Some generalizations of the Banach fixed point theorem: Single valued and multi-valued mappings

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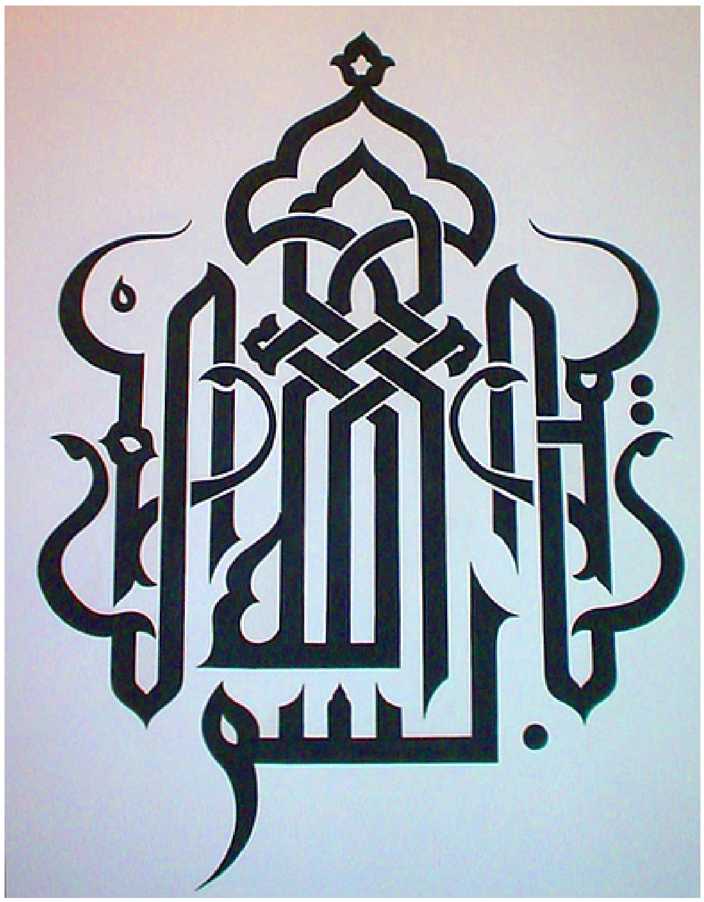

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Declaration

No portion of the work referred to in this dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Dedication

Dedicated to my mother who brought me in this world and to my father who taught me how to deal with it!

Preface

The origin of the fixed point theory, in the later part of the nineteenth century, was basically for the use of successive approximations to establish the existence and uniqueness of solutions, particularly of differential equations.

This method is associated with the names of such celebrated mathematicians as Cauchy, Liouville, Lipschitz, Peano, Fredholm and especially, Picard. Infact the basic ideas of a fixed point theoretic approach are clearly visible in the work of Picard. However, it is the Polish mathematician Banach who is credited with placing the underlying ideas into an abstract framework suitable for broad applications well beyond those for elementary differential and integral equations.

The Banach fixed point theorem [6] states that every contraction f on a complete metric space X has a unique fixed point $\xi \in X$ and starting from any point $x \in X$, the iterative sequence $\{x_{n+1} = f^{n+1}(x) = f(x_n)\}$ converges to the fixed point ξ. It has been noticed that the rate of convergence (Definition 1.1.5) of the iterative sequence $\{f^n(x)\}$ is linear. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a gauge function if it satisfies at least one of the conditions given in Definition 1.1.8. Proinov [49] used the contractive condition:

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \tag{0.0.1}$$

where φ is a gauge function to show that the function $f : D \subset X \to X$ has a fixed point $\xi \in D$ and that the iterative sequence $\{x_{n+1} = f^{n+1}(x) = f(x_n)\}$ converges to the fixed point ξ provided X is complete. In contrast to the Banach fixed point theorem [6] the rate of convergence of the iterative sequence $\{x_{n+1} = f^{n+1}(x) = f(x_n)\}$ in this case is of higher order and not necessarily linear. Furthermore, the Proinov fixed point theorem [49] also gives the error estimates.

Nadler [40] extended the Banach fixed point theorem [6] for a contraction from a complete metric space X into the space of all nonempty closed and bounded subsets of X. Suzuki [61] showed that Mizoguchi-Takahashi's fixed point theorem

[39] is indeed a generalization of Nadler fixed point theorem [40]. Inspired by the results of Mizoguchi-Takahashi [39] and using the ideas of Feng and Liu [19], Klim and Wardowski [34] obtained some fixed point theorems and showed that their results are different from Reich's fixed point theorem [53] and Mizoguchi-Takahashi's fixed point theorem [39]. Pathak and Shahzad [45] introduced a class of functions and generalized some fixed point theorems by Klim and Wardowski [34] by altering distances.

In this thesis, we have established some fixed point theorems for single valued and multi-valued mappings. The results obtained generalize Proinov fixed point theorem [49], Mizoguchi-Takahashi's fixed point theorem [39] and a fixed point theorem by Pathak and Shahzad [45]. Thus my results generalize many known fixed point theorems.

This thesis is organized as follows. Chapter **1** is devoted to some fundamental notions and related results which are in understanding and development of fixed point theory involving single valued as well as multi-valued mappings. We illustrate the concepts by some examples.

There may exist more than one metric on a given nonempty set X. If this is the case, and if X is complete with respect to one metric and a function f on X satisfies the contractive condition (0.0.1) with respect to some other metric, then it would be interesting to determine conditions which ensure that f has a fixed point. In Chapter **2**, we obtain two fixed point results for such mappings in the space with two metrics. As an application we prove a homotopy result. The results of this chapter appeared as [30] and [32].

In Chapter **3**, we obtain some fixed point theorems for multi-valued mappings. In Sections **I** and **II**, we extend the Proinov fixed point theorems [49] to the case of multi-valued mappings. The results obtained not only generalize Nadler fixed point theorem [40] but also give the higher order of convergence of the iterative sequence involved. The contractive condition involved is weaker and need not satisfy for every pair of elements from the given metric space. We have also calculated the error bounds. In Section **III**, we apply the results of Section **II** to obtain a solution to an integral inclusion. These results appeared as [29] and accepted for publication as [31] respectively. In Section **IV**, we introduce a new class of functions which is a subclass of the class of functions introduced by Pathak and Shahzad and improve some results of Pathak and Shahzad [45]. The results of this section presented as [33].

Chapter 1

Basic Concepts and Notation

The term metric fixed point theory refers to those fixed point theoretic results in which geometric conditions on the underlying spaces and/or mappings play a crucial role.

In this chapter some fundamental concepts and relevant results of metric fixed point theory are presented which are used throughout this thesis. To keep the chapter as brief as possible, the results are mentioned without proofs. All the necessary notation and the terminology used in the sequel are also introduced.

Throughout this thesis, X denotes a metric space with the metric d, unless stated otherwise. If X and Y are non-empty sets and $f : X \to Y$ is a mapping then fp denotes the image of p under f.

1.1 Fixed points of single valued mappings

1.1.1 Fixed points

Definition 1.1.1. Let f be a mapping from X into X. A point $p \in X$ is said to be a *fixed point* of f if $p = fp$.

Not all functions have fixed points. Further, if a function has a fixed point, it may not be unique.

Definition 1.1.2. A mapping $f : X \to X$ is said to be *Lipschitzian* if there is a constant $k \geq 0$ such that for all $x, y \in X$

$$d(fx, fy) \leq kd(x, y).$$

The smallest number k for which the above condition holds is called the *Lipschitz constant* of f.

Definition 1.1.3. A Lipschitzian mapping $f : X \to X$ with Lipschitz constant $k < 1$ is said to be a *contraction mapping* or simply a contraction.

1.1.2 Banach contraction principle

The Banach contraction principle, also known as the contraction mapping principle, is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of certain self maps of metric spaces and provides a constructive method to find those fixed points. The principle first appeared in explicit form in Banach's PhD thesis [6], where it was used to establish the existance of a solution to an integral equation.

Theorem 1.1.4. (Banach) Let X be a complete metric space and let f be a mapping from X into X. If there exists a real number k with $0 \leq k < 1$ satisfying

$$d(fx, fy) \leq kd(x, y)$$

for all $x, y \in X$, then f has a unique fixed point x_0. Moreover, for each $x \in X$:

(i) The iterative sequence $\{f^n x\}$ converges to x_0;

(ii) For $n \geq 1$ the following apriori estimates holds;

$$d(f^n x, x_0) \leq \frac{k^n}{1-k} d(x, fx);$$

(iii) For $n \geq 1$ the following aposteriori estimates holds;

$$d(f^{n+1}x, x_0) \leq d(f^{n+1}x, f^n x).$$

The strength of the contraction mapping principle lies in the fact that the underlying space is quite general a complete metric space while the conclusion is very strong, including even error estimates. Fig. 1.1 shows how the Banach iterative scheme works in $\mathbb{R}$. In Fig. 1.1(a) f is a contraction and the iterative sequence converges to the unique fixed point ξ. In Fig. 1.1(b) f is not a contraction, and it has a divergent iterative sequence.

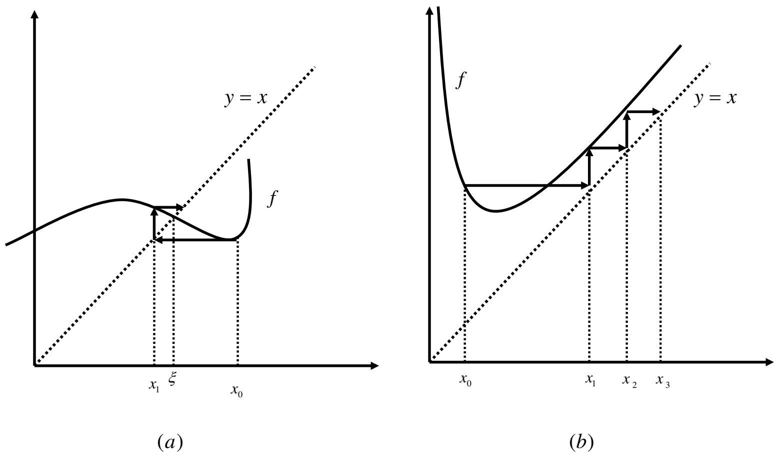

Figure 1.1: Visualization of Banach iterative scheme in $\mathbb{R}$.

Definition 1.1.5. [12] Let $\{x_n\}$ be a sequence in X that converges to ξ. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{d(x_{n+1},\xi)}{(d(x_n,\xi))^\alpha} = \lambda$$

then α is called the *rate of convergence* of the sequence $\{x_n\}$.

Remark 1.1.6. Note that the rate of convergence of the iterative sequence in the Banach contraction principle is linear.

Before proceeding further, we turn to a simple application to illustrate the usefulness of the Banach contraction principle. Consider the first order initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0. \tag{1.1.1}$$

We now state the Picard-Lindelöf theorem.

Theorem 1.1.7. (Picard) Let f be continuous on a rectangle

$$R = \{(t, x) : |t - t_0| \le a, |x - x_0| \le b\}$$

and thus bounded on R, that is

$$|f(t,x)| \leq c \qquad \text{for all } \ (t,x) \in R.$$

Suppose that f satisfies a Lipschitz condition on R with respect to x, that is for $(t,x),(t,y) \in R$

$$|f(t,x) - f(t,y)| \leq k|x - y|.$$

Then the initial value problem (1.1.1) has a unique solution on an interval $[t_0 - \beta, t_0 + \beta]$, where $\beta < \min\{a, \frac{b}{c}, \frac{1}{k}\}$.

1.1.3 Some extensions of the Banach contraction principle using gauge functions and/or semicontinuity

There have been numerous extensions of Banach contraction principle which are obtained for different collection of properties of the gauge functions (Definition 1.1.8) and/or using the notion of semicontinuity (Definiton1.1.9). Almost all such extensions prove only the convergence of iterative procedure to the fixed points of the mappings but only few of them are able to provide information on the convergence rate. Here we mention some of the well known (related to our work) extensions.

Definition 1.1.8. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *gauge function* if it satisfies at least one of the following properties:

1. φ is monotone non decreasing;
2. $\varphi(t) < t$ for all $t > 0$;
3. $\varphi(0) = 0$;
4. φ is continuous;
5. $(\varphi^n(t))$ converges to 0 for all $t > 0$;
6. $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$;
7. $t - \varphi(t) \to \infty$ as $t \to \infty$;
8. φ is subadditive.

For example, $\varphi(t) = at,\ t \in \mathbb{R}_+,\ a \in (0,1)$ is a gauge function.

Definition 1.1.9. A mapping $f : X \to X$ is said to be a *φ-contraction* if there exists a gauge function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(fx, fy) \leq \varphi(d(x,y)), \tag{1.1.2}$$

for all $x, \in X$.

Gauge means *measure.* Note from the above definition that if f is a φ-contraction then the gauge function φ provides the measure of f in the sense of inequality (1.1.2).

Definition 1.1.10. Let $f : X \to \mathbb{R}$. Then, for $\delta > 0$ *limit supremum* of f and *limit infimum* of f are defined respectively as:

$$\limsup fx := \begin{cases} \sup\{fy : |x-y| < \delta\}, & \text{if the supremum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

$$\liminf fx := \begin{cases} \inf\{fy : |x-y| < \delta\}, & \text{if the infimum exists} \\ -\infty, & \text{otherwise.} \end{cases}$$

Definition 1.1.11. A function $f : X \to \mathbb{R}$ is *upper semicontinuous* at $x \in X$ if $fx \geq \limsup fx$, and it is *lower semicontinuous* at $x \in X$ if $fx \leq \liminf fx$. We say that f is upper(lower) semicontinuous if it is upper(lower) semicontinuous at each $x \in X$. Obviously f is continuous if it is both upper and lower semi continuous.

In simple words, if f is upper semicontinuous at x, then the images of points near x under f do not exceed fx "too much", while there is no restriction on how far these images can fall below fx. Similarly, if f is lower semicontinuous at x, then the images of points near x under f do not fall below fx "too much", but they can still be very much greater than fx (Fig.1.2).

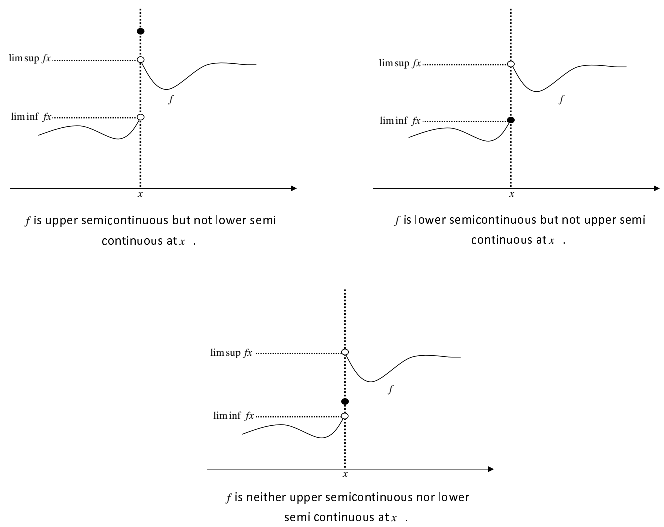

Figure 1.2: Upper and lower semicontinuity of a function f.

Rakotch [51] generalized the Banach contraction principle, using a gauge function, in the following way.

Theorem 1.1.12. [51] Let X be a complete metric space and suppose that $f : X \to X$ satisfies

$$d(fx, fy) \leq \varphi(d(x, y))d(x, y),$$

for each $x, y \in X$ where $\varphi : \mathbb{R}^+ \to [0, 1)$ is monotonically decreasing. Then f has a unique fixed point, ξ, and $\{f^n(x)\}$ converges to ξ for each $x \in X$.

Subsequently, Boyd and Wong [10] obtained a more general result.

Theorem 1.1.13. [10] Let X be a complete metric space and suppose $f : X \to X$ satisfies

$$d(fx, fy) \leq \varphi(d(x, y))$$

for each $x, y \in X$, where $\varphi : \mathbb{R}^+ \to [0, \infty)$ is upper semicontinuous from the right and satisfies $0 \leq \varphi(t) < t$ for $t > 0$. Then f has a unique fixed point, ξ, and $\{f^n(x)\}$ converges to ξ for each $x \in X$.

Since it is the explicit control over the error term that contributes so much to the wide-spread usefulness of Banach contraction principle, the following variant of the Boyd-Wong theorem due to Browder [11] is also of interest.

Theorem 1.1.14. [11] Let X be a complete metric space and let D be a bounded subset of X. Suppose $f : D \to D$ satisfies

$$d(fx, fy) \leq \varphi(d(x, y))$$

for each $x, y \in D$, where $\varphi : [0, \infty) \to [0, \infty)$ is monotone nondecreasing and continuous from the right, such that $\varphi(t) < t$ for all $t > 0$. Then there is a unique element $\xi \in D$ such that $\{f^n(x)\}$ converges to ξ for each $x \in D$. Moreover, if γ is the diameter of D, then

$$d(f^n x, \xi) \leq \varphi^n(\gamma)$$

and $\varphi^n(\gamma) \to 0$ as $n \to \infty$.

Another variant is due to Matkowski [38] wherein the continuity condition on φ is replaced with another condition.

Theorem 1.1.15. [38] Let X be a complete metric space and suppose that $f : X \to X$ satisfies

$$d(fx, fy) \leq \varphi(d(x, y))$$

for each $x, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is monotone non decreasing and satisfies $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$. Then f has a unique fixed point ξ, and $\lim_{n\to\infty} d(f^n(x), \xi) = 0$ for every $x \in X$.

1.1.4 Generalization of the Banach contraction principle using a gauge function of higher order

Proinov [49] generalized the Banach contraction principle by using a gauge function of order greater or equal to 1. Before proceeding to his results, it will be useful to establish the notations and terminology and some basic concepts about gauge functions of high orders.
Throughout the subsequent work J denotes an interval on $\mathbb{R}_+$ containing 0, i.e; an interval of the form $[0, R], [0, R)$ or $[0, \infty)$ and $S_n(t) = 1 + t + t^2 + \cdots + t^{n-1}$.

Definition 1.1.16. *(Gauge function of order $r \geq 1$).* Let $r \geq 1$. A function $\varphi : J \to J$ is said to be a *gauge function of order r* on J if it satisfies the following conditions:

1. $\varphi(\lambda t) \le \lambda^r \varphi(t)$ for all $\lambda \in (0,1)$ and $t \in J$;

2. $\varphi(t) < t$ for all $t \in J \setminus \{0\}$.

It is easy to see that the first condition of Definition 1.1.16 is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J \setminus \{0\}$.

Example 1.1.17. 1. $\varphi(t) = \lambda t$ $(0 < \lambda < 1)$ is a gauge function of the first order on $J = [0, \infty)$.

2. $\varphi(t) = ct^r$ $(c > 0, r > 1)$ is a gauge function of order r on $J = [0, R)$, where $R = (1/c)^{1/(r-1)}$.

Lemma 1.1.18. [49] Let φ be a gauge function of order $r \ge 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying

$$\varphi(t) = t\phi(t) \quad for \;\; all \;\; t \in J, \tag{1.1.3}$$

then it has the following two properties:

(i) $0 \le \phi(t) < 1$ for all $t \in J$;

(ii) $\phi(\lambda t) \le \lambda^{r-1}\phi(t)$ for all $\lambda \in (0,1)$ and $t \in J$.

Lemma 1.1.19. [49] Let φ be a gauge function of order $r \ge 1$ on J. Then for every $n \ge 0$ we have

(i) $\varphi^n(t) \le t\phi(t)^{S_n(r)}$ for all $t \in J$,

(ii) $\phi(\varphi^n(t)) \le \phi(t)^{r^n}$ for all $t \in J$,

where ϕ is a nonnegative and nondecreasing function on J satisfying (1.1.3).

Definition 1.1.20. [49] A nondecreasing function $\varphi : J \to J$ is said to be a *Bianchini-Grandolfi gauge function* [9] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for all } t \in J. \tag{1.1.4}$$

Note that Ptak [50] called a function $\varphi : J \to J$ satisfying (1.1.4) a rate of convergence on J and noticed that φ satisfies the following functional equation

$$\sigma(t) = \sigma(\varphi(t)) + t. \tag{1.1.5}$$

The following statement is an immediate consequence of the first part of Lemma 1.1.19 and the obvious inequality $S_n(r) \geq n$ for all $r \geq 1$.

Lemma 1.1.21. [49] Every gauge function of order $r \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.

Definition 1.1.22. [49] Let $f : D \subset X \to X$ be an operator satisfying

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \text{ for all } x, fx \in D \text{ with } d(x, fx) \in J,$$

with a Bianchini-Grandolfi gauge function φ on an interval J. A point $x_0 \in D$ is called an *initial point* of f if $d(x_0, fx_0) \in J$ and all of the iterates $x_0, x_1, x_2, \cdots$ are well-defined and belong to D.

From now on, $S(x_0, \rho)$ and $\overline{S}(x_0, \rho)$ denote the open and closed balls in X, respectively, with center x_0 and radius ρ.

Theorem 1.1.23. [49, Theorem 4.1] Let $f : D \subset X \to X$ be an operator on a complete metric space (X, d) satisfying

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \text{ for all } x, fx \in D \text{ with } d(x, fx) \in J,$$

with a Bianchini-Grandolfi gauge function φ on an interval J. Then, starting from an initial point x_0 of f the iterative sequence $\{x_n\}$ remains in $\overline{S}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$; $n = 0, 1, \cdots$, where $\rho_n = \sigma(d(x_n, x_{n+1}))$ and σ is defined by (1.1.4). Moreover, for each $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)).$$

If $\xi \in D$ and f is continuous at ξ, then ξ is a fixed point of f.

As a consequence of the above result Proinov obtained the following.

Theorem 1.1.24. [49, Theorem 4.2] Let $f : D \subset X \to X$ be an operator on a complete metric space (X, d) satisfying

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \text{ for all } x, fx \in D \text{ with } d(x, fx) \in J,$$

with a gauge function φ of the order $r \geq 1$ on an interval J. Suppose that $x_0 \in D$ is an initial point of f. Then the following statements hold true.

(i) The iterative sequence $\{x_n\}$ remains in $\overline{S}(x_0, \rho_0)$ and converges with rate of convergence at least r to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n), n = 0, 1, \cdots,$ where

$$\rho_n = d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))} \quad (1.1.6)$$

and ϕ is a non-negative and non-decreasing function on J satisfying (1.1.3)

(ii) For all $n \geq 0$ we have the following apriori estimate

$$d(x_n, \xi) \leq d(x_0, fx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \frac{\lambda^{S_n(r)} d(x_0, fx_0)}{1 - \lambda^{r^n}}, \quad (1.1.7)$$

where $\lambda = \phi(d(x_0, fx_0))$.

(iii) For all $n \geq 1$ we have the following aposteriori estimate

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \end{aligned} \quad (1.1.8)$$

(iv) For all $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, fx_0). \quad (1.1.9)$$

(v) If $\xi \in D$ and f is continuous at ξ, then ξ is a fixed point of f.

1.1.5 Generalization of the Banach contraction principle by introducing more than one metric on the underlying set

Another direction of extending Banach contraction principle is to consider the underlying space with two metrics. This approach is useful in obtaining homotopy

(Definition 1.1.28) results and has a significant applications in solving boundary value problem [1].

Example 1.1.25. Consider the function space $C[a,b]$, where $[a,b]$ is any given closed interval on $\mathbb{R}$. We can define two metrics on this space as follows:

$$d'(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$$

and

$$d(x,y) = \int_a^b |x(t) - y(t)| dt$$

for $x, y \in C[a,b]$. Note that $C[a,b]$ is a complete metric space with respect to d' but it is not complete with respect to d.

Whenever we consider X to be a set with two metrics d' and d, we assume that X is complete with respect to d' but may or may not complete with respect to d. In this case we denote by $\overline{S}(x_0,\rho)^{d'}$ the d'-closure of $S(x_0,\rho) = \{x \in X : d(x,x_0) < \rho\}$ and by $\overline{S}(x_0,\rho)$ the usual d-closure of $S(x_0,\rho) = \{x \in X : d(x,x_0) < \rho\}$.

Using two metrics on a given set, Agarwal et. al. [1] obtained the following results.

Theorem 1.1.26. [1, Theorem 2.1] Let (X,d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and $f : \overline{S}(x_0,r)^{d'} \to X$. Suppose that there exists a $q \in (0,1)$ such that for $x, y \in \overline{S}(x_0,r)^{d'}$ we have

$$d(fx,fy) \leq q \max\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,fy) + d(y,fx)]\}.$$

In addition assume that the following three properties hold:

$$d(x_0, fx_0) < (1-q)r \tag{1.1.10}$$

$$\text{If } d \not\geq d' \text{ then } f \text{ is uniformly continuous from } (S(x_0,r),d) \text{ into} (X,d'), \tag{1.1.11}$$

and

$$\text{If } d \neq d' \text{ then } f \text{ is continuous from } (\overline{S}(x_0,r)^{d'},d') \text{ into} (X,d'). \tag{1.1.12}$$

Then f has a fixed point i.e;, there exists $x \in \overline{S}(x_0,r)^{d'}$ with $x = fx$.

Theorem 1.1.27. [1, Theorem 2.3] Let (X, d') be a complete metric space, d another metric on X and $f : X \to X$. Suppose there exists $q \in (0, 1)$ such that for $x, y \in X$ we have

$$d(fx, fy) \leq q \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}.$$

In addition assume that the following two properties hold:

$$\text{If } d \ngeq d' \text{ then } f \text{ is uniformly continuous from } (X, d) \text{ into} (X, d'), \tag{1.1.13}$$

and

$$\text{If } d \neq d' \text{ then } f \text{ is continuous from } (X, d') \text{ into} (X, d'). \tag{1.1.14}$$

Then f has a fixed point.

Definition 1.1.28. Let (X, d) be a complete metric space, and let U be an open subset of X, $f : \bar{U} \to X$ and $g : \bar{U} \to X$ be two contractions where $\bar{U}$ denotes the closure of U in X. We say that f and g are *homotopic* if there exists $H : \bar{U} \times [0, 1] \to X$ with the following properties:

(a) $H(., 0) = g$ and $H(., 1) = f$;

(b) $x \neq H(x, t)$ for every $x \in \partial U$ and $t \in [0, 1]$ (here ∂U denotes the boundary of U in X);

(c) There exists a q, $0 \leq q < 1$, such that $d(H(x, t), H(y, t)) \leq q d(x, y)$ for every $x, y \in \bar{U}$ and $t \in [0, 1]$;

(d) There exists M, $M \geq 0$, such that $d(H(x, t), H(x, s)) \leq M|t - s|$ for every $x \in \bar{U}$ and $t, s \in [0, 1]$.

Agarwal et. al. [3] showed that the property of having a fixed point is invariant under homotopy for contractions.

Theorem 1.1.29. [3] Let (X, d) be a complete metric space and U an open subset of X. Suppose $f : \bar{U} \to X$ and $g : \bar{U} \to X$ are two homotopic contractive maps and g has a fixed point in U. Then f has a fixed point in U.

Using the Definition 1.1.28 the above Theorem can be restated as follows:

Theorem 1.1.30. Let (X, d) be a complete metric space and let U be an open subset of X.Suppose $H : \bar{U} \times [0, 1] \to X$ satisfies the following properties:

(a) $x \neq H(x,t)$ for every $x \in \partial U$ and $t \in [0,1]$ (here ∂U denotes the boundary of U in X);

(b) There exists $q,\ 0 \leq q < 1$, such that $d(H(x,t), H(y,t)) \leq qd(x,y)$ for every $x, y \in \bar{U}$ and $t \in [0,1]$;

(c) There exists $M,\ M \geq 0$, such that $d(H(x,t), H(x,s)) \leq M|t-s|$ for every $x \in \bar{U}$ and $t, s \in [0,1]$.

In addition assume that H_0 has a fixed point. Then H_1 has a fixed point in U (here $H_\lambda(.) = H(., \lambda)$).

Note that in above theorem the map H is taken as a homotopy between the contractions f and g defined in Theorem 1.1.29. Further in [1] the authors establish general continuation type theorems for generalized contractive homotopies on spaces with two metrics using Theorem 1.1.26.

Theorem 1.1.31. Let (X, d') be a complete metric space and let d be another metric on X. Let $Q \subseteq X$ be d'-closed and let $U \subseteq X$ be d-open and $U \subseteq Q$. Suppose $H : Q \times [0,1] \to X$ satisfies the following five properties:

(i) $x \neq H(x, \lambda)$ for $x \in Q \setminus U$ and $\lambda \in [0,1]$;

(ii)There exists a $q \in (0,1)$ such that for all $\lambda \in [0,1]$ and $x, y \in Q$

$$\begin{aligned} d(H(x,\lambda), H(y,\lambda)) \quad &\leq \quad q \max \Big\{ d(x,y), d(x, H(x,\lambda)), d(y, H(y,\lambda)) \\ &\qquad \frac{1}{2}[d(x, H(y,\lambda)) + d(y, H(x,\lambda))] \Big\}; \end{aligned}$$

(iii) $H(x, \lambda)$ is continuous in λ w.r.t. d, uniformly for $x \in Q$.

(iv) If $d \not\geq d'$ assume that H is uniformly continuous from $U \times [0,1]$ endowed with the metric d on U into (X, d').

(v) If $d \neq d'$ assume that H is continuous from $Q \times [0,1]$ endowed with the metric d' on Q into (X, d').

In addition assume that H_0 has a fixed point. Then for each $\lambda \in [0,1]$ we have that H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(.) = H(., \lambda)$).

1.2 Fixed points of multi-valued mappings

1.2.1 Multi-valued mappings

A multi-valued map or set valued map is a total relation; i.e; every input is associated with one or more outputs. Strictly speaking, a "well-defined" function associates one, and only one, output to any particular input. The "multi-valued mappings" are, therefore, not functions in a formal sense. Multi-valued mappings often arise from functions which are not injective. Such functions do not have an inverse function, but they do have an inverse relation.

Example 1.2.1. Let $X = [0, 1]$ and let $N(X)$ denote the family of all nonempty subsets of X. Define $T : X \to N(X)$ by:

$$Tx = [x, 1],$$

and $G : X \to N(X)$ by:

$$Gx = \begin{cases} [0, 1], & \text{if } x \neq \frac{1}{2}, \\ [0, \frac{1}{2}], & \text{if } x = \frac{1}{2}. \end{cases}$$

Then T and G are multi-valued mappings. Fig.1.3. shows graphs of T and G.

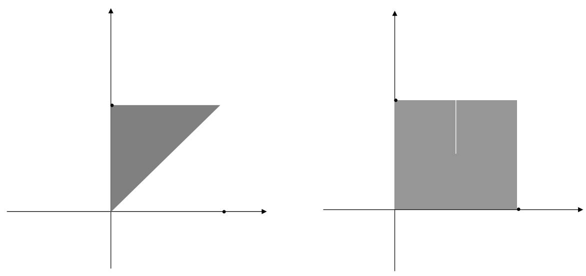

Figure 1.3: Multi-valued maps T and G .

Multi-valued maps can be observed in the study of dynamical systems and control theory.

Example 1.2.2. Consider the following control problem:

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(t) = x_0, \tag{1.2.1}$$

controlled by parameters $u(t)$, where $f : [0, a] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. In order to solve (1.2.1), we define a multi-valued map $F : [0, a] \times \mathbb{R}^n \to CL(X)$ as follows:

$$F(t, x) = \{f(t, x, u)\}_{u \in U}.$$

Then any solution of (1.2.1) is a solution of the following differential inclusion:

$$\dot{x}(t) \in F(t, x(t)), \qquad x(0) = x_0. \tag{1.2.2}$$

Thus, any control problem (1.2.1) can be transformed, by means of multi-valued maps, into problem (1.2.2).

Simple nonlinear dynamical systems and even piecewise linear systems can exhibit a completely unpredictable behavior, which might seem to be random. The physical world tend to be dissipative, if it were not for some driving force, the motion would cease. Dissipation may come from internal friction, thermodynamic losses, or loss of material, among many causes. The dissipation and the driving force tend to combine to kill out initial transients and settle the system into its typical behavior. This one part of the phase space of the dynamical system corresponding to the typical behavior is the attracting section. Attractors are parts of the phase space of the dynamical system. Until the 1960s, as evidenced by textbooks of that era, attractors were thought of as being geometrical subsets of the phase space: points, lines, surfaces, volumes. Two simple attractors are the fixed point and the limit cycle. In other words, the multi-valued map on a metric space X can be interpreted as multi-valued dynamical system and the fixed point of multi-valued map may be interpreted as the rest point or attractor of the system (see Aubin [5] for instance).

Definition 1.2.3. [16] A subset A of X is called *proximinal* if, for each $x \in X$, there is an element $a \in A$ such that $d(x, a) = d(x, A)$, where $d(x, A) = \inf\{d(x, y) : y \in A\}$.

Throughout subsequent discussion, $N(X)$ denotes the family of all nonempty subsets of X, $CL(X)$ the family of all non-empty closed subsets of X, $CB(X)$ the family of all non-empty bounded closed subsets of X ,$K(X)$ the family of all nonempty compact subsets of X and $PC(X)$ the class of all nonempty proximinal closed subsets of X without mentioning explicitly.

Definition 1.2.4. Let $T : X \to N(X)$. A point $p \in X$ is said to be a *fixed point* of T if $p \in Tp$.

Example 1.2.5. Let $X = [0, 1]$. Define $T : X \to N(X)$ by $Tx = [0, x^2]$. Then 0 and 1 are fixed points of T (see Fig 1.4.).

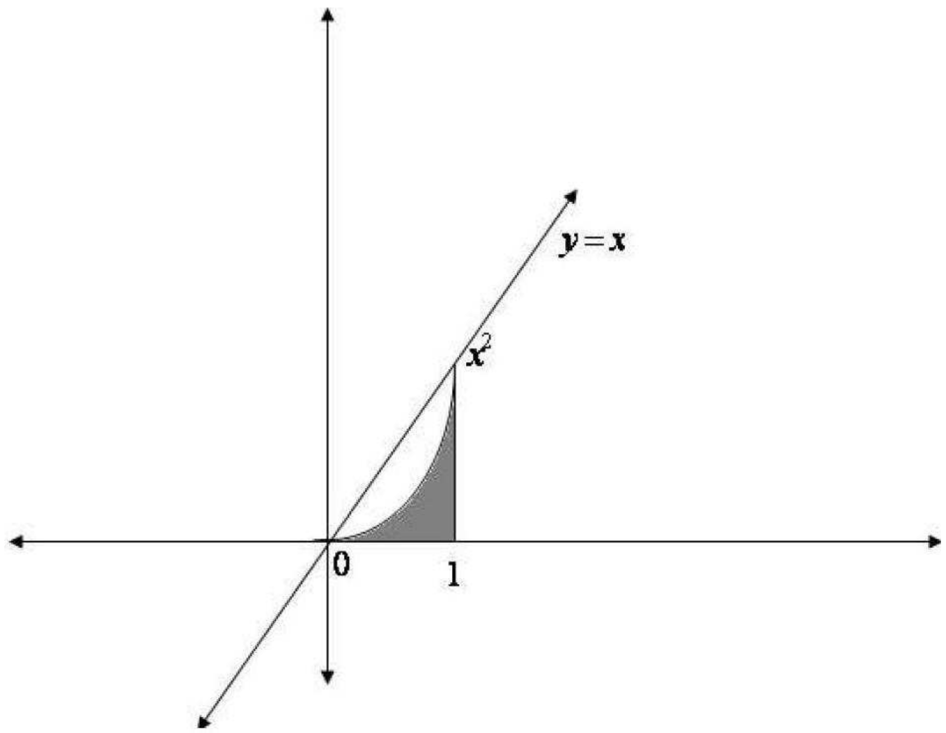

Figure 1.4: Fixed point of multi-valued map T .

For the investigation of fixed point of multi-valued maps, one needs a concept of distance(metric) between two sets. One such notion is the notion of Hausdorff metric that was introduced by Hausdorff and some of its properties explored about 80 years ago.

Definition 1.2.6. For $A, B \in CB(X)$ $H(A, B)$ defined by,

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

is a metric on $CB(X)$ and is called the *Hausdorff metric* generated by the metric d. The metric H depends heavily on the choice of the metric d defined on X. Fig.1.5 shows the geometrical interpretation of the Hausdorff metric in $\mathbb{R}$ and $\mathbb{R}^2$. By Definition, first calculate $\{d(x, B) : x \in A\}$ and $\{d(y, A) : y \in B\}$ and take supremum of these sets respectively. $H(A, B)$ is then the maximum of these two values.

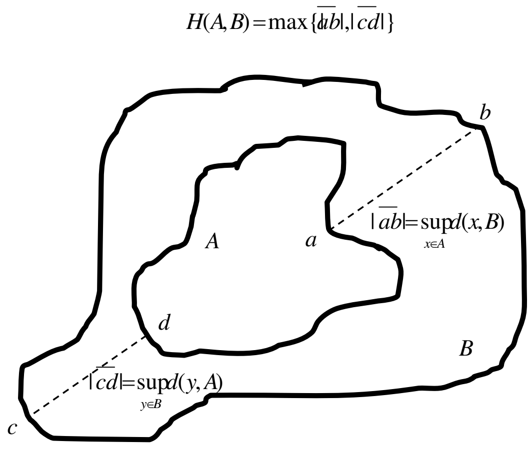

Figure 1.5: Hausdorff metric.

Definition 1.2.7. For $A, B \in CL(X)$ the function,

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x,B),\ \ \sup_{y\in B} d(y,A)\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

is called the *Generalized Hausdorff metric* generated by the metric d.

Remark 1.2.8. If (X, d) is a complete metric space, then so is the metric space $(CL(X), H)$. Further, $CB(X), K(X)$ and $PC(X)$ are all closed subsets of $CL(X)$ and thus are also complete with respect to H.

Following lemmas are needed in the sequel.

Lemma 1.2.9. [25] Let $A, B \in CB(X)$ and $\epsilon > 0$ with $H(A, B) < \epsilon$, then, for each $a \in A$, there exists an element $b \in B$ such that $d(a, b) < \epsilon$.

Lemma 1.2.10. [17] Let $A, B \in CB(X)$. Then, for each $a \in A$,

$$d(a, B) \leq H(A, B).$$

Lemma 1.2.11. [40] Let $A, B \in CB(X)$ and let $a \in A$. If $\epsilon > 0$, then there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \epsilon.$$

Lemma 1.2.12. [26] If $A, B \in K(X)$ and $a \in A$, then there exists an element $b \in B$ such that

$$d(a, b) \leq H(A, B).$$

Lemma 1.2.13. [40] Let $A, B \in CL(X)$ and $q > 1$. Then, for each $a \in A$, there exists an element $b \in B$ such that

$$d(a, b) \leq qH(A, B).$$

Definition 1.2.14. A mapping $T : X \to CL(X)$ is said to be *Lipschitzian* if there is a constant $k \geq 0$ such that for all $x, y \in X$

$$H(Tx, Ty) \leq kd(x, y).$$

The smallest number k for which the above condition holds is said to be the *Lipschitz constant* of T.

Definition 1.2.15. A Lipschitzian mapping $T : X \to CL(X)$ with Lipschitz constant $k < 1$ is said to be a *multi-valued contraction.*

1.2.2 Nadler fixed point theorem

Banach contraction principle extends nicely to multi-valued mappings, a fact first noticed by Nadler [40]. He extended the Banach contraction principle in the following way.

Theorem 1.2.16. [40] Let (X, d) be complete metric space and T is a mapping from X into $CB(X)$ such that for all $x, y \in X$,

$$H(Tx, Ty) \leq \lambda d(x, y),$$

where $0 \leq \lambda < 1$. Then T has a fixed point.

In contrast to Banach contraction principle, the preceding theorem does not assert that the fixed point is unique. Also, no information regarding error estimates has been provided.

1.2.3 Some generalizations of the Nadler fixed point theorem

Reich [53] extended the Nadler fixed point theorem in the following way:

Theorem 1.2.17. [53] If (X, d) is a complete metric space and $T : X \to K(X)$ satisfies

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \tag{1.2.3}$$

for each $x, y \in X$, where α is a function of $(0, \infty)$ into $[0, 1)$ such that

$$\limsup_{r \to t^+} \alpha(r) < 1 \tag{1.2.4}$$

for each $t \in (0, \infty)$, then T has a fixed point.

This result generalizes the fixed point theorem for single valued mappings and was proved by Boyd and Wong [10]. Reich [54] raised the question: If T satisfies the same contractive condition (1.2.3), whether or not the range of T can be relaxed. Specifically the question was whether the range of T, $K(X)$ can be replaced by $CB(X)$ or $CL(X)$. In [39] Mizoguchi and Takahashi gave the positive answer to the conjecture of Reich [53], when the inequality (1.2.4) holds also for $t = 0$, in particular they proved:

Theorem 1.2.18. [39] Let (X, d) be a complete metric space and $T : X \to CB(X)$. If α is a function of $(0, \infty)$ into $[0, 1)$ such that

$$\limsup_{r \to t^+} \alpha(r) < 1$$

for each $t \in [0, \infty)$ and if

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \text{ for each } x, y \in X, \tag{1.2.5}$$

then T has a fixed point in X.

It is worth mentioning that Suzuki [61] showed that Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler fixed point theorem. Inspiring from the result of Mizoguchi-Takahashi and using the ideas of Feng-Liu [19], Klim and Wardowski [34] obtained the following result and showed that their result is different from Theorem 1.2.17 and Theorem 1.2.18.

Definition 1.2.19. For $b \in (0, 1]$ and $x \in X$, define

$$I_b^x = \{y \in Tx : bd(x, y) \leq d(x, Tx)\}.$$

Theorem 1.2.20. [34] Let (X, d) be a complete metric space and let $T : X \to CL(X)$. If:

(i) The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semi-continuous;

(ii) there exist $\alpha : [0, \infty) \to [0, 1)$ such that

$$\forall_{t \in [0,\infty)} \{\limsup_{r \to t^+} \alpha(r) < 1\}$$

and

$$\forall_{x \in X} \exists_{y \in I_1^x} \{d(y, Ty) \leq \alpha(d(x, y))d(x, y)\};$$

then T has a fixed point.

They also established the following:

Theorem 1.2.21. [34] Let (X, d) be a complete metric space and let $T : X \to CL(X)$. Assume that the following conditions hold:

(i)The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semicontinuous;

(ii)there exist $b \in (0, 1)$ and $\alpha : [0, \infty) \to [0, b)$ such that

$$\forall_{t \in [0,\infty)} \{\limsup_{r \to t^+} \alpha(r) < b\}$$

and

$$\forall_{x \in X} \exists_{y \in I_b^x} \{d(y, Ty) \leq \alpha(d(x, y))d(x, y)\}.$$

Then T has a fixed point.

Pathak and Shahzad [45] introduced a new class of mappings $\Theta[0, A)$ as follows. Let $A \in (0, +\infty]$. $\Theta[0, A)$ [45] denotes the class of functions $\theta : [0, A) \to \mathbb{R}$ satisfy the following conditions: (i) θ is nondecreasing on $[0, A)$; (ii) $\theta(t) > 0$ for each $t \in (0, A)$; (iii) θ is subadditive in $(0, A)$; i-e., $\theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2)$ for $t_1, t_2 \in (0, A)$. Notice that (i) implies that θ is strictly inverse isotone on $(0, A)$; i-e., $\theta(t_1) < \theta(t_2) \Rightarrow t_1 < t_2, \quad t_1, t_2 \in (0, A)$. Moreover, for $b \in (0, 1]$ and $x \in X$,

$$M(b, x; \theta) = \{y \in Tx : b\theta(d(x, y)) \leq \theta(d(x, Tx))\}.$$

Pathak and Shahzad generalized Theorems 1.2.20 and 1.2.21 in the following way.

Theorem 1.2.22. [45] Let (X, d) be a complete metric space and let $T : X \to CL(X)$. If following conditions hold:

(i) The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semicontinuous;

(ii) there exists $\alpha : (0, \infty) \to [0, 1)$ such that

$$\forall t \in [0, \infty)\{\limsup_{r \to t^+} \alpha(r) < 1\}$$

and

(iii) there exists $\theta \in \Theta[0, A)$ satisfying the following conditions:

$$\text{for all } x \in X, M(1, x; \theta) \text{ is nonempty}$$

and

$$\forall x \in X \exists y \in M(1, x; \theta)\{\theta(d(y, Ty)) \leq \alpha(d(x, y))\theta(d(x, y))\};$$

then T has a fixed point.

Theorem 1.2.23. [45] Let (X, d) be a complete metric space and let $T : X \rightarrow CL(X)$. If following conditions hold:

(i) The map $f : X \rightarrow \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semi-continuous;

(ii) there exists $b \in (0, 1)$ and $\alpha : (0, \infty) \rightarrow [0, b)$ such that

$$\forall t \in [0, \infty)\{\limsup_{r \rightarrow t^+} \alpha(r) < b\}$$

and

(iii) there exists $\theta \in \Theta[0, A)$ satisfying the following condition(with b and α of *(ii)*):

$$\text{for all } x \in X, M(a, x; \theta) \text{is nonempty for any constant } a \in (0, 1)$$

and

$$\forall x \in X \exists y \in M(b, x; \theta)\{\theta(d(y, Ty)) \leq \alpha(d(x, y))\theta(d(x, y))\};$$

then T has a fixed point.

Definition 1.2.24. For $x_0 \in X$, if there is a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ then $O(f, x_0) = \{x_0, x_1, x_2, \cdots\}$ is said to be *orbit* of $f : X \rightarrow X$.

Definition 1.2.25. A mapping $g : X \rightarrow \mathbb{R}$ is said to be *f-orbitally lower semi-continuous* [24] if $\{x_n\}$ is a sequence in $O(f, x_0)$ and $x_n \rightarrow \xi$ implies $g(\xi) \leq \lim_n \inf g(x_n)$.

Example 1.2.26. Let $X = [0, 1]$ and $g : X \to \mathbb{R}$ and $f : X \to X$ be defined as:

$$gx = \begin{cases} \frac{1}{2}x, & \text{if } x \neq 0 \text{and rational}, \\ 0, & \text{if } x = 0, \\ \frac{1}{3}x, & \text{if } x \text{is irrational}. \end{cases}$$

and $fx = \frac{1}{2}x$. Then the map g is f-orbitally continuous but not continuous.

Chapter 2

Some generalizations of the Banach contraction principle

The basic motivation behind this chapter is to combine the techniques discussed in the sections 1.1.4 and 1.1.5.Therefore, the established results not only extend Banach contraction principle but also some results of Agarwal et. al. [1] and Proinov[49].

Throughout this chapter, the underlying set X is endowed with two metrics d and d' such that X is complete with respect to d' and there is a map $f : D \subset X \to X$ which is a φ-contraction with respect to d, where φ is a gauge function. The results present in sections first and second are different due to the conditions imposed on the map f. Rest of the terminology and notations are same as introduced in the section 1.1.5. In the last section, a homotopy result is obtained as an application.

2.1 Single valued φ-contractions

In this section, we establish a convergence theorem for the iterative processes of the type:

$$x_{n+1} = fx_n, \ n = 0, 1, 2, \cdots, \tag{2.1.1}$$

where $f : D \subset X \to X$ is an operator satisfying:

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \text{ for all } x \in D, fx \in D \text{ with } d(x, fx) \in J, \tag{2.1.2}$$

where φ is a gauge function on an interval J. The results of this section generalize, extend and improve some results by Agarwal and O'Regan [1] and thus generalize some results of Hardy and Rogers[22], Kannan [28], Maia [37], Precup [48] and Reich [53]. We use the machinery of Proinov [49] to prove our results.

Theorem 2.1.1. If $f : D \subset X \to X$ is an operator satisfying condition (2.1.2), with a Bianchini-Grandolfi gauge function φ on an interval J, then starting from an initial point x_0 of f the iterative sequence (2.1.1) remains in $\overline{S}(x_0, \rho_0)^d$ and converges to a point $\xi \in \overline{S}(x_0, \rho_0)^{d'}$ which is a fixed point of f provided that the following three conditions hold:

$$\text{If } d \ngeq d' \text{ then } f \text{ is uniformly continuous from } (S(x_0, \rho_0), d) \text{ into } (X, d'); \tag{2.1.3}$$

$$\text{If } d \neq d' \text{ then } f \text{ is continuous from } (\overline{S}(x_0, \rho_0)^{d'}, d') \text{ into } (X, d'); \tag{2.1.4}$$

$$\text{If } d = d' \text{ then } f \text{ is continuous at } \xi. \tag{2.1.5}$$

Proof. Since x_0 is an initial point of f, it follows from [49, Lemma 3.2] that

$$\overline{S}(x_{n+1}, \rho_{n+1}) \subset \overline{S}(x_n, \rho_n) \text{ for all } n \geq 0, \tag{2.1.6}$$

which implies that $x_n \in \overline{S}(x_0, \rho_0)$ for all $n \geq 0$. Using the definition of ρ_n, from [49, Lemma 3.6], we get

$$\begin{aligned} \rho_n = \sigma(E(x_n)) &\leq \sigma(\varphi^n(E(x_0))) \\ &= \sum_{j=0}^{\infty} \varphi^j(\varphi^n(E(x_0))) \\ &= \sum_{j=n}^{\infty} \varphi^j(E(x_0)) \text{ for all } n \geq 0, \end{aligned} \tag{2.1.7}$$

since σ is non-decreasing. As φ is Bianchini-Grandolfi gauge function on J and $E(x_0) \in J$, from (2.1.7), we get

$$\rho_n \to 0 \text{ as } n \to \infty. \tag{2.1.8}$$

From (2.1.6) and (2.1.8) it follow that $\{\overline{S}(x_n, \rho_n)^d\}$ is nested sequence of closed spheres such that $\rho_n \to 0$ as $n \to \infty$. Therefore, it follows from Cantor's Theorem that the sequence (2.1.1) is a Cauchy sequence with respect to d i.e., there exists

$N \in \{1, 2, \cdots\}$ with

$$d(x_n, x_m) < \delta \quad \text{whenever} \quad n, m \geq N. \tag{2.1.9}$$

We now claim that, (2.1.1) is a Cauchy sequence with respect to d'. If $d \geq d'$ this is trivial. Next suppose $d \ngeq d'$. Let $\epsilon > 0$ be given. Then (2.1.3) guarantees that there exists $\delta > 0$ such that

$$d'(fx, fy) < \epsilon \;\; \text{whenever} \;\; x, y \in S(x_0, \rho_0) \;\; \text{and} \;\; d(x, y) < \delta \tag{2.1.10}$$

Now (2.1.9) and (2.1.10) imply

$$d'(x_{n+1}, x_{m+1}) = d'(fx_n, fx_m) < \epsilon \quad \text{whenever} \quad n, m \geq N,$$

and as a result our claim is proved. Now since (X, d') is complete there exists $\xi \in \overline{S}(x_0, \rho_0)^{d'}$ with $d'(x_n, \xi) \to 0$ as $n \to \infty$. We claim that $\xi = f\xi$. First consider the case when $d \neq d'$.

$$d'(\xi, f\xi) \leq d'(\xi, x_n) + d'(x_n, f\xi) = d'(\xi, x_n) + d'(fx_{n-1}, f\xi) \tag{2.1.11}$$

Let $n \to \infty$ then (2.1.4) insures that $d'(\xi, x_n) \to 0$ implies $d'(fx_{n-1}, f\xi) \to 0$ and so $\xi = f\xi$. Next suppose $d = d'$.

$$d(\xi, fx_n) \leq d(\xi, x_n) + d(x_n, fx_n) = d(\xi, x_n) + d(x_n, x_{n+1}).$$

Taking limit as $n \to \infty$ we get,

$$lim_{n\to\infty} d(\xi, fx_n) \leq 0.$$

From (2.1.5) since f is continuous at ξ so we have $d(\xi, f\xi) = 0$ which simply means that $\xi = f\xi$. □

Remark 2.1.2. Theorem 2.1.1 remains true if φ is a gauge function of order $r \geq 1$.

Corollary 2.1.3. Let $f : D \subset X \to X$ is an operator satisfying

$$d(fx, fy) \leq \varphi(d(x, y)) \text{ for all } x, y, fx, fy \in D \text{ with } d(x, y) \in J, \tag{2.1.12}$$

where φ is a gauge function of order $r \geq 1$ on an interval J. If x_0 is an initial point of f such that $d(x_0, fx_0) \in J$ then the following statements hold true:

(a) The iterative sequence (2.1.1) converges to a fixed point ξ of f.

(b) The operator f has a unique fixed point in $S = \{x \in X : d(x, \xi) \in J\}$;

provided that (2.1.3) and (2.1.4) hold.

Proof. From (2.1.12) $d(fx, f^2x) \leq \varphi(d(x, fx))$ holds for all $x \in D, y = fx \in D$ and $d(x, fx) \in J$. Further, by definition of φ we have $d(fx, fy) \leq \varphi(d(x, y)) < d(x, y)$ for $x \neq y$. Hence, f is d continuous. Thus *(a)* follows immediately from Theorem 2.1.1. To show the uniqueness, i.e., *(b)*, assume that $\eta \in S$ is another fixed point of f. Then $d(\xi, \eta) \in J$. It follows from (2.1.12) that

$$d(\xi, \eta) = d(f\xi, f\eta) \leq \varphi(d(\xi, \eta)) < d(\xi, \eta)$$

Hence $\xi = \eta$. □

Corollary 2.1.4. If $f : D \subset X \to X$ is an operator satisfying

$$d(fx, fy) \leq \varphi(m(x, y)) \text{ for all } x, y, fx, fy \in D \tag{2.1.13}$$

where φ is a gauge function of order $r \geq 1$ on an interval $J = [0, \infty)$ and

$$m(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\}, \tag{2.1.14}$$

then for initial point x_0 of f, the iterative sequence (2.1.1) converges to a unique fixed point ξ of f provided that (2.1.3), (2.1.4) and (2.1.5) hold. Moreover, if φ is continuous, then continuity of f in (2.1.5) can be omitted.

Proof. First we shall prove that, $m(x, fx) = d(x, fx)$ for all $x \in D$. By (2.1.14) and triangle inequality, we get

$$\begin{aligned} m(x, fx) &= \max\{d(x, fx), d(x, fx), d(fx, f^2x), d(x, f^2x)\} \\ &= \max\{d(x, fx), d(fx, f^2x)\} = \max\{E(x), E(fx)\} \end{aligned}$$

where $E(x) = d(x, fx)$. If $x = fx$ then $m(x, fx) = E(x) = d(x, fx)$. If $x \neq fx$ then $\varphi(t) < t$ for $t > 0$ so we have from (2.1.13)

$$E(fx) = d(fx, f^2x) \leq \varphi(m(x, fx)) < m(x, fx) = \max\{E(x), E(fx)\}$$

which means $m(x, fx) = E(x)$. Thus condition (2.1.13) implies (2.1.2). Applying Theorem 2.1.1, we deduce that the iterative sequence (2.1.1) converges to the fixed point ξ of f.

If η is another fixed point then from (2.1.14) we have $m(\xi, \eta) = d(\xi, \eta)$ and hence uniqueness follows immediately from Corollary 2.1.3.

Finally, suppose that $d = d'$ and φ is continuous then it follows from (2.1.13) that

$$d(x_{n+1}, f\xi) = d(fx_n, f\xi) \le \varphi(m(x_n, \xi)) \quad \text{for all} \quad n \ge 0$$

Since $m(x_n, \xi) = d(x_n, \xi)$ so passing limit as $n \to \infty$ we obtain $d(\xi, f\xi) \le \varphi(0) = 0$ which implies $\xi = f\xi$. □

Remark 2.1.5. If $\varphi(t) = qt \quad (0 < q < 1)$, then Corollary 2.1.4 generalizes Theorem 1.1.26 (resp. Theorem 1.1.27) in the following manner.

(i) Taking $D = S(x_0, r)^{d'}$ (resp. $D = X$).

(ii) Uniform Continuity of the operator f in (1.1.11) (resp. in (1.1.13)) is required from $(S(x_0, r), d)$ into (X, d') (resp. from (X, d) into (X, d')) while it is required from $(S(x_0, \rho_0), d)$ into (X, d') in (2.1.3).

(iii)Continuity f the operator f in (1.1.12) (resp. in (1.1.14)) is required from $\overline{S(x_0, r)^{d'}}$ into (X, d') (resp. from (X, d') into (X, d')) while it is required from $(\overline{S}(x_0, \rho_0)^{d'}, d')$ into (X, d') in (2.1.3).

(iv) Corollary 2.1.4 concludes with a unique fixed point.

(v) (2.1.5) can be omitted because φ is continuous in this case.

Remark 2.1.6. Note that in the hypothesis of Corollary 2.1.4 we assumed that x_0 is an initial point of f. This assumption was not the part of the hypothesis of Theorem 1.1.26. But a closer look at condition (1.1.10) in Theorem 1.1.26 in-fact implies that x_0 is an initial point of f.

Remark 2.1.7. If $d = d'$, then Corollary 2.1.4 coincides with the first part of [49, Corollary 4.5]. In addition, if $\varphi(t) = qt \quad (0 < q < 1)$ then it reduces to [1, Corollary 2.2].

2.2 Single valued generalized φ-contractions

In this section, we establish some fixed point results for generalized φ contractions on a set with two metrics. The results generalize, extend and improve some recent results by Agarwal and O'Regan [1] and thus generalize some results of Hardy and Rogers [22], Kannan [28], Maia [37], Precup [48] and Reich [52].

Theorem 2.2.1. Let $x_0 \in X$, $\rho > 0$ and $f : S(x_0, \rho) \to X$ satisfies

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \ \forall x, fx \in \overline{S}(x_0, \rho) \text{ with } d(x, fx) \in J, \tag{2.2.1}$$

where φ is a Bianchini-Grandolfi gauge function on an interval $J = [0, \infty)$. Then, starting from x_0 the iterative sequence

$$x_{n+1} = fx_n, \quad n \geq 0 \tag{2.2.2}$$

converges to a point $\xi \in \overline{S}(x_0, \rho)^{d'}$ which will be the fixed point of f if the following conditions hold:

(i)

$$d(x_0, fx_0) < \delta, \tag{2.2.3}$$

where $\delta > 0$ is such that $\sigma(\delta) \leq \rho$.

(ii) If $d \ngeq d'$ then f is uniformly continuous from $(S(x_0, \rho), d)$ into (X, d').

(iii) If $d \neq d'$ then f is continuous from $(\overline{S}(x_0, \rho)^{d'}, d')$ into (X, d').

(iv) If $d = d'$ then f is continuous at ξ.

Proof. Let $x_1 = fx_0$. Then from *(i)*

$$d(x_0, x_1) < \delta \leq \sigma(\delta) \leq \rho.$$

Next let $x_2 = fx_1$. Then from (2.2.1) we have

$$d(x_1, x_2) = d(fx_0, f^2x_0) \leq \varphi(d(x_0, x_1)).$$

Note that $d(x_1, x_2) \in J$. Further, $x_2 \in \overline{S}(x_0, \rho)$ since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) \\ &< \delta + \sigma(\varphi(\delta)) \\ &= \sigma(\delta) \text{ (using (1.1.5))} \\ &\leq \rho. \end{aligned}$$

If $x_3 = fx_2$ then

$$d(x_2, x_3) = d(fx_1, f^2x_1) \leq \varphi(d(x_1, fx_1)) = \varphi(d(fx_0, f^2x_0)) \leq \varphi^2(d(x_0, x_1)).$$

Note that $d(x_2, x_3) \in J$. Further, $x_3 \in \overline{S}(x_0, \rho)$ since

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) + \varphi^2(d(x_0, x_1))) \\ &< \sum_{j=0}^{\infty} \varphi^j(\delta) \\ &= \sigma(\delta) \leq \rho. \end{aligned}$$

Proceeding inductively we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\overline{S}(x_0, \rho)$ such that $d(x_{n-1}, x_n) \in J$ where

$$x_n = fx_{n-1}, \tag{2.2.4}$$

and

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)). \tag{2.2.5}$$

Note that $\{x_n\}$ is a Cauchy sequence w.r.t d since for $n, p \in \mathbb{N}$, from (3.2.10) we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \varphi^{n+p-1}(d(x_0, x_1)) + \cdots + \varphi^n(d(x_0, x_1)) \\ &\leq \sum_{j=n}^{\infty} \varphi^j(d(x_0, x_1)) \to 0 \text{ (using (1.1.4))}. \end{aligned}$$

Thus, there exists $N \in \mathbb{N}$ with

$$d(x_n, x_m) < \eta \quad \text{whenever} \quad n, m \geq N. \tag{2.2.6}$$

We claim that, (2.2.2) is a Cauchy sequence with respect to d'. If $d \geq d'$ this is trivial. Next suppose $d \ngeq d'$. Let $\epsilon > 0$ be given. Then *(ii)* guarantees that there exists $\eta > 0$ such that

$$d'(fx, fy) < \epsilon \;\; \text{whenever } x, y \in S(x_0, \rho) \text{ and } d(x, y) < \eta. \tag{2.2.7}$$

Now (2.2.6) and (2.2.7) imply

$$d'(x_{n+1}, x_{m+1}) = d'(fx_n, fx_m) < \epsilon \quad \text{whenever} \quad n, m \geq N,$$

and this prove our claim. Since (X, d') is complete there exists $\xi \in \overline{S}(x_0, \rho)^{d'}$ with $d'(x_n, \xi) \to 0$ as $n \to \infty$. We claim that $\xi = f\xi$. First consider the case when $d \neq d'$.

$$d'(\xi, f\xi) \leq d'(\xi, x_n) + d'(x_n, f\xi) = d'(\xi, x_n) + d'(fx_{n-1}, f\xi) \tag{2.2.8}$$

Let $n \to \infty$ then *(iii)* insures that $d'(\xi, x_n) \to 0$ implies $d'(fx_{n-1}, f\xi) \to 0$ and so $\xi = f\xi$. Next suppose $d = d'$.

$$d(\xi, fx_n) \leq d(\xi, x_n) + d(x_n, fx_n) = d(\xi, x_n) + d(x_n, x_{n+1}).$$

Taking limit as $n \to \infty$ we get,

$$\lim_{n \to \infty} d(\xi, fx_n) \leq 0.$$

From (iv), since f is continuous at ξ, we have $d(\xi, f\xi) = 0$. This means that $\xi = f\xi$. □

Remark 2.2.2. Theorem 2.2.1 remains true if φ is a gauge function of order $r \geq 1$.

The following global result can easily be obtain from Theorem 2.2.1 and Remark 2.2.2.

Theorem 2.2.3. Let $f : X \to X$ is an operator satisfying (2.2.1) with gauge function φ of order $r \geq 1$ on an interval $J = [0, \infty)$. Then, f has a fixed point if the following conditions are satisfied:

(a) If $d \not\geq d'$ assume f is uniformly continuous from (X, d) into (X, d').

(b) If $d \neq d'$ then f is continuous from (X, d') into (X, d').

(c) If $d = d'$ then f is continuous at ξ.

Proof. Fix $x_0 \in X$. For $\delta > 0$ such that

$$d(x_0, fx_0) < \delta,$$

take $\rho = \sigma(\delta)$, where σ is given by (1.1.4). Now Theorem 2.2.1 guarantees that there exists $\xi \in \overline{S(x_0, \rho)}^{d'}$ with $\xi \in f\xi$. □

Theorem 2.2.4. Let $x_0 \in X$, $\rho > 0$ and $f : S(x_0, \rho) \to X$ is an operator satisfying

$$d(fx, fy) \leq \varphi(m(x, y)) \text{ for all } x, y, fx, fy \in S(x_0, \rho), \tag{2.2.9}$$

where φ is a gauge function of order $r \geq 1$ on an interval $J = [0, \infty)$ and

$$m(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\}. \tag{2.2.10}$$

Then, the iterative sequence (2.2.2) converges to a unique fixed point ξ of f if conditions (i) - (iv) hold. Moreover, if φ is continuous then continuity of f in (iv) can be omitted.

Proof. Let $x_1 = fx_0$. Then, from (i) we have

$$d(x_0, fx_0) < \delta \leq \sigma(\delta) \leq \rho.$$

If $x_2 = fx_1$ then from (2.2.9) we have

$$d(x_1, x_2) = d(fx_0, fx_1) \leq \varphi(m(x_0, x_1))$$

where,

$$\begin{aligned} m(x_0, x_1) &= \varphi\Big(\max\Big\{d(x_0, x_1), d(x_0, fx_0), d(x_1, fx_1), \\ &\qquad \frac{d(x_0, fx_1) + d(x_1, fx_0)}{2}\Big\}\Big) \\ &= \varphi\Big(\max\Big\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\Big\}\Big). \end{aligned}$$

We claim that

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{2.2.11}$$

Case 1. If $m(x_0, x_1) = d(x_0, x_1)$ then clearly (2.2.11) is true.
Case 2. If $m(x_0, x_1) = d(x_1, x_2)$ then we have

$$d(x_1, x_2) \leq \varphi(d(x_1, x_2)) < d(x_1, x_2),$$

which is a contradiction.
Case 3. Finally suppose $m(x_0, x_1) = \frac{d(x_0, x_2)}{2}$. Then, we have

$$d(x_1, x_2) \leq \varphi(\frac{d(x_0, x_2)}{2}) < \frac{d(x_0, x_2)}{2} \leq \frac{d(x_0, x_1) + d(x_1, x_2)}{2},$$

Thus, $d(x_1, x_2) < d(x_0, x_1)$. As a result,

$$m(x_0, x_1) \leq \frac{d(x_0, x_1) + d(x_1, x_2)}{2} < d(x_0, x_1),$$

which contradicts the definition of $m(x_0, x_1)$. This prove our claim. Proceeding inductively in a similar way as in Theorem 2.2.1 we obtain the iterative sequence (2.2.2) converges to the fixed point ξ of f. If η is another fixed point then from (2.2.9) and (3.2.26) we have $m(\xi, \eta) = d(\xi, \eta)$ and

$$d(\xi, \eta) = d(f\xi, f\eta) \leq \varphi(m(\xi, \eta)) = \varphi(d(\xi, \eta)) < d(\xi, \eta).$$

So, $\xi = \eta$. Finally, suppose $d = d'$ and φ is continuous then it follows from (2.2.9) that

$$\begin{aligned} d(x_{n+1}, f\xi) = d(fx_n, f\xi) &\leq \varphi(m(x_n, \xi)) \text{ for all } n \geq 0 \\ &= \varphi\Big(\max\Big\{d(x_n, \xi), d(x_n, fx_n), d(\xi, f\xi), \\ &\qquad \frac{d(x_n, f\xi) + d(\xi, fx_n)}{2}\Big\}\Big) \end{aligned}$$

so passing limit as $n \to \infty$ we obtain $d(\xi, f\xi) \leq \varphi(d(\xi, f\xi))$ which is possible only when $\xi = f\xi$.

The following global result can easily be obtain from Theorem 2.2.4. □

Theorem 2.2.5. Let $f : X \to X$ is an operator satisfying (2.2.9) with gauge function φ of order $r \geq 1$ on an interval $J = [0, \infty)$ and $m(x, y)$ is defined in (3.2.26). Then, f has a unique fixed point if the following conditions hold:

(a')If $d \ngeq d'$ then f is uniformly continuous from (X, d) into (X, d').

(b')If $d \neq d'$ then f is continuous from (X, d') into (X, d').

(c')If $d = d'$ then f is continuous at ξ.

Moreover, if φ is continuous, then continuity of f in *(c′)* can be omitted.

Remark 2.2.6. If $\varphi(t) = qt \quad (0 < q < 1)$, then Theorem 2.2.4 generalizes Theorem 1.1.26 ([1, Theorem 2.1]) and Theorem 2.2.5 generalizes Theorem 1.1.27 ([1, Theorem 2.2]). In addition if $d = d'$ then Theorem 2.2.4 and 2.2.5 reduces to [1, Corollary 2.2 and 2.4] respectively. Note that unlike the results of [1] our results also provide the uniqueness of fixed point.

2.3 Application: A homotopy result

In this section, we obtain a homotopy result as an application of Theorem 2.2.4. The proof of the result is inspired by the [2, Theorem 2.4]. We begin with the following Lemma whose proof is given in the proof of [2, Theorem 2.4]. The section includes its detail for completeness.

Lemma 2.3.1. If $\{s_n\}$ is a sequence of nonnegative real numbers and $\varphi : J \to J$ is a gauge function of the order $r \geq 1$ then $\liminf \varphi(s_n) \leq \varphi(\liminf s_n)$.

Proof. Fix $\epsilon > 0$ and $k \in \mathbb{N}$. Note that there exists $m > k$ with

$$s_m \leq \inf\{s_n : n > k\} + \epsilon.$$

Since φ is nondecreasing we have

$$\varphi(s_m) \leq \varphi(\inf\{s_n : n > k\} + \epsilon) \leq \varphi(\liminf s_n + \epsilon),$$

and so

$$\inf\{\varphi(s_n) : n > k\} \leq \varphi(\liminf s_n + \epsilon).$$

Hence, $\liminf \varphi(s_n) \leq \varphi(\liminf s_n)$. □

Theorem 2.3.2. Let $U \subseteq X$ be d-open subset of X. Suppose $H : \overline{U^{d'}} \times [0,1] \to X$ satisfies the following properties:

(i) $x \neq H(x,\lambda)$ for $x \in \overline{U^{d'}} \setminus U$ and $\lambda \in [0,1]$;

(ii) For every $x, y \in \overline{U^{d'}}$

$$d(H(x,\lambda), H(y,\lambda)) \leq \varphi\Big(\max\Big\{d(x,y), d(x,H(x,\lambda)), d(y,H(y,\lambda)), \frac{1}{2}[d(x,H(y,\lambda)) + d(y,H(x,\lambda))]\Big\}\Big);$$

where φ is a gauge function of the order $r \geq 1$ on an interval $J = [0,\infty)$, $\lambda \in [0,1]$ and;

(iii) $H(x,\lambda)$ is continuous in λ w.r.t. d, uniformly for $x \in \overline{U^{d'}}$.

(iv) If $d \not\geq d'$ assume H is uniformly continuous from $U \times [0,1]$ endowed with the metric d on U into (X, d').

(v) If $d \neq d'$ assume H is continuous from $\overline{U^{d'}} \times [0,1]$ endowed with the metric d' on $\overline{U^{d'}}$ into (X, d').

(vi) If $d = d'$ then f is continuous.

(vii) $\inf\{d(x, H_\lambda(x)) : x \in \overline{U^{d'}} \setminus U, \lambda \in [0,1]\} > 0$.

Moreover, if H_0 has a fixed point then, for each $\lambda \in [0,1]$, H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(.) = H(., \lambda)$).

Proof. Let

$$A = \{\lambda \in [0,1] : x = H(x,\lambda) \text{ for some } x \in U\}.$$

Since $H(.,0)$ has a fixed point and (i) holds we have that $0 \in A$, so A is nonempty. We will show A is both closed and open in $[0,1]$, and so by connectedness of $[0,1]$ we have $A = [0,1]$.
First we show A is closed in [0,1]. Let (λ_k) be a sequence in A with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition for k, there exists $x_k \in U$ with $x_k = H(x_k, \lambda_k)$. We claim that,

$$\inf_{k\geq 1} d(x_k, \overline{U^{d'}} \setminus U) > 0. \tag{2.3.1}$$

Otherwise, for a fixed $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ and $y_{n_i} \in \overline{U^{d'}} \setminus U$ with

$$d(x_{n_i}, y_{n_i}) < \frac{1}{i}.$$

Consequently, there exists a subsequence K of $\{1, 2, \cdots\}$ and a sequence $\{y_i\} \subseteq \overline{U^{d'}} \setminus U$ (for $i \in K$) with

$$d(x_i, y_i) < \frac{1}{i} \text{ for } i \in K. \tag{2.3.2}$$

This together with (vii) implies

$$0 < \inf\{d(x, H_{\lambda(x)}) : x \in \overline{U^{d'}} \setminus U, \lambda \in [0,1]\} \leq \lim_{i\to\infty} \inf_{i \in K} d(y_i, H_{\lambda_i}(y_i)). \tag{2.3.3}$$

We will now show that

$$\lim_{i\to\infty} \inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) = 0. \tag{2.3.4}$$

To see this, note that

$$\begin{aligned} \lim_{i\to\infty} \inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) &\leq \lim_{i\to\infty} \inf_{i\in K} [d(y_i, x_i) + d(x_i, H_{\lambda_i}(y_i))] \\ &\leq \lim_{i\to\infty} \inf_{i\in K} [\frac{1}{i} + d(H(x_i,\lambda_i), H(y_i,\lambda_i))] \\ &= \lim_{i\to\infty} \inf_{i\in K} d(H(x_i,\lambda_i), H(y_i,\lambda_i)) \\ &\leq \lim_{i\to\infty} \inf_{i\in K} \varphi(\max\{d(x_i,y_i), d(x_i, H(x_i,\lambda_i)), \\ &\quad d(y_i, H(y_i,\lambda_i)), \frac{1}{2}[d(x_i, H(y_i,\lambda_i)) + d(y_i, H(x_i,\lambda_i))]\}). \end{aligned}$$

Let

$$m(x_i, y_i) = \max\Big\{d(x_i, y_i), d(x_i, H(x_i, \lambda_i)), d(y_i, H(y_i, \lambda_i)), \frac{1}{2}[d(x_i, H(y_i, \lambda_i)) + d(y_i, H(x_i, \lambda_i))]\Big\}.$$

Case 1: If $m(x_i, y_i) = d(x_i, y_i)$ then

$$\begin{aligned}\lim_{i\to\infty}\inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) &\leq \lim_{i\to\infty}\inf_{i\in K} \varphi(d(x_i, y_i)) \\ &< \lim_{i\to\infty}\inf_{i\in K} d(x_i, y_i) = \lim_{i\to\infty}\inf_{i\in K} \frac{1}{i} = 0.\end{aligned}$$

Hence (2.3.4) is true in this case.

Case 2: If $m(x_i, y_i) = d(x_i, H(x_i, \lambda_i))$ then $m(x_i, y_i) = 0$ and (2.3.4) is immediate.

Case 3: If $m(x_i, y_i) = d(y_i, H(y_i, \lambda_i))$ then by Lemma 2.3.1 we have

$$\lim_{i\to\infty}\inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) \leq \lim_{i\to\infty}\inf_{i\in K} \varphi(d(y_i, H_{\lambda_i}(y_i))) \leq \varphi(\lim_{i\to\infty}\inf_{i\in K} d(y_i, H_{\lambda_i}(y_i))),$$

which implies $\lim_{i\to\infty}\inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) = 0$ since $\varphi(z) < z$ if $z > 0$. Hence (2.3.4) is true.

Case 4: Finally, if $m(x_i, y_i) = \frac{1}{2}[d(x_i, H(y_i, \lambda_i)) + d(y_i, H(x_i, \lambda_i))]\}$ then $\varphi(m(x_i, y_i)) \leq m(x_i, y_i)$ implies that

$$\begin{aligned}\lim_{i\to\infty}\inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) &\leq \lim_{i\to\infty}\inf_{i\in K} \frac{1}{2}[d(x_i, H(y_i, \lambda_i)) + d(y_i, H(x_i, \lambda_i))] \\ &\leq \lim_{i\to\infty}\inf_{i\in K} \frac{1}{2}[d(x_i, y_i) + d(y_i, H(y_i, \lambda_i)) + \\ &\qquad d(x_i, y_i) + d(x_i, H(x_i, \lambda_i))] \\ &\leq \lim_{i\to\infty}\inf_{i\in K} \frac{1}{2}[\frac{1}{i} + d(y_i, H_{\lambda_i}(y_i)) + \frac{1}{i} + 0].\end{aligned}$$

Hence (2.3.4) is immediate. Thus we have a contradiction from (2.3.3) and as a result our claim (2.3.1) is true. Hence there exists $s > 0$ with $d(x_k, z) > s$ for all $k \geq 1$ and for all $z \in \overline{U^{d'}} \setminus U$. As a result (note $x_k \in U$ for each k)

$$\overline{S}(x_k, s)^{d'} \subseteq U \ k \geq 1.$$

If we choose δ such that $\sigma(\delta) < s$ then $\overline{S}(x_k, \delta)^{d'} \subseteq U$ for $k \geq 1$. This together with (iii) implies that there exists an integer n_0 with

$$d(x_{n_0}, H_\lambda(x_{n_0})) = d(H(x_{n_0}, \lambda), H(x_{n_0}, \lambda_{n_0})) < \delta.$$

Now Theorem 2.2.4 guarantees that H_λ has a fixed point $x_{\lambda,n_0} \in \overline{S}(x_k, \delta)^{d'} \subseteq \overline{U^{d'}}$. Note from (i) that $x_{\lambda,n_0} \in U$. Consequently, $\lambda \in A$, so A is closed in $[0, 1]$.
Next we will show that A is open in $[0, 1]$. Let $\lambda_0 \in A$ and $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since U is open there exists $\epsilon > 0$ with $S(x_0, \epsilon) \subseteq U$. Now (iii) guarantees that there exists $\eta = \eta(\epsilon, \delta) > 0$ with

$$d(x_0, H(x_0, \lambda)) = d(H(x_0, \lambda_0), H(x_0, \lambda)) < \delta$$

for $\lambda \in [0, 1]$ and $|\lambda - \lambda_0| < \eta$. We may choose δ such that $\sigma(\delta) < \epsilon$. Now Theorem 2.2.4 guarantees that there exists $x_\lambda \in \overline{S}(x_0, r)^{d'} \subseteq \overline{U^{d'}}$ with $x_\lambda = H_\lambda(x_\lambda)$ for $\lambda \in [0, 1]$ and $|\lambda - \lambda_0| < \eta$. Hence A is open in $[0, 1]$. □

Remark 2.3.3. If $\varphi(t) = qt \ \ (0 < q < 1)$, then Theorem 2.3.2 extends Theorem 1.1.31 ([1, Theorem 3.1]) and thus some results of [48, 21, 41].

Chapter 3

Some generalizations of the Nadler fixed point theorem

This chapter consists on four sections. The motivation behind the first two sections is to extend the Nadler fixed point theorem by using the machinery introduced in section 1.1.4. We succeeded in obtaining error bounds and high order of convergence of the iterative scheme. In the first section, the partial generalization of Nadler theorem is obtained when the range of the multi-valued map T is $PC(X)$. In the second section, we extend the Nadler fixed point theorem by combining the techniques of Proinov[49] and Agarwal [2]. The range of the multi-valued map T in this case is $CB(X)$. Another strength of these results is their weaker contraction condition i.e; the contraction condition on the map T, which does not need to satisfy for every $x, y \in X$ like Nadler theorem demands, but only require satisfaction for every $x \in X$ and $y \in Tx$. The results contains several other important theorems which are mentioned in detail with references in the sequel. In the third section, as an application to my result, an existence and uniqueness result for the solution to a class of integral inclusions is obtained with high order of convergence. In the last section, another extension of Nadler theorem is obtained by altering the distance d on the space X by introducing a class of functions $\Theta_h[0, A)$, improving some results of Pathak and Shahzad [45].

3.1 Multi-valued φ-contractions

In this section, we present the results that extend Proinov's results Theorems 1.1.23 and 1.1.24 ([49, Theorems 4.1 and 4.2]) and thus generalize some results of

Rheinboldt [56], Gel'man [20], Ciric [14], Rus [57], Hicks and Rhoades [24], Hicks [23] and others. Theorem 3.1.15 (below) also provide a generalization of Nadler fixed point theorem [40] in the case when T is a map from X into $PC(X)$. The presented results may be useful for computational methods of solutions to optimization problems (see Polac [47] and Zangwill [64] for instance) and in system theory (see Willems [62] for instance).

Definition 3.1.1. Let $T : X \to N(X)$. A sequence $\{x_n\} \subseteq X$ satisfying $x_{n+1} \in Tx_n$ with $d(x_n, x_{n+1}) = d(x_n, Tx_n)$ for $n = 0, 1, \cdots$, is called a *proximinal orbit* of T at the point x_0.

Example 3.1.2. Let $X = [0, 1]$ with usual metric and $T : X \to N(X)$ defined by $Tx = [0, x^2]$. Take $x_0 = \frac{1}{2}$. Then, $T\frac{1}{2} = [0, \frac{1}{4}]$ and $d(\frac{1}{2}, T\frac{1}{2}) = \frac{1}{4}$. So, there is a point $x_1 = \frac{1}{4} \in T\frac{1}{2}$ such that $d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{2}, T\frac{1}{2})$. Further, $x_2 = \frac{1}{16} \in T\frac{1}{4}$ such that $d(\frac{1}{4}, \frac{1}{16}) = d(\frac{1}{4}, T\frac{1}{4})$. Continuing this procedure we obtain $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \cdots$ as a proximinal orbit of T at the point $\frac{1}{2}$.

Following Lemma is obvious.

Lemma 3.1.3. Let $T : X \to PC(X)$ and $x_0 \in X$. Then there exists a proximinal orbit $\{x_n\} \subseteq X$ of T at the point x_0, i.e.,

$$x_{n+1} \in Tx_n, \quad n = 0, 1, 2, \cdots . \tag{3.1.1}$$

with

$$d(x_n, x_{n+1}) = d(x_n, Tx_n). \tag{3.1.2}$$

Unless otherwise stated, we assume that $D \subset X$ and T is an operator from D into $PC(X)$ satisfying

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{3.1.3}$$

for all $x \in D$, $Tx \subset D$ with $d(x, y) \in J$ and for $y \in Tx$, where φ is a Bianchini-Grandolfi gauge function on the interval J. For convenience we define the function $E : D \to \mathbb{R}_+$ by $E(x) = d(x, Tx)$.

Lemma 3.1.4. Let $x, y \in X$. Suppose $d(x, y) \in J$ for $y \in Tx$. Then $E(x) \in J$.

Proof. Since $0 \in J$ and J is an interval. Therefore, $0 \leq d(x, Tx) \leq d(x, y)$ implies $E(x) \in J$. □

Lemma 3.1.5. Suppose a point $x_0 \in X$ satisfies $E(x_0) \in J$. Then, $E(x_n) \in J$ for any $n \geq 0$.

Proof. For $n = 0$, the Lemma is trivial. For $n \geq 1$, using Lemma 3.1.3 we get x_n such that $x_n \in D$, since $Tx \subset D$ for all $x \in D$. As, $0 \in J$ and $\varphi(t) \in J$ for all $t \in J$ we have, using (3.1.3), that

$$\begin{aligned} 0 \leq E(x_n) = d(x_n, Tx_n) &\leq H(Tx_{n-1}, Tx_n) \\ &\leq \varphi(E(x_{n-1})) \text{ (using (3.1.3), as } x_n \in Tx_{n-1}). \end{aligned}$$

Substituting $n = 1$, we get

$$0 \leq E(x_1) \leq \varphi(E(x_0)) \in J \Rightarrow E(x_1) \in J.$$

By induction, for any $n \geq 0$, we have $E(x_n) \in J$. □

Definition 3.1.6. Suppose $x_0 \in D$ and $E(x_0) \in J$. Then for every iterate $x_n (n \geq 0)$ which belongs to D we define the closed ball $\overline{S}(x_n, \rho_n)$ with center x_n and radius $\rho_n = \sigma(E(x_n))$, where $\sigma : J \to \mathbb{R}_+$ is defined by (1.1.4).

Lemma 3.1.7. Suppose a point $x_0 \in D$ satisfies $E(x_0) \in J$ and $\overline{S}(x_n, \rho_n) \subset D$ for some $n \geq 0$. Then $x_{n+1} \in D$ and $\overline{S}(x_{n+1}, \rho_{n+1}) \subset \overline{S}(x_n, \rho_n)$.

Proof. For each $n \geq 0$, x_{n+1} is well-defined, since $x_n \in D$. Further, Lemma 3.1.5 implies that $E(x_n) \in J$. As, $\sigma(t) \geq t$ for all $t \in J$. Therefore, using Lemma 3.1.3 we have

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) = E(x_n) \leq \sigma(E(x_n)) = \rho_n.$$

Hence, $x_{n+1} \in \overline{S}(x_n, \rho_n) \subset D$. Now we shall prove the second part of Lemma. First note that

$$E(x_{n+1}) = d(x_{n+1}, Tx_{n+1}) \leq H(Tx_n, Tx_{n+1}) \leq \varphi(E(x_n)). \tag{3.1.4}$$

Let $x \in \overline{S}(x_{n+1}, \rho_{n+1})$, then, $d(x, x_{n+1}) \leq \rho_{n+1}$. Using the triangle inequality, we have

$$\begin{aligned} d(x,x_n) &\le d(x,x_{n+1}) + d(x_{n+1},x_n) \\ &\le \rho_{n+1} + d(x_n,Tx_n) = \sigma(E(x_{n+1})) + E(x_n) \\ &\le \sigma(\varphi(E(x_n))) + E(x_n) = \rho_n \text{ (using (1.1.5 \& 3.1.4))}, \end{aligned}$$

since σ is non-decreasing. Hence, $x \in \overline{S}(x_n,\rho_n)$ and so $\overline{S}(x_{n+1},\rho_{n+1}) \subset \overline{S}(x_n,\rho_n)$. □

Definition 3.1.8. *(Initial points).* A point $x_0 \in D$ will be called an *initial point* of T if $E(x_0) \in J$ and all of the iterates $x_0, x_1, x_2, \cdots$ are well-defined and belong to D.

Lemma 3.1.9. *(Test for initial points).* Assume $x_0 \in D$ satisfies $E(x_0) \in J$ and there exists an integer $n \ge 0$ such that $\overline{S}(x_n,\rho_n) \subset D$. Then x_0 is an initial point of T.

Proof. Lemma 3.1.7 insure that $x_{n+1} \in D$ and $\overline{S}(x_{n+1},\rho_{n+1}) \subset \overline{S}(x_n,\rho_n)$ for $n \ge 0$. This implies that $x_0, x_1, x_2, \cdots$ are well-defined and belong to D, so x_0 is an initial point of T. □

Lemma 3.1.10. For every initial point $x_0 \in D$ of T and every $n \ge 0$ we have

$$E(x_{n+1}) \le \varphi(E(x_n))$$

and

$$E(x_n) \le \varphi^n(E(x_0)).$$

In addition, if φ is a gauge function of order $r \ge 1$, then

$$E(x_n) \le E(x_0)\lambda^{S_n(r)}$$

and

$$\phi(E(x_n)) \le \lambda^{r^n},$$

where $\lambda = \phi(E(x_0))$ and ϕ is a nonnegative and nondecreasing function on J satisfying (1.1.3).

Proof. Note that $x_{n+1} \in D$, since x_0 is an initial point of T. As $x_{n+1} \in Tx_n$, using (3.1.3), we get

$$E(x_{n+1}) = d(x_{n+1}, Tx_{n+1}) \leq H(Tx_n, Tx_{n+1}) \leq \varphi(E(x_n)).$$

As, φ is non-decreasing on J we have

$$\begin{aligned} E(x_n) &\leq \varphi(E(x_{n-1})) \\ &\leq \varphi^2(E(x_{n-2})) \cdots \leq \varphi^n(E(x_0)). \end{aligned}$$

If φ is a gauge function of order $r \geq 1$, then from Lemma 1.1.19-(i)

$$E(x_n) \leq \varphi^n(E(x_0)) \leq E(x_0)(\phi(E(x_0)))^{S_n(r)} = E(x_0)\lambda^{S_n(r)}.$$

Further, from Lemma 1.1.19-(ii), we have

$$\phi(E(x_0)) \leq \phi(\varphi^n(E(x_0)) \leq (\phi(E(x_0)))^{r^n} = \lambda^{r^n},$$

since ϕ is non-decreasing.
From Lemma 3.1.10 we get the following Lemma, which is muti valued version of Proinov [49, Lemma 3.7], i.e., here $E(x) = d(x, Tx)$ where T is a map from X into $PC(X)$ satisfying (3.1.3). □

Lemma 3.1.11. *(Bounds for inclusion radii)* Suppose x_0 is an initial point of T and φ is a gauge function of order $r \geq 1$. Then for the radii $\rho_n = \sigma(E(x_n));\ n = 0, 1, 2, ...$ the following estimates hold:

$$(i)\ \rho_n \leq E(x_n)\sum_{j=0}^{\infty}[\phi(E(x_n))]^{S_j(r)} \leq \frac{E(x_n)}{1-\phi(E(x_n))}; \quad (3.1.5)$$

$$(ii)\ \rho_n \leq E(x_n)\sum_{j=0}^{\infty}[\lambda^{r^n}]^{S_j(r)} \leq \frac{E(x_n)}{1-\lambda^{r^n}}; \quad (3.1.6)$$

$$(iii)\ \rho_n \leq E(x_0)\lambda^{S_n(r)}\sum_{j=0}^{\infty}[\lambda^{r^n}]^{S_j(r)} \leq \frac{\lambda^{S_n(r)}E(x_0)}{1-\lambda^{r^n}}; \quad (3.1.7)$$

$$(iv)\ \rho_{n+1} \leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(\varphi(E(x_n)))]^{S_j(r)} \leq \frac{\varphi(E(x_n))}{1-\phi(\varphi(E(x_n)))}; \quad (3.1.8)$$

$$(v)\ \rho_{n+1} \leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\lambda^{r^n}]^{S_j(r)} \leq \frac{\varphi(E(x_n))}{1-\lambda^{r^n}}; \quad (3.1.9)$$

where $\lambda = \phi(E(x_0))$.

Proof. We will make use of the following facts during the proof.

$$0 \leq \phi(t) < 1, S_j(r) \geq j \text{ and } 0 \leq \lambda^{r^n} < 1.$$

(i) Using Lemma 1.1.19-(i) and definition of ρ_n we have

$$\begin{aligned}
\rho_n = \sigma(E(x_n)) &= \sum_{j=0}^{\infty} \varphi^j(E(x_n)) \\
&\leq \sum_{j=0}^{\infty} E(x_n)[\phi(E(x_n))]^{S_j(r)} \\
&= E(x_n)\sum_{j=0}^{\infty}[\phi(E(x_n))]^{S_j(r)} \\
&\leq E(x_n)\sum_{j=0}^{\infty}[\phi(E(x_n))]^j = \frac{E(x_n)}{1-\phi(E(x_n))}.
\end{aligned}$$

(ii) From (3.1.5)

$$\begin{aligned}
\rho_n &\leq E(x_n)\sum_{j=0}^{\infty}[\phi(E(x_n))]^{S_j(r)} \\
&\leq E(x_n)\sum_{j=0}^{\infty}[\lambda^{r^n}]^{S_j(r)} \quad \text{(using Lemma 3.1.10)} \\
&\leq E(x_n)\sum_{j=0}^{\infty}[\lambda^{r^n}]^j = \frac{E(x_n)}{1-\lambda^{r^n}}.
\end{aligned}$$

(iii) From(3.1.6)

$$\begin{aligned}
\rho_n &\leq E(x_n)\sum_{j=0}^{\infty}[\lambda^{r^n}]^{S_j(r)} \\
&\leq E(x_0)\lambda^{S_n(r)}\sum_{j=0}^{\infty}[\lambda^{r^n}]^{S_j(r)} \quad \text{(using Lemma 3.1.10)} \\
&\leq E(x_0)\lambda^{S_n(r)}\sum_{j=0}^{\infty}[\lambda^{r^n}]^j \\
&\leq E(x_0)\frac{\lambda^{S_n(r)}}{1-\lambda^{r^n}}.
\end{aligned}$$

(iv) Since $E(x_{n+1}) \leq \varphi(E(x_n))$ and ϕ is non-decreasing, using Lemma 1.1.19 we have

$$\begin{aligned}
\rho_{n+1} = \sigma(E(x_{n+1})) &= \sum_{j=0}^{\infty} \varphi^j(E(x_{n+1})) \\
&\leq E(x_{n+1}) \sum_{j=0}^{\infty} [\phi(E(x_{n+1}))]^{S_j(r)} \\
&\leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\phi(\varphi(E(x_n)))]^{S_j(r)} \\
&\leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\phi(\varphi(E(x_n)))]^{j} \\
&= \frac{\varphi(E(x_n))}{1 - \phi(\varphi(E(x_n)))}.
\end{aligned}$$

(v) Using (3.1.8) we get

$$\begin{aligned}
\rho_{n+1} &\leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\phi(E(x_{n+1}))]^{S_j(r)} \\
&\leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\lambda^{r^{n+1}}]^{S_j(r)} \text{ (using Lemma 3.1.10)} \\
&= \varphi(E(x_n)) \sum_{j=0}^{\infty} [\lambda^{r^{n}}]^{rS_j(r)} \\
&\leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\lambda^{r^{n}}]^{S_{j+1}(r)-1} \text{ (using definition of } S_j(r)) \\
&\leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\lambda^{r^{n}}]^{j+1-1} \\
&= \frac{\varphi(E(x_n))}{1 - \lambda^{r^n}}.
\end{aligned}$$

□

Theorem 3.1.12. Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying (3.1.3) with a Bianchini-Grandolfi gauge function φ on an interval J. Then, starting from an initial point x_0 of T the iterative sequence $\{x_n\}$ remains in $\overline{S}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$; $n = 0, 1, \cdots$, where $\rho_n = \sigma(d(x_n, Tx_n))$ and σ

is defined by (1.1.4). Moreover, for each $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)).$$

If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

Proof. Since x_0 is an initial point of T, it follows from Lemma 3.1.7 that

$$\overline{S}(x_{n+1}, \rho_{n+1}) \subset \overline{S}(x_n, \rho_n) \text{ for all } n \geq 0, \tag{3.1.10}$$

which implies that $x_n \in \overline{S}(x_0, \rho_0)$ for all $n \geq 0$. Using the definition of ρ_n, from Lemma 3.1.10, we get

$$\begin{aligned} \rho_n = \sigma(E(x_n)) &\leq \sigma(\varphi^n(E(x_0))) \\ &= \sum_{j=0}^{\infty} \varphi^j(\varphi^n(E(x_0))) \\ &= \sum_{j=n}^{\infty} \varphi^j(E(x_0)) \text{ for all } n \geq 0, \end{aligned} \tag{3.1.11}$$

since σ is non-decreasing. As φ is Bianchini-Grandolfi gauge function on J and $E(x_0) \in J$, from (3.1.11), we get

$$\rho_n \to 0 \text{ as } n \to \infty. \tag{3.1.12}$$

From (3.1.10) and (3.1.12) it follow that $\{\overline{S}(x_n, \rho_n)\}$ is nested sequence of closed spheres such that $\rho_n \to 0$ as $n \to \infty$. Therefore, it follows from Cantor's Theorem that there exists a unique point ξ such that $\xi \in \overline{S}(x_n, \rho_n)$ for all $n \geq 0$ and $x_n \to \xi$. In other words, $lim_{n\to\infty} d(x_n, \xi) = 0$. Now,

$$d(\xi, Tx_n) \leq d(\xi, x_n) + d(x_n, Tx_n) = d(\xi, x_n) + d(x_n, x_{n+1}).$$

Taking limit as $n \to \infty$ we get,

$$lim_{n\to\infty} d(\xi, Tx_n) \leq 0.$$

If $\xi \in D$ and T is continuous at ξ then we have $d(\xi, T\xi) = 0$ which simply means that $\xi \in T\xi$. Moreover, from Lemma 3.1.10 we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, Tx_n) = E(x_n) &\leq \varphi(E(x_{n-1})) \\ &= \varphi(d(x_{n-1}, Tx_{n-1})) \\ &= \varphi(d(x_{n-1}, x_n)). \end{aligned}$$

□

Corollary 3.1.13. Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying

$$H(Tx, Ty) \leq \varphi(d(x, Tx))$$

for all $x \in D$, $Tx \subset D$ with $d(x, y) \in J$ and for $y \in Tx$, with a Bianchini-Grandolfi gauge function φ on an interval J. Then, starting from an initial point x_0 of T the iterative sequence $\{x_n\}$ remains in $\overline{S}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$; $n = 0, 1, \cdots$, where $\rho_n = \sigma(d(x_n, Tx_n))$ and σ is defined by (1.1.4). Moreover, for each $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)).$$

If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

Remark 3.1.14. Note that when T is a single valued map from $D \subset X$ into X Theorem 3.1.12 extends Theorem 1.1.23 (Proinov [49, Theorem 4.1]) and thus generalize/extend some results of Bianchini-Grandolfi [9] and Hicks [23, Theorems 1 and 2].

Theorem 3.1.15. Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying (3.1.3) with a gauge function φ of the order $r \geq 1$ on an interval J. Suppose that $x_0 \in D$ is an initial point of T. Then the following statements hold true.

(i) The iterative sequence (3.1.1) remains in $\overline{S}(x_0, \rho_0)$ and converges with rate of convergence at least r to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n), n = 0, 1, \cdots$, where

$$\rho_n = d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))} \quad (3.1.13)$$

and ϕ is a non-negative and non-decreasing function on J satisfying (1.1.3).

(ii) For all $n \geq 0$ we have the following a priori estimate

$$d(x_n, \xi) \leq d(x_0, Tx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \frac{\lambda^{S_n(r)} d(x_0, Tx_0)}{1 - \lambda^{r^n}}, \tag{3.1.14}$$

where $\lambda = \phi(d(x_0, Tx_0))$.

(iii) For all $n \geq 1$ we have the following a posteriori estimate

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \end{aligned} \tag{3.1.15}$$

(iv) For all $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, Tx_0). \tag{3.1.16}$$

(v) If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

Proof. (i) Theorem 3.1.12 insures that the iterative sequence (3.1.1) remains in $\overline{S}(x_0, \rho_0)$ and converges to ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$, $n = 0, 1, \cdots$. Further, from Lemma 3.1.11, estimate (3.1.5), we have (by using $E(x_n) = d(x_n, x_{n+1})$)

$$\rho_n \leq d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))}.$$

(ii) For $m > n$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &= E(x_n) + E(x_{n+1}) + ... + E(x_{m-1}) \\ &\leq E(x_0)[\lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + ... + \lambda^{S_{m-1}(r)}] \qquad \text{(by Lemma 3.1.10)} \\ &= E(x_0) \sum_{j=n}^{m-1} \lambda^{S_j(r)}. \end{aligned}$$

Keeping n fixed and letting $m \to \infty$, we get

$$d(x_n, \xi) \leq E(x_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} = d(x_0, Tx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)}. \tag{3.1.17}$$

Note that,

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{S_j(r)} &= \lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots \\ &= \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + \cdots] \end{aligned}$$

Since $r \geq 1$, therefore

$$r^n + r^{n+1} \geq 2r^n, \;\; r^n + r^{n+1} + r^{n+2} \geq 3r^n \cdots$$

and hence,

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n}, \;\; \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n} \cdots ,$$

since $0 < \lambda < 1$. Thus,

$$\sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + \cdots] = \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}$$

Substituting this in (3.1.17), we get

$$d(x_n, \xi) \leq E(x_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} = d(x_0, Tx_0) \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

(iii) From (3.1.14) we have for $n \geq 0$,

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} [\phi(d(x_0, x_1))]^{S_j(r)}.$$

Putting $n = 0, y_0 = x_n$ and $y_1 = x_1$ we have,

$$d(y_0, \xi) \;\; \leq \;\; d(y_0, y_1) \sum_{j=0}^{\infty} [\phi(d(y_0, y_1))]^{S_j(r)}.$$

Putting $y_0 = x_n$, and $y_1 = x_{n+1}$ we have,

$$\begin{aligned} d(x_n, \xi) &\leq d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \qquad (3.1.18) \\ &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{j} \\ &= \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))}, \qquad (3.1.19) \end{aligned}$$

since $S_j(r) \geq j$. Now by Lemma 1.1.19-(ii), we have

$$\phi(\varphi(d(x_n, x_{n-1}))) \leq [\phi(d(x_n, x_{n-1}))]^r$$

which means that,

$$\frac{1}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \leq \frac{1}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \qquad (3.1.20)$$

From (3.1.18) we get for $n \geq 1$,

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r} \quad \text{(using (3.1.20))} \end{aligned}$$

(iv) For $n \geq 1$, using Lemma 3.1.10 we have

$$\begin{aligned} d(x_{n+1}, x_n) = E(x_n) &\leq \varphi(E(x_{n-1})) \\ &= E(x_{n-1})\phi(E(x_{n-1})) \text{ (using Lemma 1.1.18)} \\ &\leq E(x_0)\lambda^{S_{n-1}(r)}\phi(E(x_{n-1})) \\ &\leq E(x_0)\lambda^{S_{n-1}(r)}\lambda^{r^{n-1}} \\ &= E(x_0)\lambda^{S_{n-1}(r)+r^{n-1}} \\ &= E(x_0)\lambda^{S_n(r)} \\ &= \lambda^{S_n(r)}d(x_0, Tx_0). \end{aligned}$$

(v) Proof is similar as in the proof of Theorem 3.1.12. □

Corollary 3.1.16. Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying

$$H(Tx, Ty) \leq \varphi(d(x, Tx))$$

for all $x \in D$, $Tx \subset D$ with $d(x, y) \in J$ and for $y \in Tx$, with a gauge function φ of the order $r \geq 1$ on an interval J. Suppose that $x_0 \in D$ is an initial point of T. Then the following statements hold true.

(i) The iterative sequence *(3.1.1)* remains in $\overline{S}(x_0, \rho_0)$ and converges with rate of convergence at least r to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n), n = 0, 1, \cdots$, where

$$\rho_n = d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))} \quad (3.1.21)$$

and ϕ is a non-negative and non-decreasing function on J satisfying (1.1.3).

(ii) For all $n \geq 0$ we have the following a priori estimate

$$d(x_n, \xi) \leq d(x_0, Tx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \frac{\lambda^{S_n(r)} d(x_0, Tx_0)}{1 - \lambda^{r^n}}, \quad (3.1.22)$$

where $\lambda = \phi(d(x_0, Tx_0))$.

(iii) For all $n \geq 1$ we have the following a posteriori estimate

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \end{aligned} \tag{3.1.23}$$

(iv) For all $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, Tx_0). \tag{3.1.24}$$

(v) If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

Remark 3.1.17. Note that when T is a single valued map from $D \subset X$ into X Theorem 3.1.12 extends Theorem 1.1.24 (Proinov [49, Theorem 4.2]).

Corollary 3.1.18. Let (X, d) be a complete metric space, $T : X \to PC(X)$ be a continuous operator satisfying

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{3.1.25}$$

for any $x \in X$, $y \in Tx$ with $d(x, y) \in J$ and φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, Tx_0) \in J$. Then, the following statements hold true.

(i) The iterative sequence *(3.1.1)* converges to a fixed point ξ of T.

(ii) The estimates *(3.1.14)* - *(3.1.16)* are valid.

Following simple example shows the generality of our result over the Nadler contraction principle, in the case when T is a map from X int $PC(X)$.

Example 3.1.19. Let $X = [0, \frac{3}{5}]$ equipped with usual metric d. Define $T : X \to PC(X)$ by $Tx = [0, x^2]$. Then, for any $x \in X$ and $y \in Tx$,

$$H(Tx, Ty) \leq \frac{24}{25} d(x, y).$$

By taking $\varphi(t) = \frac{24}{25}t$ and $J = [0, \infty)$, we see that all conditions of Corollary 3.1.18 are satisfied and 0 is fixed point of T. Note that Nadler contraction principle is not applicable here.

Corollary 3.1.20. Let (X, d) be complete metric space, $T : X \to PC(X)$ be an operator satisfying

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{3.1.26}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x, y) \in J$ and φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, Tx_0) \in J$. Then, the following statements hold true.

(i) The iterative sequence (3.1.1) converges to a fixed point $\xi \in S = \{x \in X : d(x, \xi) \in J\}$ of T.

(ii) The estimates (3.1.14) - (3.1.16) are valid.

Proof. It follows from (3.2.24) that

$$H(Tx, Ty) \leq \varphi(d(x, y)) < d(x, y) \tag{3.1.27}$$

for all $x, y \in X$ $(x \neq y)$. Hence T is continuous. Thus *(i)* and *(ii)* follow immediately from Theorem 3.1.15. □

Remark 3.1.21. In the case when T is a mapping from X into $PC(X)$, Nadler contraction principle (Theorem 1.2.16) is a special case of Corollary 3.1.20. To see this, take $\varphi(t) = \lambda t$ $(0 < \lambda < 1)$ and $J = [0, \infty)$.

Corollary 3.1.22. (Proinov [49, Corollary 4.4]) Let (X, d) be complete metric space, $T : X \to X$ be an operator satisfying

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{3.1.28}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x, y) \in J$ and φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, Tx_0) \in J$. Then, the following statements hold true.

(i) The iterative sequence $x_{n+1} = Tx_n$ converges to a fixed point ξ of T.

(ii) The operator T has a unique fixed point in $S = \{x \in X : d(x, \xi) \in J\}$.

(iii) The estimates (3.1.14) - (3.1.16) are valid.

Corollary 3.1.23. Let $T : X \to PC(X)$ be a continuous operator on a complete metric space (X, d) satisfying

$$H(Tx, Ty) \leq \varphi(m(x, y)) \tag{3.1.29}$$

for all $x \in X$, $y \in Tx$ where φ is a gauge function of the first order on $J = [0, \infty)$ and

$$m(x, y) = \max\{d(x, y), d(y, Ty)\} \tag{3.1.30}$$

Then for each $x_0 \in X$ the following statements hold true.

(i) The iterative sequence $\{x_n\}$ converges to a unique fixed point ξ of T.

(ii) For $n \geq 0$ we have the following a priori estimate

$$d(x_n, \xi) \leq \frac{\lambda^n}{1-\lambda} d(x_0, Tx_0). \tag{3.1.31}$$

(iii)For all $n \geq 1$ we have the following a posteriori estimate

$$d(x_n, \xi) \leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]}. \tag{3.1.32}$$

Proof. Let $x \in X$. It follows from (3.1.29) that

$$H(Tx, Ty) \quad \leq \quad \varphi(\max\{d(x, y), d(y, Ty)\}), \tag{3.1.33}$$

for $y \in Tx$. Suppose $\max\{d(x, y), d(y, Ty)\} = d(y, Ty)$, then we have

$$d(y, Ty) \leq H(Tx, Ty) \leq \varphi(d(y, Ty)) < d(y, Ty),$$

which is a contradiction. Therefore, from (3.1.33) we get

$$H(Tx, Ty) \leq \varphi(d(x, y)),$$

for all $x \in X$ with $d(x, y) \in J$ and for $y \in Tx$, which is nothing, but the condition (3.1.3). Applying Theorem 3.1.15 we deduce that the iterative sequence $\{x_n\}$

converges to a point $\xi \in X$. Since $r = 1$ in this case so the a priori and a posteriori estimates (3.1.14) and (3.1.15) becomes (3.1.31) and (3.1.32). The uniqueness of ξ follows from Corollary 3.1.18. □

Remark 3.1.24. Theorem 3.1.15 extends/generalize Theorem 1.1.24 ([49, Theorem 4.2]) and thus extends/generalize Rheinboldt [56], Kornstaedt [35, Satz 4.1], Hicks and Rhoades [23], Park [44, Theorem 2], Gel'man [20, Theorem 3] and Hicks [23, Theorem 3].

3.2 Multi-valued generalized φ-contractions

In the first section we extended some results of Proinov [49] to the case of multi-valued maps from a complete metric space X into the space of all nonempty proximinal closed subsets of X. In this section we present some fixed point theorems for multi-valued maps that generalize [39, Theorem 5], [16, Theorem 2.1], Theorems 3.1.12 and 3.1.15 ([**?**, Theorems 2.11 & 2.15]) and [2, Theorems,2.1 & 2.2].

Theorem 3.2.1. Let (X, d) be a complete metric space, D be a closed subset of X, φ is a Bianchini-Grandolfi gauge function on an interval J and T be a mapping from D into $CB(X)$ such that $Tx \cap D \neq \emptyset$ and

$$H(Tx \cap D, Ty \cap D) \leq \varphi(d(x, y)) \tag{3.2.1}$$

for all $x \in D$, $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $d(x, y) \neq 0$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then:

(i) there exists an orbit $\{x_n\}$ of T in D and $\xi \in D$ such that $\lim_n x_n = \xi$;

(ii) ξ is fixed point of T if and only if the function $f(x) := d(x, Tx \cap D)$ is T-orbitally lower semi continuous at ξ.

Proof. Take $x_1 = z \in Tx_0 \cap D$. We assume that $d(x_0, x_1) \neq 0$, for otherwise x_0 is fixed point of T. Define $\rho_0 = \sigma(d(x_0, x_1))$ where σ is defined by (1.1.4). Since from (1.1.5), $\sigma(t) \geq t$ so we have

$$d(x_0, x_1) \leq \rho_0. \tag{3.2.2}$$

Notice that $x_1 \in \overline{S}(x_0, \rho_0)$. It follow from (3.2.1) that $H(Tx_0 \cap D, Tx_1 \cap D) < \varphi(d(x_0, x_1))$. Choose an $\epsilon_1 > 0$ with

$$H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1 \leq \varphi(d(x_0, x_1)). \tag{3.2.3}$$

It follows from Lemma 1.2.11 that there exists $x_2 \in Tx_1 \cap D$ such that

$$d(x_1, x_2) \leq H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1, \tag{3.2.4}$$

since D is closed and Tx_1 is closed and bounded. We assume that $d(x_1, x_2) \neq 0$, for otherwise x_1 is fixed point of T. From inequalities (3.2.3) and (3.2.4) we have

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{3.2.5}$$

Note that $d(x_1, x_2) \in J$. Further, $x_2 \in \overline{S}(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) \\ &\leq d(x_0, x_1) + \sigma(\varphi(d(x_0, x_1))) \\ &= \sigma(d(x_0, x_1)) \text{ (using (1.1.5))} \\ &= \rho_0. \end{aligned}$$

Now choose $\epsilon_2 > 0$ with

$$H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2 \leq \varphi(d(x_1, x_2)). \tag{3.2.6}$$

It again follows from Lemma 1.2.11 that there exists $x_3 \in Tx_2 \cap D$ such that

$$d(x_2, x_3) \leq H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2. \tag{3.2.7}$$

We assume that $d(x_2, x_3) \neq 0$, for otherwise x_2 is fixed point of T. From inequalities (3.2.5), (3.2.6) and (3.2.7) we have

$$d(x_2, x_3) \leq \varphi^2(d(x_0, x_1)). \tag{3.2.8}$$

Note that $d(x_2, x_3) \in J$. Further, $x_3 \in \overline{S}(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) + \varphi^2(d(x_0, x_1))) \\ &\leq \sum_{j=0}^{\infty} \varphi^j(d(x_0, x_1)) \\ &= \sigma(d(x_0, x_1)) = \rho_0. \end{aligned}$$

Repeating the above argument, inductively we obtain the a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n \in Tx_{n-1} \cap D, \tag{3.2.9}$$

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)), \tag{3.2.10}$$

$$d(x_{n-1}, x_n) \in J, \text{ and } x_n \in \overline{S}(x_0, \rho_0). \tag{3.2.11}$$

We claim that $\{x_n\}$ is a Cauchy sequence. For $n, p \in \mathbb{N}$, from (3.2.10) we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \varphi^{n+p-1}(d(x_0, x_1)) + \cdots + \varphi^n(d(x_0, x_1)) \\ &\leq \sum_{j=n}^{\infty} \varphi^j(d(x_0, x_1)). \end{aligned}$$

Using (1.1.4), it follows from the above inequality that $\{x_n\}$ is a Cauchy sequence. Thus there exists $\xi \in \overline{S}(x_0, \rho_0)$ with $x_n \to \xi$. Note that $\xi \in D$, as well. Since $x_n \in Tx_{n-1} \cap D$, it follow from (3.2.1) that

$$\begin{aligned} d(x_n, Tx_n \cap D) &\leq H(Tx_{n-1} \cap D, Tx_n \cap D) \\ &\leq \varphi(d(x_{n-1}, x_n)) \\ &< d(x_{n-1}, x_n). \end{aligned} \tag{3.2.12}$$

Letting $n \to \infty$, from (3.4.16) we get

$$\lim_{n\to\infty} d(x_n, Tx_n \cap D) = 0. \tag{3.2.13}$$

Suppose $f(x) = d(x, Tx \cap D)$ is T orbitally lower continuous at ξ, then

$$d(\xi, T\xi \cap D) = f(\xi) \leq \liminf_n f(x_n) = \liminf_n d(x_n, Tx_n \cap D) = 0.$$

Hence, $\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is fixed point of T then $f(\xi) = 0 \leq \lim_n \inf f(x_n)$, since $\xi \in D$. □

Example 3.2.2. Let $X = [0, \frac{3}{5}]$ equipped with usual metric d. Define $T : X \to CB(X)$ by $Tx = [0, x^2]$. Then, for any $x, y \in X$

$$\begin{aligned} H(Tx, Ty) &= |x^2 - y^2| = |x + y||x - y| \\ &\leq |x + y| d(x, y). \end{aligned}$$

Therefore, for $x \in [0, \frac{3}{5}]$ and $y \in [0, x^2]$, we have

$$\begin{aligned} H(Tx, Ty) &\leq (\frac{3}{5} + \frac{9}{25}) d(x, y) \\ &= \frac{24}{25} d(x, y). \end{aligned}$$

By taking $\varphi(t) = \frac{24}{25}t$ and $J = [0, \infty)$, we see that all conditions of Theorem 3.2.1 are satisfied and 0 is the only fixed point of T. Observe that for $0 \leq \lambda < 1$,

$$H(T\frac{1}{2}, T\frac{3}{5}) = \frac{11}{100} \nleq \lambda \frac{1}{10} = d(\frac{1}{2}, \frac{3}{5}).$$

Therefore, T does not satisfies the hypothesis of [39, Theorem5] and [16, Theorem 2.1].

In [13, 59, 49] the following class of functions was introduced and studied.

Definition 3.2.3. [13, 59, 49] Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. The function ψ is said to satisfy the condition (Φ) (denoted by $\psi \in (\Phi)$) if

(i) $\psi(t) < t$ for all $t \in (0, \infty)$,

(ii) ψ is upper semicontinuous from the right on $(0, \infty)$ and

(iii) there exists a positive real number s such that ψ is nondecreasing on $(0, s]$ and $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for all $t \in (0, s]$.

Following lemma is an immediate consequence of the third part of Definition 3.2.3 and Definition 1.1.16.

Lemma 3.2.4. Every $\psi \in (\Phi)$ is a Bianchini-Grandolfi gauge function on $J = (0, s]$.

Remark 3.2.5. Chang [13] observed that if $k : (0, \infty) \to [0, 1)$ satisfies

$$\limsup_{s \to t^+} k(s) < 1 \text{ for every } t \in [0, \infty), \tag{3.2.14}$$

then there exists a function $\psi \in (\Phi)$ such $k(t)t \leq \psi(t)$ for all $t \in (0, \infty)$.

As a consequence of above Lemma and Remark we have the following Corollary.

Corollary 3.2.6. ([39, Theorem 5],[16, Theorem 2.1]) Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that T satisfies

$$H(Tx, Ty) \leq k(d(x, y))d(x, y) \tag{3.2.15}$$

for all $x, y \in X$ with $x \neq y$ where $k : (0, \infty) \to [0, 1)$ satisfies (3.2.14). Then T has a fixed point in X.

Remark 3.2.7. In [61], Suzuki proved that Corollary 3.2.6 is a real generalization of Nadler Theorem. As a result, we can see Theorem 3.2.1 as a generalization of Nadler Theorem.

Theorem 3.2.8. Let (X, d) be a complete metric space, D be a closed subset of X, φ is a gauge function of order $r \geq 1$ on an interval J and T be a mapping from D into $CB(X)$ such that $Tx \cap D \neq \emptyset$ and

$$H(Tx \cap D, Ty \cap D) \leq \varphi(d(x, y)) \tag{3.2.16}$$

for all $x \in D$, $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $d(x, y) \neq 0$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then:

(i) there exists an orbit $\{x_n\}$ of T in $\overline{S}(x_0, \rho_0)$ that converges with rate of convergence at least r to a point $\xi \in \overline{S}(x_0, \rho_0)$;

(ii) for all $n \geq 0$ we have the following a priori estimate

$$d(x_n, \xi) \leq \frac{\lambda^{S_n(r)} d(x_0, x_1)}{1 - \lambda^{r^n}}, \tag{3.2.17}$$

where $\lambda = \phi(d(x_0, x_1))$;

(iii) for all $n \geq 1$ we have the following a posteriori estimate

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}; \end{aligned} \tag{3.2.18}$$

(iv) for all $n \geq 1$ we have

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, x_1); \tag{3.2.19}$$

(v) ξ is fixed point of T if and only if the function $f(x) := d(x, Tx \cap D)$ is T-orbitally lower semi continuous at ξ.

Proof. (i) Theorem 3.2.1 insures the existence of an orbit $\{x_n\}$ of T in $\overline{S}(x_0, \rho_0)$ that converges to ξ which belongs to $\overline{S}(x_0, \rho_0)$.
(ii) For $m > n$, using (3.2.10) and Lemma 1.1.19-(i) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq \varphi^n(d(x_0, x_1)) + \varphi^{n+1}(d(x_0, x_1)) + \cdots + \varphi^{m-1}(d(x_0, x_1)) \\ &\leq d(x_0, x_1)[\lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + ... + \lambda^{S_{m-1}(r)}] \\ &= d(x_0, x_1) \sum_{j=n}^{m-1} \lambda^{S_j(r)}. \end{aligned}$$

Keeping n fixed and letting $m \to \infty$, we get

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} \lambda^{S_j(r)}. \tag{3.2.20}$$

Note that,

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{S_j(r)} &= \lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots \\ &= \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + \cdots] \end{aligned}$$

Since $r \geq 1$, therefore

$$r^n + r^{n+1} \geq 2r^n,\ r^n + r^{n+1} + r^{n+2} \geq 3r^n \cdots$$

and hence,

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n},\ \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n} \cdots,$$

since $0 < \lambda < 1$. Thus,

$$\sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + \cdots] = \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}$$

Substituting this in (3.2.20), we get

$$d(x_n, \xi) \leq d(x_0, x_1)\frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

(iii) From (3.2.20) we have for $n \geq 0$,

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} [\phi(d(x_0, x_1))]^{S_j(r)}.$$

Putting $n = 0, y_0 = x_n$ and $y_1 = x_1$ we have,

$$d(y_0, \xi) \quad \leq \quad d(y_0, y_1) \sum_{j=0}^{\infty} [\phi(d(y_0, y_1))]^{S_j(r)}.$$

Putting $y_0 = x_n$, and $y_1 = x_{n+1}$ we have,

$$\begin{aligned} d(x_n, \xi) &\leq d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} & (3.2.21) \\ &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{j} \\ &= \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))}, & (3.2.22) \end{aligned}$$

since $S_j(r) \geq j$. Now by Lemma 1.1.19-(ii), we have

$$\phi(\varphi(d(x_n, x_{n-1}))) \leq [\phi(d(x_n, x_{n-1}))]^r$$

which means that,

$$\frac{1}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \leq \frac{1}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \tag{3.2.23}$$

From (3.2.21) we get for $n \geq 1$,

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_0^\infty [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r} \quad \text{(using (3.2.23))} \end{aligned}$$

(iv) For $n \geq 1$, using (3.2.10) and Lemma 1.1.19 we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \varphi(d(x_n, x_{n-1})) \\ &\leq d(x_0, x_1)\lambda^{S_{n-1}(r)}\phi(d(x_{n-1}, x_n)) \\ &\leq d(x_0, x_1)\lambda^{S_{n-1}(r)}\lambda^{r^{n-1}} \\ &= d(x_0, x_1)\lambda^{S_{n-1}(r)+r^{n-1}} \\ &= d(x_0, x_1)\lambda^{S_n(r)} \end{aligned}$$

(v) Proof is similar as in the proof of Theorem 3.2.1. □

Remark 3.2.9. We can note the rate of convergence from the a priori estimate (3.2.17) through the standard technique as follows:

$$\frac{d(x_{n+1}, \xi)}{(d(x_n, \xi))^r} = \frac{\lambda^{S_{n+1}(r)}d(x_0, x_1)}{1 - \lambda^{r^{n+1}}}\Big(\frac{1 - \lambda^{r^n}}{\lambda^{S_n(r)}d(x_0, x_1)}\Big)^r = \frac{\lambda}{(d(x_0, x_1))^{r-1}}\frac{(1 - \lambda^{r^n})^r}{1 - \lambda^{r^{n+1}}}$$

Taking the limit when $n \to \infty$ we get

$$\lim_{n \to \infty} \frac{d(x_{n+1}, \xi)}{(d(x_n, \xi))^r} = \frac{\lambda}{(d(x_0, x_1))^{r-1}}$$

so the rate of convergence of the iterative sequence (x_n) is at least r with asymptotic error constant $\frac{\lambda}{(d(x_0,x_1))^{r-1}}$.

Remark 3.2.10. Theorems 3.2.1 and 3.2.8 generalize Theorems 3.1.12 and 3.1.15 ([**?**, Theorems 2.11 & 2.15]), as the range of T can be taken as $CB(X)$ instead of the space of all nonempty proximinal closed subsets of X.

Corollary 3.2.11. Let (X, d) be complete metric space, $T : X \to CB(X)$ be an operator satisfying

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{3.2.24}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x, y) \in J$ and φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, z) \in J$ for some $z \in Tx_0$. Then, the following statements hold true.

(i) There exists an orbit $\{x_n\}$ of T in X that converges to a fixed point $\xi \in S = \{x \in X : d(x, \xi) \in J\}$ of T.

(ii) The estimates (3.2.17) - (3.2.19) are valid.

Proof. It follows from (3.2.24) that

$$H(Tx, Ty) \leq \varphi(d(x, y)) < d(x, y) \tag{3.2.25}$$

for all $x, y \in X$ $(x \neq y)$. Hence T is continuous. Thus *(i)* and *(ii)* follow immediately from Theorem 3.2.8. □

Corollary 3.2.12. Let (X, d) be a complete metric space, D be a closed subset of X, and T be a continuous mapping from D into $CB(X)$ such that $Tx \cap D \neq \emptyset$ and

$$H(Tx \cap D, Ty \cap D) \leq \varphi(m(x, y)) \text{ for all } x \in D, y \in Tx \cap D \tag{3.2.26}$$

with strict inequality holds if $m(x, y) \neq 0$ where φ is a gauge function of the first order on $J = [0, \infty)$ and

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}. \tag{3.2.27}$$

Then for each $x_0 \in D$ such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$ the following statements hold true.

(i) There exists an orbit of T at x_0 in D that converges to a unique fixed point ξ of T.

(ii) For $n \geq 0$ we have the following a priori estimate

$$d(x_n, \xi) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1). \tag{3.2.28}$$

(iii)For all $n \geq 1$ we have the following a posteriori estimate

$$d(x_n, \xi) \leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]}. \tag{3.2.29}$$

Proof. Choose $x_1 = z \in Tx_0 \cap D$. Define $\rho_0 = \sigma(d(x_0, x_1))$ where σ is defined by (1.1.4). Since from (1.1.5), $\sigma(t) \geq t$ so we have

$$d(x_0, x_1) \leq \rho_0. \tag{3.2.30}$$

We may assume that $m(x_0, x_1) \neq 0$, for otherwise $d(x_0, Tx_0) \leq m(x_0, x_1) = 0$ and x_0 is the fixed point of T. From (3.2.26) we have $H(Tx_0 \cap D, Tx_1 \cap D) < \varphi(m(x_0, x_1))$ we may choose $\epsilon_1 > 0$ with

$$H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1 \leq \varphi(m(x_0, x_1)). \tag{3.2.31}$$

It follows from Lemma 1.2.11 that there exists $x_2 \in Tx_1 \cap D$ such that

$$d(x_1, x_2) \leq H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1. \tag{3.2.32}$$

From above two inequalities we get

$$d(x_1, x_2) \leq \varphi(m(x_0, x_1)) \tag{3.2.33}$$

$$\leq \varphi\Big(\max \Big\{ d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2} \Big\}\Big) \tag{3.2.34}$$

$$\leq \varphi\Big(\max \Big\{ d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2} \Big\}\Big) \tag{3.2.35}$$

We claim that

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{3.2.36}$$

Let

$$\eta_1 = \max\{d(x_0,x_1), d(x_1,Tx_1), \frac{d(x_0,Tx_1)}{2}\}. \tag{3.2.37}$$

Case 1. If $\eta_1 = d(x_0,x_1)$ then clearly (3.2.36) is true.
Case 2. If $\eta_1 = d(x_1,Tx_1)$. Since $x_2 \in Tx_1$ we have

$$d(x_1,x_2) \leq \varphi(d(x_1,Tx_1)) < d(x_1,Tx_1) \leq d(x_1,x_2),$$

which is a contradiction.
Case 3. Finally suppose $\eta_1 = \frac{d(x_0,Tx_1)}{2}$. Then we have

$$d(x_1,x_2) \leq \varphi(\frac{d(x_0,Tx_1)}{2}) < \frac{d(x_0,x_2)}{2} \leq \frac{d(x_0,x_1)+d(x_1,x_2)}{2},$$

since $x_2 \in Tx_1$. Thus, $d(x_1,x_2) < d(x_0,x_1)$. As a result,

$$\eta_1 \leq \frac{d(x_0,x_1)+d(x_1,x_2)}{2} < d(x_0,x_1),$$

which contradicts the definition of η_1. This proves our claim. Proceeding inductively in a similar way as in Theorem 3.2.1 we obtain the sequence $\{x_n\}$ in D such that $x_n \to \xi \in D$ and $\lim_n d(x_{n+1},Tx_n) = 0$. Since T is continuous so taking limit as $n \to \infty$ we have $\xi \in T\xi$. If η is another fixed point then $m(\xi,\eta) = d(\xi,\eta)$ and uniqueness follows immediately . Estimates (3.2.17) and (3.2.18) becomes (3.2.28) and (3.2.29) for $r = 1$. □

Remark 3.2.13. Corollary 3.2.12 can be observed as an improved version (with uniqueness of fixed point and error bounds) of [2, Theorem,2.1,2.2] if the continuous function ϕ in [2] is replace by the gauge function φ of order 1.

3.3 Existence and uniqueness for the solution to integral inclusions

In this section we apply Corollary 3.2.11 to establish the existence of unique solution for the class of integral inclusions. Let $\mathbb{R}$ denote the real line. We consider the following integral inclusion.

$$\begin{aligned} x(t) &\in P\int_{t_0}^{t} k(s,x(s))ds + \beta \\ &\in PK^x(t) + \beta \end{aligned} \tag{3.3.1}$$

where

$$K^x(t) := \int_{t_0}^{t} k(s,x(s))ds \tag{3.3.2}$$

P is a closed and bounded (compact) subset of $\mathbb{R}$, $\beta \in \mathbb{R}$ and $k : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies:

$$|k(t,x(t)) - k(t,y(t))| \le \frac{q}{\alpha}|x(t)-y(t)|^r \tag{3.3.3}$$

where $\alpha = \max_{p\in P}|p|$, $0 < q \le \alpha$ and $1 < r \le 2$;

$$|k(t,x)| < \frac{1}{2\alpha}(\frac{1}{q})^{r-1} \tag{3.3.4}$$

and is continuous on $R = \{(t,x) : |t-t_0| \le (\frac{1}{q})^{2-r}, |x-\beta| \le \frac{1}{2q}\}$.

Theorem 3.3.1. Under the above assumptions *(3.3.2) - (3.3.4)*, the integral inclusion (3.3.1) has a unique solution on $I = [t_0 - (\frac{1}{q})^{2-r}, t_0 + (\frac{1}{q})^{2-r}]$.

Remark 3.3.2. Note that in almost all the existence results, as for as we know, for integral inclusions or integral equations the kernel of the equation satisfies Lipschitz condition in some sense. But in this result the kernel satisfies the inequality (3.3.3), which is not Lipschitz condition, since $1 < r \le 2$. This result not only guarantees the existence but also the uniqueness of the solution and provide the iterative scheme with higher convergence rate. It also provides the domain of existence of the solution.

Proof. (Theorem 3.3.1) Consider the space of all continuous functions $C(I)$ endowed by the classical metric

$$d(x,y) = \sup_{t\in I}|x(t)-y(t)|.$$

Note that $C(I)$ is complete with respect to the metric $'d'$. Let $\widetilde{C} = \{x \in C(I) : d(x,\beta) \le \frac{1}{2q}\}$. Then, $\widetilde{C}$ is a closed subspace of $C(I)$ and thus is complete. Define an operator T by:

$$\begin{aligned} Tx(t) &= P\int_{t_0}^{t} k(s,x(s))ds + \beta \\ &= PK^x(t) + \beta. \end{aligned} \tag{3.3.5}$$

Then, (3.3.5) and (3.3.1) are equivalent problems in a sense that fixed point(s) of T are solution(s) of (3.3.1). We claim that $T : (\widetilde{C}, d) \to (K(\widetilde{C}), H)$. For this we need to prove that: (i) T is defined for each $x \in \widetilde{C}$. (ii) Tx is a compact subset of $\widetilde{C}$ for any $x \in \widetilde{C}$.

If $\tau \in I$ then $|\tau - t_0| \leq (\frac{1}{q})^{2-r}$. Hence by Definition of $\widetilde{C}$ we have, $|x(\tau) - \beta| \leq \frac{1}{2q}$. So, if $\tau \in I$ then $(\tau, x(\tau)) \in R$ and since k is continuous on R, therefore integral (3.3.2) exists in (3.3.5) and T is defined for each $x \in \widetilde{C}$. We now show that $Tx \subset \widetilde{C}$. Let $y(t) \in Tx(t)$. Then, $y(t) = pK^x(t) + \beta$ for some $p \in P$, and

$$\begin{aligned} |y(t) - \beta| &= |pK^x(t)| \\ &= |p||K^x(t)| \\ &\leq \alpha \int_{t_0}^{t} k(s,x(s))ds| \\ &\leq \alpha \int_{t_0}^{t} |k(s,x(s))|ds \\ &< \alpha|t - t_0|\frac{1}{2\alpha}\left(\frac{1}{q}\right)^{r-1} \leq \frac{1}{2q}. \end{aligned}$$

Thus, $d(y, \beta) < \frac{1}{2q}$ for any $y \in Tx$ and hence $Tx \subset \widetilde{C}$ for any $x \in \widetilde{C}$. Next we show that Tx is compact. Consider a sequence $(u_n) \subset Tx$ then, $u_n = p_n K^x(t) + \beta$, where (p_n) is a sequence in P. Since P is compact there is a subsequence (p_{n_i}) of (p_n) such that $p_{n_i} \to \overline{p} \in P$. Let $u = \overline{p}K^x(t) + \beta$. Then

$$d(u_{n_i}, u) = \sup_{t\in I}(|p_{n_i} - \overline{p}||K^x(t)|) \leq |p_{n_i} - \overline{p}| \sup_{t\in I} |K^x(t)| \to 0$$

when $n_i \to \infty$. This proves our claim. Further, note that

$$\begin{aligned} H(Tx, Ty) &= H(PK^x(t) + \beta, PK^y(t) + \beta) \\ &\leq H(PK^x(t), PK^y(t)). \end{aligned} \tag{3.3.6}$$

By definition,

$$H(PK^x(t), PK^y(t)) = \max\{\max_{a\in PK^x(t)} d(a, PK^y(t)), \max_{b\in PK^y(t)} d(b, PK^x(t))\}.$$

Consider,

$$\begin{aligned}
\max_{a\in PK^x(t)} d(a, PK^y(t)) &= \max_{a\in PK^x} \min_{b\in PK^y} d(a,b) \\
&= \max_{p\in P} \min_{p^*\in P} d(pK(t,x), p^*K(t,y)) \\
&= \max_{p\in P} \min_{p^*\in P} \sup_{t\in I} |pK(t,x) - p^*K(t,y)| \\
&\le \max_{p\in P} \min_{p^*\in P} \sup_{t\in I}[|pK(t,y) - p^*K(t,y)| + |pK(t,y) - pK(t,x)|] \\
&\le \max_{p\in P} \min_{p^*\in P}[|p| \sup_{t\in I} |K(t,y) - K(t,x)| + |p-p^*| \sup_{t\in I} |K(t,y)|] \\
&= \max_{p\in P} |p| \sup_{t\in I} |K(t,y) - K(t,x)| \\
&= \alpha \sup_{t\in I} |K(t,y) - K(t,x)|.
\end{aligned}$$

Now,

$$\begin{aligned}
|K(t,y) - K(t,x)| &\le \int_{t_0}^{t} |k(s,y(s)) - k(s,x(s))| ds \\
&\le \frac{q}{\alpha} \int_{t_0}^{t} |y(s) - x(s)|^r ds \\
&\le \frac{q}{\alpha} (\sup_{t\in I} |y(s) - x(s)|)^r \int_{t_0}^{t} ds \\
&= \frac{q}{\alpha} |t - t_0| (d(x,y))^r \\
&\le \frac{q}{\alpha} \left(\frac{1}{q}\right)^{2-r} (d(x,y))^r = \frac{q^{r-1}}{\alpha} (d(x,y))^r.
\end{aligned}$$

Hence,

$$\max_{a\in PK^x(t)} d(a, PK^y(t)) \le \alpha \frac{q^{r-1}}{\alpha} (d(x,y))^r = q^{r-1}(d(x,y))^r.$$

Changing roles of x and y we obtain,

$$\max_{b\in PK^y(t)} d(b, PK^x(t)) \le q^{r-1}(d(x,y))^r.$$

Now from (3.3.6) we have,

$$H(Tx,Ty) \le q^{r-1}(d(x,y))^r.$$

Since for all $x, y \in \widetilde{C}$ we have $d(x,y) < \frac{1}{q}$. So, if we take $\varphi(t) = q^{r-1}t^r$ for $t \in J = [0, \frac{1}{q})$, then φ is the gauge function of the order r. To prove this, note that for $\lambda \in (0,1)$ and $t \in J$ we have

$$\varphi(\lambda t) = \lambda^r q^{r-1} t^r \le \lambda^r \phi(t).$$

Further, for $t \in J - \{0\}$ we have

$$\varphi(t) = q^{r-1}t^r = q^{r-1}tt^{r-1} < q^{r-1}t(\frac{1}{q})^{r-1} = t.$$

From the above arguments we have

$$H(Tx,Ty) \le \varphi(d(x,y)) \tag{3.3.7}$$

for all $x, y \in \widetilde{C}$ with $d(x,y) \in J$. Hence starting from $x_0 = \beta$ the iterative sequence $x_n \in Tx_{n-1}; n = 1, 2, \cdots$ converges to the unique fixed point ξ of T at a rate r. □

As a consequence of the above result we have the following theorem.

Theorem 3.3.3. Consider the integral inclusion

$$x(t) \in P\int_{t_0}^{t} k(s,x(s))ds + Q \tag{3.3.8}$$

where P and Q are compact subsets of $\mathbb{R}$. Under the assumptions (3.3.2), (3.3.3) and (3.3.4) that hold on $R = \{(t,x) : |t - t_0| \le (\frac{1}{q})^{2-r}, |x - \beta| \le \frac{1}{2q}\}$ for any $\beta \in Q$ the problem (3.3.8) has a solution on $I = [t_0 - (\frac{1}{q})^{2-r}, t_0 + (\frac{1}{q})^{2-r}]$.

In the Theorem 3.3.1 if we let $P = 1$ then the integral inclusion (3.3.1) becomes

$$x(t) = \int_{t_0}^{t} k(s,x(s))ds + \beta$$

which is equivalent to the initial value problem

$$x'(t) = k(t,x(t)), \quad x(t_0) = \beta.$$

In fact, we have got the Picard's type Theorem [29, Theorem 3.1] which can also be viewed as an application of Theorem 3.1.15 as follows:

Consider the following first order initial value problem:

$$x'(t) = k(t, x(t)),\ x(t_0) = \beta. \tag{3.3.9}$$

Suppose $k(t, x(t))$ have the following properties:

(i) k is continuous,

(ii) k satisfies the condition

$$|k(t, x) - k(t, y)| \leq q|x(t) - y(t)|^r, \tag{3.3.10}$$

(iii) k is bounded in such a way that

$$|k(t, x)| < \frac{1}{2}(\frac{1}{q})^{r-1}, \tag{3.3.11}$$

on $R = \{(t, x) : |t - t_0| \leq (\frac{1}{q})^{2-r}, |x - \beta| \leq \frac{1}{2q}\}$ where $1 < r \leq 2$ and $0 < q \leq 1$.

Theorem 3.3.4. [29] Under the assumptions *(i), (ii)* and *(iii)*, the initial value problem (3.3.9) has a unique solution on $I = [t_0 - (\frac{1}{k})^{2-r}, t_0 + (\frac{1}{k})^{2-r}]$.

Remark 3.3.5. The rate of convergence of the iterative sequence obtained in Theorem 3.3.4 is at least r, as compared to the Picard's Theorem 1.1.7 in which the rate of convergence is linear.

3.4 Multi-valued contractions by altering distances

Pathak and Shahzad [45] introduced a class of functions $\Theta[0, A)$ and generalized some fixed point theorems of Klim and Wardowski by altering distances, i-e., for the mapping T (from a complete metric space (X, d) into the class of nonempty closed subsets of X). A crucial assumption in the hypothesis of Theorem 1.2.22 (Pathak and Shahzad [45]) is that $M(1, x; \theta) = \{y \in Tx : \theta(d(x, y)) \leq \theta(d(x, Tx))\}$ is nonempty. We observe that it exclude all such θ's which are strictly increasing from the class $\Theta[0, A)$. Note that we have the following inclusion.

$$K(X) \subseteq PC(X) \subseteq CB(X) \subseteq CL(X) \subseteq N(X).$$

Therefore, if T is a mapping from X into $CL(X)$ having values in $CL(X)\backslash PC(X)$ then in general for any $x \in X$ we have $d(x,y) > d(x,Tx) \forall y \in Tx$ and if θ is strictly increasing then obviously $\theta(d(x,y)) > \theta(d(x,Tx)) \forall y \in Tx$. In simple words, if θ is strictly increasing then $M(1,x;\theta)$ will be nonempty only when T have values in the set of proximinal subsets of X. Thus, the question arises that whether one can remove the condition $M(1,x;\theta) \neq \emptyset$ from the hypotheses of the Theorem 1.2.22 so that the result still holds. In this section we give an affirmative answer to this question provided the function $\theta \in \Theta[0,A)$ is positive homogenous in $[0,A)$; i-e.,

$$\text{(iv) } \theta(at) \leq a\theta(t) \quad \forall\ a > 0,\ t \in [0,A).$$

We denote by $\Theta_h[0,A)$ the class of function $\theta \in \Theta[0,A)$ satisfying condition (iv), above. The established results not only relax the conditions but also the proofs are simpler than those of [45].
We start with the following simple Lemmas.

Lemma 3.4.1. Let $B \in CL(X)$. Then for each $x \in X$ and $q > 1$ there exists an element $b \in B$ such that

$$d(x,b) \leq qd(x,B). \tag{3.4.1}$$

Proof. Let $d(x,B) = 0$ then $x \in B$, since B is closed subset of X. Further, taking $b = x$ we see that (3.4.3) holds. Now, suppose that $d(x,B) > 0$ and choose

$$\epsilon = (q-1)d(x,B). \tag{3.4.2}$$

Then using the definition of $d(x,B)$ it follow that there exists $b \in B$ such that

$$\begin{aligned} d(x,b) &\leq d(x,B) + \epsilon \\ &\leq qd(x,B) \text{ (using (3.4.2))}. \end{aligned}$$

□

Lemma 3.4.2. Let $\theta \in \Theta_h[0,A)$ and $B \in CL(X)$. Then for each $x \in X$ and $q > 1$ there exists an element $b \in B$ such that

$$\theta(d(x,b)) \leq q\theta(d(x,B)). \tag{3.4.3}$$

Theorem 3.4.3. Let (X,d) be a complete metric space and α is a function from

$(0, \infty)$ into $(0, b]$, $0 < b < 1$ such that

$$\limsup_{r \to t^+} \alpha(r) < b \text{ for each } t \in [0, \infty). \tag{3.4.4}$$

Let $T : X \to CL(X)$. Assume that the following condition holds.

$$\theta(d(y, Ty)) \leq \alpha(d(x, y))\theta(d(x, y)) \text{ for each } x \in X \text{ and } y \in Tx, \tag{3.4.5}$$

where $\theta \in \Theta_h[0, A)$. Then,

(i) for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_n x_n = \xi$;

(ii) ξ is fixed point of T if and only if the function $f(x) := d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Proof. Theorem 3.4.3 Let $x_0 \in X$. Since $Tx_0 \neq \emptyset$, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. If $x_0 = x_1$, then x_0 is fixed point of T. Let $x_0 \neq x_1$, by taking $q = \frac{1}{\sqrt{\alpha(d(x_0, x_1))}}$ it follows from Lemma 3.4.2 that there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq \frac{1}{\sqrt{\alpha(d(x_0, x_1))}}\theta(d(x_1, Tx_1)). \tag{3.4.6}$$

Repeating the above argument we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

$$\theta(d(x_n, x_{n+1})) \leq \frac{1}{\sqrt{\alpha(d(x_{n-1}, x_n))}}\theta(d(x_n, Tx_n)) \tag{3.4.7}$$

where,

$$x_n \in Tx_{n-1}, \; n = 1, 2, \cdots . \tag{3.4.8}$$

We have assumed that $x_{n-1} \neq x_n$, for otherwise x_{n-1} is fixed point of T. Using (3.4.5) it follows from (3.4.7) that

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \sqrt{\alpha(d(x_{n-1}, x_n))}\theta(d(x_{n-1}, x_n)) & (3.4.9) \\ &< \theta(d(x_{n-1}, x_n)). & (3.4.10) \end{aligned}$$

Hence $\{\theta(d(x_n, x_{n+1}))\}$ is decreasing sequence of positive real numbers bounded below by 0. Since θ is strictly inverse isotone $\{d(x_n, x_{n+1})\}$ is also a decreasing sequence of positive real numbers bounded below by 0, thus convergent. Let

$\{\theta(d(x_n, x_{n+1}))\}$ converges to some nonnegative real number r, say. We claim that $r = 0$, for otherwise, by taking limits in (3.4.9) we get

$$r \leq \sqrt{\lim_{n\to\infty} \sup \alpha(d(x_{n-1}, x_n))}\; r < br < r \text{ (using (3.4.4))},$$

which is a contradiction. We claim that $\{d(x_n, x_{n+1})\}$ also converges to 0. Suppose $d(x_n, x_{n+1}) \to \gamma > 0$. Then, for $0 < \epsilon < \gamma$, there exists a natural number n_0 such that

$$0 < \delta = \gamma - \epsilon < d(x_n, x_{n+1}) \;\forall n \geq n_0.$$

Since θ is positive and nondecreasing,

$$0 < \theta(\delta) \leq \theta(d(x_n, x_{n+1})) \;\forall n \geq n_0,$$

which is a contradiction, since $\theta(d(x_n, x_{n+1})) \to 0$. From (3.4.9), we get

$$\theta(d(x_n, x_{n+1})) \leq [\sqrt{\alpha(d(x_{n-1}, x_n))} \cdots \sqrt{\alpha(d(x_0, x_1))}]\theta(d(x_0, x_1)). \tag{3.4.11}$$

It follow from (3.4.4) that we may choose an $\epsilon > 0$ and $a \in (0, b)$ such that

$$\alpha(t) < a^2 \text{ for } t \in (0, \epsilon). \tag{3.4.12}$$

Let N be such that

$$d(x_{n-1}, x_n) < \epsilon \text{ for } n \geq N. \tag{3.4.13}$$

Then, from (3.4.11) we have

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq a^{n-(N-1)}[\sqrt{\alpha(d(x_{N-2}, x_{N-1}))} \cdots \sqrt{\alpha(d(x_0, x_1))}]\theta(d(x_0, x_1)) \\ &< a^{n-N+1}b^{N-1}\theta(d(x_0, x_1)). \end{aligned} \tag{3.4.14}$$

Therefore, for any $m \in \mathbb{N}$ we have

$$\begin{aligned} \theta(d(x_n, x_{n+m})) &\leq \theta(d(x_n, x_{n+1})) + \cdots + \theta(d(x_{n+m-1}, x_{n+m})) \\ &< a^{n-N+1}b^{N-1}[1 + a + a^2 + \cdots + a^{m-1}]\theta(d(x_0, x_1)) \\ &< b^{N-1}\frac{a^{n-N+1}}{1-a}\theta(d(x_0, x_1)). \end{aligned} \tag{3.4.15}$$

Therefore

$$\lim_{n,m\to\infty} \theta(d(x_n, x_m)) = 0.$$

We claim that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Suppose not. Then there exist $\delta > 0$ and subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ respectively such that

$$d(x_{m_i}, x_{ni}) > \delta \ \forall i.$$

Since θ is nondecreasing,

$$0 < \theta(\delta) \leq \theta(d(x_{m_i}, x_{n_i})) \to 0,$$

which is a contradiction. This proves our claim. Hence $\{x_n\}$ is Cauchy sequence in X. Since X is complete there exists $\xi \in X$ such that $x_n \to \xi$. Since $x_n \in Tx_{n-1}$, it follow from (3.4.18) that

$$\begin{aligned} \theta(d(x_n, Tx_n)) &\leq \alpha(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n)) \\ &< \theta(d(x_{n-1}, x_n)), \end{aligned} \tag{3.4.16}$$

which implies that $d(x_n, Tx_n) < d(x_{n-1}, x_n)$. Letting $n \to \infty$, from (3.4.16) we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.4.17}$$

Suppose $f(x) = d(x, Tx)$ is T orbitally lower continuous at ξ, then

$$d(\xi, T\xi) = f(\xi) \leq \lim_n \inf f(x_n) = \lim_n \inf d(x_n, Tx_n) = 0.$$

Hence, $\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is fixed point of then $f(\xi) = 0 \leq \lim_n \inf f(x_n)$.

If we take $b = 1$ then the following Theorem can be obtained as a consequence of Theorem 3.4.3. □

Theorem 3.4.4. Let (X, d) be a complete metric space and $T : X \to CL(X)$ satisfying

$$\theta(d(y, Ty)) \leq \alpha(d(x, y))\theta(d(x, y)) \text{ for each } x \in X \text{ and } y \in Tx, \tag{3.4.18}$$

where α is a function from $(0,\infty)$ into $(0,1]$ such that

$$\limsup_{r\to t^+} \alpha(r) < 1 \text{ for each } t \in [0,\infty), \tag{3.4.19}$$

and $\theta \in \Theta_h[0,A)$. Then,

(i) for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_n x_n = \xi$;

(ii) ξ is fixed point of T if and only if the function $f(x) := d(x,Tx)$ is T-orbitally lower semi continuous at ξ.

Example 3.4.5. Let l_2 denote the Hilbert space of all square summable sequences of real numbers with usual norm. Note that for $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$

$$d(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}.$$

For each $n = 1, 2, \cdots$, let e_n be the vector in l_2 with zeros in all its coordinates except the n^{th} coordinate which is equal to 1. Take $a = (-1, -\frac{1}{2}, \cdots, -\frac{1}{n}, \cdots)$, and $B = \{e_1, e_2, \cdots, e_n, \cdots\}$. Define $T : l_2 \to CL(l_2)$ by

$$Tx = B \ \forall \ x \in l_2.$$

Then, for $\theta(t) = t$ condition (3.4.18) of Theorem 3.4.4 is obviously satisfied. Further,

$$d(a,B) = \inf \|a - e_n\| = \inf(\|a\|^2 + 1 + \frac{2}{n})^{\frac{1}{2}} = (\|a\|^2 + 1)^{\frac{1}{2}}.$$

Observe that $B \in CL(X)$ and there is no e_n in B such that $\|a - e_n\| \leq d(a,B)$ and hence $M(1,x;\theta)$ is empty in this case and conditions of Theorem 1.2.22 are not satisfied. Note that, $e_1, e_2, \cdots$ are fixed points of T.

Corollary 3.4.6. Let (X,d) be a complete metric space and let $T : X \to CL(X)$ satisfying

$$\int_0^{\theta(d(y,Ty))} \psi(t)dt \leq \alpha(d(x,y)) \int_0^{\theta(d(x,y))} \psi(t)dt, \text{ for each } x \in X \text{ and } y \in Tx. \tag{3.4.20}$$

Where α is a function from $(0,\infty)$ into $(0,1]$ such that

$$\limsup_{r\to t^+} \alpha(r) < 1 \text{ for each } t \in [0,\infty), \tag{3.4.21}$$

$\theta \in \Theta_h[0,A)$ and $\Psi \in \Theta_h[0,A)$ given by $\Psi(\epsilon) = \int_0^\epsilon \psi(t)dt$ and $\psi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\epsilon \psi(t)dt > 0$ for all $\epsilon > 0$. Then T has a fixed point.

Since $d(y,Ty) \leq H(Tx,Ty)$ for $y \in Tx$. We have the following.

Corollary 3.4.7. Let (X,d) be a complete metric space and $T : X \to CL(X)$ satisfying

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y) \text{ for each } x \in X \text{ and } y \in Tx. \tag{3.4.22}$$

Where α is a function from $(0,\infty)$ into $(0,1]$ such that

$$\limsup_{r\to t^+} \alpha(r) < 1 \text{ for each } t \in [0,\infty). \tag{3.4.23}$$

Then,

(i) for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_n x_n = \xi$;

(ii) ξ is fixed point of T if and only if the function $f(x) := d(x,Tx)$ is T-orbitally lower semi continuous at ξ.

Remark 3.4.8. Note that Corollary 3.4.7 generalize Theorem 1.2.18 in the following sense:

(i) the inequality (3.4.22) is weaker than the inequality (1.2.5);

(ii) the range of T in Corollary 3.4.7 is $CL(X)$ which is general than $CB(X)$;

(iii) for the existence of fixed point we merely require that $d(x,Tx)$ is T orbitally continuous at ξ, whereas condition (1.2.5) in Theorem 1.2.18 demands that T is a continuous map from X into $CB(X)$.

We have the following simple example which shows the generality of our result.

Example 3.4.9. Let $X = (-\infty, \frac{3}{5}]$ equipped with usual metric d. Define $T : X \to CL(X)$ by

$$Tx = \begin{cases} (-\infty, \frac{1}{2}x] & \text{if } x \in (-\infty, 0), \\ (-\infty, x^2] & \text{if } x \in [0, \frac{3}{5}]. \end{cases}$$

Case 1. When $x \in (-\infty, 0)$ and $y \in Tx = (-\infty, \frac{1}{2}x]$, then we have

$$H(Tx, Ty) = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y),$$

since $y \in (-\infty, 0)$ in this case.

Case 2. When $x \in [0, \frac{3}{5}]$ and $y \in Tx$, then we have

$$H(Tx, Ty) \leq |x^2 - y^2| = (x + y)d(x, y) \leq \frac{24}{25}d(x, y).$$

Thus, for any $x \in X$ and $y \in Tx$, we have

$$d(y, Ty) \leq H(Tx, Ty) \leq \frac{24}{25}d(x, y).$$

By taking $\alpha(t) = c$, where $\frac{24}{25} \leq c < 1$, we see that all conditions of Corollary 3.4.7 are satisfied and 0 is fixed point of T. Note that T does not satisfies the hypothesis of Theorems 1.2.18 and 1.2.20.

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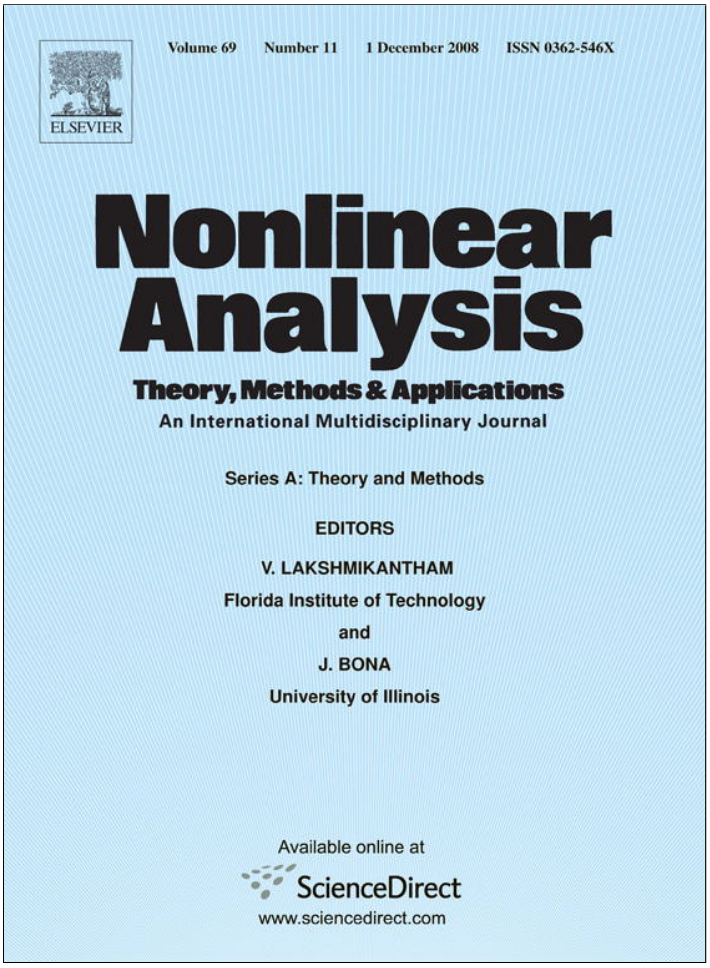
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Nadler's type principle with high order of convergence

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Abstract

Recently Proinov [P.D. Proinov, A generalization of the Banach contraction principle with high order of convergence of successive approximations, Nonlinear Analysis (2006), doi:10.1016/j.na.2006.09.008] generalized Banach contraction principle with high order of convergence. We extend some results of Proinov to the case of multi-valued maps from a complete metric space X into the space of all nonempty proximinal closed subsets of X. Our results not only generalize Nadler's fixed-point theorem (in the case when T is a mapping from a complete metric space X into the space of all nonempty proximinal closed subsets of X) but also gives high order of convergence. As an application, we obtain an existence theorem for first-order initial value problem.

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1. Introduction and preliminaries

Let (X, d) be a metric space. A subset A of X is called proximinal [7] if, for each $x \in X$, there is an element $a \in A$ such that $d(x, a) = d(x, A)$. For $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $N(X)$ the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty bounded closed subsets of X, by $PC(X)$ the class of all nonempty proximinal closed subsets of X. Let H be the generalized Hausdorff metric on $CL(X)$ generated by the metric d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for every $A, B \in CL(X)$. A point $p \in X$ is said to be a fixed point of $T : X \to CL(X)$ if $p \in Tp$. Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, A]$, $[0, A)$ or $[0, \infty)$ and $S_n(t)$ denote the polynomial $S_n(t) = 1 + t + \cdots + t^{n-1}$. We use the abbreviation φ^n for the nth iterate of a function $\varphi : J \to J$.

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Definition 1.1 (*[16]*). Let $r \geq 1$. A function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(i) $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;
(ii) (ii) $\varphi(t) < t$ for all $t \in J - \{0\}$.

It is easy to see that the first condition of Definition 1.1 is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$.

The metric fixed-point theory is very rich in fixed-point theorems given for various classes of φ-contractions, which are obtained for different collection of properties of the comparison or gauge function φ. Almost all such fixed-point theorems prove only the convergence of iterative procedure to the fixed point of the mapping but only few of them are able to provide information on the convergence rate. Banach contraction principle is one which not only discuss iterative scheme but also the convergence rate and stability. Nadler extended the Banach contraction principle in the following way.

Theorem 1.2 (*Nadler Contraction Principle [12]*). *Let (X, d) be complete metric space and T is a mapping from X into $CB(X)$ such that for all $x, y \in X$,*

$$H(Tx, Ty) \leq \lambda d(x, y)$$

where $0 < \lambda < 1$. Then T has a fixed point.

Afterward many authors obtained important fixed-point theorems, see for example, Berinde [2,3], Ciric [5,6], Petrusel [14,20], Rus [19,20] and Shahzad [21,22]. Recently, Proinov [16] generalizes the Banach contraction principle with high order of convergence. The purpose of this paper is to extend Proinov's results [16, Theorems 4.1, and 4.2] to the case of multi-valued mappings. The multi-valued map on a metric space X can be interpreted as set-valued dynamical system and the fixed point of multi-valued map may be interpreted as the rest point of the system (see Aubin [1] for instance). Our results extend Proinov's results [16, Theorems 4.1 and 4.2] and thus generalize some results of Rheinboldt [18], Gel'man [8], Ciric [5], Rus [19], Hicks and Rhoades [10], Hicks [9] and others. Theorem 2.15 (below) also provide a generalization of Nadler's fixed-point theorem [12] in the case when T is a map from X into $PC(X)$. Our results may be useful for computational methods of solutions to optimization problems (see Polac [15] and Zangwill [24] for instance) and in system theory (see Willems [23] for instance). We need the following lemmas for the sequel.

Lemma 1.3 (*[16]*). *Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying*

$$\varphi(t) = t\phi(t) \quad \text{for all } t \in J, \tag{1}$$

then it has the following two properties:

(i) $0 \leq \phi(t) < 1$ *for all* $t \in J$*;*
(ii) $\phi(\lambda t) \leq \lambda^{r-1}\phi(t)$ *for all* $\lambda \in (0, 1)$ *and* $t \in J$.

Lemma 1.4 (*[16]*). *Let φ be a gauge function of order $r \geq 1$ on J. Then for every $n \geq 0$ we have*

(i) $\varphi^n(t) \leq t\phi(t)^{S_n(r)}$ *for all* $t \in J$,
(ii) $\phi(\varphi^n(t)) \leq \phi(t)^{r^n}$ *for all* $t \in J$,

where ϕ is a nonnegative and nondecreasing function on J satisfying (1).

Definition 1.5 (*[16]*). A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini–Grandolfi gauge function [4] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \quad \text{for all } t \in J. \tag{2}$$

Note that Ptak [17] called a function $\varphi : J \to J$ satisfying (2) a rate of convergence on J and noticed that φ satisfies the following functional equation

$$\sigma(t) = \sigma(\varphi(t)) + t. \tag{3}$$

The following statement is an immediate consequence of the first part of Lemma 1.4 and the obvious inequality $S_n(r) \geq n$ for all $r \geq 1$.

Lemma 1.6 ([16]). *Every gauge function of order $r \geq 1$ on J is a Bianchini–Grandolfi gauge function on J.*

2. Main results

We begin this section by introducing the following definition.

Definition 2.1. Let (X, d) be a metric space and $T : X \to N(X)$. A sequence $\{x_n\} \subseteq X$ satisfying $x_{n+1} \in Tx_n$ with $d(x_n, x_{n+1}) = d(x_n, Tx_n)$ for $n = 0, 1, \ldots$, is called a proximinal orbit of T at the point x_0.

Following lemma is obvious.

Lemma 2.2. *Let (X, d) be a metric space, $T : X \to PC(X)$ and $x_0 \in X$. Then there exists a proximinal orbit $\{x_n\} \subseteq X$ of T at the point x_0, i.e.,*

$$x_{n+1} \in Tx_n, \quad n = 0, 1, 2, \ldots. \tag{4}$$

with

$$d(x_n, x_{n+1}) = d(x_n, Tx_n). \tag{5}$$

Unless otherwise stated, we assume that X is a metric space with metric d and T is an operator from $D \subset X$ into $PC(X)$ satisfying

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{6}$$

for all $x \in D$, $Tx \subset D$ with $d(x, y) \in J$ and for $y \in Tx$, where φ is a Bianchini–Grandolfi gauge function on the interval J. For convenience we define the function $E : D \to \mathbb{R}_+$ by $E(x) = d(x, Tx)$.

Lemma 2.3. *Let $x, y \in X$. Suppose that $d(x, y) \in J$ for $y \in Tx$. Then $E(x) \in J$.*

Proof. Since $0 \in J$ and J is an interval. Therefore, $0 \leq d(x, Tx) \leq d(x, y)$ implies $E(x) \in J$. □

Lemma 2.4. *Suppose that a point $x_0 \in X$ satisfies $E(x_0) \in J$. Then, $d(x_n, Tx_n) \in J$ for any $n \geq 0$.*

Proof. For $n = 0$, the Lemma is trivial. For $n \geq 1$, using Lemma 2.2 we obtain x_n such that $x_n \in D$, since $Tx \subset D$ for all $x \in D$. As, $0 \in J$ and $\varphi(t) \in J$ for all $t \in J$ we have, using (6), that

$$\begin{aligned} 0 \leq E(x_n) = d(x_n, Tx_n) &\leq H(Tx_{n-1}, Tx_n) \\ &\leq \varphi(E(x_{n-1})) \quad \text{(using (6), as } x_n \in Tx_{n-1}). \end{aligned}$$

Substituting $n = 1$, we obtain

$$0 \leq E(x_1) \leq \varphi(E(x_0)) \in J \Rightarrow E(x_1) \in J.$$

By induction, for any $n \geq 0$, we have $E(x_n) \in J$. □

Definition 2.5. Suppose that $x_0 \in D$ and $E(x_0) \in J$. Then for every iterate $x_n (n \geq 0)$ which belongs to D we define the closed ball $\overline{S}(x_n, \rho_n)$ with center x_n and radius $\rho_n = \sigma(E(x_n))$, where $\sigma : J \to \mathbb{R}_+$ is defined by (2).

Lemma 2.6. *Suppose that a point $x_0 \in D$ satisfies $E(x_0) \in J$ and $\overline{S}(x_n, \rho_n) \subset D$ for some $n \geq 0$. Then $x_{n+1} \in D$ and $\overline{S}(x_{n+1}, \rho_{n+1}) \subset \overline{S}(x_n, \rho_n)$.*

Proof. For each $n \geq 0$, x_{n+1} is well-defined, since $x_n \in D$. Further, Lemma 2.4 implies that $E(x_n) \in J$. As, $\sigma(t) \geq t$ for all $t \in J$. Therefore, using Lemma 2.2 we have

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) = E(x_n) \leq \sigma(E(x_n)) = \rho_n.$$

Hence, $x_{n+1} \in \overline{S}(x_n, \rho_n) \subset D$. Now we shall prove the second part of lemma. First note that

$$E(x_{n+1}) = d(x_{n+1}, Tx_{n+1}) \le H(Tx_n, Tx_{n+1}) \le \varphi(E(x_n)). \tag{7}$$

Let $x \in \overline{S}(x_{n+1}, \rho_{n+1})$, then, $d(x, x_{n+1}) \le \rho_{n+1}$. Using the triangle inequality, we have

$$\begin{aligned} d(x, x_n) &\le d(x, x_{n+1}) + d(x_{n+1}, x_n) \\ &\le \rho_{n+1} + d(x_n, Tx_n) = \sigma(E(x_{n+1})) + E(x_n) \\ &\le \sigma(\varphi(E(x_n))) + E(x_n) = \rho_n \quad \text{(using (3) \& (7))}, \end{aligned}$$

since σ is nondecreasing. Hence, $x \in \overline{S}(x_n, \rho_n)$ and so $\overline{S}(x_{n+1}, \rho_{n+1}) \subset \overline{S}(x_n, \rho_n)$. □

Definition 2.7 (*Initial Points*). A point $x_0 \in D$ will be called an initial point of T if $E(x_0) \in J$ and all of the iterates $x_0, x_1, x_2, \ldots$ are well-defined and belong to D.

Lemma 2.8 (*Test for Initial Points*). *Assume that $x_0 \in D$ satisfies $E(x_0) \in J$ and there exists an integer $n \ge 0$ such that $\overline{S}(x_n, \rho_n) \subset D$. Then x_0 is an initial point of T.*

Proof. Lemma 2.6 insures that $x_{n+1} \in D$ and $\overline{S}(x_{n+1}, \rho_{n+1}) \subset \overline{S}(x_n, \rho_n)$ for $n \ge 0$. This implies that $x_0, x_1, x_2, \ldots$ are well-defined and belong to D, so x_0 is an initial point of T. □

Lemma 2.9. *For every initial point $x_0 \in D$ of T and every $n \ge 0$ we have*

$$E(x_{n+1}) \le \varphi(E(x_n))$$

and

$$E(x_n) \le \varphi^n(E(x_0)).$$

In addition, if φ is a gauge function of order $r \ge 1$, then

$$E(x_n) \le E(x_0)\lambda^{S_n(r)}$$

and

$$\phi(E(x_n)) \le \lambda^{r^n},$$

where $\lambda = \phi(E(x_0))$ and ϕ is a nonnegative and nondecreasing function on J satisfying (1).

Proof. Note that $x_{n+1} \in D$, since x_0 is an initial point of T. As $x_{n+1} \in Tx_n$, using (6), we obtain

$$E(x_{n+1}) = d(x_{n+1}, Tx_{n+1}) \le H(Tx_n, Tx_{n+1}) \le \varphi(E(x_n)).$$

As, φ is nondecreasing on J we have

$$\begin{aligned} E(x_n) &\le \varphi(E(x_{n-1})) \\ &\le \varphi^2(E(x_{n-2})) \cdots \le \varphi^n(E(x_0)). \end{aligned}$$

If φ is a gauge function of order $r \ge 1$, then from Lemma 1.4(i)

$$E(x_n) \le \varphi^n(E(x_0)) \le E(x_0)(\phi(E(x_0)))^{S_n(r)} = E(x_0)\lambda^{S_n(r)}.$$

Further, from Lemma 1.4(ii), we have

$$\phi(E(x_0)) \le \phi(\varphi^n(E(x_0))) \le (\phi(E(x_0)))^{r^n} = \lambda^{r^n},$$

since ϕ is nondecreasing.

From Lemma 2.9 we obtain the following Lemma, which is muti-valued version of Proinov [16, Lemma 3.7], i.e., here $E(x) = d(x, Tx)$ where T is a map from X into $PC(X)$ satisfying (6). □

Lemma 2.10 (*Bounds for Inclusion Radii*). *Suppose that x_0 is an initial point of T and φ is a gauge function of order $r \geq 1$. Then for the radii $\rho_n = \sigma(E(x_n))$; $n = 0, 1, 2, \ldots$ the following estimates hold:*

$$\text{(i) } \rho_n \leq E(x_n) \sum_{j=0}^{\infty} [\phi(E(x_n))]^{S_j(r)} \leq \frac{E(x_n)}{1 - \phi(E(x_n))}, \tag{8}$$

$$\text{(ii) } \rho_n \leq E(x_n) \sum_{j=0}^{\infty} [\lambda^{r^n}]^{S_j(r)} \leq \frac{E(x_n)}{1 - \lambda^{r^n}}, \tag{9}$$

$$\text{(iii) } \rho_n \leq E(x_0)\lambda^{S_n(r)} \sum_{j=0}^{\infty} [\lambda^{r^n}]^{S_j(r)} \leq \frac{\lambda^{S_n(r)} E(x_0)}{1 - \lambda^{r^n}}, \tag{10}$$

$$\text{(iv) } \rho_{n+1} \leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\phi(\varphi(E(x_n)))]^{S_j(r)} \leq \frac{\varphi(E(x_n))}{1 - \phi(\varphi(E(x_n)))}, \tag{11}$$

$$\text{(v) } \rho_{n+1} \leq \varphi(E(x_n)) \sum_{j=0}^{\infty} [\lambda^{r^n}]^{S_j(r)} \leq \frac{\varphi(E(x_n))}{1 - \lambda^{r^n}}, \tag{12}$$

where $\lambda = \phi(E(x_0))$.

Proof. We will make use of the following facts during the proof.

$$0 \leq \phi(t) < 1, \quad S_j(r) \geq j \quad \text{and} \quad 0 \leq \lambda^{r^n} < 1.$$

(i) Using Lemma 1.4(i) and definition of ρ_n we have

$$\begin{aligned}
\rho_n = \sigma(E(x_n)) &= \sum_{j=0}^{\infty} \varphi^j(E(x_n)) \\
&\leq \sum_{j=0}^{\infty} E(x_n)[\phi(E(x_n))]^{S_j(r)} \\
&= E(x_n) \sum_{j=0}^{\infty} [\phi(E(x_n))]^{S_j(r)} \\
&\leq E(x_n) \sum_{j=0}^{\infty} [\phi(E(x_n))]^j = \frac{E(x_n)}{1 - \phi(E(x_n))}.
\end{aligned}$$

(ii) From (8)

$$\begin{aligned}
\rho_n &\leq E(x_n) \sum_{j=0}^{\infty} [\phi(E(x_n))]^{S_j(r)} \\
&\leq E(x_n) \sum_{j=0}^{\infty} [\lambda^{r^n}]^{S_j(r)} \quad \text{(using Lemma 2.9)} \\
&\leq E(x_n) \sum_{j=0}^{\infty} [\lambda^{r^n}]^j = \frac{E(x_n)}{1 - \lambda^{r^n}}.
\end{aligned}$$

(iii) From (9)

$$\begin{aligned}
\rho_n &\leq E(x_n) \sum_{j=0}^{\infty} [\lambda^{r^n}]^{S_j(r)} \\
&\leq E(x_0)\lambda^{S_n(r)} \sum_{j=0}^{\infty} [\lambda^{r^n}]^{S_j(r)} \quad \text{(using Lemma 2.9)}
\end{aligned}$$

$$\leq E(x_0)\lambda^{S_n(r)}\sum_{j=0}^{\infty}[\lambda^{r^n}]^j$$
$$\leq E(x_0)\frac{\lambda^{S_n(r)}}{1-\lambda^{r^n}}.$$

(iv) Since $E(x_{n+1}) \leq \varphi(E(x_n))$ and ϕ is nondecreasing, using Lemma 1.4 we have

$$\begin{aligned}\rho_{n+1} = \sigma(E(x_{n+1})) &= \sum_{j=0}^{\infty}\varphi^j(E(x_{n+1}))\\ &\leq E(x_{n+1})\sum_{j=0}^{\infty}[\phi(E(x_{n+1}))]^{S_j(r)}\\ &\leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(\varphi(E(x_n)))]^{S_j(r)}\\ &\leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(\varphi(E(x_n)))]^{j}\\ &= \frac{\varphi(E(x_n))}{1-\phi(\varphi(E(x_n)))}.\end{aligned}$$

(v) Using (11) we obtain

$$\begin{aligned}\rho_{n+1} &\leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(E(x_{n+1}))]^{S_j(r)}\\ &\leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\lambda^{r^{n+1}}]^{S_j(r)} \quad \text{(using Lemma 2.9)}\\ &= \varphi(E(x_n))\sum_{j=0}^{\infty}[\lambda^{r^n}]^{rS_j(r)}\\ &\leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\lambda^{r^n}]^{S_{j+1}(r)-1} \quad \text{(using definition of } S_j(r))\\ &\leq \varphi(E(x_n))\sum_{j=0}^{\infty}[\lambda^{r^n}]^{j+1-1}\\ &= \frac{\varphi(E(x_n))}{1-\lambda^{r^n}}. \quad \square\end{aligned}$$

Theorem 2.11. *Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying (6) with a Bianchini–Grandolfi gauge function φ on an interval J. Then, starting from an initial point x_0 of T the iterative sequence $\{x_n\}$ remains in $\overline{S}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$; $n = 0, 1, \ldots$, where $\rho_n = \sigma(d(x_n, Tx_n))$ and σ is defined by (2). Moreover, for each $n \geq 1$ we have*

$$d(x_n, x_{n+1}) \leq \varphi(d(x_{n-1}, x_n)).$$

If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

Proof. Since x_0 is an initial point of T, it follows from Lemma 2.6 that

$$\overline{S}(x_{n+1}, \rho_{n+1}) \subset \overline{S}(x_n, \rho_n) \quad \text{for all } n \geq 0, \tag{13}$$

which implies that $x_n \in \overline{S}(x_0, \rho_0)$ for all $n \geq 0$. Using the definition of ρ_n, from Lemma 2.9, we obtain

$$\rho_n = \sigma(E(x_n)) \le \sigma(\varphi^n(E(x_0))) = \sum_{j=0}^{\infty} \varphi^j(\varphi^n(E(x_0))) = \sum_{j=n}^{\infty} \varphi^j(E(x_0)) \quad \text{for all } n \ge 0, \tag{14}$$

since σ is nondecreasing. As φ is Bianchini–Grandolfi gauge function on J and $E(x_0) \in J$, from (14), we obtain

$$\rho_n \to 0 \quad \text{as } n \to \infty. \tag{15}$$

From (13) and (15) it follows that $\{\overline{S}(x_n, \rho_n)\}$ is nested sequence of closed spheres such that $\rho_n \to 0$ as $n \to \infty$. Therefore, it follows from Cantor's Theorem that there exists a unique point ξ such that $\xi \in \overline{S}(x_n, \rho_n)$ for all $n \ge 0$ and $x_n \to \xi$. In other words, $\lim_{n\to\infty} d(x_n, \xi) = 0$. Now,

$$d(\xi, Tx_n) \le d(\xi, x_n) + d(x_n, Tx_n) = d(\xi, x_n) + d(x_n, x_{n+1}).$$

Taking limit as $n \to \infty$ we obtain,

$$\lim_{n\to\infty} d(\xi, Tx_n) \le 0.$$

If $\xi \in D$ and T is continuous at ξ then we have $d(\xi, T\xi) = 0$ which simply means that $\xi \in T\xi$. Moreover, from Lemma 2.9 we obtain

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) = E(x_n) \le \varphi(E(x_{n-1})) = \varphi(d(x_{n-1}, Tx_{n-1})) = \varphi(d(x_{n-1}, x_n)). \quad \square$$

Corollary 2.12. *Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying*

$$H(Tx, Ty) \le \varphi(d(x, Tx))$$

for all $x \in D$, $Tx \subset D$ with $d(x, y) \in J$ and for $y \in Tx$, with a Bianchini–Grandolfi gauge function φ on an interval J. Then, starting from an initial point x_0 of T the iterative sequence $\{x_n\}$ remains in $\overline{S}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$; $n = 0, 1, \ldots$, where $\rho_n = \sigma(d(x_n, Tx_n))$ and σ is defined by (2). Moreover, for each $n \ge 1$ we have

$$d(x_n, x_{n+1}) \le \varphi(d(x_{n-1}, x_n)).$$

If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

When T is a single-valued map from $D \subset X$ into X, from the above corollary we obtain at once the following.

Corollary 2.13 (*Proinov [16, Theorem 4.1]*). *Let $T : D \subset X \to X$ be an operator on a complete metric space (X, d) satisfying*

$$d(Tx, T^2x) \le \varphi(d(x, Tx))$$

for all $x \in D$, $Tx \subset D$ with $d(x, Tx) \in J$, with a Bianchini–Grandolfi gauge function φ on an interval J. Then, starting from an initial point x_0 of T the iterative sequence $\{x_n\}$ remains in $\overline{S}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$; $n = 0, 1, \ldots$, where $\rho_n = \sigma(d(x_n, Tx_n))$ and σ is defined by (2). Moreover, for each $n \ge 1$ we have $d(x_n, x_{n+1}) \le \varphi(d(x_{n-1}, x_n))$. If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.

Remark 2.14. Note that Theorem 2.11 extends Proinov [16, Theorem 4.1] and thus generalizes/extends some results of Bianchini–Grandolfi [4] and Hicks [9, Theorems 1 and 2].

Theorem 2.15. *Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying (6) with a gauge function φ of the order $r \geq 1$ on an interval J. Suppose that $x_0 \in D$ is an initial point of T. Then the following statements hold true.*

(i) *The iterative sequence (4) remains in $\overline{S}(x_0, \rho_0)$ and converges with the rate of convergence at least r to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$, $n = 0, 1, \ldots$, where*

$$\rho_n = d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))} \tag{16}$$

and ϕ is a nonnegative and nondecreasing function on J satisfying (1).

(ii) *For all $n \geq 0$ we have the following a priori estimate*

$$d(x_n, \xi) \leq d(x_0, Tx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \frac{\lambda^{S_n(r)} d(x_0, Tx_0)}{1 - \lambda^{r^n}}, \tag{17}$$

where $\lambda = \phi(d(x_0, Tx_0))$.

(iii) *For all $n \geq 1$ we have the following a posteriori estimate*

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \end{aligned} \tag{18}$$

(iv) *For all $n \geq 1$ we have*

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, Tx_0). \tag{19}$$

(v) *If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.*

Proof. (i) Theorem 2.11 insures that the iterative sequence (4) remains in $\overline{S}(x_0, \rho_0)$ and converges to ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$, $n = 0, 1, \ldots$. Further, from Lemma 2.10, estimate (8), we have (by using $E(x_n) = d(x_n, x_{n+1})$)

$$\rho_n \leq d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))}.$$

(ii) For $m > n$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= E(x_n) + E(x_{n+1}) + \cdots + E(x_{m-1}) \\ &\leq E(x_0)[\lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots + \lambda^{S_{m-1}(r)}] \quad \text{(by Lemma 2.9)} \\ &= E(x_0) \sum_{j=n}^{m-1} \lambda^{S_j(r)}. \end{aligned}$$

Keeping n fixed and letting $m \to \infty$, we obtain

$$d(x_n, \xi) \leq E(x_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} = d(x_0, Tx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)}. \tag{20}$$

Note that,

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{S_j(r)} &= \lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots \\ &= \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + \cdots]. \end{aligned}$$

Since $r \geq 1$, therefore

$$r^n + r^{n+1} \geq 2r^n, \quad r^n + r^{n+1} + r^{n+2} \geq 3r^n \ldots$$

and hence,

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n}, \quad \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n} \ldots,$$

since $0 < \lambda < 1$. Thus,

$$\sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + \cdots] = \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

Substituting this in (20), we obtain

$$d(x_n, \xi) \leq E(x_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} = d(x_0, Tx_0) \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

(iii) From (29) we have for $n \geq 0$,

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} [\phi(d(x_0, x_1))]^{S_j(r)}.$$

Putting $n = 0$, $y_0 = x_n$ and $y_1 = x_1$ we have,

$$d(y_0, \xi) \leq d(y_0, y_1) \sum_{j=0}^{\infty} [\phi(d(y_0, y_1))]^{S_j(r)}.$$

Putting $y_0 = x_n$, and $y_1 = x_{n+1}$ we have,

$$\begin{aligned} d(x_n, \xi) &\leq d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} && (21) \\ &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{j} \\ &= \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))}, && (22) \end{aligned}$$

since $S_j(r) \geq j$. Now by Lemma 1.4(ii), we have

$$\phi(\varphi(d(x_n, x_{n-1}))) \leq [\phi(d(x_n, x_{n-1}))]^r$$

which means that,

$$\frac{1}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \leq \frac{1}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \quad (23)$$

From (21) we obtain for $n \geq 1$,

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r} \quad \text{(using (23))}. \end{aligned}$$

(iv) For $n \geq 1$, using Lemma 2.9 we have

$$\begin{aligned} d(x_{n+1}, x_n) = E(x_n) &\leq \varphi(E(x_{n-1})) \\ &= E(x_{n-1})\phi(E(x_{n-1})) \quad \text{(using Lemma 1.3)} \\ &\leq E(x_0)\lambda^{S_{n-1}(r)}\phi(E(x_{n-1})) \\ &\leq E(x_0)\lambda^{S_{n-1}(r)}\lambda^{r^{n-1}} \\ &= E(x_0)\lambda^{S_{n-1}(r)+r^{n-1}} \\ &= E(x_0)\lambda^{S_n(r)} \\ &= \lambda^{S_n(r)}d(x_0, Tx_0). \end{aligned}$$

(v) Proof is similar as in the proof of Theorem 2.11. □

Corollary 2.16. *Let $T : D \subset X \to PC(X)$ be an operator on a complete metric space (X, d) satisfying*

$$H(Tx, Ty) \leq \varphi(d(x, Tx))$$

for all $x \in D$, $Tx \subset D$ with $d(x, y) \in J$ and for $y \in Tx$, with a gauge function φ of the order $r \geq 1$ on an interval J. Suppose that $x_0 \in D$ is an initial point of T. Then the following statements hold true.

(i) *The iterative sequence (4) remains in $\overline{S}(x_0, \rho_0)$ and converges with the rate of convergence at least r to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$, $n = 0, 1, \ldots$, where*

$$\rho_n = d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))} \tag{24}$$

and ϕ is a nonnegative and nondecreasing function on J satisfying (1).

(ii) *For all $n \geq 0$ we have the following a priori estimate*

$$d(x_n, \xi) \leq d(x_0, Tx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \frac{\lambda^{S_n(r)} d(x_0, Tx_0)}{1 - \lambda^{r^n}}, \tag{25}$$

where $\lambda = \phi(d(x_0, Tx_0))$.

(iii) *For all $n \geq 1$ we have the following a posteriori estimate*

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \end{aligned} \tag{26}$$

(iv) *For all $n \geq 1$ we have*

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, Tx_0). \tag{27}$$

(v) *If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.*

From the above Corollary, we obtain at once the following.

Corollary 2.17 (*Proinov [16, Theorem 4.2]*). *Let $T : D \subset X \to X$ be an operator on a complete metric space (X, d) satisfying*

$$d(Tx, T^2y) \leq \varphi(d(x, Tx))$$

for all $x \in D$, $Tx \subset D$ with $d(x, Tx) \in J$ and for $y \in Tx$, with a gauge function φ of the order $r \geq 1$ on an interval J. Suppose that $x_0 \in D$ is an initial point of T. Then the following statements hold true.

(i) *The iterative sequence (4) remains in $\overline{S}(x_0, \rho_0)$ and converges with the rate of convergence at least r to a point ξ which belongs to each of the closed balls $\overline{S}(x_n, \rho_n)$, $n = 0, 1, \ldots$, where*

$$\rho_n = d(x_n, x_{n+1}) \sum_{j=0}^{\infty} [\phi(d(x_n, x_{n+1}))]^{S_j(r)} \leq \frac{d(x_n, x_{n+1})}{1 - \phi(d(x_n, x_{n+1}))} \tag{28}$$

and ϕ is a nonnegative and nondecreasing function on J satisfying (1).

(ii) *For all $n \geq 0$ we have the following a priori estimate*

$$d(x_n, \xi) \leq d(x_0, Tx_0) \sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \frac{\lambda^{S_n(r)} d(x_0, Tx_0)}{1 - \lambda^{r^n}}, \tag{29}$$

where $\lambda = \phi(d(x_0, Tx_0))$.

(iii) *For all $n \geq 1$ we have the following a posteriori estimate*

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}. \end{aligned} \tag{30}$$

(iv) *For all $n \geq 1$ we have*

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, Tx_0). \tag{31}$$

(v) *If $\xi \in D$ and T is continuous at ξ, then ξ is a fixed point of T.*

Corollary 2.18. *Let (X, d) be a complete metric space, $T : X \to PC(X)$ be a continuous operator satisfying*

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{32}$$

for any $x \in X$, $y \in Tx$ with $d(x, y) \in J$ and φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, Tx_0) \in J$. Then, the following statements hold true.

(i) *The iterative sequence (4) converges to a fixed point ξ of T.*
(ii) *The estimates (29)–(31) are valid.*

The following simple example shows the generality of our results over the Nadler's contraction principle, in the case when T is a map from X int $PC(X)$.

Example 2.19. Let $X = [0, \frac{3}{5}]$ equipped with usual metric d. Define $T : X \to PC(X)$ by $Tx = [0, x^2]$. Then, for any $x \in X$ and $y \in Tx$,

$$H(Tx, Ty) \leq \frac{24}{25} d(x, y).$$

By taking $\varphi(t) = \frac{24}{25}t$ and $J = [0, \infty)$, we see that all conditions of Corollary 2.18 are satisfied and 0 is a fixed point of T. Note that Nadler's contraction principle is not applicable here.

Corollary 2.20. *Let (X, d) be complete metric space, $T : X \to PC(X)$ be an operator satisfying*

$$H(Tx, Ty) \leq \varphi(d(x, y)) \tag{33}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x, y) \in J$ and φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, Tx_0) \in J$. Then, the following statements hold true.

(i) *The iterative sequence (4) converges to a fixed point ξ of T.*
(ii) *The operator T has a unique fixed point in $S = \{x \in X : d(x, \xi) \in J\}$.*
(iii) *The estimates (29)–(31) are valid.*

Proof. It follows from (33) that

$$H(Tx, Ty) \leq \varphi(d(x, y)) < d(x, y) \tag{34}$$

for all $x, y \in X$ $(x \neq y)$. Hence T is continuous. Thus (i) and (iii) follow immediately from Theorem 2.15. To show the uniqueness, that is (ii), assume that $\eta \in S$ is another fixed point of T. Then $d(\xi, \eta) \in J$. It follows from (34) that

$$d(\xi, \eta) \leq H(T\xi, T\eta) \leq \varphi(d(\xi, \eta))$$

which gives $\xi = \eta$. □

Remark 2.21. In the case when T is a mapping from X into $PC(X)$, Nadler's contraction principle (Theorem 1.2) is a special case of Corollary 2.20. To see this, take $\varphi(t) = \lambda t$ $(0 < \lambda < 1)$ and $J = [0, \infty)$.

Corollary 2.22 (*Proinov [16, Corollary 4.4]*). *Let (X, d) be complete metric space, $T : X \to X$ be an operator satisfying*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{35}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x, y) \in J$ and φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, Tx_0) \in J$. Then, the following statements hold true.

(i) *The iterative sequence $x_{n+1} = Tx_n$ converges to a fixed point ξ of T.*
(ii) *The operator T has a unique fixed point in $S = \{x \in X : d(x, \xi) \in J\}$.*
(iii) *The estimates (29)–(31) are valid.*

Corollary 2.23. *Let $T : X \to PC(X)$ be a continuous operator on a complete metric space (X, d) satisfying*

$$H(Tx, Ty) \leq \varphi(m(x, y)) \tag{36}$$

for all $x \in X$, $y \in Tx$ where φ is a gauge function of the first order on $J = [0, \infty)$ and

$$m(x, y) = \max\{d(x, y), d(y, Ty)\}. \tag{37}$$

Then for each $x_0 \in X$ the following statements hold true.

(i) *The iterative sequence $\{x_n\}$ converges to a unique fixed point ξ of T.*
(ii) *For $n \geq 0$ we have the following a priori estimate*

$$d(x_n, \xi) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, Tx_0). \tag{38}$$

(iii) *For all $n \geq 1$ we have the following a posteriori estimate*

$$d(x_n, \xi) \leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]}. \tag{39}$$

Proof. Let $x \in X$. It follows from (36) that

$$H(Tx, Ty) \leq \varphi(\max\{d(x, y), d(y, Ty)\}), \tag{40}$$

for $y \in Tx$. Suppose that $\max\{d(x, y), d(y, Ty)\} = d(y, Ty)$, then we have

$$d(y, Ty) \leq H(Tx, Ty) \leq \varphi(d(y, Ty)) < d(y, Ty),$$

which is a contradiction. Therefore, from (40) we obtain

$$H(Tx, Ty) \le \varphi(d(x, y)),$$

for all $x \in X$ with $d(x, y) \in J$ and for $y \in Tx$, which is nothing, but the condition (6). Applying Theorem 2.15 we deduce that the iterative sequence $\{x_n\}$ converges to a point $\xi \in X$. Since $r = 1$ in this case so the a priori and a posteriori estimates (29) and (30) become (38) and (39). The uniqueness of ξ follows from Corollary 2.18. □

Remark 2.24. Theorem 2.15 extends/generalizes [16, Theorem 4.2] and thus extends/generalize Rheinboldt [18], Kornstaedt [11, Satz 4.1], Hicks and Rhoades [9], Park [13, Theorem 2], Gel'man [8, Theorem 3] and Hicks [9, Theorem 3].

3. Application

In this section we apply Theorem 2.15 to establish the existence of unique solution for first-order initial value problem:

$$x'(t) = f(t, x(t)),\ \ x(t_0) = x_0. \tag{41}$$

Suppose that $f(t, x(t))$ have the following properties:

(i) f is continuous,
(ii) f satisfies the condition

$$|f(t, x) - f(t, y)| \le k|x(t) - y(t)|^r, \tag{42}$$

(iii) f is bounded in such a way that

$$|f(t, x)| < \frac{1}{2}\left(\frac{1}{k}\right)^{r-1}, \tag{43}$$

on $R = \{(t, x) : |t - t_0| \le (\frac{1}{k})^{2-r}, |x - x_0| \le \frac{1}{2k}\}$ where $1 < r \le 2$ and $0 < k \le 1$.

Theorem 3.1. *Under the assumptions* (i), (ii) *and* (iii), *the initial value problem* (41) *has a unique solution on* $I = [t_0 - (\frac{1}{k})^{2-r}, t_0 + (\frac{1}{k})^{2-r}]$.

Proof. Consider the metric space $C(I)$ of all continuous functions on I. Note that $C(I)$ is complete with respect to the metric

$$d(x, y) = \max_{t\in I} |x(t) - y(t)|. \tag{44}$$

Let $\tilde{C}$ be the subspace of $C(I)$ such that

$$d(x, x_0) \le \frac{1}{2k}. \tag{45}$$

Then, $\tilde{C}$ is a closed subspace of $C(I)$ and thus is complete. From (41) on integration, we obtain

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\mathrm{d}\tau. \tag{46}$$

Define $T : \tilde{C} \to \tilde{C}$ by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\mathrm{d}\tau. \tag{47}$$

Then (46) and (47) are equivalent problems in a sense that fixed point(s) of T are solution(s) of (46). We now show that T is defined for each $x \in \tilde{C}$. Indeed, if $\tau \in I$ then $|\tau - t_0| \le (\frac{1}{k})^{2-r}$. Hence from (45) we have $|x(\tau) - x_0| \le \frac{1}{2k}$. So, if $\tau \in I$ then $(\tau, x(\tau)) \in R$ and since f is continuous on R, therefore, the integral in (47) exists and T is defined for each $x \in \tilde{C}$.

From (47) we can write

$$\begin{aligned} Tx(t) - x_0 &= \int_{t_0}^{t} f(\tau, x(\tau))\mathrm{d}\tau \\ |Tx(t) - x_0| &= \left|\int_{t_0}^{t} f(\tau, x(\tau))\mathrm{d}\tau\right| \\ &\leq \int_{t_0}^{t} |f(\tau, x(\tau))|\mathrm{d}\tau \\ &< \frac{1}{2}\left(\frac{1}{k}\right)^{r-1} \int_{t_0}^{t} \mathrm{d}\tau \quad \text{(using (43))} \\ &= \frac{1}{2}\left(\frac{1}{k}\right)^{r-1} |t - t_0| \leq \frac{1}{2}\left(\frac{1}{k}\right)^{r-1}\left(\frac{1}{k}\right)^{2-r} = \frac{1}{2k}. \end{aligned}$$

Hence $Tx(t) \in \tilde{C}$ that is T maps $\tilde{C}$ to $\tilde{C}$.

Lastly,

$$\begin{aligned} Tx(t) - Ty(t) &= \int_{t_0}^{t} [f(\tau, x(\tau)) - f(\tau, y(\tau))]\mathrm{d}\tau \\ |Tx(t) - Ty(t)| &\leq \int_{t_0}^{t} |f(\tau, x(\tau)) - f(\tau, y(\tau))|\mathrm{d}\tau \\ &\leq k \int_{t_0}^{t} |x(\tau) - y(\tau)|^r \mathrm{d}\tau \quad \text{(using (42))} \\ &\leq k \left(\max_{t\in J} |x(t) - y(t)|\right)^r \int_{t_0}^{t} \mathrm{d}\tau \\ &\leq k|t - t_0|(d(x, y))^r \leq k\left(\frac{1}{k}\right)^{2-r} (d(x, y))^r \\ &= k^{r-1}(d(x, y))^r. \end{aligned} \tag{48}$$

Note that from (46), in fact we have for $x \in \tilde{C}$,

$$\begin{aligned} x(t) - x_0 &= \int_{t_0}^{t} f(\tau, x(\tau))\mathrm{d}\tau \\ |x(t) - x_0| &= \left|\int_{t_0}^{t} f(\tau, x(\tau))\mathrm{d}\tau\right| \\ &\leq \int_{t_0}^{t} |f(\tau, x(\tau))|\mathrm{d}\tau \\ &< \frac{1}{2}\left(\frac{1}{k}\right)^{r-1} \int_{t_0}^{t} \mathrm{d}\tau \quad \text{(using (43))} \\ &= \frac{1}{2}\left(\frac{1}{k}\right)^{r-1} |t - t_0| \leq \frac{1}{2}\left(\frac{1}{k}\right)^{r-1}\left(\frac{1}{k}\right)^{2-r} = \frac{1}{2k} \end{aligned}$$

which turns out to be $d(x, x_0) < \frac{1}{2k}$ and so for all $x, y \in \tilde{C}$ we have $d(x, y) < \frac{1}{k}$. So, if we take $\varphi(s) = k^{r-1}s^r$ for $s \in J = [0, \frac{1}{k})$, then φ is the gauge function of the order r. The proof is rather simple. Indeed, for $\lambda \in (0, 1)$ and $s \in J$ we have

$$\varphi(\lambda s) = \lambda^r k^{r-1} s^r \leq \lambda^r \phi(s).$$

Further, for $s \in J - \{0\}$ we have

$$\varphi(s) = k^{r-1}s^r = k^{r-1}ss^{r-1} < k^{r-1}s\left(\frac{1}{k}\right)^{r-1} = s$$

Hence, (48) gives

$$d(Tx, Ty) \leq \varphi(d(x, y)) \tag{49}$$

for all $x, y \in \tilde{C}$ with $d(x, y) \in J$. Therefore, all conditions of Corollary 2.22 are satisfied. Hence, the iterative sequence $x_n = Tx_{n-1}$; $n = 1, 2, \ldots$ converges to the unique fixed point of T at a rate at least r as compared to Picard's iteration in which the rate of convergence is linear. □

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Fixed point theorems for generalized contractive multi-valued maps

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ABSTRACT

In [N. Mizoguchi, W. Takahashi, Fixed point theorems for multi-valued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989) 177–188] the authors gave a positive answer to the conjecture of S. Reich concerning the existence of fixed points of multi-valued mappings that satisfy certain contractive conditions. In this paper, we establish some results for multi-valued mappings that satisfy a generalized contractive condition in a way that it contains Mizoguchi's result as one of its special cases. In addition, our results not only improve the results of Kiran and Kamran [Q. Kiran, T. Kamran, Nadler's type principle with high order of convergence, Nonlinear Anal. TMA 69 (2008) 4106–4120] and some results of Agarwal et al. [R.P. Agarwal, Jewgeni Dshalalow, Donal O'Regan, Fixed point and homotopy results for generalized contractive maps of Reich type, Appl. Anal. 82 (4) (2003) 329–350] but also provide the high order of convergence of the iterative scheme and error bounds. As an application of our results, we obtain an existence result for a class of integral inclusions.

1. Introduction and preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $N(X)$ the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty bounded closed subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. Let H be the generalized Hausdorff metric on $CB(X)$ generated by the metric d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for every $A, B \in CB(X)$. A point $p \in X$ is said to be a fixed point of $T : X \to CL(X)$ if $p \in Tp$. If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$ then $O(T, x_0) = \{x_0, x_1, x_2, \ldots\}$ is said to be orbit of $T : X \to CL(X)$. A mapping $f : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \xi$ implies $f(\xi) \le \lim_n \inf f(x_n)$. Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, A]$, $[0, A)$ or $[0, \infty)$ and $S_n(t)$ denotes the polynomial $S_n(t) = 1 + t + \cdots + t^{n-1}$. We use the abbreviation φ^n for the nth iterate of a function $\varphi : J \to J$.

Definition 1.1 (*[1]*). Let $r \ge 1$. A function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(i) $\varphi(\lambda t) \le \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;
(ii) $\varphi(t) < t$ for all $t \in J - \{0\}$.

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It is easy to see that the first condition of Definition 1.1 is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$. We are stating the following results for convenience.

Lemma 1.2 ([2]). *Let $A, B \in CB(X)$ and let $a \in A$. If $\epsilon > 0$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

Lemma 1.3 ([1]). *Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying*

$$\varphi(t) = t\phi(t) \quad \text{for all } t \in J, \tag{1}$$

then it has the following two properties:
(i) $0 \leq \phi(t) < 1$ *for all* $t \in J$;
(ii) $\phi(\lambda t) \leq \lambda^{r-1}\phi(t)$ *for all* $\lambda \in (0, 1)$ *and* $t \in J$.

Lemma 1.4 ([1]). *Let φ be a gauge function of order $r \geq 1$ on J. Then for every $n \geq 0$ we have*
(i) $\varphi^n(t) \leq t\phi(t)^{S_n(r)}$ *for all* $t \in J$,
(ii) $\phi(\varphi^n(t)) \leq \phi(t)^{r^n}$ *for all* $t \in J$,
where ϕ is a nonnegative and nondecreasing function on J satisfying (1).

Definition 1.5 ([1]). A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini–Grandolfi gauge function [3] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \quad \text{for all } t \in J. \tag{2}$$

Note that Ptak [4] called a function $\varphi : J \to J$ satisfying (2) a rate of convergence on J and noticed that φ satisfies the following functional equation

$$\sigma(t) = \sigma(\varphi(t)) + t. \tag{3}$$

The following statement is an immediate consequence of the first part of Lemma 1.4 and the obvious inequality $S_n(r) \geq n$ for all $r \geq 1$.

Lemma 1.6 ([1]). *Every gauge function of order $r \geq 1$ on J is a Bianchini–Grandolfi gauge function on J.*

Definition 1.7 ([5]). Suppose (x_n) is a sequence that converges to ξ. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{d(x_{n+1}, \xi)}{(d(x_n, \xi))^\alpha} = \lambda$$

then (x_n) is said to converge to ξ of order α, with asymptotic error constant λ.

Remark 1.8. In general, a sequence with high order of convergence converges more rapidly than a sequence with a lower order. If $\alpha = 1$, the method is called linear. If $\alpha = 2$, the method is called quadratic.

In [6], Reich proved that a mapping $T : X \to K(X)$ has a fixed point in X if it satisfies

$$H(Tx, Ty) \leq k(d(x, y))d(x, y) \tag{4}$$

for all $x, y \in X$ with $x \neq y$, where $k : (0, \infty) \to [0, 1)$ satisfies $\limsup_{s\to t^+} k(s) < 1$ for every $t \in (0, \infty)$. This result generalizes the fixed point theorem for single-valued mappings that was proved by Boyd and Wong [7]. Reich questioned in [8,9] that whether or not the range of T, $K(X)$ can be replaced by $CB(X)$. Mizoguchi and Takahashi [10], Daffer and Kaneko [11] and Tong-Huei Chang [12] gave a positive answer to the conjecture of Reich. Recently, Pathak and Shahzad [13] generalized Nadler's contraction principle in contrast to Reich's and Mizoguchi–Takahashi's theorems. More recently, Thagfi and Shahzad [14] obtained some fixed point theorems for an operator which is closely related to the Reich type contraction. The authors in [15] extended some results of Proinov [1] to the case of multi-valued maps from a complete metric space X into the space of all nonempty proximinal closed subsets of X. The purpose of this paper is to obtain some fixed point theorems for multi-valued maps which not only provide the iterative scheme with a high convergence rate but also the error bounds. Our results generalize [10, Theorem 5], [11, Theorem 2.1], [15, Theorems 2.11 & 2.15] and [16, Theorems 2.1 & 2.2].

2. Main results

Theorem 2.1. *Let (X, d) be a complete metric space, D be a closed subset of X, φ is a Bianchini–Grandolfi gauge function on an interval J and T be a mapping from D into $CB(X)$ such that $Tx \cap D \neq \emptyset$ and*

$$H(Tx \cap D, Ty \cap D) \leq \varphi(d(x, y)) \tag{5}$$

for all $x \in D$, $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $d(x, y) \neq 0$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then:
(i) *there exists an orbit $\{x_n\}$ of T in D and $\xi \in D$ such that $\lim_n x_n = \xi$;*
(ii) *ξ is a fixed point of T if and only if the function $f(x) := d(x, Tx \cap D)$ is T-orbitally lower semi-continuous at ξ.*

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Proof. Take $x_1 = z \in Tx_0 \cap D$. We assume that $d(x_0, x_1) \neq 0$, for otherwise x_0 is fixed point of T. Define $\rho_0 = \sigma(d(x_0, x_1))$ where σ is defined by (2). Since from (3), $\sigma(t) \geq t$ so we have

$$d(x_0, x_1) \leq \rho_0. \tag{6}$$

Notice that $x_1 \in \overline{S}(x_0, \rho_0)$. It follow from (5) that $H(Tx_0 \cap D, Tx_1 \cap D) < \varphi(d(x_0, x_1))$. Choose an $\epsilon_1 > 0$ with

$$H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1 \leq \varphi(d(x_0, x_1)). \tag{7}$$

It follows from Lemma 1.2 that there exists $x_2 \in Tx_1 \cap D$ such that

$$d(x_1, x_2) \leq H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1, \tag{8}$$

since D is closed and Tx_1 is closed and bounded. We assume that $d(x_1, x_2) \neq 0$, for otherwise x_1 is fixed point of T. From inequalities (7) and (8) we have

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{9}$$

Note that $d(x_1, x_2) \in J$. Further, $x_2 \in \overline{S}(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) \\ &\leq d(x_0, x_1) + \sigma(\varphi(d(x_0, x_1))) \\ &= \sigma(d(x_0, x_1)) \quad \text{(using (3))} \\ &= \rho_0. \end{aligned}$$

Now choose $\epsilon_2 > 0$ with

$$H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2 \leq \varphi(d(x_1, x_2)). \tag{10}$$

It again follows from Lemma 1.2 that there exists $x_3 \in Tx_2 \cap D$ such that

$$d(x_2, x_3) \leq H(Tx_1 \cap D, Tx_2 \cap D) + \epsilon_2. \tag{11}$$

We assume that $d(x_2, x_3) \neq 0$, for otherwise x_2 is fixed point of T. From inequalities (9), (10) and (11) we have

$$d(x_2, x_3) \leq \varphi^2(d(x_0, x_1)). \tag{12}$$

Note that $d(x_2, x_3) \in J$. Further, $x_3 \in \overline{S}(x_0, \rho_0)$, since

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) + \varphi^2(d(x_0, x_1)) \\ &\leq \sum_{j=0}^{\infty} \varphi^j(d(x_0, x_1)) \\ &= \sigma(d(x_0, x_1)) = \rho_0. \end{aligned}$$

Repeating the above argument, inductively we obtain the a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n \in Tx_{n-1} \cap D, \tag{13}$$

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)), \tag{14}$$

$$d(x_{n-1}, x_n) \in J, \quad \text{and} \quad x_n \in \overline{S}(x_0, \rho_0). \tag{15}$$

We claim that $\{x_n\}$ is a Cauchy sequence. For $n, p \in \mathbb{N}$, from (14) we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \varphi^{n+p-1}(d(x_0, x_1)) + \cdots + \varphi^n(d(x_0, x_1)) \\ &\leq \sum_{j=n}^{\infty} \varphi^j(d(x_0, x_1)). \end{aligned}$$

Using (2), it follows from the above inequality that $\{x_n\}$ is a Cauchy sequence. Thus there exists $\xi \in \overline{S}(x_0, \rho_0)$ with $x_n \to \xi$. Note that $\xi \in D$, as well. Since $x_n \in Tx_{n-1} \cap D$, it follow from (5) that

$$\begin{aligned} d(x_n, Tx_n \cap D) &\leq H(Tx_{n-1} \cap D, Tx_n \cap D) \\ &\leq \varphi(d(x_{n-1}, x_n)) \\ &< d(x_{n-1}, x_n). \end{aligned} \tag{16}$$

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Letting $n \to \infty$, from (16) we get

$$\lim_{n\to\infty} d(x_n, Tx_n \cap D) = 0. \tag{17}$$

Suppose $f(x) = d(x, Tx \cap D)$ is T orbitally lower continuous at ξ, then

$$d(\xi, T\xi \cap D) = f(\xi) \le \liminf_n f(x_n) = \liminf_n d(x_n, Tx_n \cap D) = 0.$$

Hence, $\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is fixed point of T then $f(\xi) = 0 \le \lim_n \inf f(x_n)$, since $\xi \in D$. □

Example 2.2. Let $X = \left[0, \frac{3}{5}\right]$ equipped with usual metric d. Define $T : X \to CB(X)$ by $Tx = [0, x^2]$. Then, for any $x, y \in X$

$$\begin{aligned} H(Tx, Ty) &= |x^2 - y^2| = |x + y||x - y| \\ &\le |x + y| d(x, y). \end{aligned}$$

Therefore, for $x \in \left[0, \frac{3}{5}\right]$ and $y \in [0, x^2]$, we have

$$\begin{aligned} H(Tx, Ty) &\le \left(\frac{3}{5} + \frac{9}{25}\right) d(x, y) \\ &= \frac{24}{25} d(x, y). \end{aligned}$$

By taking $\varphi(t) = \frac{24}{25}t$ and $J = [0, \infty)$, we see that all conditions of Theorem 2.1 are satisfied and 0 is the only fixed point of T. Observe that for $0 \le \lambda < 1$,

$$H\left(T\frac{1}{2}, T\frac{3}{5}\right) = \frac{11}{100} \not\le \lambda \frac{1}{10} = d\left(\frac{1}{2}, \frac{3}{5}\right).$$

Therefore, T does not satisfies the hypothesis of [10, Theorem 5] and [11, Theorem 2.1].

In [12,17,1] the following class of functions was introduced and studied.

Definition 2.3 (*[12,17,1]*). Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. The function ψ is said to satisfy the condition (Φ) (denoted by $\psi \in (\Phi)$) if

(i) $\psi(t) < t$ for all $t \in (0, \infty)$,
(ii) ψ is upper semicontinuous from the right on $(0, \infty)$ and
(iii) there exists a positive real number s such that ψ is nondecreasing on $(0, s]$ and $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for all $t \in (0, s]$.

Following lemma is an immediate consequence of the third part of Definitions 2.3 and 1.1.

Lemma 2.4. *Every $\psi \in (\Phi)$ is a Bianchini–Grandolfi gauge function on $J = (0, s]$.*

Remark 2.5. Chang [12] observed that if $k : (0, \infty) \to [0, 1)$ satisfies

$$\limsup_{s\to t^+} k(s) < 1 \quad \text{for every } t \in [0, \infty), \tag{18}$$

then there exists a function $\psi \in (\Phi)$ such $k(t)t \le \psi(t)$ for all $t \in (0, \infty)$.

As a consequence of above lemma and remark we have the following corollary.

Corollary 2.6 (*[10, Theorem 5], [11, Theorem 2.1]*). *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that T satisfies*

$$H(Tx, Ty) \le k(d(x, y))d(x, y) \tag{19}$$

for all $x, y \in X$ with $x \ne y$ where $k : (0, \infty) \to [0, 1)$ satisfies (18). Then T has a fixed point in X.

Remark 2.7. In [18], Suzuki proved that Corollary 2.6 is a real generalization of Nadler's Theorem. As a result, we can see Theorem 2.1 as a generalization of Nadler's Theorem.

Theorem 2.8. *Let (X, d) be a complete metric space, D be a closed subset of X, φ is a gauge function of order $r \ge 1$ on an interval J, $\phi : J \to \mathbb{R}^+$ is a nondecreasing function defined by (1) and T be a mapping from D into $CB(X)$ such that $Tx \cap D \ne \emptyset$ and*

$$H(Tx \cap D, Ty \cap D) \le \varphi(d(x, y)) \tag{20}$$

for all $x \in D$, $y \in Tx \cap D$ with $d(x, y) \in J$. Moreover, the strict inequality holds when $d(x, y) \ne 0$. Suppose $x_0 \in D$ is such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$. Then:

(i) *there exists an orbit $\{x_n\}$ of T in $\overline{S}(x_0, \rho_0)$ that converges with rate of convergence at least r to a point $\xi \in \overline{S}(x_0, \rho_0)$;*

(ii) *for all $n \geq 0$ we have the following a priori estimate*

$$d(x_n, \xi) \leq \frac{\lambda^{S_n(r)} d(x_0, x_1)}{1 - \lambda^{r^n}}, \tag{21}$$

where $\lambda = \phi(d(x_0, x_1))$;

(iii) *for all $n \geq 1$ we have the following a posteriori estimate*

$$\begin{aligned} d(x_n, \xi) &\leq \varphi(d(x_n, x_{n-1})) \sum_{j=0}^{\infty} [\phi(\varphi(d(x_n, x_{n-1})))]^{S_j(r)} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi(\varphi(d(x_n, x_{n-1})))} \\ &\leq \frac{\varphi(d(x_n, x_{n-1}))}{1 - [\phi(d(x_n, x_{n-1}))]^r}; \end{aligned} \tag{22}$$

(iv) *for all $n \geq 1$ we have*

$$d(x_n, x_{n+1}) \leq \varphi(d(x_n, x_{n-1})) \leq \lambda^{S_n(r)} d(x_0, x_1); \tag{23}$$

(v) *ξ is fixed point of T if and only if the function $f(x) := d(x, Tx \cap D)$ is T-orbitally lower semi-continuous at ξ.*

Proof. (i) Theorem 2.1 ensures the existence of an orbit $\{x_n\}$ of T in $\overline{S}(x_0, \rho_0)$ that converges to ξ which belongs to $\overline{S}(x_0, \rho_0)$.
(ii) For $m > n$, using (14) and Lemma 1.4-(i) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \varphi^n(d(x_0, x_1)) + \varphi^{n+1}(d(x_0, x_1)) + \cdots + \varphi^{m-1}(d(x_0, x_1)) \\ &\leq d(x_0, x_1)[\lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots + \lambda^{S_{m-1}(r)}] \\ &= d(x_0, x_1) \sum_{j=n}^{m-1} \lambda^{S_j(r)}. \end{aligned}$$

Keeping n fixed and letting $m \to \infty$, we get

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} \lambda^{S_j(r)}. \tag{24}$$

Note that,

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{S_j(r)} &= \lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots \\ &= \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + \cdots]. \end{aligned}$$

Since $r \geq 1$, therefore

$$r^n + r^{n+1} \geq 2r^n, \qquad r^n + r^{n+1} + r^{n+2} \geq 3r^n \cdots$$

and hence,

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n}, \qquad \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n} \cdots,$$

since $0 < \lambda < 1$. Thus,

$$\sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + \cdots] = \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

Substituting this in (24), we get

$$d(x_n, \xi) \leq d(x_0, x_1) \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

(iii) From (24) we have for $n \geq 0$,

$$d(x_n, \xi) \leq d(x_0, x_1) \sum_{j=n}^{\infty} [\phi(d(x_0, x_1))]^{S_j(r)}.$$

Putting $n=0$, $y_0=x_n$ and $y_1=x_1$ we have,

$$d(y_0,\xi)\le d(y_0,y_1)\sum_{j=0}^{\infty}[\phi(d(y_0,y_1))]^{S_j(r)}.$$

Putting $y_0=x_n$, and $y_1=x_{n+1}$ we have,

$$\begin{aligned} d(x_n,\xi) &\le d(x_n,x_{n+1})\sum_{j=0}^{\infty}[\phi(d(x_n,x_{n+1}))]^{S_j(r)} \qquad (25)\\ &\le \varphi(d(x_n,x_{n-1}))\sum_{j=0}^{\infty}[\phi(\varphi(d(x_n,x_{n-1})))]^{S_j(r)}\\ &\le \varphi(d(x_n,x_{n-1}))\sum_{j=0}^{\infty}[\phi(\varphi(d(x_n,x_{n-1})))]^{j}\\ &= \frac{\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))}, \qquad (26)\end{aligned}$$

since $S_j(r)\ge j$. Now by Lemma 1.4-(ii), we have

$$\phi(\varphi(d(x_n,x_{n-1})))\le[\phi(d(x_n,x_{n-1}))]^r$$

which means that,

$$\frac{1}{1-\phi(\varphi(d(x_n,x_{n-1})))}\le\frac{1}{1-[\phi(d(x_n,x_{n-1}))]^r}. \qquad (27)$$

From (25) we get for $n\ge 1$,

$$\begin{aligned} d(x_n,\xi) &\le \varphi(d(x_n,x_{n-1}))\sum_{0}^{\infty}[\phi(\varphi(d(x_n,x_{n-1})))]^{S_j(r)}\\ &\le \frac{\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))}\\ &\le \frac{\varphi(d(x_n,x_{n-1}))}{1-[\phi(d(x_n,x_{n-1}))]^r} \quad \text{(using (27))}.\end{aligned}$$

(iv) For $n\ge 1$, using (14) and Lemma 1.4 we have

$$\begin{aligned} d(x_{n+1},x_n) &\le \varphi(d(x_n,x_{n-1}))\\ &\le d(x_0,x_1)\lambda^{S_{n-1}(r)}\phi(d(x_{n-1},x_n))\\ &\le d(x_0,x_1)\lambda^{S_{n-1}(r)}\lambda^{r^{n-1}}\\ &= d(x_0,x_1)\lambda^{S_{n-1}(r)+r^{n-1}}\\ &= d(x_0,x_1)\lambda^{S_n(r)}.\end{aligned}$$

(v) Proof is similar to the proof of Theorem 2.1. □

Remark 2.9. We can note the rate of convergence from the a priori estimate (21) as follows:

$$\frac{d(x_{n+1},\xi)}{(d(x_n,\xi))^r}=\frac{\lambda^{S_{n+1}(r)}d(x_0,x_1)}{1-\lambda^{r^{n+1}}}\left(\frac{1-\lambda^{r^n}}{\lambda^{S_n(r)}d(x_0,x_1)}\right)^r=\frac{\lambda}{(d(x_0,x_1))^{r-1}}\frac{\left(1-\lambda^{r^n}\right)^r}{1-\lambda^{r^{n+1}}}.$$

Taking the limit when $n\to\infty$ we get

$$\lim_{n\to\infty}\frac{d(x_{n+1},\xi)}{(d(x_n,\xi))^r}=\frac{\lambda}{(d(x_0,x_1))^{r-1}},$$

so by Definition 1.7 the rate of convergence of the iterative sequence (x_n) is r with asymptotic error constant $\frac{\lambda}{(d(x_0,x_1))^{r-1}}$.

Remark 2.10. Theorems 2.1 and 3.4 generalize [15, Theorems 2.11 & 2.15], as the range of T can be taken as $CB(X)$ instead of the space of all nonempty proximinal closed subsets of X.

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Corollary 2.11. *Let (X, d) be a complete metric space, $T : X \to CB(X)$ be an operator satisfying*

$$H(Tx, Ty) \le \varphi(d(x, y)) \tag{28}$$

for all $x, y \in X$ $(x \neq y)$ with $d(x, y) \in J$ and φ is a gauge function of order $r \ge 1$ on an interval J. Assume that x_0 is a point in X such that $d(x_0, z) \in J$ for some $z \in Tx_0$. Then, the following statements hold true.

(i) *There exists an orbit $\{x_n\}$ of T in X that converges to a fixed point $\xi \in S = \{x \in X : d(x, \xi) \in J\}$ of T.*
(ii) *The estimates (21)–(23) are valid.*

Proof. It follows from (28) that

$$H(Tx, Ty) \le \varphi(d(x, y)) < d(x, y) \tag{29}$$

for all $x, y \in X$ $(x \neq y)$. Hence T is continuous. Thus (i) and (ii) follow immediately from Theorem 3.4. □

Corollary 2.12. *Let (X, d) be a complete metric space, D be a closed subset of X, and T be a continuous mapping from D into $CB(X)$ such that $Tx \cap D \neq \emptyset$ and*

$$H(Tx \cap D, Ty \cap D) \le \varphi(m(x, y)) \quad \text{for all } x \in D, y \in Tx \cap D \tag{30}$$

where strict inequality holds if $m(x, y) \neq 0$ where φ is a gauge function of the first order on $J = [0, \infty)$ and

$$m(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}. \tag{31}$$

Then for each $x_0 \in D$ such that $d(x_0, z) \in J$ for some $z \in Tx_0 \cap D$ the following statements hold true.

(i) *There exists an orbit of T at x_0 in D that converges to a unique fixed point ξ of T.*
(ii) *For $n \ge 0$ we have the following a priori estimate*

$$d(x_n, \xi) \le \frac{\lambda^n}{1 - \lambda} d(x_0, x_1). \tag{32}$$

(iii) *For all $n \ge 1$ we have the following a posteriori estimate*

$$d(x_n, \xi) \le \frac{\varphi(d(x_n, x_{n-1}))}{1 - \phi[\varphi(d(x_n, x_{n-1}))]}. \tag{33}$$

Proof. Choose $x_1 = z \in Tx_0 \cap D$. Define $\rho_0 = \sigma(d(x_0, x_1))$ where σ is defined by (2). Since from (3), $\sigma(t) \ge t$ so we have

$$d(x_0, x_1) \le \rho_0. \tag{34}$$

We may assume that $m(x_0, x_1) \neq 0$, for otherwise $d(x_0, Tx_0) \le m(x_0, x_1) = 0$ and x_0 is the fixed point of T. From (30) we have $H(Tx_0 \cap D, Tx_1 \cap D) < \varphi(m(x_0, x_1))$ we may choose $\epsilon_1 > 0$ with

$$H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1 \le \varphi(m(x_0, x_1)). \tag{35}$$

It follows from Lemma 1.2 that there exists $x_2 \in Tx_1 \cap D$ such that

$$d(x_1, x_2) \le H(Tx_0 \cap D, Tx_1 \cap D) + \epsilon_1. \tag{36}$$

From the above two inequalities we get

$$d(x_1, x_2) \le \varphi(m(x_0, x_1)) \tag{37}$$

$$\le \varphi\left(\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\}\right) \tag{38}$$

$$\le \varphi\left(\max\left\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2}\right\}\right). \tag{39}$$

We claim that

$$d(x_1, x_2) \le \varphi(d(x_0, x_1)). \tag{40}$$

Let

$$\eta_1 = \max\left\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2}\right\}. \tag{41}$$

Case 1. If $\eta_1 = d(x_0, x_1)$ then clearly (40) is true.

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Case 2. If $\eta_1 = d(x_1, Tx_1)$. Since $x_2 \in Tx_1$ we have

$$d(x_1, x_2) \le \varphi(d(x_1, Tx_1)) < d(x_1, Tx_1) \le d(x_1, x_2),$$

which is a contradiction.
Case 3. Finally suppose $\eta_1 = \frac{d(x_0, Tx_1)}{2}$. Then we have

$$d(x_1, x_2) \le \varphi\left(\frac{d(x_0, Tx_1)}{2}\right) < \frac{d(x_0, x_2)}{2} \le \frac{d(x_0, x_1) + d(x_1, x_2)}{2},$$

since $x_2 \in Tx_1$. Thus, $d(x_1, x_2) < d(x_0, x_1)$. As a result,

$$\eta_1 \le \frac{d(x_0, x_1) + d(x_1, x_2)}{2} < d(x_0, x_1),$$

which contradicts the definition of η_1. This proves our claim. Proceeding inductively in a similar way as in Theorem 2.1 we obtain the sequence $\{x_n\}$ in D such that $x_n \to \xi \in D$ and $\lim_n d(x_{n+1}, Tx_n) = 0$. Since T is continuous so taking the limit as $n \to \infty$ we have $\xi \in T\xi$. If η is another fixed point then $m(\xi, \eta) = d(\xi, \eta)$ and uniqueness follows immediately. Estimate (21) and (22) becomes (32) and (33) for $r = 1$. □

Remark 2.13. Corollary 2.12 can be observed as an improved version (with uniqueness of fixed point and error bounds) of [16, Theorems 2.1 and 2.2] if the continuous function ϕ in [16] is replaced by the gauge function φ of order 1.

3. Application

In this section we apply Corollary 2.11 to establish the existence of unique solution for the class of integral inclusions. Let $\mathbb{R}$ denote the real line. Consider the following integral inclusion.

$$\begin{aligned} x(t) &\in P \int_{t_0}^{t} k(s, x(s)) \mathrm{d}s + \beta \\ &\in PK^x(t) + \beta \end{aligned} \tag{42}$$

where

$$K^x(t) := \int_{t_0}^{t} k(s, x(s)) \mathrm{d}s. \tag{43}$$

P is a closed and bounded (compact) subset of $\mathbb{R}$, $\beta \in \mathbb{R}$ and $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$|k(t, x(t)) - k(t, y(t))| \le \frac{q}{\alpha} |x(t) - y(t)|^r \tag{44}$$

where $\alpha = \max_{p \in P} |p|$, $0 < q \le \alpha$ and $1 < r \le 2$

$$|k(t, x)| < \frac{1}{2\alpha} \left(\frac{1}{q}\right)^{r-1} \tag{45}$$

and is continuous on $R = \left\{(t, x) : |t - t_0| \le \left(\frac{1}{q}\right)^{2-r}, |x - \beta| \le \frac{1}{2q}\right\}$.

Theorem 3.1. *Under the above assumptions (2)–(4), the integral inclusion (42) has a unique solution on* $I = [t_0 - (\frac{1}{q})^{2-r}, t_0 + (\frac{1}{q})^{2-r}]$.

Remark 3.2. Note that in almost all the existence results, as for as we know, for integral inclusions or integral equations the kernel of the equation satisfies the Lipschitz condition in some sense. But in our result the kernel satisfies the inequality (44), which is not the Lipschitz condition, since $1 < r \le 2$. Our result not only guarantees the existence but also the uniqueness of the solution and provides the iterative scheme with a higher convergence rate. It also provides the domain of existence of the solution.

Proof of Theorem 3.1. Consider the space of all continuous functions $C(I)$ endowed by the classical metric

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|.$$

Note that $C(I)$ is complete with respect to the metric 'd'. Let $\widetilde{C} = \left\{x \in C(I) : d(x, \beta) \le \frac{1}{2q}\right\}$. Then, $\widetilde{C}$ is a closed subspace of $C(I)$ and thus is complete. Define an operator T by:

$$\begin{aligned} Tx(t) &= P\int_{t_0}^{t} k(s,x(s))\mathrm{d}s + \beta \\ &= PK^x(t) + \beta. \end{aligned} \tag{46}$$

Then, (46) and (42) are equivalent problems in a sense that fixed point(s) of T are solution(s) of (42). We claim that $T : (\widetilde{C}, d) \to (\underset{\sim}{K}(\widetilde{C}), H)$. For this we need to prove that: (i) T is defined for each $x \in \widetilde{C}$, and (ii) Tx is a compact subset of $\widetilde{C}$ for any $x \in \widetilde{C}$.

If $\tau \in I$ then $|\tau - t_0| \le \left(\frac{1}{q}\right)^{2-r}$. Hence by definition of $\widetilde{C}$ we have, $|x(\tau) - \beta| \le \frac{1}{2q}$. So, if $\tau \in I$ then $(\tau, x(\tau)) \in R$ and since k is continuous on R, therefore integral (43) exists in (46) and T is defined for each $x \in \widetilde{C}$. We now show that $Tx \subset \widetilde{C}$. Let $y(t) \in Tx(t)$. Then, $y(t) = pK^x(t) + \beta$ for some $p \in P$, and

$$\begin{aligned} |y(t) - \beta| &= |pK^x(t)| \\ &= |p|\,|K^x(t)| \\ &\le \alpha \int_{t_0}^{t} |k(s,x(s))\mathrm{d}s| \\ &\le \alpha \int_{t_0}^{t} |k(s,x(s))|\mathrm{d}s \\ &< \alpha|t - t_0|\frac{1}{2\alpha}\left(\frac{1}{q}\right)^{r-1} \le \frac{1}{2q}. \end{aligned}$$

Thus, $d(y, \beta) < \frac{1}{2q}$ for any $y \in Tx$ and hence $Tx \subset \widetilde{C}$ for any $x \in \widetilde{C}$. Next we show that Tx is compact. Consider a sequence $(u_n) \subset Tx$ then, $u_n = p_n K^x(t) + \beta$, where (p_n) is a sequence in P. Since P is compact there is a subsequence (p_{n_i}) of (p_n) such that $p_{n_i} \to \overline{p} \in P$. Let $u = \overline{p}K^x(t) + \beta$. Then

$$d(u_{n_i}, u) = \sup_{t\in I}(|p_{n_i} - \overline{p}|\,|K^x(t)|) \le |p_{n_i} - \overline{p}| \sup_{t\in I} |K^x(t)| \to 0$$

when $n_i \to \infty$. This proves our claim. Further, note that

$$\begin{aligned} H(Tx, Ty) &= H(PK^x(t) + \beta, PK^y(t) + \beta) \\ &\le H(PK^x(t), PK^y(t)). \end{aligned} \tag{47}$$

By definition,

$$H(PK^x(t), PK^y(t)) = \max\left\{\max_{a\in PK^x(t)} d(a, PK^y(t)),\ \max_{b\in PK^y(t)} d(b, PK^x(t))\right\}.$$

Consider,

$$\begin{aligned} \max_{a\in PK^x(t)} d(a, PK^y(t)) &= \max_{a\in PK^x}\min_{b\in PK^y} d(a,b) \\ &= \max_{p\in P}\min_{p^*\in P} d(pK(t,x), p^*K(t,y)) \\ &= \max_{p\in P}\min_{p^*\in P}\sup_{t\in I} |pK(t,x) - p^*K(t,y)| \\ &\le \max_{p\in P}\min_{p^*\in P}\sup_{t\in I}\left[|pK(t,y) - p^*K(t,y)| + |pK(t,y) - pK(t,x)|\right] \\ &\le \max_{p\in P}\min_{p^*\in P}\left[|p|\sup_{t\in I}|K(t,y) - K(t,x)| + |p - p^*|\sup_{t\in I}|K(t,y)|\right] \\ &= \max_{p\in P}|p|\sup_{t\in I}|K(t,y) - K(t,x)| \\ &= \alpha\sup_{t\in I}|K(t,y) - K(t,x)|. \end{aligned}$$

Now,

$$\begin{aligned} |K(t,y) - K(t,x)| &\le \int_{t_0}^{t} |k(s,y(s)) - k(s,x(s))|\mathrm{d}s \\ &\le \frac{q}{\alpha}\int_{t_0}^{t} |y(s) - x(s)|^r \mathrm{d}s \end{aligned}$$

$$\leq \frac{q}{\alpha}\left(\sup_{t\in I}|y(s)-x(s)|\right)^r \int_{t_0}^{t} \mathrm{d}s$$
$$= \frac{q}{\alpha}|t-t_0|(d(x,y))^r$$
$$\leq \frac{q}{\alpha}\left(\frac{1}{q}\right)^{2-r}(d(x,y))^r = \frac{q^{r-1}}{\alpha}(d(x,y))^r.$$

Hence,

$$\max_{a\in PK^x(t)} d(a, PK^y(t)) \leq \alpha\frac{q^{r-1}}{\alpha}(d(x,y))^r = q^{r-1}(d(x,y))^r.$$

Changing roles of x and y we obtain,

$$\max_{b\in PK^y(t)} d(b, PK^x(t)) \leq q^{r-1}(d(x,y))^r.$$

Now from (47) we have,

$$H(Tx, Ty) \leq q^{r-1}(d(x,y))^r.$$

Since for all $x, y \in \widetilde{C}$ we have $d(x,y) < \frac{1}{q}$. So, if we take $\varphi(t) = q^{r-1}t^r$ for $t \in J = [0, \frac{1}{q})$, then φ is the gauge function of the order r. To prove this, note that for $\lambda \in (0,1)$ and $t \in J$ we have

$$\varphi(\lambda t) = \lambda^r q^{r-1} t^r \leq \lambda^r \phi(t).$$

Further, for $t \in J - \{0\}$ we have

$$\varphi(t) = q^{r-1}t^r = q^{r-1}tt^{r-1} < q^{r-1}t\left(\frac{1}{q}\right)^{r-1} = t.$$

From the above arguments we have

$$H(Tx, Ty) \leq \varphi(d(x,y)) \tag{48}$$

for all $x, y \in \widetilde{C}$ with $d(x,y) \in J$. Hence starting from $x_0 = \beta$ the iterative sequence $x_n \in Tx_{n-1}$; $n = 1, 2, \ldots$ converges to the unique fixed point ξ of T at a rate r.

Remark 3.3. Note that if we take P to be $\{1\}$, then the integral inclusion (42) becomes:

$$x(t) = \int_{t_0}^{t} k(s, x(s))\mathrm{d}s + \beta$$

which is equivalent to the initial value problem

$$x'(t) = k(t, x(t)), \qquad x(t_0) = \beta. \tag{49}$$

Thus, we observe that [15, Theorem 3.1] is a special case of Theorem 3.1.

One may generalize the above result as follows.

Theorem 3.4. *Consider the integral inclusion*

$$x(t) \in P\int_{t_0}^{t} k(s, x(s))\mathrm{d}s + Q \tag{50}$$

where P and Q are compact subsets of $\mathbb{R}$. Under the assumptions (43), (44) and (45) that hold on $R = \{(t,x) : |t-t_0| \leq (\frac{1}{q})^{2-r}, |x-\beta| \leq \frac{1}{2q}\}$ for any $\beta \in Q$ the problem (50) has a solution on $I = [t_0 - (\frac{1}{q})^{2-r}, t_0 + (\frac{1}{q})^{2-r}]$.

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Generalized Contractions on Spaces with Two Metrics

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Abstract

We establish a convergence theorem for iterative processes of the type: $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$, where $T : D \subset X \to X$ is an operator on a complete metric space (X, d') satisfying:

$$d(Tx, T^2x) \leq \varphi(d(x, Tx)) \text{ for all } x \in D, Tx \in D \text{ with } d(x, Tx) \in J,$$

φ is a Bianchini-Grandolfi gauge function on an interval J and d is another metric on X.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fixed point theorems; Gauge functions

1 Introduction and Preliminaries

Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, A], [0, A)$ or $[0, \infty)$. We use the abbreviation φ^n for the nth iterate of a function $\varphi : J \to J$. Let $r \geq 1$, a function $\varphi : J \to J$ is said to be a gauge function [7, Definition 1.1] of order r on J if it satisfies the following conditions: (i) $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$; (ii) $\varphi(t) < t$ for all $t \in J - \{0\}$. It is easy to see that the first condition of this definition is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$. A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function [2, 7, Definition 2.3] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for all } t \in J. \tag{1}$$

Let (X, d') be a complete metric space, d another metric on X. For $x_0 \in X$, $r > 0$ and $S(x_0, r) = \{x \in X : d(x, x_0) < r\}$ we denote by $\overline{S(x_0, r)}^{d'}$ the d'-closure of $S(x_0, r)$ and by $\overline{S(x_0, r)}^{d}$ the d-closure of $S(x_0, r)$. In this paper we establish a convergence theorem for iterative processes of the type:

$$x_{n+1} = Tx_n, \ n = 0, 1, 2, \cdots \tag{2}$$

where $T : D \subset X \to X$ is an operator satisfying:

$$d(Tx, T^2x) \leq \varphi(d(x, Tx)) \ \text{ for all } \ x \in D, Tx \in D \ \text{ with } \ d(x, Tx) \in J, \tag{3}$$

and φ is a Bianchini-Grandolfi gauge function on an interval J. Our results generalize, extend and improve some recent results by Agarwal and O'Regan [1] and thus generalize some results of Ciric [3], Hardy and Rogers[4], Kannan [5], Maia [6], Precup [8] and Reich [9]. We use the machinery of Proinov [7] to prove our results. For convenience we define the function $E : D \to \mathbb{R}_+$ by $E(x) = d(x, Tx)$. Therefore condition (3) can be rewritten in the form

$$E(Tx) \leq \varphi(E(x)) \text{ for all } x \in D \text{ with } Tx \in D \text{ and } E(x) \in J. \tag{4}$$

Suppose $x_0 \in D$ and $E(x_0) \in J$. Then for every iterate x_n $(n \geq 0)$ which belongs to D we define the closed ball $\overline{S(x_n, \rho_n)}^d$ with center x_n and radius $\rho_n = \sigma(E(x_n))$, where $\sigma : J \to \mathbb{R}_+$ is defined by (1). A point $x_0 \in D$ will be called an initial point [7, Definition 3.2] of T if $E(x_0) \in J$ and all of the iterates $x_0, x_1, x_2, \cdots$ are well-defined and belong to D. We state following results for convenience.

Lemma 1.1 *[7, Lemma 2.4] Every gauge function of order $r \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.*

Theorem 1.2 *[1, Theorem 2.1] Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and $F : \overline{B(x_0, r)}^{d'} \to X$. Suppose there exists $q \in (0, 1)$ such that for $x, y \in \overline{B(x_0, r)}^{d'}$ we have*

$$d(Fx, Fy) \leq q \max\{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2}[d(x, Fy) + d(y, Fx)]\}.$$

In addition assume the following three properties hold:

$$d(x_0, Fx_0) < (1 - q)r \tag{5}$$

$$\textit{If } d \not\geq d' \textit{ assume } F \textit{ is uniformly continuous from } (B(x_0, r), d) \textit{ into} (X, d'), \tag{6}$$

and

$$\textit{If } d \neq d' \textit{ assume } F \textit{ is continuous from } (\overline{B(x_0, r)}^{d'}, d') \textit{ into} (X, d'), \tag{7}$$

Then F has a fixed point. That is, there exists $x \in \overline{B(x_0, r)}^{d'}$ with $x = Fx$.

Theorem 1.3 *[1, Theorem 2.3] Let (X, d') be a complete metric space, d another metric on X and $F : X \to X$. Suppose there exists $q \in (0, 1)$ such that for $x, y \in X$ we have*

$$d(Fx, Fy) \leq q \max\{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2}[d(x, Fy) + d(y, Fx)]\}.$$

In addition assume the following two properties hold:

$$\textit{If } d \not\geq d' \textit{ assume } F \textit{ is uniformly continuous from } (X, d) \textit{ into} (X, d'), \quad (8)$$

and

$$\textit{If } d \neq d' \textit{ assume } F \textit{ is continuous from } (X, d') \textit{ into} (X, d'), \quad (9)$$

Then F has a fixed point.

2 Main Results

Theorem 2.1 *Let (X, d') be a complete metric space, d another metric on X and $T : D \subset X \to X$ is an operator satisfying condition (3), with a Bianchini-Grandolfi gauge function φ on an interval J. Then starting from an initial point x_0 of T the iterative sequence (2) remains in $\overline{S(x_0, \rho_0)^d}$ and converges to a point $\xi \in \overline{S(x_0, \rho_0)^{d'}}$ which is a fixed point of T provided that the following three conditions hold:*

$$\textit{If } d \not\geq d' \textit{ then } T \textit{ is uniformly continuous from } (S(x_0, \rho_0), d) \textit{ into } (X, d') \quad (10)$$

$$\textit{If } d \neq d' \textit{ then } T \textit{ is continuous from } (\overline{S(x_0, \rho_0)^{d'}}, d') \textit{ into } (X, d'), \quad (11)$$

$$\textit{If } d = d' \textit{ then } T \textit{ is continuous at } \xi. \quad (12)$$

Proof: Since x_0 is an initial point of T, it follows from [7, Lemma 3.2] that

$$\overline{S(x_{n+1}, \rho_{n+1})^d} \subset \overline{S(x_n, \rho_n)^d} \text{ for all } n \geq 0, \quad (13)$$

which implies that $x_n \in \overline{S(x_0, \rho_0)^d}$ for all $n \geq 0$. Using the definition of ρ_n, from [7, Lemma 3.6], we get

$$\begin{aligned} \rho_n = \sigma(E(x_n)) &\leq \sigma(\varphi^n(E(x_0))) \\ &= \sum_{j=0}^{\infty} \varphi^j(\varphi^n(E(x_0))) \\ &= \sum_{j=n}^{\infty} \varphi^j(E(x_0)) \text{ for all } n \geq 0, \end{aligned} \quad (14)$$

since σ is non-decreasing. As φ is Bianchini-Grandolfi gauge function on J and $E(x_0) \in J$, from (14), we get

$$\rho_n \to 0 \text{ as } n \to \infty. \tag{15}$$

From (13) and (15) it follow that $\{\overline{S(x_n, \rho_n)^d}\}$ is nested sequence of closed spheres such that $\rho_n \to 0$ as $n \to \infty$. Therefore, it follows from Cantor's Theorem that the sequence (2) is a Cauchy sequence with respect to d i.e., there exists $N \in \{1, 2, \cdots\}$ with

$$d(x_n, x_m) < \delta \quad \text{whenever} \quad n, m \geq N. \tag{16}$$

We now claim that, (2) is a Cauchy sequence with respect to d'. If $d \geq d'$ this is trivial. Next suppose $d \ngeq d'$. Let $\epsilon > 0$ be given. Then (10) guarantees that there exists $\delta > 0$ such that

$$d'(Tx, Ty) < \epsilon \quad \text{whenever} \quad x, y \in S(x_0, \rho_0) \quad \text{and} \quad d(x, y) < \delta \tag{17}$$

Now (16) and (17) imply

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \epsilon \quad \text{whenever} \quad n, m \geq N,$$

and as a result our claim is proved. Now since (X, d') is complete there exists $\xi \in \overline{S(x_0, \rho_0)^{d'}}$ with $d'(x_n, \xi) \to 0$ as $n \to \infty$. We claim that $\xi = T\xi$. First consider the case when $d \neq d'$.

$$d'(\xi, T\xi) \leq d'(\xi, x_n) + d'(x_n, T\xi) = d'(\xi, x_n) + d'(Tx_{n-1}, T\xi) \tag{18}$$

Let $n \to \infty$ then (11) insures that $d'(\xi, x_n) \to 0$ implies $d'(Tx_{n-1}, T\xi) \to 0$ and so $\xi = T\xi$. Next suppose $d = d'$.

$$d(\xi, Tx_n) \leq d(\xi, x_n) + d(x_n, Tx_n) = d(\xi, x_n) + d(x_n, x_{n+1}).$$

Taking limit as $n \to \infty$ we get,

$$lim_{n\to\infty} d(\xi, Tx_n) \leq 0.$$

From (12) since T is continuous at ξ so we have $d(\xi, T\xi) = 0$ which simply means that $\xi = T\xi$.

Remark 2.2 *Theorem 2.1 remains true if φ is a gauge function of order $r \geq 1$.*

Corollary 2.3 *Let (X, d') be a complete metric space, d another metric on X and $T : D \subset X \to X$ is an operator satisfying*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \textit{ for all } x, y, Tx, Ty \in D \textit{ with } d(x, y) \in J \tag{19}$$

where φ is a gauge function of order $r \geq 1$ on an interval J. Assume that x_0 is an initial point of T such that $d(x_0, Tx_0) \in J$. Then the following statements hold true:

(a) The iterative sequence (2) converges to a fixed point ξ of T.

(b) The operator T has a unique fixed point in $S = \{x \in X : d(x, \xi) \in J\}$,

provided that (10) and (11) hold.

Proof: From (19) $d(Tx, T^2x) \leq \varphi(d(x, Tx))$ holds for all $x \in D, y = Tx \in D$ and $d(x, Tx) \in J$. Further, by definition of φ we have $d(Tx, Ty) \leq \varphi(d(x, y)) < d(x, y)$ for $x \neq y$. Hence, T is d continuous. Thus *(a)* follows immediately from Theorem 2.1. To show the uniqueness, i.e., *(b)*, assume that $\eta \in S$ is another fixed point of T. Then $d(\xi, \eta) \in J$. It follows from (19) that

$$d(\xi, \eta) = d(T\xi, T\eta) \leq \varphi(d(\xi, \eta)) < d(\xi, \eta)$$

Hence $\xi = \eta$.

Corollary 2.4 *Let (X, d') be a complete metric space, d another metric on X and $T : D \subset X \to X$ is an operator satisfying*

$$d(Tx, Ty) \leq \varphi(m(x, y)) \text{ for all } x, y, Tx, Ty \in D \tag{20}$$

where φ is a gauge function of order $r \geq 1$ on an interval $J = [0, \infty)$ and

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}. \tag{21}$$

Then for initial point x_0 of T, the iterative sequence (2) converges to a unique fixed point ξ of T provided that (10),(11) and (12) hold. Moreover, if φ is continuous, then continuity of T in (12) can be omitted.

Proof: First we shall prove that, $m(x, Tx) = d(x, Tx)$ for all $x \in D$. By (21) and triangle inequality, we get

$$\begin{aligned} m(x, Tx) &= \max\{d(x, Tx), d(x, Tx), d(Tx, T^2x), d(x, T^2x)\} \\ &= \max\{d(x, Tx), d(Tx, T^2x)\} = \max\{E(x), E(Tx)\} \end{aligned}$$

where $E(x) = d(x, Tx)$. If $x = Tx$ then $m(x, Tx) = E(x) = d(x, Tx)$. If $x \neq Tx$ then since $\varphi(t) < t$ for $t > 0$ so we have from (20)

$$E(Tx) = d(Tx, T^2x) \leq \varphi(m(x, Tx)) < m(x, Tx) = \max\{E(x), E(Tx)\}$$

which means $m(x, Tx) = E(x)$. Thus condition (20) implies (3). Applying Theorem 2.1, we deduce that the iterative sequence (2) converges to the fixed point ξ of T.

If η is another fixed point then from (21) we have $m(\xi, \eta) = d(\xi, \eta)$ and hence uniqueness follows immediately from Corollary 2.3.

Finally, suppose that $d = d'$ and φ is continuous then it follows from (20) that

$$d(x_{n+1}, T\xi) = d(Tx_n, T\xi) \leq \varphi(m(x_n, \xi)) \text{ for all } n \geq 0$$

Since $m(x_n, \xi) = d(x_n, \xi)$ so passing limit as $n \to \infty$ we obtain $d(\xi, T\xi) \leq \varphi(0) = 0$ which implies $\xi = T\xi$.

Remark 2.5 *If $\varphi(t) = qt \ \ (0 < q < 1)$, then Corollary 2.4 generalizes Theorem 1.2 (resp. Theorem 1.3) in the following manner.*

(i) Taking $D = B(x_0, r)^{d'}$ (resp. $D = X$).

(ii) Uniform Continuity of the operator T in (6) (resp. in (8)) is required from $(B(x_0, r), d)$ into (X, d') (resp. from (X, d) into (X, d')) while it is required from $(S(x_0, \rho_0), d)$ into (X, d') in (10).

(iii) Continuity f the operator T in (7) (resp. in (9)) is required from $\overline{B(x_0, r)^{d'}}$ into (X, d') (resp. from (X, d') into (X, d')) while it is required from $(\overline{S(x_0, \rho_0)^{d'}}, d')$ into (X, d') in (10).

(iv) Corollary 2.4 concludes with a unique fixed point.

(v) (12) can be omitted because φ is continuous in this case.

Remark 2.6 *Note that in the hypothesis of Corollary 2.4 we assumed that x_0 is an initial point of T. This assumption was not the part of the hypothesis of Theorem 1.2. But a closer look at condition (5) in Theorem 1.2 in-fact implies that x_0 is an initial point of T.*

Remark 2.7 *If $d = d'$, then Corollary 2.4 coincides with the first part of [7, Corollary 4.5]. In addition, if $\varphi(t) = qt \ \ (0 < q < 1)$ then it reduces to [1, Corollary 2.2].*

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FIXED POINT AND HOMOTOPY RESULTS FOR GENERALIZED CONTRACTIONS ON SPACES WITH TWO METRICS

Abstract. In this paper, we establish fixed point and homotopic results for generalized contractions on spaces with two metrics. Our results generalize and extend the results of Agarwal and O'Regan [R. P. Agarwal, Donal O'Regan, Fixed point theory for generalized contractions on spaces with two metrics, J. Math. Anal. Appl. 248 (2000), 402–414] and those contain therein.

1. Introduction and preliminaries

Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, A], [0, A)$ or $[0, \infty)$. We use the abbreviation φ^n for the nth iterate of a function $\varphi : J \to J$. Let $s \geq 1$, a function $\varphi : J \to J$ is said to be a gauge function [7, Definition 1.1] of order s on J if it satisfies the following conditions: (i) $\varphi(\lambda t) \leq \lambda^s \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$; (ii) $\varphi(t) < t$ for all $t \in J - \{0\}$. It is easy to see that the first condition of this definition is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^s$ is nondecreasing on $J - \{0\}$. A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function [3, 2, Definition 2.3] on J if

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for all } t \in J. \tag{1}$$

Ptak [2] noticed that a function $\varphi : J \to J$ satisfying (1) on J also satisfies the following functional equation

$$\sigma(t) = \sigma(\varphi(t)) + t. \tag{2}$$

Let (X, d') be a complete metric space, d another metric on X. For $x_0 \in X$, $r > 0$ and $S(x_0, r) = \{x \in X : d(x, x_0) < r\}$ we denote by $\overline{S(x_0, r)}^{d'}$ the

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d'-closure of $S(x_0, r)$ and by $\overline{S(x_0, r)}^d$ the d-closure of $S(x_0, r)$. Our results generalize, extend and improve some recent results by Agarwal and O'Regan [1] and thus generalize some results of Hardy and Rogers [4], Kannan [5], Maia [6], Precup [8] and Reich [9]. We state following results for convenience.

Lemma 1.1. [7, Lemma 2.4] *Every gauge function of order $s \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.*

2. Main results

Theorem 2.1. *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and $T : S(x_0, r) \to X$ satisfies*

(3) $d(Tx, T^2x) \leq \varphi(d(x, Tx))\ \forall x, Tx \in S(x_0, r)$ *with* $d(x, Tx) \in J$

where φ is a Bianchini-Grandolfi gauge function on an interval $J = [0, \infty)$. Then starting from x_0 the iterative sequence

$$x_{n+1} = Tx_n, \quad n \geq 0 \tag{4}$$

converges to a point $\xi \in \overline{S(x_0, r)}^{d'}$ which will be the fixed point of T if the following conditions are satisfied:

(i)

$$d(x_0, Tx_0) < \delta, \tag{5}$$

where $\delta > 0$ is such that $\sigma(\delta) \leq r$.

(ii) *If $d \ngeq d'$ assume T is uniformly continuous from $(S(x_0, r), d)$ into (X, d').*

(iii) *If $d \neq d'$ then T is continuous from $(\overline{S(x_0, r)}^{d'}, d')$ into (X, d').*

(iv) *If $d = d'$ then T is continuous at ξ.*

Proof. Let $x_1 = Tx_0$. Then from *(i)*

$$d(x_0, x_1) < \delta \leq \sigma(\delta) \leq r.$$

Next let $x_2 = Tx_1$. Then from (3) we have

$$d(x_1, x_2) = d(Tx_0, T^2x_0) \leq \varphi(d(x_0, x_1)).$$

Note that $d(x_1, x_2) \in J$. Further, $x_2 \in \overline{S(x_0, r)}^d$ since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) \\ &< \delta + \sigma(\varphi(\delta)) = \sigma(\delta) \text{ (using (2))} \leq r. \end{aligned}$$

Let $x_3 = Tx_2$. Then,

$$\begin{aligned} d(x_2, x_3) = d(Tx_1, T^2x_1) &\leq \varphi(d(x_1, Tx_1)) = \varphi(d(Tx_0, T^2x_0)) \\ &\leq \varphi^2(d(x_0, x_1)). \end{aligned}$$

Note that $d(x_2, x_3) \in J$. Further, $x_3 \in \overline{S(x_0, r)^d}$ since

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &\leq d(x_0, x_1) + \varphi(d(x_0, x_1)) + \varphi^2(d(x_0, x_1))) \\ &< \sum_{j=0}^{\infty} \varphi^j(\delta) = \sigma(\delta) \leq r. \end{aligned}$$

Proceeding inductively we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{S(x_0, r)^d}$ such that $d(x_{n-1}, x_n) \in J$ where

$$x_n = Tx_{n-1}, \tag{6}$$

and

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)). \tag{7}$$

Note that $\{x_n\}$ is a Cauchy sequence w.r.t d since for $n, p \in \mathbb{N}$, from (7) we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \varphi^{n+p-1}(d(x_0, x_1)) + \cdots + \varphi^n(d(x_0, x_1)) \\ &\leq \sum_{j=n}^{\infty} \varphi^j(d(x_0, x_1)) \to 0 \text{ (using (1))}. \end{aligned}$$

Thus, there exists $N \in \mathbb{N}$ with

$$d(x_n, x_m) < \eta \quad \text{whenever} \quad n, m \geq N. \tag{8}$$

We claim that, (4) is a Cauchy sequence with respect to d'. If $d \geq d'$ this is trivial. Next suppose $d \not\geq d'$. Let $\epsilon > 0$ be given. Then (ii) guarantees that there exists $\eta > 0$ such that

$$d'(Tx, Ty) < \epsilon \text{ whenever } x, y \in S(x_0, \rho_0) \text{ and } d(x, y) < \eta. \tag{9}$$

Now (8) and (9) imply

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \epsilon \quad \text{whenever} \quad n, m \geq N,$$

and this prove our claim. Since (X, d') is complete there exists $\xi \in \overline{S(x_0, \rho_0)^{d'}}$ with $d'(x_n, \xi) \to 0$ as $n \to \infty$. We claim that $\xi = T\xi$. First consider the case when $d \neq d'$

$$d'(\xi, T\xi) \leq d'(\xi, x_n) + d'(x_n, T\xi) = d'(\xi, x_n) + d'(Tx_{n-1}, T\xi). \tag{10}$$

Let $n \to \infty$ then (iii) insures that $d'(\xi, x_n) \to 0$ implies $d'(Tx_{n-1}, T\xi) \to 0$ and so $\xi = T\xi$. Next suppose $d = d'$

$$d(\xi, Tx_n) \leq d(\xi, x_n) + d(x_n, Tx_n) = d(\xi, x_n) + d(x_n, x_{n+1}).$$

Taking limit as $n \to \infty$ we get,

$$\lim_{n\to\infty} d(\xi, Tx_n) \leq 0.$$

From (iv) since T is continuous at ξ so we have $d(\xi, T\xi) = 0$ which simply means that $\xi = T\xi$.

Remark 2.2. Theorem 2.1 remains true if φ is a gauge function of order $s \geq 1$.

The following global result can easily be obtain from Theorem 2.1 and Remark 2.2.

Theorem 2.3. *Let (X, d') be a complete metric space, d another metric on X and $T : X \to X$ is an operator satisfying* (3) *with gauge function φ of order $s \geq 1$ on an interval $J = [0, \infty)$. Then T has a fixed point provided that the following conditions are satisfied:*

(a) *if $d \ngeq d'$ assume T is uniformly continuous from (X, d) into (X, d').*
(b) *if $d \neq d'$ then T is continuous from (X, d') into (X, d').*
(c) *if $d = d'$ then T is continuous at ξ.*

Proof. Fix $x_0 \in X$. For $\delta > 0$ such that

$$d(x_0, Tx_0) < \delta,$$

take $r = \sigma(\delta)$, where σ is given by (1). Now Theorem 2.1 guarantees that there exists $\xi \in \overline{S(x_0, r)^{d'}}$ with $\xi \in T\xi$.

Theorem 2.4. *Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and $T : S(x_0, r) \to X$ is an operator satisfying*

$$d(Tx, Ty) \leq \varphi(m(x, y)) \text{ for all } x, y, Tx, Ty \in S(x_0, r) \tag{11}$$

where φ is a gauge function of order $s \geq 1$ on an interval $J = [0, \infty)$ and

$$m(x, y) = \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}. \tag{12}$$

Then the iterative sequence (4) *converges to a unique fixed point ξ of T provided that* (i)–(iv) *hold. Moreover, if φ is continuous, then continuity of T in* (iv) *can be omitted.*

Proof. Let $x_1 = Tx_0$. Then from (i)

$$d(x_0, Tx_0) < \delta \leq \sigma(\delta) \leq r.$$

Next let $x_2 = Tx_1$ then from (11) we have

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leq \varphi(m(x_0, x_1))$$

where,

$$m(x_0,x_1)=\varphi\Big(\max\Big\{d(x_0,x_1),d(x_0,Tx_0),d(x_1,Tx_1),\frac{d(x_0,Tx_1)+d(x_1,Tx_0)}{2}\Big\}\Big)$$
$$=\varphi\Big(\max\Big\{d(x_0,x_1),d(x_1,x_2),\frac{d(x_0,x_2)}{2}\Big\}\Big).$$

We claim that

$$d(x_1,x_2)\le\varphi(d(x_0,x_1)). \tag{13}$$

Case 1. If $m(x_0,x_1)=d(x_0,x_1)$ then clearly (13) is true.
Case 2. If $m(x_0,x_1)=d(x_1,x_2)$. Then we have

$$d(x_1,x_2)\le\varphi(d(x_1,x_2))<d(x_1,x_2),$$

which is a contradiction.
Case 3. Finally suppose $m(x_0,x_1)=\frac{d(x_0,x_2)}{2}$. Then we have

$$d(x_1,x_2)\le\varphi\Big(\frac{d(x_0,x_2)}{2}\Big)<\frac{d(x_0,x_2)}{2}\le\frac{d(x_0,x_1)+d(x_1,x_2)}{2}.$$

Thus, $d(x_1,x_2)<d(x_0,x_1)$. As a result,

$$m(x_0,x_1)\le\frac{d(x_0,x_1)+d(x_1,x_2)}{2}<d(x_0,x_1),$$

which contradicts the definition of $m(x_0,x_1)$. This prove our claim. Proceeding inductively in a similar way as in Theorem 2.1 we obtain the iterative sequence (4) converges to the fixed point ξ of T. If η is another fixed point then from (11) and (12) we have $m(\xi,\eta)=d(\xi,\eta)$ and

$$d(\xi,\eta)=d(T\xi,T\eta)\le\varphi(m(\xi,\eta))=\varphi(d(\xi,\eta))<d(\xi,\eta).$$

So, $\xi=\eta$. Finally, suppose $d=d'$ and φ is continuous then it follows from (11) that

$$d(x_{n+1},T\xi)=d(Tx_n,T\xi)\le\varphi(m(x_n,\xi))\quad\text{for all}\quad n\ge 0$$
$$=\varphi\Big(\max\Big\{d(x_n,\xi),d(x_n,Tx_n),d(\xi,T\xi),\frac{d(x_n,T\xi)+d(\xi,Tx_n)}{2}\Big\}\Big)$$

so passing limit as $n\to\infty$ we obtain $d(\xi,T\xi)\le\varphi(d(\xi,T\xi))$ which is possible only when $\xi=T\xi$.

The following global result can easily be obtain from Theorem 2.4.

Theorem 2.5. *Let (X,d') be a complete metric space, d another metric on X and $T:X\to X$ is an operator satisfying* (11) *with gauge function φ of order $s\ge 1$ on an interval $J=[0,\infty)$ and $m(x,y)$ is defined in* (12).

Then T has a unique fixed point provided that the following conditions are satisfied:

(a′) *If $d \not\geq d'$ assume T is uniformly continuous from (X, d) into (X, d').*
(b′) *If $d \neq d'$ then T is continuous from (X, d') into (X, d').*
(c′) *If $d = d'$ then T is continuous at ξ.*

Moreover, if φ is continuous, then continuity of T in (c′) can be omitted.

Remark 2.6. If $\varphi(t) = qt \quad (0 < q < 1)$, then Theorem 2.4 generalizes [1, Theorem 2.1] and Theorem 2.5 generalizes [1, Theorem 2.2]. In addition if $d = d'$ then Theorem 2.4 and 2.5 reduces to [1, Corollary 2.2 and 2.4] respectively. Note that unlike the results of [1] our results also provide the uniqueness of fixed point.

3. Homotopy result

In this section we obtain a homotopy result as an application of Theorem 2.4. The proof of the result is inspired by the [10, Theorem 2.4]. We begin with the following Lemma whose proof is given in the proof of [10, Theorem 2.4]. We include its detail for completeness.

Lemma 3.1. *If $\{s_n\}$ is a sequence of nonnegative real numbers and $\varphi : J \to J$ is a gauge function of the order $s \geq 1$ then $\liminf \varphi(s_n) \leq \varphi(\liminf s_n)$.*

Proof. Fix $\epsilon > 0$ and $k \in \mathbb{N}$. Note that there exists $m > k$ with

$$s_m \leq \inf\{s_n : n > k\} + \epsilon.$$

Now since φ is nondecreasing we have

$$\varphi(s_m) \leq \varphi(\inf\{s_n : n > k\} + \epsilon) \leq \varphi(\liminf s_n + \epsilon),$$

and so

$$\inf\{\varphi(s_n) : n > k\} \leq \varphi(\liminf s_n + \epsilon).$$

Hence, $\liminf \varphi(s_n) \leq \varphi(\liminf s_n)$.

Theorem 3.2. *Let (X, d') be a complete metric space and d be another metric on X. Let $U \subseteq X$ be d-open subset of X. Suppose $H : \overline{U}^{d'} \times [0, 1] \to X$ satisfies the following properties:*

(i) *$x \neq H(x, \lambda)$ for $x \in \overline{U}^{d'} \setminus U$ and $\lambda \in [0, 1]$.*
(ii) *For every $x, y \in \overline{U}^{d'}$*

$$d(H(x,\lambda), H(y,\lambda)) \leq \varphi\Big(\max\Big\{d(x,y), d(x,H(x,\lambda)), d(y,H(y,\lambda)), \frac{1}{2}[d(x,H(y,\lambda)) + d(y,H(x,\lambda))]\Big\}\Big);$$

where φ is a gauge function of the order $s \geq 1$ on an interval $J = [0, \infty)$, $\lambda \in [0, 1]$ and;

(iii) *$H(x, \lambda)$ is continuous in λ w.r.t. d, uniformly for $x \in \overline{U^{d'}}$.*

(iv) *If $d \not\geq d'$ assume H is uniformly continuous from $U \times [0, 1]$ endowed with the metric d on U into (X, d').*

(v) *If $d \neq d'$ assume H is continuous from $\overline{U^{d'}} \times [0, 1]$ endowed with the metric d' on $\overline{U^{d'}}$ into (X, d').*

(vi) *If $d = d'$ then T is continuous.*

(vii) $\inf\{d(x, H_\lambda(x)) : x \in \overline{U^{d'}} \setminus U, \lambda \in [0, 1]\} > 0$.

In addition assume H_0 has a fixed point. Then for each $\lambda \in [0, 1]$ we have that H_λ has a fixed point $x_\lambda \in U$ (here $H_\lambda(.) = H(., \lambda)$).

Proof. Let

$$A = \{\lambda \in [0, 1] : x = H(x, \lambda) \text{ for some } x \in U\}.$$

Now since $H(., 0)$ has a fixed point and *(i)* holds we have that $0 \in A$, so A is nonempty. We will show A is both closed and open in $[0, 1]$, and so by connectedness of $[0, 1]$ we have $A = [0, 1]$.

First we show A is closed in [0,1]. Let (λ_k) be a sequence in A with $\lambda_k \to \lambda \in [0, 1]$ as $k \to \infty$. By definition for k, there exists $x_k \in U$ with $x_k = H(x_k, \lambda_k)$. We claim that,

$$\inf_{k \geq 1} d(x_k, \overline{U^{d'}} \setminus U) > 0. \tag{14}$$

Suppose not. For a fixed $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ and $y_{n_i} \in \overline{U^{d'}} \setminus U$ with

$$d(x_{n_i}, y_{n_i}) < \frac{1}{i}.$$

Consequently, there exists a subsequence K of $\{1, 2, \cdots\}$ and a sequence $\{y_i\} \subseteq \overline{U^{d'}} \setminus U$ (for $i \in K$) with

$$d(x_i, y_i) < \frac{1}{i} \text{ for } i \in K. \tag{15}$$

This together with (vii) implies

$$0 < \inf\{d(x, H_{\lambda(x)}) : x \in \overline{U^{d'}} \setminus U, \lambda \in [0, 1]\} \leq \lim_{i \to \infty} \inf_{i \in K} d(y_i, H_{\lambda_i}(y_i)). \tag{16}$$

We will now show that

$$\lim_{i \to \infty} \inf_{i \in K} d(y_i, H_{\lambda_i}(y_i)) = 0. \tag{17}$$

To see this, note that

$$\begin{aligned}
&\lim_{i\to\infty}\inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) \\
&\qquad\leq \lim_{i\to\infty}\inf_{i\in K}[d(y_i,x_i)+d(x_i,H_{\lambda_i}(y_i))] \\
&\qquad\leq \lim_{i\to\infty}\inf_{i\in K}[\frac{1}{i}+d(H(x_i,\lambda_i),H(y_i,\lambda_i))] \\
&\qquad= \lim_{i\to\infty}\inf_{i\in K} d(H(x_i,\lambda_i),H(y_i,\lambda_i)) \\
&\qquad\leq \lim_{i\to\infty}\inf_{i\in K}\varphi(\max\{d(x_i,y_i),d(x_i,H(x_i,\lambda_i)), \\
&\qquad\quad d(y_i,H(y_i,\lambda_i)),\frac{1}{2}[d(x_i,H(y_i,\lambda_i))+d(y_i,H(x_i,\lambda_i))]\}).
\end{aligned}$$

Let

$$\begin{aligned}
m(x_i,y_i)=\max\Big\{&d(x_i,y_i),d(x_i,H(x_i,\lambda_i)),d(y_i,H(y_i,\lambda_i)), \\
&\frac{1}{2}[d(x_i,H(y_i,\lambda_i))+d(y_i,H(x_i,\lambda_i))]\Big\}.
\end{aligned}$$

Case 1: If $m(x_i,y_i)=d(x_i,y_i)$ then,

$$\begin{aligned}
\lim_{i\to\infty}\inf_{i\in K} d(y_i,H_{\lambda_i}(y_i)) &\leq \lim_{i\to\infty}\inf_{i\in K}\varphi(d(x_i,y_i)) \\
&< \lim_{i\to\infty}\inf_{i\in K} d(x_i,y_i)=\lim_{i\to\infty}\inf_{i\in K}\frac{1}{i}=0.
\end{aligned}$$

Hence (17) is true in this case.

Case 2: If $m(x_i,y_i)=d(x_i,H(x_i,\lambda_i))$. Then $m(x_i,y_i)=0$ and (17) is immediate.

Case 3: If $m(x_i,y_i)=d(y_i,H(y_i,\lambda_i))$ then by Lemma 3.1 we have

$$\begin{aligned}
\lim_{i\to\infty}\inf_{i\in K} d(y_i,H_{\lambda_i}(y_i)) &\leq \lim_{i\to\infty}\inf_{i\in K}\varphi(d(y_i,H_{\lambda_i}(y_i))) \\
&\leq \varphi(\lim_{i\to\infty}\inf_{i\in K} d(y_i,H_{\lambda_i}(y_i))),
\end{aligned}$$

which implies $\lim_{i\to\infty}\inf_{i\in K} d(y_i,H_{\lambda_i}(y_i))=0$ since $\varphi(z)<z$ if $z>0$. Hence (17) is true.

Case 4: Finally, if $m(x_i,y_i)=\frac{1}{2}[d(x_i,H(y_i,\lambda_i))+d(y_i,H(x_i,\lambda_i))]\}$. Then $\varphi(m(x_i,y_i))\leq m(x_i,y_i)$ implies that

$$\begin{aligned}\lim_{i\to\infty}\inf_{i\in K} d(y_i, H_{\lambda_i}(y_i)) &\leq \lim_{i\to\infty}\inf_{i\in K}\frac{1}{2}[d(x_i, H(y_i,\lambda_i)) + d(y_i, H(x_i,\lambda_i))]\\ &\leq \lim_{i\to\infty}\inf_{i\in K}\frac{1}{2}[d(x_i,y_i) + d(y_i, H(y_i,\lambda_i)) + \\ &\qquad\qquad d(x_i,y_i) + d(x_i, H(x_i,\lambda_i))]\\ &\leq \lim_{i\to\infty}\inf_{i\in K}\frac{1}{2}\left[\frac{1}{i} + d(y_i, H_{\lambda_i}(y_i)) + \frac{1}{i} + 0\right].\end{aligned}$$

Hence (17) is immediate. Thus we have a contradiction from (16) and as a result our claim (14) is true. Hence there exists $r > 0$ with $d(x_k, z) > r$ for all $k \geq 1$ and for all $z \in \overline{U^{d'}} \setminus U$. As a result (note $x_k \in U$ for each k)

$$\overline{S(x_k, r)^{d'}} \subseteq U \;\; k \geq 1.$$

If we choose δ such that $\sigma(\delta) < r$ then $\overline{S(x_k,\delta)^{d'}} \subseteq U$ for $k \geq 1$. This together with (iii) implies that there exists an integer n_0 with

$$d(x_{n_0}, H_\lambda(x_{n_0})) = d(H(x_{n_0},\lambda), H(x_{n_0},\lambda)) < \delta.$$

Now Theorem 2.4 guarantees that H_λ has a fixed point $x_{\lambda,n_0} \in \overline{S(x_k,r)^{d'}} \subseteq \overline{U^{d'}}$. Note from (i) that $x_{\lambda,n_0} \in U$. Consequently, $\lambda \in A$, so A is closed in $[0,1]$.

Next we will show that A is open in $[0,1]$. Let $\lambda_0 \in A$ and $x_0 \in U$ with $x_0 = H(x_0,\lambda_0)$. Since U is open there exists $r > 0$ with $S(x_0, r) \subseteq U$. Now (iii) guarantees that there exists $\eta = \eta(r,\delta) > 0$ with

$$d(x_0, H(x_0,\lambda)) = d(H(x_0,\lambda_0), H(x_0,\lambda)) < \delta$$

for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$. We may choose δ such that $\sigma(\delta) < r$. Now Theorem 2.4 guarantees that there exists $x_\lambda \in \overline{S(x_0,r)^{d'}} \subseteq \overline{U^{d'}}$ with $x_\lambda = H_\lambda(x_\lambda)$ for $\lambda \in [0,1]$ and $|\lambda - \lambda_0| < \eta$. Hence A is open in $[0,1]$.

Remark 3.3. If $\varphi(t) = qt \;\; (0 < q < 1)$, then Theorem 3.2 extends [1, Theorem 3.1] and thus some results of [8, 11, 12].

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Coincidence and fixed points for hybrid tangential maps

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Abstract. In [9] by Liu et al. the common property $(E.A)$ for two pairs of hybrid maps is defined. Recently, O'Regan and Shahzad [16] have introduced a very general contractive condition and obtained some fixed point results for hybrid maps. We introduce a new property for pairs of hybrid maps that contains the property $(E.A)$ and obtain some coincidence and fixed point theorems that extend/generalize some results from the above-mentioned papers.

Keywords. Fixed points, hybrid maps, the property $(E.A)$, weak commutativity.

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1 Introduction

In recent years, many fixed point theorems have appeared in the literature using the notion of compatibility [4] by various authors. Compatible mappings of type (A) [5], type (B) [12] (which is the extension of type (A) compatible maps), type (P) [13] and type (C) [14] have also been the subject of interest for many researchers. It is noted that if the maps are continuous then the notions of compatibility, compatibility of type (A), (B), (C) and type (P) are all equivalent [12–14]. Jungck and Rhoades [6] introduced the notion of weak-compatibility which is the extension of compatibility and all of its types [2]. Pant [10, 11] initiated the study of non-compatible maps by introducing the concept of pointwise R-weakly commuting mappings. It is known [7, 11] that at an coincidence point, pointwise R-weak commutativity is equivalent to weak compatibility and [11] that pointwise weak commutativity is a minimal condition for the existence of fixed points. Itoh and Takahashi [3] defined the notion of (IT)-commuting maps for the setting of single-valued and multivalued maps. Note that at a coincidence point, a weakly compatible Hybrid pair (f, T) is (IT)-commuting but the converse is not true [17, Example 1]. Sastry and Murthy [15] defined the notion of tangential single-valued maps. Aamri and Moutawakil [1] rediscovered the notion of tangential maps and called it the property $(E.A)$. They obtained some common fixed point theorems for pair(s) of weakly compatible maps satisfying the property $(E.A)$. The class of maps satisfying the property $(E.A)$ has a remarkable property that it contains the class of non-compatible maps. Kamran [7] extended the notion of the property $(E.A)$ for a hybrid pair (f, T), introduced the notion of T-weak commutativity, and showed that (IT)-commutativity at a coincidence point of a hybrid pair (f, T) implies T-weak commutativity at the same point but the converse is not true in general [7, Example 3.8]. Recently, Y. Liu et al. [9] defined the common property $(E.A)$ for two pairs of hybrid maps and obtained some coincidence and fixed point results for such pairs. More recently, O'Regan and Shahzad [16] introduced a very general contractive condition and obtained some fixed point results for hybrid maps, as an application of their results they got some invariant approximation results. The purpose of our paper is to define a new property for hybrid maps that contains the common property $(E.A)$. Using this notion we obtain some coincidence and fixed point theorems. Our results extend the results of O'Regan and Shahzad [16] substantially and thus generalize many fixed point theorems in the literature.

2 Preliminaries

Let X be a metric space with metric d. Then, for $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $\mathrm{CB}(X)$ the class of all nonempty bounded closed subsets of X, by $\mathrm{CL}(X)$ the class of all nonempty closed subsets of X. Let H be the generalized Hausdorff metric on $\mathrm{CL}(X)$ generated by the metric d, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \ \sup_{y\in B} d(y, A)\right\}$$

for every $A, B \in \mathrm{CL}(X)$. A point $p \in X$ is said to be a fixed point of $T : X \to \mathrm{CL}(X)$ if $p \in Tp$. The point p is called a coincidence point of $f : X \to X$ and $T : X \to \mathrm{CL}(X)$ if $fp \in Tp$. The set of coincidence points of f and T is denoted by $C(f, T)$ and the set of fixed points of f by $F(f)$.

Definition 2.1 ([8]). Maps $f : X \to X$ and $T : X \to \mathrm{CL}(X)$ are said to be compatible if $fTx \in \mathrm{CL}(X)$ for all $x \in X$ and $H(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to A \in \mathrm{CL}(X)$ and $fx_n \to t \in A$.

Remark 2.2. Therefore, the maps $f : X \to X$ and $T : X \to \mathrm{CL}(X)$ are non-compatible if $fTx \in \mathrm{CL}(X)$ for all $x \in X$ and there exists at least one sequence $\{x_n\}$ in X such that $Tx_n \to A \in \mathrm{CL}(X)$ and $fx_n \to t \in A$ but $\lim_{n\to\infty} H(fTx_n, Tfx_n) \neq 0$ or nonexistent.

Definition 2.3 ([6]). Maps $f : X \to X$ and $T : X \to \mathrm{CL}(X)$ are weakly compatible if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx \in Tx$.

Definition 2.4 ([3, 17]). Maps $f : X \to X$ and $T : X \to \mathrm{CL}(X)$ are said to be (IT)-commuting at $x \in X$ if $fTx \subseteq Tfx$.

Example 2.5 ([17]). Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$, $T : X \to \mathrm{CL}(X)$ by $fx = 4x$ and $Tx = [3 + x, \infty)$ for all $x \in X$. Then at $x = 1$, f and T are (IT)-commuting but not weakly compatible.

Definition 2.6 ([7]). Let $T : X \to \mathrm{CL}(X)$, then $f : X \to X$ is said to be T-weakly commuting at $x \in X$ if $ffx \in Tfx$.

Example 2.7 ([7]). Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$, $T : X \to \mathrm{CL}(X)$ by $fx = 2x$ and $Tx = [1, 2x + 1]$ for all $x \in X$. Then for all $x \in X$, $fx \in Tx$, $ffx = 4x \in [1, 4x + 1] = Tfx$, $fTx = [2, 4x + 2] \nsubseteq Tfx$. Therefore f is T-weakly commuting but not (IT)-commuting. Also note that f and T are not weakly compatible.

Definition 2.8 ([7]). Maps $f : X \to X$ and $T : X \to \mathrm{CL}(X)$ are said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X, some $t \in X$ and $A \in \mathrm{CL}(X)$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$.

Definition 2.9 ([9]). Let $f, g : X \to X$ and $S, T : X \to \mathrm{CL}(X)$. The hybrid pairs (f, T) and (g, S) are said to satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X, some $t \in X$, and $A, B \in \mathrm{CL}(X)$ such that

$$\lim_{n\to\infty} Tx_n = A, \quad \lim_{n\to\infty} Sy_n = B, \quad \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B. \tag{2.1}$$

We state the following results for convenience.

Theorem 2.10 ([9]). *Let f, g be two self-maps of the metric space (X, d) and let F, G be two maps from X into $\mathrm{CB}(X)$ such that*

(1) *(f, F) and (g, G) satisfy the common property $(E.A)$;*

(2) *for all $x \neq y \in X$,*

$$H(Fx, Gy) < \max\left\{d(fx, gy), \frac{d(fx, Fx) + d(gy, Gy)}{2}, \frac{d(fx, Gy) + d(gy, Fx)}{2}\right\}. \tag{2.2}$$

If fX and gX are closed subsets of X, then

(a) *f and F have a coincidence point;*

(b) *g and G have a coincidence point;*

(c) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f, F)$;*

(d) *g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$;*

(e) *f, g, F, and G have a common fixed point provided that both* (c) *and* (d) *are true.*

Theorem 2.11 ([16]). *Let $X := (X, d)$ be a metric space, $f : X \to X$ and $T : X \to \mathrm{CL}(X)$ such that f and T satisfy the property $(E.A)$. Suppose that there exist a continuous non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ and continuous functions $\phi_i : [0,\infty) \to [0,\infty)$ $(i = 1, 2, \ldots, 7)$ satisfying $\phi_i(0) = 0$ for $i = 1, 2, 4$ and $\phi(\phi_i(z)) < z$ for $z > 0$ and $i = 3, 5, 6, 7$ and*

$$\begin{aligned} H(Tx, Ty) \le \phi\Big(&\max\{\phi_1(d(fx, fy)), \phi_2(d(fx, Tx)), \phi_3(d(fy, Ty)), \\ &\phi_4(d(fy, Tx)), \phi_5(d(fx, Ty)), \phi_6(d(fx, fy) + d(fx, Tx) \\ &+ d(fy, Tx) + d(fy, Ty)), \phi_7(d(fx, fy) + d(fx, Tx) \\ &+ d(fy, Tx) + d(fx, Ty))\}\Big), \end{aligned} \tag{2.3}$$

for all $x, y \in X$. If fX is closed, then $C(f, T) \neq \emptyset$.

We have the following definition.

Definition 2.12. Let $X := (X, d)$ be a metric space, $f, g : X \to X$ and $S, T : X \to \mathrm{CL}(X)$. The hybrid pair (f, T) is said to be g-tangential at $t \in X$ if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X, $A \in \mathrm{CL}(X)$ such that $\lim_{n\to\infty} Sy_n \in \mathrm{CL}(X)$ and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A = \lim_{n\to\infty} Tx_n. \tag{2.4}$$

Lemma 2.13. *Let $X := (X, d)$ be a metric space and the hybrid pairs (f, T) and (g, S) satisfy the common property $(E.A)$, then (f, T) is g-tangential and (g, S) is f-tangential.*

However, the following example shows that if (f, T) is g-tangential, then it is not necessary that (f, T) and (g, S) satisfy the common property $(E.A)$.

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Example 2.14. Let $X = [1, \infty)$ with the usual metric. Define $f, g : X \to X$ and $S, T : X \to \mathrm{CL}(X)$ by

$$fx = 2 + \frac{1}{2}x, \quad gx = 2 + \frac{1}{3}x, \quad Sx = [1, 2] \quad \text{and} \quad Tx = [2, 2 + x]$$

for all $x \in X$. Consider the sequences $\{x_n\} = \left\{1 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\{y_n\} = \left\{\frac{3}{2} + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \frac{5}{2} \in [2, 3] = \lim_{n\to\infty} Tx_n.$$

Therefore the hybrid pair (f, T) is g-tangential. Suppose that there exist $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Tx_n = A$, $\lim_{n\to\infty} Ty_n = B$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B = \{2\}$. This implies that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Clearly, there do not exist such sequences in X.

3 Main results

We begin with the following theorem.

Theorem 3.1. *Let $X := (X, d)$ be a metric space, $f, g : X \to X$ and $T, S : X \to CL(X)$ be such that either (f, T) is g-tangential or (g, S) is f-tangential. Suppose that there exist a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ and continuous functions $\phi_i : [0, \infty) \to [0, \infty)$ $(i = 1, 2, \ldots, 7)$ satisfying $\phi_i(0) = 0$ for $i = 1, 2, 4$ and $\phi(\phi_i(z)) < z$ for $z > 0$ and $i = 3, 5, 6, 7$ and*

$$\begin{aligned} H(Tx, Sy) \le \phi\big(&\max\{\phi_1(d(fx, gy)), \phi_2(d(fx, Tx)), \phi_3(d(gy, Sy)), \\ &\phi_4(d(gy, Tx)), \phi_5(d(fx, Sy)), \phi_6(d(fx, gy) + d(fx, Tx) \\ &+ d(gy, Tx) + d(gy, Sy)), \phi_7(d(fx, gy) + d(fx, Tx) \\ &+ d(gy, Tx) + d(fx, Sy))\}\big), \end{aligned} \tag{3.1}$$

$$\begin{aligned} H(Sx, Ty) \le \phi\big(&\max\{\phi_1(d(gx, fy)), \phi_2(d(gx, Sx)), \phi_3(d(fy, Ty)), \\ &\phi_4(d(fy, Sx)), \phi_5(d(gx, Ty)), \phi_6(d(gx, fy) + d(gx, Sx) \\ &+ d(fy, Sx) + d(fy, Ty)), \phi_7(d(gx, fy) + d(gx, Sx) \\ &+ d(fy, Sx) + d(gx, Ty))\}\big), \end{aligned} \tag{3.2}$$

for all $x, y \in X$. Further suppose that one of the following holds:

(i) *fX is a closed subset of X and $TX \subseteq gX$;*

(ii) *gX is a closed subset of X and $SX \subseteq fX$;*

(iii) *fX and gX are closed subsets of X.*

Then $C(f,T) \neq \varnothing$ and $C(g,S) \neq \varnothing$.

Proof. Suppose that the hybrid pair (f,T) is g-tangential, then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Let $\lim_{n\to\infty} Sy_n = B$. We claim that $A = B$. Suppose the contrary, i.e., $H(A,B) > 0$. Using

$$H(A,Sy_n) \leq H(A,Tx_n) + H(Tx_n,Sy_n),$$

from (3.1) we get

$$\begin{aligned} H(A,Sy_n) \leq{}& H(A,Tx_n) + \phi\Big(\max\{\phi_1(d(fx_n,gy_n)),\phi_2(d(fx_n,Tx_n)),\\ &\phi_3(d(gy_n,Sy_n)),\phi_4(d(gy_n,Tx_n)),\phi_5(d(fx_n,Sy_n)),\\ &\phi_6(d(fx_n,gy_n)+d(fx_n,Tx_n)+d(gy_n,Tx_n)\\ &+d(gy_n,Sy_n)),\phi_7(d(fx_n,gy_n)+d(fx_n,Tx_n)\\ &+d(gy_n,Tx_n)+d(fx_n,Sy_n))\}\Big)\\ \leq{}& H(A,Tx_n) + \phi\Big(\max\{\phi_1(d(fx_n,gy_n)),\phi_2(d(fx_n,Tx_n)),\\ &\phi_3(d(gy_n,A)+H(A,Sy_n)),\phi_4(d(gy_n,Tx_n)),\phi_5(d(fx_n,A)\\ &+H(A,Sy_n)),\phi_6(d(fx_n,gy_n)+d(fx_n,Tx_n)+d(gy_n,Tx_n)\\ &+d(gy_n,A)+H(A,Sy_n)),\phi_7(d(fx_n,gy_n)+d(fx_n,Tx_n)\\ &+d(gy_n,Tx_n)+d(fx_n,A)+H(A,Sy_n))\}\Big). \end{aligned}$$

Letting $n\to\infty$, we get (using $\phi_i(0)=0$ for $i=1,2,4$)

$$\begin{aligned} &H(A,B)\\ &\quad\leq \phi\Big(\max\{\phi_3(H(A,B)),\phi_5(H(A,B)),\phi_6(H(A,B)),\phi_7(H(A,B))\}\Big). \end{aligned} \tag{3.3}$$

Then, if

$$\phi_3(H(A,B)) = \max\Big\{\phi_3(H(A,B)),\phi_5(H(A,B)),\phi_6(H(A,B)),\phi_7(H(A,B))\Big\},$$

from (3.3) we have (using $\phi(\phi_3(z)) < z$ for $z > 0$)

$$H(A,B) \leq \phi(\phi_3(H(A,B))) < H(A,B),$$

which is a contradiction. Similarly, for other cases we get contradictions as well. Hence,

$$\lim_{n\to\infty} Sy_n = A.$$

Now, suppose that condition (i) holds, i.e., fX is a closed subset of X and $TX \subseteq gX$. Then, there exists $a \in X$ such that $\lim_{n\to\infty} fx_n = t = fa$. We claim that $a \in C(f,T)$. Otherwise, from (3.2) we have

$$\begin{aligned} H(Sy_n, Ta) \le \phi\big(&\max\{\phi_1(d(gy_n, fa)), \phi_2(d(gy_n, Sy_n)), \phi_3(d(fa, Ta)), \\ &\phi_4(d(fa,Sy_n)),\phi_5(d(gy_n,Ta)),\phi_6(d(gy_n,fa)+d(gy_n,Sy_n) \\ &+ d(fa, Sy_n) + d(fa, Ta)), \phi_7(d(gy_n, fa) + d(gy_n, Sy_n) \\ &+ d(fa, Sy_n) + d(gy_n, Ta))\}\big). \end{aligned}$$

Letting $n \to \infty$, from the above inequality we have

$$\begin{aligned} H(A, Ta) \le \phi\big(&\max\{\phi_1(0), \phi_2(0), \phi_3(d(fa, Ta)), \phi_4(0), \phi_5(d(fa, Ta)), \\ &\phi_6(d(fa, Ta)), \phi_7(d(fa, Ta))\}\big). \end{aligned} \tag{3.4}$$

Suppose $d(fa, Ta) > 0$. Then, if $\phi_3(d(fa, Ta)) = \max_{i=3,5,6,7} \phi_i(d(fa, Ta))$, from (3.4), we have (using $\phi(\phi_3(z)) < z$ for $z > 0$)

$$H(A, Ta) < d(fa, Ta). \tag{3.5}$$

Inequality (3.5) also holds for other cases as well. Since $fa \in A$, using the definition of Hausdorff metric, from (3.5) we get $d(fa, Ta) < d(fa, Ta)$, which is a contradiction. Thus $C(f, T) \neq \varnothing$.

Now, we shall show that $C(g, S) \neq \varnothing$. As $Ta \subseteq gX$, there exists $b \in X$ such that $fa = gb = t$. Using (3.1), we get

$$\begin{aligned} H(Ta, Sb) \le \phi\big(&\max\{\phi_1(d(fa, gb)), \phi_2(d(fa, Ta)), \phi_3(d(gb, Sb)), \\ &\phi_4(d(gb, Ta)), \phi_5(d(fa, Sb)), \phi_6(d(fa, gb) + d(fa, Ta) \\ &+ d(gb, Ta) + d(gb, Sb)), \phi_7(d(fa, gb) + d(fa, Ta) \\ &+ d(gb, Ta) + d(fa, Sb))\}\big) \\ = \phi\big(&\max\{\phi_1(0), \phi_2(0), \phi_3(d(gb, Sb)), \phi_4(0), \phi_5(d(gb, Sb)), \\ &\phi_6(d(gb, Sb)), \phi_7(d(gb, Sb))\}\big). \end{aligned} \tag{3.6}$$

Then, if $\phi_3(d(gb, Sb)) = \max_{i=3,5,6,7} \phi_i(d(gb, Sb))$, using the definition of Hausdorff metric, from (3.6) we have (using $\phi(\phi_3(z)) < z$ for $z > 0$)

$$d(gb, Sb) = d(fa, Sb) \le H(Ta, Sb) < d(gb, Sb) \tag{3.7}$$

which is a contradiction. Inequality (3.7) also holds for other cases as well. Thus, $b \in C(g, S)$, since Sb is closed. The proofs are similar if (i) is replaced by (ii) or (iii), or/and the g-tangentiality of (f, T) is replaced by the f-tangentiality of (g, S). □

Corollary 3.2. *Let $X := (X, d)$ be a metric space, $f, g : X \to X$ and $T, S : X \to \mathrm{CL}(X)$. Suppose that the pairs (f, T) and (g, S) satisfy the common property $(E.A)$, there exist a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ and continuous functions $\phi_i : [0, \infty) \to [0, \infty)$ $(i = 1, 2, \ldots, 7)$ satisfying $\phi_i(0) = 0$ for $i = 1, 2, 4$ and $\phi(\phi_i(z)) < z$ for $z > 0$ and $i = 3, 5, 6, 7$ and*

$$\begin{aligned} H(Tx, Sy) \le \phi\big(&\max\{\phi_1(d(fx, gy)), \phi_2(d(fx, Tx)), \phi_3(d(gy, Sy)),\\ &\phi_4(d(gy, Tx)), \phi_5(d(fx, Sy)), \phi_6(d(fx, gy) + d(fx, Tx)\\ &+ d(gy, Tx) + d(gy, Sy)), \phi_7(d(fx, gy) + d(fx, Tx)\\ &+ d(gy, Tx) + d(fx, Sy))\}\big), \end{aligned} \tag{3.8}$$

$$\begin{aligned} H(Sx, Ty) \le \phi\big(&\max\{\phi_1(d(gx, fy)), \phi_2(d(gx, Sx)), \phi_3(d(fy, Ty)),\\ &\phi_4(d(fy, Sx)), \phi_5(d(gx, Ty)), \phi_6(d(gx, fy) + d(gx, Sx)\\ &+ d(fy, Sx) + d(fy, Ty)), \phi_7(d(gx, fy) + d(gx, Sx)\\ &+ d(fy, Sx) + d(gx, Ty))\}\big), \end{aligned} \tag{3.9}$$

for all $x, y \in X$. Further, suppose that one of the following holds:

(i) *fX is a closed subset of X and $TX \subseteq gX$;*

(ii) *gX is a closed subset of X and $SX \subseteq fX$;*

(iii) *fX and gX are closed subsets of X.*

Then $C(f, T) \neq \varnothing$ and $C(g, S) \neq \varnothing$.

Proof. The proof follows from Theorem 3.1, using Lemma 2.13. □

Remark 3.3. Note that when $g = f$ and $T = S$, inequalities (3.1) and (3.2) reduce to inequality (2.3). Therefore, substituting $g = f$ and $T = S$ in Corollary 3.2 we obtain Theorem 2.11.

Lemma 3.4. *Let $X := (X, d)$ be a metric space, $f : X \to X$, $T : X \to \mathrm{CL}(X)$, f be T-weakly commuting at $a \in C(f, T)$ and $ffa = fa$. Then f and T have a common fixed point.*

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Proof. Let $t = fa$, then $t = ft = ffa \in Tfa = Tt$. □

We have extracted the following lemma from the proof of [16, Theorem 2.5].

Lemma 3.5. *Let $X := (X, d)$ be a metric space, $f : X \to X$, $T : X \to \mathrm{CL}(X)$ be (IT)-commuting at $a \in C(f, T)$. Then,* (I) *for each natural number n, $f^n(a)$ is a coincidence point of f and T,* (II) *if f is continuous, T is closed (that is, has a closed graph), and $\lim_{n\to\infty} f^n(a)$ exists, then f and T have a common fixed point.*

Proof.

(I) Since $fa \in Ta$, the statement is true for $n = 1$. Suppose that it is true for $n = k \in \mathbb{N}$ (that is, $f^{k+1}(a) \in Tf^k(a)$). Then $f(f^{k+1}(a)) \in fTf^k(a) \subseteq Tf^{k+1}(a)$.

(II) Using (I), we have

$$\begin{aligned} f^{n-1}Ta = f^{n-2}fTa &\subseteq f^{n-2}Tfa \\ &= f^{n-3}fTfa \\ &\subseteq f^{n-3}Tf^2a \\ &= \cdots \subseteq Tf^{n-1}a \end{aligned}$$

Thus $f^na = f^{n-1}fa \in f^{n-1}Ta \subseteq Tf^{n-1}a$. Let $t = \lim_{n\to\infty} f^na$. Then taking $n \to \infty$, we get $t \in Tt$ and from the continuity of f we get $t = f(t)$. □

Theorem 3.6. *Let $X := (X, d)$ be a metric space, $f, g : X \to X$ and $T, S : X \to \mathrm{CL}(X)$ be such that either (f, T) is g-tangential or (g, S) is f-tangential. Suppose that there exist a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ and continuous functions $\phi_i : [0, \infty) \to [0, \infty)$ $(i = 1, 2, \ldots, 7)$ satisfying $\phi_i(0) = 0$ for $i = 1, 2, 4$ and $\phi(\phi_i(z)) < z$ for $z > 0$ and $i = 3, 5, 6, 7$ and*

$$\begin{aligned} H(Tx, Sy) \le \phi\Big(\max\{&\phi_1(d(fx, gy)), \phi_2(d(fx, Tx)), \phi_3(d(gy, Sy)), \\ &\phi_4(d(gy, Tx)), \phi_5(d(fx, Sy)), \phi_6(d(fx, gy) + d(fx, Tx) \\ &+ d(gy, Tx) + d(gy, Sy)), \phi_7(d(fx, gy) + d(fx, Tx) \\ &+ d(gy, Tx) + d(fx, Sy))\}\Big), \end{aligned} \tag{3.10}$$

$$H(Sx,Ty) \le \phi\Big(\max\{\phi_1(d(gx,fy)),\phi_2(d(gx,Sx)),\phi_3(d(fy,Ty)),\\ \phi_4(d(fy,Sx)),\phi_5(d(gx,Ty)),\phi_6(d(gx,fy)+d(gx,Sx)\\ +d(fy,Sx)+d(fy,Ty)),\phi_7(d(gx,fy)+d(gx,Sx)\\ +d(fy,Sx)+d(gx,Ty))\}\Big), \qquad (3.11)$$

for all $x, y \in X$. Further, suppose that one of the following holds:

(i) *fX is a closed subset of X and $TX \subseteq gX$;*

(ii) *gX is a closed subset of X and $SX \subseteq fX$;*

(iii) *fX and gX are closed subsets of X.*

If one of the following conditions holds:

(a) *f, g are continuous, S, T are closed (that is, have a closed graph), f is (IT)-commuting at $a \in C(f,T)$, g is (IS)-commuting at $b \in C(g,S)$ and $\lim_{n\to\infty} f^n a$, $\lim_{n\to\infty} g^n b$ exist;*

(b) *f is T-weakly commuting at a and $ffa = fa$ for any $a \in C(f,T)$ and g is S-weakly commuting at b and $ggb = gb$ for any $b \in C(g,S)$;*

then $F(f) \cap F(T) \neq \emptyset$ and $F(g) \cap F(S) \neq \emptyset$.

Proof. Theorem 3.1 guarantees that $C(f,T) \neq \emptyset$ and $C(g,S) \neq \emptyset$. Thus there exist $a, b \in X$ such that $fa \in Ta$ and $gb \in Sb$. The rest of the proof follows from Lemmas 3.4 and 3.5. □

Example 3.7. Let $X = [0,\infty)$ with the usual metric. Define $f, g : X \to X$ and $S, T : x \to \mathrm{CL}(X)$ by

$$fx = 2 + \frac{1}{2}x, \quad gy = 2 + \frac{1}{4}y, \quad Tx = [x, 2+x] \quad \text{and} \quad Sy = \left[\frac{y}{2}, 2 + \frac{1}{2}y\right]$$

for all $s, y \in X$. Consider the sequences $x_n = \{1 + \frac{1}{n}\}$ and $y_n = \{2 + \frac{1}{n}\}$ in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \frac{5}{2} = [1,3] = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n.$$

Therefore the pair (f, T) is g-tangential and the pair (g, S) is f-tangential. Also,

$$\begin{aligned} H(Tx, Sy) &= \left|x - \frac{y}{2}\right| \\ &= 2d(fx, gy) \\ &\leq \phi\big(\max\{\phi_1(d(fx, gy)), \phi_2(d(fx, Tx)), \phi_3(d(gy, Sy)), \\ &\qquad \phi_4(d(gy, Tx)), \phi_5(d(fx, Sy)), \phi_6(d(fx, gy) + d(fx, Tx) \\ &\qquad + d(gy, Tx) + d(gy, Sy)), \phi_7(d(fx, gy) + d(fx, Tx) \\ &\qquad + d(gy, Tx) + d(fx, Sy))\}\big), \end{aligned}$$

and

$$\begin{aligned} H(Sx, Ty) &= \left|y - \frac{x}{2}\right| \\ &= 2d(gx, fy) \\ &\leq \phi\big(\max\{\phi_1(d(gx, fy)), \phi_2(d(gx, Sx)), \phi_3(d(fy, Ty)), \\ &\qquad \phi_4(d(fy, Sx)), \phi_5(d(gx, Ty)), \phi_6(d(gx, fy) + d(gx, Sx) \\ &\qquad + d(fy, Sx) + d(fy, Ty)), \phi_7(d(gx, fy) + d(gx, Sx) \\ &\qquad + d(fy, Sx) + d(gx, Ty))\}\big), \end{aligned}$$

for all $x, y \in X$, here $\phi(t) = 2t$, $\phi_i(t) = t, i = 1, 2, 4$ and $\phi_i(t) = t, i = 3, 5, 6, 7$. Therefore all hypotheses of Theorem 3.1 are satisfied. Note that $4 \in C(f, T)$ and $\frac{8}{3} \in C(g, S)$. Moreover, f is T-weakly commuting at $4 \in X$, g is S-weakly commuting at $\frac{8}{3}$, $ff4 = 4$ and $gg\frac{8}{3} = \frac{8}{3}$. Therefore all hypotheses of Theorem 3.6 are satisfied. Indeed, $4 \in F(f) \cap F(T)$ and $\frac{8}{3} \in F(g) \cap F(S)$. Furthermore, Theorem 2.10 is not applicable here.

Remark 3.8. As applications of our results one can extend the invariant approximation results in O'Regan and Shahzad [16] to pairs of hybrid maps.

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Fixed Point Theorems for Multi-valued Mappings by Altering Distances

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Abstract

Recently T. Suzuki showed that Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's fixed point theorem. Inspiring from the result of Mizoguchi-Takahashi and using the ideas of Feng and Liu, Klim and Wardowski obtained some fixed point Theorems and showed that their results are different from Reich point theorem and Mizoguchi-Takahashi's fixed point theorem. Very recently, Pathak and Shahzad introduced a class of functions and generalized some fixed point theorems by Klim and Wardowski by altering distances, i-e., for the mapping T (from a complete metric space (X, d) into the class of nonempty closed subsets of X). We notice that results of Pathak and Shahzad actually holds only when T takes values in the set of proximinal subsets of X. In this paper we introduce a new class of functions which is the subclass of the class introduced by Pathak and Shahzad and improve some results of Pathak and Shahzad by allowing T to have values in closed subsets of X, indeed.

Key words and phrases. Complete metric space, Hausdorff metric, Nadler's Theorem, Reich's Theorem.

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1 Preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. A subset A of X is called proximinal [2] if, for each $x \in X$, there is an element $a \in A$ such that $d(x, a) = d(x, A)$. We denote by $N(X)$ the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $PC(X)$ the class of all nonempty proximinal subsets of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $K(X)$ the class of all nonempty compact subsets of X . Let H be the Hausdorff metric on $CL(X)$ generated by the metric d, that is,

$$H(A, B) = \begin{cases} \max\{\sup_{x\in A} d(x, B),\ \ \sup_{y\in B} d(y, A)\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

for every $A, B \in CL(X)$. A point $p \in X$ is said to be a fixed point of $T : X \to CL(X)$ if $p \in Tp$. If, for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$ then $O(T, x_0) = \{x_0, x_1, x_2, \cdots\}$ is said to be orbit of $T : X \to CL(X)$. A mapping $f : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous [7] if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \xi$ implies $f(\xi) \le \lim_n \inf f(x_n)$.

Let $T : X \to N(X)$. For $b \in (0, 1]$ and $x \in X$, we define $I_b^x = \{y \in Tx : bd(x, y) \leq d(x, Tx)\}$, and $M(b, x; \theta) = \{y \in Tx : b\theta(d(x, y)) \leq \theta(d(x, Tx))\}$. Let $A \in (0, +\infty]$ and $\mathbb{R}$ denote the set of real numbers. $\Theta[0, A)$ [3] denoted the class of functions $\theta : [0, A) \to \mathbb{R}$ satisfy the following conditions: (i) θ is nondecreasing on $[0, A)$; (ii) $\theta(t) > 0$ for each $t \in (0, A)$; (iii) θ is subadditive in $(0, A)$; i-e., $\theta(t_1 + t_2) \leq \theta(t_1) + \theta(t_2)$ for $t_1, t_2 \in (0, A)$. Notice that (i) implies that θ is strictly inverse isotone on $(0, A)$; i-e., $\theta(t_1) < \theta(t_2) \Rightarrow t_1 < t_2, \quad t_1, t_2 \in (0, A)$.

2 Introduction

Nadler [10] initiated the study of fixed points for multivalued maps by extending the Banach contraction principle in the following way.

Theorem 2.1 *(Nadler [10]). Let (X, d) be a complete metric space and T is a mapping from X into $CB(X)$ such that for all $x, y \in X$,*

$$H(Tx, Ty) \leq \lambda d(x, y)$$

where, $0 \leq \lambda < 1$. Then T has a fixed point.

Next, S. Reich [11] established the following:

Theorem 2.2 *(Reich [11]) If (X, d) be a complete metric space and $T : X \to K(X)$ satisfies*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \tag{1}$$

for each $x, y \in X$, where α is a function of $(0, \infty)$ into $[0, 1)$ such that

$$\limsup_{r \to t^+} \alpha(r) < 1 \tag{2}$$

for each $t \in (0, \infty)$, then T has a fixed point.

Reich [12] raised the question: If T satisfies the same contractive condition (1), whether or not the range of T can be relaxed. Specifically the question was whether the range of T, $K(X)$ can be replaced by $CB(X)$ or $CL(X)$. In [9] Mizoguchi and Takahashi gave the positive answer to the conjecture of Reich [11], when the inequality (2) holds also for $t = 0$, in particular they proved:

Theorem 2.3 *(Mizoguchi-Takahashi[9]) Let (X, d) be a complete metric space and $T : X \to CB(X)$. If α is a function of $(0, \infty)$ into $[0, 1)$ such that $\limsup_{r \to t^+} \alpha(r) < 1$ for each $t \in [0, \infty)$ and if*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \text{ for each } x, y \in X, \tag{3}$$

then T has a fixed point in X.

It is worth mentioning to note that recently T. Suzuki [13] showed that Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's fixed point theorem. Inspiring from the result of Mizoguchi-Takahashi and using the ideas of Feng-Liu [1], Klim and Wardowski [8] obtained the following result and showed that their result is different from Theorem 2.2 and Theorem 2.3.

Theorem 2.4 *(Klim, Wardowski [8]) Let (X, d) be a complete metric space and let $T : X \to CL(X)$. Assume that the following conditions hold:*

(i)The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semi-continuous;

(ii)there exist $\alpha : [0, \infty) \to [0, 1)$ such that

$$\forall_{t\in[0,\infty)}\{\limsup_{r\to t^+} \alpha(r) < 1\}$$

and

$$\forall_{x\in X}\exists_{y\in I_1^x}\{d(y, Ty) \le \alpha(d(x, y))d(x, y)\}.$$

Then T has a fixed point.

They also established the following result:

Theorem 2.5 *(Klim, Wardowski [8]) Let (X, d) be a complete metric space and let $T : X \to CL(X)$. Assume that the following conditions hold:*

(i)The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semi-continuous;

(ii)there exist $b \in (0, 1)$ and $\alpha : [0, \infty) \to [0, b)$ such that

$$\forall_{t\in[0,\infty)}\{\limsup_{r\to t^+} \alpha(r) < b\}$$

and

$$\forall_{x\in X}\exists_{y\in I_b^x}\{d(y, Ty) \le \alpha(d(x, y))d(x, y)\}.$$

Then T has a fixed point.

Recently, H. K. Pathak and N. Shahzad [3] introduced a new class of mappings $\Theta[0, A)$ and generalizes Theorem 2.4 and Theorem 2.5.

Theorem 2.6 *(Pathak, Shahzad [3]) Let (X, d) be a complete metric space and let $T : X \to CL(X)$. Assume that the following conditions hold:*

(i)The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semi-continuous;

(ii)there exists $\alpha : (0, \infty) \to [0, 1)$ such that

$$\forall t \in [0, \infty)\{\limsup_{r\to t^+} \alpha(r) < 1\}$$

and

(iii)there exists $\theta \in \Theta[0, A)$ satisfying the following condition:

$$\textit{for all } x \in X, M(1, x; \theta) \textit{is nonempty}$$

and

$$\forall x \in X \exists y \in M(1, x; \theta)\{\theta(d(y, Ty)) \le \alpha(d(x, y))\theta(d(x, y))\}.$$

Then T has a fixed point.

Theorem 2.7 *(Pathak, Shahzad [3]) Let (X, d) be a complete metric space and let $T : X \to CL(X)$. Assume that the following conditions hold:*

(i) The map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, Tx), x \in X$, is lower semi-continuous;

(ii) there exists $b \in (0, 1)$ and $\alpha : (0, \infty) \to [0, b)$ such that

$$\forall t \in [0, \infty)\{\limsup_{r \to t^+} \alpha(r) < b\}$$

and

(iii) there exists $\theta \in \Theta[0, A)$ satisfying the following condition(with b and α of (ii)):

$$\text{for all } x \in X, M(a, x; \theta) \text{is nonempty for any constant } a \in (0, 1)$$

and

$$\forall x \in X \exists y \in M(b, x; \theta)\{\theta(d(y, Ty)) \le \alpha(d(x, y))\theta(d(x, y))\}.$$

Then T has a fixed point.

3 Motivation

A crucial assumption in the hypothesis of Theorem 2.6 is that $M(1, x; \theta) = \{y \in Tx : \theta(d(x, y)) \le \theta(d(x, Tx))\}$ is nonempty. We observe that it exclude all such θ's which are strictly increasing from the class $\Theta[0, A)$. Note that we have the following inclusion.

$$K(X) \subseteq CB(X) \subseteq PC(X) \subseteq CL(X) \subseteq N(X).$$

Therefore, if T is a mapping from X into $CL(X)$ having values in $CL(X) \setminus PC(X)$ then in general for any $x \in X$ we have $d(x, y) > d(x, Tx) \forall y \in Tx$ and if θ is strictly increasing then obviously $\theta(d(x, y)) > \theta(d(x, Tx)) \forall y \in Tx$. In simple words, if θ is strictly increasing then $M(1, x; \theta)$ will be nonempty only when T have values in the set of proximinal subsets of X. Thus, the question arises that whether we can remove the condition $M(1, x; \theta) \neq \emptyset$ from the hypotheses of the Theorem 2.6 so that the result still holds. In this paper we give an affirmative answer to this question provided the function $\theta \in \Theta[0, A)$ is positive homogenous in $[0, A)$; i-e.,

$$\text{(iv) } \theta(at) \le a\theta(t) \quad \forall\ a > 0,\ t \in [0, A).$$

We denote by $\Theta_h[0, A)$ the class of function $\theta \in \Theta[0, A)$ satisfying condition (iv), above. Our results not only relax the conditions but also the proofs are simpler than those of [3].

4 Main Result

We start with the following simple Lemmas.

Lemma 4.1 *Let (X, d) be a metric space and $B \in CL(X)$. Then for each $x \in X$ and $q > 1$ there exists an element $b \in B$ such that*

$$d(x, b) \leq qd(x, B). \tag{4}$$

Proof. Let $d(x, B) = 0$ then $x \in B$, since B is closed subset of X. Further, taking $b = x$ we see that (6) holds. Now, suppose that $d(x, B) > 0$ and choose

$$\epsilon = (q - 1)d(x, B). \tag{5}$$

Then using the definition of $d(x, B)$ it follow that there exists $b \in B$ such that

$$\begin{aligned} d(x, b) &\leq d(x, B) + \epsilon \\ &\leq qd(x, B) \text{ (using (5))}. \end{aligned}$$

Lemma 4.2 *Let (X, d) be a metric space $\theta \in \Theta_h[0, A)$ and $B \in CL(X)$. Then for each $x \in X$ and $q > 1$ there exists an element $b \in B$ such that*

$$\theta(d(x, b)) \leq q\theta(d(x, B)). \tag{6}$$

Theorem 4.3 *Let (X, d) be a complete metric space and α is a function from $(0, \infty)$ into $(0, b]$, $0 < b < 1$ such that*

$$\limsup_{r \to t^+} \alpha(r) < b \text{ for each } t \in [0, \infty). \tag{7}$$

Let $T : X \to CL(X)$. Assume that the following condition holds.

$$\theta(d(y, Ty)) \leq \alpha(d(x, y))\theta(d(x, y)) \text{ for each } x \in X \text{ and } y \in Tx, \tag{8}$$

where $\theta \in \Theta_h[0, A)$. Then,

(i) for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_n x_n = \xi$;

(ii) ξ is fixed point of T if and only if the function $f(x) := d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Proof of Theorem 4.3 Let $x_0 \in X$. Since $Tx_0 \neq \emptyset$, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. If $x_0 = x_1$, then x_0 is fixed point of T. Let $x_0 \neq x_1$, by taking $q = \frac{1}{\sqrt{\alpha(d(x_0, x_1))}}$ it follows from Lemma 4.2 that there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq \frac{1}{\sqrt{\alpha(d(x_0, x_1))}}\theta(d(x_1, Tx_1)). \tag{9}$$

Repeating the above argument we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that

$$\theta(d(x_n, x_{n+1})) \leq \frac{1}{\sqrt{\alpha(d(x_{n-1}, x_n))}}\theta(d(x_n, Tx_n)) \tag{10}$$

where,

$$x_n \in Tx_{n-1},\ n = 1, 2, \cdots. \tag{11}$$

We have assumed that $x_{n-1} \neq x_n$, for otherwise x_{n-1} is fixed point of T. Using (8) it follows from (10) that

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq \sqrt{\alpha(d(x_{n-1}, x_n))}\theta(d(x_{n-1}, x_n)) && (12)\\ &< \theta(d(x_{n-1}, x_n)). && (13) \end{aligned}$$

Hence $\{\theta(d(x_n, x_{n+1}))\}$ is decreasing sequence of positive real numbers bounded below by 0. Since θ is strictly inverse isotone $\{d(x_n, x_{n+1})\}$ is also a decreasing sequence of positive real numbers bounded below by 0, thus convergent. Let $\{\theta(d(x_n, x_{n+1}))\}$ converges to some nonnegative real number r, say. We claim that $r = 0$, for otherwise, by taking limits in (12) we get

$$r \leq \sqrt{\lim_{n\to\infty} \sup \alpha(d(x_{n-1}, x_n))}\ r < br < r \text{ (using (7))},$$

which is a contradiction. we claim that $\{d(x_n, x_{n+1})\}$ also converges to 0. Suppose $d(x_n, x_{n+1}) \to \gamma > 0$. Then, for $0 < \epsilon < \gamma$, there exists a natural number n_0 such that

$$0 < \delta = \gamma - \epsilon < d(x_n, x_{n+1})\ \forall n \geq n_0.$$

Since θ is positive and nondecreasing,

$$0 < \theta(\delta) \leq \theta(d(x_n, x_{n+1}))\ \forall n \geq n_0,$$

which is a contradiction, since $\theta(d(x_n, x_{n+1})) \to 0$. From (12), we get

$$\theta(d(x_n, x_{n+1})) \leq [\sqrt{\alpha(d(x_{n-1}, x_n))} \cdots \sqrt{\alpha(d(x_0, x_1))}]\theta(d(x_0, x_1)). \tag{14}$$

It follow from (7) that we may choose an $\epsilon > 0$ and $a \in (0, b)$ such that

$$\alpha(t) < a^2 \text{ for } t \in (0, \epsilon). \tag{15}$$

Let N be such that

$$d(x_{n-1}, x_n) < \epsilon \text{ for } n \geq N. \tag{16}$$

Then, from (14) we have

$$\begin{aligned} \theta(d(x_n, x_{n+1})) &\leq a^{n-(N-1)}[\sqrt{\alpha(d(x_{N-2}, x_{N-1}))} \cdots \sqrt{\alpha(d(x_0, x_1))}]\theta(d(x_0, x_1))\\ &< a^{n-N+1}b^{N-1}\theta(d(x_0, x_1)). && (17) \end{aligned}$$

Therefore, for any $m \in \mathbb{N}$ we have

$$\begin{aligned} \theta(d(x_n, x_{n+m})) &\leq \theta(d(x_n, x_{n+1})) + \cdots + \theta(d(x_{n+m-1}, x_{n+m}))\\ &< a^{n-N+1}b^{N-1}[1 + a + a^2 + \cdots + a^{m-1}]\theta(d(x_0, x_1))\\ &< b^{N-1}\frac{a^{n-N+1}}{1-a}\theta(d(x_0, x_1)). && (18) \end{aligned}$$

Therefore

$$\lim_{n,m\to\infty} \theta(d(x_n, x_m)) = 0.$$

We claim that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Suppose not. Then there exist $\delta > 0$ and subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ respectively such that

$$d(x_{m_i}, x_{ni}) > \delta \ \forall i.$$

Since θ is nondecreasing,

$$0 < \theta(\delta) \leq \theta(d(x_{m_i}, x_{n_i})) \to 0,$$

which is a contradiction. This proves our claim. Hence $\{x_n\}$ is Cauchy sequence in X. Since X is complete there exists $\xi \in X$ such that $x_n \to \xi$. Since $x_n \in Tx_{n-1}$, it follow from (21) that

$$\begin{aligned} \theta(d(x_n, Tx_n)) &\leq \alpha(d(x_{n-1}, x_n))\theta(d(x_{n-1}, x_n)) \\ &< \theta(d(x_{n-1}, x_n)), \end{aligned} \tag{19}$$

which implies that $d(x_n, Tx_n) < d(x_{n-1}, x_n)$. Letting $n \to \infty$, from (19) we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{20}$$

Suppose $f(x) = d(x, Tx)$ is T orbitally lower continuous at ξ, then

$$d(\xi, T\xi) = f(\xi) \leq \lim_n \inf f(x_n) = \lim_n \inf d(x_n, Tx_n) = 0.$$

Hence, $\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is fixed point of then $f(\xi) = 0 \leq \lim_n \inf f(x_n)$.

If we take $b = 1$ then the following Theorem can be obtained as a consequence of Theorem 4.3.

Theorem 4.4 *Let (X, d) be a complete metric space and $T : X \to CL(X)$ satisfying*

$$\theta(d(y, Ty)) \leq \alpha(d(x, y))\theta(d(x, y)) \text{ for each } x \in X \text{ and } y \in Tx, \tag{21}$$

where α is a function from $(0, \infty)$ into $(0, 1]$ such that

$$\limsup_{r\to t^+} \alpha(r) < 1 \text{ for each } t \in [0, \infty), \tag{22}$$

and $\theta \in \Theta_h[0, A)$. Then,

(i) for each $x_0 \in X$, there exists an orbit $\{x_n\}$ of T and $\xi \in X$ such that $\lim_n x_n = \xi$;

(ii) ξ is fixed point of T if and only if the function $f(x) := d(x, Tx)$ is T-orbitally lower semi continuous at ξ.

Example 4.5 *Let l_2 denote the Hilbert space of all square summable sequences of real numbers with usual norm. Note that for $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$*

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}.$$

For each $n = 1, 2, \cdots$, *let* e_n *be the vector in* l_2 *with zeros in all its coordinates except the* n^{th} *coordinate which is equal to* 1. *Take* $a = (-1, -\frac{1}{2}, \cdots, -\frac{1}{n}, \cdots)$, *and* $B = \{e_1, e_2, \cdots, e_n, \cdots\}$. *Define* $T : l_2 \to CL(l_2)$ *by*

$$Tx = B \ \forall \ x \in l_2.$$

Then, for $\theta(t) = t$ *condition (21) of Theorem 4.4 is obviously satisfied. Further,*

$$d(a, B) = \inf \|a - e_n\| = \inf(\|a\|^2 + 1 + \frac{2}{n})^{\frac{1}{2}} = (\|a\|^2 + 1)^{\frac{1}{2}}.$$

Observe that $B \in CL(X)$ *and there is no* e_n *in* B *such that* $\|a - e_n\| \leq d(a, B)$ *and hence* $M(1, x; \theta)$ *is empty in this case and conditions of Theorem 2.6 are not satisfied. Note that,* $e_1, e_2, \cdots$ *are fixed points of* T.

Corollary 4.6 *Let* (X, d) *be a complete metric space and let* $T : X \to CL(X)$ *satisfying*

$$\int_0^{\theta(d(y,Ty))} \psi(t)dt \leq \alpha(d(x, y)) \int_0^{\theta(d(x,y))} \psi(t)dt, \text{ for each } x \in X \text{ and } y \in Tx. \tag{23}$$

Where α *is a function from* $(0, \infty)$ *into* $(0, 1]$ *such that*

$$\limsup_{r \to t^+} \alpha(r) < 1 \text{ for each } t \in [0, \infty), \tag{24}$$

$\theta \in \Theta_h[0, A)$ *and* $\Psi \in \Theta_h[0, A)$ *given by* $\Psi(\epsilon) = \int_0^\epsilon \psi(t)dt$ *and* $\psi : [0, \infty) \to [0, \infty)$ *is a Lebesgue-integrable mapping satisfying* $\int_0^\epsilon \psi(t)dt > 0$ *for all* $\epsilon > 0$. *Then* T *has a fixed point.*

Since $d(y, Ty) \leq H(Tx, Ty)$ for $y \in Tx$. We have the following.

Corollary 4.7 *Let* (X, d) *be a complete metric space and* $T : X \to CL(X)$ *satisfying*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \text{ for each } x \in X \text{ and } y \in Tx. \tag{25}$$

Where α *is a function from* $(0, \infty)$ *into* $(0, 1]$ *such that*

$$\limsup_{r \to t^+} \alpha(r) < 1 \text{ for each } t \in [0, \infty). \tag{26}$$

Then,

(i) for each $x_0 \in X$, *there exists an orbit* $\{x_n\}$ *of* T *and* $\xi \in X$ *such that* $\lim_n x_n = \xi$;

(ii) ξ *is fixed point of* T *if and only if the function* $f(x) := d(x, Tx)$ *is* T*-orbitally lower semi continuous at* ξ.

Remark 4.8 *Note that Corollary 4.7 generalize Theorem 2.3 in the following sense:*

(i) the inequality (25) is weaker than the inequality (3);

(ii) the range of T *in Corollary 4.7 is* $CL(X)$ *which is general than* $CB(X)$*;*

(iii) for the existence of fixed point we merely require that $d(x, Tx)$ is T orbitally continuous at ξ, whereas condition (3) in Theorem 2.3 demands that T is a continuous map from X into $CB(X)$.

we have the following simple example which shows the generality of our result.

Example 4.9 *Let $X = (-\infty, \frac{3}{5}]$ equipped with usual metric d. Define $T : X \to CL(X)$ by*

$$Tx = \begin{cases} (-\infty, \frac{1}{2}x] & \text{if } x \in (-\infty, 0), \\ (-\infty, x^2] & \text{if } x \in [0, \frac{3}{5}]. \end{cases}$$

Case 1. When $x \in (-\infty, 0)$ and $y \in Tx = (-\infty, \frac{1}{2}x]$, then we have

$$H(Tx, Ty) = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y),$$

since $y \in (-\infty, 0)$ in this case.
Case 2. When $x \in [0, \frac{3}{5}]$ and $y \in Tx$, then we have

$$H(Tx, Ty) \leq |x^2 - y^2| = (x + y)d(x, y) \leq \frac{24}{25}d(x, y).$$

Thus, for any $x \in X$ and $y \in Tx$, we have

$$d(y, Ty) \leq H(Tx, Ty) \leq \frac{24}{25}d(x, y).$$

By taking $\alpha(t) = c$, where $\frac{24}{25} \leq c < 1$, we see that all conditions of Corollary 4.7 are satisfied and 0 is fixed point of T. Note that T does not satisfies the hypothesis of Theorems 2.3 and 2.4.

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