# Existence and Multiplicity Results for Systems of Singular Boundary Value Problems

by

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A thesis submitted to the
Centre for Advanced Mathematics and Physics,
National University of Sciences and Technology, Islamabad,
Pakistan
for the degree of
Doctor of Philosophy

March 2011

Dedicated

***To My Parents***

# Acknowledgements

I express my profound gratitude to Dr. Rahmat Ali Khan, who supervised my PhD research work. His deep enthusiasm in various aspects of Mathematical Sciences always fascinated me. Working under his supervision over the last three years transformed my orientation towards research and the role of Analysis in Differential Equations. He gave me tremendous freedom in pursuing this research, and was always willing to discuss new ideas. I am really grateful to him for his guidance in my research and dissertation.

I would like to thank the National University of Sciences and Technology for providing financial support for my studies. I am obliged to Prof. Asghar Qadir, Director General of CAMP for providing good research environment. I thank him for his valuable suggestions for my thesis. I would like to thank the CAMP administration for their help during my graduate studies.

I sincerely thank Drs. Tayyab Kamran, Asghar Qadir and Brig (R) Muhammad Rafique for officiating as the Graduate Examination Committee (GEC). I express my deep appreciation to Prof. Paul W. Eloe, University of Dayton, USA, and Prof. Johnny Henderson, Baylor University, USA, for their collaboration in the research.

I am very grateful to my colleagues for their unflinching support. My friends Sajid Ali, Sajid Hussain and Mujeeb ur Rehman are very dear to me for their honest support on various occasions. I am also thankful to my friends Yasir Ali, Muhammad Umar Farooq and Riaz Ahmad Khan for their company.

Finally, I would like to acknowledge the continuous support and encouragement of my family. I fondly remember my late father Taj ud Din who inculcated the tradition of lifelong learning and hard work in our family. May Allah bestow him heavenly abode. I thank my mother, sister and brothers for their kind prayers and motivation in this endeavor.

Naseer Ahmad Asif

# Preface

In science and engineering, many problems engage convoluted phenomena. Most of these complicated phenomena result in nonlinear ordinary differential equations (ODEs) along with some initial and/or boundary conditions [41]. Moreover, many of these nonlinear ODEs are singular. Singularity of these nonlinear ODEs is distinguished into two types—singularity in independent and singularity in dependent variable. Singular boundary value problems (SBVPs) arise in various fields of Mathematics, Engineering and Physics such as boundary layer theory, gas dynamics, nuclear physics, nonlinear optics, etc, [7, 40, 70, 113, 117, 119, 133]. In the 1990's some new results in analysis and fixed point theory were used to provide a general existence theory for SBVPs. This thesis is devoted to systems of SBVPs for ODEs. It presents existence theory for a variety of problems having unbounded nonlinearities in regions where their solutions are searched for. The main attention is concentrated on the positive solutions. The results are based on regularization and sequential procedure. The impact of our results is verified by descriptive examples. Essentially, the thesis is divided into six chapters. A brief description of each chapter is as follows.

In Chapter 1, first we describe the motivation of the study of SBVPs and present several examples of SBVPs which model real world phenomena. Then in Section 1.1, we present some terminologies and previously studied results. These definitions and results are very useful for our work. In Section 1.2, we define degree for finite and infinite dimensional spaces and present some of its properties. Then, we give properties of fixed point index and some known fixed point results. A relation between fixed point index and degree is also given. We also include Scahuder's fixed point theorem and the Guo–Krasnosel'skii fixed point theorem.

In Chapter 2, we establish some results for the existence of positive solutions to the system of second–order singular ODEs

$$\begin{aligned} -x''(t) &= f(t, y(t)), \quad t \in (0,1), \\ -y''(t) &= g(t, x(t)), \quad t \in (0,1), \end{aligned} \tag{0.0.1}$$

and

$$\begin{aligned} -x''(t) &= f(t, x(t), y(t)), \quad t \in (0,1), \\ -y''(t) &= g(t, x(t), y(t)), \quad t \in (0,1), \end{aligned} \tag{0.0.2}$$

subject to the following three–point boundary conditions (BCs)

$$\begin{aligned} x(0) &= 0,\ x(1) = \alpha x(\eta), \\ y(0) &= 0,\ y(1) = \alpha y(\eta), \end{aligned} \tag{0.0.3}$$

where $\eta \in (0,1)$, $0 < \alpha < 1/\eta$, $f, g \in C((0,1) \times (0,\infty), (0,\infty))$. Moreover, $f$ and $g$ are allowed to be singular at $t = 0$, $t = 1$, $x = 0$ and $y = 0$. We provide various results by employing new assumptions on the nonlinear functions $f$ and $g$, [14,15]. For this purpose, we consider a sequence of modified nonsingular problems on compact subset of $(0,1)$, we construct a cone of positive functions and then make use of the Guo–Krasnosel'skii fixed point theorem to show the existence of solution for each modified problem. Furthermore, we show that these solutions of the modified problems converges to solution of SBVPs. Our results are more general than the previous existence results where the nonlinear terms are singular with respect to $t = 0$ and $t = 1$ only [95,129].

In Chapter 3, we establish existence and multiplicity results to systems of SBVPs with nonlinear functions dependent on first derivative [16–18,24]. In this chapter, we study the following coupled systems of ODEs

$$\begin{aligned} -x''(t) &= p(t) f(t, y(t), x'(t)), \quad t \in (0,1), \\ -y''(t) &= q(t) g(t, x(t), y'(t)), \quad t \in (0,1), \end{aligned} \tag{0.0.4}$$

subject to the following set of two–point BCs

$$x(0) = y(0) = x'(1) = y'(1) = 0, \tag{0.0.5}$$

and

$$\begin{aligned} a_1 x(0) - b_1 x'(0) &= x'(1) = 0, \\ a_2 y(0) - b_2 y'(0) &= y'(1) = 0, \end{aligned} \tag{0.0.6}$$

where the nonlinearities $f, g : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous and are allowed to be singular at $x' = 0$, $y' = 0$. Moreover, $p, q \in C(0,1)$ and positive on $(0,1)$, and the real constants $a_i$ $(i = 1,2) > 0$, $b_i$ $(i = 1,2) > 0$. In Section 3.1, we prove the existence of at least one $C^1$-positive solution for the system of SBVPs (0.0.4), (0.0.5). In Section 3.2, we formulate conditions which guarantee the existence of at least two positive solutions for the system of SBVPs (0.0.4), (0.0.5) by using the theory of fixed point index. For this purpose we construct a cone in a special Banach space. Further in Section 3.3, we provide sufficient conditions for the existence of at least one positive solution for the system of SBVPs (0.0.4), (0.0.6). Moreover in Section 3.4, we provide sufficient conditions for the existence of at least two positive solutions for the system of SBVPs (0.0.4), (0.0.6) by employing fixed point index theory.

In Chapter 4, we develop an existence theory for systems of ODEs with coupled BCs [19, 20]. In Section 4.1, we discuss the existence of positive solution for the following coupled singular system of ODEs subject to four–point coupled BCs

$$\begin{aligned} -x''(t) &= f(t, x(t), y(t)), \quad t \in (0,1), \\ -y''(t) &= g(t, x(t), y(t)), \quad t \in (0,1), \\ x(0) &= 0,\ x(1) = \alpha y(\xi), \\ y(0) &= 0,\ y(1) = \beta x(\eta), \end{aligned} \tag{0.0.7}$$

where the parameters $\alpha$, $\beta$, $\xi$, $\eta$ satisfy $\xi, \eta \in (0,1)$, $0 < \alpha\beta\xi\eta < 1$. We assume that $f, g : (0,1) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and allowed to be singular at $t = 0$ and $t = 1$. We introduce notion of the Green's functions for the linear system corresponding to system of boundary value problems (BVPs) (0.0.7). We construct a positive cone and then define a completely continuous map via Green's functions. By employing the Guo–Krasnosel'skii fixed point theorem, we show that the completely continuous map has a fixed point in the positive cone. Moreover in Section 4.2, we study the existence of $C^1$-positive solutions to the following system subject to two–point coupled BCs

$$\begin{aligned} -x''(t) &= p(t)f(t, x(t), y(t), x'(t)), \quad t \in (0,1), \\ -y''(t) &= q(t)g(t, x(t), y(t), y'(t)), \quad t \in (0,1), \\ a_1 y(0) - b_1 x'(0) &= 0,\ y'(1) = 0, \\ a_2 x(0) - b_2 y'(0) &= 0,\ x'(1) = 0, \end{aligned} \tag{0.0.8}$$

where $f, g : [0,1] \times [0,\infty) \times [0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous and are allowed to be singular at $x' = 0$, $y' = 0$; $p, q \in C(0,1)$, $p > 0$ and $q > 0$ on $(0,1)$; $a_i, b_i$ $(i = 1,2)$ are positive real constants.

In Chapter 5, we study systems of SBVPs with sign–changing nonlinear functions [21–23]. In Section 5.1, we develop the notion of upper and lower solutions and prove the existence of $C^1$-positive solutions for the following system of SBVPs

$$\begin{aligned} -x''(t) &= p_1(t)f_1(t, x(t), y(t), x'(t)), \quad t \in (0,1), \\ -y''(t) &= p_2(t)f_2(t, x(t), y(t), y'(t)), \quad t \in (0,1), \\ x(0) &= x(1) = y(0) = y(1) = 0, \end{aligned} \tag{0.0.9}$$

where the functions $f_1, f_2 : [0,1] \times (0,\infty) \times (0,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous and allowed to be singular at $x = 0$ and $y = 0$. Moreover, $p_1, p_2 \in C(0,1)$ are positive on $(0,1)$. Sections 5.2 and 5.3 presents an existence theory for the systems of SBVPs on an infinite domain. BVPs over infinite intervals arises in the study of plasma physics; in determining the electrical potential in an isolated neutral atom; in the theory of shallow membrane caps; in the theory of colloids; in the flow and heat transfer over a stretching sheet; in the unsteady flow of a gas through a semi–infinite porous medium; in the theory of draining flows, etc. In Sections 5.2 and 5.3, we establish the existence of $C^1$-positive solutions to the following coupled system of ODEs

$$\begin{aligned} -x''(t) &= p_1(t)f_1(t, x(t), y(t), x'(t)), \quad t \in \mathbb{R}_0^+, \\ -y''(t) &= p_2(t)f_2(t, x(t), y(t), y'(t)), \quad t \in \mathbb{R}_0^+, \end{aligned} \tag{0.0.10}$$

subject to the following set of BCs

$$x(0) = y(0) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} x'(t) = 0, \tag{0.0.11}$$

and

$$\begin{aligned} a_1x(0) - b_1x'(0) &= \lim_{t\to\infty} x'(t) = 0, \\ a_2y(0) - b_2y'(0) &= \lim_{t\to\infty} y'(t) = 0, \end{aligned} \tag{0.0.12}$$

where $f_1, f_2 : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}_0 \to \mathbb{R}$ are continuous and allowed to change sign. We allow the nonlinear functions $f_i$ $(i = 1, 2)$ to be singular at $x' = 0$ and $y' = 0$. Also, $p_i \in C(\mathbb{R}_0^+)$, $p_i > 0$ on $\mathbb{R}_0^+$ and the constants $a_i, b_i > 0$, $i = 1, 2$; here $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}_0^+ = \mathbb{R}^+ \setminus \{0\}$. To establish the existence theory for a system on an infinite domain, first we consider the system on a finite domain and prove its existence. Then, we employ a diagnalization argument to establish the existence of positive solution for the system on an infinite domain.

Finally Chapter 6, is devoted to remarks about hypothesis that we established in Chapters 2–5.

# List of Publications

[1] N.A. Asif, P.W. Eloe and R.A. Khan, Positive solutions for a system of singular second–order nonlocal boundary value problems, *J. Korean Math. Soc.* **47**(5)(2010) 985–1000.

[2] N.A. Asif and R.A. Khan, Positive solutions for a class of coupled system of singular three–point boundary value problems, *Boundary Value Problems* **2009**, Article ID 273063, 18 pages.

[3] N.A. Asif and R.A. Khan, Multiplicity results for positive solutions of a coupled system of singular boundary value problems, *Communications on Applied Nonlinear Analysis* **17**(2)(2010) 53–68.

[4] N.A. Asif, R.A. Khan and P.W. Eloe, Existence of positive solutions to a singular system of boundary value problems, Dynamics of Continuous, Discrete and Impulsive Systems (accepted).

[5] N.A. Asif, R.A. Khan and J. Henderson, Existence of positive solutions to a system of singular boundary value problems, *Dynam. Systems Appl.* **19**(2010) 395–404.

# List of Norms

| | |
|---|---|
| $\|\cdot\|$ | For $x \in C[0,1]$, $\|x\| = \max_{t\in[0,1]} |x(t)|$. |
| $\|\cdot\|_1$ | For $x \in C^1(0,1]$, $\|x\|_1 = \sup_{t\in(0,1]} t|x'(t)|$. |
| $\|\cdot\|_2$ | For $x \in \mathcal{E} := \{x \in C[0,1] \cap C^1(0,1] : \|x\|_1 < \infty\}$, $\|x\|_2 = \max\{\|x\|, \|x\|_1\}$. |
| $\|\cdot\|_3$ | For $x \in C^1[0,1]$, $\|x\|_3 = \max\{\|x\|, \|x'\|\}$. |
| $\|\cdot\|_4$ | For $(x,y) \in \mathcal{E} \times \mathcal{E}$, $\|(x,y)\|_4 = \|x\|_2 + \|y\|_2$. |
| $\|\cdot\|_5$ | For $(x,y) \in C^1[0,1] \times C^1[0,1]$, $\|(x,y)\|_5 = \|x\|_3 + \|y\|_3$. |
| $\|\cdot\|_6$ | For $(x,y) \in C[0,1] \times C[0,1]$, $\|(x,y)\|_6 = \|x\| + \|y\|$. |
| $\|\cdot\|_{7,m}$ | For $x \in C[0,m]$, $\|x\|_{7,m} = \max_{t\in[0,m]} |x(t)|$, where $m \in \{1,2,\cdots\}$. |
| $\|\cdot\|_{\mathcal{E}_n}$ | For $x \in \mathcal{E}_n := C[\frac{1}{n}, 1-\frac{1}{n}]$, $\|x\|_{\mathcal{E}_n} = \max_{t\in[\frac{1}{n},1-\frac{1}{n}]} |x(t)|$, where $n \in \{3,4,\cdots\}$. |
| $\|\cdot\|_{\mathcal{E}_n\times\mathcal{E}_n}$ | For $(x,y) \in \mathcal{E}_n \times \mathcal{E}_n$, $\|(x,y)\|_{\mathcal{E}_n\times\mathcal{E}_n} = \|x\|_{\mathcal{E}_n} + \|y\|_{\mathcal{E}_n}$, where $n \in \{3,4,\cdots\}$. |

# Contents

# Chapter 1

# Introduction and Preliminaries

Many problems in applied sciences can be modeled by SBVPs. For example, many problems in the study of rotating flow [65]; in the theory of viscous fluids [31]; in the study of pseudoplastic fluids [33, 110]; in boundary layer theory [32, 115, 124, 125]; the theory of shallow membrane caps [25, 44, 75]; in pre–breakdown of gas discharge [46]; the turbulent flow of a gas in a porous medium [53]; can be represented by SBVPs. In view of the above mentioned areas, the study of SBVPs becomes a fundamental. In order to demonstrate the importance and applications of SBVPs, we include below few models of SBVPs emerging from different applications.

A shallow membrane cap which is rotationally symmetric in its undeformed state and whose undeformed profile is given in cylindrical coordinates by $z(r) = C(1 - r^\gamma)$, $\gamma > 1$. When radial stress is applied on the boundary and a small uniform vertical pressure $P$ is applied to the membrane, the shape of the cap is described by a nonlinear model. If the deformed membrane is rotationally symmetric, then, under the assumptions of small strain, the radial stress $S$ on the membrane is given by the following singular ODE

$$-S''(r) = \frac{1}{8(S(r))^2} - \frac{\alpha\nu}{S(r)} + \frac{3}{r}S'(r) - \frac{\beta^2}{2}r^{2(\gamma-2)}, \quad 0 < r \leq 1, \tag{1.0.1}$$

where $\alpha$, $\beta$ are positive constants depending on the pressure $P$, the thickness of the membrane and Young's modulus or the Poisson ratio $\nu$ satisfies $0 \leq \nu < 0.5$. In case of stress problem, the BC for $r = 1$ is specified by $S(1) = A$, with $A > 0$. However, in case of the displacement problem, the radial displacement at the boundary is given by $(1 - \nu)S(1) + S'(1) = B$, where $B$ is any real number. Further, one may require a BC at the singular end $r = 0$ as well and therefore, one may assume that $S(r)$ is bounded as $r \to 0^+$. For more detailed study of the model (1.0.1) and other problems related to the circular membrane cap, we refer the readers to [25, 43, 44, 75, 118].

SBVPs also occur in the study of BVPs on infinite domain. In 1927, L.H. Thomas [121] and E. Fermi [55] independently studied the electrical potential in an isolated neutral atom and derived the following singular BVP

$$\begin{aligned} &\chi''(r) = r^{-1/2}(\chi(r))^{3/2}, \quad 0 < r < \infty, \\ &\chi(0) = 1, \ \lim_{r\to\infty} \chi(r) = 0, \end{aligned} \tag{1.0.2}$$

where $\chi$ is a semiclassical description of the charge density in atoms of high atomic number. In study of mass transfer on a rotating disk in non–Newtonian fluid [60], the concentration $C$ of the diffusing species is given by the infinite domain SBVP as follows

$$\begin{aligned} -C''(t) &= \frac{1}{9}(\frac{5\alpha+7}{2\alpha+2} + \frac{6}{t})C'(t), \quad 0 < t < \infty, \\ C(0) &= 0, \ \lim_{t\to\infty} C(t) = C_\infty, \end{aligned}$$

where $\alpha$ and $C_\infty$ are some constants.

Most of the mathematical models of various applications from nuclear physics, plasma physics, nonlinear optics, fluid mechanics and chemical reactor theory are systems of time dependent partial differential equations (PDEs) subject to initial and/or boundary conditions. In the investigation of stationary solutions, many of these models of systems of PDEs can be reduced to systems of BVPs for singular ODEs. The following are some singular models from numerous applications.

The system of second–order PDEs also arises in chemical reactor theory. Let $\Omega$ be a bounded reactor in $\mathbb{R}^n$, where $n = 1, 2, 3$. The autocatalytic chemical reaction of reactant $A$ and autocatalyst $B$ is described by the following system of PDEs [139],

$$\begin{aligned} \frac{\partial a}{\partial t} &= D_A \Delta a - ab^p, \quad t \geq 0, \, x \in \Omega, \\ \frac{\partial b}{\partial t} &= D_B \Delta b + ab^p, \quad t \geq 0, \, x \in \Omega, \end{aligned} \tag{1.0.3}$$

where $D_A$ and $D_B$ are the diffusion coefficients of $A$ and $B$, respectively, $a$ and $b$ are the concentrations of $A$ and $B$, and $p \geq 1$ is the order of the reaction with respect to the autocatalytic species. In applications, the chemicals $A$ and $B$ can diffuse from a reservoir of constant composition across the boundary $\partial\Omega$ into $\Omega$, therefore the BCs of chemicals $A$ and $B$ may be of the type

$$a(x,t) = a_0 > 0, \, b(x,t) = b_0 \geq 0, \quad t \geq 0, \, x \in \partial\Omega. \tag{1.0.4}$$

The steady state solution of the system of BVPs (1.0.3) and (1.0.4) satisfy

$$\begin{aligned} &D_A \Delta a - ab^p = 0, && x \in \Omega, \\ &D_B \Delta b + ab^p = 0, && x \in \Omega, \\ &a(x) = a_0, \, b(x) = b_0, && x \in \partial\Omega. \end{aligned} \tag{1.0.5}$$

The addition of the two equations in (1.0.5), leads to $\Delta(D_A a + D_B b)(x) = 0$ for $x \in \Omega$, which implies that $D_A a(x) + D_B b(x) = 0$ for $x \in \Omega$ and $D_A a(x) + D_B b(x) = D_A a_0 + D_B b_0$

for $x \in \partial\Omega$. By the uniqueness of the solution of Laplace equation, one have $D_A a(x) + D_B b(x) = D_A a_0 + D_B b_0$ for $x \in \Omega$. Hence, the system of BVPs (1.0.5) reduced to the following scalar BVP

$$\begin{aligned} -D_A D_B \Delta b &= (D_A a_0 + D_B b_0 - D_B b) b^p, \quad x \in \Omega, \\ b(x) &= b_0, \quad x \in \partial\Omega. \end{aligned} \tag{1.0.6}$$

Let $v(x) = b(x)/(D_A D_B^{-1} a_0 + b_0)$, then using (1.0.6), $v(x)$ satisfies

$$\begin{aligned} -\Delta v &= \lambda(1-v)v^p, \quad x \in \Omega, \\ v(x) &= k, \qquad\qquad x \in \partial\Omega, \end{aligned} \tag{1.0.7}$$

where $\lambda = D_A^{-1}(D_A D_B^{-1} a_0 + b_0)^p$ and $k = D_B b_0/(D_A a_0 + D_B b_0)$. Since $v \geq 0$, by the maximum principle, $k \leq v(x) \leq 1$. Now set $u(x) = v(x) - k$, then (1.0.7) takes the form

$$\begin{aligned} -\Delta u &= \lambda((u+k)^p - (u+k)^{p+1}), \quad x \in \Omega, \\ u(x) &= 0, \qquad\qquad\qquad\qquad x \in \partial\Omega. \end{aligned} \tag{1.0.8}$$

In case the reactor $\Omega$ is a unit ball and the reaction is of high order, that is, $p > 1$, the positive solution of (1.0.8) is radially symmetric and decreasing along the radial direction. Therefore, the following SBVP arises

$$\begin{aligned} -u''(r) &= \frac{n-1}{r}u'(r) + \lambda((u(r)+k)^p - (u(r)+k)^{p+1}), \quad r \in (0,1), \\ u'(0) &= u(1) = 0, \end{aligned} \tag{1.0.9}$$

where $u(r) > 0$ and $u'(r) < 0$ for $r \in (0,1)$.

The problem of Hagen–Poiseuille flow in hydrodynamics arises the study of linear stability of incompressible flow in a circular pipe subject to non–axisymmetric disturbances. The problem is singular system of ODEs of the form

$$\begin{aligned} \mathcal{T}_r(r^2(k(r))^2\mathcal{T}_r)\Phi(r) + i\alpha R(u(r)-c)\mathcal{T}_r\Phi(r) + i\alpha R\frac{1}{r}\left(\frac{u'(r)}{r(k(r))^2}\right)'\Phi(r) + 2\alpha n\mathcal{T}_r\Omega(r) = 0, \\ 2\alpha n\mathcal{T}_r\Phi(r) - inR\frac{u'(r)}{r}\Phi(r) + \mathcal{S}_r\Omega(r) + i\alpha Rr^2(k(r))^2(u(r)-c)\Omega(r) = 0, \end{aligned} \tag{1.0.10}$$

on the interval $(0,1]$ subject to the BCs of the type

$$\begin{aligned} &\lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Phi'(r) = \Phi(1) = \Phi'(1) = \Omega(1) = 0, \text{ if } n = 0, \\ &\lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Omega(r) = \Phi(1) = \Phi'(1) = \Omega(1) = 0, \lim_{r\to 0}\Phi'(r) \text{ is finite, if } n = \pm 1, \\ &\lim_{r\to 0}\Phi(r) = \lim_{r\to 0}\Phi'(r) = \lim_{r\to 0}\Omega(r) = \Phi(1) = \Phi'(1) = \Omega(1) = 0, \text{ if } |n| \geq 2, \end{aligned} \tag{1.0.11}$$

where

$$\mathcal{T}_r = \frac{1}{r^2} - \frac{1}{r}\frac{d}{dr}\left(\frac{1}{r(k(r))^2}\frac{d}{dr}\right), \ \mathcal{S}_r = r^2(k(r))^4 - \frac{1}{r}\frac{d}{dr}\left(r^3(k(r))^2\frac{d}{dr}\right), \ k(r) = \alpha^2 + \frac{n^2}{r^2},$$

$R \geq 0$ is the Reynolds number and $\alpha \in \mathbb{R} \setminus \{0\}$ is the stream–wise wave number, $c$ is a complex wave speed resulted from $e^{i(\alpha x + n\phi - \alpha ct)}$, where $(x, \phi, t)$ are axial–angular–time coordinates. The axial mean flow $u : [0,1] \to \mathbb{R}$ is a twice differentiable function. For more detail about (1.0.10), (1.0.11), see [103, 116] and references therein.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. The interaction of two substances, activator $u$ and inhibitor $v$, is represented by a singular system of reaction–diffusion equations

$$\begin{aligned} &\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} - r_u u + \frac{\varrho u^2}{v} + \varrho b_u, && t \geq 0,\, x \in \Omega, \\ &\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} - r_v v + \varrho u^2 + \varrho b_v, && t \geq 0,\, x \in \Omega, \\ &u(t,x) = 0,\, v(t,x) = 0, && t \geq 0,\, x \in \partial\Omega, \end{aligned} \tag{1.0.12}$$

where $b_u$, $b_v$, $r_u$, $r_v$, $D_u$, $D_v$ are the respective basic production rates, decay rates, diffusion coefficients and $\varrho$ represents the ability of the cells to perform autocatalysis. The terms $\varrho u^2$ describe the autocatalysis and crosscatalysis of the activator, $1/v$ covers the action of the inhibitor, the remaining terms are degradation and source terms. The singular system (1.0.12) is an example of a large class of models which generate patterns by a combination of short–range activation due to the catalytic nonlinearity and a long–range inhibition. Also, many of the activator–inhibitor models for the formation of tropical shell patterns are variations of the singular system (1.0.12). Further details about activator–inhibitor models is available in [29, 54].

Gierer–Meinhardt equations [38,58,85] are mathematical models for pattern formations of spatial tissue structures of morphogenesis. These are system of elliptic PDEs based on an interaction between activators and inhibitors. Specifically, under the assumption that the activators, $u(t,x)$, and the inhibitors, $v(t,x)$, are acting proportionally on some powers of $u$ and $v$ in the source term and are having source distributions $\rho_1$ and $\rho_2$, respectively. The generalized Gierer–Meinhardt model is presented in the form of following singular system of BVPs

$$\begin{aligned} &\frac{\partial u}{\partial t} = d_1 \Delta u - \alpha_1 u + c_1 \rho_1 \frac{u^p}{v^q} + \gamma \rho_1, && t \geq 0,\, x \in \Omega, \\ &\frac{\partial v}{\partial t} = d_2 \Delta v - \alpha_2 v + c_2 \rho_2 \frac{u^r}{v^s}, && t \geq 0,\, x \in \Omega, \\ &u(t,x) = 0,\, v(t,x) = 0, && t \geq 0,\, x \in \partial\Omega, \end{aligned} \tag{1.0.13}$$

with $d_1$ and $d_2$ are diffusion constants, $\alpha_1$, $\alpha_2$ and $\gamma$ are positive parameters, $p$, $q$, $r$, $s$ are positive constants. For $q \neq s$, the activator and inhibitor sources are said to be different. Similar equations also occur in certain models of predator–prey interactions [123].

In the scenario of the above mentioned models of various phenomenon, the theory of SBVPs has become much more important. In this dissertation, we present existence results for positive solutions to various systems of BVPs for nonlinear ODEs. We provide sufficient conditions for the existence and multiplicity results corresponding to singular systems of nonlinear ODEs subject to various type of BCs both on finite and infinite domains. We use the classical tools of functional analysis including fixed point theory and the theory of the fixed point index. The rest of this chapter is devoted to the basic study of these notions. In Section 1.1, we present some definitions from functional analysis and known results. In Section 1.2, the concept of topological degree and fixed point index theory is described in detail. Moreover, some famous fixed point results such as Schauder's fixed point theorem and the Guo–Krasnosel'skii fixed point theorem are also included in this section.

## 1.1 Some basic definitions and known results

In this section, we present some basic definitions and known results from functional analysis. For details, we refer the reader to [45, 48, 71].

**Definition 1.1.1. (Compact):** A subset $\Omega$ of a Banach space $\mathcal{B}$ is said to be compact if every open covering of $\Omega$ can be reduced to a finite open covering of $\Omega$, that is, if $\Omega \subset \cup_{\lambda\in\Lambda}\Omega_\lambda$, where $\Omega_\lambda$ is an open subset of $\mathcal{B}$ for every $\lambda$ belongs to the indexed set $\Lambda$, then there exist a finite subset $\{\Omega_{\lambda_i} : i = 1, 2, \cdots, m\}$ of $\{\Omega_\lambda : \lambda \in \Lambda\}$ such that $\Omega \subset \cup_{i=1}^{m}\Omega_{\lambda_i}$. Equivalently, $\Omega$ is compact if and only if every sequence $\{x_n\} \subset \Omega$ has a convergent subsequence with limit in $\Omega$. Moreover, $\Omega$ is relatively compact if its closure $\overline{\Omega}$ is compact.

**Definition 1.1.2. (Compact map):** Let $\Omega$ be a subset of a Banach space $\mathcal{B}$. A map $T : \Omega \to \mathcal{B}$ is compact if $T$ maps every bounded subset of $\Omega$ into a relatively compact subset of $\mathcal{B}$. $T$ is said to be completely continuous if $T$ is continuous and compact.

**Definition 1.1.3. (Retract):** A nonempty subset $K$ of a Banach space $\mathcal{B}$ is a retract of $\mathcal{B}$ if there exist a continuous map $r : \mathcal{B} \to K$, a retraction, such that $r|_K = I_K$, where $I_K$ is identity map on $K$. Every closed and convex subset of $\mathcal{B}$ is a retract. However, every retract of $\mathcal{B}$ is closed but not necessarily convex.

**Definition 1.1.4. (Cone):** Let $\mathcal{B}$ be a real Banach space. A nonempty, closed and convex set $P \subset \mathcal{B}$ is said to be a cone if the following are satisfied:

$(\mathcal{P}_1)$ $\alpha\, x \in P$ for all $x \in P$ and $\alpha \geq 0$,

$(\mathcal{P}_2)$ $x, -x \in P$ implies $x = 0$.

A cone $P$ in a real Banach space $\mathcal{B}$ induces an ordering $\preceq$ in $\mathcal{B}$ which is defined by $x \preceq y$ if and only if $y - x \in P$. Clearly, $P = \{x \in \mathcal{B} : x \succeq 0\}$. The elements in the set $P \setminus \{0\} = \{x \in \mathcal{B} : x \succ 0\}$ are positive and $P$ is said to be positive cone of ordering.

Consequently, for every Banach space $\mathcal{B}$ there is a one–to–one correspondence between the family of linear orderings and the family of cones in $\mathcal{B}$.

**Example 1.1.5.** Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^n$ of positive measure. For each $p$ with $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the space of all real–valued measurable functions defined on $\Omega$ whose $p$–th powers are integrable. For each $x \in L^p(\Omega)$ the norm is $\|x\|_{L^p(\Omega)} = (\int_\Omega |x(t)|^p dt)^{1/p}$. Let $L^\infty(\Omega)$ denote the space of all real–valued, essentially bounded and measurable functions defined on $\Omega$ with the norm $\|x\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}\{|x(t)| : t \in \Omega\}$. Clearly, for each $0 \leq p \leq \infty$, $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space. Further, the Banach space $L^p(\Omega)$, $1 \leq p \leq \infty$, has a positive cone

$$K^p_\Omega = \{x \in L^p(\Omega) : x \geq 0 \text{ almost everywhere on } \Omega\}.$$

**Theorem 1.1.6.** ***(Arzelà–Ascoli theorem):*** *Let $\Omega$ be a compact subset of $\mathbb{R}^n$. A set $M$ of continuous functions on $\Omega$ is relatively compact in $C(\Omega)$ if and only if $M$ is a family of uniformly bounded and equicontinuous functions.*

**Theorem 1.1.7.** ***(Dugundji's theorem):*** *Every nonempty closed and convex subset of a Banach space $\mathcal{B}$ is a retract of $\mathcal{B}$.*

Now, we present a special Banach space and some other known results, details are available in [6, 131, 132]. For each $x \in C[0,1] \cap C^1(0,1]$, we write $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $\|x\|_1 = \sup_{t\in(0,1]} t|x'(t)|$. Moreover, for each $x \in \mathcal{E} := \{x \in C[0,1] \cap C^1(0,1] : \|x\|_1 < +\infty\}$, we write $\|x\|_2 = \max\{\|x\|, \|x\|_1\}$. Further, for each $x \in C^1[0,1]$, we write $\|x\|_3 = \max\{\|x\|, \|x'\|\}$.

**Lemma 1.1.8.** $(\mathcal{E}, \|\cdot\|_2)$ *is a Banach space.*

*Proof.* Clearly, $\mathcal{E}$ is a linear space and $\theta(t) \equiv 0$ for $t \in [0,1]$ is a zero element of $\mathcal{E}$. Further, $\|\cdot\|_2$ is a norm on $\mathcal{E}$. Now, we show that $(\mathcal{E}, \|\cdot\|_2)$ is a complete space. Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{E}$. Then, for any $\epsilon > 0$, there exists an $n^* > 0$ such that

$$\|x_n - x_m\|_2 < \epsilon \text{ for all } n > n^*,\ m > n^*.$$

Since $\|x_n - x_m\| \leq \|x_n - x_m\|_2$, therefore $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $C[0,1]$. Moreover, $(C[0,1], \|\cdot\|)$ is a complete space. So, there exist $x_0 \in C[0,1]$ such that

$$\lim_{n\to+\infty} \|x_n - x_0\| = 0. \tag{1.1.1}$$

Now, we show that $x_0$ is continuously differentiable on $(0,1]$. For any $\delta \in (0,1)$, consider

$$\begin{aligned} \delta \max_{t\in[\delta,1]} |x'_n(t) - x'_m(t)| &\leq \max_{t\in[\delta,1]} t|x'_n(t) - x'_m(t)| \leq \sup_{t\in(0,1]} t|x'_n(t) - x'_m(t)| \\ &= \|x'_n - x'_m\|_1 \leq \|x'_n - x'_m\|_2, \end{aligned}$$

which implies that

$$\max_{t\in[\delta,1]} |x_n'(t) - x_m'(t)| \le \frac{1}{\delta}\|x_n' - x_m'\|_2.$$

Consequently, $x_0$ is continuously differentiable on $[\delta, 1]$ and $x_n'$ converges uniformly to $x_0'$ on $[\delta, 1]$. Since $\delta$ is arbitrary, $x_0$ is continuously differentiable on $(0, 1]$.

From $\sup_{t\in(0,1]} t|x_n'(t) - x_m'(t)| \le \|x_n - x_m\|_2$ and taking $\lim_{m\to+\infty}$, we have

$$\sup_{t\in(0,1]} t|x_n'(t) - x_0'(t)| \le \epsilon, \tag{1.1.2}$$

which implies that $x_0 \in \mathcal{E}$. Also, from (1.1.1) and (1.1.2), we obtain

$$\lim_{n\to+\infty} \|x_n - x_0\|_2 = 0,$$

which shows that $x_n$ converges to $x_0$. Hence, $(\mathcal{E}, \|\cdot\|_2)$ is a Banach space. □

**Lemma 1.1.9.** *If $x \in \mathcal{E}$, then $|x'(t)| \le \frac{\|x\|_2}{t}$ for all $t \in (0, 1]$.*

*Proof.* For $t \in (0, 1]$, consider

$$t|x'(t)| \le \sup_{t\in(0,1]} t|x'(t)| = \|x\|_1 \le \|x\|_2,$$

which implies that

$$|x'(t)| \le \frac{\|x\|_2}{t}, \quad t \in (0, 1].$$

□

**Lemma 1.1.10.** *If $x \in P := \{x \in \mathcal{E} : x(t) \ge t\|x\| \,\forall\, t \in [0, 1], x(1) \ge \|x\|_1\}$, then $\|x\|_2 = \|x\|$.*

*Proof.* For $x \in P$, we have

$$\|x\| = \max_{t\in[0,1]} |x(t)| \ge x(1) \ge \|x\|_1.$$

Then,

$$\|x\|_2 = \max\{\|x\|, \|x\|_1\} = \|x\|.$$

□

**Lemma 1.1.11.** *Let $\sigma \in C(0, 1)$ and $\sigma > 0$ on $(0, 1)$ with $\int_0^1 \sigma(t)dt < +\infty$. Then,*

$$t \max_{\tau\in[0,1]} \int_0^1 G(\tau, s)\sigma(s)ds \le \int_0^1 G(t, s)\sigma(s)ds \text{ for } t \in [0, 1],$$
$$\sup_{\tau\in(0,1]} \tau \int_\tau^1 \sigma(s)ds \le \max_{t\in[0,1]} \int_0^1 G(t, s)\sigma(s)ds,$$

*where*

$$G(t,s)=\begin{cases} s, & 0\le s\le t\le 1,\\ t, & 0\le t\le s\le 1.\end{cases}$$

*Proof.* Let $\max_{\tau\in[0,1]} G(\tau,s)=G(\kappa,s)$, $s\in[0,1]$. For $t,s\in[0,1]$, we have

$$\frac{G(t,s)}{G(\kappa,s)}=\begin{cases} \frac{t}{s}\ge t, & t\le s\le\kappa,\\ \frac{s}{\kappa}\ge 1, & \kappa\le s\le t,\\ \frac{t}{\kappa}\ge t, & t,\kappa\le s,\\ \frac{s}{s}=1, & t,\kappa\ge s,\end{cases} \quad \ge t,$$

which implies that $G(t,s)\ge tG(\kappa,s)$. Hence, for $t\in[0,1]$, we have

$$\int_0^1 G(t,s)\sigma(s)ds\ge t\int_0^1 G(\kappa,s)\sigma(s)ds=t\max_{\tau\in[0,1]}\int_0^1 G(\tau,s)\sigma(s)ds.$$

Moreover, for $\tau\in[0,1]$, we have

$$\tau\int_\tau^1\sigma(s)ds\le\int_0^\tau s\sigma(s)ds+\int_\tau^1\tau\sigma(s)ds=\int_0^1 G(\tau,s)\sigma(s)ds\le\max_{t\in[0,1]}\int_0^1 G(t,s)\sigma(s)ds,$$

which implies that

$$\sup_{\tau\in(0,1]}\tau\int_\tau^1\sigma(s)ds\le\max_{t\in[0,1]}\int_0^1 G(t,s)\sigma(s)ds.$$

□

**Lemma 1.1.12.** *If* $x\in P:=\{x\in C^1[0,1]: x(t)\ge\gamma\|x\|,\ \forall\, t\in[0,1],\ x(0)\ge\frac{b}{a}\|x'\|\}$, *then* $x(t)\ge\gamma\rho\|x\|_3$ *for all* $t\in[0,1]$, *where* $\gamma=\frac{b}{a+b}$, $\varrho=\frac{1}{\max\{1,\frac{a}{b}\}}$, $a,b>0$.

*Proof.* For $x\in P$, we have

$$x(t)\ge\gamma\|x\|,\quad t\in[0,1]. \tag{1.1.3}$$

Then,

$$\begin{aligned}\|x\|_3&=\max\{\|x\|,\|x'\|\}\le\max\{\|x\|,\frac{a}{b}x(0)\}\\ &\le\max\{\|x\|,\frac{a}{b}\|x\|\}=\max\{1,\frac{a}{b}\}\|x\|=\varrho^{-1}\|x\|.\end{aligned}$$

This together with (1.1.3) implies that

$$x(t)\ge\gamma\|x\|\ge\gamma\varrho\|x\|_3.$$

□

**Lemma 1.1.13.** *Let $\sigma \in C(0,1)$ and $\sigma > 0$ on $(0,1)$ with $\int_0^1 \sigma(t)dt < +\infty$. Then,*

$$\gamma \max_{\tau\in[0,1]} \int_0^1 G(\tau,s)\sigma(s)ds \le \int_0^1 G(t,s)\sigma(s)ds \text{ for } t \in [0,1],\ \gamma = \frac{b}{a+b},\ a,b>0,$$
$$\frac{b}{a} \max_{\tau\in[0,1]} \int_\tau^1 \sigma(s)ds = \int_0^1 G(0,s)\sigma(s)ds,$$

*where*

$$G(t,s) = \frac{1}{a}\begin{cases} b+as, & 0 \le s \le t \le 1,\\ b+at, & 0 \le t \le s \le 1.\end{cases}$$

*Proof.* Let $\max_{\tau\in[0,1]} G(\tau,s) = G(\kappa,s)$, $s \in [0,1]$. For $t,s \in [0,1]$, we have

$$\frac{G(t,s)}{G(\kappa,s)} = \begin{cases} \frac{\frac{1}{a}(b+at)}{\frac{1}{a}(b+as)} \ge \frac{\frac{1}{a}(b+at)}{\frac{1}{a}(b+a)}, & t \le s \le \kappa,\\[2ex] \frac{\frac{1}{a}(b+as)}{\frac{1}{a}(b+a\kappa)} \ge 1, & \kappa \le s \le t,\\[2ex] \frac{\frac{1}{a}(b+at)}{\frac{1}{a}(b+a\kappa)} \ge \frac{\frac{1}{a}(b+at)}{\frac{1}{a}(b+a)}, & t,\kappa \le s,\\[2ex] \frac{\frac{1}{a}(b+as)}{\frac{1}{a}(b+as)} = 1, & t,\kappa \ge s,\end{cases} \ge \frac{b}{a+b},$$

which shows that $G(t,s) \ge \frac{b}{a+b}G(\kappa,s)$. Now, for $t \in [0,1]$, we have

$$\int_0^1 G(t,s)\sigma(s)ds \ge \frac{b}{a+b}\int_0^1 G(\kappa,s)\sigma(s)ds = \gamma \max_{\tau\in[0,1]} \int_0^1 G(\tau,s)\sigma(s)ds.$$

Moreover,

$$\max_{\tau\in[0,1]} \int_\tau^1 \sigma(s)ds = \int_0^1 \sigma(s)ds = \frac{a}{b}\int_0^1 G(0,s)\sigma(s)ds,$$

which implies that

$$\frac{b}{a} \max_{\tau\in[0,1]} \int_\tau^1 \sigma(s)ds = \int_0^1 G(0,s)\sigma(s)ds.$$

□

**Lemma 1.1.14.** *Let $x \in C^1[0,1] \cap C^2(0,1)$ satisfies $x'' < 0$ on $(0,1)$, $x(0) = 0$, $x'(1) = a \ge 0$. Then, $x(t) \ge tx(1)$ for $t \in [0,1]$.*

*Proof.* Let $y(t) = x(t) - tx(1)$. Then, $y(0) = y(1) = 0$ and $y''(t) < 0$ on $(0,1)$. Consequently, $y(t) \ge 0$ for $t \in [0,1]$. Hence, $x(t) \ge tx(1)$ for $t \in [0,1]$. □

## 1.2 Topological degree and fixed point index

Let $\Omega$ be a bounded and open subset of a topological space $X$ and $f : \Omega \to X$ be a continuous map. The degree of $f$ in $\Omega$ at point $y$, $\deg(f, \Omega, y)$, is a device that describes the number of solutions to the equation $f(x) = y$ in $\Omega$, where $y \notin f(\partial\Omega)$.

### 1.2.1 The Brouwer degree

Now, we recall the notion of Brouwer degree for continuous maps in finite dimensional spaces. Let $\Omega$ be a bounded and open subset of $\mathbb{R}^n$ and $f : \overline{\Omega} \to \mathbb{R}^n$ be a continuous map. A point $x_0 \in \Omega$ is a critical point of $f$ if $J_f(x_0) = 0$, where $J_f(x) := \det f'(x)$ is the Jacobean of $f$ at $x$. If $x_0$ is a critical point of $f$, then $f(x_0)$ is called a critical value. Otherwise, $f(x_0)$ is called a regular value. An integer $\deg(f, \Omega, y)$ which, roughly speaking, corresponds to the number of solutions $x \in \Omega$ of the equation $f(x) = y$ is called degree of $f$ in $\Omega$ at point $y$, where $y \notin f(\partial\Omega)$. If $y$ is not a critical value for $f$, the Brouwer degree is defined as follows:

Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $f \in C^1(\Omega) \cap C(\overline{\Omega})$ and $y \notin f(\partial\Omega \cup S_f)$, where $S_f = \{x \in \Omega : J_f(x) = 0\}$. Then,

$$\deg_{\mathrm{B}}(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x).$$

However, if $y$ is a critical value, $f \in C^2(\Omega) \cap C(\overline{\Omega})$ and $y \notin f(\partial\Omega)$. Then,

$$\deg_{\mathrm{B}}(f, \Omega, y) = \deg_{\mathrm{B}}(f, \Omega, z) = \sum_{x \in f^{-1}(z)} \operatorname{sgn} J_f(x),$$

where $z$ is any regular value of $f$ such that $|z - y| < \operatorname{dist}(y, f(\partial\Omega))$. Further, if $f \in C(\overline{\Omega})$,

$$\deg_{\mathrm{B}}(f, \Omega, y) = \deg_{\mathrm{B}}(g, \Omega, y),$$

where $g \in C^2(\Omega) \cap C(\overline{\Omega})$ is any map such that $\sup_{x \in \Omega} |f(x) - g(x)| < \operatorname{dist}(y, f(\partial\Omega))$. For further detail see [45, 56].

The Brouwer degree satisfies the following basic properties which are listed in the following theorem.

**Theorem 1.2.1.** *Let $M = \{(f, \Omega, y) : \Omega \subset \mathbb{R}^n$ bounded and open, $f \in C(\overline{\Omega}), y \notin f(\partial\Omega)\}$. Then, the Brouwer degree $\deg_{\mathrm{B}} : M \to \mathbb{Z}$ satisfies the following properties.*

($\mathcal{D}_1$) ***Normalization:*** *$\deg_{\mathrm{B}}(id, \Omega, y) = 1$ for $y \in \Omega$, where $id$ denotes the identity mapping of $\mathbb{R}^n$.*

($\mathcal{D}_2$) ***Additivity:*** *$\deg_{\mathrm{B}}(f, \Omega, y) = \deg_{\mathrm{B}}(f, \Omega_1, y) + \deg_{\mathrm{B}}(f, \Omega_2, y)$, whenever $\Omega_1$ and $\Omega_2$ are disjoint open subsets of $\Omega$ such that $y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.*

$(\mathcal{D}_3)$ ***Homotopy invariance:*** $\deg_{\mathrm{B}}(h(t,\cdot),\Omega,y(t))$ *is independent of* $t\in[0,1]$, *whenever* $h:[0,1]\times\overline{\Omega}\to\mathbb{R}^n$ *and* $y:[0,1]\to\mathbb{R}^n$ *are continuous and* $y(t)\notin h(t,\partial\Omega)$ *for all* $t\in[0,1]$.

$(\mathcal{D}_4)$ ***Solution property:*** $\deg_{\mathrm{B}}(f,\Omega,y)\neq 0$ *implies* $f^{-1}(y)\neq\emptyset$.

$(\mathcal{D}_5)$ ***Boundary dependence:*** $\deg_{\mathrm{B}}(f,\Omega,y)=\deg_{\mathrm{B}}(g,\Omega,y)$ *whenever* $f|_{\partial\Omega}=g|_{\partial\Omega}$.

$(\mathcal{D}_6)$ ***Excision:*** $\deg_{\mathrm{B}}(f,\Omega,y)=\deg_{\mathrm{B}}(f,\Omega_1,y)$ *for every open subset* $\Omega_1$ *of* $\Omega$ *such that* $y\notin f(\overline{\Omega}\setminus\Omega_1)$.

### 1.2.2 The Leray–Schauder degree

In 1934, an important extension of the Brouwer degree to infinite dimensional spaces was revealed by J. Leray and J. Schauder [87]. This requires continuity and some kind of compactness. They proved that there is a complete analog of finite–dimensional degree theory for the class of "compact perturbation of the identity", that is for map of the form $I-T$, where $I$ is the identity map and $T$ is a completely continuous map.

We define the Leray–Schauder degree by means of the following theorem on the approximation of a compact mapping with finite–dimensional mappings.

**Theorem 1.2.2.** *Let* $(\mathcal{B},\|\cdot\|)$ *be a real Banach space. Assume that* $\Omega$ *is a bounded, open subset of* $\mathcal{B}$ *and* $T:\overline{\Omega}\to\mathcal{B}$ *is a completely continuous map. Then, for every* $\varepsilon>0$, *there exist a finite–dimensional space* $\mathbb{B}$ *and a continuous map* $T_\varepsilon:\overline{\Omega}\to\mathbb{B}$ *such that* $\|T(x)-T_\varepsilon(x)\|<\varepsilon$ *for every* $x\in\overline{\Omega}$.

In view of Theorem 1.2.2, we have $\|T(x)-T_\varepsilon(x)\|<\varepsilon:=\rho(y,(I-T)(\partial\Omega))$, where $\rho$ is the metric associated with the norm $\|\cdot\|$. Thus, we define the Leray–Schauder degree for the map $I-T$ as follows:

Let $\Omega$ be a bounded and open subset of a Banach space $\mathcal{B}$. Let $T:\overline{\Omega}\to\mathcal{B}$ be a completely continuous mapping and $y\notin(I-T)(\partial\Omega)$. The Leray–Schauder degree of $I-T$ over $\Omega$ at point $y$ is defined by

$$\deg_{\mathrm{LS}}(I-T,\Omega,y)=\deg_{\mathrm{B}}(I-T_\varepsilon,\Omega,y),$$

where $T_\varepsilon$ is given in Theorem 1.2.2. Essentially, all the properties of Brouwer degree $(\mathcal{D}_1)-(\mathcal{D}_6)$ are also satisfied by the Leray–Schauder degree for the map $f=I-T$, for details see [45, 56].

### 1.2.3 Fixed point index theory

Let $K$ be a retract of a real Banach space $\mathcal{B}$ and $\Omega$ is an open subset of $K$. Let $T:\overline{\Omega}\to K$ be a completely continuous map such that $0\notin(I-T)(\partial\Omega)$. For a retraction $r:\mathcal{B}\to K$, the degree $\deg_{\mathrm{LS}}(I - T \circ r, r^{-1}(\Omega), 0)$ is defined and it follows from the homotopy invariance and excision properties of Leray–Schauder degree that this integer is same for all retractions from $\mathcal{B}$ onto $K$. We call this number the fixed point index of the map $T$ over $\Omega$ with respect to $K$ and is denoted by $\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K)$.

If the positive cone does not have the interior point, the Leray–Schauder degree is not immediately applicable. However, since positive cone is a retract of the Banach space, fixed point index is defined in the positive cone for completely continuous maps. In this section, we present the basic properties of fixed point index for completely continuous map. Further, we show that the fixed point index is equivalent to the Leray–Schauder degree, for detail see [9, 45]. The most significant properties of the fixed point index for a completely continuous map are presented in the following theorem.

**Theorem 1.2.3.** *Let $K$ be a retract of a Banach apace $\mathcal{B}$. For every open subset $\Omega$ of $K$ and every completely continuous map $T : \overline{\Omega} \to K$ which has no fixed point on $\partial\Omega$, there exist an integer $\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K)$, the fixed point index of $T$ over $\Omega$ with respect to $K$, satisfying the following properties:*

($\mathcal{I}_1$) ***Normalization:*** *For every constant map $T$ mapping $\overline{\Omega}$ into $\Omega$, $\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K) = 1$.*

($\mathcal{I}_2$) ***Additivity:*** *For every pair of disjoint open subsets $\Omega_1$ and $\Omega_2$ of $\Omega$ such that $T$ has no fixed point on $\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$,*

$$\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K) = \mathrm{ind}_{\mathrm{FP}}(T, \Omega_1, K) + \mathrm{ind}_{\mathrm{FP}}(T, \Omega_2, K),$$

*where $\mathrm{ind}_{\mathrm{FP}}(T, \Omega_i, K) := \mathrm{ind}_{\mathrm{FP}}(T|_{\overline{\Omega}_i}, \Omega_i, K)$, $i = 1, 2$.*

($\mathcal{I}_3$) ***Homotopy invariance:*** *For every compact interval $\Lambda \subset \mathbb{R}$ and every compact map $h : \Lambda \times \overline{\Omega} \to K$ such that $h(\lambda, x) \neq x$ for $(\lambda, x) \in \Lambda \times \partial\Omega$,*

$$\mathrm{ind}_{\mathrm{FP}}(h(\lambda, \cdot), \Omega, K)$$

*is well defined and is independent of $\lambda \in \Lambda$.*

($\mathcal{I}_4$) ***Solution property:*** *If $\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K) \neq 0$, then $T$ has at least one fixed point in $\Omega$.*

($\mathcal{I}_5$) ***Permanence:*** *If $K_1$ is a retract of $K$ and $T(\overline{\Omega}) \subset K_1$, then*

$$\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K) = \mathrm{ind}_{\mathrm{FP}}(T, \Omega \cap K_1, K_1),$$

*where $\mathrm{ind}_{\mathrm{FP}}(T, \Omega \cap K_1, K_1) := \mathrm{ind}_{\mathrm{FP}}(T|_{\overline{\Omega \cap K_1}}, \Omega \cap K_1, K_1)$.*

($\mathcal{I}_6$) ***Excision:*** *For every open set $\Omega_1 \subset \Omega$ such that $T$ has no fixed point in $\overline{\Omega} \setminus \Omega_1$,*

$$\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K) = \mathrm{ind}_{\mathrm{FP}}(T, \Omega_1, K).$$

Now, we prove that the fixed point index $\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K)$ is equivalent to the Leray–Schauder degree and is uniquely determined by the properties $(\mathcal{I}_1) - (\mathcal{I}_3)$, $(\mathcal{I}_5)$.

If $K = \mathcal{B}$, the conditions $(\mathcal{I}_1) - (\mathcal{I}_3)$ are specifically the properties which illustrate the Leray–Schauder degree. Hence,

$$\mathrm{ind}_{\mathrm{FP}}(I - T, \Omega, \mathcal{B}) = \deg_{\mathrm{LS}}(I - T, \Omega, 0),$$

where $\deg_{\mathrm{LS}}(I - T, \Omega, 0)$ denotes the Leray–Schauder degree with respect to zero of the closure of the compact vector field $I - T$, which is defined on the closure of the open subset $\Omega$ of $\mathcal{B}$.

Now, let $K$ be any retract of $\mathcal{B}$ and $r_0 : \mathcal{B} \to K$ be a retraction. Then, in view of permeance property $(\mathcal{I}_5)$, we have

$$\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K) = \mathrm{ind}_{\mathrm{FP}}(T \circ r_0,\, r_0^{-1}(\Omega),\, \mathcal{B}) = \deg_{\mathrm{LS}}(I - T \circ r_0,\, r_0^{-1}(\Omega),\, 0),$$

which shows that every fixed point index of $T$ over $\Omega$ with respect to $K$ is equal to $\deg_{\mathrm{LS}}(I - T \circ r_0,\, r_0^{-1}(\Omega),\, 0)$ and therefore is unique. Thus, by the above uniqueness proof, we define

$$\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K) = \deg_{\mathrm{LS}}(I - T \circ r_0,\, r_0^{-1}(\Omega),\, 0). \tag{1.2.1}$$

Now, we show that the fixed point index defined by (1.2.1) is well–defined, that is, definition (1.2.1) is independent of choice of retraction $r_0 : \mathcal{B} \to K$. Let $r_1 : \mathcal{B} \to K$ be another retraction. Then, by the excision property of the Leray–Schauder degree, we have

$$\deg_{\mathrm{LS}}(I - T \circ r_i,\, r_i^{-1}(\Omega),\, 0) = \deg_{\mathrm{LS}}(I - T \circ r_i,\, r_0^{-1}(\Omega) \cap r_1^{-1}(\Omega),\, 0), \quad i = 0, 1.$$

Let $h : [0, 1] \times \overline{\Omega} \to K$ be a compact map defined by

$$h(\lambda, x) = r_0((1 - \lambda)T(r_0(x)) + \lambda T(r_1(x))).$$

Then, by the homotopy invariance property of the Leray–Schauder degree, we have

$$\deg_{\mathrm{LS}}(I - T \circ r_0,\, r_0^{-1}(\Omega) \cap r_1^{-1}(\Omega),\, 0) = \deg_{\mathrm{LS}}(I - T \circ r_1,\, r_0^{-1}(\Omega) \cap r_1^{-1}(\Omega),\, 0).$$

Consequently, $\mathrm{ind}_{\mathrm{FP}}(T, \Omega, K)$ is independent of the choice of retraction and, therefore, (1.2.1) is well–defined.

We need the following results for our work. These results are useful and have significant role in proving the existence of solutions of operator equations and, in particular, in establishing the existence of multiple solutions of operator equations. For further details and many other useful results we refer the reader to [9, 62].

**Theorem 1.2.4.** ***(Schauder's fixed point theorem):*** *Let $C$ be a nonempty, closed, bounded and convex subset of a Banach space $\mathcal{B}$ and $T : C \to C$ be a completely continuous map. Then, $T$ has a fixed point in $C$.*

*Proof.* By Theorem 1.1.7, $C$ is a retract of $\mathcal{B}$. Hence, $\text{ind}_{\text{FP}}(T, C, C)$ is well–defined. Let $x_0 \in C$ and $h : [0, 1] \times C \to C$ be a compact map defined by

$$h(\lambda, x) = (1 - \lambda)x_0 + \lambda Tx.$$

Then, the homotopy invariance property $(\mathcal{I}_3)$ together with the normalization property $(\mathcal{I}_1)$ implies that

$$\text{ind}_{\text{FP}}(T, C, C) = \text{ind}_{\text{FP}}(x_0, C, C) = 1,$$

and by the solution property $(\mathcal{I}_4)$, $T$ has a fixed point in $C$. □

**Lemma 1.2.5.** *Let $\Omega$ be a bounded and open set in a real Banach space $\mathcal{B}$, $P$ be a cone of $\mathcal{B}$, $0 \in \Omega$ and $T : \overline{\Omega} \cap P \to P$ be a completely continuous map. Suppose $x \neq \lambda Tx$, for any $x \in \partial\Omega \cap P$, $\lambda \in (0, 1]$. Then, the fixed point index* $\text{ind}_{\text{FP}}(T, \Omega \cap P, P) = 1$.

*Proof.* Define a compact map $h : [0, 1] \times (\overline{\Omega} \cap P) \to P$ by $h(\tau, x) = \tau Tx$. Then, the homotopy invariance property $(\mathcal{I}_3)$ together with normalization property $(\mathcal{I}_1)$, implies that

$$\text{ind}_{\text{FP}}(T, \Omega \cap P, P) = \text{ind}_{\text{FP}}(0, \Omega \cap P, P) = 1.$$

□

**Lemma 1.2.6.** *Let $\Omega$ be a bounded and open set in a real Banach space $(\mathcal{B}, \|\cdot\|)$, $P$ be a cone of $\mathcal{B}$, $0 \in \Omega$ and $T : \overline{\Omega} \cap P \to P$ be a completely continuous map. Suppose there exist a $v \in P \setminus \{0\}$ with $x \neq Tx + \delta v$ for every $\delta > 0$ and $x \in \partial\Omega \cap P$. Then, the fixed point index* $\text{ind}_{\text{FP}}(T, \Omega \cap P, P) = 0$.

*Proof.* Let $\mu = \sup\{\|Tx\| : x \in \Omega \cap P\}$ and $\rho = \sup\{\|x\| : x \in \Omega\}$. Choose $\delta_1 > (\mu + \rho)/\|v\|$ and define a compact map $h : [0, 1] \times (\overline{\Omega} \cap P) \to P$ by

$$h(\tau, x) = T(x) + \tau\delta_1 v.$$

Then, by the homotopy invariance property $(\mathcal{I}_3)$, we obtain

$$\text{ind}_{\text{FP}}(T, \Omega \cap P, P) = \text{ind}_{\text{FP}}(T + \delta_1 v, \Omega \cap P, P).$$

Now, if $\text{ind}_{\text{FP}}(T, \Omega \cap P, P) \neq 0$, then there exist an element $x \in \Omega \cap P$ such that $x = Tx + \delta_1 v$. Consequently,

$$\|x\| = \|Tx + \delta_1 v\| \geq \delta_1\|v\| - \|Tx\| \geq \delta_1\|v\| - \mu > \rho,$$

a contradiction. Hence, $\text{ind}_{\text{FP}}(T, \Omega \cap P, P) = 0$. □

**Lemma 1.2.7.** *Let $\Omega$ be a bounded and open set in a real Banach space $(\mathcal{B}, \|\cdot\|)$, $P$ be a cone of $\mathcal{B}$, $0 \in \Omega$ and $T : \overline{\Omega} \cap P \to P$ be a completely continuous map. Suppose $Tx \nleq x$, for any $x \in \partial\Omega \cap P$. Then, the fixed point index* $\text{ind}_{\text{FP}}(T, \Omega \cap P, P) = 0$.

*Proof.* Geometrically, by $Tx \nleq x$ for any $x \in \partial\Omega \cap P$ means that no $x \in \partial\Omega \cap P$ can be mapped radially toward the origin under the map $T$. We claim that for $v \in P \setminus \{0\}$, $x \neq Tx + \delta v$ for every $\delta > 0$ and $x \in \partial\Omega \cap P$. Suppose there exist some $\delta_0 > 0$ and $x_0 \in \partial\Omega \cap P$ such that $x_0 = Tx_0 + \delta_0 v$. Then, $\|x_0\| = \|Tx_0 + \delta_0 v\| > \|Tx_0\|$, that is, $x_0$ is mapped toward origin, a contradiction. Hence, by Lemma 1.2.6, $\text{ind}_{\text{FP}}(T, \Omega \cap P, P) = 0$. □

**Theorem 1.2.8.** ***(Guo–Krasnosel'skii fixed point theorem):*** *Let $P$ be a cone of a real Banach space $\mathcal{B}$. Let $\Omega_1$, $\Omega_2$ be open and bounded neighborhoods of $0 \in \mathcal{B}$ such that $\Omega_1 \subset \Omega_2$. Suppose that $T : (\overline{\Omega}_2 \setminus \Omega_1) \cap P \to P$ is completely continuous such that one of the following conditions holds:*

*(i)* $\|Tx\| \leq \|x\|$ *for* $x \in \partial\Omega_1 \cap P$, $\|Tx\| \geq \|x\|$ *for* $x \in \partial\Omega_2 \cap P$;

*(ii)* $\|Tx\| \leq \|x\|$ *for* $x \in \partial\Omega_2 \cap P$, $\|Tx\| \geq \|x\|$ *for* $x \in \partial\Omega_1 \cap P$.

*Then, $T$ has a fixed point in $(\overline{\Omega}_2 \setminus \Omega_1) \cap P$.*

*Proof.* Assume that $(i)$ holds. First we show that $x \neq Tx$ for all $x \in \partial\Omega_1 \cap P$. Suppose, $x_1 = Tx_1$ for some $x_1 \in \partial\Omega_1 \cap P$. Then, $\|x_1\| = \|Tx_1\|$, a contradiction whenever $\|Tx_1\| < \|x_1\|$. Now, we show that $x \neq \lambda Tx$ for $\lambda \in (0,1)$ and $x \in \partial\Omega_1 \cap P$. Suppose, $x_1 = \lambda_1 Tx_1$ for some $\lambda_1 \in (0,1)$ and $x_1 \in \partial\Omega_1 \cap P$. Then, $\|x_1\| = \lambda_1\|Tx_1\| < \|Tx_1\| \leq \|x_1\|$, again a contradiction. Consequently, $x \neq \lambda Tx$ for $\lambda \in (0,1]$ and $x \in \partial\Omega_1 \cap P$. Hence, by Lemma 1.2.5, the fixed point index $\text{ind}_{\text{FP}}(T, \Omega_1 \cap P, P) = 1$.

We claim that there exist a $v \in P \setminus \{0\}$ with $x \neq Tx + \delta v$ for $\delta > 0$ and $x \in \partial\Omega_2 \cap P$. Suppose, $x_2 = Tx_2 + \delta_2 v$ for some $\delta_2 > 0$ and $x_2 \in \partial\Omega_2 \cap P$. Then, $\|x_2\| = \|Tx_2 + \delta_2 v\| > \|Tx_2\| \geq \|x_2\|$, a contradiction. Therefore, by Lemma 1.2.6, the fixed point index $\text{ind}_{\text{FP}}(T, \Omega_2 \cap P, P) = 0$.

Thus, by the additivity property of fixed point index $(\mathcal{I}_2)$, we obtain

$$\text{ind}_{\text{FP}}(T, (\Omega_2 \setminus \overline{\Omega}_1) \cap P, P) = \text{ind}_{\text{FP}}(T, \Omega_2 \cap P, P) - \text{ind}_{\text{FP}}(T, \Omega_1 \cap P, P) = 0 - 1 = -1,$$

which shows that $T$ has a fixed point in $(\overline{\Omega}_2 \setminus \Omega_1) \cap P$. The proof for $(ii)$ is similar. □

# Chapter 2

# Singular Systems with Nonlocal Boundary Conditions

Nonlocal BVPs arise in different areas of applied mathematics and physics. For example, the vibration of a guy wire composed of $N$ parts with a uniform cross section and different densities in different parts can be modeled as a nonlocal BVP [108]; problems in the theory of elastic stability can also be modeled as nonlocal BVPs [122].

The study of nonlocal BVPs for linear second–order ODEs was initiated by Il'in and Moiseev in [72,73] and extended to nonlocal linear elliptic BVPs by Bitsadze and Samarskiĭ, [26–28]. Existence theory for nonlinear three–point BVPs was initiated by Gupta [63]. Since then the study of nonlinear regular multi–point BVPs has attracted the attention of many researchers; see for example, [30, 79, 92, 94, 96, 101, 108, 126, 138] for scalar equations, and for systems of ODEs, see [37, 42, 77].

Recently, the study of SBVPs has also attracted some attention. An excellent resource with an extensive bibliography was produced by Agarwal and O'Regan [3]. More recently, S. Xie and J. Zhu [129] applied topological degree theory in a cone to study the following two–point BVPs for a coupled system of nonlinear fourth–order ODEs

$$\begin{aligned} -x^{(4)}(t) &= f_1(t, y(t)), \quad t \in (0,1), \\ -y''(t) &= f_2(t, x(t)), \quad t \in (0,1), \\ x(0) &= x(1) = x''(0) = x''(1) = 0, \\ y(0) &= y(1) = 0, \end{aligned}$$

where the nonlinear functions $f_i \in C((0,1) \times [0,\infty), [0,\infty))$ satisfy $f_i(t,0) \equiv 0$ $(i = 1,2)$ and are allowed to be singular at $t = 0$ or $t = 1$.

Y. Zhou and Y. Xu [140] studied the following nonlocal BVPs for a system of second–

order regular ODEs

$$\begin{aligned} -x''(t) &= f(t, y(t)), \quad t \in (0,1), \\ -y''(t) &= g(t, x(t)), \quad t \in (0,1), \\ x(0) &= 0,\ x(1) = \alpha x(\eta), \\ y(0) &= 0,\ y(1) = \alpha y(\eta), \end{aligned} \tag{2.0.1}$$

where $\eta \in (0,1)$, $0 < \alpha < 1/\eta$, $f, g \in C([0,1] \times [0,\infty), [0,\infty))$ and $f(t,0) \equiv 0$, $g(t,0) \equiv 0$. The system of BVPs (2.0.1) was extended to the singular case by B. Liu, L. Liu and Y. Wu [95], where the functions $f$, $g$ were assumed to be singular at $t = 0$ or $t = 1$ together with the assumption that $f(t,0) \equiv 0$, $g(t,0) \equiv 0$, $t \in (0,1)$.

In this chapter, we study the following singular systems of ODEs

$$\begin{aligned} -x''(t) &= f(t, y(t)), \quad t \in (0,1), \\ -y''(t) &= g(t, x(t)), \quad t \in (0,1), \end{aligned} \tag{2.0.2}$$

and

$$\begin{aligned} -x''(t) &= f(t, x(t), y(t)), \quad t \in (0,1), \\ -y''(t) &= g(t, x(t), y(t)), \quad t \in (0,1), \end{aligned} \tag{2.0.3}$$

subject to nonlocal BCs

$$x(0) = y(0) = 0,\ x(1) = \alpha x(\eta),\ y(1) = \alpha y(\eta), \tag{2.0.4}$$

where $\eta \in (0,1)$, $0 < \alpha < 1/\eta$. For the system of ODEs (2.0.2), we assume that $f, g : (0,1) \times (0,\infty) \to (0,\infty)$ are continuous and $f(t,0)$, $g(t,0)$ are not identically 0. Similarly, for the system of ODEs (2.0.3), we assume that $f, g : (0,1) \times (0,\infty) \times (0,\infty) \to (0,\infty)$ are continuous and $f(t,0,0)$, $g(t,0,0)$ are not identically 0. Moreover, $f$ and $g$ are allowed to be singular at $t = 0$, $t = 1$, $x = 0$ and $y = 0$. By singularity we mean that the nonlinear functions $f$ and $g$ are allowed to be unbounded at $t = 0$, $t = 1$, $x = 0$ or $y = 0$. By applying Theorem 1.2.8, we obtain sufficient conditions for the existence of positive solutions to the systems of ODEs (2.0.2) and (2.0.3) subject to BCs (2.0.4), [14, 15]. In general, the assumption that there exist singularities with respect to the dependent variable is not new; see [3, 37, 93], for example. However, in the case of nonlocal BCs and coupled systems of ODEs, we believe this assumption is new.

Let $n_0 > \max\{\frac{1}{\eta}, \frac{1}{1-\eta}, \frac{2-\alpha}{1-\alpha\eta}\}$ be a fixed positive integer. For each $u \in \mathcal{E}_n := C[\frac{1}{n}, 1-\frac{1}{n}]$, we write $\|u\|_{\mathcal{E}_n} = \max\{|u(t)| : t \in [\frac{1}{n}, 1-\frac{1}{n}]\}$, where $n \in \{n_0, n_0+1, n_0+2, \cdots\}$. Clearly, $\mathcal{E}_n$ with the norm $\|\cdot\|_{\mathcal{E}_n}$ is a Banach space. Define a cone $K_n$ of $\mathcal{E}_n$ as

$$K_n = \{u \in \mathcal{E}_n : u \geq 0 \text{ and concave on } [\frac{1}{n}, 1-\frac{1}{n}]\}.$$

For any real constant $r > 0$, define an open neighborhood of $0 \in \mathcal{E}_n$ of radius $r$ by

$$\Omega_r = \{u \in \mathcal{E}_n : \|u\|_{\mathcal{E}_n} < r\}.$$

For each $(x, y) \in \mathcal{E}_n \times \mathcal{E}_n$, we write $\|(x, y)\|_{\mathcal{E}_n \times \mathcal{E}_n} = \|x\|_{\mathcal{E}_n} + \|y\|_{\mathcal{E}_n}$. Clearly, $(\mathcal{E}_n \times \mathcal{E}_n, \|\cdot\|_{\mathcal{E}_n \times \mathcal{E}_n})$ is a Banach space and $K_n \times K_n$ is a cone of $\mathcal{E}_n \times \mathcal{E}_n$. Further, for any real constant $r > 0$, we define an open neighborhood of $(0, 0) \in \mathcal{E}_n \times \mathcal{E}_n$ as

$$\mathcal{O}_r = \{(x, y) \in \mathcal{E}_n \times \mathcal{E}_n : \|(x, y)\|_{\mathcal{E}_n \times \mathcal{E}_n} < r\}.$$

## 2.1 Green's function for the associated linear problem

**Lemma 2.1.1.** *For $z \in \mathcal{E}_n$, the linear BVP*

$$\begin{aligned} -u''(t) &= z(t), \quad t \in [\frac{1}{n}, 1 - \frac{1}{n}], \\ u(\frac{1}{n}) &= 0, \; u(1 - \frac{1}{n}) = \alpha u(\eta), \end{aligned} \tag{2.1.1}$$

*has a unique solution*

$$u(t) = \int_{1/n}^{1-1/n} H_n(t, s) z(s) ds, \tag{2.1.2}$$

*where $H_n : [\frac{1}{n}, 1 - \frac{1}{n}] \times [\frac{1}{n}, 1 - \frac{1}{n}] \to [0, \infty)$ is an associated Green's function and is defined by*

$$H_n(t, s) = \begin{cases} \frac{(t-\frac{1}{n})((1-\frac{1}{n}-s)-\alpha(\eta-s))}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t - s), & \frac{1}{n} \le s \le t \le 1 - \frac{1}{n}, \; s \le \eta, \\ \frac{(t-\frac{1}{n})((1-\frac{1}{n}-s)-\alpha(\eta-s))}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}, & \frac{1}{n} \le t \le s \le 1 - \frac{1}{n}, \; s \le \eta, \\ \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}, & \frac{1}{n} \le t \le s \le 1 - \frac{1}{n}, \; s \ge \eta, \\ \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t - s), & \frac{1}{n} \le s \le t \le 1 - \frac{1}{n}, \; s \ge \eta. \end{cases} \tag{2.1.3}$$

*Proof.* Integrating (2.1.1) from $1/n$ to $t$, we have

$$u'(t) = u'(\frac{1}{n}) - \int_{1/n}^{t} z(s) ds,$$

again integrating from $1/n$ to $t$ and using the BCs (2.1.1), leads to

$$u(t) = u'(\frac{1}{n})(t - \frac{1}{n}) - \int_{1/n}^{t} (t - s) z(s) ds. \tag{2.1.4}$$

Using the BCs (2.1.1), we obtain

$$u'(\frac{1}{n}) = \frac{1}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta} \int_{1/n}^{1-1/n} (1 - \frac{1}{n} - s) z(s) ds - \frac{\alpha}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta} \int_{1/n}^{\eta} (\eta - s) z(s) ds. \tag{2.1.5}$$

Now using (2.1.5) in (2.1.4), we get

$$\begin{aligned} u(t) =& \frac{t-\frac{1}{n}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{1-1/n}(1-\frac{1}{n}-s)z(s)ds - \frac{\alpha(t-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{\eta}(\eta-s)z(s)ds \\ & - \int_{1/n}^{t}(t-s)z(s)ds, \end{aligned}$$

which can also be written as (2.1.2). □

We note that $H_n(t,s) \to H(t,s)$ as $n \to \infty$, where

$$H(t,s) = \begin{cases} \frac{t(1-s)}{1-\alpha\eta} - \frac{\alpha t(\eta-s)}{1-\alpha\eta} - (t-s), & 0 \le s \le t \le 1,\ s \le \eta, \\ \frac{t(1-s)}{1-\alpha\eta} - \frac{\alpha t(\eta-s)}{1-\alpha\eta}, & 0 \le t \le s \le 1,\ s \le \eta, \\ \frac{t(1-s)}{1-\alpha\eta}, & 0 \le t \le s \le 1,\ s \ge \eta, \\ \frac{t(1-s)}{1-\alpha\eta} - (t-s), & 0 \le s \le t \le 1,\ s \ge \eta, \end{cases}$$

is the Green's function corresponding to the BVP

$$\begin{aligned} -u''(t) = 0, \qquad & t \in [0,1], \\ u(0) = 0,\ u(1) = \alpha u(\eta). & \end{aligned}$$

**Lemma 2.1.2.** *The function $H_n$ satisfies*

*(i)* $H_n(t,s) \le \mu_n(s-\frac{1}{n})(1-\frac{1}{n}-s), \quad (t,s) \in [\frac{1}{n}, 1-\frac{1}{n}] \times [\frac{1}{n}, 1-\frac{1}{n}]$,

*(ii)* $H_n(t,s) \ge \nu_n(s-\frac{1}{n})(1-\frac{1}{n}-s), \quad (t,s) \in [\eta, 1-\frac{1}{n}] \times [\frac{1}{n}, 1-\frac{1}{n}]$,

*where*

$$\mu_n := \frac{\max\{1,\alpha\}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} > 0,\ \nu_n := \frac{\min\{1,\alpha\}\min\{\eta-\frac{1}{n}, 1-\frac{1}{n}-\eta\}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} > 0.$$

*Proof.* First we prove $(i)$. For $(t,s) \in [\frac{1}{n}, 1-\frac{1}{n}] \times [\frac{1}{n}, 1-\frac{1}{n}]$, we discuss various cases.
**Case 1:** $s \le \eta$, $t \ge s$; using (2.1.3), we obtain

$$H_n(t,s) = \frac{(t-\frac{1}{n})((1-\frac{1}{n}-s)-\alpha(\eta-s))}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t-s) = s - \frac{1}{n} + (\alpha-1)\frac{(t-\frac{1}{n})(s-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}.$$

If $\alpha > 1$, the maximum of $H_n(t,s)$ occurs at $t = 1-\frac{1}{n}$, hence

$$\begin{aligned} H_n(t,s) \le H_n(1-\frac{1}{n}, s) = \alpha\frac{(s-\frac{1}{n})(1-\frac{1}{n}-\eta)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \alpha\frac{(s-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \\ \le \mu_n(s-\frac{1}{n})(1-\frac{1}{n}-s), \end{aligned}$$

and if $\alpha \le 1$, the maximum of $H_n(t,s)$ occurs at $t = s$, hence

$$\begin{aligned} H_n(t,s) \le H_n(s,s) = \frac{(s-\frac{1}{n})(1-\frac{1}{n}-s+\alpha(s-\eta))}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \frac{(s-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \\ \le \mu_n(s-\frac{1}{n})(1-\frac{1}{n}-s). \end{aligned}$$

**Case 2:** $s \le \eta$, $t \le s$; using (2.1.3), we have

$$H_n(t,s) = \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - \alpha\frac{(t-\frac{1}{n})(\eta-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}$$
$$\le \frac{(s-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \mu_n(s-\frac{1}{n})(1-\frac{1}{n}-s).$$

**Case 3:** $s \ge \eta$, $t \le s$; using (2.1.3), we get

$$H_n(t,s) = \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \frac{(s-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \mu_n(s-\frac{1}{n})(1-\frac{1}{n}-s).$$

**Case 4:** $s \ge \eta$, $t \ge s$; using (2.1.3), we get

$$H_n(t,s) = \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t-s) = s-\frac{1}{n}+(t-\frac{1}{n})\frac{\alpha(\eta-\frac{1}{n})-(s-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}.$$

If $\alpha(\eta-\frac{1}{n}) > s-\frac{1}{n}$, the maximum of $H_n(t,s)$ occurs at $t = 1-\frac{1}{n}$, hence

$$H_n(t,s) \le H_n(1-\frac{1}{n},s) = \alpha\frac{(\eta-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \alpha\frac{(s-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}$$
$$\le \mu_n(s-\frac{1}{n})(1-\frac{1}{n}-s),$$

and if $\alpha(\eta-\frac{1}{n}) \le s-\frac{1}{n}$, the maximum of $H_n(t,s)$ occurs at $t = s$, so

$$H_n(t,s) \le H_n(s,s) = \frac{(s-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \le \mu_n(s-\frac{1}{n})(1-\frac{1}{n}-s).$$

Now, we prove $(ii)$. Here, for $(t,s) \in [\eta, 1-\frac{1}{n}] \times [\frac{1}{n}, 1-\frac{1}{n}]$, we discuss different cases.
**Case 1:** $s \le \eta$, $t \ge s$; using (2.1.3), we obtain

$$H_n(t,s) = \frac{(t-\frac{1}{n})((1-\frac{1}{n}-s)-\alpha(\eta-s))}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t-s) = s-\frac{1}{n}+(\alpha-1)\frac{(t-\frac{1}{n})(s-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}.$$

If $\alpha < 1$, the minimum of $H_n(t,s)$ occurs at $t = 1-\frac{1}{n}$, hence

$$H_n(t,s) \ge H_n(1-\frac{1}{n},s) = s-\frac{1}{n}+(\alpha-1)\frac{(1-\frac{2}{n})(s-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} = \alpha\frac{(s-\frac{1}{n})(1-\frac{1}{n}-\eta)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}$$
$$\ge \nu_n(s-\frac{1}{n})(1-\frac{1}{n}-s),$$

and if $\alpha \ge 1$, the minimum of $H_n(t,s)$ occurs at $t = \eta$, hence

$$H_n(t,s) \ge H_n(\eta,s) = s-\frac{1}{n}+(\alpha-1)\frac{(\eta-\frac{1}{n})(s-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} = \frac{(s-\frac{1}{n})(1-\frac{1}{n}-\eta)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}$$
$$\ge \nu_n(s-\frac{1}{n})(1-\frac{1}{n}-s).$$

**Case 2:** $s \ge \eta$, $t \le s$; using (2.1.3), we have

$$H_n(t,s) = \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \ge \frac{(\eta-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \ge \nu_n(s-\frac{1}{n})(1-\frac{1}{n}-s).$$

**Case 3:** $s \geq \eta$, $t \geq s$; using (2.1.3), we have

$$H_n(t,s) = \frac{(t-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} - (t-s) = s - \frac{1}{n} - (t-\frac{1}{n})\frac{(s-\frac{1}{n})-\alpha(\eta-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}.$$

If $s-\frac{1}{n} > \alpha(\eta-\frac{1}{n})$, the minimum of $H_n(t,s)$ occurs at $t = 1-\frac{1}{n}$, hence

$$H_n(t,s) \geq H_n(1-\frac{1}{n}, s) = \alpha\frac{(\eta-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \geq \nu_n(s-\frac{1}{n})(1-\frac{1}{n}-s),$$

and if $s-\frac{1}{n} \leq \alpha(\eta-\frac{1}{n})$, the minimum of $H_n(t,s)$ occurs at $t = s$, therefore

$$H_n(t,s) \geq H_n(s,s) = \frac{(s-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \geq \frac{(\eta-\frac{1}{n})(1-\frac{1}{n}-s)}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} \geq \nu_n(s-\frac{1}{n})(1-\frac{1}{n}-s).$$

□

## 2.2 Sufficient conditions for the existence of at least one solution

In this section, we establish existence of positive solution to the system of BVPs (2.0.2), (2.0.4). We say $(x,y)$ is a positive solution to system of BVPs (2.0.2), (2.0.4) if $(x,y) \in (C[0,1] \cap C^2(0,1)) \times (C[0,1] \cap C^2(0,1))$, $x > 0$ and $y > 0$ on $(0,1]$, $(x,y)$ satisfies (2.0.2) and (2.0.4). For this purpose, we consider the system of nonlinear non–singular BVPs

$$\begin{aligned}
&-x''(t) = f(t, \max\{y(t)+\frac{1}{n}, \frac{1}{n}\}), \quad t \in [\frac{1}{n}, 1-\frac{1}{n}],\\
&-y''(t) = g(t, \max\{x(t)+\frac{1}{n}, \frac{1}{n}\}), \quad t \in [\frac{1}{n}, 1-\frac{1}{n}],\\
&x(\frac{1}{n}) = 0,\ x(1-\frac{1}{n}) = \alpha x(\eta),\\
&y(\frac{1}{n}) = 0,\ y(1-\frac{1}{n}) = \alpha y(\eta).
\end{aligned} \tag{2.2.1}$$

We write (2.2.1) as an equivalent system of integral equations

$$\begin{aligned}
x(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \max\{y(s)+\frac{1}{n}, \frac{1}{n}\})ds, \quad t \in [\frac{1}{n}, 1-\frac{1}{n}],\\
y(t) &= \int_{1/n}^{1-1/n} H_n(t,s) g(s, \max\{x(s)+\frac{1}{n}, \frac{1}{n}\})ds, \quad t \in [\frac{1}{n}, 1-\frac{1}{n}].
\end{aligned} \tag{2.2.2}$$

Thus, $(x_n, y_n)$ is a solution of (2.2.1) if and only if $(x_n, y_n) \in \mathcal{E}_n \times \mathcal{E}_n$ and $(x_n, y_n)$ is a solution of (2.2.2).

Define operators $A_n, B_n, T_n : \mathcal{E}_n \to K_n$ by

$$\begin{aligned}
(A_n y)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \max\{y(s)+\frac{1}{n}, \frac{1}{n}\})ds, \quad t \in [\frac{1}{n}, 1-\frac{1}{n}],\\
(B_n x)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) g(s, \max\{x(s)+\frac{1}{n}, \frac{1}{n}\})ds, \quad t \in [\frac{1}{n}, 1-\frac{1}{n}],\\
(T_n x)(t) &= (A_n(B_n x))(t), \quad t \in [\frac{1}{n}, 1-\frac{1}{n}].
\end{aligned} \tag{2.2.3}$$

If $u_n \in K_n$ is a fixed point of $T_n$; then the system of BVPs (2.2.1) has a solution $(x_n, y_n)$ given by

$$\begin{cases} x_n(t) = u_n(t), & t \in [\frac{1}{n}, 1 - \frac{1}{n}], \\ y_n(t) = (B_n u_n)(t), & t \in [\frac{1}{n}, 1 - \frac{1}{n}]. \end{cases}$$

Assume that the following holds:

($\mathcal{H}_1$) there exist $K, L \in C((0,1),(0,\infty))$ and $F, G \in C((0,\infty),(0,\infty))$ such that

$$f(t,u) \le K(t)F(u), \quad g(t,u) \le L(t)G(u), \quad t \in (0,1),\ u \in (0,\infty),$$

where

$$a := \int_0^1 t(1-t)K(t)dt < +\infty, \quad b := \int_0^1 t(1-t)L(t)dt < +\infty.$$

**Lemma 2.2.1.** *Under the hypothesis ($\mathcal{H}_1$), the operator $T_n : \overline{\Omega}_r \cap K_n \to K_n$ is completely continuous.*

*Proof.* Firstly, we show that the operator $T_n(K_n) \subseteq K_n$. Clearly, for any $u \in K_n$, we have $(T_n u)(t) \ge 0$ and $(T_n u)''(t) \le 0$ for $t \in [\frac{1}{n}, 1 - \frac{1}{n}]$. Consequently, $T_n u \in K_n$ for all $u \in K_n$. Now, we show that $T_n : \overline{\Omega}_r \cap K_n \to K_n$ is uniformly bounded and equicontinuous. We introduce

$$\begin{aligned} d_n &= b\,\mu_n \max_{u\in[0,r]} G(u + \frac{1}{n}), \\ \omega_n &= \int_{1/n}^{1-1/n} f(t, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) g(s, u(s) + \frac{1}{n}) ds) dt. \end{aligned} \tag{2.2.4}$$

For any $u \in \overline{\Omega}_r \cap K_n$, using (2.2.3), ($\mathcal{H}_1$) and ($i$) of Lemma 2.1.2, we have

$$\begin{aligned} (T_n u)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\ &\le \int_{1/n}^{1-1/n} H_n(t,s) K(s) F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\ &\le \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) K(s) F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds. \end{aligned}$$

But, using ($\mathcal{H}_1$), ($i$) of Lemma 2.1.2 and (2.2.4),

$$\begin{aligned} 0 &\le \int_{1/n}^{1-1/n} H_n(t,s) g(s, u(s) + \frac{1}{n}) ds \le \int_{1/n}^{1-1/n} H_n(t,s) L(s) G(u(s) + \frac{1}{n}) ds \\ &\le \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) L(s) G(u(s) + \frac{1}{n}) ds \le \mu_n \max_{u\in[0,r]} G(u + \frac{1}{n}) \\ &\int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) L(s) ds \le b\mu_n \max_{u\in[0,r]} G(u + \frac{1}{n}) = d_n. \end{aligned}$$

Therefore,

$$(T_n u)(t) \le \mu_n \max_{u\in[0,d_n]} F(u+\frac{1}{n}) \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)K(s)ds \le a\mu_n \max_{u\in[0,d_n]} F(u+\frac{1}{n}),$$

which implies that

$$\|T_n u\|_{\mathcal{E}_n} \le a\mu_n \max_{u\in[0,d_n]} F(u+\frac{1}{n}),$$

that is, $T_n(\overline{\Omega}_r \cap K_n)$ is uniformly bounded. To show $T_n(\overline{\Omega}_r \cap K_n)$ is equicontinuous, let $t_1, t_2 \in [\frac{1}{n}, 1-\frac{1}{n}]$. Since $H_n$ is uniformly continuous on $[\frac{1}{n}, 1-\frac{1}{n}] \times [\frac{1}{n}, 1-\frac{1}{n}]$, for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta$ implies

$$|H_n(t_1,s) - H_n(t_2,s)| < \frac{\varepsilon}{\omega_n} \text{ for } s \in [\frac{1}{n}, 1-\frac{1}{n}]. \tag{2.2.5}$$

For any $u \in \overline{\Omega}_r \cap K_n$, using (2.2.3), (2.2.5) and (2.2.4), we obtain

$$|(T_n u)(t_1) - (T_n u)(t_2)| \le \int_{1/n}^{1-1/n} |H_n(t_1,s) - H_n(t_2,s)| f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)$$
$$g(\tau, u(\tau) + \frac{1}{n})d\tau)ds < \frac{\varepsilon}{\omega_n} \int_{1/n}^{1-1/n} f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n})d\tau)ds = \varepsilon.$$

Hence,

$$|(T_n u)(t_1) - (T_n u)(t_2)| < \varepsilon \text{ for all } u \in \overline{\Omega}_r \cap K_n,\ |t_1 - t_2| < \delta,$$

which implies that $T_n(\overline{\Omega}_r \cap K_n)$ is equicontinuous. By Theorem 1.1.6, $T_n(\overline{\Omega}_r \cap K_n)$ is relatively compact. Hence, $T_n$ is a compact operator.

Now, we show that $T_n$ is continuous. Let $u_m, u \in \overline{\Omega}_r \cap K_n$ such that

$$\|u_m - u\|_{\mathcal{E}_n} \to 0 \text{ as } m \to +\infty.$$

Using (2.2.3) and $(i)$ of Lemma 2.1.2, we have

$$|(T_n u_m)(t) - (T_n u)(t)| = \Big| \int_{1/n}^{1-1/n} H_n(t,s)\big(f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u_m(\tau) + \frac{1}{n})d\tau)$$
$$-f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n})d\tau)\big)ds \Big| \le \mu_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)$$
$$\Big| f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u_m(\tau) + \frac{1}{n})d\tau) - f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n})d\tau) \Big| ds.$$

Consequently,

$$\|T_n u_m - T_n u\|_{\mathcal{E}_n} \le \mu_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s) \Big| f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)$$
$$g(\tau, u_m(\tau) + \frac{1}{n})d\tau) - f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n})d\tau) \Big| ds.$$

By the Lebesgue dominated convergent theorem, it follows that

$$\|T_n u_m - T_n u\|_{\mathcal{E}_n} \to 0 \text{ as } m \to +\infty,$$

that is, $T_n : \overline{\Omega}_r \cap K_n \to K_n$ is a continuous. Hence, $T_n : \overline{\Omega}_r \cap K_n \to K_n$ is completely continuous. □

Assume that

$(\mathcal{H}_2)$ there exist $\alpha_1, \alpha_2 \in (0, \infty)$ with $\alpha_1\alpha_2 \leq 1$ such that

$$\lim_{u\to\infty} \frac{F(u)}{u^{\alpha_1}} \to 0, \quad \lim_{u\to\infty} \frac{G(u)}{u^{\alpha_2}} \to 0;$$

$(\mathcal{H}_3)$ there exist $\beta_1, \beta_2 \in (0, \infty)$ with $\beta_1\beta_2 \geq 1$ such that

$$\liminf_{u\to 0^+} \min_{t\in[\eta,1]} \frac{f(t,u)}{u^{\beta_1}} > 0, \quad \liminf_{u\to 0^+} \min_{t\in[\eta,1]} \frac{g(t,u)}{u^{\beta_2}} > 0.$$

**Theorem 2.2.2.** *Under the hypothesis* $(\mathcal{H}_1) - (\mathcal{H}_3)$, *the system of BVPs* (2.0.2), (2.0.4) *has at least one positive solution.*

*Proof.* By $(\mathcal{H}_2)$, there exist real constants $c_1, c_2, c_3, c_4 > 0$ such that

$$2^{2\alpha_1+\alpha_1\alpha_2} ab^{\alpha_1} \mu_n^{\alpha_1+1} c_1 c_3^{\alpha_1} < 1, \tag{2.2.6}$$

and

$$F(u + \frac{1}{n}) \leq c_1(u + \frac{1}{n})^{\alpha_1} + c_2, \; G(u + \frac{1}{n}) \leq c_3(u + \frac{1}{n})^{\alpha_2} + c_4 \text{ for } u \geq 0. \tag{2.2.7}$$

In view of (2.2.6), we choose a real constant $R > 0$ such that

$$R \geq \frac{a\mu_n c_2 + 2^{2\alpha_1} ab^{\alpha_1}\mu_n^{\alpha_1+1} c_1 c_4^{\alpha_1} + 2^{\alpha_1} a\mu_n n^{-\alpha_1} c_1 + 2^{2\alpha_1+\alpha_1\alpha_2} ab^{\alpha_1}\mu_n^{\alpha_1+1} n^{-\alpha_1\alpha_2} c_1 c_3^{\alpha_1}}{1 - 2^{2\alpha_1+\alpha_1\alpha_2} ab^{\alpha_1}\mu_n^{\alpha_1+1} c_1 c_3^{\alpha_1}}. \tag{2.2.8}$$

For any $u \in \partial\Omega_R \cap K_n$, using (2.2.3), $(\mathcal{H}_1)$ and (2.2.7), it follows that

$$\begin{aligned}
(T_n u)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\
&\leq \int_{1/n}^{1-1/n} H_n(t,s) K(s) F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\
&\leq \int_{1/n}^{1-1/n} H_n(t,s) K(s) (c_1(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau)^{\alpha_1} + c_2) ds \\
&= c_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s) (\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau)^{\alpha_1} ds \\
&+ c_2 \int_{1/n}^{1-1/n} H_n(t,s) K(s) ds.
\end{aligned}$$

Again using $(\mathcal{H}_1)$ and (2.2.7), we obtain

$$(T_n u)(t) \le c_1 \int_{1/n}^{1-1/n} H_n(t,s)K(s)(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)L(\tau)G(u(\tau)+\frac{1}{n})d\tau)^{\alpha_1} ds$$
$$+ c_2 \int_{1/n}^{1-1/n} H_n(t,s)K(s)ds \le c_1 \int_{1/n}^{1-1/n} H_n(t,s)K(s)(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)L(\tau)$$
$$(c_3(u(\tau)+\frac{1}{n})^{\alpha_2} + c_4)d\tau)^{\alpha_1} ds + c_2 \int_{1/n}^{1-1/n} H_n(t,s)K(s)ds \le c_1 \int_{1/n}^{1-1/n} H_n(t,s)K(s)ds$$
$$(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)L(\tau)d\tau(c_3(R+\frac{1}{n})^{\alpha_2} + c_4))^{\alpha_1} + c_2 \int_{1/n}^{1-1/n} H_n(t,s)K(s)ds.$$

Employing $(i)$ of Lemma 2.1.2 and $(\mathcal{H}_1)$, leads to

$$(T_n u)(t) \le c_1\mu_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)K(s)ds(\frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau-\frac{1}{n})(1-\frac{1}{n}-\tau)$$
$$L(\tau)d\tau(c_3(R+\frac{1}{n})^{\alpha_2}+c_4))^{\alpha_1} + c_2\mu_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)K(s)ds$$
$$\le a\mu_n c_1(\frac{1}{n} + b\,\mu_n(c_3(R+\frac{1}{n})^{\alpha_2}+c_4))^{\alpha_1} + a\mu_n c_2.$$

But,

$$(\frac{1}{n} + b\,\mu_n(c_3(R+\frac{1}{n})^{\alpha_2}+c_4))^{\alpha_1} \le 2^{\alpha_1}(\frac{1}{n^{\alpha_1}} + b^{\alpha_1}\,\mu_n^{\alpha_1}(c_3(R+\frac{1}{n})^{\alpha_2}+c_4)^{\alpha_1}).$$

Therefore,

$$(T_n u)(t) \le 2^{\alpha_1} a\mu_n c_1(\frac{1}{n^{\alpha_1}} + b^{\alpha_1}\,\mu_n^{\alpha_1}(c_3(R+\frac{1}{n})^{\alpha_2}+c_4)^{\alpha_1}) + a\mu_n c_2.$$

Also,

$$(c_3(R+\frac{1}{n})^{\alpha_2}+c_4)^{\alpha_1} \le 2^{\alpha_1}(c_3^{\alpha_1}(R+\frac{1}{n})^{\alpha_1\alpha_2} + c_4^{\alpha_1})$$
$$\le 2^{\alpha_1}(2^{\alpha_1\alpha_2}c_3^{\alpha_1}(R^{\alpha_1\alpha_2} + \frac{1}{n^{\alpha_1\alpha_2}}) + c_4^{\alpha_1}).$$

Consequently,

$$(T_n u)(t) \le 2^{\alpha_1} a\mu_n c_1(\frac{1}{n^{\alpha_1}} + 2^{\alpha_1}b^{\alpha_1}\,\mu_n^{\alpha_1}(2^{\alpha_1\alpha_2}c_3^{\alpha_1}(R^{\alpha_1\alpha_2} + \frac{1}{n^{\alpha_1\alpha_2}}) + c_4^{\alpha_1})) + a\mu_n c_2$$
$$= 2^{\alpha_1} a\mu_n n^{-\alpha_1} c_1 + 2^{2\alpha_1} ab^{\alpha_1}\mu_n^{\alpha_1+1} c_1(2^{\alpha_1\alpha_2}c_3^{\alpha_1}(R^{\alpha_1\alpha_2} + \frac{1}{n^{\alpha_1\alpha_2}}) + c_4^{\alpha_1}) + a\mu_n c_2$$
$$= 2^{\alpha_1} a\mu_n n^{-\alpha_1} c_1 + 2^{2\alpha_1+\alpha_1\alpha_2} ab^{\alpha_1}\mu_n^{\alpha_1+1} c_1 c_3^{\alpha_1}(R^{\alpha_1\alpha_2} + \frac{1}{n^{\alpha_1\alpha_2}}) + 2^{2\alpha_1} ab^{\alpha_1}\mu_n^{\alpha_1+1} c_1 c_4^{\alpha_1}$$
$$+ a\mu_n c_2 = 2^{\alpha_1} a\mu_n n^{-\alpha_1} c_1 + 2^{2\alpha_1+\alpha_1\alpha_2} ab^{\alpha_1}\mu_n^{\alpha_1+1} c_1 c_3^{\alpha_1} R^{\alpha_1\alpha_2}$$
$$+ 2^{2\alpha_1+\alpha_1\alpha_2} ab^{\alpha_1}\mu_n^{\alpha_1+1} n^{-\alpha_1\alpha_2} c_1 c_3^{\alpha_1} + 2^{2\alpha_1} ab^{\alpha_1}\mu_n^{\alpha_1+1} c_1 c_4^{\alpha_1} + a\mu_n c_2.$$

Using (2.2.8), we obtain

$$\|T_n u\|_{\mathcal{E}_n} \le \|u\|_{\mathcal{E}_n} \text{ for all } u \in \partial\Omega_R \cap K_n. \tag{2.2.9}$$

Now, by $(\mathcal{H}_3)$, there exist constants $c_5, c_6 > 0$ and $\rho \in (0, R)$ such that

$$f(t,x) \geq c_5 x^{\beta_1},\ g(t,x) \geq c_6 x^{\beta_2} \text{ for } x \in [0,\rho],\ t \in [\eta,1]. \tag{2.2.10}$$

Choose

$$r_n = \min\{\rho, \nu_n^{\beta_1+1} n^{-\beta_1\beta_2} c_5 c_6^{\beta_1} (\int_\eta^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)ds)^{\beta_1+1}\}. \tag{2.2.11}$$

For any $u \in \partial\Omega_{r_n} \cap K_n$ and $t \in [\eta, 1-\frac{1}{n}]$, using (2.2.3) and (2.2.10), we have

$$\begin{aligned}
(T_n u)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\
&\geq c_5 \int_{1/n}^{1-1/n} H_n(t,s)(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau)^{\beta_1} ds \\
&\geq c_5 \int_{1/n}^{1-1/n} H_n(t,s)(\int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau)^{\beta_1} ds \\
&\geq c_5 c_6^{\beta_1} \int_{1/n}^{1-1/n} H_n(t,s)(\int_{1/n}^{1-1/n} H_n(s,\tau)(u(\tau) + \frac{1}{n})^{\beta_2} d\tau)^{\beta_1} ds \\
&\geq n^{-\beta_1\beta_2} c_5 c_6^{\beta_1} \int_{1/n}^{1-1/n} H_n(t,s)(\int_{1/n}^{1-1/n} H_n(s,\tau) d\tau)^{\beta_1} ds \\
&\geq n^{-\beta_1\beta_2} c_5 c_6^{\beta_1} \int_{\eta}^{1-1/n} H_n(t,s)(\int_{\eta}^{1-1/n} H_n(s,\tau) d\tau)^{\beta_1} ds.
\end{aligned}$$

Employing $(ii)$ of Lemma 2.1.2, we get

$$\begin{aligned}
(T_n u)(t) &\geq \nu_n^{\beta_1+1} n^{-\beta_1\beta_2} c_5 c_6^{\beta_1} \int_\eta^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s) ds (\int_\eta^{1-1/n} (\tau - \frac{1}{n}) \\
&(1-\frac{1}{n}-\tau) d\tau)^{\beta_1} = \nu_n^{\beta_1+1} n^{-\beta_1\beta_2} c_5 c_6^{\beta_1} (\int_\eta^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s) ds)^{\beta_1+1}.
\end{aligned}$$

Using (2.2.11), we obtain

$$\|T_n u\|_{\mathcal{E}_n} \geq \|u\|_{\mathcal{E}_n} \text{ for all } u \in \partial\Omega_{r_n} \cap K_n. \tag{2.2.12}$$

In view of (2.2.9), (2.2.12) and by Theorem 1.2.8, $T_n$ has a fixed point $u_n \in (\overline{\Omega}_R \backslash \Omega_{r_n}) \cap K_n$. Note that

$$r_n \leq \|u_n\|_{\mathcal{E}_n} \leq R \tag{2.2.13}$$

and $r_n \to 0$ as $n \to \infty$. Thus, we have exhibited a uniform bound for each $u_n \in \mathcal{E}_n$, and $\{u_m\}_{m \geq n}$ is uniformly bounded on $[\frac{1}{n}, 1-\frac{1}{n}]$.

Now, we show that $\{u_m\}_{m \geq n}$, is equicontinuous on $[\frac{1}{n}, 1-\frac{1}{n}]$. For $t \in [\frac{1}{n}, 1-\frac{1}{n}]$, consider the integral equation

$$u_m(t) = u_m(\frac{1}{n}) + \frac{u_m(1-\frac{1}{n}) - \alpha u_m(\eta) - (1-\alpha) u_m(\frac{1}{n})}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}(t - \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(t,s) \tilde{f}(s) ds,$$

where $\tilde{f}(t) = f(t, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s)g(s, \frac{1}{n} + u_m(s))ds)$.

Which can also be written as

$$u_m(t) = u_m(\frac{1}{n}) + \frac{u_m(1-\frac{1}{n}) - \alpha u_m(\eta) - (1-\alpha)u_m(\frac{1}{n})}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}(t - \frac{1}{n}) + \frac{t - \frac{1}{n}}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}$$
$$\int_{1/n}^{1-1/n}(1 - \frac{1}{n} - s)\tilde{f}(s)ds - \frac{\alpha(t-\frac{1}{n})}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}\int_{1/n}^{\eta}(\eta - s)\tilde{f}(s)ds - \int_{1/n}^{t}(t-s)\tilde{f}(s)ds.$$

Differentiating with respect to $t$, we get

$$u_m'(t) = \frac{u_m(1-\frac{1}{n}) - \alpha u_m(\eta) - (1-\alpha)u_m(\frac{1}{n})}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta} + \frac{1}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}\int_{1/n}^{1-1/n}(1 - \frac{1}{n} - s)$$
$$\tilde{f}(s)ds - \frac{\alpha}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}\int_{1/n}^{\eta}(\eta - s)\tilde{f}(s)ds - \int_{1/n}^{t}\tilde{f}(s)ds,$$

which implies that

$$\|u_m'\|_{\mathcal{E}_n} \le \frac{2(1+\alpha)R}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta} + \frac{1}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}\int_{1/n}^{1-1/n}(1 - \frac{1}{n} - s)\tilde{f}(s)ds$$
$$+ \frac{\alpha}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha\eta}\int_{1/n}^{\eta}(\eta - s)\tilde{f}(s)ds + \int_{1/n}^{1-1/n}\tilde{f}(s)ds.$$

Hence, $\{u_m\}_{m\ge n}$ is equicontinuous on $[\frac{1}{n}, 1 - \frac{1}{n}]$.

For $m \ge n$, we define

$$v_m(t) = \begin{cases} u_m(\frac{1}{n}), & 0 \le t \le \frac{1}{n}, \\ u_m(t), & \frac{1}{n} \le t \le 1 - \frac{1}{n}, \\ \alpha u_m(\eta), & 1 - \frac{1}{n} \le t \le 1. \end{cases}$$

Since $v_m$ is a constant extension of $u_m$ to $[0,1]$, the sequence $\{v_m\}$ is uniformly bounded and equicontinuous on $[0,1]$. Thus, there exists a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ converging uniformly to $v \in C[0,1]$.

We introduce the notation

$$x_{m_k}(t) = v_{m_k}(t),\ y_{m_k}(t) = \int_{1/m_k}^{1-1/m_k} H_{m_k}(t,s)g(s, v_{m_k}(s) + \frac{1}{m_k})ds,$$
$$x(t) = \lim_{m_k\to\infty} x_{m_k}(t),\ y(t) = \lim_{m_k\to\infty} y_{m_k}(t).$$

For $t \in [0,1]$ consider the integral equations

$$x_{m_k}(t) = \int_{1/m_k}^{1-1/m_k} H_{m_k}(t,s)f(t, y_{m_k}(s) + \frac{1}{m_k})ds,$$
$$y_{m_k}(t) = \int_{1/m_k}^{1-1/m_k} H_{m_k}(t,s)g(t, x_{m_k}(s) + \frac{1}{m_k})ds.$$

Letting $m_k \to \infty$, we have

$$x(t) = \int_0^1 H(t,s)f(t,y(s))ds, \quad t \in [0,1],$$
$$y(t) = \int_0^1 H(t,s)g(s,x(s))ds, \quad t \in [0,1].$$

Moreover,

$$x(0) = 0,\ x(1) = \alpha x(\eta),\ y(0) = 0,\ y(1) = \alpha y(\eta).$$

Hence, $(x, y)$ is a solution of the system of BVPs (2.0.2), (2.0.4). Since

$$f, g : (0,1) \times (0,\infty) \to (0,\infty),$$

$f(t, 0)$, $g(t, 0)$ are not identically 0 and $H$ is of fixed sign on $(0, 1) \times (0, 1)$, it follows that $x > 0$ and $y > 0$ on $(0, 1]$. □

**Example 2.2.3.** Let

$$f(t,y) = \frac{1}{t(1-t)}\left(\frac{1}{y} + 3y^{1/3}\right), \quad g(t,x) = \frac{1}{t(1-t)}\left(\frac{1}{x} + 4x\right)$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{y} + 3y^{1/3}, \quad G(x) = \frac{1}{x} + 4x,$$

and $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 2$, $\beta_1 = \beta_2 = 1$. Clearly, $(\mathcal{H}_1) - (\mathcal{H}_3)$ are satisfied. Hence, by Theorem 2.2.2, the system of BVPs (2.0.2), (2.0.4) has a positive solution.

Assume that

$(\mathcal{H}_4)$ $f(t, u)$, $G(u)$ are non–increasing with respect to $u$ and for each fixed $n \in \{n_0, n_0 + 1, n_0 + 2, \cdots\}$, there exists a constant $\rho_n > 0$ such that

$$f(t, \frac{1}{n} + b\,\mu_n G(\frac{1}{n})) \geq \rho_n(\nu_n \int_\eta^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds)^{-1}, \quad t \in [\frac{1}{n}, 1 - \frac{1}{n}].$$

**Theorem 2.2.4.** *Under the hypothesis $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$, the system of BVPs* (2.0.2), (2.0.4) *has at least one positive solution.*

*Proof.* For any $u \in \partial\Omega_{\rho_n} \cap K_n$, using (2.2.3), $(i)$ of Lemma 2.1.2 and $(\mathcal{H}_1)$, we have

$$\begin{aligned}(T_n u)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\
&\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\
&\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) L(\tau) G(u(\tau) + \frac{1}{n}) d\tau) ds \\
&\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n G(\frac{1}{n}) \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) L(\tau) d\tau) ds \\
&\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + b\, \mu_n\, G(\frac{1}{n})) ds.\end{aligned}$$

Now in view of $(\mathcal{H}_4)$, we have

$$\begin{aligned}(T_n u)(t) &\geq \rho_n \int_{1/n}^{1-1/n} H_n(t,s) ds (\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) d\tau)^{-1} \\
&\geq \rho_n \nu_n \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) ds (\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) d\tau)^{-1} = \rho_n,\end{aligned}$$

which implies that

$$\|T_n u\|_{\mathcal{E}_n} \geq \|u\|_{\mathcal{E}_n} \text{ for all } u \in \partial\Omega_{\rho_n} \cap K_n. \tag{2.2.14}$$

In view of $(\mathcal{H}_2)$, we can choose $R > \rho_n$ such that (2.2.9) holds. Hence, in view of (2.2.9), (2.2.14) and by Theorem 1.2.8, $T_n$ has a fixed point $u_n \in (\overline{\Omega}_R \setminus \Omega_{\rho_n}) \cap K_n$. Now, by the same process as done in Theorem 2.2.2, the system of BVPs (2.0.2), (2.0.4) has a positive solution. □

**Example 2.2.5.** Let

$$f(t,y) = \frac{e^{\frac{1}{y}}}{t(1-t)}, \quad g(t,x) = \frac{e^{\frac{1}{x}}}{t(1-t)}$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = e^{\frac{1}{y}}, \quad G(x) = e^{\frac{1}{x}}.$$

Choose $\rho_n \leq \frac{4(n-3)}{n} e^{\frac{n}{1+6ne^n}} \int_{1/3}^{1-1/n} (s - 1/n)(1 - 1/n - s) ds$. Then $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$ are satisfied. Hence, by Theorem 2.2.4, the system of BVPs (2.0.2), (2.0.4) has a positive solution.

Assume that

$(\mathcal{H}_5)$ $F(u)$, $g(t,u)$ are non–increasing with respect to $u$ and for each fixed $n \in \{n_0, n_0 + 1, n_0 + 2, \cdots\}$, there exists a constant $M > 0$ such that

$$a\, \mu_n F(\nu_n \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) g(s, M + \frac{1}{n}) ds) \leq M.$$

**Theorem 2.2.6.** *Under the hypothesis* $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ *and* $(\mathcal{H}_5)$, *the system of BVPs* (2.0.2), (2.0.4) *has at least one positive solution.*

*Proof.* For any $u \in \partial\Omega_M \cap K_n$, using (2.2.3), $(\mathcal{H}_1)$ and $(\mathcal{H}_5)$, we obtain

$$\begin{aligned}
(T_nu)(t) &= \int_{1/n}^{1-1/n} H_n(t,s)f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)g(\tau,u(\tau)+\frac{1}{n})d\tau)ds \\
&\le \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s,\tau)g(\tau,u(\tau)+\frac{1}{n})d\tau)ds \\
&\le \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\int_{1/n}^{1-1/n} H_n(s,\tau)g(\tau,u(\tau)+\frac{1}{n})d\tau)ds \\
&\le \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\int_{1/n}^{1-1/n} H_n(s,\tau)g(\tau,M+\frac{1}{n})d\tau)ds \\
&\le \int_{1/n}^{1-1/n} H_n(t,s)K(s)F(\int_{\eta}^{1-1/n} H_n(s,\tau)g(\tau,M+\frac{1}{n})d\tau)ds.
\end{aligned}$$

Employing $(ii)$ of Lemma 2.1.2 and $(\mathcal{H}_5)$, leads to

$$(T_nu)(t) \le F(\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1-\frac{1}{n}-\tau)g(\tau,M+\frac{1}{n})d\tau) \int_{1/n}^{1-1/n} H_n(t,s)K(s)ds.$$

Now, using $(i)$ of Lemma 2.1.2, $(\mathcal{H}_1)$ and $(\mathcal{H}_5)$, we obtain

$$\begin{aligned}
&(T_nu)(t) \le \mu_n F(\nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1-\frac{1}{n}-\tau)g(\tau,M+\frac{1}{n})d\tau) \int_{1/n}^{1-1/n} (s-\frac{1}{n}) \\
&(1-\frac{1}{n}-s)K(s)ds \le a\mu_n F(\nu_n \int_{\eta}^{1-1/n} (\tau-\frac{1}{n})(1-\frac{1}{n}-\tau)g(\tau,M+\frac{1}{n})d\tau) \le M,
\end{aligned}$$

which implies that

$$\|T_nu\|_{\mathcal{E}_n} \le \|u\|_{\mathcal{E}_n} \text{ for all } u \in \partial\Omega_M \cap K_n. \tag{2.2.15}$$

By $(\mathcal{H}_3)$, we can choose $r_n \in (0,M)$ such that (2.2.12) holds. Hence, in view of (2.2.15), (2.2.12) and by Theorem 1.2.8, $T_n$ has a fixed point $u_n \in (\overline{\Omega}_M \setminus \Omega_{r_n}) \cap K_n$. By the same process as done in Theorem 2.2.2, the system of BVPs (2.0.2), (2.0.4) has a positive solution. □

**Example 2.2.7.** Let

$$f(t,y) = \begin{cases} \frac{ye^{\frac{1}{y}}}{t(1-t)}, & y \le 1, \\ \frac{e}{t(1-t)}, & y > 1, \end{cases} \qquad g(t,x) = \begin{cases} \frac{xe^{\frac{1}{x}}}{t(1-t)}, & x \le 1, \\ \frac{e}{t(1-t)}, & x > 1, \end{cases}$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \begin{cases} ye^{\frac{1}{y}}, & y \le 1, \\ e, & y > 1, \end{cases} \quad G(x) = \begin{cases} xe^{\frac{1}{x}}, & x \le 1, \\ e, & x > 1, \end{cases}$$

and $\beta_1 = \beta_2 = 1$. Choose a constant $M > 0$ such that

$$M \geq \max\{1,\, 6\, F(e(1-\frac{3}{n})\int_{1/3}^{1-1/n}\frac{(t-\frac{1}{n})(1-\frac{1}{n}-t)}{t(1-t)}ds)\}.$$

Then $(\mathcal{H}_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_5)$ are satisfied. Hence, by Theorem 2.2.6, the system of BVPs (2.0.2), (2.0.4) has a positive solution.

**Theorem 2.2.8.** *Under the hypothesis* $(\mathcal{H}_1)$, $(\mathcal{H}_4)$ *and* $(\mathcal{H}_5)$, *the system of BVPs* (2.0.2), (2.0.4) *has at least one positive solution.*

*Proof.* By $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$, we obtain (2.2.14). By $(\mathcal{H}_5)$ we can choose a constant $M > \rho_n$ such that (2.2.15) holds. Then by Theorem 1.2.8, $T_n$ has a fixed point $u_n \in (\overline{\Omega}_M \backslash \Omega_{\rho_n}) \cap K_n$. By the same process as done in Theorem 2.2.2, the system of BVPs (2.0.2), (2.0.4) has a positive solution. □

**Example 2.2.9.** Let

$$f(t,y) = \frac{1}{t(1-t)}\frac{1}{\sqrt{y}}, \quad g(t,x) = \frac{1}{t(1-t)}\frac{1}{x^2}$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{\sqrt{y}}, \quad G(x) = \frac{1}{x^2}.$$

Choose constants $\rho_n$ and $M$ such that

$$\begin{aligned} \rho_n &\leq \frac{4(n-3)}{\sqrt{n(6n^3+1)}}\int_{1/3}^{1-1/n}(t-\frac{1}{n})(1-\frac{1}{n}-t)dt,\\ M &\geq \frac{1}{n}\big(\frac{1}{6}\big((1-\frac{3}{n})\int_{1/3}^{1-1/n}\frac{(t-\frac{1}{n})(1-\frac{1}{n}-t)}{t(1-t)}dt\big)^{1/2}-1\big)^{-1}. \end{aligned}$$

Then $(\mathcal{H}_1)$, $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ are satisfied. Hence, by Theorem 2.2.8, the system of BVPs (2.0.2), (2.0.4) has a positive solution.

## 2.3 Sufficient conditions for the existence of at least one solution of more general systems

In this section, we establish the existence of positive solution to the system of BVPs (2.0.3), (2.0.4). We say $(x, y)$ is a positive solution to the system of BVPs (2.0.3), (2.0.4) if $(x,y) \in (C[0,1] \cap C^2(0,1)) \times (C[0,1] \cap C^2(0,1))$, $x > 0$ and $y > 0$ on $(0,1]$, $(x,y)$ satisfies (2.0.3) and (2.0.4). Therefore, we consider the nonlinear non–singular system of BVPs

$$\begin{aligned} -x''(t) &= f(t, \max\{x(t)+\frac{1}{n}, \frac{1}{n}\}, \max\{y(t)+\frac{1}{n}, \frac{1}{n}\}), \quad t \in [\frac{1}{n}, 1-\frac{1}{n}],\\ -y''(t) &= g(t, \max\{x(t)+\frac{1}{n}, \frac{1}{n}\}, \max\{y(t)+\frac{1}{n}, \frac{1}{n}\}), \quad t \in [\frac{1}{n}, 1-\frac{1}{n}],\\ x(\frac{1}{n}) &= y(\frac{1}{n}) = 0,\ x(1-\frac{1}{n}) = \alpha x(\eta),\ y(1-\frac{1}{n}) = \alpha y(\eta). \end{aligned} \tag{2.3.1}$$

We write (2.3.1) as an equivalent system of integral equations

$$\begin{aligned} x(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \max\{x(s)+\frac{1}{n},\frac{1}{n}\}, \max\{y(s)+\frac{1}{n},\frac{1}{n}\})ds, \\ y(t) &= \int_{1/n}^{1-1/n} H_n(t,s) g(s, \max\{x(s)+\frac{1}{n},\frac{1}{n}\}, \max\{y(s)+\frac{1}{n},\frac{1}{n}\})ds. \end{aligned} \tag{2.3.2}$$

By a solution of the system of BVPs (2.3.1), we mean a solution of the corresponding system of integral equations (2.3.2). Define an operator $T_n : \mathcal{E}_n \times \mathcal{E}_n \to K_n \times K_n$ by

$$T_n(x,y) = (A_n(x,y), B_n(x,y)), \tag{2.3.3}$$

where the operators $A_n, B_n : \mathcal{E}_n \times \mathcal{E}_n \to K_n$ are defined by

$$\begin{aligned} A_n(x,y)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, \max\{x(s)+\frac{1}{n},\frac{1}{n}\}, \max\{y(s)+\frac{1}{n},\frac{1}{n}\})ds, \\ B_n(x,y)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) g(s, \max\{x(s)+\frac{1}{n},\frac{1}{n}\}, \max\{y(s)+\frac{1}{n},\frac{1}{n}\})ds. \end{aligned} \tag{2.3.4}$$

Clearly, if $(x_n, y_n) \in \mathcal{E}_n \times \mathcal{E}_n$ is a fixed point of $T_n$; then $(x_n, y_n)$ also satisfies the system of BVPs (2.3.1).

Assume that the following holds:

($\mathcal{H}_6$) there exist $K, L \in C((0,1),(0,\infty))$ and $F, G \in C((0,\infty)\times(0,\infty),(0,\infty))$ such that

$$f(t,x,y) \le K(t)F(x,y), \quad g(t,x,y) \le L(t)G(x,y), \quad t \in (0,1),\ x,y \in (0,\infty),$$

where

$$a := \int_0^1 t(1-t)K(t)dt < +\infty, \quad b := \int_0^1 t(1-t)L(t)dt < +\infty.$$

**Lemma 2.3.1.** *Under the hypothesis ($\mathcal{H}_6$), the operator $T_n : \overline{\mathcal{O}}_r \cap (K_n \times K_n) \to K_n \times K_n$ is completely continuous.*

*Proof.* First we show that $T_n(K_n \times K_n) \subseteq K_n \times K_n$. Clearly, for any $(u,v) \in K_n \times K_n$, $A_n(u,v)(t) \ge 0$, $B_n(u,v)(t) \ge 0$, $A_n(u,v)''(t) \le 0$ and $B_n(u,v)''(t) \le 0$ for $t \in [\frac{1}{n}, 1-\frac{1}{n}]$. Consequently, $T_n(u,v) \in K_n \times K_n$ for all $(u,v) \in K_n \times K_n$. Now, we show that the operator $A_n : \overline{\mathcal{O}}_r \cap (K_n \times K_n) \to K_n$ is uniformly bounded and equicontinuous. We introduce

$$\omega_n = \max\{\max_{(u,v)\in\overline{\mathcal{O}}_r\cap(K_n\times K_n)} F(u+\frac{1}{n}, v+\frac{1}{n}), \max_{(u,v)\in\overline{\mathcal{O}}_r\cap(K_n\times K_n)} G(u+\frac{1}{n}, v+\frac{1}{n})\}. \tag{2.3.5}$$

For $(u,v) \in \overline{\mathcal{O}}_r \cap (K_n \times K_n)$, using (2.3.4), ($\mathcal{H}_6$) and (2.3.5), we have

$$\begin{aligned} A_n(u,v)(t) &= \int_{1/n}^{1-1/n} H_n(t,s) f(s, u(s)+\frac{1}{n}, v(s)+\frac{1}{n})ds \le \int_{1/n}^{1-1/n} H_n(t,s)K(s) \\ &F(u(s)+\frac{1}{n}, v(s)+\frac{1}{n})ds \le \mu_n\omega_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)K(s)ds \\ &\le \mu_n\omega_n \int_{1/n}^{1-1/n} s(1-s)K(s)ds \le a\mu_n\omega_n \end{aligned}$$

which implies that

$$\|A_n(u,v)\|_{\mathcal{E}_n} \leq a\mu_n\omega_n < +\infty,$$

that is, $A_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is uniformly bounded. Similarly, using (2.3.4), $(\mathcal{H}_6)$ and (2.3.5), we can show that $B_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is also uniformly bounded. Thus, $T_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is uniformly bounded. To show $A_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is equicontinuous, let $t_1, t_2 \in [\frac{1}{n}, 1-\frac{1}{n}]$ with $t_1 \leq t_2$. Since $H_n$ is uniformly continuous on $[\frac{1}{n}, 1-\frac{1}{n}] \times [\frac{1}{n}, 1-\frac{1}{n}]$, for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta$ implies

$$|H_n(t_1,s) - H_n(t_2,s)| < \min\{\frac{1}{a}, \frac{1}{b}\}\frac{\varepsilon}{\omega_n} \text{ for } s \in [\frac{1}{n}, 1-\frac{1}{n}]. \tag{2.3.6}$$

For $(u,v) \in \overline{\mathcal{O}}_r \cap (K_n \times K_n)$, using (2.3.4), $(\mathcal{H}_6)$, (2.3.5) and (2.3.6), we have

$$|A_n(u,v)(t_1) - A_n(u,v)(t_2)| = \left|\int_{1/n}^{1-1/n} (H_n(t_1,s) - H_n(t_2,s))f(s, u(s)+\frac{1}{n}, v(s)+\frac{1}{n})ds\right|$$
$$\leq \int_{1/n}^{1-1/n} |H_n(t_1,s) - H_n(t_2,s)| f(s, u(s)+\frac{1}{n}, v(s)+\frac{1}{n})ds$$
$$\leq \int_{1/n}^{1-1/n} |H_n(t_1,s) - H_n(t_2,s)| K(s)F(u(s)+\frac{1}{n}, v(s)+\frac{1}{n})ds$$
$$\leq \omega_n \int_{1/n}^{1-1/n} |H_n(t_1,s) - H_n(t_2,s)| K(s)ds < \omega_n \frac{\varepsilon}{a\omega_n}\int_{1/n}^{1-1/n} K(s)ds \leq \frac{\varepsilon}{a}\int_0^1 K(s)ds = \varepsilon.$$

Hence,

$$|A_n(u,v)(t_1) - A_n(u,v)(t_2)| < \varepsilon \text{ for all } (x,y) \in \overline{\Omega}_r \cap (K_n \times K_n),\ |t_1 - t_2| < \delta,$$

which implies that $A_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is equicontinuous. Similarly, using (2.3.4), we can show that $B_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is also equicontinuous. Thus, $T_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is equicontinuous. This together with uniform boundedness of $T_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ and by Theorem 1.1.6, it follows that $T_n(\overline{\mathcal{O}}_r \cap (K_n \times K_n))$ is relatively compact. Hence, $T_n$ is a compact operator.

Now, we show that $T_n$ is continuous. Let $(u_m, v_m), (u,v) \in K_n \times K_n$ such that

$$\|(u_m, v_m) - (x,y)\|_{\mathcal{E}_n \times \mathcal{E}_n} \to 0 \text{ as } m \to +\infty.$$

Using (2.3.4) and $(i)$ of Lemma 2.1.2, we have

$$|A_n(u_m,v_m)(t) - A_n(u,v)(t)| = \left|\int_{1/n}^{1-1/n} H_n(t,s)(f(s, u_m(s)+\frac{1}{n}, v_m(s)+\frac{1}{n})\right.$$
$$\left. - f(s, u(s)+\frac{1}{n}, v(s)+\frac{1}{n}))ds\right| \leq \int_{1/n}^{1-1/n} H_n(t,s)\left|f(s, u_m(s)+\frac{1}{n}, v_m(s)+\frac{1}{n})\right.$$
$$\left. - f(s, u(s)+\frac{1}{n}, v(s)+\frac{1}{n})\right| ds \leq \mu_n \int_{1/n}^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)\left|f(s, u_m(s)+\frac{1}{n}, v_m(s)+\frac{1}{n})\right.$$
$$\left. - f(s, u(s)+\frac{1}{n}, v(s)+\frac{1}{n})\right| ds.$$

Consequently,

$$\|A_n(u_m, v_m) - A_n(u, v)\|_{\mathcal{E}_n} \le \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) \left| f(s, u_m(s) + \frac{1}{n}, v_m(s) + \frac{1}{n}) - f(s, u(s) + \frac{1}{n}, v(s) + \frac{1}{n}) \right| ds.$$

By Lebesgue dominated convergence theorem, it follows that

$$\|A_n(u_m, v_m) - A_n(u, v)\|_{\mathcal{E}_n} \to 0 \text{ as } m \to +\infty. \tag{2.3.7}$$

Similarly, by using (2.3.4) and $(i)$ of Lemma 2.1.2, we have

$$\|B_n(u_m, v_m) - B_n(u, v)\|_{\mathcal{E}_n} \to 0 \text{ as } m \to +\infty. \tag{2.3.8}$$

From (2.3.7), (2.3.8) and (2.3.3), it follows that

$$\|T_n(u_m, v_m) - T_n(u, v)\|_{\mathcal{E}_n \times \mathcal{E}_n} \to 0 \text{ as } m \to +\infty,$$

that is, $T_n : \overline{\mathcal{O}}_r \cap (K_n \times K_n) \to K_n \times K_n$ is continuous. Hence, $T_n : \overline{\mathcal{O}}_r \cap (K_n \times K_n) \to K_n \times K_n$ is completely continuous. □

For $(x, y) \in \mathbb{R}^2$, we write $|(x, y)| = |x| + |y|$. Assume that

$(\mathcal{H}_7)$ There exist real constants $\tilde{\beta}$, $\tilde{\rho}$ with $0 < \tilde{\beta}, \tilde{\rho} \le 1$ such that

$$\lim_{|(x,y)|\to\infty} \frac{F(x, y)}{|(x, y)|^{\tilde{\beta}}} \to 0, \qquad \lim_{|(x,y)|\to\infty} \frac{G(x, y)}{|(x, y)|^{\tilde{\rho}}} \to 0;$$

$(\mathcal{H}_8)$ There exist real constants $\alpha_i$, $\beta_i$ with $\alpha_i \le 0 \le \beta_i < 1$, $i = 1, 2$; such that for all $t \in (0, 1)$, $x, y \in (0, \infty)$,

$$\begin{aligned}
c^{\beta_1} f(t, x, y) &\le f(t, cx, y) \le c^{\alpha_1} f(t, x, y), && \text{if } 0 < c \le 1,\\
c^{\alpha_1} f(t, x, y) &\le f(t, cx, y) \le c^{\beta_1} f(t, x, y), && \text{if } c \ge 1,\\
c^{\beta_2} f(t, x, y) &\le f(t, x, cy) \le c^{\alpha_2} f(t, x, y), && \text{if } 0 < c \le 1,\\
c^{\alpha_2} f(t, x, y) &\le f(t, x, cy) \le c^{\beta_2} f(t, x, y), && \text{if } c \ge 1;
\end{aligned}$$

$(\mathcal{H}_9)$ There exist real constants $\gamma_i$, $\rho_i$ with $\gamma_i \le 0 \le \rho_i < 1$, $i = 1, 2$; such that for all $t \in (0, 1)$, $x, y \in (0, \infty)$,

$$\begin{aligned}
c^{\rho_1} g(t, x, y) &\le g(t, cx, y) \le c^{\gamma_1} g(t, x, y), && \text{if } 0 < c \le 1,\\
c^{\gamma_1} g(t, x, y) &\le g(t, cx, y) \le c^{\rho_1} g(t, x, y), && \text{if } c \ge 1,\\
c^{\rho_2} g(t, x, y) &\le g(t, x, cy) \le c^{\gamma_2} g(t, x, y), && \text{if } 0 < c \le 1,\\
c^{\gamma_2} g(t, x, y) &\le g(t, x, cy) \le c^{\rho_2} g(t, x, y), && \text{if } c \ge 1.
\end{aligned}$$

**Theorem 2.3.2.** *Under the hypothesis* $(\mathcal{H}_6) - (\mathcal{H}_9)$*, the system of BVPs* (2.0.3), (2.0.4) *has at least one positive solution.*

*Proof.* By $(\mathcal{H}_7)$, there exist real constants $c_7, c_8, c_9, c_{10} > 0$ such that

$$a\mu_n 2^{1+\tilde{\beta}} c_7 < 1,\ b\mu_n 2^{1+\tilde{\rho}} c_9 < 1 \tag{2.3.9}$$

and

$$F(x+\frac{1}{n}, y+\frac{1}{n}) \leq c_7(x+y+\frac{2}{n})^{\tilde{\beta}} + c_8,\ G(x+\frac{1}{n}, y+\frac{1}{n}) \leq c_9(x+y+\frac{2}{n})^{\tilde{\rho}} + c_{10} \text{ for } |(x,y)| \geq 0. \tag{2.3.10}$$

In view of (2.3.9), we choose a real constant $R > 0$ such that

$$R \geq \max\left\{\frac{a\mu_n 2^{1+2\tilde{\beta}} n^{-\tilde{\beta}} c_7 + a\mu_n c_8}{1 - a\mu_n 2^{1+\tilde{\beta}} c_7}, \frac{b\mu_n 2^{1+2\tilde{\rho}} n^{-\tilde{\rho}} c_9 + b\mu_n c_{10}}{1 - b\mu_n 2^{1+\tilde{\rho}} c_9}\right\}. \tag{2.3.11}$$

For any $(u,v) \in \partial\mathcal{O}_R \cap (K_n \times K_n)$, using (2.3.4), $(\mathcal{H}_6)$, (2.3.10) and $(i)$ of Lemma 2.1.2, we have

$$\begin{aligned}
&A_n(u,v)(t) = \int_{1/n}^{1-1/n} H_n(t,s) f(s, u(s)+\frac{1}{n}, v(s)+\frac{1}{n}) ds \leq \int_{1/n}^{1-1/n} H_n(t,s) K(s) \\
&F(u(s)+\frac{1}{n}, v(s)+\frac{1}{n}) ds \leq \int_{1/n}^{1-1/n} H_n(t,s) K(s)(c_7(u(s)+v(s)+\frac{2}{n})^{\tilde{\beta}} + c_8) ds \\
&\leq \int_{1/n}^{1-1/n} H_n(t,s) K(s)(c_7(R+\frac{2}{n})^{\tilde{\beta}} + c_8) ds \leq \mu_n (c_7(R+\frac{2}{n})^{\tilde{\beta}} + c_8) \int_{1/n}^{1-1/n} (s-\frac{1}{n}) \\
&(1-\frac{1}{n}-s) K(s) ds \leq \mu_n (c_7(R+\frac{2}{n})^{\tilde{\beta}} + c_8) \int_{1/n}^{1-1/n} s(1-s) K(s) ds \\
&\leq \mu_n (c_7(R+\frac{2}{n})^{\tilde{\beta}} + c_8) \int_0^1 s(1-s) K(s) ds = a\mu_n (c_7(R+\frac{2}{n})^{\tilde{\beta}} + c_8).
\end{aligned}$$

But,

$$(R+\frac{2}{n})^{\tilde{\beta}} \leq 2^{\tilde{\beta}}(R^{\tilde{\beta}} + \frac{2^{\tilde{\beta}}}{n^{\tilde{\beta}}}).$$

Therefore, in view of (2.3.11), we have

$$A_n(u,v)(t) \leq a\mu_n(2^{\tilde{\beta}} c_7 (R^{\tilde{\beta}} + \frac{2^{\tilde{\beta}}}{n^{\tilde{\beta}}}) + c_8) = a\mu_n 2^{\tilde{\beta}} R^{\tilde{\beta}} c_7 + a\mu_n 2^{2\tilde{\beta}} n^{-\tilde{\beta}} c_7 + a\mu_n c_8 \leq \frac{R}{2}.$$

Thus,

$$\|A_n(u,v)\|_{\mathcal{E}_n} \leq \frac{\|(u,v)\|_{\mathcal{E}_n \times \mathcal{E}_n}}{2} \text{ for all } (u,v) \in \partial\mathcal{O}_R \cap (K_n \times K_n). \tag{2.3.12}$$

Similarly, using (2.3.4), $(\mathcal{H}_6)$, (2.3.10), $(i)$ of Lemma 2.1.2, (2.3.11), we obtain

$$\|B_n(u,v)\|_{\mathcal{E}_n} \leq \frac{\|(u,v)\|_{\mathcal{E}_n \times \mathcal{E}_n}}{2} \text{ for all } (u,v) \in \partial\mathcal{O}_R \cap (K_n \times K_n). \tag{2.3.13}$$

From (2.3.12), (2.3.13) and (2.3.3), it follows that

$$\|T_n(u,v)\|_{\mathcal{E}_n\times\mathcal{E}_n} \le \|(u,v)\|_{\mathcal{E}_n\times\mathcal{E}_n} \text{ for all } (u,v)\in\partial\mathcal{O}_R\cap(K_n\times K_n). \tag{2.3.14}$$

Choose a real constant $r_n\in(0,R)$ such that

$$r_n+\frac{1}{n}\le\min\left\{1,\frac{1}{n}+\frac{2\nu_n}{n^{\alpha_1^2+\alpha_2^2-\alpha_1\beta_1-\alpha_2\beta_2+\beta_1+\beta_2}}\int_\eta^{1-1/n}(s-\frac{1}{n})(1-\frac{1}{n}-s)f(s,1,1)ds,\right.$$
$$\left.\frac{1}{n}+\frac{2\nu_n}{n^{\gamma_1^2+\gamma_2^2-\gamma_1\rho_1-\gamma_2\rho_2+\gamma_1+\rho_2}}\int_\eta^{1-1/n}(s-\frac{1}{n})(1-\frac{1}{n}-s)g(s,1,1)ds\right\}. \tag{2.3.15}$$

Choose constants $c_{11}=(r_n+\frac{1}{n})^{-\alpha_1}$ and $c_{12}=(r_n+\frac{1}{n})^{-\alpha_2}$. For any $(u,v)\in\partial\mathcal{O}_{r_n}\cap(K_n\times K_n)$, using (2.3.4), $(\mathcal{H}_8)$ and $(\mathcal{H}_9)$, we have

$$\begin{aligned}
A_n(u,v)(t) &= \int_{1/n}^{1-1/n}H_n(t,s)f(s,u(s)+\frac{1}{n},v(s)+\frac{1}{n})ds\\
&= \int_{1/n}^{1-1/n}H_n(t,s)f(s,c_{11}\frac{u(s)+\frac{1}{n}}{c_{11}},c_{12}\frac{v(s)+\frac{1}{n}}{c_{12}})ds\\
&\ge (\frac{1}{c_{11}})^{\alpha_1}\int_{1/n}^{1-1/n}H_n(t,s)f(s,c_{11}(u(s)+\frac{1}{n}),c_{12}\frac{v(s)+\frac{1}{n}}{c_{12}})ds\\
&\ge (\frac{1}{c_{11}})^{\alpha_1}(\frac{1}{c_{12}})^{\alpha_2}\int_{1/n}^{1-1/n}H_n(t,s)f(s,c_{11}(u(s)+\frac{1}{n}),c_{12}(v(s)+\frac{1}{n}))ds\\
&\ge c_{11}^{\beta_1-\alpha_1}c_{12}^{-\alpha_2}\int_{1/n}^{1-1/n}H_n(t,s)(u(s)+\frac{1}{n})^{\beta_1}f(s,1,c(v(s)+\frac{1}{n}))ds\\
&\ge c_{11}^{\beta_1-\alpha_1}c_{12}^{\beta_2-\alpha_2}\int_{1/n}^{1-1/n}H_n(t,s)(u(s)+\frac{1}{n})^{\beta_1}(v(s)+\frac{1}{n})^{\beta_2}f(s,1,1)ds\\
&\ge c_{11}^{\beta_1-\alpha_1}c_{12}^{\beta_2-\alpha_2}\frac{1}{n^{\beta_1+\beta_2}}\int_{1/n}^{1-1/n}H_n(t,s)f(s,1,1)ds\\
&\ge c_{11}^{\beta_1-\alpha_1}c_{12}^{\beta_2-\alpha_2}\frac{1}{n^{\beta_1+\beta_2}}\int_{\eta}^{1-1/n}H_n(t,s)f(s,1,1)ds.
\end{aligned}$$

For $t\in[\eta,1-\frac{1}{n}]$, using $(ii)$ of Lemma 2.1.2, we obtain

$$\begin{aligned}
A_n(u,v)(t) &\ge c_{11}^{\beta_1-\alpha_1}c_{12}^{\beta_2-\alpha_2}\frac{\nu_n}{n^{\beta_1+\beta_2}}\int_\eta^{1-1/n}(s-\frac{1}{n})(1-\frac{1}{n}-s)f(s,1,1)ds\\
&= (r_n+\frac{1}{n})^{-\alpha_1(\beta_1-\alpha_1)}(r_n+\frac{1}{n})^{-\alpha_2(\beta_2-\alpha_2)}\frac{\nu_n}{n^{\beta_1+\beta_2}}\int_\eta^{1-1/n}(s-\frac{1}{n})(1-\frac{1}{n}-s)f(s,1,1)ds\\
&= (r_n+\frac{1}{n})^{\alpha_1^2+\alpha_2^2-\alpha_1\beta_1-\alpha_2\beta_2}\frac{\nu_n}{n^{\beta_1+\beta_2}}\int_\eta^{1-1/n}(s-\frac{1}{n})(1-\frac{1}{n}-s)f(s,1,1)ds\\
&\ge \frac{\nu_n}{n^{\alpha_1^2+\alpha_2^2-\alpha_1\beta_1-\alpha_2\beta_2+\beta_1+\beta_2}}\int_\eta^{1-1/n}(s-\frac{1}{n})(1-\frac{1}{n}-s)f(s,1,1)ds,
\end{aligned}$$

which implies that

$$\|A_n(u,v)\|_{\mathcal{E}_n}\ge\frac{\nu_n}{n^{\alpha_1^2+\alpha_2^2-\alpha_1\beta_1-\alpha_2\beta_2+\beta_1+\beta_2}}\int_\eta^{1-1/n}(s-\frac{1}{n})(1-\frac{1}{n}-s)f(s,1,1)ds,$$

Thus, in view of (2.3.15), it follows that

$$\|A_n(u,v)\|_{\mathcal{E}_n} \geq \frac{\|(u,v)\|_{\mathcal{E}_n\times\mathcal{E}_n}}{2} \text{ for all } (u,v)\in\partial\mathcal{O}_{r_n}\cap(K_n\times K_n). \tag{2.3.16}$$

Similarly, using (2.3.4), $(\mathcal{H}_8)$, $(\mathcal{H}_9)$, $(ii)$ of Lemma 2.1.2 and (2.3.15), we get

$$\|B_n(u,v)\|_{\mathcal{E}_n} \geq \frac{\|(u,v)\|_{\mathcal{E}_n\times\mathcal{E}_n}}{2} \text{ for all } (u,v)\in\partial\mathcal{O}_{r_n}\cap(K_n\times K_n). \tag{2.3.17}$$

From (2.3.16), (2.3.17) and (2.3.3), we obtain

$$\|T_n(u,v)\|_{\mathcal{E}_n\times\mathcal{E}_n} \geq \|(u,v)\|_{\mathcal{E}_n\times\mathcal{E}_n} \text{ for all } (u,v)\in\partial\mathcal{O}_{r_n}\cap(K_n\times K_n). \tag{2.3.18}$$

In view of (2.3.14), (2.3.18) and by Theorem 1.2.8, $T_n$ has a fixed point $(u_n,v_n)\in(\overline{\mathcal{O}}_R\setminus\mathcal{O}_{r_n})\cap(K_n\times K_n)$. Note that

$$r_n \leq \|(u_n,v_n)\|_{\mathcal{E}_n\times\mathcal{E}_n} \leq R, \tag{2.3.19}$$

where $r_n\to 0$ as $n\to\infty$. Thus, we have exhibited a uniform bound for each $(u_n,v_n)\in\mathcal{E}_n\times\mathcal{E}_n$, and $\{(u_m,v_m)\}_{m\geq n}$ is uniformly bounded on $[\frac{1}{n},1-\frac{1}{n}]$.

Now, we show that $\{(u_m,v_m)\}_{m\geq n}$, is equicontinuous on $[\frac{1}{n},1-\frac{1}{n}]$. For $t\in[\frac{1}{n},1-\frac{1}{n}]$, consider the integral equations

$$u_m(t) = u_m(\frac{1}{n}) + \frac{u_m(1-\frac{1}{n})-\alpha u_m(\eta)-(1-\alpha)u_m(\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}(t-\frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(t,s)\tilde{f}(s)ds,$$
$$v_m(t) = v_m(\frac{1}{n}) + \frac{v_m(1-\frac{1}{n})-\alpha v_m(\eta)-(1-\alpha)v_m(\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}(t-\frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(t,s)\tilde{g}(s)ds,$$

where $\tilde{f}(s) = f(s,u_m(s)+\frac{1}{n},v_m(s)+\frac{1}{n})$ and $\tilde{g}(s) = g(s,u_m(s)+\frac{1}{n},v_m(s)+\frac{1}{n})$.

Which can also be written as

$$u_m(t) = u_m(\frac{1}{n}) + \frac{u_m(1-\frac{1}{n})-\alpha u_m(\eta)-(1-\alpha)u_m(\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}(t-\frac{1}{n}) + \frac{t-\frac{1}{n}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}$$
$$\int_{1/n}^{1-1/n}(1-\frac{1}{n}-s)\tilde{f}(s)ds - \frac{\alpha(t-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{\eta}(\eta-s)\tilde{f}(s)ds - \int_{1/n}^{t}(t-s)\tilde{f}(s)ds,$$
$$v_m(t) = v_m(\frac{1}{n}) + \frac{v_m(1-\frac{1}{n})-\alpha v_m(\eta)-(1-\alpha)v_m(\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}(t-\frac{1}{n}) + \frac{t-\frac{1}{n}}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}$$
$$\int_{1/n}^{1-1/n}(1-\frac{1}{n}-s)\tilde{g}(s)ds - \frac{\alpha(t-\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{\eta}(\eta-s)\tilde{g}(s)ds - \int_{1/n}^{t}(t-s)\tilde{g}(s)ds.$$

Differentiating with respect to $t$, we have

$$u_m'(t) = \frac{u_m(1-\frac{1}{n})-\alpha u_m(\eta)-(1-\alpha)u_m(\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} + \frac{1}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{1-1/n}(1-\frac{1}{n}-s)$$
$$\tilde{f}(s)ds - \frac{\alpha}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{\eta}(\eta-s)\tilde{f}(s)ds - \int_{1/n}^{t}\tilde{f}(s)ds,$$
$$v_m'(t) = \frac{v_m(1-\frac{1}{n})-\alpha v_m(\eta)-(1-\alpha)v_m(\frac{1}{n})}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} + \frac{1}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{1-1/n}(1-\frac{1}{n}-s)$$
$$\tilde{g}(s)ds - \frac{\alpha}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{\eta}(\eta-s)\tilde{g}(s)ds - \int_{1/n}^{t}\tilde{g}(s)ds,$$

which implies that

$$\begin{aligned}
\|u'_m\|_{\mathcal{E}_n} \leq & \frac{2(1+\alpha)R}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} + \frac{1}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{1-1/n}(1-\frac{1}{n}-s)\tilde{f}(s)ds \\
& + \frac{\alpha}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{\eta}(\eta-s)\tilde{f}(s)ds + \int_{1/n}^{1-1/n}\tilde{f}(s)ds, \\
\|v'_m\|_{\mathcal{E}_n} \leq & \frac{2(1+\alpha)R}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta} + \frac{1}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{1-1/n}(1-\frac{1}{n}-s)\tilde{g}(s)ds \\
& + \frac{\alpha}{1-\frac{2}{n}+\frac{\alpha}{n}-\alpha\eta}\int_{1/n}^{\eta}(\eta-s)\tilde{g}(s)ds + \int_{1/n}^{1-1/n}\tilde{g}(s)ds.
\end{aligned}$$

Hence, $\{(u_m, v_m)\}_{m\geq n}$ is equicontinuous on $[\frac{1}{n}, 1-\frac{1}{n}]$.

For $m \geq n$, we define

$$x_m(t) = \begin{cases} u_m(\frac{1}{n}), & 0 \leq t \leq \frac{1}{n}, \\ u_m(t), & \frac{1}{n} \leq t \leq 1-\frac{1}{n}, \\ \alpha u_m(\eta), & 1-\frac{1}{n} \leq t \leq 1, \end{cases} \quad y_m(t) = \begin{cases} v_m(\frac{1}{n}), & 0 \leq t \leq \frac{1}{n}, \\ v_m(t), & \frac{1}{n} \leq t \leq 1-\frac{1}{n}, \\ \alpha v_m(\eta), & 1-\frac{1}{n} \leq t \leq 1. \end{cases}$$

Since $x_m$ and $y_m$ is a constant extension of $u_m$ and $v_m$, respectively, over the interval $[0, 1]$. Therefore, the sequence $\{(x_m, y_m)\}$ is uniformly bounded and equicontinuous on $[0, 1]$. Thus, there exists a subsequence $\{(x_{m_k}, y_{m_k})\}$ of $\{(x_m, y_m)\}$ converging uniformly to $(x, y) \in C[0, 1] \times C[0, 1]$.

For $t \in [0, 1]$, consider the integral equations

$$\begin{aligned}
x_{m_k}(t) &= \int_{1/m_k}^{1-1/m_k} H_{m_k}(t,s)f(t, x_{m_k}(s) + \frac{1}{m_k}, y_{m_k}(s) + \frac{1}{m_k})ds, \\
y_{m_k}(t) &= \int_{1/m_k}^{1-1/m_k} H_{m_k}(t,s)g(t, x_{m_k}(s) + \frac{1}{m_k}, y_{m_k}(s) + \frac{1}{m_k})ds.
\end{aligned}$$

Letting $m_k \to \infty$, we have

$$\begin{aligned}
x(t) &= \int_0^1 H(t,s)f(t, x(s), y(s))ds, \quad t \in [0,1], \\
y(t) &= \int_0^1 H(t,s)g(s, x(s), y(s))ds, \quad t \in [0,1].
\end{aligned}$$

Moreover,

$$x(0) = 0,\ x(1) = \alpha x(\eta),\ y(0) = 0,\ y(1) = \alpha y(\eta).$$

Hence, $(x, y)$ is a solution of the system of BVPs (2.0.3), (2.0.4). Since

$$f, g : (0,1) \times (0,\infty) \times (0,\infty) \to (0,\infty),$$

$f(t, 0, 0)$, $g(t, 0, 0)$ are not identically 0 and $H$ is of fixed sign on $(0, 1) \times (0, 1)$, it follows that $x > 0$ and $y > 0$ on $(0, 1]$. □

**Example 2.3.3.** Let

$$f(t,x,y) = \frac{1}{t^{1/2}(1-t)^{3/2}}(x^{\alpha_1} + x^{\beta_1})(y^{\alpha_2} + y^{\beta_2}),$$
$$g(t,x,y) = \frac{1}{t^{3/2}(1-t)^{1/2}}(x^{\gamma_1} + x^{\rho_1})(y^{\gamma_2} + y^{\rho_2}),$$

where $\alpha_i \leq 0 \leq \beta_i < 1$, $\gamma_i \leq 0 \leq \rho_i < 1$, $i = 1, 2$. Choose real constants $\tilde{\beta} = \tilde{\rho} = 1$. Clearly, $f$ and $g$ satisfy assumptions $(\mathcal{H}_6) - (\mathcal{H}_9)$. Hence, by Theorem 2.3.2, the system of BVPs (2.0.3), (2.0.4) has a positive solution.

# Chapter 3

# Singular Systems of Two–Point Boundary Value Problems

The existence of positive solutions for second–order nonlinear two–point BVPs has received much attention; see for example the case of regular nonlinear terms, [51, 52, 69, 80, 90], and the case of singular nonlinear terms, see [1, 4, 39]. However, these results are for the case when nonlinear functions are independent of the first derivative. The BVPs involving the first derivative with regular nonlinear functions can be seen in [61, 68, 136]. Though, the SBVPs with nonlinear functions dependent on first derivative not have many results, see [81, 120].

In [3, Section 2.4], Agarwal and O'Regan studied the existence of at least one positive solution for the following BVP with $\alpha = 1$ and $\beta = 0$,

$$\begin{aligned} -y''(t) &= q(t)f(t, y(t), y'(t)), \quad t \in (0, 1), \\ \alpha y(0) - \beta y'(0) &= y'(1) = 0, \end{aligned} \tag{3.0.1}$$

where $f : [0, 1] \times [0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous and is allowed to be singular at $y' = 0$; $q \in C(0, 1)$ and $q > 0$ on $(0, 1)$.

The existence of multiple positive solutions for second–order BVPs has also invited many authors, [68, 74, 82, 83, 89, 96, 99, 104, 137, 138]. B. Yan et al. [131] have studied the existence of multiple positive solutions of the BVP (3.0.1) with $\alpha = 1$ and $\beta = 0$. The method they used is based on the theory of fixed point index on a cone of an ordered Banach space. Further, they generalized these results and established the existence of at least two positive solutions for BVP (3.0.1) with $\alpha, \beta > 0$, [132].

In this chapter, we study the existence and multiplicity of positive solutions to the following coupled system of ODEs

$$\begin{aligned} -x''(t) &= p(t)f(t, y(t), x'(t)), \quad t \in (0, 1), \\ -y''(t) &= q(t)g(t, x(t), y'(t)), \quad t \in (0, 1), \end{aligned} \tag{3.0.2}$$

subject to the following set of two–point BCs

$$x(0) = y(0) = x'(1) = y'(1) = 0, \tag{3.0.3}$$

and

$$a_1x(0) - b_1x'(0) = a_2y(0) - b_2y'(0) = x'(1) = y'(1) = 0, \tag{3.0.4}$$

where the functions $f, g : [0, 1] \times [0, \infty) \times (0, \infty) \to [0, \infty)$ are continuous and are allowed to be singular at $x' = 0$, $y' = 0$. Moreover, $p, q \in C(0, 1)$ and positive on $(0, 1)$, and the real constants $a_i$ $(i = 1, 2) > 0$, $b_i$ $(i = 1, 2) > 0$. By singularity of $f$ and $g$, we mean that the functions $f(t, x, y)$ and $g(t, x, y)$ are allowed to be unbounded at $y = 0$. In literature, there are several results on existence of positive solutions for system of BVPs; see for example, [36, 66, 67, 95, 111, 127, 134]. Further, there are some results on multiplicity of positive solutions for system of BVPs, [36, 77, 98, 100, 105, 128]. However, there are only a few results for systems with nonlinear functions dependent on the first derivative. In general, the assumption that there exist singularities with respect to the first derivative is not new; see [3, 35, 120]. However, in the case of systems of ODEs, we believe this assumption is new. Some of the results of this chapter has been published in international journals [17, 24] and some are submitted [16, 18].

System of differential equations with BCs of the type (3.0.4) appear naturally when we study an $n$–th order reaction in an adiabatic tabular reactor with Arrhenius temperature dependence. Assume that a reactant is flowing through the tube of unit length and is reacting some product. Moreover, the flow is ideal and the concentration $c$, temperature $T$ of the reactant depend only on the time $t$ and the distance $z$ from the inlet plane at $z = 0$. Then, the model of adiabatic tabular reactor for an $n$–th order reaction with Arrhenius temperature dependence [12] is represented by the system of BVPs

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \beta_1 \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial z} + \mu(1-u)^n e^{\gamma \frac{v}{v+1}}, \quad t \geq 0,\ 0 \leq z \leq 1,\\
&\frac{\partial v}{\partial t} = \beta_2 \frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z} + \nu(1-u)^n e^{\gamma \frac{v}{v+1}}, \quad t \geq 0,\ 0 \leq z \leq 1,\\
&u(t,0) - \beta_1 \frac{\partial u}{\partial z}(t,0) = 0,\ \frac{\partial u}{\partial z}(t,1) = 0,\\
&v(t,0) - \beta_2 \frac{\partial v}{\partial z}(t,0) = 0,\ \frac{\partial v}{\partial z}(t,1) = 0,
\end{aligned} \tag{3.0.5}$$

where the constants $\mu$, $\gamma$, $\beta_1$ and $\beta_2$ are positive, while $\nu > 0$ if heat is generated (exothermal reaction) and $\nu < 0$ if heat is consumed (endothermal reaction). Clearly, steady state solution of (3.0.5) admits BCs of the type (3.0.4).

For each $x \in C[0, 1] \cap C^1(0, 1]$, we write $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $\|x\|_1 = \sup_{t\in(0,1]} t|x'(t)|$. Moreover, for each $x \in \mathcal{E} := \{x \in C[0, 1] \cap C^1(0, 1] : \|x\|_1 < +\infty\}$, we write $\|x\|_2 = \max\{\|x\|, \|x\|_1\}$. By Lemma 1.1.8, $(\mathcal{E}, \|\cdot\|_2)$ is a Banach space. Moreover, for each $x \in C^1[0, 1]$, we write $\|x\|_3 = \max\{\|x\|, \|x'\|\}$. Clearly, $(C^1[0, 1], \|\cdot\|_3)$ is a Banach space.

## 3.1 Existence of $C^1$-positive solutions

In this section, we establish sufficient conditions for the existence of $C^1$-positive solutions to the system of BVPs (3.0.2), (3.0.3). By a $C^1$-positive solution to the system of BVPs (3.0.2), (3.0.3), we mean that $(x, y) \in (C^1[0,1] \cap C^2(0,1)) \times (C^1[0,1] \cap C^2(0,1))$ satisfying (3.0.2) and (3.0.3), $x > 0$ and $y > 0$ on $(0,1]$, $x' > 0$ and $y' > 0$ on $[0,1)$.

Assume that the following holds:

($\mathcal{H}_{10}$) $p, q \in C(0,1)$, $p, q > 0$ on $(0,1)$, $\int_0^1 p(t)dt < +\infty$ and $\int_0^1 q(t)dt < +\infty$;

($\mathcal{H}_{11}$) $f, g : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous with $f(t,x,y) > 0$ and $g(t,x,y) > 0$ on $[0,1] \times (0,\infty) \times (0,\infty)$;

($\mathcal{H}_{12}$) $f(t,x,y) \leq k_1(x)(u_1(y) + v_1(y))$ and $g(t,x,y) \leq k_2(x)(u_2(y) + v_2(y))$, where $u_i(i = 1,2) > 0$ are continuous and nonincreasing on $(0,\infty)$, $k_i(i = 1,2) \geq 0$, $v_i(i = 1,2) \geq 0$ are continuous and nondecreasing on $[0,\infty)$;

($\mathcal{H}_{13}$)

$$\sup_{c\in(0,\infty)} \frac{c}{I^{-1}(k_1(J^{-1}(k_2(c)\int_0^1 q(s)ds))\int_0^1 p(s)ds)} > 1,$$
$$\sup_{c\in(0,\infty)} \frac{c}{J^{-1}(k_2(I^{-1}(k_1(c)\int_0^1 p(s)ds))\int_0^1 q(s)ds)} > 1,$$

where $I(\mu) = \int_0^\mu \frac{d\tau}{u_1(\tau)+v_1(\tau)}$, $J(\mu) = \int_0^\mu \frac{d\tau}{u_2(\tau)+v_2(\tau)}$, for $\mu \in (0,\infty)$;

($\mathcal{H}_{14}$) $I(\infty) = \infty$ and $J(\infty) = \infty$;

($\mathcal{H}_{15}$) for real constants $E > 0$ and $F > 0$, there exist continuous functions $\varphi_{EF}$ and $\psi_{EF}$ defined on $[0,1]$ and positive on $(0,1)$, and constants $0 \leq \delta_1, \delta_2 < 1$ such that

$$f(t,x,y) \geq \varphi_{EF}(t)x^{\delta_1}, \; g(t,x,y) \geq \psi_{EF}(t)x^{\delta_2} \text{ on } [0,1] \times [0,E] \times [0,F];$$

($\mathcal{H}_{16}$) $\int_0^1 p(t)u_1(C\int_t^1 s^{\delta_1}p(s)\varphi_{EF}(s)ds)dt < +\infty$ and $\int_0^1 q(t)u_2(C\int_t^1 s^{\delta_2}q(s)\psi_{EF}(s)ds)dt < +\infty$ for any real constant $C > 0$.

**Remark 3.1.1.** *Since $I$, $J$ are continuous, $I(0) = 0$, $I(\infty) = \infty$, $J(0) = 0$, $J(\infty) = \infty$, and they are monotone increasing. Hence, $I$ and $J$ are invertible. Moreover, $I^{-1}$ and $J^{-1}$ are also monotone increasing.*

**Theorem 3.1.2.** *Under the hypothesis ($\mathcal{H}_{10}$) – ($\mathcal{H}_{16}$), the system of BVPs (3.0.2), (3.0.3) has at least one $C^1$-positive solution.*

*Proof.* In view of ($\mathcal{H}_{13}$), we can choose real constants $M_1 > 0$ and $M_2 > 0$ such that

$$\frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1)\int_0^1 q(s)ds))\int_0^1 p(s)ds)} > 1,$$

$$\frac{M_2}{J^{-1}(k_2(I^{-1}(k_1(M_2)\int_0^1 p(s)ds))\int_0^1 q(s)ds)} > 1.$$

From the continuity of $k_1$, $k_2$, $I$ and $J$, choose $\varepsilon > 0$ small enough such that

$$\frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon))} > 1, \tag{3.1.1}$$

$$\frac{M_2}{J^{-1}(k_2(I^{-1}(k_1(M_2)\int_0^1 p(s)ds + I(\varepsilon)))\int_0^1 q(s)ds + J(\varepsilon))} > 1. \tag{3.1.2}$$

Choose real constants $L_1 > 0$ and $L_2 > 0$ such that

$$I(L_1) > k_1(M_2)\int_0^1 p(s)ds + I(\varepsilon), \tag{3.1.3}$$

$$J(L_2) > k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon). \tag{3.1.4}$$

Choose $n_0 \in \{1, 2, \cdots\}$ such that $\frac{1}{n_0} < \varepsilon$. For each fixed $n \in \{n_0, n_0+1, \cdots\}$, define retractions $\theta_i : \mathbb{R} \to [0, M_i]$ and $\rho_i : \mathbb{R} \to [\frac{1}{n}, L_i]$ by

$$\theta_i(x) = \max\{0, \min\{x, M_i\}\} \text{ and } \rho_i(x) = \max\{\frac{1}{n}, \min\{x, L_i\}\},\ i = 1, 2.$$

Consider the modified system of BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t, \theta_2(y(t)), \rho_1(x'(t))), \quad t \in (0,1), \\ -y''(t) &= q(t)g(t, \theta_1(x(t)), \rho_2(y'(t))), \quad t \in (0,1), \\ x(0) &= y(0) = 0,\ x'(1) = y'(1) = \frac{1}{n}. \end{aligned} \tag{3.1.5}$$

Since $f(t, \theta_2(y(t)), \rho_1(x'(t)))$, $g(t, \theta_1(x(t)), \rho_2(y'(t)))$ are continuous and bounded on $[0,1]\times\mathbb{R}^2$, by Theorem 1.2.4, it follows that the modified system of BVPs (3.1.5) has a solution $(x_n, y_n) \in (C^1[0,1] \cap C^2(0,1)) \times (C^1[0,1] \cap C^2(0,1))$.

Using (3.1.5) and $(\mathcal{H}_{11})$, we obtain

$$x_n''(t) \le 0 \text{ and } y_n''(t) \le 0 \text{ for } t \in (0,1),$$

which on integration from $t$ to 1, using the BCs (3.1.5), implies that

$$x_n'(t) \ge \frac{1}{n} \text{ and } y_n'(t) \ge \frac{1}{n} \text{ for } t \in [0,1]. \tag{3.1.6}$$

Integrating (3.1.6) from 0 to $t$, using the BCs (3.1.5), we have

$$x_n(t) \ge \frac{t}{n} \text{ and } y_n(t) \ge \frac{t}{n} \text{ for } t \in [0,1]. \tag{3.1.7}$$

From (3.1.6) and (3.1.7), it follows that

$$\|x_n\| = x_n(1) \text{ and } \|y_n\| = y_n(1).$$

Now, we show that

$$x_n'(t) < L_1, \ y_n'(t) < L_2, \quad t \in [0,1]. \tag{3.1.8}$$

First, we prove $x_n'(t) < L_1$ for $t \in [0,1]$. Suppose $x_n'(t_1) \geq L_1$ for some $t_1 \in [0,1]$. Using (3.1.5) and $(\mathcal{H}_{12})$, we have

$$-x_n''(t) \leq p(t)k_1(\theta_2(y_n(t)))(u_1(\rho_1(x_n'(t))) + v_1(\rho_1(x_n'(t)))), \quad t \in (0,1),$$

which implies that

$$\frac{-x_n''(t)}{u_1(\rho_1(x_n'(t))) + v_1(\rho_1(x_n'(t)))} \leq k_1(M_2)p(t), \quad t \in (0,1).$$

Integrating from $t_1$ to 1, using the BCs (3.1.5), we obtain

$$\int_{\frac{1}{n}}^{x_n'(t_1)} \frac{dz}{u_1(\rho_1(z)) + v_1(\rho_1(z))} \leq k_1(M_2) \int_{t_1}^{1} p(t)dt,$$

which can also be written as

$$\int_{\frac{1}{n}}^{L_1} \frac{dz}{u_1(z) + v_1(z)} + \int_{L_1}^{x_n'(t_1)} \frac{dz}{u_1(L_1) + v_1(L_1)} \leq k_1(M_2) \int_0^1 p(t)dt.$$

Using the increasing property of $I$, we obtain

$$I(L_1) + \frac{x_n'(t_1) - L_1}{u_1(L_1) + v_1(L_1)} \leq k_1(M_2) \int_0^1 p(t)dt + I(\varepsilon),$$

a contradiction to (3.1.3). Hence, $x_n'(t) < L_1$ for $t \in [0,1]$.

Similarly, we can show that $y_n'(t) < L_2$ for $t \in [0,1]$.

Now, we show that

$$x_n(t) < M_1, \ \ y_n(t) < M_2, \quad t \in [0,1]. \tag{3.1.9}$$

Suppose $x_n(t_2) \geq M_1$ for some $t_2 \in [0,1]$. From (3.1.5), (3.1.8) and $(\mathcal{H}_{12})$, it follows that

$$\begin{aligned} -x_n''(t) &\leq p(t)k_1(\theta_2(y_n(t)))(u_1(x_n'(t)) + v_1(x_n'(t))), \quad t \in (0,1), \\ -y_n''(t) &\leq q(t)k_2(\theta_1(x_n(t)))(u_2(y_n'(t)) + v_2(y_n'(t))), \quad t \in (0,1), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{-x_n''(t)}{u_1(x_n'(t)) + v_1(x_n'(t))} &\leq k_1(\theta_2(\|y_n\|))p(t), \quad t \in (0,1), \\ \frac{-y_n''(t)}{u_2(y_n'(t)) + v_2(y_n'(t))} &\leq k_2(M_1)q(t), \qquad\qquad t \in (0,1). \end{aligned}$$

Integrating from $t$ to 1, using the BCs (3.1.5), we obtain

$$\int_{\frac{1}{n}}^{x_n'(t)} \frac{dz}{u_1(z)+v_1(z)} \le k_1(\theta_2(\|y_n\|)) \int_t^1 p(s)ds, \quad t \in [0,1],$$
$$\int_{\frac{1}{n}}^{y_n'(t)} \frac{dz}{u_2(z)+v_2(z)} \le k_2(M_1) \int_t^1 q(s)ds, \qquad t \in [0,1],$$

which implies that

$$I(x_n'(t)) - I(\frac{1}{n}) \le k_1(\theta_2(\|y_n\|)) \int_0^1 p(s)ds, \quad t \in [0,1],$$
$$J(y_n'(t)) - J(\frac{1}{n}) \le k_2(M_1) \int_0^1 q(s)ds, \qquad t \in [0,1].$$

The increasing property of $I$ and $J$ leads to

$$x_n'(t) \le I^{-1}(k_1(\theta_2(\|y_n\|)) \int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0,1], \tag{3.1.10}$$

$$y_n'(t) \le J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon)), \qquad t \in [0,1]. \tag{3.1.11}$$

Integrating (3.1.10) from 0 to $t_2$ and (3.1.11) from 0 to 1, using the BCs (3.1.5), we obtain

$$M_1 \le x_n(t_2) \le I^{-1}(k_1(\theta_2(\|y_n\|)) \int_0^1 p(s)ds + I(\varepsilon)), \tag{3.1.12}$$

$$\|y_n\| \le J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon)). \tag{3.1.13}$$

Either we have $\|y_n\| < M_2$ or $\|y_n\| \ge M_2$. If $\|y_n\| < M_2$, then from (3.1.12), we have

$$M_1 \le I^{-1}(k_1(\|y_n\|) \int_0^1 p(s)ds + I(\varepsilon)). \tag{3.1.14}$$

Now, by using (3.1.13) in (3.1.14) and the increasing property of $k_1$ and $I^{-1}$, we obtain

$$M_1 \le I^{-1}(k_1(J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon))) \int_0^1 p(s)ds + I(\varepsilon)),$$

which implies that

$$\frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon))) \int_0^1 p(s)ds + I(\varepsilon))} \le 1,$$

a contradiction to (3.1.1).

On the other hand, if $\|y_n\| \ge M_2$, then from (3.1.12) and (3.1.13), we have

$$M_1 \le x_n(t_2) \le I^{-1}(k_1(M_2) \int_0^1 p(s)ds + I(\varepsilon)), \tag{3.1.15}$$

$$M_2 \le J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon)). \tag{3.1.16}$$

Using (3.1.16) in (3.1.15) and the increasing property of $k_1$ and $I^{-1}$, leads to

$$M_1 \le I^{-1}(k_1(J^{-1}(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon)),$$

which implies that

$$\frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon))} \le 1,$$

a contradiction to (3.1.1). Hence, $x_n(t) < M_1$ for $t \in [0,1]$.

Similarly, we can show that $y_n(t) < M_2$ for $t \in [0,1]$.

Thus, in view of (3.1.5)–(3.1.9), $(x_n, y_n)$ is a solution of the following coupled system of BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t, y(t), x'(t)), \quad t \in (0,1),\\ -y''(t) &= q(t)g(t, x(t), y'(t)), \quad t \in (0,1),\\ x(0) &= y(0) = 0,\ x'(1) = y'(1) = \frac{1}{n}, \end{aligned} \tag{3.1.17}$$

satisfying

$$\begin{aligned} \frac{t}{n} &\le x_n(t) < M_1,\ \frac{1}{n} \le x_n'(t) < L_1, \quad t \in [0,1],\\ \frac{t}{n} &\le y_n(t) < M_2,\ \frac{1}{n} \le y_n'(t) < L_2, \quad t \in [0,1]. \end{aligned} \tag{3.1.18}$$

Now, in view of $(\mathcal{H}_{15})$, there exist continuous functions $\varphi_{M_2L_1}$ and $\psi_{M_1L_2}$ defined on $[0,1]$ and positive on $(0,1)$, and real constants $0 \le \delta_1, \delta_2 < 1$ such that

$$\begin{aligned} f(t, y_n(t), x_n'(t)) &\ge \varphi_{M_2L_1}(t)(y_n(t))^{\delta_1}, \quad (t, y_n(t), x_n'(t)) \in [0,1]\times[0,M_2]\times[0,L_1],\\ g(t, x_n(t), y_n'(t)) &\ge \psi_{M_1L_2}(t)(x_n(t))^{\delta_2}, \quad (t, x_n(t), y_n'(t)) \in [0,1]\times[0,M_1]\times[0,L_2]. \end{aligned} \tag{3.1.19}$$

We claim that

$$x_n'(t) \ge C_2^{\delta_1}\int_t^1 s^{\delta_1}p(s)\varphi_{M_2L_1}(s)ds, \tag{3.1.20}$$

$$y_n'(t) \ge C_1^{\delta_2}\int_t^1 s^{\delta_2}q(s)\psi_{M_1L_2}(s)ds, \tag{3.1.21}$$

where

$$\begin{aligned} C_1 &= \left(\int_0^1 s^{\delta_2+1}q(s)\psi_{M_1L_2}(s)ds\right)^{\frac{\delta_1}{1-\delta_1\delta_2}}\left(\int_0^1 s^{\delta_1+1}p(s)\varphi_{M_2L_1}(s)ds\right)^{\frac{1}{1-\delta_1\delta_2}},\\ C_2 &= \left(\int_0^1 s^{\delta_1+1}p(s)\varphi_{M_2L_1}(s)ds\right)^{\frac{\delta_2}{1-\delta_1\delta_2}}\left(\int_0^1 s^{\delta_2+1}q(s)\psi_{M_1L_2}(s)ds\right)^{\frac{1}{1-\delta_1\delta_2}}. \end{aligned}$$

To prove (3.1.20), consider the following relation

$$x_n(t) = \frac{t}{n} + \int_0^t sp(s)f(s, y_n(s), x_n'(s))ds + \int_t^1 tp(s)f(s, y_n(s), x_n'(s))ds, \tag{3.1.22}$$

which implies that

$$x_n(1) \geq \int_0^1 sp(s)f(s, y_n(s), x_n'(s))ds.$$

Using (3.1.19) and Lemma 1.1.14, we obtain

$$x_n(1) \geq (y_n(1))^{\delta_1} \int_0^1 s^{\delta_1+1}p(s)\varphi_{M_2L_1}(s)ds. \tag{3.1.23}$$

Similarly, using (3.1.19) and Lemma 1.1.14, we obtain

$$y_n(1) \geq (x_n(1))^{\delta_2} \int_0^1 s^{\delta_2+1}q(s)\psi_{M_1L_2}(s)ds,$$

which in view of (3.1.23) implies that

$$y_n(1) \geq (y_n(1))^{\delta_1\delta_2} \left(\int_0^1 s^{\delta_1+1}p(s)\varphi_{M_2L_1}(s)ds\right)^{\delta_2} \int_0^1 s^{\delta_2+1}q(s)\psi_{M_1L_2}(s)ds.$$

Hence,

$$y_n(1) \geq C_2. \tag{3.1.24}$$

Now, from (3.1.22), it follows that

$$x_n'(t) \geq \int_t^1 p(s)f(s, y_n(s), x_n'(s))ds.$$

Using (3.1.19), Lemma 1.1.14 and (3.1.24), we obtain (3.1.20).

Similarly, we can prove (3.1.21).

Now, using (3.1.17), $(\mathcal{H}_{12})$, (3.1.18), (3.1.20) and (3.1.21), we have

$$\begin{aligned} 0 \leq -x_n''(t) \leq k_1(M_2)p(t)(u_1(C_2^{\delta_1} \int_t^1 s^{\delta_1}p(s)\varphi_{M_2L_1}(s)ds) + v_1(L_1)), \quad t \in (0,1), \\ 0 \leq -y_n''(t) \leq k_2(M_1)q(t)(u_2(C_1^{\delta_2} \int_t^1 s^{\delta_2}q(s)\psi_{M_1L_2}(s)ds) + v_2(L_2)), \quad t \in (0,1). \end{aligned} \tag{3.1.25}$$

In view of (3.1.18), (3.1.25), $(\mathcal{H}_{10})$ and $(\mathcal{H}_{16})$, it follows that the sequences $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, 1]$. Hence, by Theorem 1.1.6, there exist subsequences $\{(x_{n_k}^{(j)}, y_{n_k}^{(j)})\}$ $(j = 0, 1)$ of $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j = 0, 1)$ and $(x, y) \in C^1[0, 1] \times C^1[0, 1]$ such that $(x_{n_k}^{(j)}, y_{n_k}^{(j)})$ converges uniformly to $(x^{(j)}, y^{(j)})$ on $[0, 1]$ $(j = 0, 1)$. Also, $x(0) = y(0) = x'(1) = y'(1) = 0$. Moreover, from (3.1.20) and (3.1.21), with $n_k$ in place of $n$ and taking $\lim_{n_k\to+\infty}$, we have

$$\begin{aligned} x'(t) &\geq C_2^{\delta_1} \int_t^1 s^{\delta_1}p(s)\varphi_{M_2L_1}(s)ds, \\ y'(t) &\geq C_1^{\delta_2} \int_t^1 s^{\delta_2}q(s)\psi_{M_1L_2}(s)ds, \end{aligned}$$

which shows that $x' > 0$ and $y' > 0$ on $[0,1)$, $x > 0$ and $y > 0$ on $(0,1]$. Further, $(x_{n_k}, y_{n_k})$ satisfy

$$x'_{n_k}(t) = x'_{n_k}(0) - \int_0^t p(s)f(s, y_{n_k}(s), x'_{n_k}(s))ds, \quad t \in [0,1],$$
$$y'_{n_k}(t) = y'_{n_k}(0) - \int_0^t q(s)g(s, x_{n_k}(s), y'_{n_k}(s))ds, \quad t \in [0,1].$$

Passing to the limit as $n_k \to \infty$, we obtain

$$x'(t) = x'(0) - \int_0^t p(s)f(s, y(s), x'(s))ds, \quad t \in [0,1],$$
$$y'(t) = y'(0) - \int_0^t q(s)g(s, x(s), y'(s))ds, \quad t \in [0,1],$$

which implies that

$$-x''(t) = p(t)f(t, y(t), x'(t)), \quad t \in (0,1),$$
$$-y''(t) = q(t)g(t, x(t), y'(t)), \quad t \in (0,1).$$

Hence, $(x, y)$ is a $C^1$-positive solution of the system of BVPs (3.0.2), (3.0.3). □

**Example 3.1.3.** Consider the following coupled system of SBVPs

$$\begin{aligned} -x''(t) &= t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}(y(t))^{\frac{1}{3}}(x'(t))^{-\beta_1}, \quad t \in (0,1), \\ -y''(t) &= t^{-\frac{2}{3}}(1-t)^{-\frac{1}{3}}(x(t))^{\frac{2}{3}}(y'(t))^{-\beta_2}, \quad t \in (0,1), \\ x(0) &= y(0) = x'(1) = y'(1) = 0, \end{aligned} \tag{3.1.26}$$

where $0 < \beta_1 < 1$ and $0 < \beta_2 < \frac{1}{2}$.

Taking $p(t) = t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}$, $q(t) = t^{-\frac{2}{3}}(1-t)^{-\frac{1}{3}}$, $k_1(x) = x^{\frac{1}{3}}$, $k_2(x) = x^{\frac{2}{3}}$, $u_1(x) = x^{-\beta_1}$, $u_2(x) = x^{-\beta_2}$ and $v_1(x) = v_2(x) = 0$. Then, $I(z) = \frac{z^{\beta_1+1}}{\beta_1+1}$, $J(z) = \frac{z^{\beta_2+1}}{\beta_2+1}$, $I^{-1}(z) = (\beta_1+1)^{\frac{1}{\beta_1+1}} z^{\frac{1}{\beta_1+1}}$ and $J^{-1}(z) = (\beta_2+1)^{\frac{1}{\beta_2+1}} z^{\frac{1}{\beta_2+1}}$.

Then, $\int_0^1 p(t)dt = \int_0^1 q(t)dt = \frac{2\pi}{\sqrt{3}}$. Also,

$$\sup_{c\in(0,\infty)} \frac{c}{I^{-1}(k_1(J^{-1}(k_2(c)\int_0^1 q(s)ds))\int_0^1 p(s)ds)} =$$
$$\sup_{c\in(0,\infty)} \frac{c}{(\frac{2\pi}{\sqrt{3}})^{\frac{3\beta_2+4}{3(\beta_1+1)(\beta_2+1)}} (\beta_1+1)^{\frac{1}{\beta_1+1}} (\beta_2+1)^{\frac{1}{3(\beta_1+1)(\beta_2+1)}} c^{\frac{2}{9(\beta_1+1)(\beta_2+1)}}} = \infty \text{ and}$$
$$\sup_{c\in(0,\infty)} \frac{c}{J^{-1}(k_2(I^{-1}(k_1(c)\int_0^1 p(s)ds))\int_0^1 q(s)ds)} =$$
$$\sup_{c\in(0,\infty)} \frac{c}{(\frac{2\pi}{\sqrt{3}})^{\frac{3\beta_1+5}{3(\beta_1+1)(\beta_2+1)}} (\beta_2+1)^{\frac{1}{\beta_2+1}} (\beta_1+1)^{\frac{2}{3(\beta_1+1)(\beta_2+1)}} c^{\frac{2}{9(\beta_1+1)(\beta_2+1)}}} = \infty.$$

Clearly, $(\mathcal{H}_{10}) - (\mathcal{H}_{14})$ are satisfied. Choose $\delta_1 = \frac{1}{3}$, $\delta_2 = \frac{2}{3}$, $\varphi_{EF}(t) = F^{-\beta_1}$ and $\psi_{EF}(t) = F^{-\beta_2}$, then $(\mathcal{H}_{15})$ holds. Moreover,

$$\begin{aligned}
\int_0^1 p(t)u_1(C\int_t^1 s^{\delta_1}p(s)\varphi_{EF}(s)ds)dt &= 3^{-\beta_1}C^{-\beta_1}F^{\beta_1^2}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{\beta_1+2}{3}}dt \\
&= 3^{-\beta_1}C^{-\beta_1}F^{\beta_1^2}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1-\beta_1}{3})}{\Gamma(1-\frac{\beta_1}{3})}, \\
\int_0^1 q(t)u_2(C\int_t^1 s^{\delta_2}q(s)\psi_{EF}(s)ds)dt &= (\frac{3}{2})^{-\beta_2}C^{-\beta_2}F^{\beta_2^2}\int_0^1 t^{-\frac{2}{3}}(1-t)^{-\frac{2\beta_2+1}{3}}dt \\
&= (\frac{3}{2})^{-\beta_2}C^{-\beta_2}F^{\beta_2^2}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1-2\beta_2}{3})}{\Gamma(1-\frac{2\beta_2}{3})},
\end{aligned}$$

shows that $(\mathcal{H}_{16})$ also holds.

Since, $(\mathcal{H}_{10}) - (\mathcal{H}_{16})$ are satisfied. Therefore, by Theorem 3.1.2, the system of BVPs (3.1.26) has at least one $C^1$-positive solution.

## 3.2 Existence of at least two positive solutions

In this section, we establish sufficient conditions for the existence of at least two positive solutions of the system of BVPs (3.0.2), (3.0.3). By a positive solution $(x, y)$ of the system of BVPs (3.0.2), (3.0.3), we mean that $(x, y) \in \mathcal{E} \times \mathcal{E}$ satisfies (3.0.2) and (3.0.3), $x > 0$ and $y > 0$ on $(0, 1]$, $x' > 0$ and $y' > 0$ on $[0, 1)$. Define a cone $P$ of $\mathcal{E}$ by

$$P = \{x \in \mathcal{E} : x(t) \geq t\|x\| \text{ for all } t \in [0, 1], x(1) \geq \|x\|_1\}.$$

For each $(x, y) \in \mathcal{E}\times\mathcal{E}$ we write $\|(x, y)\|_4 = \|x\|_2 + \|y\|_2$. Clearly, $(\mathcal{E}\times\mathcal{E}, \|\cdot\|_4)$ is a Banach space and $P \times P$ is a cone of $\mathcal{E}\times\mathcal{E}$. We define a partial ordering in $\mathcal{E}$, by $x \leq y$ if and only if $x(t) \leq y(t)$, $t \in [0, 1]$. We define a partial ordering in $\mathcal{E}\times\mathcal{E}$, by $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. For any real constant $r > 0$, we define an open neighborhood of $(0, 0) \in \mathcal{E}\times\mathcal{E}$ as

$$\mathcal{O}_r = \{(x, y) \in \mathcal{E}\times\mathcal{E} : \|(x, y)\|_4 < r\}.$$

In view of $(\mathcal{H}_{13})$, there exist real constants $R_1 > 0$ and $R_2 > 0$ such that

$$\frac{R_1}{I^{-1}(k_1(J^{-1}(k_2(R_1)\int_0^1 q(s)ds))\int_0^1 p(s)ds)} > 1, \tag{3.2.1}$$

$$\frac{R_2}{J^{-1}(k_2(I^{-1}(k_1(R_2)\int_0^1 p(s)ds))\int_0^1 q(s)ds)} > 1. \tag{3.2.2}$$

From the continuity of $k_1$, $k_2$, $I$ and $J$, we choose $\varepsilon > 0$ small enough such that

$$\frac{R_1}{I^{-1}(k_1(J^{-1}(k_2(R_1+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon)) + \varepsilon)\int_0^1 p(s)ds + I(\varepsilon))} > 1, \tag{3.2.3}$$

$$\frac{R_2}{J^{-1}(k_2(I^{-1}(k_1(R_2+\varepsilon)\int_0^1 p(s)ds + I(\varepsilon))+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon))} > 1. \tag{3.2.4}$$

Choose $n_0 \in \{1, 2, \cdots\}$ such that $\frac{1}{n_0} < \varepsilon$ and for each fixed $n \in \{n_0, n_0+1, \cdots\}$, consider the system of non–singular BVPs

$$\begin{aligned}
-x''(t) &= p(t)f(t, y(t) + \frac{t}{n}, |x'(t)| + \frac{1}{n}), \quad t \in (0,1),\\
-y''(t) &= q(t)g(t, x(t) + \frac{t}{n}, |y'(t)| + \frac{1}{n}), \quad t \in (0,1),\\
x(0) &= x'(1) = y(0) = y'(1) = 0.
\end{aligned} \tag{3.2.5}$$

We write (3.2.5) as an equivalent system of integral equations

$$\begin{aligned}
x(t) &= \int_0^1 G(t,s)p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds, \quad t \in [0,1],\\
y(t) &= \int_0^1 G(t,s)q(s)f(s, x(s) + \frac{s}{n}, |y'(s)| + \frac{1}{n})ds, \quad t \in [0,1],
\end{aligned} \tag{3.2.6}$$

where the Green's function $G$ is defined as

$$G(t,s) = \begin{cases} s, & 0 \le s \le t \le 1,\\ t, & 0 \le t \le s \le 1. \end{cases}$$

By a solution of the system of BVPs (3.2.5), we mean a solution of the corresponding system of integral equations (3.2.6).

Define an operator $T_n : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ by

$$T_n(x,y) = (A_n(x,y), B_n(x,y)), \tag{3.2.7}$$

where the operators $A_n, B_n : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ are defined by

$$\begin{aligned}
A_n(x,y)(t) &= \int_0^1 G(t,s)p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds, \quad t \in [0,1],\\
B_n(x,y)(t) &= \int_0^1 G(t,s)q(s)f(s, x(s) + \frac{s}{n}, |y'(s)| + \frac{1}{n})ds, \quad t \in [0,1].
\end{aligned} \tag{3.2.8}$$

Clearly, if $(x_n, y_n) \in \mathcal{E} \times \mathcal{E}$ is a fixed point of $T_n$; then $(x_n, y_n)$ is a solution of the system of BVPs (3.2.5).

Assume that

($\mathcal{H}_{17}$) for any real constant $E > 0$, there exist continuous functions $\varphi_E$ and $\psi_E$ defined on $[0,1]$ and positive on $(0,1)$, and constants $0 \le \delta_1, \delta_2 < 1$ such that

$$f(t,x,y) \ge \varphi_E(t)x^{\delta_1},\ g(t,x,y) \ge \psi_E(t)x^{\delta_2} \text{ on } [0,1]\times[0,E]\times[0,\infty);$$

($\mathcal{H}_{18}$) for any real constant $C > 0$, $\int_0^1 p(t)v_1(\frac{C}{t})dt < +\infty$, $\int_0^1 q(t)v_2(\frac{C}{t})dt < +\infty$, $\int_0^1 p(t)u_1(C\int_t^1 s^{\delta_1}p(s)\varphi_E(s)ds)dt < +\infty$ and $\int_0^1 q(t)u_2(C\int_t^1 s^{\delta_2}q(s)\psi_E(s)ds)dt < +\infty$.

**Lemma 3.2.1.** *Under the hypothesis* $(\mathcal{H}_{10}) - (\mathcal{H}_{12})$ *and* $(\mathcal{H}_{18})$, *the operator* $T_n : \overline{\mathcal{O}}_r \cap (P \times P) \to P \times P$ *is completely continuous.*

*Proof.* Firstly, we show that $T_n(P \times P) \subseteq P \times P$. For $(x, y) \in P \times P$, $t \in [0, 1]$, using (3.2.8) and Lemma 1.1.11, we obtain

$$\begin{aligned} A_n(x,y)(t) &= \int_0^1 G(t,s)p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds \\ &\geq t \max_{\tau\in[0,1]} \int_0^1 G(\tau,s)p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds = t\|A_n(x,y)\| \end{aligned} \tag{3.2.9}$$

and

$$\begin{aligned} &\|A_n(x,y)\|_1 = \sup_{\tau\in(0,1]} \tau|A_n(x,y)'(\tau)| = \sup_{\tau\in(0,1]} \tau \int_\tau^1 p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds \\ &\leq \max_{t\in[0,1]} \int_0^1 G(t,s)p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds \leq A_n(x,y)(1). \end{aligned} \tag{3.2.10}$$

From (3.2.9) and (3.2.10), $A_n(x, y) \in P$ for every $(x, y) \in P \times P$, that is, $A_n(P \times P) \subseteq P$. Similarly, by using (3.2.8) and Lemma 1.1.11, we can show that $B_n(P \times P) \subseteq P$. Hence, $T_n(P \times P) \subseteq P \times P$.

Now, we show that $T_n : \overline{\mathcal{O}}_r \cap (P \times P) \to P \times P$ is uniformly bounded. For any $(x, y) \in \overline{\mathcal{O}}_r \cap (P \times P)$, using (3.2.8), $(\mathcal{H}_{12})$, Lemma 1.1.9, $(\mathcal{H}_{10})$ and $(\mathcal{H}_{18})$, we have

$$\begin{aligned} \|A_n(x,y)\| &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds \right| \\ &\leq \int_0^1 p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds \\ &\leq \int_0^1 p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(\frac{\|x\|_2}{s} + \frac{1}{n}))ds \\ &\leq \int_0^1 p(s)k_1(y(s) + \frac{s}{n})(u_1(\frac{1}{n}) + v_1(\frac{r}{s} + \frac{1}{n}))ds \\ &\leq k_1(r + \frac{1}{n}) \int_0^1 p(s)(u_1(\frac{1}{n}) + v_1((r + \frac{1}{n})\frac{1}{s}))ds < +\infty. \end{aligned} \tag{3.2.11}$$

Also, for $(x, y) \in \overline{\mathcal{O}}_r \cap (P \times P)$, using (3.2.8), Lemma 1.1.11 and $(\mathcal{H}_{12})$, we have

$$\begin{aligned} &\|A_n(x,y)\|_1 = \sup_{\tau\in(0,1]} \tau|A_n(x,y)'(\tau)| = \sup_{\tau\in(0,1]} \tau \int_\tau^1 p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds \\ &\leq \max_{t\in[0,1]} \int_t^1 G(t,s)p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds \leq \int_0^1 p(s)f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds \\ &\leq \int_0^1 p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds. \end{aligned}$$

Now, using Lemma 1.1.9, $(\mathcal{H}_{10})$ and $(\mathcal{H}_{18})$, we obtain

$$\begin{aligned}\|A_n(x,y)\|_1 &\le \int_0^1 p(s)k_1(y(s)+\frac{s}{n})(u_1(\frac{1}{n})+v_1(\frac{\|x\|_2}{s}+\frac{1}{n}))ds\\ &\le \int_0^1 p(s)k_1(y(s)+\frac{s}{n})(u_1(\frac{1}{n})+v_1(\frac{r}{s}+\frac{1}{n}))ds\\ &\le k_1(r+\frac{1}{n})\int_0^1 p(s)(u_1(\frac{1}{n})+v_1((r+\frac{1}{n})\frac{1}{s}))ds<+\infty.\end{aligned} \tag{3.2.12}$$

From (3.2.11) and (3.2.12), it follows that $A_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is uniformly bounded under the norm $\|\cdot\|_2$. Similarly, by using (3.2.8), Lemma 1.1.9, Lemma 1.1.11, $(\mathcal{H}_{10})$, $(\mathcal{H}_{12})$ and $(\mathcal{H}_{18})$, we can show that $B_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is uniformly bounded under the norm $\|\cdot\|_2$. Hence, $T_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is uniformly bounded.

Now, we show that $T_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is equicontinuous. For $(x,y)\in\overline{\mathcal{O}}_r\cap(P\times P)$, $t_1,t_2\in[0,1]$, using (3.2.8), $(\mathcal{H}_{12})$ and Lemma 1.1.9, we have

$$\begin{aligned}&|A_n(x,y)(t_1)-A_n(x,y)(t_2)| = \left|\int_0^1 (G(t_1,s)-G(t_2,s))p(s)f(s,y(s)+\frac{s}{n},|x'(s)|+\frac{1}{n})ds\right|\\ &\le \int_0^1 |G(t_1,s)-G(t_2,s)|p(s)f(s,y(s)+\frac{s}{n},|x'(s)|+\frac{1}{n})ds\\ &\le \int_0^1 |G(t_1,s)-G(t_2,s)|p(s)k_1(y(s)+\frac{s}{n})(u_1(|x'(s)|+\frac{1}{n})+v_1(|x'(s)|+\frac{1}{n}))ds\\ &\le k_1(r+\frac{1}{n})\int_0^1 |G(t_1,s)-G(t_2,s)|p(s)(u_1(\frac{1}{n})+v_1(\frac{\|x\|_2}{s}+\frac{1}{n}))ds\\ &\le k_1(r+\frac{1}{n})\int_0^1 |G(t_1,s)-G(t_2,s)|p(s)(u_1(\frac{1}{n})+v_1(r+\frac{1}{n})\frac{1}{s})ds,\end{aligned} \tag{3.2.13}$$

and

$$\begin{aligned}&|A_n(x,y)'(t_1)-A_n(x,y)'(t_2)| = \left|\int_{t_1}^{t_2} p(s)f(s,y(s)+\frac{s}{n},|x'(s)|+\frac{1}{n})ds\right|\\ &\le \int_{t_1}^{t_2} p(s)k_1(y(s)+\frac{s}{n})(u_1(|x'(s)|+\frac{1}{n})+v_1(|x'(s)|+\frac{1}{n}))ds\\ &\le k_1(r+\frac{1}{n})\int_{t_1}^{t_2} p(s)(u_1(\frac{1}{n})+v_1(\frac{\|x\|_2}{s}+\frac{1}{n}))ds \le k_1(r+\frac{1}{n})\int_{t_1}^{t_2} p(s)\\ &(u_1(\frac{1}{n})+v_1(\frac{r}{s}+\frac{1}{n}))ds \le k_1(r+\frac{1}{n})\int_{t_1}^{t_2} p(s)(u_1(\frac{1}{n})+v_1((r+\frac{1}{n})\frac{1}{s}))ds.\end{aligned} \tag{3.2.14}$$

From (3.2.13), (3.2.14), $(\mathcal{H}_{10})$ and $(\mathcal{H}_{18})$, it follows that $A_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is equicontinuous under the norm $\|\cdot\|_3$. But, the norm $\|\cdot\|_3$ is equivalent to the norm $\|\cdot\|_2$. Hence, $A_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is equicontinuous under $\|\cdot\|_2$.

Similarly, using (3.2.8), $(\mathcal{H}_{12})$ and Lemma 1.1.9, we can show that $B_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is equicontinuous under the norm $\|\cdot\|_2$. Consequently, $T_n(\overline{\mathcal{O}}_r\cap(P\times P))$ is equicontinuous.

Hence, by Theorem 1.1.6, $T_n(\overline{\mathcal{O}}_r \cap (P \times P))$ is relatively compact which implies that $T_n$ is a compact operator.

Now, we show that $T_n$ is continuous. Let $(x_m, y_m), (x, y) \in \overline{\mathcal{O}}_r \cap (P \times P)$ such that $\|(x_m, y_m) - (x, y)\|_4 \to 0$ as $m \to +\infty$. Using $(\mathcal{H}_{12})$ and Lemma 1.1.9, we have

$$\begin{aligned}
&\left| f(t, y_m(t) + \frac{t}{n}, |x_m'(t)| + \frac{1}{n}) \right| \leq k_1(y_m(t) + \frac{t}{n})(u_1(|x_m'(t)| + \frac{1}{n}) + v_1(|x_m'(t)| + \frac{1}{n})) \\
&\leq k_1(r + \frac{1}{n})(u_1(\frac{1}{n}) + v_1(\frac{\|x_m\|_2}{t} + \frac{1}{n})) \leq k_1(r + \frac{1}{n})(u_1(\frac{1}{n}) + v_1(\frac{r}{t} + \frac{1}{n})) \\
&\leq k_1(r + \frac{1}{n})(u_1(\frac{1}{n}) + v_1((r + \frac{1}{n})\frac{1}{t})).
\end{aligned}$$

Using (3.2.8) and Lemma 1.1.11, we have

$$\begin{aligned}
&\|A_n(x_m, y_m) - A_n(x, y)\| = \max_{t \in [0,1]} \left| \int_0^1 G(t, s)p(s)(f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, \right. \\
&\left. |x'(s)| + \frac{1}{n}))ds \right| \leq \int_0^1 p(s) \left| f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right| ds
\end{aligned} \tag{3.2.15}$$

and

$$\begin{aligned}
&\|A_n(x_m, y_m)' - A_n(x, y)'\|_1 = \sup_{\tau \in (0,1]} \tau \left| \int_\tau^1 p(s)(f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| \right. \\
&\left. + \frac{1}{n}))ds \right| \leq \max_{t \in [0,1]} \int_0^1 G(t, s)p(s) \left| f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right| ds \\
&\leq \int_0^1 p(s) \left| f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right| ds.
\end{aligned} \tag{3.2.16}$$

From (3.2.15) and (3.2.16), using the Lebesgue dominated convergence theorem, it follows that

$$\|A_n(x_m, y_m) - A_n(x, y)\| \to 0, \;\; \|A_n(x_m, y_m)' - A_n(x, y)'\|_1 \to 0 \text{ as } m \to +\infty.$$

Hence, $\|A_n(x_m, y_m) - A_n(x, y)\|_2 \to 0$ as $m \to \infty$.

Similarly, we can show that $\|B_n(x_m, y_m) - B_n(x, y)\|_2 \to 0$ as $m \to \infty$. Consequently, $\|T_n(x_m, y_m) - T_n(x, y)\|_4 \to 0$ as $m \to +\infty$, that is, $T_n : \overline{\mathcal{O}}_r \cap (P \times P) \to P \times P$ is continuous. Hence, $T_n : \overline{\mathcal{O}}_r \cap (P \times P) \to P \times P$ is completely continuous. □

Assume that

$(\mathcal{H}_{19})$ there exist $h_1, h_2 \in C([0, \infty) \times (0, \infty), [0, \infty))$ with $f(t, x, y) \geq h_1(x, y)$ and $g(t, x, y) \geq h_2(x, y)$ on $[0, 1] \times [0, \infty) \times (0, \infty)$ such that

$$\lim_{x \to +\infty} \frac{h_i(x, y)}{x} = +\infty, \text{ uniformly for } y \in (0, \infty), \; i = 1, 2.$$

**Theorem 3.2.2.** *Under the hypothesis* $(\mathcal{H}_{10}) - (\mathcal{H}_{14})$ *and* $(\mathcal{H}_{17}) - (\mathcal{H}_{19})$, *the system of BVPs* (3.0.2), (3.0.3) *has at least two positive solutions.*

*Proof.* Let $R_0 = R_1 + R_2$ and define $\mathcal{O}_{R_0} = \Omega_{R_1} \times \Omega_{R_2}$ where

$$\Omega_{R_1} = \{x \in E : \|x\|_2 < R_1\},\ \Omega_{R_2} = \{x \in E : \|x\|_2 < R_2\}.$$

We claim that

$$(x, y) \neq \lambda T_n(x, y),\ \text{for } \lambda \in (0, 1], (x, y) \in \partial\mathcal{O}_{R_0} \cap (P \times P). \tag{3.2.17}$$

Suppose there exist $(x_0, y_0) \in \partial\mathcal{O}_R \cap (P \times P)$ and $\lambda_0 \in (0, 1]$ such that $(x_0, y_0) = \lambda_0 T_n(x_0, y_0)$. Then,

$$\begin{aligned} -x_0''(t) &= \lambda_0 p(t) f(t, y_0(t) + \frac{t}{n}, |x_0'(t)| + \frac{1}{n}), \quad t \in (0, 1), \\ -y_0''(t) &= \lambda_0 q(t) g(t, x_0(t) + \frac{t}{n}, |y_0'(t)| + \frac{1}{n}), \quad t \in (0, 1), \\ x_0(0) &= x_0'(1) = y_0(0) = y_0'(1) = 0. \end{aligned} \tag{3.2.18}$$

From (3.2.18) and $(\mathcal{H}_{11})$, we have $x_0'' \leq 0$ and $y_0'' \leq 0$ on $(0, 1)$, integrating from $t$ to 1, using the BCs (3.2.18), we obtain $x_0'(t) \geq 0$ and $y_0'(t) \geq 0$ for $t \in [0, 1]$. From (3.2.18) and $(\mathcal{H}_{12})$, we have

$$\begin{aligned} -x_0''(t) &\leq p(t) k_1(y_0(t) + \frac{t}{n})(u_1(x_0'(t) + \frac{1}{n}) + v_1(x_0'(t) + \frac{1}{n})), \quad t \in (0, 1), \\ -y_0''(t) &\leq q(t) k_2(x_0(t) + \frac{t}{n})(u_2(y_0'(t) + \frac{1}{n}) + v_2(y_0'(t) + \frac{1}{n})), \quad t \in (0, 1), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{-x_0''(t)}{u_1(x_0'(t) + \frac{1}{n}) + v_1(x_0'(t) + \frac{1}{n})} &\leq p(t) k_1(y_0(t) + \frac{t}{n}) \leq k_1(R_2 + \varepsilon) p(t), \quad t \in (0, 1), \\ \frac{-y_0''(t)}{u_2(y_0'(t) + \frac{1}{n}) + v_2(y_0'(t) + \frac{1}{n})} &\leq q(t) k_2(x_0(t) + \frac{t}{n}) \leq k_2(R_1 + \varepsilon) q(t), \quad t \in (0, 1). \end{aligned}$$

Integrating from $t$ to 1, using the BCs (3.2.18), we obtain

$$\begin{aligned} I(x_0'(t) + \frac{1}{n}) - I(\frac{1}{n}) &\leq k_1(R_2 + \varepsilon) \int_t^1 p(s) ds, \quad t \in [0, 1], \\ J(y_0'(t) + \frac{1}{n}) - J(\frac{1}{n}) &\leq k_2(R_1 + \varepsilon) \int_t^1 q(s) ds, \quad t \in [0, 1], \end{aligned}$$

which implies that

$$\begin{aligned} x_0'(t) &\leq I^{-1}(k_1(R_2 + \varepsilon) \int_0^1 p(s) ds + I(\varepsilon)), \quad t \in [0, 1], \\ y_0'(t) &\leq J^{-1}(k_2(R_1 + \varepsilon) \int_0^1 q(s) ds + J(\varepsilon)), \quad t \in [0, 1], \end{aligned}$$

which on integration from 0 to 1, using the BCs (3.2.18) and Lemma 1.1.10, leads to

$$R_1 \leq I^{-1}(k_1(R_2 + \varepsilon) \int_0^1 p(s) ds + I(\varepsilon)), \tag{3.2.19}$$

$$R_2 \leq J^{-1}(k_2(R_1+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon)). \tag{3.2.20}$$

Now, using (3.2.20) in (3.2.19) together with increasing property of $k_1$ and $I^{-1}$, we have

$$\frac{R_1}{I^{-1}(k_1(J^{-1}(k_2(R_1+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon))+\varepsilon)\int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (3.2.3). Similarly, using (3.2.19) in (3.2.20) together with increasing property of $k_2$ and $J^{-1}$, we have

$$\frac{R_2}{J^{-1}(k_2(I^{-1}(k_1(R_2+\varepsilon)\int_0^1 p(s)ds + I(\varepsilon))+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon))} \leq 1,$$

a contradiction to (3.2.4). Hence, (3.2.17) is true and by Lemma 1.2.5, the fixed point index

$$\mathrm{ind}_{\mathrm{FP}}(T_n, \mathcal{O}_{R_0} \cap (P \times P), P \times P) = 1. \tag{3.2.21}$$

Now, choose a $t_0 \in (0,1)$ and define

$$N_1 = \left(t_0 \min_{t\in[t_0,1]} \int_{t_0}^1 G(t,s)p(s)ds\right)^{-1} + 1 \text{ and } N_2 = \left(t_0 \min_{t\in[t_0,1]} \int_{t_0}^1 G(t,s)q(s)ds\right)^{-1} + 1. \tag{3.2.22}$$

By $(\mathcal{H}_{19})$, there exist real constants with $R_1^* > R_1$ and $R_2^* > R_2$ such that

$$\begin{aligned} h_1(x,y) &\geq N_1 x, \text{ for } x \geq R_1^*, y \in (0,\infty), \\ h_2(x,y) &\geq N_2 x, \text{ for } x \geq R_2^*, y \in (0,\infty). \end{aligned} \tag{3.2.23}$$

Let $R^* = \frac{R_1^*+R_2^*}{t_0}$ and define $\mathcal{O}_{R^*} = \Omega_{R_1^*} \times \Omega_{R_2^*}$, where

$$\Omega_{R_1^*} = \{x \in E : \|x\|_2 < \frac{R_1^*}{t_0}\},\ \Omega_{R_2^*} = \{x \in E : \|x\|_2 < \frac{R_2^*}{t_0}\}.$$

We show that

$$T_n(x,y) \not\preceq (x,y), \text{ for } (x,y) \in \partial\mathcal{O}_{R^*} \cap (P \times P). \tag{3.2.24}$$

Suppose $T_n(x_0,y_0) \preceq (x_0,y_0)$ for some $(x_0,y_0) \in \partial\mathcal{O}_{R^*} \cap (P\times P)$. Then,

$$x_0(t) \geq A_n(x_0,y_0)(t) \text{ and } y_0(t) \geq B_n(x_0,y_0)(t) \text{ for } t \in [0,1]. \tag{3.2.25}$$

Note that, by Lemma 1.1.10, we have

$$x_0(t) \geq t\|x_0\| \geq t_0\|x_0\|_2 = t_0\frac{R_1^*}{t_0} = R_1^* \text{ for } t \in [t_0,1].$$

Similarly, $y_0(t) \geq R_2^*$ for $t \in [t_0,1]$. Hence,

$$|x_0(t)| + \frac{t}{n} \geq R_1^* \text{ and } |y_0(t)| + \frac{t}{n} \geq R_2^* \text{ for } t \in [t_0,1].$$

Now, using (3.2.25), (3.2.23) and ($\mathcal{H}_{19}$), we have

$$\begin{aligned} x_0(t) &\geq A_n(x_0,y_0)(t) = \int_0^1 G(t,s)p(s)f(s,y_0(s)+\frac{s}{n},|x_0'(s)|+\frac{1}{n})ds \\ &\geq \int_{t_0}^1 G(t,s)p(s)h_1(y_0(s)+\frac{s}{n},|x_0'(s)|+\frac{1}{n})ds \geq \int_{t_0}^1 G(t,s)p(s)N_1(y_0(s)+\frac{s}{n})ds \\ &\geq \int_{t_0}^1 G(t,s)p(s)dsN_1R_2^* \geq \min_{t\in[t_0,1]}\int_{t_0}^1 G(t,s)p(s)dsN_1R_2^* > \frac{R_2^*}{t_0}, \end{aligned}$$

which implies that $\|x_0\|_2 = \|x_0\| > \frac{R_2^*}{t_0}$. Similarly, using (3.2.23), (3.2.25) and ($\mathcal{H}_{19}$), we have $\|y_0\|_2 > \frac{R_1^*}{t_0}$. Consequently, it follows that, $\|(x_0,y_0)\|_4 = \|x_0\|_2 + \|y_0\|_2 > R^*$, a contradiction. Hence, (3.2.24) is true and by Lemma 1.2.7, the fixed point index

$$\mathrm{ind}_{\mathrm{FP}}(T_n,\mathcal{O}_{R^*}\cap(P\times P),P\times P)=0. \tag{3.2.26}$$

From (3.2.21) and (3.2.26), it follows that

$$\mathrm{ind}_{\mathrm{FP}}(T_n,(\mathcal{O}_{R^*}\setminus\overline{\mathcal{O}}_R)\cap(P\times P),P\times P)=-1. \tag{3.2.27}$$

Thus, in view of (3.2.21) and (3.2.27), there exist $(x_{n,1},y_{n,1}) \in \mathcal{O}_R\cap(P\times P)$ and $(x_{n,2},y_{n,2}) \in (\mathcal{O}_{R^*}\setminus\overline{\mathcal{O}}_R)\cap(P\times P)$ such that $(x_{n,j},y_{n,j}) = T_n(x_{n,j},y_{n,j})$, $(j=1,2)$ which implies that

$$\begin{aligned} x_{n,j}(t) &= \int_0^1 G(t,s)p(s)f(t,y_{n,j}(s)+\frac{s}{n},|x_{n,j}'(s)|+\frac{1}{n})ds, \quad t\in[0,1], \\ y_{n,j}(t) &= \int_0^1 G(t,s)q(s)g(s,x_{n,j}(s)+\frac{s}{n},|y_{n,j}'(s)|+\frac{1}{n})ds, \quad t\in[0,1],\ j=1,2. \end{aligned} \tag{3.2.28}$$

Using ($\mathcal{H}_{17}$) there exist continuous functions $\varphi_{R_2+\varepsilon}$ and $\psi_{R_1+\varepsilon}$ defined on $[0,1]$ and positive on $(0,1)$ and real constants $0\leq\delta_1,\delta_2<1$ such that

$$\begin{aligned} f(t,x,y) &\geq \varphi_{R_2+\varepsilon}(t)x^{\delta_1}, \quad (t,x,y)\in[0,1]\times[0,R_2+\varepsilon]\times[0,\infty), \\ g(t,x,y) &\geq \psi_{R_1+\varepsilon}(t)x^{\delta_2}, \quad (t,x,y)\in[0,1]\times[0,R_1+\varepsilon]\times[0,\infty). \end{aligned} \tag{3.2.29}$$

By the definition of $P$, we have $x_{n,1}(t)\geq t\|x_{n,1}\|$ and $y_{n,1}(t)\geq t\|y_{n,1}\|$ for $t\in[0,1]$. We show that

$$x_{n,1}'(t) \geq C_4^{\delta_1}\int_t^1 s^{\delta_1}p(s)\varphi_{R_2+\varepsilon}(s)ds, \quad t\in[0,1], \tag{3.2.30}$$

$$y_{n,1}'(t) \geq C_3^{\delta_2}\int_t^1 s^{\delta_2}q(s)\psi_{R_1+\varepsilon}(s)ds, \quad t\in[0,1], \tag{3.2.31}$$

where

$$\begin{aligned} C_3 &= \left(\int_0^1 s^{\delta_2+1}q(s)\psi_{R_1+\varepsilon}(s)ds\right)^{\frac{\delta_1}{1-\delta_1\delta_2}}\left(\int_0^1 s^{\delta_1+1}p(s)\varphi_{R_2+\varepsilon}(s)ds\right)^{\frac{1}{1-\delta_1\delta_2}}, \\ C_4 &= \left(\int_0^1 s^{\delta_1+1}p(s)\varphi_{R_2+\varepsilon}(s)ds\right)^{\frac{\delta_2}{1-\delta_1\delta_2}}\left(\int_0^1 s^{\delta_2+1}q(s)\psi_{R_1+\varepsilon}(s)ds\right)^{\frac{1}{1-\delta_1\delta_2}}. \end{aligned}$$

In order to prove (3.2.30), using (3.2.28) and (3.2.29), we consider

$$x_{n,1}(t) = \int_0^1 G(t,s)p(s)f(s, y_{n,1}(s) + \frac{s}{n}, |x'_{n,1}(s)| + \frac{1}{n})ds$$
$$\geq \int_0^1 G(t,s)p(s)\varphi_{R_2+\varepsilon}(s)(y_{n,1}(s) + \frac{s}{n})^{\delta_1}ds \geq \|y_{n,1}\|^{\delta_1} \int_0^1 G(t,s)s^{\delta_1}p(s)\varphi_{R_2+\varepsilon}(s)ds,$$

which shows that

$$\|x_{n,1}\| \geq \|y_{n,1}\|^{\delta_1} \int_0^1 s^{\delta_1+1}p(s)\varphi_{R_2+\varepsilon}(s)ds. \tag{3.2.32}$$

Similarly, from (3.2.28) and (3.2.29), we have

$$\|y_{n,1}\| \geq \|x_{n,1}\|^{\delta_2} \int_0^1 s^{\delta_2+1}q(s)\psi_{R_1+\varepsilon}(s)ds. \tag{3.2.33}$$

Using (3.2.33) in (3.2.32), we have

$$\|y_{n,1}\| \geq \left(\|y_{n,1}\|^{\delta_1} \int_0^1 s^{\delta_1+1}p(s)\varphi_{R_2+\varepsilon}(s)ds\right)^{\delta_2} \int_0^1 s^{\delta_2+1}q(s)\psi_{R_1+\varepsilon}(s)ds,$$

which implies that

$$\|y_{n,1}\| \geq \left(\int_0^1 s^{\delta_1+1}p(s)\varphi_{R_2+\varepsilon}(s)ds\right)^{\frac{\delta_2}{1-\delta_1\delta_2}} \left(\int_0^1 s^{\delta_2+1}q(s)\psi_{R_1+\varepsilon}(s)ds\right)^{\frac{1}{1-\delta_1\delta_2}} = C_4. \tag{3.2.34}$$

Using (3.2.29) and (3.2.34) in the following relation

$$x'_{n,1}(t) = \int_t^1 p(s)f(s, y_{n,1}(s) + \frac{s}{n}, |x'_{n,1}(s)| + \frac{1}{n})ds,$$

we obtain (3.2.30). Similarly, we can prove (3.2.31).

Now, differentiating (3.2.28), using ($\mathcal{H}_{12}$), (3.2.30), (3.2.31) and Lemma 1.1.9, we have

$$\begin{aligned} 0 \leq -x''_{n,1}(t) &\leq p(t)k_1(R_2+\varepsilon)(u_1(C_4^{\delta_1} \int_t^1 s^{\delta_1}p(s)\varphi_{R_2+\varepsilon}(s)ds) + v_1(\frac{R_1+1}{t})), \quad t \in (0,1), \\ 0 \leq -y''_{n,1}(t) &\leq q(t)k_2(R_1+\varepsilon)(u_2(C_3^{\delta_2} \int_t^1 s^{\delta_2}q(s)\psi_{R_1+\varepsilon}(s)ds) + v_2(\frac{R_2+1}{t})), \quad t \in (0,1). \end{aligned} \tag{3.2.35}$$

Integration from $t$ to 1, using the BCs (3.2.5), leads to

$$x'_{n,1}(t) \leq k_1(R_2+\varepsilon)\int_t^1 p(s)(u_1(C_4^{\delta_1} \int_s^1 \tau^{\delta_1}p(\tau)\varphi_{R_2+\varepsilon}(\tau)d\tau) + v_1(\frac{R_1+1}{s}))ds, \quad t \in [0,1],$$
$$y'_{n,1}(t) \leq k_2(R_1+\varepsilon)\int_t^1 q(s)(u_2(C_3^{\delta_2} \int_s^1 \tau^{\delta_2}q(\tau)\psi_{R_1+\varepsilon}(\tau)d\tau) + v_2(\frac{R_2+1}{s}))ds, \quad t \in [0,1],$$

which implies that

$$\begin{aligned} x'_{n,1}(t) &\leq k_1(R_2+\varepsilon)\int_0^1 p(s)(u_1(C_4^{\delta_1} \int_s^1 \tau^{\delta_1}p(\tau)\varphi_{R_2+\varepsilon}(\tau)d\tau) + v_1(\frac{R_1+1}{s}))ds, \quad t \in [0,1], \\ y'_{n,1}(t) &\leq k_2(R_1+\varepsilon)\int_0^1 q(s)(u_2(C_3^{\delta_2} \int_s^1 \tau^{\delta_2}q(\tau)\psi_{R_1+\varepsilon}(\tau)d\tau) + v_2(\frac{R_2+1}{s}))ds, \quad t \in [0,1]. \end{aligned} \tag{3.2.36}$$

In view of (3.2.30), (3.2.31), (3.2.36), (3.2.35), $(\mathcal{H}_{10})$ and $(\mathcal{H}_{18})$, the sequences $\{(x_{n,1}^{(j)}, y_{n,1}^{(j)})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, 1]$. Thus, by Theorem 1.1.6, there exist subsequences $\{(x_{n_k,1}^{(j)}, y_{n_k,1}^{(j)})\}$ $(j = 0, 1)$ of $\{(x_{n,1}^{(j)}, y_{n,1}^{(j)})\}$ and functions $(x_{0,1}, y_{0,1}) \in \mathcal{E} \times \mathcal{E}$ such that $(x_{n_k,1}^{(j)}, y_{n_k,1}^{(j)})$ converges uniformly to $(x_{0,1}^{(j)}, y_{0,1}^{(j)})$ on $[0, 1]$. Also, $x_{0,1}(0) = y_{0,1}(0) = x'_{0,1}(1) = y'_{0,1}(1) = 0$. Moreover, from (3.2.30) and (3.2.31), with $n_k$ in place of $n$ and taking $\lim_{n_k \to +\infty}$, we have

$$x'_{0,1}(t) \geq C_4^{\delta_1} \int_t^1 s^{\delta_1} p(s) \varphi_{R_2+\varepsilon}(s) ds,$$
$$y'_{0,1}(t) \geq C_3^{\delta_2} \int_t^1 s^{\delta_2} q(s) \psi_{R_1+\varepsilon}(s) ds,$$

which implies that $x'_{0,1} > 0$ and $y'_{0,1} > 0$ on $[0, 1)$, $x_{0,1} > 0$ and $y_{0,1} > 0$ on $(0, 1]$. Further,

$$\begin{aligned} \left| f(t, y_{n_k,1}(t) + \frac{t}{n}, x'_{n_k,1}(t) + \frac{1}{n_k}) \right| &\leq k_1(R_2 + \varepsilon)(u_1(C_4^{\delta_1} \int_t^1 s^{\delta_1} p(s) \varphi_{R_2+\varepsilon}(s) ds) + v_1(\frac{R_1 + 1}{t})), \\ \left| g(t, x_{n_k,1}(t) + \frac{t}{n}, y'_{n_k,1}(t) + \frac{1}{n_k}) \right| &\leq k_2(R_1 + \varepsilon)(u_2(C_3^{\delta_2} \int_t^1 s^{\delta_2} q(s) \psi_{R_1+\varepsilon}(s) ds) + v_2(\frac{R_2 + 1}{t})), \end{aligned} \tag{3.2.37}$$

$$\begin{aligned} \lim_{n_k \to \infty} f(t, y_{n_k,1}(t) + \frac{t}{n_k}, x'_{n_k,1}(t) + \frac{1}{n_k}) &= f(t, y_{0,1}(t), x'_{0,1}(t)), \quad t \in (0, 1], \\ \lim_{n_k \to \infty} g(t, x_{n_k,1}(t) + \frac{t}{n_k}, y'_{n_k,1}(t) + \frac{1}{n_k}) &= g(t, x_{0,1}(t), y'_{0,1}(t)), \quad t \in (0, 1]. \end{aligned} \tag{3.2.38}$$

Moreover, $(x_{n_k,1}, y_{n_k,1})$ satisfies

$$x_{n_k,1}(t) = \int_0^1 G(t, s) p(s) f(s, y_{n_k,1}(s) + \frac{s}{n_k}, x'_{n_k,1}(s) + \frac{1}{n_k}) ds, \quad t \in [0, 1],$$
$$y_{n_k,1}(t) = \int_0^1 G(t, s) q(s) g(s, x_{n_k,1}(s) + \frac{s}{n_k}, y'_{n_k,1}(s) + \frac{1}{n_k}) ds, \quad t \in [0, 1],$$

in view of (3.2.37), $(\mathcal{H}_{18})$, (3.2.38), the Lebesgue dominated convergence theorem and taking $\lim_{n_k \to +\infty}$, we have

$$x_{0,1}(t) = \int_0^1 G(t, s) p(s) f(s, y_{0,1}(s), x'_{0,1}(s)) ds, \quad t \in [0, 1],$$
$$y_{0,1}(t) = \int_0^1 G(t, s) q(s) g(s, x_{0,1}(s), y'_{0,1}(s)) ds, \quad t \in [0, 1],$$

which implies that $(x_{0,1}, y_{0,1}) \in C^2(0, 1) \times C^2(0, 1)$ and

$$-x''_{0,1}(t) = p(t) f(t, y_{0,1}(t), x'_{0,1}(t)), \quad t \in (0, 1),$$
$$-y''_{0,1}(t) = q(t) g(t, x_{0,1}(t), y'_{0,1}(t)), \quad t \in (0, 1).$$

Moreover, by (3.2.1) and (3.2.2), we have $\|x_{0,1}\|_2 < R_1$ and $\|y_{0,1}\|_2 < R_2$, that is, $\|(x_{0,1}, y_{0,1})\|_3 < R_0$. By a similar proof the sequence $\{(x_{n,2}, y_{n,2})\}$ has a convergent subsequence $\{(x_{n_k,2}, y_{n_k,2})\}$ converging uniformly to $(x_{0,2}, y_{0,2}) \in \mathcal{E} \times \mathcal{E}$ on $[0, 1]$. Moreover, $(x_{0,2}, y_{0,2})$ is a solution to the system (3.0.2), (3.0.3) with $x_{0,2} > 0$ and $y_{0,2} > 0$ on $(0, 1]$, $x'_{0,2} > 0$ and $y'_{0,2} > 0$ on $[0, 1)$, $R_0 < \|(x_{0,2}, y_{0,2})\|_4 < R^*$. □

**Example 3.2.3.** Consider the following coupled system of SBVPs

$$
\begin{aligned}
&-x''(t) = \mu_1(1 + (y(t))^{\delta_1} + (y(t))^{\eta_1})(1 + (x'(t))^{\alpha_1} + (x'(t))^{-\beta_1}), \quad t \in (0,1),\\
&-y''(t) = \mu_2(1 + (x(t))^{\delta_2} + (x(t))^{\eta_2})(1 + (y'(t))^{\alpha_2} + (y'(t))^{-\beta_2}), \quad t \in (0,1),\\
&\quad x(0) = y(0) = x'(1) = y'(1) = 0,
\end{aligned}
\tag{3.2.39}
$$

where $0 \le \delta_i < 1$, $\eta_i > 1$, $0 < \alpha_i < 1$, $0 < \beta_i < 1$, and $\mu_i > 0$, $i = 1,2$.

Taking $p(t) = \mu_1$, $q(t) = \mu_2$, $k_i(x) = 1 + x^{\delta_i} + x^{\eta_i}$, $u_i(x) = x^{-\beta_i}$ and $v_i(x) = 1 + x^{\alpha_i}$, $i = 1,2$. We choose $\varphi_E(t) = \mu_1$, $\psi_E(t) = \mu_2$ and $h_i(x,y) = \mu_i(1 + x^{\eta_i})$, $i = 1,2$.

Assume that $\mu_1$ is arbitrary and

$$
\mu_2 < \min\{\inf_{c\in(0,\infty)} \frac{J(c)}{k_2(I^{-1}(\mu_1 k_1(c)))}, \inf_{c\in(0,\infty)} \frac{J((\mu_1^{-1} I(c))^{\delta_1^{-1}})}{k_2(c)}, \inf_{c\in(0,\infty)} \frac{J((\mu_1^{-1} I(c))^{\eta_1^{-1}})}{k_2(c)}\}.
$$

Then,

$$
\begin{aligned}
&\sup_{c\in(0,\infty)} \frac{c}{I^{-1}(k_1(J^{-1}(k_2(c)\int_0^1 q(s)ds))\int_0^1 p(s)ds)}\\
&= \sup_{c\in(0,\infty)} \frac{c}{I^{-1}(\mu_1 k_1(J^{-1}(\mu_2 k_2(c))))}\\
&\ge \frac{c}{I^{-1}(\mu_1 k_1(J^{-1}(\mu_2 k_2(c))))}, \qquad c \in (0,\infty)\\
&= \frac{c}{I^{-1}(\mu_1(1 + (J^{-1}(\mu_2 k_2(c)))^{\delta_1} + (J^{-1}(\mu_2 k_2(c)))^{\eta_1}))}, \quad c \in (0,\infty)\\
&> 1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{c\in(0,\infty)} \frac{c}{J^{-1}(k_2(I^{-1}(k_1(c)\int_0^1 p(s)ds))\int_0^1 q(s)ds)}\\
&= \sup_{c\in(0,\infty)} \frac{c}{J^{-1}(\mu_2 k_2(I^{-1}(\mu_1 k_1(c))))}\\
&= \frac{c}{J^{-1}(\mu_2 k_2(I^{-1}(\mu_1 k_1(c))))}, \quad c \in (0,\infty)\\
&> 1.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_0^1 p(t) v_1(\frac{C}{t})dt = \mu_1(1 + \frac{C^{\alpha_1}}{1-\alpha_1}) < +\infty,\\
&\int_0^1 p(t) u_1(C\int_t^1 s^{\delta_1} p(s)\varphi_E(s)ds)dt \le \mu_1^{1-2\beta_1} C^{-\beta_1}(\delta_1+1)^{\beta_1}\int_0^1 (1-t)^{-\beta_1}dt\\
&\qquad = \mu_1^{1-2\beta_1} C^{-\beta_1}(\delta_1+1)^{\beta_1}(1-\beta_1)^{-1} < +\infty, \text{ etc.}
\end{aligned}
$$

Also,

$$
\lim_{x\to+\infty} \frac{h_i(x,y)}{x} = \lim_{x\to+\infty} \frac{\mu_i(1+x^{\eta_i})}{x} = +\infty,\ i = 1,2.
$$

Clearly, $(\mathcal{H}_{10}) - (\mathcal{H}_{14})$ and $(\mathcal{H}_{17}) - (\mathcal{H}_{19})$ are satisfied. Hence, by Theorem 3.2.2, the system of BVPs (3.2.39) has at least two positive solutions.

## 3.3 Existence of $C^1$-positive solutions with more general BCs

In this section, we study the system of BVPs (3.0.2), (3.0.4) and establish sufficient conditions for the existence of $C^1$-positive solutions, [24]. By a $C^1$-positive solution to the system of BVPs (3.0.2), (3.0.4), we mean $(x,y) \in (C^1[0,1] \cap C^2(0,1)) \times (C^1[0,1] \cap C^2(0,1))$ satisfying (3.0.2) and (3.0.4), $x > 0$ and $y > 0$ on $[0,1]$, $x' > 0$ and $y' > 0$ on $[0,1)$.

Assume that

$(\mathcal{H}_{20})$

$$\sup_{c\in(0,\infty)} \frac{c}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(c)\int_0^1 q(t)dt))\int_0^1 p(t)dt)} > 1,$$
$$\sup_{c\in(0,\infty)} \frac{c}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(c)\int_0^1 p(t)dt))\int_0^1 q(t)dt)} > 1,$$

where $I(\mu) = \int_0^\mu \frac{d\tau}{u_1(\tau)+v_1(\tau)}$, $J(\mu) = \int_0^\mu \frac{d\tau}{u_2(\tau)+v_2(\tau)}$, for $\mu \in (0,\infty)$;

$(\mathcal{H}_{21})$ $\int_0^1 p(t)u_1(C\int_t^1 p(s)\varphi_{EF}(s)ds)dt < +\infty$ and $\int_0^1 q(t)u_2(C\int_t^1 q(s)\psi_{EF}(s)ds)dt < +\infty$ for any real constant $C > 0$.

**Theorem 3.3.1.** *Under the hypothesis* $(\mathcal{H}_{10})-(\mathcal{H}_{12})$, $(\mathcal{H}_{14})$, $(\mathcal{H}_{15})$, $(\mathcal{H}_{20})$ *and* $(\mathcal{H}_{21})$, *the system of BVPs* (3.0.2), (3.0.4) *has at least one* $C^1$*-positive solution.*

*Proof.* In view of $(\mathcal{H}_{20})$, we can choose real constants $M_3 > 0$ and $M_4 > 0$ such that

$$\frac{M_3}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(t)dt))\int_0^1 p(t)dt)} > 1,$$

$$\frac{M_4}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(M_4)\int_0^1 p(t)dt))\int_0^1 q(t)dt)} > 1.$$

From the continuity of $k_1$, $k_2$, $I$ and $J$, we choose $\varepsilon > 0$ small enough such that

$$\frac{M_3}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(t)dt + J(\varepsilon)))\int_0^1 p(t)dt + I(\varepsilon))} > 1, \quad (3.3.1)$$

$$\frac{M_4}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(M_4)\int_0^1 p(t)dt + I(\varepsilon)))\int_0^1 q(t)dt + J(\varepsilon))} > 1. \quad (3.3.2)$$

Choose real constants $L_3 > 0$ and $L_4 > 0$ such that

$$I(L_3) > k_1(M_4)\int_0^1 p(t)dt + I(\varepsilon), \quad (3.3.3)$$

$$J(L_4) > k_2(M_3)\int_0^1 q(t)dt + J(\varepsilon). \quad (3.3.4)$$

Choose $n_0 \in \{1,2,\cdots\}$ such that $\frac{1}{n_0} < \varepsilon$. For each fixed $n \in \{n_0, n_0+1, \cdots\}$, define retractions $\theta_i : \mathbb{R} \to [0, M_i]$ and $\rho_i : \mathbb{R} \to [\frac{1}{n}, L_i]$ by

$$\theta_i(x) = \max\{0, \min\{x, M_i\}\} \text{ and } \rho_i(x) = \max\{\frac{1}{n}, \min\{x, L_i\}\}, i = 3,4.$$

Consider the modified system of BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t,\theta_4(y(t)),\rho_3(x'(t))), \quad t\in(0,1),\\ -y''(t) &= q(t)g(t,\theta_3(x(t)),\rho_4(y'(t))), \quad t\in(0,1),\\ a_1x(0)-b_1x'(0)&=0,\ x'(1)=\frac{1}{n},\\ a_2y(0)-b_2y'(0)&=0,\ y'(1)=\frac{1}{n}. \end{aligned} \tag{3.3.5}$$

Since $f(t,\theta_4(y(t)),\rho_3(x'(t)))$, $g(t,\theta_3(x(t)),\rho_4(y'(t)))$ are continuous and bounded on $[0,1]\times\mathbb{R}^2$, by Theorem 1.2.4, it follows that the modified system of BVPs (3.3.5) has a solution $(x_n,y_n)\in(C^1[0,1]\cap C^2(0,1))\times(C^1[0,1]\cap C^2(0,1))$.

Using (3.3.5) and $(\mathcal{H}_{11})$, we obtain

$$x_n''(t)\le 0 \text{ and } y_n''(t)\le 0 \text{ for } t\in(0,1),$$

which on integration from $t$ to 1, and using the BCs (3.3.5), yields

$$x_n'(t)\ge\frac{1}{n} \text{ and } y_n'(t)\ge\frac{1}{n} \text{ for } t\in[0,1]. \tag{3.3.6}$$

Integrating (3.3.6) from 0 to $t$, using the BCs (3.3.5) and (3.3.6), we have

$$x_n(t)\ge(t+\frac{b_1}{a_1})\frac{1}{n} \text{ and } y_n(t)\ge(t+\frac{b_2}{a_2})\frac{1}{n} \text{ for } t\in[0,1]. \tag{3.3.7}$$

From (3.3.6) and (3.3.7), it follows that

$$\|x_n\|=x_n(1) \text{ and } \|y_n\|=y_n(1).$$

Now, we show that

$$x_n'(t)<L_3,\ y_n'(t)<L_4, \quad t\in[0,1]. \tag{3.3.8}$$

First, we prove $x_n'(t)<L_3$ for $t\in[0,1]$. Suppose $x_n'(t_1)\ge L_3$ for some $t_1\in[0,1]$. Using (3.3.5) and $(\mathcal{H}_{12})$, we have

$$-x_n''(t)\le p(t)k_1(\theta_4(y_n(t)))(u_1(\rho_3(x_n'(t)))+v_1(\rho_3(x_n'(t)))), \quad t\in(0,1),$$

which implies that

$$\frac{-x_n''(t)}{u_1(\rho_3(x_n'(t)))+v_1(\rho_3(x_n'(t)))}\le k_1(M_4)p(t), \quad t\in(0,1).$$

Integrating from $t_1$ to 1, using the BCs (3.3.5), we obtain

$$\int_{\frac{1}{n}}^{x_n'(t_1)}\frac{dz}{u_1(\rho_3(z))+v_1(\rho_3(z))}\le k_1(M_4)\int_{t_1}^1 p(t)dt,$$

which can also be written as

$$\int_{\frac{1}{n}}^{L_3}\frac{dz}{u_1(\rho_3(z))+v_1(\rho_3(z))}+\int_{L_3}^{x_n'(t_1)}\frac{dz}{u_1(\rho_3(z))+v_1(\rho_3(z))}\le k_1(M_4)\int_0^1 p(t)dt.$$

Using the increasing property of $I$, we obtain

$$I(L_3) + \frac{x_n'(t_1) - L_3}{u_1(L_3) + v_1(L_3)} \le k_1(M_4)\int_0^1 p(t)dt + I(\varepsilon),$$

a contradiction to (3.3.3). Hence, $x_n'(t) < L_3$ for $t \in [0,1]$.

Similarly, we can show that $y_n'(t) < L_4$ for $t \in [0,1]$.

Now, we show that

$$x_n(t) < M_3,\ y_n(t) < M_4, \quad t \in [0,1]. \tag{3.3.9}$$

Suppose $x_n(t_2) \ge M_3$ for some $t_2 \in [0,1]$. From (3.3.5), (3.3.8) and $(\mathcal{H}_{12})$, it follows that

$$\begin{aligned} -x_n''(t) &\le p(t)k_1(\theta_4(y_n(t)))(u_1(x_n'(t)) + v_1(x_n'(t))), \quad t \in (0,1),\\ -y_n''(t) &\le q(t)k_2(\theta_3(x_n(t)))(u_2(y_n'(t)) + v_2(y_n'(t))), \quad t \in (0,1), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{-x_n''(t)}{u_1(x_n'(t)) + v_1(x_n'(t))} &\le k_1(\theta_4(\|y_n\|))p(t), \quad t \in (0,1),\\ \frac{-y_n''(t)}{u_2(y_n'(t)) + v_2(y_n'(t))} &\le k_2(M_3)q(t), \qquad\qquad t \in (0,1). \end{aligned}$$

Integrating from $t$ to 1, using the BCs (3.3.5), we obtain

$$\begin{aligned} \int_{\frac{1}{n}}^{x_n'(t)} \frac{dz}{u_1(z) + v_1(z)} &\le k_1(\theta_4(\|y_n\|))\int_t^1 p(s)ds, \quad t \in [0,1],\\ \int_{\frac{1}{n}}^{y_n'(t)} \frac{dz}{u_2(z) + v_2(z)} &\le k_2(M_3)\int_t^1 q(s)ds, \qquad\qquad t \in [0,1], \end{aligned}$$

which implies that

$$\begin{aligned} I(x_n'(t)) - I(\frac{1}{n}) &\le k_1(\theta_4(\|y_n\|))\int_0^1 p(s)ds, \quad t \in [0,1],\\ J(y_n'(t)) - J(\frac{1}{n}) &\le k_2(M_3)\int_0^1 q(s)ds, \qquad\qquad t \in [0,1]. \end{aligned}$$

The increasing property of $I$ and $J$ leads to

$$x_n'(t) \le I^{-1}(k_1(\theta_4(\|y_n\|))\int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0,1], \tag{3.3.10}$$

$$y_n'(t) \le J^{-1}(k_2(M_3)\int_0^1 q(s)ds + J(\varepsilon)), \qquad t \in [0,1]. \tag{3.3.11}$$

Integrating (3.3.10) from 0 to $t_2$ and (3.3.11) from 0 to 1, using the BCs (3.3.5), (3.3.10) and (3.3.11), we obtain

$$M_3 \le x_n(t_2) \le (1 + \frac{b_1}{a_1})I^{-1}(k_1(\theta_4(\|y_n\|))\int_0^1 p(s)ds + I(\varepsilon)), \tag{3.3.12}$$

$$\|y_n\| \leq (1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(s)ds + J(\varepsilon)). \tag{3.3.13}$$

Either we have $\|y_n\| < M_4$ or $\|y_n\| \geq M_4$. If $\|y_n\| < M_4$, then from (3.3.12), we have

$$M_3 \leq (1+\frac{b_1}{a_1})I^{-1}(k_1(\|y_n\|)\int_0^1 p(s)ds + I(\varepsilon)), \tag{3.3.14}$$

Now, by using (3.3.13) in (3.3.14) and the increasing property of $k_1$ and $I^{-1}$, we obtain

$$M_3 \leq (1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon)),$$

which implies that

$$\frac{M_3}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (3.3.1).

On the other hand, if $\|y_n\| \geq M_4$, then from (3.3.12) and (3.3.13), we have

$$M_3 \leq (1+\frac{b_1}{a_1})I^{-1}(k_1(M_4)\int_0^1 p(s)ds + I(\varepsilon)), \tag{3.3.15}$$

$$M_4 \leq (1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(s)ds + J(\varepsilon)). \tag{3.3.16}$$

Using (3.3.16) in (3.3.15) and the increasing property of $k_1$ and $I^{-1}$, we obtain

$$M_3 \leq (1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon)),$$

which implies that

$$\frac{M_3}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(M_3)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (3.3.1). Hence, $x_n(t) < M_3$ for $t \in [0, 1]$.

Similarly, we can show that $y_n(t) < M_4$ for $t \in [0, 1]$.

Thus, in view of (3.3.5)–(3.3.9), $(x_n, y_n)$ is a solution of the following coupled system of BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t, y(t), x'(t)), \quad t \in (0, 1),\\ -y''(t) &= q(t)g(t, x(t), y'(t)), \quad t \in (0, 1),\\ a_1x(0) - b_1x'(0) &= 0,\ x'(1) = \frac{1}{n},\\ a_2y(0) - b_2y'(0) &= 0,\ y'(1) = \frac{1}{n}, \end{aligned} \tag{3.3.17}$$

satisfying

$$\begin{aligned} (t+\frac{b_1}{a_1})\frac{1}{n} &\leq x_n(t) < M_3,\ \frac{1}{n} \leq x_n'(t) < L_3, \quad t \in [0, 1],\\ (t+\frac{b_2}{a_2})\frac{1}{n} &\leq y_n(t) < M_4,\ \frac{1}{n} \leq y_n'(t) < L_4, \quad t \in [0, 1]. \end{aligned} \tag{3.3.18}$$

Now, in view of $(\mathcal{H}_{15})$, there exist continuous functions $\varphi_{M_4L_3}$ and $\psi_{M_3L_4}$ defined on $[0,1]$ and positive on $(0,1)$, and real constants $0 \le \delta_1, \delta_2 < 1$ such that

$$\begin{aligned} f(t,y_n(t),x_n'(t)) &\ge \varphi_{M_4L_3}(t)(y_n(t))^{\delta_1}, \ \ (t,y_n(t),x_n'(t)) \in [0,1]\times[0,M_4]\times[0,L_3],\\ g(t,x_n(t),y_n'(t)) &\ge \psi_{M_3L_4}(t)(x_n(t))^{\delta_2}, \ \ (t,x_n(t),y_n'(t)) \in [0,1]\times[0,M_3]\times[0,L_4]. \end{aligned} \tag{3.3.19}$$

We claim that

$$x_n'(t) \ge C_6^{\delta_1}\int_t^1 p(s)\varphi_{M_4L_3}(s)ds, \tag{3.3.20}$$

$$y_n'(t) \ge C_5^{\delta_2}\int_t^1 q(s)\psi_{M_3L_4}(s)ds, \tag{3.3.21}$$

where

$$C_5 = \left(\frac{b_1}{a_1}\right)^{\frac{1}{1-\delta_1\delta_2}}\left(\frac{b_2}{a_2}\right)^{\frac{\delta_1}{1-\delta_1\delta_2}}\left(\int_0^1 p(t)\varphi_{M_4L_3}(t)dt\right)^{\frac{1}{1-\delta_1\delta_2}}\left(\int_0^1 q(t)\psi_{M_3L_4}(t)dt\right)^{\frac{\delta_1}{1-\delta_1\delta_2}},$$

$$C_6 = \left(\frac{b_1}{a_1}\right)^{\frac{\delta_2}{1-\delta_1\delta_2}}\left(\frac{b_2}{a_2}\right)^{\frac{1}{1-\delta_1\delta_2}}\left(\int_0^1 p(t)\varphi_{M_4L_3}(t)dt\right)^{\frac{\delta_2}{1-\delta_1\delta_2}}\left(\int_0^1 q(t)\psi_{M_3L_4}(t)dt\right)^{\frac{1}{1-\delta_1\delta_2}}.$$

To prove (3.3.20), consider the following relation

$$\begin{aligned} x_n(t) =&(t+\frac{b_1}{a_1})\frac{1}{n} + \frac{1}{a_1}\int_0^t (a_1 s + b_1)p(s)f(s,y_n(s),x_n'(s))ds\\ &+\frac{1}{a_1}\int_t^1 (a_1 t + b_1)p(s)f(s,y_n(s),x_n'(s))ds, \quad t\in[0,1], \end{aligned} \tag{3.3.22}$$

which implies that

$$x_n(0) = \frac{b_1}{a_1}\frac{1}{n} + \frac{b_1}{a_1}\int_0^1 p(s)f(s,y_n(s),x_n'(s))ds.$$

Using (3.3.19) and (3.3.18), we obtain

$$x_n(0) \ge \frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_4L_3}(s)(y_n(s))^{\delta_1}ds \ge (y_n(0))^{\delta_1}\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_4L_3}(s)ds. \tag{3.3.23}$$

Similarly, using (3.3.19) and (3.3.18), we obtain

$$y_n(0) \ge (x_n(0))^{\delta_2}\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_3L_4}(s)ds,$$

which in view of (3.3.23) implies that

$$y_n(0) \ge (y_n(0))^{\delta_1\delta_2}\left(\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_4L_3}(s)ds\right)^{\delta_2}\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_3L_4}(s)ds.$$

Hence,

$$y_n(0) \ge C_6. \tag{3.3.24}$$

Now, from (3.3.22), it follows that

$$x_n'(t) \geq \int_t^1 p(s)f(s, y_n(s), x_n'(s))ds,$$

and using (3.3.19) and (3.3.24), we obtain (3.3.20).

Similarly, we can prove (3.3.21).

Now, using (3.3.17), $(\mathcal{H}_{12})$, (3.3.18), (3.3.20) and (3.3.21), we have

$$\begin{aligned} 0 \leq -x_n''(t) \leq k_1(M_4)p(t)(u_1(C_6^{\delta_1} \int_t^1 p(s)\varphi_{M_4L_3}(s)ds) + v_1(L_3)), \quad t \in (0,1), \\ 0 \leq -y_n''(t) \leq k_2(M_3)q(t)(u_2(C_5^{\delta_2} \int_t^1 q(s)\psi_{M_3}(s)ds) + v_2(L_4)), \quad t \in (0,1). \end{aligned} \tag{3.3.25}$$

In view of (3.3.18), (3.3.25), $(\mathcal{H}_{10})$ and $(\mathcal{H}_{21})$, it follows that the sequences $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, 1]$. Hence, by Theorem (1.1.6), there exist subsequences $\{(x_{n_k}^{(j)}, y_{n_k}^{(j)})\}$ $(j = 0, 1)$ of $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j = 0, 1)$ and $(x, y) \in C^1[0, 1] \times C^1[0, 1]$ such that $(x_{n_k}^{(j)}, y_{n_k}^{(j)})$ converges uniformly to $(x^{(j)}, y^{(j)})$ on $[0, 1]$ $(j = 0, 1)$. Also, $a_1x(0) - b_1x'(0) = a_2y(0) - b_2y'(0) = x'(1) = y'(1) = 0$. Moreover, from (3.3.20) and (3.3.21), with $n_k$ in place of $n$ and taking $\lim_{n_k \to +\infty}$, we have

$$\begin{aligned} x'(t) \geq & C_6^{\delta_1} \int_t^1 p(s)\varphi_{M_4L_3}(s)ds, \\ y'(t) \geq & C_5^{\delta_2} \int_t^1 q(s)\psi_{M_3L_4}(s)ds, \end{aligned}$$

which shows that $x' > 0$ and $y' > 0$ on $[0, 1)$, $x > 0$ and $y > 0$ on $[0, 1]$. Further, $(x_{n_k}, y_{n_k})$ satisfy

$$\begin{aligned} x_{n_k}'(t) = & x_{n_k}'(0) - \int_0^t p(s)f(s, y_{n_k}(s), x_{n_k}'(s))ds, \quad t \in [0, 1], \\ y_{n_k}'(t) = & y_{n_k}'(0) - \int_0^t q(s)f(s, x_{n_k}(s), y_{n_k}'(s))ds, \quad t \in [0, 1]. \end{aligned}$$

Passing to the limit as $n_k \to \infty$, we obtain

$$\begin{aligned} x'(t) = & x'(0) - \int_0^t p(s)f(s, y(s), x'(s))ds, \quad t \in [0, 1], \\ y'(t) = & y'(0) - \int_0^t q(s)f(s, x(s), y'(s))ds, \quad t \in [0, 1], \end{aligned}$$

which implies that

$$\begin{aligned} -x''(t) = & p(t)f(t, y(t), x'(t)), \quad t \in (0, 1), \\ -y''(t) = & q(t)f(t, x(t), y'(t)), \quad t \in (0, 1). \end{aligned}$$

Hence, $(x, y)$ is a $C^1$-positive solution of the system of BVPs (3.0.2), (3.0.4). □

**Example 3.3.2.** Consider the following coupled system of singular BVPs

$$\begin{aligned} -x''(t) &= (1-t)^{-\frac{3}{4}}(y(t))^{\frac{1}{4}}(x'(t))^{-\beta_1}, \quad t\in(0,1),\\ -y''(t) &= (1-t)^{-\frac{1}{4}}(x(t))^{\frac{3}{4}}(y'(t))^{-\beta_2}, \quad t\in(0,1),\\ x(0)-x'(0) &= y(0)-y'(0)=x'(1)=y'(1)=0, \end{aligned} \tag{3.3.26}$$

where $0<\beta_1<1$ and $0<\beta_2<1$.

Taking $p(t)=(1-t)^{-\frac{3}{4}}$, $q(t)=(1-t)^{-\frac{1}{4}}$, $k_1(x)=x^{\frac{1}{4}}$, $k_2(x)=x^{\frac{3}{4}}$, $u_1(x)=x^{-\beta_1}$, $u_2(x)=x^{-\beta_2}$ and $v_1(x)=v_2(x)=0$. Then, $I(z)=\frac{z^{\beta_1+1}}{\beta_1+1}$, $J(z)=\frac{z^{\beta_2+1}}{\beta_2+1}$, $I^{-1}(z)=(\beta_1+1)^{\frac{1}{\beta_1+1}}z^{\frac{1}{\beta_1+1}}$ and $J^{-1}(z)=(\beta_2+1)^{\frac{1}{\beta_2+1}}z^{\frac{1}{\beta_2+1}}$.

Then, $\int_0^1 p(t)dt=4$ and $\int_0^1 q(t)dt=\frac{4}{3}$. Choose $\delta_1=\frac{1}{4}$, $\delta_2=\frac{3}{4}$, $\varphi_{EF}(t)=F^{-\beta_1}$ and $\psi_{EF}(t)=F^{-\beta_2}$. Clearly, $(\mathcal{H}_{10})-(\mathcal{H}_{12})$, $(\mathcal{H}_{14})$ and $(\mathcal{H}_{15})$ are satisfied. Moreover,

$$\sup_{c\in(0,\infty)}\frac{c}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(c)\int_0^1 q(t)dt))\int_0^1 p(t)dt)}=$$

$$\sup_{c\in(0,\infty)}\frac{c}{2^{1+\frac{9}{4(\beta_1+1)}}(\frac{4}{3})^{\frac{1}{4(\beta_1+1)(\beta_2+1)}}(\beta_1+1)^{\frac{1}{\beta_1+1}}(\beta_2+1)^{\frac{1}{4(\beta_1+1)(\beta_2+1)}}c^{\frac{3}{16(\beta_1+1)(\beta_2+1)}}}=\infty,$$

$$\sup_{c\in(0,\infty)}\frac{c}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(c)\int_0^1 p(t)dt))\int_0^1 q(t)dt)}=$$

$$\sup_{c\in(0,\infty)}\frac{c}{2^{1+\frac{3}{4(\beta_2+1)}(1+\frac{2}{\beta_1+1})}(\frac{4}{3})^{\frac{1}{\beta_2+1}}(\beta_2+1)^{\frac{1}{\beta_2+1}}(\beta_1+1)^{\frac{3}{4(\beta_1+1)(\beta_2+1)}}c^{\frac{3}{16(\beta_1+1)(\beta_2+1)}}}=\infty,$$

$$\int_0^1 p(t)u_1(C\int_t^1 p(s)\varphi_{EF}(s)ds)dt=4^{1-\beta_1}C^{-\beta_1}F^{\beta_1^2}\int_0^1(1-t)^{-\frac{\beta_1+3}{4}}dt=\frac{4^{1-\beta_1}C^{-\beta_1}F^{\beta_1^2}}{1-\beta_1},$$

$$\begin{aligned}\int_0^1 q(t)u_2(C\int_t^1 q(s)\psi_{EF}(s)ds)dt &= (\frac{4}{3})^{-\beta_2}C^{-\beta_2}F^{\beta_2^2}\int_0^1(1-t)^{-\frac{3\beta_2+1}{4}}dt\\ &= \frac{(4/3)^{1-\beta_2}C^{-\beta_2}F^{\beta_2^2}}{1-\beta_2},\end{aligned}$$

which shows that $(\mathcal{H}_{20})$ and $(\mathcal{H}_{21})$ also holds.

Since, $(\mathcal{H}_{10})-(\mathcal{H}_{12})$, $(\mathcal{H}_{14})$, $(\mathcal{H}_{15})$, $(\mathcal{H}_{20})$ and $(\mathcal{H}_{21})$ are satisfied. Therefore, by Theorem 3.3.1, the system of BVPs (3.3.26) has at least one $C^1$-positive solution.

## 3.4 Existence of at least two positive solutions with more general BCs

In this section, we establish at lest two $C^1$-positive solutions to the system of BVPs (3.0.2), (3.0.4). For each $(x,y)\in C^1[0,1]\times C^1[0,1]$, we write $\|(x,y)\|_5=\|x\|_3+\|y\|_3$. Clearly, $(C^1[0,1]\times C^1[0,1],\|\cdot\|_5)$ is a Banach space. We define a partial ordering in $C^1[0,1]$, by $x\le y$ if and only if $x(t)\le y(t)$, $t\in[0,1]$. We define a partial ordering in $C^1[0,1]\times C^1[0,1]$, by $(x_1,y_1)\preceq(x_2,y_2)$ if and only if $x_1\le x_2$ and $y_1\le y_2$. Let

$$P_i=\{x\in C^1[0,1]:x(t)\ge\gamma_i\|x\| \text{ for all } t\in[0,1], x(0)\ge\frac{b_i}{a_i}\|x'\|\},$$

where $\gamma_i = \frac{b_i}{a_i+b_i}$, $i = 1, 2$. Clearly, $P_i$ $(i = 1, 2)$ are cones of $C^1[0, 1]$ and $P_1 \times P_2$ is a cone of $C^1[0, 1] \times C^1[0, 1]$. For any real constant $r > 0$, we define an open neighborhood of $(0, 0) \in C^1[0, 1] \times C^1[0, 1]$ as

$$\mathcal{O}_r = \{(x, y) \in C^1[0, 1] \times C^1[0, 1] : \|(x, y)\|_5 < r\}.$$

In view of $(\mathcal{H}_{20})$, there exist real constants $R_3 > 0$ and $R_4 > 0$ such that

$$\frac{R_3}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(R_3)\int_0^1 q(t)dt))\int_0^1 p(t)dt)} > 1, \tag{3.4.1}$$

$$\frac{R_4}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(R_4)\int_0^1 p(t)dt))\int_0^1 q(t)dt)} > 1. \tag{3.4.2}$$

From the continuity of $k_1$, $k_2$, $I$ and $J$, we choose $\varepsilon > 0$ small enough such that

$$\frac{R_3}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(R_3+\varepsilon)\int_0^1 q(t)dt + J(\varepsilon)) + \varepsilon)\int_0^1 p(t)dt + I(\varepsilon))} > 1, \tag{3.4.3}$$

$$\frac{R_4}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(R_4+\varepsilon)\int_0^1 p(t)dt + I(\varepsilon)) + \varepsilon)\int_0^1 q(t)dt + J(\varepsilon))} > 1. \tag{3.4.4}$$

Choose $n_0 \in \{1, 2, \cdots\}$ such that $\max\{\frac{1}{n_0}(1+\frac{b_1}{a_1}), \frac{1}{n_0}(1+\frac{b_2}{a_2})\} < \varepsilon$ and for each fixed $n \in \{n_0, n_0+1, \cdots\}$, consider the system of non–singular BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t, y(t) + \frac{1}{n}(t+\frac{b_2}{a_2}), |x'(t)| + \frac{1}{n}), \quad t \in (0, 1),\\ -y''(t) &= q(t)g(t, x(t) + \frac{1}{n}(t+\frac{b_1}{a_1}), |y'(t)| + \frac{1}{n}), \quad t \in (0, 1),\\ a_1x(0) - b_1x'(0) &= a_2y(0) - b_2y'(0) = x'(1) = y'(1) = 0. \end{aligned} \tag{3.4.5}$$

We write (3.4.5) as an equivalent system of integral equations

$$\begin{aligned} x(t) &= \int_0^1 G_1(t, s)p(s)f(s, y(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds, \quad t \in [0, 1],\\ y(t) &= \int_0^1 G_2(t, s)q(s)f(s, x(s) + \frac{1}{n}(s+\frac{b_1}{a_1}), |y'(s)| + \frac{1}{n})ds, \quad t \in [0, 1], \end{aligned} \tag{3.4.6}$$

where

$$G_i(t, s) = \frac{1}{a_i}\begin{cases} b_i + a_is, & 0 \le s \le t \le 1,\\ b_i + a_it, & 0 \le t \le s \le 1, \ i = 1, 2. \end{cases}$$

By a solution of the system of BVPs (3.4.5), we mean a solution of the corresponding system of integral equations (3.4.6).

Define an operator $T_n : C^1[0, 1] \times C^1[0, 1] \to C^1[0, 1] \times C^1[0, 1]$ by

$$T_n(x, y) = (A_n(x, y), B_n(x, y)), \tag{3.4.7}$$

where the operators $A_n, B_n : C^1[0,1] \times C^1[0,1] \to C^1[0,1]$ are defined by

$$\begin{aligned} A_n(x,y)(t) &= \int_0^1 G_1(t,s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds, \quad t \in [0,1], \\ B_n(x,y)(t) &= \int_0^1 G_2(t,s)q(s)f(s, x(s) + \frac{1}{n}(s + \frac{b_1}{a_1}), |y'(s)| + \frac{1}{n})ds, \quad t \in [0,1]. \end{aligned} \tag{3.4.8}$$

Clearly, if $(x_n, y_n) \in C^1[0,1] \times C^1[0,1]$ is a fixed point of $T_n$; then $(x_n, y_n)$ is a solution of the system of BVPs (3.4.5).

**Lemma 3.4.1.** *Under the hypothesis* $(\mathcal{H}_{10}) - (\mathcal{H}_{12})$, *the operator* $T_n : \overline{\mathcal{O}}_r \cap (P_1 \times P_2) \to P_1 \times P_2$ *is completely continuous.*

*Proof.* Firstly, we show that $T_n(P_1 \times P_2) \subseteq P_1 \times P_2$. For $(x,y) \in P_1 \times P_2$, $t \in [0,1]$, using (3.4.8) and Lemma 1.1.13, we obtain

$$\begin{aligned} &A_n(x,y)(t) = \int_0^1 G_1(t,s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds \\ &\geq \gamma_1 \max_{\tau \in [0,1]} \int_0^1 G_1(\tau,s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds = \gamma_1 \|A_n(x,y)\| \end{aligned} \tag{3.4.9}$$

and

$$\begin{aligned} A_n(x,y)(0) &= \int_0^1 G_1(0,s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds \\ &= \frac{b_1}{a_1} \max_{\tau \in [0,1]} \int_\tau^1 p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds \\ &= \frac{b_1}{a_1} \max_{\tau \in [0,1]} |A_n(x,y)'(\tau)| = \frac{b_1}{a_1}\|A_n(x,y)'\|. \end{aligned} \tag{3.4.10}$$

From (3.4.9) and (3.4.10), $A_n(x,y) \in P_1$ for every $(x,y) \in P_1 \times P_2$, that is, $A_n(P_1 \times P_2) \subseteq P_1$. Similarly, by using (3.4.8) and Lemma 1.1.13, we can show that $B_n(P_1 \times P_2) \subseteq P_2$. Hence, $T_n(P_1 \times P_2) \subseteq P_1 \times P_2$.

Now, we show that $T_n : \overline{\mathcal{O}}_r \cap (P_1 \times P_2) \to P_1 \times P_2$ is uniformly bounded. For any $(x,y) \in \overline{\mathcal{O}}_r \cap (P_1 \times P_2)$, using (3.4.8), Lemma 1.1.13, $(\mathcal{H}_{10})$ and $(\mathcal{H}_{12})$, we have

$$\begin{aligned} &\|A_n(x,y)\| = \max_{t \in [0,1]} \left| \int_0^1 G_1(t,s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds \right| \\ &\leq \frac{1}{a_1} \int_0^1 (a_1 s + b_1)p(s)k_1(y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}))(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds \\ &\leq \frac{1}{a_1} k_1(r + \frac{1}{n}(1 + \frac{b_2}{a_2}))(u_1(\frac{1}{n}) + v_1(r + \frac{1}{n})) \int_0^1 (a_1 s + b_1)p(s)ds < +\infty, \end{aligned} \tag{3.4.11}$$

$$\|A_n(x,y)'\| = \max_{\tau\in[0,1]} |A_n(x,y)'(\tau)| = \max_{\tau\in[0,1]} \int_\tau^1 p(s)f(s,y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}),|x'(s)|+\frac{1}{n})ds$$
$$= \int_0^1 p(s)f(s,y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}),|x'(s)|+\frac{1}{n})ds \le \int_0^1 p(s)k_1(y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}))$$
$$(u_1(|x'(s)|+\frac{1}{n})+v_1(|x'(s)|+\frac{1}{n}))ds \le k_1(r+\frac{1}{n}(1+\frac{b_2}{a_2}))(u_1(\frac{1}{n})+v_1(r+\frac{1}{n}))\int_0^1 p(s)ds$$
$$< +\infty. \tag{3.4.12}$$

From (3.4.11) and (3.4.12), it follows that $A_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is uniformly bounded under $\|\cdot\|_3$. Similarly, by using (3.4.8), Lemma 1.1.13, $(\mathcal{H}_{10})$ and $(\mathcal{H}_{12})$, we can show that $B_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is uniformly bounded under $\|\cdot\|_3$. Hence, $T_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is uniformly bounded.

Now, we show that $T_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is equicontinuous. For any $(x,y) \in \overline{\mathcal{O}}_r \cap (P_1 \times P_2)$ and $t_1, t_2 \in [0,1]$, using (3.4.8) and $(\mathcal{H}_{12})$, we have

$$|A_n(x,y)(t_1) - A_n(x,y)(t_2)| = \left| \int_0^1 (G_1(t_1,s)-G_1(t_2,s))p(s)f(s,y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}),\right.$$
$$\left.|x'(s)|+\frac{1}{n})ds\right| \le \int_0^1 |G_1(t_1,s)-G_1(t_2,s)|p(s)f(s,y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}),|x'(s)|+\frac{1}{n})ds$$
$$\le \int_0^1 |G_1(t_1,s)-G_1(t_2,s)|p(s)k_1(y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}))(u_1(|x'(s)|+\frac{1}{n})+v_1(|x'(s)|+\frac{1}{n}))ds$$
$$\le k_1(r+\frac{1}{n}(1+\frac{b_2}{a_2}))(u_1(\frac{1}{n})+v_1(r+\frac{1}{n}))\int_0^1 |G(t_1,s)-G(t_2,s)|p(s)ds, \tag{3.4.13}$$

$$|A_n(x,y)'(t_1) - A_n(x,y)'(t_2)| = \left| \int_{t_1}^{t_2} p(s)f(s,y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}),|x'(s)|+\frac{1}{n})ds \right|$$
$$\le \int_{t_1}^{t_2} p(s)k_1(y(s)+\frac{1}{n}(s+\frac{b_2}{a_2}))(u_1(|x'(s)|+\frac{1}{n})+v_1(|x'(s)|+\frac{1}{n}))ds \tag{3.4.14}$$
$$\le k_1(r+\frac{1}{n}(1+\frac{b_2}{a_2}))(u_1(\frac{1}{n})+v_1(r+\frac{1}{n}))\int_{t_1}^{t_2} p(s)ds.$$

From (3.4.13), (3.4.14) and $(\mathcal{H}_{10})$, it follows that $A_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is equicontinuous under the norm $\|\cdot\|_3$. Similarly, using (3.4.8) and $(\mathcal{H}_{12})$, we can show that $B_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is equicontinuous under $\|\cdot\|_3$. Consequently, $T_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is equicontinuous. Hence, by Theorem 1.1.6, $T_n(\overline{\mathcal{O}}_r \cap (P_1 \times P_2))$ is relatively compact which implies that $T_n$ is a compact operator.

Further, we show that $T_n$ is continuous. Let $(x_m, y_m), (x,y) \in \overline{\mathcal{O}}_r \cap (P_1 \times P_2)$ such that

$$\|(x_m,y_m)-(x,y)\|_5 \to 0 \text{ as } m \to +\infty.$$

Using $(\mathcal{H}_{12})$, we have

$$\left|f(t, y_m(t) + \frac{1}{n}(t + \frac{b_2}{a_2}), |x'_m(t)| + \frac{1}{n})\right| \leq k_1(y_m(t) + \frac{1}{n}(t + \frac{b_2}{a_2}))$$
$$(u_1(|x'_m(t)| + \frac{1}{n}) + v_1(|x'_m(t)| + \frac{1}{n})) \leq k_1(r + \frac{1}{n}(1 + \frac{b_2}{a_2}))(u_1(\frac{1}{n}) + v_1(r + \frac{1}{n})).$$

Using (3.4.8) and Lemma 1.1.13, we have

$$\|A_n(x_m, y_m) - A_n(x, y)\| = \max_{t\in[0,1]} \left|\int_0^1 G_1(t, s)p(s)(f(s, y_m(s) + \frac{1}{n}(s + \frac{b_2}{a_2}),\right.$$
$$\left.|x'_m(s)| + \frac{1}{n}) - f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}))ds\right| \leq \frac{1}{a_1}\int_0^1 (a_1 s + b_1)p(s) \quad (3.4.15)$$
$$\left|f(s, y_m(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'_m(s)| + \frac{1}{n}) - f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})\right| ds$$

and

$$\|A_n(x_m, y_m)' - A_n(x, y)'\| = \max_{\tau\in[0,1]} \left|\int_\tau^1 p(s)(f(s, y_m(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'_m(s)| + \frac{1}{n})\right.$$
$$\left.-f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}))ds\right|$$
$$\leq \int_0^1 p(s)\left|f(s, y_m(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'_m(s)| + \frac{1}{n}) - f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})\right| ds. \quad (3.4.16)$$

From (3.4.15) and (3.4.16), using Lebesgue dominated convergence theorem, it follows that

$$\|A_n(x_m, y_m) - A_n(x, y)\| \to 0, \ \ \|A_n(x_m, y_m)' - A_n(x, y)'\| \to 0, \text{ as } m \to +\infty.$$

Hence, $\|A_n(x_m, y_m) - A_n(x, y)\|_3 \to 0$ as $m \to \infty$.

Similarly, we can show that $\|B_n(x_m, y_m) - B_n(x, y)\|_3 \to 0$ as $m \to \infty$. Consequently, $\|T_n(x_m, y_m) - T_n(x, y)\|_5 \to 0$ as $m \to +\infty$, that is, $T_n : \overline{\mathcal{O}}_r \cap (P_1 \times P_2) \to P_1 \times P_2$ is continuous. Hence, $T_n : \overline{\mathcal{O}}_r \cap (P_1 \times P_2) \to P_1 \times P_2$ is completely continuous. □

Assume that

$(\mathcal{H}_{22})$ for any real constant $C > 0$, $\int_0^1 p(t)u_1(C\int_t^1 p(s)\varphi_E(s)ds)dt < +\infty$ and $\int_0^1 q(t)u_2(C\int_t^1 q(s)\psi_E(s)ds)dt < +\infty$.

**Theorem 3.4.2.** *Under the hypothesis* $(\mathcal{H}_{10}) - (\mathcal{H}_{12})$*,* $(\mathcal{H}_{14})$*,* $(\mathcal{H}_{17})$*,* $(\mathcal{H}_{19})$*,* $(\mathcal{H}_{20})$ *and* $(\mathcal{H}_{22})$*, the system of BVPs* (3.0.2), (3.0.4) *has at least two* $C^1$*-positive solutions.*

*Proof.* Let $R_5 = R_3 + R_4$ and define $\mathcal{O}_{R_5} = \Omega_{R_3} \times \Omega_{R_4}$ where

$$\Omega_{R_3} = \{x \in C^1[0, 1] : \|x\|_3 < R_3\}, \, \Omega_{R_4} = \{x \in C^1[0, 1] : \|x\|_3 < R_4\}.$$

We claim that

$$(x,y) \neq \lambda T_n(x,y), \text{ for } \lambda \in (0,1], (x,y) \in \partial\mathcal{O}_{R_5} \cap (P_1 \times P_2). \tag{3.4.17}$$

Suppose there exist $(x_0, y_0) \in \partial\mathcal{O}_{R_5} \cap (P_1 \times P_2)$ and $\lambda_0 \in (0,1]$ such that

$$(x_0, y_0) = \lambda_0 T_n(x_0, y_0).$$

Then,

$$\begin{aligned} &-x_0''(t) = \lambda_0 p(t) f(t, y_0(t) + \frac{1}{n}(t + \frac{b_2}{a_2}), |x_0'(t)| + \frac{1}{n}), \quad t \in (0,1), \\ &-y_0''(t) = \lambda_0 q(t) g(t, x_0(t) + \frac{1}{n}(t + \frac{b_1}{a_1}), |y_0'(t)| + \frac{1}{n}), \quad t \in (0,1), \\ &a_1 x(0) - b_1 x'(0) = a_2 y(0) - b_2 y'(0) = x'(1) = y'(1) = 0. \end{aligned} \tag{3.4.18}$$

From (3.4.18) and $(\mathcal{H}_{11})$, we have $x_0'' \leq 0$ and $y_0'' \leq 0$ on $(0,1)$, integrating from $t$ to 1, using the BCs (3.4.18), we obtain $x_0'(t) \geq 0$ and $y_0'(t) \geq 0$ for $t \in [0,1]$. From (3.4.18) and $(\mathcal{H}_{12})$, we have

$$\begin{aligned} -x_0''(t) &\leq p(t) k_1(y_0(t) + \frac{1}{n}(t + \frac{b_2}{a_2}))(u_1(x_0'(t) + \frac{1}{n}) + v_1(x_0'(t) + \frac{1}{n})), \quad t \in (0,1), \\ -y_0''(t) &\leq q(t) k_2(x_0(t) + \frac{1}{n}(t + \frac{b_1}{a_1}))(u_2(y_0'(t) + \frac{1}{n}) + v_2(y_0'(t) + \frac{1}{n})), \quad t \in (0,1), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{-x_0''(t)}{u_1(x_0'(t) + \frac{1}{n}) + v_1(x_0'(t) + \frac{1}{n})} &\leq p(t) k_1(y_0(t) + \frac{1}{n}(t + \frac{b_2}{a_2})) \leq k_1(R_4 + \varepsilon) p(t), \quad t \in (0,1), \\ \frac{-y_0''(t)}{u_2(y_0'(t) + \frac{1}{n}) + v_2(y_0'(t) + \frac{1}{n})} &\leq q(t) k_2(x_0(t) + \frac{1}{n}(t + \frac{b_1}{a_1})) \leq k_2(R_3 + \varepsilon) q(t), \quad t \in (0,1). \end{aligned}$$

Integrating from $t$ to 1, using the BCs (3.4.18), we obtain

$$\begin{aligned} I(x_0'(t) + \frac{1}{n}) - I(\frac{1}{n}) &\leq k_1(R_4 + \varepsilon) \int_t^1 p(s) ds, \quad t \in [0,1], \\ J(y_0'(t) + \frac{1}{n}) - J(\frac{1}{n}) &\leq k_2(R_3 + \varepsilon) \int_t^1 q(s) ds, \quad t \in [0,1], \end{aligned}$$

which implies that

$$x_0'(t) \leq I^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(s) ds + I(\varepsilon)), \quad t \in [0,1], \tag{3.4.19}$$

$$y_0'(t) \leq J^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(s) ds + J(\varepsilon)), \quad t \in [0,1]. \tag{3.4.20}$$

Integrating from 0 to $t$, using the BCs (3.4.18), leads to

$$\begin{aligned} x_0(t) &\leq \frac{b_1}{a_1} x_0'(0) + I^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(s) ds + I(\varepsilon)), \quad t \in [0,1], \\ y_0(t) &\leq \frac{b_2}{a_2} y_0'(0) + J^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(s) ds + J(\varepsilon)), \quad t \in [0,1]. \end{aligned}$$

Using (3.4.19) and (3.4.20), we have

$$x_0(t) \leq (1+\frac{b_1}{a_1})I^{-1}(k_1(R_4+\varepsilon)\int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0,1], \tag{3.4.21}$$

$$y_0(t) \leq (1+\frac{b_2}{a_2})J^{-1}(k_2(R_3+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0,1]. \tag{3.4.22}$$

From (3.4.19)–(3.4.22), it follows that

$$R_3 \leq (1+\frac{b_1}{a_1})I^{-1}(k_1(R_4+\varepsilon)\int_0^1 p(s)ds + I(\varepsilon)), \tag{3.4.23}$$

$$R_4 \leq (1+\frac{b_2}{a_2})J^{-1}(k_2(R_3+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon)). \tag{3.4.24}$$

Now, using (3.4.24) in (3.4.23) together with increasing property of $k_1$ and $I^{-1}$, we have

$$\frac{R_3}{(1+\frac{b_1}{a_1})I^{-1}(k_1((1+\frac{b_2}{a_2})J^{-1}(k_2(R_3+\varepsilon)\int_0^1 q(s)ds + J(\varepsilon)) + \varepsilon)\int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (3.4.3). Similarly, using (3.4.23) in (3.4.24) together with increasing property of $k_2$ and $J^{-1}$, we have

$$\frac{R_4}{(1+\frac{b_2}{a_2})J^{-1}(k_2((1+\frac{b_1}{a_1})I^{-1}(k_1(R_4+\varepsilon)\int_0^1 p(s)ds + I(\varepsilon)) + \varepsilon)\int_0^1 q(s)ds + J(\varepsilon))} \leq 1,$$

a contradiction to (3.4.4). Hence, (3.4.17) is true and by Lemma 1.2.5, the fixed point index

$$\text{ind}_{\text{FP}}(T_n, \mathcal{O}_{R_5} \cap (P_1 \times P_2), P_1 \times P_2) = 1. \tag{3.4.25}$$

Now, choose

$$N_3 = \frac{1+\gamma_2^{-1}\varrho_2^{-1}}{\max_{t\in[0,1]}\int_0^1 G_1(t,s)p(s)ds} \text{ and } N_4 = \frac{1+\gamma_1^{-1}\varrho_1^{-1}}{\max_{t\in[0,1]}\int_0^1 G_2(t,s)q(s)ds}. \tag{3.4.26}$$

By $(\mathcal{H}_{19})$, there exist real constants with $R_3^* > R_3$ and $R_4^* > R_4$ such that

$$\begin{aligned} h_1(x,y) &\geq N_3 x, \text{ for } x \geq R_3^*, y \in (0,\infty), \\ h_2(x,y) &\geq N_4 x, \text{ for } x \geq R_4^*, y \in (0,\infty). \end{aligned} \tag{3.4.27}$$

Let $R^{**} = \frac{R_3^*}{\gamma_1\varrho_1} + \frac{R_4^*}{\gamma_2\varrho_2}$ and define $\mathcal{O}_{R^{**}} = \Omega_{R_3^*} \times \Omega_{R_4^*}$, where

$$\Omega_{R_3^*} = \{x \in C^1[0,1] : \|x\|_3 < \frac{R_3^*}{\gamma_1\varrho_1}\}, \ \Omega_{R_4^*} = \{x \in C^1[0,1] : \|x\|_3 < \frac{R_4^*}{\gamma_2\varrho_2}\}.$$

We show that

$$T_n(x,y) \npreceq (x,y), \text{ for } (x,y) \in \partial\mathcal{O}_{R^{**}} \cap (P_1 \times P_2). \tag{3.4.28}$$

Suppose $T_n(x_0,y_0) \preceq (x_0,y_0)$ for some $(x_0,y_0) \in \partial\mathcal{O}_{R^{**}} \cap (P_1 \times P_2)$. Then,

$$x_0(t) \geq A_n(x_0,y_0)(t) \text{ and } y_0(t) \geq B_n(x_0,y_0)(t) \text{ for } t \in [0,1]. \tag{3.4.29}$$

Note that, by Lemma 1.1.12, we have

$$x_0(t) \geq \gamma_1 \varrho_1 \|x_0\|_3 = R_3^*, \quad t \in [0,1].$$

Similarly, $y_0(t) \geq R_4^*$ for $t \in [0,1]$. Hence,

$$|x_0(t)| + \frac{1}{n}(t + \frac{b_1}{a_1}) \geq R_3^* \text{ and } |y_0(t)| + \frac{1}{n}(t + \frac{b_2}{a_2}) \geq R_4^* \text{ for } t \in [0,1].$$

Now, using (3.4.29), ($\mathcal{H}_{19}$) and (3.4.27), we have

$$\begin{aligned}
&\|x_0\| \geq x_0(t) \geq A_n(x_0, y_0)(t) = \int_0^1 G_1(t,s)p(s)f(s, y_0(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x_0'(s)| + \frac{1}{n})ds \\
&\geq \int_0^1 G_1(t,s)p(s)h_1(y_0(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x_0'(s)| + \frac{1}{n})ds \\
&\geq N_3 \int_0^1 G_1(t,s)p(s)(y_0(s) + \frac{1}{n}(s + \frac{b_2}{a_2}))ds \geq N_3 R_4^* \int_0^1 G_1(t,s)p(s)ds, \quad t \in [0,1],
\end{aligned}$$

in view of (3.4.27) we have

$$\|x_0\| \geq N_3 R_4^* \max_{t \in [0,1]} \int_0^1 G_1(t,s)p(s)ds > \frac{R_4^*}{\gamma_2 \varrho_2}.$$

Thus $\|x_0\|_3 \geq \|x_0\| > \frac{R_4^*}{\gamma_2 \varrho_2}$. Similarly, using (3.4.29), ($\mathcal{H}_{19}$), (3.4.27) and (3.4.26), we have $\|y_0\|_3 > \frac{R_3^*}{\gamma_1 \varrho_1}$. Consequently, it follows that, $\|(x_0, y_0)\|_5 = \|x_0\|_3 + \|y_0\|_3 > R^{**}$, a contradiction. Hence, (3.4.28) is true and by Lemma 1.2.7, the fixed point index

$$\text{ind}_{\text{FP}}(T_n, \mathcal{O}_{R^{**}} \cap (P_1 \times P_2), P_1 \times P_2) = 0. \tag{3.4.30}$$

From (3.4.25) and (3.4.30), it follows that

$$\text{ind}_{\text{FP}}(T_n, (\mathcal{O}_{R^{**}} \setminus \overline{\mathcal{O}}_R) \cap (P_1 \times P_2), P_1 \times P_2) = -1. \tag{3.4.31}$$

Thus, in view of (3.4.25) and (3.4.31), there exist $(x_{n,1}, y_{n,1}) \in \mathcal{O}_R \cap (P_1 \times P_2)$ and $(x_{n,2}, y_{n,2}) \in (\mathcal{O}_{R^{**}} \setminus \overline{\mathcal{O}}_R) \cap (P_1 \times P_2)$ such that $(x_{n,j}, y_{n,j}) = T_n(x_{n,j}, y_{n,j})$, $(j = 1,2)$ which implies that

$$\begin{aligned}
x_{n,j}(t) &= \int_0^1 G_1(t,s)p(s)f(t, y_{n,j}(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x_{n,j}'(s)| + \frac{1}{n})ds, \quad t \in [0,1], \\
y_{n,j}(t) &= \int_0^1 G_2(t,s)q(s)g(s, x_{n,j}(s) + \frac{1}{n}(s + \frac{b_1}{a_1}), |y_{n,j}'(s)| + \frac{1}{n})ds, \quad t \in [0,1],\ j = 1,2.
\end{aligned} \tag{3.4.32}$$

Using ($\mathcal{H}_{17}$) there exist continuous functions $\varphi_{R_4+\varepsilon}$ and $\psi_{R_3+\varepsilon}$ defined on $[0,1]$ and positive on $(0,1)$ and real constants $0 \leq \delta_1, \delta_2 < 1$ such that

$$\begin{aligned}
f(t,x,y) &\geq \varphi_{R_4+\varepsilon}(t)x^{\delta_1}, \quad (t,x,y) \in [0,1] \times [0, R_4 + \varepsilon] \times [0, R_4 + \varepsilon], \\
g(t,x,y) &\geq \psi_{R_3+\varepsilon}(t)x^{\delta_2}, \quad (t,x,y) \in [0,1] \times [0, R_3 + \varepsilon] \times [0, R_3 + \varepsilon].
\end{aligned} \tag{3.4.33}$$

By construction of the cones $P_1$ and $P_2$, we have $x_{n,1}(t) \geq \gamma_1 \|x_{n,1}\|$ and $y_{n,1}(t) \geq \gamma_2 \|y_{n,1}\|$ for $t \in [0,1]$. We show that

$$x'_{n,1}(t) \geq C_8^{\delta_1} \gamma_2^{\delta_1} \int_t^1 p(s)\varphi_{R_4+\varepsilon}(s)ds, \quad t \in [0,1], \tag{3.4.34}$$

$$y'_{n,1}(t) \geq C_7^{\delta_2} \gamma_1^{\delta_2} \int_t^1 q(s)\psi_{R_3+\varepsilon}(s)ds, \quad t \in [0,1], \tag{3.4.35}$$

where

$$C_7 = \gamma_1^{\frac{1+\delta_1\delta_2}{1-\delta_1\delta_2}} \gamma_2^{\frac{2\delta_1}{1-\delta_1\delta_2}} \left( \max_{\tau\in[0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds \right)^{\frac{1}{1-\delta_1\delta_2}}$$
$$\left( \max_{\tau\in[0,1]} \int_0^1 G_2(\tau,s)q(s)\psi_{R_3+\varepsilon}(s)ds \right)^{\frac{\delta_1}{1-\delta_1\delta_2}},$$
$$C_8 = \gamma_1^{\frac{2\delta_2}{1-\delta_1\delta_2}} \gamma_2^{\frac{1+\delta_1\delta_2}{1-\delta_1\delta_2}} \left( \max_{\tau\in[0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds \right)^{\frac{\delta_2}{1-\delta_1\delta_2}}$$
$$\left( \max_{\tau\in[0,1]} \int_0^1 G_2(\tau,s)q(s)\psi_{R_3+\varepsilon}(s)ds \right)^{\frac{1}{1-\delta_1\delta_2}}.$$

To prove (3.4.34), using (3.4.32) and (3.4.33), we have

$$\begin{aligned} x_{n,1}(t) &= \int_0^1 G_1(t,s)p(s)f(s, y_{n,1}(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'_{n,1}(s)| + \frac{1}{n})ds \\ &\geq \int_0^1 G_1(t,s)p(s)\varphi_{R_4+\varepsilon}(s)(y_{n,1}(s) + \frac{1}{n}(s+\frac{b_2}{a_2}))^{\delta_1} ds \\ &\geq \gamma_2^{\delta_1} \|y_{n,1}\|^{\delta_1} \int_0^1 G_1(t,s)p(s)\varphi_{R_4+\varepsilon}(s)ds, \quad t \in [0,1], \end{aligned}$$

which implies that

$$x_{n,1}(t) \geq \gamma_1 \gamma_2^{\delta_1} \|y_{n,1}\|^{\delta_1} \max_{\tau\in[0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds, \quad t \in [0,1].$$

Hence,

$$\|x_{n,1}\| \geq \gamma_1 \gamma_2^{\delta_1} \|y_{n,1}\|^{\delta_1} \max_{\tau\in[0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds. \tag{3.4.36}$$

Similarly, from (3.4.32) and (3.4.33), we have

$$\|y_{n,1}\| \geq \gamma_1^{\delta_2} \gamma_2 \|x_{n,1}\|^{\delta_2} \max_{\tau\in[0,1]} \int_0^1 G_2(\tau,s)q(s)\psi_{R_3+\varepsilon}(s)ds. \tag{3.4.37}$$

Using (3.4.36) in (3.4.37), we have

$$\|y_{n,1}\| \geq \gamma_1^{2\delta_2} \gamma_2^{1+\delta_1\delta_2} \|y_{n,1}\|^{\delta_1\delta_2} \left( \max_{\tau\in[0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds \right)^{\delta_2}$$
$$\max_{\tau\in[0,1]} \int_0^1 G_2(\tau,s)q(s)\psi_{R_3+\varepsilon}(s)ds,$$

which implies that

$$\|y_{n,1}\| \geq \gamma_1^{\frac{2\delta_2}{1-\delta_1\delta_2}} \gamma_2^{\frac{1+\delta_1\delta_2}{1-\delta_1\delta_2}} \left( \max_{\tau\in[0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds \right)^{\frac{\delta_2}{1-\delta_1\delta_2}} \left( \max_{\tau\in[0,1]} \int_0^1 G_2(\tau,s)q(s)\psi_{R_3+\varepsilon}(s)ds \right)^{\frac{1}{1-\delta_1\delta_2}} = C_8. \quad (3.4.38)$$

Using (3.4.33) and (3.4.38) in the following relation

$$x'_{n,1}(t) = \int_t^1 p(s)f(s, y_{n,1}(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'_{n,1}(s)| + \frac{1}{n})ds$$

we obtain (3.4.34). Similarly, we can prove (3.4.35).

Now, differentiating (3.4.32), using $(\mathcal{H}_{12})$, (3.4.34) and (3.4.35), we have

$$\begin{aligned} 0 &\leq -x''_{n,1}(t) \leq p(t)k_1(R_4+\varepsilon)(u_1(C_8^{\delta_1}\gamma_2^{\delta_1}\int_t^1 p(s)\varphi_{R_4+\varepsilon}(s)ds) + v_1(R_3+\varepsilon)), \quad t\in(0,1), \\ 0 &\leq -y''_{n,1}(t) \leq q(t)k_2(R_3+\varepsilon)(u_2(C_7^{\delta_2}\gamma_1^{\delta_2}\int_t^1 q(s)\psi_{R_3+\varepsilon}(s)ds) + v_2(R_4+\varepsilon)), \quad t\in(0,1), \end{aligned} \quad (3.4.39)$$

which on integration from $t$ to 1, using the BCs (3.4.5), leads to

$$\begin{aligned} x'_{n,1}(t) &\leq k_1(R_4+\varepsilon)\int_t^1 p(s)(u_1(C_8^{\delta_1}\gamma_2^{\delta_1}\int_s^1 p(\tau)\varphi_{R_4+\varepsilon}(\tau)d\tau) + v_1(R_3+\varepsilon))ds, \quad t\in[0,1], \\ y'_{n,1}(t) &\leq k_2(R_3+\varepsilon)\int_t^1 q(s)(u_2(C_7^{\delta_2}\gamma_1^{\delta_2}\int_s^1 q(\tau)\psi_{R_3+\varepsilon}(\tau)d\tau) + v_2(R_4+\varepsilon))ds, \quad t\in[0,1], \end{aligned}$$

which implies that

$$\begin{aligned} x'_{n,1}(t) &\leq k_1(R_4+\varepsilon)\int_0^1 p(s)(u_1(C_8^{\delta_1}\gamma_2^{\delta_1}\int_s^1 p(\tau)\varphi_{R_4+\varepsilon}(\tau)d\tau) + v_1(R_3+\varepsilon))ds, \quad t\in[0,1], \\ y'_{n,1}(t) &\leq k_2(R_3+\varepsilon)\int_0^1 q(s)(u_2(C_7^{\delta_2}\gamma_1^{\delta_2}\int_s^1 q(\tau)\psi_{R_3+\varepsilon}(\tau)d\tau) + v_2(R_4+\varepsilon))ds, \quad t\in[0,1]. \end{aligned} \quad (3.4.40)$$

In view of (3.4.34), (3.4.35), (3.4.40), (3.4.39), $(\mathcal{H}_{10})$ and $(\mathcal{H}_{22})$, the sequences $\{(x^{(j)}_{n,1}, y^{(j)}_{n,1})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, 1]$. Thus, by Theorem 1.1.6, there exist subsequences $\{(x^{(j)}_{n_k,1}, y^{(j)}_{n_k,1})\}$ $(j = 0, 1)$ of $\{(x^{(j)}_{n,1}, y^{(j)}_{n,1})\}$ and functions $(x_{0,1}, y_{0,1})$ $\in C^1[0, 1] \times C^1[0, 1]$ such that $(x^{(j)}_{n_k,1}, y^{(j)}_{n_k,1})$ converges uniformly to $(x^{(j)}_{0,1}, y^{(j)}_{0,1})$ on $[0, 1]$. Also, $a_1x_{0,1}(0) - b_1x'_{0,1}(0) = a_2y_{0,1}(0) - b_2y'_{0,1}(0) = x'_{0,1}(1) = y'_{0,1}(1) = 0$. Moreover, from (3.4.34) and (3.4.35), with $n_k$ in place of $n$ and taking $\lim_{n_k\to+\infty}$, we have

$$\begin{aligned} x'_{0,1}(t) &\geq C_8^{\delta_1}\gamma_2^{\delta_1}\int_t^1 p(s)\varphi_{R_4+\varepsilon}(s)ds, \\ y'_{0,1}(t) &\geq C_7^{\delta_2}\gamma_1^{\delta_2}\int_t^1 q(s)\psi_{R_3+\varepsilon}(s)ds, \end{aligned}$$

which implies that $x'_{0,1} > 0$ and $y'_{0,1} > 0$ on $[0,1)$, $x_{0,1} > 0$ and $y_{0,1} > 0$ on $[0,1]$. Further,

$$\left|f(t, y_{n_k,1}(t) + \frac{1}{n_k}(t + \frac{b_2}{a_2}), x'_{n_k,1}(t) + \frac{1}{n_k})\right| \leq p(t)k_1(R_4 + \varepsilon)(u_1(C_8^{\delta_1}\gamma_2^{\delta_1}\int_t^1 p(s)\varphi_{R_4+\varepsilon}(s)ds) + v_1(R_3 + \varepsilon)),$$

$$\left|g(t, x_{n_k,1}(t) + \frac{1}{n_k}(t + \frac{b_1}{a_1}), y'_{n_k,1}(t) + \frac{1}{n_k})\right| \leq q(t)k_2(R_3 + \varepsilon)(u_2(C_7^{\delta_2}\gamma_1^{\delta_2}\int_t^1 q(s)\psi_{R_3+\varepsilon}(s)ds) + v_2(R_4 + \varepsilon)), \tag{3.4.41}$$

$$\begin{aligned}
&\lim_{n_k\to\infty} f(t, y_{n_k,1}(t) + \frac{1}{n_k}(t + \frac{b_2}{a_2}), x'_{n_k,1}(t) + \frac{1}{n_k}) = f(t, y_{0,1}(t), x'_{0,1}(t)), \quad t \in (0,1),\\
&\lim_{n_k\to\infty} g(t, x_{n_k,1}(t) + \frac{1}{n_k}(t + \frac{b_1}{a_1}), y'_{n_k,1}(t) + \frac{1}{n_k}) = g(t, x_{0,1}(t), y'_{0,1}(t)), \quad t \in (0,1).
\end{aligned} \tag{3.4.42}$$

Moreover, $(x_{n_k,1}, y_{n_k,1})$ satisfies

$$\begin{aligned}
x_{n_k,1}(t) &= \int_0^1 G_1(t,s)p(s)f(s, y_{n_k,1}(s) + \frac{1}{n_k}(s + \frac{b_2}{a_2}), x'_{n_k,1}(s) + \frac{1}{n_k})ds, \quad t \in [0,1],\\
y_{n_k,1}(t) &= \int_0^1 G_2(t,s)q(s)g(s, x_{n_k,1}(s) + \frac{1}{n_k}(s + \frac{b_1}{a_1}), y'_{n_k,1}(s) + \frac{1}{n_k})ds, \quad t \in [0,1],
\end{aligned}$$

in view of (3.4.41), $(\mathcal{H}_{22})$, (3.4.42), the Lebesgue dominated convergence theorem and taking $\lim_{n_k\to+\infty}$, we have

$$\begin{aligned}
x_{0,1}(t) &= \int_0^1 G_1(t,s)p(s)f(s, y_{0,1}(s), x'_{0,1}(s))ds, \quad t \in [0,1],\\
y_{0,1}(t) &= \int_0^1 G_2(t,s)q(s)g(s, x_{0,1}(s), y'_{0,1}(s))ds, \quad t \in [0,1],
\end{aligned}$$

which implies that $(x_{0,1}, y_{0,1}) \in C^2(0,1) \times C^2(0,1)$ and

$$\begin{aligned}
-x''_{0,1}(t) &= p(t)f(t, y_{0,1}(t), x'_{0,1}(t)), \quad t \in (0,1),\\
-y''_{0,1}(t) &= q(t)g(t, x_{0,1}(t), y'_{0,1}(t)), \quad t \in (0,1).
\end{aligned}$$

Moreover, by (3.4.1) and (3.4.2), we have $\|x_{0,1}\|_3 < R_3$ and $\|y_{0,1}\|_3 < R_4$, that is, $\|(x_{0,1}, y_{0,1})\|_5 < R_5$. By a similar proof the sequence $\{(x_{n,2}, y_{n,2})\}$ has a convergent subsequence $\{(x_{n_k,2}, y_{n_k,2})\}$ converging uniformly to $(x_{0,2}, y_{0,2}) \in C^1[0,1] \times C^1[0,1]$ on $[0,1]$. Moreover, $(x_{0,2}, y_{0,2})$ is a solution to the system (3.0.2), (3.0.4) with $x_{0,2} > 0$ and $y_{0,2} > 0$ on $[0,1]$, $x'_{0,2} > 0$ and $y'_{0,2} > 0$ on $[0,1)$, $R_5 < \|(x_{0,2}, y_{0,2})\|_5 < R^{**}$. □

**Example 3.4.3.** Consider the following coupled system of SBVPs

$$\begin{aligned}
-x''(t) &= \mu_1(1 + (y(t))^{\delta_1} + (y(t))^{\eta_1})(1 + (x'(t))^{\alpha_1} + (x'(t))^{-\beta_1}), \quad t \in (0,1),\\
-y''(t) &= \mu_2(1 + (x(t))^{\delta_2} + (x(t))^{\eta_2})(1 + (y'(t))^{\alpha_2} + (y'(t))^{-\beta_2}), \quad t \in (0,1),\\
&x(0) - x'(0) = y(0) - y'(0) = x'(1) = y'(1) = 0,
\end{aligned} \tag{3.4.43}$$

where $0 \le \delta_i < 1$, $\eta_i > 1$, $0 < \alpha_i < 1$, $0 < \beta_i < 1$, and $\mu_i > 0$, $i = 1, 2$.

Let $p(t) = \mu_1$, $q(t) = \mu_2$, $k_i(x) = 1 + x^{\delta_i} + x^{\eta_i}$, $u_i(x) = x^{-\beta_i}$ and $v_i(x) = 1 + x^{\alpha_i}$, $i = 1, 2$. Assume that $\mu_1$ is arbitrary and

$$\mu_2 < \min\{\inf_{c\in(0,\infty)} \frac{J(2^{-1}(\mu_1^{-1}I(\frac{c}{2}))^{\delta_1^{-1}})}{k_2(c)}, \inf_{c\in(0,\infty)} \frac{J(2^{-1}(\mu_1^{-1}I(\frac{c}{2}))^{\eta_1^{-1}})}{k_2(c)}, \inf_{c\in(0,\infty)} \frac{J(\frac{c}{2})}{k_2(2I^{-1}(\mu_1 k_1(c)))}\}.$$

We choose $\varphi_E(t) = \mu_1$, $\psi_E(t) = \mu_2$ and $h_i(x, y) = \mu_i(1 + x^{\eta_i})$, $i = 1, 2$. Then,

$$\lim_{x\to+\infty} \frac{h_i(x, y)}{x} = \lim_{x\to+\infty} \frac{\mu_i(1 + x^{\eta_i})}{x} = +\infty,\ i = 1, 2.$$

Moreover,

$$\begin{aligned}
&\sup_{c\in(0,\infty)} \frac{c}{(1 + \frac{b_1}{a_1})I^{-1}(k_1((1 + \frac{b_2}{a_2})J^{-1}(k_2(c)\int_0^1 q(t)dt))\int_0^1 p(t)dt)} \\
&= \sup_{c\in(0,\infty)} \frac{c}{2I^{-1}(\mu_1 k_1(2J^{-1}(\mu_2 k_2(c))))} \\
&\ge \frac{c}{2I^{-1}(\mu_1 k_1(2J^{-1}(\mu_2 k_2(c))))}, \qquad c \in (0, \infty) \\
&= \frac{c}{2I^{-1}(\mu_1(1 + (2J^{-1}(\mu_2 k_2(c)))^{\delta_1} + (2J^{-1}(\mu_2 k_2(c)))^{\eta_1}))}, \quad c \in (0, \infty) \\
&> 1,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{c\in(0,\infty)} \frac{c}{(1 + \frac{b_2}{a_2})J^{-1}(k_2((1 + \frac{b_1}{a_1})I^{-1}(k_1(c)\int_0^1 p(t)dt))\int_0^1 q(t)dt)} \\
&= \sup_{c\in(0,\infty)} \frac{c}{2J^{-1}(\mu_2 k_2(2I^{-1}(\mu_1 k_1(c))))} \\
&= \frac{c}{2J^{-1}(\mu_2 k_2(2I^{-1}(\mu_1 k_1(c))))}, \quad c \in (0, \infty) \\
&> 1.
\end{aligned}$$

Further,

$$\int_0^1 p(t)u_1(C\int_t^1 p(s)\varphi_E(s)ds)dt = \mu_1^{1-2\beta_1}C^{-\beta_1}\int_0^1 (1 - t)^{-\beta_1}dt = \frac{\mu_1^{1-2\beta_1}C^{-\beta_1}}{1 - \beta_1},$$
$$\int_0^1 q(t)u_2(C\int_t^1 q(s)\psi_E(s)ds)dt = \mu_2^{1-2\beta_2}C^{-\beta_2}\int_0^1 (1 - t)^{-\beta_2}dt = \frac{\mu_1^{1-2\beta_2}C^{-\beta_2}}{1 - \beta_2}.$$

Clearly, $(\mathcal{H}_{10}) - (\mathcal{H}_{12})$, $(\mathcal{H}_{14})$, $(\mathcal{H}_{17})$, $(\mathcal{H}_{19})$, $(\mathcal{H}_{20})$ and $(\mathcal{H}_{22})$ are satisfied. Hence, by Theorem 3.4.2, the system of BVPs (3.4.43) has at least two $C^1$-positive solutions.

# Chapter 4

# Singular Systems with Coupled Boundary Conditions

Coupled BCs arises in the study of reaction–diffusion equations and Sturm–Liouvillie problems, see [10, 11, 88] and [135, Chapter 13]. The study of elliptic systems with coupled BCs was initiated by Agmon and coauthors [8]. In [8], the authors studied elliptic systems with the following type of coupled BCs

$$\mathcal{D}_1 f|_{\partial\Omega} + \mathcal{D}_2 \frac{\partial f}{\partial \nu} = 0,$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are differential operators from $L^2(\Omega; W)$ to $L^2(\partial\Omega; W)$, $\Omega \subset \mathbb{R}^n$ and $W$ is a separable Hilbert space. Mehmeti [106], Mehmeti, Nicaise [107] and Nicaise [112] have been studied coupled BCs in the study of interaction problems and elliptic operators on polygonal domains.

Coupled BCs have also some applications in mathematical biology. For example, Leung [88] studied the following reaction–diffusion system for prey–predator interaction:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \sigma_1 \Delta u + u(a + f(u,v)), \qquad t \geq 0, x \in \Omega \subset \mathbb{R}^n, \\ \frac{\partial v}{\partial t}(t,x) &= \sigma_2 \Delta v + v(-r + g(u,v)), \quad t \geq 0, x \in \Omega \subset \mathbb{R}^n, \end{aligned}$$

subject to the coupled BCs

$$\frac{\partial u}{\partial \eta} = 0, \ \frac{\partial v}{\partial \eta} - p(u) - q(v) = 0 \text{ on } \partial\Omega,$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $a$, $r$, $\sigma_1$, $\sigma_2$ are positive constants, $f, g : \mathbb{R}^2 \to \mathbb{R}$ have Hölder continuous partial derivatives up to second–order in compact sets, $\eta$ is a unit outward normal at $\partial\Omega$, $p$ and $q$ have Hölder continuous first derivatives in compact subsets of $[0, \infty)$. The functions $u(t,x)$, $v(t,x)$ respectively represent the density of prey and predator at time $t \geq 0$ and at position $x = (x_1, \cdots, x_n)$. Similar coupled BCs are also studied in [13] for biochemical system.

Krstic et al. [86, Section 8.3] presented the Timoshenko beam model with free–end BCs

$$
\begin{aligned}
\varepsilon\frac{\partial^2 u}{\partial t^2} &= (1+d\frac{\partial}{\partial t})(\frac{\partial^2 u}{\partial z^2}-\frac{\partial \theta}{\partial z}), && t\geq 0,\, 0\leq z\leq 1,\\
\mu\varepsilon\frac{\partial^2 \theta}{\partial t^2} &= (1+d\frac{\partial}{\partial t})(\varepsilon\frac{\partial^2 \theta}{\partial z^2}+a(\frac{\partial u}{\partial z}-\theta)), && t\geq 0,\, 0\leq z\leq 1,\\
\frac{\partial u}{\partial z}(t,0) &= \theta(t,0),\ \frac{\partial \theta}{\partial z}(t,0)=0,
\end{aligned}
$$

where $u(t,z)$ denotes the displacement and $\theta(t,z)$ denotes the angle of rotation due to the bending. The positive constants $a$ and $\mu$ are proportional to the nondimensional cross–sectional area, and the nondimensional moment of inertia of the beam, respectively. The parameter $\varepsilon$ is inversely proportional to the nondimensional shear modulus of the beam. The coefficient $d$ denotes the possible presence of Kelvin–Voigt damping. The meaning of the first BC is that zero force is being applied at the tip, whereas the meaning of the second BC is that zero moment is being applied at the tip. A backstepping boundary control design applied to the most complex beam model. The model is controlled at $z=1$ through the conditions on $u(t,1)$ and $\theta(t,1)$.

In Section 4.1, we present existence result for the following coupled singular system of ODEs subject to four–point coupled BCs

$$
\begin{aligned}
-x''(t) &= f(t,x(t),y(t)), \quad t\in(0,1),\\
-y''(t) &= g(t,x(t),y(t)), \quad t\in(0,1),\\
x(0) &= 0,\ x(1)=\alpha y(\xi),\\
y(0) &= 0,\ y(1)=\beta x(\eta),
\end{aligned}
\tag{4.0.1}
$$

where the parameters $\alpha,\ \beta,\ \xi,\ \eta$ satisfy $\xi,\eta\in(0,1)$, $0<\alpha\beta\xi\eta<1$. We assume that the nonlinearities $f,g:(0,1)\times[0,\infty)\times[0,\infty)\to[0,\infty)$ are continuous and allowed to be singular at $t=0$ or $t=1$.

Further in Section 4.2, we study more general coupled system of ODEs than the system (3.0.2) and prove the existence of $C^1$-positive solution to the following system of ODEs subject to two–point coupled BCs

$$
\begin{aligned}
-x''(t) &= p(t)f(t,x(t),y(t),x'(t)), \quad t\in(0,1),\\
-y''(t) &= q(t)g(t,x(t),y(t),y'(t)), \quad t\in(0,1),\\
a_1y(0)-b_1x'(0) &= 0,\ y'(1)=0,\\
a_2x(0)-b_2y'(0) &= 0,\ x'(1)=0,
\end{aligned}
\tag{4.0.2}
$$

where the nonlinearities $f,g:[0,1]\times[0,\infty)\times[0,\infty)\times(0,\infty)\to[0,\infty)$ are continuous and are allowed to be singular at $x'=0$, $y'=0$. Moreover, $p,q\in C(0,1)$, $p>0$ and $q>0$ on $(0,1)$.

To the best of our knowledge, we believe that coupled system of nonlinear ODEs subject to coupled BCs have never been studied previously. Results of this chapter are submitted [19, 20].

## 4.1 Singular systems of ODEs with four–point coupled BCs

In this section, we establish the existence of positive solutions for the system of BVPs (4.0.1). By a positive solution to the system of BVPs (4.0.1), we mean that $(x, y) \in (C[0,1] \cap C^2(0,1)) \times (C[0,1] \cap C^2(0,1))$, $(x, y)$ satisfies (4.0.1), $x > 0$ and $y > 0$ on $(0, 1]$. For each $u \in C[0,1]$ we write $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let

$$P = \{u \in C[0,1] : \min_{t\in[\max\{\xi,\eta\},1]} u(t) \geq \gamma\|u\|\},$$

where

$$0 < \gamma := \frac{\min\{1, \alpha\xi, \alpha\beta\xi, \beta\eta, \alpha\beta\eta\} \min\{\xi, \eta, 1-\xi, 1-\eta\}}{\max\{1, \alpha, \beta, \alpha\beta\xi, \alpha\beta\eta\}} < 1.$$

Clearly, $(C[0,1], \|\cdot\|)$ is a Banach space and $P$ is a cone of $C[0,1]$. Similarly, for each $(x, y) \in C[0,1] \times C[0,1]$ we write $\|(x, y)\|_6 = \|x\| + \|y\|$. Clearly, $(C[0,1] \times C[0,1], \|\cdot\|_6)$ is a Banach space and $P \times P$ is a cone of $C[0,1] \times C[0,1]$. For any real constant $r > 0$, define $\mathcal{O}_r = \{(x, y) \in C[0,1] \times C[0,1] : \|(x, y)\|_6 < r\}$.

**Lemma 4.1.1.** *Let $u, v \in C[0,1]$, then the system of BVPs*

$$\begin{aligned} -x''(t) &= u(t), \quad t \in [0,1],\\ -y''(t) &= v(t), \quad t \in [0,1],\\ x(0) &= 0,\ x(1) = \alpha y(\xi),\\ y(0) &= 0,\ y(1) = \beta x(\eta), \end{aligned} \tag{4.1.1}$$

*has integral representation*

$$\begin{aligned} x(t) &= \int_0^1 F_{\xi\eta}(t,s)u(s)ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)v(s)ds,\\ y(t) &= \int_0^1 F_{\eta\xi}(t,s)v(s)ds + \int_0^1 G_{\beta\alpha\eta\xi}(t,s)u(s)ds, \end{aligned} \tag{4.1.2}$$

*where*

$$F_{\xi\eta}(t,s) = \begin{cases} \frac{t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha\beta\xi t(\eta-s)}{1-\alpha\beta\xi\eta} - (t-s), & 0 \leq s \leq t \leq 1,\ s \leq \eta,\\ \frac{t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha\beta\xi t(\eta-s)}{1-\alpha\beta\xi\eta}, & 0 \leq t \leq s \leq 1,\ s \leq \eta,\\ \frac{t(1-s)}{1-\alpha\beta\xi\eta} - (t-s), & 0 \leq s \leq t \leq 1,\ s \geq \eta,\\ \frac{t(1-s)}{1-\alpha\beta\xi\eta}, & 0 \leq t \leq s \leq 1,\ s \geq \eta, \end{cases} \tag{4.1.3}$$

$$G_{\alpha\beta\xi\eta}(t,s) = \begin{cases} \frac{\alpha\xi t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha t(\xi-s)}{1-\alpha\beta\xi\eta}, & 0 \leq s, t \leq 1,\ s \leq \xi,\\ \frac{\alpha\xi t(1-s)}{1-\alpha\beta\xi\eta}, & 0 \leq s, t \leq 1,\ s \geq \xi. \end{cases} \tag{4.1.4}$$

*Proof.* Integrating the system of BVPs (4.1.1), we have

$$\begin{aligned} x(t) &= c_1 + c_3 t - \int_0^t (t-s)u(s)ds, \\ y(t) &= c_2 + c_4 t - \int_0^t (t-s)v(s)ds, \end{aligned} \tag{4.1.5}$$

where $c_i$, $i = 1, \cdots, 4$, are constants. Now, employing the BCs (4.1.1), we obtain

$$\begin{aligned} &c_1 = 0,\ c_2 = 0, \\ &c_3 - c_4\alpha\xi = \int_0^1 (1-s)u(s)ds - \alpha\int_0^\xi (\xi - s)v(s)ds, \\ &c_3\beta\eta - c_4 = \beta\int_0^\eta (\eta - s)u(s)ds - \int_0^1 (1-s)v(s)ds. \end{aligned}$$

Solving for $c_3$ and $c_4$, we get

$$\begin{aligned} c_3 &= \frac{1}{1-\alpha\beta\xi\eta}\int_0^1 (1-s)u(s)ds - \frac{\alpha\beta\xi}{1-\alpha\beta\xi\eta}\int_0^\eta (\eta-s)u(s)ds \\ &+ \frac{\alpha\xi}{1-\alpha\beta\xi\eta}\int_0^1 (1-s)v(s)ds - \frac{\alpha}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds, \\ c_4 &= \frac{\beta\eta}{1-\alpha\beta\xi\eta}\int_0^1 (1-s)u(s)ds - \frac{\beta}{1-\alpha\beta\xi\eta}\int_0^\eta (\eta-s)u(s)ds \\ &+ \frac{1}{1-\alpha\beta\xi\eta}\int_0^1 (1-s)v(s)ds - \frac{\alpha\beta\eta}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds. \end{aligned}$$

Thus, the system (4.1.5) becomes

$$\begin{aligned} x(t) &= \frac{1}{1-\alpha\beta\xi\eta}\int_0^1 t(1-s)u(s)ds - \frac{\alpha\beta\xi}{1-\alpha\beta\xi\eta}\int_0^\eta t(\eta-s)u(s)ds - \int_0^t (t-s)u(s)ds \\ &+ \frac{\alpha\xi}{1-\alpha\beta\xi\eta}\int_0^1 t(1-s)v(s)ds - \frac{\alpha}{1-\alpha\beta\xi\eta}\int_0^\xi t(\xi-s)v(s)ds, \\ y(t) &= \frac{1}{1-\alpha\beta\xi\eta}\int_0^1 t(1-s)v(s)ds - \frac{\alpha\beta\eta}{1-\alpha\beta\xi\eta}\int_0^\xi t(\xi-s)v(s)ds - \int_0^t (t-s)v(s)ds \\ &+ \frac{\beta\eta}{1-\alpha\beta\xi\eta}\int_0^1 t(1-s)u(s)ds - \frac{\beta}{1-\alpha\beta\xi\eta}\int_0^\eta t(\eta-s)u(s)ds, \end{aligned}$$

which is equivalent to the integral representation (4.1.2). □

**Lemma 4.1.2.** *The functions $F_{\xi\eta}$ and $G_{\alpha\beta\xi\eta}$ satisfies*

(i) $F_{\xi\eta}(t,s) \le \frac{\max\{1,\alpha\beta\xi\}}{1-\alpha\beta\xi\eta} s(1-s), \quad t,s \in [0,1]$,

(ii) $G_{\alpha\beta\xi\eta}(t,s) \le \frac{\alpha}{1-\alpha\beta\xi\eta} s(1-s), \quad t,s \in [0,1]$.

*Proof.* For $(t,s) \in [0,1] \times [0,1]$, we discuss various cases.
**Case 1:** $s \le \eta$, $t \ge s$; using (4.1.3), we obtain

$$F_{\xi\eta}(t,s) = \frac{t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha\beta\xi t(\eta-s)}{1-\alpha\beta\xi\eta} - (t-s) = s + (\alpha\beta\xi - 1)\frac{ts}{1-\alpha\beta\xi\eta}.$$

If $\alpha\beta\xi > 1$, the maximum occurs at $t = 1$, hence

$$F_{\xi\eta}(t,s) \le F_{\xi\eta}(1,s) = \alpha\beta\xi\frac{s(1-\eta)}{1-\alpha\beta\xi\eta} \le \alpha\beta\xi\frac{s(1-s)}{1-\alpha\beta\xi\eta} \le \frac{\max\{1,\alpha\beta\xi\}}{1-\alpha\beta\xi\eta}s(1-s),$$

and if $\alpha \le 1$, the maximum occurs at $t = s$, hence

$$F_{\xi\eta}(t,s) \le F_{\xi\eta}(s,s) = \frac{s(1-s+\alpha\beta\xi(s-\eta))}{1-\alpha\beta\xi\eta} \le \frac{s(1-s)}{1-\alpha\beta\xi\eta} \le \frac{\max\{1,\alpha\beta\xi\}}{1-\alpha\beta\xi\eta}s(1-s).$$

**Case 2:** $s \le \eta$, $t \le s$; using (4.1.3), we have

$$F_{\xi\eta}(t,s) = \frac{t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha\beta\xi t(\eta-s)}{1-\alpha\beta\xi\eta} \le \frac{t(1-s)}{1-\alpha\beta\xi\eta} \le \frac{s(1-s)}{1-\alpha\beta\xi\eta} \le \frac{\max\{1,\alpha\beta\xi\}}{1-\alpha\beta\xi\eta}s(1-s).$$

**Case 3:** $s \ge \eta$, $t \ge s$; using (4.1.3), we get

$$F_{\xi\eta}(t,s) = \frac{t(1-s)}{1-\alpha\beta\xi\eta} - (t-s) = s + \frac{t(\alpha\beta\xi\eta - s)}{1-\alpha\beta\xi\eta}.$$

If $\alpha\beta\xi\eta > s$, the maximum occurs at $t = 1$, hence

$$F_{\xi\eta}(t,s) \le F_{\xi\eta}(1,s) = \alpha\beta\xi\frac{\eta(1-s)}{1-\alpha\beta\xi\eta} \le \alpha\beta\xi\frac{s(1-s)}{1-\alpha\beta\xi\eta} \le \frac{\max\{1,\alpha\beta\xi\}}{1-\alpha\beta\xi\eta}s(1-s),$$

and if $\alpha\beta\xi\eta \le s$, the maximum occurs at $t = s$, so

$$F_{\xi\eta}(t,s) \le F_{\xi\eta}(s,s) = \frac{s(1-s)}{1-\alpha\beta\xi\eta} \le \frac{\max\{1,\alpha\beta\xi\}}{1-\alpha\beta\xi\eta}s(1-s).$$

**Case 4:** $s \ge \eta$, $t \le s$; using (4.1.3), we get

$$F_{\xi\eta}(t,s) = \frac{t(1-s)}{1-\alpha\beta\xi\eta} \le \frac{s(1-s)}{1-\alpha\beta\xi\eta} \le \frac{\max\{1,\alpha\beta\xi\}}{1-\alpha\beta\xi\eta}s(1-s).$$

Now we prove $(ii)$. For $(t,s) \in [0,1]\times[0,1]$, we discuss two cases.

**Case 1:** $s \le \xi$; using (4.1.4), we obtain

$$G_{\alpha\beta\xi\eta}(t,s) = \frac{\alpha\xi t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha t(\xi-s)}{1-\alpha\beta\xi\eta} = \frac{\alpha ts(1-\xi)}{1-\alpha\beta\xi\eta} \le \frac{\alpha}{1-\alpha\beta\xi\eta}s(1-s).$$

**Case 2:** $s \ge \xi$; using (4.1.4), we have

$$G_{\alpha\beta\xi\eta}(t,s) = \frac{\alpha\xi t(1-s)}{1-\alpha\beta\xi\eta} \le \frac{\alpha}{1-\alpha\beta\xi\eta}s(1-s).$$

□

**Remark 4.1.3.** *In view of Lemma 4.1.2, we have*

$$F_{\eta\xi}(t,s) \le \frac{\max\{1,\alpha\beta\eta\}}{1-\alpha\beta\xi\eta}s(1-s), \quad t,s \in [0,1],$$
$$G_{\beta\alpha\eta\xi}(t,s) \le \frac{\beta}{1-\alpha\beta\xi\eta}s(1-s), \qquad t,s \in [0,1].$$

**Lemma 4.1.4.** *The functions $F_{\xi\eta}$ and $G_{\alpha\beta\xi\eta}$ satisfies*

(*i*) $F_{\xi\eta}(t,s) \geq \frac{\min\{1,\alpha\beta\xi\}\min\{\eta,1-\eta\}}{1-\alpha\beta\xi\eta}s(1-s), \quad (t,s) \in [\eta,1]\times[0,1],$

(*ii*) $G_{\alpha\beta\xi\eta}(t,s) \geq \frac{\alpha\xi\ \min\{\xi,1-\xi\}}{1-\alpha\beta\xi\eta}s(1-s), \quad (t,s) \in [\xi,1]\times[0,1].$

*Proof.* Here for $(t,s) \in [\eta,1]\times[0,1]$, we discuss different cases.
**Case 1:** $s \leq \eta$, $t \geq s$; using (4.1.3), we obtain

$$F_{\xi\eta}(t,s) = \frac{t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha\beta\xi t(\eta-s)}{1-\alpha\beta\xi\eta} - (t-s) = s + (\alpha\beta\xi-1)\frac{ts}{1-\alpha\beta\xi\eta}.$$

If $\alpha\beta\xi < 1$, the minimum occurs at $t=1$, hence

$$F_{\xi\eta}(t,s) \geq F_{\xi\eta}(1,s) = \alpha\beta\xi\frac{s(1-\eta)}{1-\alpha\beta\xi\eta} = \frac{\alpha\beta\xi s(1-\eta)}{1-\alpha\beta\xi\eta} \geq \frac{\min\{1,\alpha\beta\xi\}\min\{\eta,1-\eta\}}{1-\alpha\beta\xi\eta}s(1-s),$$

and if $\alpha\beta\xi \geq 1$, the minimum occurs at $t=\eta$, then

$$F_{\xi\eta}(t,s) \geq F_{\xi\eta}(\eta,s) = \frac{s(1-\eta)}{1-\alpha\beta\xi\eta} \geq \frac{\min\{1,\alpha\beta\xi\}\min\{\eta,1-\eta\}}{1-\alpha\beta\xi\eta}s(1-s).$$

**Case 2:** $s \geq \eta$, $t \geq s$; using (4.1.3), we have

$$F_{\xi\eta}(t,s) = \frac{t(1-s)}{1-\alpha\beta\xi\eta} - (t-s) = s - \frac{t(s-\alpha\beta\xi\eta)}{1-\alpha\beta\xi\eta}.$$

If $s > \alpha\beta\xi\eta$, the minimum occurs at $t=1$, hence

$$F_{\xi\eta}(t,s) \geq F_{\xi\eta}(1,s) = \frac{\alpha\beta\xi\eta(1-s)}{1-\alpha\beta\xi\eta} \geq \frac{\min\{1,\alpha\beta\xi\}\min\{\eta,1-\eta\}}{1-\alpha\beta\xi\eta}s(1-s),$$

and if $s \leq \alpha\beta\xi\eta$, the minimum occurs at $t=s$, therefore

$$F_{\xi\eta}(t,s) \geq F_{\xi\eta}(s,s) = \frac{s(1-s)}{1-\alpha\beta\xi\eta} \geq \frac{\eta(1-s)}{1-\alpha\beta\xi\eta} \geq \frac{\min\{1,\alpha\beta\xi\}\min\{\eta,1-\eta\}}{1-\alpha\beta\xi\eta}s(1-s).$$

**Case 3:** $s \geq \eta$, $t \leq s$; using (4.1.3), we have

$$H_n(t,s) = \frac{t(1-s)}{1-\alpha\beta\xi\eta} \geq \frac{\eta(1-s)}{1-\alpha\beta\xi\eta} \geq \frac{\min\{1,\alpha\beta\xi\}\min\{\eta,1-\eta\}}{1-\alpha\beta\xi\eta}s(1-s).$$

Now we prove (*ii*). For $(t,s) \in [\xi,1]\times[0,1]$, we discuss two cases.
**Case 1:** $s \leq \xi$; using (4.1.4), we have

$$\begin{aligned} G_{\alpha\beta\xi\eta}(t,s) &= \frac{\alpha\xi t(1-s)}{1-\alpha\beta\xi\eta} - \frac{\alpha t(\xi-s)}{1-\alpha\beta\xi\eta} = \frac{\alpha ts(1-\xi)}{1-\alpha\beta\xi\eta} \geq \frac{\alpha\xi s(1-\xi)}{1-\alpha\beta\xi\eta} \\ &\geq \frac{\alpha\xi\ \min\{\xi,1-\xi\}}{1-\alpha\beta\xi\eta}s(1-s). \end{aligned}$$

**Case 2:** $s \geq \xi$; using (4.1.4), we get

$$G_{\alpha\beta\xi\eta}(t,s) = \frac{\alpha\xi t(1-s)}{1-\alpha\beta\xi\eta} \geq \frac{\alpha\xi\xi(1-s)}{1-\alpha\beta\xi\eta} \geq \frac{\alpha\xi\ \min\{\xi,1-\xi\}}{1-\alpha\beta\xi\eta}s(1-s).$$

□

**Remark 4.1.5.** *In view of Lemma 4.1.4, we have*

$$
\begin{aligned}
F_{\eta\xi}(t,s) &\geq \frac{\min\{1,\alpha\beta\eta\}\min\{\xi,1-\xi\}}{1-\alpha\beta\xi\eta}s(1-s), \quad (t,s)\in[\xi,1]\times[0,1],\\
G_{\beta\alpha\eta\xi}(t,s) &\geq \frac{\beta\eta\,\min\{\eta,1-\eta\}}{1-\alpha\beta\xi\eta}s(1-s), \qquad (t,s)\in[\eta,1]\times[0,1].
\end{aligned}
$$

**Remark 4.1.6.** *From Lemma 4.1.2 and Remark 4.1.3, for $t,s\in[0,1]$, we have*

$$
\begin{aligned}
F_{\xi\eta}(t,s) \leq \mu s(1-s), F_{\eta\xi}(t,s) \leq \mu s(1-s),\\
G_{\alpha\beta\xi\eta}(t,s) \leq \mu s(1-s), G_{\beta\alpha\eta\xi}(t,s) \leq \mu s(1-s),
\end{aligned}
$$

*where $\mu=\frac{\max\{1,\alpha,\beta,\alpha\beta\xi,\alpha\beta\eta\}}{1-\alpha\beta\xi\eta}$. Similarly, from Lemma 4.1.4 and Remark 4.1.5, for $(t,s)\in[\max\{\xi,\eta\},1]\times[0,1]$, we have*

$$
\begin{aligned}
F_{\xi\eta}(t,s) \geq \nu s(1-s), F_{\eta\xi}(t,s) \geq \nu s(1-s),\\
G_{\alpha\beta\xi\eta}(t,s) \geq \nu s(1-s), G_{\beta\alpha\eta\xi}(t,s) \geq \nu s(1-s),
\end{aligned}
$$

*where $\nu=\frac{\min\{1,\alpha\xi,\alpha\beta\xi,\beta\eta,\alpha\beta\eta\}\min\{\xi,\eta,1-\xi,1-\eta\}}{1-\alpha\beta\xi\eta}$.*

**Lemma 4.1.7.** *The Green's functions $F_{\xi\eta}$ and $G_{\alpha\beta\xi\eta}$ can be expressed as*

$$
\begin{aligned}
F_{\xi\eta}(t,s) &= H(t,s)+\frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}H(\eta,s),\\
G_{\alpha\beta\xi\eta}(t,s) &= \frac{\alpha t}{1-\alpha\beta\xi\eta}H(\xi,s),
\end{aligned}
\tag{4.1.6}
$$

*where*

$$
H(t,s)=\begin{cases} s(1-t), & 0\leq s\leq t\leq 1,\\ t(1-s), & 0\leq t\leq s\leq 1.\end{cases}
$$

*Proof.* From (4.1.2), consider the integral equation

$$
\begin{aligned}
x(t) &= \int_0^1 F_{\xi\eta}(t,s)u(s)ds+\int_0^1 G_{\alpha\beta\xi\eta}(t,s)v(s)ds\\
&= \frac{t}{1-\alpha\beta\xi\eta}\int_0^1(1-s)u(s)ds-\frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta(\eta-s)u(s)ds-\int_0^t(t-s)u(s)ds\\
&+\frac{\alpha\xi t}{1-\alpha\beta\xi\eta}\int_0^1(1-s)v(s)ds-\frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi(\xi-s)v(s)ds\\
&= \int_0^1 H(t,s)u(s)ds-\int_0^t s(1-t)u(s)ds-\int_t^1 t(1-s)u(s)ds\\
&+\frac{1}{1-\alpha\beta\xi\eta}\int_0^1 t(1-s)u(s)ds-\frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta(\eta-s)u(s)ds-\int_0^t(t-s)u(s)ds\\
&+\frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1\xi(1-s)v(s)ds-\frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi(\xi-s)v(s)ds
\end{aligned}
$$

$$= \int_0^1 H(t,s)u(s)ds - \int_0^t s(1-t)u(s)ds - \int_t^1 t(1-s)u(s)ds$$
$$+ \frac{1}{1-\alpha\beta\xi\eta}\int_0^t t(1-s)u(s)ds + \frac{1}{1-\alpha\beta\xi\eta}\int_t^1 t(1-s)u(s)ds - \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta (\eta-s)u(s)ds$$
$$- \int_0^t (t-s)u(s)ds + \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1 \xi(1-s)v(s)ds - \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds$$
$$= \int_0^1 H(t,s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^t \eta(1-s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_t^1 \eta(1-s)u(s)ds$$
$$- \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta (\eta-s)u(s)ds + \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1 \xi(1-s)v(s)ds - \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds$$
$$= \int_0^1 H(t,s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^1 \eta(1-s)u(s)ds - \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta (\eta-s)u(s)ds$$
$$+ \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1 \xi(1-s)v(s)ds - \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds$$
$$= \int_0^1 H(t,s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta s(1-\eta)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_\eta^1 \eta(1-s)u(s)ds$$
$$+ \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1 \xi(1-s)v(s)ds - \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds$$
$$= \int_0^1 H(t,s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^1 H(\eta,s)u(s)ds + \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1 \xi(1-s)v(s)ds$$
$$- \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds$$
$$= \int_0^1 H(t,s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta H(\eta,s)u(s)ds + \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi \xi(1-s)v(s)ds$$
$$+ \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_\xi^1 \xi(1-s)v(s)ds - \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi (\xi-s)v(s)ds$$
$$= \int_0^1 H(t,s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta H(\eta,s)u(s)ds + \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi s(1-\xi)v(s)ds$$
$$+ \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_\xi^1 \xi(1-s)v(s)ds$$
$$= \int_0^1 H(t,s)u(s)ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^\eta H(\eta,s)u(s)ds + \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^\xi H(\xi,s)v(s)ds$$
$$= \int_0^1 \left(H(t,s) + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}H(\eta,s)\right)u(s)ds + \int_0^1 \frac{\alpha t}{1-\alpha\beta\xi\eta}H(\xi,s)v(s)ds.$$

This proves (4.1.6). □

Employing Lemma 4.1.1, the system of BVPs (4.0.1) can be expressed as

$$\begin{aligned} x(t) &= \int_0^1 F_{\xi\eta}(t,s)f(s,x(s),y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)g(s,x(s),y(s))ds, \quad t\in[0,1],\\ y(t) &= \int_0^1 F_{\eta\xi}(t,s)g(s,x(s),y(s))ds + \int_0^1 G_{\beta\alpha\eta\xi}(t,s)f(s,x(s),y(s))ds, \quad t\in[0,1]. \end{aligned} \tag{4.1.7}$$

By a solution of the system of BVPs (4.0.1), we mean a solution of the corresponding system of integral equations (4.1.7). Define an operator $T : P \times P \to P \times P$ by

$$T(x, y) = (A(x, y), B(x, y)), \tag{4.1.8}$$

where the operators $A, B : P \times P \to P$ are defined by

$$\begin{aligned} A(x, y)(t) &= \int_0^1 F_{\xi\eta}(t, s)f(s, x(s), y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t, s)g(s, x(s), y(s))ds, \quad t \in [0, 1], \\ B(x, y)(t) &= \int_0^1 F_{\eta\xi}(t, s)g(s, x(s), y(s))ds + \int_0^1 G_{\beta\alpha\eta\xi}(t, s)f(s, x(s), y(s))ds, \quad t \in [0, 1]. \end{aligned} \tag{4.1.9}$$

Clearly, if $(x, y) \in P \times P$ is a fixed point of $T$, then $(x, y)$ is a solution of the system of BVPs (4.0.1).

Assume that the following holds:

($\mathcal{H}_{23}$) $f(\cdot, 1, 1), g(\cdot, 1, 1) \in C((0, 1), (0, \infty))$ and satisfy

$$a := \int_0^1 t(1 - t)f(t, 1, 1)dt < +\infty, \quad b := \int_0^1 t(1 - t)g(t, 1, 1)dt < +\infty.$$

($\mathcal{H}_{24}$) There exist real constants $\alpha_i, \beta_i$ with $0 \leq \alpha_i \leq \beta_i < 1$, $i = 1, 2$; $\beta_1 + \beta_2 < 1$, such that for all $t \in (0, 1)$, $x, y \in [0, \infty)$,

$$\begin{aligned} c^{\beta_1} f(t, x, y) &\leq f(t, cx, y) \leq c^{\alpha_1} f(t, x, y), \quad 0 < c \leq 1, \\ c^{\alpha_1} f(t, x, y) &\leq f(t, cx, y) \leq c^{\beta_1} f(t, x, y), \quad c \geq 1, \\ c^{\beta_2} f(t, x, y) &\leq f(t, x, cy) \leq c^{\alpha_2} f(t, x, y), \quad 0 < c \leq 1, \\ c^{\alpha_2} f(t, x, y) &\leq f(t, x, cy) \leq c^{\beta_2} f(t, x, y), \quad c \geq 1. \end{aligned}$$

($\mathcal{H}_{25}$) There exist real constants $\gamma_i, \rho_i$ with $0 \leq \gamma_i \leq \rho_i < 1$, $i = 1, 2$; $\rho_1 + \rho_2 < 1$, such that for all $t \in (0, 1)$, $x, y \in [0, \infty)$,

$$\begin{aligned} c^{\rho_1} g(t, x, y) &\leq g(t, cx, y) \leq c^{\gamma_1} g(t, x, y), \quad 0 < c \leq 1, \\ c^{\gamma_1} g(t, x, y) &\leq g(t, cx, y) \leq c^{\rho_1} g(t, x, y), \quad c \geq 1, \\ c^{\rho_2} g(t, x, y) &\leq g(t, x, cy) \leq c^{\gamma_2} g(t, x, y), \quad 0 < c \leq 1, \\ c^{\gamma_2} g(t, x, y) &\leq g(t, x, cy) \leq c^{\rho_2} g(t, x, y), \quad c \geq 1. \end{aligned}$$

**Lemma 4.1.8.** *Under the hypothesis* ($\mathcal{H}_{23}$)$-$($\mathcal{H}_{25}$)*, the operator* $T : \overline{\mathcal{O}}_r \cap (P \times P) \to P \times P$ *is completely continuous.*

*Proof.* First we show that $T(P \times P) \subseteq P \times P$. For any $(x, y) \in P \times P$, $t \in [0, 1]$, using (4.1.9) and Remark 4.1.6, we have

$$\begin{aligned} A(x, y)(t) &= \int_0^1 F_{\xi\eta}(t, s)f(s, x(s), y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t, s)g(s, x(s), y(s))ds \\ &\leq \mu \int_0^1 s(1 - s)f(s, x(s), y(s))ds + \mu \int_0^1 s(1 - s)g(s, x(s), y(s))ds, \end{aligned}$$

which implies that

$$\|A(x,y)\| \leq \mu \int_0^1 s(1-s)f(s,x(s),y(s))ds + \mu \int_0^1 s(1-s)g(s,x(s),y(s))ds. \quad (4.1.10)$$

Also, for $(x,y) \in P \times P$ and $t \in [\max\{\xi,\eta\},1]$, using (4.1.9), Remark 4.1.6 and (4.1.10), we obtain

$$\begin{aligned} A(x,y)(t) &= \int_0^1 F_{\xi\eta}(t,s)f(s,x(s),y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)g(s,x(s),y(s))ds \\ &\geq \nu \int_0^1 s(1-s)f(s,x(s),y(s))ds + \nu \int_0^1 s(1-s)g(s,x(s),y(s))ds \\ &= \gamma\mu \int_0^1 s(1-s)f(s,x(s),y(s))ds + \gamma\mu \int_0^1 s(1-s)g(s,x(s),y(s))ds \geq \gamma\|A(x,y)\|. \end{aligned}$$

Consequently, $A(x,y) \in P$ for all $(x,y) \in P \times P$. Thus, $A(P \times P) \subseteq P$. Similarly, we can show that $B(P \times P) \subseteq P$. Hence, $T(P \times P) \subseteq (P \times P)$. Now, we show that the operator $T : \overline{\mathcal{O}}_r \cap (P \times P) \to P \times P$ is uniformly bounded. Choose a real constant $c \in (0,1]$ such that $cr \leq 1$. For $(x,y) \in \overline{\mathcal{O}}_r \cap (P \times P)$, $t \in [0,1]$, using (4.1.9), Remark 4.1.6 and $(\mathcal{H}_{23}) - (\mathcal{H}_{25})$, we have

$$\begin{aligned} A(x,y)(t) &= \int_0^1 F_{\xi\eta}(t,s)f(s,x(s),y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)g(s,x(s),y(s))ds \\ &\leq \mu \int_0^1 s(1-s)f(s,x(s),y(s))ds + \mu \int_0^1 s(1-s)g(s,x(s),y(s))ds \\ &= \mu \int_0^1 s(1-s)f(s,\frac{c\,x(s)}{c},\frac{c\,y(s)}{c})ds + \mu \int_0^1 s(1-s)g(s,\frac{c\,x(s)}{c},\frac{c\,y(s)}{c})ds \\ &\leq \mu c^{-\beta_1} \int_0^1 s(1-s)f(s,c\,x(s),\frac{c\,y(s)}{c})ds + \mu c^{-\rho_1} \int_0^1 s(1-s)g(s,c\,x(s),\frac{c\,y(s)}{c})ds \\ &\leq \mu c^{-\beta_1-\beta_2} \int_0^1 s(1-s)f(s,c\,x(s),c\,y(s))ds + \mu c^{-\rho_1-\rho_2} \int_0^1 s(1-s)g(s,c\,x(s),c\,y(s))ds \\ &\leq \mu c^{\alpha_1-\beta_1-\beta_2} \int_0^1 s(1-s)(x(s))^{\alpha_1} f(s,1,c\,y(s))ds + \mu c^{\gamma_1-\rho_1-\rho_2} \int_0^1 s(1-s)(x(s))^{\gamma_1} \\ &\quad g(s,1,c\,y(s))ds \leq \mu c^{\alpha_1+\alpha_2-\beta_1-\beta_2} \int_0^1 s(1-s)(x(s))^{\alpha_1}(y(s))^{\alpha_2} f(s,1,1)ds + \mu c^{\gamma_1+\gamma_2-\rho_1-\rho_2} \\ &\quad \int_0^1 s(1-s)(x(s))^{\gamma_1}(y(s))^{\gamma_2} g(s,1,1)ds \leq \mu\, a\, c^{\alpha_1+\alpha_2-\beta_1-\beta_2} r^{\alpha_1+\alpha_2} + \mu\, b\, c^{\gamma_1+\gamma_2-\rho_1-\rho_2} r^{\gamma_1+\gamma_2}, \end{aligned}$$

which implies that $A(\overline{\mathcal{O}}_r \cap (P \times P))$ is uniformly bounded. Similarly, using (4.1.9), Remark 4.1.6 and $(\mathcal{H}_{23}) - (\mathcal{H}_{25})$, we can show that $B(\overline{\mathcal{O}}_r \cap (P \times P))$ is also uniformly bounded. Thus, $T(\overline{\mathcal{O}}_r \cap (P \times P))$ is uniformly bounded.

Now, we show that $A(\overline{\mathcal{O}}_r \cap (P \times P))$ is equicontinuous. For any $(x,y) \in \overline{\mathcal{O}}_r \cap (P \times P)$, $t \in [0,1]$, using (4.1.9) and Lemma 4.1.7, we have

$$A(x,y)(t) = \int_0^1 F_{\xi\eta}(t,s)f(s,x(s),y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)g(s,x(s),y(s))ds$$

$$= \int_0^1 H(t,s)f(s,x(s),y(s))ds + \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^1 H(\eta,s)f(s,x(s),y(s))ds$$
$$+ \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1 H(\xi,s)g(s,x(s),y(s))ds$$
$$= \int_0^t s(1-t)f(s,x(s),y(s))ds + \int_t^1 t(1-s)f(s,x(s),y(s))ds$$
$$+ \frac{\alpha\beta\xi t}{1-\alpha\beta\xi\eta}\int_0^1 H(\eta,s)f(s,x(s),y(s))ds + \frac{\alpha t}{1-\alpha\beta\xi\eta}\int_0^1 H(\xi,s)g(s,x(s),y(s))ds.$$

Differentiating with respect to $t$, we obtain

$$A(x,y)'(t) = -\int_0^t sf(s,x(s),y(s))ds + \int_t^1 (1-s)f(s,x(s),y(s))ds$$
$$+ \frac{\alpha\beta\xi}{1-\alpha\beta\xi\eta}\int_0^1 H(\eta,s)f(s,x(s),y(s))ds + \frac{\alpha}{1-\alpha\beta\xi\eta}\int_0^1 H(\xi,s)g(s,x(s),y(s))ds,$$

which implies that

$$|A(x,y)'(t)| \le \int_0^t sf(s,x(s),y(s))ds + \int_t^1 (1-s)f(s,x(s),y(s))ds$$
$$+ \frac{\alpha\beta\xi}{1-\alpha\beta\xi\eta}\int_0^1 s(1-s)f(s,x(s),y(s))ds + \frac{\alpha}{1-\alpha\beta\xi\eta}\int_0^1 s(1-s)g(s,x(s),y(s))ds.$$

Now, using $(\mathcal{H}_{23}) - (\mathcal{H}_{25})$, we have

$$|A(x,y)'(t)| = \int_0^t sf(s,\frac{c\,x(s)}{c},\frac{c\,y(s)}{c})ds + \int_t^1 (1-s)f(s,\frac{c\,x(s)}{c},\frac{c\,y(s)}{c})ds$$
$$+ \frac{\alpha\beta\xi}{1-\alpha\beta\xi\eta}\int_0^1 s(1-s)f(s,\frac{c\,x(s)}{c},\frac{c\,y(s)}{c})ds + \frac{\alpha}{1-\alpha\beta\xi\eta}\int_0^1 s(1-s)g(s,\frac{c\,x(s)}{c},\frac{c\,y(s)}{c})ds$$
$$\le c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2}(\int_0^t sf(s,1,1)ds + \int_t^1 (1-s)f(s,1,1)ds)$$
$$+ \frac{\alpha}{1-\alpha\beta\xi\eta}(\beta\xi a\, c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2} + b\, c^{\gamma_1+\gamma_2-\rho_1-\rho_2}r^{\gamma_1+\gamma_2}).$$

Let

$$h(t) = c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2}\left(\int_0^t sf(s,1,1)ds + \int_t^1 (1-s)f(s,1,1)ds\right)$$
$$+ \frac{\alpha}{1-\alpha\beta\xi\eta}(\beta\xi a\, c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2} + b\, c^{\gamma_1+\gamma_2-\rho_1-\rho_2}r^{\gamma_1+\gamma_2}).$$

Integrating from 0 to 1 and using $(\mathcal{H}_{23})$, we have

$$\int_0^1 h(t)dt = c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2}(\int_0^1\int_0^t sf(s,1,1)dsdt + \int_0^1\int_t^1 (1-s)$$
$$f(s,1,1)dsdt) + \frac{\alpha}{1-\alpha\beta\xi\eta}(\beta\xi a\, c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2} + b\, c^{\gamma_1+\gamma_2-\rho_1-\rho_2}r^{\gamma_1+\gamma_2})$$
$$= c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2}(\int_0^1 s(1-s)f(s,1,1)ds + \int_0^1 s(1-s)f(s,1,1)ds) \tag{4.1.11}$$
$$+ \frac{\alpha}{1-\alpha\beta\xi\eta}(\beta\xi a\, c^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2} + b\, c^{\gamma_1+\gamma_2-\rho_1-\rho_2}r^{\gamma_1+\gamma_2})$$
$$= \frac{(2+\alpha\beta\xi-2\alpha\beta\xi\eta)ac^{\alpha_1+\alpha_2-\beta_1-\beta_2}r^{\alpha_1+\alpha_2} + \alpha bc^{\gamma_1+\gamma_2-\rho_1-\rho_2}r^{\gamma_1+\gamma_2}}{1-\alpha\beta\xi\eta}.$$

Thus, for any given $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$ and $(x,y) \in \overline{\mathcal{O}}_r \cap (P \times P)$, we have

$$|A(x,y)(t_1) - A(x,y)(t_2)| = \left| \int_{t_1}^{t_2} A(x,y)'(t)dt \right| \leq \int_{t_1}^{t_2} h(t)dt,$$

this together with (4.1.11), it follows that $A(\overline{\mathcal{O}}_r \cap (P \times P))$ is equicontinuous on $[0,1]$. Similarly, we can show that $B(\overline{\mathcal{O}}_r \cap (P \times P))$ is also equicontinuous. Thus, $T(\overline{\mathcal{O}}_r \cap (P \times P))$ is equicontinuous. Thus, by Theorem 1.1.6, it follows that $T(\overline{\mathcal{O}}_r \cap (P \times P))$ is relatively compact. Hence, $T$ is a compact operator.

Now, we show that $T$ is continuous. Let $(x_m, y_m), (x,y) \in \overline{\mathcal{O}}_r \cap (P \times P)$ such that $\|(x_m, y_m) - (x,y)\|_6 \to 0$ as $m \to +\infty$. Then by using (4.1.9) and Remark 4.1.6, we have

$$\begin{aligned}
&|A(x_m, y_m)(t) - A(x,y)(t)| = \left| \int_0^1 F_{\xi\eta}(t,s)(f(s, x_m(s), y_m(s)) - f(s, x(s), y(s)))ds \right. \\
&\left. + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)(g(s, x_m(s), y_m(s)) - g(s, x(s), y(s)))ds \right| \leq \int_0^1 F_{\xi\eta}(t,s)|f(s, x_m(s), y_m(s)) \\
&- f(s, x(s), y(s))|ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)|g(s, x_m(s), y_m(s)) - g(s, x(s), y(s))|ds \\
&\leq \mu \int_0^1 s(1-s)|f(s, x_m(s), y_m(s)) - f(s, x(s), y(s))|ds \\
&+ \mu \int_0^1 s(1-s)|g(s, x_m(s), y_m(s)) - g(s, x(s), y(s))|ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|A(x_m, y_m) - A(x,y)\| \leq{}& \mu \int_0^1 s(1-s)|f(s, x_m(s), y_m(s)) - f(s, x(s), y(s))|ds \\
&+ \mu \int_0^1 s(1-s)|g(s, x_m(s), y_m(s)) - g(s, x(s), y(s))|ds.
\end{aligned}$$

By Lebesgue dominated convergence theorem, it follows that

$$\|A(x_m, y_m) - A(x,y)\| \to 0 \text{ as } m \to +\infty. \tag{4.1.12}$$

Similarly, by using (4.1.9) and Remark 4.1.6, we have

$$\|B(x_m, y_m) - B(x,y)\| \to 0 \text{ as } m \to +\infty. \tag{4.1.13}$$

From (4.1.8), (4.1.12) and (4.1.13), it follows that

$$\|T(x_m, y_m) - T(x,y)\|_6 \to 0 \text{ as } m \to +\infty,$$

that is, $T : P \times P \to P \times P$ is continuous. Hence, $T : P \times P \to P \times P$ is completely continuous. □

**Theorem 4.1.9.** *Under the hypothesis* $(\mathcal{H}_{23}) - (\mathcal{H}_{25})$*, the system of BVPs* (4.0.1) *has at least one positive solution.*

*Proof.* Choose a constant $R > 0$ such that

$$R \geq \max\{1, (4a\mu)^{\frac{1}{1-\beta_1-\beta_2}}, (4b\mu)^{\frac{1}{1-\rho_1-\rho_2}}\}. \tag{4.1.14}$$

Let $cR = 1$, for some real constants $c$. Then, for any $(x, y) \in \partial\mathcal{O}_R \cap (P \times P)$, $t \in [0, 1]$, using (4.1.9), Remark 4.1.6, $(\mathcal{H}_{23}) - (\mathcal{H}_{25})$, we have

$$\begin{aligned}
A(x,y)(t) &= \int_0^1 F_{\xi\eta}(t,s)f(s,x(s),y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)g(s,x(s),y(s))ds \\
&\leq \mu \int_0^1 s(1-s)f(s,x(s),y(s))ds + \mu \int_0^1 s(1-s)g(s,x(s),y(s))ds \\
&= \mu \int_0^1 s(1-s)f(s,\frac{cx(s)}{c},\frac{cy(s)}{c})ds + \mu \int_0^1 s(1-s)g(s,\frac{cx(s)}{c},\frac{cy(s)}{c})ds \\
&\leq \mu c^{\alpha_1-\beta_1}R^{\alpha_1} \int_0^1 s(1-s)f(s,1,\frac{cy(s)}{c})ds + \mu c^{\gamma_1-\rho_1}R^{\gamma_1} \int_0^1 s(1-s)g(s,1,\frac{cy(s)}{c})ds \\
&\leq \mu c^{\alpha_1-\beta_1+\alpha_2-\beta_2}R^{\alpha_1+\alpha_2} \int_0^1 s(1-s)f(s,1,1)ds + \mu c^{\gamma_1-\rho_1+\gamma_2-\rho_2}R^{\gamma_1+\gamma_2} \\
&\int_0^1 s(1-s)g(s,1,1)ds = a\mu R^{\beta_1+\beta_2} + b\mu R^{\rho_1+\rho_2}.
\end{aligned}$$

Thus, in view of (4.1.14), we have

$$\|A(x,y)\| \leq \frac{\|(x,y)\|_6}{2} \text{ for all } (x,y) \in \partial\mathcal{O}_R \cap (P \times P). \tag{4.1.15}$$

Similarly, using (4.1.9), Remark 4.1.6, $(\mathcal{H}_{23}) - (\mathcal{H}_{25})$, we have

$$\|B(x,y)\| \leq \frac{\|(x,y)\|_6}{2} \text{ for all } (x,y) \in \partial\mathcal{O}_R \cap (P \times P). \tag{4.1.16}$$

From (4.1.8), (4.1.15) and (4.1.16), it follows that

$$\|T(x,y)\|_6 \leq \|(x,y)\|_6 \text{ for all } (x,y) \in \partial\mathcal{O}_R \cap (P \times P). \tag{4.1.17}$$

Choose a real constant $r \in (0, R)$ such that

$$\begin{aligned}
r \leq \min\big\{1, (4\nu\gamma^{\beta_1+\beta_2} &\int_{\max\{\xi,\eta\}}^1 s(1-s)f(s,1,1)ds)^{\frac{1}{1-\beta_1-\beta_2}}, \\
(4\nu\gamma^{\rho_1+\rho_2} &\int_{\max\{\xi,\eta\}}^1 s(1-s)g(s,1,1)ds)^{\frac{1}{1-\rho_1-\rho_2}}\big\}.
\end{aligned} \tag{4.1.18}$$

Then, for any $(x, y) \in \partial\mathcal{O}_r \cap (P \times P)$, $t \in [\max\{\xi, \eta\}, 1]$, using (4.1.9), Remark 4.1.6, $(\mathcal{H}_{24}) - (\mathcal{H}_{25})$, we have

$$\begin{aligned}
A(x,y)(t) &= \int_0^1 F_{\xi\eta}(t,s)f(s,x(s),y(s))ds + \int_0^1 G_{\alpha\beta\xi\eta}(t,s)g(s,x(s),y(s))ds \\
&\geq \nu \int_0^1 s(1-s)f(s,x(s),y(s))ds + \nu \int_0^1 s(1-s)g(s,x(s),y(s))ds \\
&\geq \nu \int_0^1 s(1-s)(x(s))^{\beta_1}f(s,1,y(s))ds + \nu \int_0^1 s(1-s)g(s,x(s),y(s))ds
\end{aligned}$$

$$\geq \nu \int_0^1 s(1-s)(x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1)ds + \nu \int_0^1 s(1-s)g(s,x(s),y(s))ds$$
$$\geq \nu \int_0^1 s(1-s)(x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1)ds + \nu \int_0^1 s(1-s)(x(s))^{\rho_1} g(s,1,y(s))ds$$
$$\geq \nu \int_0^1 s(1-s)(x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1)ds + \nu \int_0^1 s(1-s)(x(s))^{\rho_1}(y(s))^{\rho_2} g(s,1,1)ds$$
$$= \nu \int_0^1 s(1-s)((x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1) + (x(s))^{\rho_1}(y(s))^{\rho_2} g(s,1,1))ds$$
$$\geq \nu \int_{\max\{\xi,\eta\}}^1 s(1-s)((x(s))^{\beta_1}(y(s))^{\beta_2} f(s,1,1) + (x(s))^{\rho_1}(y(s))^{\rho_2} g(s,1,1))ds$$
$$\geq \nu \gamma^{\beta_1+\beta_2} r^{\beta_1+\beta_2} \int_{\max\{\xi,\eta\}}^1 s(1-s)f(s,1,1)ds$$
$$+ \nu \gamma^{\rho_1+\rho_2} r^{\rho_1+\rho_2} \int_{\max\{\xi,\eta\}}^1 s(1-s)g(s,1,1)ds.$$

Thus in view of (4.1.18), we have

$$\|A(x,y)\| \geq \frac{\|(x,y)\|_6}{2} \text{ for all } (x,y) \in \partial\mathcal{O}_r \cap (P \times P). \tag{4.1.19}$$

Similarly, using (4.1.9), Remark 4.1.6, $(\mathcal{H}_{24}) - (\mathcal{H}_{25})$, in view of (4.1.18), we have

$$\|B(x,y)\| \geq \frac{\|(x,y)\|_6}{2} \text{ for all } (x,y) \in \partial\mathcal{O}_r \cap (P \times P). \tag{4.1.20}$$

From (4.1.8), (4.1.19) and (4.1.20), it follows that

$$\|T(x,y)\|_6 \geq \|(x,y)\|_6 \text{ for all } (x,y) \in \partial\mathcal{O}_r \cap (P \times P). \tag{4.1.21}$$

Hence, in view of (4.1.17), (4.1.21) and by Theorem 1.2.8, $T$ has a fixed point $(x,y) \in (\overline{\mathcal{O}}_R \setminus \mathcal{O}_r) \cap (P \times P)$. That is, $x = A(x,y)$ and $y = B(x,y)$. Moreover, $(x,y)$ is positive. In fact, by concavity of $x$ and by construction of the cone $P$, we have

$$x(1) \geq \min_{t\in[\max\{\xi,\eta\},1]} x(t) \geq \gamma\|x\| > 0,$$

which implies that $x(t) > 0$ for all $t \in (0,1]$. Similarly, $y(t) > 0$ for all $t \in (0,1]$. Hence, $(x,y)$ is a positive solution of the system of BVPs (4.0.1). □

**Example 4.1.10.**

$$f(t,x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} t^{p_i}(1-t)^{q_j} x^{r_i} y^{s_j},$$
$$g(t,x,y) = \sum_{k=1}^{m'}\sum_{l=1}^{n'} t^{p'_k}(1-t)^{q'_l} x^{r'_k} y^{s'_l},$$

where the real constants $p_i, q_j, r_i, s_j$ satisfy $p_i, q_j > -2$, $0 \leq r_i, s_j < 1, i = 1,2,\cdots,m; j = 1,2,\cdots,n$, with $\max_{1\leq i\leq m} r_i + \max_{1\leq j\leq n} s_j < 1$, and the real constants $p'_k, q'_l, r'_k, s'_l$ satisfy $p'_k, q'_l > -2$, $0 \leq r'_k, s'_l < 1, k = 1,2,\cdots,m'; l = 1,2,\cdots,n'$, with $\max_{1\leq k\leq m'} r'_k + \max_{1\leq l\leq n'} s'_l < 1$. Clearly, $f$ and $g$ satisfy assumptions $(\mathcal{H}_{23}) - (\mathcal{H}_{25})$. Hence, by Theorem 4.1.9, the system of BVPs (4.0.1) has a positive solution.

## 4.2 Singular systems of ODEs with two–point coupled BCs

In this section, we establish existence of at least one $C^1$-positive solution for the system of BVPs (4.0.2). By a $C^1$-positive solution to the system of BVPs (4.0.2), we mean that $(x,y) \in (C^1[0,1] \cap C^2(0,1)) \times (C^1[0,1] \cap C^2(0,1))$, $(x,y)$ satisfies (4.0.2), $x > 0$ and $y > 0$ on $[0,1]$, $x' > 0$ and $y' > 0$ on $[0,1)$.

Assume that the following holds:

$(\mathcal{H}_{26})$ hypothesis $(\mathcal{H}_{10})$ of Chapter 3, page 42;

$(\mathcal{H}_{27})$ hypothesis $(\mathcal{H}_{11})$ of Chapter 3, page 42;

$(\mathcal{H}_{28})$ hypothesis $(\mathcal{H}_{12})$ of Chapter 3, page 42;

$(\mathcal{H}_{29})$

$$\sup_{c\in(0,\infty)} \frac{c}{(1+\frac{b_1}{a_1})I^{-1}(h_1(c)k_1(c)\int_0^1 p(t)dt) + (1+\frac{b_2}{a_2})J^{-1}(h_2(c)k_2(c)\int_0^1 q(t)dt)} > 1,$$

where $I(\mu) = \int_0^\mu \frac{d\tau}{u_1(\tau)+v_1(\tau)}$, $J(\mu) = \int_0^\mu \frac{d\tau}{u_2(\tau)+v_2(\tau)}$, for $\mu > 0$;

$(\mathcal{H}_{30})$ hypothesis $(\mathcal{H}_{14})$ of Chapter 3, page 42;

$(\mathcal{H}_{31})$ for real constants $M > 0$ and $L > 0$ there exist continuous functions $\varphi_{ML}$ and $\psi_{ML}$ defined on $[0,1]$ and positive on $(0,1)$, and constants $0 \leq \gamma_1, \delta_1, \gamma_2, \delta_2 < 1$ satisfying $(1-\gamma_1)(1-\gamma_2) \neq \delta_1\delta_2$, such that $f(t,x,y,z) \geq \varphi_{ML}(t)x^{\gamma_1}y^{\delta_1}$ and $g(t,x,y,z) \geq \psi_{ML}(t)x^{\gamma_2}y^{\delta_2}$ on $[0,1]\times[0,M]\times[0,M]\times[0,L]$;

$(\mathcal{H}_{32})$ hypothesis $(\mathcal{H}_{21})$ of Chapter 3, page 60, with $E = M$ and $F = L$.

**Theorem 4.2.1.** *Under the hypothesis* $(\mathcal{H}_{26}) - (\mathcal{H}_{32})$*, the system of BVPs* (4.0.2) *has at least one* $C^1$*-positive solution.*

*Proof.* In view of $(\mathcal{H}_{29})$, we can choose real constant $M_5 > 0$ such that

$$\frac{M_5}{(1+\frac{b_1}{a_1})I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds) + (1+\frac{b_2}{a_2})J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds)} > 1.$$

From the continuity of $I$ and $J$, we choose $\varepsilon > 0$ small enough such that

$$\frac{M_5}{(1+\frac{b_1}{a_1})I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds + I(\varepsilon)) + (1+\frac{b_2}{a_2})J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds + J(\varepsilon))} > 1. \tag{4.2.1}$$

Choose a real constant $L_5 > 0$ such that

$$L_5 > \max\{I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(t)dt + I(\varepsilon)), J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(t)dt + J(\varepsilon))\} \tag{4.2.2}$$

Choose $n_0 \in \{1, 2, \cdots\}$ such that $\frac{1}{n_0} < \varepsilon$. For each $n \in \{n_0, n_0+1, \cdots\}$, define retractions $\theta_5 : \mathbb{R} \to [0, M_5]$ and $\rho_5 : \mathbb{R} \to [\frac{1}{n}, L_5]$ by

$$\theta_5(x) = \max\{0, \min\{x, M_5\}\} \text{ and } \rho_5(x) = \max\{\frac{1}{n}, \min\{x, L_5\}\}.$$

Consider the modified system of BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(x'(t))), \quad t \in (0,1), \\ -y''(t) &= q(t)g(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(y'(t))), \quad t \in (0,1), \\ & a_1 y(0) - b_1 x'(0) = 0,\ y'(1) = \frac{1}{n} \\ & a_2 x(0) - b_2 y'(0) = 0,\ x'(1) = \frac{1}{n}. \end{aligned} \tag{4.2.3}$$

Since $f(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(x'(t)))$, $g(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(y'(t)))$ are continuous and bounded on $[0,1] \times \mathbb{R}^3$, by Theorem 1.2.4, it follows that the modified system of BVPs (4.2.3) has a solution $(x_n, y_n) \in (C^1[0,1] \cap C^2(0,1)) \times (C^1[0,1] \cap C^2(0,1))$.

Using (4.2.3) and $(\mathcal{H}_{27})$, we obtain

$$x_n''(t) \le 0 \text{ and } y_n''(t) \le 0 \text{ for } t \in (0,1),$$

which on integration from $t$ to 1, using the BCs (4.2.3), implies that

$$x_n'(t) \ge \frac{1}{n} \text{ and } y_n'(t) \ge \frac{1}{n} \text{ for } t \in [0,1]. \tag{4.2.4}$$

Integrating (4.2.4) from 0 to $t$, using the BCs (4.2.3) and (4.2.4), we have

$$x_n(t) \ge (t + \frac{b_2}{a_2})\frac{1}{n} \text{ and } y_n(t) \ge (t + \frac{b_1}{a_1})\frac{1}{n} \text{ for } t \in [0,1]. \tag{4.2.5}$$

From (4.2.4) and (4.2.5), it follows that

$$\|x_n\| = x_n(1) \text{ and } \|y_n\| = y_n(1). \tag{4.2.6}$$

Now, we show that

$$x_n'(t) < L_5,\ y_n'(t) < L_5, \quad t \in [0,1]. \tag{4.2.7}$$

First, we prove $x_n'(t) < L_5$ for $t \in [0,1]$. Suppose $x_n'(t_1) \ge L_5$ for some $t_1 \in [0,1]$. Using (4.2.3) and $(\mathcal{H}_{28})$, we have

$$-x_n''(t) \le p(t)h_1(\theta_5(x_n(t)))k_1(\theta_5(y_n(t)))(u_1(\rho_5(x_n'(t))) + v_1(\rho_5(x_n'(t)))), \quad t \in (0,1),$$

which implies that

$$\frac{-x_n''(t)}{u_1(\rho_5(x_n'(t))) + v_1(\rho_5(x_n'(t)))} \le h_1(M_5)k_1(M_5)p(t), \quad t \in (0,1).$$

Integrating from $t_1$ to 1, using the BCs (4.2.3), we obtain

$$\int_{\frac{1}{n}}^{x_n'(t_1)} \frac{dz}{u_1(\rho_5(z)) + v_1(\rho_5(z))} \leq h_1(M_5)k_1(M_5)\int_{t_1}^{1} p(t)dt,$$

which can also be written as

$$\int_{\frac{1}{n}}^{L_5} \frac{dz}{u_1(z)+v_1(z)} + \int_{L_5}^{x_n'(t_1)} \frac{dz}{u_1(L_5)+v_1(L_5)} \leq h_1(M_5)k_1(M_5)\int_0^1 p(t)dt.$$

Using the increasing property of $I$, we obtain

$$I(L_5) + \frac{x_n'(t_1) - L_5}{u_1(L_5)+v_1(L_5)} \leq h_1(M_5)k_1(M_5)\int_0^1 p(t)dt + I(\varepsilon),$$

and using the increasing property of $I^{-1}$, leads to

$$L_5 \leq I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(t)dt + I(\varepsilon)).$$

Which is a contradiction to (4.2.2). Hence, $x_n'(t) < L_5$ for $t \in [0,1]$.

Similarly, we can show that $y_n'(t) < L_5$ for $t \in [0,1]$.

Now, we show that

$$\|x_n\| + \|y_n\| < M_5. \tag{4.2.8}$$

Suppose $\|x_n\| + \|y_n\| \geq M_5$. From (4.2.3), (4.2.4), (4.2.7) and $(\mathcal{H}_{28})$, it follows that

$$\begin{aligned}
-x_n''(t) &\leq p(t)h_1(\theta_5(x_n(t)))k_1(\theta_5(y_n(t)))(u_1(x_n'(t)) + v_1(x_n'(t))), \quad t \in (0,1),\\
-y_n''(t) &\leq q(t)h_2(\theta_5(x_n(t)))k_2(\theta_5(y_n(t)))(u_2(y_n'(t)) + v_2(y_n'(t))), \quad t \in (0,1),
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{-x_n''(t)}{u_1(x_n'(t)) + v_1(x_n'(t))} &\leq h_1(M_5)k_1(M_5)p(t), \quad t \in (0,1),\\
\frac{-y_n''(t)}{u_2(y_n'(t)) + v_2(y_n'(t))} &\leq h_2(M_5)k_2(M_5)q(t), \quad t \in (0,1).
\end{aligned}$$

Integrating from $t$ to 1, using the BCs (4.2.3), we obtain

$$\begin{aligned}
\int_{\frac{1}{n}}^{x_n'(t)} \frac{dz}{u_1(z)+v_1(z)} &\leq h_1(M_5)k_1(M_5)\int_t^1 p(s)ds, \quad t \in [0,1],\\
\int_{\frac{1}{n}}^{y_n'(t)} \frac{dz}{u_2(z)+v_2(z)} &\leq h_2(M_5)k_2(M_5)\int_t^1 q(s)ds, \quad t \in [0,1],
\end{aligned}$$

which can also be written as

$$\begin{aligned}
I(x_n'(t)) - I(\frac{1}{n}) &\leq h_1(M_5)k_1(M_5)\int_0^1 p(s)ds, \quad t \in [0,1],\\
J(y_n'(t)) - J(\frac{1}{n}) &\leq h_2(M_5)k_2(M_5)\int_0^1 q(s)ds, \quad t \in [0,1].
\end{aligned}$$

The increasing property of $I$ and $J$ leads to

$$\begin{aligned} x_n'(t) &\le I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0,1], \\ y_n'(t) &\le J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0,1]. \end{aligned} \tag{4.2.9}$$

Integrating from 0 to $t$, using the BCs (4.2.3) and (4.2.9), we obtain

$$\begin{aligned} x_n(t) &\le I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds + I(\varepsilon)) \\ &+ \frac{b_2}{a_2}J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0,1], \\ y_n(t) &\le \frac{b_1}{a_1}I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds + I(\varepsilon)) \\ &+ J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds + J(\varepsilon)), \qquad t \in [0,1]. \end{aligned} \tag{4.2.10}$$

From (4.2.10) and (4.2.6), it follows that

$$\begin{aligned} M_5 \le \|x_n\| + \|y_n\| &\le (1 + \frac{b_1}{a_1})I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds + I(\varepsilon)) \\ &+ (1 + \frac{b_2}{a_2})J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds + J(\varepsilon)), \end{aligned}$$

which implies that

$$\frac{M_5}{(1+\frac{b_1}{a_1})I^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds + I(\varepsilon)) + (1+\frac{b_2}{a_2})J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds + J(\varepsilon))} \le 1,$$

a contradiction to (4.2.1). Hence, $\|x_n\| + \|y_n\| < M_5$.

Thus, in view of (4.2.3)–(4.2.8), $(x_n, y_n)$ is a solution of the following coupled system of BVPs

$$\begin{aligned} -x''(t) &= p(t)f(t, x(t), y(t), x'(t)), \quad t \in (0,1), \\ -y''(t) &= q(t)g(t, x(t), y(t), y'(t)), \quad t \in (0,1), \\ a_1y(0) - b_1x'(0) &= 0,\ x'(1) = \frac{1}{n}, \\ a_2x(0) - b_2y'(0) &= 0,\ x'(1) = \frac{1}{n}, \end{aligned} \tag{4.2.11}$$

satisfying

$$\begin{aligned} (t + \frac{b_2}{a_2})\frac{1}{n} \le x_n(t) < M_5,\ \frac{1}{n} \le x_n'(t) < L_5, \quad t \in [0,1], \\ (t + \frac{b_1}{a_1})\frac{1}{n} \le y_n(t) < M_5,\ \frac{1}{n} \le y_n'(t) < L_5, \quad t \in [0,1]. \end{aligned} \tag{4.2.12}$$

We claim that

$$x_n'(t) \ge C_9^{\gamma_1}C_{10}^{\delta_1}\int_t^1 p(s)\varphi_{M_5L_5}(s)ds, \tag{4.2.13}$$

$$y_n'(t) \geq C_9^{\gamma_2} C_{10}^{\delta_2} \int_t^1 q(s)\psi_{M_5L_5}(s)ds, \tag{4.2.14}$$

where

$$C_9 = \left(\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_5L_5}(s)ds\right)^{\frac{\delta_1}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}} \left(\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_5L_5}(s)ds\right)^{\frac{1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}},$$

$$C_{10} = \left(\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_5L_5}(s)ds\right)^{\frac{1-\gamma_1}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}} \left(\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_5L_5}(s)ds\right)^{\frac{\delta_2}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}}.$$

To prove (4.2.13), consider the following relation

$$\begin{aligned} x_n(t) = {}& (t+\frac{b_2}{a_2})\frac{1}{n} + \int_0^t sp(s)f(s,x_n(s),y_n(s),x_n'(s))ds \\ &+ \int_t^1 tp(s)f(s,x_n(s),y_n(s),x_n'(s))ds + \frac{b_2}{a_2}\int_t^1 q(s)g(s,x_n(s),y_n(s),y_n'(s))ds, \end{aligned} \tag{4.2.15}$$

which implies that

$$x_n(0) = \frac{b_2}{a_2}\frac{1}{n} + \frac{b_2}{a_2}\int_0^1 q(s)g(s,x_n(s),y_n'(s))ds.$$

Using $(\mathcal{H}_{31})$ and (4.2.12), we obtain

$$x_n(0) \geq \frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_5L_5}(s)(x_n(s))^{\gamma_1}(y_n(s))^{\delta_1}ds \geq (x_n(0))^{\gamma_1}(y_n(0))^{\delta_1}\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_5L_5}(s)ds,$$

which implies that

$$x_n(0) \geq (y_n(0))^{\frac{\delta_1}{1-\gamma_1}}\left(\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_5L_5}(s)ds\right)^{\frac{1}{1-\gamma_1}}. \tag{4.2.16}$$

Similarly, using $(\mathcal{H}_{31})$ and (4.2.12), we obtain

$$y_n(0) \geq (x_n(0))^{\frac{\delta_2}{1-\gamma_2}}\left(\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_5L_5}(s)ds\right)^{\frac{1}{1-\gamma_2}}. \tag{4.2.17}$$

Now, using (4.2.17) in (4.2.16), we have

$$x_n(0) \geq (x_n(0))^{\frac{\delta_1\delta_2}{(1-\gamma_1)(1-\gamma_2)}}\left(\frac{b_1}{a_1}\int_0^1 p(s)\varphi_{M_5L_5}(s)ds\right)^{\frac{\delta_1}{(1-\gamma_1)(1-\gamma_2)}}\left(\frac{b_2}{a_2}\int_0^1 q(s)\psi_{M_5L_5}(s)ds\right)^{\frac{1}{1-\gamma_1}}.$$

Hence,

$$x_n(0) \geq C_9. \tag{4.2.18}$$

Similarly, using (4.2.16) in (4.2.17), we obtain

$$y_n(0) \geq C_{10}. \tag{4.2.19}$$

Now, from (4.2.15), it follows that

$$x_n'(t) \geq \int_t^1 p(s)f(s,x_n(s),y_n(s),x_n'(s))ds.$$

and using $(\mathcal{H}_{31})$, (4.2.12), (4.2.18) and (4.2.19), we obtain (4.2.13).

Similarly, we can prove (4.2.14).

Now, using (4.2.11), $(\mathcal{H}_{28})$, (4.2.12), (4.2.13) and (4.2.14), we have

$$\begin{aligned}
0 \le -x_n''(t) &\le h_1(M_5)k_1(M_5)p(t)(u_1(C_9^{\gamma_1}C_{10}^{\delta_1}\int_t^1 p(s)\varphi_{M_5L_5}(s)ds) + v_1(L_5)), \quad t \in (0,1),\\
0 \le -y_n''(t) &\le h_2(M_5)k_2(M_5)q(t)(u_2(C_9^{\gamma_2}C_{10}^{\delta_2}\int_t^1 q(s)\psi_{M_5L_5}(s)ds) + v_2(L_5)), \quad t \in (0,1).
\end{aligned} \tag{4.2.20}$$

In view of (4.2.12), (4.2.20), $(\mathcal{H}_{26})$ and $(\mathcal{H}_{32})$, it follows that the sequences $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j = 0,1)$ are uniformly bounded and equicontinuous on $[0,1]$. Hence, by Theorem (1.1.6), there exist subsequences $\{(x_{n_k}^{(j)}, y_{n_k}^{(j)})\}$ $(j = 0,1)$ of $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j = 0,1)$ and $(x,y) \in C^1[0,1]\times C^1[0,1]$ such that $(x_{n_k}^{(j)}, y_{n_k}^{(j)})$ converges uniformly to $(x^{(j)}, y^{(j)})$ on $[0,1]$ $(j = 0,1)$. Also, $a_2x(0) - b_2y'(0) = a_1y(0) - b_1x'(0) = x'(1) = y'(1) = 0$. Moreover, from (4.2.13) and (4.2.14), with $n_k$ in place of $n$ and taking $\lim_{n_k\to+\infty}$, we have

$$\begin{aligned}
x'(t) &\ge C_9^{\gamma_1}C_{10}^{\delta_1}\int_t^1 p(s)\varphi_{M_5L_5}(s)ds,\\
y'(t) &\ge C_9^{\gamma_2}C_{10}^{\delta_2}\int_t^1 q(s)\psi_{M_5L_5}(s)ds,
\end{aligned}$$

which shows that $x' > 0$ and $y' > 0$ on $[0,1)$, $x > 0$ and $y > 0$ on $[0,1]$. Further, $(x_{n_k}, y_{n_k})$ satisfy

$$\begin{aligned}
x_{n_k}'(t) &= x_{n_k}'(0) - \int_0^t p(s)f(s, x_{n_k}(s), y_{n_k}(s), x_{n_k}'(s))ds, \quad t \in [0,1],\\
y_{n_k}'(t) &= y_{n_k}'(0) - \int_0^t q(s)g(s, x_{n_k}(s), y_{n_k}(s), y_{n_k}'(s))ds, \quad t \in [0,1].
\end{aligned}$$

Passing to the limit as $n_k \to \infty$, we obtain

$$\begin{aligned}
x'(t) &= x'(0) - \int_0^t p(s)f(s, x(s), y(s), x'(s))ds, \quad t \in [0,1],\\
y'(t) &= y'(0) - \int_0^t q(s)g(s, x(s), y(s), y'(s))ds, \quad t \in [0,1],
\end{aligned}$$

which implies that

$$\begin{aligned}
-x''(t) &= p(t)f(t, x(t), y(t), x'(t)), \quad t \in (0,1),\\
-y''(t) &= q(t)g(t, x(t), y(t), y'(t)), \quad t \in (0,1).
\end{aligned}$$

Hence, $(x,y)$ is a $C^1$-positive solution of the system of BVPs (4.0.2). □

**Example 4.2.2.** Consider the following coupled system of SBVPs

$$\begin{aligned}
-x''(t) &= \nu^{\beta_1+1}(x(t))^{\gamma_1}(y(t))^{\delta_1}(x'(t))^{-\beta_1}, \quad t \in (0,1),\\
-y''(t) &= \nu^{\beta_2+1}(x(t))^{\gamma_2}(y(t))^{\delta_2}(y'(t))^{-\beta_2}, \quad t \in (0,1),\\
x(0) - y'(0) &= y(0) - x'(0) = x'(1) = y'(1) = 0,
\end{aligned} \tag{4.2.21}$$

where $0 \leq \gamma_1, \gamma_2, \delta_1, \delta_2 < 1$ satisfying $(1-\gamma_1)(1-\gamma_2) \neq \delta_1\delta_2$, $0 < \beta_1 < 1$, $0 < \beta_2 < 1$ and $\nu > 0$ such that

$$\nu < \sup_{c\in(0,\infty)} \frac{c}{2\sum_{i=1}^{2}(\beta_i+1)^{\frac{1}{\beta_i+1}} c^{\frac{\gamma_i+\delta_i}{\beta_i+1}}}.$$

Take $p(t) = q(t) = 1$, $h_1(x) = \nu^{\beta_1+1}x^{\gamma_1}$, $h_2(x) = \nu^{\beta_2+1}x^{\gamma_2}$, $k_1(x) = x^{\delta_1}$, $k_2(x) = x^{\delta_2}$, $u_1(x) = x^{-\beta_1}$, $u_2(x) = x^{-\beta_2}$ and $v_1(x) = v_2(x) = 0$. Then, $I(\nu) = \frac{\nu^{\beta_1+1}}{\beta_1+1}$, $J(\nu) = \frac{\nu^{\beta_2+1}}{\beta_2+1}$, $I^{-1}(\nu) = (\beta_1+1)^{\frac{1}{\beta_1+1}}\nu^{\frac{1}{\beta_1+1}}$ and $J^{-1}(\nu) = (\beta_2+1)^{\frac{1}{\beta_2+1}}z^{\frac{1}{\beta_2+1}}$.

Choose $\varphi_{ML}(t) = L^{-\beta_1}$, $\psi_{ML}(t) = L^{-\beta_2}$. Then,

$$\sup_{c\in(0,\infty)} \frac{c}{(1+\frac{b_1}{a_1})I^{-1}(h_1(c)k_1(c)\int_0^1 p(t)dt) + (1+\frac{b_2}{a_2})J^{-1}(h_2(c)k_2(c)\int_0^1 q(t)dt)} =$$
$$\sup_{c\in(0,\infty)} \frac{c}{2\nu\sum_{i=1}^{2}(\beta_i+1)^{\frac{1}{\beta_i+1}} c^{\frac{\gamma_i+\delta_i}{\beta_i+1}}} > 1.$$

Clearly, $(\mathcal{H}_{26})-(\mathcal{H}_{32})$ are satisfied. Hence, by Theorem 4.2.1, the system of BVPs (4.2.21) has at least one $C^1$-positive solution.

# Chapter 5

# Singular Systems with Sign–Changing Nonlinear Functions on Finite and Infinite Intervals

BVPs defined on a half–line frequently occur in the study of radially symmetric solutions of nonlinear elliptic PDEs [49, 57, 64, 78]. Moreover, second–order BVPs on infinite intervals model many physical phenomenon such as unsteady flow of gasses through semi–infinite porous media [84]; mass transfer on a rotating disc in non–Newtonian fluids [109]; heat transfer in radial flow between parallel circular discs [109]; phase change of solids with temperature dependent thermal conductivity [109]; plasma physics [2, 59]; electrical potential in an isolated neutral atom [2, 34] and so on. In all these applications, positive solutions are meaningful only. Recently, the theory on existence of solutions to nonlinear BVPs on unbounded domain has attracted the attention of many authors, see for example [5, 47, 50, 76, 91, 97, 102, 130] and the references therein. For BVPs defined on half–line, an excellent resource is produced by Agarwal and O'Regan [5] that have been received considerable attentions.

Agarwal and O'Regan [3, Section 2.10] have developed the method of upper and lower solutions for the following two–point BVP

$$\begin{aligned} -y''(t) &= q(t)f(t, y(t), y'(t)), \quad t \in (0, 1), \\ y(0) &= y(1) = 0, \end{aligned} \tag{5.0.1}$$

where $f : [0, 1] \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and singular at $y = 0$ and the function $q \in C(0, 1)$ is positive on $(0, 1)$. Further, they have presented the method of upper and lower solutions for more general problems in [114]. The main idea in both is to approximate

(5.0.1) as a sequence of non–singular problems such that each element of the sequence has a lower solution $\alpha_n$ or $\rho_n$ and an upper solution $\beta_n$.

In [3, Section 2.15], they studied the existence of positive solutions to the following BVP

$$\begin{aligned} -x''(t) &= \phi(t)f(t,x(t)), \quad t \in (0,\infty), \\ x(0) &= 0,\ \lim_{t\to\infty} x'(t) = 0, \end{aligned} \tag{5.0.2}$$

where $f(t,x)$ is singular at $x = 0$. Further, in [5, Section 1.11] they establish the existence results for (5.0.2) when $f$ includes first derivative also. In [102], Ma studied existence of $C^1[0,\infty)$ positive solutions to the BVP

$$\begin{aligned} -y''(t) &= g(t,y(t),y'(t)),\ \text{for a.e. } t \in (0,\infty), \\ y(0) &= 0,\ y \text{ is bounded on } [0,\infty), \end{aligned} \tag{5.0.3}$$

under certain growth on the nonlinear function $g$. However in [36, 127], it was assumed that the nonlinear functions are positive which lead to a concave solution.

In Section 5.1, we study the existence of $C^1$-positive solutions for the following system of SBVPs [23],

$$\begin{aligned} -x''(t) &= p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t \in (0,1), \\ -y''(t) &= p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t \in (0,1), \\ x(0) &= x(1) = y(0) = y(1) = 0, \end{aligned} \tag{5.0.4}$$

where $f_1, f_2 : [0,1]\times(0,\infty)\times(0,\infty)\times\mathbb{R} \to \mathbb{R}$ are continuous. Moreover, $f_1$, $f_2$ are allowed to change sign and may be singular at $x = 0$, $y = 0$. Also, $p_1, p_2 \in C(0,1)$ are positive on $(0,1)$.

Further in Sections 5.2 and 5.3, we study the existence of $C^1$-positive solutions to the following coupled systems of ODEs

$$\begin{aligned} -x''(t) &= p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t \in \mathbb{R}_0^+, \\ -y''(t) &= p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t \in \mathbb{R}_0^+, \end{aligned} \tag{5.0.5}$$

subject to the following set of BCs

$$x(0) = y(0) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} x'(t) = 0 \tag{5.0.6}$$

and

$$\begin{aligned} a_1x(0) - b_1x'(0) &= \lim_{t\to\infty} x'(t) = 0, \\ a_2y(0) - b_2y'(0) &= \lim_{t\to\infty} y'(t) = 0, \end{aligned} \tag{5.0.7}$$

where the functions $f_i : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}_0 \to \mathbb{R}$ are continuous and allowed to change sign. Further, the nonlinear functions $f_i$ $(i = 1,2)$ are allowed to be singular at $x' = 0$ and

$y' = 0$. Also, $p_i \in C(\mathbb{R}_0^+)$, $p_i(i = 1, 2) > 0$ on $\mathbb{R}_0^+$ and the constants $a_i, b_i(i = 1, 2) > 0$; here $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}_0^+ = \mathbb{R}^+ \setminus \{0\}$.

To the best of our knowledge, the existence of positive solutions to the systems of BVPs defined on infinite intervals with nonlinear functions explicitly dependent on first derivative have never been studied previously. Since an infinite interval is noncompact, the study of BVPs on infinite intervals is much more complicated. Therefore to prove existence results for a system on an infinite interval, first, we establish the existence of positive solution to systems of BVPs posed on finite intervals. Then, we apply diagonalization argument and establish existence result for the system on an infinite domain. The results of Sections 5.2 and 5.3 are submitted [21, 22].

## 5.1 Existence of at least one $C^1$-positive solutions

In this section, we establish existence of at least one $C^1$-positive solution of the system of BVPs (5.0.4). By a $C^1$-positive solution to the system of BVPs (5.0.4), we means $(x, y) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1))$ satisfying (5.0.4), $x > 0$ and $y > 0$ on $(0, 1)$.

Let $\{\rho_n\}_{n=1}^{\infty}$ be a nonincreasing sequence of real constants such that $\lim_{n\to\infty} \rho_n = 0$. Assume that the following holds:

($\mathcal{H}_{33}$) hypothesis ($\mathcal{H}_{10}$) of Chapter 3, page 42, with $p = p_1$ and $q = p_2$;

($\mathcal{H}_{34}$) $f_i : [0, 1] \times (0, \infty) \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous, $i = 1, 2$;

($\mathcal{H}_{35}$) there exist $(\beta_1, \beta_2) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1))$ and $n_0 \in \{1, 2, \cdots\}$ such that $\beta_1(t) \geq \rho_{n_0}$, $\beta_2(t) \geq \rho_{n_0}$ for $t \in [0, 1]$ and

$$-\beta_1''(t) \geq p_1(t)f_1(t, \beta_1(t), \beta_2(t), \beta_1'(t)), \quad t \in (0, 1),$$
$$-\beta_2''(t) \geq p_2(t)f_2(t, \beta_1(t), \beta_2(t), \beta_2'(t)), \quad t \in (0, 1);$$

($\mathcal{H}_{36}$) there exist $(\alpha_1, \alpha_2) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1))$ with $\alpha_1(0) = \alpha_1(1) = \alpha_2(0) = \alpha_2(1) = 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ on $(0, 1)$ such that for $(t, x, y) \in (0, 1) \times \{x \in (0, \infty) : x < \alpha_1(t)\} \times \{y \in (0, \infty) : y \leq \beta_2(t)\}$,

$$-\alpha_1''(t) < p_1(t)f_1(t, x, y, \alpha_1'(t)),$$

for $(t, x, y) \in (0, 1) \times \{x \in (0, \infty) : x \leq \beta_1(t)\} \times \{y \in (0, \infty) : y < \alpha_2(t)\}$,

$$-\alpha_2''(t) < p_2(t)f_2(t, x, y, \alpha_2'(t));$$

($\mathcal{H}_{37}$) for each $n \in \{n_0, n_0 + 1, \cdots\}$, $0 \leq t \leq 1$, $\rho_n \leq x \leq \beta_1(t)$, $\rho_n \leq y \leq \beta_2(t)$, we have $f_1(t, \rho_n, y, 0) \geq 0$ and $f_2(t, x, \rho_n, 0) \geq 0$;

($\mathcal{H}_{38}$) $|f_i(t,x,y,z)| \leq (h_i(x)+k_i(x))(u_i(y)+v_i(y))\psi_i(|z|)$, where $h_i, u_i > 0$ are continuous and nonincreasing on $(0,\infty)$, $k_i, v_i \geq 0$, $\psi_i > 0$ are continuous on $[0,\infty)$ with $\frac{k_i}{h_i}$, $\frac{v_i}{u_i}$ nondecreasing on $(0,\infty)$, $i = 1,2$;

($\mathcal{H}_{39}$) $\int_0^1 p_i(t)h_i(\alpha_1(t))u_i(\alpha_2(t))dt < +\infty$, $i = 1,2$;

($\mathcal{H}_{40}$)

$$\int_0^\infty \frac{du}{\psi_i(u)} > \left[1+\frac{k_i(b_1)}{h_i(b_1)}\right]\left[1+\frac{v_i(b_2)}{u_i(b_2)}\right]\int_0^1 p_i(t)h_i(\alpha_1(t))u_i(\alpha_2(t))dt,$$

where $b_i = \max\{\beta_i(t) : t \in [0,1]\}$, $i = 1,2$.

**Theorem 5.1.1.** *Under the hypothesis* ($\mathcal{H}_{33}$) $-$ ($\mathcal{H}_{40}$), *the system of BVPs* (5.0.4) *has at least one $C^1$-positive solution.*

*Proof.* In view of ($\mathcal{H}_{40}$), we choose a real constant $M > 0$ such that

$$M > \max\big\{\max_{t\in[0,1]}|\alpha_1'(t)|, \max_{t\in[0,1]}|\alpha_2'(t)|, \max_{t\in[0,1]}|\beta_1'(t)|, \max_{t\in[0,1]}|\beta_2'(t)|\big\}, \tag{5.1.1}$$

and

$$\int_0^M \frac{du}{\psi_i(u)} > \left[1+\frac{k_i(b_1)}{h_i(b_1)}\right]\left[1+\frac{v_i(b_2)}{u_i(b_2)}\right]\int_0^1 p_1(t)h_i(\alpha_1(t))u_i(\alpha_2(t))dt,\ i = 1,2. \tag{5.1.2}$$

Define a retraction $z^* : \mathbb{R} \to [-M,M]$ by $z^*(z) = \max\{-M, \min\{z,M\}\}$. Define a radial retraction $r : \mathbb{R} \to [-1,1]$ by

$$r(x) = \begin{cases} x, & |x| \leq 1, \\ \frac{x}{|x|}, & |x| > 1. \end{cases}$$

For each fixed $n \in \{n_0, n_0+1, \cdots\}$, we construct the modification of $f_i$ ($i = 1,2$) as follows:

$$f_1^*(t,x,y,z) = \begin{cases} f_1(t,\beta_1(t),\beta_2(t),z^*(z)) + r(\beta_1(t)-x), & x \geq \beta_1(t), y \geq \beta_2(t), \\ f_1(t,x,\beta_2(t),z^*(z)), & \rho_n \leq x \leq \beta_1(t), y \geq \beta_2(t), \\ f_1(t,\beta_1(t),y,z^*(z)) + r(\beta_1(t)-x), & x \geq \beta_1(t), \rho_n \leq y \leq \beta_2(t), \\ f_1(t,x,y,z^*(z)), & \rho_n \leq x \leq \beta_1(t), \rho_n \leq y \leq \beta_2(t), \\ f_1(t,\rho_n,y,z^*(z)) + r(\rho_n - x), & x < \rho_n, \rho_n \leq y \leq \beta_2(t), \\ f_1(t,x,\rho_n,z^*(z)), & \rho_n \leq x \leq \beta_1(t), y < \rho_n, \\ f_1(t,\rho_n,\rho_n,z^*(z)) + r(\rho_n - x), & x < \rho_n, y < \rho_n, \end{cases} \tag{5.1.3}$$

$$f_2^*(t,x,y,z) = \begin{cases} f_2(t,\beta_1(t),\beta_2(t),z^*(z)) + r(\beta_2(t)-y), & x \geq \beta_1(t), y \geq \beta_2(t), \\ f_2(t,x,\beta_2(t),z^*(z)) + r(\beta_2(t)-y), & \rho_n \leq x \leq \beta_1(t), y \geq \beta_2(t), \\ f_2(t,\beta_1(t),y,z^*(z)), & x \geq \beta_1(t), \rho_n \leq y \leq \beta_2(t), \\ f_2(t,x,y,z^*(z)), & \rho_n \leq x \leq \beta_1(t), \rho_n \leq y \leq \beta_2(t), \\ f_2(t,\rho_n,y,z^*(z)), & x < \rho_n, \rho_n \leq y \leq \beta_2(t), \\ f_2(t,x,\rho_n,z^*(z)) + r(\rho_n - y), & \rho_n \leq x \leq \beta_1(t), y < \rho_n, \\ f_2(t,\rho_n,\rho_n,z^*(z)) + r(\rho_n - y), & x < \rho_n, y < \rho_n. \end{cases} \tag{5.1.4}$$

We note that $f_1^*$ and $f_2^*$ are continuous and bounded on $[0,1]\times\mathbb{R}^3$. Consider the modified system of BVPs

$$\begin{aligned} -x''(t) &= p_1(t) f_1^*(t,x(t),y(t),x'(t)), \quad t \in (0,1), \\ -y''(t) &= p_2(t) f_2^*(t,x(t),y(t),y'(t)), \quad t \in (0,1), \\ x(0) &= x(1) = y(0) = y(1) = \rho_n. \end{aligned} \tag{5.1.5}$$

By Theorem 1.2.4, it follows that the system of BVPs (5.1.5) has a solution $(x_n, y_n) \in (C^1[0,1]\cap C^2(0,1)) \times (C^1[0,1]\cap C^2(0,1))$. Now, we show that

$$x_n(t) \geq \rho_n, \; y_n(t) \geq \rho_n, \quad t \in [0,1]. \tag{5.1.6}$$

First, we prove $x_n(t) \geq \rho_n$ for $t \in [0,1]$. Suppose $x_n - \rho_n$ has a negative absolute minimum at some $t_0 \in (0,1)$. Then, $x_n'(t_0) = 0$ and $x_n''(t_0) \geq 0$. However, in view of (5.1.5) and (5.1.3), we have

$$\begin{aligned} -x_n''(t_0) =& p_1(t_0) f_1^*(t_0, x_n(t_0), y_n(t_0), 0) \\ =& \begin{cases} p_1(t_0)(f_1(t_0,\rho_n,y_n(t_0),0) + r(\rho_n - x_n(t_0))), & \rho_n \leq y_n(t_0) \leq \beta_2(t_0), \\ p_1(t_0)(f_1(t_0,\rho_n,\rho_n,0) + r(\rho_n - x_n(t_0))), & y_n(t_0) < \rho_n, \end{cases} \end{aligned}$$

which shows that $x_n''(t_0) < 0$, a contradiction. Hence, $x_n(t) \geq \rho_n$ for $t \in [0,1]$.

Similarly, we can show that $y_n(t) \geq \rho_n$ for $t \in [0,1]$.

Now, we show that

$$x_n(t) \leq \beta_1(t), \; y_n(t) \leq \beta_2(t), \quad t \in [0,1]. \tag{5.1.7}$$

First, we prove $x_n(t) \leq \beta_1(t)$ for $t \in [0,1]$. Suppose, $x_n - \beta_1$ has a positive absolute maximum at some $t_1 \in (0,1)$. Then, $x_n'(t_1) = \beta_1'(t_1)$ and $x_n''(t_1) \leq \beta_1''(t_1)$. But, in view of (5.1.5) and (5.1.3), we obtain

$$\begin{aligned} -x_n''(t_1) =& p_1(t_1) f_1^*(t_1, x_n(t_1), y_n(t_1), x_n'(t_1)) \\ =& \begin{cases} p_1(t_1)(f_1(t_1,\beta_1(t_1),\beta_2(t_1),\beta_1'(t_1)) + r(\beta_1(t_1) - x_n(t_1))), & y_n(t_1) \geq \beta_2(t_1) \\ p_1(t_1)(f_1(t_1,\beta_1(t_1),y_n(t_1),\beta_1'(t_1)) + r(\beta_1(t_1) - x_n(t_1))), & \rho_n \leq y_n(t_1) \leq \beta_2(t_1), \end{cases} \end{aligned}$$

which shows that $x_n''(t_1) > \beta_1''(t_1)$, a contradiction. Thus, $x_n(t) \leq \beta_1(t)$ for $t \in [0,1]$.

Similarly, we can show that $y_n(t) \leq \beta_2(t)$ for $t \in [0,1]$.

From (5.1.6) and (5.1.7), we have

$$\rho_n \leq x_n(t) \leq \beta_1(t),\ \rho_n \leq y_n(t) \leq \beta_2(t), \quad t \in [0,1]. \tag{5.1.8}$$

Now, we show that

$$x_n(t) \geq \alpha_1(t),\ y_n(t) \geq \alpha_2(t), \quad t \in [0,1]. \tag{5.1.9}$$

First, we show that $x_n(t) \geq \alpha_1(t)$ for $t \in [0,1]$. Suppose, $x_n - \alpha_1$ has a negative absolute minimum at some $t_2 \in (0,1)$. Then, $x_n'(t_2) = \alpha_1'(t_2)$ and $x_n''(t_2) \geq \alpha_1''(t_2)$. From (5.1.5), (5.1.8), (5.1.1) and $(\mathcal{H}_{36})$, it follows that

$$\begin{aligned}-x_n''(t_2) =& p_1(t_2)f_1^*(t_2, x_n(t_2), y_n(t_2), x_n'(t_2))\\ =& p_1(t_2)f_1(t_2, x_n(t_2), y_n(t_2), \alpha_1'(t_2)) > -\alpha_1''(t_2),\end{aligned}$$

a contradiction. Thus, $x_n(t) \geq \alpha_1(t)$ for $t \in [0,1]$.

Similarly, we can show that $y_n(t) \geq \alpha_2(t)$ for $t \in [0,1]$.

From (5.1.7) and (5.1.9), it follows that

$$\alpha_1(t) \leq x_n(t) \leq \beta_1(t),\ \alpha_2(t) \leq y_n(t) \leq \beta_2(t), \quad t \in [0,1]. \tag{5.1.10}$$

Next, we show that

$$|x_n'(t)| \leq M,\ |y_n'(t)| \leq M, \quad t \in [0,1]. \tag{5.1.11}$$

First, we prove $|x_n'(t)| \leq M$ for $t \in [0,1]$. Since $x_n(0) = \rho_n$ and $x_n(1) = \rho_n$. So, there exist $t_3 \in (0,1)$ such that $x_n'(t_3) = 0$. Suppose there exist $t_* \in [0,1]$ such that $x_n'(t_*) > M$. Either we have $t_* \in [0,t_3]$ or $t_* \in [t_3,1]$. If $t_* \in [0,t_3]$, then there exist a maximal interval $[t_4,t_5]$ in $[0,t_3]$, containing $t_*$ such that $x_n'(t) \geq 0$ on $[t_4,t_5]$ and $x_n'(t_5) = 0$. Let $M_1 = \max\{x_n'(t) : t \in [t_4,t_5]\} = x_n'(t_6) > M$. Then, $x_n''(t) \leq 0$ for $t \in [t_6,t_5]$. For $t \in [t_6,t_5]$, using (5.1.5), (5.1.8), (5.1.10) and $(\mathcal{H}_{38})$, we obtain

$$\begin{aligned}-x_n''(t) &= p_1(t)|f_1^*(t, x_n(t), y_n(t), x_n'(t))|\\ &= p_1(t)|f_1(t, x_n(t), y_n(t), z^*(x_n'(t)))|\\ &\leq \left[1 + \frac{k_1(x_n(t))}{h_1(x_n(t))}\right]\left[1 + \frac{v_1(y_n(t))}{u_1(y_n(t))}\right] p_1(t)h_1(x_n(t))u_1(y_n(t))\psi_1(|z^*(x_n'(t))|)\\ &\leq \left[1 + \frac{k_1(b_1)}{h_1(b_1)}\right]\left[1 + \frac{v_1(b_2)}{u_1(b_2)}\right] p_1(t)h_1(\alpha_1(t))u_1(\alpha_2(t))\psi_1(|z^*(x_n'(t))|),\end{aligned}$$

which implies that

$$\frac{-x_n''(t)}{\psi_1(|z^*(x_n'(t))|)} \leq \left[1 + \frac{k_1(b_1)}{h_1(b_1)}\right]\left[1 + \frac{v_1(b_2)}{u_1(b_2)}\right] p_1(t)h_1(\alpha_1(t))u_1(\alpha_2(t)).$$

Integrating from $t_6$ to $t_5$, we obtain

$$\int_{t_6}^{t_5} \frac{-x_n''(t)dt}{\psi_1(|z^*(x_n'(t))|)} \le \left[1+\frac{k_1(b_1)}{h_1(b_1)}\right]\left[1+\frac{v_1(b_2)}{u_1(b_2)}\right]\int_{t_6}^{t_5} p_1(t)h_1(\alpha_1(t))u_1(\alpha_2(t))dt,$$

which implies that

$$\int_0^{M_1} \frac{du}{\psi_1(z^*(u))} \le \left[1+\frac{k_1(b_1)}{h_1(b_1)}\right]\left[1+\frac{v_1(b_2)}{u_1(b_2)}\right]\int_0^1 p_1(t)h_1(\alpha_1(t))u_1(\alpha_2(t))dt,$$

which can also be written as

$$\int_0^{M} \frac{du}{\psi_1(u)} + \frac{M_1-M}{\psi_1(M)} \le \left[1+\frac{k_1(b_1)}{h_1(b_1)}\right]\left[1+\frac{v_1(b_2)}{u_1(b_2)}\right]\int_0^1 p_1(t)h_1(\alpha_1(t))u_1(\alpha_2(t))dt,$$

a contradiction to (5.1.2). Similarly, for $t_* \in [t_3, 1]$ we can show a similar contradiction. Thus, $x_n'(t) \le M$ for $t \in [0,1]$. Similarly, we can prove that $x_n'(t) \ge -M$ for $t \in [0,1]$. Hence, $|x_n'(t)| \le M$ for $t \in [0,1]$.

In a similar way, we can show that $|y_n'(t)| \le M$ for $t \in [0,1]$.

Thus, in view of (5.1.5), (5.1.8), (5.1.11), (5.1.3) and (5.1.4), $(x_n, y_n)$ is a solution of the following coupled system of BVPs

$$\begin{aligned}-x''(t) &= p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t\in(0,1),\\ -y''(t) &= p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t\in(0,1),\\ x(0) &= x(1)=y(0)=y(1)=\rho_n.\end{aligned} \tag{5.1.12}$$

Using (5.1.12), $(\mathcal{H}_{38})$, (5.1.10) and (5.1.11), we obtain

$$\begin{aligned}|x_n''(t)| &\le \left[1+\frac{k_1(b_1)}{h_1(b_1)}\right]\left[1+\frac{v_1(b_2)}{u_1(b_2)}\right]\left[\max_{z\in[0,M]}\psi_1(z)\right]p_1(t)h_1(\alpha_1(t))u_1(\alpha_2(t)), \quad t\in(0,1),\\ |y_n''(t)| &\le \left[1+\frac{k_2(b_1)}{h_2(b_1)}\right]\left[1+\frac{v_2(b_2)}{u_2(b_2)}\right]\left[\max_{z\in[0,M]}\psi_2(z)\right]p_2(t)h_2(\alpha_1(t))u_2(\alpha_2(t)), \quad t\in(0,1).\end{aligned} \tag{5.1.13}$$

In view of (5.1.10), (5.1.11), (5.1.13) and $(\mathcal{H}_{39})$, it follows that the sequences $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j=0,1)$ are uniformly bounded and equicontinuous on $[0,1]$. Hence, by Theorem 1.1.6, there exist subsequences $\{(x_{n_k}^{(j)}, y_{n_k}^{(j)})\}$ $(j=0,1)$ of $\{(x_n^{(j)}, y_n^{(j)})\}$ $(j=0,1)$ and $(x,y)\in C^1[0,1]\times C^1[0,1]$ such that $(x_{n_k}^{(j)}, y_{n_k}^{(j)})$ converges uniformly to $(x^{(j)}, y^{(j)})$ on $[0,1]$ $(j=0,1)$. Also, $x(0)=x(1)=y(0)=y(1)=0$, $\alpha_1(t)\le x(t)\le \beta_1(t)$, $\alpha_2(t)\le y(t)\le\beta_2(t)$, $|x'(t)|\le M$ and $|y'(t)|\le M$ for $t\in[0,1]$. Further, $(x_{n_k}, y_{n_k})$ satisfy

$$\begin{aligned}x_{n_k}'(t) &= x_{n_k}'(1/2) + \int_t^{1/2} p_1(s)f_1(s,x_{n_k}(s),y_{n_k}(s),x_{n_k}'(s))ds, \quad t\in[0,1],\\ y_{n_k}'(t) &= y_{n_k}'(1/2) + \int_t^{1/2} p_2(s)f_2(s,x_{n_k}(s),y_{n_k}(s),y_{n_k}'(s))ds, \quad t\in[0,1].\end{aligned}$$

Passing to the limit as $n_k \to \infty$, we obtain

$$x'(t) = x'(1/2) + \int_t^{1/2} p_1(s) f_1(s, x(s), y(s), x'(s))ds, \quad t \in [0,1],$$
$$y'(t) = y'(1/2) + \int_t^{1/2} p_2(s) f_2(t, x(s), y(s), y'(s))ds, \quad t \in [0,1],$$

which implies that

$$-x''(t) = p_1(t) f_1(t, x(t), y(t), x'(t)), \quad t \in (0,1),$$
$$-y''(t) = p_2(t) f_2(t, x(t), y(t), y'(t)), \quad t \in (0,1).$$

Hence, $(x, y)$ is a $C^1$-positive solution of the system of BVPs (5.0.4). □

**Example 5.1.2.** Let

$$p_i(t) = \frac{1}{t^{1/4}(1-t)^{1/4}}, \; i = 1, 2,$$
$$f_1(t, x, y, z) = (x^{-1/3} - C)\{y^{-1/3} - \big(C^{-3} + \frac{1}{2n_0}\big)^{-1/3} + 1\}(1 - z),$$
$$f_2(t, x, y, z) = \{x^{-1/3} - \big(C^{-3} + \frac{1}{2n_0}\big)^{-1/3} + 1\}(y^{-1/3} - C)(1 - z),$$

where $C$ is a positive real constant. Choose a fixed $n_0 \in \{1, 2, \cdots\}$ such that $n_0 \geq C^3$. Let $\rho_n = \frac{1}{n+n_0}$ and $\beta_1(t) = \beta_2(t) = C^{-3} + \frac{1}{2n_0}$. Clearly $\beta_1(t) \geq \rho_{n_0}$ and $\beta_2(t) \geq \rho_{n_0}$ for $t \in [0,1]$. Now,

$$\begin{aligned} p_1(t) f_1(t, \beta_1(t), \beta_2(t), \beta_1'(t)) + \beta_1''(t) =& \frac{1}{t^{1/4}(1-t)^{1/4}}((C^{-3} + \frac{1}{2n_0})^{-1/3} - C) \\ \leq& \frac{1}{t^{1/4}(1-t)^{1/4}}(C - C) = 0, \end{aligned}$$

Similarly, $p_2(t) f_2(t, \beta_1(t), \beta_2(t), \beta_2'(t)) + \beta_2''(t) \leq 0$. Consequently, $(\mathcal{H}_{35})$ is satisfied.

Let $\alpha_1(t) = \alpha_2(t) = \nu t(1-t)$, where $\nu > 0$ satisfying $(2^{2/3}\nu^{-1/3} - C)(1-\nu) + 2^{1/2}\nu \geq 0$. Then, for $(t, x, y) \in (0,1) \times \{x \in (0, \infty) : x < \alpha_1(t)\} \times \{y \in (0, \infty) : y \leq \beta_2(t)\}$, we have

$$\begin{aligned} p_1(t) f_1(t, x, y, \alpha_1'(t)) + \alpha_1''(t) =& \frac{1}{t^{1/4}(1-t)^{1/4}}(x^{-1/3} - C)\{y^{-1/3} - \big(C^{-3} + \frac{1}{2n_0}\big)^{-1/3} + 1\} \\ & (1 - \nu + 2\nu t) + 2\nu \\ \geq& 2^{1/2}(x^{-1/3} - C)(1 - \nu + 2\nu t) + 2\nu \\ >& 2^{1/2}(2^{2/3}\nu^{-1/3} - C)(1 - \nu) + 2\nu \geq 0. \end{aligned}$$

Similarly, for $(t, x, y) \in (0,1) \times \{x \in (0, \infty) : x \leq \beta_1(t)\} \times \{y \in (0, \infty) : y < \alpha_2(t)\}$, we have

$$p_2(t) f_2(t, x, y, \alpha_2'(t)) + \alpha_2''(t) < 0,$$

which shows that $(\mathcal{H}_{36})$ is satisfied.

For each $n \in \{n_0, n_0+1, \cdots\}$, $0 \le t \le 1$ and $\rho_n \le y \le \beta_2(t)$, we have

$$\begin{aligned} f_1(t, \rho_n, y, 0) =& (\rho_{n_0}^{-1/3} - C)(y^{-1/3} - \big(C^{-3} + \frac{1}{2n_0}\big)^{-1/3} + 1) \\ \ge& ((n+n_0)^{1/3} - C) \ge (n_0^{1/3} - C) \ge 0. \end{aligned}$$

Similarly, for each $n \in \{n_0, n_0+1, \cdots\}$, $0 \le t \le 1$, and $\rho_n \le x \le \beta_1(t)$, we have $f_2(t, x, \rho_n, 0) \ge 0$, that is, $(\mathcal{H}_{37})$ is satisfied. Choose, $h_1(x) = h_2(x) = u_1(x) = u_2(x) = x^{-1/3}$, $k_1(x) = v_2(x) = C$, $k_2(x) = v_1(x) = (C^{-3} + \frac{1}{2n_0})^{-1/3} + 1$ and $\psi_1(z) = \psi_2(z) = 1 + z$. Also,

$$\int_0^1 p_i(t) h_i(\alpha_1(t)) u_i(\alpha_2(t)) dt = \frac{[\Gamma(1/12)]^2}{\nu^{2/3}\Gamma(1/6)}, \ i = 1, 2,$$

which shows that $(\mathcal{H}_{39})$ also holds. Clearly, $(\mathcal{H}_{33}) - (\mathcal{H}_{40})$ are satisfied. Hence, by Theorem 5.1.1, the system of BVPs (5.0.4) has at least one $C^1$-positive solution.

## 5.2 Singular systems of BVPs on infinite intervals

In this section, we establish the existence of $C^1$-positive solutions for the system of BVPs (5.0.5), (5.0.6). We say, $(x, y) \in (C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}_0^+)) \times (C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}_0^+))$ is a $C^1$-positive solution of the system of BVPs (5.0.5), (5.0.6), if $(x, y)$ satisfies (5.0.5) and (5.0.6), $x > 0$ and $y > 0$ on $\mathbb{R}_0^+$, $x' > 0$ and $y' > 0$ on $\mathbb{R}^+$. Here, we study existence of positive solutions under weaker hypothesis as compared to the results studied in Chapters 3 and 4.

Assume that the following holds:

$(\mathcal{H}_{41})$ $p_i \in C(\mathbb{R}_0^+)$, $p_i > 0$ on $\mathbb{R}_0^+$, $\int_0^\infty p_i(t)dt < +\infty$, $i = 1, 2$;

$(\mathcal{H}_{42})$ $f_i : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}_0 \to \mathbb{R}$ is continuous, $i = 1, 2$;

$(\mathcal{H}_{43})$ $|f_i(t, x, y, z)| \le h_i(|x|) k_i(|y|)(u_i(|z|) + v_i(|z|))$, where $u_i > 0$ is continuous and nonincreasing on $\mathbb{R}_0^+$, $h_i$, $k_i$, $v_i \ge 0$ are continuous and nondecreasing on $\mathbb{R}^+$, $i = 1, 2$;

$(\mathcal{H}_{44})$ there exist a constant $M > 0$ such that $\frac{M}{\omega(M)} > 1$, where $\omega(M) = \lim_{\varepsilon \to 0} \omega_\varepsilon(M)$,

$$\begin{aligned} \omega_\varepsilon(M) =& \sum_{i=1}^{2} \int_0^\infty [J_i^{-1}\big(h_i(M)k_i(M)\int_t^\infty p_i(s)ds + J_i(\varepsilon)\big)]dt \\ &+ \sum_{i=1}^{2} J_i^{-1}\big(h_i(M)k_i(M)\int_0^\infty p_i(s)ds + J_i(\varepsilon)\big), \\ J_i(\mu) =& \int_0^\mu \frac{d\tau}{u_i(\tau) + v_i(\tau)}, \ \text{for } \mu > 0, \ i = 1, 2; \end{aligned}$$

$(\mathcal{H}_{45})$ hypothesis $(\mathcal{H}_{14})$ of Chapter 3, page 42, with $I = J_1$ and $J = J_2$;

$(\mathcal{H}_{46})$ $f_i$ is positive on $\mathbb{R}^+ \times (0, M]^3$, $i = 1, 2$;

($\mathcal{H}_{47}$) there exist continuous functions $\varphi_M$ and $\psi_M$ defined on $\mathbb{R}^+$ and positive on $\mathbb{R}_0^+$, and constants $0 \leq \gamma_1, \gamma_2, \delta_1, \delta_2 < 1$ satisfying $(1-\gamma_1)(1-\gamma_2) \neq \delta_1\delta_2$, such that $f_1(t,x,y,z) \geq \varphi_M(t)x^{\gamma_1}y^{\delta_1}$ and $f_2(t,x,y,z) \geq \psi_M(t)x^{\gamma_2}y^{\delta_2}$ on $\mathbb{R}^+ \times [0,M]^3$.

### 5.2.1 Existence of positive solutions on finite intervals

Choose $m \in N_0 \setminus \{0\}$, where $N_0 := \{0,1,\cdots\}$, and consider the following system of BVPs on finite interval

$$\begin{aligned} -x''(t) &= p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t \in (0,m), \\ -y''(t) &= p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t \in (0,m), \\ x(0) &= y(0) = x'(m) = y'(m) = 0. \end{aligned} \tag{5.2.1}$$

First, we show that system of BVPs (5.2.1) has a $C^1$-positive solution. We say, $(x,y) \in (C^1[0,m] \cap C^2(0,m)) \times (C^1[0,m] \cap C^2(0,m))$, a $C^1$-positive solution of the system of BVPs (5.2.1), if $(x,y)$ satisfies (5.2.1), $x > 0$ and $y > 0$ on $(0,m]$, $x' > 0$ and $y' > 0$ on $[0,m)$.

**Theorem 5.2.1.** *Under the hypothesis* ($\mathcal{H}_{41}$) $-$ ($\mathcal{H}_{47}$)*, the system of BVPs* (5.2.1) *has at least one $C^1$-positive solution.*

*Proof.* In view of ($\mathcal{H}_{44}$), we choose $\varepsilon > 0$ small enough such that

$$\frac{M}{\omega_\varepsilon(M)} > 1. \tag{5.2.2}$$

Choose $n_0 \in \{1,2,\cdots\}$ such that $\frac{1}{n_0} < \varepsilon$. For each $n \in N := \{n_0, n_0+1, \cdots\}$, define retractions $\theta : \mathbb{R} \to [0,M]$ and $\rho : \mathbb{R} \to [\frac{1}{n}, M]$ as

$$\theta(x) = \max\{0, \min\{x,M\}\} \text{ and } \rho(x) = \max\{\frac{1}{n}, \min\{x,M\}\}.$$

Consider the modified system of BVPs

$$\begin{aligned} -x''(t) &= p_1(t)f_1^*(t,x(t),y(t),x'(t)), \quad t \in (0,m), \\ -y''(t) &= p_2(t)f_2^*(t,x(t),y(t),x'(t)), \quad t \in (0,m), \\ x(0) &= y(0) = 0,\ x'(m) = y'(m) = \frac{1}{n}, \end{aligned} \tag{5.2.3}$$

where $f_1^*(t,x,y,x') = f_1(t,\theta(x),\theta(y),\rho(x'))$ and $f_2^*(t,x,y,y') = f_2(t,\theta(x),\theta(y),\rho(y'))$. Clearly, $f_i^*$ $(i=1,2)$ are continuous and bounded on $[0,m] \times \mathbb{R}^3$. Hence, by Theorem 1.2.4, the modified system of BVPs (5.2.3) has a solution $(x_{m,n}, y_{m,n}) \in (C^1[0,m] \cap C^2(0,m)) \times (C^1[0,m] \cap C^2(0,m))$.

Using (5.2.3), ($\mathcal{H}_{41}$) and ($\mathcal{H}_{46}$), we obtain

$$x''_{m,n} \leq 0 \text{ and } y''_{m,n} \leq 0 \text{ on } \in (0,m).$$

Integrating from $t$ to $m$ and using the BCs (5.2.3), we obtain

$$x'_{m,n}(t) \geq \frac{1}{n} \text{ and } y'_{m,n}(t) \geq \frac{1}{n} \text{ for } t \in [0,m]. \tag{5.2.4}$$

Integrating (5.2.4) from 0 to $t$, using the BCs (5.2.3), we have

$$x_{m,n}(t) \geq \frac{t}{n} \text{ and } y_{m,n}(t) \geq \frac{t}{n} \text{ for } t \in [0,m]. \tag{5.2.5}$$

From (5.2.4) and (5.2.5), it follows that

$$\|x_{m,n}\|_{7,m} = x_{m,n}(m) \text{ and } \|y_{m,n}\|_{7,m} = y_{m,n}(m), \text{ where } \|u\|_{7,m} = \max_{t\in[0,m]} |u(t)|.$$

Now, we show that the following hold

$$\|x'_{m,n}\|_{7,m} < M \text{ and } \|y'_{m,n}\|_{7,m} < M. \tag{5.2.6}$$

Suppose $x'_{m,n}(t_1) \geq M$ for some $t_1 \in [0,m]$. Using (5.2.3) and $(\mathcal{H}_{43})$, we have

$$-x''_{m,n}(t) \leq p_1(t)h_1(\theta(x_{m,n}(t)))k_1(\theta(y_{m,n}(t)))(u_1(\rho(x'_{m,n}(t))) + v_1(\rho(x'_{m,n}(t)))),\ t \in (0,m),$$

which implies that

$$\frac{-x''_{m,n}(t)}{u_1(\rho(x'_{m,n}(t))) + v_1(\rho(x'_{m,n}(t)))} \leq h_1(M)k_1(M)p_1(t), \quad t \in (0,m).$$

Integrating from $t_1$ to $m$, using the BCs (5.2.3), we obtain

$$\int_{\frac{1}{n}}^{x'_{m,n}(t_1)} \frac{dz}{u_1(\rho(z)) + v_1(\rho(z))} \leq h_1(M)k_1(M)\int_{t_1}^{m} p_1(t)dt,$$

which can also be written as

$$\int_{\frac{1}{n}}^{M} \frac{dz}{u_1(z) + v_1(z)} + \int_{M}^{x'_{m,n}(t_1)} \frac{dz}{u_1(M) + v_1(M)} \leq h_1(M)k_1(M)\int_{0}^{\infty} p_1(t)dt.$$

Using the increasing property of $J_1$, we obtain

$$J_1(M) + \frac{x'_{m,n}(t_1) - M}{u_1(M) + v_1(M)} \leq h_1(M)k_1(M)\int_0^\infty p_1(t)dt + J_1(\varepsilon),$$

and the increasing property of $J_1^{-1}$ yields

$$M \leq J_1^{-1}(h_1(M)k_1(M)\int_0^\infty p_1(t)dt + J_1(\varepsilon)) \leq \omega_\varepsilon(M),$$

a contradiction to (5.2.2). Hence, $\|x'_{m,n}\|_{7,m} < M$.

Similarly, we can show that $\|y'_{m,n}\|_{7,m} < M$.

Now, we show that

$$\|x_{m,n}\|_{7,m} < M \text{ and } \|y_{m,n}\|_{7,m} < M. \tag{5.2.7}$$

Suppose $\|x_{m,n}\|_{7,m} \geq M$. From (5.2.3), (5.2.4), (5.2.6) and ($\mathcal{H}_{43}$), it follows that

$$-x''_{m,n}(t) \leq p_1(t)h_1(\theta(x_{m,n}(t)))k_1(\theta(y_{m,n}(t)))(u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))),$$

which implies that

$$\frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))} \leq h_1(M)k_1(M)p_1(t), \quad t \in (0, m).$$

Integrating from $t$ to $m$, using the BCs (5.2.3), we obtain

$$\int_{\frac{1}{n}}^{x'_{m,n}(t)} \frac{dz}{u_1(z) + v_1(z)} \leq h_1(M)k_1(M) \int_t^m p_1(s)ds, \quad t \in [0, m],$$

which can also be written as

$$J_1(x'_{m,n}(t)) - J_1(\frac{1}{n}) \leq h_1(M)k_1(M) \int_t^\infty p_1(s)ds, \quad t \in [0, m].$$

The increasing property of $J_1$ and $J_1^{-1}$, leads to

$$x'_{m,n}(t) \leq J_1^{-1}(h_1(M)k_1(M) \int_t^\infty p_1(s)ds + J_1(\varepsilon)), \quad t \in [0, m].$$

Now, integrating from 0 to $m$, using the BCs (5.2.3), we obtain

$$M \leq \|x_{m,n}\|_{7,m} \leq \int_0^m [J_1^{-1}(h_1(M)k_1(M) \int_t^\infty p_1(s)ds + J_1(\varepsilon))]dt,$$

which implies that

$$M \leq \int_0^\infty [J_1^{-1}(h_1(M)k_1(M) \int_t^\infty p_1(s)ds + J_1(\varepsilon))]dt \leq \omega_\varepsilon(M),$$

a contradiction to (5.2.2). Therefore, $\|x_{m,n}\|_{7,m} < M$.

Similarly, we can show that $\|y_{m,n}\|_{7,m} < M$.

Hence, in view of (5.2.3)–(5.2.7), $(x_{m,n}, y_{m,n})$ is a solution of the following coupled system of BVPs

$$\begin{aligned} -x''(t) &= p_1(t)f_1(t, x(t), y(t), x'(t)), \quad t \in (0, m), \\ -y''(t) &= p_2(t)f_2(t, x(t), y(t), y'(t)), \quad t \in (0, m), \\ x(0) &= y(0) = 0, \; x'(m) = y'(m) = \frac{1}{n}, \end{aligned} \tag{5.2.8}$$

satisfying

$$\begin{aligned} &\frac{t}{n} \leq x_{m,n}(t) < M, \; \frac{1}{n} \leq x'_{m,n}(t) < M, \quad t \in [0, m], \\ &\frac{t}{n} \leq y_{m,n}(t) < M, \; \frac{1}{n} \leq y'_{m,n}(t) < M, \quad t \in [0, m]. \end{aligned} \tag{5.2.9}$$

Now, we show that

$$\{x'_{m,n}\}_{n\in N} \text{ and } \{y'_{m,n}\}_{n\in N} \text{ are equicontinuous on } [0, m]. \tag{5.2.10}$$

From (5.2.8), (5.2.9) and $(\mathcal{H}_{43})$, it follows that

$$\begin{aligned}-x''_{m,n}(t) \leq& p_1(t)h_1(M)k_1(M)(u_1(x'_{m,n}(t))+v_1(x'_{m,n}(t))), \quad t\in(0,m),\\ -y''_{m,n}(t) \leq& p_2(t)h_2(M)k_2(M)(u_2(y'_{m,n}(t))+v_2(y'_{m,n}(t))), \quad t\in(0,m),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t))+v_1(x'_{m,n}(t))} &\leq h_1(M)k_1(M)p_1(t), \quad t\in(0,m),\\ \frac{-y''_{m,n}(t)}{u_2(y'_{m,n}(t))+v_2(y'_{m,n}(t))} &\leq h_2(M)k_2(M)p_2(t), \quad t\in(0,m).\end{aligned}$$

Thus for $t_1, t_2 \in [0,m]$, we have

$$\begin{aligned}|J_1(x'_{m,n}(t_1)) - J_1(x'_{m,n}(t_2))| &\leq h_1(M)k_1(M)\left|\int_{t_1}^{t_2} p_1(t)dt\right|,\\ |J_2(y'_{m,n}(t_1)) - J_2(y'_{m,n}(t_2))| &\leq h_2(M)k_2(M)\left|\int_{t_1}^{t_2} p_2(t)dt\right|.\end{aligned} \tag{5.2.11}$$

In view of (5.2.11), $(\mathcal{H}_{41})$, uniform continuity of $J_i^{-1}$ over $[0, J_i(M)]$ $(i=1,2)$ and

$$\begin{aligned}|x'_{m,n}(t_1) - x'_{m,n}(t_2)| &= |J_1^{-1}(J_1(x'_{m,n}(t_1))) - J_1^{-1}(J_1(x'_{m,n}(t_2)))|,\\ |y'_{m,n}(t_1) - y'_{m,n}(t_2)| &= |J_2^{-1}(J_2(y'_{m,n}(t_1))) - J_2^{-1}(J_2(y'_{m,n}(t_2)))|,\end{aligned}$$

we obtain (5.2.10).

From (5.2.9) and (5.2.10), it follows that the sequences $\{(x^{(j)}_{m,n}, y^{(j)}_{m,n})\}_{n\in N}$ $(j=0,1)$ are uniformly bounded and equicontinuous on $[0,m]$. Hence, by Theorem (1.1.6), there exist subsequence $N_*$ of $N$ and $(x_m, y_m) \in C^1[0,m]\times C^1[0,m]$ such that for each $j=0,1$, the sequences $(x^{(j)}_{m,n}, y^{(j)}_{m,n})$ converges uniformly to $(x^{(j)}_m, y^{(j)}_m)$ on $[0,m]$ as $n\to\infty$ through $N_*$. From the BCs (5.2.8), we have $x_m(0) = y_m(0) = x'_m(m) = y'_m(m) = 0$. Next, we show that $x_m > 0$ and $y_m > 0$ on $(0,m]$, $x'_m > 0$ and $y'_m > 0$ on $[0,m)$.

We claim that

$$x_{m,n}(t) \geq C_{11}^{\gamma_1}C_{12}^{\delta_1}\int_0^{\min\{t,1\}} \tau^{1+\gamma_1+\delta_1}p_1(\tau)\varphi_M(\tau)d\tau \equiv \Phi_M(t), \ \ t\in[0,m], \tag{5.2.12}$$

$$y_{m,n}(t) \geq C_{11}^{\gamma_2}C_{12}^{\delta_2}\int_0^{\min\{t,1\}} \tau^{1+\gamma_2+\delta_2}p_2(\tau)\psi_M(\tau)d\tau \equiv \Psi_M(t), \ \ t\in[0,m], \tag{5.2.13}$$

$$x'_{m,n}(t) \geq \int_t^m p_1(s)\varphi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1}ds, \quad t\in[0,m], \tag{5.2.14}$$

$$y'_{m,n}(t) \geq \int_t^m p_2(s)\psi_M(s)(\Phi_M(s))^{\gamma_2}(\Psi_M(s))^{\delta_2}ds, \quad t\in[0,m], \tag{5.2.15}$$

where

$$C_{11} = \left(\int_0^1 \tau^{1+\gamma_1+\delta_1}p_1(\tau)\varphi_M(\tau)d\tau\right)^{\frac{1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}}\left(\int_0^1 \tau^{1+\gamma_2+\delta_2}p_2(\tau)\psi_M(\tau)d\tau\right)^{\frac{\delta_1}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}},$$

$$C_{12} = \left(\int_0^1 \tau^{1+\gamma_1+\delta_1}p_1(\tau)\varphi_M(\tau)d\tau\right)^{\frac{\delta_2}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}}\left(\int_0^1 \tau^{1+\gamma_2+\delta_2}p_2(\tau)\psi_M(\tau)d\tau\right)^{\frac{1-\gamma_1}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}}.$$

First we prove (5.2.12). Let $z(t) = x_{m,n}(t) - tx_{m,n}(1)$ for $t \in [0, 1]$. Then, $z(0) = z(1) = 0$, $z''(t) \leq 0$ for $t \in [0, 1]$. So, $z(t) \geq 0$ for $t \in [0, 1]$, that is

$$x_{m,n}(t) \geq tx_{m,n}(1), \quad t \in [0, 1]. \tag{5.2.16}$$

Similarly,

$$y_{m,n}(t) \geq ty_{m,n}(1), \quad t \in [0, 1]. \tag{5.2.17}$$

Now, consider the following relation

$$\begin{aligned} x_{m,n}(t) = \frac{t}{n} + \int_0^t sp_1(s)f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds \\ + \int_t^m tp_1(s)f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds, \quad t \in [0, m]. \end{aligned} \tag{5.2.18}$$

In view of $(\mathcal{H}_{47})$, using (5.2.16) and (5.2.17), for $t \in [0, m]$, we have

$$\begin{aligned} x_{m,n}(t) &\geq \int_0^t sp_1(s)f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds \\ &\geq \int_0^{\min\{t,1\}} sp_1(s)f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds \\ &\geq \int_0^{\min\{t,1\}} sp_1(s)(x_{m,n}(s))^{\gamma_1}(y_{m,n}(s))^{\delta_1}\varphi_M(s)ds \\ &\geq (x_{m,n}(1))^{\gamma_1}(y_{m,n}(1))^{\delta_1} \int_0^{\min\{t,1\}} s^{1+\gamma_1+\delta_1}p_1(s)\varphi_M(s)ds, \end{aligned} \tag{5.2.19}$$

which implies that

$$x_{m,n}(1) \geq (y_{m,n}(1))^{\frac{\delta_1}{1-\gamma_1}} \left( \int_0^1 s^{1+\gamma_1+\delta_1}p_1(s)\varphi_M(s)ds \right)^{\frac{1}{1-\gamma_1}}. \tag{5.2.20}$$

Similarly,

$$y_{m,n}(1) \geq (x_{m,n}(1))^{\frac{\delta_2}{1-\gamma_2}} \left( \int_0^1 s^{1+\gamma_2+\delta_2}p_2(s)\psi_M(s)ds \right)^{\frac{1}{1-\gamma_2}}. \tag{5.2.21}$$

Now, using (5.2.21) in (5.2.20), we have

$$\begin{aligned} (x_{m,n}(1))^{1-\frac{\delta_1\delta_2}{(1-\gamma_1)(1-\gamma_2)}} \geq &\left( \int_0^1 s^{1+\gamma_1+\delta_1}p_1(s)\varphi_M(s)ds \right)^{\frac{1}{1-\gamma_1}} \\ &\left( \int_0^1 s^{1+\gamma_2+\delta_2}p_2(s)\psi_M(s)ds \right)^{\frac{\delta_1}{(1-\gamma_1)(1-\gamma_2)}}. \end{aligned}$$

Hence,

$$x_{m,n}(1) \geq C_{11}. \tag{5.2.22}$$

Similarly, using (5.2.20) in (5.2.21), we obtain

$$y_{m,n}(1) \geq C_{12}. \tag{5.2.23}$$

Thus, from (5.2.19), using (5.2.22) and (5.2.23), we get (5.2.12).

Similarly, we can prove (5.2.13).

Now, we prove (5.2.14). From (5.2.18), it follows that

$$x'_{m,n}(t) \geq \int_t^m p_1(s) f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds.$$

Using $(\mathcal{H}_{47})$, (5.2.12) and (5.2.13), we obtain (5.2.14).

Similarly, we can prove (5.2.15).

From (5.2.12)–(5.2.15), passing to the limit $n \to \infty$ through $N_*$, we obtain

$$\begin{aligned} & x_m(t) \geq \Phi_M(t),\ y_m(t) \geq \Psi_M(t), && t \in [0, m], \\ & x'_m(t) \geq \int_t^m p_1(s)\varphi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1} ds, && t \in [0, m], \\ & y'_m(t) \geq \int_t^m p_2(s)\psi_M(s)(\Phi_M(s))^{\gamma_2}(\Psi_M(s))^{\delta_2} ds, && t \in [0, m]. \end{aligned} \tag{5.2.24}$$

Consequently, $x_m > 0$ and $y_m > 0$ on $(0, m]$, $x'_m > 0$ and $y'_m > 0$ on $[0, m)$.

Moreover, $(x_{m,n}, y_{m,n})$ satisfy

$$\begin{aligned} & x'_{m,n}(t) = x'_{m,n}(0) - \int_0^t p_1(s) f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds, \quad t \in [0, m], \\ & y'_{m,n}(t) = y'_{m,n}(0) - \int_0^t p_2(s) f_2(s, x_{m,n}(s), y_{m,n}(s), y'_{m,n}(s))ds, \quad t \in [0, m]. \end{aligned}$$

Letting $n \to \infty$ through $N_*$, we obtain

$$\begin{aligned} & x'_m(t) = x'_m(0) - \int_0^t p_1(s) f_1(s, x_m(s), y_m(s), x'_m(s))ds, \quad t \in [0, m], \\ & y'_m(t) = y'_m(0) - \int_0^t p_2(s) f_2(s, x_m(s), y_m(s), y'_m(s))ds, \quad t \in [0, m], \end{aligned}$$

which imply that

$$\begin{aligned} & -x''_m(t) = p_1(t) f_1(t, x_m(t), y_m(t), x'_m(t)), \quad t \in (0, m), \\ & -y''_m(t) = p_2(t) f_2(t, x_m(t), y_m(t), y'_m(t)), \quad t \in (0, m). \end{aligned} \tag{5.2.25}$$

Hence, $(x_m, y_m)$ is a $C^1$-positive solution of the system of BVPs (5.2.1). □

### 5.2.2 Existence of positive solutions on an infinite interval

**Theorem 5.2.2.** *Under the hypothesis* $(\mathcal{H}_{41}) - (\mathcal{H}_{47})$, *the system of BVPs* (5.0.5), (5.0.6) *has at least one* $C^1$*-positive solution.*

*Proof.* By Theorem 5.2.1, for each $m \in N_0 \setminus \{0\}$, the system of BVPs (5.2.1) has a $C^1$-positive solution $(x_m, y_m)$ defined on $[0, m]$. By applying diagonalization argument we will show that the system of BVPs (5.0.5), (5.0.6) has a $C^1$-positive solution. For this purpose we define a continuous extension $(\overline{x}_m, \overline{y}_m)$ of $(x_m, y_m)$ by

$$\overline{x}_m(t) = \begin{cases} x_m(t), & t \in [0, m], \\ x_m(m), & t \in [m, \infty), \end{cases} \qquad \overline{y}_m(t) = \begin{cases} y_m(t), & t \in [0, m], \\ y_m(m), & t \in [m, \infty). \end{cases} \tag{5.2.26}$$

Clearly, $\overline{x}_m, \overline{y}_m \in C^1[0, \infty)$ and satisfy

$$\begin{aligned} &0 \le \overline{x}_m(t) < M,\ 0 \le \overline{x}'_m(t) < M, \quad t \in [0, \infty), \\ &0 \le \overline{y}_m(t) < M,\ 0 \le \overline{y}'_m(t) < M, \quad t \in [0, \infty). \end{aligned} \tag{5.2.27}$$

We claim that

$$\{\overline{x}'_m\}_{m \in N_0 \setminus \{0\}} \text{ and } \{\overline{y}'_m\}_{m \in N_0 \setminus \{0\}} \text{ are equicontinuous on } [0, 1]. \tag{5.2.28}$$

Using (5.2.25), (5.2.26), (5.2.27) and $(\mathcal{H}_{43})$, we obtain

$$\begin{aligned} -\overline{x}''_m(t) &\le p_1(t)h_1(M)k_1(M)(u_1(\overline{x}'_m(t)) + v_1(\overline{x}'_m(t))), \quad t \in (0, 1), \\ -\overline{y}''_m(t) &\le p_2(t)h_2(M)k_2(M)(u_2(\overline{y}'_m(t)) + v_2(\overline{y}'_m(t))), \quad t \in (0, 1), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{-\overline{x}''_m(t)}{u_1(\overline{x}'_m(t)) + v_1(\overline{x}'_m(t))} &\le h_1(M)k_1(M)p_1(t), \quad t \in (0, 1), \\ \frac{-\overline{y}''_m(t)}{u_2(\overline{y}'_m(t)) + v_2(\overline{y}'_m(t))} &\le h_2(M)k_2(M)p_2(t), \quad t \in (0, 1). \end{aligned}$$

Hence, for $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} |J_1(\overline{x}'_m(t_1)) - J_1(\overline{x}'_m(t_2))| &\le h_1(M)k_1(M)\left|\int_{t_1}^{t_2} p_1(t)dt\right|, \\ |J_2(\overline{y}'_m(t_1)) - J_2(\overline{y}'_m(t_2))| &\le h_2(M)k_2(M)\left|\int_{t_1}^{t_2} p_2(t)dt\right|. \end{aligned} \tag{5.2.29}$$

In view of (5.2.29), $(\mathcal{H}_{41})$, uniform continuity of $J_i^{-1}$ over $[0, J_i(L)]$ $(i = 1, 2)$ and

$$\begin{aligned} |\overline{x}'_m(t_1) - \overline{x}'_m(t_2)| &= |J_1^{-1}(J_1(\overline{x}'_m(t_1))) - J_1^{-1}(J_1(\overline{x}'_m(t_2)))|, \\ |\overline{y}'_m(t_1) - \overline{y}'_m(t_2)| &= |J_2^{-1}(J_2(y'_m(t_1))) - J_2^{-1}(J_2(y'_m(t_2)))|, \end{aligned}$$

we establish (5.2.28).

From (5.2.27) and (5.2.28), it follows that the sequences $\{(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, 1]$. Hence, by Theorem 1.1.6, there exist subsequence $N_1$ of $N_0 \setminus \{0\}$ and $(u_1, v_1) \in C^1[0, 1] \times C^1[0, 1]$ such that for each $j = 0, 1$, the sequence $(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})$ converges uniformly to $(u_1^{(j)}, v_1^{(j)})$ on $[0, 1]$ as $m \to \infty$ through $N_1$. Also from BCs (5.2.1), we have $u_1(0) = v_1(0) = 0$.

Moreover, from (5.2.24) and (5.2.26), for each $m \in N_0 \setminus \{0\}$, we have

$$\begin{aligned}
&\overline{x}_m(t) \geq \Phi_M(t),\ \overline{y}_m(t) \geq \Psi_M(t), && t \in [0,1],\\
&\overline{x}'_m(t) \geq \int_t^1 p_1(s)\varphi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1}ds, && t \in [0,1],\\
&\overline{y}'_m(t) \geq \int_t^1 p_2(s)\psi_M(s)(\Phi_M(s))^{\gamma_2}(\Psi_M(s))^{\delta_2}ds, && t \in [0,1].
\end{aligned}$$

Passing to the limit $m \to \infty$ through $N_1$, we obtain

$$\begin{aligned}
&u_1(t) \geq \Phi_M(t),\ v_1(t) \geq \Psi_M(t), && t \in [0,1],\\
&u'_1(t) \geq \int_t^1 p_1(s)\varphi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1}ds, && t \in [0,1],\\
&v'_1(t) \geq \int_t^1 p_2(s)\psi_M(s)(\Phi_M(s))^{\gamma_2}(\Psi_M(s))^{\delta_2}ds, && t \in [0,1],
\end{aligned}$$

which shows that $u_1 > 0$ and $v_1 > 0$ on $(0,1]$, $u'_1 > 0$ and $v'_1 > 0$ on $[0,1)$.

By the same process as above, we can show that

$$\{\overline{x}'_m\}_{m\in N_1\setminus\{1\}} \text{ and } \{\overline{y}'_m\}_{m\in N_1\setminus\{1\}} \text{ are equicontinuous families on } [0,2]. \tag{5.2.30}$$

Further, in view of (5.2.27) and (5.2.30), it follows that the sequences $\{(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})\}$ $(j = 0,1)$ are uniformly bounded and equicontinuous on $[0,2]$. Hence, by Theorem 1.1.6, there exist subsequence $N_2$ of $N_1 \setminus \{1\}$ and $(u_2, v_2) \in C^1[0,2] \times C^1[0,2]$ such that for each $j = 0,1$, the sequence $(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})$ converges uniformly to $(u_2^{(j)}, v_2^{(j)})$ on $[0,2]$ as $m \to \infty$ through $N_2$. Also from BCs (5.2.1), $u_2(0) = v_2(0) = 0$. Moreover, in view of (5.2.24) and (5.2.26), for each $m \in N_1 \setminus \{1\}$, we have

$$\begin{aligned}
&\overline{x}_m(t) \geq \Phi_M(t),\ \overline{y}_m(t) \geq \Psi_M(t), && t \in [0,2],\\
&\overline{x}'_m(t) \geq \int_t^2 p_1(s)\varphi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1}ds, && t \in [0,2],\\
&\overline{y}'_m(t) \geq \int_t^2 p_2(s)\psi_M(s)(\Phi_M(s))^{\gamma_2}(\Psi_M(s))^{\delta_2}ds, && t \in [0,2].
\end{aligned}$$

Now, the $\lim_{m\to\infty}$ through $N_2$ leads to

$$\begin{aligned}
&u_2(t) \geq \Phi_M(t),\ v_2(t) \geq \Psi_M(t), && t \in [0,2],\\
&u'_2(t) \geq \int_t^2 p_1(s)\varphi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1}ds, && t \in [0,2],\\
&v'_2(t) \geq \int_t^2 p_2(s)\psi_M(s)(\Phi_M(s))^{\gamma_2}(\Psi_M(s))^{\delta_2}ds, && t \in [0,2],
\end{aligned}$$

which shows that $u_2 > 0$ and $v_2 > 0$ on $(0,2]$, $u'_2 > 0$ and $v'_2 > 0$ on $[0,2)$. Note that, $u_2 = u_1$ and $v_2 = v_1$ on $[0,1]$ as $N_2 \subseteq N_1$.

In general, for each $k \in N_0 \setminus \{0\}$, there exists a subsequence $N_k$ of $N_{k-1} \setminus \{k-1\}$ and $(u_k, v_k) \in C^1[0,k] \times C^1[0,k]$ such that $(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})$ converges uniformly to $(u_k^{(j)}, v_k^{(j)})$

$(j = 0, 1)$ on $[0, k]$, as $m \to \infty$ through $N_k$. Also, $u_k(0) = v_k(0) = 0$, $u_k = u_{k-1}$ and $v_k = v_{k-1}$ on $[0, k-1]$ as $N_k \subseteq N_{k-1}$. Moreover,

$$\begin{aligned}
&u_k(t) \geq \Phi_M(t),\ v_k(t) \geq \Psi_M(t), && t \in [0, k],\\
&u_k'(t) \geq \int_t^k p_1(s)\varphi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1}ds, && t \in [0, k],\\
&v_k'(t) \geq \int_t^k p_2(s)\psi_M(s)(\Phi_M(s))^{\gamma_1}(\Psi_M(s))^{\delta_1}ds, && t \in [0, k],
\end{aligned}$$

which shows that $u_k > 0$ and $v_k > 0$ on $(0, k]$, $u_k' > 0$ and $v_k' > 0$ on $[0, k)$.

Define functions $x, y : \mathbb{R}^+ \to \mathbb{R}^+$ as:

For fixed $\tau \in \mathbb{R}_0^+$ and $k \in N_0 \setminus \{0\}$ with $\tau \leq k$, $x(\tau) = u_k(\tau)$ and $y(\tau) = v_k(\tau)$. Then, $x$ and $y$ are well defined as, $x(t) = u_k(t) > 0$ and $y(t) = v_k(t) > 0$ for $t \in (0, k]$. We can do this for each $\tau \in \mathbb{R}_0^+$. Thus, $(x, y) \in C^1(\mathbb{R}^+) \times C^1(\mathbb{R}^+)$ with $x > 0$ and $y > 0$ on $\mathbb{R}_0^+$, $x' > 0$ and $y' > 0$ on $\mathbb{R}^+$.

Now, we show that $(x, y)$ is a solution of system of BVPs (5.0.5), (5.0.6). Choose a fixed $\tau \in \mathbb{R}^+$ and $k \in N_0 \setminus \{0\}$ such that $k \geq \tau$. Then, $(\overline{x}_m(\tau), \overline{y}_m(\tau))$ where $m \in N_k$, satisfy

$$\begin{aligned}
\overline{x}_m'(\tau) &= \overline{x}_m'(0) - \int_0^\tau p_1(s)f_1(s, \overline{x}_m(s), \overline{y}_m(s), \overline{x}_m'(s))ds,\\
\overline{y}_m'(\tau) &= \overline{y}_m'(0) - \int_0^\tau p_2(s)f_2(s, \overline{x}_m(s), \overline{y}_m(s), \overline{y}_m'(s))ds.
\end{aligned}$$

Passing to the limit $m \to \infty$ through $N_k$, we obtain

$$\begin{aligned}
u_k'(\tau) &= u_k'(0) - \int_0^\tau p_1(s)f_1(s, u_k(s), v_k(s), u_k'(s))ds,\\
v_k'(\tau) &= v_k'(0) - \int_0^\tau p_2(s)f_2(s, u_k(s), v_k(s), v_k'(s))ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
x'(\tau) &= x'(0) - \int_0^\tau p_1(s)f_1(s, x(s), y(s), x'(s))ds,\\
y'(\tau) &= y'(0) - \int_0^\tau p_2(s)f_2(s, x(s), y(s), y'(s))ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
-x''(\tau) &= p_1(\tau)f_1(\tau, x(\tau), y(\tau), x'(\tau)),\\
-y''(\tau) &= p_2(\tau)f_2(\tau, x(\tau), y(\tau), y'(\tau)).
\end{aligned}$$

We can do this for each $\tau \in \mathbb{R}^+$. Consequently,

$$\begin{aligned}
-x''(t) &= p_1(t)f_1(t, x(t), y(t), x'(t)), \quad t \in \mathbb{R}_0^+,\\
-y''(t) &= p_2(t)f_2(t, x(t), y(t), y'(t)), \quad t \in \mathbb{R}_0^+.
\end{aligned}$$

Thus, $(x, y) \in C^2(\mathbb{R}_0^+) \times C^2(\mathbb{R}_0^+)$, $x(0) = y(0) = 0$.

It remains to show that

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} y'(t) = 0.$$

First, we show that $\lim_{t\to\infty} x'(t) = 0$. Suppose $\lim_{t\to\infty} x'(t) = \varepsilon_0$, for some $\varepsilon_0 > 0$. Then, $x'(t) \geq \varepsilon_0$ for all $t \in [0, \infty)$. Choose $k \in N_0 \setminus \{0\}$, then for $m \in N_k$, in view of (5.2.26), we have

$$x'(t) = u_k'(t) = \lim_{m\to\infty} \overline{x}_m'(t) = \lim_{m\to\infty} x_m'(t), \quad t \in [0, k],$$

which leads to

$$x'(k) = \lim_{m\to\infty} x_m'(k).$$

Thus for every $\varepsilon > 0$, there exist $m^* \in N_k$ such that $|x_m'(k) - x'(k)| < \varepsilon$ for all $m \geq m^*$. Without loss of generality assume that $m^* = k$, then $|x_k'(k) - x'(k)| < \varepsilon$, that is, $|x'(k)| < \varepsilon$. Which is a contradiction whenever $\varepsilon = \varepsilon_0$. Hence, $\lim_{t\to\infty} x'(t) = 0$. Similarly, we can prove $\lim_{t\to\infty} y'(t) = 0$. Thus, $(x, y)$ is a $C^1$-positive solution of system of BVPs (5.0.5), (5.0.6). □

**Example 5.2.3.** Let

$$f_i(t, x, y, z) = \nu^{\alpha_i+1} e^{-t}(M + 1 - x)(M + 1 - y)|x|^{\gamma_i}|y|^{\delta_i}|z|^{-\alpha_i},\ i = 1, 2,$$

where $\nu > 0$, $M > 0$, $\alpha_i > 0$, $0 \leq \gamma_i, \delta_i < 1$, $i = 1, 2$.

Assume that $(1 - \gamma_1)(1 - \gamma_2) \neq \delta_1\delta_2$ and

$$\nu < \frac{M}{\sum_{i=1}^{2}(\alpha_i + 2)(\alpha_i + 1)^{\frac{1}{\alpha_i+1}}(2M + 1)^{\frac{2}{\alpha_i+1}} M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}}.$$

Taking $p_i(t) = e^{-t}$, $h_i(x) = \nu^{\alpha_i+1}(M + 1 + x)x^{\gamma_i}$, $k_i(y) = (M + 1 + y)y^{\delta_i}$, $u_i(z) = z^{-\alpha_i}$ and $v_i(z) = 0$, $i = 1, 2$. Choose $\varphi_M(t) = \nu^{\alpha_1+1}M^{-\alpha_1}e^{-t}$ and $\psi_M(t) = \nu^{\alpha_2+1}M^{-\alpha_2}e^{-t}$. Then, $J_i(\mu) = \frac{\mu^{\alpha_i+1}}{\alpha_i+1}$ and $J_i^{-1}(\mu) = (\alpha_i + 1)^{\frac{1}{\alpha_i+1}}\mu^{\frac{1}{\alpha_i+1}}$, $i = 1, 2$.

Also,

$$\begin{aligned}
\frac{M}{\omega(M)} &= \frac{M}{\sum_{i=1}^{2}\int_0^\infty J_i^{-1}\big(h_i(M)k_i(M)\int_t^\infty p_i(s)ds\big)dt + \sum_{i=1}^{2} J_i^{-1}\big(h_i(M)k_i(M)\int_0^\infty p_i(s)ds\big)} \\
&= \frac{M}{\sum_{i=1}^{2}\int_0^\infty J_i^{-1}\big(\nu^{\alpha_i+1}(2M+1)^2M^{\gamma_i+\delta_i}e^{-t}\big)dt + \sum_{i=1}^{2} J_i^{-1}\big(\nu^{\alpha_i+1}(2M+1)^2M^{\gamma_i+\delta_i}\big)} \\
&= \frac{M}{\nu\sum_{i=1}^{2}(\alpha_i + 2)(\alpha_i + 1)^{\frac{1}{\alpha_i+1}}(2M + 1)^{\frac{2}{\alpha_i+1}} M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}} > 1.
\end{aligned}$$

Clearly, $(\mathcal{H}_{41}) - (\mathcal{H}_{47})$ are satisfied. Hence, by Theorem 5.2.2, the system of BVPs (5.0.5), (5.0.6) has at least one $C^1$-positive solution.

## 5.3 Singular systems of BVPs on infinite intervals with more general BCs

We say, $(x,y) \in (C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}_0^+)) \times (C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}_0^+))$ is a $C^1$-positive solution of the system of BVPs (5.0.5), (5.0.7), if $(x,y)$ satisfies (5.0.5) and (5.0.7), $x>0$, $y>0$, $x'>0$ and $y'>0$ on $\mathbb{R}^+$.

Assume that

$(\mathcal{H}_{48})$ there exist a constant $M>0$ such that $\frac{M}{\omega(M)} > 1$, where $\omega(M) = \lim_{\varepsilon\to 0}\omega_\varepsilon(M)$,

$$\begin{aligned}\omega_\varepsilon(M) = &\sum_{i=1}^{2}\int_0^\infty [J_i^{-1}\big(h_i(M)k_i(M)\int_t^\infty p_i(s)ds + J_i(\varepsilon)\big)]dt\\ &+\sum_{i=1}^{2}\Big(1+\frac{b_i}{a_i}\Big)J_i^{-1}\big(h_i(M)k_i(M)\int_0^\infty p_i(s)ds + J_i(\varepsilon)\big),\\ J_i(\mu) = &\int_0^\mu \frac{dz}{u_i(z)+v_i(z)},\ \text{ for } \mu>0,\ i=1,2.\end{aligned}$$

### 5.3.1 Existence of positive solutions on finite intervals

Choose $m \in N_0 \setminus \{0\}$, where $N_0 := \{0,1,\cdots\}$, and consider the system of BVPs on finite interval

$$\begin{aligned}&-x''(t) = p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t\in(0,m),\\ &-y''(t) = p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t\in(0,m),\\ &a_1x(0) - b_1x'(0) = x'(m) = 0,\\ &a_2y(0) - b_2y'(0) = y'(m) = 0.\end{aligned} \tag{5.3.1}$$

First we show that the system of BVPs (5.3.1) has a $C^1$-positive solution. We say, $(x,y) \in (C^1[0,m]\cap C^2(0,m)) \times (C^1[0,m]\cap C^2(0,m))$, a $C^1$-positive solution of the system of BVPs (5.3.1), if $(x,y)$ satisfies (5.3.1), $x>0$ and $y>0$ on $[0,m]$, $x'>0$ and $y'>0$ on $[0,m)$.

**Theorem 5.3.1.** *Under the hypothesis* $(\mathcal{H}_{41})-(\mathcal{H}_{43})$ *and* $(\mathcal{H}_{45})-(\mathcal{H}_{48})$*, the system of BVPs* (5.3.1) *has at least one* $C^1$*-positive solution.*

*Proof.* In view of $(\mathcal{H}_{48})$, we choose $\varepsilon>0$ small enough such that

$$\frac{M}{\omega_\varepsilon(M)} > 1. \tag{5.3.2}$$

Choose $n_0 \in \{1,2,\cdots\}$ such that $\frac{1}{n_0} < \varepsilon$. For each $n \in N := \{n_0, n_0+1,\cdots\}$, define retractions $\theta:\mathbb{R}\to[0,M]$ and $\rho:\mathbb{R}\to[\frac{1}{n},M]$ as

$$\theta(x) = \max\{0,\min\{x,M\}\} \text{ and } \rho(x) = \max\{\frac{1}{n},\min\{x,M\}\}.$$

Consider the modified system of BVPs

$$\begin{aligned}
-x''(t) &= p_1(t)f_1^*(t, x(t), y(t), x'(t)), \quad t \in (0, m),\\
-y''(t) &= p_2(t)f_2^*(t, x(t), y(t), x'(t)), \quad t \in (0, m),\\
a_1x(0) - b_1x'(0) &= 0,\ x'(m) = \frac{1}{n},\\
a_2y(0) - b_2y'(0) &= 0,\ y'(m) = \frac{1}{n},
\end{aligned} \tag{5.3.3}$$

where $f_1^*(t, x, y, x') = f_1(t, \theta(x), \theta(y), \rho(x'))$ and $f_2^*(t, x, y, y') = f_2(t, \theta(x), \theta(y), \rho(y'))$. Clearly, $f_i^*$ $(i = 1, 2)$ are continuous and bounded on $[0, m] \times \mathbb{R}^3$. Hence, by Theorem 1.2.4, the modified system of BVPs (5.3.3) has a solution $(x_{m,n}, y_{m,n}) \in (C^1[0, m] \cap C^2(0, m)) \times (C^1[0, m] \cap C^2(0, m))$.

Using (5.3.3), $(\mathcal{H}_{41})$ and $(\mathcal{H}_{46})$, we obtain

$$x''_{m,n} \le 0 \text{ and } y''_{m,n} \le 0 \text{ on } \in (0, m).$$

Integrating from $t$ to $m$ and using the BCs (5.3.3), we obtain

$$x'_{m,n}(t) \ge \frac{1}{n} \text{ and } y'_{m,n}(t) \ge \frac{1}{n} \text{ for } t \in [0, m]. \tag{5.3.4}$$

Integrating (5.3.4) from 0 to $t$, using the BCs (5.3.3) and (5.3.4), we have

$$x_{m,n}(t) \ge (t + \frac{b_1}{a_1})\frac{1}{n} \text{ and } y_{m,n}(t) \ge (t + \frac{b_2}{a_2})\frac{1}{n} \text{ for } t \in [0, m]. \tag{5.3.5}$$

From (5.3.4) and (5.3.5), it follows that

$$\|x_{m,n}\|_{7,m} = x_{m,n}(m) \text{ and } \|y_{m,n}\|_{7,m} = y_{m,n}(m).$$

Now, we show that the following hold

$$\|x'_{m,n}\|_{7,m} < M \text{ and } \|y'_{m,n}\|_{7,m} < M. \tag{5.3.6}$$

Suppose $x'_{m,n}(t_1) \ge M$ for some $t_1 \in [0, m]$. Using (5.3.3) and $(\mathcal{H}_{43})$, we have

$$-x''_{m,n}(t) \le p_1(t)h_1(\theta(x_{m,n}(t)))k_1(\theta(y_{m,n}(t)))(u_1(\rho(x'_{m,n}(t))) + v_1(\rho(x'_{m,n}(t)))),\ t \in (0, m),$$

which implies that

$$\frac{-x''_{m,n}(t)}{u_1(\rho(x'_{m,n}(t))) + v_1(\rho(x'_{m,n}(t)))} \le h_1(M)k_1(M)p_1(t), \quad t \in (0, m).$$

Integrating from $t_1$ to $m$, using the BCs (5.3.3), we obtain

$$\int_{\frac{1}{n}}^{x'_{m,n}(t_1)} \frac{dz}{u_1(\rho(z)) + v_1(\rho(z))} \le h_1(M)k_1(M)\int_{t_1}^{m} p_1(t)dt,$$

which can also be written as

$$\int_{\frac{1}{n}}^{M}\frac{dz}{u_1(z)+v_1(z)}+\int_{M}^{x'_{m,n}(t_1)}\frac{dz}{u_1(M)+v_1(M)}\leq h_1(M)k_1(M)\int_0^\infty p_1(t)dt.$$

Using the increasing property of $J_1$, we obtain

$$J_1(M)+\frac{x'_{m,n}(t_1)-M}{u_1(M)+v_1(M)}\leq h_1(M)k_1(M)\int_0^\infty p_1(t)dt+J_1(\varepsilon),$$

and the increasing property of $J_1^{-1}$ yields

$$M\leq J_1^{-1}(h_1(M)k_1(M)\int_0^\infty p_1(t)dt+J_1(\varepsilon))\leq\omega_\varepsilon(M)$$

a contradiction to (5.3.2). Hence, $\|x'_{m,n}\|_{7,m}<M$.

Similarly, we can show that $\|y'_{m,n}\|_{7,m}<M$.

Now, we show that

$$\|x_{m,n}\|_{7,m}<M \text{ and } \|y_{m,n}\|_{7,m}<M. \tag{5.3.7}$$

Suppose $\|x_{m,n}\|_{7,m}\geq M$. From (5.3.3), (5.3.4), (5.3.6) and ($\mathcal{H}_{43}$), it follows that

$$-x''_{m,n}(t)\leq p_1(t)h_1(\theta(x_{m,n}(t)))k_1(\theta(y_{m,n}(t)))(u_1(x'_{m,n}(t))+v_1(x'_{m,n}(t))),\quad t\in(0,m),$$

which implies that

$$\frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t))+v_1(x'_{m,n}(t))}\leq h_1(M)k_1(M)p_1(t),\quad t\in(0,m).$$

Integrating from $t$ to $m$, using the BCs (5.3.3), we obtain

$$\int_{\frac{1}{n}}^{x'_{m,n}(t)}\frac{dz}{u_1(z)+v_1(z)}\leq h_1(M)k_1(M)\int_t^m p_1(s)ds,\quad t\in[0,m],$$

which can also be written as

$$J_1(x'_{m,n}(t))-J_1(\frac{1}{n})\leq h_1(M)k_1(M)\int_t^\infty p_1(s)ds,\quad t\in[0,m].$$

The increasing property of $J_1$ and $J_1^{-1}$, leads to

$$x'_{m,n}(t)\leq J_1^{-1}(h_1(M)k_1(M)\int_t^\infty p_1(s)ds+J_1(\varepsilon)),\quad t\in[0,m], \tag{5.3.8}$$

Now, integrating from 0 to $m$, using the BCs (5.3.3) and (5.3.8), we obtain

$$\begin{aligned}M\leq\|x_{m,n}\|_{7,m}\leq&\int_0^m[J_1^{-1}(h_1(M)k_1(M)\int_t^\infty p_1(s)ds+J_1(\varepsilon))]dt\\&+\frac{b_1}{a_1}J_1^{-1}(h_1(M)k_1(M)\int_0^\infty p_1(s)ds+J_1(\varepsilon)),\end{aligned}$$

which implies that

$$M \le \int_0^\infty [J_1^{-1}(h_1(M)k_1(M)\int_t^\infty p_1(s)ds + J_1(\varepsilon))]dt$$
$$+ \frac{b_1}{a_1}J_1^{-1}(h_1(M)k_1(M)\int_0^\infty p_1(s)ds + J_1(\varepsilon)) \le \omega_\varepsilon(M),$$

a contradiction to (5.3.2). Therefore, $\|x_{m,n}\|_{7,m} < M$.

Similarly, we can show that $\|y_{m,n}\|_{7,m} < M$.

Hence, in view of (5.3.3)–(5.3.7), $(x_{m,n}, y_{m,n})$ is a solution of the following coupled system of BVPs

$$\begin{aligned} -x''(t) &= p_1(t)f_1(t, x(t), y(t), x'(t)), \quad t \in (0, m),\\ -y''(t) &= p_2(t)f_2(t, x(t), y(t), y'(t)), \quad t \in (0, m),\\ a_1x(0) - b_1x'(0) &= 0,\ x'(m) = \frac{1}{n},\\ a_2y(0) - b_2y'(0) &= 0,\ y'(m) = \frac{1}{n}, \end{aligned} \tag{5.3.9}$$

satisfying

$$\begin{aligned} (t + \frac{b_1}{a_1})\frac{1}{n} \le x_{m,n}(t) < M,\ \frac{1}{n} \le x'_{m,n}(t) < M, \quad t \in [0, m],\\ (t + \frac{b_2}{a_2})\frac{1}{n} \le y_{m,n}(t) < M,\ \frac{1}{n} \le y'_{m,n}(t) < M, \quad t \in [0, m]. \end{aligned} \tag{5.3.10}$$

Now, we show that

$$\{x'_{m,n}\}_{n\in N} \text{ and } \{y'_{m,n}\}_{n\in N} \text{ are equicontinuous on } [0, m]. \tag{5.3.11}$$

From (5.3.9), (5.3.10) and $(\mathcal{H}_{43})$, it follows that

$$\begin{aligned} -x''_{m,n}(t) \le& p_1(t)h_1(M)k_1(M)(u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))), \quad t \in (0, m),\\ -y''_{m,n}(t) \le& p_2(t)h_2(M)k_2(M)(u_2(y'_{m,n}(t)) + v_2(y'_{m,n}(t))), \quad t \in (0, m), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))} \le h_1(M)k_1(M)p_1(t), \quad t \in (0, m),\\ \frac{-y''_{m,n}(t)}{u_2(y'_{m,n}(t)) + v_2(y'_{m,n}(t))} \le h_2(M)k_2(M)p_2(t), \quad t \in (0, m). \end{aligned}$$

Thus for $t_1, t_2 \in [0, m]$, we have

$$\begin{aligned} |J_1(x'_{m,n}(t_1)) - J_1(x'_{m,n}(t_2))| \le h_1(M)k_1(M)\left|\int_{t_1}^{t_2} p_1(t)dt\right|,\\ |J_2(y'_{m,n}(t_1)) - J_2(y'_{m,n}(t_2))| \le h_2(M)k_2(M)\left|\int_{t_1}^{t_2} p_2(t)dt\right|. \end{aligned} \tag{5.3.12}$$

In view of (5.3.12), ($\mathcal{H}_{41}$), uniform continuity of $J_i^{-1}$ over $[0, J_i(M)]$ $(i = 1, 2)$ and

$$|x'_{m,n}(t_1) - x'_{m,n}(t_2)| = |J_1^{-1}(J_1(x'_{m,n}(t_1))) - J_1^{-1}(J_1(x'_{m,n}(t_2)))|,$$
$$|y'_{m,n}(t_1) - y'_{m,n}(t_2)| = |J_2^{-1}(J_2(y'_{m,n}(t_1))) - J_2^{-1}(J_2(y'_{m,n}(t_2)))|,$$

we obtain (5.3.11).

From (5.3.10) and (5.3.11), it follows that the sequences $\{(x^{(j)}_{m,n}, y^{(j)}_{m,n})\}_{n\in N}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, m]$. Hence, by Theorem (1.1.6), there exist subsequence $N_*$ of $N$ and $(x_m, y_m) \in C^1[0, m] \times C^1[0, m]$ such that for each $j = 0, 1$ the sequence $(x^{(j)}_{m,n}, y^{(j)}_{m,n})$ converges uniformly to $(x^{(j)}_m, y^{(j)}_m)$ on $[0, m]$ as $n \to \infty$ through $N_*$. From the BCs (5.3.9), we have $a_1x_m(0) - b_1x'_m(0) = a_2y_m(0) - b_2y'_m(0) = x'_m(m) = y'_m(m) = 0$. Next, we show that $x_m > 0$ and $y_m > 0$ on $[0, m]$, $x'_m > 0$ and $y'_m > 0$ on $[0, m)$.

We claim that

$$x'_{m,n}(t) \geq C_{13}^{\gamma_1} C_{14}^{\delta_1} \int_t^m p_1(s)\varphi_M(s)ds, \quad t \in [0, m], \tag{5.3.13}$$

$$y'_{m,n}(t) \geq C_{13}^{\gamma_2} C_{14}^{\delta_2} \int_t^m p_2(s)\psi_M(s)ds, \quad t \in [0, m], \tag{5.3.14}$$

where

$$C_{13} = \left(\frac{b_1}{a_1}\int_0^1 p_1(s)\varphi_M(s)ds\right)^{\frac{1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}} \left(\frac{b_2}{a_2}\int_0^1 p_2(s)\psi_M(s)ds\right)^{\frac{\delta_1}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}},$$
$$C_{14} = \left(\frac{b_1}{a_1}\int_0^1 p_1(s)\varphi_M(s)ds\right)^{\frac{\delta_2}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}} \left(\frac{b_2}{a_2}\int_0^1 p_2(s)\psi_M(s)ds\right)^{\frac{1-\gamma_1}{(1-\gamma_1)(1-\gamma_2)-\delta_1\delta_2}}.$$

To prove (5.3.13), consider the following relation

$$\begin{aligned} x_{m,n}(t) = (t + \frac{b_1}{a_1})\frac{1}{n} + \frac{1}{a_1}\int_0^t (a_1s + b_1)p_1(s)f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds \\ + \frac{1}{a_1}\int_t^m (a_1t + b_1)p_1(s)f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds, \quad t \in [0, m], \end{aligned} \tag{5.3.15}$$

which implies that

$$x_{m,n}(0) = \frac{b_1}{a_1}\frac{1}{n} + \frac{b_1}{a_1}\int_0^m p_1(s)f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds.$$

Using ($\mathcal{H}_{47}$) and (5.3.10), we obtain

$$\begin{aligned} x_{m,n}(0) \geq &(x_{m,n}(0))^{\gamma_1}(y_{m,n}(0))^{\delta_1}\frac{b_1}{a_1}\int_0^m p_1(s)\varphi_M(s)ds \\ \geq &(x_{m,n}(0))^{\gamma_1}(y_{m,n}(0))^{\delta_1}\frac{b_1}{a_1}\int_0^1 p_1(s)\varphi_M(s)ds, \end{aligned}$$

which implies that

$$x_{m,n}(0) \geq (y_{m,n}(0))^{\frac{\delta_1}{1-\gamma_1}}\left(\frac{b_1}{a_1}\int_0^1 p_1(s)\varphi_M(s)ds\right)^{\frac{1}{1-\gamma_1}}. \tag{5.3.16}$$

Similarly,

$$y_{m,n}(0) \geq (x_{m,n}(0))^{\frac{\delta_2}{1-\gamma_2}} \left( \frac{b_2}{a_2} \int_0^1 p_2(s)\psi_M(s)ds \right)^{\frac{1}{1-\gamma_2}}. \tag{5.3.17}$$

Now, using (5.3.17) in (5.3.16), we have

$$(x_{m,n}(0))^{1-\frac{\delta_1\delta_2}{(1-\gamma_1)(1-\gamma_2)}} \geq \left( \frac{b_1}{a_1} \int_0^1 p_1(s)\varphi_M(s)ds \right)^{\frac{1}{1-\gamma_1}} \left( \frac{b_2}{a_2} \int_0^1 p_2(s)\psi_M(s)ds \right)^{\frac{\delta_1}{(1-\gamma_1)(1-\gamma_2)}}.$$

Hence,

$$x_{m,n}(0) \geq C_{13}. \tag{5.3.18}$$

Similarly, using (5.3.16) in (5.3.17), we obtain

$$y_{m,n}(0) \geq C_{14}. \tag{5.3.19}$$

Now, from (5.3.15), it follows that

$$x'_{m,n}(t) \geq \int_t^m p_1(s) f_1(s, y_{m,n}(s), x'_{m,n}(s))ds.$$

Using ($\mathcal{H}_{46}$), (5.3.10), (5.3.18) and (5.3.19), we obtain (5.3.13).

Similarly, we can prove (5.3.14).

From (5.3.13) and (5.3.14), passing to the limit $n \to \infty$ through $N_*$, we obtain

$$\begin{aligned} x'_m(t) &\geq C_{13}^{\gamma_1} C_{14}^{\delta_1} \int_t^m p_1(s)\varphi_M(s)ds, \quad t \in [0, m], \\ y'_m(t) &\geq C_{13}^{\gamma_2} C_{14}^{\delta_2} \int_t^m p_2(s)\psi_M(s)ds, \quad t \in [0, m]. \end{aligned} \tag{5.3.20}$$

Consequently, $x'_m > 0$, $y'_m > 0$ on $[0, m)$ and $x_m > 0$, $y_m > 0$ on $[0, m]$.

Moreover, $x_{m,n}, y_{m,n}$ satisfy

$$\begin{aligned} x'_{m,n}(t) &= x'_{m,n}(0) - \int_0^t p_1(s) f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s))ds, \quad t \in [0, m], \\ y'_{m,n}(t) &= y'_{m,n}(0) - \int_0^t p_2(s) f_2(s, x_{m,n}(s), y_{m,n}(s), y'_{m,n}(s))ds, \quad t \in [0, m]. \end{aligned}$$

Letting $n \to \infty$ through $N_*$, we obtain

$$\begin{aligned} x'_m(t) &= x'_m(0) - \int_0^t p_1(s) f_1(s, x_m(s), y_m(s), x'_m(s))ds, \quad t \in [0, m], \\ y'_m(t) &= y'_m(0) - \int_0^t p_2(s) f_2(s, x_m(s), y_m(s), y'_m(s))ds, \quad t \in [0, m], \end{aligned}$$

which imply that

$$\begin{aligned} -x''_m(t) &= p_1(t) f_1(t, x_m(t), y_m(t), x'_m(t)), \quad t \in (0, m), \\ -y''_m(t) &= p_2(t) f_2(t, x_m(t), y_m(t), y'_m(t)), \quad t \in (0, m). \end{aligned} \tag{5.3.21}$$

Hence, $(x_m, y_m)$ is a $C^1$-positive solution of (3.4.1). □

### 5.3.2 Existence of positive solutions on an infinite interval

**Theorem 5.3.2.** *Under the hypothesis* $(\mathcal{H}_{41}) - (\mathcal{H}_{43})$ *and* $(\mathcal{H}_{45}) - (\mathcal{H}_{48})$*, the system of BVPs* (5.0.5)*,* (5.0.7) *has at least one* $C^1$*-positive solution.*

*Proof.* By Theorem 5.3.1, for each $m \in N_0 \setminus \{0\}$, the system of BVPs (5.3.1) has a $C^1$-positive solution $(x_m, y_m)$ defined on $[0, m]$. By applying diagonalization argument we will show that the system of BVPs (5.0.5), (5.0.7) has a $C^1$-positive solution. For this we define a continuous extension $(\overline{x}_m, \overline{y}_m)$ of $(x_m, y_m)$ by

$$\overline{x}_m(t) = \begin{cases} x_m(t), & t \in [0, m], \\ x_m(m), & t \in [m, \infty), \end{cases} \qquad \overline{y}_m(t) = \begin{cases} y_m(t), & t \in [0, m], \\ y_m(m), & t \in [m, \infty). \end{cases} \tag{5.3.22}$$

Clearly, $\overline{x}_m, \overline{y}_m \in C^1[0, \infty)$ and satisfy,

$$\begin{aligned} &0 \le \overline{x}_m(t) < M,\ 0 \le \overline{x}'_m(t) < M, \quad t \in [0, \infty), \\ &0 \le \overline{y}_m(t) < M,\ 0 \le \overline{y}'_m(t) < M, \quad t \in [0, \infty). \end{aligned} \tag{5.3.23}$$

We claim that

$$\{\overline{x}'_m\}_{m \in N_0 \setminus \{0\}} \text{ and } \{\overline{y}'_m\}_{m \in N_0 \setminus \{0\}} \text{ are equicontinuous on } [0, 1]. \tag{5.3.24}$$

Using (5.3.21), (5.3.22), (5.3.23) and $(\mathcal{H}_{43})$, we obtain

$$\begin{aligned} -\overline{x}''_m(t) &\le p_1(t) h_1(M) k_1(M) (u_1(\overline{x}'_m(t)) + v_1(\overline{x}'_m(t))), \quad t \in (0, 1), \\ -\overline{y}''_m(t) &\le p_2(t) h_2(M) k_2(M) (u_2(\overline{y}'_m(t)) + v_2(\overline{y}'_m(t))), \quad t \in (0, 1), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{-\overline{x}''_m(t)}{u_1(\overline{x}'_m(t)) + v_1(\overline{x}'_m(t))} &\le h_1(M) k_1(M) p_1(t), \quad t \in (0, 1), \\ \frac{-\overline{y}''_m(t)}{u_2(\overline{y}'_m(t)) + v_2(\overline{y}'_m(t))} &\le h_2(M) k_2(M) p_2(t), \quad t \in (0, 1). \end{aligned}$$

Hence, for $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} |J_1(\overline{x}'_m(t_1)) - J_1(\overline{x}'_m(t_2))| &\le h_1(M) k_1(M) \left| \int_{t_1}^{t_2} p_1(t) dt \right|, \\ |J_2(\overline{y}'_m(t_1)) - J_2(\overline{y}'_m(t_2))| &\le h_2(M) k_2(M) \left| \int_{t_1}^{t_2} p_2(t) dt \right|. \end{aligned} \tag{5.3.25}$$

In view of (5.3.25), $(\mathcal{H}_{41})$, uniform continuity of $J_i^{-1}$ over $[0, J_i(L)]$ $(i = 1, 2)$, and

$$\begin{aligned} |\overline{x}'_m(t_1) - \overline{x}'_m(t_2)| &= |J_1^{-1}(J_1(\overline{x}'_m(t_1))) - J_1^{-1}(J_1(\overline{x}'_m(t_2)))|, \\ |\overline{y}'_m(t_1) - \overline{y}'_m(t_2)| &= |J_2^{-1}(J_2(y'_m(t_1))) - J_2^{-1}(J_2(y'_m(t_2)))|, \end{aligned}$$

we establish (5.3.24).

From (5.3.23) and (5.3.24), it follows that the sequences $\{(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, 1]$. Hence, by Theorem (1.1.6), there exist subsequence $N_1$ of $N_0 \setminus \{0\}$ and $(u_1, v_1) \in C^1[0, 1] \times C^1[0, 1]$ such that for each $j = 0, 1$, the sequence $(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})$ converges uniformly to $(u_1^{(j)}, v_1^{(j)})$ on $[0, 1]$ as $m \to \infty$ through $N_1$. Also from BCs (5.3.1), we have $a_1u_1(0) - b_1u_1'(0) = a_2v_1(0) - b_2v_1'(0) = 0$.

Moreover, from (5.3.20) and (5.3.22), for each $m \in N_0 \setminus \{0\}$, we have

$$\overline{x}_m'(t) \geq C_{13}^{\gamma_1} C_{14}^{\delta_1} \int_t^1 p_1(s)\varphi_M(s)ds, \quad t \in [0, 1],$$
$$\overline{y}_m'(t) \geq C_{13}^{\gamma_2} C_{14}^{\delta_2} \int_t^1 p_2(s)\psi_M(s)ds, \quad t \in [0, 1],$$

as limit $m \to \infty$ through $N_1$, we obtain

$$u_1'(t) \geq C_{13}^{\gamma_1} C_{14}^{\delta_1} \int_t^1 p_1(s)\varphi_M(s)ds, \quad t \in [0, 1],$$
$$v_1'(t) \geq C_{13}^{\gamma_2} C_{14}^{\delta_2} \int_t^1 p_2(s)\psi_M(s)ds, \quad t \in [0, 1],$$

which shows that $u_1' > 0$ and $v_1' > 0$ on $[0, 1)$, $u_1 > 0$ and $v_1 > 0$ on $[0, 1]$.

By the same process as above, we can show that

$$\{\overline{x}_m'\}_{m \in N_1 \setminus \{1\}} \text{ and } \{\overline{y}_m'\}_{m \in N_1 \setminus \{1\}} \text{ are equicontinuous families on } [0, 2]. \tag{5.3.26}$$

Now, in view of (5.3.23) and (5.3.26), it follows that the sequences $\{(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})\}$ $(j = 0, 1)$ are uniformly bounded and equicontinuous on $[0, 2]$. Hence, by Theorem (1.1.6), there exist subsequence $N_2$ of $N_1 \setminus \{1\}$ and $(u_2, v_2) \in C^1[0, 2] \times C^1[0, 2]$ such that for each $j = 0, 1$, the sequence $(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})$ converges uniformly to $(u_2^{(j)}, v_2^{(j)})$ on $[0, 2]$ as $m \to \infty$ through $N_2$. Also, $a_1u_2(0) - b_1u_2'(0) = a_2v_2(0) - b_2v_2'(0) = 0$. Moreover, in view of (5.3.20) and (5.3.22), for each $m \in N_1 \setminus \{1\}$, we have

$$\overline{x}_m'(t) \geq C_{13}^{\gamma_1} C_{14}^{\delta_1} \int_t^2 p_1(s)\varphi_M(s)ds, \quad t \in [0, 2],$$
$$\overline{y}_m'(t) \geq C_{13}^{\gamma_2} C_{14}^{\delta_2} \int_t^2 p_2(s)\psi_M(s)ds, \quad t \in [0, 2].$$

Now, the $\lim_{m \to \infty}$ through $N_2$ leads to

$$u_2'(t) \geq C_{13}^{\gamma_1} C_{14}^{\delta_1} \int_t^2 p_1(s)\varphi_M(s)ds, \quad t \in [0, 2],$$
$$v_1'(t) \geq C_{13}^{\gamma_2} C_{14}^{\delta_2} \int_t^2 p_2(s)\psi_M(s)ds, \quad t \in [0, 2],$$

which shows that $u_2' > 0$ and $v_2' > 0$ on $[0, 2)$, $u_2 > 0$ and $v_2 > 0$ on $[0, 2]$. Note that, $u_2 = u_1$ and $v_2 = v_1$ on $[0, 1]$ as $N_2 \subseteq N_1$.

In general, for each $k \in N_0 \setminus \{0\}$, there exists a subsequence $N_k$ of $N_{k-1} \setminus \{k - 1\}$ and $(u_k, v_k) \in C^1[0, k] \times C^1[0, k]$ such that $(\overline{x}_m^{(j)}, \overline{y}_m^{(j)})$ converges uniformly to $(u_k^{(j)}, v_k^{(j)})$

$(j = 0, 1)$ on $[0, k]$, as $m \to \infty$ through $N_k$. Also, $a_1 u_k(0) - b_1 u_k'(0) = a_2 v_k(0) - b_2 v_k'(0) = 0$, $u_k = u_{k-1}$ and $v_k = v_{k-1}$ on $[0, k-1]$ as $N_k \subseteq N_{k-1}$. Moreover,

$$u_k'(t) \geq C_{13}^{\gamma_1} C_{14}^{\delta_1} \int_t^k p_1(s)\varphi_M(s)ds, \quad t \in [0, k],$$
$$v_k'(t) \geq C_{13}^{\gamma_2} C_{14}^{\delta_2} \int_t^k p_2(s)\psi_M(s)ds, \quad t \in [0, k],$$

which shows that $u_k' > 0$ and $v_k' > 0$ on $[0, k)$, $u_k > 0$ and $v_k > 0$ on $[0, k]$.

Define functions $x, y : \mathbb{R}^+ \to \mathbb{R}^+$ as:

For fixed $\tau \in \mathbb{R}_0^+$ and $k \in N_0 \setminus \{0\}$ with $\tau \leq k$, $x(\tau) = u_k(\tau)$ and $y(\tau) = v_k(\tau)$. Then, $x$ and $y$ are well defined as, $x(t) = u_k(t) > 0$ and $y(t) = v_k(t) > 0$ for $t \in [0, k]$. We can do this for each $\tau \in \mathbb{R}_0^+$. Thus, $(x, y) \in C^1(\mathbb{R}^+) \times C^1(\mathbb{R}^+)$ with $x > 0$, $y > 0$, $x' > 0$ and $y' > 0$ on $\mathbb{R}^+$.

Now, we show that $(x, y)$ is a solution of system of BVPs (5.0.5), (5.0.7). Choose a fixed $\tau \in \mathbb{R}^+$ and $k \in N_0 \setminus \{0\}$ such that $k \geq \tau$. Then, $(\overline{x}_m(\tau), \overline{y}_m(\tau))$ where $m \in N_k$, satisfy

$$\overline{x}_m'(\tau) = \overline{x}_m'(0) - \int_0^\tau p_1(s) f_1(s, \overline{x}_m(s), \overline{y}_m(s), \overline{x}_m'(s))ds,$$
$$\overline{y}_m'(\tau) = \overline{y}_m'(0) - \int_0^\tau p_2(s) f_2(s, \overline{x}_m(s), \overline{y}_m(s), \overline{y}_m'(s))ds.$$

Passing to the limit $m \to \infty$, we obtain

$$u_k'(\tau) = u_k'(0) - \int_0^\tau p_1(s) f_1(s, u_k(s), v_k(s), u_k'(s))ds,$$
$$v_k'(\tau) = v_k'(0) - \int_0^\tau p_2(s) f_2(s, u_k(s), v_k(s), v_k'(s))ds.$$

Hence,

$$x'(\tau) = x'(0) - \int_0^\tau p_1(s) f_1(s, x(s), y(s), x'(s))ds,$$
$$y'(\tau) = y'(0) - \int_0^\tau p_2(s) f_2(s, x(s), y(s), y'(s))ds,$$

which implies that

$$-x''(\tau) = p_1(\tau) f_1(\tau, x(\tau), y(\tau), x'(\tau)),$$
$$-y''(\tau) = p_2(\tau) f_2(\tau, x(\tau), y(\tau), y'(\tau)).$$

We can do this for each $\tau \in \mathbb{R}^+$. Consequently,

$$-x''(t) = p_1(t) f_1(t, x(t), y(t), x'(t)), \quad t \in \mathbb{R}_0^+,$$
$$-y''(t) = p_2(t) f_2(t, x(t), y(t), y'(t)), \quad t \in \mathbb{R}_0^+.$$

Thus, $(x, y) \in C^2(\mathbb{R}_0^+) \times C^2(\mathbb{R}_0^+)$, $a_1x(0) - b_1x'(0) = a_2y(0) - b_2y'(0) = 0$.

It remains to show that

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} y'(t) = 0.$$

First, we show that $\lim_{t\to\infty} x'(t) = 0$. Suppose $\lim_{t\to\infty} x'(t) = \varepsilon_0$, for some $\varepsilon_0 > 0$. Then, $x'(t) \geq \varepsilon_0$ for all $t \in [0, \infty)$. Choose $k \in N_0 \setminus \{0\}$, then for $m \in N_k$, in view of (5.3.22), we have

$$x'(t) = u_k'(t) = \lim_{m\to\infty} \overline{x}_m'(t) = \lim_{m\to\infty} x_m'(t), \quad t \in [0, k],$$

which leads to

$$x'(k) = \lim_{m\to\infty} x_m'(k).$$

Thus for every $\varepsilon > 0$, there exist $m^* \in N_k$ such that $|x_m'(k) - x'(k)| < \varepsilon$ for all $m \geq m^*$. Without loss of generality assume that $m^* = k$, then $|x_k'(k) - x'(k)| < \varepsilon$, that is, $|x'(k)| < \varepsilon$. Which is a contradiction whenever $\varepsilon = \varepsilon_0$. Hence, $\lim_{t\to\infty} x'(t) = 0$. Similarly, we can prove $\lim_{t\to\infty} y'(t) = 0$. Thus, $(x, y)$ is a $C^1$-positive solution of the system of BVPs (5.0.5), (5.0.7). □

**Example 5.3.3.** Let

$$f_i(t, x, y, z) = \nu^{\alpha_i+1} e^{-t}(M + 1 - x)(M + 1 - y)|x|^{\gamma_i}|y|^{\delta_i}|z|^{-\alpha_i},\ i = 1, 2,$$

where $\nu > 0$, $M > 0$, $\alpha_i > 0$, $0 \leq \gamma_i, \delta_i < 1$, $i = 1, 2$.

Assume that $(1 - \gamma_1)(1 - \gamma_2) \neq \delta_1\delta_2$ and

$$\nu < \frac{M}{\sum_{i=1}^2 \left(\frac{b_i}{a_i} + \alpha_i + 2\right)(\alpha_i + 1)^{\frac{1}{\alpha_i+1}}(2M + 1)^{\frac{2}{\alpha_i+1}} M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}}.$$

Taking $p_i(t) = e^{-t}$, $h_i(x) = \nu^{\alpha_i+1}(M + 1 + x)x^{\gamma_i}$, $k_i(y) = (M + 1 + y)y^{\delta_i}$, $u_i(z) = z^{-\alpha_i}$ and $v_i(z) = 0$, $i = 1, 2$. Choose $\varphi_M(t) = \nu^{\alpha_1+1}M^{-\alpha_1}e^{-t}$ and $\psi_M(t) = \nu^{\alpha_2+1}M^{-\alpha_2}e^{-t}$. Then, $J_i(\mu) = \frac{\mu^{\alpha_i+1}}{\alpha_i+1}$ and $J_i^{-1}(\mu) = (\alpha_i + 1)^{\frac{1}{\alpha_i+1}}\mu^{\frac{1}{\alpha_i+1}}$, $i = 1, 2$.

Also,

$$\begin{aligned}\frac{M}{\omega(M)} &= \frac{M}{\sum_{i=1}^2 \int_0^\infty J_i^{-1}(h_i(M)k_i(M)\int_t^\infty p_i(s)ds)dt + \sum_{i=1}^2(1 + \frac{b_i}{a_i})J_i^{-1}(h_i(M)k_i(M)\int_0^\infty p_i(s)ds)} \\ &= \frac{M}{\sum_{i=1}^2 \int_0^\infty J_i^{-1}(\nu^{\alpha_i+1}(2M + 1)^2M^{\gamma_i+\delta_i}e^{-t})dt + \sum_{i=1}^2(1 + \frac{b_i}{a_i})J_i^{-1}(\nu^{\alpha_i+1}(2M + 1)^2M^{\gamma_i+\delta_i})} \\ &= \frac{M}{\nu\sum_{i=1}^2(\frac{b_i}{a_i} + \alpha_i + 2)(\alpha_i + 1)^{\frac{1}{\alpha_i+1}}(2M + 1)^{\frac{2}{\alpha_i+1}}M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}} > 1.\end{aligned}$$

Clearly, $(\mathcal{H}_{41}) - (\mathcal{H}_{43})$ and $(\mathcal{H}_{45}) - (\mathcal{H}_{48})$ are satisfied. Hence, by Theorem 5.3.2, the system of BVPs (5.0.5), (5.0.7) has at least one $C^1$-positive solution.

# Chapter 6

# Concluding Remarks

In Chapter 2, Section 2.2, we studied the system of ODEs (2.0.2) subject to three–point BCs (2.0.4). We have established four different results (Theorem 2.2.2, Theorem 2.2.4, Theorem 2.2.6 and Theorem 2.2.8) for the existence of at least one positive solutions to the system of SBVPs (2.0.2), (2.0.4) under the new hypothesis on the nonlinear functions $f$ and $g$. In Theorem 2.2.2, we provide the existence of at least one positive solution for the system of SBVPs (2.0.2), (2.0.4) under the hypothesis $(\mathcal{H}_1) - (\mathcal{H}_3)$, where $(\mathcal{H}_1)$ is about integrability condition while $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are natural assumptions satisfied by a class of singular nonlinear functions. Our next result, Theorem 2.2.4, is obtained by replacing $(\mathcal{H}_3)$ with $(\mathcal{H}_4)$ in Theorem 2.2.2. Theorem 2.2.6 is obtained by replacing $(\mathcal{H}_2)$ with $(\mathcal{H}_5)$ in Theorem 2.2.2. Moreover, Theorem 2.2.8 can be obtained either by replacing $(\mathcal{H}_2)$ with $(\mathcal{H}_5)$ in Theorem 2.2.4 or by replacing $(\mathcal{H}_3)$ with $(\mathcal{H}_4)$ in Theorem 2.2.6. Further in Section 2.3, Theorem 2.3.2, the existence of positive solutions to the system of ODEs (2.0.3) subject to BCs (2.0.4) is provided under the hypothesis $(\mathcal{H}_6) - (\mathcal{H}_9)$, where the hypothesis $(\mathcal{H}_6)$ is an extension of $(\mathcal{H}_1)$ while $(\mathcal{H}_7)$ corresponds to $(\mathcal{H}_2)$ in two–dimensional case. The hypothesis $(\mathcal{H}_8)$ and $(\mathcal{H}_9)$ are sublinear conditions on the nonlinear functions $f$ and $g$.

In Chapter 3, Section 3.1, we establish the existence results for a coupled system of SBVPs (3.0.2), (3.0.3). In Theorem 3.1.2, we prove the existence of at least one $C^1$-positive solution for the system of SBVPs (3.0.2), (3.0.3) under the hypothesis $(\mathcal{H}_{10}) - (\mathcal{H}_{16})$. The hypothesis $(\mathcal{H}_{10})$ and $(\mathcal{H}_{16})$ are some integrability conditions, $(\mathcal{H}_{11})$ is necessary because, otherwise, positive solution $(x, y)$ will not satisfy the condition $x' > 0$ and $y' > 0$ on $[0, 1)$, and therefore, $(x, y) \notin C^2(0, 1) \cap C^2(0, 1)$, $(\mathcal{H}_{12})$ is a natural assumption when $f(t, x, y)$ and $g(t, x, y)$ have singularity at $y = 0$, $(\mathcal{H}_{13})$ is required to bound the solution, whereas $(\mathcal{H}_{14})$ is necessary for invertibility of the maps $I$ and $J$, and the solution is positive due to $(\mathcal{H}_{15})$. By replacing the hypothesis $(\mathcal{H}_{15})$ and $(\mathcal{H}_{16})$ of Theorem 3.1.2 with $(\mathcal{H}_{17})$ and $(\mathcal{H}_{18})$, and including one more hypothesis $(\mathcal{H}_{19})$, we obtained the existence of at least two positive solutions for the system of ODEs (3.0.2) subject to two–point BCs (3.0.3), that is, Theorem 3.2.2 of Section 3.2. The hypothesis $(\mathcal{H}_{19})$ is required for the existence of at least two solutions. By replacing the hypothesis $(\mathcal{H}_{13})$ and $(\mathcal{H}_{16})$ of Theorem 3.1.2 with $(\mathcal{H}_{20})$ and $(\mathcal{H}_{21})$ we get our next result, that is, Theorem 3.3.1 of Section 3.3, which provide existence of at least one $C^1$-positive solutions to the SBVPs (3.0.2), (3.0.4). Theorem 3.4.2 of Section 3.4 is obtained by replacing the hypothesis $(\mathcal{H}_{13})$ and $(\mathcal{H}_{18})$ of Theorem 3.2.2 with $(\mathcal{H}_{20})$ and $(\mathcal{H}_{22})$, which is a criteria for the existence of at least two $C^1$-positive solutions for the system of SBVPs (3.0.2), (3.0.4).

In Chapter 4, Section 4.1, we discuss the four–point coupled BCs and studied the system of SBVPs (4.0.1). In Theorem 4.1.9, by employing the Guo–Krasnosel'skii fixed point theorem for a completely continuous map on a positive cone, it is shown that the system (4.0.1) has a positive solution under the assumptions $(\mathcal{H}_{23})-(\mathcal{H}_{25})$, where $(\mathcal{H}_{23})$ is about integrability condition while $(\mathcal{H}_{24})$ and $(\mathcal{H}_{25})$ are sublinear conditions on nonlinear functions $f$ and $g$. Moreover in Section 4.2, Theorem 4.2.1, we studied the existence of $C^1$-positive solutions to the system of SBVPs (4.0.2) under the hypothesis $(\mathcal{H}_{10})-(\mathcal{H}_{12})$, $(\mathcal{H}_{14})$, $(\mathcal{H}_{29})$, $(\mathcal{H}_{31})$ and $(\mathcal{H}_{32})$. The hypothesis $(\mathcal{H}_{29})$ is a replacement of $(\mathcal{H}_{20})$ in the case of two–point coupled BCs (4.0.2), $(\mathcal{H}_{31})$ is a generalization of $(\mathcal{H}_{15})$ in case system of SBVPs (4.0.2) while $(\mathcal{H}_{32})$ is nothing but $(\mathcal{H}_{16})$ for $E = M$ and $F = L$.

In Chapter 5, Section 5.1, we develop the notion of upper and lower solutions for the system of SBVPs (5.0.4). Theorem 5.1.1 guarantees the existence of $C^1$-positive solutions for the system (5.0.4) under the hypothesis $(\mathcal{H}_{33}) - (\mathcal{H}_{40})$, where $(\mathcal{H}_{33})$ is equivalent to $(\mathcal{H}_{10})$, $(\mathcal{H}_{34})$ is just a continuity condition on nonlinear functions $f_i$ $(i = 1, 2)$, $(\mathcal{H}_{35})$ and $(\mathcal{H}_{36})$ defines upper and lower solutions, $(\mathcal{H}_{37})$ is a condition about the concavity of solutions, $(\mathcal{H}_{38})$ is a natural assumption when the functions $f$, $g$ are singular with respect to $x = 0$ and $y = 0$, $(\mathcal{H}_{39})$ is about integrability condition and $(\mathcal{H}_{40})$ is desired to bound the derivative of solution. Further in Section 5.2, we establish the existence of $C^1$-positive solutions to the coupled system of ODEs (5.0.5) subject to BCs (5.0.6). Theorem 5.2.2 offer $C^1$-positive solutions to the system of SBVPs (5.0.5), (5.0.6) under the hypothesis $(\mathcal{H}_{41}) - (\mathcal{H}_{47})$, where $(\mathcal{H}_{41})$ is about integrability condition on $p_i$ $(i = 1, 2)$, $(\mathcal{H}_{42})$ and $(\mathcal{H}_{46})$ are weaker than $(\mathcal{H}_{11})$, $(\mathcal{H}_{43})$ is more general than $(\mathcal{H}_{12})$ when nonlinear functions are sign–changing, $(\mathcal{H}_{44})$ is required to bound the solution which is much simpler than $(\mathcal{H}_{13})$, $(\mathcal{H}_{45})$ is just $(\mathcal{H}_{14})$ for $I = J_1$ and $J = J_2$, and $(\mathcal{H}_{47})$ is required to prove that the solution is positive. By replacing the hypothesis $(\mathcal{H}_{44})$ in Theorem 5.2.2 with $(\mathcal{H}_{48})$ we obtain Theorem 5.3.2 of Section 5.3, which is a criteria for the existence of at least one $C^1$-positive solutions to the system of ODEs (5.0.5) subject to BCs (5.0.7).

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