of a set and convexity of functions have been the major topic of the research studies. Hölder [39], Jensen [51] and Minkowski [61, 62] are the initial contributors of convex analysis. Convex functions can be minimise so are used for solving convex optimization problems that has many applications like circuit design and controller design etc. Convex functions also have many important applications in engineering. Although, convexity in this field does not appear in its natural form. But it appears in its extended form. Many generalizations of convex functions have been established.

Let $\aleph \subseteq \mathbb{R}^n$. Then $\aleph$ is said to be convex set, if

$$rc_1 + (1-r)c_2 \in \aleph,$$

for all $c_1, c_2 \in \aleph$ and $0 < r < 1$.

Every real interval is a convex set. We now define a convex function on real interval.

A function $\Psi : \aleph \to \mathbb{R}$, where $\aleph$ is an interval of real numbers, is called convex, if

$$\Psi(rc_1 + (1-r)c_2) \leq r\Psi(c_1) + (1-r)\Psi(c_2), \tag{1.1}$$

for all $c_1, c_2 \in \aleph$ and $r \in (0,1)$. Function Ψ is called concave if $-\Psi$ is convex.

In above definition Ψ is said to be strictly convex function if the inequality (1.1) is strict for every two nonequal points of $\aleph$. Convex functions are continuous in interior of its domain. For more detail and properties of convex functions we refer [78].

Now we define some generalized convex functions.

In 1961, Orlicz [71] and, in 1994, Hudzik and Maligranda [40] gave the following generalizations of a convex function.

Definition 1.1.1 ([71])**.** *Let $s \in (0,1]$. A function $\Psi : [0,\infty) \to \mathbb{R}$, is called s-convex in first sense or $\Psi \in K_s^1$, if*

$$\Psi(rc_1 + (1-r)c_2) \leq r^s\Psi(c_1) + (1-r)^s\Psi(c_2), \tag{1.2}$$

for all $c_1, c_2 \in [0,\infty)$ and $r \in [0,1]$ with $r^s + (1-r)^s = 1$.

Definition 1.1.2 ([40])**.** *Let $s \in (0,1]$. A function $\Psi : \aleph \subset [0,\infty) \to [0,\infty)$, is called s-convex in second sense or $\Psi \in K_s^2$, if*

$$\Psi(rc_1 + (1-r)c_2) \leq r^s\Psi(c_1) + (1-r)^s\Psi(c_2), \tag{1.3}$$

for all $c_1, c_2 \in \aleph$ and $r \in [0,1]$.

In 1984, Toader [87] defined m-convex functions as follows:

Definition 1.1.3 ([87])**.** *Let $m \in [0,1]$. A function $\Psi : [0,b] \to \mathbb{R}$ is called m-convex, if*

$$\Psi(rc_1 + m(1-r)c_2) \leq r\Psi(c_1) + m(1-r)\Psi(c_2), \tag{1.4}$$

for all $c_1, c_2 \in [0,b]$ and $r \in [0,1]$.

Contrary to convex functions m-convex functions may not be continuous at the interior points. For example, consider a function defined by

$$\Psi(u) = \begin{cases} \frac{1}{2}u, & \text{if } 0 \leq u < 1, \\ \frac{3}{2}u - \frac{1}{2}, & \text{if } 1 \leq u \leq 2; \end{cases}$$

is m-convex for every $m \in (0, \frac{1}{2}]$ and is discontinuous at $u = 1$, see [64].

Dragomir et. al. [29] define following simple class of convex functions.

Definition 1.1.4 ([29])**.** *A non-negative mapping $\Psi : \aleph \subseteq \mathbb{R} \to \mathbb{R}$ belongs to the class $P(\aleph)$, if it satisfies*

$$\Psi(rc_1 + (1-r)c_2) \leq \Psi(c_1) + \Psi(c_2), \tag{1.5}$$

for all $c_1, c_2 \in \aleph$ and $r \in [0,1]$.

In 2014, Iscan [43] defined harmonically convex functions as:

Definition 1.1.5 ([43])**.** *Let $\aleph \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $\Psi : \aleph \to \mathbb{R}$ is said to be harmonically convex, if*

$$\Psi\left(\frac{c_1c_2}{rc_1 + (1-r)c_2}\right) \leq r\Psi(c_2) + (1-r)\Psi(c_1), \tag{1.6}$$

for all $c_1, c_2 \in \aleph$ and $r \in [0,1]$. If the inequality in (1.6) is reversed then Ψ is said to be harmonically concave.

Consider the functions $\Psi : (0,\infty) \to \mathbb{R}$, $\Psi(u) = u$, and $\varphi : (-\infty, 0) \to \mathbb{R}$, $\varphi(u) = u$. Then Ψ is harmonically convex while φ is harmonically concave function, respectively.

In 2016, Iscan [45, 46] defined the p-convex function as follows:

Definition 1.1.6 ([45],[46]). *Let $\aleph \subset (0,\infty) = \mathbb{R}_+$ be an interval and $p \in \mathbb{R}\backslash\{0\}$. A function $\varPsi : \aleph \to \mathbb{R}$ is said to be p-convex, if*

$$\varPsi\left(\left(rc_1^p + (1-r)c_2^p\right)^{\frac{1}{p}}\right) \leq r\varPsi(c_1) + (1-r)\varPsi(c_2), \tag{1.7}$$

for all $c_1, c_2 \in \aleph$ and $r \in [0,1]$. If the inequality (1.7) *is reversed then $\varPsi$ is called p-concave function.*

A function $\varPsi : (0,\infty) \to \mathbb{R}$ defined as $\varPsi(u) = u^p$ is both p-convex and p-concave for all $p \in \mathbb{R}\backslash\{0\}$.

In 2018, Awan et al. [14] introduced a new class of convex functions called exponentially convex functions.

Definition 1.1.7 ([14]). *A function $\varPsi : \aleph \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex function, if*

$$\varPsi(rc_1 - (1-r)c_2) \leq r\frac{\varPsi(c_1)}{e^{\alpha c_1}} + (1-r)\frac{\varPsi(c_2)}{e^{\alpha c_2}}, \tag{1.8}$$

for all $c_1, c_2 \in \aleph$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality (1.8) *is reversed then $\varPsi$ is called exponentially concave function.*

Every convex function is exponentially convex but not conversely. For more generalisations of convex functions and explanations see [33, 34, 36, 66, 67, 87, 88, 89].

1.2 Convex functions on co-ordinates and some generalizations

In 2001, Dragomir [23] introduced new modification of convex functions known as co-ordinated convex functions.

Definition 1.2.1 ([23]). *Let $\varDelta = [c_1, c_2] \times [d_1, d_2]$ be a bidimensional interval $\mathbb{R}^2$ such that $c_1 < c_2$ and $d_1 < d_2$. Then a function $\varPsi : \varDelta \to \mathbb{R}$ is called convex on the co-ordinates $\varDelta$, if the partial mappings:*

$$\varPsi_y : [c_1, c_2] \to \mathbb{R},\ \varPsi_y(u) = \varPsi(u, y) \text{ and } \varPsi_x : [d_1, d_2] \to \mathbb{R},\ \varPsi_x(v) = \varPsi(x, v),$$

are convex for all $x \in [c_1, c_2]$ and $y \in [d_1, d_2]$.

In other words, a function $\Psi : \Delta \to \mathbb{R}$ is called convex on the co-ordinates Δ, if the following inequality holds:

$$\begin{aligned} &\Psi(r_1 x + (1-r_1)y, r_2 w + (1-r_2)z) \\ &\leq r_1 r_2 \Psi(x,w) + r_1(1-r_2)\Psi(x,z) + r_2(1-r_1)\Psi(y,w) \\ &\quad + (1-r_2)(1-r_1)\Psi(y,z), \end{aligned} \tag{1.9}$$

for all $(x, y), (w, z) \in \Delta$ and $r_1, r_2 \in [0, 1]$.

Since a function $\Psi : \Delta \to \mathbb{R}$ is called convex on Δ, if the following inequality holds:

$$\Psi(rx + (1-r)y, rw + (1-r)z) \leq r\Psi(x,w) + (1-r)\Psi(y,z), \tag{1.10}$$

for all $(x, y), (w, z) \in \Delta$ and $r \in [0, 1]$. Therefore, Dragomir [23] proved that every convex function $\Psi : \Delta \to \mathbb{R}$ is co-ordinated convex but not conversely.

In 2008, Alomari and Darus [4, 5] introduced the concept of s-convex functions on co-ordinates in both sense.

Definition 1.2.2 ([4, 5])**.** *Let $s \in (0, 1]$. A function $\Psi : \Delta \to \mathbb{R}$ is called s-convex in first sense (in second sense) on the co-ordinates Δ, if the partial mappings:*

$$\Psi_y : [c_1, c_2] \to \mathbb{R},\ \Psi_y(u) = \Psi(u, y) \text{ and } \Psi_x : [d_1, d_2] \to \mathbb{R},\ \Psi_x(v) = \Psi(x, v),$$

are s-convex in first sense (in second sense) for all $x \in [c_1, c_2]$ and $y \in [d_1, d_2]$.

Since a function $\Psi : \Delta \subseteq [0, \infty)^2 \to \mathbb{R}$ is called s-convex in first sense (in second sense) on Δ, if the following inequality holds:

$$\Psi(rx + (1-r)y, rw + (1-r)z) \leq r^s\Psi(x,w) + (1-r)^s\Psi(y,z), \tag{1.11}$$

for all $(x, y), (w, z) \in \Delta$ and $r \in [0, 1]$. Therefore, Alomari and Darus [4] proved that every s-convex function $\Psi : \Delta \to \mathbb{R}$ is co-ordinated s-convex but not conversely.

In 2011, Ozdemir et. al. [74] defined the concept of m-convex on co-ordinates.

Definition 1.2.3 ([74]). *Let $m \in [0,1]$. Let $\varDelta = [0,c] \times [0,d]$ be a bidimensional interval $\mathbb{R}^2$. Then a function $\varPsi : \varDelta \to \mathbb{R}$ is called m-convex on the co-ordinates $\varDelta$, if the partial mappings:*

$$\varPsi_y : [0,c] \to \mathbb{R},\ \varPsi_y(u) = \varPsi(u,y) \text{ and } \varPsi_x : [0,d] \to \mathbb{R},\ \varPsi_x(v) = \varPsi(x,v),$$

are m-convex for all $x \in [0,c]$ and $y \in [0,d]$.

A function $\varPsi : \varDelta \to \mathbb{R}$ is called m-convex on $\varDelta$, if the following inequality holds,

$$\varPsi(rx + (1-r)y, rw + m(1-r)z) \leq r\varPsi(x,w) + m(1-r)\varPsi(y,z), \tag{1.12}$$

for all $(x,y),(w,z) \in \varDelta$ and $r \in [0,1]$.

Ozdemir et. al. [74] proved that every m-convex mapping $\varPsi : \varDelta \to \mathbb{R}$ is m-convex on the co-ordinates.

In 2015, Noor et. al. [69] defined following definition for harmonically convex functions on co-ordinates.

Definition 1.2.4 ([69]). *A function $\varPsi : \varDelta = [c_1,c_2] \times [d_1,d_2] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is called co-ordinated harmonically convex on $\varDelta$ with $c_1 < c_2$ and $d_1 < d_2$, if*

$$\begin{aligned}&\varPsi\left(\frac{xz}{r_1x + (1-r_1)z}, \frac{yw}{r_2y + (1-r_2)w}\right)\\ &\leq r_1r_2\varPsi(x,y) + r_1(1-r_2)\varPsi(x,w) + (1-r_1)r_2\varPsi(z,y) + (1-r_1)(1-r_2)\varPsi(z,w),\end{aligned}$$

for all $r_1, r_2 \in [0,1]$ and $(x,y),(z,w) \in \varDelta$.

Clearly, a function $\varPsi : \varDelta = [c_1,c_2] \times [d_1,d_2] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ is called harmonically convex on the co-ordinate with $c_1 < c_2$ and $d_1 < d_2$, if the partial functions:

$$\varPsi_y : [c_1,c_2] \to \mathbb{R},\ \varPsi_y(a) = \varPsi(a,y) \text{ and } \varPsi_x : [d_1,d_2] \to \mathbb{R},\ \varPsi_x(b) = \varPsi(x,b),$$

are harmonically convex for all $x \in [c_1,c_2]$ and $y \in [d_1,d_2]$.

In 2016, Noor et. al. [68] defined pq-convex functions on co-ordinates.

Definition 1.2.5 ([68]). *Let $\Delta = [c_1, c_2] \times [d_1, d_2]$ be a bidimensional interval of $\mathbb{R}^2$ such that $c_1 < c_2$ and $d_1 < d_2$. A function $\Psi : \Delta \to \mathbb{R}$ is called two dimensional pq-convex function on Δ, if the following inequality holds:*

$$\begin{aligned}&\Psi\left((r_1x^p + (1-r_1)y^p)^{\frac{1}{p}}, (r_2w^q + (1-r_2)z^q)^{\frac{1}{q}}\right)\\&\leq r_1r_2\Psi(x,w) + r_1(1-r_2)\Psi(x,z) + r_2(1-r_1)\Psi(y,w) + (1-r_2)(1-r_1)\Psi(y,z),\end{aligned} \tag{1.13}$$

for all $(x,y),(w,z) \in \Delta$ and $r_1, r_2 \in [0,1]$.

1.3 Convex functions and inequalities

Mathematical inequalities plays important role in the development of many results. These inequalities has outstanding applications in various fields of science. According to American Mathematical Society, there are above $63,000$ references and applications of inequalities. Due to importance of inequalities many researchers plays role for establishment of new and generalized type of inequalities. Among many researchers A. M. Fink [35], Hardy [37], Bechenbach and Bellman [16] and Mitrinovic [63] are well known to study the inequalities and their applications. Most common and significant inequalities are Hölder's inequality, Minkowski's inequality, power mean inequality and Jensen's inequality. There are many integral inequalities. Some of the integral inequalities are Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, Ostrowski inequality, Gauss inequality etc.

Our interest is to study the Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities for convex functions and their generalizations. The Hermite-Hadamard inequality [38] for a convex function $\Psi : \aleph \to \mathbb{R}$ on an interval $\aleph$ is defined as:

$$\Psi\left(\frac{c_1+c_2}{2}\right) \leq \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)du \leq \frac{\Psi(c_1)+\Psi(c_2)}{2}, \tag{1.14}$$

for all $c_1, c_2 \in \aleph$ with $c_1 < c_2$. Many authors made generalizations to the inequality (1.14). For more generalized results and detail see [13, 14, 20, 21, 42, 43, 45, 54, 91].

Fejer [32] introduced the weighted generalization of (1.14) as follows:

$$\Psi\left(\frac{c_1+c_2}{2}\right)\int_{c_1}^{c_2}\phi(u)du \le \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\phi(u)du \le \frac{\Psi(c_1)+\Psi(c_2)}{2}\int_{c_1}^{c_2}\phi(u)du, \tag{1.15}$$

where $\phi : [c_1, c_2] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(c_1 + c_2)/2$.

These two inequalities are then generalized in many ways. For more results and generalizations see [22, 44, 57, 58, 66, 67, 72]. These inequalities are further extended in other fractional integrals like Riemann−Liouville fractional integrals [77], conformable fractional integrals [1], new conformable fractional integrals [50] and Katugampola fractional integrals [52] etc. For extra detail see [17, 18, 48, 80, 82, 85, 84].

In 2001, Dragomir [23] gave co-ordinated version of inequalities (1.14) as follows:

Theorem 1.3.1 ([23])**.** *Let $\Psi : \Delta = [c_1, c_2] \times [d_1, d_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on the co-ordiantes on Δ with $c_1 < c_2$ and $d_1 < d_2$. Then one has the following inequalities:*

$$\begin{aligned}
&\Psi\left(\frac{c_1+c_2}{2}, \frac{d_1+d_2}{2}\right)\\
&\le \frac{1}{2}\left[\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi\left(u, \frac{d_1+d_2}{2}\right)du + \frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi\left(\frac{c_1+c_2}{2}, v\right)dv\right]\\
&\le \frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)dudv\\
&\le \frac{1}{4}\left[\frac{1}{c_2-c_1}\left(\int_{c_1}^{c_2}\Psi(u,d_1)du + \int_{c_1}^{c_2}\Psi(u,d_2)du\right)\right.\\
&\quad \left. + \frac{1}{d_2-d_1}\left(\int_{d_1}^{d_2}\Psi(c_1,v)dv + \int_{d_1}^{d_2}\Psi(c_2,v)dv\right)\right]\\
&\le \frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}.
\end{aligned} \tag{1.16}$$

In 2016, Farid et. al. [31] gave Fejér-Hadamard inequality for convex functions on co-ordinates.

Theorem 1.3.2 ([31])**.** *Let $\Psi : \Delta = [c_1, c_2] \times [d_1, d_2] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on the co-ordinates in Δ. Let $\phi_1 : [c_1, c_2] \to (0, \infty)$ and $\phi_2 : [d_1, d_2] \to (0, \infty)$ be two integrable and symmetric functions about $(c_1 + c_2)/2$ and $(d_1 + d_2)/2$ respectively. Then one has* *the following inequalities:*

$$\begin{aligned}
&\Psi\left(\frac{c_1+c_2}{2},\frac{d_1+d_2}{2}\right)\\
&\leq \frac{1}{2}\left[\frac{1}{F_1}\int_{c_1}^{c_2}\Psi\left(u,\frac{d_1+d_2}{2}\right)\phi_1(u)du+\frac{1}{F_2}\int_{d_1}^{d_2}\Psi\left(\frac{c_1+c_2}{2},v\right)\phi_2(v)dv\right]\\
&\leq \frac{1}{F_1F_2}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\phi_1(u)\phi_2(y)dudv\\
&\leq \frac{1}{4}\left[\frac{1}{F_1}\left(\int_{c_1}^{c_2}\Psi(u,d_1)\phi_1(u)du+\int_{c_1}^{c_2}\Psi(u,d_2)\phi_1(u)du\right)\right.\\
&\quad\left.+\frac{1}{F_2}\left(\int_{d_1}^{d_2}\Psi(c_1,v)\phi_2(v)dv+\int_{d_1}^{d_2}\Psi(c_2,v)\phi_2(v)dv\right)\right]\\
&\leq \frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4},
\end{aligned} \tag{1.17}$$

where

$$F_1=\int_{c_1}^{c_2}\phi_1(u)du \text{ and } F_2=\int_{d_1}^{d_2}\phi_2(v)dv.$$

These inequalities are sharp.

The inequalities (1.16) and (1.17) are then further generalized. For more knowledge see [2, 3, 32, 69, 70, 81, 90].

1.4 Cauchy's means and Jensen's inequality

Now a days Cauchy's type means has become more significant in the field of research. Number of researchers established results in this notion. Mercer [60], and Pečarić [75] made connection between cauchy's type means and Jensen's inequality. These are given both in discrete as well as in integral form and has many applications. For more detail see [7, 8, 9, 10, 11, 60, 75, 76].

Caucy mean value theorem states that:
Let Ψ and ϕ be real and continuous functions on a closed interval $\aleph=[c_1,c_2]$ and also are differentiable on $\aleph^o$(interior of $\aleph$), then there exists a number $\upsilon\in\aleph$ such that

$$\frac{\Psi'(\upsilon)}{\phi'(\upsilon)}=\frac{\Psi(c_1)-\Psi(c_2)}{\phi(c_1)-\phi(c_2)}.$$

The number υ is unique in the case when the function $\frac{\Psi'}{\phi'}$ is invertible. Also following holds:

$$\upsilon = \left(\frac{\Psi'}{\phi'}\right)^{-1} \left(\frac{\Psi(c_1) - \Psi(c_2)}{\phi(c_1) - \phi(c_2)}\right).$$

Then number υ the cauchy's mean number of the numbers c_1, c_2.

Whereas the Jensen inequality for a convex function is defined as:

Let $\aleph \subseteq \mathbb{R}$ be an interval and $\Psi : \aleph \to \mathbb{R}$ be a convex function. If

$$\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \aleph^n, \ \mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbb{R}_+^n \text{ and } D_n = \sum_{i=1}^{n} d_i,$$

then following holds:

$$\Psi\left(\frac{1}{D_n}\sum_{i=1}^{n} d_i c_i\right) \leq \frac{1}{D_n}\sum_{i=1}^{n} d_i \Psi(c_i). \tag{1.18}$$

1.5 Thesis outline

Aim of the thesis is to extend the inequalities (1.14) and (1.15) in various generalized convex functions that have already presented in literature and also in some new type of convex functions. These extensions are not only based on classical integrals but also for Riemann−Liouville fractional integrals, conformable fractional integrals, new conformable fractional integrals and Katugampola fractional integrals as well. Some mean value theorem are also establish.
Plan of the present thesis is as follows:

In Chapter 2, We give Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for p-convex functions via conformable and new conformable fractional integrals.

In Chapter 3, we give some Hermite-Hadamard inequalities for s- and m-convex functions via Katugampola fractional integrals.

In Chapter 4, we establish some mean value theorem for p-convex functions and s-convex functions.

In Chapter 5, we find some integral inequalities via exponentially p- and exponentially s-convex functions in classical integrals.

In Chapter 6, we construct the integral inequalities for (s, p)-convex functions via Riemann−Liouville fractional integrals in first section and in the second section we built integral inequalities for (s, p)-convex functions on co-ordinates.

In Chapter 7, we establish Hermite-Hadamard inequalities for harmonically convex functions via Riemann−Liouville and Katugampola fractional integrals. Also we give some inequalities of Fejér type for harmonically convex functions via Katugampola fractional integrals.

In Chapter 8, we summed up our work and give ideas for future work.

Chapter 2

Inequalities for p-convex function via some generalized fractional integrals

In this chapter, we obtain the Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for p-convex functions via conformable fractional integrals and via new fractional conformable integral operators. We also establish some new Hermite-Hadamard-Fejér type inequalities for convex functions and harmonically convex functions via conformable fractional integrals and via new fractional conformable integral operators.

2.1 Introduction

The classical Beta function and Hypergeometric function are given as:

1. The Beta function:

   $\mathcal{B}(c_1, c_2) = \int_0^1 u^{c_1-1}(1-u)^{c_2-1}du$

2. The Hypergeometric function:

   ${}_2F_1(c_1, c_2; x; y) = \frac{1}{\mathcal{B}(c_2, x-c_2)} \int_0^1 u^{c_2-1}(1-u)^{x-c_2-1}(1-yu)^{-c_1}du, x > c_2 > 0,\ |y| < 1.$

The incomplete Beta function is defined as:

$$\mathcal{B}_r(c_1,c_2)=\int_0^r u^{c_1-1}(1-u)^{c_2-1}du,\ r\in[0,1].$$

Relationship between classical Beta function and incomplete Beta function is given as:

$$\mathcal{B}(c_1,c_2)=\mathcal{B}_r(c_1,c_2)+\mathcal{B}_{1-r}(c_1,c_2).$$

Further,

$$\mathcal{B}_r(c_1+1,c_2)=\frac{c_1\mathcal{B}_r(c_1,c_2)-(\frac{1}{2})^{c_1+c_2}}{c_1+c_2},$$

and

$$\mathcal{B}_r(c_1,c_2+1)=\frac{c_2\mathcal{B}_r(c_1,c_2)-(\frac{1}{2})^{c_1+c_2}}{c_1+c_2}.$$

Abdeljawad [1] defined the left conformable fractional integral as:

Definition 2.1.1 ([1])**.** *Let $\alpha\in(n,n+1]$ and $\gamma=\alpha-n$. Then the left conformable fractional integrals starting at c_1 of order $\alpha>0$ is defined by*

$$J_\alpha^{c_1}\Psi(u)=\frac{1}{n!}\int_{c_1}^u(u-t)^n(t-c_1)^{\gamma-1}\Psi(t)dt.$$

Similarly, the right conformable fractional integrals of any order $\alpha>0$ is defined by

$$J_\alpha^{c_2}\Psi(u)=\frac{1}{n!}\int_u^{c_2}(t-u)^n(c_2-t)^{\gamma-1}\Psi(t)dt.$$

Jarad [50] defined the left and right fractional conformable integral operators as:

Definition 2.1.2 ([50])**.** *Let $\beta\in\mathbb{C}$. Then the left and right sided fractional conformable integral operators has defined, respectively, as follows:*

$$ {}^{\beta}_{c_1}\mathcal{J}^{\alpha}\Psi(u)=\frac{1}{\Gamma(\beta)}\int_{c_1}^{u}\left(\frac{(u-c_1)^\alpha-(t-c_1)^\alpha}{\alpha}\right)^{\beta-1}\frac{\Psi(t)}{(t-c_1)^{1-\alpha}}dt, \tag{2.1}$$

$$ {}^{\beta}\mathcal{J}^{\alpha}_{c_2}\Psi(u)=\frac{1}{\Gamma(\beta)}\int_{u}^{c_2}\left(\frac{(c_2-u)^\alpha-(c_2-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{\Psi(t)}{(c_2-t)^{1-\alpha}}dt. \tag{2.2}$$

2.2 Integral inequalities via conformable fractional integrals

We first calculate some integral inequalities via conformable fractional integrals.

2.2.1 Hermite-Hadamard type inequalities

In this subsection, we prove some Hermite-Hadamard type inequalities for p-convex functions via conformable fractional integrals.

Theorem 2.2.1. *Let $\varPsi : [c_1, c_2] \subset (0, \infty) \to \mathbb{R}$ be a p-convex function such that $\varPsi \in L_1[c_1, c_2]$. Then*
(i) for $p > 0$, we get

$$\begin{aligned}&\varPsi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}\right)\\ &\le \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha-n)(c_2^p - c_1^p)^\alpha}\left[J_\alpha^{c_1^p}(\varPsi \circ \mu)(c_2^p) + J_\alpha^{c_2^p}(\varPsi \circ \mu)(c_1^p)\right]\\ &\le \frac{\varPsi(c_1^p) + \varPsi(c_2^p)}{2},\end{aligned} \tag{2.3}$$

where $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_1^p, c_2^p]$.
(ii) for $p < 0$, we get

$$\begin{aligned}&\varPsi\left(\left[\frac{c_1^p + c_2^p}{2}\right]^{1/p}\right)\\ &\le \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha-n)(c_1^p - c_2^p)^\alpha}\left[J_\alpha^{c_1^p}(\varPsi \circ \mu)(c_2^p) + J_\alpha^{c_2^p}(\varPsi \circ \mu)(c_1^p)\right]\\ &\le \frac{\varPsi(c_1^p) + \varPsi(c_2^p)}{2},\end{aligned} \tag{2.4}$$

here $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_2^p, c_1^p]$.

Proof. (*i*) Since $\varPsi$ is p-convex function on $[c_1, c_2]$, we have

$$\varPsi\left(\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{\varPsi(u) + \varPsi(v)}{2}.$$

Taking $u^p = rc_1^p + (1-r)c_2^p$ and $v^p = (1-r)c_1^p + rc_2^p$ with $r \in [0, 1]$, we get

$$\varPsi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{\varPsi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) + \varPsi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)}{2}. \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{1}{n!}r^n(1-r)^{\alpha-n-1}$ with $r \in (0,1)$, $\alpha > 0$ and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{n!}\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right)\int_0^1 r^n(1-r)^{\alpha-n-1}dr \\
&\leq \frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&\quad+\frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left(((1-r)c_1^p+rc_2^p)^{\frac{1}{p}}\right)dr \\
&= I_1+I_2.
\end{aligned} \tag{2.6}$$

By setting $u = rc_1^p + (1-r)c_2^p$, we have

$$\begin{aligned}
I_1 &= \frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&= \frac{1}{n!}\int_{c_2^p}^{c_1^p}\left(\frac{u-c_2^p}{c_1^p-c_2^p}\right)^n\left(1-\frac{u-c_2^p}{c_1^p-c_2^p}\right)^{\alpha-n-1}(\Psi\circ\mu)(u)\frac{du}{c_1^p-c_2^p} \\
&= \frac{1}{n!\,(c_2^p-c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}\left(c_2^p-u\right)^n\left(u-c_1^p\right)^{\alpha-n-1}(\Psi\circ\mu)(u)du \\
&= \frac{1}{(c_2^p-c_1^p)^\alpha}J_\alpha^{a^p}(\Psi\circ\mu)(c_2^p).
\end{aligned} \tag{2.7}$$

Similarly, by setting $u = rc_2^p + (1-r)c_1^p$, we have

$$\begin{aligned}
I_2 &= \frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left((rc_2^p+(1-r)c_1^p)^{\frac{1}{p}}\right)dr \\
&= \frac{1}{n!}\int_{c_1^p}^{c_2^p}\left(\frac{u-c_1^p}{c_2^p-a^p}\right)^n\left(1-\frac{u-c_1^p}{c_2^p-c_1^p}\right)^{\alpha-n-1}(\Psi\circ\mu)(u)\frac{du}{c_2^p-c_1^p} \\
&= \frac{1}{n!\,(c_2^p-c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}\left(u-c_1^p\right)^n\left(c_2^p-u\right)^{\alpha-n-1}(\Psi\circ\mu)(u)du \\
&= \frac{1}{(c_2^p-c_1^p)^\alpha}J_\alpha^{c_2^p}(\Psi\circ\mu)(c_1^p).
\end{aligned} \tag{2.8}$$

Thus by putting values of I_1 and I_2 in (2.6), we get

$$\frac{2\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{1}{(c_2^p-c_1^p)^\alpha}\left[J_\alpha^{c_1^p}(\Psi\circ\mu)(c_2^p)+J_\alpha^{c_2^p}(\Psi\circ\mu)(c_1^p)\right]. \tag{2.9}$$

This completes the first inequality of (2.3). For second inequality, we know that

$$\varPsi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) + \varPsi\left((rc_2^p + (1-r)c_1^p)^{\frac{1}{p}}\right) \leq \left[\varPsi(c_1) + \varPsi(c_2)\right]. \tag{2.10}$$

Multiplying both sides of (2.10) by $\frac{1}{n!}r^n(1-r)^{\alpha-n-1}$ with $r \in (0,1)$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\frac{1}{(c_2^p - c_1^p)^\alpha}\left[J_\alpha^{c_1^p}(\varPsi \circ \mu)(c_2^p) + J_\alpha^{c_2^p}(\varPsi \circ \mu)(c_1^p)\right] \leq \frac{\Gamma(\alpha - n)}{\Gamma(\alpha + 1)}\left[\varPsi(c_1) + \varPsi(c_2)\right]. \tag{2.11}$$

This completes the second inequality of (2.3).

(ii) Proof is similar to (i). □

Remark 1. *In Theorem* 2.2.1*,*

1. *If we take* $p = 1$*, we get Theorem* 2.1 *in* [82].
2. *If we take* $p = -1$*, we get Theorem* 2.1 *in* [15].
3. *If we take* $p = 1$ *and* $\alpha = n + 1$*, we get Theorem* 2 *in* [80].
4. *If we take* $p = -1$ *and* $\alpha = n + 1$*, we get Theorem* 4 *in* [48].

Lemma 2.2.1. *Let* $\varPsi : [c_1, c_2] \subset (0, \infty) \to \mathbb{R}$ *be a differentiable function on* (c_1, c_2) *with* $c_1 < c_2$ *such that* $\psi' \in L_1[c_1, c_2]$*. Then*

(i) *for* $p > 0$*, we get*

$$\begin{aligned}&{}_1\Delta_\varPsi(c_1, c_2; \alpha; \mathcal{B}; J)\\ &= \frac{c_2^p - c_1^p}{2p}\int_0^1 \left[\mathcal{B}_{1-r}(n+1, \alpha - n) - \mathcal{B}_r(n+1, \alpha - n)\right] A_r^{\frac{1}{p}-1}\varPsi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr,\end{aligned} \tag{2.12}$$

where $A_r = [rc_1^p + (1-r)c_2^p]$ *and*

$$\begin{aligned}&{}_1\Delta_\varPsi(c_1, c_2; \alpha; \mathcal{B}; J)\\ &= \mathcal{B}(n+1, \alpha - n)\left(\frac{\varPsi(c_1) + \varPsi(c_2)}{2}\right) - \frac{n!}{2(c_2^p - c_1^p)^\alpha}\left[J_\alpha^{c_1^p}(\varPsi \circ \mu)(c_2^p) + J_\alpha^{c_2^p}(\varPsi \circ \mu)(c_1^p)\right].\end{aligned}$$

(ii) *for* $p < 0$ *we have*

$$\begin{aligned}&{}_2\Delta_\varPsi(c_1, c_2; \alpha; \mathcal{B}; J)\\ &= \frac{c_1^p - c_2^p}{2p}\int_0^1 \left(\mathcal{B}_r(n+1, \alpha - n) - \mathcal{B}_{1-r}(n+1, \alpha - n)\right) B_r^{\frac{1}{p}-1}\varPsi'\left([rc_2^p + (1-r)c_1^p]^{\frac{1}{p}}\right) \mathrm{d}r,\end{aligned} \tag{2.13}$$

here $B_r = [rc_2^p + (1-r)c_1^p]$ *and*

$$
\begin{aligned}
&{}_2\Delta_{\Psi}(c_1, c_2; \alpha; \mathcal{B}; J)\\
&= \mathcal{B}(n+1, \alpha-n)\left(\frac{\Psi(c_1)+\Psi(c_2)}{2}\right) - \frac{n!}{2(c_1^p - c_2^p)^{\alpha}}\left[J_{\alpha}^{c_1^p}(\Psi\circ\mu)(c_2^p) + J_{\alpha}^{c_2^p}(\Psi\circ\mu)(c_1^p)\right].
\end{aligned}
$$

Proof. (i) Consider,

$$
\begin{aligned}
&\int_0^1 \left[\mathcal{B}_{1-r}(n+1,\alpha-n) - \mathcal{B}_r(n+1,\alpha-n)\right] A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\\
&= \int_0^1 \mathcal{B}_{1-r}(n+1,\alpha-n) A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\\
&\quad - \int_0^1 \mathcal{B}_r(n+1,\alpha-n) A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\\
&= I_1 - I_2.
\end{aligned}
\tag{2.14}
$$

Then by integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^1 \mathcal{B}_{1-r}(n+1,\alpha-n) A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\\
&= \int_0^1 \left(\int_0^{1-r} u^n(1-u)^{\alpha-n-1}du\right) A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\\
&= \frac{p}{c_2^p - c_1^p}\mathcal{B}(n+1,\alpha-n)\Psi(b)\\
&\quad - \frac{p}{c_2^p - c_1^p}\int_0^1 (1-r)^n r^{\alpha-n-1}\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\\
&= \frac{p}{c_2^p - c_1^p}\mathcal{B}(n+1,\alpha-n)\Psi(c_2)\\
&\quad - \frac{p}{c_2^p - c_1^p}\int_{c_2^p}^{c_1^p}\left(1 - \frac{x-c_2^p}{c_1^p - c_2^p}\right)^n \left(\frac{x-c_2^p}{c_1^p - c_2^p}\right)^{\alpha-n-1}\frac{(\Psi\circ\mu)(x)}{c_1^p - c_2^p}dx\\
&= \frac{p}{c_2^p - c_1^p}\mathcal{B}(n+1,\alpha-n)\Psi(c_2) - \frac{n!}{(c_2^p - c_1^p)^{\alpha+1}}J_{\alpha}^{c_2^p}(\Psi\circ\mu)(c_1^p).
\end{aligned}
\tag{2.15}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \int_0^1 \mathcal{B}_r(n+1,\alpha-n)\Psi'\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&= \int_0^1 \left(\int_0^r u^n(1-u)^{\alpha-n-1}du\right)\Psi'\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&= -\frac{p}{c_2^p-c_1^p}\mathcal{B}(n+1,\alpha-n)\Psi(c_1) \\
&\quad + \frac{p}{c_2^p-c_1^p}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr \qquad (2.16) \\
&= -\frac{p}{c_2^p-c_1^p}\mathcal{B}(n+1,\alpha-n)\Psi(c_1) \\
&\quad + \frac{p}{c_2^p-c_1^p}\int_{c_2^p}^{c_1^p}\left(\frac{x-c_2^p}{c_1^p-c_2^p}\right)^n\left(1-\frac{x-c_2^p}{c_1^p-c_2^p}\right)^{\alpha-n-1}\frac{(\Psi\circ\mu)(x)}{c_1^p-c_2^p}dx \\
&= -\frac{p}{c_2^p-c_1^p}\mathcal{B}(n+1,\alpha-n)\Psi(c_1)+\frac{n!}{(c_2^p-c_1^p)^{\alpha+1}}J_{\alpha}^{c_1^p}(\Psi\circ\mu)(c_2^p).
\end{aligned}
$$

By substituting values of I_1 and I_2 in (2.14) and then multiplying by $\frac{c_2^p-c_1^p}{2}$ we get (2.12).

(*ii*) Proof is similar to (*i*). □

Remark 2. *By taking $p=-1$ in above Lemma 2.2.1, we obtain Lemma 2.1 in [15].*

Theorem 2.2.2. *Let $\Psi:[c_1,c_2]\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on (c_1,c_2) with $c_1<c_2$ such that $\Psi'\in L_1[c_1,c_2]$. Let $|\Psi'|^q$, where $q\geq 1$, is p-convex function. Then*

(i) for $p>0$, we have

$$
|{}_1\Delta_{\Psi}(c_1,c_2;\alpha;\mathcal{B};J)|\leq\frac{c_2^p-c_1^p}{2p}\lambda^{1-1/q}\left(\lambda_1|\Psi'(c_1)|^q+\lambda_2|\Psi'(c_2)|^q\right)^{1/q}, \qquad (2.17)
$$

where

$$
\lambda=\mathcal{B}(n+1,\alpha-n+1)-\mathcal{B}(n+1,\alpha-n)+\mathcal{B}(n+2,\alpha-n),
$$

$$
\lambda_1=\frac{c_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{c_1^p}{c_2^p}\right)\text{ and }\lambda_2=\frac{c_2^{1-p}}{2}\,{}_2F_1\left(1-\frac{1}{p},1;3;1-\frac{c_1^p}{c_2^p}\right).
$$

(ii) for $p<0$ we have

$$
|{}_2\Delta_{\Psi}(c_1,c_2;\alpha;\mathcal{B};J)|\leq\frac{c_1^p-c_2^p}{2p}\lambda_3^{1-1/q}\left(\lambda_4|\Psi'(c_1)|^q+\lambda_5|\Psi'(c_2)|^q\right)^{1/q}, \qquad (2.18)
$$

here

$$\lambda_3 = \mathcal{B}(n+1, \alpha-n+1) - \mathcal{B}(n+2, \alpha-n),$$

$$\lambda_4 = \frac{c_2^{p-1}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1-\frac{c_2^p}{c_1^p}\right) \ \textit{and}\ \lambda_5 = \frac{c_2^{p-1}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 2; 3; 1-\frac{c_2^p}{c_1^p}\right).$$

Proof. (i) Let $A_r = (rc_1^p + (1-r)c_2^p)$. Using Lemma 2.2.1, property of modulus, power mean inequality and the fact that $|\Psi'|^q$ is p-convex function, we have

$$\begin{aligned}
&|{}_1\Delta_\Psi(c_1, c_2; \alpha; \mathcal{B}; J)| \\
&= \left|\frac{c_2^p - c_1^p}{2p}\int_0^1 \{\mathcal{B}_{1-r}(n+1, \alpha-n) - \mathcal{B}_t(n+1, \alpha-n)\}\, A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\right| \\
&\le \frac{c_2^p - c_1^p}{2p}\left(\int_0^1 \{\mathcal{B}_{1-r}(n+1, \alpha-n) - \mathcal{B}_r(n+1, \alpha-n)\}\, dr\right)^{1-1/q} \\
&\quad \times \left(\int_0^1 A_r^{\frac{1}{p}-1}\left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right|^q dr\right)^{1/q} \\
&\le \frac{c_2^p - c_1^p}{2p}\lambda^{1-1/q}\left(\int_0^1 A_r^{\frac{1}{p}-1}\left[r|\Psi'(c_1)|^q + (1-r)|\Psi'(c_2)|^q\right] dr\right)^{1/q} \\
&= \frac{c_2^p - c_1^p}{2p}\lambda^{1-1/q}\left(\lambda_1|\Psi'(c_1)|^q + \lambda_2|\Psi'(c_2)|^q\right)^{1/q},
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
\lambda &= \int_0^1 \left(\mathcal{B}_{1-r}(n+1, \alpha-n) - \mathcal{B}_r(n+1, \alpha-n)\right) dr \\
&= \int_0^1 \left(\int_0^{1-r} u^n(1-u)^{\alpha-n-1}du\right) dr + \int_0^1 \left(\int_0^{r} u^n(1-u)^{\alpha-n-1}du\right) dr \\
&= r\left(\int_0^{1-r} u^n(1-u)^{\alpha-n-1}du\right)\Big|_0^1 + \int_0^1 r(1-r)^n r^{\alpha-n-1}dr \\
&\quad + r\left(\int_0^{r} u^n(1-u)^{\alpha-n-1}du\right)\Big|_0^1 + \int_0^1 r^{n+1}(1-r)^{\alpha-n-1}dr \\
&= \mathcal{B}(n+1, \alpha-n+1) - \mathcal{B}(n+1, \alpha-n) + \mathcal{B}(n+2, \alpha-n),
\end{aligned}$$

$$\lambda_1 = \int_0^1 rA_r^{\frac{1}{p}-1}dr = \frac{c_2^{1-p}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 2; 3; 1-\frac{c_1^p}{c_2^p})\right),$$

and

$$\lambda_2 = \int_0^1 (1-r)A_r^{\frac{1}{p}-1}dr = \frac{c_2^{1-p}}{2}\ {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1-\frac{c_1^p}{c_2^p}\right).$$

Hence the proof is completed.

(*ii*) Proof is similar to (*i*). □

Remark 3. *By letting $p = -1$ in above Theorem 2.2.2 we obtain Theorem 2.2 in [15].*

Now for the next two theorems we consider the case when $p > 0$ and leave the case when $p < 0$ for the reader.

Theorem 2.2.3. *Let $\Psi : [c_1, c_2] \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on (c_1, c_2) with $c_1 < c_2$ such that $\Psi' \in L_1[c_1, c_2]$. Let $|\Psi'|^q$, where $q \geq 1$, is p-convex function. then for $p > 0$ we have*

$$\begin{aligned} &|{}_1\Delta_\Psi(c_1, c_2; \alpha; \mathcal{B}; J)| \\ &\leq \frac{c_2^p - c_1^p}{2p} \sigma^{1-1/q} \left((\sigma_1 - \sigma_2)|\Psi'(c_1)|^q + (\sigma_3 - \sigma_4)|\Psi'(c_2)|^q \right)^{1/q}, \end{aligned} \tag{2.20}$$

where

$$\sigma = \frac{c_2^{1-p}}{2} \, {}_2F_1 \left(1 - \frac{1}{p}, 1; 2; 1 - \frac{c_1^p}{c_2^p} \right),$$

$$\sigma_1 = \frac{1}{2} \mathcal{B}(n+1, \alpha - n + 2),$$

$$\sigma_2 = \frac{1}{2} \left(\mathcal{B}(n+1, \alpha - n) - \mathcal{B}(n+3, \alpha - n) \right),$$

$$\sigma_3 = \mathcal{B}(n+2, \alpha - n + 1) - \frac{1}{2} \mathcal{B}(n+1, \alpha - n + 2),$$

and

$$\sigma_4 = \frac{1}{2} \mathcal{B}(n+1, \alpha - n) + \frac{1}{2} \mathcal{B}(n+3, \alpha - n) - \mathcal{B}(n+2, \alpha - n).$$

Proof. Using Lemma 2.2.1, property of modulus, power mean inequality and the fact that $|\Psi'|^q$ is p-convex function, we have

$$\begin{aligned} &|{}_1\Delta_\Psi(c_1, c_2; \alpha; \mathcal{B}; J)| \\ &= \left| \frac{c_2^p - c_1^p}{2p} \int_0^1 \{ \mathcal{B}_{1-r}(n+1, \alpha - n) - \mathcal{B}_r(n+1, \alpha - n) \} A_r^{\frac{1}{p}-1} \Psi' \left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \right| \\ &\leq \frac{c_2^p - c_1^p}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-1/q} \\ &\quad \times \left(\int_0^1 \{ \mathcal{B}_{1-r}(n+1, \alpha - n) - \mathcal{B}_r(n+1, \alpha - n) \} \left| \Psi' \left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) \right|^q dr \right)^{1/q} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{c_2^p - c_1^p}{2p}\lambda^{1-1/q}\left(\int_0^1 \{\mathcal{B}_{1-r}(n+1,\alpha-n) - \mathcal{B}_r(n+1,\alpha-n)\}\,[r|\Psi'(c_1)|^q \right.\\
&\quad \left. + (1-r)|\Psi'(c_2)|^q]dr\right)^{1/q} \qquad (2.21)\\
&= \frac{c_2^p - c_1^p}{2p}\sigma^{1-1/q}\left((\sigma_1-\sigma_2)|\Psi'(c_1)|^q + (\sigma_3-\sigma_4)|\Psi'(c_2)|^q\right)^{1/q},
\end{aligned}
$$

where

$$
\sigma = \int_0^1 A_r^{\frac{1}{p}-1}dr = \frac{c_2^{1-p}}{2}\ {}_2F_1\left(1-\frac{1}{p},1;2;1-\frac{c_1^p}{c_2^p}\right),
$$

$$
\sigma_1 = \int_0^1 r\mathcal{B}_{1-r}(n+1,\alpha-n)dr = \frac{1}{2}\mathcal{B}(n+1,\alpha-n+2),
$$

$$
\sigma_2 = \int_0^1 r\mathcal{B}_r(n+1,\alpha-n) = \frac{1}{2}\left(\mathcal{B}(n+1,\alpha-n) - \mathcal{B}(n+3,\alpha-n)\right),
$$

$$
\begin{aligned}
\sigma_3 &= \int_0^1 (1-r)\mathcal{B}_{1-r}(n+1,\alpha-n)dr\\
&= \mathcal{B}(n+2,\alpha-n+1) - \frac{1}{2}\mathcal{B}(n+1,\alpha-n+2),
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_4 &= \int_0^1 (1-r)\mathcal{B}_r(n+1,\alpha-n)dr\\
&= \frac{1}{2}\mathcal{B}(n+1,\alpha-n) + \frac{1}{2}\mathcal{B}(n+3,\alpha-n) - \mathcal{B}(n+2,\alpha-n).
\end{aligned}
$$

Hence the proof is completed. □

Theorem 2.2.4. *Let $\Psi : [c_1, c_2] \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on (c_1, c_2) with $c_1 < c_2$ such that $\Psi' \in L_1[c_1, c_2]$. Let $|\Psi'|^q$, where $q > 1$ and $\frac{1}{l} + \frac{1}{q} = 1$, is p-convex function, then*

$$
\begin{aligned}
&|{}_1\Delta_{\Psi}(c_1, c_2; \alpha; \mathcal{B}; J)|\\
&\leq \frac{c_2^p - c_1^p}{2p}\nu^{\frac{1}{l}}\left(\nu_1|\Psi'(c_1)|^q + \nu_2|\Psi'(c_2)|^q\right)^{1/q}, \qquad (2.22)
\end{aligned}
$$

where

$$\nu = 2\int_0^{\frac{1}{2}} \left(\int_r^{1-r} u^n(1-u)^{\alpha-n-1}du\right)^l dr,$$

$$\nu_1 = \frac{c_2^{q(1-l)}}{2}\ {}_2F_1\left(q\left(1-\frac{1}{p}\right), 2; 3; 1-\frac{c_1^p}{c_2^p}\right),$$

$$\nu_2 = \frac{c_2^{q(1-p)}}{2}\ {}_2F_1\left(q\left(1-\frac{1}{p}\right), 1; 3; 1-\frac{c_1^p}{c_2^p}\right).$$

Proof. Let $A_r = (rc_1^p + (1-r)c_2^p)$. Using Lemma 2.2.1, property of modulus, Hölder's inequality and the fact that $|\Psi'|^q$ is p-convex function, we have

$$\begin{aligned}
&|{}_1\Delta_{\Psi}(c_1, c_2; \alpha; \mathcal{B}; J)| \\
&= \left|\frac{c_2^p - c_1^p}{2p}\int_0^1 \{\mathcal{B}_{1-r}(n+1, \alpha-n) - \mathcal{B}_r(n+1, \alpha-n)\}\, A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr\right| \\
&\leq \frac{c_2^p - c_1^p}{2p}\left(\int_0^1 |\mathcal{B}_{1-r}(n+1, \alpha-n) - \mathcal{B}_r(n+1, \alpha-n)|^l\, dr\right)^{\frac{1}{l}} \\
&\quad \times \left(\int_0^1 A_r^{q(\frac{1}{p}-1)}\left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right|^q dr\right)^{1/q} \\
&\leq \frac{c_2^p - c_1^p}{2p}\nu^{\frac{1}{l}}\left(\int_0^1 A_r^{q(\frac{1}{p}-1)}\left[r|\Psi'(c_1)|^q + (1-r)|\Psi'(c_2)|^q\right] dr\right)^{1/q} \\
&= \frac{c_2^p - c_1^p}{2p}\nu^{\frac{1}{l}}\left(\nu_1|\Psi'(c_1)|^q + \nu|\Psi'(c_2)|^q\right)^{1/q},
\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
\nu &= \int_0^1 |\mathcal{B}_{1-r}(n+1, \alpha-n) - \mathcal{B}_r(n+1, \alpha-n)|^l\, dr \\
&= \int_0^{\frac{1}{2}} (\mathcal{B}_{1-r}(n+1, \alpha-n) - \mathcal{B}_r(n+1, \alpha-n))^l\, dr \\
&\quad + \int_{\frac{1}{2}}^1 (\mathcal{B}_r(n+1, \alpha-n) - \mathcal{B}_{1-r}(n+1, \alpha-n))^l\, dr \\
&= \int_0^{\frac{1}{2}}\left(\int_r^{1-r} u^n(1-u)^{\alpha-n-1}du\right)^l dr + \int_{\frac{1}{2}}^1\left(\int_{1-r}^r u^n(1-u)^{\alpha-n-1}du\right)^l dr \\
&= 2\int_0^{\frac{1}{2}}\left(\int_r^{1-r} u^n(1-u)^{\alpha-n-1}du\right)^l dr,
\end{aligned}$$

$$\nu_1 = \int_0^1 r A_r^{\frac{1}{p}-1} dr = \frac{c_2^{q(1-p)}}{2} \, {}_2F_1 \left(q \left(1 - \frac{1}{p} \right), 2; 3; 1 - \frac{c_1^p}{c_2^p} \right),$$

and

$$\nu_2 = \int_0^1 (1-r) A_r^{\frac{1}{p}-1} dr = \frac{c_2^{q(1-p)}}{2} \, {}_2F_1 \left(q \left(1 - \frac{1}{p} \right), 1; 3; 1 - \frac{c_1^p}{c_2^p} \right).$$

Hence the proof is completed. □

2.2.2 Hermite-Hadamard-Fejér type inequalities

In this subsection, we prove some Hermite-Hadamard-Fejér type inequalities for p-convex functions via conformable fractional integral. Kunt and Iscan [56] defined following useful definition.

Definition 2.2.2 ([56])**.** *Let $p \in \mathbb{R}\backslash\{0\}$. A function $\phi : [c_1, c_2] \subseteq (0, \infty) \to \mathbb{R}$ is called p-symmetric with respect to $\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}$, if*

$$\phi(x) = \phi\left((c_1^p + c_2^p - x^p)^{\frac{1}{p}} \right)$$

holds, for all $x \in [c_1, c_2]$.

In order to give result involving Hermite-Hadamard-Fejér type Inequality we need following lemma.

Lemma 2.2.3. *Let $p \in \mathbb{R}\backslash\{0\}$ and $\phi : [c_1, c_2] \subseteq (0, \infty) \to \mathbb{R}$ is integrable and p-symmetric with respect to $\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}$. Then*
(i) for $p > 0$ we have

$$J_\alpha^{c_1^p}(\phi \circ \mu)(c_2^p) = J_\alpha^{c_2^p}(\phi \circ \mu)(c_1^p) = \frac{1}{2}\left[J_\alpha^{c_1^p}(\phi \circ \mu)(c_2^p) + J_\alpha^{c_1^p}(\phi \circ \mu)(c_1^p) \right], \tag{2.24}$$

with $\alpha > 0$ and $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_1^p, c_2^p]$.
(ii) for $p < 0$ we have

$$J_\alpha^{c_2^p}(\phi \circ \mu)(c_1^p) = J_\alpha^{c_1^p}(\phi \circ \mu)(c_2^p) = \frac{1}{2}\left[J_\alpha^{c_2^p}(\phi \circ \mu)(c_1^p) + J_\alpha^{c_1^p}(\phi \circ \mu)(c_2^p) \right], \tag{2.25}$$

with $\alpha > 0$ and $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_2^p, c_1^p]$.

Proof. (*i*) Since ϕ is p-symmetric with respect to $\left(\frac{c_1^p+c_2^p}{2}\right)^{1/p}$, then by definition we have $\phi(u^{\frac{1}{p}}) = \phi\left([c_1^p + c_2^p - u]^{\frac{1}{p}}\right)$, for all $u \in [c_1^p, c_2^p]$. In the following integral, by setting $x = c_1^p + c_2^p - u$ gives

$$\begin{aligned} J_\alpha^{c_1^p}(\phi \circ \mu)(c_2^p) &= \frac{1}{n!}\int_{c_1^p}^{c_2^p} (c_2^p - x)^n \, (x - c_1^p)^{\alpha-n-1} \, \phi(x^{\frac{1}{p}})dx \\ &= \frac{1}{n!}\int_{c_1^p}^{c_2^p} (u - c_1^p)^n \, (c_2^p - u)^{\alpha-n-1} \, \phi\left([c_1^p + c_2^p - u]^{\frac{1}{p}}\right) du \qquad (2.26) \\ &= \frac{1}{n!}\int_{c_1^p}^{c_2^p} (u - c_1^p)^n \, (c_2^p - u)^{\alpha-n-1} \, \phi\left(u^{\frac{1}{p}}\right) du = J_\alpha^{c_2^p}(\phi \circ \mu)(c_1^p). \end{aligned}$$

This completes the proof.

(*ii*) Proof is similar to (*i*). □

Remark 4. *In Lemma* 2.2.3*,*

1. *If we take* $\alpha = n + 1$*, we get Lemma* 1 *in* [57]*.*
2. *If we take* $\alpha = n + 1$ *and* $p = 1$*, we get Lemma* 3 *in* [44]*.*

Corollary 2.2.4. *Under the assumptions of Lemma* 2.2.3*,*

1. *If* $p = 1$ *in* (*i*)*, then we get*

$$J_\alpha^{c_1}\phi(c_2) = J_\alpha^{c_2}\phi(c_1) = \frac{1}{2}\left[J_\alpha^{c_1}\phi(c_2) + J_\alpha^{c_2}\phi(c_1)\right]. \qquad (2.27)$$

2. *If* $p = -1$ *in* (*ii*)*, then we get*

$$J_\alpha^{1/c_1}(\phi \circ \mu)(1/c_2) = J_\alpha^{1/c_2}(\phi \circ \mu)(1/c_1) = \frac{1}{2}\left[J_\alpha^{1/c_1}(\phi \circ \mu)(1/c_2) + J_\alpha^{1/c_2}(\phi \circ \mu)(1/c_1)\right]. \qquad (2.28)$$

Theorem 2.2.5. *Let* $\varPsi : [c_1, c_2] \subset (0, \infty) \to \mathbb{R}$ *be a* p*-convex function with* $c_1 < c_2$ *and* $\varPsi \in L_1[c_1, c_2]$*. Let* $\phi : [c_1, c_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *is non-negative, integrable and* p*-symmetric with respect to* $\left(\frac{c_1^p+c_2^p}{2}\right)^{1/p}$*. Then* *(i) for $p > 0$ we have*

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}\right)\left[J_\alpha^{c_1^p}(\phi\circ\mu)(c_2^p) + J_\alpha^{c_2^p}(\phi\circ\mu)(c_1^p)\right] \\
&\le \left[J_\alpha^{c_1^p}(\Psi\phi\circ\mu)(c_2^p) + J_\alpha^{c_2^p}(\Psi\phi\circ\mu)(c_1^p)\right] \\
&\le \frac{\Psi(c_1)+\Psi(c_2)}{2}\left[J_\alpha^{c_1^p}(\phi\circ\mu)(c_2^p) + J_\alpha^{c_2^p}(\phi\circ\mu)(c_1^p)\right],
\end{aligned} \tag{2.29}$$

with $\alpha > 0$ and $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_1^p, c_2^p]$.

(ii) for $p < 0$, we have

$$\begin{aligned}
&\Psi\left(\left[\frac{c_1^p + c_2^p}{2}\right]^{1/p}\right)\left[J_\alpha^{w_2^p}\phi\circ\mu(c_1^p) + J_\alpha^{c_1^p}\phi\circ\mu(c_2^p)\right] \\
&\le \left[J_\alpha^{c_2^p}(\Psi\phi\circ\mu)(c_1^p) + J_\alpha^{c_1^p}(\Psi\phi\circ\mu)(c_2^p)\right] \\
&\le \frac{\Psi(c_1)+\Psi(c_2)}{2}\left[J_\alpha^{c_2^p}(\phi\circ\mu)(c_1^p) + J_\alpha^{c_1^p}(\phi\circ\mu)(c_2^p)\right],
\end{aligned} \tag{2.30}$$

with $\alpha > 0$ and $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_2^p, c_1^p]$.

Proof. *(i)* Since Ψ is p-convex function on $[c_1, c_2]$, we have

$$\Psi\left(\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{\Psi(u)+\Psi(v)}{2}.$$

Taking $u^p = rc_1^p + (1-r)c_2^p$ and $v^p = (1-r)c_1^p + rc_2^p$ with $r \in [0,1]$, we get

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) + \Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)}{2}. \tag{2.31}$$

Multiplying both sides of (2.31) by $\frac{1}{n!}r^n(1-r)^{\alpha-n-1}\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{n!}\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right)\int_0^1 r^n(1-r)^{\alpha-n-1}\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&\le \frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&\quad + \frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)dr.
\end{aligned} \tag{2.32}$$

Since ϕ is non-negative, integrable and p-symmetric with respect to $\left(\frac{c_1^p+c_2^p}{2}\right)^{1/p}$, then

$$\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)=\phi\left((rc_2^p+(1-r)c_1^p)^{\frac{1}{p}}\right).$$

Also choosing $u=rc_1^p+(1-r)c_2^p$

$$\begin{aligned}
&\frac{2}{n!\,(c_2^p-c_1^p)^{\alpha}}\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right)\int_{c_1^p}^{c_2^p}(c_2^p-u)^n\,(u-c_1^p)^{\alpha-n-1}\,\phi(u^{\frac{1}{p}})du\\
&\leq\frac{1}{n!\,(c_2^p-c_1^p)^{\alpha}}\Big[\int_{c_1^p}^{c_2^p}(c_2^p-u)^n\,(u-c_1^p)^{\alpha-n-1}\,\Psi(u^{\frac{1}{p}})\phi(u^{\frac{1}{p}})du\\
&\quad+\int_{c_1^p}^{c_2^p}(c_2^p-u)^n\,(u-c_1^p)^{\alpha-n-1}\,\Psi\left((c_1^p+c_2^p-u)^{\frac{1}{p}}\right)\phi(u^{\frac{1}{p}})du\Big] \qquad (2.33)\\
&=\frac{1}{n!\,(c_2^p-c_1^p)^{\alpha}}\Big[\int_{c_1^p}^{c_2^p}(c_2^p-u)^n\,(u-c_1^p)^{\alpha-n-1}\,\Psi(u^{\frac{1}{p}})\phi(u^{\frac{1}{p}})du\\
&\quad+\int_{c_1^p}^{c_2^p}(u-c_1^p)^n\,(c_2^p-u)^{\alpha-n-1}\,\Psi(u^{\frac{1}{p}})\phi\left((c_1^p+c_2^p-u)^{\frac{1}{p}}\right)du\Big].
\end{aligned}$$

Therefore, by Lemma 2.2.3 we have

$$\begin{aligned}
&\frac{1}{(c_2^p-c_1^p)^{\alpha}}\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right)\left[J_{\alpha}^{c_1^p}(\phi\circ\mu)(c_2^p)+J_{\alpha}^{c_2^p}(\phi\circ\mu)(c_1^p)\right]\\
&\leq\frac{1}{(c_2^p-c_1^p)^{\alpha}}\left[J_{\alpha}^{c_1^p}(\Psi\phi\circ\mu)(c_2^p)+J_{\alpha}^{c_2^p}(\Psi\phi\circ\mu)(c_1^p)\right].
\end{aligned} \qquad (2.34)$$

This completes the first inequality of (2.29). For second inequality, we first note that if Ψ is p-convex function, then we have

$$\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)+\Psi\left((rc_2^p+(1-r)c_1^p)^{\frac{1}{p}}\right)\leq[\Psi(c_1)+\Psi(c_2)]. \qquad (2.35)$$

Multiplying both sides of (2.35) by $\frac{1}{n!}r^n(1-r)^{\alpha-n-1}\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)$, $\alpha>0$, and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr\\
&\quad+\frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\Psi\left((rc_2^p+(1-r)c_1^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr \qquad (2.36)\\
&\leq[\Psi(c_1)+\Psi(c_2)]\,\frac{1}{n!}\int_0^1 r^n(1-r)^{\alpha-n-1}\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr.
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{1}{(c_2^p - c_1^p)^\alpha}\left[J_\alpha^{c_1^p}(\Psi\phi\circ\mu)(c_2^p) + J_\alpha^{c_2^p}(\Psi\phi\circ\mu)(c_1^p)\right] \\
&\leq \frac{1}{(c_2^p - c_1^p)^\alpha}\left[\frac{\Psi(c_1)+\Psi(c_2)}{2}\right]\left[J_\alpha^{c-1^p}(\phi\circ\mu)(c_2^p) + J_\alpha^{c_2^p}(\phi\circ\mu)(c_1^p)\right].
\end{aligned} \tag{2.37}$$

This completes the proof.

(ii) Proof is similar to (i). □

Remark 5. *In Theorem* 2.2.5*,*

1. *If we take* $\alpha = n+1$*, we get Theorem* 9 *in* [57]*.*
2. *If we take* $\alpha = n+1$ *and* $p = 1$*, we get Theorem* 4 *in* [44]*.*

Corollary 2.2.5. *Under the assumptions of Theorem* 2.2.5*,*

(i) *If* $p = 1$*, then*

$$\begin{aligned}
&\Psi\left(\frac{c_1+c_2}{2}\right)\left[J_\alpha^{c_1}\phi(c_2) + J_\alpha^{c_2}\phi(c_1)\right] \\
&\leq \left[J_\alpha^{c_1}\Psi\phi(c_2) + J_\alpha^{c_2}\Psi\phi(c_1)\right] \\
&\leq \frac{\Psi(c_1)+\Psi(c_2)}{2}\left[J_\alpha^{c_1}\phi(c_2) + J_\alpha^{c_2}\phi(c_1)\right].
\end{aligned} \tag{2.38}$$

2. *If* $p = -1$*, then*

$$\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2}\right)\left[J_\alpha^{1/c_1}(\phi\circ\mu)(1/c_2) + J_\alpha^{1/c_2}\phi\circ\mu(1/c_1)\right] \\
&\leq \left[J_\alpha^{1/c_1}(\Psi\phi\circ\mu)(1/c_2) + J_\alpha^{1/c_2}(\Psi\phi\circ\mu)(1/c_1)\right] \\
&\leq \frac{\Psi(c_1)+\Psi(c_2)}{2}\left[J_\alpha^{1/c_1}(\phi\circ\mu)(1/c_2) + J_\alpha^{1/c_2}(\phi\circ\mu)(1/c_1)\right].
\end{aligned} \tag{2.39}$$

Remark 6. *In Corollary* 2.2.5*, if we take* $\alpha = n+1$*, we get equation* (1.15)*.*

Lemma 2.2.6. *Let* $\Psi : [c_1, c_2] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable mapping and* $\Psi' \in L_1[c_1, c_2]$*. Let* $\phi : [c_1, c_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *is non-negative, integrable and p-symmetric with respect to* $\left[\frac{c_1^p+c_2^p}{2}\right]^{1/p}$*. Then the following inequalities hold:*

(i) *for* $p > 0$

$$\frac{\Psi(c_1)+\Psi(c_2)}{2}\left[J_{\alpha}^{c_1^p}(\phi\circ\mu)(c_2^p)+J_{\alpha}^{c_2^p}(\phi\circ\mu)(c_1^p)\right]-\left[J_{\alpha}^{c_1^p}(\Psi\phi\circ\mu)(c_2^p)+J_{\alpha}^{c_2^p}(\Psi\phi\circ\mu)(c_1^p)\right]$$
$$\leq \frac{1}{n!}\int_{c_1^p}^{c_2^p}\Big[\int_{c_1^p}^{t}(c_2^p-s)^n(s-c_1^p)^{\alpha-n-1}(\phi\circ\mu)(s)ds$$
$$-\int_{t}^{c_2^p}(s-c_1^p)^n(c_2^p-s)^{\alpha-n-1}(\phi\circ\mu)(s)ds\Big](\Psi\circ\mu)'(t)dt, \tag{2.40}$$

where $\mu(u) = u^{1/p}$, *for all* $u \in [c_1^p, c_2^p]$.

(ii) *for* $p < 0$

$$\frac{\Psi(c_1)+\Psi(c_2)}{2}\left[J_{\alpha}^{c_2^p}(\phi\circ\mu)(c_1^p)+J_{\alpha}^{c_1^p}(\phi\circ\mu)(c_2^p)\right]-\left[J_{\alpha}^{c_2^p}(\Psi\phi\circ\mu)(c_1^p)+J_{\alpha}^{c_1^p}(\Psi\phi\circ\mu)(c_2^p)\right]$$
$$\leq \frac{1}{n!}\int_{c_2^p}^{c_1^p}\Big[\int_{c_2^p}^{t}(c_1^p-s)^n(s-c_2^p)^{\alpha-n-1}(\phi\circ\mu)(s)ds$$
$$-\int_{t}^{c_1^p}(s-c_2^p)^n(c_1^p-s)^{\alpha-n-1}(\phi\circ\mu)(s)ds\Big](\Psi\circ\mu)'(t)dt, \tag{2.41}$$

where $\mu(u) = u^{1/p}$, *for all* $x \in [c_2^p, c_1^p]$.

Proof. (i) Note that,

$$\begin{aligned} I &= \int_{c_1^p}^{c_2^p}\left(\int_{c_1^p}^{t}(c_2^p-s)^n(s-c_1^p)^{\alpha-n-1}(\phi\circ\mu)(s)ds\right)(\Psi\circ\mu)'(t)dt \\ &\quad -\int_{c_1^p}^{c_2^p}\left(\int_{t}^{c_2^p}(s-c_1^p)^n(c_2^p-s)^{\alpha-n-1}(\phi\circ\mu)(s)ds\right)(\Psi\circ\mu)'(t)dt \\ &= I_1 - I_2. \end{aligned} \tag{2.42}$$

Integrating by parts and using Lemma 2.2.3, we get

$$\begin{aligned} I_1 &= \left(\int_{c_1^p}^{t}(c_2^p-s)^n(s-c_1^p)^{\alpha-n-1}(\phi\circ\mu)(s)ds\right)(\Psi\circ\mu)(t)\Big|_{c_1^p}^{c_2^p} \\ &\quad -\int_{c_1^p}^{c_2^p}(c_2^p-t)^n(t-c_1^p)^{\alpha-n-1}(\phi\circ\mu)(t)(\Psi\circ\mu)(t)dt \\ &= n!\left[(\Psi\circ\mu)(c_2^p)J_{\alpha}^{c_1^p}(\phi\circ\mu)(c_2^p)-J_{\alpha}^{c_1^p}(\Psi\phi\circ\mu)(c_2^p)\right] \\ &= n!\left[\frac{(\Psi\circ\mu)(c_2^p)}{2}\left\{J_{\alpha}^{c_2^p}(\phi\circ\mu)(c_1^p)+J_{\alpha}^{c_1^p}(\phi\circ\mu)(c_2^p)\right\}-J_{\alpha}^{c_1^p}(\Psi\phi\circ\mu)(c_2^p)\right]. \end{aligned} \tag{2.43}$$

Similarly,

$$
\begin{aligned}
I_2 &= \left(\int_t^{c_2^p} (s-c_1^p)^n (c_2^p - s)^{\alpha-n-1} (\phi \circ \mu)(s) ds \right) (\Psi \circ \mu)(t) \Big|_{c_1^p}^{c_2^p} \\
&\quad + \int_{c_1^p}^{c_2^p} (t-c_1^p)^n (c_2^p - t)^{\alpha-n-1} (\phi \circ \mu)(t) (\Psi \circ \mu)(t) dt \\
&= n! \left[-(\Psi \circ \mu)(c_1^p) J_\alpha^{c_2^p} (\phi \circ \mu)(c_1^p) - J_\alpha^{c_2^p} (\Psi\phi \circ \mu)(c_1^p) \right] \\
&= n! \left[\frac{-(\Psi \circ \mu)(c_1^p)}{2} \left\{ J_\alpha^{c_2^p} (\phi \circ \mu)(c_1^p) + J_\alpha^{c_1^p} (\phi \circ \mu)(c_2^p) \right\} + J_\alpha^{c_2^p} (\Psi\phi \circ \mu)(c_1^p) \right].
\end{aligned}
\tag{2.44}
$$

Thus from (2.43) and (2.44) we get

$$
\begin{aligned}
I &= I_1 - I_2 \\
&= n! \Bigg[\frac{\Psi(c_1) + \Psi(c_2)}{2} \left[J_\alpha^{c_1^p} (\phi \circ \mu)(c_2^p) + J_\alpha^{c_2^p} (\phi \circ \mu)(c_1^p) \right] \\
&\quad - \left[J_\alpha^{c_1^p} (\Psi\phi \circ \mu)(c_2^p) + J_\alpha^{c_2^p} (\Psi\phi \circ \mu)(c_1^p) \right] \Bigg].
\end{aligned}
\tag{2.45}
$$

Multiplying (2.45) by $n!$ we obtain (2.40).

(ii) Proof is similar to (i). □

Remark 7. *In Lemma* 2.2.6*,*

1. *If we take* $\alpha = n+1$*, we get Lemma* 2 *in* [57]*.*

2. *If we take* $\alpha = n+1$ *and* $p = 1$*, we get Lemma* 4 *in* [44]*.*

Lemma 2.2.6 is also hold for convex and harmonically convex functions just by letting $p = 1$ and $p = -1$, respectively. Also, from Lemma 2.2.6 we can establish more useful results.

2.3 Integral inequalities via new fractional conformable integral operators.

Throughout this subsection, we consider $p \in \mathbb{R} \setminus 0$. We calculate some new inequalities via new fractional conformable integral operators.

2.3.1 Hermite-Hadamard type inequalities

We start our result with the following theorem.

Theorem 2.3.1. *Let $\Psi : [c_1, c_2] \subset (0, \infty) \to \mathbb{R}$ be a p-convex function such that $\Psi \in L_1[c_1, c_2]$.*
(i) Then for $p > 0$ we have

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}\right) \leq \frac{\alpha^\beta \Gamma(\beta+1)}{2(c_2^p - c_1^p)^{\alpha\beta}} \left[{}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi \circ \mu)(c_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi \circ \mu)(c_1^p)\right] \leq \frac{\Psi(c_1) + \Psi(c_2)}{2}, \tag{2.46}$$

where $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_1^p, c_2^p]$.
(ii) Then for $p < 0$ we have

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}\right) \leq \frac{\alpha^\beta \Gamma(\beta+1)}{2(c_1^p - c_2^p)^{\alpha\beta}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{c_1^p}(\Psi \circ \mu)(c_2^p) + {}^{\beta}_{c_2^p}\mathcal{J}^{\alpha}(\Psi \circ \mu)(c_1^p)\right] \leq \frac{\Psi(c_1) + \Psi(c_2)}{2}, \tag{2.47}$$

where $\mu(u) = u^{\frac{1}{p}}$, for all $u \in [c_2^p, c_1^p]$.

Proof. (i) Since Ψ is p-convex function on $[c_1, c_2]$, we have

$$\Psi\left(\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi(u) + \Psi(v)}{2}.$$

Taking $u^p = rc_1^p + (1-r)c_2^p$ and $v^p = (1-r)c_1^p + rc_2^p$ with $r \in [0, 1]$, we get

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) + \Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)}{2}. \tag{2.48}$$

Multiplying both sides of (2.48) by $\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}$ with $r \in (0, 1)$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0, 1]$, we obtain

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1} dr \\
&\leq \int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1} \Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) dr \\
&\quad + \int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1} \Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right) dr \\
&= I_1 + I_2.
\end{aligned} \tag{2.49}$$

By setting $u = rc_1^p + (1-r)c_2^p$, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&= \int_{c_2^p}^{c_1^p} \left(\frac{1-\left(\frac{u-c_2^p}{c_1^p-c_2^p}\right)^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{u-c_2^p}{c_1^p-c_2^p}\right)^{\alpha-1} (\Psi\circ\mu(u))\frac{du}{c_1^p-c_2^p} \\
&= \frac{1}{(c_2^p-c_1^p)^{\alpha\beta}} \int_{c_1^p}^{c_2^p} \left(\frac{(c_2^p-c_1^p)^\alpha-(c_2^p-u)^\alpha}{\alpha}\right)^{\beta-1} (c_2^p-u)^{\alpha-1}(\Psi\circ\mu)(u)du \\
&= \frac{\Gamma(\beta)}{(c_2^p-c_1^p)^{\alpha\beta}}\ {}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi\circ\mu)(c_1^p).
\end{aligned} \tag{2.50}$$

Similarly, by setting $u = rc_2^p + (1-r)c_1^p$, we have

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\Psi\left([(1-r)c_1^p+rc_2^p]^{\frac{1}{p}}\right)dr \\
&= \int_{c_1^p}^{c_2^p} \left(\frac{1-\left(\frac{u-c_1^p}{c_2^p-c_1^p}\right)^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{u-c_1^p}{c_2^p-c_1^p}\right)^{\alpha-1} (\Psi\circ\mu)(u)\frac{du}{c_2^p-c_1^p} \\
&= \frac{1}{(c_2^p-c_1^p)^{\alpha\beta}} \int_{c_1^p}^{c_2^p} \left(\frac{(c_2^p-c_1^p)^\alpha-(u-c_1^p)^\alpha}{\alpha}\right)^{\beta-1} (u-c_1^p)^{\alpha-1}(\Psi\circ\mu)(u)du \\
&= \frac{\Gamma(\beta)}{(c_2^p-c_1^p)^{\alpha\beta}}\ {}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi\circ\mu)(c_2^p).
\end{aligned} \tag{2.51}$$

Thus by putting values of I_1 and I_2 in (2.49), we get

$$\frac{1}{\alpha^\beta\beta}\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right) \le \frac{\Gamma(\beta)}{(c_2^p-c_1^p)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi\circ\mu)(c_1^p) + {}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi\circ\mu)(c_2^p)\right]. \tag{2.52}$$

This completes the first inequality of (2.46). For second inequality, we know that

$$\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right) + \Psi\left((rc_2^p+(1-r)c_1^p)^{\frac{1}{p}}\right) \le [\Psi(c_1)+\Psi(c_2)]. \tag{2.53}$$

Multiplying both sides of (2.53) by $\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}$ with $r \in (0,1)$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\frac{\Gamma(\beta)}{(c_2^p-c_1^p)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi\circ\mu)(c_1^p) + {}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi\circ\mu)(c_2^p)\right] \le \frac{1}{\alpha^\beta\beta}(\Psi(c_1)+\Psi(c_2)). \tag{2.54}$$

This completes the second inequality of (2.46). Hence the proof.

(*ii*) The proof is similar to (*i*). □

Remark 8. *In Theorem* 2.3.1*,*

1. *If we take* $p = 1$*, we get Theorem* 2.1 *in* [83]*.*

2. *If we take* $p = 1$ *and* $\alpha = 1$*, we get Theorem* 2 *in* [80]*.*

3. *If we take* $p = -1$ *and* $\alpha = 1$*, we get Theorem* 4 *in* [48]*.*

Corollary 2.3.1. *Under the assumption of Theorem* 2.3.1*, if we take* $p = -1$*, then we get*

$$\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2}\right) \\
&\leq \frac{(c_1c_2)^{\alpha\beta}\alpha^\beta\Gamma(\beta+1)}{2(c_2-c_1)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{1/c_1}(\Psi\circ\mu)\left(\frac{1}{c_2}\right) + {}^{\beta}_{1/c_2}\mathcal{J}^{\alpha}(\Psi\circ\mu)\left(\frac{1}{c_1}\right)\right] \\
&\leq \frac{\Psi(c_1)+\Psi(c_2)}{2},
\end{aligned} \tag{2.55}$$

where $\mu(u) = \frac{1}{u}$ *for all* $u \in [\frac{1}{c_2}, \frac{1}{c_1}]$*.*

Lemma 2.3.2. *Let* $\Psi : [c_1, c_2] \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* (c_1, c_2) *with* $c_1 < c_2$ *such that* $\Psi' \in L_1[c_1, c_2]$*, then*

(i) for $p > 0$

$$\begin{aligned}
&{}_1\Delta_\Psi(c_1,c_2;\alpha;\beta;\mathcal{J}) \\
&= \frac{(c_2^p-c_1^p)\alpha^\beta}{2p}\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-r)^\alpha}{\alpha}\right)^\beta\right]A_r^{\frac{1}{p}-1}\Psi'\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr,
\end{aligned} \tag{2.56}$$

where $A_r = (rc_1^p + (1-r)c_2^p)$ *and*

$$\begin{aligned}
&{}_1\Delta_\Psi(c_1,c_2;\alpha;\beta;\mathcal{J}) \\
&= \left(\frac{\Psi(c_1^p)+\Psi(c_2^p)}{2}\right) - \frac{\Gamma(\beta+1)\alpha^\beta}{2(c_2^p-c_1^p)^{\alpha\beta}}\left[{}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi\circ\mu)(c_2^p) + {}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi\circ\mu)(c_1^p)\right].
\end{aligned}$$

(ii) for $p < 0$

$$\begin{aligned}
&{}_2\Delta_\Psi(c_1,c_2;\alpha;\beta;\mathcal{J}) \\
&= \frac{(c_1^p-c_2^p)\alpha^\beta}{2p}\int_0^1\left[\left(\frac{1-r^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-r)^\alpha}{\alpha}\right)^\beta\right]B_r^{\frac{1}{p}-1}\psi'\left((rc_2^p+(1-r)c_1^p)^{\frac{1}{p}}\right)dr,
\end{aligned} \tag{2.57}$$

where $B_r = (rc_2^p + (1-r)c_1^p)$ *and*

$$
\begin{aligned}
&{}_2\Delta_{\Psi}(c_1, c_2; \alpha; \beta; \mathcal{J}) \\
&= \left(\frac{\Psi(c_1^p) + \Psi(c_2^p)}{2}\right) - \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(c_1^p - c_2^p)^{\alpha\beta}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{c_1^p}(\Psi \circ \mu)(c_2^p) + {}^{\beta}_{c_2^p}\mathcal{J}^{\alpha}(\Psi \circ \mu)(c_1^p) \right].
\end{aligned}
$$

Proof. (i) Consider

$$
\begin{aligned}
&\int_0^1 \left[\left(\frac{1-r^{\alpha}}{\alpha}\right)^{\beta} - \left(\frac{1-(1-r)^{\alpha}}{\alpha}\right)^{\beta} \right] A_r^{\frac{1}{p}-1} \Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \\
&= \int_0^1 \left(\frac{1-r^{\alpha}}{\alpha}\right)^{\beta} A_r^{\frac{1}{p}-1} \Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \\
&\quad - \int_0^1 \left(\frac{1-(1-r)^{\alpha}}{\alpha}\right)^{\beta} A_r^{\frac{1}{p}-1} \Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \\
&= I_1 - I_2.
\end{aligned}
\tag{2.58}
$$

Then by integration by parts, we have

$$
\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1-r^{\alpha}}{\alpha}\right)^{\beta} A_r^{\frac{1}{p}-1} \Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \\
&= \left(\frac{1-r^{\alpha}}{\alpha}\right)^{\beta} \frac{p}{c_1^p - c_2^p} \Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) \Big|_0^1 \\
&\quad - \frac{p}{c_2^p - c_1^p} \int_0^1 \beta \left(\frac{1-r^{\alpha}}{\alpha}\right)^{\beta-1} r^{\alpha-1} \Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \\
&= \frac{p}{\alpha^{\beta}(c_2^p - c_1^p)} \Psi(c_2^p) - \frac{p\beta}{c_2^p - c_1^p} \frac{\Gamma(\beta)}{(c_2^p - c_1^p)^{\alpha\beta}}\, {}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi \circ \mu)(c_1^p) \\
&= \frac{p}{c_2^p - c_1^p} \left[\frac{\Psi(c_2^p)}{\alpha^{\beta}} - \frac{\Gamma(\beta+1)}{(c_2^p - c_1^p)^{\alpha\beta}}\, {}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi \circ \mu)(c_1^p) \right].
\end{aligned}
\tag{2.59}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1-(1-r)^{\alpha}}{\alpha}\right)^{\beta} A_r^{\frac{1}{p}-1} \Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \\
&= \left(\frac{1-(1-r)^{\alpha}}{\alpha}\right)^{\beta} \frac{p}{c_1^p - c_2^p} \Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) \Big|_0^1 \\
&\quad - \frac{p}{c_1^p - c_2^p} \int_0^1 \beta \left(\frac{1-(1-r)^{\alpha}}{\alpha}\right)^{\beta-1} (1-r)^{\alpha-1} \Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}} \right) dr \\
&= -\frac{p}{\alpha^{\beta}(c_2^p - c_1^p)} \Psi(c_1^p) + \frac{p\beta}{c_2^p - c_1^p} \frac{\Gamma(\beta)}{(c_2^p - c_1^p)^{\alpha\beta}}\, {}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi \circ \mu)(c_2^p) \\
&= -\frac{p}{c_2^p - c_1^p} \left[\frac{\Psi(c_2^p)}{\alpha^{\beta}} - \frac{\Gamma(\beta+1)}{(c_2^p - c_1^p)^{\alpha\beta}}\, {}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi \circ \mu)(c_2^p) \right],
\end{aligned}
\tag{2.60}
$$

where we used the change of variable with $x = 1 - r$. Thus by adding I_1, $-I_2$ and then by multiplying both sides by $\frac{\alpha^\beta(c_2^p - c_1^p)}{2p}$, we get the required result (2.56).

(ii) The proof is similar to (i). □

Remark 9. *In Lemma* 2.3.2*,*

1. *If we take* $p = 1$ *we get Lemma* 3.1 *in* [83].

2. *If we take* $p = 1$ *and* $\alpha = 1$ *we get Lemma* 2 *in* [80].

Theorem 2.3.2. *Let* $\Psi : [c_1, c_2] \subset (0, \infty) \to \mathbb{R}$ *be a differentiable function on* (c_1, c_2) *with* $c_1 < c_2$ *such that* $\Psi' \in L_1[c_1, c_2]$. *Let* $|\Psi'|^q$*, where* $q \geq 1$*, is* p*-convex function, then*

(i) *for* $p > 0$

$$\begin{aligned} &| \, {}_1\Delta_\Psi(c_1, c_2; \alpha; \beta; \mathcal{J})| \\ &\leq \frac{(c_2^p - c_1^p)\alpha^\beta}{2p} \left(\frac{c_2^{1-p}}{2} \, {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{c_1^p}{c_2^p}\right)\right)^{1-\frac{1}{q}} \\ &\quad \times \left(\frac{1}{\alpha^{\beta+1}} \mathcal{B}\left(\frac{2}{\alpha}, \beta + 1\right) [|\Psi'(c_1)|^q + |\Psi'(c_2)|^q] \right)^q, \end{aligned} \tag{2.61}$$

(ii) *for* $p < 0$

$$\begin{aligned} &| \, {}_2\Delta_\Psi(c_1, c_2; \alpha; \beta; \mathcal{J})| \\ &\leq \frac{(c_1^p - c_2^p)\alpha^\beta}{2p} \left(\frac{c_1^{1-p}}{2} \, {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{c_2^p}{c_1^p}\right)\right)^{1-\frac{1}{q}} \\ &\quad \times \left(\frac{1}{\alpha^{\beta+1}} \mathcal{B}\left(\frac{2}{\alpha}, \beta + 1\right) [|\Psi'(c_1)|^q + |\Psi'(c_2)|^q] \right)^q, \end{aligned} \tag{2.62}$$

where $\mathcal{B}$ *and* ${}_2F_1$ *are classical Beta function and Hypergeometric function, respectively.*

Proof. (i) Using Lemma 2.3.2, property of modulus, power mean inequality and the fact that $|\Psi'|^q$ is p-convex function, we have

$$\begin{aligned} &| \, {}_1\Delta_\Psi(c_1, c_2; \alpha; \beta; \mathcal{J})| \\ &= \frac{(c_2^p - c_1^p)\alpha^\beta}{2p} \left| \int_0^1 \left[\left(\frac{1 - r^\alpha}{\alpha}\right)^\beta - \left(\frac{1 - (1 - r)^\alpha}{\alpha}\right)^\beta \right] A_r^{\frac{1}{p}-1} \Psi'\left((rc_1^p + (1 - r)c_2^p)^{\frac{1}{p}}\right) dr \right| \\ &\leq \frac{(c_2^p - c_1^p)\alpha^\beta}{2p} \left| \int_0^1 \left[\left(\frac{1 - r^\alpha}{\alpha}\right)^\beta + \left(\frac{1 - (1 - r)^\alpha}{\alpha}\right)^\beta \right] A_r^{\frac{1}{p}-1} \Psi'\left((rc_1^p + (1 - r)c_2^p)^{\frac{1}{p}}\right) \right| dr \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(c_2^p - c_1^p)\alpha^\beta}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}} \times \\
&\left(\int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^\beta + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^\beta \right] |\Psi' \left((r c_1^p + (1-r) c_2^p)^{\frac{1}{p}} \right) |^q dr \right)^q \\
&\leq \frac{(c_2^p - c_1^p)\alpha^\beta}{2p} \left(\int_0^1 A_r^{\frac{1}{p}-1} dr \right)^{1-\frac{1}{q}} \times \\
&\left(\int_0^1 \left[\left(\frac{1-r^\alpha}{\alpha} \right)^\beta + \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^\beta \right] \left(r|\Psi'(c_1)|^q + (1-r)|\Psi'(c_2)|^q \right) dr \right)^q \qquad (2.63) \\
&= \frac{(c_2^p - c_1^p)\alpha^\beta}{2p} \Upsilon^{1-\frac{1}{q}} \left(|\Psi'(c_1)|^q \int_0^1 \left[r \left(\frac{1-r^\alpha}{\alpha} \right)^\beta + r \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^\beta \right] dr \right. \\
&\left. + |\Psi'(c_2)|^q \int_0^1 \left[(1-r) \left(\frac{1-r^\alpha}{\alpha} \right)^\beta + (1-r) \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^\beta \right] dr \right)^q,
\end{aligned}
$$

where

$$\Upsilon = \int_0^1 A_r^{\frac{1}{p}-1} dr = \frac{c_2^{1-p}}{2} \, {}_2F_1 \left(1 - \frac{1}{p}, 1; 2; 1 - \frac{c_1^p}{c_2^p} \right),$$

and by using change of variable as $x = r^\alpha$ and $y = (1-r)^\alpha$, we get

$$\int_0^1 r \left(\frac{1-r^\alpha}{\alpha} \right)^\beta dr = \frac{1}{\alpha^{\beta+1}} \mathcal{B} \left(\frac{2}{\alpha}, \beta+1 \right),$$

$$\int_0^1 r \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^\beta dr = \frac{1}{\alpha^{\beta+1}} \left[\mathcal{B} \left(\frac{1}{\alpha}, \beta+1 \right) - \mathcal{B} \left(\frac{2}{\alpha}, \beta+1 \right) \right],$$

$$\int_0^1 (1-r) \left(\frac{1-r^\alpha}{\alpha} \right)^\beta dr = \frac{1}{\alpha^{\beta+1}} \frac{1}{\alpha^{\beta+1}} \left[\mathcal{B} \left(\frac{1}{\alpha}, \beta+1 \right) - \mathcal{B} \left(\frac{2}{\alpha}, \beta+1 \right) \right],$$

$$\int_0^1 (1-r) \left(\frac{1-(1-r)^\alpha}{\alpha} \right)^\beta dr = \frac{1}{\alpha^{\beta+1}} \mathcal{B} \left(\frac{2}{\alpha}, \beta+1 \right).$$

Thus by substituting all above equalities in (2.63), we get the inequality (2.61).

(ii) Proof is similar to (i). □

Corollary 2.3.3. *Under the assumptions of Theorem* 2.3.2. *If we take* $p=-1$*, then we get*

$$\left|\left(\frac{\Psi\left(\frac{1}{c_1}\right)+\Psi\left(\frac{1}{c_2}\right)}{2}\right)-\frac{(c_1c_2)^{\alpha\beta}\Gamma(\beta+1)\alpha^{\beta}}{2(c_2-c_1)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{1}{c_1}}(\Psi\circ\mu)\left(\frac{1}{c_2}\right)+{}^{\beta}_{\frac{1}{c_2}}\mathcal{J}^{\alpha}(\Psi\circ\mu)\left(\frac{1}{c_1}\right)\right]\right|$$
$$\leq\frac{(c_2-c_1)\alpha^{\beta}}{-2c_1c_2}\left(\frac{c_1^2}{2}\,{}_2F_1\left(2,1;2;1-\frac{c_1}{c_2}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\alpha^{\beta+1}}\mathcal{B}\left(\frac{2}{\alpha},\beta+1\right)\left[|\Psi'(c_1)|^q+|\Psi'(c_2)|^q\right]\right)^{q}, \tag{2.64}$$

where $\mu(u)=1/u$ *for all* $u\in[\frac{1}{c_2},\frac{1}{c_1}]$.

2.3.2 Hermite-Hadamard-Fejér type inequalities

In order to give result involving Hermite-Hadamard-Fejér type inequality we need following lemma.

Lemma 2.3.4. *Let* $p\in\mathbb{R}\backslash\{0\}$ *and* $\phi:[c_1,c_2]\subseteq(0,\infty)\to\mathbb{R}$ *is integrable and p-symmetric with respect to* $\left[\frac{c_1^p+c_2^p}{2}\right]^{1/p}$*, then*
(i) for $p>0$

$${}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\phi\circ\mu)(c_2^p)={}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\phi\circ\mu)(c_1^p)=\frac{1}{2}\left[{}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\phi\circ\mu)(c_2^p)+{}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\phi\circ\mu)(c_1^p)\right], \tag{2.65}$$

with $\alpha>0$ *and* $\mu(u)=u^{\frac{1}{p}}$*, for all* $u\in[c_1^p,c_2^p]$.
(ii) for $p<0$

$${}^{\beta}_{c_2^p}\mathcal{J}^{\alpha}(\phi\circ\mu)(c_1^p)={}^{\beta}\mathcal{J}^{\alpha}_{c_1^p}(\phi\circ\mu)(c_2^p)=\frac{1}{2}\left[{}^{\beta}_{c_2^p}\mathcal{J}^{\alpha}(\phi\circ\mu)(c_1^p)+{}^{\beta}\mathcal{J}^{\alpha}_{c_1^p}(\phi\circ\mu)(c_2^p)\right], \tag{2.66}$$

with $\alpha>0$ *and* $\mu(u)=u^{\frac{1}{p}}$*, for all* $u\in[c_2^p,c_1^p]$.

Proof. (*i*) Since ϕ is p-symmetric with respect to $\left[\frac{c_1^p+c_2^p}{2}\right]^{1/p}$, then by definition we have $\phi(u^{\frac{1}{p}})=\phi\left((c_1^p+c_2^p-u)^{\frac{1}{p}}\right)$, for all $u\in[c_1^p,c_2^p]$. In the following integral, by setting

$x = c_1^p + c_2^p - u$ gives

$$\begin{aligned}
{}_{c_1^p}^{\beta}\mathcal{J}^{\alpha}(\phi \circ \mu)(c_2^p) &= \frac{1}{\Gamma(\beta)} \int_{c_1^p}^{c_2^p} \left(\frac{(c_2^p - c_1^p)^{\alpha} - (x - c_1^p)^{\alpha}}{\alpha} \right)^{\beta-1} (x - c_1^p)^{\alpha-1} \phi(x^{\frac{1}{p}}) dx \\
&= \frac{1}{\Gamma(\beta)} \int_{c_1^p}^{c_2^p} \left(\frac{(c_2^p - c_1^p)^{\alpha} - (c_2^p - u)^{\alpha}}{\alpha} \right)^{\beta-1} \\
&\quad \times (c_2^p - u)^{\alpha-1} \phi\left([c_1^p + c_2^p - u]^{\frac{1}{p}} \right) du \\
&= \frac{1}{\Gamma(\beta)} \int_{c_1^p}^{c_2^p} \left(\frac{(c_2^p - c_1^p)^{\alpha} - (c_2^p - u)^{\alpha}}{\alpha} \right)^{\beta-1} (c_2^p - u)^{\alpha-1} \phi\left(u^{\frac{1}{p}} \right) du \\
&= {}^{\beta}\mathcal{J}_{c_2^p}^{\alpha}(\phi \circ \mu)(c_1^p).
\end{aligned} \tag{2.67}$$

This completes the proof.

(*ii*) The proof is similar to (*i*). □

Remark 10. *In Lemma* 2.3.4*,*

1. *If we take* $\alpha = 1$*, we get Lemma* 1 *in* [57].
2. *If we take* $\alpha = 1$ *and* $p = 1$*, we get Lemma* 3 *in* [44].

Corollary 2.3.5. *Under the assumption of Lemma* 2.3.4*,*

1. *If we take* $p = 1$ *we get similar result for convex function.*
2. *If we take* $p = -1$ *we get similar result for harmonically convex function.*

Theorem 2.3.3. *Let* $p \in \mathbb{R}\backslash\{0\}$*. Let* $\Psi : [c_1, c_2] \subset (0, \infty) \to \mathbb{R}$ *be a p-convex function with* $c_1 < c_2$ *and* $\Psi \in L_1[c_1, c_2]$*. Let* $\phi : [c_1, c_2] \subseteq (0, \infty) \to \mathbb{R}$ *is non-negative, integrable and p-symmetric with respect to* $\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}$*, then*

(*i*) *for* $p > 0$

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}\right) \left[{}_{c_1^p}^{\beta}\mathcal{J}^{\alpha}(\phi \circ \mu)(c_2^p) + {}^{\beta}\mathcal{J}_{c_2^p}^{\alpha}(\phi \circ \mu)(c_1^p)\right] \\
&\leq \left[{}_{c_1^p}^{\beta}\mathcal{J}^{\alpha}(\Psi\phi \circ \mu)(c_2^p) + {}^{\beta}\mathcal{J}_{c_2^p}^{\alpha}(\Psi\phi \circ \mu)(c_1^p)\right] \\
&\leq \frac{\Psi(c_1) + \Psi(c_2)}{2} \left[{}_{c_1^p}^{\beta}\mathcal{J}^{\alpha}(\phi \circ \mu)(c_2^p) + {}^{\beta}\mathcal{J}_{c_2^p}^{\alpha}(\phi \circ \mu)(c_1^p)\right],
\end{aligned} \tag{2.68}$$

with $\alpha > 0$ *and* $\mu(u) = u^{\frac{1}{p}}$, *for all* $u \in [c_1^p, c_2^p]$.
(ii) *for* $p < 0$

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{1/p}\right)\left[{}^{\beta}_{c_2^p}\mathcal{J}^{\alpha}(\phi \circ \mu)(c_1^p) + {}^{\beta}\mathcal{J}^{\alpha}_{c_1^p}(\phi \circ \mu)(c_2^p)\right] \\
&\leq \left[{}^{\beta}_{c_2^p}\mathcal{J}^{\alpha}(\Psi\phi \circ \mu)(c_1^p) + {}^{\beta}\mathcal{J}^{\alpha}_{c_1^p}(\Psi\phi \circ \mu)(c_2^p)\right] \\
&\leq \frac{\Psi(c_1) + \Psi(c_2)}{2}\left[{}^{\beta}_{c_2^p}\mathcal{J}^{\alpha}(\phi \circ \mu)(c_1^p) + {}^{\beta}\mathcal{J}^{\alpha}_{c_1^p}(\phi \circ \mu)(c_2^p)\right],
\end{aligned} \tag{2.69}$$

with $\alpha > 0$ *and* $\mu(u) = u^{\frac{1}{p}}$, *for all* $u \in [c_2^p, c_1^p]$.

Proof. (i) Since Ψ is p-convex function on $[c_1, c_2]$, we have

$$\Psi\left(\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi(u) + \Psi(v)}{2}.$$

Taking $u^p = rc_1^p + (1-r)c_2^p$ and $v^p = (1-r)c_1^p + rc_2^p$ with $r \in [0,1]$, we get

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) + \Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)}{2}. \tag{2.70}$$

Multiplying both sides of (2.70) by $\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)$ with $r \in (0,1)$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned}
&2\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right)\int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&\leq \int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)dr \\
&\quad + \int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)dr.
\end{aligned} \tag{2.71}$$

Since ϕ is non-negative, integrable and p-symmetric with respect to $\left(\frac{c_1^p+c_2^p}{2}\right)^{1/p}$, then

$$\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) = \phi\left((rc_2^p + (1-r)c_1^p)^{\frac{1}{p}}\right).$$

Also choosing $u = rc_1^p + (1-r)c_2^p$

$$\begin{aligned}
&\frac{2}{(c_2^p - c_1^p)^{\alpha\beta}}\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \\
&\quad \times \int_{c_1^p}^{c_2^p}\left(\frac{(c_2^p - c_1^p)^\alpha - (c_2^p - u)^\alpha}{\alpha}\right)^{\beta-1}(c_2^p - u)^{\alpha-1}\phi\left(u^{\frac{1}{p}}\right)du \\
&\leq \frac{1}{(c_2^p - c_1^p)^{\alpha\beta}}\int_{c_1^p}^{c_2^p}\left(\frac{(c_2^p - c_1^p)^\alpha - (c_2^p - u)^\alpha}{\alpha}\right)^{\beta-1}(c_2^p - u)^{\alpha-1}\Psi\left(u^{\frac{1}{p}}\right)\phi\left(u^{\frac{1}{p}}\right)du \\
&\quad + \frac{1}{(c_2^p - c_1^p)^{\alpha\beta}}\int_{c_1^p}^{c_2^p}\left(\frac{(c_2^p - c_1^p)^\alpha - (c_2^p - u)^\alpha}{\alpha}\right)^{\beta-1} \\
&\quad \times (c_2^p - u)^{\alpha-1}\Psi\left([c_1^p + c_2^p - u]^{\frac{1}{p}}\right)\phi\left(u^{\frac{1}{p}}\right)du \\
&= \frac{1}{(c_2^p - c_1^p)^{\alpha\beta}}\Big[\int_{c_1^p}^{c_2^p}\left(\frac{(c_2^p - c_1^p)^\alpha - (c_2^p - u)^\alpha}{\alpha}\right)^{\beta-1}(c_2^p - u)^{\alpha-1}\Psi\left(u^{\frac{1}{p}}\right)\phi\left(u^{\frac{1}{p}}\right)du \\
&\quad + \int_{c_1^p}^{c_2^p}\left(\frac{(c_2^p - c_1^p)^\alpha - (u - c_1^p)^\alpha}{\alpha}\right)^{\beta-1}(u - c_1^p)^{\alpha-1}\Psi\left(u^{\frac{1}{p}}\right)\phi\left([c_1^p + c_2^p - u]^{\frac{1}{p}}\right)\Big]du \\
&= \frac{1}{(c_2^p - c_1^p)^{\alpha\beta}}\Big[\int_{c_1^p}^{c_2^p}\left(\frac{(c_2^p - c_1^p)^\alpha - (c_2^p - u)^\alpha}{\alpha}\right)^{\beta-1}(c_2^p - u)^{\alpha-1}\Psi\left(u^{\frac{1}{p}}\right)\phi\left(u^{\frac{1}{p}}\right)du \\
&\quad + \int_{c_1^p}^{c_2^p}\left(\frac{(c_2^p - c_1^p)^\alpha - (u - c_1^p)^\alpha}{\alpha}\right)^{\beta-1}(u - c_1^p)^{\alpha-1}\Psi\left(u^{\frac{1}{p}}\right)\phi\left(u^{\frac{1}{p}}\right)\Big]du.
\end{aligned} \tag{2.72}$$

Therefore, by Lemma 2.3.4 we have

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right)\left[{}^{\beta}_{c_1^p}\mathcal{J}^\alpha(\phi\circ\mu)(c_2^p) + {}^{\beta}\mathcal{J}^\alpha_{c_2^p}(\phi\circ\mu)(c_1^p)\right] \\
&\leq \left[{}^{\beta}_{c_1^p}\mathcal{J}^\alpha(\Psi\phi\circ\mu)(c_2^p) + {}^{\beta}\mathcal{J}^\alpha_{c_2^p}(\Psi\phi\circ\mu)(c_1^p)\right].
\end{aligned} \tag{2.73}$$

This completes the first inequality of (2.68). For second inequality, we first note that if Ψ is p-convex function, then we have

$$\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) + \Psi\left((rc_2^p + (1-r)c_1^p)^{\frac{1}{p}}\right) \leq [\Psi(c_1) + \Psi(c_2)]. \tag{2.74}$$

Multiplying both sides of (2.74) by $\left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\phi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)$ with $r \in (0,1)$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over

$[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr\\
&\quad+\int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\Psi\left(((1-r)c_1^p+rc_2^p)^{\frac{1}{p}}\right)\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr\\
&\leq [\Psi(c_1)+\Psi(c_2)]\int_0^1 \left(\frac{1-r^\alpha}{\alpha}\right)^{\beta-1} r^{\alpha-1}\phi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr.
\end{aligned}\tag{2.75}$$

That is,

$$\begin{aligned}
&\frac{1}{(c_2^p-c_1^p)^{\alpha\beta}}\left[{}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\Psi\phi\circ\mu)(c_2^p)+{}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\Psi\phi\circ\mu)(c_1^p)\right]\\
&\leq \frac{1}{(c_2^p-c_1^p)^{\alpha\beta}}\left[{}^{\beta}_{c_1^p}\mathcal{J}^{\alpha}(\phi\circ\mu)(c_2^p)+{}^{\beta}\mathcal{J}^{\alpha}_{c_2^p}(\phi\circ\mu)(c_1^p)\right]\left[\frac{\Psi(c_1)+\Psi(c_2)}{2}\right].
\end{aligned}\tag{2.76}$$

This completes the proof.

(ii) Proof is similar to (i). □

Remark 11. *In Theorem* 2.3.3,

1. *If we take* $\alpha=1$, *we get Theorem* 9 *in* [57].
2. *If we take* $\alpha=1$ *and* $p=1$, *we get Theorem* 4 *in* [44].

Corollary 2.3.6. *Under the similar assumption of Theorem* 2.3.3,

1. *If we take* $p=1$, *we get*

$$\begin{aligned}
&\Psi\left(\frac{c_1+c_2}{2}\right)\left[{}^{\beta}_{c_1}\mathcal{J}^{\alpha}\phi(c_2)+{}^{\beta}\mathcal{J}^{\alpha}_{c_2}\phi(c_1)\right]\\
&\leq \left[{}^{\beta}_{c_1}\mathcal{J}^{\alpha}\Psi\phi(c_2)+{}^{\beta}\mathcal{J}^{\alpha}_{c_2}\Psi\phi(c_1)\right]\\
&\leq \frac{\Psi(c_1)+\Psi(c_2)}{2}\left[{}^{\beta}_{c_1}\mathcal{J}^{\alpha}\phi(c_2)+{}^{\beta}\mathcal{J}^{\alpha}_{c_2}\phi(c_1)\right].
\end{aligned}\tag{2.77}$$

2. *If we take* $p=-1$, *we get*

$$\begin{aligned}
&\Psi\left(\frac{c_1+c_2}{2c_1c_2}\right)\left[{}^{\beta}_{1/c_2}\mathcal{J}^{\alpha}(\phi\circ\mu)\left(\frac{1}{c_1}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{1/c_1}(\phi\circ\mu)\left(\frac{1}{c_2}\right)\right]\\
&\leq \left[{}^{\beta}_{1/c_2}\mathcal{J}^{\alpha}(\Psi\phi\circ\mu)\left(\frac{1}{c_1}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{1/c_1}(\Psi\phi\circ\mu)\left(\frac{1}{c_2}\right)\right]\\
&\leq \frac{\Psi(c_1)+\Psi(c_2)}{2}\left[{}^{\beta}_{1/c_2}\mathcal{J}^{\alpha}(\phi\circ\mu)\left(\frac{1}{c_1}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{1/c_1}(\phi\circ\mu)\left(\frac{1}{c_2}\right)\right].
\end{aligned}\tag{2.78}$$

where $\mu(u)=1/u$ *for all* $u\in\left[\frac{1}{c_2},\frac{1}{c_1}\right]$.

Chapter 3

Integral inequalities for s- and m-convex functions via Katugampola fractional integral

In this chapter, we obtain the Hermite-Hadamard type inequalities for s-convex functions and m-convex functions via generalized fractional integral, known as Katugampola fractional integral.

3.1 Introduction

Katugampola fractional integral was introduced by Katugampola, in 2014, which is extension of Riemann−Liouville fractional integral and Hadamard fractional integral. Now first we define these integrals one by one.

Definition 3.1.1 ([77]). *Let $\alpha > 0$ with $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $1 < u < c_2$. The left- and right-side Riemann−Liouville fractional integrals of order α of function Ψ are given by*

$$J^{\alpha}_{c_1+}\Psi(u) = \frac{1}{\Gamma(\alpha)} \int_{c_1}^{u} (u-t)^{\alpha-1}\Psi(t)dt,$$

and

$$J^{\alpha}_{c_2-}\Psi(u) = \frac{1}{\Gamma(\alpha)} \int_{u}^{c_2} (t-u)^{\alpha-1}\Psi(t)dt,$$

respectively, where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

Definition 3.1.2 ([79])**.** *Let $\alpha > 0$ with $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $1 < u < c_2$. The left- and right-side Hadamard fractional integrals of order α of function Ψ are given by*

$$H^{\alpha}_{c_1+}\Psi(u) = \frac{1}{\Gamma(\alpha)} \int_{c_1}^{u} \left(\ln \frac{u}{t}\right)^{\alpha-1} \frac{\Psi(t)}{t} dt,$$

and

$$H^{\alpha}_{c_2-}\Psi(u) = \frac{1}{\Gamma(\alpha)} \int_{u}^{c_2} \left(\ln \frac{t}{u}\right)^{\alpha-1} \frac{\Psi(t)}{t} dt.$$

Definition 3.1.3 ([52])**.** *Let $[c_1, c_2] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha > 0$ of $\Psi \in X^p_c(c_1, c_2)$ are defined by,*

$${}^{\rho}I^{\alpha}_{c_1+}\Psi(u) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{c_1}^{u} (u^{\rho} - t^{\rho})^{\alpha-1} t^{\rho-1} \Psi(t) dt,$$

and

$${}^{\rho}I^{\alpha}_{c_1-}\Psi(u) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{u}^{c_2} (t^{\rho} - u^{\rho})^{\alpha-1} t^{\rho-1} \Psi(t) dt,$$

with $c_1 < u < c_2$ and $\rho > 0$. Here $X^p_c(c_1, c_2)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty\ )$ is the space of those complex valued Lebesgue measurable functions Ψ on $[c_1, c_2]$ for which $\|\Psi\|_{X^p_c} < \infty$, where the norm is defined by,

$$\|\Psi\|_{X^p_c} = \left(\int_{c_1}^{c_2} |t^c \Psi(t)|^p \frac{dt}{t}\right)^{1/p} < \infty,$$

for $1 \leq p < \infty$, $c \in \mathbb{R}$ and for the case $p = \infty$,

$$\|\Psi\|_{X^{\infty}_c} = ess\ sup_{c_1 \leq t \leq c_2}[t^c |\Psi(t)|].$$

Here ess sup stands for essential supremum.

From above definitions we have following obvious result.

Theorem 3.1.1 ([52])**.** *Let $\alpha > 0$ and $\rho > 0$. Then for $u > c_1$,*

1. $\lim_{\rho \to 1} {}^{\rho}I^{\alpha}_{c_1+}\Psi(u) = J^{\alpha}_{c_1+}\Psi(u)$,
2. $\lim_{\rho \to 0+} {}^{\rho}I^{\alpha}_{c_1+}\Psi(u) = H^{\alpha}_{c_1+}\Psi(u)$.

3.2 Hermite-Hadamard type inequalities for s-convex function

Here we give Hermite-Hadamard type inequalities for s-convex function.

Theorem 3.2.1. *Let $\alpha, \rho > 0$ and $\Psi : [c_1^\rho, c_2^\rho] \subset [0, \infty) \to \mathbb{R}$ be a positive function with $0 \le c_1 < c_2$ and $\Psi \in X_c^p(c_1^\rho, c_2^\rho)$. Let Ψ is also a s-convex function on $[c_1^\rho, c_2^\rho]$, then the following inequalities hold:*

$$\begin{aligned} &2^{s-1}\Psi\left(\frac{c_1^\rho + c_2^\rho}{2}\right) \\ &\le \frac{\rho^\alpha \Gamma(\alpha+1)}{2(c_2^\rho - c_1^\rho)^\alpha}\left[{}^\rho I_{c_1+}^\alpha \Psi(c_2^\rho) + {}^\rho I_{c_2-}^\alpha \Psi(c_1^\rho)\right] \\ &\le \left[\frac{\alpha}{\alpha+s} + \alpha\mathcal{B}(\alpha, s+1)\right]\frac{\Psi(c_1^\rho) + \Psi(c_2^\rho)}{2}, \end{aligned} \tag{3.1}$$

where $\mathcal{B}$ is a beta function.

Proof. Let $r \in [0, 1]$. Consider $u, v \in [c_1, c_2]$, $c_1 \ge 0$, defined by $u^\rho = r^\rho c_1^\rho + (1 - r^\rho)c_2^\rho$, $v^\rho = r^\rho c_2^\rho + (1 - r^\rho)c_1^\rho$. Since Ψ is s-convex function on $[c_1^\rho, c_2^\rho]$, we have

$$\Psi\left(\frac{u^\rho + v^\rho}{2}\right) \le \frac{\Psi(u^\rho) + \Psi(v^\rho)}{2^s}.$$

Then we have

$$2^s\Psi\left(\frac{c_1^\rho + c_2^\rho}{2}\right) \le \Psi(r^\rho c_1^\rho + (1 - r^\rho)c_2^\rho) + \Psi(r^\rho c_2^\rho + (1 - r^\rho)c_1^\rho). \tag{3.2}$$

Multiplying both sides of (3.2) by $r^{\alpha\rho-1}$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0, 1]$, we obtain

$$\begin{aligned} \frac{2^s}{\alpha\rho}\Psi\left(\frac{c_1^\rho + c_2^\rho}{2}\right) &\le \int_0^1 r^{\alpha\rho-1}\Psi(r^\rho c_1^\rho + (1 - r^\rho)c_2^\rho)dr + \int_0^1 r^{\alpha\rho-1}\Psi(r^\rho c_2^\rho + (1 - r^\rho)c_1^\rho)dr \\ &= \int_{c_2}^{c_1}\left(\frac{c_2^\rho - u^\rho}{c_2^\rho - c_1^\rho}\right)^{\alpha-1}\Psi(u^\rho)\frac{u^{\rho-1}}{c_1^\rho - c_2^\rho}du \\ &\quad + \int_{c_1}^{c_2}\left(\frac{v^\rho - c_1^\rho}{c_2^\rho - c_1^\rho}\right)^{\alpha-1}\Psi(v^\rho)\frac{v^{\rho-1}}{c_2^\rho - c_1^\rho}dv \\ &= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho - c_1^\rho)^\alpha}\left[{}^\rho I_{c_1+}^\alpha \Psi(c_2^\rho) + {}^\rho I_{c_2-}^\alpha \Psi(c_1^\rho)\right]. \end{aligned} \tag{3.3}$$

This establishes the first inequality. For the second inequality of (3.1), using s-convexity of Ψ, we have

$$\Psi(r^\rho c_1^\rho + (1-r^\rho)c_2^\rho) \leq (r^\rho)^s \Psi(c_1^\rho) + (1-r^\rho)^s \Psi(c_2^\rho),$$

and

$$\Psi(r^\rho c_2^\rho + (1-r^\rho)c_1^\rho) \leq (r^\rho)^s \Psi(c_2^\rho) + (1-r^\rho)^s \Psi(c_1^\rho).$$

By adding above two inequalities, we get

$$\Psi(r^\rho c_1^\rho + (1-r^\rho)c_2^\rho) + \Psi(r^\rho c_2^\rho + (1-r^\rho)c_1^\rho) \leq ((r^\rho)^s + (1-r^\rho)^s)\left[\Psi(c_1^\rho) + \Psi(c_2^\rho)\right]. \quad (3.4)$$

Multiplying both sides of (3.4) by $r^{\alpha\rho-1}$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho - c_1^\rho)^\alpha}\left[{}^\rho I_{c_1+}^\alpha \Psi(c_2^\rho) + {}^\rho I_{c_2-}^\alpha \Psi(c_1^\rho)\right] \leq \int_0^1 r^{\alpha\rho-1}((r^\rho)^s + (1-r^\rho)^s)\left[\Psi(c_1^\rho) + \Psi(c_2^\rho)\right] dr. \quad (3.5)$$

Since

$$\int_0^1 r^{\alpha\rho+s\rho-1} dr = \frac{1}{\rho(\alpha+s)},$$

and applying change of variable $r^\rho = z$, we have

$$\int_0^1 r^{\alpha\rho-1}(1-r^\rho)^s dr = \frac{\mathcal{B}(\alpha, s+1)}{\rho}.$$

Thus (3.5) becomes

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho - c_1^\rho)^\alpha}\left[{}^\rho I_{c_1+}^\alpha \Psi(c_2^\rho) + {}^\rho I_{c_2-}^\alpha \Psi(c_1^\rho)\right] \leq \frac{1}{\rho}\left[\frac{1}{\alpha+s} + \mathcal{B}(\alpha, s+1)\right](\Psi(c_1^\rho) + \Psi(c_2^\rho)). \quad (3.6)$$

Thus (3.3) and (3.6) gives (3.1). □

Remark 12. *By letting $\rho \to 1$ in (3.1) of Theorem 3.2.1 we get Theorem 3 of [86].*

Theorem 3.2.2. *Let $\alpha, \rho > 0$ and $\Psi : [c_1^\rho, c_2^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on (c_1^ρ, c_2^ρ) with $0 \leq c_1 < c_2$. Let $|\Psi'|$ is s-convex on $[c_1^\rho, c_2^\rho]$, then the following inequality hold:*

$$\left|\frac{\Psi(c_1^\rho) + \Psi(c_2^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(c_2^\rho - c_1^\rho)^\alpha}\left[{}^\rho I_{c_1+}^\alpha \Psi(c_2^\rho) + {}^\rho I_{c_2-}^\alpha \Psi(c_1^\rho)\right]\right|$$
$$\leq \frac{c_2^\rho - c_1^\rho}{2}\left[\frac{1}{\alpha+s+1} + \mathcal{B}(\alpha+1, s+1)\right](|\Psi'(c_1^\rho)| + |\Psi'(c_2^\rho)|). \quad (3.7)$$

Proof. From (3.3) one can have

$$\begin{aligned}&\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho-c_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{c_1+}\Psi(c_2^\rho)+{}^\rho I^\alpha_{c_2-}\Psi(c_1^\rho)\right]\\&=\int_0^1 r^{\alpha\rho-1}\Psi(r^\rho c_1^\rho+(1-r^\rho)c_2^\rho)dr+\int_0^1 r^{\alpha\rho-1}\Psi(r^\rho c_2^\rho+(1-r^\rho)c_1^\rho)dr.\end{aligned}\tag{3.8}$$

By integrating by parts, we then get

$$\begin{aligned}&\frac{\Psi(c_1^\rho)+\Psi(c_2^\rho)}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho-c_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{c_1+}\Psi(c_2^\rho)+{}^\rho I^\alpha_{c_2-}\Psi(c_1^\rho)\right]\\&\quad=\frac{c_2^\rho-c_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}\left[\Psi'(r^\rho c_2^\rho+(1-r^\rho)c_1^\rho)-\Psi'(r^\rho c_1^\rho+(1-r^\rho)c_2^\rho)\right]dr.\end{aligned}\tag{3.9}$$

By using triangle inequality, s-convexity of $|\Psi'|$ and change of variable $r^\rho=z$, we obtain

$$\begin{aligned}&\left|\frac{\Psi(c_1^\rho)+\Psi(c_2^\rho}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho-c_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{c_1+}\Psi(c_2^\rho)+{}^\rho I^\alpha_{c_2-}\Psi(c_1^\rho)\right]\right|\\&\le\frac{c_2^\rho-c_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}\left|\left[\Psi'(r^\rho c_2^\rho+(1-r^\rho)c_1^\rho)-\Psi'(r^\rho c_1^\rho+(1-r^\rho)c_2^\rho)\right]\right|dr\\&\le\frac{c_2^\rho-c_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}\left[|\Psi'(r^\rho c_2^\rho+(1-r^\rho)c_1^\rho)|+|\Psi'(r^\rho c_1^\rho+(1-r^\rho)c_2^\rho)|\right]dr\\&\le\frac{c_2^\rho-c_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}[(r^\rho)^s|\Psi'(c_2^\rho)|+(1-r^\rho)^s|\Psi'(c_1^\rho)|\\&\quad+(r^\rho)^s|\Psi'(c_1^\rho)|+(1-r^\rho)^s|\Psi'(c_2^\rho)|]dr\\&=\frac{c_2^\rho-c_1^\rho}{\alpha}\int_0^1 r^{\rho(\alpha+1)-1}\left[(r^\rho)^s+(1-r^\rho)^s\right]\left[|\Psi'(c_1^\rho)|+|\Psi'(c_2^\rho)|\right]dr\\&=\frac{c_2^\rho-c_1^\rho}{\alpha\rho}\left[\frac{1}{\alpha+s+1}+\mathcal{B}(\alpha+1,s+1)\right]\left[|\Psi'(c_1^\rho)|+|\Psi'(c_2^\rho)|\right].\end{aligned}\tag{3.10}$$

□

Corollary 3.2.1. *Under the same assumptions of Theorem* 3.2.2*,*

1. *If* $\rho=1$*, then*

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_2-c_1)^\alpha}\left[J^\alpha_{c_1+}\Psi(c_2)+J^\alpha_{c_2-}\Psi(c_1)\right]\right|\\&\le\frac{c_2-c_1}{2}\left[\frac{1}{\alpha+s+1}+\mathcal{B}(\alpha+1,s+1)\right](|\Psi'(c_1)|+|\Psi'(c_2)|).\end{aligned}\tag{3.11}$$

2. *If* $\rho = s = 1$, *then*

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2-c_1)^{\alpha}}\left[J^{\alpha}_{c_1+}\Psi(c_2) + J^{\alpha}_{c_2-}\Psi(c_1)\right]\right| \\ &\le \frac{c_2-c_1}{2}\left[\frac{1}{\alpha+2} + \mathcal{B}(\alpha+1,2)\right]\left(|\Psi'(c_1)| + |\Psi'(c_2)|\right).\end{aligned} \tag{3.12}$$

3. *If* $\rho = s = \alpha = 1$, *then*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(x)dx\right| \le \frac{c_2-c_1}{4}\left(|\Psi'(c_1)| + |\Psi'(c_2)|\right). \tag{3.13}$$

For more results we need the following lemma.

Lemma 3.2.2. *Let* $\alpha, \rho > 0$ *and* $\Psi : [c_1^{\rho}, c_2^{\rho}] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (c_1^{ρ}, c_2^{ρ}) *such that* $\Psi' \in [c_1, c_2]$ *with* $0 \le c_1 < c_2$*. Then the following equality holds if the fractional integrals exist:*

$$\begin{aligned}&\frac{\Psi(c_1^{\rho})+\Psi(c_2^{\rho})}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(c_2^{\rho}-c_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{c_1+}\Psi(c_2^{\rho}) +^{\rho} I^{\alpha}_{c_2-}\Psi(c_1^{\rho})\right] \\ &= \frac{\rho(c_2^{\rho}-c_1^{\rho})}{2}\int_0^1\left[(1-r^{\rho})^{\alpha} - (r^{\rho})^{\alpha}\right]r^{\rho-1}\Psi'(r^{\rho}c_1^{\rho} + (1-r^{\rho})c_2^{\rho})dr.\end{aligned} \tag{3.14}$$

Proof. Applying similar arguments as in the proof of Lemma 2 in [80]. First consider,

$$\begin{aligned}&\int_0^1(1-r^{\rho})^{\alpha}r^{\rho-1}\Psi'(r^{\rho}c_1^{\rho} + (1-r^{\rho})c_2^{\rho})dr \\ &= \frac{(1-r^{\rho})^{\alpha}\Psi(r^{\rho}a^{\rho} + (1-r^{\rho})c_2^{\rho})}{\rho(c_1^{\rho}-c_2^{\rho})}\Big|_0^1 + \frac{\alpha}{c_1^{\rho}-c_2^{\rho}}\int_0^1(1-r^{\rho})^{\alpha-1}r^{\rho-1}\Psi(r^{\rho}c_1^{\rho} + (1-r^{\rho})c_2^{\rho})dr \\ &= \frac{\Psi(c_2^{\rho})}{\rho(c_2^{\rho}-c_1^{\rho})} - \frac{\alpha}{c_2^{\rho}-c_1^{\rho}}\int_{c_2}^{c_1}\left(\frac{u^{\rho}-c_1^{\rho}}{c_2^{\rho}-c_1^{\rho}}\right)^{\alpha-1}\cdot\frac{u^{\rho-1}\Psi(u^{\rho})}{c_1^{\rho}-c_2^{\rho}}du \\ &= \frac{\Psi(c_2^{\rho})}{\rho(c_2^{\rho}-c_1^{\rho})} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(c_2^{\rho}-c_1^{\rho})^{\alpha+1}}\cdot^{\rho} I^{\alpha}_{c_2-}\Psi(u^{\rho})\Big|_{u=c_1}.\end{aligned} \tag{3.15}$$

Similarly, we can show that

$$\int_0^1 r^{\rho\alpha}\cdot r^{\rho-1}\Psi'(r^{\rho}c_1^{\rho} + (1-r^{\rho})c_2^{\rho})dr = -\frac{\Psi(c_1^{\rho})}{\rho(c_2^{\rho}-c_1^{\rho})} + \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(c_2^{\rho}-c_1^{\rho})^{\alpha+1}}\cdot^{\rho} I^{\alpha}_{c_1+}\Psi(u^{\rho})\Big|_{u=c_2}. \tag{3.16}$$

Thus from (3.15) and (3.16) we get (3.14). □

Remark 13. *By taking $\rho = 1$ in (3.14) of Lemma 3.2.2, we get Lemma 2 in [80].*

Through out all other results we denote,

$$I_{\Psi}(\alpha,\rho,c_1,c_2) = \frac{\Psi(c_1^{\rho}) + \Psi(c_2^{\rho})}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(c_2^{\rho}-c_1^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{c_1+}\Psi(c_2^{\rho}) + {}^{\rho}I^{\alpha}_{c_2-}\Psi(c_1^{\rho})\right].$$

Theorem 3.2.3. *Let $\alpha, \rho > 0$ and $\Psi : [c_1^{\rho}, c_2^{\rho}] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (c_1^{ρ}, c_2^{ρ}) such that $\Psi' \in L_1[c_1,c_2]$ with $0 \le c_1 < c_2$. Let $|\Psi'|^q$ is s-convex on $[c_1^{\rho}, c_2^{\rho}]$ for some fixed $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}
&|I_{\Psi}(\alpha,\rho,c_1,c_2)| \\
&\le \frac{\rho(c_2^{\rho}-c_1^{\rho})}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-1/q} \\
&\times \Bigg(\Big({}^{\rho}\gamma(s+1,\alpha+1) + \frac{1}{\rho(\alpha+s+1)}\Big)|\Psi'(c_1^{\rho})|^q \\
&+ \Big({}^{\rho}\gamma(1,\alpha+s+1) + {}^{\rho}\gamma(\alpha+1,s+1)\Big)|\Psi'(c_2^{\rho})|^q\Bigg)^{1/q},
\end{aligned} \tag{3.17}$$

where

$${}^{\rho}\gamma(c_1,c_2) = \int_0^1 (u^{\rho})^{c_1-1}(1-u^{\rho})^{c_2-1}u^{\rho-1}du.$$

Proof. Using Lemma 3.2.2 and power mean inequality and s-convexity of $|\Psi'|^q$, we obtain

$$\begin{aligned}
&|I_{\Psi}(\alpha,\rho,c_1,c_2)| \\
&= \left|\frac{\rho(c_2^{\rho}-c_1^{\rho})}{2}\int_0^1 \{(1-r^{\rho})^{\alpha} - (r^{\rho})^{\alpha}\} r^{\rho-1}\Psi'(r^{\rho}c_1^{\rho} + (1-r^{\rho})c_2^{\rho})dr\right| \\
&\le \frac{\rho(c_2^{\rho}-c_1^{\rho})}{2}\left(\int_0^1 |(1-r^{\rho})^{\alpha} - (r^{\rho})^{\alpha}|\, r^{\rho-1}dr\right)^{1-1/q} \\
&\times \left(\int_0^1 |(1-r^{\rho})^{\alpha} - (r^{\rho})^{\alpha}|\, r^{\rho-1}|\Psi'(r^{\rho}c_1^{\rho} + (1-r^{\rho})c_2^{\rho})|^q dr\right)^{1/q} \\
&\le \frac{\rho(c_2^{\rho}-c_1^{\rho})}{2}\left(\int_0^1 \{(1-r^{\rho})^{\alpha} + (r^{\rho})^{\alpha}\} r^{\rho-1}dr\right)^{1-1/q} \\
&\times \left(\int_0^1 \{(1-r^{\rho})^{\alpha} + (r^{\rho})^{\alpha}\} r^{\rho-1}\left[(r^{\rho})^s|\Psi'(c_1^{\rho})|^q + (1-r^{\rho})^s|\Psi'(c_2^{\rho})|^q\right]dr\right)^{1/q} \\
&= \frac{\rho(c_2^{\rho}-c_1^{\rho})}{2}\Big(\frac{1}{\rho(\alpha+1)}\Big)^{1-1/q} \times \Bigg(\Big({}^{\rho}\gamma(s+1,\alpha+1) + \frac{1}{\rho(\alpha+s+1)}\Big)|\Psi'(c_1^{\rho})|^q \\
&+ \Big({}^{\rho}\gamma(1,\alpha+s+1) + {}^{\rho}\gamma(\alpha+1,s+1)\Big)|\Psi'(c_2^{\rho})|^q\Bigg)^{1/q}.
\end{aligned} \tag{3.18}$$

Hence the proof is completed. □

Corollary 3.2.3. *Under the similar conditions of Theorem 3.2.3,*

1. *If* $\rho = 1$, *then*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2-c_1)^{\alpha}}\left[J^{\alpha}_{c_1+}\Psi(c_2) + J^{\alpha}_{c_2-}\Psi(c_1)\right]\right|$$
$$\leq \frac{(c_2-c_1)}{2}\frac{1}{(\alpha+1)^{1-1/q}} \times \Big(\Big(\ \mathcal{B}(s+1,\alpha+1) + \frac{1}{(\alpha+s+1)}\Big)|\Psi'(c_1)|^q$$
$$+ \big(\mathcal{B}(1,\alpha+s+1) +\ \mathcal{B}(\alpha+1,s+1)\big)|\Psi'(c_2)|^q\Big)^{1/q}.$$

2. *If* $\rho = s = 1$, *then*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2-c_1)^{\alpha}}\left[J^{\alpha}_{c_1+}\Psi(c_2) + J^{\alpha}_{c_2-}\Psi(c_1)\right]\right|$$
$$\leq \frac{(c_2-c_1)}{2}\frac{1}{(\alpha+1)^{1-1/q}} \times \Big(\Big(\ \mathcal{B}(2,\alpha+1) + \frac{1}{(\alpha+2)}\Big)|\Psi'(c_1)|^q$$
$$+ \big(\mathcal{B}(1,\alpha+2) +\ \mathcal{B}(\alpha+1,2)\big)|\Psi'(c_2)|^q\Big)^{1/q}.$$

3. *If* $\rho = s = \alpha = 1$, *then*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)du\right| \leq \frac{(c_2-c_1)}{2^{2-1/q}} \times \left(\frac{|\Psi'(c_1)|^q + |\Psi'(c_2)|^q}{2}\right)^{1/q}.$$

Theorem 3.2.4. *Let* $\alpha, \rho > 0$ *and* $\Psi : [c_1^{\rho}, c_2^{\rho}] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (c_1^{ρ}, c_2^{ρ}) *such that* $\Psi' \in L_1[c_1, c_2]$ *with* $0 \leq c_1 < c_2$. *Let* $|\Psi'|^q$ *is s-convex on* $[c_1^{\rho}, c_2^{\rho}]$ *for some fixed* $q \geq 1$, *then the following inequality holds:*

$$|I_{\Psi}(\alpha,\rho,c_1,c_2)|$$
$$\leq \frac{\rho^{\frac{1}{q}}(c_2^{\rho}-c_1^{\rho})}{2}\left(\left[\mathcal{B}(s+1,\alpha+1) + \frac{1}{\alpha+s+1}\right]\left[|\Psi'(c_1^{\rho})|^q + |\Psi'(c_2^{\rho})|^q\right]\right)^{1/q}. \tag{3.19}$$

Proof. Using Lemma 3.2.2, property of modulus, power mean inequality and the fact that $|\Psi'|^q$ is s-convex function, we have

$$
\begin{aligned}
&|I_\Psi(\alpha,\rho,c_1,c_2)| \\
&\leq \left|\frac{\rho(c_2^\rho-c_1^\rho)}{2}\int_0^1 \{(1-r^\rho)^\alpha-(r^\rho)^\alpha\}\, r^{\rho-1}|\Psi'(r^\rho c_1^\rho+(1-r^\rho)c_2^\rho)|dr\right| \\
&\leq \frac{\rho(c_2^\rho-c_1^\rho)}{2}\left(\int_0^1 r^{\rho-1}dr\right)^{1-1/q}\left(\int_0^1 \{(1-r^\rho)^\alpha-(r^\rho)^\alpha\}\,|\Psi'(r^\rho c_1^\rho+(1-r^\rho)c_2^\rho)|^q dr\right)^{1/q} \\
&\leq \frac{\rho(c_2^\rho-c_1^\rho)}{2}\frac{1}{\rho^{1-1/q}}\left(\int_0^1 \{(1-r^\rho)^\alpha+(r^\rho)^\alpha\}\left[(r^\rho)^s|\Psi'(c_1^\rho)|^q+(1-r^\rho)^s|\Psi'(c_2^\rho)|^q\right]dr\right)^{1/q} \\
&= \frac{\rho^{\frac{1}{q}}(c_2^\rho-c_1^\rho)}{2}\Big(|\Psi'(c_1^\rho)|^q\int_0^1 \{(1-r^\rho)^\alpha(r^\rho)^s+(r^\rho)^\alpha(r^\rho)^s\}\,dr \\
&\quad+|\Psi'(c_2^\rho)|^q\int_0^1 \{(1-r^\rho)^\alpha(1-r^\rho)^s+(r^\rho)^\alpha(1-r^\rho)^s\}\,dr\Big)^{1/q} \\
&= \frac{\rho^{\frac{1}{q}}(c_2^\rho-c_1^\rho)}{2}\left(A|\Psi'(c_1^\rho)|^q+B|\Psi'(c_2^\rho)|^q\right)^{1/q}.
\end{aligned}
\tag{3.20}
$$

By using change of variable $r^\rho = z$, we get

$$A=\int_0^1 \{(1-r^\rho)^\alpha(r^\rho)^s+(r^\rho)^\alpha(r^\rho)^s\}\,dr=\mathcal{B}(s+1,\alpha+1)+\frac{1}{\alpha+s+1},$$

and

$$B=\int_0^1 \{(1-r^\rho)^\alpha(1-r^\rho)^s+(r^\rho)^\alpha(1-r^\rho)^s\}\,dr=\mathcal{B}(\alpha+1,s+1)+\frac{1}{\alpha+s+1}.$$

Thus substituting the values of A and B in (3.20) and applying the fact that $\mathcal{B}(c_1,c_2)=\mathcal{B}(c_2,c_1)$, we get the desired result. □

Corollary 3.2.4. *Under the similar conditions of Theorem* 3.2.4*,*

1. *If* $\rho=1$*, then*

$$
\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_2-c_1)^\alpha}\left[J^\alpha_{c_1+}\Psi(c_2)+J^\alpha_{c_2-}\Psi(c_1)\right]\right| \\
&\leq \frac{(c_2-c_1)}{2}\left(\left[\mathcal{B}(s+1,\alpha+1)+\frac{1}{\alpha+s+1}\right]\left[|\Psi'(c_1)|^q+|\Psi'(c_2)|^q\right]\right)^{1/q}.
\end{aligned}
$$

2. *If $\rho = s = 1$, then*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_1-c_2)^\alpha}\left[J^\alpha_{c_1+}\Psi(c_2) + J^\alpha_{c_2-}\Psi(c_1)\right]\right| \leq \frac{(c_2-c_1)}{2}\left(\left[\mathcal{B}(2,\alpha+1)+\frac{1}{\alpha+2}\right]\left[|\Psi'(c_1)|^q + |\Psi'(c_2)|^q\right]\right)^{1/q}.$$

3. *If $\rho = s = \alpha = 1$, then*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)du\right| \leq \frac{(c_2-c_1)}{2}\left(\frac{|\Psi'(c_1)|^q + |\Psi'(c_2)|^q}{2}\right)^{1/q}.$$

3.3 Hermite-Hadamard type inequalities for m-convex function

In this section we give Hermite-Hadamard type inequalities for m-convex function.

Theorem 3.3.1. *Let $\alpha, \rho > 0$ and $\Psi : [c_1^\rho, c_2^\rho] \subset [0,\infty) \to \mathbb{R}$ be a positive function with $0 \leq c_1 < c_2$ and $\Psi \in X_c^p(c_1^\rho, c_2^\rho)$. Let Ψ is also a m-convex function on $[c_1^\rho, c_2^\rho]$, then the following inequalities hold:*

$$\begin{aligned}
&\Psi\left(\frac{m^\rho(c_1^\rho + c_2^\rho)}{2}\right) \\
&\leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2((mc_2)^\rho - (mc_1)^\rho)^\alpha}\ {}^\rho I^\alpha_{mc_1+}\Psi\left((mc_2)^\rho\right) + \frac{m^\rho\rho^\alpha\Gamma(\alpha+1)}{2(c_2^\rho - c_1^\rho)^\alpha}\ {}^\rho I^\alpha_{c_2-}\Psi\left(c_1^\rho\right) \qquad (3.21)\\
&\leq \frac{m^\rho}{2}\left(\Psi(c_1^\rho) + \Psi(c_2^\rho)\right).
\end{aligned}$$

Proof. Since Ψ is m-convex, we have

$$\Psi\left(\frac{u^\rho + m^\rho v^\rho}{2}\right) \leq \frac{\Psi(u^\rho) + m^\rho\Psi(v^\rho)}{2}.$$

Let $u^\rho = m^\rho r^\rho c_1^\rho + m^\rho(1-r^\rho)c_2^\rho$, $v^\rho = r^\rho c_2^\rho + (1-r^\rho)c_1^\rho$ with $r \in [0,1]$. Then we obtain

$$\Psi\left(\frac{m^\rho(c_1^\rho + c_2^\rho)}{2}\right) \leq \frac{\Psi(m^\rho r^\rho c_1^\rho + m^\rho(1-r^\rho)c_2^\rho) + m^\rho\Psi(r^\rho c_2^\rho + (1-r^\rho)c_1^\rho)}{2}. \qquad (3.22)$$

Multiplying both sides of (3.22) by $r^{\alpha\rho-1}$, $\alpha > 0$, and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{\rho\alpha}\Psi\left(\frac{m^\rho(c_1^\rho+c_2^\rho)}{2}\right)\\
&\leq \int_0^1 r^{\alpha\rho-1}\Psi(m^\rho r^\rho c_1^\rho + m^\rho(1-r^\rho)c_2^\rho)dr + m^\rho\int_0^1 r^{\alpha\rho-1}\Psi(r^\rho c_2^\rho+(1-r^\rho)c_1^\rho)dr\\
&= \int_{mc_2}^{mc_1}\left(\frac{u^\rho-(mc_2)^\rho}{(mc_1)^\rho-(mc_2)^\rho}\right)^{\alpha-1} u^{\rho-1}\frac{\Psi(u^\rho)du}{(mc_1)^\rho-(mc_2)^\rho}\\
&\quad + m^\rho\int_{c_1}^{c_2}\left(\frac{v^\rho-c_1^\rho}{c_2^\rho-c_1^\rho}\right)^{\alpha-1} v^{\rho-1}\frac{\Psi(v^\rho)dv}{c_2^\rho-c_1^\rho}\\
&= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{((mc_2)^\rho-(mc_1)^\rho)^\alpha}\ {}^\rho I^\alpha_{mc_1+}\Psi\left((mc_2)^\rho\right) + \frac{m^\rho\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho-c_1^\rho)^\alpha}\ {}^\rho I^\alpha_{c_2-}\Psi\left(c_1^\rho\right).
\end{aligned}\tag{3.23}$$

Now by multiplying both sides of (3.23) by $\frac{\alpha\rho}{2}$, we get first inequality of (3.21). For second inequality using m-convexity of Ψ, we have

$$\Psi(m^\rho r^\rho c_1^\rho + m^\rho(1-r^\rho)c_2^\rho) + m^\rho\Psi((1-r^\rho)c_1^\rho + r^\rho c_2^\rho) \leq m^\rho\left[\Psi(c_1^\rho)+\Psi(c_2^\rho)\right]. \tag{3.24}$$

Multiplying both sides of (3.24) by $r^{\alpha\rho-1}$, $\alpha > 0$ and then integrating the resulting inequality with respect to r over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{\rho^{\alpha-1}\Gamma(\alpha)}{((mc_2)^\rho-(mc_1)^\rho)^\alpha}\ {}^\rho I^\alpha_{mc_1+}\Psi\left((mc_2)^\rho\right) + \frac{m^\rho\rho^{\alpha-1}\Gamma(\alpha)}{(c_2^\rho-c_1^\rho)^\alpha}\ {}^\rho I^\alpha_{c_2-}\Psi\left(c_1^\rho\right)\\
&\leq \frac{m^\rho}{\rho\alpha}\left(\Psi(c_1^\rho)+\Psi(c_2^\rho)\right).
\end{aligned}\tag{3.25}$$

Now by multiplying both sides of (3.25) by $\frac{\alpha\rho}{2}$, we achieve the second inequality of (3.21). □

Corollary 3.3.1. *Under the assumptions of Theorem* 3.3.1*, we have*

1. *For* $\rho = 1$*, then*

$$\begin{aligned}
&\Psi\left(\frac{m(c_1+c_2)}{2}\right)\\
&\leq \frac{\Gamma(\alpha+1)}{2(mc_2-mc_1)^\alpha}\ J^\alpha_{mc_1+}\Psi(mc_2) + \frac{m\Gamma(\alpha+1)}{2(c_2-c_1)^\alpha}\ J^\alpha_{c_2-}\Psi\left(c_1\right)\\
&\leq \frac{m}{2}\left(\Psi(c_1)+\Psi(c_2)\right).
\end{aligned}\tag{3.26}$$

2. *For $\rho = \alpha = 1$, then*

$$\Psi\left(\frac{m(c_1+c_2)}{2}\right) \leq \frac{1}{2(mc_2-mc_1)}\int_{mc_1}^{mc_2}\Psi(u)du + \frac{m}{2(c_2-c_1)}\int_{c_1}^{c_2}\Psi(u)du \leq \frac{m}{2}\left(\Psi(c_1)+\Psi(c_2)\right). \tag{3.27}$$

Remark 14. *If we take $m = 1$ in (3.27) of Corollary (3.3.1)(2), then we get (1.14).*

Theorem 3.3.2. *Let $\alpha, \rho > 0$ and $\Psi : [c_1^\rho, c_2^\rho] \subset [0, \infty) \to \mathbb{R}$ be a positive function with $0 \leq c_1 < c_2$ and $\Psi \in X_c^p(c_1^\rho, c_2^\rho)$. Let Ψ is also a m-convex function on $[c_1^\rho, c_2^\rho]$. Let $G(x^\rho, y^\rho)_{r^\rho} : [0,1] \to \mathbb{R}$ are defined as:*

$$G(x^\rho, y^\rho)_{r^\rho} = \frac{1}{2}\left[\Psi(r^\rho x^\rho + m^\rho(1-r^\rho)y^\rho) + \Psi((1-r^\rho)x^\rho + m^\rho r^\rho y^\rho)\right].$$

Then, we have

$$\frac{1}{(c_2^\rho - c_1^\rho)^\alpha}\int_{c_1}^{c_2}(c_2^\rho - u^\rho)^{\alpha-1}u^{\rho-1}G\left(u^\rho, \frac{c_1^\rho + c_2^\rho}{2}\right)_{\left(\frac{c_2^\rho - u^\rho}{c_2^\rho - c_1^\rho}\right)} du \leq \frac{\rho^{\alpha-1}\Gamma(\alpha)}{2(c_2^\rho - c_1^\rho)^\alpha}\, {}^\rho I_{c_1+}^\alpha \Psi(c_2^\rho) + \frac{m}{2\rho\alpha}\Psi\left(\frac{c_1^\rho + c_2^\rho}{2}\right). \tag{3.28}$$

Proof. Since Ψ is m-convex function, we have

$$G(x^\rho, y^\rho)_{r^\rho} \leq \frac{1}{2}\left[r^\rho\Psi(x^\rho) + m^\rho(1-r^\rho)\Psi(y^\rho) + (1-r^\rho)\Psi(x^\rho) + m^\rho r^\rho\Psi(y^\rho)\right] = \frac{1}{2}\left[\Psi(x^\rho) + m^\rho\Psi(y^\rho)\right],$$

and also

$$G\left(x^\rho, \frac{c_1^\rho + c_2^\rho}{2}\right)_{r^\rho} \leq \frac{1}{2}\left[\Psi(x^\rho) + m^\rho\Psi\left(\frac{c_1^\rho + c_2^\rho}{2}\right)\right].$$

Take $x^\rho = r^\rho c_1^\rho + (1-r^\rho)c_2^\rho$, we have

$$G\left(r^\rho c_1^\rho + (1-r^\rho)c_2^\rho, \frac{c_1^\rho + c_2^\rho}{2}\right)_{r^\rho} \leq \frac{1}{2}\left[\Psi(r^\rho c_1^\rho + (1-r^\rho)c_2^\rho) + m^\rho\Psi\left(\frac{c_1^\rho + c_2^\rho}{2}\right)\right]. \tag{3.29}$$

Multiplying both sides of (3.29) by $r^{\alpha\rho-1}$, $\alpha > 0$ and after that integrating the resulting inequality with respect to r over $[0, 1]$, we obtain

$$\int_0^1 r^{\alpha\rho-1}G\left(r^\rho c_1^\rho + (1-r^\rho)c_2^\rho, \frac{c_1^\rho + c_2^\rho}{2}\right)_{r^\rho} dr \leq \frac{1}{2}\int_0^1 r^{\alpha\rho-1}\left[\Psi(r^\rho c_1^\rho + (1-r^\rho)c_2^\rho) + m^\rho\Psi\left(\frac{c_1^\rho + c_2^\rho}{2}\right)\right] dr. \tag{3.30}$$

Thus by change of variable $u^\rho = r^\rho c_1^\rho + (1-r^\rho)c_2^\rho$, we achieve our desirous, that is, the inequality (3.28). □

Remark 15. *By taking* $\rho = 1$ *in* (3.28) *of Theorem* 3.3.2*, we get Theorem* 6 *in* [86].

3.4 Applications to special means

In this section, we consider some applications to our results. Consider the following means:

(1) The Arithmetic mean:

$$A(c_1, c_2) = \frac{c_1 + c_2}{2}; \ c_1, c_2 \in \mathbb{R}.$$

(2) The Logarithmic mean:

$$L(c_1, c_2) = \frac{\ln|c_2| - \ln|c_1|}{c_2 - c_1}; \ c_1, c_2 \in \mathbb{R}, \ |c_1| \neq |c_2|, c_1, c_2 \neq 0.$$

(3) The Generalized log mean:

$$L_n(c_1, c_2) = \left[\frac{c_2^{n+1} - c_1^{n+1}}{(n+1)(c_2 - c_1)}\right]^{1/n}; \ c_1, c_2 \in \mathbb{R}, \ n \in \mathbb{Z} \setminus \{-1, 0\}, c_1, c_2 \neq 0.$$

Proposition 1. *Let* $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$ *and* $n \in \mathbb{Z}$, $|n| \geq 2$, *then*

$$\left|A(c_1^n, c_2^n) - \frac{c_2 - c_1}{c_2 + c_1} L_n^n(c_1, c_2)\right| \leq \frac{|n|(c_2 - c_1)}{2} A\left(|c_1|^{n-1}, |c_2|^{n-1}\right). \tag{3.31}$$

Proof. By taking $\Psi(u) = u^n$ in Corollary 3.2.1(3), we get the required result. □

Proposition 2. *Let* $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$ *and* $n \in \mathbb{Z}$, $|n| \geq 2$. *Then for* $q \geq 1$, *we have*

$$\left|A(c_1^n, c_2^n) - \frac{c_2 - c_1}{c_1 + c_2} L_n^n(c_1, c_2)\right| \leq \frac{|n|(c_2 - c_1)}{2^{2-1/q}} A^{1/q}\left(|c_1|^{q(n-1)}, |c_2|^{q(n-1)}\right). \tag{3.32}$$

Proof. By taking $\Psi(u) = u^n$ in Corollary 3.2.3(3), we get the required result. □

Proposition 3. *Let* $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$ *and* $n \in \mathbb{Z}$, $|n| \geq 2$. *Then for* $q \geq 1$, *we have*

$$\left|A(c_1^n, c_2^n) - \frac{c_2 - c_1}{c_1 + c_2} L_n^n(c_1, c_2)\right| \leq \frac{|n|(c_2 - c_1)}{2} A^{1/q}\left(|c_1|^{q(n-1)}, |c_2|^{q(n-1)}\right). \tag{3.33}$$

Proof. By taking $\Psi(u) = u^n$ in Corollary 3.2.4(3), we get the required result. □

Proposition 4. *Let $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$ and $n \in \mathbb{Z}$, $|n| \geq 2$, $m \in [0,1]$, then we have*

$$\Psi(mA(c_1, c_2)) \leq \frac{1}{2} L_n^n(mc_1, mc_2) + \frac{m}{2} L_n^n(c_1, c_2) \leq mA(c_1^n, c_2^n). \tag{3.34}$$

Proof. By taking $\Psi(u) = u^n$ in Corollary 3.3.1(2), we get the required result. □

Proposition 5. *Let $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$, then*

$$\left| A(c_1^{-1}, c_2^{-1}) - \frac{c_2 - c_1}{c_1 + c_2} L(c_1, c_2) \right| \leq \frac{c_2 - c_1}{2} A\left(|c_1|^{-2}, |c_2|^{-2} \right). \tag{3.35}$$

Proof. By taking $\Psi(u) = \frac{1}{u}$ in Corollary 3.2.1(3), we get the required result. □

Proposition 6. *Let $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$. Then for $q \geq 1$, we have*

$$\left| A(c_1^{-1}, c_2^{-1}) - \frac{c_2 - c_1}{c_1 + c_2} L(c_1, c_2) \right| \leq \frac{c_2 - c_1}{2^{2-1/q}} A^{1/q}\left(|c_1|^{-2q}, |c_2|^{-2q} \right). \tag{3.36}$$

Proof. By taking $\Psi(u) = \frac{1}{u}$ in Corollary 3.2.3(3), we get the required result. □

Proposition 7. *Let $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$. Then for $q \geq 1$, we have*

$$\left| A(c_1^{-1}, c_2^{-1}) - \frac{c_2 - c_1}{c_1 + c_2} L(c_1, c_2) \right| \leq \frac{c_2 - c_1}{2} A^{1/q}\left(|c_1|^{-2q}, |c_2|^{-2q} \right). \tag{3.37}$$

Proof. By taking $\Psi(u) = \frac{1}{u}$ in Corollary 3.2.4(3), we get the required result. □

Proposition 8. *Let $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, $0 \notin [c_1, c_2]$ and $m \in [0,1]$, then we have*

$$\Psi(mA(c_1, c_2)) \leq \frac{1}{2} L(mc_1, mc_2) + \frac{m}{2} L(c_1, c_2) \leq mA(c_1^{-1}, c_2^{-1}). \tag{3.38}$$

Proof. By taking $\Psi(u) = \frac{1}{u}$ in Corollary 3.3.1(2), we get the required result. □

Chapter 4

Mean value theorem for p-convex functions and s-convex functions

In this chapter we give some mean value theorem p and s-convex functions in Jensen as well as in Hermite-Hadamard sense.

4.1 Introduction

A generalized mean value theorem for convex functions is given by Anwar and Pečarić [11]. Following result is given in [65] which involve the Jensen's inequality both in numerator and denominator.

Theorem 4.1.1 ([65]). *Let $r_i > 0$ for all $1 \leq i \leq n$ such that $\Sigma_{i=1}^{n} r_i = T_n$ and $c_1, ..., c_n \in I$ not all same. Consider the twice differentiable functions $\Psi_1, \Psi_2 : \aleph \to \mathbb{R}$ such that*

$$0 \leq l \leq \Psi_1^{''}(c) \leq L \quad \textit{and} \quad 0 \leq m \leq \Psi_2^{''}(x) \leq M \quad \textit{for all} \quad c \in \aleph.$$

Then

$$\frac{l}{M} \leq \frac{\frac{1}{T_n}\Sigma_{i=1}^{n} r_i \Psi_1(c_i) - \Psi_1(\frac{1}{T_n}\Sigma_{i=1}^{n} r_i c_i)}{\frac{1}{T_n}\Sigma_{i=1}^{n} r_i \Psi_2(c_i) - \Psi_2(\frac{1}{T_n}\Sigma_{i=1}^{n} r_i c_i)} \leq \frac{L}{m}. \tag{4.1}$$

For more results also see [75, 76].

4.2 Mean value theorem for p-convex functions

In order to state the mean value theorem for p-convex functions in Jensen sense we need the following lemma.

Lemma 4.2.1. *Let $\aleph \subset (0, \infty)$ be a p-convex set, $p \in \mathbb{R}\backslash\{0\}$, and $\Psi : \aleph \to \mathbb{R}$ be a p-convex function. Let $(\sum_1^n r_i c_i^p)^{\frac{1}{p}} \in \aleph$ be p-convex combinations of points $c_i \in \aleph$ with coefficients $r_i \in [0,1]$. Then each p-convex function $\Psi : \aleph \to \mathbb{R}$ satisfies the following inequality:*

$$\Psi\left(\left(\sum_1^n r_i c_i^p\right)^{\frac{1}{p}}\right) \leq \sum_1^n r_i \Psi(c_i), \tag{4.2}$$

where $\sum_1^n r_i = 1$.

Proof. We use mathematical induction to prove the result. We apply induction on the number of points in p-convex combination.

Basis step: for $n = 1$ the result holds, since

$$\Psi\left((r_1 c_1^p)^{\frac{1}{p}}\right) \leq r_1 \Psi(c_1),$$

where $r_1 = 1$.

Induction step: suppose that (4.2) holds for all p-convex combination of the points containing less than or equal to $n-1$ points. Let $r_n \neq 1$ and

$$u = \left(\sum_1^{n-1} \frac{r_i}{1-r_n} c_i^p\right)^{\frac{1}{p}},$$

where the sum $\left(\sum_1^{n-1} \frac{r_i}{1-r_n} c_i^p\right)^{\frac{1}{p}} \in \aleph$. Then by induction hypothesis, we have

$$\Psi(u) \leq \sum_1^{n-1} \frac{r_i}{1-r_n} \Psi(c_i). \tag{4.3}$$

By using (1.7) and (4.3), we get

$$\begin{aligned}\Psi\left(\left(\sum_1^n r_i c_i^p\right)^{\frac{1}{p}}\right) &= \Psi\left(((1-r_n)u^p + r_n c_n^p)^{\frac{1}{p}}\right) \\ &\leq (1-r_n)\Psi(u) + r_n\Psi(c_n) \\ &\leq (1-r_n)\sum_1^{n-1}\frac{r_i}{1-r_n}\Psi(c_i) + r_n\Psi(c_n) \\ &= \sum_1^n r_i\Psi(c_i).\end{aligned} \tag{4.4}$$

Thus the inequality (4.2) holds. □

Theorem 4.2.1. *Let $\aleph \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $r_i \in [0,1]$. Let $\Psi_1, \Psi_2 \in C^2(\aleph)$ be p-convex functions. Then there exist some $\xi \in \aleph$ such that the following equality holds:*

$$\frac{\sum_1^n r_i\Psi_1(c_i) - \Psi_1\left((\sum_1^n r_i c_i^p)^{\frac{1}{p}}\right)}{\sum_1^n r_i\Psi_2(c_i) - \Psi_2\left((\sum_1^n r_i c_i^p)^{\frac{1}{p}}\right)} = \frac{\Psi_1''(\xi)}{\Psi_2''(\xi)}, \tag{4.5}$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i = 1$ and provided that the denominators are non-zero.

Proof. Let us define

$$H := \left(\sum_1^n r_i c_i^p\right)^{\frac{1}{p}},$$

and

$$(T\Psi_1)(\lambda) := \sum_1^n r_i\Psi_1(\lambda c_i + (1-\lambda)H) - \Psi_1(H),$$

where $\lambda \in [0,1]$. Analogously, we define $(T\Psi_2)(\lambda)$.

Note that

$$(T\Psi_1)'(\lambda) := \sum_1^n r_i(c_i - H)\Psi_1'(\lambda c_i + (1-\lambda)H),$$

and

$$(T\Psi_1)''(\lambda) := \sum_1^n r_i(c_i - H)^2\Psi_1''(\lambda c_i + (1-\lambda)H).$$

Now consider a function $Q(\lambda)$ defined as:

$$Q(\lambda) = (T\Psi_2)(1)(T\Psi_1)(\lambda) - (T\Psi_1)(1)(T\Psi_2)(\lambda),$$

such that we have

$$Q(0) = Q(1) = Q^{'}(0) = 0.$$

Then from two applications of mean value theorem, we have $\upsilon \in \aleph$ so that

$$Q^{''}(\upsilon) = 0.$$

It implies that,

$$\sum_{i=1}^{n} r_i(c_i - H)^2[(T\Psi_2)(1)\Psi_1^{''}(\upsilon c_i + (1-\upsilon)H) - (T\Psi_1)(1)\Psi_2^{''}(\upsilon c_i + (1-\upsilon)H)] = 0. \quad (4.6)$$

For some fixed υ, the expression in the square brackets in (4.6) is a continuous function of c_i, so it vanishes. Corresponding to that value of c_i, we can have a number

$$\xi = \upsilon c_i + (1-\upsilon)H,$$

such that

$$(T\Psi_2)(1).\Psi_1^{''}(\xi) - (T\Psi_1)(1).\Psi_2^{''}(\xi) = 0.$$

This gives equality (4.5). □

Corollary 4.2.2. *Let $\aleph \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $r_i \in [0,1]$. Let $\Psi_1, \Psi_2 \in C^2(\aleph)$ be p-convex functions such that $\frac{\Psi_1^{''}}{\Psi_2^{''}}$ is invertible. Then there exist some $\xi \in \aleph$ such that the following equality holds:*

$$\xi = \left(\frac{\Psi_1^{''}}{\Psi_2^{''}}\right)^{-1} \left(\frac{\sum_1^n r_i\Psi_1(c_i) - \Psi_1\left((\sum_1^n r_i c_i^p)^{\frac{1}{p}}\right)}{\sum_1^n r_i\Psi_2(c_i) - \Psi_2\left((\sum_1^n r_i c_i^p)^{\frac{1}{p}}\right)}\right), \quad (4.7)$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i = 1$ and provided that the denominators are non-zero.

Corollary 4.2.3. *Let $\aleph \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $r_i \in [0,1]$. Let $\Psi \in C^2(\aleph)$ be p-convex function. Then there exist some $\xi \in \aleph$ such that the following equality holds:*

$$\sum_1^n r_i\Psi(c_i) - \Psi\left(\left(\sum_1^n r_i c_i^p\right)^{\frac{1}{p}}\right) = \frac{\Psi''(\xi)}{2}\left(\sum_1^n r_i c_i^2 - \left(\left(\sum_1^n r_i c_i^p\right)^{\frac{1}{p}}\right)^2\right), \tag{4.8}$$

with each $r_i \in [0,1]$ such that $\Sigma_1^n r_i = 1$.

Proof. By letting $\Psi_1 = \Psi$ and $\Psi_2(u) = u^2$, where $u \in (0,\infty)$, in Theorem 4.2.1, we achieve the equality (4.8). □

4.3 Mean value theorem for p-convex functions in Hermite-Hadamard sense

Let $\Psi : \aleph \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $c_1, c_2 \in \aleph$ with $c_1 < c_2$. If $\Psi \in L_1[c_1,c_2]$ then we have (e.g. see [45])

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{\Psi(u)}{u^{1-p}}du \leq \frac{\Psi(c_1)+\Psi(c_2)}{2}. \tag{4.9}$$

By using the right half of the inequality (4.9), we have following result.

Theorem 4.3.1. *Let $\aleph \subset (0,\infty)$ be an interval, $p \in \mathbb{R}\backslash\{0\}$, and $c_1, c_2 \in \aleph$ with $c_1 < c_2$. Let $\Psi_1, \Psi_2 \in C^2(\aleph)$ be p-convex functions. Then there exists some $\xi \in \aleph$ such that the following equality holds:*

$$\frac{\frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{\Psi_1(u)}{u^{1-p}}du - \Psi_1\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right)}{\frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{\Psi_2(u)}{u^{1-p}}du - \Psi_2\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right)} = \frac{\Psi_1''(\xi)}{\Psi_2''(\xi)}, \tag{4.10}$$

provided that the denominators are non-zero.

Proof. Let

$$H := \left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}},$$

and

$$(T\Psi_1)(\lambda) := \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{\Psi_1(\lambda u + (1-\lambda)H)}{u^{1-p}}du - \Psi_1(H),$$

where $\lambda \in [0,1]$. Similarly, we can define $(T\Psi_2)(\lambda)$.

Observe that

$$(T\Psi_1)^{'}(\lambda) := \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} (u-H)\frac{\Psi_1^{'}(\lambda u + (1-\lambda)H)}{u^{1-p}}du,$$

and

$$(T\Psi_1)^{''}(\lambda) := \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} (u-H)^2\frac{\Psi_1^{''}(\lambda u + (1-\lambda)H)}{u^{1-p}}du.$$

Now consider the function $Q(\lambda)$ defined by

$$Q(\lambda) = (T\Psi_2)(1)(T\Psi_1)(\lambda) - (T\Psi_1)(1)(T\Psi_2)(\lambda),$$

such that we have

$$Q(0) = Q(1) = Q^{'}(0) = 0.$$

Then from two applications of mean value theorem, we find $\upsilon \in \aleph$ such that

$$Q^{''}(\upsilon) = 0.$$

It implies,

$$\frac{p}{c_2^p - c_1^p}\int_{[c_1,c_2]} (u-H)^2[(T\Psi_2)(1)\Psi_1^{''}(u\upsilon - (1-\upsilon)H) - (T\Psi_1)(1)\Psi_2^{''}(u\upsilon - (1-\upsilon)H)] = 0. \tag{4.11}$$

For any fixed υ, the expression in the square brackets in (4.11) is a continuous function of u, so it vanishes. Corresponding to that value of u, we get a number

$$\xi = u\upsilon + (1-\upsilon)H,$$

such that

$$(T\Psi_2)(1).\Psi_1^{''}(\xi) - (T\Psi_1)(1).\Psi_2^{''}(\xi) = 0.$$

This gives equality (4.10). □

Remark 16. *If $\frac{\Psi_1''}{\Psi_2''}$ is invertible, then we have*

$$\xi = \left(\frac{\Psi_1''(\xi)}{\Psi_2''(\xi)}\right)^{-1} \left(\frac{\frac{p}{c_2^p - c_1^p} \int_{c_1}^{c_2} \frac{\Psi_1(u)}{u^{1-p}} du - \Psi_1 \left(\left(\frac{c_1^p + c_2^p}{2} \right)^{\frac{1}{p}} \right)}{\frac{p}{c_2^p - c_1^p} \int_{c_1}^{c_2} \frac{\Psi_2(u)}{u^{1-p}} du - \Psi_2 \left(\left(\frac{c_1^p + c_2^p}{2} \right)^{\frac{1}{p}} \right)} \right). \tag{4.12}$$

Remark 17. *By taking $\Psi_2(u) = u^2$ and $\Psi_1 = \Psi$ in Theorem 4.3.1, we have*

$$\begin{aligned} &\frac{p}{c_2^p - c_1^p} \int_{c_1}^{c_2} \frac{\Psi(u)}{u^{1-p}} du - \Psi \left(\left(\frac{c_1^p + c_2^p}{2} \right)^{\frac{1}{p}} \right) \\ &= \frac{\Psi''(\xi)}{2} \left[\frac{p}{c_2^p - c_1^p} \left(\frac{c_2^{p+2} - c_1^{p+2}}{p+2} \right) - \left(\frac{c_1^p + c_2^p}{2} \right)^{\frac{2}{p}} \right]. \end{aligned} \tag{4.13}$$

4.4 Mean value theorem for s-convex functions

In this section we prove mean value theorem for s-convex function. First consider the following lemma.

Lemma 4.4.1. *Let $s \in (0,1]$ and $\Psi : \aleph \subset \mathbb{R}^+ \to \mathbb{R}$ be a s-convex function. Let $\sum_1^n r_i c_i$ be convex combinations of points $c_i \in \aleph$ with coefficients $r_i \in [0,1]$. Then for each s-convex function (in first sense) satisfies the inequality*

$$\Psi\left(\sum_1^n r_i c_i\right) \leq \sum_1^n r_i^s \Psi(c_i), \tag{4.14}$$

where $\sum_1^n r_i^s = 1$.

Proof. We apply induction on the number of points in convex combination.

Basis step: for $n = 1$ the equality (4.14) is true since

$$\Psi(r_1 c_1) \leq r_1^s \Psi(c_1),$$

where $r_1^s = 1$ since $r_1 = 1$.

Induction step: suppose that (4.14) holds for all convex combination of points containing less than or equal to $n - 1$ points. Let $r_n \neq 1$ and

$$u = \sum_1^{n-1} \frac{r_i}{1 - r_n} c_i,$$

where the sum $\sum_1^{n-1}\left(\frac{r_i}{1-r_n}\right)c_i \in \aleph$. Then by induction hypothesis, we have

$$\Psi(u) \leq \sum_1^{n-1}\left(\frac{r_i}{1-r_n}\right)^s \Psi(c_i). \tag{4.15}$$

By using (1.2) and (4.15), we get

$$\begin{aligned}\Psi\left(\sum_1^n r_i c_i\right) &= \Psi((1-r_n)u + r_n c_n)\\ &\leq (1-r_n)^s \Psi(u) + r_n^s \Psi(c_n)\\ &\leq (1-r_n)^s \sum_1^{n-1}\left(\frac{r_i}{1-r_n}\right)^s \Psi(c_i) + r_n^s\Psi(c_n)\\ &= \sum_1^n r_i^s \Psi(c_i).\end{aligned} \tag{4.16}$$

Thus we get (4.14). □

Theorem 4.4.1. *Let $s \in (0,1]$ and $r_i \in [0,1]$. Let $\Psi_1, \Psi_2 \in C^2(\aleph \subset [0,\infty))$ be s-convex functions (in first sense). Then there exist some $\xi \in \aleph$ such that the following equality holds:*

$$\frac{\sum_1^n r_i^s \Psi_1(c_i) - \Psi_1(\sum_1^n r_i c_i)}{\sum_1^n r_i^s \Psi_2(c_i) - \Psi_2(\sum_1^n r_i c_i)} = \frac{\Psi_1^{''}(\xi)}{\Psi_2^{''}(\xi)}, \tag{4.17}$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i^s = 1$ and provided that the denominators are non-zero.

Proof. Define

$$H := \sum_1^n r_i c_i,$$

and

$$(T\Psi_1)(\lambda) := \sum_1^n r_i^s \Psi_1(\lambda c_i + (1-\lambda)H) - \Psi_1(H),$$

where $\lambda \in [0,1]$. Accordingly, we can define $(T\Psi_2)(\lambda)$.

Note that

$$(T\Psi_1)^{'}(\lambda) := \sum_1^n r_i^s (c_i - H)\Psi_1^{'}(\lambda c_i + (1-\lambda)H),$$

and

$$(T\Psi_1)^{''}(\lambda) := \sum_1^n r_i^s (c_i - H)^2 \Psi_1^{''}(\lambda c_i + (1-\lambda)H).$$

Now consider the function $Q(\lambda)$ defined by

$$Q(\lambda) = (T\Psi_2)(1)(T\Psi_1)(\lambda) - (T\Psi_1)(1)(T\Psi_2)(\lambda),$$

such that we have

$$Q(0) = Q(1) = Q^{'}(0) = 0.$$

Then from two applications of mean value theorem, we find $\upsilon \in \aleph$ such that

$$Q^{''}(\upsilon) = 0.$$

It follows that

$$\sum_{i=1}^n r_i^s (c_i - H)^2 [(T\Psi_2)(1).\Psi_1^{''}(\upsilon c_i + (1-\upsilon)H) - (T\Psi_1)(1).\Psi_2^{''}(\upsilon c_i + (1-\upsilon)H)] = 0. \quad (4.18)$$

For any fixed υ, the expression in the square brackets in (4.18) is a continuous function of c_i, so it vanishes. Corresponding to that value of c_i, we get a number

$$\xi = \upsilon + (1-\upsilon)H,$$

so that

$$(T\Psi_2)(1).\Psi_1^{''}(\xi) - (T\Psi_1)(1).\Psi_2^{''}(\xi) = 0.$$

This gives equality (4.17). □

Corollary 4.4.2. *Let $s \in (0,1]$. Let $\Psi_1, \Psi_2 \in C^2(I \subset [0,\infty))$ be s-convex functions (in first sense) such that $\frac{\Psi_1^{''}}{\Psi_2^{''}}$ is invertible. Then there exist some $\xi \in \aleph$ such that the following equality holds:*

$$\xi = \left(\frac{\Psi_1^{''}}{\Psi_2^{''}}\right)^{-1} \left(\frac{\sum_1^n r_i^s \Psi_1(c_i) - \Psi_1(\sum_1^n r_i c_i)}{\sum_1^n r_i^s \Psi_2(c_i) - \Psi_2(\sum_1^n r_i c_i)}\right), \quad (4.19)$$

with each $r_i \in [0,1]$ such that $\sum_1^n r_i^s = 1$ and provided that the denominators are non-zero.

Remark 18. *Let $s_1, s_2 \in (0,1)$. Let $\Psi_1, \Psi_2 \in C^2((0,\infty))$ be s_1-convex function and s_2-convex function (in first sense), respectively, defined as $\Psi_1(u) = u^{s_1}$ and $\Psi_2(u) = u^{s_2}$. Then from Theorem 4.4.1, we get*

$$\frac{\sum_1^n r_i^{s_1}(c_i)^{s_1} - \left(\sum_1^n r_i c_i\right)^{s_1}}{\sum_1^n r_i^{s_2}(c_i)^{s_2} - \left(\sum_1^n r_i c_i\right)^{s_2}} = \frac{s_1(s_1-1)}{s_2(s_2-1)}(\xi)^{s_1-s_2}. \tag{4.20}$$

4.5 Mean value theorem for s-convex functions in Hermite-Hadamard sense

Drgomir and Fitzpatrick [25] gave the following result:

Theorem 4.5.1. *Suppose that $\Psi : [0,\infty) \to \mathbb{R}$ be an s-convex function in first sense, where $s \in (0,1)$ and let $c_1, c_2 \in [0,\infty)$, $c_1 \leq c_2$. Then the following inequality holds:*

$$\Psi\left(\frac{c_1+c_2}{2}\right) \leq \frac{1}{c_2-c_1}\int_{c_1}^{c_2} \Psi(u)du \leq \frac{\Psi(c_1)+s\Psi(c_2)}{s+1}. \tag{4.21}$$

The above inequalities are sharp.

Then from inequality (4.21) we give following result.

Theorem 4.5.2. *Suppose that $\Psi_1, \Psi_2 : [0,\infty) \to \mathbb{R}$ be an s-convex function in first sense, where $s \in (0,1)$ and let $c_1, c_2 \in [0,\infty)$, $c_1 \leq c_2$. Let $\Psi_1, \Psi_2 \in C^2([c_1, c_2])$. Then there exist some $\xi \in [c_1, c_2]$ such that the following equality holds:*

$$\frac{\frac{1}{c_2-c_1}\int_{c_1}^{c_2} \Psi_1(u)du - \Psi_1(\frac{c_1+c_2}{2})}{\frac{1}{c_2-c_1}\int_{c_1}^{c_2} \Psi_2(u)du - \Psi_2(\frac{c_1+c_2}{2})} = \frac{\Psi_1''(\xi)}{\Psi_2''(\xi)}, \tag{4.22}$$

provided that the denominators are non-zero.

Proof. Let

$$H := \frac{c_1+c_2}{2},$$

and

$$(T\Psi)(\lambda) := \frac{1}{c_2-c_1}\int_{c_1}^{c_2} \Psi_1(\lambda u + (1-\lambda)H)du - \Psi_1(H),$$

where $\lambda \in [0,1]$. Accordingly, we can define $(T\Psi_2)(\lambda)$.
We can have

$$(T\Psi_1)^{'}(\lambda) := \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} (u - H)\Psi_1^{'}(\lambda u + (1-\lambda)H)du,$$

and

$$(T\Psi_1)^{''}(\lambda) := \frac{1}{c_2 - c_1} \int_{c_1}^{c_2} (u - H)^2\Psi_1^{''}(\lambda u + (1-\lambda)H)du.$$

Now consider the function $Q(\lambda)$ defined by

$$Q(\lambda) = (T\Psi_2)(1)(T\Psi_1)(\lambda) - (T\Psi_1)(1)(T\Psi_2)(\lambda),$$

such that we have

$$Q(0) = Q(1) = Q^{'}(0) = 0.$$

Then from two applications of mean value theorem, we find $\upsilon \in [c_1, c_2]$ such that

$$Q^{''}(\upsilon) = 0.$$

It implies

$$\frac{1}{c_2 - c_1} \int_{[c_1,c_2]} (u-H)^2[(T\Psi_2)(1).\Psi_1^{''}(u\upsilon-(1-\upsilon)H)-(T\Psi_1)(1).\Psi_2^{''}(u\upsilon-(1-\upsilon)H)] = 0. \tag{4.23}$$

For some fixed υ, the expression in the square brackets in (4.23) is a continuous function of u, so it vanishes. Corresponding to that value of u, we get a number

$$\xi = u\upsilon + (1-\upsilon)H,$$

such that

$$(T\Psi_2)(1).\Psi_1^{''}(\xi) - (T\Psi_1)(1).\Psi_2^{''}(\xi) = 0.$$

Thus we get (4.22). □

Remark 19. *If* $\frac{\Psi_1^{''}}{\Psi_2^{''}}$ *is invertible, then we have*

$$\xi = \left(\frac{\Psi_1^{''}(\xi)}{\Psi_2^{''}(\xi)}\right)^{-1} \left(\frac{\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi_1(u)du - \Psi_1(\frac{c_1+c_2}{2})}{\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi_2(u)du - \Psi_2(\frac{c_1+c_2}{2})}\right). \tag{4.24}$$

Remark 20. *Let $s_1, s_2 \in (0,1)$. By taking $\Psi_1(u) = u^{s_1}$ and $\Psi_2(u) = u^{s_2}$, where $u \in (0,\infty)$, in Theorem 4.5.2 we have*

$$\frac{\frac{c_2^{s_1+1}-c_1^{s_1+1}}{(s_1+1)(c_2-c_1)} - \left(\frac{c_1+c_2}{2}\right)^{s_1}}{\frac{c_2^{s_2+1}-c_1^{s_2+1}}{(s_2+1)(c_2-c_1)} - \left(\frac{c_1+c_2}{2}\right)^{s_2}} = \frac{s_1(s_1-1)}{s_2(s_2-1)}(\xi)^{s_1-s_2}. \tag{4.25}$$

Now we define the following definition.

Definition 4.5.1. *Let $s \in (0,1)$ and $c_1, c_2 \in [0,\infty)$, $c_1 \le c_2$. Then quasi-arithmetic mean for strictly monotonic function φ defined on $[c_1, c_2]$ as:*

$$\widehat{M}_\varphi(c_1,c_2) = \varphi^{-1}\left(\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\varphi(w)dw - \varphi\left(\frac{c_1+c_2}{2}\right)\right). \tag{4.26}$$

Theorem 4.5.3. *Let $s \in (0,1)$ and $c_1, c_2 \in [0,\infty)$, $c_1 \le c_2$. Let $\varphi_1, \varphi_2, \varphi_3 \in C^2([c_1, c_2])$ be strictly monotonic real valued functions. Then*

$$\frac{\varphi_1\left(\widehat{M}_{\varphi_1}(c_1,c_2)\right) - \varphi_1\left(\widehat{M}_{\varphi_3}(c_1,c_2)\right)}{\varphi_2\left(\widehat{M}_{\varphi_2}(c_1,c_2)\right) - \varphi_2\left(\widehat{M}_{\varphi_3}(c_1,c_2)\right)} = \frac{\varphi_1''(\upsilon)\varphi_3'(\upsilon) - \varphi_1'(\upsilon)\varphi_3''(\upsilon)}{\varphi_2''(\upsilon)\varphi_3'(\upsilon) - \varphi_2'(\eta)\varphi_3''(\upsilon)}, \tag{4.27}$$

for some υ, provided that the denominators are non-zero.

Proof. Let us choose functions $\Psi_1 = \varphi_1 \circ \varphi_3^{-1}$, $\Psi_2 = \varphi_2 \circ \varphi_3^{-1}$, $u = \varphi_3(u)$ and $\frac{c_1+c_2}{2} = \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\varphi_3(u)du$ in Theorem 4.5.2, we observe that there exists some $\upsilon \in [c_1, c_2]$ such that

$$\begin{aligned}&\frac{\varphi_1\left(\widehat{M}_{\varphi_1}(c_1,c_2)\right) - \varphi_1\left(\widehat{M}_{\varphi_3}(c_1,c_2)\right)}{\varphi_2\left(\widehat{M}_{\varphi_2}(c_1,c_2)\right) - \varphi_2\left(\widehat{M}_{\varphi_3}(c_1,c_2)\right)}\\ &= \frac{\varphi_1''(\varphi_3^{-1}(\xi))\varphi_3'(\varphi_3^{-1}(\xi)) - \varphi_1'(\varphi_3^{-1}(\xi))\varphi_3''(\varphi_3^{-1}(\xi))}{\varphi_2''(\varphi_3^{-1}(\xi))\varphi_3'(\varphi_3^{-1}(\xi)) - \varphi_2'(\varphi_3^{-1}(\xi))\varphi_3''(\varphi_3^{-1}(\xi))}.\end{aligned} \tag{4.28}$$

Then by letting $\varphi_3^{-1}(\xi) = \upsilon$, we notice that we have $\upsilon \in [c_1, c_2]$, such that

$$\frac{\varphi_1\left(\widehat{M}_{\varphi_1}(c_1,c_2)\right) - \varphi_1\left(\widehat{M}_{\varphi_3}(c_1,c_2)\right)}{\varphi_2\left(\widehat{M}_{\varphi_2}(c_1,c_2)\right) - \varphi_2\left(\widehat{M}_{\varphi_3}(c_1,c_2)\right)} = \frac{\varphi_1''(\upsilon)\varphi_3'(\upsilon) - \varphi_1'(\upsilon)\varphi_3''(\upsilon)}{\varphi_2''(\upsilon)\varphi_3'(\upsilon) - \varphi_2'(\upsilon)\varphi_3''(\upsilon)}. \tag{4.29}$$

□

Again from inequality (4.21) we have following result.

Theorem 4.5.4. *Suppose that $\Psi_1, \Psi_2 : [0, \infty) \to \mathbb{R}$ be an s-convex function in first sense, where $s \in (0, 1)$ and let $c_1, c_2 \in [0, \infty)$, $c_1 \leq c_2$. Let $\Psi_1, \Psi_2 \in C^2([c_1, c_2])$. Then there exist some $\xi \in [c_1, c_2]$ such that the following equality holds:*

$$\frac{\frac{\Psi_1(c_1)+s\Psi_1(c_2)}{s+1} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi_1(u)du}{\frac{\Psi_2(c_1)+s\Psi_2(c_2)}{s+1} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi_2(u)du} = \frac{\Psi_1''(\xi)}{\Psi_2''(\xi)}, \tag{4.30}$$

provided that the denominators are non-zero.

Proof. Consider the function

$$(T\Psi_1)(u) = \frac{s\Psi_1(u) + \Psi_1(c_1)}{s+1}(u - c_1) - \int_{c_1}^{u}\Psi_1(x)dx. \tag{4.31}$$

Similarly, we can define $T\Psi_2(u)$.
Note that

$$(T\Psi_1)^{'}(u) = \frac{s\Psi_1^{'}(u)}{s+1}(u - c_1) - \frac{\Psi_1(u) - \Psi_1(c_1)}{s+1}, \tag{4.32}$$

and

$$(T\Psi_1)^{''}(u) = \frac{s\Psi_1^{''}(u)}{s+1}(u - c_1). \tag{4.33}$$

We observe that

$$(T\Psi_1)(c_1) = (T\Psi_1)^{'}(c_1) = (T\Psi_1)^{''}(c_1) = 0.$$

Now we define $D(u)$ as:

$$D(u) = (T\Psi_2)(c_2)(T\Psi_1)(u) - (T\Psi_1)(c_2)(T\Psi_2)(u). \tag{4.34}$$

Then note that

$$D(c_1) = D^{'}(c_2) = D^{''}(c_1) = D(c_2) = 0.$$

Thus by application of the mean-value theorem we get

$$D^{''}(\xi) = 0,$$

for some $\xi \in [c_1, c_2]$. Consequently, this completes the proof of the theorem. □

Remark 21. *If $\frac{\Psi_1''}{\Psi_2''}$ is invertible, then we have*

$$\xi = \left(\frac{\Psi_1''(\xi)}{\Psi_2''(\xi)}\right)^{-1} \left(\frac{\frac{\Psi_1(c_1)+s\Psi_1(c_2)}{s+1} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi_1(u)du}{\frac{\Psi_2(c_1)+s\Psi_2(c_2)}{s+1} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi_2(u)du}\right). \tag{4.35}$$

Remark 22. *Let $s_1, s_2 \in (0,1)$. By taking $\Psi_1(u) = u^{s_1}$ and $\Psi_2(u) = u^{s_2}$, where $u \in (0,\infty)$, in Theorem 4.5.4, we have*

$$\frac{(c_1^{s_1} + s_1 c_2^{s_1}) - \left(\frac{c_2^{s_1+1} - c_1^{s_1+1}}{c_2-c_1}\right)}{(c_1^{s_2} + s_2 c_2^{s_2}) - \left(\frac{c_2^{s_2+1} - c_1^{s_2+1}}{c_2-c_1}\right)} = \frac{s_1(s_1-1)(s_2+1)}{s_2(s_2-1)(s_1+1)}(\xi)^{s_1-s_2}. \tag{4.36}$$

Chapter 5

Integral inequalities of some generalized convex functions with applications

In the running chapter, we introduce the notion of an exponentially p-convex function and an exponentially s-convex function in second sense. We give some Hermite-Hadamard type inequalities via exponentially p-convex functions and exponentially s-convex functions in second sense and also give some applications.

5.1 Introduction

Iscan [45] gave following results for p-convex functions.

Theorem 5.1.1 ([45]). *Let $p \in \mathbb{R}\backslash\{0\}$, $\aleph \subset (0, \infty)$ be an interval and $\Psi : \aleph \to \mathbb{R}$ be a p-convex function, $c_1, c_2 \in \aleph$ with $c_1 < c_2$. such that $\Psi \in L_1[c_1, c_2]$, then we have*

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}du \leq \frac{\Psi(c_1) + \Psi(c_2)}{2}. \tag{5.1}$$

Lemma 5.1.1 ([45]). *Let $\Psi : \aleph \to \mathbb{R}$ be a differentiable function on $\aleph^\circ$, where $\aleph^\circ$ is interior of $\aleph$, and $c_1, c_2 \in \aleph$ with $c_1 < c_2$ and $p \in \mathbb{R}\backslash\{0\}$. Let $\Psi' \in L_1[c_1, c_2]$, then we* *have*

$$
\begin{aligned}
&\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{p}{c_2^p-c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}du \\
&=\frac{c_2^p-c_1^p}{2p}\int_0^1\frac{1-2r}{(rc_1^p+(1-r)c_2^p)^{1-\frac{1}{p}}}\Psi'\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)dr.
\end{aligned}
\tag{5.2}
$$

Dragomir [24, 25] gave important results for s-convex function.

Theorem 5.1.2 ([25])**.** *Suppose that $\Psi : [0,\infty) = \mathbb{R}_0 \to \mathbb{R}_0$ be an s-convex function in second sense, where $s \in (0,1)$ and let $c_1, c_2 \in [0,\infty)$, $c_1 \le c_2$. If $\Psi \in L_1[c_1,c_2]$, then the following inequalities hold:*

$$
2^{s-1}\Psi\left(\frac{c_1+c_2}{2}\right)\le\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)du\le\frac{\Psi(c_1)+\Psi(c_2)}{s+1}. \tag{5.3}
$$

Lemma 5.1.2 ([24])**.** *Let $\Psi : \aleph^\circ \to \mathbb{R}$ be a differentiable mapping on $\aleph^\circ$, where $\aleph^\circ$ is interior of $\aleph$, and $c_1, c_2 \in \aleph$ with $c_1 < c_2$. Let $\Psi' \in L_1[c_1,c_2]$ then following equality holds:*

$$
\begin{aligned}
&\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)du \\
&=\frac{c_2-c_1}{2}\int_0^1(1-2r)\Psi'(rc_1+(1-r)c_2)dr.
\end{aligned}
\tag{5.4}
$$

Awan et al. [14] introduced a new class of convex functions called exponentially convex function.

Definition 5.1.3 ([14])**.** *A function $\Psi : \aleph \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex, if*

$$
\Psi(rc_1-(1-r)c_2)\le r\frac{\Psi(c_1)}{e^{\alpha c_1}}+(1-r)\frac{\Psi(c_2)}{e^{\alpha c_2}}, \tag{5.5}
$$

for all $c_1, c_2 \in \aleph$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality (5.5) *is reversed the Ψ is called exponentially concave function.*

Now we define an exponentially p-convex function and an exponentially s-convex function in second sense as:

Definition 5.1.4. *Let $\aleph \subset (0,\infty)$ be an interval and $p \in \mathbb{R}\backslash\{0\}$. A function $\Psi : \aleph \to \mathbb{R}$ is said to be exponentially p-convex, if*

$$\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) \leq r\frac{\Psi(c_1)}{e^{\alpha c_1}} + (1-r)\frac{\Psi(c_2)}{e^{\alpha c_2}}, \tag{5.6}$$

for all $c_1, c_2 \in \aleph$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality (5.6) is in reversed order then Ψ is said to be exponentially p-concave function.

Definition 5.1.5. *Let $s \in (0,1]$ and $\aleph \subset \mathbb{R}_0$ be an interval. A function $\Psi : \aleph \to \mathbb{R}$ is said to be exponentially s-convex in the second sense, if*

$$\Psi(rc_1 + (1-r)c_2) \leq r^s\frac{\Psi(c_1)}{e^{\alpha c_1}} + (1-r)^s\frac{\Psi(c_2)}{e^{\alpha c_2}}, \tag{5.7}$$

for all $c_1, c_2 \in \aleph$, $r \in [0,1]$ and $\alpha \in \mathbb{R}$. If the inequality (5.7) is in reversed order then Ψ is said to be exponentially s-concave function.

Observe that, for $\alpha = 0$, exponentially p-convex function and exponentially s-convex function becomes p-convex function and s-convex function, respectively.

5.2 Integral inequalities for exponentially p-convex functions

We start with the following theorem.

Theorem 5.2.1. *Let $\Psi : \aleph \to \mathbb{R}$ be an integrable exponentially p-convex function. Let $c_1, c_2 \in \aleph$ with $c_1 < c_2$, Then following inequalities hold:*

$$\Psi\left(\left[\frac{c_1^p + c_2^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}e^{\alpha u}}du \leq A_1(r)\frac{\Psi(c_1)}{e^{\alpha c_1}} + A_2(r)\frac{\Psi(c_2)}{e^{\alpha c_2}}, \tag{5.8}$$

where

$$A_1(r) = \int_0^1 \frac{rdr}{e^{\alpha(rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}}} \quad \text{and} \quad A_2(r) = \int_0^1 \frac{(1-r)dr}{e^{\alpha(rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}}}.$$

Proof. By using exponentially p-convexity of Ψ, we have

$$2\Psi\left(\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi(u)}{e^{\alpha u}} + \frac{\Psi(v)}{e^{\alpha v}}. \tag{5.9}$$

Let $u^p = rc_1^p + (1-r)c_2^p$ and $v^p = (1-r)c_1^p + rc_2^p$, we get

$$2\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)}{e^{\alpha(rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}}} + \frac{\Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)}{e^{\alpha((1-r)c_1^p+rc_2^p)^{\frac{1}{p}}}}. \tag{5.10}$$

Integrating both sides with respect to $r \in [0,1]$ and applying change of variable, we find

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{\Psi(u)}{u^{1-p}e^{\alpha u}}du. \tag{5.11}$$

Hence the first inequality of (5.8) is complete. For next inequality, again using exponentially p-convexity of Ψ, we have

$$\frac{\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)}{e^{\alpha(rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}}} \leq \frac{r\frac{\Psi(c_1)}{e^{\alpha c_1}} + (1-r)\frac{\Psi(c_2)}{e^{\alpha c_2}}}{e^{\alpha(rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}}}. \tag{5.12}$$

Integrating with respect to $r \in [0,1]$, we get

$$\frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{\Psi(u)}{u^{1-p}e^{\alpha u}}du \leq \frac{\Psi(c_1)}{e^{\alpha c_1}}\int_0^1 \frac{rdr}{e^{\alpha(rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}}} + \frac{\Psi(c_2)}{e^{\alpha c_2}}\int_0^1 \frac{(1-r)dr}{e^{\alpha(rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}}}. \tag{5.13}$$

By combining (5.11) and (5.13), we get (5.8). □

Remark 23. *In Theorem* 5.2.1*, if one takes* $\alpha = 0$*, then one gets inequality* (1.11) *in Theorem* 6 *in* [45].

Theorem 5.2.2. *Let* $\Psi : \aleph \to \mathbb{R}$ *be a differentiable function on* $\aleph^\circ$ *and* $c_1, c_2 \in \aleph$ *with* $c_1 < c_2$ *and* $\Psi' \in L_1[c_1, c_2]$*. Let* $|\Psi'|^q$ *is exponentially p-convex on* $[c_1, c_2]$ *for* $q \geq 1$*, then following inequality holds:*

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}du\right| \\ &\leq \frac{c_2^p - c_1^p}{2p}B_1^{1-\frac{1}{q}}\left[B_2\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + B_3\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} B_1 =& B_1(c_1, c_2; p) = \frac{1}{4}\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}-1} \\ &\times \left[\, {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{c_1^p - c_2^p}{c_1^p + c_2^p}\right) + \, {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{c_2^p - c_1^p}{c_1^p + c_2^p}\right)\right], \end{aligned}$$

$$B_2 = B_2(c_1, c_2; p) = \frac{1}{24}\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}-1}\left[\ {}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{c_1^p - c_2^p}{c_1^p + c_2^p}\right)\right.$$
$$\left. + 6\ {}_2F_1\left(1-\frac{1}{p}, 2; 3; \frac{c_2^p - c_1^p}{c_1^p + c_2^p}\right) + \ {}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{c_2^p - c_1^p}{c_1^p + c_2^p}\right)\right],$$
$$B_3 = B_3(c_1, c_2; p) = B_1 - B_2.$$

Proof. Applying power mean inequality on (5.2) of Lemma 5.1.1, we get

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}du\right| \\
&\leq \frac{c_2^p - c_1^p}{2p}\int_0^1 \left|\frac{1-2r}{[rc_1^p + (1-r)c_2^p]^{1-\frac{1}{p}}}\right| \left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right| dr \\
&\leq \frac{c_2^p - c_1^p}{2p}\left(\int_0^1 \frac{|1-2r|}{[rc_1^p + (1-r)c_2^p]^{1-\frac{1}{p}}}dr\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \frac{|1-2r|}{[rc_1^p + (1-r)c_2^p]^{1-\frac{1}{p}}}\left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right|^q dr\right)^{\frac{1}{q}}.
\end{aligned} \tag{5.15}$$

Since $|\Psi'|^q$ is exponentially p-convex on $[c_1, c_2]$, we have

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}du\right| \\
&\leq \frac{c_2^p - c_1^p}{2p}\left(\int_0^1 \frac{|1-2r|}{[rc_1^p + (1-r)c_2^p]^{1-\frac{1}{p}}}dr\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 \frac{||1-2r||\left[r\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + (1-r)\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]}{[rc_1^p + (1-r)c_2^p]^{1-\frac{1}{p}}}dr\right)^{\frac{1}{q}} \\
&\leq \frac{c_2^p - c_1^p}{2p}B_1^{1-\frac{1}{q}}\left[B_2\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + B_3\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{5.16}$$

It is easy to note that,

$$\int_0^1 \frac{|1-2r|}{[rc_1^p + (1-r)c_2^p]^{1-\frac{1}{p}}}dr = B_1(c_1, c_2; p),$$

$$\int_0^1 \frac{|1-2r|r}{[rc_1^p + (1-r)c_2^p]^{1-\frac{1}{p}}}dr = B_2(c_1, c_2; p),$$

$$\int_0^1 \frac{|1-2r|(1-r)}{[rc_1^p+(1-r)c_2^p]^{1-\frac{1}{p}}}\mathrm{d}r = B_1(c_1,c_2;p) - B_2(c_1,c_2;p).$$

Hence the proof is completed. □

Remark 24. *In Theorem* 5.2.2*,*

(a) if one takes $\alpha = 0$*, then one gets the Theorem* 7 *in* [45]*.*

(b) if one takes $p = 1$*, then one gets the Theorem* 5 *in* [14]*.*

Corollary 5.2.1. *Let* $\Psi : \aleph \to \mathbb{R}$ *be a differentiable function on* $\aleph^\circ$ *and* $c_1, c_2 \in \aleph$ *with* $c_1 < c_2$ *and* $\Psi' \in L_1[c_1, c_2]$*. Let* $|\Psi'|$ *is exponentially p-convex on* $[c_1, c_2]$*, then following inequality holds:*

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}\mathrm{d}u\right| \\ &\leq \frac{c_2^p - c_1^p}{2p}\left[B_2\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right| + B_3\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right], \end{aligned} \tag{5.17}$$

where B_2 *and* B_3 *are given in Theorem* 5.2.2*.*

Remark 25. *In Corollary* 5.2.1*,*

1. *If one takes* $\alpha = 0$*, then one gets the Corollary* 1 *in* [45]*.*
2. *If one takes* $p = 1$*, then one gets the Theorem* 3 *in* [14]*.*

Theorem 5.2.3. *Let* $\Psi : \aleph \to \mathbb{R}$ *be a differentiable function on* $\aleph^\circ$ *and* $c_1, c_2 \in \aleph$ *with* $c_1 < c_2$ *and* $\Psi' \in L_1[c_1, c_2]$*. Let* $|\Psi'|^q$ *is exponentially p-convex on* $[c_1, c_2]$ *for* $q > 1$*,* $1/l + 1/q = 1$*, then following inequality holds:*

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}\mathrm{d}u\right| \\ &\leq \frac{c_2^p - c_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}}\left[B_4\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + B_5\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} B_4 &= B_4(c_1,c_2;p;q) \\ &= \begin{cases} \frac{1}{2c_1^{qp-q}}\ {}_2F_1\left(q-\frac{q}{p},1;3;1-(\frac{c_2}{c_1})^p\right), & p<0 \\ \frac{1}{2c_2^{qp-q}}\ {}_2F_1\left(q-\frac{q}{p},2;3;1-(\frac{c_1}{c_2})^p\right), & p>0, \end{cases} \end{aligned}$$

$$B_5 = B_5(c_1, c_2; p; q)$$
$$= \begin{cases} \frac{1}{2c_1^{qp-q}} \ {}_2F_1\left(q - \frac{q}{p}, 2; 3; 1 - (\frac{c_2}{c_1})^p\right), & p<0 \\ \frac{1}{2c_2^{qp-q}} \ {}_2F_1\left(q - \frac{q}{p}, 1; 3; 1 - (\frac{c_1}{c_2})^p\right), & p>0 \ . \end{cases}$$

Proof. Using Hölder's inequality on (5.2) of Lemma 5.1.1 and then using exponential p-convexity of $|\Psi'|^q$ on $[c_1, c_2]$, we get

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{\Psi(u)}{u^{1-p}} du\right| \\ &\leq \frac{c_2^p - c_1^p}{2p}\left(\int_0^1 |1-2r|^l \mathrm{d}r\right)^{\frac{1}{l}} \\ &\quad \times \left(\int_0^1 \frac{1}{[rc_1^p + (1-r)c_2^p]^{q(1-\frac{1}{p})}} \left|\Psi'\left([rc_1^p + (1-r)c_2^p]^{\frac{1}{p}}\right)\right|^q \mathrm{d}r\right)^{\frac{1}{q}} \qquad (5.19) \\ &\leq \frac{c_2^p - c_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}} \left(\int_0^1 \frac{r\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + (1-r)\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q}{[rc_1^p + (1-r)c_2^p]^{q-\frac{q}{p}}} \mathrm{d}r\right)^{\frac{1}{q}} \\ &\leq \frac{c_2^p - c_1^p}{2p}\left(\frac{1}{l+1}\right)^{\frac{1}{l}} \left[B_4\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + B_5\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]^{\frac{1}{q}}, \end{aligned}$$

where after calculations, we have

$$B_4 = \int_0^1 \frac{r}{[rc_1^p + (1-r)c_2^p]^{q-\frac{q}{p}}} \mathrm{d}r$$
$$= \begin{cases} \frac{1}{2c_1^{qp-q}} \ {}_2F_1\left(q - \frac{q}{p}, 1; 3; 1 - (\frac{c_2}{c_1})^p\right), & p<0 \\ \frac{1}{2c_2^{qp-q}} \ {}_2F_1\left(q - \frac{q}{p}, 2; 3; 1 - (\frac{c_1}{c_2})^p\right), & p>0 \ , \end{cases}$$

$$B_5 = \int_0^1 \frac{1-r}{[rc_1^p + (1-r)c_2^p]^{q-\frac{q}{p}}} \mathrm{d}r$$
$$= \begin{cases} \frac{1}{2c_1^{qp-q}} \ {}_2F_1\left(q - \frac{q}{p}, 2; 3; 1 - (\frac{c_2}{c_1})^p\right), & p<0 \\ \frac{1}{2c_2^{qp-q}} \ {}_2F_1\left(q - \frac{q}{p}, 1; 3; 1 - (\frac{c_1}{c_2})^p\right), & p>0 \ . \end{cases}$$

□

Remark 26. *In Theorem* 5.2.3*,*

1. *If one takes* $\alpha = 0$*, then one gets the Theorem* 8 *in* [45]*.*
2. *If one takes* $p = 1$*, then one gets the Theorem* 4 *in* [14]*.*

Theorem 5.2.4. *Let $\Psi : \aleph \to \mathbb{R}$ be a differentiable function on $\aleph^\circ$ and $c_1, c_2 \in \aleph$ with $c_1 < c_2$ and $\Psi' \in L_1[c_1, c_2]$. Let $|\Psi'|^q$ is exponentially p-convex on $[c_1, c_2]$ for $q > 1$, $1/l + 1/q = 1$, then we have*

$$\begin{aligned} &\left| \frac{\Psi(c_1) + \Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p} \int_{c_1}^{c_2} \frac{\Psi(u)}{u^{1-p}} \mathrm{d}u \right| \\ &\leq \frac{c_2^p - c_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{\left| \frac{\Psi'(c_1)}{e^{\alpha c_1}} \right|^q + \left| \frac{\Psi'(c_2)}{e^{\alpha c_2}} \right|^q}{2} \right)^{\frac{1}{q}}, \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} B_6 &= B_6(c_1, c_2; p; l) \\ &= \begin{cases} \frac{1}{2c_1^{pl-l}} \ {}_2F_1 \left(l - \frac{l}{p}, 1; 2; 1 - (\frac{c_2}{c_1})^p \right), & p<0 \\ \frac{1}{2c_2^{pl-l}} \ {}_2F_1 \left(l - \frac{l}{p}, 1; 2; 1 - (\frac{c_1}{c_2})^p \right), & p>0 \ . \end{cases} \end{aligned}$$

Proof. Using Hölder's inequality on (5.2) of Lemma 5.1.1 and then using exponential p-convexity of $|\Psi'|^q$ on $[c_1, c_2]$, we get

$$\begin{aligned} &\left| \frac{\Psi(c_1) + \Psi(c_2)}{2} - \frac{p}{c_2^p - c_1^p} \int_{c_1}^{c_2} \frac{\Psi(u)}{u^{1-p}} \mathrm{d}u \right| \\ &\leq \frac{c_2^p - c_1^p}{2p} \left(\int_0^1 \frac{1}{[rc_1^p + (1-r)c_2^p]^{l - \frac{l}{p}}} \mathrm{d}r \right)^{\frac{1}{l}} \\ &\quad \times \left(\int_0^1 |1 - 2r|^q \left| \Psi' \left([rc_1^p + (1-r)c_2^p]^{\frac{1}{p}} \right) \right|^q \mathrm{d}r \right)^{\frac{1}{q}} \\ &\leq \frac{c_2^p - c_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{\left| \frac{\Psi'(c_1)}{e^{\alpha c_1}} \right|^q + \left| \frac{\Psi'(c_2)}{e^{\alpha c_2}} \right|^q}{2} \right)^{\frac{1}{q}}, \end{aligned} \tag{5.21}$$

where a simple calculation implies

$$\begin{aligned} B_6(c_1, c_2; p; l) &= \int_0^1 \frac{1}{[rc_1^p + (1-r)c_1^p]^{l - \frac{l}{p}}} \mathrm{d}r \\ &= \begin{cases} \frac{1}{2c_1^{pl-l}} \ {}_2F_1 \left(l - \frac{l}{p}, 1; 2; 1 - (\frac{c_2}{c_1})^p \right), & p < 0 \\ \frac{1}{2c_2^{pl-l}} \ {}_2F_1 \left(l - \frac{l}{p}, 1; 2; 1 - (\frac{c_1}{c_2})^p \right), & p > 0 \ . \end{cases} \end{aligned} \tag{5.22}$$

and

$$\int_0^1 r|1-2r|^q \mathrm{d}r = \int_0^1 (1-r)|1-2r|^q \mathrm{d}r = \frac{1}{2(q+1)}. \tag{5.23}$$

By substituting (5.22) and (5.23) in (5.21), we get the required inequality (5.20). Hence the proof. □

Remark 27. *In Theorem* 5.2.4*, if one takes* $\alpha = 0$*, then one gets the Theorem* 9 *in* [45]*.*

5.2.1 Applications

Consider the following special means of two positive numbers c_1, c_2 with $c_1 < c_2$:

(1). The arithmetic mean

$$A = A(c_1, c_2) = \frac{c_1 + c_2}{2}.$$

(2). The harmonic mean

$$H = H(c_1, c_2) = \frac{2c_1c_2}{c_1 + c_2}.$$

(3). The p-logarithmic mean

$$L_p = L_p(c_1, c_2) = \left(\frac{c_2^{p+1} - c_1^{p+1}}{(p+1)(c_2 - c_1)}\right)^{\frac{1}{p}}, \ p \in \mathbb{R} \setminus \{-1, 0\}.$$

In the coming three propositions we take $0 < c_1 < c_2$ and $q > 1$.

Proposition 9. *Let* $\alpha \in \mathbb{R}$ *and* $p < 1$*. Then we have*

$$\left|L_{p-1}^{p-1} - HL_{p-2}^{p-2}\right| \leq \frac{c_2^p - c_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left|\frac{1}{c_1^2 e^{\alpha c_1}}\right|^q, \left|\frac{1}{c_2^2 e^{\alpha c_2}}\right|^q\right) HL_{p-1}^{p-1},$$

where B_6 *is defined as in Theorem* 5.2.4*.*

Proof. The proof follows from Theorem 5.2.4, for $\Psi : (0, \infty) \to \mathbb{R}$, $\Psi(u) = \frac{1}{u}$. Where $|\Psi'(u)|^q = |\frac{1}{u^2}|^q$ is exponentially p-convex for all $p < 1$ and $\alpha \in \mathbb{R}$. □

Proposition 10. *Let $\alpha \leq 0$ and $p > 1$. Then we have*

$$\left|L_{p-1}^{p-1}A(c_1^p, c_2^p) - L_{2p-1}^{2p-1}\right| \leq \frac{c_2^p - c_1^p}{2} B_6^{\frac{1}{l}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left|\frac{1}{c_1^{p-1}e^{\alpha c_1}}\right|^q, \left|\frac{1}{c_2^{p-1}e^{\alpha c_2}}\right|^q\right) L_{p-1}^{p-1},$$

where B_6 is defined as in Theorem 5.2.4.

Proof. The proof follows from Theorem 5.2.4, for $\Psi : (0, \infty) \to \mathbb{R}$, $\Psi(u) = u^p$. Here $|\Psi'(u)|^q = |pu^{p-1}|^q$ is exponentially p-convex for all $p > 1$ and $\alpha \leq 0$. □

Proposition 11. *Let $\alpha \leq 0$ and $p > 1$. Then we have*

$$\left|L_{p-1}^{p-1}A - L_p^p\right| \leq \frac{u_2^p - u_1^p}{2p} B_6^{\frac{1}{l}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(\left|\frac{1}{e^{\alpha u_1}}\right|^q, \left|\frac{1}{e^{\alpha u_2}}\right|^q\right) L_{p-1}^{p-1},$$

where B_6 is defined as in Theorem 5.2.4.

Proof. The proof follows from Theorem 5.2.4, for $\Psi : (0, \infty) \to \mathbb{R}$, $\Psi(u) = u$. Here note that $|\Psi'(u)|^q = 1$ is exponentially p-convex for all $p > 1$ and $\alpha \leq 0$. □

5.3 Integral inequalities for exponentially s-convex functions

Through out the section, we denote $\aleph \subset \mathbb{R}_0$ be an interval with interior $\aleph^\circ$ and $s \in (0, 1]$. We start our main results for exponentially s-convex functions in second sense from the following theorem.

Theorem 5.3.1. *Let $\Psi : \aleph \subset \mathbb{R}_0 \to \mathbb{R}$ be an integrable exponentially s-convex function in second sense on $\aleph^\circ$. Then for $c_1, c_2 \in \aleph$ with $c_1 < c_2$, we have*

$$2^{s-1}\Psi\left(\frac{c_1 + c_2}{2}\right) \leq \frac{1}{c_2 - c_1}\int_{c_1}^{c_2} \frac{\Psi(u)}{e^{\alpha u}}\mathrm{d}u \leq A_3(r)\frac{\Psi(c_1)}{e^{\alpha c_1}} + A_4(r)\frac{\Psi(c_2)}{e^{\alpha c_2}}, \tag{5.24}$$

where

$$A_3(r) = \int_0^1 \frac{r^s \mathrm{d}r}{e^{\alpha(rc_1 + (1-r)c_2)}} \quad \text{and} \quad A_4(r) = \int_0^1 \frac{(1-r)^s \mathrm{d}r}{e^{\alpha(rc_1 + (1-r)c_2)}}.$$

Proof. Applying exponentially s-convexity of Ψ, we have

$$2^s\Psi\left(\frac{u+v}{2}\right) \leq \frac{\Psi(u)}{e^{\alpha u}} + \frac{\Psi(v)}{e^{\alpha v}}. \tag{5.25}$$

Let $u = rc_1 + (1-r)c_2$ and $v = (1-r)c_1 + rc_2$, we get

$$2^s\Psi\left(\frac{c_1+c_2}{2}\right) \leq \frac{\Psi(rc_1+(1-r)c_2)}{e^{\alpha rc_1+(1-r)c_2}} + \frac{\Psi\left((1-r)c_1+rc_2\right)}{e^{\alpha(1-r)c_1+rc_2}}. \tag{5.26}$$

Integrating both sides with respect to $r \in [0,1]$ and applying changes of variable, we find

$$2^{s-1}\Psi\left(\frac{c_1+c_2}{2}\right) \leq \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\frac{\Psi(u)}{e^{\alpha u}}\mathrm{d}u. \tag{5.27}$$

Hence the first inequality of (5.24) is complete. For next inequality, again using exponentially s-convexity of Ψ, we have

$$\frac{\Psi(rc_1+(1-r)c_2)}{e^{\alpha(rc_1+(1-r)c_2)}} \leq \frac{r^s\frac{\Psi(c_1)}{e^{\alpha c_1}} + (1-r)^s\frac{\Psi(c_2)}{e^{\alpha c_2}}}{e^{\alpha(rc_1+(1-r)c_2)}}. \tag{5.28}$$

Integrating with respect to $r \in [0,1]$, we get

$$\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\frac{\Psi(u)}{e^{\alpha u}}\mathrm{d}u \leq \frac{\Psi(c_1)}{e^{\alpha c_1}}\int_0^1\frac{r^s\mathrm{d}r}{e^{\alpha(rc_1+(1-r)c_2)}} + \frac{\Psi(c_2)}{e^{\alpha c_2}}\int_0^1\frac{(1-r)^s\mathrm{d}r}{e^{\alpha(rc_1+(1-r)c_2)}}. \tag{5.29}$$

By combining (5.27) and (5.29), we get (5.24). □

Remark 28. *In Theorem* 5.3.1*, if one takes* $\alpha = 0$*, then one gets inequality* (5.3) *in Theorem* 5.1.2.

Theorem 5.3.2. *Let* $\Psi : \aleph \to \mathbb{R}$ *be a differentiable function on* $\aleph^\circ$ *and* $c_1, c_2 \in \aleph$ *with* $c_1 < c_2$ *and* $\Psi' \in L_1[c_1, c_2]$*. Let* $|\Psi'|$ *is exponentially s-convex in second sense on* $[c_1, c_2]$*, then we have*

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &\leq \frac{c_2-c_1}{2(s+1)(s+2)}\left[(3s+4)\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right| + (s+4)\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right].\end{aligned} \tag{5.30}$$

Proof. From Lemma 5.1.2, we have

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\
&=\frac{c_2-c_1}{2}\left|\int_0^1(1-2r)\Psi'(rc_1+(1-r)c_2)\mathrm{d}r\right| \\
&\leq\frac{c_2-c_1}{2}\int_0^1|1-2r||\Psi'(rc_1+(1-r)c_2)|\mathrm{d}r.
\end{aligned} \tag{5.31}$$

Using exponential s-convexity of Ψ', we get

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\
&\leq\frac{c_2-c_1}{2}\int_0^1|1-2r|\left[r^s\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|+(1-r)^s\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right]\mathrm{d}r \\
&\leq\frac{c_2-c_1}{2}\int_0^1(1+2r)\left[r^s\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|+(1-r)^s\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right]\mathrm{d}r \\
&=\frac{c_2-c_1}{2}\int_0^1\left[(1+2r)r^s\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|+(1+2r)(1-r)^s\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right]\mathrm{d}r.
\end{aligned} \tag{5.32}$$

Since

$$\int_0^1(1+2r)r^s\mathrm{d}r=\frac{3s+4}{(s+1)(s+2)}, \tag{5.33}$$

$$\int_0^1(1+2r)(1-r)^s\mathrm{d}r=\frac{s+4}{(s+1)(s+2)}. \tag{5.34}$$

By substituting equalities (5.33) and (5.34) in (5.32), we get inequality (5.30). □

Corollary 5.3.1. *Under the assumptions of Theorem* 5.3.2*,*

1. *If* $s=1$*, then*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right|\leq\frac{c_2-c_1}{12}\left[7\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|+5\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right]. \tag{5.35}$$

2. *If* $\alpha=0$*, then*

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\
&\leq\frac{c_2-c_1}{2(s+1)(s+2)}\left[(3s+4)\left|\Psi'(c_1)\right|+(s+4)\left|\Psi'(c_2)\right|\right].
\end{aligned} \tag{5.36}$$

Theorem 5.3.3. *Let $\Psi : \aleph \to \mathbb{R}$ be a differentiable function on $\aleph^\circ$ and $c_1, c_2 \in \aleph$ with $c_1 < c_2$ and $\Psi' \in L_1[c_1, c_2]$. Let $|\Psi'|$ is exponentially s-convex in second sense on $[c_1, c_2]$, then we have*

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &\le \frac{c_2-c_1}{2}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right| + \left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right]. \end{aligned} \tag{5.37}$$

Proof. From Lemma 5.1.2, we have

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &= \frac{c_2-c_1}{2}\left|\int_0^1 (1-2r)\Psi'(rc_1+(1-r)c_2)\mathrm{d}r\right| \\ &\le \frac{c_2-c_1}{2}\int_0^1 |1-2r||\Psi'(rc_1+(1-r)c_2)|\mathrm{d}r. \end{aligned} \tag{5.38}$$

Using exponential s-convexity of Ψ', we get

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &\le \frac{c_2-c_1}{2}\int_0^1 |1-2r|\left[r^s\left|\frac{\Psi(c_1)}{e^{\alpha c_1}}\right| + (1-r)^s\left|\frac{\Psi(c_2)}{e^{\alpha c_2}}\right|\right]\mathrm{d}r \\ &= \frac{c_2-c_1}{2}\int_0^1 \left[|1-2r|r^s\left|\frac{\Psi(c_1)}{e^{\alpha c_1}}\right| + |1-2r|(1-r)^s\left|\frac{\Psi(c_2)}{e^{\alpha c_2}}\right|\right]\mathrm{d}r \\ &= \frac{c_2-c_1}{2}\left[C_1(s)\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right| + C_2(s)\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right]. \end{aligned} \tag{5.39}$$

It is easily seen that

$$C_1(s) = \int_0^1 |1-2r|r^s\mathrm{d}r = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}, \tag{5.40}$$

$$C_2(s) = \int_0^1 |1-2r|(1-r)^s\mathrm{d}r = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}. \tag{5.41}$$

Thus by substituting equalities (5.40) and (5.41) in (5.39), we achieve inequality (5.37). □

Remark 29. *In Theorem* 5.3.3*,*

1. *If we take $\alpha = 0$,then we obtain Theorem* 1*, for $q = 1$, in* [55].
2. *If we take $s = 1$, then we get Theorem* 3 *in* [14].

Theorem 5.3.4. *Let $\Psi : \aleph \to \mathbb{R}$ be a differentiable function on $\aleph^\circ$ and $c_1, c_2 \in \aleph$ with $c_1 < c_2$ and $\Psi' \in L_1[c_1, c_2]$. Let $|\Psi'|^q$ is exponentially s-convex in second sense on $[c_1, c_2]$ with $q \geq 1$, then we have*

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &\leq \frac{c_2-c_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{s+\frac{1}{2^s}}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left[\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + \left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]^{\frac{1}{q}}.\end{aligned} \tag{5.42}$$

Proof. From Lemma 5.1.2, we have

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &= \frac{c_2-c_1}{2}\left|\int_0^1 (1-2r)\Psi'(rc_1+(1-r)c_2)\mathrm{d}r\right| \\ &\leq \frac{c_2-c_1}{2}\int_0^1 |1-2r||\Psi'(rc_1+(1-r)c_2)|\mathrm{d}r.\end{aligned} \tag{5.43}$$

Applying power-mean inequality, we find

$$\begin{aligned}&\frac{c_2-c_1}{2}\int_0^1 |1-2r||\Psi'(rc_1+(1-r)c_2)|\mathrm{d}r \\ &\leq \frac{c_2-c_1}{2}\left(\int_0^1 |1-2r|dr\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2r||\Psi'(rc_1+(1-r)c_2)|^q\mathrm{d}r\right)^{\frac{1}{q}}.\end{aligned} \tag{5.44}$$

Since $|\Psi'|^q$ is exponentially s-convex, we get

$$\begin{aligned}&\int_0^1 |1-2r||\Psi'(rc_1+(1-r)c_2)|^q\mathrm{d}r \\ &\leq \int_0^1 |1-2r|\left[r^s\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + (1-r)^s\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]\mathrm{d}r \\ &= \left[C_1(s)\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + C_2(s)\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right],\end{aligned} \tag{5.45}$$

where

$$C_1(s) = \int_0^1 |1-2r|r^s\mathrm{d}r = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}, \tag{5.46}$$

$$C_2(s) = \int_0^1 |1-2r|(1-r)^s\mathrm{d}r = \frac{s}{(s+1)(s+2)} + \frac{1}{2^s(s+1)(s+2)}, \tag{5.47}$$

$$\int_0^1 |1-2r|\mathrm{d}r = \frac{1}{2}. \tag{5.48}$$

Using (5.44)−(5.48) in (5.43), we get (5.42). □

Remark 30. *In Theorem* 5.3.4,

1. *If one takes* $\alpha = 0$, *then one gets Theorem* 1, *for* $q > 1$, *in* [55].

2. *If one takes* $s = 1$, *then one gets the Theorem* 5 *in* [14].

Theorem 5.3.5. *Let* $\Psi : \aleph \to \mathbb{R}$ *be a differentiable function on* $\aleph^\circ$ *and* $c_1, c_2 \in \aleph$ *with* $c_1 < c_2$ *and* $\Psi' \in L_1[c_1, c_2]$. *Let* $|\Psi'|^q$ *is exponentially* s*-convex in second sense on* $[c_1, c_2]$ *with* $q > 1$ *and* $\frac{1}{l} + \frac{1}{q} = 1$, *then we have*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \leq \frac{c_2-c_1}{2(l+1)^{\frac{1}{l}}}\left[\frac{\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + \left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q}{s+1}\right]^{\frac{1}{q}}. \tag{5.49}$$

Proof. From Lemma 5.1.2 and using Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &\leq \frac{c_2-c_1}{2}\left(\int_0^1 |1-2r|^l dr\right)^{\frac{1}{l}}\left(\int_0^1 |\Psi'(rc_1+(1-r)c_2)|^q \mathrm{d}r\right)^{\frac{1}{q}}.\end{aligned} \tag{5.50}$$

Since $|\Psi'|^q$ is exponentially s-convex, we get

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \\ &\leq \frac{c_2-c_1}{2}\left(\int_0^1 |1-2r|^l dr\right)^{\frac{1}{l}}\left(\int_0^1 \left[r^s\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + (1-r)^s\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q\right]\mathrm{d}r\right)^{\frac{1}{q}} \\ &= \frac{c_2-c_1}{2(l+1)^{\frac{1}{l}}}\left[\frac{\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|^q + \left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|^q}{s+1}\right]^{\frac{1}{q}}.\end{aligned} \tag{5.51}$$

Hence the proof is completed. □

Remark 31. *In Theorem* 5.3.5,

1. *If one takes* $\alpha = 0$, *then one gets*

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u\right| \leq \frac{c_2-c_1}{2(l+1)^{\frac{1}{l}}}\left[\frac{|\Psi'(c_1)|^q + |\Psi'(c_2)|^q}{s+1}\right]^{\frac{1}{q}}. \tag{5.52}$$

2. *If one takes* $s = 1$, *then one gets the Theorem* 4 *in* [14].

5.3.1 Applications

Suppose d be a division of the interval $[c_1, c_2]$, that is, $d : c_1 = u_0 < u_1 < \cdots < u_{m-1} < u_m = c_2$, and the trapezoidal formula is given as:

$$T(\Psi, d) = \sum_{k=0}^{m-1} \frac{\Psi(u_k) + \Psi(u_{k+1})}{2}(u_{k+1} - u_k).$$

It is known that if the mapping $\Psi : [c_1, c_2] \to \mathbb{R}$ is twice differentiable on (c_1, c_2) and $\mathcal{M} = max_{j\in(c_1,c_2)}|\Psi''(j)| < \infty$, then

$$\int_{c_1}^{c_2} \Psi(u)\mathrm{d}u = T(\Psi, d) + R(\Psi, d), \tag{5.53}$$

where the remainder term is given as:

$$|R(\Psi, d)| \le \frac{\mathcal{M}}{12} \sum_{k=0}^{m-1} (u_{k+1} - u_k)^3. \tag{5.54}$$

It is noticed that if Ψ is not twice differentiable or Ψ'' is not bounded, then (5.53) is invalid. However, Dragomir and Wang [26, 27, 28] have shown that the term $R(\Psi, d)$ can be obtained by using first derivative only. These estimates surely have several applications. In this section, we estimate the remainder term $R(\Psi, d)$ in some new way.

Proposition 12. *Let $\Psi : \aleph \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on $\aleph^\circ$ and $c_1, c_2 \in \aleph$ with $c_1 < c_2$. Let $|\Psi'|$ is exponentially s-convex in second sense on $[c_1, c_2]$ and $s \in (0, 1]$, then in (5.53), for every division d of $[c_1, c_2]$, we have*

$$\begin{aligned} |R(\Psi, d)| &\le \frac{1}{2}\frac{1}{(s+1)(s+2)}\left(s + \frac{1}{2^s}\right)\sum_{k=0}^{m-1}(u_{k+1} - u_k)^2\left[\left|\frac{\Psi'(u_k)}{e^{\alpha u_n}}\right| + \left|\frac{\Psi'(u_{k+1})}{e^{\alpha u_{k+1}}}\right|\right] \\ &\le max\left\{\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|, \left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right\}\frac{1}{(s+1)(s+2)}\left(s + \frac{1}{2^s}\right)\sum_{k=0}^{m-1}(u_{k+1} - u_k)^2. \end{aligned} \tag{5.55}$$

Proof. Applying Theorem 5.3.3 on the subinterval $[u_k, u_{k+1}]$ $(k = 0, 1, \ldots, m-1)$ of the division d, we obtain

$$\begin{aligned} &\left|\frac{\Psi(u_k) + \Psi(u_{k+1})}{2}(u_{k+1} - u_k) - \int_{u_k}^{u_{k+1}} \Psi(u)\mathrm{d}u\right| \\ &\le \frac{(u_{k+1} - u_k)^2}{2}\frac{1}{(s+1)(s+2)}\left(s + \frac{1}{2^s}\right)\left[\left|\frac{\Psi'(u_k)}{e^{\alpha u_k}}\right| + \left|\frac{\Psi'(u_{k+1})}{e^{\alpha u_{k+1}}}\right|\right]. \end{aligned} \tag{5.56}$$

Applying summation over k from 0 to $m-1$, we get

$$\begin{aligned}&\left|T(\Psi, d) - \int_{c_1}^{c_2} \Psi(u)\mathrm{d}u\right| \\ &\leq \frac{1}{2}\sum_{k=0}^{m-1}(u_{k+1}-u_k)^2 \frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\left[\left|\frac{\Psi'(u_k)}{e^{\alpha u_k}}\right| + \left|\frac{\Psi'(u_{k+1})}{e^{\alpha u_{k+1}}}\right|\right] \\ &\leq max\left\{\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|, \left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|\right\}\frac{1}{(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)\sum_{k=0}^{m-1}(u_{k+1}-u_k)^2.\end{aligned} \tag{5.57}$$

□

Proposition 13. *Let $\Psi : \aleph \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on $\aleph^\circ$ and $c_1, c_2 \in \aleph$ with $c_1 < c_2$. Let $|\Psi'|^q$ is exponentially s-convex in second sense on $[c_1, c_2]$ and $s \in (0, 1]$ and $q > 1$ such that $\frac{1}{l} + \frac{1}{q} = 1$, then in (5.53), for every division d of $[c_1, c_2]$, we have*

$$\begin{aligned}|R(\Psi, d)| &\leq \frac{1}{2(l+1)^{\frac{1}{l}}}\sum_{k=0}^{m-1}(u_{k+1}-u_k)^2\left[\frac{\left|\frac{\Psi'(u_k)}{e^{\alpha u_k}}\right|^q + \left|\frac{\Psi'(u_{k+1})}{e^{\alpha u_{k+1}}}\right|^q}{s+1}\right]^{\frac{1}{q}} \\ &\leq \frac{max\left\{\frac{2\left|\frac{\Psi'(c_1)}{e^{\alpha c_1}}\right|}{s+1}, \frac{2\left|\frac{\Psi'(c_2)}{e^{\alpha c_2}}\right|}{s+1}\right\}}{2(l+1)^{\frac{1}{l}}}\sum_{k=0}^{m-1}(u_{k+1}-u_k)^2.\end{aligned} \tag{5.58}$$

Proof. Using Theorem 5.3.5 and similar arguments used in Proposition 12, we get the required result. □

Chapter 6

Some integral properties of (s,p)-convex functions

In this chapter, we give some Hermite-Hadamard type inequalities for (s,p)-convex functions via fractional integrals. This chapter also includes some integral inequalities for (s,p)-convex functions on co-ordinates.

6.1 Introduction

The Hermite-Hadamard inequalities for harmonically convex functions, p-convex functions and s-convex functions are given in the following theorems, respectively.

Theorem 6.1.1 ([43]). *Let $\Psi : \aleph \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $c_1, c_2 \in \aleph$ with $c_1 < c_2$. If $\Psi \in L_1[c_1, c_2]$ then the following inequalities hold:*

$$\Psi\left(\frac{2c_1c_2}{c_1+c_2}\right) \leq \frac{c_1c_2}{c_2-c_1}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^2}\mathrm{d}u \leq \frac{\Psi(c_1)+\Psi(c_2)}{2}. \tag{6.1}$$

Theorem 6.1.2 ([45]). *Let $\Psi : \aleph \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\setminus\{0\}$, and $c_1, c_2 \in \aleph$ with $c_1 < c_2$. If $\Psi \in L_1[c_1, c_2]$ then we have*

$$\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{p}{c_2^p-c_1^p}\int_{c_1}^{c_2}\frac{\Psi(u)}{u^{1-p}}\mathrm{d}u \leq \frac{\Psi(c_1)+\Psi(c_2)}{2}. \tag{6.2}$$

Theorem 6.1.3 ([25]). *Suppose that $\Psi : \mathbb{R}_0 \to \mathbb{R}_0$ be an s-convex function in second sense, where $s \in (0,1)$ and let $c_1, c_2 \in [0,\infty)$, $c_1 \leq c_2$. If $\Psi \in L_1[c_1, c_2]$, then the* *following inequalities hold:*

$$2^{s-1}\Psi\left(\frac{c_1+c_2}{2}\right) \leq \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi(u)\mathrm{d}u \leq \frac{\Psi(c_1)+\Psi(c_2)}{s+1}. \tag{6.3}$$

We recall the Riemann−Liouville fractional integrals.

Definition 6.1.1 ([53]). *Let $\Psi \in L_1[c_1, c_2]$. The right-hand side and left-hand side Riemann−Liouville fractional integrals $\mathcal{J}^{\alpha}_{c_1+}\Psi$ and $\mathcal{J}^{\alpha}_{c_2-}\Psi$ of order $\alpha > 0$ with $c_2 > c_1 \geq 0$ are defined by*

$$\mathcal{J}^{\alpha}_{c_1+}\Psi(u) = \frac{1}{\Gamma(\alpha)}\int_{c_1}^{u}(u-t)^{\alpha-1}\Psi(t)\mathrm{d}t, \ u > c_1,$$

and

$$\mathcal{J}^{\alpha}_{c_2-}\Psi(u) = \frac{1}{\Gamma(\alpha)}\int_{u}^{c_2}(t-u)^{\alpha-1}\Psi(t)\mathrm{d}t, \ u < c_2,$$

respectively, here $\Gamma(\cdot)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\mathrm{d}t$.

Fang and Shi [30] introduced the notion of (p, h)-convex function as follows:

Definition 6.1.2 ([30]). *Let J is a real interval and $\aleph$ is p-convex set. Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. Then $\Psi : \aleph \to \mathbb{R}$ is called a (p, h)-convex function, if Ψ is non-negative and*

$$\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) \leq h(r)\Psi(c_1) + h(1-r)\Psi(c_2) \tag{6.4}$$

for all $c_1, c_2 \in \aleph$ and $r \in (0, 1)$.

Now we define subclass of (p, h)-convex function as:

Definition 6.1.3. *Let $s \in (0, 1]$, $p \in \mathbb{R} \setminus \{0\}$ and $\aleph \subset (0, \infty)$ be an interval. Then a function $\Psi : \aleph \to (0, \infty)$ is said to be (s, p)-convex, if*

$$\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) \leq r^s\Psi(c_1) + (1-r)^s\Psi(c_2) \tag{6.5}$$

for all $c_1, c_2 \in \aleph$ with $c_1 < c_2$ and $r \in [0, 1]$.

Remark 32. *In Definition* 6.1.3*,*

1. *If we take $p = 1$ then we have Definition* 1.1.2*.*
2. *If we take $s = 1$ then we have Definition* 1.1.6*.*
3. *If we take $s = 1$ and $p = 1$ then we have definition of convex function .*

6.2 Integral inequalities of (s,p)-convex functions via fractional integrals

We begin from the following result.

Theorem 6.2.1. *Let $s \in (0,1]$, $r \in [0,1]$ and $p \in \mathbb{R}\backslash\{0\}$. Let $\Psi : \aleph \subset (0,\infty) \to (0,\infty)$ be a (s,p)-convex function, where $\aleph$ is an interval, such that $\Psi \in L_1[c_1, c_2]$ for $c_1, c_2 \in \aleph$, where $\aleph^o$ is the interior of $\aleph$, with $c_1 < c_2$ and $\alpha > 0$. Then the following inequalities for fractional integrals hold:*

(i) If $p > 0$,

$$\begin{aligned} 2^{s-1}\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) &\leq \frac{\Gamma(\alpha+1)}{2(c_2^p - c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p) + \mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right] \\ &\leq \left[\frac{\alpha}{\alpha+s} + \alpha\beta(\alpha, s+1)\right]\frac{\Psi(c_1)+\Psi(c_2)}{2}, \end{aligned} \tag{6.6}$$

where $\mu(u) = u^{\frac{1}{p}}$ for all $u \in [c_1^p, c_2^p]$.

(ii) If $p < 0$,

$$\begin{aligned} 2^{s-1}\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) &\leq \frac{\Gamma(\alpha+1)}{2(c_1^p - c_2^p)^\alpha}\left[\mathcal{J}^\alpha_{c_2^p+}(\Psi\circ\mu)(c_1^p) + \mathcal{J}^\alpha_{c_1^p-}(\Psi\circ\mu)(c_2^p)\right] \\ &\leq \left[\frac{\alpha}{\alpha+s} + \alpha\beta(\alpha, s+1)\right]\frac{\Psi(c_1)+\Psi(c_2)}{2}, \end{aligned} \tag{6.7}$$

where $\mu(u) = u^{\frac{1}{p}}$ for all $u \in [c_2^p, c_1^p]$.

Proof. (i) Let $p > 0$. Since Ψ is (s,p)-convex then for all $u, v \in \aleph$, we have

$$\Psi\left(\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi(u) + \Psi(v)}{2^s}.$$

Taking $u = (rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}$ and $v = ((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}$ with $r \in [0,1]$, we get

$$\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Psi\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right) + \Psi\left(((1-r)c_1^p + rc_2^p)^{\frac{1}{p}}\right)}{2^s}. \tag{6.8}$$

Multiplying both sides of (6.8) by $r^{\alpha-1}$ and integrating with respect to r over [0,1], implies

$$\frac{1}{\alpha}\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \frac{\Gamma(\alpha)}{2^s(c_2^p-c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p)+\mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]. \tag{6.9}$$

Which is the left hand side of (6.6). Again applying (s,p)-convexity of Ψ, for all $r\in[0,1]$, we have

$$\frac{\Psi\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)+\Psi\left(((1-r)c_1^p+rc_2^p)^{\frac{1}{p}}\right)}{2^s} \leq \frac{(r^s+(1-r)^s)\left(\Psi(c_1)+\Psi(c_2)\right)}{2^s}. \tag{6.10}$$

Multiplying both sides of (6.10) by $r^{\alpha-1}$ and integrating with respect to r over [0,1], yields

$$\begin{aligned}&\frac{\Gamma(\alpha)}{2^s(c_2^p-c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p)+\mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\\ &\leq\left[\frac{1}{\alpha+s}+\beta(\alpha,s+1)\right]\frac{\Psi(c_1)+\Psi(c_2)}{2^s}.\end{aligned} \tag{6.11}$$

Thus from (6.9) and (6.11) we get (6.6).

(ii) The proof is similar with (i). □

Remark 33. *Under the assumptions of Theorem* 6.2.1*,*

1. *If* $p=1$*, we get Theorem* 3 *in* [86].
2. *If* $p=1$ *and* $s=1$*, we get Theorem* 2 *in* [80].
3. *If* $s=1$ *and* $\alpha=1$*, we get Theorem* 6 *in* [45].
4. *If* $p=1$ *and* $\alpha=1$*, we get* (6.3).
5. *If* $p=1$ *and* $\alpha=1$ *and* $s=1$*, we get* (1.14).
6. *If* $p=-1$ *and* $\alpha=1$ *and* $s=1$*, we get* (6.1).

Lemma 6.2.1. *Let* $p\in\mathbb{R}\setminus\{0\}$*. Let* $\Psi:\aleph\subset(0,\infty)\to(0,\infty)$*, where* $\aleph$ *is an interval, be a differentiable function on* $\aleph^o$ *such that* $\Psi'\in L_1[c_1,c_2]$ *for* $c_1,c_2\in\aleph^o$ *with* $c_1<c_2$ *and* $\alpha>0$*. Then the following equalities for fractional integrals hold:*

(*i*) *If* $p > 0$,

$$\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2^p - c_1^p)^\alpha}\left[\mathcal{J}^{\alpha}_{c_1^p+}(\Psi\circ\mu)(c_2^p) + \mathcal{J}^{\alpha}_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]$$
$$= \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(u-c_1^p)^\alpha(\Psi\circ\mu)'(u)\mathrm{d}u - \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(c_2^p-u)^\alpha(\Psi\circ\mu)'(u)\mathrm{d}u, \tag{6.12}$$

where $\mu(u) = u^{\frac{1}{p}}$ *for all* $u \in [c_1^p, c_2^p]$.

(*ii*) *If* $p < 0$,

$$\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_1^p - c_2^p)^\alpha}\left[\mathcal{J}^{\alpha}_{c_2^p+}(\Psi\circ\mu)(c_1^p) + \mathcal{J}^{\alpha}_{c_1^p-}(\Psi\circ\mu)(c_2^p)\right]$$
$$= \frac{1}{2(c_1^p - c_2^p)^\alpha}\int_{c_2^p}^{c_1^p}(u-c_2^p)^\alpha(\Psi\circ\mu)'(u)\mathrm{d}u - \frac{1}{2(c_1^p - c_2^p)^\alpha}\int_{c_2^p}^{c_1^p}(c_1^p-w)^\alpha(\Psi\circ\mu)'(u)\mathrm{d}u, \tag{6.13}$$

where $\mu(u) = u^{\frac{1}{p}}$ *for all* $u \in [c_2^p, c_1^p]$.

Proof. Consider,

$$I = \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(u-c_1^p)^\alpha(\Psi\circ\mu)'(u)\mathrm{d}u - \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(c_2^p-u)^\alpha(\Psi\circ\mu)'(u)\mathrm{d}u$$
$$= I_1 - I_2. \tag{6.14}$$

Then by integrating by parts, we get

$$\begin{aligned} I_1 &= \frac{1}{2(c_2^p - c_1^p)^\alpha}\left[(u-c_1^p)^\alpha(\Psi\circ\mu)(u)\Big|_{c_1^p}^{c_2^p} - \alpha\int_{c_1^p}^{c_2^p}(u-c_1^p)^{\alpha-1}(\Psi\circ\mu)(u)\mathrm{d}u\right] \\ &= \frac{1}{2}(\Psi\circ\mu)(c_2^p) - \frac{\Gamma(\alpha+1)}{2(c_2^p - c_1^p)^\alpha}\mathcal{J}^{\alpha}_{c_2^p-}(\Psi\circ\mu)(c_1^p). \end{aligned} \tag{6.15}$$

Similarly, we have

$$\begin{aligned} I_2 &= \frac{1}{2(c_2^p - c_1^p)^\alpha}\left[(c_2^p-u)^\alpha(\Psi\circ\mu)(u)\Big|_{c_1^p}^{c_2^p} + \alpha\int_{c_1^p}^{c_2^p}(c_2^p-u)^{\alpha-1}(\Psi\circ\mu)(u)\mathrm{d}u\right] \\ &= -\frac{1}{2}(\Psi\circ\mu)(c_1^p) + \frac{\Gamma(\alpha+1)}{2(c_2^p - c_1^p)^\alpha}\mathcal{J}^{\alpha}_{c_1^p+}(\Psi\circ\mu)(c_2^p). \end{aligned} \tag{6.16}$$

Thus from (6.14), (6.15) and (6.16), we achieve (6.12).

(*ii*) The proof is similar with (*i*). □

Remark 34. *In above Lemma* 6.2.1*, if we take* $p = 1$ *then we have*

$$\begin{aligned}&\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_2-c_1)^{\alpha}}[\mathcal{J}^{\alpha}_{c_1+}\Psi(c_2)+\mathcal{J}^{\alpha}_{c_2-}\Psi(c_1)]\\&=\frac{1}{2(c_2-c_1)^{\alpha}}\int_{c-1}^{c_2}(u-c_1)^{\alpha}\Psi'(u)\mathrm{d}u-\frac{1}{2(c_2-c_1)^{\alpha}}\int_{c-1}^{c_2}(c_2-u)^{\alpha}\Psi'(u)\mathrm{d}u.\end{aligned}\tag{6.17}$$

Theorem 6.2.2. *Let* $s \in (0,1]$*,* $r \in [0,1]$ *and* $p \in \mathbb{R}\backslash\{0\}$*. Let* $\Psi : \aleph \subset (0,\infty) \to (0,\infty)$*, where* $\aleph$ *is an interval, be a differentiable function on* $\aleph^o$ *such that* $\Psi' \in L_1[c_1, c_2]$ *for* $c_1, c_2 \in \aleph^o$ *with* $c_1 < c_2$ *and* $\alpha > 0$*. Let* $|\Psi'|$ *is* (s,p)*-convex function on* $[c_1, c_2]$*, then the following inequalities for fractional integrals hold:*
(i) *If* $p > 0$*,*

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_2^p-c_1^p)^{\alpha}}\left[\mathcal{J}^{\alpha}_{c_1^p+}(\Psi\circ\mu)(c_2^p)+\mathcal{J}^{\alpha}_{c_2^p-}(\psi\circ\mu)(c_1^p)\right]\right|\\&\leq\frac{c_2^p-c_1^p}{2}\left[B_1(\alpha,s,p)|\psi'(c_1)|+B_2(\alpha,s,p)|\Psi'(c_2)|\right],\end{aligned}\tag{6.18}$$

where $\mu(u) = u^{\frac{1}{p}}$ *for all* $u \in [c_1^p, c_2^p]$*.*
(ii) *If* $p < 0$*,*

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_1^p-c_2^p)^{\alpha}}\left[\mathcal{J}^{\alpha}_{c_2^p+}(\Psi\circ\mu)(c_1^p)+\mathcal{J}^{\alpha}_{c_1^p-}(\Psi\circ\mu)(c_2^p)\right]\right|\\&\leq\frac{c_1^p-c_2^p}{2}\left[B_1(\alpha,s,p)|\Psi'(c_1)|+B_2(\alpha,s,p)|\Psi'(c_2)|\right],\end{aligned}\tag{6.19}$$

with $\mu(u) = u^{\frac{1}{p}}$ *for all* $u \in [c_2^p, c_1^p]$*. Here*

$$B_1=\int_0^1\frac{r^s(1-r)^{\alpha}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r+\int_0^1\frac{r^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r,$$

and

$$B_2=\int_0^1\frac{(1-r)^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r+\int_0^1\frac{r^{\alpha}(1-r)^s}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r.$$

Proof. (i) Let $p > 0$. Using Lemma 6.2.1(i), we get

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2^p - c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p) + \mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\right| \\
&\le \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(u-c_1^p)^\alpha|(\Psi\circ\mu)'(u)|\mathrm{d}u \\
&\quad + \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(c_2^p - u)^\alpha|(\Psi\circ\mu)'(u)|\mathrm{d}u \\
&= \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(u-c_1^p)^\alpha\frac{1}{pu^{1-\frac{1}{p}}}|\Psi'(u^{\frac{1}{p}})|\mathrm{d}u \\
&\quad + \frac{1}{2(c_2^p - c_1^p)^\alpha}\int_{c_1^p}^{c_2^p}(c_2^p - u)^\alpha\frac{1}{pu^{1-\frac{1}{p}}}|\Psi'(u^{\frac{1}{p}})|\mathrm{d}u.
\end{aligned} \tag{6.20}$$

Setting $u = rc_1^p + (1-r)c_2^p$, we find

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2^p - c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p) + \mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\right| \\
&\le \frac{1}{2(c_2^p - c_1^p)^\alpha}\Big[\int_{c_1^p}^{c_2^p}\frac{(rc_1^p + (1-r)c_2^p - c_1^p)^\alpha}{p(rc_1^p + (1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}})\right|\mathrm{d}r \\
&\quad + \int_{c_1^p}^{c_2^p}\frac{(c_2^p - rc_1^p - (1-r)c_2^p)^\alpha}{p(rc_1^p + (1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}})\right|\mathrm{d}r\Big] \\
&= \frac{c_2^p - c_1^p}{2}\Big[\int_0^1\frac{(1-r)^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}})\right|\mathrm{d}r \\
&\quad + \int_0^1\frac{r^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}})\right|\mathrm{d}r\Big].
\end{aligned} \tag{6.21}$$

Since $\Psi'|$ is (s,p)-convex function on $[c_1, c_2]$, we have

$$\left|\Psi'((rc_1^p + (1-r)c-2^p)^{\frac{1}{p}})\right| \le r^s\,|\Psi'(c_1)| + (1-r)^s|\Psi'(c_2)|.$$

Then (6.21) gives

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_1)}{2}-\frac{\Gamma(\alpha+1)}{2(c_2^p-c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p)+\mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\right| \\
&\leq \frac{c_2^p-c_1^p}{2}\Bigg[\int_0^1 \frac{(1-r)^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}(r^s|\Psi'(c_1)|+(1-r)^s|\Psi'(c_1)|)dr \\
&\quad+\int_0^1 \frac{r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}(r^s|\Psi'(c_1)|+(1-r)^s|\Psi'(c_2)|)dr\Bigg] \\
&= \frac{c_2^p-c_1^p}{2}\Bigg[\left\{\int_0^1 \frac{r^s(1-r)^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r+\int_0^1 \frac{r^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r\right\}|\Psi'(c_1)| \\
&\quad+\left\{\int_0^1 \frac{(1-r)^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r+\int_0^1 \frac{r^\alpha(1-r)^s}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r\right\}|\Psi'(c_2)|\Bigg] \\
&= \frac{c_2^p-c_1^p}{2}\left[B_1(\alpha,s,p)|\Psi'(c_1)|+B_2(\alpha,s,p)|\Psi'(c_2)|\right].
\end{aligned} \tag{6.22}$$

Hence (i) is proved.

(ii) The proof is similar with (i) by using Lemma 6.2.1(ii). □

Theorem 6.2.3. *Let $s\in(0,1]$, $r\in[0,1]$ and $p\in\mathbb{R}\backslash\{0\}$. Let $\Psi:\aleph\subset(0,\infty)\to(0,\infty)$, where $\aleph$ is an interval, be a differentiable function on $\aleph$ such that $\Psi'\in L_1[c_1,c_2]$ for $c_1,c_2\in\aleph$ with $c_1<c_2$ and $\alpha>0$. Let $|\Psi'|^q$, $q\geq 1$, is (s,p)-convex function on $[c_1,c_2]$, then the following inequalities for fractional integrals hold:*
(i) *If $p>0$,*

$$\begin{aligned}
&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_2^p-c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p)+\mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\right| \\
&\leq \frac{c_2^p-c_1^p}{2}\Big[(B_3(\alpha,s,p))^{1-\frac{1}{q}}\left\{|B_4\Psi'(u)|^q+B_5(\alpha,s,p)|\Psi'(c_2)|^q\right\}^{\frac{1}{q}} \\
&\quad+(B_6(\alpha,s,p))^{1-\frac{1}{q}}\left\{B_7|\Psi'(c_1)|^q+B_8(\alpha,s,p)|\Psi'(c_2)|^q\right\}^{\frac{1}{q}}\Big],
\end{aligned} \tag{6.23}$$

where $\mu(u)=u^{\frac{1}{p}}$ for all $u\in[c_1^p,c_2^p]$.

(ii) If $p<0$,

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_1^p-c_2^p)^\alpha}\left[\mathcal{J}^\alpha_{c_2^p+}(\Psi\circ\mu)(c_1^p)+\mathcal{J}^\alpha_{c_1^p-}(\Psi\circ\mu)(c_2^p)\right]\right|$$
$$\le\frac{c_1^p-c_2^p}{2}\Big[\left(B_3(\alpha,s,p)\right)^{1-\frac{1}{q}}\left\{B_4|\Psi'(c_1)|^q+B_5(\alpha,s,p)|\Psi'(c_2)|^q\right\}^{\frac{1}{q}} \tag{6.24}$$
$$+\left(B_6(\alpha,s,p)\right)^{1-\frac{1}{q}}\left\{B_7|\Psi'(c_1)|^q+B_8(\alpha,s,p)|\Psi'(c_2)|^q\right\}^{\frac{1}{q}}\Big],$$

with $\mu(u)=u^{\frac{1}{p}}$ *for all* $u\in[c_2^p,c_1^p]$*. Where*

$$B_3=\int_0^1\frac{(1-r)^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r,\ B_4=\int_0^1\frac{(1-r)^\alpha r^s}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r,$$

$$B_5=\int_0^1\frac{(1-r)^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r,\ B_6=\int_0^1\frac{r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r,$$

and

$$B_7=\int_0^1\frac{r^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r,\ B_8=\int_0^1\frac{r^\alpha(1-r)^s}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r.$$

Proof. (i) Let $p>0$. Using (6.21), power mean inequality and the (s,p)-convexity of $|\Psi'|^q$, we find

$$\left|\frac{\Psi(c_1)+\Psi(c_2)}{2}-\frac{\Gamma(\alpha+1)}{2(c_2^p-c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p)+\mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\right|$$
$$\le\frac{c_2^p-c_1^p}{2}\left[\int_0^1\frac{(1-r)^\alpha}{p\left(rc_1^p-(1-r)c_2^p\right)^{1-\frac{1}{p}}}\left|\Psi'\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)\right|\mathrm{d}r\right.$$
$$\left.+\int_0^1\frac{r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)\right|\mathrm{d}r\right]$$
$$\le\frac{c_2^p-c_1^p}{2}\left[\left(\int_0^1\frac{(1-r)^\alpha}{p\left(rc_1^p-(1-r)c_2^p\right)^{1-\frac{1}{p}}}\mathrm{d}r\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^1\frac{(1-r)^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'\left((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}}\right)\right|^q\mathrm{d}r\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\frac{r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\int_0^1\frac{r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'((rc_1^p+(1-r)c_2^p)^{\frac{1}{p}})\right|^q\mathrm{d}r\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{c_2^p - c_1^p}{2}\Bigg[\left(\int_0^1 \frac{(1-r)^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{(1-r)^\alpha r^s}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}|\Psi'(c_1)|^q\mathrm{d}r + \int_0^1 \frac{(1-r)^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}|\Psi'(c_2)|^q\mathrm{d}r\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \frac{r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\mathrm{d}r\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{r^{\alpha+s}}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}|\Psi'(c_1)|^q\mathrm{d}r + \int_0^1 \frac{(1-r)^s r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}|\Psi'(c_2)|^q\mathrm{d}r\right)^{\frac{1}{q}}\Bigg]$$

$$= \frac{c_2^p - c_1^p}{2}\Big[(B_3(\alpha,s,p))^{1-\frac{1}{q}}\left\{|B_4 f'(a)|^q + B_5(\alpha,s,p)|\Psi'(c_2)|^q\right\}^{\frac{1}{q}}$$

$$+(B_6(\alpha,s,p))^{1-\frac{1}{q}}\left\{B_7|\Psi'(c_1)|^q + B_8(\alpha,s,p)|\Psi'(c_2)|^q\right\}^{\frac{1}{q}}\Big]. \tag{6.25}$$

This completes the proof of (i).

(ii) The proof is similar with (i). □

Theorem 6.2.4. *Let $s \in (0,1]$, $r \in [0,1]$ and $p \in \mathbb{R}\backslash\{0\}$. Let $\Psi : \aleph \subset (0,\infty) \to (0,\infty)$, where $\aleph$ is an interval, be a differentiable function on $\aleph^o$ such that $\Psi' \in L_1[c_1,c_2]$ for $c_1, c_2 \in \aleph^o$ with $c_1 < c_2$ and $\alpha > 0$. Let $|\Psi'|^q$, where $q, l > 1$ such that $\frac{1}{l}+\frac{1}{q}=1$, is (s,p)-convex function on $[c_1,c_2]$, then the following inequalities for fractional integrals hold:*

(i) If $p > 0$,

$$\begin{aligned}&\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2^p-c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p) + \mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\right|\\ &\leq \frac{c_2^p-c_1^p}{2}\left[(B_9(\alpha,p,l))^{\frac{1}{l}} + (B_{10}(\alpha,p,l))^{\frac{1}{l}}\right]\left[\frac{|\Psi'(c_1)|^q+|\Psi'(c_2)|^q}{s+1}\right]^{\frac{1}{q}},\end{aligned} \tag{6.26}$$

where $\mu(u) = u^{\frac{1}{p}}$ for all $u \in [c_1^p, c_2^p]$.

(ii) if $p < 0$,

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_1^p - c_2^p)^\alpha}\left[\mathcal{J}^\alpha_{c_2^p+}(\Psi\circ\mu)(c_1^p) + \mathcal{J}^\alpha_{c_1^p-}(\Psi\circ\mu)(c_2^p)\right]\right| \\ &\leq \frac{c_1^p - c_2^p}{2}\left[(B_9(\alpha,p,l))^{\frac{1}{l}} + (B_{10}(\alpha,p,l))^{\frac{1}{l}}\right]\left[\frac{|\Psi'(c_1)|^q + |\Psi'(c_2)|^q}{s+1}\right]^{\frac{1}{q}}, \end{aligned} \tag{6.27}$$

with $\mu(u) = u^{\frac{1}{p}}$ for all $u \in [c_1^p, c_2^p]$. Where

$$B_9 = \int_0^1 \left(\frac{(1-r)^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\right)^l \mathrm{d}r, \;\; B_{10} = \int_0^1 \left(\frac{r^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\right)^l \mathrm{d}r.$$

Proof. (i) Let $p > 0$. Using (6.21), Hölder's inequality and the (s,p)-convexity of $|\Psi'|^q$ implies,

$$\begin{aligned} &\left|\frac{\Psi(c_1)+\Psi(c_2)}{2} - \frac{\Gamma(\alpha+1)}{2(c_2^p - c_1^p)^\alpha}\left[\mathcal{J}^\alpha_{c_1^p+}(\Psi\circ\mu)(c_2^p) + \mathcal{J}^\alpha_{c_2^p-}(\Psi\circ\mu)(c_1^p)\right]\right| \\ &\leq \frac{c_2^p - c_1^p}{2}\Bigg[\int_0^1 \frac{(1-r)^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right| \mathrm{d}r \\ &\quad + \int_0^1 \frac{r^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right| \mathrm{d}r\Bigg] \\ &\leq \frac{c_2^p - c_1^p}{2}\Bigg[\left(\int_0^1 \left(\frac{(1-r)^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\right)^l \mathrm{d}r\right)^{\frac{1}{l}} \\ &\quad \times \left(\int_0^1 \left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right|^q \mathrm{d}r\right)^{\frac{1}{q}} \\ &\quad + \left(\int_0^1 \left(\frac{r^\alpha}{p(rc_1^p - (1-r)c_2^p)^{1-\frac{1}{p}}}\right)^l \mathrm{d}r\right)^{\frac{1}{l}} \\ &\quad \times \left(\int_0^1 \left|\Psi'\left((rc_1^p + (1-r)c_2^p)^{\frac{1}{p}}\right)\right|^q \mathrm{d}r\right)^{\frac{1}{q}}\Bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{c_2^p - c_1^p}{2}\Bigg[\left(\int_0^1 \left(\frac{(1-r)^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\right)^l \mathrm{d}r\right)^{\frac{1}{l}} \\
&\quad \times \left(\int_0^1 r^s|\Psi'(c_1)|^q\mathrm{d}r + \int_0^1 (1-r)^s\,|\Psi'(c_2)|^q\,\mathrm{d}r\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left(\frac{r^\alpha}{p(rc_1^p-(1-r)c_2^p)^{1-\frac{1}{p}}}\right)^l \mathrm{d}r\right)^{\frac{1}{l}} \\
&\quad \times \left(\int_0^1 r^s|\Psi'(c_1)|^q\mathrm{d}r + \int_0^1 (1-r)^s|\Psi'(c_2)|^q\mathrm{d}r\right)^{\frac{1}{q}}\Bigg] \\
&= \frac{c_2^p - c_1^p}{2}\left[(B_9(\alpha,p,l))^{\frac{1}{l}} + (B_{10}(\alpha,p,l))^{\frac{1}{l}}\right]\left[\frac{|\Psi'(c_1)|^q + |\Psi'(c_2)|^q}{s+1}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{6.28}
$$

This completes the proof of (i).

(ii) The proof is similar with (i). □

6.3 Integral inequalities of product of two (s,p)-convex functions via fractional integrals

Chen and Wu in [20] and [21] gave some integral results includes the products of two h- and two s-convex functions in fractional form, respectively. We generalize this idea to the products of two (s,p)-convex functions.

Theorem 6.3.1. *Let $s_1, s_2 \in (0,1]$, $r_1, r_2 \in [0,1]$ and $p_1, p_2 \in \mathbb{R} \setminus \{0\}$. Let $\Psi_1, \Psi_2 : \aleph \subset (0,\infty) \to (0,\infty)$ be (s_1,p_1)-convex and (s_2,p_2)-convex functions, respectively, such that $\Psi_1, \Psi_2 \in L_1[c_1,c_2]$ for $c_1, c_2 \in \aleph^o$ with $c_1 < c_2$ and $\alpha_1, \alpha_2 > 0$. Then the following* *inequalities for fractional integral hold:*

$$\begin{aligned}
&\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(c_2^{p_1}-c_1^{p_1})^{\alpha_1}(c_2^{p_2}-c_1^{p_2})^{\alpha_2}}\Big[\mathcal{J}^{\alpha_1}_{c_1^{p_1}+}(\Psi_1\circ\mu_1)(c_2^{p_1})\mathcal{J}^{\alpha_2}_{c_1^{p_2}+}(\Psi_2\circ\mu_2)(c_2^{p_2})\\
&\quad+\mathcal{J}^{\alpha_1}_{c_2^{p_1}-}(\Psi_1\circ\mu_1)(u^{p_1})\mathcal{J}^{\alpha_2}_{c_2^{p_2}-}(\Psi_2\circ\mu_2)(c_1^{p_2})\Big]\\
&\le\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\mathcal{B}(\alpha_1,s_1+1)\mathcal{B}(\alpha_2,s_2+1)\right]M(c_1,c_2)\\
&\quad+\left[\frac{\mathcal{B}(\alpha_1,s_1+1)}{\alpha_2+s_2}+\frac{\mathcal{B}(\alpha_2,s_2+1)}{\alpha_1+s_1}\right]N(c_1,c_2),
\end{aligned}\tag{6.29}$$

where $\mu_1(u)=u^{\frac{1}{p_1}}$ *for all* $u\in[c_1^{p_1},c_2^{p_1}]$, $\mu_2(v)=v^{\frac{1}{p_2}}$ *for all* $v\in[c_1^{p_2},c_2^{p_2}]$, *and* $M(c_1,c_2)=\Psi_1(c_1)\Psi_2(c_1)+\Psi_1(c_2)\Psi_2(c_2)$, $N(c_1,c_2)=\Psi_1(c_1)\Psi_2(c_2)+\Psi_1(c_2)\Psi_2(c_1)$.

Proof. Since Ψ_1 is (s_1,p_1)-convex and Ψ_2 is (s_2,p_2)-convex, then for $r_1,r_2\in[0,1]$, we have

$$\Psi_1\left((r_1c_1^{p_1}+(1-r_1)c_2^{p_1})^{\frac{1}{p_1}}\right)\le r_1^{s_1}\Psi_1(c_1)+(1-r_1)^{s_1}\Psi_1(c_2),$$

and

$$\Psi_2\left((r_2c_1^{p_2}+(1-r_2)c_2^{p_2})^{\frac{1}{p_2}}\right)\le r_2^{s_2}\Psi_2(c_1)+(1-r_2)^{s_2}\Psi_2(c_2).$$

Then from above, we obtain

$$\begin{aligned}
&\Psi_1\left((r_1c_1^{p_1}+(1-r_1)v^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_1^{p_2}+(1-r_2)c_2^{p_2})^{\frac{1}{p_2}}\right)\\
&\le r_1^{s_1}r_2^{s_2}\Psi_1(c_1)\Psi_2(c_1)+(1-r_1)^{s_1}(1-r_2)^{s_2}\Psi_1(c_2)\Psi_2(c_2)\\
&\quad+r_1^{s_1}(1-r_2)^{s_2}\Psi_1(c_1)\Psi_2(c_2)+(1-r_1)^{s_1}r_2^{s_2}\Psi_1(c_2)\Psi_2(c_1).
\end{aligned}\tag{6.30}$$

Similarly,

$$\begin{aligned}
&\Psi_1\left((r_1c_2^{p_1}+(1-r_1)c_1^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_2^{p_2}+(1-r_2)c_1^{p_2})^{\frac{1}{p_2}}\right)\\
&\le r_1^{s_1}r_2^{s_2}\Psi_1(c_2)\Psi_2(c_2)+(1-r_1)^{s_1}(1-r_2)^{s_2}\Psi_1(c_1)\Psi_2(c_1)\\
&\quad+r_1^{s_1}(1-r_2)^{s_2}\Psi_1(c_2)\Psi_2(c_1)+(1-r_1)^{s_1}r_2^{s_2}\Psi_1(c_1)\Psi_2(c_2).
\end{aligned}\tag{6.31}$$

Then by combining, we get

$$\begin{aligned}
&\Psi_1\left((r_1c_1^{p_1}+(1-r_1)v^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_1^{p_2}+(1-r_2)c_2^{p_2})^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left((r_1c_2^{p_1}+(1-r_1)c_1^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_2^{p_2}+(1-r_2)c_1^{p_2})^{\frac{1}{p_2}}\right)\\
&\le\left(r_1^{s_1}r_2^{s_2}+(1-r_1)^{s_1}(1-r_2)^{s_2}\right)[\Psi_1(c_1)\Psi_2(c_1)+\Psi_1(c_2)\Psi_2(c_2)]\\
&\quad+\left(r_1^{s_1}(1-r_2)^{s_2}+(1-r_1)^{s_1}r_2^{s_2}\right)[\Psi_1(c_2)\Psi_2(c_1)+\Psi_1(c_1)\Psi_2(c_2)].
\end{aligned}\tag{6.32}$$

By multiplying (6.32) by $r_1^{\alpha_1-1}r_2^{\alpha_2-1}$ and integrating with respect to r_1 and r_2 over $[0,1]\times[0,1]$, we obtain

$$\begin{aligned}&\int_0^1\int_0^1 r_1^{\alpha_1-1}r_2^{\alpha_2-1}\Big[\Psi_1\left((r_1c_1^{p_1}+(1-r_1)c_2^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_1^{p_2}+(1-r_2)c_2^{p_2})^{\frac{1}{p_2}}\right)\\&\quad+\Psi_1\left((r_1c_2^{p_1}+(1-r_1)c_1^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_2^{p_2}+(1-r_2)c_1^{p_2})^{\frac{1}{p_2}}\right)\Big]\mathrm{d}r_1\mathrm{d}r_2\\&\le\int_0^1\int_0^1 r_1^{\alpha_1-1}r_2^{\alpha_2-1}\Big[\left(r_1^{s_1}r_2^{s_2}+(1-r_1)^{s_1}(1-r_2)^{s_2}\right)[\Psi_1(c_1)\Psi_2(c_1)+\Psi_1(c_2)\Psi_2(c_2)]\\&\quad+\left(r_1^{s_1}(1-r_2)^{s_2}+(1-r_1)^{s_1}r_2^{s_2}\right)[\Psi_1(c_2)\Psi_2(c_1)+\Psi_1(c_1)\Psi_2(c_2)]\Big]\mathrm{d}r_1\mathrm{d}r_2.\end{aligned} \tag{6.33}$$

By letting $u=r_1c_1^{p_1}+(1-r_1)c_2^{p_1}$ and $v=r_2c_1^{p_2}+(1-r_2)c_2^{p_2}$, we get

$$\begin{aligned}&\int_0^1\int_0^1 r_1^{\alpha_1-1}r_2^{\alpha_2-1}\Psi_1\left((r_1c_1^{p_1}+(1-r_1)c_2^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_1^{p_2}+(1-r_2)c_2^{p_2})^{\frac{1}{p_2}}\right)\mathrm{d}r_1\mathrm{d}r_2\\&=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(c_2^{p_1}-c_1^{p_1})^{\alpha_1}(c_2^{p_2}-c_1^{p_2})^{\alpha_2}}\mathcal{J}^{\alpha_1}_{c_1^{p_1}+}(\Psi_1\circ\mu_1)(c_2^{p_1})\mathcal{J}^{\alpha_2}_{c_1^{p_2}+}(\Psi_2\circ\mu_2)(c_2^{p_2}).\end{aligned} \tag{6.34}$$

Similarly, by letting $u=r_1c_2^{p_1}+(1-r_1)c_1^{p_1}$ and $v=r_2c_2^{p_2}+(1-r_2)c_1^{p_2}$, we find

$$\begin{aligned}&\int_0^1\int_0^1 r_1^{\alpha_1-1}r_2^{\alpha_2-1}\Psi_1\left((r_1c_2^{p_1}+(1-r_1)c_1^{p_1})^{\frac{1}{p_1}}\right)\Psi_2\left((r_2c_2^{p_2}+(1-r_2)c_1^{p_2})^{\frac{1}{p_2}}\right)\mathrm{d}r_1\mathrm{d}r_2\\&=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(c_2^{p_1}-c_1^{p_1})^{\alpha_1}(c_2^{p_2}-c_1^{p_2})^{\alpha_2}}\mathcal{J}^{\alpha_1}_{c_2^{p_1}-}(\Psi_1\circ\mu_1)(c_1^{p_1})\mathcal{J}^{\alpha_2}_{c_2^{p_2}-}(\Psi_2\circ\mu_2)(c_1^{p_2}).\end{aligned} \tag{6.35}$$

Also note that,

$$\begin{aligned}&\int_0^1\int_0^1 r_1^{\alpha_1-1}r_2^{\alpha_2-1}\left(r_1^{s_1}r_2^{s_2}+(1-r_1)^{s_1}(1-r_2)^{s_2}\right)[\Psi_1(c_1)\Psi_2(c_1)+\Psi_1(c_2)\Psi_2(c_2)]\,\mathrm{d}r_1\mathrm{d}r_2\\&=\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\mathcal{B}(\alpha_1,s_1+1)\mathcal{B}(\alpha_2,s_2+1)\right]M(c_1,c_2),\end{aligned} \tag{6.36}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 r_1^{\alpha_1-1}r_2^{\alpha_2-1}(r_1^{s_1}(1-r_2)^{s_2}+(1-r_1)^{s_1}r_2^{s_2})[\Psi_1(c_2)\Psi_2(c_1)+\Psi_1(c_1)\Psi_2(c_2)]\mathrm{d}r_1\mathrm{d}r_2\\&=\left[\frac{\mathcal{B}(\alpha_2,s_2+1)}{\alpha_1+s_1}+\frac{\mathcal{B}(\alpha_1,s_1+1)}{\alpha_2+s_2}\right]N(c_1,c_2).\end{aligned} \tag{6.37}$$

Hence by substituting values of (6.34)−(6.37) in (6.33), we get (6.29). □

Corollary 6.3.1. *Under the assumptions of Theorem* 6.3.1,

1. *If* $p_1 = p_2 = 1$, *then*

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(c_2-c_1)^{\alpha_1+\alpha_2}}\left[\mathcal{J}^{\alpha_1}_{c_1+}\Psi_1(c_2)\mathcal{J}^{\alpha_2}_{c_1+}\Psi_2(c_2)+\mathcal{J}^{\alpha_1}_{c_2-}\Psi_1(c_1)\mathcal{J}^{\alpha_2}_{c_2-}\varphi(c_1)\right]$$
$$\leq\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\mathcal{B}(\alpha_1,s_1+1)\mathcal{B}(\alpha_2,s_2+1)\right]M(c_1,c_2) \tag{6.38}$$
$$+\left[\frac{\mathcal{B}(\alpha_1,s_1+1)}{\alpha_2+s_2}+\frac{\mathcal{B}(\alpha_2,s_2+1)}{\alpha_1+s_1}\right]N(c_1,c_2).$$

2. *If* $s_1 = s_2 = 1$, *then*

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(c_2^{p_1}-c_1^{p_1})^{\alpha_1}(c_2^{p_2}-c_1^{p_2})^{\alpha_2}}\Big[\mathcal{J}^{\alpha_1}_{c_1^{p_1}+}(\Psi_1\circ\mu_1)(c_2^{p_1})\mathcal{J}^{\alpha_2}_{c_1^{p_2}+}(\Psi_2\circ\mu_2)(c_2^{p_2})$$
$$+\mathcal{J}^{\alpha_1}_{c_2^{p_1}-}(\Psi_1\circ\mu_1)(c_1^{p_1})\mathcal{J}^{\alpha_2}_{c_2^{p_2}-}(\Psi_2\circ\mu_2)(c_1^{p_2})\Big]$$
$$\leq\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\mathcal{B}(\alpha_1,2)\mathcal{B}(\alpha_2,2)\right]M(c_1,c_2)+\left[\frac{\mathcal{B}(\alpha_1,2)}{\alpha_2+1}+\frac{\mathcal{B}(\alpha_2,2)}{\alpha_1+1}\right]N(c_1,c_2). \tag{6.39}$$

3. *If* $p_1 = p_2 = 1$ *and* $s_1 = s_2 = 1$, *then*

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{(c_2-c_1)^{\alpha_1+\alpha_2}}\left[\mathcal{J}^{\alpha_1}_{c_1+}\Psi_1(c-2)\mathcal{J}^{\alpha_2}_{c_1+}\Psi_1(c_2)+\mathcal{J}^{\alpha_1}_{c_2-}\Psi_1(c_1)J^{\alpha_2}_{c_2-}\Psi_2(c_1)\right]$$
$$\leq\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\mathcal{B}(\alpha_1,2)\mathcal{B}(\alpha_2,2)\right]M(c_1,c_2)+\left[\frac{\mathcal{B}(\alpha_1,2)}{\alpha_2+1}+\frac{\mathcal{B}(\alpha_2,2)}{\alpha_1+1}\right]N(c_1,c_2). \tag{6.40}$$

4. *If* $p_1 = p_2 = 1$, $s_1 = s_2 = 1$ *and* $\alpha_1 = \alpha_2 = 1$, *then*

$$\frac{2}{(c_2-c_1)^2}\int_{c_1}^{c_2}\Psi_1(x)\mathrm{d}x\int_{c_1}^{c_2}\Psi_2(y)\mathrm{d}y\leq\frac{M(c_1,c_2)+N(c_1,c_2)}{2}. \tag{6.41}$$

Theorem 6.3.2. *Let* $s_1, s_2 \in (0,1]$, $r_1, r_2 \in [0,1]$ *and* $p_1, p_2 \in \mathbb{R}\setminus\{0\}$. *Let* $\Psi_1, \Psi_2 : \aleph \subset (0,\infty) \to (0,\infty)$ *be* (s_1,p_1)*-convex and* (s_2,p_2)*-convex functions, respectively, such that* $\Psi_1, \Psi_2 \in L_1[c_1,c_2]$ *for* $c_1, c_2 \in \aleph^o$ *with* $c_1 < c_2$ *and* $\alpha_1, \alpha_2 > 0$. *Then the following* *inequalities for fractional integral hold:*

$$\begin{aligned}
&2^{s_1+s_2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&\leq \frac{\Gamma(\alpha_1\alpha_2+1)}{(c_2^{p_1}-c_1^{p_1})^{\alpha_1}(c_2^{p_2}-c_1^{p_2})^{\alpha_2}}\Big[\mathcal{J}^{\alpha_1}_{c_1^{p_1}+}(\Psi_1\circ\mu_1)(c_2^{p_1})\mathcal{J}^{\alpha_2}_{c_1^{p_2}+}(\Psi_2\circ\mu_2)(c_2^{p_2})\\
&\quad+\mathcal{J}^{\alpha_1}_{c_2^{p_1}-}(\Psi\circ\mu_1)(c_1^{p_1})\mathcal{J}^{\alpha_2}_{c_2^{p_2}-}(\Psi_2\circ\mu_2)(c_1^{p_2})\Big]\\
&\quad+\alpha_1\alpha_2\left[\frac{\mathcal{B}(\alpha_1,s_1+1)}{\alpha_2+s_2}+\frac{\mathcal{B}(\alpha_2,s_2+1)}{\alpha_1+s_1}\right]M(c_1,c_2)\\
&\quad+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\mathcal{B}(\alpha_1,s_1+1)\mathcal{B}(\alpha_2,s_2+1)\right]N(c_1,c_2),
\end{aligned}\tag{6.42}$$

where $\mu_1(u)=u^{\frac{1}{p_1}}$ *for all* $u\in[c_1^{p_1},c_2^{p_1}]$, $\mu_2(v)=v^{\frac{1}{p_2}}$ *for all* $v\in[c_1^{p_2},c_2^{p_2}]$ *and* $M(c_1,c_2)=\Psi_1(c_1)\Psi_2(c_1)+\Psi_1(c_2)\Psi_2(c_2)$, $N(c_1,c_2)=\Psi_1(c_1)\Psi_2(c_2)+\Psi_1(c_2)\Psi_2(c_1)$.

Proof. Since Ψ_1 is (s_1,p_1)-convex and Ψ_2 is (s_2,p_2)-convex, then

$$\begin{aligned}
&\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&=\Psi_1\left(\left(\frac{r_1c_1^{p_1}+(1-r_1)c_2^{p_1}}{2}+\frac{r_1c_2^{p_1}+(1-r_1)c_1^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\\
&\quad\times\Psi_2\left(\left(\frac{r_2c_1^{p_2}+(1-r_2)c_2^{p_2}}{2}+\frac{r_2c_2^{p_2}+(1-r_2)c_1^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&\leq\frac{1}{2^{s_1+s_2}}\left[\Psi_1\left((r_1c_1^{p_1}+(1-r_1)c_2^{p_1})^{\frac{1}{p_1}}\right)+\Psi_1\left((r_1c_2^{p_1}+(1-r_1)c_1^{p_1})^{\frac{1}{p_1}}\right)\right]\\
&\quad\times\left[\Psi_1\left((r_2c_1^{p_2}+(1-r_2)c_2^{p_2})^{\frac{1}{p_2}}\right)+\Psi_2\left((r_2c_2^{p_2}+(1-r_2)c_1^{p_2})^{\frac{1}{p_2}}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{s_1+s_2}}\Big[\Psi_1\left(\left(r_1c_1^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_1^{p_2}+(1-r_2)c_2^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left(\left(r_1c_2^{p_1}+(1-r_1)c_1^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_2^{p_2}+(1-r_2)c_1^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left(\left(r_1c_1^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_2^{p_2}+(1-r_2)c_1^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left(\left(r_1c_2^{p_1}+(1-r_1)c_1^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_1^{p_2}+(1-r_2)c_2^{p_2}\right)^{\frac{1}{p_2}}\right)\Big]\\
&\le \frac{1}{2^{s_1+s_2}}\Big[\Psi_1\left(\left(r_1c_1^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_1^{p_2}+(1-r_2)c_2^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left(\left(r_1c_2^{p_1}+(1-r_1)c_1^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_2^{p_2}+(1-r_2)c_1^{p_2}\right)^{\frac{1}{p_2}}\right)\Big]\\
&\quad+\frac{1}{2^{s_1+s_2}}\Big[\Psi_1\left(\left(r_1c_1^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_2^{p_2}+(1-r_2)c_1^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left(\left(r_1c_2^{p_1}+(1-r_1)c_1^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_1^{p_2}+(1-r_2)c_2^{p_2}\right)^{\frac{1}{p_2}}\right)\Big]\\
&\le \frac{1}{2^{s_1+s_2}}\Big[\Psi_1\left(\left(r_1c_1^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_1^{p_2}+(1-r_2)c_2^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi\left(\left(r_1c_2^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_2^{p_2}+(1-r_2)c_1^{p_2}\right)^{\frac{1}{p_2}}\right)\Big]\\
&\quad+\frac{1}{2^{s_1+s_2}}\Big[(r_1^{s_1}\Psi_1(c_1)+(1-r_1)^{s_1}\Psi_1(c_2))((1-r_2)^{s_2}\Psi_2(c_1)+r_2^{s_2}\Psi_2(c_2))\\
&\quad+(r_1^{s_1}\Psi_1(c_2)+(1-r_1)^{s_1}\Psi_1(c_1))((1-r_2)^{s_2}\Psi_2(c_2)+r_2^{s_2}\Psi_2(c_1))]\\
&= \frac{1}{2^{s_1+s_2}}\Big[\Psi_1\left(\left(r_1c_1^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_1^{p_2}+(1-r_2)c_2^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left(\left(r_1c_2^{p_1}+(1-r_1)c_1^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_2^{p_2}+(1-r_2)c_1^{p_2}\right)^{\frac{1}{p_2}}\right)\Big]\\
&\quad+\frac{1}{2^{s_1+s_2}}\Big[\left(r_1^{s_1}(1-r_2)^{s_2}+(1-r_1)^{s_1}r_2^{s_2}\right)\left(\Psi_1(c_1)\Psi_2(c_1)\right.\\
&\quad\left.+\Psi_1(c_2)\Psi_2(c_2)\right)+\left((1-r_1)^{s_1}(1-r_2)^{s_2}+r_1^{s_1}r_2^{s_2}\right)\left(\Psi_1(c_1)\Psi_2(c_2)+\Psi_1(c_2)\Psi_2(c_1)\right)\Big].
\end{aligned}
\tag{6.43}
$$

Summing up, we have

$$
\begin{aligned}
&2^{s_1+s_2}\Psi\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&\le\Big[\Psi_1\left(\left(r_1c_1^{p_1}+(1-r_1)c_2^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_1^{p_2}+(1-r_2)c_2^{p_2}\right)^{\frac{1}{p_2}}\right)\\
&\quad+\Psi_1\left(\left(r_1c_2^{p_1}+(1-r_1)c_1^{p_1}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(r_2c_2^{p_2}+(1-r_2)c_1^{p_2}\right)^{\frac{1}{p_2}}\right)\Big]
\end{aligned}
$$

$$+\Big[(r_1^{s_1}(1-r_2)^{s_2}+(1-r_1)^{s_1}r_2^{s_2})(\Psi_1(c_1)\Psi_2(c_1)+\Psi_1(c_2)\Psi_2(c_2)) + ((1-r_1)^{s_1}(1-r_2)^{s_2}+r_1^{s_1}r_2^{s_2})\left(\Psi_1(c_1)\Psi_2(c_2)+\Psi_1(c_2)\Psi_2(c_1)\right)\Big]. \tag{6.44}$$

Multiplying above inequality (6.44) by $r_1^{\alpha_1-1}r_2^{\alpha_2-1}$ and integrating with respect to r_1 and r_2 over $[0,1]\times[0,1]$, we get (6.42). Hence the proof is completed. □

Corollary 6.3.2. *Under the assumptions of Theorem* 6.3.2,

1. *If* $p_1=p_2=1$, *then*

$$\begin{aligned}
&2^{s_1+s_2}\Psi_1\left(\left(\frac{c_1+c_2}{2}\right)\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right)\right)\\
&\le \frac{\Gamma(\alpha_1\alpha_2+1)}{(c_2-c_1)^{\alpha_1+\alpha_2}}\left[\mathcal{J}_{c_1+}^{\alpha_1}\Psi_1(c_2)\mathcal{J}_{c_1+}^{\alpha_2}\Psi_2(c_2)+\mathcal{J}_{c_2-}^{\alpha_1}\Psi_1(c_1)\mathcal{J}_{c_2-}^{\alpha_2}\Psi_2(c_1)\right]\\
&\quad+\alpha_1\alpha_2\left[\frac{\mathcal{B}(\alpha_1,s_1+1)}{\alpha_2+s_2}+\frac{\mathcal{B}(\alpha_2,s_2+1)}{\alpha_1+s_1}\right]M(c_1,c_2)\\
&\quad+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+s_1)(\alpha_2+s_2)}+\mathcal{B}(\alpha_1,s_1+1)\mathcal{B}(\alpha_2,s_2+1)\right]N(c_1,c_2).
\end{aligned} \tag{6.45}$$

2. *If* $s_1=s_2=1$, *then*

$$\begin{aligned}
&2^{s_1+s_2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}\right)\\
&\le \frac{\Gamma(\alpha_1\alpha_2+1)}{(c_2^{p_1}-c_1^{p_1})^{\alpha_1}(c_2^{p_2}-c_1^{p_2})^{\alpha_2}}\Big[\mathcal{J}_{c_1^{p_1}+}^{\alpha_1}(\Psi_1\circ\mu_1)(c_2^{p_1})\mathcal{J}_{c_1^{p_2}+}^{\alpha_2}(\Psi_2\circ\mu_2)(c_2^{p_2})\\
&\quad+\mathcal{J}_{c_2^{p_1}-}^{\alpha_1}(\Psi_1\circ\mu_1)(c_1^{p_1})\mathcal{J}_{c_2^{p_2}-}^{\alpha_2}(\Psi_2\circ\mu_2)(c_1^{p_2})\Big]+\alpha_1\alpha_2\left[\frac{\mathcal{B}(\alpha_1,2)}{\alpha_2+1}+\frac{\mathcal{B}(\alpha_2,2)}{\alpha_1+1}\right]M(c_1,c_2)\\
&\quad+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\mathcal{B}(\alpha_1,2)\mathcal{B}(\alpha_2,2)\right]N(c_1,c_2).
\end{aligned} \tag{6.46}$$

3. *If* $p_1 = p_2 = 1$ *and* $s_1 = s_2 = 1$, *then*

$$\begin{aligned}
&4\Psi_1\left(\left(\frac{c_1+c_2}{2}\right)\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right)\right)\\
&\le \frac{\Gamma(\alpha_1\alpha_2+1)}{(c_2-c_1)^{\alpha_1+\alpha_2}}\left[\mathcal{J}^{\alpha_1}_{c_1+}\Psi_1(c_2)\mathcal{J}^{\alpha_2}_{c_1+}\Psi_2(c_2)+\mathcal{J}^{\alpha_1}_{c_2-}\Psi_1(c_1)\mathcal{J}^{\alpha_2}_{c_2-}\Psi_2(c_2)\right]\\
&\quad+\alpha_1\alpha_2\left[\frac{\mathcal{B}(\alpha_1,2)}{\alpha_2+1}+\frac{\mathcal{B}(\alpha_2,2)}{\alpha_1+1}\right]M(c_1,c_2)\\
&\quad+\alpha_1\alpha_2\left[\frac{1}{(\alpha_1+1)(\alpha_2+1)}+\mathcal{B}(\alpha_1,2)\mathcal{B}(\alpha_2,2)\right]N(c_1,c_2).
\end{aligned}\tag{6.47}$$

4. *If* $p_1 = p_2 = 1$, $s_1 = s_2 = 1$ *and* $\alpha_1 = \alpha_2 = 1$, *then*

$$\begin{aligned}
&4\Psi_1\left(\left(\frac{c_1+c_2}{2}\right)\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right)\right)\\
&\le \frac{2}{(c_2-c_1)^2}\int_{c_1}^{c_2}\Psi_1(x)\mathrm{d}x\int_{c_1}^{c_2}\Psi_2(y)\mathrm{d}y+\frac{M(c_1,c_2)+N(c_1,c_2)}{2}.
\end{aligned}\tag{6.48}$$

6.4 Integral inequalities of (s,p)-convex functions on co-ordinates

In this section, we obtain some integral inequalities for (s,p)-convex functions on co-ordinates. First we define (s,p)-convex functions on co-ordinates.

Definition 6.4.1. *Let* $s \in [-1,1]$ *and* $p \in \mathbb{R}\backslash\{0\}$, *a function* $\Psi : \Delta = [c_1,c_2]\times[d_1,d_2] \subseteq \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is called co-ordinate* (s,p)*-convex on* Δ *with* $c_1 < c_2$ *and* $d_1 < d_2$, *if*

$$\begin{aligned}
&\Psi\left((r_1x^p+(1-r_1)z^p)^{\frac{1}{p}}, r_2y+(1-r_2)w\right)\\
\le& r_1^s[\Psi(x,y)+\Psi(x,w)]+(1-r_1)^s[\Psi(z,y)+\Psi(z,w)]
\end{aligned}\tag{6.49}$$

holds, for all $(x,y),(z,w) \in \Delta$ *and* $r_1 \in (0,1)$, $r_2 \in [0,1]$.

On the basis of above Definition 6.4.1 we give our new results.

Theorem 6.4.1. *Let* $s \in (-1,1]$ *and* $p \in \mathbb{R}\setminus\{0\}$. *Let* $\Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *be a co-ordinated* (s,p)*-convex function on* $\Delta = [c_1,c_2]\times[d_1,d_2]$ *with* $c_1 < c_2$ *and* $d_1 < d_2$. *If*

$\varPsi \in L_1([c_1, c_2] \times [d_1, d_2])$, *then*

$$2^{s-2}\varPsi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1 + d_2}{2}\right) \leq \frac{p}{(c_1^p - c_2^p)(d_2 - d_1)} \int_{d_1}^{d_2} \int_{c_1}^{c_2} u^{p-1}\varPsi(u, v)\mathrm{d}u\mathrm{d}v$$
$$\leq \left[\frac{\varPsi(c_1, d_1) + \varPsi(c_1, d_2) + \varPsi(c_2, d_1) + \varPsi(c_2, d_2)}{s + 1}\right]. \tag{6.50}$$

Proof. By considering $u_1^p = r_1c_1^p + (1 - r_1)c_2^p$ and $u_2^p = (1 - r_1)c_1^p + r_2c_2^p$ and using co-ordinated (s, p)-convexity of $\varPsi$, we have

$$\begin{aligned}
&\varPsi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1 + d_2}{2}\right) \\
&= \varPsi\left(\left(\frac{r_1c_1^p + (1 - r_1)c_2^p}{2} + \frac{(1 - r_1)c_1^p + r_1c_2^p}{2}\right)^{\frac{1}{p}}, \frac{r_2d_1 + (1 - r_2)d_2}{2} + \frac{(1 - r_2)d_1 + r_2d_2}{2}\right) \\
&= \varPsi\left(\left(\frac{u_1^p}{2} + \frac{u_2^p}{2}\right)^{\frac{1}{p}}, \frac{r_2d_1 + (1 - r_2)d_2}{2} + \frac{(1 - r_2)d_1 + r_2d_2}{2}\right) \\
&\leq \frac{1}{2^s}\big[\varPsi(u_1, r_2d_1 + (1 - r_2)d_2) + \varPsi(u_1, (1 - r_2)d_1 + r_2d_2) \\
&\quad + \varPsi(u_2, r_2d_1 + (1 - r_2)d_2) + \varPsi(u_2, (1 - r_2)d_1 + r_2d_2)\big] \\
&= \frac{1}{2^s}\Big[\varPsi\left((r_1c_1^p + (1 - r_1)c_2^p)^{\frac{1}{p}}, r_2d_1 + (1 - r_2)d_2\right) \\
&\quad + \varPsi\left((r_1c_1^p + (1 - r_1)c_2^p)^{\frac{1}{p}}, (1 - r_2)d_1 + r_2d_2\right) \\
&\quad + \varPsi\left(((1 - r_1)c_1^p + r_1c_2^p)^{\frac{1}{p}}, r_2d_1 + (1 - r_2)d_2\right) \\
&\quad + \varPsi\left(((1 - r_1)c_1^p + r_1c_2^p)^{\frac{1}{p}}, (1 - r_2)d_1 + r_2d_2\right)\Big],
\end{aligned} \tag{6.51}$$

for all $r_1, r_2 \in [0, 1]^2$. By integrating with respect to r_1 and r_2 over $[0, 1] \times [0, 1]$, we obtain

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}},\frac{d_1+d_2}{2}\right)\\
&\leq\frac{1}{2^s}\int_0^1\int_0^1\Big[\Psi((r_1c_1^p+(1-r_1)c_2^p)^{\frac{1}{p}},r_2d_1+(1-r_2)d_2)\\
&\quad+\Psi\left((r_1c_1^p+(1-r_1)c_2^p)^{\frac{1}{p}},(1-r_2)d_1+r_2d_2\right)\\
&\quad+\Psi\left(((1-r_1)c_1^p+r_1c_2^p)^{\frac{1}{p}},r_2d_1+(1-r_2)d_2\right)\\
&\quad+\Psi\left(((1-r_1)c_1^p+r_1c_2^p)^{\frac{1}{p}},(1-r_2)d_1+r_2d_2\right)\Big]\mathrm{d}r_1\mathrm{d}r_2\\
&=\frac{p}{2^{s-2}(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}u^{p-1}\Psi(u,v)\mathrm{d}u\mathrm{d}v.
\end{aligned}\tag{6.52}$$

Again using similar arguments, we have

$$\begin{aligned}
&\frac{p}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}u^{p-1}\Psi(u,v)\mathrm{d}u\mathrm{d}v\\
&=\int_0^1\int_0^1\Psi((r_1c_1^p+(1-r_1)c_2^p)^{\frac{1}{p}},r_2d_1+(1-r_2)d_2)\mathrm{d}r_1\mathrm{d}r_2\\
&\leq\int_0^1\int_0^1[r_1^s\{\Psi(r_1,d_1)+\Psi(c_1,d_2)\}+(1-r_1)^s\{\Psi(c_2,d_1)+\Psi(c_2,d_2)\}]\mathrm{d}r_1\mathrm{d}r_2\\
&=\frac{1}{s+1}[\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)].
\end{aligned}\tag{6.53}$$

From (6.52) and (6.53) we get (6.50). Thus Theorem 6.4.1 is proved. □

Corollary 6.4.2. *Under the same considerations of Theorem* 6.4.1*, if we take* $p=1$*, then*

$$\begin{aligned}
2^{s-2}\Psi\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)&\leq\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\mathrm{d}u\mathrm{d}v\\
&\leq\left[\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{s+1}\right].
\end{aligned}\tag{6.54}$$

Remark 35. *Under the same consideration of Corollary* 6.4.2*, taking* $\Psi(u,v)=\Psi(v)$ *for all* $(u,v)\in\Delta$ *and* $s=1$ *in the inequality* (6.54) *and assuming* $\Psi\in P([d_1,d_2])$*, then Theorem* 3.1 *in* [29] *is achieved.*

Theorem 6.4.2. *Let $s \in [-1,1]$ and $p \in \mathbb{R} \setminus \{0\}$. Let $\Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a co-ordinated (s,p)-convex function on $\Delta = [c_1, c_2] \times [d_1, d_2]$ with $c_1 < c_2$ and $d_1 < d_2$. If $\Psi \in L_1([c_1, c_2] \times [d_1, d_2])$, then*

$$\begin{aligned} &2^{2s-2}\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right) \\ &\leq 2^{s-1}\left[\frac{1}{c_2^p - c_1^p}\int_{c_1}^{c_2} \frac{p}{u^{1-p}}\Psi\left(u, \frac{d_1+d_2}{2}\right)\mathrm{d}u + \frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}, v\right)\mathrm{d}v\right] \\ &\leq \frac{p}{(c_2^p - c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2} u^{p-1}\Psi(u,v)\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{6.55}$$

Proof. By considering $u_1^p = r_1c_1^p + (1-r_1)c_2^p$ and $u_2^p = (1-r_1)c_1^p + r_1c_2^p$ and using co-ordinated (s,p)-convexity of Ψ, we have

$$\begin{aligned} &\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right) \\ &= \Psi\left(\left(\frac{r_1c_1^p + (1-r_1)c_2^p}{2} + \frac{(1-r_1)c_1^p + r_1c_2^p}{2}\right)^{\frac{1}{p}}, \frac{\frac{d_1+d_2}{2}}{2} + \frac{\frac{d_1+d_2}{2}}{2}\right) \\ &= \Psi\left(\left(\frac{u_1^p}{2} + \frac{u_2^p}{2}\right)^{\frac{1}{p}}, \frac{\frac{d_1+d_2}{2}}{2} + \frac{\frac{d_1+d_2}{2}}{2}\right) \\ &\leq \frac{1}{2^s}\left[\Psi\left(u_1, \frac{c+d}{2}\right) + \Psi\left(u_1, \frac{c+d}{2}\right) + \Psi\left(u_2, \frac{c+d}{2}\right) + \Psi\left(u_2, \frac{c+d}{2}\right)\right] \\ &= \frac{1}{2^{s-1}}\left[\Psi\left((r_1c_1^p + (1-r_1)c_2^p)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right) + \Psi\left(((1-r_1)c_1^p + r_1c_2^p)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right)\right]. \end{aligned} \tag{6.56}$$

By integrating this with respect to r_1 over $[0,1]$ implies

$$\begin{aligned} &\Psi\left(\left(\frac{c_1^p + c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right) \\ &\leq \frac{1}{2^{s-1}}\int_0^1\left[\Psi\left((r_1c_1^p + (1-r_1)c_2^p)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right)\right. \\ &\quad\left.+ \Psi\left(((1-r_1)c_1^p + r_1c_2^p)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right)\right]\mathrm{d}r_1 \\ &= \frac{1}{2^{s-2}}\frac{p}{c_2^p - c_1^p}\int_{c_1}^{c_2} x^{p-1}\Psi\left(u, \frac{d_1+d_2}{2}\right)\mathrm{d}u. \end{aligned} \tag{6.57}$$

Further we note that,

$$\begin{aligned}
&\Psi\left(u, \frac{d_1+d_2}{2}\right) \\
&= \Psi\left(\left(\frac{u^p}{2}+\frac{u^p}{2}\right)^{\frac{1}{p}}, \frac{r_2 d_1+(1-r_2)d_2}{2}+\frac{(1-r_2)d_1+r_2 d_2}{2}\right) \\
&\leq \frac{1}{2^{s-1}}\left[\Psi(u, r_2 d_1+(1-r_2)d_2)+\Psi(u,(1-r_2)d_1+r_2 d_2)\right].
\end{aligned} \tag{6.58}$$

Substituting (6.58) in (6.57) and then integrating with respect to r_2 over $[0,1]$ leads to

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right) \\
&\leq \frac{1}{2^{s-1}}\int_0^1\left[\Psi\left((r_1c_1^p+(1-r_1)c_2^p)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right)\right. \\
&\quad \left.+\Psi\left(((1-r_1)c_1^p+r_1c_2^p)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right)\right]\mathrm{d}r_1 \\
&= \frac{1}{2^{s-2}}\frac{p}{c_2^p-c_1^p}\int_{c_1}^{c_2} u^{p-1}\Psi\left(u, \frac{d_1+d_2}{2}\right)\mathrm{d}u \\
&\leq \frac{1}{2^{2s-3}}\frac{p}{c_1^p-c_1^p}\int_0^1\int_{c_1}^{c_2} x^{p-1}[\Psi(u, r_2 d_1+(1-r_2)d_2) \\
&\quad +\Psi\left(u,(1-r_2)d_1+r_2 d_2\right)]\mathrm{d}u\mathrm{d}r_2 \\
&= \frac{p}{2^{2s-2}(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2} x^{p-1}\Psi(u,v)\mathrm{d}u\mathrm{d}v.
\end{aligned} \tag{6.59}$$

By considering $u_1^p = u_2^p = \frac{c_1^p+c_2^p}{2}$ and using similar arguments to Ψ, we can have

$$\begin{aligned}
&\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right) \\
&\leq \frac{1}{2^{s-1}}\int_0^1\left[\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, r_2 d_1+(1-r_2)d_2\right)\right. \\
&\quad \left.+\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, (1-r_2)d_1+r_2 d_2\right)\right]\mathrm{d}r_2 \\
&= \frac{1}{2^{s-2}(d_2-d_1)}\int_{d_1}^{d_2}\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, v\right)\mathrm{d}v \\
&= \frac{1}{2^{s-2}(d_2-d_1)}\int_{d_1}^{d_2}\int_0^1\Psi\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, v\right)\mathrm{d}r_1\mathrm{d}v
\end{aligned}$$

$$\leq \frac{1}{2^{2s-3}(d_2-d_1)}\int_{d_1}^{d_2}\int_0^1 \left[\Psi\left((r_1c_1^p+(1-r_1)c_2^p)^{\frac{1}{p}},v\right)\right.$$

$$\left.+\Psi\left(((1-r_1)c_1^p+r_1c_2^p)^{\frac{1}{p}},v\right)\right]\mathrm{d}r_1\mathrm{d}v \tag{6.60}$$

$$=\frac{p}{2^{2s-2}(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}u^{p-1}\Psi(u,v)\mathrm{d}u\mathrm{d}v.$$

By adding (6.59) and (6.60) we get (6.55). Hence Theorem 6.4.2 is proved. □

Corollary 6.4.3. *Under the similar assumptions of Theorem* 6.4.2, *we have*

1. *If* $p=1$, *then*

$$2^{2s-2}\Psi\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)$$

$$\leq 2^{s-1}\left[\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi\left(\left(\frac{c_1+c_2}{2}\right),v\right)\mathrm{d}v\right] \tag{6.61}$$

$$\leq \frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\mathrm{d}u\mathrm{d}v.$$

2. *If* $p=s=1$, *then*

$$\Psi\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)$$

$$\leq \left[\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi\left(\left(\frac{c_1+c_2}{2}\right),v\right)\mathrm{d}v\right] \tag{6.62}$$

$$\leq \frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\mathrm{d}u\mathrm{d}v.$$

3. *If* $p=1$ *and* $s=0$, *then*

$$\frac{1}{4}\Psi\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)$$

$$\leq \frac{1}{2}\left[\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\Psi\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi\left(\left(\frac{c_1+c_2}{2}\right),v\right)\mathrm{d}v\right] \tag{6.63}$$

$$\leq \frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\mathrm{d}u\mathrm{d}v.$$

Theorem 6.4.3. *Let $s \in (-1,1]$ and $p \in \mathbb{R} \setminus \{0\}$. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a co-ordinated (s,p)-convex function on $\Delta = [c_1, c_2] \times [d_1, d_2]$ with $c_1 < c_2$ and $d_1 < d_2$. If $\Psi \in L_1([c_1, c_2] \times [d_1, d_2])$, then*

$$\begin{aligned}
&\frac{p}{(c_2^p - c_1^p)(d_2 - d_1)} \int_{d_1}^{d_2} \int_{c_1}^{c_2} u^{p-1} \Psi(u,v) \mathrm{d}u \mathrm{d}v \\
&\leq \frac{1}{(c_2^p - c_1^p)} \int_{c_1}^{c_2} \frac{p}{u^{p-1}} \frac{\Psi(u,d_1) + \Psi(u,d_2)}{2^s} \mathrm{d}u + \frac{1}{d_2 - d_1} \int_{d_1}^{d_2} [\frac{\Psi(c_1,v) + \Psi(c_2,v)}{s+1}] \mathrm{d}v \\
&\leq \frac{\Psi(c_1,d_1) + (c_1,d_2) + \Psi(c_2,d_1) + \Psi(c_2,d_2)}{2^{s-2}(s+1)}.
\end{aligned} \tag{6.64}$$

Proof. By co-ordinated (s,p)-convexity of Ψ, we observe

$$\begin{aligned}
&\Psi(u, r_2 d_1 + (1-r_2) d_2) \\
&= \Psi\left(\left(\frac{u^p}{2} + \frac{u^p}{2}\right)^{\frac{1}{p}}, r_2 d_1 + (1-r_2) d_2\right) \\
&\leq \frac{1}{2^s}[2\Psi(u,d_1) + 2\Psi(u,d_2)] \\
&= \frac{1}{2^{s-1}}[\Psi(u,d_1) + \Psi(u,d_2)],
\end{aligned} \tag{6.65}$$

for all $r_2 \in [0,1]$ and $u \in [c_1, c_2]$. Consider

$$\begin{aligned}
&\frac{p}{(c_2^p - c_1^p)(d_2 - d_1)} \int_{d_1}^{d_2} \int_{c_1}^{c_2} u^{p-1} \Psi(u,v) \mathrm{d}u \mathrm{d}v \\
&= \frac{p}{c_2^p - c_1^p} \int_0^1 \int_{c_1}^{c_2} u^{p-1} \Psi(u, r_2 d_1 + (1-r_2) d_2) \mathrm{d}u \mathrm{d}r_2 \\
&\leq \frac{p}{2^{s-1}(c_2^p - c_1^p)} \int_{c_1}^{c_2} u^{p-1} [\Psi(u,d_1) + \Psi(u,d_2)] \mathrm{d}u.
\end{aligned} \tag{6.66}$$

Since by letting $u^p = r_1 c_1^p + (1-r_1) c_2^p$, we have

$$\begin{aligned}
\Psi(u,d_1) &= \Psi((r_1 c_1^p (1-r_1) c_2^p)^{\frac{1}{p}}, r_2 d_1 + (1-r_2) d_1) \\
&\leq 2 r_1^s \Psi(c_1,d_1) + 2(1-r_1)^s \Psi(c_2,d_1) \\
&= 2\{r_1^s \Psi(c_1,d_1) + (1-r_1)^s \Psi(c_2,d_1)\}.
\end{aligned} \tag{6.67}$$

Similarly,

$$\begin{aligned}\Psi(u,d_2) &= \Psi((r_1c_1^p(1-r_1)c_2^p)^{\frac{1}{p}}, r_2d_2+(1-r_2)d_2)\\ &\leq 2r_1^s\Psi(c_1,d_2)+2(1-r_1)^s\Psi(c_2,d_2)\\ &= 2\{r_1^s\Psi(c_1,d_2)+(1-r_1)^s\Psi(c_2,d_2)\}.\end{aligned} \tag{6.68}$$

From above two inequalities the inequality (6.57) becomes

$$\begin{aligned}&\frac{p}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}u^{p-1}\Psi(u,v)\mathrm{d}u\mathrm{d}v\\ &\leq \frac{p}{2^{s-1}(c_2^p-c_1^p)}\int_{c_1}^{c_2}u^{p-1}[\Psi(u,d_1)+\Psi(u,d_2)]\mathrm{d}u\\ &\leq \frac{1}{2^{s-1}}\int_0^1[2\{r_1^s\Psi(c_1,d_1)+(1-r_1)^s\Psi(c_2,d_1)\}+2\{r^s\Psi(c_1,d_2)+(1-r_1)^s\Psi(c_2,d_2)\}]\mathrm{d}r_1\\ &= \frac{1}{2^{s-2}(s+1)}[\Psi(c_1,d_1)+\Psi(c_2,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_2)].\end{aligned} \tag{6.69}$$

Again by similar arguments we have

$$\begin{aligned}&\frac{p}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}u^{p-1}\Psi(u,v)\mathrm{d}u\mathrm{d}v\\ &\leq \frac{2}{d_2-d_1}\int_{d_1}^{d_2}\int_0^1[r-1^s\Psi(c_1,v)+(1-r_1)^s\Psi(c_2,v)]\mathrm{d}r_1\mathrm{d}v\\ &= \frac{2}{(s+1)(d_2-d_1)}\int_{d_1}^{d_2}[\Psi(c_1,v)+\Psi(c_2,v)]\mathrm{d}v\\ &= \frac{1}{2^{s-2}(s+1)}[\Psi(c_1,d_1)+\Psi(c_2,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_2)].\end{aligned} \tag{6.70}$$

By adding (6.69) and (6.70) we get (6.64). □

Corollary 6.4.4. *Under the assumptions of Theorem* 6.4.3*, we have*

1. *If* $p=1$*, then*

$$\begin{aligned}&\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\mathrm{d}u\mathrm{d}v\\ &\leq \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\frac{\Psi(u,d_1)+\Psi(u,d_2)}{2^s}\mathrm{d}u+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\frac{\Psi(c_1,v)+\Psi(c_2,v)}{s+1}\mathrm{d}v\\ &\leq \frac{[\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)]}{2^{s-2}(s+1)}.\end{aligned} \tag{6.71}$$

2. *If $p = s = 1$, then*

$$\begin{aligned}
&\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\mathrm{d}u\mathrm{d}v \\
&\leq \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\frac{\Psi(u,d_1)+\Psi(u,d_2)}{2}\mathrm{d}u + \frac{1}{d_2-d_1}\int_{d_1}^{d_2}\frac{\Psi(c_1,v)+\Psi(c_2,v)}{2}\mathrm{d}v \\
&\leq [\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)].
\end{aligned} \tag{6.72}$$

3. *If $p = 1$ and $s = 0$, then*

$$\begin{aligned}
&\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi(u,v)\mathrm{d}u\mathrm{d}v \\
&\leq \frac{1}{c_2-c_1}\int_{c_1}^{c_2}[\Psi(u,d_1)+\Psi(u,d_2)]\mathrm{d}u + \frac{1}{d_2-d_1}\int_{d_1}^{d_2}[\Psi(c_1,v)+\Psi(c_2,v)]\mathrm{d}v \\
&\leq 4[\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)].
\end{aligned} \tag{6.73}$$

Theorem 6.4.4. *Let $\Psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a co-ordinated (s,p)-convex function on $\Delta = [c_1,c_2] \times [d_1,d_2]$ for $p \in \mathbb{R} \setminus \{0\}$ and $s = -1$ with $c_1 < c_2$ and $d_1 < d_2$. If $\Psi \in L_1([c_1,c_2] \times [d_1,d_2])$, then*

$$\begin{aligned}
&\frac{p}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}u^{p-1}\varphi(u)\Psi(u,v)\mathrm{d}u\mathrm{d}v \\
&\leq \frac{1}{2}[\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)],
\end{aligned} \tag{6.74}$$

where $\varphi(u) = \frac{(c_2^p-u^p)(u^p-c_1^p)}{(c_2^p-c_1^p)^2}$ for $u \in [c_1,c_2]$.

Proof. By taking $u^p = r_1c_1^p + (1-r_1)c_2^p$ and $v = r_2d_1 + (1-r_2)d_2$ for $0 < r_1 < 1$ and $0 \leq r_2 \leq 1$ and using the co-ordinated (s,p)-convexity of Ψ implies

$$\begin{aligned}
&\frac{p}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}u^{p-1}\varphi(u)\Psi(u,v)\mathrm{d}u\mathrm{d}v \\
&= \int_0^1\int_0^1 r_1(1-r_1)\Psi((r_1c_1^p+(1-r_1)c_2^p)^{\frac{1}{p}}, r_2d_1+(1-r_2)d_2)\mathrm{d}r_1\mathrm{d}r_2 \\
&\leq \int_0^1\int_0^1 [(1-r_1)\{\Psi(c_1,d_1)+\Psi(c_1,d_2)\} + r_1\{\Psi(c_2,d_1)+\Psi(c_2,d_2)\}]\mathrm{d}r_1\mathrm{d}r_2 \\
&\leq \frac{1}{2}[\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)].
\end{aligned} \tag{6.75}$$

Hence required inequality (6.74) is proved. □

6.5 Integral inequalities for product of two (s,p)-convex functions

Ozdemir et al. [73] established inequalities for product of two s-convex functions on the co-ordinates. In this section, we establish inequalities for product of two (s,p)-convex functions on the co-ordinates.

Theorem 6.5.1. *Let $s_1, s_2 \in (-1,1]$ and $p_1, p_2 \in \mathbb{R}\setminus\{0\}$. Let $\Psi_1, \Psi_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be co-ordinated (s_1,p_1)-,(s_2,p_2)-convex functions on $\Delta = [c_1,c_2]\times[d_1,d_2]$, respectively, with $c_1 < c_2$ and $d_1 < d_2$. If $\Psi_1, \Psi_2 \in L_1([c_1,c_2]\times[d_1,d_2])$, then*

$$\begin{aligned}
&2^{s_1+s_2-2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}, \frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}, \frac{d_1+d_2}{2}\right)\\
&\leq \Bigg[\frac{1}{(c_2^{p_1}-c_1^{p_1})(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{p_1}{u_1^{1-p_1}}\Psi_1\left(u_1, \frac{d_1+d_2}{2}\right)\\
&\quad\times\{\Psi_2(c_1,v)+\Psi(c_2,v)\}du_1dv\\
&\quad+\frac{1}{(c_2^{p_2}-c_1^{p_2})(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{2p_2}{u_2^{1-p_2}}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}, v\right)\\
&\quad\times\Psi_2\left(u_2, \frac{d_1+d_2}{2}\right)du_2dv\Bigg] \qquad (6.76)\\
&\leq \frac{1}{(s_1+1)(s_2+1)}\Bigg[4\Bigg\{\left(\Psi_1\left(c_1, \frac{d_1+d_2}{2}\right)+\Psi_1\left(c_2, \frac{d_1+d_2}{2}\right)\right)\\
&\quad\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))\Bigg\}\\
&\quad+2^{s_1}\Bigg\{(\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)))\\
&\quad\times\left(\Psi_2\left(c_1, \frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2, \frac{d_1+d_2}{2}\right)\right)\Bigg\}\Bigg].
\end{aligned}$$

Proof. Since Ψ_1 is an co-ordinated (s_1,p_1)-convex functions on Δ, then by Theorem

6.4.2 we have

$$2^{2s_1-2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},\frac{d_1+d_2}{2}\right)$$

$$\le 2^{s_1-1}\left[\frac{1}{c_2^{p_1}-c_1^{p_1}}\int_{c_1}^{c_2}\frac{p_1}{u_1^{1-p_1}}\Psi_1\left(u_1,\frac{d_1+d_2}{2}\right)\mathrm{d}u_1\right. \tag{6.77}$$

$$\left.+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},v\right)\mathrm{d}v\right].$$

Multiplying above inequality by $\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)$ gives

$$2^{2s_1-2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)$$

$$\le 2^{s_1-1}\left[\frac{1}{c_2^{p_1}-c_1^{p_1}}\int_{c_1}^{c_2}\frac{p_1}{u_1^{1-p_1}}\Psi_1\left(u_1,\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)\mathrm{d}u_1\right.$$

$$\left.+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},v\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)\mathrm{d}v\right]. \tag{6.78}$$

By integrating with respect to r_1 and r_2 over $[0,1]\times[0,1]$, we obtain

$$2^{2s_1-2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)$$

$$\le 2^{s_1-1}\left[\frac{1}{c_2^{p_1}-c_1^{p_1}}\int_0^1\int_{c_1}^{c_2}\frac{p_1}{u_1^{1-p_1}}\Psi_1\left(u_1,\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)\mathrm{d}u_1\mathrm{d}r_2\right.$$

$$\left.+\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\int_0^1\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},v\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)\mathrm{d}r_1\mathrm{d}v\right]. \tag{6.79}$$

Further, note that

$$\begin{aligned}
&\int_0^1 \Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}, \frac{d_1+d_2}{2}\right)\mathrm{d}r_2 \\
&= \int_0^1 \Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}, \frac{r_2d_1+(1-r_2)d_2}{2}+\frac{(1-r_2)d_1+r_2d_2}{2}\right)\mathrm{d}r_2 \\
&\le \frac{1}{2^{s_2}}\int_0^1 [\Psi_2(c_1, r_2d_1+(1-r_2)d_2)+\Psi_2(c_1,(1-r_2)d_1+r_2d_2) \\
&\quad +\Psi_2(c_2, r_2d_1+(1-r_2)d_2)+\Psi_2(c_2,(1-r_2)d_1+r_2d)_2]\mathrm{d}r_2 \\
&= \frac{1}{2^{s_2-1}(d_2-d_1)}\int_{d_1}^{d_2}[\Psi_2(c_1,v)+\Psi_2(c_2,v)]\mathrm{d}v.
\end{aligned} \tag{6.80}$$

Similarly, by taking $z_1^{p_2} = r_1c_1^{p_2}+(1-r_1)c_2^{p_2}$ and $z_2^{p_2} = (1-r_1)c_1^{p_2}+r_1c_2^{p_2}$, we get

$$\begin{aligned}
&\int_0^1 \Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}, \frac{d_1+d_2}{2}\right)\mathrm{d}r_1 \\
&= \int_0^1 \Psi_2\left(\left(\frac{r_1c_1^{p_2}+(1-r_1)c_2^{p_2}}{2}+\frac{(1-r_1)c_1^{p_2}+r_1c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}, \frac{\frac{d_1+d_2}{2}}{2}+\frac{\frac{d_1+d_2}{2}}{2}\right)\mathrm{d}r_1 \\
&= \int_0^1 \Psi_2\left(\left(\frac{z_1^{p_2}}{2}+\frac{z_2^{p_2}}{2}\right)^{\frac{1}{p_2}}, \frac{\frac{d_1+d_2}{2}}{2}+\frac{\frac{d_1+d_2}{2}}{2}\right)\mathrm{d}r_1 \\
&\le \frac{1}{2^{s_2}}\int_0^1 \left[\Psi_2\left(z_1,\frac{d_1+d_2}{2}\right)+\Psi_2\left(z_1,\frac{d_1+d_2}{2}\right)\right. \\
&\quad \left.+\Psi_2\left(z_2,\frac{d_1+d_2}{2}\right)+\Psi_2\left(z_2,\frac{d_1+d_2}{2}\right)\right]\mathrm{d}r_1 \\
&= \frac{1}{2^{s_2-1}}\int_0^1 \left[\Psi_2\left((r_1c_1^{p_2}+(1-r_1)c_2^{p_2})^{\frac{1}{p_2}}, \frac{d_1+d_2}{2}\right)\right. \\
&\quad \left.+\Psi_2\left(((1-r_1)c_1^{p_2}+r_1c_2^{p_2})^{\frac{1}{p_2}}, \frac{d_1+d_2}{2}\right)\right]\mathrm{d}r_1 \\
&= \frac{p_2}{2^{s_2-2}(c_2^{p_2}-c_1^{p_2})}\int_{c_1}^{c_2} u_2^{p_2-1}\Psi_2\left(u_2,\frac{d_1+d_2}{2}\right)\mathrm{d}u_2.
\end{aligned} \tag{6.81}$$

Substituting (6.80) and (6.81) in (6.79), we get

$$
\begin{aligned}
&2^{2s_1-2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}},\frac{d_1+d_2}{2}\right)\\
&\leq\left[\frac{2^{s_1-s_2+1}}{(c_2^{p_1}-c_1^{p_1})(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{p_1}{u_1^{1-p_1}}\Psi_1\left(u_1,\frac{d_1+d_2}{2}\right)\frac{\Psi_2(c_1,v)+\Psi_2(c_2,v)}{2}\mathrm{d}u_1\mathrm{d}v\right.\\
&\quad+\frac{2^{s_1-s_2+1}}{(c_2^{p_2}-c_1^{p_2})(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{p_2}{u_2^{1-p_2}}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},v\right)\\
&\quad\left.\times\Psi_2\left(u_2,\frac{d_1+d_2}{2}\right)\mathrm{d}u_2\mathrm{d}v\right].
\end{aligned}
\tag{6.82}
$$

Since by letting $u_1^{p_1}=r_1c_1^{p_1}+(1-r_1)c_2^{p_1}$, we have

$$
\begin{aligned}
\Psi_1\left(u_1,\frac{d_1+d_2}{2}\right)&=\Psi_1\left((r_1c_1^{p_1}+(1-r_1)c_2^{p_1})^{\frac{1}{p_1}},r_2\frac{d_1+d_2}{2}+(1-r_2)\frac{d_1+d_2}{2}\right)\\
&\leq 2r_1^{s_1}\Psi_1\left(c_1,\frac{d_1+d_2}{2}\right)+2(1-r_1)^{s_1}\Psi_1\left(c_2,\frac{d_1+d_2}{2}\right)\\
&=2\left[r_1^{s_1}\Psi_1\left(c_1,\frac{d_1+d_2}{2}\right)+(1-r_1)^{s_1}\Psi_1\left(c_2,\frac{d_1+d_2}{2}\right)\right].
\end{aligned}
\tag{6.83}
$$

Then we have

$$
\begin{aligned}
&\frac{p_1}{c_2^{p_1}-c_1^{p_1}}\int_{c_1}^{c_2}u_1^{p_1-1}\Psi_1\left(u_1,\frac{d_1+d_2}{2}\right)\mathrm{d}u_1\\
&\leq 2\int_0^1\left[r_1^{s_1}\Psi_1\left(c_1,\frac{d_1+d_2}{2}\right)+(1-r_1)^{s_1}\Psi_1\left(c_2,\frac{d_1+d_2}{2}\right)\right]\mathrm{d}r_1\\
&=\frac{2}{s_1+1}\left[\Psi_1\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_1\left(c_2,\frac{d_1+d_2}{2}\right)\right].
\end{aligned}
\tag{6.84}
$$

Similarly,

$$
\begin{aligned}
&\frac{p_2}{c_2^{p_2}-c_1^{p_2}}\int_{c_1}^{c_2}u_2^{p_1-1}\Psi_2\left(u_2,\frac{d_1+d_2}{2}\right)\mathrm{d}u_2\\
&\leq 2\int_0^1\left[r_1^{s_2}\Psi_2\left(c_1,\frac{d_1+d_2}{2}\right)+(1-r_1)^{s_2}\Psi_2\left(c_2,\frac{d_1+d_2}{2}\right)\right]\mathrm{d}r_2\\
&=\frac{2}{s_2+1}\left[\Psi_2\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2,\frac{d_1+d_2}{2}\right)\right].
\end{aligned}
\tag{6.85}
$$

Further, by taking $v = r_2 d_1 + (1 - r_2) d_2$

$$\begin{aligned}\Psi_2(c_1, v) &= \Psi_2((r_1 c_1^{p_2} + (1-r_1)c_1^{p_2})^{\frac{1}{p_2}}, r_2 d_1 + (1-r_2)d_2) \\ &\le 2r_1^{s_2}\Psi_2(c_1, d_1) + 2(1-r_1)^{s_2}\Psi_2(c_1, d_2) \\ &= 2\{r_1^{s_2}\Psi_2(c_1, d_1) + (1-r_1)^{s_2}\Psi_2(c_1, d_2)\},\end{aligned} \tag{6.86}$$

and

$$\begin{aligned}\Psi_2(c_2, v) &= \Psi_2((r_1 c_2^{p_2} + (1-r_1)c_2^{p_2})^{\frac{1}{p_2}}, r_2 d_1 + (1-r_2)d_2) \\ &\le 2r_1^{s_2}\Psi_2(c_2, d_1) + 2(1-r_1)^{s_2}\Psi_2(c_2, d_2) \\ &= 2\{r_1^{s_2}\Psi_2(c_2, d_1) + (1-r_1)^{s_2}\Psi_2(c_2, d_2)\}.\end{aligned} \tag{6.87}$$

Then

$$\begin{aligned}&\frac{1}{d_2 - d_1}\int_{d_1}^{d_2} \frac{\Psi_2(c_1, v) + \Psi_2(c_2, v)}{2} \mathrm{d}v \\ &\le \int_0^1 \{r_1^{s_2}[\Psi_2(c_1, d_1) + \Psi_2(c_2, d_1)] + (1-r_1)^{s_2}[\Psi_2(c_1, d_2) + \Psi_2(c_2, d_2)]\}\mathrm{d}r_2 \\ &= \int_0^1 \{r_1^{s_2}[\Psi_2(c_1, d_1) + \Psi_2(c_2, d_1)] + (1-r_1)^{s_2}[\Psi_2(c_1, d_2) + \Psi_2(c_2, d_2)]\}\mathrm{d}r_1 \\ &= \frac{1}{s_2+1}[\Psi_2(c_1, d_1) + \Psi_2(c_1, d_2) + \Psi_2(c_2, d_1) + \Psi_2(c_2, d_2)].\end{aligned} \tag{6.88}$$

Also, note that by taking $v = r_2 d_1 + (1 - r_2) d_2$

$$\begin{aligned}\Psi_1\left(\left(\frac{c_1^{p_1} + c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}, v\right) &= \Psi_1\left(\left(\frac{c_1^{p_1} + c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}, \frac{v}{2} + \frac{v}{2}\right) \\ &\le 2^{s_1-1}[\Psi_1(c_1, v) + \Psi_1(c_2, v)] \\ &= 2^{s_1-1}[\Psi_1(c_1, r_2 d_1 + (1-r_2)d_2) + \Psi_1(c_2, r_2 d_1 + (1-r_2)d_2)] \\ &\le 2^{s_1-2}[\{r_1^{s_1}\Psi_1(c_1, d_1) + (1-r_1)^{s_1}\Psi_1(c_1, d_2)\} \\ &\quad + \{r_1^{s_1}\Psi_1(c_2, d_1) + (1-r_1)^{s_1}\Psi_1(c_2, d_2)\}].\end{aligned} \tag{6.89}$$

Then

$$\begin{aligned}
&\frac{1}{d_2-d_1}\int_{d_1}^{d_2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}},v\right)\mathrm{d}v \\
&\le 2^{s_1-2}\int_0^1 [\{r_1^{s_1}\Psi_1(c_1,d_1)+(1-r_1)^{s_1}\Psi_1(c_1,d_2)\} \\
&\quad+\{r_1^{s_1}\Psi_1(c_2,d_1)+(1-r_1)^{s_1}\Psi_1(c_2,d_2)\}]\mathrm{d}r_2 \\
&= 2^{s_1-2}\int_0^1 [\{r_1^{s_1}\Psi_1(c_1,d_1)+(1-r_1)^{s_1}\Psi_1(c_1,d_2)\} \\
&\quad+\{r_1^{s_1}\Psi_1(c_2,d_1)+(1-r_1)^{s_1}\Psi_1(c_2,d_2)\}]\mathrm{d}r_1 \\
&= \frac{2^{s_1-2}}{s_1+1}[\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)].
\end{aligned} \tag{6.90}$$

By using inequalities (6.84), (6.85), (6.88) and (6.90) the inequality (6.82) implies (6.76) after some calculations. Hence Theorem (6.5.1) is proved. □

Corollary 6.5.1. *Under similar assumptions of Theorem* 6.5.1,

1. *If* $p_1=p_2=p$, *then*

$$\begin{aligned}
&2^{s_1+s_2-2}\Psi_1\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}},\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}},\frac{d_1+d_2}{2}\right) \\
&\le\left[\frac{1}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{p}{u^{1-p}}\Psi_1\left(u,\frac{d_1+d_2}{2}\right)\{\Psi_2(c_1,v)+\Psi(c_2,v)\}\mathrm{d}u\mathrm{d}v\right. \\
&\quad\left.+\frac{1}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{2p}{u^{1-p}}\Psi_1\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}},v\right)\Psi_2\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v\right] \\
&\le\frac{1}{(s_1+1)(s_2+1)}\left[4\left\{\left(\Psi_1\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_1\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right.\right. \\
&\quad\left.\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))\right\} \\
&\quad+2^{s_1}\left\{(\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)))\right. \\
&\quad\left.\left.\times\left(\Psi_2\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right\}\right].
\end{aligned} \tag{6.91}$$

2. *If* $p_1 = p_2 = p = 1$, *then*

$$
\begin{aligned}
&2^{s_1+s_2-2}\Psi_1\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)\\
&\leq\left[\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi_1\left(u,\frac{d_1+d_2}{2}\right)\{\Psi_2(c_1,v)+\Psi(c_2,v)\}\mathrm{d}u\mathrm{d}v\right.\\
&\quad\left.+\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}2\Psi_1\left(\left(\frac{c_1+c_2}{2}\right)^{\frac{1}{p}},v\right)\Psi_2\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v\right]\\
&\leq\frac{1}{(s_1+1)(s_2+1)}\left[4\left\{\left(\Psi_1\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_1\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right.\right.\\
&\quad\left.\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))\right\}\\
&\quad+2^{s_1}\left\{(\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)))\right.\\
&\quad\left.\left.\times\left(\Psi_2\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right\}\right].
\end{aligned}
\tag{6.92}
$$

3. *If* $p_1 = p_2 = p = 1$ *and* $s_1 = s_2 = 1$, *then*

$$
\begin{aligned}
&\Psi_1\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right),\frac{d_1+d_2}{2}\right)\\
&\leq\left[\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi_1\left(u,\frac{d_1+d_2}{2}\right)\{\Psi_2(c_1,v)+\Psi(c_2,v)\}\mathrm{d}u\mathrm{d}v\right.\\
&\quad\left.+\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}2\Psi_1\left(\left(\frac{c_1+c_2}{2}\right)^{\frac{1}{p}},v\right)\Psi_2\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v\right]\\
&\leq\left[\left\{\left(\Psi_1\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_1\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right.\right.\\
&\quad\left.\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))\right\}\\
&\quad+\frac{1}{2}\left\{(\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)))\right.\\
&\quad\left.\left.\times\left(\Psi_2\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right\}\right].
\end{aligned}
\tag{6.93}
$$

4. *If $p_1 = p_2 = p = 1$ and $s_1 = s_2 = 0$, then*

$$\frac{1}{4}\Psi_1\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right)$$
$$\leq \Bigg[\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\Psi_1\left(u, \frac{d_1+d_2}{2}\right)\{\Psi_2(c_1,v)+\Psi(c_2,v)\}\mathrm{d}u\mathrm{d}v$$
$$+\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}2\Psi_1\left(\left(\frac{c_1+c_2}{2}\right)^{\frac{1}{p}}, v\right)\Psi_2\left(u, \frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v\Bigg]$$
$$\leq \Bigg[4\Bigg\{\left(\Psi_1\left(c_1, \frac{d_1+d_2}{2}\right)+\Psi_1\left(c_2, \frac{d_1+d_2}{2}\right)\right) \tag{6.94}$$
$$\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))\Bigg\}$$
$$+\Bigg\{(\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)))$$
$$\times\left(\Psi_2\left(c_1, \frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2, \frac{d_1+d_2}{2}\right)\right)\Bigg\}\Bigg].$$

Theorem 6.5.2. *Let $s_1, s_2 \in (-1,1]$ and $p_1, p_2 \in \mathbb{R}\setminus\{0\}$. Let $\Psi_1, \Psi_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be co-ordinated (s_1,p_1)-,(s_2,p_2)-convex functions on $\Delta = [c_1,c_2]\times[d_1,d_2]$, respectively, with $c_1 < c_2$ and $d_1 < d_2$. If $\Psi_1, \Psi_2 \in L_1([c_1,c_2]\times[d_1,d_2])$, then*

$$2^{s_1+s_2-2}\Psi_1\left(\left(\frac{c_1^{p_1}+c_2^{p_1}}{2}\right)^{\frac{1}{p_1}}, \frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^{p_2}+c_2^{p_2}}{2}\right)^{\frac{1}{p_2}}, \frac{d_1+d_2}{2}\right)$$
$$\leq \Bigg[\frac{1}{(c_2^{p_1}-c_1^{p_1})(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{p_1}{u_1^{1-p_1}}\frac{\Psi_1(u_1,d_1)+\Psi_1(u_1,d_2)}{2^{s_1-1}}\{\Psi_2(c_1,v)+\Psi_2(c_2,v)\}\mathrm{d}u_1\mathrm{d}v$$
$$+\frac{1}{(c_2^{p_2}-c_1^{p_2})(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{2p_2}{u_2^{1-p_2}}\frac{\Psi_1(c_1,v)+\Psi_1(c_2,v)}{s_1+1}\Psi_2\left(u_2, \frac{d_1+d_2}{2}\right)]\mathrm{d}u_2\mathrm{d}v$$
$$\leq \frac{4}{(s_1+1)(s_2+1)}\Bigg[\frac{\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)}{2^{s_1-1}}$$
$$\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))$$
$$+\frac{\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)}{s_1+1}$$
$$\times\left(\Psi_2\left(c_1, \frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2, \frac{d_1+d_2}{2}\right)\right)\Bigg]. \tag{6.95}$$

Proof. By using Theorem 6.4.1 and Theorem 6.4.3 and applying similar procedure of Theorem 6.5.1 we get the required result. □

Corollary 6.5.2. *Under similar assumptions of Theorem* 6.5.2*,*

1. *If* $p_1 = p_2 = p$*, then*

$$\begin{aligned}
&2^{s_1+s_2-2}\Psi_1\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1^p+c_2^p}{2}\right)^{\frac{1}{p}}, \frac{d_1+d_2}{2}\right)\\
&\leq \left[\frac{1}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{p}{x^{1-p}}\frac{\Psi_1(u,d_1)+\Psi_1(u,d_2)}{2^{s_1-1}}\{\Psi_2(c_1,v)+\Psi_2(c_2,v)\}\mathrm{d}u\mathrm{d}v\right.\\
&\quad\left.+\frac{1}{(c_2^p-c_1^p)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{2p}{u^{1-p}}\frac{\Psi_1(c_1,v)+\Psi_1(c_2,v)}{s_1+1}\Psi_2\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v\right.\\
&\leq \frac{4}{(s_1+1)(s_2+1)}\left[\frac{\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)}{2^{s_1-1}}\right.\\
&\quad\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))\\
&\quad+\frac{\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)}{s_1+1}\\
&\quad\left.\times\left(\Psi_2\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right].
\end{aligned} \tag{6.96}$$

2. *If* $p_1 = p_2 = p = 1$*, then*

$$\begin{aligned}
&2^{s_1+s_2-2}\Psi_1\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right)\\
&\leq \left[\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{\Psi_1(u,d_1)+\Psi_1(u,d_2)}{2^{s_1-1}}\{\Psi_2(c_1,v)+\Psi_2(c_2,v)\}\mathrm{d}u\mathrm{d}v\right.\\
&\quad\left.+\frac{2}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}\frac{\Psi_1(c_1,v)+\Psi_1(c_2,v)}{s_1+1}\Psi_2\left(u,\frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v\right.\\
&\leq \frac{4}{(s_1+1)(s_2+1)}\left[\frac{\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)}{2^{s_1-1}}\right.\\
&\quad\times(\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2))\\
&\quad+\frac{\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)}{s_1+1}\\
&\quad\left.\times\left(\Psi_2\left(c_1,\frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2,\frac{d_1+d_2}{2}\right)\right)\right].
\end{aligned} \tag{6.97}$$

3. *If* $p_1 = p_2 = p = 1$ *and* $s_1 = s_2 = 1$, *then*

$$\begin{aligned}
&\Psi_1\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right) \\
&\leq \left[\frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}(\Psi_1(u,d_1)+\Psi_1(u,d_2))\{\Psi_2(c_1,v)+\Psi_2(c_2,v)\}\mathrm{d}u\mathrm{d}v\right. \\
&\quad + \frac{1}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}(\Psi_1(c_1,v)+\Psi_1(c_2,v))\Psi_2\left(u, \frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v \\
&\leq \Big[(\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)) \\
&\quad \times (\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2)) \\
&\quad + \frac{\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)}{2} \\
&\quad \times \left.\left(\Psi_2\left(c_1, \frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2, \frac{d_1+d_2}{2}\right)\right)\right].
\end{aligned} \tag{6.98}$$

4. *If* $p_1 = p_2 = p = 1$ *and* $s_1 = s_2 = 0$, *then*

$$\begin{aligned}
&\frac{1}{4}\Psi_1\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right)\Psi_2\left(\left(\frac{c_1+c_2}{2}\right), \frac{d_1+d_2}{2}\right) \\
&\leq \left[\frac{2}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}(\Psi_1(u,d_1)+\Psi_1(u,d_2))\{\Psi_2(c_1,v)+\Psi_2(c_2,v)\}\mathrm{d}u\mathrm{d}v\right. \\
&\quad + \frac{2}{(c_2-c_1)(d_2-d_1)}\int_{d_1}^{d_2}\int_{c_1}^{c_2}(\Psi_1(c_1,v)+\Psi_1(c_2,v))\Psi_2\left(u, \frac{d_1+d_2}{2}\right)\mathrm{d}u\mathrm{d}v \\
&\leq 4\Big[2(\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)) \\
&\quad \times (\Psi_2(c_1,d_1)+\Psi_2(c_1,d_2)+\Psi_2(c_2,d_1)+\Psi_2(c_2,d_2)) \\
&\quad + (\Psi_1(c_1,d_1)+\Psi_1(c_1,d_2)+\Psi_1(c_2,d_1)+\Psi_1(c_2,d_2)) \\
&\quad \times \left.\left(\Psi_2\left(c_1, \frac{d_1+d_2}{2}\right)+\Psi_2\left(c_2, \frac{d_1+d_2}{2}\right)\right)\right].
\end{aligned} \tag{6.99}$$

Chapter 7

Integral inequalities for co-ordinated harmonically convex functions

In this chapter, we obtain Hermite-Hadamard type inequalities for co-ordinated harmonically convex functions via Riemann−Liouville fractional integrals and Katugampola fractional integrals. We also find some Fejér type inequalities via Katugampola frasctional integrals.

7.1 Introduction

Noor et. al. [69] gave following Hermite-Hadamard type inequalities for co-ordinated harmonically convex functions.

Theorem 7.1.1 ([69])**.** *Let $\Psi : \Delta = [c_1, c_2] \times [d_1, d_2] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ be co-ordinated harmonically convex on Δ with $c_1 < c_2$ and $d_1 < d_2$. Then*

$$\begin{aligned} \Psi\left(\frac{2c_1c_2}{c_1+c_2}, \frac{2d_1d_2}{d_1+d_2}\right) &\leq \frac{(c_1c_2)(d_1d_2)}{(c_2-c_1)(d_2-d_1)} \int_{c_1}^{c_2} \int_{d_1}^{d_2} \frac{\Psi(u,v)}{u^2v^2} dv du \\ &\leq \frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}. \end{aligned} \tag{7.1}$$

Theorem 7.1.2 ([48])**.** *Let $\Psi : \aleph \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $\Psi \in L_1(c_1, c_2)$ where $c_1, c_2 \in \aleph$ with $c_1 < c_2$. If Ψ is harmonically convex function on $[c_1, c_2]$, then* *following inequality for fractional integral holds:*

$$\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2}\right)\\
&\le \frac{\Gamma(\alpha+1)}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[J^{\alpha}_{1/c_1-}(\Psi\circ\mu)\left(\frac{1}{c_2}\right)+J^{\alpha}_{1/c_2+}(\Psi\circ\mu)\left(\frac{1}{c_1}\right)\right] \qquad (7.2)\\
&\le \frac{\Psi(c_1)+\Psi(c_2)}{2},
\end{aligned}$$

where $\alpha > 0$ *and* $\mu(u) = \frac{1}{u}$.

The Riemann−Liouville fractional integrals and Katugampola fractional integrals on co-ordinates are defined as:

Definition 7.1.1 ([81])**.** *Let* $\Psi \in L_1([c_1, c_2] \times [d_1, d_2])$*. The Riemann−Liouville integrals* $J^{\alpha,\beta}_{c_1+,d_1+}$, $J^{\alpha,\beta}_{c_1+,d_2-}$, $J^{\alpha,\beta}_{c_2-,d_1+}$ *and* $J^{\alpha,\beta}_{c_2-,d_2-}$ *of order* $\alpha, \beta > 0$ *with* $c_1, d_1 \ge 0$ *are defined by*

$$J^{\alpha,\beta}_{c_1+,d_1+}\Psi(u,v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{c_1}^{u}\int_{d_1}^{v}(u-t)^{\alpha-1}(v-s)^{\beta-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t,\ u > c_1\ v > d_1,$$

$$J^{\alpha,\beta}_{c_1+,d_2-}\Psi(u,v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{c_1}^{u}\int_{v}^{d_2}(u-t)^{\alpha-1}(v-s)^{\beta-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t,\ u > c_1\ v < d_2,$$

$$J^{\alpha,\beta}_{c_2-,d_1+}\Psi(u,v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{u}^{c_2}\int_{d_1}^{v}(u-t)^{\alpha-1}(v-s)^{\beta-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t,\ u < c_2\ v > c_1,$$

and

$$J^{\alpha,\beta}_{c_2-,d_2-}\Psi(u,v) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{u}^{c_2}\int_{v}^{d_2}(u-t)^{\alpha-1}(v-s)^{\beta-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t,\ u < c_2\ v < d_2,$$

respectively. Here Γ *is the Gamma function.*

Definition 7.1.2. *Let* $\Psi \in L_1([c_1, c_2] \times [d_1, d_2])$*. The Katugampola fractional integrals* ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_1+,d_1+}$, ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_1+,d_2-}$, ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_2-,d_1+}$ *and* ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_2-,d_2-}$ *of order* $\alpha, \beta > 0$ *with* $c_1, d_1 \ge 0$ *are defined by*

$${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_1+,d_1+}\Psi(u,v) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_{c_1}^{u}\int_{d_1}^{v}(u^{\rho_1}-t^{\rho_1})^{\alpha-1}(v^{\rho_2}-s^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t,$$

with $u > c_1$ $v > d_1$,

$$ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_1+,d_2-}\Psi(u,v) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{c_1}^{u}\int_{v}^{d_2}(u^{\rho_1}-t^{\rho_1})^{\alpha-1}(s^{\rho_2}-v^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t, $$

with $u > c_1$ $v < d_2$,

$$ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_2-,d_1+}\Psi(u,v) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{u}^{c_2}\int_{d_1}^{v}(t^{\rho_1}-u^{\rho_1})^{\alpha-1}(v^{\rho_2}-s^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t, $$

with $u < c_2$ $v > d_1$, and

$$ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{c_2-,d_2-}\Psi(u,v) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{u}^{c_2}\int_{v}^{d_2}(t^{\rho_1}-u^{\rho_1})^{\alpha-1}(s^{\rho_2}-v^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}\Psi(t,s)\mathrm{d}s\mathrm{d}t, $$

with $u < c_2$ $v < d_2$, respectively. Here $\Gamma(\cdot)$ is the gamma function.

In first section we first find the respected inequalities via Riemann−Liouville fractional integrals and in second and third sections we find inequalities via Katugampola fractional integrals.

7.2 Hermite-Hadamard type inequalities via Riemann−Liouville fractional integrals

In this section we find inequalities via Riemann−Liouville fractional integrals.

Theorem 7.2.1. *Let $\Psi : \Delta = [c_1, c_2] \times [d_1, d_2] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ be harmonically convex on Δ with $c_1 < c_2$ and $d_1 < d_2$ and $\Psi \in L_1(\Delta)$. Then*

$$ \begin{aligned} &\Psi\left(\frac{2c_1c_2}{c_1+c_2}, \frac{2d_1d_2}{d_1+d_2}\right) \\ &\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta} \\ &\quad\times\Bigg[J^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_2}\right) + J^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right) \\ &\quad + J^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right) + J^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)\Bigg] \\ &\leq \frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}, \end{aligned} \tag{7.3} $$

where $\mu(u,v) = \left(\frac{1}{u},\frac{1}{v}\right)$ for all $(u,v) \in \left(\left[\frac{1}{c_2},\frac{1}{c_1}\right],\left[\frac{1}{d_2},\frac{1}{d_1}\right]\right)$.

Proof. Let $(x,y),(z,w)\in\Delta$ and $r_1,r_2\in[0,1]$. Since Ψ is co-ordinated harmonically convex on Δ, we have

$$\begin{aligned}&\Psi\left(\frac{xz}{r_1x+(1-r_1)z},\frac{yw}{r_2y+(1-r_2)w}\right)\\ &\le r_1r_2\Psi(x,y)+r_1(1-r_2)\Psi(x,w)+(1-r_1)r_2\Psi(z,y)+(1-r_1)(1-r_2)\Psi(z,w).\end{aligned}\tag{7.4}$$

By taking $x=\frac{c_1c_2}{r_1c_1+(1-r_1)c_2}$, $z=\frac{c_1c_2}{r_1c_2+(1-r_1)c_1}$, $y=\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}$, $w=\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}$ and $r_1=r_2=\frac{1}{2}$ in (7.4), we get

$$\begin{aligned}&\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{2d_1d_2}{d_1+d_2}\right)\\ &\le\frac{1}{4}\Bigg[\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\ &\quad+\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}\right)\\ &\quad+\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\ &\quad+\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}\right)\Bigg].\end{aligned}\tag{7.5}$$

Multiplying both sides of (7.5) by $r_1^{\alpha-1}r_2^{\beta-1}$ and then integrating with respect to (r_1,r_2) over $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\frac{1}{\alpha\beta}\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{2d_1d_2}{d_1+d_2}\right)\\ &\le\frac{1}{4}\Bigg[\int_0^1\int_0^1\Bigg\{\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\ &\quad+\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}\right)\Bigg\}r_1^{\alpha-1}r_2^{\beta-1}\mathrm{d}r_1\mathrm{d}r_2\\ &\quad+\int_0^1\int_0^1\Bigg\{\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\ &\quad+\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}\right)\Bigg\}r_1^{\alpha-1}r_2^{\beta-1}\mathrm{d}r_1\mathrm{d}r_2\Bigg].\end{aligned}\tag{7.6}$$

Applying change of variable, we find

$$
\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{2d_1d_2}{d_1+d_2}\right)\\
&\leq \frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\Bigg[\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\Bigg\{\left(\frac{1}{c_1}-u\right)^{\alpha-1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\\
&\quad+\left(\frac{1}{c_1}-u\right)^{\alpha-1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\Bigg\}\mathrm{d}u\mathrm{d}v\\
&\quad+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\Bigg\{\left(u-\frac{1}{c_2}\right)^{\alpha-1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\\
&\quad+\left(u-\frac{1}{c_2}\right)^{\alpha-1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\Bigg\}\mathrm{d}u\mathrm{d}v\Bigg].
\end{aligned}
\tag{7.7}
$$

Then by multiplying and dividing by $\Gamma(\alpha)\Gamma(\beta)$ on right hand side of inequality (7.7), we get the first inequality of (7.3). For the second inequality of (7.3), we consider

$$
\begin{aligned}
&\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\
&\leq r_1r_2\Psi(c_1,d_1)+r_1(1-r_2)\Psi(c_1,d_2)+(1-r_1)r_2\Psi(c_2,d_1)\\
&\quad+(1-r_1)(1-r_2)\Psi(c-2,d_2),
\end{aligned}
$$

$$
\begin{aligned}
&\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}\right)\\
&\leq r_1r_2\Psi(c_1,d_2)+t(1-r_2)\Psi(c_1,d-1)+(1-r_1)r_2\Psi(c_2,d_2)\\
&\quad+(1-r_1)(1-r_2)\Psi(c_2,d_1),
\end{aligned}
$$

$$
\begin{aligned}
&\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\
&\leq r_1r_2\Psi(c_2,d_1)+r_1(1-r_2)\Psi(c_2,d_2)+(1-r_1)r_2\Psi(c_1,d_1)\\
&\quad+(1-r_1)(1-r_2)\Psi(c_1,d_2).
\end{aligned}
$$

and

$$\begin{aligned}&\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d+(1-r_2)c}\right)\\&\le r_1r_2\Psi(c_2,d_2)+r_1(1-r_2)\Psi(c_2,d_1)+(1-r_1)r_2\Psi(c_1,d_2)\\&\quad+(1-r_1)(1-r_2)\Psi(c_1,d_1).\end{aligned}$$

Thus by adding above inequalities, we get

$$\begin{aligned}&\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\&+\Psi\left(\frac{c_1c_2}{r_1c_1+(1-r_1)c_2},\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}\right)\\&+\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d_1+(1-r_2)d_2}\right)\\&+\Psi\left(\frac{c_1c_2}{r_1c_2+(1-r_1)c_1},\frac{d_1d_2}{r_2d_2+(1-r_2)d_1}\right)\\&\le\Psi(c_1,d_1)+\Psi(c_2,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_2).\end{aligned}\tag{7.8}$$

Thus by multiplying (7.8) by $r_1^{\alpha-1}r_2^{\beta-1}$ and then integrating with respect to (r_1, r_2) over $[0,1]\times[0,1]$, we get the second inequality of (7.3). Hence the proof is completed. □

Remark 36. *In Theorem* 7.2.1, *if one takes* $\alpha=\beta=1$ *and using change of variable* $x=1/u$ *and* $y=1/v$, *then one has Theorem* 3.1 *in* [69].

Theorem 7.2.2. *Let* $\Psi:\Delta=[c_1,c_2]\times[d_1,d_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be harmonically* *convex on Δ with $c_1 < c_2$ and $d_1 < d_2$ and $\Psi \in L_1(\Delta)$. Then*

$$\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{2d_1d_2}{d_1+d_2}\right)\\
&\leq \frac{\Gamma(\alpha+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},\frac{2d_1d_2}{d_1+d_2}\right)+J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},\frac{2d_1d_2}{d_1+d_2}\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[J^{\beta}_{1/d_2+}(\Psi\circ\mu_2)\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{d_1}\right)+J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{d_2}\right)\right]\\
&\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[J^{\alpha,\beta}_{c_1+,d_1+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{c_1+,d_2-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\right.\\
&\quad\left.+J^{\alpha,\beta}_{c_2-,d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{c_2-,d_2-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)\right]\\
&\leq \frac{\Gamma(\alpha+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},d_2\right)+J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},d_1\right)\right.\\
&\quad\left.+J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},d_1\right)+J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},d_1\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\alpha}\left[J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_1,\frac{1}{d_2}\right)+J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_2,\frac{1}{d_2}\right)\right.\\
&\quad\left.+J^{\alpha}_{1/d_2+}(\Psi\circ\mu_2)\left(c_1,\frac{1}{d_1}\right)+J^{\alpha}_{1/d_2+}(\Psi\circ\mu_2)\left(c_2,\frac{1}{d_1}\right)\right]\\
&\leq \frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4},
\end{aligned} \tag{7.9}$$

where $\mu(u,v) = \left(\frac{1}{u},\frac{1}{v}\right)$, $\mu_1(u,v) = \left(\frac{1}{u},v\right)$ and $\mu_2(u,v) = \left(u,\frac{1}{v}\right)$ for all $(u,v) \in ([\frac{1}{c_2},\frac{1}{c_1}],[\frac{1}{d_2},\frac{1}{d_1}])$.

Proof. Since Ψ is co-ordinated harmonically convex on Δ then we have $\Psi_{\frac{1}{u}} : [d_1,d_2] \to \mathbb{R}$, $\Psi_{\frac{1}{u}}(v) = \Psi(\frac{1}{u},v)$, is harmonically convex on $[d_1,d_2]$ for all $u \in [\frac{1}{c_2},\frac{1}{c_1}]$. Then from inequality (7.2), we have

$$\begin{aligned}
&\Psi_{\frac{1}{u}}\left(\frac{2d_1d_2}{d_1+d_2}\right)\\
&\le \frac{\Gamma(\beta+1)}{2}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\left[J^{\beta}_{1/c-}(\Psi_{\frac{1}{u}}\circ\mu_2)\left(\frac{1}{d_2}\right)+J^{\beta}_{1/d_2+}(\Psi_{\frac{1}{u}}\circ\mu_2)\left(\frac{1}{d_1}\right)\right] \qquad (7.10)\\
&\le \frac{\Psi_{\frac{1}{u}}(d_1)+\Psi_{\frac{1}{u}}(d_2)}{2}.
\end{aligned}$$

In other words,

$$\begin{aligned}
&\Psi\left(\frac{1}{u},\frac{2d_1d_2}{d_1+d_2}\right)\\
&\le \frac{\beta}{2}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\left[\int_{1/d_2}^{1/d_1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\right.\\
&\quad\left.+\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\right] \qquad (7.11)\\
&\le \frac{\Psi\left(\frac{1}{u},d_1\right)+\Psi\left(\frac{1}{u},d_2\right)}{2},
\end{aligned}$$

for all $u\in[\frac{1}{c_2},\frac{1}{c_1}]$. Now by multiplying (7.11) by $\frac{\alpha(u-1/c_2)^{\alpha-1}}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}$ and $\frac{\alpha(1/c_1-u)^{\alpha-1}}{2}\times\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}$, and then integrating with respect to u over $[1/c_2,1/c_1]$, respectively, we find

$$\begin{aligned}
&\frac{\alpha}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\int_{1/c_2}^{1/c_1}\left(u-\frac{1}{c_2}\right)^{\alpha-1}\Psi\left(\frac{1}{u},\frac{2d_1d_2}{d_1+d_2}\right)\mathrm{d}u\\
&\le \frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u-\frac{1}{c_2}\right)^{\alpha-1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u-\frac{1}{c_2}\right)^{\alpha-1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right] \qquad (7.12)\\
&\le \frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[\int_{1/c_2}^{1/c_1}\left(u-\frac{1}{c_2}\right)^{\alpha-1}\Psi\left(\frac{1}{u},d_1\right)\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\left(u-\frac{1}{c_2}\right)^{\alpha-1}\Psi\left(\frac{1}{u},d_2\right)\mathrm{d}u\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\alpha}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\Psi\left(\frac{1}{u},\frac{2d_1d_2}{d_1+d_2}\right)\mathrm{d}u\\
&\le\frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right]\\
&\le\frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\Psi\left(\frac{1}{u},d_1\right)\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\Psi\left(\frac{1}{u},d_2\right)\mathrm{d}u\right].
\end{aligned}\tag{7.13}$$

Again by similar arguments for $\Psi_{\frac{1}{v}}:[c_1,c_2]\to\mathbb{R}$, $\Psi_{\frac{1}{v}}(u)=\Psi(u,\frac{1}{v})$, we get

$$\begin{aligned}
&\frac{\beta}{2}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\int_{1/d_2}^{1/d_1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{v}\right)\mathrm{d}v\\
&\le\frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u-\frac{1}{c_2}\right)^{\alpha-1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right]\\
&\le\frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[\int_{1/d_2}^{1/d_1}\left(v-\frac{1}{d_2}\right)^{\alpha-1}\Psi\left(c_1,\frac{1}{v}\right)\mathrm{d}v\right.\\
&\quad\left.+\int_{1/d_2}^{1/d_1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(c_2,\frac{1}{v}\right)\mathrm{d}v\right],
\end{aligned}\tag{7.14}$$

and

$$
\begin{aligned}
&\frac{\beta}{2}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{v}\right)\mathrm{d}v\\
&\leq\frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u-\frac{1}{c_2}\right)^{\alpha-1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{1}{v}\right)\mathrm{d}v\mathrm{d}u\right]\\
&\leq\frac{\alpha\beta}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1}-v\right)^{\alpha-1}\Psi\left(c_1,\frac{1}{v}\right)\mathrm{d}v\right.\\
&\quad\left.+\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(c_2,\frac{1}{v}\right)\mathrm{d}v\right].
\end{aligned}
\tag{7.15}
$$

By adding inequalites (7.12)−(7.15), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},\frac{2d_1d_2}{d_1+d_2}\right)+J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},\frac{2d_1d_2}{d_1+d_2}\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[J^{\beta}_{1/d_2+}(\Psi\circ\mu_2)\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{d_1}\right)+J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{d_2}\right)\right]\\
&\le\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[J^{\alpha,\beta}_{c_1+,d_1+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{c_1+,d_2-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\right.\\
&\quad\left.+J^{\alpha,\beta}_{c_2-,d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{c_2-,d_2-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)\right]\\
&\le\frac{\Gamma(\alpha+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},d_2\right)+J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},d_1\right)\right.\\
&\quad\left.+J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},d_1\right)+J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},d_1\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\alpha}\left[J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_1,\frac{1}{d_2}\right)+J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_2,\frac{1}{d_2}\right)\right.\\
&\quad\left.+J^{\alpha}_{1/d_2+}(\Psi\circ\mu_2)\left(c_1,\frac{1}{d_1}\right)+J^{\alpha}_{1/d_2+}(\Psi\circ\mu_2)\left(c_2,\frac{1}{d_1}\right)\right].
\end{aligned}\tag{7.16}$$

This completes the second and third inequality of (7.9). Now again using (7.2), we have

$$\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{2d_1d_2}{d_1+d_2}\right)\\
&\le\frac{\alpha}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[\int_{1/c_2}^{1/c-1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\Psi\left(\frac{1}{u},\frac{2d_1d_2}{d_1+d_2}\right)\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\left(u-\frac{1}{c_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},\frac{2d_1d_2}{d_1+d_2}\right)\mathrm{d}u\right],
\end{aligned}\tag{7.17}$$

$$\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{2d_1d_2}{d_1+d_2}\right)\\
&\leq \frac{\beta}{2}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\left[\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{v}\right)\mathrm{d}v\right.\\
&\quad\left.+\int_{1/d_2}^{1/d_1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{v}\right)\mathrm{d}v\right].
\end{aligned}\tag{7.18}$$

Adding (7.17) and (7.18), we get

$$\begin{aligned}
&\Psi\left(\frac{2c_1c_2}{c_1+c_2},\frac{2d_1d_2}{d_1+d_2}\right)\\
&\leq \frac{\Gamma(\alpha+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\\
&\quad\times\left[J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},\frac{2d_1d_2}{d_1+d_2}\right)+J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},\frac{2d_1d_2}{d_1+d_2}\right)\right]\\
&\quad+\frac{\Gamma(\beta+1)}{4}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[J^{\beta}_{1/d_2+}(\Psi\circ\mu_2)\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{d_1}\right)+J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(\frac{2c_1c_2}{c_1+c_2},\frac{1}{d_2}\right)\right].
\end{aligned}\tag{7.19}$$

This completes the first inequality of (7.9). For the last inequality by using (7.2), we have

$$\begin{aligned}
&\frac{\alpha}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\Psi\left(\frac{1}{u},d_1\right)\mathrm{d}u+\int_{1/c_2}^{1/c_1}\left(u-\frac{1}{c_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},d_1\right)\mathrm{d}u\right]\\
&\leq\frac{\Psi(c_1,d_1)+\Psi(c_2,d_1)}{2},
\end{aligned}$$

$$\begin{aligned}
&\frac{\alpha}{2}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left[\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1}-u\right)^{\alpha-1}\Psi\left(\frac{1}{u},d_2\right)\mathrm{d}u+\int_{1/c_2}^{1/c_1}\left(u-\frac{1}{c_2}\right)^{\beta-1}\Psi\left(\frac{1}{u},d_2\right)\mathrm{d}u\right]\\
&\leq\frac{\Psi(c_1,d_2)+\Psi(c_2,d_2)}{2},
\end{aligned}$$

$$\begin{aligned}
&\frac{\beta}{2}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\left[\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(c_1,\frac{1}{v}\right)\mathrm{d}v+\int_{1/d_2}^{1/d_1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(c_1,\frac{1}{v}\right)\mathrm{d}v\right]\\
&\leq\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)}{2},
\end{aligned}$$

$$\frac{\beta}{2}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\left[\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1}-v\right)^{\beta-1}\Psi\left(c_2,\frac{1}{v}\right)\mathrm{d}v+\int_{1/d_2}^{1/d_1}\left(v-\frac{1}{d_2}\right)^{\beta-1}\Psi\left(c_2,\frac{1}{v}\right)\mathrm{d}v\right]$$
$$\leq\frac{\Psi(c_2,d_1)+\Psi(c_2,d_2)}{2}.$$

Thus by adding all above inequalities, we get the last inequality of (7.9). Hence the proof is completed. □

Lemma 7.2.1. *Let $\Psi:\Delta=[c_1,c_2]\times[d_1,d_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0<c_1<c_2$ and $0<d_1<d_2$. If $\partial^2\Psi/\partial r_1\partial r_2\in L_1(\Delta)$, then following holds:*

$$\begin{aligned}
&\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}\\
&\quad+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[J^{\alpha,\beta}_{1/c_2+,1/d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\right.\\
&\quad\left.+J^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_2}\right)\right]-\Xi\\
&=\frac{c_1c_2d_1d_2(c_2-c_1)(d_2-d_1)}{4}\left[\int_0^1\int_0^1\frac{r_1^{\alpha}r_2^{\beta}}{A_{r_1}^2B_{r_2}^2}\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\mathrm{d}r_2\mathrm{d}r_1\right.\\
&\quad-\int_0^1\int_0^1\frac{(1-r_1)^{\alpha}r_2^{\beta}}{A_{r_1}^2B_{r_2}^2}\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\mathrm{d}r_2\mathrm{d}r_1\\
&\quad-\int_0^1\int_0^1\frac{r_1^{\alpha}(1-r_2)^{\beta}}{A_{r_1}^2B_{r_2}^2}\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\mathrm{d}r_2\mathrm{d}r_1\\
&\quad\left.+\int_0^1\int_0^1\frac{(1-r_1)^{\alpha}(1-r_2)^{\beta}}{A_{r_1}^2B_{r_2}^2}\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\mathrm{d}r_2\mathrm{d}r_1\right],
\end{aligned}\tag{7.20}$$

where

$$\begin{aligned}
\Xi = \frac{\Gamma(\alpha+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\Bigg[J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},d_2\right) + J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},d_2\right)\\
+ J^{\alpha}_{1/c_2+}(\Psi\circ\mu_1)\left(\frac{1}{c_1},d_1\right) + J^{\alpha}_{1/c_1-}(\Psi\circ\mu_1)\left(\frac{1}{c_2},d_1\right)\Bigg]\\
+ \frac{\Gamma(\beta+1)}{4}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\Bigg[J^{\beta}_{1/d_2+}(\Psi\circ\mu_2)\left(c_2,\frac{1}{d_1}\right)\\
+ J^{\beta}_{1/d_2+}(\Psi\circ\mu_2)\left(c_1,\frac{1}{d_1}\right) + J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_2,\frac{1}{d_2}\right) + J^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_1,\frac{1}{d_2}\right)\Bigg],
\end{aligned} \tag{7.21}$$

and $A_{r_1} = r_1a+(1-r_1)b$, $B_{r_2} = r_2c+(1-r_2)d$. *Also,* $\mu(u,v) = (\frac{1}{u},\frac{1}{v})$, $\mu_1(u,v) = (\frac{1}{u},v)$, *and* $\mu_2(u,v) = (u,\frac{1}{v})$ *for all* $(u,v)\in\Delta$.

Proof. By integration by parts and using the change of variable $u = \frac{A_{r_1}}{c_1c_2}$ and $v = \frac{B_{r_2}}{d_1d_2}$, we find that

$$
\begin{aligned}
I_1 &= \int_0^1 \int_0^1 \frac{r_1^\alpha r_2^\beta}{A_{r_1}^2 B_{r_2}^2} \frac{\partial^2 \Psi}{\partial r_1 \partial r_2} \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \mathrm{d}r_1 \\
&= \int_0^1 \frac{r_2^\beta}{B_{r_2}^2} \left\{ \frac{r_1^\alpha}{c_1 c_2 (c_2 - c_1)} \frac{\partial \Psi}{\partial r_2} \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \Bigg|_0^1 \right. \\
&\quad \left. - \frac{\alpha}{c_1 c_2 (c_2 - c_1)} \int_0^1 r_1^{\alpha - 1} \frac{\partial \Psi}{\partial r_2} \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_1 \right\} \mathrm{d}r_2 \\
&= \frac{1}{c_1 c_2 (c_2 - c_1)} \int_0^1 \frac{r_2^\beta}{B_{r_2}^2} \frac{\partial \Psi}{\partial r_2} \left(c_2, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \\
&\quad - \frac{\alpha}{c_1 c_2 (c_2 - c_1)} \int_0^1 r_1^{\alpha - 1} \left\{ \int_0^1 \frac{r_2^\beta}{B_{r_2}^2} \frac{\partial \Psi}{\partial r_2} \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \right\} \mathrm{d}r_1 \\
&= \frac{1}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \Psi(c_2, d_2) \\
&\quad - \frac{\beta}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \int_0^1 r_2^{\beta - 1} \Psi \left(c_2, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \\
&\quad - \frac{\alpha}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \int_0^1 r_1^{\alpha - 1} \Psi \left(\frac{c_1 c_2}{A_{r_1}}, d \right) \mathrm{d}r_1 \\
&\quad + \frac{\alpha \beta}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \int_0^1 r_1^{\alpha - 1} r_2^{\beta - 1} \Psi \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \\
&= \frac{1}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \\
&\quad \times \left[\Psi(c_2, d_2) - \Gamma(\beta + 1) \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^\beta_{1/d_2+} (\Psi \circ \mu_2) \left(c_2, \frac{1}{d_1} \right) \right. \\
&\quad - \Gamma(\alpha + 1) \left(\frac{c_1 c_2}{c_2 - c_1} \right)^\alpha J^\alpha_{1/c_2+} (\Psi \circ \mu_1) \left(\frac{1}{c_1}, d_2 \right) + \Gamma(\alpha + 1) \Gamma(\beta + 1) \\
&\quad \left. \times \left(\frac{c_1 c_2}{c_2 - c_1} \right)^\alpha \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^{\alpha, \beta}_{1/c_2+, 1/d_2+} (\Psi \circ \mu) \left(\frac{1}{c_1}, \frac{1}{d_1} \right) \right].
\end{aligned}
\tag{7.22}
$$

Similarly, we can have

$$\begin{aligned}
I_2 &= \int_0^1 \int_0^1 \frac{(1-r_1)^\alpha r_2^\beta}{A_{r_1}^2 B_{r_2}^2} \frac{\partial^2 \Psi}{\partial r_1 \partial r_2} \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \mathrm{d}r_1 \\
&= \frac{1}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \Bigg[-\Psi(c_1, d_1) \\
&\quad + \Gamma(\beta+1) \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^\beta_{1/d_2+} (\Psi \circ \mu_2) \left(c_1, \frac{1}{d_1} \right) \\
&\quad + \Gamma(\alpha+1) \left(\frac{c_1 c_2}{c_2 - c_1} \right)^\alpha J^\alpha_{1/c_1-} (\Psi \circ \mu_1) \left(\frac{1}{c_2}, d_2 \right) - \Gamma(\alpha+1)\Gamma(\beta+1) \\
&\quad \times \left(\frac{c_1 c_2}{c_2 - c_1} \right)^\alpha \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^{\alpha,\beta}_{1/c_2+,1/d_2+} (\Psi \circ \mu) \left(\frac{1}{c_1}, \frac{1}{d_1} \right) \Bigg].
\end{aligned} \tag{7.23}$$

$$\begin{aligned}
I_3 &= \int_0^1 \int_0^1 \frac{r_1^\alpha (1-r_2)^\beta}{A_{r_1}^2 B_{r_2}^2} \frac{\partial^2 \Psi}{\partial r_1 \partial r_2} \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \mathrm{d}r_1 \\
&= \frac{1}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \Bigg[-\Psi(c_2, d_1) \\
&\quad + \Gamma(\beta+1) \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^\beta_{1/d_1-} (\Psi \circ \mu_2) \left(c_2, \frac{1}{d_2} \right) \\
&\quad + \Gamma(\alpha+1) \left(\frac{c_1 c_2}{c_2 - c_1} \right)^\alpha J^\alpha_{1/c_2+} (\Psi \circ \mu_1) \left(\frac{1}{c_1}, d_1 \right) - \Gamma(\alpha+1)\Gamma(\beta+1) \\
&\quad \times \left(\frac{c_1 c_2}{c_2 - c_1} \right)^\alpha \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^{\alpha,\beta}_{1/c_2+,1/d_1-} (\Psi \circ \mu) \left(\frac{1}{c_1}, \frac{1}{d_2} \right) \Bigg].
\end{aligned} \tag{7.24}$$

$$\begin{aligned}
I_4 &= \int_0^1 \int_0^1 \frac{(1-r_1)^\alpha (1-r_2)^\beta}{A_{r_1}^2 B_{r_2}^2} \frac{\partial^2 \Psi}{\partial r_1 \partial r_2} \left(\frac{c_1 c_2}{A_{r_1}}, \frac{d_1 d_2}{B_{r_2}} \right) \mathrm{d}r_2 \mathrm{d}r_1 \\
&= \frac{1}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \Bigg[\Psi(c_1, d_2) \\
&\quad - \Gamma(\beta+1) \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^\beta_{1/d_1-} (\Psi \circ \mu_2) \left(c_1, \frac{1}{d_2} \right) \\
&\quad - \Gamma(\alpha+1) \left(\frac{c_1 c_2}{c_1 - c_1} \right)^\alpha J^\alpha_{1/c_1-} (\Psi \circ \mu_1) \left(\frac{1}{c_2}, d_1 \right) + \Gamma(\alpha+1)\Gamma(\beta+1) \\
&\quad \times \left(\frac{c_1 c_2}{c_1 - c_1} \right)^\alpha \left(\frac{d_1 d_2}{d_2 - d_1} \right)^\beta J^{\alpha,\beta}_{1/c_1-,1/d_1-} (\Psi \circ \mu) \left(\frac{1}{c_2}, \frac{1}{d_2} \right) \Bigg].
\end{aligned} \tag{7.25}$$

Thus from equalities (7.22)−(7.25), we have

$$
\begin{aligned}
& I_1 - I_2 - I_3 + I_4 \\
& = \frac{\Psi(c_2, d_2) + \Psi(c_1, d_1) + \Psi(c_2, d_1) + \Psi(c_1, d_2)}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \\
& \quad - \frac{\Gamma(\beta + 1)}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \left(\frac{d_1 d_2}{d_2 - d_1} \right)^{\beta} \\
& \quad \times \left[J^{\beta}_{1/d_2+} (\Psi \circ \mu_2) \left(c_2, \frac{1}{d_1} \right) + J^{\beta}_{1/d_2+} (\Psi \circ \mu_2) \left(c_1, \frac{1}{d_1} \right) \right. \\
& \quad \left. + J^{\beta}_{1/d_1-} (\Psi \circ \mu_2) \left(c_2, \frac{1}{d_2} \right) + J^{\beta}_{1/d_1-} (\Psi \circ \mu_2) \left(c_1, \frac{1}{d_2} \right) \right] \\
& \quad - \frac{\Gamma(\alpha + 1)}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \left(\frac{c_1 c_2}{c_2 - c_1} \right)^{\alpha} \\
& \quad \times \left[J^{\alpha}_{1/c_2+} (\Psi \circ \mu_1) \left(\frac{1}{c_1}, d_2 \right) + J^{\alpha}_{1/c_1-} (\Psi \circ \mu_1) \left(\frac{1}{c_2}, d_2 \right) \right. \\
& \quad \left. + J^{\alpha}_{1/c_2+} (\Psi \circ \mu_1) \left(\frac{1}{c_1}, d_1 \right) + J^{\alpha}_{1/c_1-} (\Psi \circ \mu_1) \left(\frac{1}{c_2}, d_1 \right) \right] \\
& \quad + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)} \\
& \quad \times \left[J^{\alpha,\beta}_{1/c_2+,1/d_2+} (\Psi \circ \mu) \left(\frac{1}{c_1}, \frac{1}{d_1} \right) + J^{\alpha,\beta}_{1/c_1-,1/d_2+} (\Psi \circ \mu) \left(\frac{1}{c_2}, \frac{1}{d_1} \right) \right. \\
& \quad \left. + J^{\alpha,\beta}_{1/c_2+,1/d_1-} (\Psi \circ \mu) \left(\frac{1}{c_1}, \frac{1}{d_2} \right) + J^{\alpha,\beta}_{1/c_1-,1/d_1-} (\Psi \circ \mu) \left(\frac{1}{c_2}, \frac{1}{d_2} \right) \right].
\end{aligned}
\tag{7.26}
$$

Multiplying both sides of equality (7.26) by $\frac{c_1 c_2 d_1 d_2 (c_2 - c_1)(d_2 - d_1)}{4}$, we get the desired equality (7.20). □

Theorem 7.2.3. *Let $\Psi : \Delta = [c_1, c_2] \times [d_1, d_2] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ be a partial differentiable mapping on Δ with $0 < c_1 < c_2$ and $0 < d_1 < d_2$. If $|\partial^2 \Psi / \partial r_1 \partial r_2|$ is a* *harmonically convex on the co-ordinates on Δ, then following holds:*

$$\begin{aligned}
&\left|\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}\right.\\
&\quad+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\left[J^{\alpha,\beta}_{1/c_2+,1/d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\right.\\
&\quad\left.\left.+J^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_2}\right)\right]-\Xi\right|\\
&\le\frac{c_1d_1(c_2-c_1)(d_2-d_1)}{4c_2d_2(\alpha+1)(\beta+1)(\alpha+2)(\beta+2)}\left[\vartheta_1\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_1)\right|+\vartheta_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_2)\right|\right.\\
&\quad\left.+\vartheta_3\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_1)\right|+\vartheta_4\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_2)\right|\right],
\end{aligned}\tag{7.27}$$

where

$$\begin{aligned}
\vartheta_1&=(\alpha+1)(\beta+1)\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{d_1}{d_2}\right)\\
&\quad+(\beta+1)\ {}_2F_1\left(2,2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{d_1}{d_2}\right)\\
&\quad+\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,2;\beta+3;1-\frac{d_1}{d_2}\right)\\
&\quad+\ {}_2F_1\left(2,2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,2;\beta+3;1-\frac{d_1}{d_2}\right),
\end{aligned}\tag{7.28}$$

$$\begin{aligned}
\vartheta_2&=(\beta+1)\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{d_1}{d_2}\right)\\
&\quad+(\alpha+1)(\beta+1)\ {}_2F_1\left(2,1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+2;\beta+3;1-\frac{d_1}{d_2}\right)\\
&\quad+\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,2;\beta+3;1-\frac{d_1}{d_2}\right)\\
&\quad+\ {}_2F_1\left(2,1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,2;\beta+3;1-\frac{d_1}{d_2}\right),
\end{aligned}\tag{7.29}$$

$$\begin{aligned}\vartheta_3 = (\alpha+1)\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{d_1}{d_2}\right)\\ +(\beta+1)\ {}_2F_1\left(2,2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{d_1}{d_2}\right)\\ +(\beta+1)\ {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,1;\beta+3;1-\frac{d_1}{d_2}\right)\\ +\ {}_2F_1\left(2,2;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,1;\beta+3;1-\frac{d_1}{d_2}\right),\end{aligned} \tag{7.30}$$

$$\begin{aligned}\vartheta_4 = \ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{d_1}{d_2}\right)\\ +(\alpha+1)\ {}_2F_1\left(2,1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,\beta+1;\beta+3;1-\frac{d_1}{d_2}\right)\\ +(\beta+1)\ {}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,1;\beta+3;1-\frac{d_1}{d_2}\right)\\ +(\alpha+1)(\beta+1)\ {}_2F_1\left(2,1;\alpha+3;1-\frac{c_1}{c_2}\right)\ {}_2F_1\left(2,1;\beta+3;1-\frac{d_1}{d_2}\right).\end{aligned} \tag{7.31}$$

Proof. Using Lemma 7.2.1, we have

$$\begin{aligned}&\Bigg|\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}\\ &+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\ &\times\Bigg[J^{\alpha,\beta}_{1/c_2+,1/d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\\ &+J^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_2}\right)\Bigg]-\Xi\Bigg|\\ &\leq\frac{c_1c_2d_1d_2(c_2-c_1)(d_2-d_1)}{4}\Bigg[\int_0^1\int_0^1\frac{r_1^{\alpha}r_2^{\beta}}{A_{r_1}^2B_{r_2}^2}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1\\ &+\int_0^1\int_0^1\frac{(1-r_1)^{\alpha}r_2^{\beta}}{A_{r_1}^2B_{r_2}^2}\left|\frac{\partial^2\Psi}{\partial t\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1\\ &+\int_0^1\int_0^1\frac{r_1^{\alpha}(1-r_2)^{\beta}}{A_{r_1}^2B_{r_2}^2}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1\\ &+\int_0^1\int_0^1\frac{(1-r_1)^{\alpha}(1-r_2)^{\beta}}{A_{r_1}^2B_{r_2}^2}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1\Bigg].\end{aligned} \tag{7.32}$$

Since $\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\right|$ co-ordinated harmonically convex, we get

$$\begin{aligned}
&\Bigg|\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}\\
&\quad+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\Bigg[J^{\alpha,\beta}_{1/c_2+,1/d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\\
&\quad+J^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_2}\right)\Bigg]-\Xi\Bigg|\\
&\le\frac{c_1c_2d_1d_2(c_2-c_1)(d_2-d_1)}{4}\Bigg[\int_0^1\int_0^1\Bigg\{\frac{r_1^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{r_1^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}\\
&\quad+\frac{(1-r_1)^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}\Bigg\}\Bigg\{r_1r_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_1)\right|+(1-r_1)r_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_1)\right|\\
&\quad+r_1(1-r_2)\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_2)\right|+(1-r_1)(1-r_2)\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_2)\right|\Bigg\}\mathrm{d}r_2\mathrm{d}r_1\Bigg]\\
&=\frac{c_1c_2d_1d_2(c_2-c_1)(d_2-d_1)}{4}\\
&\quad\times\Bigg[\int_0^1\int_0^1 r_1r_2\Bigg\{\frac{r_1^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{r_1^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}\Bigg\}\\
&\qquad\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_1)\right|\mathrm{d}r_1\mathrm{d}r_2\\
&\quad+\int_0^1\int_0^1(1-r_1)r_2\Bigg\{\frac{r_1^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{r_1^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}\Bigg\}\\
&\qquad\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_1)\right|\mathrm{d}r_1\mathrm{d}r_2\\
&\quad+\int_0^1\int_0^1 r_1(1-r_2)\Bigg\{\frac{r_1^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{r_1^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}\Bigg\}\\
&\qquad\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_2)\right|\mathrm{d}r_1\mathrm{d}r_2\\
&\quad+\int_0^1\int_0^1(1-r_1)(1-r_2)\Bigg\{\frac{r_1^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{(1-r_1)^{\alpha}r_2^{\beta}}{A^2_{r_1}B^2_{r_2}}+\frac{r_1^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}\\
&\qquad\qquad+\frac{(1-r_1)^{\alpha}(1-r_2)^{\beta}}{A^2_{r_1}B^2_{r_2}}\Bigg\}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_2)\right|\mathrm{d}r_1\mathrm{d}r_2\Bigg].
\end{aligned}\tag{7.33}$$

After calculating above integrations, we get the required result. $\square$

Theorem 7.2.4. *Let $\Psi : \Delta = [c_1, c_2] \times [d_1, d_2] \subseteq (0, \infty) \times (0, \infty) \to \mathbb{R}$ be a partial differentiable mapping on Δ with $0 < c_1 < c_2$ and $0 < d_1 < d_2$. If $|\partial^2\Psi/\partial r_1 \partial r_2|^q$, $q > 1$ such that $1/q + 1/l = 1$, is a harmonically convex on the co-ordinates on Δ, then following holds:*

$$\begin{aligned}
&\Bigg| \frac{\Psi(c_1, d_1) + \Psi(c_1, d_2) + \Psi(c_2, d_1) + \Psi(c_2, d_2)}{4} \\
&\quad + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4} \left(\frac{c_1 c_2}{c_2 - c_1} \right)^{\alpha} \left(\frac{d_1 d_2}{d_2 - d_1} \right)^{\beta} \\
&\quad \times \Bigg[J^{\alpha,\beta}_{1/c_2+,1/d_1+} (\Psi \circ \mu) \left(\frac{1}{c_1}, \frac{1}{d_1} \right) + J^{\alpha,\beta}_{1/c_1-,1/d_2+} (\Psi \circ \mu) \left(\frac{1}{c_2}, \frac{1}{d_1} \right) \\
&\quad + J^{\alpha,\beta}_{1/c_2+,1/d_1-} (\Psi \circ \mu) \left(\frac{1}{c_1}, \frac{1}{d_2} \right) + J^{\alpha,\beta}_{1/c_1-,1/d_1-} (\Psi \circ \mu) \left(\frac{1}{c_2}, \frac{1}{d_2} \right) \Bigg] - \Xi \Bigg| \\
&\leq \frac{c_1 d_1 (c_2 - c_1)(d_2 - d_1)}{4 c_2 d_2 [(l\alpha + 1)(l\beta + 1)]^{1/l}} \left[\sigma_1^{1/l} + \sigma_2^{1/l} + \sigma_3^{1/l} + \sigma_4^{1/l} \right] \\
&\quad \times \left(\frac{\left| \frac{\partial^2 \Psi}{\partial r_1 \partial r_2}(c_1, d_1) \right|^q + \left| \frac{\partial^2 \Psi}{\partial r_1 \partial r_2}(c_1, d_2) \right|^q + \left| \frac{\partial^2 \Psi}{\partial r_1 \partial r_2}(c_2, d_1) \right|^q + \left| \frac{\partial^2 \Psi}{\partial r_1 \partial r_2}(c_2, d_2) \right|^q}{4} \right)^{1/q},
\end{aligned} \tag{7.34}$$

where

$$\sigma_1 = {}_2F_1 \left(2l, l\alpha + 1; l\alpha + 2; 1 - \frac{c_1}{c_2} \right) {}_2F_1 \left(2l, l\beta + 1; l\beta + 2; 1 - \frac{d_1}{d_2} \right), \tag{7.35}$$

$$\sigma_2 = {}_2F_1 \left(2l, 1; l\alpha + 2; 1 - \frac{c_1}{c_2} \right) {}_2F_1 \left(2l, l\beta + 1; l\beta + 2; 1 - \frac{d_1}{d_2} \right), \tag{7.36}$$

$$\sigma_3 = {}_2F_1 \left(2l, l\alpha + 1; l\alpha + 2; 1 - \frac{c_1}{c_2} \right) {}_2F_1 \left(2l, 1; l\beta + 2; 1 - \frac{d_1}{d_2} \right), \tag{7.37}$$

$$\sigma_4 = {}_2F_1 \left(2l, 1; l\alpha + 2; 1 - \frac{c_1}{c_2} \right) {}_2F_1 \left(2l, 1; l\beta + 2; 1 - \frac{d_1}{d_2} \right). \tag{7.38}$$

Proof. Applying the Hölder's inequality for double integrals in (7.32), we get

$$\begin{aligned}
&\Bigg|\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}\\
&\quad+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\Bigg[J^{\alpha,\beta}_{1/c_2+,1/d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\\
&\quad+J^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_2}\right)\Bigg]-\Xi\Bigg|\\
&\leq\frac{c_1c_2d_1d_2(c_2-c_1)(d_2-d_1)}{4}\Bigg[\left(\int_0^1\int_0^1\frac{r_1^{p\alpha}r_2^{p\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/p}\\
&\quad+\left(\int_0^1\int_0^1\frac{(1-r_1)^{l\alpha}r_2^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}+\left(\int_0^1\int_0^1\frac{r_1^{l\alpha}(1-r_2)^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}\\
&\quad+\left(\int_0^1\int_0^1\frac{(1-r_1)^{l\alpha}(1-r_2)^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}\Bigg]\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1c_2}{A_{r_1}},\frac{d_1d_2}{B_{r_2}}\right)\right|^q\mathrm{d}r_1\mathrm{d}r_2\right)^{1/q}.
\end{aligned}\tag{7.39}$$

Using co-ordinated harmonically convexity of $\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\right|^q$, we get

$$\begin{aligned}
&\Bigg|\frac{\Psi(c_1,d_1)+\Psi(c_1,d_2)+\Psi(c_2,d_1)+\Psi(c_2,d_2)}{4}\\
&\quad+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1c_2}{c_2-c_1}\right)^{\alpha}\left(\frac{d_1d_2}{d_2-d_1}\right)^{\beta}\\
&\quad\times\Bigg[J^{\alpha,\beta}_{1/c_2+,1/d_1+}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_1}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_1}\right)\\
&\quad+J^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1},\frac{1}{d_2}\right)+J^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2},\frac{1}{d_2}\right)\Bigg]-\Xi\Bigg|\\
&\leq\frac{c_1c_2d_1d_2(c_2-c_1)(d_2-d_1)}{4}\\
&\quad\times\Bigg[\left(\int_0^1\int_0^1\frac{r_1^{l\alpha}r_2^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}+\left(\int_0^1\int_0^1\frac{(1-r_1)^{l\alpha}r_2^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1\int_0^1 \frac{r_1^{l\alpha}(1-r_2)^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}+\left(\int_0^1\int_0^1 \frac{(1-r_1)^{l\alpha}(1-r_2)^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}\Bigg]\\
&\times\Bigg(\int_0^1\int_0^1\Bigg\{r_1r_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_1)\right|^q+(1-r_1)r_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_1)\right|^q\\
&+r_1(1-r_2)\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_2)\right|^q+(1-r_1)(1-r_2)\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_2)\right|^q\Bigg\}\mathrm{d}r_2\mathrm{d}r_1\Bigg)^{1/q}\\
=&\frac{c_1c_2d_1d_2(c_2-c_1)(d_2-d_1)}{4}\\
&\times\Bigg[\left(\int_0^1\int_0^1 \frac{r_1^{l\alpha}r_2^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}+\left(\int_0^1\int_0^1 \frac{(1-r_1)^{l\alpha}r_2^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}\\
&+\left(\int_0^1\int_0^1 \frac{r_1^{l\alpha}(1-r_2)^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}+\left(\int_0^1\int_0^1 \frac{(1-r_1)^{l\alpha}(1-r_2)^{l\beta}}{A_{r_1}^{2l}B_{r_2}^{2l}}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/l}\Bigg]\\
&\times\left(\frac{\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_1)\right|^q+\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1,d_2)\right|^q+\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_1)\right|^q+\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2,d_2)\right|^q}{4}\right)^{1/q}.
\end{aligned}
\tag{7.40}
$$

By calculating all integrals, we get the required result (7.34). □

7.3 Hermite-Hadamard type inequalities via Katugampola fractional integrals

In this section we find inequalities via Katugampola fractional integrals.

Theorem 7.3.1. *Let $\alpha,\beta>0$ and $\rho_1,\rho_2>0$. Let $\Psi:\Delta=[c_1^{\rho_1},c_2^{\rho_1}]\times[d_1^{\rho_2},d_2^{\rho_2}]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a function with $0<c_1<c_2$, $0<d_1<d_2$. If Ψ is also*

co-ordinated harmonically convex on $\varDelta$ and $\Psi \in L_1(\varDelta)$. Then

$$\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\\
&\le \frac{\rho_1^{\alpha}\rho_2^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\\
&\quad\times\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]\\
&\le \frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{4},
\end{aligned}\tag{7.41}$$

where $\mu(u^{\rho_1},v^{\rho_2})=\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)$.

Proof. Let $(x^{\rho_1},y^{\rho_2}),(z^{\rho_1},w^{\rho_2})\in\varDelta$ and $r_1,r_2\in[0,1]$. Since Ψ is co-ordinated harmonically convex on $\varDelta$, we have

$$\begin{aligned}
&\Psi\left(\frac{x^{\rho_1}z^{\rho_1}}{r_1^{\rho_1}x^{\rho_1}+(1-r_1^{\rho_1})z^{\rho_1}},\frac{y^{\rho_2}w^{\rho_2}}{r_2^{\rho_2}y^{\rho_2}+(1-r_2^{\rho_2})w^{\rho_2}}\right)\\
&\le r_1^{\rho_1}r_2^{\rho_2}\Psi(x^{\rho_1},y^{\rho_2})+r_1^{\rho_1}(1-r_2^{\rho_2})\Psi(x^{\rho_1},w^{\rho_2})\\
&\quad+(1-r_1^{\rho_1})r_2^{\rho_1}\Psi(z^{\rho_1},y^{\rho_2})+(1-r_1^{\rho_1})(1-r_2^{\rho_2})\Psi(z^{\rho_1},w^{\rho_2}).
\end{aligned}\tag{7.42}$$

By taking $x^{\rho_1}=\frac{a^{\rho_1}b^{\rho_1}}{r_1^{\rho_1}a^{\rho_1}+(1-r_1^{\rho_1})b^{\rho_1}}$, $z^{\rho_1}=\frac{a^{\rho_1}b^{\rho_1}}{r_1^{\rho_1}b^{\rho_1}+(1-r_1^{\rho_1})a^{\rho_1}}$, $y^{\rho_2}=\frac{c^{\rho_2}d^{\rho_2}}{r_2^{\rho_2}c^{\rho_2}+(1-r_2^{\rho_2})d^{\rho_2}}$, $w^{\rho_2}=\frac{c^{\rho_2}d^{\rho_2}}{r_2^{\rho_2}d^{\rho_2}+(1-r_2^{\rho_2})c^{\rho_2}}$ and $r_1^{\rho_1}=r_2^{\rho_2}=\frac{1}{2}$ in (7.42), we get

$$\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\\
&\le\frac{1}{4}\Bigg[\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\Bigg].
\end{aligned}\tag{7.43}$$

Multiplying both sides of (7.43) by $r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}$ and then integrating with respect to (r_1, r_2) over $[0,1]\times[0,1]$, we get

$$\begin{aligned}
&\frac{1}{\rho_1\rho_2\alpha\beta}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\\
&\le\frac{1}{4}\left[\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\right.\\
&\quad+\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\\
&\quad+\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\\
&\quad\left.+\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\right].
\end{aligned}\tag{7.44}$$

Applying change of variable, we find

$$\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\\
&\le\frac{\rho_1\rho_2\alpha\beta}{4}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\\
&\quad\times\left[\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\right.\\
&\quad+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\\
&\quad+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\\
&\quad\left.+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\right].
\end{aligned}\tag{7.45}$$

Then by multiplying and dividing by $\rho_1^{1-\alpha}\rho_2^{1-\beta}\Gamma(\alpha)\Gamma(\beta)$ on right hand side of inequality (7.45), we get the first inequality of (7.41). For the second inequality of (7.41), we consider

$$\begin{aligned}
&\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\leq r_1^{\rho_1}r_2^{\rho_2}\Psi(c_1^{\rho_1}, d_1^{\rho_2}) + r_1^{\rho_1}(1-r_2^{\rho_2})\Psi(c_1^{\rho_1}, d_2^{\rho_2})\\
&\quad + (1-r_1^{\rho_1})r_2^{\rho_1}\Psi(c_2^{\rho_1}, d_1^{\rho_2}) + (1-r_1^{\rho_1})(1-r_2^{\rho_2})\Psi(c_2^{\rho_1}, d_2^{\rho_2}),
\end{aligned}$$

$$\begin{aligned}
&\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\leq r_1^{\rho_1}r_2^{\rho_2}\Psi(c_1^{\rho_1}, d_2^{\rho_2}) + r_1^{\rho_1}(1-r_2^{\rho_2})\Psi(c_1^{\rho_1}, d_1^{\rho_2})\\
&\quad + (1-r_1^{\rho_1})r_2^{\rho_1}\Psi(c_2^{\rho_1}, d_2^{\rho_2}) + (1-r_1^{\rho_1})(1-r_2^{\rho_2})\Psi(c_2^{\rho_1}, d_1^{\rho_2}),
\end{aligned}$$

$$\begin{aligned}
&\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\leq r_1^{\rho_1}r_2^{\rho_2}\Psi(c_2^{\rho_1}, d_1^{\rho_2}) + r_1^{\rho_1}(1-r_2^{\rho_2})\Psi(c_2^{\rho_1}, d_2^{\rho_2})\\
&\quad + (1-r_1^{\rho_1})r_2^{\rho_1}\Psi(c_1^{\rho_1}, d_1^{\rho_2}) + (1-r_1^{\rho_1})(1-r_2^{\rho_2})\Psi(c_1^{\rho_1}, d_2^{\rho_2}),
\end{aligned}$$

and

$$\begin{aligned}
&\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\leq r_1^{\rho_1}r_2^{\rho_2}\Psi(c_2^{\rho_1}, d_2^{\rho_2}) + r_1^{\rho_1}(1-r_2^{\rho_2})\Psi(c_2^{\rho_1}, d_1^{\rho_2})\\
&\quad + (1-r_1^{\rho_1})r_2^{\rho_1}\Psi(c_1^{\rho_1}, d_2^{\rho_2}) + (1-r_1^{\rho_1})(1-r_2^{\rho_2})\Psi(c_1^{\rho_1}, d_1^{\rho_2}).
\end{aligned}$$

Thus by adding above inequalities, we get

$$\begin{aligned}
&\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\leq \Psi(c_1^{\rho_1}, d_1^{\rho_2}) + \Psi(c_1^{\rho_1}, d_2^{\rho_2}) + \Psi(c_2^{\rho_1}, d_1^{\rho_2}) + \Psi(c_2^{\rho_1}, d_2^{\rho_2}).
\end{aligned} \tag{7.46}$$

Thus by multiplying (7.46) by $r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}$ and then integrating with respect to (r_1, r_2) over $[0,1]\times[0,1]$, we get the second inequality of (7.41). Hence the proof is completed.

□

Remark 37. *In inequality (7.41), if one takes $\rho_1 = \rho_2 = 1$, $\alpha = \beta = 1$ and using change of variable $x = 1/u$ and $y = 1/v$, then one has inequality (7.1).*

Lemma 7.3.1. *Let $\Psi : \Delta = [c_1^{\rho_1}, c_2^{\rho_1}] \times [d_1^{\rho_2}, d_2^{\rho_2}] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a partial differentiable mapping on Δ with $0 < c_1 < c_2$ and $0 < d_1 < d_2$. If $\partial^2\Psi/\partial r_1 \partial r_2 \in L_1(\Delta)$, then following equality holds:*

$$\begin{aligned}
&\frac{\Psi(c_1^{\rho_1}, d_1^{\rho_2}) + \Psi(c_1^{\rho_1}, d_2^{\rho_2}) + \Psi(c_2^{\rho_1}, d_1^{\rho_2}) + \Psi(c_2^{\rho_1}, d_2^{\rho_2})}{4} + \frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4} \\
&\quad \times \left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}}\right)^{\alpha} \left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}}\right)^{\beta} \Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi \circ \mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi \circ \mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi \circ \mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi \circ \mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) \Bigg] - \Xi \\
&= \frac{\rho_1\rho_2 c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_1}(c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_1})}{4} \\
&\quad \times \Bigg[\int_0^1\int_0^1 \frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2 B_{r_2}^2} r_1^{\rho_1-1}r_2^{\rho_2-1} \frac{\partial^2\Psi}{\partial r_1 \partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right) \mathrm{d}r_2\mathrm{d}r_1 \\
&\quad - \int_0^1\int_0^1 \frac{(1-r_1^{\rho_1})^{\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2 B_{r_2}^2} r_1^{\rho_1-1}r_2^{\rho_2-1} \frac{\partial^2\Psi}{\partial r_1 \partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right) \mathrm{d}r_2\mathrm{d}r_1 \\
&\quad - \int_0^1\int_0^1 \frac{r_1^{\rho_1\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2 B_{r_2}^2} r_1^{\rho_1-1}r_2^{\rho_2-1} \frac{\partial^2\Psi}{\partial r_1 \partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right) \mathrm{d}r_2\mathrm{d}r_1 \\
&\quad + \int_0^1\int_0^1 \frac{(1-r_1^{\rho_1})^{\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2 B_{r_2}^2} r_1^{\rho_1-1}r_2^{\rho_2-1} \frac{\partial^2\Psi}{\partial r_1 \partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right) \mathrm{d}r_2\mathrm{d}r_1 \Bigg],
\end{aligned} \tag{7.47}$$

where

$$\begin{aligned}
\Xi = {} & \frac{\rho_1^{\alpha+1}\Gamma(\alpha+1)}{4}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}}\right)^{\alpha} \Bigg[{}^{\rho_1}I^{\alpha}_{1/c_2+}(\Psi \circ \mu_1)\left(\frac{1}{c_1^{\rho_1}}, d_2^{\rho_2}\right) \\
& + {}^{\rho_1}I^{\alpha}_{1/c_2+}(\Psi \circ \mu_1)\left(\frac{1}{c_1^{\rho_1}}, d_1^{\rho_2}\right) + {}^{\rho_1}I^{\alpha}_{1/c_1-}(\Psi \circ \mu_1)\left(\frac{1}{c_2^{\rho_1}}, d_2^{\rho_2}\right) \\
& + {}^{\rho_1}I^{\alpha}_{1/c_1-}(\Psi \circ \mu_1)\left(\frac{1}{c_2^{\rho_1}}, d_1^{\rho_2}\right) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\rho_2^{\beta+1}\Gamma(\beta+1)}{4}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Bigg[\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_1^{\rho_1},\frac{1}{d_2^{\rho_2}}\right)\\
&+\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\Psi\circ\mu_2)\left(c_2^{\rho_1},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_2}I^{\alpha}_{1/d_2+}(\Psi\circ\mu_2)\left(c_1^{\rho_1},\frac{1}{d_1^{\rho_2}}\right) \\
&+\ {}^{\rho_2}I^{\alpha}_{1/d_2+}(\Psi\circ\mu_2)\left(c_2^{\rho_1},\frac{1}{d_1^{\rho_2}}\right)\Bigg],
\end{aligned}
\tag{7.48}
$$

and $A_{r_1}=r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}$, $B_{r_2}=r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}$. *Also,* $\mu(u^{\rho_1},v^{\rho_2})=(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}})$, $\mu_1(u^{\rho_1},v^{\rho_2})=(\frac{1}{u^{\rho_1}},v^{\rho_2})$ *and* $\mu_2(u^{\rho_1},v^{\rho_2})=(u^{\rho_1},\frac{1}{v^{\rho_2}})$.

Proof. By integration by parts and using the change of variable $u^{\rho_1}=\frac{A_{r_1}}{c_1^{\rho_1}c_2^{\rho_1}}$ and $v^{\rho_2}=\frac{B_{r_2}}{d_1^{\rho_2}d_2^{\rho_2}}$, we find that

$$
\begin{aligned}
I_1 &= \int_0^1\int_0^1 \frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}r_1^{\rho_1-1}r_2^{\rho_2-1}\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\mathrm{d}r_2\mathrm{d}r_1\\
&=\int_0^1\frac{r_2^{\rho_2\beta}}{B_{r_2}^2}r_2^{\rho_2-1}\Bigg\{\frac{r_1^{\rho_1\alpha}}{\rho_1c_1^{\rho_1}c_2^{\rho_1}(c_2^{\rho_1}-c_1^{\rho_1})}\frac{\partial\Psi}{\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\Bigg|_0^1\\
&\quad-\frac{\alpha}{c_1^{\rho_1}c_2^{\rho_1}(c_2^{\rho_1}-c_1^{\rho_1})}\int_0^1 r_1^{\rho_1\alpha-1}\frac{\partial\Psi}{\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\mathrm{d}r_1\Bigg\}\mathrm{d}r_2\\
&=\frac{1}{\rho_1c_1^{\rho_1}c_2^{\rho_1}(c_2^{\rho_1}-c_1^{\rho_1})}\int_0^1\frac{r_2^{\rho_2\beta}}{B_{r_2}^2}r_2^{\rho_2-1}\frac{\partial\Psi}{\partial r_2}\left(c_2^{\rho_1},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\mathrm{d}r_2\\
&\quad-\frac{\alpha}{c_1^{\rho_1}c_2^{\rho_1}(c_2^{\rho_1}-c_1^{\rho_1})}\int_0^1 r_1^{\rho_1\alpha-1}\left\{\int_0^1\frac{r_2^{\rho_2\beta}}{B_{r_2}^2}r_2^{\rho_2-1}\frac{\partial\Psi}{\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\mathrm{d}r_2\right\}\mathrm{d}r_1\\
&=\frac{1}{\rho_1\rho_2c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}\Psi(c_2^{\rho_1},d_2^{\rho_2})\\
&\quad-\frac{\beta}{\rho_1c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}\int_0^1 r_2^{\beta-1}\Psi\left(c_2^{\rho_1},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\mathrm{d}r_2\\
&\quad-\frac{\alpha}{\rho_2c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}\int_0^1 r_1^{\rho_1\alpha-1}\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},d_2^{\rho_2}\right)\mathrm{d}r_1\\
&\quad+\frac{\alpha\beta}{c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}\int_0^1\int_0^1 r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\Psi\left(\frac{a^{\rho_1}b^{\rho_1}}{A_{r_1}},\frac{c^{\rho_2}d^{\rho_2}}{B_{r_2}}\right)\mathrm{d}r_2\mathrm{d}r_1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2} (c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \Bigg[\frac{\Psi(c_2^{\rho_1}, d_2^{\rho_2})}{\rho_1 \rho_2} \\
&\quad - \frac{\rho_2^\beta \Gamma(\beta+1)}{\rho_1} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^\beta \, {}^{\rho_2} I^\alpha_{1/d_2+} (\Psi \circ \mu_2) \left(c_2^{\rho_1}, \frac{1}{d_1^{\rho_2}} \right) \\
&\quad - \frac{\rho_1^\alpha \Gamma(\alpha+1)}{\rho_2} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^\alpha \, {}^{\rho_1} I^\alpha_{1/c_2+} (\Psi \circ \mu_1) \left(\frac{1}{c_1^{\rho_1}}, d_2^{\rho_2} \right) + \rho_1^\alpha \rho_2^\beta \Gamma(\alpha+1) \Gamma(\beta+1) \\
&\quad \times \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^\alpha \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^\beta \, {}^{\rho_1, \rho_2} I^{\alpha,\beta}_{1/c_2+, 1/d_2+} (\Psi \circ \mu) \left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}} \right) \Bigg].
\end{aligned}
\tag{7.49}
$$

Analogously, we have

$$
\begin{aligned}
I_2 &= \int_0^1 \int_0^1 \frac{(1-t^{\rho_1})^\alpha \lambda^{\rho_2 \beta}}{A_{r_1}^2 B_{r_2}^2} r_1^{\rho_1 - 1} r_2^{\rho_2 - 1} \frac{\partial^2 \Psi}{\partial r_1 \partial r_2} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2} d_2^{\rho_2}}{B_{r_2}} \right) \mathrm{d}r_2 \mathrm{d}r_1 \\
&= \frac{1}{c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2} (c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \Bigg[- \frac{\Psi(c_1^{\rho_1}, d_1^{\rho_2})}{\rho_1 \rho_2} \\
&\quad + \frac{\rho_2^\beta \Gamma(\beta+1)}{\rho_1} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^\beta \, {}^{\rho_2} I^\alpha_{1/d_2+} (\Psi \circ \mu_2) \left(c_1^{\rho_1}, \frac{1}{d_1^{\rho_2}} \right) \\
&\quad + \frac{\rho_1^\alpha \Gamma(\alpha+1)}{\rho_2} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^\alpha \, {}^{\rho_1} I^\alpha_{1/c_1-} (\Psi \circ \mu_1) \left(\frac{1}{c_2^{\rho_1}}, d_2^{\rho_2} \right) - \rho_1^\alpha \rho_2^\beta \Gamma(\alpha+1) \Gamma(\beta+1) \\
&\quad \times \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^\alpha \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^\beta \, {}^{\rho_1, \rho_2} I^{\alpha,\beta}_{1/c_1-, 1/d_2+} (\Psi \circ \mu) \left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}} \right) \Bigg],
\end{aligned}
\tag{7.50}
$$

$$
\begin{aligned}
I_3 &= \int_0^1 \int_0^1 \frac{t^{\rho_1 \alpha}(1-\lambda^{\rho_2})^{\beta}}{A_{r_1}^2 B_{r_2}^2} r_1^{\rho_1 - 1} r_2^{\rho_2 - 1} \frac{\partial^2 \Psi}{\partial r_1 \partial r_2} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2} d_2^{\rho_2}}{B_{r_2}} \right) \mathrm{d}r_2 \mathrm{d}r_1 \\
&= \frac{1}{c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2} (c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \Bigg[- \frac{\Psi(c_2^{\rho_1}, d_1^{\rho_2})}{\rho_1 \rho_2} \\
&\quad + \frac{\rho_2^{\beta} \Gamma(\beta+1)}{\rho_1} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^{\beta} \, {}^{\rho_2} I_{1/d_1 -}^{\alpha} (\Psi \circ \mu_2) \left(c_2^{\rho_1}, \frac{1}{d_2^{\rho_2}} \right) \\
&\quad + \frac{\rho_1^{\alpha} \Gamma(\alpha+1)}{\rho_2} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^{\alpha} \, {}^{\rho_1} I_{1/c_2 +}^{\alpha} (\Psi \circ \mu_1) \left(\frac{1}{c_1^{\rho_1}}, d_1^{\rho_2} \right) - \rho_1^{\alpha} \rho_2^{\beta} \Gamma(\alpha+1) \Gamma(\beta+1) \\
&\quad \times \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^{\alpha} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^{\beta} \, {}^{\rho_1, \rho_2} I_{1/c_2 +, 1/d_1 -}^{\alpha, \beta} (\Psi \circ \mu) \left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}} \right) \Bigg],
\end{aligned}
\tag{7.51}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1 \int_0^1 \frac{(1-t^{\rho_1})^{\alpha}(1-\lambda^{\rho_2})^{\beta}}{A_{r_1}^2 B_{r_2}^2} r_1^{\rho_1 - 1} r_2^{\rho_2 - 1} \frac{\partial^2 \Psi}{\partial r_1 \partial r_2} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2} d_2^{\rho_2}}{B_{r_2}} \right) \mathrm{d}r_2 \mathrm{d}r_1 \\
&= \frac{1}{c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2} (c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \Bigg[\frac{\Psi(c_1^{\rho_1}, d_2^{\rho_2})}{\rho_1 \rho_2} \\
&\quad - \frac{\rho_2^{\beta} \Gamma(\beta+1)}{\rho_1} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^{\beta} \, {}^{\rho_2} I_{1/d_1 -}^{\alpha} (\Psi \circ \mu_2) \left(c_1^{\rho_1}, \frac{1}{d_2^{\rho_2}} \right) \\
&\quad - \frac{\rho_1^{\alpha} \Gamma(\alpha+1)}{\rho_2} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^{\alpha} \, {}^{\rho_1} I_{1/c_1 -}^{\alpha} (\Psi \circ \mu_1) \left(\frac{1}{c_2^{\rho_1}}, d_1^{\rho_2} \right) + \rho_1^{\alpha} \rho_2^{\beta} \Gamma(\alpha+1) \Gamma(\beta+1) \\
&\quad \times \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}} \right)^{\alpha} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}} \right)^{\beta} \, {}^{\rho_1, \rho_2} I_{1/c_1 -, 1/d_1 -}^{\alpha, \beta} (\Psi \circ \mu) \left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}} \right) \Bigg].
\end{aligned}
\tag{7.52}
$$

Then from equalities (7.49)−(7.52), we find

$$
\begin{aligned}
&I_1 - I_2 - I_3 + I_4 \\
&= \frac{\Psi(c_1^{\rho_1}, d_1^{\rho_2}) + \Psi(c_1^{\rho_1}, d_2^{\rho_2}) + \Psi(c_2^{\rho_1}, d_1^{\rho_2}) + \Psi(c_2^{\rho_1}, d_2^{\rho_2})}{\rho_1\rho_2 c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2}(c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \\
&\quad - \frac{\rho_2^{\beta}\Gamma(\beta+1)}{\rho_1 c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2}(c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}}\right)^{\beta} \\
&\quad \times \left[{}^{\rho_2}I^{\beta}_{1/d_1-}(\Psi \circ \mu_2)\left(c_1^{\rho_1}, \frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_2}I^{\beta}_{1/d_1-}(\Psi \circ \mu_2)\left(c_2^{\rho_1}, \frac{1}{d_2^{\rho_2}}\right) \right. \\
&\quad \left. + {}^{\rho_2}I^{\alpha}_{1/d_2+}(\Psi \circ \mu_2)\left(c_1^{\rho_1}, \frac{1}{d_1^{\rho_2}}\right) + {}^{\rho_2}I^{\alpha}_{1/d_2+}(\Psi \circ \mu_2)\left(c_2^{\rho_1}, \frac{1}{d_1^{\rho_2}}\right) \right] \\
&\quad - \frac{\rho_1^{\alpha}\Gamma(\alpha+1)}{\rho_2 c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2}(c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}}\right)^{\alpha} \\
&\quad \times \left[{}^{\rho_1}I^{\alpha}_{1/c_2+}(\Psi \circ \mu_1)\left(\frac{1}{c_1^{\rho_1}}, d_2^{\rho_2}\right) + {}^{\rho_1}I^{\alpha}_{1/c_2+}(\Psi \circ \mu_1)\left(\frac{1}{c_1^{\rho_1}}, d_1^{\rho_2}\right) \right. \\
&\quad \left. + {}^{\rho_1}I^{\alpha}_{1/c_1-}(\Psi \circ \mu_1)\left(\frac{1}{c_2^{\rho_1}}, d_2^{\rho_2}\right) + {}^{\rho_1}I^{\alpha}_{1/c_1-}(\Psi \circ \mu_1)\left(\frac{1}{c_2^{\rho_1}}, d_1^{\rho_2}\right) \right] \\
&\quad + \frac{\rho_1^{\alpha}\rho_2^{\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2}(c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})} \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}}\right)^{\alpha} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}}\right)^{\beta} \\
&\quad \times \left[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi \circ \mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi \circ \mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) \right. \\
&\quad \left. + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi \circ \mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi \circ \mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) \right].
\end{aligned}
\tag{7.53}
$$

Multiplying by $\frac{\rho_1\rho_2 c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_2}(c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_2})}{4}$ on both sides of the equality (7.53), we get the required equality (7.47). □

Theorem 7.3.2. *Let* $\Psi : \Delta = [c_1^{\rho_1}, c_2^{\rho_1}] \times [d_1^{\rho_2}, d_2^{\rho_2}] \subseteq (0,\infty) \times (0,\infty) \to \mathbb{R}$ *be a partial differentiable mapping on* Δ *with* $0 < c_1 < c_2$ *and* $0 < d_1 < d_2$. *If* $|\partial^2\Psi/\partial r_1 \partial r_2|$ *is a* *harmonically convex on the co-ordinates on Δ, then following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\\
&\quad\times\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]-\Xi\Bigg|\\
&\le\frac{c_1^{\rho_1}d_1^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}{4c_2^{\rho_1}d_2^{\rho_2}(\alpha+1)(\beta+1)(\alpha+2)(\beta+2)}\left[\vartheta_1\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1^{\rho_1},d_1^{\rho_2})\right|+\vartheta_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1^{\rho_1},d_2^{\rho_2})\right|\right.\\
&\quad\left.+\vartheta_3\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_1^{\rho_2})\right|+\vartheta_4\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_2^{\rho_2})\right|\right],
\end{aligned}\tag{7.54}$$

where

$$\begin{aligned}
\vartheta_1&=(\alpha+1)(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+2;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\
&\quad+(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_2}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+2;\beta+3;1-\frac{d_1^{\rho_1}}{d_2^{\rho_1}}\right)\\
&\quad+(\alpha+1)\ {}^{\rho_1}_{2}G_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,2;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1}_{2}G_1\left(2,2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,2;\beta+3;1-\frac{d_1^{\rho_1}}{d_2^{\rho_1}}\right),
\end{aligned}\tag{7.55}$$

$$\begin{aligned}
\vartheta_2&=(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+2;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\
&\quad+(\alpha+1)(\beta+1)\ {}^{\rho_1}_{2}G_1\left(2,1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,\beta+2;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1}_{2}G_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,2;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\
&\quad+(\alpha+1)\ {}^{\rho_1}_{2}G_1\left(2,1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}^{\rho_2}_{2}G_1\left(2,2;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right),
\end{aligned}\tag{7.56}$$

$$\begin{aligned}\vartheta_3 = &(\alpha+1)\ {}_2^{\rho_1}G_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,\beta+1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\ &+\ {}_2^{\rho_1}G_1\left(2,2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,\beta+1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\ &+(\alpha+1)(\beta+1)\ {}_2^{\rho_1}G_1\left(2,\alpha+2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\ &+(\beta+1)\ {}_2^{\rho_1}G_1\left(2,2;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right),\end{aligned} \tag{7.57}$$

$$\begin{aligned}\vartheta_4 = &\ {}_2^{\rho_1}G_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,\beta+1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\ &+(\alpha+1)\ {}_2^{\rho_1}G_1\left(2,1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,\beta+1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\ &+(\beta+1)\ {}_2^{\rho_1}G_1\left(2,\alpha+1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right)\\ &+(\alpha+1)(\beta+1)\ {}_2^{\rho_1}G_1\left(2,1;\alpha+3;1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)\ {}_2^{\rho_2}G_1\left(2,1;\beta+3;1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right).\end{aligned} \tag{7.58}$$

Here

$$ {}_2^{\rho}G_1(c_1,c_2;x;y)=\frac{1}{{}^{\rho}\gamma(c_2,x-c_2)}\int_0^1(u^\rho)^{c_2-1}(1-u^\rho)^{x-c_2-1}(1-yu^\rho)^{-c_1}u^{\rho-1}\mathrm{d}u,$$

for $x>c_2>0,\ |y|<1,$ *and*

$${}^{\rho}\gamma(c_1,c_2)=\int_0^1(u^\rho)^{c_1-1}(1-u^\rho)^{c_2-1}u^{\rho-1}\mathrm{d}u.$$

Proof. Using Lemma 7.3.1 and triangular inequality, we have

$$\begin{aligned}
&\Bigg|\frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\\
&\quad\times\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&\quad+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&\quad+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]-\Xi\Bigg|\\
&\leq\frac{\rho_1\rho_2c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}{4}\Bigg[\int_0^1\int_0^1\frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}r_1^{\rho_1-1}r_2^{\rho_2-1}\\
&\quad\times\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1-\int_0^1\int_0^1\frac{(1-r_1^{\rho_1})^{\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}r_1^{\rho_1-1}r_2^{\rho_2-1}\\
&\quad\times\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1-\int_0^1\int_0^1\frac{r_1^{\rho_1\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}r_1^{\rho_1-1}r_2^{\rho_2-1}\\
&\quad\times\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1+\int_0^1\int_0^1\frac{(1-r_1^{\rho_1})^{\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}r_1^{\rho_1-1}r_2^{\rho_2-1}\\
&\quad\times\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\right|\mathrm{d}r_2\mathrm{d}r_1\Bigg].
\end{aligned}\tag{7.59}$$

Now since $\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\right|$ co-ordinated harmonically convex, we get

$$\left|\frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\right.$$
$$\times\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\left[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right.$$
$$+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)$$
$$\left.\left.+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]-\Xi\right|$$
$$\leq\frac{\rho_1\rho_2c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}{4}\left[\int_0^1\int_0^1\left[\frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}\right.\right.$$
$$\left.+\frac{(1-r_1^{\rho_1})^{\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{r_1^{\rho_1\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}\right]r_1^{\rho_1-1}r_2^{\rho_2-1}$$
$$\times\left\{r_1^{\rho_1}r_2^{\rho_2}\left|\frac{\partial^2\Psi}{\partial r_1\partial}(c_1^{\rho_1},d_1^{\rho_2})\right|+(1-r_1^{\rho_1})r_2^{\rho_2}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_1^{\rho_2})\right|\right.$$
$$\left.\left.+r_1^{\rho_1}(1-r_2^{\rho_2})\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1^{\rho_1},d_2^{\rho_2})\right|+(1-r_1^{\rho_1})(1-r_2^{\rho_2})\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_2^{\rho_2})\right|\right\}\mathrm{d}r_2\mathrm{d}r_1\right]$$
$$=\frac{\rho_1\rho_2c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}{4}\left[\int_0^1\int_0^1 r_1^{\rho_1}r_2^{\rho_2}\left\{\frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}\right.\right.$$
$$\left.+\frac{r_1^{\rho_1\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}\right\}r_1^{\rho_1-1}r_2^{\rho_2-1}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1^{\rho_1},d_1^{\rho_2})\right|\mathrm{d}r_2\mathrm{d}r_1$$
$$+\int_0^1\int_0^1(1-r_1^{\rho_1})r_2^{\rho_2}\left\{\frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}\right.$$

$$
\begin{aligned}
&+\frac{r_1^{\rho_1\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}\Big\}r_1^{\rho_1-1}r_2^{\rho_2-1}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_1^{\rho_2})\right|\mathrm{d}r_2\mathrm{d}r_1\\
&+\int_0^1\int_0^1 r_1^{\rho_1}(1-r_2^{\rho_2})\Big\{\frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}\\
&+\frac{r_1^{\rho_1\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}\Big\}r_1^{\rho_1-1}r_2^{\rho_2-1}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1^{\rho_1},d_2^{\rho_2})\right|\mathrm{d}r_2\mathrm{d}r_1 \qquad (7.60)\\
&+\int_0^1\int_0^1 (1-r_1^{\rho_1})(1-r_2^{\rho_2})\Big\{\frac{r_1^{\rho_1\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}r_2^{\rho_2\beta}}{A_{r_1}^2B_{r_2}^2}\\
&+\frac{r_1^{\rho_1\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}+\frac{(1-r_1^{\rho_1})^{\alpha}(1-r_2^{\rho_2})^{\beta}}{A_{r_1}^2B_{r_2}^2}\Big\}r_1^{\rho_1-1}r_2^{\rho_2-1}\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_2^{\rho_2})\right|\mathrm{d}r_2\mathrm{d}r_1\Bigg].
\end{aligned}
$$

After some calculations we get the desired result. □

Theorem 7.3.3. *Let $\Psi:\Delta=[c_1^{\rho_1},c_2^{\rho_1}]\times[d_1^{\rho_2},d_2^{\rho_2}]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0<c_1<c_2$ and $0<d_1<d_2$. If $|\partial^2\Psi/\partial r_1\partial r_2|^q$, $q>1$, is a harmonically convex on the co-ordinates on Δ, then following holds:*

$$
\begin{aligned}
&\Bigg|\frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{4}+\frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4}\\
&\times\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]-\Xi\Bigg|\\
&\leq\frac{\rho_1\rho_2c_1^{\rho_1}d_1^{\rho_2}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_2})}{4c_2^{\rho_1}d_2^{\rho_2}[\rho_1\rho_2(p\alpha+1)(p\beta+1)]^{1/p}}\left[\upsilon_1^{1/p}+\upsilon_2^{1/p}+\upsilon_3^{1/p}+\upsilon_4^{1/p}\right]\\
&\Bigg(\frac{\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1^{\rho_1},d_1^{\rho_2})\right|^q+\rho_1\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_1^{\rho_1},d_2^{\rho_2})\right|^q}{(\rho_1+1)(\rho_2+1)}\\
&+\frac{\rho_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_1^{\rho_2})\right|^q+\rho_1\rho_2\left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}(c_2^{\rho_1},d_2^{\rho_2})\right|^q}{(\rho_1+1)(\rho_2+1)}\Bigg)^{1/q},
\end{aligned}
\qquad (7.61)
$$

where

$$\upsilon_1 = {}_2^{\rho_1}G_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right) {}_2^{\rho_2}G_1\left(2p, p\beta+1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right), \quad (7.62)$$

$$\upsilon_2 = {}_2^{\rho_1}G_1\left(2p, 1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_2}}\right) {}_2^{\rho_2}G_1\left(2p, p\beta+1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right), \quad (7.63)$$

$$\upsilon_3 = {}_2^{\rho_1}G_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right) {}_2^{\rho_2}G_1\left(2p, 1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right), \quad (7.64)$$

$$\upsilon_4 = {}_2^{\rho_1}G_1\left(2p, 1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right) {}_2^{\rho_2}G_1\left(2p, 1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right). \quad (7.65)$$

Where ${}_2^{\rho_i}G_1(c_1, c_2; u; v)$ *is defined as in Theorem 7.3.2.*

Proof. Applying the Hölder's inequality for double integrals in (7.59), we get

$$\begin{aligned}
&\Bigg|\frac{\Psi(c_1^{\rho_1}, d_1^{\rho_2}) + \Psi(c_1^{\rho_1}, d_2^{\rho_2}) + \Psi(c_2^{\rho_1}, d_1^{\rho_2}) + \Psi(c_2^{\rho_1}, d_2^{\rho_2})}{4} + \frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4} \\
&\quad\times \left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad+ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad+ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\Bigg] - \Xi\Bigg| \\
&\le \frac{\rho_1\rho_2 c_1^{\rho_1}c_2^{\rho_1}d_1^{\rho_2}d_2^{\rho_1}(c_2^{\rho_1}-c_1^{\rho_1})(d_2^{\rho_2}-d_1^{\rho_1})}{4} \\
&\quad\times\Bigg[\left(\int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha}r_2^{p\rho_2\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/p} \\
&\quad+\left(\int_0^1\int_0^1 \frac{(1-r_1^{\rho_1})^{p\alpha}r_2^{p\rho_2\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/p} \\
&\quad+\left(\int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha}(1-r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}\times r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/p} \\
&\quad+\left(\int_0^1\int_0^1 \frac{(1-r_1^{\rho_1})^{p\alpha}(1-r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1\right)^{1/p}\Bigg] \\
&\quad\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2\Psi}{\partial r_1\partial r_2}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{A_{r_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{B_{r_2}}\right)\right|^q \mathrm{d}r_2\mathrm{d}r_1\right)^{1/q}.
\end{aligned} \quad (7.66)$$

Using co-ordinated harmonically convexity of $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$, we get

$$\begin{aligned}
&\Bigg|\frac{\Psi(c_1^{\rho_1}, d_1^{\rho_2}) + \Psi(c_1^{\rho_1}, d_2^{\rho_2}) + \Psi(c_2^{\rho_1}, d_1^{\rho_2}) + \Psi(c_2^{\rho_1}, d_2^{\rho_2})}{4} + \frac{\rho_1^{\alpha+1}\rho_2^{\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{4} \\
&\quad \times \left(\frac{c_1^{\rho_1} c_2^{\rho_1}}{c_2^{\rho_1} - c_1^{\rho_1}}\right)^{\alpha} \left(\frac{d_1^{\rho_2} d_2^{\rho_2}}{d_2^{\rho_2} - d_1^{\rho_2}}\right)^{\beta} \Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi \circ \mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi \circ \mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi \circ \mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi \circ \mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\Bigg] - \Xi\Bigg| \\
&\le \frac{\rho_1\rho_2 c_1^{\rho_1} c_2^{\rho_1} d_1^{\rho_2} d_2^{\rho_1}(c_2^{\rho_1} - c_1^{\rho_1})(d_2^{\rho_2} - d_1^{\rho_1})}{4} \\
&\quad \times \Bigg[\left(\int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha} r_2^{p\rho_2\beta}}{A_{r_1}^{2p} B_{r_2}^{2p}} r_1^{p(\rho_1-1)} r_2^{p(\rho_2-1)} \mathrm{d}r_2 \mathrm{d}r_1\right)^{1/p} \\
&\quad + \left(\int_0^1\int_0^1 \frac{(1 - r_1^{\rho_1})^{p\alpha} r_2^{p\rho_2\beta}}{A_{r_1}^{2p} B_{r_2}^{2p}} r_1^{p(\rho_1-1)} r_2^{p(\rho_2-1)} \mathrm{d}r_2 \mathrm{d}r_1\right)^{1/p} \\
&\quad + \left(\int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha}(1 - r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p} B_{r_2}^{2p}} r_1^{p(\rho_1-1)} r_2^{p(\rho_2-1)} \mathrm{d}r_2 \mathrm{d}r_1\right)^{1/p} \\
&\quad + \left(\int_0^1\int_0^1 \frac{(1 - r_1^{\rho_1})^{p\alpha}(1 - r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p} B_{r_2}^{2p}} r_1^{p(\rho_1-1)} r_2^{p(\rho_2-1)} \mathrm{d}r_2 \mathrm{d}r_1\right)^{1/p}\Bigg] \\
&\quad \times \Bigg(\int_0^1\int_0^1 \Bigg\{ r_1^{\rho_1} r_1^{\rho_2} \left|\frac{\partial^2 \Psi}{\partial r_1 \partial r_2}(c_1^{\rho_1}, d_1^{\rho_2})\right|^q + (1 - r_1^{\rho_1}) r_2^{\rho_2} \left|\frac{\partial^2 \Psi}{\partial r_1 \partial r_2}(c_2^{\rho_1}, d_1^{\rho_2})\right|^q \\
&\quad + r_1^{\rho_1}(1 - r_2^{\rho_2}) \left|\frac{\partial^2 f}{\partial r_1 \partial r_2}(c_1^{\rho_1}, d_2^{\rho_2})\right|^q \\
&\quad + (1 - r_1^{\rho_1})(1 - r_2^{\rho_2}) \left|\frac{\partial^2 \Psi}{\partial r_1 \partial r_2}(c_2^{\rho_1}, d_2^{\rho_2})\right|^q \Bigg\} \mathrm{d}r_2 \mathrm{d}r_1\Bigg)^{1/q}.
\end{aligned} \tag{7.67}$$

Here

$$\int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha}r_2^{p\rho_2\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1$$
$$\leq \int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha}r_2^{p\rho_2\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{\rho_1-1}r_2^{\rho_2-1}\mathrm{d}r_2\mathrm{d}r_1$$
$$= \frac{c_2^{-2p}d_2^{-2p}}{\rho_1\rho_2(p\alpha+1)(p\beta+1)}\ {}_2^{\rho_1}G_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)$$
$$\times\ {}_2^{\rho_2}G_1\left(2p, p\beta+1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right),$$

$$\int_0^1 \frac{(1-r_1^{\rho_1})^{p\alpha}r_2^{p\rho_2\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1$$
$$\leq \int_0^1\int_0^1 \frac{(1-r_1^{\rho_1})^{p\alpha}r_2^{p\rho_2\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{\rho_1-1}r_2^{\rho_2-1}\mathrm{d}r_2\mathrm{d}r_1$$
$$= \frac{c_2^{-2p}d_2^{-2p}}{\rho_1\rho_2(p\alpha+1)(p\beta+1)} \times\ {}_2^{\rho_1}G_1\left(2p, 1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)$$
$$\times\ {}_2^{\rho_2}G_1\left(2p, p\beta+1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right),$$

$$\int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha}(1-r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1$$
$$\leq \int_0^1\int_0^1 \frac{r_1^{p\rho_1\alpha}(1-r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{\rho_1-1}r_2^{\rho_2-1}\mathrm{d}r_2\mathrm{d}r_1$$
$$= \frac{c_2^{-2p}d_2^{-2p}}{\rho_1\rho_2(p\alpha+1)(p\beta+1)} \times\ {}_2^{\rho_1}G_1\left(2p, p\alpha+1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)$$
$$\times\ {}_2^{\rho_2}G_1\left(2p, 1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right),$$

$$\int_0^1\int_0^1 \frac{(1-r_1^{\rho_1})^{p\alpha}(1-r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{p(\rho_1-1)}r_2^{p(\rho_2-1)}\mathrm{d}r_2\mathrm{d}r_1$$
$$\leq \int_0^1\int_0^1 \frac{(1-r_1^{\rho_1})^{p\alpha}(1-r_2^{\rho_2})^{p\beta}}{A_{r_1}^{2p}B_{r_2}^{2p}} r_1^{\rho_1-1}r_2^{\rho_2-1}\mathrm{d}r_2\mathrm{d}r_1$$
$$= \frac{c_2^{-2p}d^{-2p}}{\rho_1\rho_2(p\alpha+1)(p\beta+1)} \times\ {}_2^{\rho_1}G_1\left(2p, 1; p\alpha+2; 1-\frac{c_1^{\rho_1}}{c_2^{\rho_1}}\right)$$
$$\times\ {}_2^{\rho_2}G_1\left(2p, 1; p\beta+2; 1-\frac{d_1^{\rho_2}}{d_2^{\rho_2}}\right),$$

$$\int_0^1\int_0^1 r_1^{\rho_1} r_2^{\rho_2} \mathrm{d}r_2 \mathrm{d}r_1 = \frac{1}{(\rho_1+1)(\rho_2+1)},$$
$$\int_0^1\int_0^1 (1-r_1^{\rho_1}) r_2^{\rho_2} \mathrm{d}r_2 \mathrm{d}r_1 = \frac{\rho_1}{(\rho_1+1)(\rho_2+1)},$$
$$\int_0^1\int_0^1 r_1^{\rho_1}(1-r_2^{\rho_2}) \mathrm{d}r_2 \mathrm{d}r_1 = \frac{\rho_2}{(\rho_1+1)(\rho_2+1)},$$
$$\int_0^1\int_0^1 (1-r_1^{\rho_1})(1-r_2^{\rho_2}) \mathrm{d}r_2 \mathrm{d}r_1 = \frac{\rho_1\rho_2}{(\rho_1+1)(\rho_2+1)}.$$

Putting the values of above integrals in (7.67) and after some simplification, we get the required result (7.61). □

Remark 38. *By taking $\rho_1 = \rho_2 = 1$ in Theorem 7.3.1, Lemma 7.3.1, Theorem 7.3.2 and in Theorem 7.3.3, we get similar results for co-ordinated harmonically convex functions via Riemann−Liouville fractional integrals.*

7.4 Hermite-Hadamard-Fejér type inequalities via Katugampola fractional integrals

In this section, we find some Hermite-Hadamard-Fejér type inequalities for co-ordinated harmonically convex functions via Katugampola fractional integrals.

Definition 7.4.1 ([59]). *A function $\phi : [c_1, c_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $2c_1c_2/(c_1+c_2)$, if*

$$\phi(u) = \phi\left(\frac{1}{\frac{1}{c_1}+\frac{1}{c_2}-\frac{1}{u}}\right)$$

holds, for all $u \in [c_1, c_2]$.

For the results on co-ordinates we need some results on one dimension.

Lemma 7.4.2. *Let $\rho > 0$. If $\phi : [c_1^\rho, c_2^\rho] \subseteq (0,\infty) \to \mathbb{R}$ is integrable and harmonically symmetric with respect to $2c_1^\rho c_2^\rho/(c_1^\rho + c_2^\rho)$, then*

$$\begin{aligned} {}^\rho I^\alpha_{1/c_2+}(\phi\circ\mu)(1/c_1^\rho) &= \frac{1}{2}\left[{}^\rho I^\alpha_{1/c_2+}(\phi\circ\mu)(1/c_1^\rho) + {}^\rho I^\alpha_{1/c_1-}(\phi\circ\mu)(1/c_2^\rho)\right] \\ &= {}^\rho I^\alpha_{1/c_1-}(\phi\circ\mu)(1/c_2^\rho), \end{aligned} \tag{7.68}$$

with $\alpha > 0$ and $\mu(u^\rho) = 1/u^\rho$.

Proof. Since ϕ is harmonically symmetric with respect to $2c_1^\rho c_2^\rho/(c_1^\rho + c_2^\rho)$, then by definition we have $\phi(\frac{1}{u^\rho}) = \phi\left(\frac{1}{\frac{1}{c_1^\rho}+\frac{1}{c_2^\rho}-u^\rho}\right)$, for all $u^\rho \in \left[\frac{1}{c_2^\rho}, \frac{1}{c_1^\rho}\right]$. In the following integral, by setting $x^\rho = \frac{1}{c_1^\rho} + \frac{1}{c_2^\rho} - u^\rho$ gives

$$
\begin{aligned}
{}^\rho I^\alpha_{1/c_2+}(\phi \circ \mu)(1/c_1^\rho) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\frac{1}{c_2}}^{\frac{1}{c_1}} \left(\frac{1}{c_1^\rho} - x^\rho\right)^{\alpha-1} x^{\rho-1} c\left(\frac{1}{x^\rho}\right) \mathrm{d}x \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\frac{1}{c_2}}^{\frac{1}{c_1}} \left(u^\rho - \frac{1}{c_2^\rho}\right)^{\alpha-1} u^{\rho-1} \phi\left(\frac{1}{\frac{1}{c_1^\rho}+\frac{1}{c_2^\rho}-u^\rho}\right) \mathrm{d}u \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\frac{1}{c_2}}^{\frac{1}{c_1}} \left(u^\rho - \frac{1}{c_2^\rho}\right)^{\alpha-1} u^{\rho-1} \phi\left(\frac{1}{u^\rho}\right) \mathrm{d}u = {}^\rho I^\alpha_{1/c_1-}(\phi \circ \mu)(1/c_2^\rho).
\end{aligned}
\tag{7.69}
$$

This completes the proof. □

Remark 39. *In Lemma* 7.4.2 *as* $\rho \longmapsto 0$*, we have Lemma* 2 *in* [47].

Theorem 7.4.1. *Let* $\rho > 0$*. Let* $\Psi : [c_1^\rho, c_2^\rho] \subseteq (0,\infty) \to \mathbb{R}$ *be a harmonically convex with* $c_1 < c_2$ *and* $\Psi \in L_1[c_1, c_2]$*. If* $\phi : [c_1^\rho, c_2^\rho] \subseteq (0,\infty) \to \mathbb{R}$ *is nonnegative and harmonically symmetric with respect to* $2c_1^\rho c_2^\rho/(c_1^\rho + c_2^\rho)$*, then the following inequalities hold:*

$$
\begin{aligned}
&\Psi\left(\frac{2c_1^\rho c_2^\rho}{c_1^\rho + c_2^\rho}\right)\left[{}^\rho I^\alpha_{1/c_1-}(\phi \circ \mu)(1/c_2^\rho) +^\rho I^\alpha_{1/c_2+}(\phi \circ \mu)(1/c_1^\rho)\right] \\
&\le \left[{}^\rho I^\alpha_{1/c_1-}(\Psi\phi \circ \mu)(1/c_2^\rho) +^\rho I^\alpha_{1/c_2+}(\Psi\phi \circ \mu)(1/c_1^\rho)\right] \\
&\le \frac{\Psi(c_1^\rho) + \Psi(c_2^\rho)}{2}\left[{}^\rho I^\alpha_{1/c_1-}(\phi \circ \mu)(1/c_2^\rho) +^\rho I^\alpha_{1/c_2+}(\phi \circ \mu)(1/c_1^\rho)\right],
\end{aligned}
\tag{7.70}
$$

with $\alpha > 0$ *and* $\mu(u^\rho) = 1/u^\rho$*.*

Proof. Since Ψ is harmonically convex on $[c_1^\rho, c_2^\rho]$, we have for all $r \in [0,1]$

$$
\begin{aligned}
\Psi\left(\frac{2c_1^\rho c_2^\rho}{c_1^\rho + c_2^\rho}\right) &= \Psi\left(\frac{2c_1^\rho c_2^\rho}{(r^\rho c_1^\rho + (1-r^\rho)c_2^\rho) + (r^\rho c_2^\rho + (1-r^\rho)c_1^\rho)}\right) \\
&\le \frac{\Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_1^\rho + (1-r^\rho)c_2^\rho}\right) + \Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho + (1-r^\rho)c_1^\rho}\right)}{2}.
\end{aligned}
\tag{7.71}
$$

Multiplying (7.71) by $r^{\alpha\rho-1}\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)$ on both sides and integrate with respect to $[0,1]$, we get

$$\begin{aligned}
&2\Psi\left(\frac{2c_1^\rho c2^\rho}{c_1^\rho+c_2^\rho}\right)\int_0^1 r^{\alpha\rho-1}\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\mathrm{d}r\\
&\leq \int_0^1 r^{\alpha\rho-1}\Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_1^\rho+(1-r^\rho)c_2^\rho}\right)\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\mathrm{d}r \qquad (7.72)\\
&\quad+\int_0^1 r^{\alpha\rho-1}\Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\mathrm{d}r.
\end{aligned}$$

Since ϕ is harmonically symmetric with respect to $2c_1^\rho c_2^\rho/(c_1^\rho+c_2^\rho)$. By setting $u^\rho = \frac{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}{c_1^\rho c_2^\rho}$, we get

$$\begin{aligned}
&2\left(\frac{c_1^\rho c_2^\rho}{c_2^\rho-c_1^\rho}\right)^\alpha\Psi\left(\frac{2c_1^\rho c_2^\rho}{c_1^\rho+c_2^\rho}\right)\int_{1/c_2}^{1/c_1}\left(u^p-\frac{1}{c_2^\rho}\right)^{\alpha-1}\phi\left(\frac{1}{u^\rho}\right)\mathrm{d}u\\
&\leq\left(\frac{c_1^\rho c_2^\rho}{c_2^\rho-c_1^\rho}\right)^\alpha\left[\int_{1/c_2}^{1/c_1}\left(u^p-\frac{1}{c_2^\rho}\right)^{\alpha-1}\Psi\left(\frac{1}{\frac{1}{c_1^\rho}+\frac{1}{c_2^\rho}-u^\rho}\right)\phi\left(\frac{1}{u^\rho}\right)\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\left(u^p-\frac{1}{c_2^\rho}\right)^{\alpha-1}\Psi\left(\frac{1}{u^\rho}\right)\phi\left(\frac{1}{u^\rho}\right)\mathrm{d}u\right] \qquad (7.73)\\
&=\left(\frac{c_1^\rho c_2^\rho}{c_2^\rho-c_1^\rho}\right)^\alpha\left[\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1^\rho}-u^p\right)^{\alpha-1}\Psi\left(\frac{1}{u^\rho}\right)\phi\left(\frac{1}{\frac{1}{c_1^\rho}+\frac{1}{c_2^\rho}-u^\rho}\right)\mathrm{d}u\right.\\
&\quad\left.+\int_{1/c_2}^{1/c_1}\left(u^p-\frac{1}{c_2^\rho}\right)^{\alpha-1}\Psi\left(\frac{1}{u^\rho}\right)\phi\left(\frac{1}{u^\rho}\right)\mathrm{d}u\right].
\end{aligned}$$

Therefore, by Lemma 7.4.2, we have

$$\begin{aligned}
&\left(\frac{c_1^\rho c_2^\rho}{c_2^\rho-c_1^\rho}\right)^\alpha\rho^{\alpha-1}\Gamma(\alpha)\Psi\left(\frac{2c_1^\rho c_2^\rho}{c_1^\rho+c_2^\rho}\right)\left[{}^\rho I^\alpha_{1/c_1-}(\phi\circ\mu)(1/c_2^\rho)+^\rho I^\alpha_{1/c_2+}(\phi\circ\mu)(1/c_1^\rho)\right]\\
&\leq\left(\frac{c_1^\rho c_2^\rho}{c_2^\rho-c_1^\rho}\right)^\alpha\rho^{\alpha-1}\Gamma(\alpha)\left[{}^\rho I^\alpha_{1/c_1-}(\Psi\phi\circ\mu)(1/c_2^\rho)+^\rho I^\alpha_{1/c_2+}(\Psi\phi\circ\mu)(1/c_1^\rho)\right].
\end{aligned} \qquad (7.74)$$

This completes the first inequality.

For second inequality, we first note that if Ψ is harmonically convex function, then we have

$$\Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_1^\rho+(1-r^\rho)c_2^\rho}\right)+\Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\leq\Psi(c_1^\rho)+\Psi(c_2^\rho). \qquad (7.75)$$

Multiplying (7.75) by $r^{\alpha\rho-1}\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)$ on both sides and integrate with respect to $r \in [0,1]$, we get

$$\begin{aligned}
&\int_0^1 r^{\alpha\rho-1}\Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_1^\rho+(1-r^\rho)c_2^\rho}\right)\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\mathrm{d}r \\
&\quad+\int_0^1 r^{\alpha\rho-1}\Psi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\mathrm{d}r \\
&\leq (\Psi(c_1^\rho)+\Psi(c_2^\rho))\int_0^1 r^{\alpha\rho-1}\phi\left(\frac{c_1^\rho c_2^\rho}{r^\rho c_2^\rho+(1-r^\rho)c_1^\rho}\right)\mathrm{d}r,
\end{aligned} \tag{7.76}$$

i.e.,

$$\begin{aligned}
&\left(\frac{c_1^\rho c_2^\rho}{c_2^\rho-c_1^\rho}\right)^\alpha \rho^{\alpha-1}\Gamma(\alpha)\left[{}^\rho I^\alpha_{1/c_1-}(\Psi\phi\circ\mu)(1/c_2^\rho)+{}^\rho I^\alpha_{1/c_2+}(\Psi\phi\circ\mu)(1/c_1^\rho)\right] \\
&\leq \left(\frac{c_1^\rho c_2^\rho}{c_2^\rho-c_1^\rho}\right)^\alpha \rho^{\alpha-1}\Gamma(\alpha)\frac{\Psi(c_1^\rho)+\Psi(c_2^\rho)}{2}\left[{}^\rho I^\alpha_{1/c_1-}(\phi\circ\mu)(1/c_2^\rho)+{}^\rho I^\alpha_{1/c_2+}(\phi\circ\mu)(1/c_1^\rho)\right].
\end{aligned} \tag{7.77}$$

This completes the proof. □

Remark 40. *In Theorem* 7.4.1*,*

1. *If* $\rho \to 1$*, we get Theorem* 5 *in* [47].
2. *If* $\rho \to 1$ *and* $\alpha = 1$*, we find Theorem* 8 *in* [19].

Now we are able to give result on co-ordinates.

Theorem 7.4.2. *Let* $\alpha, \beta > 0$ *and* $\rho_1, \rho_2 > 0$*. Let* $\Psi : \Delta = [c_1^{\rho_1}, c_2^{\rho_1}] \times [d_1^{\rho_2}, d_2^{\rho_2}] \subseteq (0,\infty)\times(0,\infty) \to \mathbb{R}$ *be a co-ordinated harmonically convex on* Δ*, with* $0 < c_1 < c_2$*,* $0 < d_1 < d_2$*. If* $\phi : \Delta \to \mathbb{R}$ *is nonnegative and harmonically symmetric with respect to* $\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}$ *on* Δ*. Then*

$$\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\Bigg]
\end{aligned}$$

$$\leq \frac{1}{4}\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)$$
$$+ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]$$
$$\leq \frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{4}$$
$$\times\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)$$
$$+ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg] \tag{7.78}$$

holds, where $\mu(u^{\rho_1},v^{\rho_2}) = \left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)$.

Proof. Since Ψ is co-ordinated harmonically convex on Δ, we have

$$\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\\
&\leq \frac{1}{4}\Bigg[\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\Bigg].
\end{aligned} \tag{7.79}$$

Multiplying (7.79) by $r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\phi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)$ on both sides and then integrating with respect to (r_1, r_2) over $[0,1]\times[0,1]$, we get

$$
\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\int_0^1\int_0^1 r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\\
&\quad\times\phi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\mathrm{d}r_1\mathrm{d}r_2\\
&\le\frac{1}{4}\Bigg[\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad\times\phi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\\
&\quad+\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\quad\times\phi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\\
&\quad+\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad\times\phi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\\
&\quad+\int_0^1\int_0^1\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\quad\times\phi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}, \frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\mathrm{d}r_1\mathrm{d}r_2\Bigg].
\end{aligned}
\tag{7.80}
$$

By change of variables $u^{\rho_1} = \frac{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}}{c_1^{\rho_1}c_2^{\rho_1}}$ and $v^{\rho_2} = \frac{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}{d_1^{\rho_2}d_2^{\rho_2}}$ and using the symmetric property of ϕ, we find

$$
\begin{aligned}
&\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\times\\
&\quad\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{c^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\\
&\le\frac{1}{4}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\left[\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\right.\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{\frac{1}{c_1^{\rho_1}}+\frac{1}{c_2^{\rho_1}}-u^{\rho_1}}, \frac{1}{\frac{1}{d_1^{\rho_2}}+\frac{2}{d_2^{\rho_2}}-v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\times c^{\rho_1-1}d^{\rho_2-1}\phi\left(\frac{1}{\frac{1}{c_1^{\rho_1}}+\frac{1}{c_2^{\rho_1}}-u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\\
&+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{\frac{1}{d_1^{\rho_2}}+\frac{2}{d_1^{\rho_2}}-v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\\
&+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\Bigg]\\
=&\frac{1}{4}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Bigg[\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{\frac{1}{c_1^{\rho_1}}+\frac{1}{c_2^{\rho_1}}-u^{\rho_1}},\frac{1}{\frac{1}{d_1^{\rho_2}}+\frac{2}{d_2^{\rho_2}}-v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\\
&+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{\frac{1}{c_1^{\rho_1}}+\frac{1}{c_2^{\rho_1}}-u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)dudv\\
&+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{\frac{1}{d_1^{\rho_2}}+\frac{2}{d_2^{\rho_2}}-v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\\
&+\int_{1/d_2}^{1/d_1}\int_{1/c_2}^{1/c_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}u\mathrm{d}v\Bigg].
\end{aligned}
\tag{7.81}
$$

Thus, we get

$$\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{\rho_1^{1-\alpha}\rho_2^{1-\beta}}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\\
&\quad\times\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]\\
&\le\frac{\Gamma(\alpha)\Gamma(\beta)}{4\rho_1^{1-\alpha}\rho_2^{1-\beta}}\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{c_2^{\rho_1}-c_1^{\rho_1}}\right)^{\alpha}\left(\frac{d_1^{\rho_2}d_2^{\rho_2}}{d_2^{\rho_2}-d_1^{\rho_2}}\right)^{\beta}\\
&\quad\times\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right).\Bigg]
\end{aligned}\tag{7.82}$$

This completes the first inequality of (7.78). For the second inequality of (7.78), we consider

$$\begin{aligned}
&\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_1^{\rho_1}+(1-r_1^{\rho_1})c_2^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_1^{\rho_2}+(1-r_2^{\rho_2})d_2^{\rho_2}}\right)\\
&\quad+\Psi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)\\
&\le\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2}).
\end{aligned}\tag{7.83}$$

Thus by multiplying (7.83) by $r_1^{\rho_1\alpha-1}r_2^{\rho_2\beta-1}\phi\left(\frac{c_1^{\rho_1}c_2^{\rho_1}}{r_1^{\rho_1}c_2^{\rho_1}+(1-r_1^{\rho_1})c_1^{\rho_1}},\frac{d_1^{\rho_2}d_2^{\rho_2}}{r_2^{\rho_2}d_2^{\rho_2}+(1-r_2^{\rho_2})d_1^{\rho_2}}\right)$ and then integrating with respect to (r_1, r_2) over $[0,1]\times[0,1]$, we get the second inequality of (7.78). Hence the proof is completed. □

Theorem 7.4.3. *Let $\alpha,\beta>0$ and $\rho_1,\rho_2>0$. Let $\Psi:\Delta=[c_1^{\rho_1},c_2^{\rho_1}]\times[d_1^{\rho_2},d_2^{\rho_2}]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a co-ordinated harmonically convex on Δ, with $0<c_1<c_2$, $0<d_1<d_2$ and $\Psi\in L_1[\Delta]$. If $\phi:\Delta\to\mathbb{R}$ is nonnegative and harmonically symmetric*

with respect to $\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}$ *on* Δ. *Then the following inequalities hold:*

$$
\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\Bigg] \\
&\leq {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{u_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}}, \frac{2d_1^{\rho_1}d_2^{\rho_1}}{d_1^{\rho_1}+d_2^{\rho_1}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\right] \\
&\leq 2\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\Bigg] \\
&\leq {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}}, d_1^{\rho_2}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}}, d_2^{\rho_2}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}}, \frac{1}{d_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}}, d_1^{\rho_2}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}}, \frac{1}{d_1^{\rho_2}}\right)\right]
\end{aligned}
\tag{7.84}
$$

$$
\begin{aligned}
&+ {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},d_2^{\rho_2}\right) {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(c_1^{\rho_1},\frac{1}{d_2^{\rho_2}}\right) {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(c_2^{\rho_1},\frac{1}{d_2^{\rho_2}}\right) {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(c_1^{\rho_1},\frac{1}{d_1^{\rho_2}}\right) {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(c_2^{\rho_1},\frac{1}{d_1^{\rho_2}}\right) {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
\le{}& \frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_1^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{4}\\
&\times\left[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right.\\
&\left.+ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right].
\end{aligned}
$$

Where $\mu(u^{\rho_1},v^{\rho_2})=\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)$, $\mu_1(u^{\rho_1},v^{\rho_2})=\left(\frac{1}{u^{\rho_1}},v^{\rho_2}\right)$ *and* $\mu_2(u^{\rho_1},v^{\rho_2})=\left(u^{\rho_1},\frac{1}{v^{\rho_2}}\right)$, *respectively.*

Proof. Since Ψ is co-ordinated harmonically convex on Δ, then we have $\Psi_{1/u^{\rho_1}}:[d_1^{\rho_2},d_2^{\rho_2}]\to\mathbb{R}$, $\Psi_{1/u^{\rho_1}}(v^{\rho_2})=\Psi(\frac{1}{u^{\rho_1}},v^{\rho_2})$, is harmonically convex on $[d_1^{\rho_2},d_2^{\rho_2}]$ for all $u^{\rho_1}\in\left[\frac{1}{c_2^{\rho_1}},\frac{1}{c_1^{\rho_1}}\right]$. Then from inequality (7.70), we have

$$
\begin{aligned}
&\frac{\rho_2^{1-\beta}}{\Gamma(\beta)}\Psi\left(\frac{1}{u^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\left[\int_{1/d_2}^{1/d_1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\right.\\
&\quad\left.+\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\right]\\
&\le\frac{\rho_2^{1-\beta}}{\Gamma(\beta)}\left[\int_{1/d_2}^{1/d_1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}v^{\rho_2-1}\Psi\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\right.\\
&\quad\left.+\int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}v^{\rho_2-1}\Psi\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\right]
\end{aligned}
$$

$$\leq \frac{\Psi\left(\frac{1}{u^{\rho_1}}, d_1^{\rho_2}\right) + \Psi\left(\frac{1}{u^{\rho_1}}, d_2^{\rho_2}\right)}{2}\Bigg[\int_{1/d_2}^{1/d_1}\left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}}\right)^{\beta-1} v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v$$
$$+ \int_{1/d_2}^{1/d_1}\left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2}\right)^{\beta-1} v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\Bigg]. \tag{7.85}$$

Multiplying both sides of (7.85) by $\frac{u^{\rho_1-1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}}{\rho_1^{\alpha-1}\Gamma(\alpha)}$ and $\frac{u^{\rho_1-1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}}{\rho_1^{\alpha-1}\Gamma(\alpha)}$, and integrating with respect to u over $\left[\frac{1}{c_2}, \frac{1}{c_1}\right]$, respectively, we get

$$\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}}\right)^{\beta-1} u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)$$
$$\times \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u + \int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2}\right)^{\beta-1} u^{\rho_1-1}v^{\rho_2-1}$$
$$\times \Psi\left(\frac{1}{u^{\rho_1}}, \frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg]$$
$$\leq \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}}\right)^{\beta-1} u^{\rho_1-1}v^{\rho_2-1}$$
$$\times \Psi\left(\frac{1}{u^{\rho_1}}, \frac{1}{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u + \int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2}\right)^{\beta-1}$$
$$\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg]$$
$$\leq \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}}\right)^{\beta-1} u^{\rho_1-1}v^{\rho_2-1}$$
$$\times \Psi\left(\frac{1}{u^{\rho_1}}, d_1^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u + \int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2}\right)^{\beta-1}$$
$$\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}}, d_1^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u + \int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}$$
$$\times \left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}}\right)^{\beta-1} u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}}, d_2^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u$$
$$+ \int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2}\right)^{\beta-1} u^{\rho_1-1}v^{\rho_2-1}$$
$$\times \Psi\left(\frac{1}{u^{\rho_1}}, d_2^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg], \tag{7.86}$$

and

$$
\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\\
&\quad\times\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\bigg]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\bigg]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},d_1^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},d_1^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\\
&\quad\times\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},d_2^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\\
&\quad+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},d_2^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\bigg].
\end{aligned}
\tag{7.87}
$$

Using similar arguments for the mapping $\Psi_{\frac{1}{v^{\rho_2}}}:[c_1^{\rho_1},c_2^{\rho_1}]\to\mathbb{R}$, $\Psi_{\frac{1}{v^{\rho_2}}}(u^{\rho_1})=\Psi(u^{\rho_1},\frac{1}{v^{\rho_2}})$, we have

$$
\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\\
&\quad\times\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(c_1^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(c_1^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\\
&\quad\times\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(c_2^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\\
&\quad+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(c_2^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg],
\end{aligned}
\tag{7.88}
$$

and

$$
\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}y^{\rho_2-1}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\\
&\quad\times\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)dvdu\Bigg]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(c_1^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(c_1^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\\
&\quad\times\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\Psi\left(c_2^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\\
&\quad+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(c_2^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg].
\end{aligned}
\tag{7.89}
$$

By adding the inequalities (7.86)−(7.89), we get

$$
\begin{aligned}
&{}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{u_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{2d_1^{\rho_1}d_2^{\rho_1}}{d_1^{\rho_1}+d_2^{\rho_1}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\le 2\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\Psi\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\Psi\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\Psi\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\Psi\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]\\
&\le\ {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},d_1^{\rho_2}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},d_2^{\rho_2}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},d_1^{\rho_2}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},d_2^{\rho_2}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(c_1^{\rho_1},\frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(c_2^{\rho_1},\frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(c_1^{\rho_1},\frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(c_2^{\rho_1},\frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right].
\end{aligned}
$$

This completes the second and third inequality of (7.84).

Now, Using the first inequality of (7.70), we find

$$
\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\left[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\right.\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times u^{\rho_1-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\right]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\\
&\quad+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg].
\end{aligned}
\tag{7.90}
$$

$$
\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\left[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}\right.\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\quad\left.\times u^{\rho_1-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\right]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\\
&\quad+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg].
\end{aligned}
\tag{7.91}
$$

Adding (7.90) and (7.91) and using the fact that ϕ is symmetric, we get

$$\begin{aligned}
&\Psi\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\Bigg[\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_1-,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_1-}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\\
&\quad+\ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/c_2+,1/d_2+}(\phi\circ\mu)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\Bigg]\\
&\leq\ {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{u_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_1-}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{2d_1^{\rho_2}d_2^{\rho_2}}{d_1^{\rho_2}+d_2^{\rho_2}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_1-}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_1}I^{\alpha}_{1/c_2+}\left[(\Psi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{2d_1^{\rho_1}d_2^{\rho_1}}{d_1^{\rho_1}+d_2^{\rho_1}}\right)\ {}^{\rho_2}I^{\beta}_{1/d_2+}(\phi\circ\mu_2)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_1-}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_2^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_1-}(\phi\circ\mu_1)\left(\frac{1}{c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right]\\
&\quad+\ {}^{\rho_2}I^{\beta}_{1/d_2+}\left[(\Psi\circ\mu_2)\left(\frac{2c_1^{\rho_1}c_2^{\rho_1}}{c_1^{\rho_1}+c_2^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\ {}^{\rho_1}I^{\alpha}_{1/c_2+}(\phi\circ\mu_1)\left(\frac{1}{c_1^{\rho_1}},\frac{1}{d_1^{\rho_2}}\right)\right].
\end{aligned}$$

This completes the first inequality of (7.84). Now, to achieve the last inequality of (7.84), applying the second inequality of (7.70) as:

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(u^{\rho_1}-\frac{1}{c_2^{\rho_1}}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\\
&\quad\times\Psi\left(\frac{1}{u^{\rho_1}},d_2^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}\\
&\quad\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(\frac{1}{u^{\rho_1}},d_2^{\rho_2}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\Bigg]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Psi(c_1^{\rho_1},d_2^{\rho_2})+\Psi(c_2^{\rho_1},d_2^{\rho_2})}{2}
\end{aligned}$$

$$\times \left[\int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(\frac{1}{c_1^{\rho_1}} - u^{\rho_1} \right)^{\alpha-1} \left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right.$$

$$\left. + \int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}} \right)^{\alpha-1} \left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \times \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right], \tag{7.92}$$

$$\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)} \left[\int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}} \right)^{\alpha-1} \left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \right.$$

$$\times \Psi\left(\frac{1}{u^{\rho_1}}, d_1^{\rho_2} \right) \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u + \int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(\frac{1}{c_1^{\rho_1}} - u^{\rho_1} \right)^{\alpha-1} \left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2} \right)^{\beta-1}$$

$$\left. \times u^{\rho_1-1} v^{\rho_2-1} \Psi\left(\frac{1}{u^{\rho_1}}, d_1^{\rho_2} \right) \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right]$$

$$\leq \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Psi(c_1^{\rho_1}, d_1^{\rho_2}) + \Psi(c_2^{\rho_1}, d_1^{\rho_2})}{2}$$

$$\times \left[\int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(\frac{1}{c_1^{\rho_1}} - u^{\rho_1} \right)^{\alpha-1} \left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right.$$

$$\left. + \int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}} \right)^{\alpha-1} \left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \times \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right], \tag{7.93}$$

$$\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)} \left[\int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}} \right)^{\alpha-1} \left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \right.$$

$$\times \Psi\left(c_2^{\rho_1}, \frac{1}{v^{\rho_2}} \right) \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u + \int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}} \right)^{\alpha-1} \left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2} \right)^{\beta-1}$$

$$\left. \times u^{\rho_1-1} v^{\rho_2-1} \Psi\left(c_2^{\rho_1}, \frac{1}{v^{\rho_2}} \right) \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right]$$

$$\leq \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Psi(c_2^{\rho_1}, d_1^{\rho_2}) + \Psi(c_2^{\rho_1}, d_2^{\rho_2})}{2}$$

$$\times \left[\int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}} \right)^{\alpha-1} \left(v^{\rho_2} - \frac{1}{d_2^{\rho_2}} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right.$$

$$\left. + \int_{1/c_2}^{1/c_1} \int_{1/d_2}^{1/d_1} \left(u^{\rho_1} - \frac{1}{c_2^{\rho_1}} \right)^{\alpha-1} \left(\frac{1}{d_1^{\rho_2}} - v^{\rho_2} \right)^{\beta-1} u^{\rho_1-1} v^{\rho_2-1} \times \phi\left(\frac{1}{u^{\rho_1}}, \frac{1}{v^{\rho_2}} \right) \mathrm{d}v \mathrm{d}u \right], \tag{7.94}$$

and

$$\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}$$
$$\times\Psi\left(c_1^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}$$
$$\times u^{\rho_1-1}v^{\rho_2-1}\Psi\left(c_1^{\rho_1},\frac{1}{v^{\rho_2}}\right)\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\bigg]$$
$$\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Psi(c_1^{\rho_1},d_1^{\rho_2})+\Psi(c_1^{\rho_1},d_2^{\rho_2})}{2}$$
$$\times\bigg[\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(v^{\rho_2}-\frac{1}{d_2^{\rho_2}}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u$$
$$+\int_{1/c_2}^{1/c_1}\int_{1/d_2}^{1/d_1}\left(\frac{1}{c_1^{\rho_1}}-u^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{d_1^{\rho_2}}-v^{\rho_2}\right)^{\beta-1}u^{\rho_1-1}v^{\rho_2-1}\times\phi\left(\frac{1}{u^{\rho_1}},\frac{1}{v^{\rho_2}}\right)\mathrm{d}v\mathrm{d}u\bigg]. \tag{7.95}$$

By adding the inequalities (7.92)−(7.95), we get the last inequality of (7.84). □

Chapter 8

Conclusion

Convex functions play important role in the advancement of inequalities and in optimization problems. On the other hand, Fractional integral inequalities are helpful in establishing the uniqueness of solutions for certain fractional partial differential equations. These inequalities also provide upper as well as lower bounds for solutions of the fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators.

In this dissertation, several integral inequalities, includes Hermite-Hadamard and Hermite-Hadamard-Fejér, for generalized convex functions are proved via classical and fractional integrals including Riemann−Liouville fractional integrals, conformable, new conformable fractional integrals and also via Katugampola fractional integrals. Some Cauchy's type means are given for p- and s-convex functions. Also some co-ordinated integral inequalities are discussed for harmonically convex functions.

Due to nature of our work, it has flexibility to extend for more useful results. For instance, one can extend the results given in Chapter 5 to the co-ordinate and fractional integrals. At the same time, one can extend the results given in Chapter 7 to conformable and new conformable fractional integrals. We also suggest to the new researchers to explore new results by using generalized convex functins discussed in this thesis via Caputo fractional derivatives and AB-fractional integrals as well.

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