Jensen Inequality on Time Scales

By

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Preface

We present a series of Jensen's type inequalities for time scales integrals using convex functions and superquadratic functions.

The study of inequalities reflects the different aspects of modern mathematics. On one hand, there is the systematic search for the basic principles, such as the deeper understanding of monotonicity and convexity. On the other hand, finding the solutions to an inequality requires often new ideas. Some of them have become standard tools in mathematics. In view of the wide-ranging research related to inequalities, several recent mathematical periodicals have been devoted exclusively to this topic.

Convex functions are closely related to the theory of inequalities (see for example the monographs [62, 82, 87, 93]). The theory of convex functions is part of the general subject of convexity since a convex function is one whose epigraph is a convex set. Like differentiability, convexity is a natural and powerful property of functions that plays a significant role in many areas of mathematics, both pure and applied. The concept of superquadratic functions in one variable is recently introduced by S. Abramovich et al. [5, 6], as a generalization of the class of convex functions.

Time scales theory provides a platform to study discrete and continuous analysis simultaneously. Many results concerning differential equations carry over quite easily to corresponding results for difference equations while other results seem to be completely different from their continuous counterpart. The study of dynamic equations on time scales reveals such discrepancies and helps avoid proving results twice. Integral inequalities play a fundamental role in the development of time scale calculus. For example, when the time scale is the set of all real numbers, the time scales integral is an ordinary integral; when the time scale is the set of all integers, the time scales integral is a sum; and when the time scale is the set of all integer powers of a fixed number, the time scales integral is a Jackson integral.

In Chapter 1 we give some introduction to Jensen's inequality and some classical inequalities related to Jensen's inequality.

In Chapter 2 we give some notions and preliminary results which are used in our main results throughout the dissertation. It contains convex functions, exponentially and n–exponentially convex functions, superquadratic functions, and time scales calculus.

In Chapter 3 we show that the time scales Cauchy delta, Cauchy nabla, diamond-α, multiple Riemann, and multiple Lebesgue integrals all are indeed isotonic linear functionals. We then apply the theory of isotonic linear functionals and derive a series of classical inequalities concerning convex functions and superquadratic functions.

In Chapter 4 we consider Jensen's functionals deduced from the Jensen inequality on time scales and analyze the properties of Jensen's functionals concerning superadditivity and monotonicity. We then define weighted generalized and power means on time scales and apply the properties of Jensen's functionals to these means. It yields refinements and conversions of numerous classical inequalities such as arithmetic mean-geometric mean inequality, Young's inequality, Hölder's inequality.

In Chapter 5 we give generalizations of the results obtain in Chapter 4 for several variables.

In Chapter 6 we give generalizations of the Jensen–Steffensen inequality and its converse on time scales. We give mean value theorems of the Lagrange and Cauchy type for the functionals obtained by taking the difference of the left and right-hand side of these new inequalities. Further we investigate exponential convexity, n–exponential convexity and logarithmic convexity of the functionals obtained. Finally we present several families of exponentially convex functions which fulfil the conditions of our results. These results generalize the results given in [71, 92] for continuous and discrete cases.

In Chapter 7 we give improvements of converses of Jensen's inequality given in Chapter 3 and as a consequence improvements of the generalization of the Hermite–Hadamard inequality. We also discuss log-convexity, n–exponential convexity and exponential convexity of the differences obtained from new results.

In Chapter 8 we define Jensen's functionals by using Jensen type inequalities for superquadratic functions given in Chapter 3. We give mean value theorems and introduce related Cauchy type means by using these functionals and show the monotonicity of these means. Finally we show that these functionals are exponentially convex and give some applications of them by using the log-convexity and exponential convexity.

In Chapter 9 we obtain integral Minkowski and Beckenbach–Dresher inequalities on time scales. We analyze properties concerning superadditivity and monotonicity of several mappings arisen from the integral Minkowski and Beckenbach–Dresher inequalities. We also investigate a converse of integral Minkowski inequality on time scales.

Chapter 1

Introduction

Inequalities are used everywhere in mathematics. In 1934, Hardy, Littlewood, and Pólya [62] published a book on inequalities, since that times the theory of inequalities becomes an important branch of mathematics. Among the inequalities Jensen's inequality is one of the most important and extensively used inequality in various fields of modern mathematics, especially in mathematical analysis and statistics. It is a powerful tool of producing a large class of classical inequalities, e.g., the arithmetic mean-geometric mean-harmonic mean inequality, Young's inequality, Hölder's inequality, Minkowki's inequality, Beckenbach–Dresher inequality, the positivity of relative entropy in information theory, Shannon's inequality, Ky Fan's inequaliy, Levinson's inequality, etc. The improvements and generalizations of Jensen's inequality implies the improvements and generalizations of a whole series of other classical inequalities. A simple search in MathSciNet database of the American Mathematical Society with the key words "Jensen" and "inequalities" in the title reveals that there are more than 300 items intimately devoted to this word. However, the number of papers where this inequality is used a lot larger and far more difficult to find.

In the following we give a brief introduction to the Jensen and some of its related classical inequalities.

Jensen's inequality:

Let I be an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ a convex function on I. If

$$\mathbf{x} = (x_1, \ldots, x_n) \in I^n, \quad \mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n, \quad \text{and} \quad P_n = \sum_{i=1}^{n} p_i,$$

then we have

$$\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \Phi(x_i). \tag{1.0.1}$$

If Φ is strictly convex then (1.0.1) is strict unless $x_i = c$ (constant) for all $i \in \{j : p_j > 0\}$.

The inequality (1.0.1) is known as Jensen's inequality. Now we quote some history about the Jensen inequality from [83]. The Jensen inequality was proved under the assumption that Φ is a J-convex function by J. L. W. V. Jensen (see [68, 69] or for example [93]). He applied the famous inductive method used by Cauchy (1821) in the proof of the arithmetic mean-geometric mean inequality. However, inequality (1.0.1) appears, under different assumptions, much earlier. Jensen himself mentioned in the appendix to his paper that O. Hölder proved inequality (1.0.1) in 1889, supposing that Φ is a twice differentiable function on $[a, b]$ such that $\Phi''(x) \geq 0$ on that interval. This supposition is in the case of twice differentiable functions equivalent with the supposition that Φ is convex. The above inequality was proved, after Hölder, using the same assumptions by R. Henderson in 1895. However, as far back as 1875 a particular case of the above inequality, the case when $p_1 = \ldots = p_n$ was proved by J. Grolous by an application of the centroid method. This is, as far as we could find, the first inequality for convex functions to appear in the mathematical literature. J. Grolous introduced the assumption that $\Phi''(x) \geq 0$, but it can be seen from the text itself that it is enough to assume that Φ is a convex function, in the geometric sense (see, for instance, D. S. Mitrinović [81]).

The original Jensen's inequality for integrals can be stated as follows.

Theorem 1.0.1 (See [68, Formula (5′)]). *Let $a, b \in \mathbb{R}$ with $a < b$ and suppose $I \subset \mathbb{R}$ is an interval. If $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex and $f \in \mathrm{C}([a, b], I)$, then*

$$\Phi\left(\frac{\int_a^b f(t)\mathrm{d}t}{b-a}\right) \leq \frac{\int_a^b \Phi(f(t))\mathrm{d}t}{b-a}. \tag{1.0.2}$$

Note that in Jensen's inequality we have nonnegative weights. It is reasonable to ask whether the condition, $\mathbf{p}$ is a nonnegative n-tuple, can be relaxed at the expense of restricting $\mathbf{x}$ more severely. An answer to this question was given by Steffensen in [99] (see also [93]).

Theorem 1.0.2 (See [93, Theorem 2.19]). *Let I be an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{p} = (p_1, \ldots, p_n)$ a real n-tuple such that*

$$0 \leq P_k \leq P_n, \quad k \in \{1, \ldots, n-1\}, \quad P_n > 0$$

is satisfied, where $P_k = \sum_{i=1}^{k} p_i$, $k \in \{1, \ldots, n\}$, then (1.0.1) *holds. If Φ is strictly convex, then inequality* (1.0.1) *is strict unless $x_1 = \ldots = x_n$.*

Inequality (1.0.1) under conditions from Theorem 1.0.2 is called the Jensen–Steffensen inequality. The integral version of Jensen–Steffensen inequality is given by Boas [38] (see also [93]). Furthermore, for different refinements and generalizations of the Jensen–Steffensen inequality see [48, 55, 93].

B. Jessen in 1931 (see [70] or see for example [93]) give the generalization of Jensen's inequality for convex functions which involves positive normalized linear functionals. In 1937 E. J. McShane give the generalization of Jessen's inequality for several variables (see [80] or see for example [93]). S. Banić and S. Varošanec [27] refine the Jessen's inequality for superquadratic functions.

In 2003, A. McD. Mercer in [79] gives a variant of Jensen's inequality, so called the Jensen–Mercer inequality, later W. S. Cheung et al. generalize the Jensen–Mercer inequality for isotonic linear functionals which is called Jessen–Mercer inequality (see [45]) . Further in [2], S. Abramovich et al. give the refinement of Jessen–Mercer inequality for superquadratic functions.

There are also various generalizations of Jensen's inequality are given by the time scales theory, see in Section 3.2.

Hermite–Hadamard inequality:

The Hermite–Hadamard inequality is strongly related to the Jensen inequality. It is also known as the first fundamental inequality for convex functions. It gives us an estimate for the integral arithmetic mean:

$$(b-a)\Phi\left(\frac{a+b}{2}\right) \leq \int_a^b \Phi(t)\mathrm{d}t \leq (b-a)\frac{\Phi(a)+\Phi(b)}{2}, \tag{1.0.3}$$

where $a, b \in \mathbb{R}$ with $a < b$ and $\Phi : [a, b] \to \mathbb{R}$ is a convex function. It was first established by Hermite in 1881. Also, Beckenbach, a leading expert on the history and theory of complex functions, wrote that the first inequality in (1.0.3) was proved in 1893 by Hadamard who apparently was not aware of Hermite's result (see [93]). In general, (1.0.3) is now known as Hermite–Hadamard inequality.

Note that the first inequality in (1.0.3) is a Jensen's inequality (1.0.2) when $f(t) = t$ and the second one gives a converse of Jensen's inequality. Various generalizations and refinements of the Hermite-Hadamard inequality and converses of Jensen's inequality are given in the literature for convex functions, superquadratic functions, as well as in time scales theory, see e.g., [3, 72, 73, 27, 47, 48, 82, 87, 93].

The first inequality in (1.0.3) is stronger than the second one: if Φ is convex on $[a, b]$, then

$$\frac{1}{(b-a)}\int_a^b \Phi(t)\mathrm{d}t - \Phi\left(\frac{a+b}{2}\right) \leq \frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{(b-a)}\int_a^b \Phi(t)\mathrm{d}t. \tag{1.0.4}$$

A geometric proof of (1.0.4) is given in [61] and analytic one in [43] (see also [93]). The inequality (1.0.4) is known as the Hammer–Bullen inequality.

Cauchy, Hölder and Minkowski inequalities:

The three inequalities are basic to all studies of power means (see for example [44]) and well-known in mathematics.

Augustin-Louis Cauchy published his famous inequality in 1821. Then in 1859, Viktor Yakovlevich Bunyakovsky derived a corresponding inequality for integrals and in 1885 Hermann Schwarz proved a corresponding version for inner-product spaces. Therefore the Cauchy inequality sometimes also shows up under the name Schwarz inequality, or Cauchy-Schwarz inequality, or Cauchy-Bunyakovsky-Schwarz inequality. Hölder's generalization appeared in 1889. The Minkowski inequality was established in 1896 by Hermann Minkowski in his book Geometrie der Zahlen (Geometry of Numbers).

There are various versions of these inequalities given in the literature. For isotonic linear functionals some generalizations and conversions of the Hölder and Minkowski inequalities can be found in [93]. In any case, the discrete versions of Hölder's and Minkowski's inequalities are stated in the following two theorems respectively.

Theorem 1.0.3 (See [44]). *For $p \neq 1$ define q by $q = p/(p-1)$. If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are two positive n-tuples and $p > 1$, then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}. \tag{1.0.5}$$

If $p < 1$, $p \neq 0$, then the above inequality holds in reverse order.

In the above theorem if $p = q = 2$, then (1.0.5) becomes Cauchy's inequality.

Theorem 1.0.4 (See [44]). *If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are two positive n-tuples and $p > 1$, then*

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}}. \tag{1.0.6}$$

If $p < 1$, $p \neq 0$, then the above inequality holds in reverse order.

Beckenbach–Dresher inequality:

In 1950 E. F. Beckenbach published an inequality which has aroused interest until nowdays. He proved that for positive real numbers $x_i, y_i > 0,\ i \in \{1, \ldots, n\}$ and for $1 \leq p \leq 2$ the following inequality is valid.

$$\frac{\sum\limits_{i=1}^{n}(x_i + y_i)^p}{\sum\limits_{i=1}^{n}(x_i + y_i)^{p-1}} \leq \frac{\sum\limits_{i=1}^{n} x_i^p}{\sum\limits_{i=1}^{n} x_i^{p-1}} + \frac{\sum\limits_{i=1}^{n} y_i^p}{\sum\limits_{i=1}^{n} y_i^{p-1}}. \tag{1.0.7}$$

If $0 \leq p \leq 1$, then the inequality is reversed.

Few years later M. Dresher investigated moment spaces and stated that an integral analogue of the previous result holds. In recent literature this inequality is called the Beckenbach–Dresher inequality. Some history and recent results about Beckenbach–Dresher inequality can be found in [57, 101].

Chapter 2

Preliminaries

In this chapter we give some preliminary definitions and results of convex functions, exponentially convex functions, superquadratic functions, and time scales calculus. These results are used in the rest of this thesis.

2.1 Convex functions

Historically, logically, and pedagogically, the study of convex functions begins in the context of real-valued functions of a real variable. In this section, we give some of the fundamental concepts of convex functions (see [93]). We take our functions $\Phi : I \to \mathbb{R}$ to be defined on some interval of the real line $\mathbb{R}$. We mean to allow I to be open, half-open, or closed, finite or infinite. Hence we have $\alpha x + (1-\alpha)y \in I$, whenever $x, y \in I$ and $\alpha \in [0,1]$.

Definition 2.1.1. (a) A function $\Phi : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and all $\alpha \in [0,1]$,

$$\Phi(\alpha x + (1-\alpha)y) \leq \alpha\Phi(x) + (1-\alpha)\Phi(y) \tag{2.1.1}$$

holds. If (2.1.1) is strict for all $x \neq y$ and $\alpha \in (0,1)$, then Φ is said to be strictly convex.

(b) If the inequality in (2.1.1) is reversed, then Φ is said to be concave. If it is strict for all $x \neq y$ and $\alpha \in (0,1)$, then Φ is said to be strictly concave.

There are several equivalent ways to define convex functions, sometimes it is better to define a convex function in one way than the other.

Remark 2.1.2. (a) For $x, y \in I$, $p, q \geq 0$, $p+q > 0$, (2.1.1) is equivalent to

$$\Phi\left(\frac{px+qy}{p+q}\right) \leq \frac{p\Phi(x)+q\Phi(y)}{p+q}.$$

(b) Let x_1, x_2, x_3 be three points in I such that $x_1 < x_2 < x_3$. Then (2.1.1) is equivalent to

$$\begin{vmatrix} x_1 & \Phi(x_1) & 1 \\ x_2 & \Phi(x_2) & 1 \\ x_3 & \Phi(x_3) & 1 \end{vmatrix} = (x_3 - x_2)\Phi(x_1) + (x_1 - x_3)\Phi(x_2) + (x_2 - x_1)\Phi(x_3) \geq 0,$$

which is further equivalent to

$$\Phi(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3}\Phi(x_1) + \frac{x_1 - x_2}{x_1 - x_3}\Phi(x_3). \tag{2.1.2}$$

More symmetrically and without the condition of monotonicity on x_1, x_2, x_3, we can write

$$\frac{\Phi(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{\Phi(x_2)}{(x_2 - x_3)(x_2 - x_1)} + \frac{\Phi(x_3)}{(x_3 - x_1)(x_3 - x_2)} \geq 0.$$

(c) Φ is both convex and concave if and only if

$$\Phi(x) = \lambda x + c$$

for some $\lambda, c \in \mathbb{R}$.

(d) Another way of writing (2.1.2) is instructive:

$$\frac{\Phi(x_1) - \Phi(x_2)}{x_1 - x_2} \leq \frac{\Phi(x_2) - \Phi(x_3)}{x_2 - x_3}, \tag{2.1.3}$$

where $x_1 < x_3$ and $x_1, x_3 \neq x_2$. So that the following result is valid:

A function Φ is convex on I if and only if for every point $c \in I$ the function $(\Phi(x) - \Phi(c))/(x - c)$ is increasing on I $(x \neq c)$.

(e) By using (2.1.3) we can easily prove the following result:

If Φ is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:

$$\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \leq \frac{\Phi(y_2) - \Phi(y_1)}{y_2 - y_1}.$$

The derivative of a convex function is best studied in terms of the left and right derivatives defined by

$$\Phi'_{-}(x) = \lim_{y \uparrow x} \frac{\Phi(y) - \Phi(x)}{y - x}, \quad \Phi'_{+}(x) = \lim_{y \downarrow x} \frac{\Phi(y) - \Phi(x)}{y - x}.$$

The following two theorems concern derivatives of convex functions.

Theorem 2.1.3 (see [93, Theorem 1.3])**.** *Let I be an interval in $\mathbb{R}$ and $\Phi : I \to \mathbb{R}$ be convex. Then*

(i) *Φ'_+ and Φ'_- exist and are increasing in I, and $\Phi'_- \leq \Phi'_+$ (if Φ is strictly convex, then these derivatives are strictly increasing); and*

(ii) *Φ' exists, except possibly on a countable set, and on the complement of which it is continuous.*

Remark 2.1.4. In the above theorem if Φ'' exists on I, then Φ is convex if and only if $\Phi''(x) \geq 0$. If $\Phi''(x) > 0$, then Φ is strictly convex.

Theorem 2.1.5 (see [93, Theorem 1.6])**.** *Let I be an open interval in $\mathbb{R}$.*

(a) *$\Phi : I \to \mathbb{R}$ is convex if and only if there is at least one line of support for Φ at each $x_0 \in I$, i.e.,*

$$\Phi(x) \geq \Phi(x_0) + \lambda(x - x_0) \quad \text{for all} \quad x \in I,$$

where λ depends on x_0 and is given by $\lambda = \Phi'(x_0)$ when Φ' exists, and $\lambda \in [\Phi'_-(x_0), \Phi'_+(x_0)]$ when $\Phi'_-(x_0) \neq \Phi'_+(x_0)$.

(b) *$\Phi : I \to \mathbb{R}$ is convex if the function $\Phi(x) - \Phi(x_0) - \lambda(x - x_0)$ (the difference between the function and its support) is decreasing for $x < x_0$ and increasing for $x > x_0$.*

When dealing with functions with different degree of smoothness, divided differences are found to be very useful.

Definition 2.1.6. Let Φ be a real-valued function defined on $[a, b] \subset \mathbb{R}$. A kth ($k \in \mathbb{N}$) order divided difference of Φ at $k+1$ distinct points $x_0, \ldots, x_k$ in $[a, b]$ may be defined recursively by

$$[x_i; \Phi] = \Phi(x_i), \quad i \in \{0, 1, \ldots, k\}$$

and

$$[x_0, \ldots, x_k; \Phi] = \frac{[x_1, \ldots, x_k; \Phi] - [x_0, \ldots, x_{k-1}; \Phi]}{x_k - x_0}.$$

Remark 2.1.7. In the above definition the value $[x_0, \ldots, x_k; \Phi]$ is independent of the order of the points $x_0, \ldots, x_k$. This definition may be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq \ldots \leq x_k$ and letting

$$[\underbrace{x, \ldots, x}_{(j+1\, times)}; \Phi] = \Phi^{(j)}(x)/j!,$$

provided that $\Phi^{(j)}$ exists.

Definition 2.1.8. A real-valued function Φ defined on $[a, b] \subset \mathbb{R}$ is said to be n-convex, $n \geq 0$ on $[a, b]$ if and only if for all choices of $(n+1)$ distinct points in $[a, b]$,

$$[x_0, \ldots, x_n; \Phi] \geq 0.$$

Remark 2.1.9. A function $\Phi : [a, b] \to \mathbb{R}$ is convex if and only if for every choice of three mutually different points $x_0, x_1, x_2 \in I$, $[x_0, x_1, x_2; \Phi] \geq 0$ holds.

The definition of a convex function has a very natural generalization to real-valued function defined on $\mathbb{R}^n$. Here we merely require that the domain U of Φ be convex, i.e., $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in U$ whenever $\mathbf{x}, \mathbf{y} \in U$ and $\alpha \in [0, 1]$.

Definition 2.1.10. Let U be a convex set in $\mathbb{R}^n$. Then $\Phi : U \to \mathbb{R}$ is said to be convex if for all $\mathbf{x}, \mathbf{y} \in U$ and all $\alpha \in [0, 1]$, we have

$$\Phi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha\Phi(\mathbf{x}) + (1-\alpha)\Phi(\mathbf{y}). \tag{2.1.4}$$

J-convex function:

In the theory of convex functions the most known case is that of J-convex functions, which deals with the arithmetic mean.

Definition 2.1.11. Let $I \subset \mathbb{R}$ be an interval. A function $\Phi : I \to \mathbb{R}$ is called convex in the Jensen sense (or J-convex) on I if for all $x, y \in I$ the inequality

$$\Phi\left(\frac{x+y}{2}\right) \leq \frac{\Phi(x) + \Phi(y)}{2} \tag{2.1.5}$$

holds. A J-convex function Φ is said to be strictly J-convex if for all pairs of points (x, y), $x \neq y$, strict inequality holds in (2.1.5).

Remark 2.1.12. (i) It can be easily seen that a convex function is J-convex. If a function is continuous and J-convex, then it is also convex.

(ii) The inequality (2.1.5) can easily be extended to the convex combination of finitely many points and next to random variables associated to arbitrary probability spaces. These extensions are known as the discrete Jensen inequality and integral Jensen inequality respectively.

Log-convex function:

An important sub-class of convex functions is that of log-convex functions.

Definition 2.1.13. A function $\Phi : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be log-convex, or multiplicative convex if $\log \Phi$ is convex, or equivalently if for all $x, y \in I$ and all $\alpha \in [0, 1]$,

$$\Phi(\alpha x + (1-\alpha)y) \leq \Phi(x)^{\alpha}\Phi(y)^{1-\alpha}. \tag{2.1.6}$$

It is said to be log-concave if the inequality in (2.1.6) is reversed.

Remark 2.1.14. (*a*) If we take $\alpha = 1/2$ in (2.1.6), it becomes

$$\Phi\left(\frac{x+y}{2}\right)^2 \leq \Phi(x)\Phi(y),$$

and the function Φ is said to be log-convex in the Jensen sense. If the function Φ is log-convex in the Jensen sense and is continuous then Φ is also log-convex.

(*b*) If $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$, then (2.1.6) is equivalent to

$$[\Phi(x_2)]^{(x_3-x_1)} \leq [\Phi(x_1)]^{(x_3-x_2)}[\Phi(x_3)]^{(x_2-x_1)}.$$

Furthermore, if $x_1, x_2, y_1, y_2 \in I$ such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then

$$\left(\frac{\Phi(x_2)}{\Phi(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{\Phi(y_2)}{\Phi(y_1)}\right)^{\frac{1}{y_2-y_1}}.$$

(*c*) $\Phi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\Phi(x) + 2\alpha\beta\Phi\left(\frac{x+y}{2}\right) + \beta^2\Phi(y) \geq 0$$

for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$.

2.2 Exponential and n–exponential convexity

Exponentially convex functions are introduced by S. N. Bernstein [32] over eighty years ago and later by D. V. Widder [103]. The notion of n–exponential convexity is introduced by J. Pečarić and J. Perić in [90] (see also [73, 67, 71]). Now we quote some definitions and results about exponential and n–exponential convexity.

Definition 2.2.1. A function $\Phi : I \to \mathbb{R}$ is n–exponentially convex in the Jensen sense on I, where I is an interval in $\mathbb{R}$, if

$$\sum_{i,j=1}^{n} \xi_i\xi_j\Phi\left(\frac{x_i+x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i \in \{1, \ldots, n\}$.

A function $\Phi : I \to \mathbb{R}$ is n–exponentially convex if it is n–exponentially convex in the Jensen sense and continuous on I.

Remark 2.2.2. It is clear from the definition that 1–exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n–exponentially convex functions in the Jensen sense are k–exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.

A real symmetric matrix A is positive semi-definite if $\mathbf{x}A\mathbf{x}^t \geq 0$ for all row vectors $\mathbf{x}$. By definition of positive semi-definite matrices and some basic linear algebra, we have the following proposition.

Proposition 2.2.3 (See [90, Proposition 3.1])**.** *If Φ is an n–exponentially convex function in the Jensen sense, then the matrix $\left[\Phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite for all $k \in \mathbb{N},\ k \leq n$. Particularly,* $\det\left[\Phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$ *for all $k \in \mathbb{N},\ k \leq n$.*

Definition 2.2.4. A function $\Phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I, if it is n–exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\Phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 2.2.5 (See [21, Proposition 1])**.** *Let $\Phi : (a,b) \to \mathbb{R}$. The following are equivalent:*

(i) *Φ is exponentially convex.*

(ii) *Φ is continuous and*

$$\sum_{i,j=1}^{n} v_i v_j \Phi\left(x_i + x_j\right) \geq 0$$

for all $n \in \mathbb{N}$, $v_i \in \mathbb{R}$ and $x_i + x_j \in (a,b)$, $1 \leq i,j \leq n$.

(iii) *Φ is continuous and*

$$\det\left[\Phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0, \quad 1 \leq m \leq n$$

for all $n \in \mathbb{N}$ and for every $x_i \in (a,b)$, $i \in \{1,\ldots,n\}$.

Remark 2.2.6. Some examples of exponentially convex functions are:

(i) $\Phi : I \to \mathbb{R}$ defined by $\Phi(x) = ce^{kx}$, where $c \geq 0$ and $k \in \mathbb{R}$.

(ii) $\Phi : \mathbb{R}^+ \to \mathbb{R}$ defined by $\Phi(x) = x^{-k}$, where $k > 0$.

(iii) $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\Phi(x) = e^{-k\sqrt{x}}$, where $k > 0$.

Remark 2.2.7. From Remark 2.1.14(c) it follows that a positive function is log-convex in the Jensen sense if and only if it is 2–exponentially convex in the Jensen sense. Also, using basic convexity theory, it follows that a positive function is log-convex if and only if it is 2–exponentially convex.

2.3 Superquadratic functions

Here we quote some relevant results about superquadratic functions. The reader is referred to [1, 26, 27] for more examples and properties of superquadratic functions.

Definition 2.3.1. A function $\Psi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$\Psi(y) \geq \Psi(x) + C(x)(y-x) + \Psi(|y-x|) \tag{2.3.1}$$

for all $y \geq 0$. We say that Ψ is subquadratic if $-\Psi$ is superquadratic.

If for all $x, y > 0$ with $x \neq y$, there is strict inequality in (2.3.1), then Ψ is called strictly superquadratic.

For example, the function $\Psi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in (0,2]$. Note that this function is both superquadratic and subquadratic for $p = 2$.

The following lemma shows essentially that positive superquadratic functions are also convex functions.

Lemma 2.3.2 (see [6, Lemma 2.2])**.** *Let Ψ be a superquadratic function with $C(x)$ as in Definition* 2.3.1. *Then*

(i) $\Psi(0) \leq 0$*;*

(ii) *if $\Psi(0) = \Psi'(0) = 0$, then $C(x) = \Psi'(x)$ whenever Ψ is differentiable at $x > 0$;*

(iii) *if $\Psi \geq 0$, then Ψ is convex and $\Psi(0) = \Psi'(0) = 0$.*

In the following theorem some characterizations of superquadratic functions are given analogous to the well known characterizations of the convex functions.

Theorem 2.3.3 (see [27, Theorem 9])**.** *For the function $\Psi : [0,\infty) \to \mathbb{R}$ the following conditions are equivalent:*

(i) *The function Ψ is a superquadratic function, i.e., equation* (2.3.1) *holds.*

(ii) *For any two nonnegative n-tuples* $(x_1, \ldots, x_n)$ *and* $(p_1, \ldots, p_n)$ *such that* $P_n = \sum_{i=1}^{n} p_i > 0$ *the following inequality*

$$\Psi(\overline{x}) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \Psi(x_i) - \frac{1}{P_n} \sum_{i=1}^{n} p_i \Psi(|x_i - \overline{x}|)$$

holds, where $\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$.

(iii) *The following inequality*

$$\Psi(\lambda y_1 + (1-\lambda) y_2) \leq \lambda \Psi(y_1) + (1-\lambda)\Psi(y_2) \\ - \lambda \Psi((1-\lambda)|y_1 - y_2|) - (1-\lambda)\Psi(\lambda |y_1 - y_2|)$$

holds for all $y_1, y_2 \geq 0$ *and* $\lambda \in [0,1]$.

(iv) *For all* $x, y_1, y_2 \geq 0$*, such that* $y_1 < x < y_2$ *we have*

$$\Psi(x) \leq \frac{y_2 - x}{y_2 - y_1} \left(\Psi(y_1) - \Psi(x - y_1) \right) + \frac{x - y_1}{y_2 - y_1} \left(\Psi(y_2) - \Psi(y_2 - x) \right)$$

or equivalently

$$\frac{\Psi(y_1) - \Psi(x) - \Psi(x - y_1)}{y_1 - x} \leq \frac{\Psi(y_2) - \Psi(x) - \Psi(y_2 - x)}{y_2 - x}.$$

In the following, for any function $\Psi \in \mathrm{C}^1([0,\infty), \mathbb{R})$, we define an associated function $\overline{\Psi} \in \mathrm{C}^1((0,\infty), \mathbb{R})$ by

$$\overline{\Psi}(x) = \frac{\Psi'(x)}{x} \quad \text{for all} \quad x > 0. \tag{2.3.2}$$

Lemma 2.3.4 (See [6, Lemma 3.1])**.** *Let* $\Psi \in \mathrm{C}^1([0,\infty), \mathbb{R})$ *such that* $\Psi(0) \leq 0$*. If* $\overline{\Psi}$ *is increasing (strictly increasing) or* Ψ' *is superadditive (strictly superadditive), then* Ψ *is superquadratic (strictly superquadratic).*

Lemma 2.3.5 (See [4, Lemma 1])**.** *Let* $\Psi \in \mathrm{C}^2([0,\infty), \mathbb{R})$ *be such that*

$$m_1 \leq \frac{x\Psi''(x) - \Psi'(x)}{x^2} \leq M_1 \quad \text{for all} \quad x > 0.$$

Let the functions ϑ_1, ϑ_2 *be defined by*

$$\vartheta_1(x) = \frac{M_1 x^3}{3} - \Psi(x), \quad \vartheta_2(x) = \Psi(x) - \frac{m_1 x^3}{3}. \tag{2.3.3}$$

Then $\overline{\vartheta_1}, \overline{\vartheta_2}$ *are increasing. If also* $\Psi(0) = 0$ *then* ϑ_1, ϑ_2 *are superquadratic.*

Lemma 2.3.6 (See [4, Lemma 2]). *Let $s > 0$ and $\Psi_s : [0, \infty) \to \mathbb{R}$ be defined by*

$$\Psi_s(x) = \begin{cases} \dfrac{x^s}{s(s-2)}, & s \neq 2, \\[2ex] \dfrac{x^2}{2} \log x, & s = 2. \end{cases} \tag{2.3.4}$$

Then Ψ_s is superquadratic, with the convention $0 \log 0 := 0$.

Lemma 2.3.7 (See [4, Lemma 4]). *Let $s \in \mathbb{R}$ and $\varphi_s : [0, \infty) \to \mathbb{R}$ be defined by*

$$\varphi_s(x) = \begin{cases} \dfrac{sxe^{sx} - e^{sx} + 1}{s^3}, & s \neq 0, \\[2ex] \dfrac{x^3}{3}, & s = 0. \end{cases} \tag{2.3.5}$$

Then φ_s is superquadratic.

2.4 Time scales theory

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [63] in 1988. It is unifying integral and differential calculus with the calculus of finite differences, extending to cases "in between", and offering a formalism for studying hybrid discrete-continuous dynamic systems. It has applications in any field that requires simultaneous modelling of discrete and continuous data. Now, we briefly introduce the time scales calculus and refer to [15, 64, 65, 95] and the monograph [41] for further details.

By a time scale $\mathbb{T}$ we mean any nonempty closed subset of $\mathbb{R}$. The two most popular examples of time scales are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$. Since the time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf \{s \in \mathbb{T} : \ s > t\}$$

and the backward jump operator by

$$\rho(t) = \sup \{s \in \mathbb{T} : \ s < t\}.$$

In this definition, the convention is $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, then we say that t is right-scattered, and if $\rho(t) < t$, then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $\sigma(t) = t$, then t is said to be right-dense, and if $\rho(t) = t$, then

t is said to be left-dense. Points that are simultaneously right-dense and left-dense are called dense.

If $\mathbb{T}$ has a left-scattered maximum M_1, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M_1\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum M_2, then we define $\mathbb{T}_\kappa = \mathbb{T} \setminus \{M_2\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally we define $\mathbb{T}^* = \mathbb{T}^\kappa \bigcap \mathbb{T}_\kappa$.

The mappings $\mu, \nu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t)$$

are called the forward and backward graininess functions respectively.

In the following considerations, a time scale interval is denoted by $I_\mathbb{T} = I \cap \mathbb{T}$, where I is an interval in $\mathbb{R}$.

Definition 2.4.1. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U_\mathbb{T}$ of t such that

$$\left|(f(\sigma(t)) - f(s)) - f^\Delta(t)\left[\sigma(t) - s\right]\right| \leq \varepsilon \left|\sigma(t) - s\right| \quad \text{for all} \quad s \in U_\mathbb{T}.$$

We call $f^\Delta(t)$ the delta derivative of f at t. We say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 2.4.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then we define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U_\mathbb{T}$ of t such that

$$\left|(f(\rho(t)) - f(s)) - f^\nabla(t)\left[\rho(t) - s\right]\right| \leq \varepsilon \left|\rho(t) - s\right| \quad \text{for all} \quad s \in U_\mathbb{T}.$$

We call $f^\nabla(t)$ the nabla derivative of f at t. We say that f is nabla differentiable on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

Example 2.4.3. If $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f^\nabla(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t)$$

is the forward difference operator, while

$$f^\nabla(t) = f(t) - f(t-1)$$

is the backward difference operator.

Let $h > 0$. If $\mathbb{T} = h\mathbb{Z}$, then

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h} \quad \text{and} \quad f^{\nabla}(t) = \frac{f(t) - f(t-h)}{h}$$

are the h-derivatives.

Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, then

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{and} \quad f^{\nabla}(t) = \frac{q(f(t) - f(t/q))}{(q-1)t}$$

are the q-derivatives.

Definition 2.4.4. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}_{\kappa}$. Then we define $f^{\diamond_\alpha}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U_{\mathbb{T}}$ of t such that

$$\begin{aligned}|\alpha\,(f\,(\sigma(t)) - f(s))\,[\rho(t) - s] + (1-\alpha)\,(f\,(\rho(t)) - f(s))\,[\sigma(t) - s] \\ -f^{\diamond_\alpha}(t)\,[\rho(t) - s]\,[\sigma(t) - s]| \le \varepsilon\,|[\rho(t) - s]\,[\sigma(t) - s]| \quad \text{for all} \quad s \in U_{\mathbb{T}}.\end{aligned}$$

We call $f^{\diamond_\alpha}(t)$ the diamond-α derivative of f at t. We say that f is diamond-α differentiable on $\mathbb{T}^{\kappa}_{\kappa}$ provided $f^{\diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}^{\kappa}_{\kappa}$. If $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}$ in the sense of Δ and ∇, then f is diamond-α differentiable at $t \in \mathbb{T}^{\kappa}_{\kappa}$, and the diamond-$\alpha$ derivative is given by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Remark 2.4.5. From the above definition it is clear that f is diamond-α differentiable for $0 \le \alpha \le 1$ if and only if f is Δ and ∇ differentiable. It is obvious that for $\alpha = 1$ the diamond-α derivative reduces to the standard Δ derivative and for $\alpha = 0$ the diamond-α derivative reduces to the standard ∇ derivative.

For all $t \in \mathbb{T}^{\kappa}$, we have the following properties:

(i) If f is delta differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta differentiable at t with $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t iff $\lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s}$ exists as a finite number. In this case, $f^{\Delta}(t) = \lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

In the same manner, for all $t \in \mathbb{T}_\kappa$ we have the following properties:

(i) If f is nabla differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is left-scattere, then f is nabla differentiable at t with $f^\nabla(t) = \frac{f(t)-f(\rho(t))}{\nu(t)}$.

(iii) If t is left-dense, then f is nabla differentiable at t if and only if $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^\nabla(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is nabla differentiable at t, then $f(\rho(t)) = f(t) + \nu(t)f^\nabla(t)$.

Definition 2.4.6. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by $\mathrm{C_{rd}}$ the set of all rd-continuous functions. We say that f is rd-continuously delta differentiable (and write $f \in \mathrm{C}^1_{\mathrm{rd}}$) if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and $f^\Delta \in \mathrm{C_{rd}}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous if it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits are finite at all right-dense points in $\mathbb{T}$. We denote by $\mathrm{C_{ld}}$ the set of all ld-continuous functions. We say that f is ld-continuously nabla differentiable (and write $f \in \mathrm{C}^1_{\mathrm{ld}}$) if $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$ and $f^\nabla \in \mathrm{C_{ld}}$.

The set of all continuous functions on $\mathbb{T}$ contains both $\mathrm{C_{rd}}$ and $\mathrm{C_{ld}}$.

Definition 2.4.7. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $G^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_\kappa$. Then we define the nabla integral by

$$\int_a^t f(s)\nabla s = G(t) - G(a).$$

The importance of rd-continuous and ld-continuous functions is revealed by the following result.

Theorem 2.4.8 (See [93, Theorem 1.74, Theorem 8.45])**.** *Every rd-continuous function has a delta antiderivative and every ld-continuous function has a nabla antiderivative.*

Definition 2.4.9. Let $f : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$. Then the diamond-α integral of f from a to b is defined by

$$\int_a^b f(t)\diamond_\alpha t = \alpha \int_a^b f(t)\Delta t + (1-\alpha) \int_a^b f(t)\nabla t, \quad 0 \leq \alpha \leq 1.$$

Now we give some properties of the delta integral.

Theorem 2.4.10 (See [41, Theorem 1.77]). *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in \mathrm{C_{rd}}$, then*

(i) $\int\limits_a^b (f(t) + g(t))\, \Delta t = \int\limits_a^b f(t)\Delta t + \int\limits_a^b g(t)\Delta t,$

(ii) $\int\limits_a^b \alpha f(t)\Delta t = \alpha \int\limits_a^b f(t)\Delta t,$

(iii) $\int\limits_a^b f(t)\Delta t = -\int\limits_b^a f(t)\Delta t,$

(iv) $\int\limits_a^a f(t)\Delta t = 0,$

(v) $\int\limits_a^b f(t)\Delta t = \int\limits_a^c f(t)\Delta t + \int\limits_c^b f(t)\Delta t,$

(vi) *if $f(t) \geq 0$ for all t, then* $\int\limits_a^b f(t)\Delta t \geq 0.$

A similar theorem works for the nabla integral, for $f, g \in \mathrm{C_{ld}}$, and for diamond-α integral, for $f, g \in \mathrm{C}$.

Regarding integral calculus on time scales the literature includes, among others, the Cauchy nabla integral [24, 41]; the Riemann delta integral [42, 59, 60]; the Riemann nabla integral [59]; the Cauchy diamond-alpha integral [15, 95], which is convex combination of delta and nabla integrals; the Riemann diamond-alpha integral [77]; the Lebesgue delta and nabla integrals [42, 58]; the multiple Riemann and multiple Lebesgue delta, nabla and diamond-alpha integrals [39, 40].

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, \ldots, n\}$, let $\mathbb{T}_i$ denote a time scale and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) :\ t_i \in \mathbb{T}_i,\ 1 \leq i \leq n\} \tag{2.4.1}$$

an n-dimensional time scale. Let μ_Δ be the σ-additive Lebesgue Δ-measure on Λ^n and $\mathcal{F}$ be the family of Δ-measurable subsets of Λ^n. Let $\mathcal{E} \subset \Lambda^n$ and $(\mathcal{E}, \mathcal{F}, \mu_\Delta)$ be a time scale measure space. Then for a Δ-measurable function $f : \mathcal{E} \to \mathbb{R}$, the corresponding Δ-integral of f over $\mathcal{E}$ is denoted according to [40, (3.18)] by

$$\int_{\mathcal{E}} f(t_1, \dots, t_n)\Delta_1 t_1 \dots \Delta_n t_n, \quad \int_{\mathcal{E}} f(t)\Delta t, \quad \int_{\mathcal{E}} f \mathrm{d}\mu_\Delta, \quad \text{or} \quad \int_{\mathcal{E}} f(t)\mathrm{d}\mu_\Delta(t).$$

By [40, Section 3], all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ-integrals on Λ^n. Here we state Fubini's theorem for multiple Lebesgue Δ-integrals on time scales. It is used in Chapter 9.

Theorem 2.4.11. *Let $(X, \mathcal{K}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be two finite-dimensional time scale measure spaces. If $f : X \times Y \to \mathbb{R}$ is a Δ-integrable function and if we define the functions*

$$\varphi(y) = \int_X f(x,y)\mathrm{d}\mu_\Delta(x) \quad \textit{for a.e.} \quad y \in Y$$

and

$$\psi(x) = \int_Y f(x,y)\mathrm{d}\nu_\Delta(y) \quad \textit{for a.e.} \quad x \in X,$$

then φ is Δ-integrable on Y and ψ is Δ-integrable on X and

$$\int_X \mathrm{d}\mu_\Delta(x) \int_Y f(x,y)\mathrm{d}\nu_\Delta(y) = \int_Y \mathrm{d}\nu_\Delta(y) \int_X f(x,y)\mathrm{d}\mu_\Delta(x).$$

Chapter 3

Jensen Type Inequalities for Convex and Superquadratic Functions

In this chapter we apply the theory of isotonic linear functionals to derive a series of known inequalities, extensions of known inequalities, and new inequalities in the theory of dynamic equations on time scales. (See [18, 28]).

3.1 Isotonic linear functionals and time scales integrals

We recall the following definition from [93].

Definition 3.1.1. Let E be a nonempty set and L be a linear class of real-valued functions $f : E \to \mathbb{R}$ having the following properties:

(L_1) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $(af + bg) \in L$.

(L_2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

(A_1) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $A(af + bg) = aA(f) + bA(g)$.

(A_2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

When we use the approach of isotonic linear functionals as given in Definition 3.1.1, it is not necessary to know many details from the calculus of dynamic equations on time scales. We only need to know that the time scales integral is such an isotonic linear functional.

Theorem 3.1.2. *Let $\mathbb{T}$ be a time scale. For $a, b \in \mathbb{T}$ with $a < b$, let*

$$E = [a, b)_{\mathbb{T}} \quad \text{and} \quad L = \mathrm{C}_{\mathrm{rd}}([a, b)_{\mathbb{T}}, \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t)\Delta t,$$

where the integral is the Cauchy delta time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from Theorem 2.4.10. □

Now we give few examples of Cauchy delta time scales integral.

Example 3.1.3. If $\mathbb{T} = \mathbb{R}$ in Theorem 3.1.2, then $L = \mathrm{C}([a, b), \mathbb{R})$ and

$$A(f) = \int_a^b f(t)\mathrm{d}t.$$

If $\mathbb{T} = \mathbb{Z}$ in Theorem 3.1.2, then L consists of all real-valued functions defined on $[a, b-1] \cap \mathbb{Z}$ and

$$A(f) = \sum_{t=a}^{b-1} f(t).$$

Let $h > 0$. If $\mathbb{T} = h\mathbb{Z}$ in Theorem 3.1.2, then L consists of all real-valued functions defined on $[a, b-h] \cap h\mathbb{Z}$ and

$$A(f) = h \sum_{k=a/h}^{b/h-1} f(kh).$$

Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0}$ in Theorem 3.1.2, then L consists of all real-valued functions defined on $[a, b/q] \cap q^{\mathbb{N}_0}$ and

$$A(f) = (q-1) \sum_{k=\log_q(a)}^{\log_q(b)-1} q^k f(q^k).$$

Note that Theorem 3.1.2 also has corresponding versions for the nabla and diamond-α integral, which are given next for completeness.

Theorem 3.1.4. *Let $\mathbb{T}$ be a time scale. For $a, b \in \mathbb{T}$ with $a < b$, let*

$$E = (a, b]_{\mathbb{T}} \quad \text{and} \quad L = \mathrm{C}_{\mathrm{ld}}((a, b]_{\mathbb{T}}, \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t) \nabla t,$$

where the integral is the Cauchy nabla time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [41, Definition 8.43 and Theorem 8.47]. □

Theorem 3.1.5. *Let $\mathbb{T}$ be a time scale. For $a, b \in \mathbb{T}$ with $a < b$, let*

$$E = [a, b]_{\mathbb{T}} \quad \text{and} \quad L = \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R}).$$

Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_a^b f(t) \diamond_\alpha t,$$

where the integral is the Cauchy diamond-α time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [95, Definition 3.2 and Theorem 3.7]. □

Multiple Riemann integration on time scales was introduced in [39]. The Riemann integral introduced there is also an isotonic linear functional.

Theorem 3.1.6. *Let $\mathbb{T}_1, \ldots, \mathbb{T}_n$ be time scales. For $a_i, b_i \in \mathbb{T}_i$ with $a_i < b_i$, $1 \le i \le n$, let*

$$\mathcal{E} \subset [a_1, b_1)_{\mathbb{T}_1} \times \ldots \times [a_n, b_n)_{\mathbb{T}_n}$$

be Jordan Δ-measurable and let L be the set of all bounded Δ-integrable functions from $\mathcal{E}$ to $\mathbb{R}$. Then (L_1) and (L_2) are satisfied. Moreover, let

$$A(f) = \int_{\mathcal{E}} f(t) \Delta t,$$

where the integral is the multiple Riemann delta time scales integral. Then (A_1) and (A_2) are satisfied.

Proof. This follows from [39, Definition 4.13 and Theorem 3.4]. □

From [39, Remark 2.18], it is also clear that a theorem similar to Theorem 3.1.6 is also true for the nabla and diamond-α integrals in the multiple variable case.

Multiple Lebesgue integration on time scales was introduced in [40]. The Lebesgue integral introduced there is also an isotonic linear functional.

Theorem 3.1.7. *Let $\mathcal{E}$ be a Δ-measurable subset of Λ^n, defined as in* (2.4.1) *and let L be the set of all Δ-measurable functions from $\mathcal{E}$ to $\mathbb{R}$. Then* (L_1) *and* (L_2) *are satisfied. Moreover, let*

$$A(f) = \int_{\mathcal{E}} f(t)\mathrm{d}\mu_\Delta(t),$$

where the integral is the multiple Lebesgue delta time scales integral. Then (A_1) *and* (A_2) *are satisfied.*

Proof. This follows from [40, Section 3]. □

Theorem 3.1.8. *Under the assumptions of Theorem* 3.1.7*, let $A(f)$ be replaced by*

$$A(f) = \frac{\int_{\mathcal{E}} h(t)f(t)\mathrm{d}\mu_\Delta(t)}{\int_{\mathcal{E}} h(t)\mathrm{d}\mu_\Delta(t)},$$

where $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\mathrm{d}\mu_\Delta(t) > 0$. Then A is an isotonic linear functional satisfying $A(1) = 1$.

The monograph [93] contains numerous classical inequalities that are proved for isotonic linear functionals. Since the time scales integral is in fact an isotonic linear functional, the results from [93] can be applied to this setting. Our work shows that it is not necessary to prove such kinds of inequalities "from scratch" in the time scales setting as they can all be obtained easily from well-known inequalities for isotonic linear functionals.

For simplicity, in what follows, we use the following notations: $\mathcal{E}$ as Δ-measurable subset of Λ^n,

$$L_\Delta(f) = \int_{\mathcal{E}} f(t)\mathrm{d}\mu_\Delta(t) \quad \text{and} \quad \overline{L}_\Delta(f,h) = \frac{\int_{\mathcal{E}} f(t)h(t)\mathrm{d}\mu_\Delta(t)}{\int_{\mathcal{E}} h(t)\mathrm{d}\mu_\Delta(t)},$$

where $f : \mathcal{E} \to \mathbb{R}$ is Δ-integrable and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{\mathcal{E}} h(t)\mathrm{d}\mu_\Delta(t) > 0$. Also we assume throughout the thesis that I and $[m, M]$ are nonempty intervals in $\mathbb{R}$ such that $-\infty < m < M < \infty$.

3.2 Jensen's inequality

B. Jessen in [70] gave the following generalization of Jensen's inequality for isotonic linear functionals.

Theorem 3.2.1 (See [93, Theorem 2.4])**.** *Let L satisfy properties (L_1) and (L_2). Assume $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex. If A satisfy (A_1) and (A_2) such that $A(1) = 1$, then for all $f \in L$ such that $\Phi(f) \in L$, we have $A(f) \in I$ and*

$$\Phi(A(f)) \leq A(\Phi(f)). \tag{3.2.1}$$

Now our first result is the following generalization of Jensen's inequality.

Theorem 3.2.2. *Assume $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex, $f : \mathcal{E} \to I$ is Δ-integrable and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\Phi\left(\overline{L}_\Delta(f, h)\right) \leq \overline{L}_\Delta(\Phi(f), h). \tag{3.2.2}$$

Proof. The inequality (3.2.2) follows from Theorem 3.2.1 and Theorem 3.1.8. □

Remark 3.2.3. Known results from time scales theory, which were proved by using time scales calculus, follow from Theorem 3.2.1 in the same way as Theorem 3.2.2 does. Note also that a similar theorem for the multiple Riemann integral can be stated and proved using Theorem 3.1.6. This will be the case for all inequalities stated in this section and the following sections; however, we only explicitly state each time the case for the multiple Lebesgue integral.

The Jensen inequality for Cauchy delta integrals has been obtained by Agarwal, Bohner and Peterson [8].

Theorem 3.2.4 (See [8, Theorem 4.1])**.** *Let $a, b \in \mathbb{T}$ with $a < b$. If $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex and $f \in \mathrm{C_{rd}}([a, b)_\mathbb{T}, I)$, then*

$$\Phi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{\int_a^b \Phi(f(t))\Delta t}{b-a}.$$

Remark 3.2.5. When $\mathbb{T} = \mathbb{R}$ in Theorem 3.2.4, then we obtain Theorem 1.0.1. When $\mathbb{T} = \mathbb{Z}$ in Theorem 3.2.4, then we get the discrete Jensen inequality (1.0.1).

The following result is given by Wong, Yeh and Lian in [106]. When $h(t) \equiv 1$ in Theorem 3.2.6 below, then we obtain Theorem 3.2.4.

Theorem 3.2.6 (See [106, Theorem 2.2])**.** *Let $a, b \in \mathbb{T}$ with $a < b$. Assume $h \in \mathrm{C_{rd}}([a, b)_\mathbb{T}, \mathbb{R})$ satisfies $\int_a^b |h(t)|\Delta t > 0$. If $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex and $f \in \mathrm{C_{rd}}([a, b)_\mathbb{T}, I)$, then*

$$\Phi\left(\frac{\int_a^b |h(t)| f(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right) \leq \frac{\int_a^b |h(t)|\Phi(f(t))\Delta t}{\int_a^b |h(t)|\Delta t}.$$

Proof. This follows from Theorem 3.2.1 and Theorem 3.1.2. □

In [88], Özkan, Sarikaya and Yildirim proved that Theorem 3.2.6 is also true if we use the nabla integral (see [41, Section 8.4]) instead of the delta integral. In [95], Sheng, Fagan, Henderson and Davis introduced the so-called diamond-α integral, where $0 \le \alpha \le 1$. It is a convex combination of the delta integral and the nabla integral. When $\alpha = 1$, we get the usual delta integral, and when $\alpha = 0$, we get the usual nabla integral. The following result concerning the diamond-α integral is given by Ammi, Ferreira and Torres in [15] (see also [88]).

Theorem 3.2.7 (See [15, Theorem 3.3])**.** *Let $\alpha \in [0,1]$. Let $a, b \in \mathbb{T}$ with $a < b$. Assume $h \in \mathrm{C}([a,b]_{\mathbb{T}}, \mathbb{R})$ satisfies $\int_a^b |h(t)| \diamond_\alpha t > 0$. If $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex and $f \in \mathrm{C}([a,b]_{\mathbb{T}}, I)$, then*

$$\Phi\left(\frac{\int_a^b |h(t)| f(t) \diamond_\alpha t}{\int_a^b |h(t)| \diamond_\alpha t}\right) \le \frac{\int_a^b |h(t)| \Phi(f(t)) \diamond_\alpha t}{\int_a^b |h(t)| \diamond_\alpha t}.$$

Proof. This follows from Theorem 3.2.1 and Theorem 3.1.5. □

3.3 Hermite–Hadamard inequality

P. Beesack and J. Pečarić in [31] gave the following generalization of the converse of Jensen's inequality for isotonic linear functionals.

Theorem 3.3.1 (See [93, Theorem 3.37])**.** *Let L satisfy properties (L_1) and (L_2). Assume $\Phi : I \to \mathbb{R}$ is convex, where $I = [m, M]$. If A satisfy (A_1) and (A_2) such that $A(1) = 1$, then for all $f \in L$ such that $\Phi(f) \in L$, we have*

$$A(\Phi(f)) \le \frac{M - A(f)}{M - m}\Phi(m) + \frac{A(f) - m}{M - m}\Phi(M). \tag{3.3.1}$$

In the following theorem we give generalization of the converse of Jensen's inequality on time scales.

Theorem 3.3.2. *Assume $\Phi : I \to \mathbb{R}$ is convex, $f : \mathcal{E} \to [m, M]$ is Δ-integrable, where $[m, M] \subseteq I$, and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\overline{L}_\Delta(\Phi(f), h) \le \frac{M - \overline{L}_\Delta(f, h)}{M - m}\Phi(m) + \frac{\overline{L}_\Delta(f, h) - m}{M - m}\Phi(M). \tag{3.3.2}$$

Proof. The inequality (3.3.2) follows from Theorem 3.3.1 and Theorem 3.1.8. □

Remark 3.3.3. If Φ is continuous in Theorem 3.3.2, then by combining this theorem with the Theorem 3.2.2, we obtain generalization of the Hermite–Hadamard inequality (1.0.3).

$$\Phi\left(\overline{L}_\Delta(f,h)\right) \leq \overline{L}_\Delta(\Phi(f),h) \leq \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m) + \frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M). \tag{3.3.3}$$

Note that the known result [47, Theorem 3.14] (see also [17, 48]) follows from Theorem 3.3.1 in the same way as Theorem 3.3.2 does, this time applying Theorem 3.1.5.

A combination of Theorem 3.2.1 and Theorem 3.3.1 in a slightly different form is given by Pečarić and Beesack in [89] as follows.

Theorem 3.3.4 (See [93, Theorem 5.13])**.** *Let L satisfy properties (L_1) and (L_2). Assume $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex, where $[m,M] \subseteq I$. Suppose A satisfy (A_1) and (A_2) such that $A(1) = 1$. Let $f \in L$ such that $f(E) \subseteq [m,M]$ and $\Phi(f) \in L$, and define $p,q \geq 0$ such that $p+q > 0$ and*

$$A(f) = \frac{pm+qM}{p+q}$$

holds. Then

$$\Phi\left(\frac{pm+qM}{p+q}\right) \leq A(\Phi(f)) \leq \frac{p\Phi(m)+q\Phi(M)}{p+q}.$$

Theorem 3.3.5. *Assume $\Phi \in \mathrm{C}(I,\mathbb{R})$ is convex, $f : \mathcal{E} \to [m,M]$ is Δ-integrable, where $[m,M] \subseteq I$, and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Let $p,q \geq 0$ be such that $p+q > 0$ and*

$$\overline{L}_\Delta(f,h) = \frac{pm+qM}{p+q}$$

holds. Then

$$\Phi\left(\frac{pm+qM}{p+q}\right) \leq \overline{L}_\Delta(\Phi(f),h) \leq \frac{p\Phi(m)+q\Phi(M)}{p+q}. \tag{3.3.4}$$

Proof. The inequality (3.3.4) follows from Theorem 3.3.4 and Theorem 3.1.8. □

3.4 Related inequalities to the Jensen inequality

In this section we present some of the related inequalities of Jensen's inequalities.

Hölder's inequality:

We first recall Hölder's inequality for isotonic linear functionals as given in [93].

Theorem 3.4.1 (See [93, Theorem 4.12])**.** *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \neq 1$, define $q = p/(p-1)$. Assume w, f, g are nonnegative functions on E and $wf^p, wg^q, wfg \in L$. If $p > 1$, then*

$$A(wfg) \leq A^{1/p}(wf^p)A^{1/q}(wg^q).$$

This inequality is reversed if $0 < p < 1$ and $A(wg^q) > 0$, and it is also reversed if $p < 0$ and $A(wf^p) > 0$.

In the following theorem we give the generalization of Hölder's inequality on time scales.

Theorem 3.4.2. *For $p \neq 1$, define $q = p/(p-1)$. Assume w, f, g are nonnegative functions on $\mathcal{E}$ and wf^p, wg^q, wfg are Δ-integrable on $\mathcal{E}$. If $p > 1$, then*

$$L_\Delta(wfg) \leq L_\Delta^{\frac{1}{p}}(wf^p)L_\Delta^{\frac{1}{q}}(wg^q). \tag{3.4.1}$$

This inequality is reversed if $0 < p < 1$ and $L_\Delta(wg^q) > 0$, and it is also reversed if $p < 0$ and $L_\Delta(wf^p) > 0$.

Proof. The inequality (3.4.1) follows from Theorem 3.4.1 and Theorem 3.1.7. □

Remark 3.4.3. Note that the known results from the time scales literature follow from Theorem 3.4.1 in the same way as Theorem 3.4.2 does: [41, Theorem 6.13] follows as in Theorem 3.1.2 and [15, Theorem 4.1] (see also [53, 16]) follows as in Theorem 3.1.5.

From Hölder's inequality follows the Cauchy–Schwarz inequality given in next theorem.

Theorem 3.4.4. *If w, f, g are nonnegative functions on $\mathcal{E}$ and wf^2, wg^2, wfg are Δ-integrable on $\mathcal{E}$, then*

$$L_\Delta(wfg) \leq \sqrt{L_\Delta(wf^2)L_\Delta(wg^2)}. \tag{3.4.2}$$

Proof. The inequality (3.4.2) follows from Theorem 3.4.2 by taking $p = 2$. □

Minkowski's inequality:

We first recall Minkowski's inequality for isotonic linear functionals as given in [93].

Theorem 3.4.5 (See [93, Theorem 4.13])**.** *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \in \mathbb{R}$, assume w, f, g are nonnegative functions on E and $wf^p, wg^p, w(f+g)^p \in L$. If $p > 1$, then*

$$A^{1/p}\left(w(f+g)^p\right) \leq A^{1/p}(wf^p) + A^{1/p}(wg^p).$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ provided $A(wf^p) > 0$ and $A(wg^p) > 0$ hold.

In the following theorem we give generalization of the Minkowski inequality on time scales.

Theorem 3.4.6. *For $p \in \mathbb{R}$, assume w, f, g are nonnegative functions on $\mathcal{E}$ and $wf^p, wg^p, w(f+g)^p$ are Δ-integrable on $\mathcal{E}$. If $p > 1$, then*

$$L_{\Delta}^{\frac{1}{p}}\left(w(f+g)^p\right) \leq L_{\Delta}^{\frac{1}{p}}\left(wf^p\right) + L_{\Delta}^{\frac{1}{p}}\left(wg^p\right). \tag{3.4.3}$$

This inequality is reversed for $0 < p < 1$ or $p < 0$ provided each of the two terms on the right-hand side are positive.

Proof. The inequality (3.4.3) follows from Theorem 3.4.5 and Theorem 3.1.7. □

Remark 3.4.7. Note that the known results from the time scales literature follow from Theorem 3.4.5 in the same way as Theorem 3.4.6 does: [41, Theorem 6.16] follows as in Theorem 3.1.2 and [15, Theorem 4.4] (see also [53, 16]) follows as in Theorem 3.1.5.

Dresher's inequality:

If $n = 2$ in the following result, then we have the Dresher inequality (see [51, Section 7]). We first present the generalization of this inequality for isotonic linear functionals as given in [93].

Theorem 3.4.8 (See [93, Theorem 4.21])**.** *Let E and L be such that (L_1), (L_2) are satisfied and suppose that both A and B satisfy (A_1), (A_2). If f_i, u_i are nonnegative functions on E and $wf_i^p, w\left(\sum_{i=1}^n f_i\right)^p, wg_i^r, w\left(\sum_{i=1}^n g_i\right)^r \in L$, where $p \geq 1 > r > 0$ and $A(wg_i^r) > 0$ for $1 \leq i \leq n$, then*

$$\left(\frac{A\left(w\left(\sum\limits_{i=1}^n f_i\right)^p\right)}{B\left(w\left(\sum\limits_{i=1}^n g_i\right)^r\right)}\right)^{\frac{1}{p-r}} \leq \sum_{i=1}^n \left(\frac{A(wf_i^p)}{B(wg_i^r)}\right)^{\frac{1}{p-r}}.$$

In the following theorem we give the Dresher inequality on time scales.

Theorem 3.4.9. *If f_i, u_i are nonnegative functions on $\mathcal{E}$ and wf_i^p, $w\left(\sum_{i=1}^{n} f_i\right)^p$, wg_i^r, $w\left(\sum_{i=1}^{n} g_i\right)^r$ are Δ-integrable on $\mathcal{E}$, where $p \geq 1 > r > 0$ and $L_\Delta(wg_i^r) > 0$ for $1 \leq i \leq n$, then*

$$\left(\frac{L_\Delta\left(w\left(\sum_{i=1}^{n} f_i\right)^p\right)}{L_\Delta\left(w\left(\sum_{i=1}^{n} g_i\right)^r\right)}\right)^{\frac{1}{p-r}} \leq \sum_{i=1}^{n}\left(\frac{L_\Delta(wf_i^p)}{L_\Delta(wg_i^r)}\right)^{\frac{1}{p-r}}. \tag{3.4.4}$$

Proof. The inequality (3.4.4) follows from Theorem 3.4.8 and Theorem 3.1.7. □

Remark 3.4.10. Dresher's inequality on time scales is new even for the cases of a single variable Cauchy delta and nabla integral and also for the diamond-α integral.

Popoviciu's inequality:

We first recall Popoviciu's inequality for isotonic linear functionals as given in [93].

Theorem 3.4.11 (See [93, Theorem 4.27])**.** *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \neq 1$, define $q = p/(p-1)$. Assume f, g are nonnegative functions on E and $f^p, g^q, fg \in L$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - A(f^p) > 0 \quad \text{and} \quad g_0^q - A(g^q) > 0.$$

If $p > 1$, then

$$(f_0^p - A(f^p))^{1/p}(g_0^q - A(g^q))^{1/q} \leq f_0 g_0 - A(fg).$$

This inequality is reversed if $0 < p < 1$ and $A(g^q) > 0$, or if $p < 0$ and $A(f^p) > 0$.

In the following theorem we give the Popoviciu inequality on time scales.

Theorem 3.4.12. *For $p \neq 1$, define $q = p/(p-1)$. Assume f, g are nonnegative functions on $\mathcal{E}$ and f^p, g^q, fg are Δ-integrable on $\mathcal{E}$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - L_\Delta(f^p) > 0 \quad \text{and} \quad g_0^q - L_\Delta(g^q) > 0.$$

If $p > 1$, then

$$f_0 g_0 - L_\Delta(fg) \geq (f_0^p - L_\Delta(f^p))^{\frac{1}{p}}(g_0^q - L_\Delta(g^q))^{\frac{1}{q}}. \tag{3.4.5}$$

This inequality is reversed if $0 < p < 1$ and $L_\Delta(g^q) > 0$, or if $p < 0$ and $L_\Delta(f^p) > 0$.

Proof. The inequality (3.4.5) follows from Theorem 3.4.11 and Theorem 3.1.7. □

From Popoviciu's inequality follows the Aczél inequality given in next theorem.

Theorem 3.4.13. *Assume f, g are nonnegative functions on $\mathcal{E}$ and f^2, g^2, fg are Δ-integrable on $\mathcal{E}$. If $f_0, g_0 > 0$ are such that*

$$f_0^2 - L_\Delta(f^2) > 0 \quad \text{and} \quad g_0^2 - L_\Delta(g^2) > 0,$$

then

$$f_0 g_0 - L_\Delta(fg) \geq \sqrt{(f_0^2 - L_\Delta(f^2))\,(g_0^2 - L_\Delta(g^2))}. \tag{3.4.6}$$

Proof. The inequality (3.4.6) follows from Theorem 3.4.12 by taking $p = 2$. □

Remark 3.4.14. The Aczél and Popoviciu inequalities on time scales are new even for the cases of a single variable Cauchy delta and nabla integral and also for the diamond-α integral. The original Aczél's inequality can be found in [7]. For a version of Aczél's inequality for isotonic linear functionals, we refer to [93, Theorem 4.26].

Bellman's inequality:

We first recall Bellman's inequality for isotonic linear functionals as given in [93].

Theorem 3.4.15 (See [93, Theorem 4.29])**.** *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \in \mathbb{R}$, assume f, g are nonnegative functions on E and $f^p, g^p, (f+g)^p \in L$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - A(f^p) > 0 \quad \text{and} \quad g_0^p - A(g^p) > 0.$$

If $p > 1$, then

$$\left((f_0^p - A(f^p))^{1/p} + (g_0^p - A(g^p))^{1/p} \right)^p \leq (f_0 + g_0)^p - A((f+g)^p).$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ and $A(f^p) > 0$.

In the following theorem we give the Bellman inequality on time scales.

Theorem 3.4.16. *For $p \in \mathbb{R}$, assume f, g are nonnegative functions on $\mathcal{E}$ and $f^p, g^p, (f+g)^p$ are Δ-integrable on $\mathcal{E}$. Suppose $f_0, g_0 > 0$ are such that*

$$f_0^p - L_\Delta(f^p) > 0 \quad \text{and} \quad g_0^p - L_\Delta(g^p) > 0.$$

If $p > 1$, then

$$\left((f_0^p - L_\Delta(f^p))^{\frac{1}{p}} + (g_0^p - L_\Delta(g^p))^{\frac{1}{p}} \right)^p \leq (f_0 + g_0)^p - L_\Delta((f+g)^p). \tag{3.4.7}$$

This inequality is reversed if $0 < p < 1$ or $p < 0$ and $L_\Delta(f^p) > 0$.

Proof. The inequality (3.4.7) follows from Theorem 3.4.15 and Theorem 3.1.7. □

Diaz–Metcalf inequality:

If $p = q = 2$ and $w = 1$ in the following result, then we have the Diaz–Metcalf inequality. We first present the generalization of this inequality for isotonic linear functionals as given in [93].

Theorem 3.4.17 (See [93, Theorem 4.14])**.** *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. For $p \neq 1$, let $q = p/(p-1)$. Assume w, f, g are nonnegative functions on E such that $wf^p, wg^q, wfg \in L$ and, if $p \neq 0$,*

$$0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all} \quad t \in E.$$

If $p > 1$, or if $p < 0$ and $A(wf^p) + A(wg^q) > 0$, then

$$(M - m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \leq (M^p - m^p)A(wfg).$$

This inequality is reversed if $0 < p < 1$ and $A(wf^p) + A(wg^q) > 0$.

In the following theorem we give the Diaz–Metcalf inequality on time scales.

Theorem 3.4.18. *For $p \neq 1$, let $q = p/(p-1)$. Assume w, f, g are nonnegative functions on $\mathcal{E}$ such that wf^p, wg^q, wfg are Δ-integrable on $\mathcal{E}$ and, if $p \neq 0$,*

$$0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all} \quad t \in \mathcal{E}.$$

If $p > 1$, or if $p < 0$ and at least one of the two integrals on the left-hand side of the following inequality is positive, then

$$(M - m)L_\Delta(wf^p) + (mM^p - Mm^p)L_\Delta(wg^q) \leq (M^p - m^p)L_\Delta(wfg). \tag{3.4.8}$$

This inequality is reversed if $0 < p < 1$ and at least one of the two integrals on the left-hand side is positive.

Proof. The inequality (3.4.8) follows from Theorem 3.4.17 and Theorem 3.1.7. □

The following two inequalities follow from [93, Theorem 4.16 and Theorem 4.18] in the same way as Theorem 3.4.18 follows from Theorem 3.4.17.

Theorem 3.4.19. *Let $\mathcal{E}, p, q, w, f, g, m, M$ be as in Theorem 3.4.18. If $p > 1$, then*

$$L_\Delta(wfg) \geq K(p, m, M) \left(L_\Delta(wf^p)\right)^{\frac{1}{p}} \left(L_\Delta(wg^q)\right)^{\frac{1}{q}}. \tag{3.4.9}$$

where

$$K(p, m, M) = |p|^{1/p}|q|^{1/q} \frac{(M - m)^{1/p}|mM^p - Mm^p|^{1/q}}{|M^p - m^p|}. \tag{3.4.10}$$

The inequality (3.4.9) is reversed if $p < 0$ or $0 < p < 1$, provided at least one of the two integrals on the right-hand side is positive.

Proof. The inequality (3.4.9) follows from [93, Theorem 4.16] and Theorem 3.1.7. □

Theorem 3.4.20. *Let $\mathcal{E}, p, q, w, f, g, m, M$ be as in Theorem 3.4.18 and assume*

$$0 < m < F(t) \le M \quad \text{and} \quad 0 \le G(t) \le M \quad \text{for all} \quad t \in \mathcal{E},$$

where $F = f(f+g)^{-q/p}$ and $G = g(f+g)^{-q/p}$. Let $K(p,m,M)$ be defined as in (3.4.10). If $p > 1$, then

$$L_\Delta^{\frac{1}{p}}(w(f+g)^p) \ge |p|^{1/p}|q|^{1/q}\frac{(M-m)^{1/p}(mM^p - Mm^p)^{1/q}}{|M^p - m^p|}\times \\ \times\left\{(L_\Delta(wf^p))^{\frac{1}{p}} + L_\Delta^{\frac{1}{p}}(wg^p)\right\}. \quad (3.4.11)$$

This inequality is reversed if $0 < p < 1$, or if $p < 0$ and the integral on the left-hand side is positive.

Proof. The inequality (3.4.11) follows from [93, Theorem 4.18] and Theorem 3.1.7. □

3.5 Further converses of the Jensen inequality

Some converses of Jensen's inequality are obtained in the previous sections. This section is concerned with some further converses of Jensen's inequality. The five theorems presented follow from the specified results in [93] in the same way as Theorem 3.3.2 follows from Theorem 3.3.1.

Theorem 3.5.1. (*a*) *Assume $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex, where $I = [m, M]$, such that $\Phi''(x) \ge 0$ with equality for at most isolated points of I. Assume further that either*

- (i) $\Phi(x) > 0$ *for all $x \in I$, or*
- (i′) $\Phi(x) > 0$ *for all $m < x < M$ with either $\Phi(m) = 0$, $\Phi'(m) \ne 0$, or $\Phi(M) = 0$, $\Phi'(M) \ne 0$, or*
- (ii) $\Phi(x) < 0$ *for all $x \in I$, or*
- (ii′) $\Phi(x) < 0$ *for all $m < x < M$ with precisely one of $\Phi(m) = 0$, $\Phi(M) = 0$.*

Suppose f is Δ-integrable on $\mathcal{E}$ such that $f(\mathcal{E}) = I$ and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then

$$\overline{L}_\Delta(\Phi(f), h) \le \lambda\Phi\left(\overline{L}_\Delta(f, h)\right)$$

holds for some $\lambda > 1$ *in cases* (i), (i′), *or* $\lambda \in (0,1)$ *in cases* (ii), (ii′). *More precisely, a value of* λ, *depending only on* m, M, Φ, *may be determined as follows: Define* $\nu = (\Phi(M) - \Phi(m))/(M - m)$. *If* $\nu = 0$, *let* $\tilde{x} \in (m, M)$ *be the unique solution of the equation* $\Phi'(x) = 0$; *then* $\lambda = \Phi(m)/\Phi(\tilde{x})$. *If* $\nu \neq 0$, *let* $\tilde{x} \in [m, M]$ *be the unique solution of the equation* $\nu\Phi(x) - \Phi'(x)(\Phi(m) + \nu(x - m)) = 0$; *then* $\lambda = \nu/\Phi'(\tilde{x})$. *Moreover, we have* $\tilde{x} \in (m, M)$ *in the cases* (i), (ii).

(*b*) *Let all the hypotheses of* (*a*) *hold except that* Φ *is concave on* I *with* $\Phi''(x) \leq 0$ *with equality for at most isolated points of* I. *Then*

$$\overline{L}_\Delta(\Phi(f), h) \geq \lambda\Phi\left(\overline{L}_\Delta(f, h)\right),$$

where λ *is determined as in* (*a*). *Furthermore,* $\lambda > 1$ *holds if* $\Phi(x) < 0$ *for all* $x \in (m, M)$, *and* $0 < \lambda < 1$ *holds if* $\Phi(x) > 0$ *for all* $x \in (m, M)$.

Proof. This follows from [93, Theorem 3.39] and Theorem 3.1.8. □

Theorem 3.5.2. (*a*) *Let* f, I, m, M, h, ν *be as in Theorem* 3.5.1 *and* $\Phi \in \mathrm{C}(I, \mathbb{R})$ *be differentiable such that* Φ' *is strictly increasing on* I. *Then*

$$\overline{L}_\Delta(\Phi(f), h) \leq \lambda + \Phi\left(\overline{L}_\Delta(f, h)\right)$$

for $\lambda = \Phi(m) - \Phi(\tilde{x}) + \nu(\tilde{x} - m) \in (0, (M - m)(\nu - \Phi'(m)))$, *where* $\tilde{x} \in (m, M)$ *is the unique solution of the equation* $\Phi'(x) = \nu$.

(*b*) *Let all the hypotheses of* (*a*) *hold except that* Φ' *is strictly decreasing on* I. *Then*

$$\Phi\left(\overline{L}_\Delta(f, h)\right) \leq \lambda + \overline{L}_\Delta(\Phi(f), h)$$

for $\lambda = \Phi(\tilde{x}) - \Phi(m) - \nu(\tilde{x} - m) \in (0, (M - m)(\Phi'(m) - \nu))$ *with* $\tilde{x}$ *given in* (*a*).

Proof. This follows from [93, Theorem 3.41] and Theorem 3.1.8. □

Theorem 3.5.3. *In addition to the assumptions of Theorem* 3.3.2, *let* $J \subset \mathbb{R}$ *be an interval such that* $J \supset \Phi(I)$ *and assume that* $F : J \times J \to \mathbb{R}$ *is increasing in the first variable. Then*

$$\begin{aligned} &F(\overline{L}_\Delta(\Phi(f), h), \Phi\left(\overline{L}_\Delta(f, h)\right)) \\ &\leq \max_{x \in [m,M]} F\left(\frac{M - x}{M - m}\Phi(m) + \frac{x - m}{M - m}\Phi(M), \Phi(x)\right) \\ &= \max_{\sigma \in [0,1]} F(\sigma\Phi(m) + (1 - \sigma)\Phi(M), \Phi(\sigma m + (1 - \sigma)M)), \end{aligned}$$

and the right-hand side of the inequality is an increasing function of M *and a decreasing function of* m.

Proof. This follows from [93, Theorem 3.42] and Theorem 3.1.8. □

Remark 3.5.4. The discrete version of Theorem 3.5.3 can be found in [82, Theorem 8].

Remark 3.5.5. If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 3.5.3 it follows

$$\overline{L}_\Delta(\Phi(f),h) - \Phi\left(\overline{L}_\Delta(f,h)\right) \leq \max_{\sigma\in[0,1]} \left(\sigma\Phi(m) + (1-\sigma)\Phi(M) - \Phi(\sigma m + (1-\sigma)M)\right). \quad (3.5.1)$$

On the other hand, if we choose $F(x,y) = \frac{x}{y}$, then we get

$$\frac{\overline{L}_\Delta(\Phi(f),h)}{\Phi\left(\overline{L}_\Delta(f,h)\right)} \leq \max_{\sigma\in[0,1]} \left(\frac{\sigma\Phi(m) + (1-\sigma)\Phi(M)}{\Phi(\sigma m + (1-\sigma)M)}\right). \quad (3.5.2)$$

The inequalities (3.5.1) and (3.5.2) are the generalizations of the results given in [96, 97, 98].

Theorem 3.5.6. *Under the same hypotheses as in Theorem* 3.5.3 *except that F is decreasing in its first variable, we have*

$$\begin{aligned} &F\left(\overline{L}_\Delta(\Phi(f),h), \Phi\left(\overline{L}_\Delta(f,h)\right)\right) \\ \geq\ & \min_{x\in[m,M]} F\left(\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M), \Phi(x)\right) \\ =\ & \min_{\sigma\in[0,1]} F\left(\sigma\Phi(m) + (1-\sigma)\Phi(M), \Phi\left(\sigma m + (1-\sigma)M\right)\right). \end{aligned}$$

Moreover, right-hand side of the above inequality is a decreasing function of M and an increasing function of m.

Proof. This follows from [93, Theorem 3.42′] and Theorem 3.1.8. □

Theorem 3.5.7. *Assume $\Phi : I \to \mathbb{R}$ is convex and $f : \mathcal{E} \to I$ is Δ-integrable. Let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $0 < L_\Delta(h) < \alpha$ for some $\alpha \in \mathbb{R}$. If hf and $h(\Phi \circ f)$ are Δ-integrable on $\mathcal{E}$ and $a \in I$ is such that*

$$\frac{\alpha a - L_\Delta(hf)}{\alpha - L_\Delta(h)} \in I,$$

then

$$\Phi\left(\frac{\alpha a - L_\Delta(hf)}{\alpha - L_\Delta(h)}\right) \geq \frac{\alpha\Phi(a) - L_\Delta(h\Phi(f))}{\alpha - L_\Delta(h)}.$$

Proof. This follows from [93, Lemma 4.25] and Theorem 3.1.7. □

3.6 Jensen type inequalities for superquadratic functions

In this section, all the inequalities obtained are given for Cauchy delta time scales integrals but they also hold for many other time scales integrals, such as Cauchy, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue delta, nabla, and diamond-α time scales integrals as we know that these integrals are isotonic linear functionals.

Jensen's inequality:

First we quote the following result of S. Banić and S. Varošanec.

Theorem 3.6.1 (See [27, Theorem 10]). *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Suppose that $h \in L$ with $h \geq 0$ and $A(h) > 0$ and that $\Psi : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function. Then for all nonnegative $f \in L$ such that $hf, h\Psi(f), h\Psi\left(\left|f - \frac{A(hf)}{A(h)} \cdot 1\right|\right) \in L$, we have*

$$\Psi\left(\frac{A(hf)}{A(h)}\right) \leq \frac{A(h\Psi(f)) - A\left(h\Psi\left(\left|f - \frac{A(hf)}{A(h)} \cdot 1\right|\right)\right)}{A(h)}.$$

If Ψ is a subquadratic function, then a reversed inequality holds.

Now we will demonstrate how Jensen's inequality on time scales for a superquadratic functions can be proved by two completely different approaches: The first approach uses the methods and techniques of time scales calculus and the second one follows from Theorem 3.6.1. According to the conclusion that comes out from the second way of proving Jensen's inequality, in the rest of this section, some new inequalities with delta integrals will be obtained.

In next theorem we present the Jensen inequality on time scales for superquadratic functions.

Theorem 3.6.2. *Let $a, b \in \mathbb{T}$. Assume $f \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, [0, \infty))$ and $\Psi \in \mathrm{C}([0, \infty), \mathbb{R})$ is superquadratic. Then*

$$\Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{1}{b-a}\int_a^b \left[\Psi\left(f(s)\right) - \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right]\Delta s. \tag{3.6.1}$$

Moreover, if Ψ is subquadratic then (3.6.1) *holds in reverse order.*

First Proof of Theorem 3.6.2. Let $\Psi : [0,\infty) \to \mathbb{R}$ be a superquadratic function and let $x_0 \in [0,\infty)$. According to (2.3.1), there is a constant $C(x_0)$ such that

$$\Psi(y) \geq \Psi(x_0) + C(x_0)(y - x_0) + \Psi\left(|y - x_0|\right). \tag{3.6.2}$$

Since f is rd-continuous,

$$x_0 = \frac{\int_a^b f(t)\Delta t}{b-a} \tag{3.6.3}$$

is well defined. The function $\Psi \circ f$ is also rd-continuous, so we may apply (3.6.2) with $y = f(s)$ and (3.6.3) to obtain

$$\begin{aligned} &\Psi(f(s)) \\ &\geq \Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) + C(x_0)\left(f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right) + \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right). \end{aligned} \tag{3.6.4}$$

Integrating (3.6.4) from a to b, we get

$$\begin{aligned} &\int_a^b \left[\Psi(f(s)) - \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right] \Delta s - (b-a)\Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \\ &= \int_a^b \Psi(f(s))\Delta s - \int_a^b \Psi\left(\left|f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\Delta s - \int_a^b \Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)\Delta s \\ &\geq C(x_0)\int_a^b \left[f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right]\Delta s \\ &= C(x_0)\left[\int_a^b f(s)\Delta s - (b-a)\cdot x_0\right] \\ &= 0, \end{aligned}$$

from which (3.6.1) follows. If Ψ is subquadratic then reverse inequality in (3.6.1) can be obtained in a similar way. □

Second Proof of Theorem 3.6.2. Substituting A from Theorem 3.1.2 into Theorem 3.6.1 and using $k(t) = 1$ for all $t \in [a,b)_{\mathbb{T}}$, we get inequality (3.6.1). □

Remark 3.6.3. Note that if Ψ is strictly superquadratic in Theorem 3.6.2, then strict inequality in (3.6.1) holds.

Remark 3.6.4. In the case when Ψ is a nonnegative superquadratic function and therefore (by Lemma 2.3.2) a convex one too, the result of Theorem 3.6.2 refines the result given in Theorem 3.2.4.

Hölder's inequality:

Let us recall the following refinement of the functional Hölder inequality.

Theorem 3.6.5 (See [27, Theorem 13])**.** *For $p \neq 1$, define $q = p/(p-1)$. Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. If $p \geq 2$, then for all nonnegative functions $f, g \in L$ such that $fg, f^p, g^q, \left|f - g^{q-1}\frac{A(fg)}{A(g^q)}\right|^p \in L$, and $A(g^q) > 0$, the inequality*

$$A(fg) \leq \left[A\left(f^p\right) - A\left(\left|f - g^{q-1}\frac{A(fg)}{A(g^q)}\right|^p\right)\right]^{\frac{1}{p}} A^{\frac{1}{q}}(g^q) \tag{3.6.5}$$

holds. In the case $0 < p < 1$ or $1 < p < 2$, the inequality in (3.6.5) *is reversed.*

Now Hölder's inequality on time scales (see [8] and [41, Theorem 6.13]) can be refined as follows.

Theorem 3.6.6. *For $p \neq 1$, define $q = p/(p-1)$. Let $a, b \in \mathbb{T}$. If $p \geq 2$, then for $f, g \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$, the inequality*

$$\begin{aligned} &\int_a^b (fg)(t)\Delta t \\ &\leq \left[\int_a^b f^p(t)\Delta t - \int_a^b \left(\left|f(s) - g^{q-1}(s)\frac{\int_a^b (fg)(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right|^p\right)\Delta s\right]^{\frac{1}{p}} \left(\int_a^b g^q(t)\Delta t\right)^{\frac{1}{q}} \end{aligned} \tag{3.6.6}$$

holds. If $0 < p < 1$ or $1 < p < 2$, the inequality (3.6.6) *holds in reverse order.*

Proof. The inequality (3.6.6) follows from Theorem 3.6.5 and Theorem 3.1.2. □

Remark 3.6.7. Since the delta integral is an isotonic linear functional, we have

$$\int_a^b g^q(t)\Delta t \geq 0 \quad \text{and} \quad \int_a^b \left(\left|f(s) - g^{q-1}(s)\frac{\int_a^b (fg)(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right|^p\right)\Delta s \geq 0,$$

so the inequality (3.6.6) represents a refinement of the classical Hölder inequality on time scales for nonnegative functions f and g.

Taking $p = q = 2$ in Theorem 3.6.6 gives the following special case of the above Hölder inequality that we can name the refinement of the Cauchy–Schwarz inequality on time scales.

Theorem 3.6.8. *Let $a, b \in \mathbb{T}$. For $f, g \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$ with $\int_a^b g^2(t)\Delta t > 0$, the inequality*

$$\int_a^b (fg)(t)\Delta t$$

$$\leq \left[\int_a^b f^2(t)\Delta t - \int_a^b \left(\left|f(s) - g(s)\frac{\int_a^b (fg)(t)\Delta t}{\int_a^b g^2(t)\Delta t}\right|^2\right)\Delta s\right]^{\frac{1}{2}} \left(\int_a^b g^2(t)\Delta t\right)^{\frac{1}{2}} \tag{3.6.7}$$

holds.

Minkowski's Inequality:

First, we quote the functional Minkowski inequality for superquadratic functions.

Theorem 3.6.9 (See [27, Theorem 14])**.** *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. If $p \geq 2$, then for all nonnegative functions f, g on E such that $(f+g)^p, f^p, g^p \in L$ and $A(f+g)^p > 0$, the inequality*

$$A^{\frac{1}{p}}\left((f+g)^p\right) \leq \left(A\left(f^p\right) - A\left(\left|f - (f+g)\frac{A(f(f+g)^{p-1})}{A(f+g)^p}\right|^p\right)\right)^{\frac{1}{p}} + \left(A(g^p) - A\left(\left|g - (f+g)\frac{A(g(f+g)^{p-1})}{A(f+g)^p}\right|^p\right)\right)^{\frac{1}{p}}$$

holds.

Now, Minkowski's inequality on time scales (see [8] and [41, Theorem 6.16]) can be refined as follows.

Theorem 3.6.10. *Let $a, b \in \mathbb{T}$ and $p \geq 2$. For $f, g \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, [0, \infty))$ with $\int_a^b (f(s)+g(s))^p \Delta s > 0$, the inequality*

$$\left(\int_a^b (f(t)+g(t))^p \,\Delta t\right)^{\frac{1}{p}} \tag{3.6.8}$$

$$\leq \left(\int_a^b f^p(t)\Delta t - \int_a^b \left|f(t) - (f(t)+g(t))\,\frac{\int_a^b f(s)\,(f(s)+g(s))^{p-1}\,\Delta s}{\int_a^b (f(s)+g(s))^p \Delta s}\right|^p \Delta t\right)^{\frac{1}{p}}$$

$$+ \left(\int_a^b g^p(t)\Delta t - \int_a^b \left|g(t) - (f(t)+g(t))\frac{\int_a^b g(s)(f(s)+g(s))^{p-1}\Delta s}{\int_a^b (f(s)+g(s))^p \Delta s}\right|^p \Delta t\right)^{\frac{1}{p}}$$

is valid.

Proof. The inequality (3.6.8) follows directly from Theorem 3.6.9 and Theorem 3.1.2. □

Remark 3.6.11. If the functions f and g in Theorem 3.6.10 are nonnegative, then inequality (3.6.8) represents a refinement of Minkowski's inequality on time scales as established in [8, Theorem 3.3].

Jensen–Mercer inequality:

A variant of Jensen's inequality of Mercer's type for superquadratic functions and isotonic linear functionals is given in the following theorem.

Theorem 3.6.12 (See [2, Theorem 2.3]). *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Assume $\Psi : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, and let $0 \le m < M < \infty$. If $f \in L$ is such that $m \le f(t) \le M$ for all $t \in E$ and such that*

$$\Psi(f), \Psi(m + M - f), (M - f)\Psi(f - m), (f - m)\Psi(M - f) \in L,$$

then we have

$$\begin{aligned}\Psi(m + M - A(f)) &\le \Psi(m) + \Psi(M) - A(\Psi(f)) \\ &\quad - \frac{2}{M - m} A((f - m)\Psi(M - f) + (M - f)\Psi(f - m)) - A(\Psi(|f - A(f)|)).\end{aligned}$$

If the function Ψ is subquadratic, then the above inequality is reversed.

Next, we state the time scales version of Jensen's inequality of Mercer's type for superquadratic functions and isotonic linear functionals which we will call the Jensen–Mercer inequality for superquadratic functions on time scales.

Theorem 3.6.13. *Let $a, b \in \mathbb{T}$. Assume $f \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, [m, M])$, where $0 \le m < M < \infty$, and $\Psi \in \mathrm{C}([0, \infty), \mathbb{R})$ is superquadratic. Then*

$$\begin{aligned}(b - a)\Psi\left(m + M - \frac{1}{b - a}\int_a^b f(t)\Delta t\right) \\ \le (b - a)(\Psi(m) + \Psi(M)) - \int_a^b \Psi(f(t))\Delta t - K, \quad (3.6.9)\end{aligned}$$

where

$$\begin{aligned}K = \frac{2}{M - m}\int_a^b [(f(t) - m)\Psi(M - f(t)) + (M - f(t))\Psi(f(t) - m)]\,\Delta t \\ + \int_a^b \Psi\left(\left|f(u) - \frac{1}{b - a}\int_a^b f(t)\Delta t\right|\right)\Delta u. \quad (3.6.10)\end{aligned}$$

Moreover, if Ψ is subquadratic then (3.6.9) *holds in reverse order.*

Proof. The result follows from Theorem 3.1.2 and Theorem 3.6.12. □

Remark 3.6.14. Note that if Ψ is strictly superquadratic in Theorem 3.6.13, then strict inequality in (3.6.9) holds.

Converses of Jensen's Inequality:

In the following theorem, a functional version of the converse of Jensen's inequality for superquadratic functions is recalled.

Theorem 3.6.15 (See [27, Theorem 15]). *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Let $h \in L$ be a nonnegative function. Suppose that $\Psi : [0, \infty) \to \mathbb{R}$ is a superquadratic function. Then for every $f \in L$, $f : E \to [m, M] \subseteq [0, \infty)$ such that $hf, h(\Psi \circ f) \in L$, we have*

$$A(h\Psi(f)) + \Delta_c \leq \frac{MA(h) - A(hf)}{M - m}\Psi(m) + \frac{A(hf) - mA(h)}{M - m}\Psi(M),$$

where

$$\Delta_c = \frac{1}{M - m}A((Mk - hf)\Psi(f - m \cdot 1) + (hf - mk)\Psi(M \cdot 1 - f)).$$

Now, we give a converse of Jensen's inequality for superquadratic functions on time scales.

Theorem 3.6.16. *Let $a, b \in \mathbb{T}$. Assume $f \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, [m, M])$, where $0 \leq m < M < \infty$, and $\Psi \in \mathrm{C}([0, \infty), \mathbb{R})$ is superquadratic. Then*

$$\int_a^b \Psi\left(f(t)\right)\Delta t + R \leq \frac{M(b - a) - \int_a^b f(t)\Delta t}{M - m}\Psi(m) + \frac{\int_a^b f(t)\Delta t - m(b - a)}{M - m}\Psi(M), \quad (3.6.11)$$

where

$$R = \frac{1}{M - m}\int_a^b \left[(f(t) - m)\Psi(M - f(t)) + (M - f(t))\Psi(f(t) - m)\right]\Delta t. \qquad (3.6.12)$$

Proof. Inequality (3.6.11) follows directly from Theorem 3.1.2 and Theorem 3.6.15 with $h(t) = 1$ for all $t \in [a, b)_{\mathbb{T}}$. □

Remark 3.6.17. Note that if Ψ is strictly superquadratic in Theorem 3.6.16, then strict inequality in (3.6.11) holds.

Slater's Inequality:

A functional Slater type inequality for superquadratic functions, which gives another estimate of the expression $A(\Psi(f))$, is given next.

Theorem 3.6.18 (See [27, Theorem 17]). *Let E, L and A be such that (L_1), (L_2), (A_1), (A_2) are satisfied. Suppose that $\Psi : [0, \infty) \to \mathbb{R}$ is a superquadratic function, C is as in Definition* 2.3.1*, and $h, f \in L$ are nonnegative functions such that $h\Psi(f), hC(f), hfC(f), h\Psi(|f - S \cdot 1|) \in L$. If*

$$S = \frac{A(hfC(f))}{A(hC(f))} \geq 0,$$

then

$$A(h\Psi(f)) \leq \Psi(S)A(h) - A(h\Psi(|f - S \cdot 1|)).$$

Now, we can state the Slater type inequality for superquadratic functions on time scales.

Theorem 3.6.19. *Let $a, b \in \mathbb{T}$. Assume $\Psi : [0, \infty) \to \mathbb{R}$ is a superquadratic function, C is as in Definition* 2.3.1 *and $f : [a, b)_{\mathbb{T}} \to [0, \infty)$ such that $f, \Psi, C \in \mathrm{C_{rd}}$. If C is a nonnegative function, then*

$$\int_a^b \Psi(f(t))\Delta t \leq \Psi(S)(b - a) - \int_a^b \Psi(|f(t) - S|)\Delta t, \tag{3.6.13}$$

where

$$S = \frac{\int_a^b f(t)C(f(t))\Delta t}{\int_a^b C(f(t))\Delta t}.$$

Proof. Inequality (3.6.13) follows directly from Theorem 3.1.2 and Theorem 3.6.18 with $h(t) = 1$ for all $t \in [a, b)_{\mathbb{T}}$. □

Remark 3.6.20. Weighted version of all Theorems, given in this section, also hold, i.e., we can take the weighted mean $\frac{\int_a^b h(t)f(t)\Delta t}{\int_a^b h(t)\Delta t}$ instead of $\frac{\int_a^b f(t)\Delta t}{b - a}$, where

$$h \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, [0, \infty)) \quad \text{is such that} \quad \int_a^b h(t)\Delta t > 0.$$

Chapter 4

Jensen's Functionals, their Properties and Applications

In this chapter we consider Jensen's functionals on time scales and discuss its properties and applications. Further, we define weighted generalized and power means on time scales. By applying the properties of Jensen's functionals on these means, we obtain several refinements and converses of Hölder's inequality on time scales. (See [19]).

We give all the results for Lebesgue Δ-integrals but they also hold for many other time scales integrals, such as Cauchy, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue delta, nabla, and diamond-α time scales integrals in a similar way. We use the same notations as in [42, Chapter 5].

4.1 Properties of Jensen's functionals

First we recall Jensen's inequality on time scales for Lebesgue Δ-integrals.

Theorem 4.1.1. *Assume $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex, $f : [a, b)_{\mathbb{T}} \to I$ is Δ-integrable and $p : [a, b)_{\mathbb{T}} \to \mathbb{R}$ is nonnegative Δ-integrable such that $\int_{[a,b)} p \mathrm{d}\mu_\Delta > 0$. Then*

$$\Phi\left(\frac{\int_{[a,b)} pf \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta}\right) \leq \frac{\int_{[a,b)} p(\Phi \circ f) \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta}. \tag{4.1.1}$$

Definition 4.1.2 (Jensen's functional)**.** Under the assumptions of Theorem 4.1.1 we define Jensen's functional on time scales by

$$\mathcal{J}_\Delta(\Phi, f, p) = \int_{[a,b)} p(\Phi \circ f) \mathrm{d}\mu_\Delta - \int_{[a,b)} p \mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)} pf \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta}\right). \tag{4.1.2}$$

Remark 4.1.3. By Theorem 4.1.1, the following statements are obvious. If Φ is convex, then

$$\mathcal{J}_\Delta(\Phi, f, p) \geq 0,$$

while if Φ is concave, then

$$\mathcal{J}_\Delta(\Phi, f, p) \leq 0.$$

Theorem 4.1.4. *Assume $\Phi \in \mathrm{C}(I, \mathbb{R})$ and $f : [a,b)_\mathbb{T} \to I$ is Δ-integrable. Let $p, q : [a,b)_\mathbb{T} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$ and $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. If Φ is convex, then $\mathcal{J}_\Delta(\Phi, f, \cdot)$ is superadditive, i.e.,*

$$\mathcal{J}_\Delta(\Phi, f, p+q) \geq \mathcal{J}_\Delta(\Phi, f, p) + \mathcal{J}_\Delta(\Phi, f, q), \tag{4.1.3}$$

and $\mathcal{J}_\Delta(\Phi, f, \cdot)$ is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ implies

$$\mathcal{J}_\Delta(\Phi, f, p) \geq \mathcal{J}_\Delta(\Phi, f, q). \tag{4.1.4}$$

Moreover, if Φ is concave, then $\mathcal{J}_\Delta(\Phi, f, \cdot)$ is subadditive and decreasing, i.e., (4.1.3) and (4.1.4) hold in reverse order.

Proof. Let Φ be convex. Because the time scales integral is linear, it follows from Definition 4.1.2 that

$$\begin{aligned}
&\mathcal{J}_\Delta(\Phi, f, p+q) \\
&= \int_{[a,b)} (p+q)(\Phi \circ f)\mathrm{d}\mu_\Delta - \int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)}(p+q)f\mathrm{d}\mu_\Delta}{\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta}\right) \\
&= \int_{[a,b)} (p+q)(\Phi \circ f)\mathrm{d}\mu_\Delta - \int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta \times \\
&\quad \times \Phi\left(\frac{\int_{[a,b)} p\mathrm{d}\mu_\Delta}{\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta}\frac{\int_{[a,b)} pf\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} + \frac{\int_{[a,b)} q\mathrm{d}\mu_\Delta}{\int_{[a,b)}(p+q)\mathrm{d}\mu_\Delta}\frac{\int_{[a,b)} qf\mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta}\right) \\
&\geq \int_{[a,b)} p(\Phi \circ f)\mathrm{d}\mu_\Delta + \int_{[a,b)} q(\Phi \circ f)\mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)} pf\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right) \\
&\quad - \int_{[a,b)} q\mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)} qf\mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta}\right) \\
&= \mathcal{J}_\Delta(\Phi, f, p) + \mathcal{J}_\Delta(\Phi, f, q).
\end{aligned}$$

If $p \geq q$, we have $p - q \geq 0$. Now, because Jensen's functional is superadditive (see above) and nonnegative, we have

$$\mathcal{J}_\Delta(\Phi, f, p) = \mathcal{J}_\Delta(\Phi, f, q + p - q)$$

$$\geq \mathcal{J}_\Delta(\Phi, f, q) + \mathcal{J}_\Delta(\Phi, f, p-q)$$
$$\geq \mathcal{J}_\Delta(\Phi, f, q).$$

On the other hand, if Φ is concave, then the reversed inequalities of (4.1.3) and (4.1.4) can be obtained in a similar way. □

Superadditivity (subadditivity) and monotonicity of Jensen's functional are very important properties, considering the numerous applications of the associated inequality. Regarding monotonicity property, in the following corollaries we give some consequences of Theorem 4.1.4.

Corollary 4.1.5. *Let Φ, f, p, q satisfy the hypotheses of Theorem 4.1.4. Further, suppose there exist nonnegative constants m and M such that*

$$Mq(t) \geq p(t) \geq mq(t) \quad \textit{for all} \quad t \in [a,b)_{\mathbb{T}}$$

and

$$M \int_{[a,b)} q \mathrm{d}\mu_\Delta > \int_{[a,b)} p \mathrm{d}\mu_\Delta > m \int_{[a,b)} q \mathrm{d}\mu_\Delta.$$

If Φ is convex, then

$$M\mathcal{J}_\Delta(\Phi, f, q) \geq \mathcal{J}_\Delta(\Phi, f, p) \geq m\mathcal{J}_\Delta(\Phi, f, q), \tag{4.1.5}$$

while if Φ is concave, then the inequalities in (4.1.5) *hold in reverse order.*

Proof. By using Definition 4.1.2, we have

$$\mathcal{J}_\Delta(\Phi, f, mq) = m\mathcal{J}_\Delta(\Phi, f, q)$$

and

$$\mathcal{J}_\Delta(\Phi, f, Mq) = M\mathcal{J}_\Delta(\Phi, f, q).$$

Now the result follows from the second property of Theorem 4.1.4. □

Corollary 4.1.6. *Let Φ, f, p satisfy the hypotheses of Theorem 4.1.4. Further, assume that p attains its minimum value and its maximum value on its domain. If Φ is convex, then*

$$\left[\max_{t\in[a,b)_{\mathbb{T}}} p(t)\right] \mathfrak{J}_\Delta(\Phi, f) \geq \mathcal{J}_\Delta(\Phi, f, p) \geq \left[\min_{t\in[a,b)_{\mathbb{T}}} p(t)\right] \mathfrak{J}_\Delta(\Phi, f), \tag{4.1.6}$$

where

$$\mathfrak{J}_\Delta(\Phi, f) = \int_{[a,b)} (\Phi \circ f) \mathrm{d}\mu_\Delta - (b-a)\Phi\left(\frac{\int_{[a,b)} f \mathrm{d}\mu_\Delta}{b-a}\right).$$

Moreover, if Φ is concave, then the inequalities in (4.1.6) *hold in reverse order.*

Proof. Let p attain its minimum value $\underline{p}$ and its maximum value $\overline{p}$ on its domain $[a,b)_{\mathbb{T}}$. Then

$$\overline{p} = \max_{t\in[a,b)_{\mathbb{T}}} p(t) \geq p(x) \geq \min_{t\in[a,b)_{\mathbb{T}}} p(t) = \underline{p}.$$

By Definition 4.1.2, we have

$$\mathcal{J}_{\Delta}(\Phi, f, \overline{p}) = \overline{p}\mathfrak{J}_{\Delta}(\Phi, f)$$

and

$$\mathcal{J}_{\Delta}(\Phi, f, \underline{p}) = \underline{p}\mathfrak{J}_{\Delta}(\Phi, f).$$

Now the result follows from the second property of Theorem 4.1.4. □

Remark 4.1.7. The first inequality in (4.1.6) gives a converse of Jensen's inequality on time scales and the second one gives a refinement of the observed inequality.

Example 4.1.8. Let us take the discrete form of Jensen's functional (4.1.2). For this, let $\mathbb{T} = \mathbb{Z}$, $n \in \mathbb{N}$, $a = 1$, $b = n+1$, and $f(i) = x_i$, $p(i) = p_i$ for $i \in [a,b)_{\mathbb{T}} = \{1, 2, \ldots, n\}$. Then (4.1.2) becomes

$$\mathcal{J}_n(\Phi, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi(x_i) - P_n \Phi\left(\frac{\sum\limits_{i=1}^{n} p_i x_i}{P_n}\right), \tag{4.1.7}$$

where

$$\mathbf{x} = (x_1, \ldots, x_n) \in I^n, \quad \mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n, \quad \text{and} \quad P_n = \sum_{i=1}^{n} p_i. \tag{4.1.8}$$

Under these notations, (4.1.6) takes the form

$$\max_{1\leq i\leq n} \{p_i\} \, \mathfrak{J}_n(\Phi, \mathbf{x}) \geq \mathcal{J}_n(\Phi, \mathbf{x}, \mathbf{p}) \geq \min_{1\leq i\leq n} \{p_i\} \, \mathfrak{J}_n(\Phi, \mathbf{x}), \tag{4.1.9}$$

where

$$\mathfrak{J}_n(\Phi, \mathbf{x}) = \sum_{i=1}^{n} \Phi(x_i) - n\Phi\left(\frac{\sum\limits_{i=1}^{n} x_i}{n}\right).$$

In addition to the above notations, let $q(i) = q_i > 0$ for $i \in [a,b)_{\mathbb{T}} = \{1, 2, \ldots, n\}$ and put $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. Using

$$m = \min_{1\leq i\leq n} \left\{\frac{p_i}{q_i}\right\} \quad \text{and} \quad M = \max_{1\leq i\leq n} \left\{\frac{p_i}{q_i}\right\}$$

in Corollary 4.1.5, (4.1.5) becomes

$$\max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\mathcal{J}_\Delta(\Phi,\mathbf{x},\mathbf{q})\ge\mathcal{J}_\Delta(\Phi,\mathbf{x},\mathbf{p})\ge\min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\mathcal{J}_\Delta(\Phi,\mathbf{x},\mathbf{q})\ge 0. \tag{4.1.10}$$

Dragomir et al., [49], investigated the properties of discrete Jensen's functional (4.1.7) concerning superadditivity and monotonicity property of discrete Jensen's functional (see also [82]). In [50], Dragomir investigated boundedness of normalized Jensen's functional, i.e., functional (4.1.7) satisfying $\sum_{i=1}^{n} p_i = 1$. He obtained the lower and upper bound for normalized functional given in (4.1.10).

Example 4.1.9. Suppose $\mathbb{T}=\mathbb{R}$ and $a,b\in\mathbb{R}$. Then Jensen's functional (4.1.2) becomes

$$\int_{[a,b)} h(t)\Phi(f(t))\mathrm{d}\mu(t)-\int_{[a,b)} h(t)\mathrm{d}\mu(t)\Phi\left(\frac{\int_{[a,b)} h(t)f(t)\mathrm{d}\mu(t)}{\int_{[a,b)} h(t)\mathrm{d}\mu(t)}\right).$$

4.2 Applications to weighted generalized means

Definition 4.2.1 (Weighted generalized mean)**.** Assume $\chi\in\mathrm{C}(I,\mathbb{R})$ is strictly monotone and $f:[a,b)_\mathbb{T}\to I$ is Δ-integrable. Let $p:[a,b)_\mathbb{T}\to\mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta>0$. Then we define the weighted generalized mean on time scales by

$$\mathcal{M}_\Delta(\chi,f,p)=\chi^{-1}\left(\frac{\int_{[a,b)} p\chi\circ f\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right). \tag{4.2.1}$$

Theorem 4.2.2. *Assume $\chi,\psi\in\mathrm{C}(I,\mathbb{R})$ are strictly monotone and $f:[a,b)_\mathbb{T}\to I$ is Δ-integrable. Let $p,q:[a,b)_\mathbb{T}\to\mathbb{R}$ be nonnegative and Δ-integrable such that the functional*

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta\left[\chi\left(\mathcal{M}_\Delta(\chi,f,p)\right)-\chi\left(\mathcal{M}_\Delta(\psi,f,p)\right)\right] \tag{4.2.2}$$

is well defined. If $\chi\circ\psi^{-1}$ is convex, then (4.2.2) *is superadditive, i.e.,*

$$\begin{aligned}\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta&\left[\chi\left(\mathcal{M}_\Delta(\chi,f,p+q)\right)-\chi\left(\mathcal{M}_\Delta(\psi,f,p+q)\right)\right]\\ &\ge\int_{[a,b)} p\mathrm{d}\mu_\Delta\left[\chi\left(\mathcal{M}_\Delta(\chi,f,p)\right)-\chi\left(\mathcal{M}_\Delta(\psi,f,p)\right)\right]\\ &\quad+\int_{[a,b)} q\mathrm{d}\mu_\Delta\left[\chi\left(\mathcal{M}_\Delta(\chi,f,q)\right)-\chi\left(\mathcal{M}_\Delta(\psi,f,q)\right)\right],\end{aligned} \tag{4.2.3}$$

and (4.2.2) *is increasing, i.e.,* $p \geq q$ *with* $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ *implies*

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathcal{M}_\Delta(\chi, f, p)\right) - \chi\left(\mathcal{M}_\Delta(\psi, f, p)\right)\right]$$
$$\geq \int_{[a,b)} q\mathrm{d}\mu_\Delta \left[\chi\left(\mathcal{M}_\Delta(\chi, f, q)\right) - \chi\left(\mathcal{M}_\Delta(\psi, f, q)\right)\right]. \quad (4.2.4)$$

Moreover, if $\chi \circ \psi^{-1}$ *is concave, then* (4.2.2) *is subadditive and decreasing, i.e.,* (4.2.3) *and* (4.2.4) *hold in reverse order.*

Proof. The functional defined in (4.2.2) is obtained by replacing Φ with $\chi \circ \psi^{-1}$ and f with $\psi \circ f$ in Jensen's functional (4.1.2), i.e.,

$$\begin{aligned}
&\mathcal{J}_\Delta(\chi \circ \psi^{-1}, \psi \circ f, p)\\
&= \int_{[a,b)} p(\chi \circ \psi^{-1} \circ \psi \circ f)\mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \left(\chi \circ \psi^{-1}\left(\frac{\int_{[a,b)} p(\psi \circ f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\right)\\
&= \int_{[a,b)} p(\chi \circ f)\mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \chi\left(\mathcal{M}_\Delta(\psi, f, p)\right)\\
&= \int_{[a,b)} p\mathrm{d}\mu_\Delta \chi\left(\mathcal{M}_\Delta(\chi, f, p)\right) - \int_{[a,b)} p\mathrm{d}\mu_\Delta \chi\left(\mathcal{M}_\Delta(\psi, f, p)\right)\\
&= \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathcal{M}_\Delta(\chi, f, p)\right) - \chi\left(\mathcal{M}_\Delta(\psi, f, p)\right)\right].
\end{aligned}$$

Now, all claims follow immediately from Theorem 4.1.4. □

Corollary 4.2.3. *Let* f, p, χ, ψ *satisfy the hypotheses of Theorem* 4.2.2*. Further, assume that* p *attains its minimum value and its maximum value on its domain. If* $\chi \circ \psi^{-1}$ *is convex, then*

$$\left[\max_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi(\mathfrak{M}_\Delta(\chi, f)) - \chi(\mathfrak{M}_\Delta(\psi, f))\right]$$
$$\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left[\chi\left(\mathcal{M}_\Delta(\chi, f, p)\right) - \chi\left(\mathcal{M}_\Delta(\psi, f, p)\right)\right]$$
$$\geq \left[\min_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi(\mathfrak{M}_\Delta(\chi, f)) - \chi(\mathfrak{M}_\Delta(\psi, f))\right], \quad (4.2.5)$$

where

$$\mathfrak{M}_\Delta(\eta, f) = \eta^{-1}\left(\frac{\int_{[a,b)}(\eta \circ f)\mathrm{d}\mu_\Delta}{b-a}\right), \quad \eta \in \{\chi, \psi\}.$$

Moreover, if $\chi \circ \psi^{-1}$ *is concave, then the inequalities in* (4.2.5) *hold in reverse order.*

Proof. The proof is omitted as it is similar to the proof of Corollary 4.1.6. □

4.3 Applications to weighted generalized power means

Definition 4.3.1 (Weighted generalized power mean)**.** Let $r \in \mathbb{R}$. Assume $f : [a, b)_{\mathbb{T}} \to I$ is positive and Δ-integrable. Let $p : [a, b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p \mathrm{d}\mu_\Delta > 0$. Then we define the weighted generalized power mean on time scales by

$$\mathcal{M}_\Delta^{[r]}(f,p) = \begin{cases} \left(\frac{\int_{[a,b)} p f^r \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta} \right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\frac{\int_{[a,b)} p \ln(f) \mathrm{d}\mu_\Delta}{\int_{[a,b)} p \mathrm{d}\mu_\Delta} \right), & r = 0. \end{cases} \tag{4.3.1}$$

Remark 4.3.2. The weighted generalized power mean defined in (4.3.1) follows from the weighted generalized mean defined in (4.2.1) by taking $\chi(x) = x^r$ $(x > 0)$ in the weighted generalized mean.

Theorem 4.3.3. *Let $r, s \in \mathbb{R}$ with $r \neq 0$. Assume $f : [a, b)_{\mathbb{T}} \to I$ is positive and Δ-integrable. Let $p, q : [a, b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that the functional*

$$\int_{[a,b)} p \mathrm{d}\mu_\Delta \left\{ \left[\mathcal{M}_\Delta^{[s]}(f,p) \right]^s - \left[\mathcal{M}_\Delta^{[r]}(f,p) \right]^s \right\} \tag{4.3.2}$$

is well defined. If $\min\{0, r\} > s > \max\{0, r\}$*, then* (4.3.2) *is superadditive (also if $r = 0$), i.e.,*

$$\begin{aligned} \int_{[a,b)} (p+q) \mathrm{d}\mu_\Delta & \left\{ \left[\mathcal{M}_\Delta^{[s]}(f,p+q) \right]^s - \left[\mathcal{M}_\Delta^{[r]}(f,p+q) \right]^s \right\} \\ & \geq \int_{[a,b)} p \mathrm{d}\mu_\Delta \left\{ \left[\mathcal{M}_\Delta^{[s]}(f,p) \right]^s - \left[\mathcal{M}_\Delta^{[r]}(f,p) \right]^s \right\} \\ & \quad + \int_{[a,b)} q \mathrm{d}\mu_\Delta \left\{ \left[\mathcal{M}_\Delta^{[s]}(f,q) \right]^s - \left[\mathcal{M}_\Delta^{[r]}(f,q) \right]^s \right\}, \end{aligned} \tag{4.3.3}$$

and (4.3.2) *is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p \mathrm{d}\mu_\Delta > \int_{[a,b)} q \mathrm{d}\mu_\Delta$ implies*

$$\begin{aligned} \int_{[a,b)} p \mathrm{d}\mu_\Delta & \left\{ \left[\mathcal{M}_\Delta^{[s]}(f,p) \right]^s - \left[\mathcal{M}_\Delta^{[r]}(f,p) \right]^s \right\} \\ & \geq \int_{[a,b)} q \mathrm{d}\mu_\Delta \left\{ \left[\mathcal{M}_\Delta^{[s]}(f,q) \right]^s - \left[\mathcal{M}_\Delta^{[r]}(f,q) \right]^s \right\}. \end{aligned} \tag{4.3.4}$$

Moreover, if $r > s > 0$ or $0 > s > r$, then (4.3.2) *is subadditive and decreasing, i.e.,* (4.3.3) *and* (4.3.4) *hold in reverse order.*

Proof. If $r \neq 0$, then let $\chi(x) = x^s$ and $\psi(x) = x^r$ $(x > 0)$ in Theorem 4.2.2. Then $(\chi \circ \psi^{-1})(x) = x^{\frac{s}{r}}$ and therefore

$$(\chi \circ \psi^{-1})''(x) = \frac{s(s-r)}{r^2} x^{\frac{s}{r}-2}.$$

Thus $\chi \circ \psi^{-1}$ is convex if $\min\{0, r\} > s > \max\{0, r\}$ and concave if $r > s > 0$ or $0 > s > r$. If, however, $r = 0$, then let $\chi(x) = x^s$ and $\psi(x) = \ln(x)$ $(x > 0)$ in Theorem 4.2.2. Then $(\chi \circ \psi^{-1})(x) = e^{sx}$. Thus $\chi \circ \psi^{-1}$ is convex for $s \neq 0$. In either case the result follows now immediately from Theorem 4.2.2. □

Corollary 4.3.4. *Let* r, s, f, p *satisfy the hypotheses of Theorem* 4.3.3. *Further, assume that* p *attains its minimum value and its maximum value on its domain. If* $\min\{0, r\} > s > \max\{0, r\}$*, then*

$$\begin{aligned}\left[\max_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)&\left\{\left[\mathfrak{M}_\Delta^{[s]}(f)\right]^s - \left[\mathfrak{M}_\Delta^{[r]}(f)\right]^s\right\}\\ &\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{\left[\mathcal{M}_\Delta^{[s]}(f,p)\right]^s - \left[\mathcal{M}_\Delta^{[r]}(f,p)\right]^s\right\}\\ &\geq \left[\min_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\left\{\left[\mathfrak{M}_\Delta^{[s]}(f)\right]^s - \left[\mathfrak{M}_\Delta^{[r]}(f)\right]^s\right\},\end{aligned} \quad (4.3.5)$$

where

$$\mathfrak{M}_\Delta^{[u]}(f) = \begin{cases} \left(\frac{\int_{[a,b)} f^u \mathrm{d}\mu_\Delta}{b-a}\right)^{\frac{1}{u}}, & u \in \mathbb{R}\setminus\{0\},\\ \exp\left(\frac{\int_{[a,b)} \ln(f)\mathrm{d}\mu_\Delta}{b-a}\right), & u = 0.\end{cases} \quad (4.3.6)$$

Moreover, if $r > s > 0$ *or* $0 > s > r$*, then the inequalities in* (4.3.5) *hold in reverse order.*

Proof. The proof is omitted as it is similar to the proof of Corollary 4.1.6 followed by Theorem 4.3.3. □

Example 4.3.5. From the discrete form of Corollary 4.3.4, i.e., by using $\mathbb{T} = \mathbb{Z}$, we get a refinement and a converse of the arithmetic-geometric mean inequality. Using the notation as introduced in Example 4.1.8, let $x_i > 0$ for all $i \in [a, b)_{\mathbb{T}}$ and $s = 1$, $r = 0$. Then (4.3.5) becomes

$$\begin{aligned} n \max_{1\leq i\leq n}\{p_i\}\left[A_n(\mathbf{x}) - G_n(\mathbf{x})\right] &\geq P_n[\mathcal{M}_n^{[1]}(\mathbf{x},\mathbf{p}) - \mathcal{M}_n^{[0]}(\mathbf{x},\mathbf{p})]\\ &\geq n \min_{1\leq i\leq n}\{p_i\}\left[A_n(\mathbf{x}) - G_n(\mathbf{x})\right] \geq 0, \end{aligned} \quad (4.3.7)$$

where

$$\mathcal{M}_n^{[r]}(\mathbf{x},\mathbf{p}) = \begin{cases} \left(\frac{\sum_{i=1}^{n} p_i x_i^r}{P_n}\right)^{\frac{1}{r}}, & r \in \mathbb{R}\setminus\{0\}, \\ \left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0, \end{cases}$$

$$A_n(\mathbf{x}) = \frac{\sum_{i=1}^{n} x_i}{n}, \quad \text{and} \quad G_n(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

The first inequality in (4.3.7) gives a converse and the second one gives a refinement of the arithmetic-geometric mean inequality of $\mathcal{M}_n^{[1]}(\mathbf{x},\mathbf{p})$ and $\mathcal{M}_n^{[0]}(\mathbf{x},\mathbf{p})$. Some variants of inequalities in (4.3.7) were recently studied in paper [13] of Aldaz (see also [9, 10, 11, 14, 12]).

Theorem 4.3.6. *Let r, f, p, q satisfy the hypotheses of Theorem 4.3.3. Suppose that the functional*

$$\int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathcal{M}_\Delta^{[r]}(f,p)\right)\right\} \tag{4.3.8}$$

is well defined. If $r < 0$, then (4.3.8) is superadditive, i.e.,

$$\begin{aligned} \int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta &\left\{ \frac{\int_{[a,b)} (p+q)\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} (p+q)\mathrm{d}\mu_\Delta} - \ln\left(\mathcal{M}_\Delta^{[r]}(f,p+q)\right)\right\} \\ &\geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathcal{M}_\Delta^{[r]}(f,p)\right)\right\} \\ &\quad + \int_{[a,b)} q\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} q\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} - \ln\left(\mathcal{M}_\Delta^{[r]}(f,q)\right)\right\}, \end{aligned} \tag{4.3.9}$$

and (4.3.8) is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ implies

$$\begin{aligned} \int_{[a,b)} p\mathrm{d}\mu_\Delta &\left\{ \frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathcal{M}_\Delta^{[r]}(f,p)\right)\right\} \\ &\geq \int_{[a,b)} q\mathrm{d}\mu_\Delta \left\{ \frac{\int_{[a,b)} q\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} q\mathrm{d}\mu_\Delta} - \ln\left(\mathcal{M}_\Delta^{[r]}(f,q)\right)\right\}. \end{aligned} \tag{4.3.10}$$

Moreover, if $r > 0$, then (4.3.8) is subadditive and decreasing, i.e., (4.3.9) and (4.3.10) hold in reverse order.

Proof. Let $\chi(x) = \ln(x)$ and $\psi(x) = x^r$ in Theorem 4.2.2. Then $\chi \circ \psi^{-1}(x) = \frac{1}{r}\ln(x)$. Thus $\chi \circ \psi^{-1}$ is convex if $r < 0$ and concave if $r > 0$. Now the rest of the proof follows immediately from Theorem 4.2.2. □

Corollary 4.3.7. *Let r, f, p satisfy the hypotheses of Theorem 4.3.3. Further, assume that p attains its minimum value and its maximum value on its domain. If $r < 0$, then*

$$\begin{aligned}
&\left[\max_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\left\{\frac{\int_{[a,b)} \ln(f)\mathrm{d}\mu_\Delta}{b-a} - \ln\left(\mathfrak{M}_\Delta^{[r]}(f)\right)\right\} \\
&\quad \geq \int_{[a,b)} p\mathrm{d}\mu_\Delta \left\{\frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \ln\left(\mathcal{M}_\Delta^{[r]}(f,p)\right)\right\} \\
&\quad \geq \left[\min_{t\in[a,b)_{\mathbb{T}}} p(t)\right](b-a)\left\{\frac{\int_{[a,b)} \ln(f)\mathrm{d}\mu_\Delta}{b-a} - \ln\left(\mathfrak{M}_\Delta^{[r]}(f)\right)\right\}, \quad (4.3.11)
\end{aligned}$$

where $\mathfrak{M}_\Delta^{[r]}(f)$ is defined in (4.3.6). *Moreover, if $r > 0$, then the inequalities in* (4.3.11) *hold in reverse order.*

Proof. The proof is omitted as it is similar to the proof of Corollary 4.1.6 followed by Theorem 4.3.6. □

Example 4.3.8. Again we consider $\mathbb{T} = \mathbb{Z}$. Using the notation as introduced in Example 4.1.8, the term $\frac{\int_{[a,b)} p\ln(f)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}$ takes the form

$$\frac{\sum_{i=1}^{n} p_i \ln(x_i)}{\sum_{i=1}^{n} p_i} = \ln\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{\frac{1}{P_n}} = \ln\left(\mathcal{M}_n^{[0]}(\mathbf{x},\mathbf{p})\right),$$

and (4.3.11) becomes

$$\left[\frac{G_n(\mathbf{x})}{A_n(\mathbf{x})}\right]^{n \max\limits_{1\le i\le n}\{p_i\}} \geq \left[\frac{\mathcal{M}_n^{[0]}(\mathbf{x},\mathbf{p})}{\mathcal{M}_n^{[1]}(\mathbf{x},\mathbf{p})}\right]^{P_n} \geq \left[\frac{G_n(\mathbf{x})}{A_n(\mathbf{x})}\right]^{n \min\limits_{1\le i\le n}\{p_i\}}. \quad (4.3.12)$$

The inequalities in (4.3.12) provide a refinement and a converse of the arithmetic-geometric mean inequality in quotient form.

Example 4.3.9. The relations (4.3.7) and (4.3.12) also yield refinements and converses of Young's inequality. To see this, consider again $\mathbb{T} = \mathbb{Z}$. Using the notation as introduced in Example 4.1.8, define

$$\mathbf{x}^{\mathbf{p}} = (x_1^{p_1}, x_2^{p_2}, \ldots, x_n^{p_n}) \quad \text{and} \quad \mathbf{p}^{-1} = \left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right),$$

where $\mathbf{x}$ and $\mathbf{p}$ are positive n-tuples such that $\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then, (4.3.7) and (4.3.12) become

$$n \max_{1\leq i\leq n} \left\{\frac{1}{p_i}\right\} [A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})] \geq \mathcal{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) - \mathcal{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1})$$
$$\geq n \min_{1\leq i\leq n} \left\{\frac{1}{p_i}\right\} [A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})], \quad (4.3.13)$$

and

$$\left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})}\right]^{n \max_{1\leq i\leq n}\left\{\frac{1}{p_i}\right\}} \geq \frac{\mathcal{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1})}{\mathcal{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1})} \geq \left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})}\right]^{n \min_{1\leq i\leq n}\left\{\frac{1}{p_i}\right\}}. \quad (4.3.14)$$

The inequalities in (4.3.13) and (4.3.14) provide the refinements and converses of Young's inequality in difference and quotient form.

4.4 Improvements of Hölder's inequality

Let us recall Hölder's inequality for Lebesgue Δ-integrals.

Theorem 4.4.1. *For $p \neq 1$, define $q = \frac{p}{p-1}$. Let w, f, g be nonnegative functions such that wf^p, wg^q, wfg are Δ-integrable on $[a,b)_{\mathbb{T}}$. If $p > 1$, then*

$$\int_{[a,b)} wfg \mathrm{d}\mu_\Delta \leq \left(\int_{[a,b)} wf^p \mathrm{d}\mu_\Delta\right)^{\frac{1}{p}} \left(\int_{[a,b)} wg^q \mathrm{d}\mu_\Delta\right)^{\frac{1}{q}}. \quad (4.4.1)$$

If $0 < p < 1$ and $\int_{[a,b)} wg^q \mathrm{d}\mu_\Delta > 0$, or if $p < 0$ and $\int_{[a,b)} wf^p \mathrm{d}\mu_\Delta > 0$, then (4.4.1) is reversed.

Let $n \in \mathbb{N}$ and let $f_i : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be Δ-integrable for all $i \in \{1, 2, \ldots, n\}$. Assume $p_i > 1$ for all $i \in \{1, 2, \ldots, n\}$ are conjugate exponents, i.e., $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, and $\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}$ is Δ-integrable on $[a,b)_{\mathbb{T}}$. Hölder's inequality on time scales (Theorem 4.4.1) asserts that

$$\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \mathrm{d}\mu_\Delta \leq \prod_{i=1}^{n} \left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}.$$

It is well known from the literature (see [82, 93]) that Hölder's inequality can easily be obtained from Young's inequality. Therefore, it is natural to expect that relations (4.3.13) and (4.3.14) also provide refinements and conversions of Hölder's inequality.

The first in a series of results refers to relation (4.3.13), i.e., refinement and conversion of Hölder's inequality in difference form.

Theorem 4.4.2. *Let $p_i > 1$, $i \in \{1,2,\dots,n\}$, be conjugate exponents. Let f_i, $i \in \{1,2,\dots,n\}$, be nonnegative Δ-integrable functions such that $\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}$ and $\prod_{i=1}^{n} f_i^{\frac{1}{n}}$ are nonnegative and Δ-integrable. Then the following inequalities hold:*

$$\begin{aligned}
& n \max_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}\right. \\
&\qquad\qquad \left. -\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}-\frac{1}{n}}\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}}\mathrm{d}\mu_\Delta\right)\right] \\
\ge & \prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}} - \left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\mathrm{d}\mu_\Delta\right) \\
\ge & n \min_{1\le i\le n}\left\{\frac{1}{p_i}\right\}\left[\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}\right. \\
&\qquad\qquad \left. -\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}-\frac{1}{n}}\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}}\mathrm{d}\mu_\Delta\right)\right].
\end{aligned}$$

Proof. Let $x_i = \left[\frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right]^{\frac{1}{p_i}}$, $i \in \{1,2,\dots,n\}$, in Example 4.3.9. Then the expressions in (4.3.13) become

$$\mathcal{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - \mathcal{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) = \sum_{i=1}^{n}\frac{f_i}{p_i\int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \prod_{i=1}^{n}\frac{f_i^{\frac{1}{p_i}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}}$$

and

$$A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}}) = \frac{1}{n}\sum_{i=1}^{n}\frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \prod_{i=1}^{n}\frac{f_i^{\frac{1}{n}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{n}}}.$$

Now, by applying the Δ-integral to the last two equations, we get

$$\begin{aligned}
& \int_{[a,b)}\left[\mathcal{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1}) - \mathcal{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}},\mathbf{p}^{-1})\right]\mathrm{d}\mu_\Delta \\
&= \sum_{i=1}^{n}\frac{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{p_i\int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \frac{\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right)\mathrm{d}\mu_\Delta}{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}} \\
&= 1 - \frac{\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right)\mathrm{d}\mu_\Delta}{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}}
\end{aligned}$$

and

$$\int_{[a,b)} [A_n(\mathbf{x}^{\mathbf{p}}) - G_n(\mathbf{x}^{\mathbf{p}})]\, \mathrm{d}\mu_\Delta = \frac{1}{n}\sum_{i=1}^{n} \frac{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta} - \frac{\int_{[a,b)} \left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right) \mathrm{d}\mu_\Delta}{\prod_{i=1}^{n} \left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{n}}}$$

$$= 1 - \frac{\int_{[a,b)} \left(\prod_{i=1}^{n} f_i^{\frac{1}{n}}\right) \mathrm{d}\mu_\Delta}{\prod_{i=1}^{n} \left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{n}}}.$$

By applying the Δ-integral to the series of inequalities in (4.3.13), we obtain the required inequalities. □

Remark 4.4.3. The first inequality in Theorem 4.4.2 gives a converse and the second one gives a refinement of Hölder's inequality on time scales,

Now we give refinement and conversion of Hölder's inequality in quotient form, deduced from relation (4.3.14).

Theorem 4.4.4. *Under the same assumption as in Theorem* 4.4.2*, the following inequalities hold:*

$$\left[\frac{n^n}{\prod_{i=1}^{n} \int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right]^{\min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \times$$

$$\times \int_{[a,b)} \left[\sum_{i=1}^{n} \frac{f_i}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right] \left[\frac{\prod_{i=1}^{n} f_i^{\frac{1}{n}}}{\sum_{i=1}^{n} \frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}}\right]^{n \min\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \mathrm{d}\mu_\Delta$$

$$\ge \frac{\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \mathrm{d}\mu_\Delta}{\prod_{i=1}^{n} \left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta\right)^{\frac{1}{p_i}}}$$

$$\ge \left[\frac{n^n}{\prod_{i=1}^{n} \int_{[a,b)} f_i \mathrm{d}\mu_\Delta}\right]^{\max\limits_{1\le i\le n}\left\{\frac{1}{p_i}\right\}} \times$$

$$\times \int_{[a,b)} \left[\sum_{i=1}^{n} \frac{f_i}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta} \right] \left[\frac{\prod_{i=1}^{n} f_i^{\frac{1}{n}}}{\sum_{i=1}^{n} \frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}} \right]^{n \max_{1 \le i \le n} \left\{ \frac{1}{p_i} \right\}} \mathrm{d}\mu_\Delta$$

provided that all expressions are well defined.

Proof. We consider relation (4.3.14) in the same settings as in Theorem 4.4.2. By inverting, (4.3.14) can be rewritten in the form

$$\mathcal{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) \left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})} \right]^{n \min_{1 \le i \le n} \left\{ \frac{1}{p_i} \right\}} \ge \mathcal{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1})$$
$$\ge \mathcal{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) \left[\frac{G_n(\mathbf{x}^{\mathbf{p}})}{A_n(\mathbf{x}^{\mathbf{p}})} \right]^{n \max_{1 \le i \le n} \left\{ \frac{1}{p_i} \right\}}. \quad (4.4.2)$$

Now, if we consider the n-tuple $\mathbf{x} = (x_1, x_2, \dots, x_n)$, where

$$x_i = \left[\frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta} \right]^{\frac{1}{p_i}} \quad \text{for all} \quad i \in \{1, 2, \dots, n\},$$

then the expressions that represent the means in (4.4.2) become

$$\mathcal{M}_n^{[1]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) = \sum_{i=1}^{n} \frac{f_i}{p_i \int_{[a,b)} f_i \mathrm{d}\mu_\Delta}, \quad \mathcal{M}_n^{[0]}(\mathbf{x}^{\mathbf{p}}, \mathbf{p}^{-1}) = \prod_{i=1}^{n} \frac{f_i^{\frac{1}{p_i}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta \right)^{\frac{1}{p_i}}}$$

and

$$A_n(\mathbf{x}^{\mathbf{p}}) = \frac{1}{n} \sum_{i=1}^{n} \frac{f_i}{\int_{[a,b)} f_i \mathrm{d}\mu_\Delta}, \quad G_n(\mathbf{x}^{\mathbf{p}}) = \prod_{i=1}^{n} \frac{f_i^{\frac{1}{n}}}{\left(\int_{[a,b)} f_i \mathrm{d}\mu_\Delta \right)^{\frac{1}{n}}}.$$

Now, by taking the Δ-integral on (4.4.2) in described setting, we obtain the required inequalities. □

Remark 4.4.5. The first inequality in Theorem 4.4.4 gives a refinement and the second one gives a converse of Hölder's inequality on time scales.

Corollary 4.4.6. *Let $r, s \in \mathbb{R}$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Further, assume that f, g are positive and Δ-integrable such that f attains its minimum value and its maximum value on its domain. If $r > 1$, then*

$$\left[\max_{t\in[a,b)_{\mathbb{T}}} f(t)\right]\left[(b-a)^{\frac{1}{r}}\left(\int_{[a,b)}\frac{g}{f}\mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}-\int_{[a,b)}\left(\frac{g}{f}\right)^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right]$$
$$\geq\left(\int_{[a,b)} f\mathrm{d}\mu_\Delta\right)^{\frac{1}{r}}\left(\int_{[a,b)} g\mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}-\int_{[a,b)} f^{\frac{1}{r}}g^{\frac{1}{s}}\mathrm{d}\mu_\Delta$$
$$\geq\left[\min_{t\in[a,b)_{\mathbb{T}}} f(t)\right]\left[(b-a)^{\frac{1}{r}}\left(\int_{[a,b)}\frac{g}{f}\mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}-\int_{[a,b)}\left(\frac{g}{f}\right)^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right]. \quad (4.4.3)$$

Moreover, if $0<r<1$*, then the inequalities in* (4.4.3) *hold in reverse order.*

Proof. The result follows from Corollary 4.1.6 by replacing f with $\frac{g}{f}$, p with f, and letting $\Phi(x)=-rsx^{\frac{1}{s}}$. Then Φ is convex on $(0,\infty)$, and we have

$$\begin{aligned}\mathcal{J}_\Delta(\Phi,\frac{g}{f},f)&=\int_{[a,b)} f\Phi\left(\frac{g}{f}\right)\mathrm{d}\mu_\Delta-\int_{[a,b)} f\mathrm{d}\mu_\Delta\Phi\left(\frac{\int_{[a,b)} g\mathrm{d}\mu_\Delta}{\int_{[a,b)} f\mathrm{d}\mu_\Delta}\right)\\&=rs\left[\left(\int_{[a,b)} f\mathrm{d}\mu_\Delta\right)^{1-\frac{1}{s}}\left(\int_{[a,b)} g\mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}-\int_{[a,b)} f^{1-\frac{1}{s}}g^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right]\\&=rs\left[\left(\int_{[a,b)} f\mathrm{d}\mu_\Delta\right)^{\frac{1}{r}}\left(\int_{[a,b)} g\mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}-\int_{[a,b)} f^{\frac{1}{r}}g^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right]\end{aligned}$$

and

$$\begin{aligned}\mathfrak{J}_\Delta(\Phi,\frac{g}{f})&=\int_{[a,b)}\Phi\left(\frac{g}{f}\right)\mathrm{d}\mu_\Delta-(b-a)\Phi\left(\frac{\int_{[a,b)}\frac{g}{f}\mathrm{d}\mu_\Delta}{b-a}\right)\\&=rs\left[(b-a)^{1-\frac{1}{s}}\left(\int_{[a,b)}\frac{g}{f}\mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}-\int_{[a,b)}\left(\frac{g}{f}\right)^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right]\\&=rs\left[(b-a)^{\frac{1}{r}}\left(\int_{[a,b)}\frac{g}{f}\mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}-\int_{[a,b)}\left(\frac{g}{f}\right)^{\frac{1}{s}}\mathrm{d}\mu_\Delta\right].\end{aligned}$$

If $r>1$, then by substituting $\mathcal{J}_\Delta(\Phi,\frac{g}{f},f)$ and $\mathfrak{J}_\Delta(\Phi,\frac{g}{f})$ in (4.1.6), we get (4.4.3). If $0<r<1$, then $rs<0$, and since the expressions $\mathcal{J}_\Delta(\Phi,\frac{g}{f},f)$ and $\mathfrak{J}_\Delta(\Phi,\frac{g}{f})$ contain the factor rs, we conclude that the inequalities in (4.4.3) hold in reverse order in that case. □

Remark 4.4.7. The first inequality in (4.4.3) gives a converse and the second one gives a refinement of Hölder's inequality on time scales,

Since Hölder's inequality can directly be deduced from Jensen's inequality in the case of two functions (see [82]). That means that Corollary 4.1.6 also provide another class of refinements and conversions of Hölder's inequality.

Corollary 4.4.8. *Let $r, s \in \mathbb{R}$ such that $r > 0$ and $\frac{1}{r} + \frac{1}{s} = 1$. Further, assume that f, g are positive and Δ-integrable such that f attains its minimum value and its maximum value on its domain. Then*

$$\begin{aligned}
&\left[\max_{t\in[a,b)_{\mathbb{T}}} f(t)\right] \times \\
&\times \left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{s-1} \int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta - \left(\frac{\int_{[a,b)} f \mathrm{d}\mu_\Delta}{b-a}\right)^{s-1} \left(\int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right)^{s}\right] \\
\geq &\left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{\frac{1}{r}} \left(\int_{[a,b)} g \mathrm{d}\mu_\Delta\right)^{\frac{1}{s}}\right]^{s} - \left[\int_{[a,b)} f^{\frac{1}{r}} g^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right]^{s} \\
\geq &\left[\min_{t\in[a,b)_{\mathbb{T}}} f(t)\right] \times \\
&\times \left[\left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{s-1} \int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta - \left(\frac{\int_{[a,b)} f \mathrm{d}\mu_\Delta}{b-a}\right)^{s-1} \left(\int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right)^{s}\right].
\end{aligned}$$

Proof. In Corollary 4.1.6, replace f with $\left(\frac{g}{f}\right)^{\frac{1}{s}}$, p with f, and let $\Phi(x) = \frac{x^s}{s(s-1)}$. Then Φ is convex on $(0, \infty)$. We get

$$\begin{aligned}
&\mathcal{J}_\Delta\left(\Phi, \left(\frac{g}{f}\right)^{\frac{1}{s}}, f\right) \\
&= \int_{[a,b)} f\Phi\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right) \mathrm{d}\mu_\Delta - \int_{[a,b)} f \mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)} f^{\frac{1}{r}} g^{\frac{1}{s}} \mathrm{d}\mu_\Delta}{\int_{[a,b)} f \mathrm{d}\mu_\Delta}\right) \\
&= \frac{1}{s(s-1)}\left[\int_{[a,b)} g \mathrm{d}\mu_\Delta - \left(\int_{[a,b)} f \mathrm{d}\mu_\Delta\right)^{1-s} \left(\int_{[a,b)} f^{\frac{1}{r}} g^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right)^{s}\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{J}_\Delta\left(\Phi, \left(\frac{g}{f}\right)^{\frac{1}{s}}\right) &= \int_{[a,b)} \Phi\left(\left(\frac{g}{f}\right)^{\frac{1}{s}}\right) \mathrm{d}\mu_\Delta - (b-a)\Phi\left(\frac{\int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta}{b-a}\right) \\
&= \frac{1}{s(s-1)}\left[\int_{[a,b)} \frac{g}{f} \mathrm{d}\mu_\Delta - (b-a)^{1-s} \left(\int_{[a,b)} \left(\frac{g}{f}\right)^{\frac{1}{s}} \mathrm{d}\mu_\Delta\right)^{s}\right].
\end{aligned}$$

Now, the result follows immediately from (4.1.6). □

Remark 4.4.9. Similarly as in Chapter 3, we can apply the theory of isotonic linear functionals. The related results for isotonic linear functionals are given in [74].

Chapter 5

Jensen's Functionals for Several Variables, their Properties and Applications

In this chapter we define the Jensen functional and concerned generalized means for several variables on time scales. We derive properties of the Jensen functionals and apply them to generalized means. In this setting, we obtain generalizations, refinements, and conversions of many remarkable inequalities. (See [20]).

In case of single variable the obtained results coincide with the results given in Chapter 4. Moreover, we give all the results for Lebesgue Δ-integrals but they also hold for many other time scales integrals, such as Cauchy, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue delta, nabla, and diamond-α time scales integrals in a similar way. We use the same notations as in Chapter 4.

5.1 Jensen's inequality and Jensen's functionals

Let $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t))$ be n-tuple of functions such that $f_1, \ldots, f_n$, Δ-integrable on $[a,b)_{\mathbb{T}}$. Then $\int_{[a,b)} \mathbf{f} \mathrm{d}\mu_\Delta$ denotes the n-tuple

$$\left(\int_{[a,b)} f_1 \mathrm{d}\mu_\Delta, \ldots, \int_{[a,b)} f_n \mathrm{d}\mu_\Delta \right),$$

i.e., Δ-integral acts on each component of $\mathbf{f}$.

Theorem 5.1.1. *Assume $\Phi \in \mathrm{C}(U, \mathbb{R})$ is convex, where $U \subseteq \mathbb{R}^n$ is a closed convex set. Suppose f_i, $i \in \{1, \ldots, n\}$, are Δ-integrable on $[a,b)_{\mathbb{T}}$ such that $\mathbf{f}(t) =$*

$(f_1(t),\ldots,f_n(t)) \in U$ *for all* $t \in [a,b)_{\mathbb{T}}$. *Moreover, let* $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative and* Δ*-integrable such that* $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. *Then*

$$\Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right) \le \frac{\int_{[a,b)} p\Phi(\mathbf{f})\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}. \tag{5.1.1}$$

Proof. Since Φ is convex on $U \subset \mathbb{R}^n$, for every point $\mathbf{x}_0 \in U$ there exists a point $\lambda \in \mathbb{R}^n$ (see [93, Theorem 1.31]) such that

$$\Phi(\mathbf{x}) - \Phi(\mathbf{x}_0) \ge \langle \lambda, \mathbf{x} - \mathbf{x}_0 \rangle. \tag{5.1.2}$$

Let $\lambda = (\lambda_1,\ldots,\lambda_n)$. By (5.1.2), we get

$$\begin{aligned}
\frac{\int_{[a,b)} p\Phi(\mathbf{f})\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} - \Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right) &= \frac{\int_{[a,b)} p\left\{\Phi(\mathbf{f}) - \Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\right\}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \\
&\ge \frac{\int_{[a,b)} p\left\langle \lambda, \mathbf{f} - \frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right\rangle\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \\
&= \frac{\int_{[a,b)} p\sum_{i=1}^{n} \lambda_i\left(f_i - \frac{\int_{[a,b)} pf_i\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right)\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta} \\
&= 0,
\end{aligned}$$

and hence the proof is completed. □

Remark 5.1.2. By using the fact that time scale integral is an isotonic linear functional, Theorem 5.1.1 can also be obtained from [93, Theorem 2.6].

Definition 5.1.3 (Jensen's functional)**.** Assume $\Phi \in \mathrm{C}(U,\mathbb{R})$, where $U \subseteq \mathbb{R}^n$ is a closed convex set. Suppose f_i, $i \in \{1,\ldots,n\}$, are Δ-integrable functions on $[a,b)_{\mathbb{T}}$ such that $\mathbf{f}(t) = (f_1(t),\ldots,f_n(t)) \in U$ for all $t \in [a,b)_{\mathbb{T}}$. Moreover, let $p : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$. Then we define the Jensen functional on time scales for several variables by;

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) = \int_{[a,b)} p\Phi(\mathbf{f})\mathrm{d}\mu_\Delta - \int_{[a,b)} p\mathrm{d}\mu_\Delta \Phi\left(\frac{\int_{[a,b)} p\mathbf{f}\mathrm{d}\mu_\Delta}{\int_{[a,b)} p\mathrm{d}\mu_\Delta}\right). \tag{5.1.3}$$

Remark 5.1.4. By Theorem 5.1.1, the following statements are obvious. If Φ is continuous and convex, then

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) \ge 0.$$

While if Φ is continuous and concave, then

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) \le 0.$$

Example 5.1.5. Let $[a,b)_{\mathbb{T}} = \{1,2,\ldots,n\}$, $f_1(i) = x_{1_i},\ldots,f_n(i) = x_{n_i}$, $p(i) = p_i$, $i \in \{1,\ldots,n\}$, in (5.1.3). Then Jensen's functional (5.1.3) becomes,

$$\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) = \sum_{i=1}^{n} p_i\Phi(\mathbf{x}_i) - P_n\Phi\left(\frac{\sum\limits_{i=1}^{n} p_i\mathbf{x}_i}{P_n}\right),$$

where $\mathbf{X} = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$ with $\mathbf{x}_i = (x_{i_1},x_{i_2},\ldots,x_{i_n})$, $\mathbf{p} = (p_1,\ldots,p_n)$, and $P_n = \sum\limits_{i=1}^{n} p_i > 0$.

Example 5.1.6. If $[a,b)_{\mathbb{T}} = [a,b)$, then Jensen's functional (5.1.3) becomes

$$\int_{[a,b)} p(t)\Phi(f_1(t),f_2(t),\ldots,f_n(t))d\mu(t)$$
$$-\int_{[a,b)} p(t)\mathrm{d}\mu_\Delta\Phi\left(\frac{\int_{[a,b)} p(t)f_1(t)d\mu(t)}{\int_{[a,b)} p(t)d\mu(t)},\frac{\int_{[a,b)} p(t)f_2(t)d\mu(t)}{\int_{[a,b)} p(t)d\mu(t)},\ldots,\frac{\int_{[a,b)} p(t)f_n(t)d\mu(t)}{\int_{[a,b)} p(t)d\mu(t)}\right).$$

5.2 Properties of Jensen's functionals

In the following theorem we give our main result concerning the properties of the Jensen functional (5.1.3).

Theorem 5.2.1. *Assume $\Phi \in \mathrm{C}(U,\mathbb{R})$, where $U \subseteq \mathbb{R}^n$ is a closed convex set. Suppose f_i, $i \in \{1,\ldots,n\}$, are Δ-integrable on $[a,b)_{\mathbb{T}}$ such that $\mathbf{f}(t) = (f_1(t),\ldots,f_n(t)) \in U$ for all $t \in [a,b)_{\mathbb{T}}$. Let $p,q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative and Δ-integrable such that $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$ and $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. If Φ is convex, then $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is superadditive, i.e.,*

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p+q) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) + \mathbf{J}_\Delta(\Phi,\mathbf{f},q), \tag{5.2.1}$$

and $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is increasing, i.e., $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$ implies

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},q). \tag{5.2.2}$$

Moreover, if Φ is concave, then $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is subadditive and decreasing, i.e., (5.2.1) and (5.2.2) hold in reverse order.

Proof. We omit the proof because it is similar to the proof of Theorem 4.1.4. □

Corollary 5.2.2. *Let $\Phi,\mathbf{f},p,q$ satisfy the hypotheses of Theorem 5.2.1. Further, suppose there exist nonnegative constants m and M such that*

$$Mq(t) \geq p(t) \geq mq(t) \quad \textit{for all} \quad t \in [a,b)_{\mathbb{T}}$$

and

$$M\int_{[a,b)} q\mathrm{d}\mu_\Delta > \int_{[a,b)} p\mathrm{d}\mu_\Delta > m\int_{[a,b)} q\mathrm{d}\mu_\Delta.$$

If Φ is convex, then

$$M\mathbf{J}_\Delta(\Phi,\mathbf{f},q) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq m\mathbf{J}_\Delta(\Phi,\mathbf{f},q), \tag{5.2.3}$$

while if Φ is concave, then the inequalities in (5.2.3) *hold in reverse order.*

Proof. The proof is similar to the proof of Corollary 4.1.5. □

Corollary 5.2.3. *Let $\Phi, \mathbf{f}, p$ satisfy the hypotheses of Theorem* 5.2.1. *Further, assume that p attains its minimum value and its maximum value on its domain. If Φ is convex, then*

$$\left[\max_{t\in[a,b)_\mathbb{T}} p(t)\right]\mathbf{j}_\Delta(\Phi,\mathbf{f}) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq \left[\min_{t\in[a,b)_\mathbb{T}} p(t)\right]\mathbf{j}_\Delta(\Phi,\mathbf{f}), \tag{5.2.4}$$

where

$$\mathbf{j}_\Delta(\Phi,\mathbf{f}) = \int_{[a,b)} \Phi(\mathbf{f})d\mu_\Delta - (b-a)\Phi\left(\frac{\int_{[a,b)}\mathbf{f}d\mu_\Delta}{b-a}\right).$$

Moreover, if Φ is concave, then the inequalities in (5.2.4) *hold in reverse order.*

Proof. The proof is similar to the proof of Corollary 4.1.6. □

Example 5.2.4. Let the functional $\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p})$ be defined as in Example 5.1.5. Let $\mathbf{q} = (q_1,\ldots,q_n)$ with $q_i \geq 0$ and $\sum\limits_{i=1}^n q_i = Q_n > 0$. If Φ is convex, then Theorem 5.2.1 implies: $\mathbf{J}_n(\Phi,\mathbf{X},.)$ is superadditive, i.e.,

$$\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}+\mathbf{q}) \geq \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) + \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{q}), \tag{5.2.5}$$

and $\mathbf{J}_n(\Phi,\mathbf{X},.)$ is increasing, i.e., if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$ then

$$\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) \geq \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{q}). \tag{5.2.6}$$

Moreover, if Φ is concave, then inequalities in (5.2.5) and (5.2.6) hold in reverse order. If $\mathbf{p}$ attains minimum and maximum value on its domain, then Corollary 5.2.3 yields

$$\max_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi,\mathbf{X}) \geq \mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\}\mathbf{j}_n(\Phi,\mathbf{X}), \tag{5.2.7}$$

where

$$\mathbf{j}_n(\Phi,\mathbf{X}) = \sum_{i=1}^n \Phi(\mathbf{x}_i) - n\Phi\left(\frac{\sum\limits_{i=1}^n \mathbf{x}_i}{n}\right),$$

if Φ is convex. Further, inequalities in (5.2.7) hold in reverse order if Φ is concave.

5.3 Applications to weighted generalized means

We start this section by applying obtained results on the properties of Jensen's functional to weighted generalized means. In the sequel, $U \subseteq \mathbb{R}^n$ is closed and convex.

Definition 5.3.1. Assume $\chi \in \mathrm{C}(I, \mathbb{R})$ is strictly monotone and $\varphi : U \to I$ is a function of n variables. Suppose f_i, $i \in \{1, \ldots, n\}$, are Δ-integrable on $[a, b)_{\mathbb{T}}$ such that $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t)) \in U$ for all $t \in [a, b)_{\mathbb{T}}$. Let $p : [a, b)_{\mathbb{T}} \to \mathbb{R}$ be a nonnegative Δ-integrable function such that $p\chi(\varphi(\mathbf{f}))$ is Δ-integrable and $\int_{[a,b)} p \mathrm{d}\mu_\Delta > 0$. Then we define the weighted generalized mean on time scales by

$$\mathbf{M}_\Delta(\chi, \varphi(\mathbf{f}), p) = \chi^{-1}\left(\frac{\int_{[a,b)} p\chi(\varphi(\mathbf{f}))d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right). \tag{5.3.1}$$

Theorem 5.3.2. *Assume $\chi, \psi_i \in \mathrm{C}(I, \mathbb{R})$, $i \in \{1, \ldots, n\}$, are strictly monotone and $\varphi : U \to I \subseteq \mathbb{R}$ is a function of n variables. Suppose $f_i : [a, b)_{\mathbb{T}} \to I$, $i \in \{1, \ldots, n\}$, are Δ-integrable such that $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t)) \in U$ for all $t \in [a, b)_{\mathbb{T}}$. Let $p, q : [a, b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative Δ-integrable such that $p\chi(\varphi(\mathbf{f})), q\chi(\varphi(\mathbf{f})), p\psi_i(f_i)), q\psi_i(f_i))$, $i \in \{1, \ldots, n\}$, are Δ-integrable and $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. If H defined by*

$$H(s_1, \ldots, s_n) = \chi \circ \varphi(\psi_1^{-1}(s_1), \ldots, \psi_n^{-1}(s_n))$$

is convex, then the functional

$$\int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, \varphi(\mathbf{f}), p)\right) - \chi \circ \varphi\left(\mathbf{M}_\Delta(\psi_1, f_1, p), \ldots, \mathbf{M}_\Delta(\psi_n, f_n, p)\right)\right] \tag{5.3.2}$$

satisfies

$$\begin{aligned}
&\int_{[a,b)} (p+q) d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, \varphi(\mathbf{f}), p+q)\right)\right. \\
&\qquad\qquad \left. - \chi \circ \varphi\left(\mathbf{M}_\Delta(\psi_1, f_1, p+q), \ldots, \mathbf{M}_\Delta(\psi_n, f_n, p+q)\right)\right] \\
&\quad \geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, \varphi(\mathbf{f}), p)\right) - \chi \circ \varphi\left(\mathbf{M}_\Delta(\psi_1, f_1, p), \ldots, \mathbf{M}_\Delta(\psi_n, f_n, p)\right)\right] \\
&+ \int_{[a,b)} q d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, \varphi(\mathbf{f}), q)\right) - \chi \circ \varphi\left(\mathbf{M}_\Delta(\psi_1, f_1, q), \ldots, \mathbf{M}_\Delta(\psi_n, f_n, q)\right)\right].
\end{aligned} \tag{5.3.3}$$

If $p \geq q$ with $\int_{[a,b)} p\mathrm{d}\mu_\Delta > \int_{[a,b)} q\mathrm{d}\mu_\Delta$, then

$$\int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, \varphi(\mathbf{f}), p)\right) - \chi \circ \varphi\left(\mathbf{M}_\Delta(\psi_1, f_1, p), \ldots, \mathbf{M}_\Delta(\psi_n, f_n, p)\right)\right]$$

$$\geq \int_{[a,b)} q d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),q)\right) - \chi\circ\varphi\left(\mathbf{M}_\Delta(\psi_1,f_1,q),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,q)\right)\right]. \quad (5.3.4)$$

Moreover, if H is concave, then (5.3.3) *and* (5.3.4) *hold in reverse order. i.e., the functional*

$$\int_{[a,b)} \cdot d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),\cdot)\right) - \chi\circ\varphi\left(\mathbf{M}_\Delta(\psi_1,f_1,\cdot),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,\cdot)\right)\right]$$

is subadditive and decreasing.

Proof. The functional defined in (5.3.2) is obtained by replacing Φ with H and f_i with $\psi_i(f_i)$, $i \in \{1,\ldots,n\}$, in the Jensen functional (5.1.3), and let $\Psi(\mathbf{f}) = (\psi_1(f_1),\ldots,\psi_n(f_n))$, i.e.,

$$\begin{aligned}
&\mathbf{J}_\Delta(H,\Psi(\mathbf{f}),p)\\
&= \int_{[a,b)} p\chi\circ\varphi(f_1,\ldots,f_n)d\mu_\Delta\\
&\quad - \int_{[a,b)} p d\mu_\Delta H\left(\frac{\int_{[a,b)} p\psi_1(f_1)d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta},\ldots,\frac{\int_{[a,b)} p\psi_n(f_n)d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\\
&= \int_{[a,b)} p d\mu_\Delta \chi(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p))\\
&\quad - \int_{[a,b)} p d\mu_\Delta \chi\circ\varphi\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\\
&= \int_{[a,b)} p d\mu_\Delta \left[\chi(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p)) - \chi\circ\varphi\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\right].
\end{aligned}$$

Now, all claims follow immediately from Theorem 5.2.1. □

Corollary 5.3.3. *Let $H,\varphi,\mathbf{f},p,\chi,f_i,\psi_i$, $i \in \{1,\ldots,n\}$, satisfy the hypothesis of Theorem* 5.3.2. *Further, assume that p attains its minimum value and its maximum value on its domain. If H is convex, then*

$$\begin{aligned}
&\left[\max_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi(\mathbf{m}_\Delta(\chi,\varphi(\mathbf{f}))) - \chi\circ\varphi\left(\mathbf{m}_\Delta(\psi_1,f_1),\ldots,\mathbf{m}_\Delta(\psi_n,f_n)\right)\right] \quad (5.3.5)\\
&\geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi,\varphi(\mathbf{f}),p)\right) - \chi\circ\varphi\left(\mathbf{M}_\Delta(\psi_1,f_1,p),\ldots,\mathbf{M}_\Delta(\psi_n,f_n,p)\right)\right]\\
&\geq \left[\min_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi(\mathbf{m}_\Delta(\chi,\varphi(\mathbf{f}))) - \chi\circ\varphi\left(\mathbf{m}_\Delta(\psi_1,f_1),\ldots,\mathbf{m}_\Delta(\psi_n,f_n)\right)\right].
\end{aligned}$$

where

$$\mathbf{m}_\Delta(\chi,\varphi(\mathbf{f})) = \chi^{-1}\left(\frac{\int_{[a,b)} \chi(\varphi(\mathbf{f}))d\mu_\Delta}{b-a}\right). \quad (5.3.6)$$

Moreover, if H is concave, then the inequalities in (5.3.5) *hold in reverse order.*

Proof. The proof is similar to the proof of Corollary 4.1.6. □

Remark 5.3.4. If we take the discrete form of weighted generalized mean (5.3.1) with $\int_{[a,b)} p\mathrm{d}\mu_\Delta = 1$, then we obtain the quasi-arithmetic mean. Namely, let $\psi : I \subseteq \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be continuous and strictly monotone function, $\mathbf{a} = (a_1, \dots, a_n)$ with $a_k \in I$, $k \in \{1, \dots, n\}$, and $\mathbf{w} = (w_1, \dots, w_n)$ with $w_k \geq 0$ and $\sum\limits_{k=1}^{n} w_k = 1$. Then quasi-arithmetic mean of $\mathbf{a}$ with weight $\mathbf{w}$ is defined by;

$$\mathbf{M}_n = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(a_k)\right). \tag{5.3.7}$$

Now the following examples connects the quasi-arithmetic mean (5.3.7) and the properties of Jensen functionals.

Example 5.3.5. Let $\mathbf{w}$ and ψ be defined as in Remark 5.3.4 such that ψ be strictly increasing, strictly convex function with continuous derivatives of second order and $\frac{\psi'}{\psi''}$ is a concave function. Further, let $\mathbf{X}, \mathbf{p}, \mathbf{x_i}$, $i \in \{1, \dots, n\}$, be defined as in Example 5.1.5, and $\mathbf{q} = (q_1, \dots, q_n)$ with $q_i \geq 0$, $i \in \{1, \dots, n\}$, and $\sum\limits_{i=1}^{n} q_i = Q_n > 0$. Then, $\Phi_{M_n}(\mathbf{x_i}) = \psi^{-1}\left(\sum\limits_{k=1}^{n} w_k \psi(x_{i_k})\right)$ is a convex function (see [82, Theorem 1]). Hence by Theorem 5.2.1, the functional

$$\mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi_{\mathbf{M}_n}(\mathbf{x_i}) - P_n \Phi_{\mathbf{M}_n}\left(\frac{\sum\limits_{i=1}^{n} p_i \mathbf{x_i}}{P_n}\right)$$

is superadditive, i.e.,

$$\mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p} + \mathbf{q}) \geq \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}) + \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{q}),$$

and increasing, i.e., if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n(\Phi_{M_n}, \mathbf{X}, \mathbf{p}) \geq \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{q}).$$

Also, by Corollary 5.2.3, we have

$$\max_{1 \leq i \leq n}\{p_i\} \mathbf{j}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}) \geq \mathbf{J}_n(\Phi_{M_n}, \mathbf{X}, \mathbf{p}) \geq \min_{1 \leq i \leq n}\{p_i\} \mathbf{j}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}),$$

where

$$\mathbf{j}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}) = \sum_{i=1}^{n} \Phi_{\mathbf{M}_n}(\mathbf{x_i}) - n \Phi_{\mathbf{M}_n}\left(\frac{\sum\limits_{i=1}^{n} \mathbf{x_i}}{n}\right).$$

Example 5.3.6. Consider (5.3.7), but with different conditions on ψ and $\mathbf{w}$. Namely, if

(i) $w_i \geq 1$, for $i \in \{1, \ldots, n\}$;

(ii) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$;

(iii) $\lim_{x\to 0} \psi(x) = +\infty$ or $\lim_{x\to\infty} \psi(x) = +\infty$,

then we have the following definition:

$$\widetilde{\mathbf{M}}_n = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(a_k)\right).$$

Let $\mathbf{X}, \mathbf{p}, \mathbf{x_i}$, $i \in \{1, \ldots, n\}$, be defined as in Example 5.1.5 and $\mathbf{q} = (q_1, \ldots, q_n)$ with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. Let ψ be strictly increasing and strictly convex with continuous derivatives of second order, such that $\frac{\psi}{\psi'}$ is convex. Then $\Phi_{\widetilde{\mathbf{M}}_n}(\mathbf{x_i}) = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(x_{i_k})\right)$ is a convex function (see [82, Theorem 2]). Hence by Theorem 5.2.1, the functional

$$\mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi_{\widetilde{\mathbf{M}}_n}(x_{i_k}) - P_n \Phi_{\widetilde{\mathbf{M}}_n}\left(\frac{\sum_{i=1}^{n} p_i x_{i_k}}{P_n}\right)$$

is superadditive, i.e.,

$$\mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}+\mathbf{q}) \geq \mathbf{J}_n(\Phi_{\widetilde{M}_n}, \mathbf{X}, \mathbf{p}) + \mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{q}),$$

and increasing, i.e., if $p \geq q$, then

$$\mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}) \geq \mathbf{J}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{q}).$$

Also, by Corollary 5.2.3, we have

$$\max_{1\leq i\leq n}\{p_i\} \mathbf{j}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}) \geq \mathbf{J}_n(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\} \mathbf{j}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}),$$

where

$$\mathbf{j}_n(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}) = \sum_{i=1}^{n} \Phi_{\widetilde{\mathbf{M}}_n}(x_{i_k}) - n\Phi_{\widetilde{\mathbf{M}}_n}\left(\frac{\sum_{i=1}^{n} x_{i_k}}{n}\right).$$

Example 5.3.7. Let $\mathbf{X}, \mathbf{p}, \mathbf{x_i}$, $i \in \{1, \ldots, n\}$, be defined as in Example 5.1.5 and $\mathbf{q} = (q_1, \ldots, q_n)$, with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. Let $\varphi : I \to \mathbb{R}$ be a $(n+1)$-convex function, where I is a closed and bounded interval in $\mathbb{R}$. Then by Theorem 5.2.1, for $\Phi_G(\mathbf{x_i}) = [x_{i_1}, \ldots, x_{i_n}; \varphi]$, the functional

$$\mathbf{J}_n(\Phi_G, \mathbf{X}, \mathbf{p}) = \sum_{i=1}^{n} p_i \Phi_G(\mathbf{x_i}) - P_n \Phi_G \left(\frac{\sum_{i=1}^{n} p_i \mathbf{x_i}}{P_n} \right)$$

is superadditive, i.e.,

$$\mathbf{J}_n(\Phi_G, \mathbf{X}, \mathbf{p} + \mathbf{q}) \geq \mathbf{J}_n(\Phi_G, \mathbf{X}, \mathbf{p}) + \mathbf{J}_n(\Phi_G, \mathbf{X}, \mathbf{q}),$$

and increasing, i.e., if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n(\Phi_G, \mathbf{X}, \mathbf{p}) \geq \mathbf{J}_n(\Phi_G, \mathbf{X}, \mathbf{q}).$$

Also, by Corollary 5.2.3, we have

$$\max_{1 \leq i \leq n} \{p_i\} \mathbf{j}_n(\Phi_G, \mathbf{X}) \geq \mathbf{J}_n(\Phi_G, \mathbf{X}, \mathbf{p}) \geq \min_{1 \leq i \leq n} \{p_i\} \mathbf{j}_n(\Phi_G, \mathbf{X}),$$

where

$$\mathbf{j}_n(\Phi_G, \mathbf{X}) = \sum_{i=1}^{n} \Phi_G(\mathbf{x}_i) - n \Phi_G \left(\frac{\sum_{i=1}^{n} \mathbf{x}_i}{n} \right).$$

5.4 Applications to additive and multiplicative type inequalities

In this section we give some applications of Theorem 5.2.1 to additive and multiplicative type mean inequalities.

Corollary 5.4.1. *Assume* $\chi, \psi_1, \psi_2 \in \mathrm{C}^2(I, \mathbb{R})$ *are strictly monotone. Suppose* $f_1, f_2 : [a,b)_{\mathbb{T}} \to I$ *are* Δ*-integrable such that* $f_1(t) + f_2(t) \in I$ *for all* $t \in [a,b)_{\mathbb{T}}$*. Let* $p, q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative* Δ*-integrable such that* $p\chi(f_1+f_2), q\chi(f_1+f_2), p\psi_i(f_i)), q\psi_i(f_i))$*,* $i = 1, 2$*, are* Δ*-integrable and* $\int_{[a,b)} p \mathrm{d}\mu_\Delta > 0$*,* $\int_{[a,b)} q \mathrm{d}\mu_\Delta > 0$*. Furthermore, let*

$$E = \frac{\psi_1'}{\psi_1''}, \quad F = \frac{\psi_2'}{\psi_2''}, \quad G = \frac{\chi'}{\chi''}.$$

If ψ_1', ψ_2', χ' are positive and $\psi_1'', \psi_2'', \chi''$ are negative, then the functional

$$\int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1+f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) + \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \tag{5.4.1}$$

is superadditive, i.e.,

$$\begin{aligned} &\int_{[a,b)} (p+q) d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1+f_2, p+q)\right)\right. \\ &\quad \left.-\chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p+q) + \mathbf{M}_\Delta(\psi_2, f_2, p+q)\right)\right] \\ &\geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1+f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) + \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\ &\quad + \int_{[a,b)} q d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1+f_2, q)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, q) + \mathbf{M}_\Delta(\psi_2, f_2, q)\right)\right]. \end{aligned} \tag{5.4.2}$$

If $p \geq q$ such that $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, then

$$\begin{aligned} &\int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1+f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) + \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\ &\quad \geq \int_{[a,b)} q d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1+f_2, q)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, q) + \mathbf{M}_\Delta(\psi_2, f_2, q)\right)\right] \end{aligned} \tag{5.4.3}$$

if and only if $G(x+y) \leq E(x) + F(y)$. If p attains minimum and maximum value on its domain $[a,b)_\mathbb{T}$, then (5.4.3) *yields*

$$\begin{aligned} &\left[\max_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi\left(\mathbf{m}_\Delta(\chi, f_1+f_2)\right) - \chi\left(\mathbf{m}_\Delta(\psi_1, f_1) + \mathbf{m}_\Delta(\psi_2, f_2)\right)\right] \\ &\geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1+f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) + \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\ &\geq \left[\min_{t\in[a,b)_\mathbb{T}} p(t)\right](b-a)\left[\chi\left(\mathbf{m}_\Delta(\chi, f_1+f_2)\right) - \chi\left(\mathbf{m}_\Delta(\psi_1, f_1) + \mathbf{m}_\Delta(\psi_2, f_2)\right)\right]. \end{aligned} \tag{5.4.4}$$

Moreover, if $\psi_1', \psi_2', \chi', \psi_1'', \psi_2'', \chi''$ are all positive, then inequalities in (5.4.2), (5.4.3) *and* (5.4.4) *are reversed if and only if $G(x+y) \geq E(x) + F(y)$.*

Proof. Let $n = 2$ in Theorem 5.3.2, by setting $\varphi(x,y) = x + y$ we have

$$H(s_1, s_2) = \chi(\psi_1^{-1}(s_1) + \psi_2^{-1}(s_2)).$$

If ψ_1', ψ_2', χ' are positive and $\psi_1'', \psi_2'', \chi''$ are negative, then H is convex if and only if $G(x+y) \leq E(x) + F(y)$ (see [30]). While if $\psi_1', \psi_2', \chi', \psi_1'', \psi_2'', \chi''$ are all positive, then H is concave if and only if $G(x+y) \geq E(x) + F(y)$ (see [30]). Now, all claims follow immediately from Theorem 5.3.2. □

Corollary 5.4.2. *Assume $\chi, \psi_1, \psi_2 \in \mathrm{C}^2(I, \mathbb{R})$ are strictly monotone. Suppose $f_1, f_2 : [a,b)_{\mathbb{T}} \to I$ are Δ-integrable such that $f_1(t)f_2(t) \in I$ for all $t \in [a,b)_{\mathbb{T}}$. Let $p, q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative Δ-integrable such that $p\chi(f_1 \cdot f_2), q\chi(f_1 \cdot f_2), p\psi_i(f_i)), q\psi_i(f_i)),$ $i = 1, 2$, are Δ-integrable and $\int_{[a,b)} \mathrm{p} \mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} \mathrm{q} \mathrm{d}\mu_\Delta > 0$. Furthermore, let*

$$A(t) = \frac{\psi_1'(t)}{\psi_1'(t) + t\psi_1''(t)}, \quad B(t) = \frac{\psi_2'(t)}{\psi_2'(t) + t\psi_2''(t)}, \quad C(t) = \frac{\chi'(t)}{\chi'(t) + t\chi''(t)}.$$

If ψ_1', ψ_2', χ' are positive and A, B, C are negative, then the functional

$$\int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \tag{5.4.5}$$

is superadditive, i.e.,

$$\begin{aligned} &\int_{[a,b)} (p+q) d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p+q)\right)\right. \\ &\quad \left.-\chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p+q) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p+q)\right)\right] \\ &\geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\ &\quad + \int_{[a,b)} q d\mu_\Delta \left[\chi\left(M(\chi, f_1 \cdot f_2, q)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, q) \cdot \mathbf{M}_\Delta(\psi_2, f_2, q)\right)\right]. \end{aligned} \tag{5.4.6}$$

If $p \geq q$ such that $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, then

$$\begin{aligned} &\int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\ &\qquad \geq \int_{[a,b)} q d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, q)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, q) \cdot \mathbf{M}_\Delta(\psi_2, f_2, q)\right)\right] \end{aligned} \tag{5.4.7}$$

if and only if $C(x.y) \leq A(x) + B(y)$. If p attains minimum and maximum value on its domain $[a,b)_{\mathbb{T}}$, then (5.4.7) *yields*

$$\begin{aligned} &\left[\max_{t \in [a,b)_{\mathbb{T}}} p(t)\right] (b-a) \left[\chi\left(\mathbf{m}_\Delta(\chi, f_1 \cdot f_2)\right) - \chi\left(\mathbf{m}_\Delta(\psi_1, f_1) \cdot \mathbf{m}_\Delta(\psi_2, f_2)\right)\right] \\ &\geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p)\right) - \chi\left(\mathbf{M}_\Delta(\psi_1, f_1, p) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p)\right)\right] \\ &\geq \left[\min_{t \in [a,b)_{\mathbb{T}}} p(t)\right] (b-a) \left[\chi\left(\mathbf{m}_\Delta(\chi, f_1 \cdot f_2)\right) - \chi\left(\mathbf{m}_\Delta(\psi_1, f_1) \cdot \mathbf{m}_\Delta(\psi_2, f_2)\right)\right]. \end{aligned} \tag{5.4.8}$$

If $\psi_1', \psi_2', \chi', A, B, C$ are all positive, then inequalities in (5.4.6), (5.4.7) *and* (5.4.8) *are reversed if and only if $C(x.y) \geq A(x) + B(y)$.*

Proof. Let $n = 2$ in Theorem 5.3.2. By setting $\varphi(x, y) = x \cdot y$, we have

$$H(s_1, s_2) = \chi(\psi_1^{-1}(s_1) \cdot \psi_2^{-1}(s_2)).$$

If ψ_1', ψ_2', χ' are positive and A, B, C are negative, then H is convex if and only if $C(x.y) \leq A(x) + B(y)$. While if $\psi_1', \psi_2', \chi', A, B, C$ are all positive, then H is concave if and only if $C(x.y) \geq A(x) + B(y)$ (see [30]). Now, all claims follow immediately from Theorem 5.3.2. □

Corollary 5.4.3. *Let $\lambda, \omega, \nu \in \mathbb{R}$ be such that*

(a) $\lambda < 0 < \omega, \nu$, *or* $\omega, \nu < 0 < \lambda$;

(b) $\lambda < \omega, \nu < 0$, *or* $\nu < 0 < \omega < \lambda$, *or* $\omega < 0 < \nu < \lambda$, *for* $\frac{1}{\lambda} \leq \frac{1}{\omega} + \frac{1}{\nu}$;

(c) $\lambda < \omega < 0 < \nu$, *or* $\lambda < \nu < 0 < \omega$, *for* $\frac{1}{\lambda} \geq \frac{1}{\omega} + \frac{1}{\nu}$.

Suppose $f_1, f_2 : [a, b)_{\mathbb{T}} \to \mathbb{R}$ are Δ-integrable. Let $p, q : [a, b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative Δ-integrable such that $p \cdot f_1^\lambda \cdot f_2^\lambda$, $q \cdot f_1^\lambda \cdot f_2^\lambda$, $pf_1^\omega, qf_1^\omega, pf_2^\nu, qf_2^\nu$ are Δ-integrable and $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. Then the functional

$$\int_{[a,b)} p \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta - \int_{[a,b)} pd\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \tag{5.4.9}$$

is superadditive, i.e.,

$$\begin{aligned}
&\int_{[a,b)} (p+q) \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta \\
&\qquad - \int_{[a,b)} (p+q)d\mu_\Delta \left[\left(\frac{\int_{[a,b)} (p+q) f_1^\omega d\mu_\Delta}{\int_{[a,b)} (p+q)d\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} (p+q) f_2^\nu d\mu_\Delta}{\int_{[a,b)} (p+q)d\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&\qquad \geq \int_{[a,b)} p \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta - \int_{[a,b)} pd\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\
&+ \int_{[a,b)} q \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta - \int_{[a,b)} qd\mu_\Delta \left[\left(\frac{\int_{[a,b)} qf_1^\omega d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta} \right)^{\frac{1}{\omega}} \left(\frac{\int_{[a,b)} qf_2^\nu d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda .
\end{aligned} \tag{5.4.10}$$

If $p \geq q$ such that $\int_{[a,b)} pd\mu_\Delta > \int_{[a,b)} qd\mu_\Delta$, then

$$\int_{[a,b)} p\cdot f_1^\lambda\cdot f_2^\lambda d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta\left[\left(\frac{\int_{[a,b)} p f_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} p f_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda$$

$$\geq \int_{[a,b)} q\cdot f_1^\lambda\cdot f_2^\lambda d\mu_\Delta - \int_{[a,b)} q d\mu_\Delta\left[\left(\frac{\int_{[a,b)} q f_1^\omega d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} q f_2^\nu d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda. \quad (5.4.11)$$

If p attains minimum and maximum value on its domain then

$$\max_{t\in[a,b)_\mathbb{T}} p(t)\left[\int_{[a,b)} f_1^\lambda\cdot f_2^\lambda d\mu_\Delta - (b-a)\left[\left(\frac{\int_{[a,b)} f_1^\omega d\mu_\Delta}{b-a}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} f_2^\nu d\mu_\Delta}{b-a}\right)^{\frac{1}{\nu}}\right]^\lambda\right] \quad (5.4.12)$$

$$\geq \int_{[a,b)} p\cdot f_1^\lambda\cdot f_2^\lambda d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta\left[\left(\frac{\int_{[a,b)} p f_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} p f_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda$$

$$\geq \min_{t\in[a,b)_\mathbb{T}} p(t)\left[\int_{[a,b)} f_1^\lambda\cdot f_2^\lambda d\mu_\Delta - (b-a)\left[\left(\frac{\int_{[a,b)} f_1^\omega d\mu_\Delta}{b-a}\right)^{\frac{1}{\omega}}\left(\frac{\int_{[a,b)} f_2^\nu d\mu_\Delta}{b-a}\right)^{\frac{1}{\nu}}\right]^\lambda\right].$$

Moreover, inequalities in (5.4.10), (5.4.11) *and* (5.4.12) *are reversed provided*

(a′) $\omega, \nu > \lambda > 0$, *for* $\frac{1}{\lambda} \geq \frac{1}{\omega} + \frac{1}{\nu}$;

(b′) $\omega, \nu < \lambda < 0$, *for* $\frac{1}{\lambda} \leq \frac{1}{\omega} + \frac{1}{\nu}$.

Proof. Let $n = 2$ in Theorem 5.3.2. By setting $\varphi(x,y) = x\cdot y$, $\chi(t) = t^\lambda$, $\psi_1(t) = t^\omega$, and $\psi_2(t) = t^\nu$, we have

$$H(s_1,s_2) = \chi(\psi_1^{-1}(s_1)\cdot\psi_2^{-1}(s_2)) = \left(s_1^{\frac{1}{\omega}} s_2^{\frac{1}{\nu}}\right)^\lambda.$$

Now, H is convex if and only if $d^2H \geq 0$, which implies

$$\frac{\lambda}{\omega}\left(\frac{\lambda}{\omega}-1\right) \geq 0, \quad \frac{\lambda}{\nu}\left(\frac{\lambda}{\nu}-1\right) \geq 0 \quad \text{and} \quad \frac{\lambda^3}{\omega\nu}\left(\frac{1}{\lambda}-\frac{1}{\omega}-\frac{1}{\nu}\right) \geq 0,$$

and these are satisfied if λ, ω and ν satisfy conditions $(a), (b)$ and (c). H is concave if and only if $d^2H \leq 0$, and this implies

$$\frac{\lambda}{\omega}\left(\frac{\lambda}{\omega}-1\right) \leq 0, \quad \frac{\lambda}{\nu}\left(\frac{\lambda}{\nu}-1\right) \leq 0 \quad \text{and} \quad \frac{\lambda^3}{\omega\nu}\left(\frac{1}{\lambda}-\frac{1}{\omega}-\frac{1}{\nu}\right) \geq 0.$$

These are satisfied if λ, ω and ν satisfy conditions $(a')$and (b'). Now, all claims follow immediately from Theorem 5.3.2. □

Corollary 5.4.4. *Let $\lambda, \omega, \nu \in \mathbb{R}$ be such that $\lambda, \omega, \nu > 0$, $\lambda, \omega, \nu \neq 1$ and*

(a) $\lambda < 1 < \omega, \nu$, *or* $\omega, \nu < 1 < \lambda$;

(b) $\lambda < \omega, \nu < 1$, *or* $\nu < 1 < \omega < \lambda$, *or* $\omega < 1 < \nu < \lambda$, *for* $\frac{1}{\log \lambda} \leq \frac{1}{\log \omega} + \frac{1}{\log \nu}$;

(c) $\lambda < \omega < 1 < \nu$, *or* $\lambda < \nu < 1 < \omega$, *for* $\frac{1}{\log \lambda} \geq \frac{1}{\log \omega} + \frac{1}{\log \nu}$.

Suppose $f_1, f_2 : [a,b)_{\mathbb{T}} \to \mathbb{R}$ are Δ-integrable. Let $p, q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ be nonnegative Δ-integrable such that $p\lambda^{f_1+f_2}, q\lambda^{f_1+f_2}, p\omega^{f_1}, q\omega^{f_1}, p\nu^{f_2}, q\nu^{f_2}$ are Δ-integrable and $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. Then the functional

$$\int_{[a,b)} p\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} pd\mu_\Delta \lambda^{\log_\omega \frac{\int_{[a,b)} p\cdot\omega^{f_1} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} + \log_\nu \frac{\int_{[a,b)} p\cdot\nu^{f_2} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}} \tag{5.4.13}$$

is superadditive, i.e.,

$$\begin{aligned}
&\int_{[a,b)} (p+q)\lambda^{f_1+f_2} d\mu_\Delta \\
&\qquad - \int_{[a,b)} (p+q) d\mu_\Delta \lambda^{\log_\omega \frac{\int_{[a,b)} (p+q)\cdot\omega^{f_1} d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta} + \log_\nu \frac{\int_{[a,b)} (p+q)\cdot\nu^{f_2} d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta}} \\
&\geq \int_{[a,b)} p\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} pd\mu_\Delta \lambda^{\log_\omega \frac{\int_{[a,b)} p\cdot\omega^{f_1} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} + \log_\nu \frac{\int_{[a,b)} p\cdot\nu^{f_2} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}} \\
&\quad + \int_{[a,b)} q\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} qd\mu_\Delta \lambda^{\log_\omega \frac{\int_{[a,b)} q\cdot\omega^{f_1} d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta} + \log_\nu \frac{\int_{[a,b)} q\cdot\nu^{f_2} d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta}}.
\end{aligned} \tag{5.4.14}$$

If $p \geq q$ such that $\int_{[a,b)} pd\mu_\Delta > \int_{[a,b)} qd\mu_\Delta$, then

$$\begin{aligned}
&\int_{[a,b)} p\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} pd\mu_\Delta \lambda^{\log_\omega \frac{\int_{[a,b)} p\cdot\omega^{f_1} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} + \log_\nu \frac{\int_{[a,b)} p\cdot\nu^{f_2} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}} \\
&\quad \geq \int_{[a,b)} q\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} qd\mu_\Delta \lambda^{\log_\omega \frac{\int_{[a,b)} q\cdot\omega^{f_1} d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta} + \log_\nu \frac{\int_{[a,b)} q\cdot\nu^{f_2} d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta}}.
\end{aligned} \tag{5.4.15}$$

If p attains minimum and maximum value on its domain then

$$\begin{aligned}
&\max_{t\in[a,b)_{\mathbb{T}}} p(t) \left[\int_{[a,b)} \lambda^{f_1+f_2} d\mu_\Delta - (b-a)\lambda^{\log_\omega \frac{\int_{[a,b)} \omega^{f_1} d\mu_\Delta}{b-a} + \log_\nu \frac{\int_{[a,b)} \nu^{f_2} d\mu_\Delta}{b-a}} \right] \\
&\quad \geq \int_{[a,b)} p\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} pd\mu_\Delta \lambda^{\log_\omega \frac{\int_{[a,b)} p\cdot\omega^{f_1} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta} + \log_\nu \frac{\int_{[a,b)} p\cdot\nu^{f_2} d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}}
\end{aligned}$$

$$\geq \min_{t\in[a,b)_{\mathbb{T}}} p(t)\left[\int_{[a,b)} \lambda^{f_1+f_2} d\mu_\Delta - (b-a)\lambda^{\log_\omega \frac{\int_{[a,b)} \omega^{f_1} d\mu_\Delta}{b-a} + \log_\nu \frac{\int_{[a,b)} \nu^{f_2} d\mu_\Delta}{b-a}}\right]. \quad (5.4.16)$$

Moreover, inequalities in (5.4.14), (5.4.15) *and* (5.4.16) *are reversed provided*

(a′) $\omega, \nu > \lambda > 1$, *for* $\frac{1}{\log\lambda} \geq \frac{1}{\log\omega} + \frac{1}{\log\nu}$;

(b′) $\omega, \nu < \lambda < 0$, *for* $\frac{1}{\log\lambda} \leq \frac{1}{\log\omega} + \frac{1}{\log\nu}$.

Proof. Let $n = 2$ in Theorem 5.3.2, by setting $\varphi(x, y) = x + y$, $\chi(t) = \lambda^t$, $\psi_1(t) = \omega^t$, and $\psi_2(t) = \nu^t$ we have

$$H(s_1, s_2) = \left(s_1^{\frac{1}{\log\omega}} \cdot s_2^{\frac{1}{\log\nu}}\right)^{\log\lambda}.$$

Now, the proof is similar to the proof of Corollary 5.4.3. □

Corollary 5.4.5. *Let* $\lambda, \omega, \nu \in \mathbb{R}$ *be such that*

(a) $0 < \omega, \nu \leq \lambda < 1$, *for all* $f_1, f_2 > 0$;

(b) $0 < \nu \leq \lambda \leq \omega < 1$, *for* $f_2 \geq \frac{(\omega-\lambda)(1-\nu)}{(\lambda-\nu)(1-\omega)} f_1 \geq 0$;

(c) $0 < \omega \leq \lambda \leq \nu < 1$, *for* $\frac{(\lambda-\omega)(1-\nu)}{(\nu-\lambda)(1-\omega)} f_1 \geq f_2 \geq 0$.

Suppose $f_1, f_2 : [a,b)_{\mathbb{T}} \to \mathbb{R}$ *are* Δ*-integrable. Let* $p, q : [a,b)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative* Δ*-integrable such that* $p(f_1+f_2)^\lambda, q(f_1+f_2)^\lambda, pf_1^\omega, qf_1^\omega, pf_2^\nu, qf_2^\nu$ *are* Δ*-integrable and* $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. *Then the functional*

$$\int_{[a,b)} p\cdot(f_1+f_2)^\lambda d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda \quad (5.4.17)$$

is superadditive, i.e.,

$$\int_{[a,b)} (p+q)\cdot(f_1+f_2)^\lambda d\mu_\Delta \quad (5.4.18)$$
$$- \int_{[a,b)} (p+q) d\mu_\Delta \left[\left(\frac{\int_{[a,b)} (p+q)f_1^\omega d\mu_\Delta}{\int_{[a,b)} (p+q)d\mu_\Delta}\right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} (p+q)f_2^\nu d\mu_\Delta}{\int_{[a,b)} (p+q)d\mu_\Delta}\right)^{\frac{1}{\nu}}\right]^\lambda$$
$$\geq \int_{[a,b)} p\cdot(f_1+f_2)^\lambda d\mu_\Delta$$

$$- \int_{[a,b)} p d\mu_\Delta \left[\left(\frac{\int_{[a,b)} p f_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda$$

$$+ \int_{[a,b)} q \cdot (f_1 + f_2)^\lambda d\mu_\Delta$$

$$- \int_{[a,b)} q d\mu_\Delta \left[\left(\frac{\int_{[a,b)} q f_1^\omega d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} q f_2^\nu d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda .$$

If $p \geq q$ such that $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, then

$$\begin{aligned} &\int_{[a,b)} p \cdot (f_1 + f_2)^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} p d\mu_\Delta \left[\left(\frac{\int_{[a,b)} p f_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\ &\geq \int_{[a,b)} q \cdot (f_1 + f_2)^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} q d\mu_\Delta \left[\left(\frac{\int_{[a,b)} q f_1^\omega d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} q f_2^\nu d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda . \end{aligned} \tag{5.4.19}$$

If p attains minimum and maximum value on its domain then

$$\begin{aligned} &\max_{t \in [a,b)_{\mathbb{T}}} p(t) \left[\int_{[a,b)} (f_1 + f_2)^\lambda d\mu_\Delta \right. \\ &\quad \left. -(b-a) \left[\left(\frac{\int_{[a,b)} p f_1^\omega d\mu_\Delta}{b-a} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu d\mu_\Delta}{b-a} \right)^{\frac{1}{\nu}} \right]^\lambda \right] \\ &\geq \int_{[a,b)} p \cdot (f_1 + f_2)^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} p d\mu_\Delta \left[\left(\frac{\int_{[a,b)} p f_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{\frac{1}{\nu}} \right]^\lambda \\ &\geq \min_{t \in [a,b)_{\mathbb{T}}} p(t) \left[\int_{[a,b)} (f_1 + f_2)^\lambda d\mu_\Delta \right. \\ &\quad \left. -(b-a) \left[\left(\frac{\int_{[a,b)} p f_1^\omega d\mu_\Delta}{b-a} \right)^{\frac{1}{\omega}} + \left(\frac{\int_{[a,b)} p f_2^\nu d\mu_\Delta}{b-a} \right)^{\frac{1}{\nu}} \right]^\lambda \right] . \end{aligned} \tag{5.4.20}$$

Moreover, inequalities in (5.4.18), (5.4.19) *and* (5.4.20) *are reversed provided*

(a′) $1 < \lambda \leq \omega, \nu$, *for all* $f_1, f_2 > 0$;

(b′) $1 < \nu \leq \lambda \leq \omega$, *for* $0 \leq f_2 \leq \frac{(\omega-\lambda)(\nu-1)}{(\lambda-\nu)(\omega-1)} f_1$;

(b′) $1 < \omega \leq \lambda \leq \nu$, *for* $f_2 \geq \frac{(\lambda-\omega)(\nu-1)}{(\nu-\lambda)(\omega-1)} f_1 \geq 0$.

Proof. Let $n = 2$ in Theorem 5.3.2. By setting $\varphi(x, y) = x + y$, $\chi(t) = t^{\lambda}$, $\psi_1(t) = t^{\omega}$, and $\psi_2(t) = t^{\nu}$ we have

$$H(s_1, s_2) = \left(s_1^{\frac{1}{\omega}} + s_2^{\frac{1}{\nu}}\right)^{\lambda}.$$

Now, the proof is similar to the proof of Corollary 5.4.1, with some extra considerations of definitions of functions E, F and G. □

Corollary 5.4.6. *Suppose* $f_1, f_2 : [a, b)_{\mathbb{T}} \to [0, \frac{\pi}{4}]$ *are* Δ*-integrable. Let* $p, q : [a, b)_{\mathbb{T}} \to \mathbb{R}$ *be nonnegative* Δ*-integrable such that* $p\cos(f_1 + f_2), q\cos(f_1 + f_2), p\cos(f_i), q\cos(f_i)$, $i = 1, 2$, *are* Δ*-integrable and* $\int_{[a,b)} p\mathrm{d}\mu_\Delta > 0$, $\int_{[a,b)} q\mathrm{d}\mu_\Delta > 0$. *Then the functional*

$$\int_{[a,b)} pd\mu_\Delta \cdot \cos\left[\arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_1) d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right) + \arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_2) d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right] - \int_{[a,b)} p\cos(f_1 + f_2) d\mu_\Delta \quad (5.4.21)$$

is subadditive, i.e.,

$$\begin{aligned}
&\int_{[a,b)} (p+q) d\mu_\Delta \cdot \cos\left[\arccos\left(\frac{\int_{[a,b)} (p+q) \cdot \cos(f_1) d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta}\right)\right. \qquad (5.4.22)\\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} (p+q) \cdot \cos(f_2) d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta}\right)\right] - \int_{[a,b)} (p+q) \cdot \cos(f_1 + f_2) d\mu_\Delta \\
&\leq \int_{[a,b)} pd\mu_\Delta \cdot \cos\left[\arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_1) d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right. \\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_2) d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right] - \int_{[a,b)} p \cdot \cos(f_1 + f_2) d\mu_\Delta \\
&+ \int_{[a,b)} qd\mu_\Delta \cdot \cos\left[\arccos\left(\frac{\int_{[a,b)} q \cdot \cos(f_1) d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta}\right)\right. \\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} q \cdot \cos(f_2) d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta}\right)\right] - \int_{[a,b)} q \cdot \cos(f_1 + f_2) d\mu_\Delta.
\end{aligned}$$

If $p \geq q$ such that $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, then

$$\begin{aligned}
&\int_{[a,b)} p d\mu_\Delta \cdot \cos\left[\arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_1) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)\right. \\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_2) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)\right] - \int_{[a,b)} p \cdot \cos(f_1 + f_2) d\mu_\Delta \\
&\leq \int_{[a,b)} q d\mu_\Delta \cdot \cos\left[\arccos\left(\frac{\int_{[a,b)} q \cdot \cos(f_1) d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)\right. \\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} q \cdot \cos(f_2) d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)\right] - \int_{[a,b)} q \cdot \cos(f_1 + f_2) d\mu_\Delta.
\end{aligned} \tag{5.4.23}$$

If p attains minimum and maximum value on its domain then

$$\begin{aligned}
&\left[\max_{t \in [a,b)_{\mathbb{T}}} p(t)\right] (b-a) \cos\left[\arccos\left(\frac{\int_{[a,b)} \cos(f_1) d\mu_\Delta}{b-a}\right)\right. \\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} \cos(f_2) d\mu_\Delta}{b-a}\right)\right] - \int_{[a,b)} \cos(f_1 + f_2) d\mu_\Delta \\
&\leq \int_{[a,b)} p d\mu_\Delta \cdot \cos\left[\arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_1) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)\right. \\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} p \cdot \cos(f_2) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)\right] - \int_{[a,b)} p \cdot \cos(f_1 + f_2) d\mu_\Delta \\
&\leq \left[\min_{t \in [a,b)_{\mathbb{T}}} p(t)\right] (b-a) \cos\left[\arccos\left(\frac{\int_{[a,b)} \cos(f_1) d\mu_\Delta}{b-a}\right)\right. \\
&\quad \left. + \arccos\left(\frac{\int_{[a,b)} \cos(f_2) d\mu_\Delta}{b-a}\right)\right] - \int_{[a,b)} \cos(f_1 + f_2) d\mu_\Delta.
\end{aligned} \tag{5.4.24}$$

Proof. Let $n = 2$ in Theorem 5.3.2, by setting $\varphi(x, y) = x + y$ and $\chi(t) = \psi_1(t) = \psi_2(t) = -\cos(t)$, we have

$$H(s_1, s_2) = -\cos(\arccos(-s_1) + \arccos(-s_2)).$$

Now, the proof is similar to the proof of Corollary 5.4.1. □

5.5 Applications to Hölder's inequality

Suppose f_i, $i \in \{1, \ldots, n\}$, are nonnegative Δ-integrable functions on $[a,b)_{\mathbb{T}}$ such that $\prod_{i=1}^{n} f_i^{\alpha_i}$ is Δ-integrable, where $\alpha_i \geq 0$, $i \in \{1, \ldots, n\}$, are such that $\sum_{i=1}^{n} \alpha_i = 1$. Then, by using Theorem 4.4.1 (Hölder's inequality for Lebesgue Δ-integrals), we have

$$\int_{[a,b)} \prod_{i=1}^{n} f_i^{\alpha_i} d\mu_\Delta \leq \prod_{i=1}^{n} \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{\alpha_i}. \tag{5.5.1}$$

If $\sum_{i=1}^{n} \alpha_i = \mathcal{A}_n > 0$, then (5.5.1) implies

$$\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathcal{A}_n}} d\mu_\Delta \leq \prod_{i=1}^{n} \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{\frac{\alpha_i}{\mathcal{A}_n}} \tag{5.5.2}$$

or

$$\left(\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathcal{A}_n}} d\mu_\Delta \right)^{\mathcal{A}_n} \leq \prod_{i=1}^{n} \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{\alpha_i}. \tag{5.5.3}$$

In this section we discuss properties of the functional, deduced from the Hölder inequality (5.5.3), defined in the following way.

Definition 5.5.1. Suppose $\mathbf{f} = (f_1, \ldots, f_n)$ is such that f_i, $i \in \{1, \ldots, n\}$, are nonnegative Δ-integrable functions on $[a,b)_{\mathbb{T}}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be such that $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = \mathcal{A}_n > 0$. Then we define the functional $\mathbf{H}_\Delta$ by

$$\mathbf{H}_\Delta(\mathbf{f}, \alpha) = \frac{\prod_{i=1}^{n} \left(\int_{[a,b)} f_i d\mu_\Delta \right)^{\alpha_i}}{\left(\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathcal{A}_n}} d\mu_\Delta \right)^{\mathcal{A}_n}}. \tag{5.5.4}$$

Theorem 5.5.2. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be real n-tuples with $\alpha_i \geq 0$, $\beta_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = \mathcal{A}_n > 0$, $\sum_{i=1}^{n} \beta_i = \mathcal{B}_n > 0$. Suppose f_i, $i \in \{1, \ldots, n\}$, are nonnegative Δ-integrable functions on $[a,b)_{\mathbb{T}}$ such that $\prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathcal{A}_n}}$ and $\prod_{i=1}^{n} f_i^{\frac{\beta_i}{\mathcal{B}_n}}$ are Δ-integrable. Then*

$$\mathbf{H}_\Delta(\mathbf{f}, \alpha + \beta) \geq \mathbf{H}_\Delta(\mathbf{f}, \alpha) \cdot \mathbf{H}_\Delta(\mathbf{f}, \beta), \tag{5.5.5}$$

and $\mathbf{H}_\Delta(\mathbf{f}, \cdot)$ is increasing, i.e., if $\alpha \geq \beta$ such that $\mathcal{A}_n > \mathcal{B}_n$, then

$$\mathbf{H}_\Delta(\mathbf{f}, \alpha) \geq \mathbf{H}_\Delta(\mathbf{f}, \beta). \tag{5.5.6}$$

Proof. By Definition 5.5.1, we have

$$\mathbf{H}_\Delta(\mathbf{f},\alpha+\beta)=\frac{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\alpha_i+\beta_i}}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\alpha_i+\beta_i}{\mathcal{A}_n+\mathcal{B}_n}} d\mu_\Delta\right)^{\mathcal{A}_n+\mathcal{B}_n}} \tag{5.5.7}$$

where

$$\begin{aligned}
&\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\alpha_i+\beta_i}{\mathcal{A}_n+\mathcal{B}_n}} d\mu_\Delta\right)^{\mathcal{A}_n+\mathcal{B}_n} \\
&=\left[\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathcal{A}_n}}\right)^{\frac{\mathcal{A}_n}{\mathcal{A}_n+\mathcal{B}_n}}\left(\prod_{i=1}^{n} f_i^{\frac{\beta_i}{\mathcal{B}_n}}\right)^{\frac{\mathcal{B}_n}{\mathcal{A}_n+\mathcal{B}_n}} d\mu_\Delta\right]^{\mathcal{A}_n+\mathcal{B}_n} \\
&\le\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathcal{A}_n}} d\mu_\Delta\right)^{\mathcal{A}_n}\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\beta_i}{\mathcal{B}_n}} d\mu_\Delta\right)^{\mathcal{B}_n}.
\end{aligned} \tag{5.5.8}$$

Now, by combining (5.5.7) and (5.5.8), we have

$$\begin{aligned}
\mathbf{H}_\Delta(\mathbf{f},\alpha+\beta) &\ge \frac{\prod_{i=1}^{n}\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\alpha_i}\prod_{i=1}^{n}\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\beta_i}}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\alpha_i}{\mathcal{A}_n}} d\mu_\Delta\right)^{\mathcal{A}_n}\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{\beta_i}{\mathcal{B}_n}} d\mu_\Delta\right)^{\mathcal{B}_n}} \\
&= \mathbf{H}_\Delta(\mathbf{f},\alpha)\cdot\mathbf{H}_\Delta(\mathbf{f},\beta).
\end{aligned}$$

If $\alpha\ge\beta$, it implies $\alpha-\beta\ge 0$ and therefore

$$\begin{aligned}
\mathbf{H}_\Delta(\mathbf{f},\alpha) &= \mathbf{H}_\Delta(\mathbf{f},(\alpha-\beta)+\beta) \\
&\ge \mathbf{H}_\Delta(\mathbf{f},\alpha-\beta)\cdot\mathbf{H}_\Delta(\mathbf{f},\beta) \\
&\ge \mathbf{H}_\Delta(\mathbf{f},\beta).
\end{aligned}$$

□

Corollary 5.5.3. *Let* $\mathbf{f}$ *and* α *satisfy the hypothesis of Theorem* 5.5.2. *Then*

$$\left[\frac{\prod_{i=1}^{n}\int_{[a,b)} f_i d\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}} d\mu_\Delta\right)^{n}}\right]^{\max\limits_{1\le i\le n}\{\alpha_i\}} \ge \mathbf{H}_\Delta(\mathbf{f},\alpha) \ge \left[\frac{\prod_{i=1}^{n}\int_{[a,b)} f_i d\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\frac{1}{n}} d\mu_\Delta\right)^{n}}\right]^{\min\limits_{1\le i\le n}\{\alpha_i\}}. \tag{5.5.9}$$

Proof. Let

$$\alpha_{\max} = \left(\max_{1\le i\le n}\{\alpha_i\}, \ldots, \max_{1\le i\le n}\{\alpha_i\} \right) \quad \text{and} \quad \alpha_{\min} = \left(\min_{1\le i\le n}\{\alpha_i\}, \ldots, \min_{1\le i\le n}\{\alpha_i\} \right),$$

now by Definition 5.5.1, we have

$$\mathbf{H}_\Delta(\mathbf{f}, \alpha_{\max}) = \left[\frac{\prod_{i=1}^{n} \int_{[a,b)} f_i d\mu_\Delta}{\left(\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{n}} d\mu_\Delta \right)^n} \right]^{\max_{1\le i\le n}\{\alpha_i\}}$$

and

$$\mathbf{H}_\Delta(\mathbf{f}, \alpha_{\min}) = \left[\frac{\prod_{i=1}^{n} \int_{[a,b)} f_i d\mu_\Delta}{\left(\int_{[a,b)} \prod_{i=1}^{n} f_i^{\frac{1}{n}} d\mu_\Delta \right)^n} \right]^{\min_{1\le i\le n}\{\alpha_i\}}.$$

Since $\alpha_{\max} \geq \alpha \geq \alpha_{\min}$, the result follows from the second property of Theorem 5.5.2. □

Corollary 5.5.4. *Let* $\mathbf{f}, \alpha$ *and* β *satisfy the hypothesis of Theorem* 5.5.2 *with* $\mathcal{A}_n = \mathcal{B}_n = 1$. *If there exists constants* $M > 1 > m$ *such that* $M\beta \geq \alpha \geq m\beta$, *then*

$$\mathbf{H}_\Delta(\mathbf{f}, M\beta) \geq \mathbf{H}_\Delta(\mathbf{f}, \alpha) \geq \mathbf{H}_\Delta(\mathbf{f}, m\beta). \tag{5.5.10}$$

Proof. By Definition 5.5.1, we have

$$\mathbf{H}_\Delta(\mathbf{f}, M\beta) = M\mathbf{H}_\Delta(\mathbf{f}, \beta) \quad \text{and} \quad \mathbf{H}_\Delta(\mathbf{f}, m\beta) = m\mathbf{H}_\Delta(\mathbf{f}, \beta).$$

Now result follows from the second property of Theorem 5.5.2. □

Remark 5.5.5. Similarly as in Chapter 3, we can apply the theory of isotonic linear functionals. The related results for isotonic linear functionals are given in [75].

Chapter 6

Improvements of the Jensen–Steffensen Inequality and Its Converse

In this chapter we give a generalization of the Jensen–Steffensen inequality and its converse on time scales. These results also generalize the Jensen–Steffensen inequality and its converse given for discrete and continuous case. Further, we investigate the exponential and logarithmic convexity of the functionals defined as differences of the left-hand and the right-hand side of these inequalities. Finally we present several families of functions for which these results can be applied. (See [37]).

6.1 α-SP and α-HH weights

In order to give a better version of the Jensen inequality (Theorem 3.2.7) on time scales, C. Dinu in [48] gives the definition of α-Steffensen–Popoviciu (α-SP) weight.

Definition 6.1.1 (α-SP weight)**.** Let $g \in \mathrm{C}(\mathbb{T}, \mathbb{R})$. Then $w \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ is an α-Steffensen–Popoviciu (α-SP) weight for g on $[a, b]_{\mathbb{T}}$ if

$$\int_a^b w(t)\diamond_\alpha t > 0 \quad \text{and} \quad \int_a^b \Phi(g(t))^+ w(t)\diamond_\alpha t \geq 0, \tag{6.1.1}$$

for every convex function $\Phi \in \mathrm{C}([m, M], \mathbb{R})$, where

$$m = \inf_{t\in[a,b]_{\mathbb{T}}} g(t) \quad \text{and} \quad M = \sup_{t\in[a,b]_{\mathbb{T}}} g(t).$$

In the following lemma he gives a characterization for α-SP weight for a nondecreasing function g on time scales.

Lemma 6.1.2 (see [48, Lemma 2])**.** *Let $w \in \mathrm{C}(\mathbb{T},\mathbb{R})$ such that $\int_a^b w(t)\diamond_\alpha t > 0$. Then w is an α-SP weight for a nondecreasing function $g \in \mathrm{C}([a,b]_\mathbb{T},\mathbb{R})$ if and only if it verifies the following condition:*

$$\int_a^s (g(s)-g(t))w(t)\diamond_\alpha t \geq 0 \quad \textit{and} \quad \int_s^b (g(t)-g(s))w(t)\diamond_\alpha t \geq 0, \tag{6.1.2}$$

for every $s \in [a,b]_\mathbb{T}$.

If the following stronger (but more suitable) condition holds

$$0 \leq \int_a^s w(t)\diamond_\alpha t \leq \int_a^b w(t)\diamond_\alpha t \quad \textit{for every} \quad s \in [a,b]_\mathbb{T}, \tag{6.1.3}$$

then w is also an α-SP weight for the nondecreasing continuous function g.

As given in [48], all positive weights are α-SP weights, for any continuous function g and every $\alpha \in [0,1]$. But there are some α-SP weights that are allowed to take the negative values. The Jensen inequality on time scales (where is allowed that the weight function takes some negative values) is given in the followig theorem from [48].

Theorem 6.1.3 (see [48, Theorem 2])**.** *Let $g \in \mathrm{C}([a,b]_\mathbb{T},[m,M])$ and let $w \in \mathrm{C}([a,b]_\mathbb{T},\mathbb{R})$ such that $\int_a^b w(t)\diamond_\alpha t > 0$. Then the following two statements are equivalent:*

(i) *w is an α-SP weight for g on $[a,b]_\mathbb{T}$;*

(ii) *for every convex function $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, it holds*

$$\Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \leq \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}. \tag{6.1.4}$$

Remark 6.1.4. Let g be a nondecreasing function. If $\mathbb{T} = \mathbb{N}$, then Theorem 6.1.3 is equivalent to Theorem 1.0.2 (Jensen–Steffensen inequality). On the other hand if we take $\mathbb{T} = \mathbb{R}$ in Theorem 6.1.3, we obtain the integral version of Jensen–Steffensen inequality given by Boas [38].

Considering the converse of the Jensen inequality, C. Dinu [48] gives the following definition of α-Hermite–Hadamard (α-HH) weight. He gives the characterization for a nondeceasing function g on time scales and the improvement of the converse of the Jensen inequality for some negative weights.

Definition 6.1.5 (α-HH weight)**.** Let $g \in \mathrm{C}(\mathbb{T},\mathbb{R})$. Then $w \in \mathrm{C}(\mathbb{T},\mathbb{R})$ is an α-Hermite–Hadamard (α-HH) weight for g on $[a,b]_\mathbb{T}$ if

$$\int_a^b w(t)\diamond_\alpha t > 0 \quad \text{and} \quad \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{M-\overline{g}_\alpha}{M-m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M-m}\Phi(M),$$

for every convex function $\Phi \in \mathrm{C}([m,M],\mathbb{R})$, where

$$m = \inf_{t\in[a,b]_{\mathbb{T}}} g(t),\ M = \sup_{t\in[a,b]_{\mathbb{T}}} g(t) \quad \text{and} \quad \overline{g}_\alpha = \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}.$$

Lemma 6.1.6 (see [48, Lemma 3]). *Let $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be such that $\int_a^b w(t)\diamond_\alpha t > 0$. Then w is an α-HH weight for a nondecreasing function $g \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ if and only if it verifies the following condition:*

$$\frac{g(b)-g(s)}{g(b)-g(a)}\int_a^s (g(t)-g(a))w(t)\diamond_\alpha t + \frac{g(s)-g(a)}{g(b)-g(a)}\int_s^b (g(b)-g(t))w(t)\diamond_\alpha t \geq 0. \quad (6.1.5)$$

for every $s \in [a,b]_{\mathbb{T}}$.

In the next result C. Dinu [48] gives the connection between these two classes of weights on a time scale .

Theorem 6.1.7 (see [48, Theorem 3]). *Let $g \in \mathrm{C}(\mathbb{T},\mathbb{R})$. Then every α-SP weight for g on $[a,b]_{\mathbb{T}}$ is an α-HH weight for g on $[a,b]_{\mathbb{T}}$, for all $\alpha \in [0,1]$.*

Corollary 6.1.8 (see [48, Corollary 3]). *Let $g \in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$. Let $\Phi \in \mathrm{C}([m,M],\mathbb{R})$ be a convex function and $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ an α-SP weight for g on $[a,b]_{\mathbb{T}}$. Then*

$$\Phi\left(\overline{g}_\alpha\right) \leq \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{M-\overline{g}_\alpha}{M-m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M-m}\Phi(M), \quad (6.1.6)$$

where $\overline{g}_\alpha = \dfrac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}$.

6.2 Jensen–Steffensen inequality

Let $m, M \in \mathbb{R}$, where $m \neq M$. Consider the Green function $G : [m,M]\times[m,M] \to \mathbb{R}$ defined by

$$G(x,y) = \begin{cases} \dfrac{(x-M)(y-m)}{M-m}, & m \leq y \leq x, \\[2ex] \dfrac{(y-M)(x-m)}{M-m}, & x \leq y \leq M. \end{cases} \quad (6.2.1)$$

The function G is convex and continuous with respect to both x and y.

Remark 6.2.1. Note that the condition (6.1.5) is equivalent to:

$$\int_a^b G(g(t), g(s))w(t)\diamond_\alpha t \leq 0,$$

where the function G is defined in (6.2.1).

It is well known that (see for example [71, 87, 92, 104]) any function $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$ can be represented by

$$\Phi(x) = \frac{M - x}{M - m}\Phi(m) + \frac{x - m}{M - m}\Phi(M) + \int_m^M G(x, y)\Phi''(y)\mathrm{d}y, \tag{6.2.2}$$

where the function G is defined in (6.2.1). Using (6.2.2) we now derive several interesting results concerning the Jensen type inequalities.

Firstly, we give generalization of the Jensen inequality on time scales, where negative weights are also allowed.

Theorem 6.2.2. *Assume $g \in \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ such that $g([a, b]_{\mathbb{T}}) \subseteq [m, M]$. Let $w \in \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ be such that $\int_a^b w(t)\diamond_\alpha t \neq 0$ and $\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \in [m, M]$. Then the following two statements are equivalent:*

(i) *For every convex function $\Phi \in \mathrm{C}([m, M], \mathbb{R})$*

$$\Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \leq \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \tag{6.2.3}$$

holds.

(ii) *For all $y \in [m, M]$*

$$G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right) \leq \frac{\int_a^b G(g(t), y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \tag{6.2.4}$$

holds, where $G : [m, M] \times [m, M] \to \mathbb{R}$ is defined in (6.2.1).

Furthermore the statements (i) *and* (ii) *are also equivalent if we change the sign of inequality in both* (6.2.3) *and* (6.2.4).

Proof. (i) $\Rightarrow$ (ii): Let (i) holds. As the function $G(\cdot, y)$, where $y \in [m, M]$, is also continuous and convex, it follows that (6.2.4) holds.

(ii) $\Rightarrow$ (i): Let (ii) holds. Let $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$. By using (6.2.2), we get

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)$$
$$= \int_m^M \left[\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right)\right] \Phi''(y)\mathrm{d}y. \quad (6.2.5)$$

If the function Φ is also convex, then $\Phi''(y) \geq 0$ for all $y \in [m, M]$, and hence it follows that for every convex function $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$ inequality (6.2.3) holds. Moreover, it is not necessary to demand the existence of the second derivative of the function Φ (see [93]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The last part of our theorem can be proved analogously. □

Remark 6.2.3. Let the conditions of Theorem 6.2.2 hold. Then the following two statements are equivalent:

(i′) For every concave function $\Phi \in \mathrm{C}([m, M], \mathbb{R})$ the reverse inequality in (6.2.3) holds.

(ii′) For all $y \in [m, M]$ inequality (6.2.4) holds.

Moreover, the statements (i′) and (ii′) are also equivalent if we change the sign of inequality in both statements (i′) and (ii′).

Remark 6.2.4. Consider (6.2.5). Suppose that g is nondecreasing and that it has the first derivative. Let $m = g(a)$, $M = g(b)$ and make the substitution $y = g(s)$. Then we get

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right)$$
$$= \int_a^b \left[\frac{\int_a^b G(g(t),g(s))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, g(s)\right)\right] \Phi''(g(s))g'(s)\mathrm{d}s. \quad (6.2.6)$$

Since g is nondecreasing, it is $g'(s) \geq 0$. If $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$ is convex, then $\Phi''(g(s)) \geq 0$, for all $s \in [a, b]_{\mathbb{T}}$. Hence if and only if for all $s \in [a, b]_{\mathbb{T}}$

$$G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, g(s)\right) \leq \frac{\int_a^b G(g(t),g(s))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}$$

holds, then for every continuous convex function Φ inequality (6.2.3) holds.

Combining the result from Theorem 6.2.2 with Theorem 6.1.3 and Lemma 6.1.2, we get the following two corollaries.

Corollary 6.2.5. *Let* $g \in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$ *and* $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ *such that* $\int_a^b w(t)\diamond_\alpha t > 0$. *Then* w *is an* α*-SP weight for* g *on* $[a,b]_{\mathbb{T}}$ *if and only if for all* $y \in [m,M]$ (6.2.4) *holds.*

Corollary 6.2.6. *Let* $g \in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$ *is nondecreasing and* $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ *be such that* $\int_a^b w(t)\diamond_\alpha t > 0$. *Then for all* $s \in [a,b]_{\mathbb{T}}$ (6.1.2) *holds, if and only if for all* $y \in [m,M]$ (6.2.4) *holds.*

To shorten the notation, in the sequel we will use the notation

$$\overline{g}_\alpha = \frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}.$$

Under the assumptions of the Theorem 6.2.2, we define the following functional $\mathcal{J}_{\alpha 1}(g,\Phi)$:

$$\mathcal{J}_{\alpha 1}(g,\Phi) = \begin{cases} \dfrac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\dfrac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \\ \quad \text{if for all} \quad y \in [m,M] \quad \text{the inequality (6.2.4) holds,} \\ \\ \Phi\left(\dfrac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) - \dfrac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\ \quad \text{if for all} \quad y \in [m,M] \quad \text{the reverse inequality in (6.2.4) holds,} \end{cases} \tag{6.2.7}$$

where the function Φ is defined on $[m,M]$. Clearly, if Φ is continuous and convex, then $\mathcal{J}_{\alpha 1}(g,\Phi)$ is nonnegative.

Theorem 6.2.7. *Let* g, w *and* $\overline{g}_\alpha$ *satisfy the assumptions of Theorem* 6.2.2, *let* $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$, *and let* $\mathcal{J}_{\alpha 1}$ *be the functional defined in* (6.2.7). *Then there exists some* $\xi \in [m,M]$ *such that*

$$\mathcal{J}_{\alpha 1}(g,\Phi) = \Phi''(\xi)\mathcal{J}_{\alpha 1}(g,\Phi_0) \tag{6.2.8}$$

holds, where $\Phi_0(x) = \frac{x^2}{2}$.

Proof. The function Φ'' is continuous and

$$\frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right)$$

doesn't change it's positivity on $[m, M]$. For our function Φ equality (6.2.5) holds, and now applying the integral mean value theorem we get that there exists some $\xi \in [m, M]$ such that

$$\begin{aligned}
&\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \\
&\qquad = \Phi''(\xi)\int_m^M \left[\frac{\int_a^b G(g(t), y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - G\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}, y\right)\right] \mathrm{d}y. \quad (6.2.9)
\end{aligned}$$

As in [92], it can be easily checked that it holds

$$\begin{aligned}
\int_m^M G(x, y)\mathrm{d}y &= \int_m^x \frac{(x-M)(y-m)}{M-m}\mathrm{d}y + \int_x^M \frac{(y-M)(x-m)}{M-m}\mathrm{d}y \\
&= \frac{1}{2}(x-m)(x-M).
\end{aligned}$$

Calculating the integral on the right side in (6.2.9) we get

$$\begin{aligned}
&\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \Phi\left(\frac{\int_a^b g(t)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right) \\
&\quad = \Phi''(\xi)\left[\frac{1}{\int_a^b w(t)\diamond_\alpha t}\int_a^b \left(\int_m^M G(g(t), y)\mathrm{d}y\right) w(t)\diamond_\alpha t - \int_m^M G(\overline{g}_\alpha, y)\mathrm{d}y\right] \\
&\quad = \Phi''(\xi)\left[\frac{1}{\int_a^b w(t)\diamond_\alpha t}\int_a^b \frac{1}{2}(g(t)-m)(g(t)-M)w(t)\diamond_\alpha t - \frac{1}{2}(\overline{g}_\alpha - m)(\overline{g}_\alpha - M)\right] \\
&\quad = \frac{1}{2}\Phi''(\xi)\left[\frac{\int_a^b (g(t))^2 w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \overline{g}_\alpha^2\right]
\end{aligned}$$

and the proof is completed. □

Remark 6.2.8. Theorem 6.2.7 can also be proved by using the following two convex functions

$$\phi_1(x) = \frac{\Phi^*}{2}x^2 - \Phi(x) \quad \text{and} \quad \phi_2(x) = \Phi(x) - \frac{\Phi_*}{2}x^2$$

where

$$\Phi_* = \min_{x\in[m,M]} \Phi''(x) \quad \text{and} \quad \Phi^* = \max_{x\in[m,M]} \Phi''(x).$$

Since ϕ_1 and ϕ_2 are continuous and convex, we have

$$\mathcal{J}_{\alpha 1}(g, \phi_1) \geq 0 \quad \text{and} \quad \mathcal{J}_{\alpha 1}(g, \phi_2) \geq 0.$$

This implies that

$$\Phi_* \mathcal{J}_{\alpha 1}(g, \Phi_0) \leq \mathcal{J}_{\alpha 1}(g, \Phi) \leq \Phi^* \mathcal{J}_{\alpha 1}(g, \Phi_0).$$

Hence, as the function Φ'' is continuous, there exists some $\xi \in [m, M]$ such that (6.2.8) holds.

Theorem 6.2.9. *Let g, w and $\overline{g}_\alpha$ satisfy the assumptions of Theorem* 6.2.2. *Let $\Phi, \Psi \in C^2([m, M], \mathbb{R})$ and $\mathcal{J}_{\alpha 1}$ be the functional defined in* (6.2.7). *Then there exists some $\xi \in [m, M]$ such that*

$$\frac{\mathcal{J}_{\alpha 1}(g, \Phi)}{\mathcal{J}_{\alpha 1}(g, \Psi)} = \frac{\Phi''(\xi)}{\Psi''(\xi)}, \tag{6.2.10}$$

provided that the denominator in the left-hand side of (6.2.10) *is nonzero.*

Proof. Consider the following function χ, defined as the linear combination of functions Φ and Ψ by

$$\chi(x) = \mathcal{J}_{\alpha 1}(g, \Psi)\Phi(x) - \mathcal{J}_{\alpha 1}(g, \Phi)\Psi(x).$$

It is $\chi \in C^2([m, M], \mathbb{R})$. By applying Theorem 6.2.7 on χ, it follows that there exists some $\xi \in [m, M]$ such that

$$\mathcal{J}_{\alpha 1}(g, \chi) = \chi''(\xi)\mathcal{J}_{\alpha 1}(g, \Phi_0). \tag{6.2.11}$$

After a short calculation we get that $\mathcal{J}_{\alpha 1}(g, \chi) = 0$ and by hypothesis $\mathcal{J}_{\alpha 1}(g, \Phi_0) \neq 0$ (otherwise we have a contradiction with $\mathcal{J}_{\alpha 1}(g, \Psi) \neq 0$) it follows that

$$\chi''(\xi) = 0$$

which is equivalent to (6.2.10). □

Remark 6.2.10. In Theorem 6.2.9, if the inverse of the function $\frac{\Phi''}{\Psi''}$ exists, then (6.2.10) gives

$$\xi = \left(\frac{\Phi''}{\Psi''}\right)^{-1} \left(\frac{\mathcal{J}_{\alpha 1}(g, \Phi)}{\mathcal{J}_{\alpha 1}(g, \Psi)}\right).$$

Remark 6.2.11. Note that setting the function Ψ as $\Psi(x) = \frac{x^2}{2}$ in Theorem 6.2.9, we get the statement of Theorem 6.2.7.

As a consequence of the above two mean value theorems, the following corollaries easily follow.

Corollary 6.2.12. *Let $g \in C([a, b]_{\mathbb{T}}, [m, M])$, $\Phi, \Psi : [m, M] \to \mathbb{R}$ and let $w \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be an α-SP weight for g. Let $\mathcal{J}_{\alpha 1}$ be the functional defined in* (6.2.7). *Then the following two statements hold:*

(i) *If $\Phi \in C^2([m, M], \mathbb{R})$, then there exists some $\xi \in [m, M]$ such that* (6.2.8) *holds.*

(ii) *If* $\Phi, \Psi \in \mathrm{C}^2([m, M], \mathbb{R})$, *then there exists some* $\xi \in [m, M]$ *such that* (6.2.10) *holds.*

Proof. The statement (i) (statement (ii), respectively) directly follows from Theorem 6.2.7 (Theorem 6.2.9, respectively) and Corollary 6.2.5. □

Corollary 6.2.13. *Let* $g \in \mathrm{C}([a, b]_{\mathbb{T}}, [m, M])$ *be monotone function,* $\Phi, \Psi : [m, M] \to \mathbb{R}$ *and* $w \in \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ *such that* $\int_a^b w(t)\diamond_\alpha t > 0$. *Let for all* $s \in [a, b]_{\mathbb{T}}$, (6.1.2) *hold. Let* $\mathcal{J}_{\alpha 1}$ *be the functional defined in* (6.2.7). *Then the following two statements hold:*

(i) *If* $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$, *then there exists some* $\xi \in [m, M]$ *such that* (6.2.8) *holds.*

(ii) *If* $\Phi, \Psi \in \mathrm{C}^2([m, M], \mathbb{R})$, *then there exists some* $\xi \in [m, M]$ *such that* (6.2.10) *holds.*

Proof. The statement (i) (statement (ii), respectively) directly follows from Theorem 6.2.7 (Theorem 6.2.9, respectively) and Corollary 6.2.6. □

6.3 Converse of Jensen–Steffensen inequality

Using the similar method as in previous section, in the following theorem we obtain the generalization of the converse of Jensen–Steffensen inequality on time scales, where negative weights are also allowed.

Theorem 6.3.1. *Let* $g \in \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ *be such that* $g([a, b]_{\mathbb{T}}) \subseteq [m, M]$ *and let* $c, d \in [m, M]$, *where* $c \neq d$, *be such that* $c \leq g(t) \leq d$ *for all* $t \in [a, b]_{\mathbb{T}}$. *Let* $w \in \mathrm{C}([a, b]_{\mathbb{T}}, \mathbb{R})$ *be such that* $\int_a^b w(t)\diamond_\alpha t \neq 0$. *Then the following two statements are equivalent:*

(i) *For every convex function* $\Phi \in \mathrm{C}([m, M], \mathbb{R})$

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{d - \overline{g}_\alpha}{d - c}\Phi(c) + \frac{\overline{g}_\alpha - c}{d - c}\Phi(d) \tag{6.3.1}$$

holds.

(ii) *For all* $y \in [m, M]$

$$\frac{\int_a^b G(g(t), y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{d - \overline{g}_\alpha}{d - c}G(c, y) + \frac{\overline{g}_\alpha - c}{d - c}G(d, y) \tag{6.3.2}$$

holds, where the function $G : [m, M] \times [m, M] \to \mathbb{R}$ *is defined in* (6.2.1).

Furthermore, the statements (i) *and* (ii) *are also equivalent if we change the sign of inequality in both* (6.3.1) *and* (6.3.2)*.*

Proof. The idea of the proof is very similar to the proof of the Theorem 6.2.2.

(i) $\Rightarrow$ (ii) : Let (i) holds. As the function $G(\cdot, y)$, where ($y \in [m, M]$), is also continuous and convex, it follows that (6.3.2) holds.

(ii) $\Rightarrow$ (i) : Let (ii) holds. Let $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$. Then by using (6.2.2) we get

$$\begin{aligned} &\frac{d-\overline{g}_\alpha}{d-c}\Phi(c) + \frac{\overline{g}_\alpha - c}{d-c}\Phi(d) - \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\ &\quad = \int_m^M \left[\frac{d-\overline{g}_\alpha}{d-c}G(c,y) + \frac{\overline{g}_\alpha - c}{d-c}G(d,y) - \frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right]\Phi''(y)\mathrm{d}y. \quad (6.3.3) \end{aligned}$$

If the function Φ is also convex, then $\Phi''(y) \geq 0$ for all $y \in [m, M]$, and hence it follows that for every convex function $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$ inequality (6.3.1) holds. Moreover, it is not necessary to demand the existence of the second derivative of the function Φ (see [93]). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The last part of our theorem can be proved analogously. □

Remark 6.3.2. Let the conditions of Theorem 6.3.1 hold. Then the following two statements are equivalent:

(i′) For every concave function $\Phi \in \mathrm{C}([m, M], \mathbb{R})$ the reverse inequality in (6.3.1) holds.

(ii′) For all $y \in [m, M]$ inequality (6.3.2) holds.

Moreover, the statements (i′) and (ii′) are also equivalent if we change the sign of inequality in both statements (i′) and (ii′).

Remark 6.3.3. Note that in all the results in this section we allow that the mean value $\overline{g}_\alpha$ goes out of the interval $[m, M]$, while in the results from the previous section we demanded that $\overline{g}_\alpha \in [m, M]$.

Seting $c = m$ and $d = M$ in Theorem 6.3.1, we get the following result.

Corollary 6.3.4. *Let* $g \in \mathrm{C}([a, b]_\mathbb{T}, \mathbb{R})$ *be such that* $g([a, b]_\mathbb{T}) \subseteq [m, M]$*. Let* $w \in \mathrm{C}([a, b]_\mathbb{T}, \mathbb{R})$ *be such that* $\int_a^b w(t)\diamond_\alpha t \neq 0$*. Then the following two statements are equivalent:*

(i) *For every convex function* $\Phi \in \mathrm{C}([m, M], \mathbb{R})$

$$\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq \frac{M - \overline{g}_\alpha}{M - m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M - m}\Phi(M) \tag{6.3.4}$$

holds.

(ii) *For all* $y \in [m, M]$

$$\frac{\int_a^b G(g(t), y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \leq 0 \tag{6.3.5}$$

holds, where the function $G : [m, M] \times [m, M] \to \mathbb{R}$ *is defined in* (6.2.1).

Furthermore, the statements (i) *and* (ii) *are also equivalent if we change the sign of inequality in both* (6.3.4) *and* (6.3.5).

Remark 6.3.5. As a consequence of Corollary 6.3.4, we get the result from Lemma 6.1.6.

Let $c = m$ and $d = M$. Then (6.3.3) transforms into

$$\frac{M - \overline{g}_\alpha}{M - m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M - m}\Phi(M) - \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} = -\int_m^M \frac{\int_a^b G(g(t), y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\Phi''(y)\mathrm{d}y. \tag{6.3.6}$$

Let $\int_a^b w(t)\diamond_\alpha t > 0$, and suppose that g is nondecreasing and that it has the first derivative.

Now, similarly as in [48], we derive the result from Lemma 6.1.6.

Let $m = g(a)$, $M = g(b)$, and make the substitution $y = g(s)$. Then we get

$$\frac{M - \overline{g}_\alpha}{M - m}\Phi(m) + \frac{\overline{g}_\alpha - m}{M - m}\Phi(M) - \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} = -\frac{1}{\int_a^b w(t)\diamond_\alpha t}\int_a^b \left(\int_a^b G(g(t), g(s))w(t)\diamond_\alpha t\right)\Phi''(g(s))g'(s)\mathrm{d}s. \tag{6.3.7}$$

Since g is nondecreasing, it is $g'(s) \geq 0$. If $\Phi \in \mathrm{C}^2([m, M], \mathbb{R})$ is convex, then $\Phi''(g(s)) \geq 0$, for all $s \in [a, b]_\mathbb{T}$. Hence, if and only if for all $s \in [a, b]_\mathbb{T}$

$$\int_a^b G(g(t), g(s))w(t)\diamond_\alpha t \leq 0$$

holds, then for every continuous convex function Φ inequality (6.3.4) holds.

Corollary 6.3.6. *Let $g \in \mathrm{C}([a,b]_{\mathbb{T}},[m,M])$ and $w \in \mathrm{C}([a,b]_{\mathbb{T}},\mathbb{R})$ be an α-SP weight for g on $[a,b]_{\mathbb{T}}$. Then for all $y \in [m,M]$ (6.3.5) holds.*

Proof. The proof follows directly from Theorem 6.1.7 and Corollary 6.3.4. □

Under the assumptions of Theorem 6.3.1, we define the following functional $\mathcal{J}_{\alpha 2}(g,\Phi)$:

$$\mathcal{J}_{\alpha 2}(g,\Phi) = \begin{cases} \dfrac{d-\overline{g}_\alpha}{d-c}\Phi(c) + \dfrac{\overline{g}_\alpha - c}{d-c}\Phi(d) - \dfrac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\ \qquad \text{if for all } \; y \in [m,M] \; \text{ the inequality (6.3.2) holds,} \\[2ex] \dfrac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} - \dfrac{d-\overline{g}_\alpha}{d-c}\Phi(c) - \dfrac{\overline{g}_\alpha - c}{d-c}\Phi(d) \\ \qquad \text{if for all } \; y \in [m,M] \; \text{ the reverse inequality in (6.3.2) holds,} \end{cases} \tag{6.3.8}$$

where the function Φ is defined on $[m,M]$. Clearly, if Φ is continuous and convex, then $\mathcal{J}_{\alpha 2}(g,\Phi)$ is nonnegative.

Theorem 6.3.7. *Let c, d, g and w satisfy the assumptions of Theorem 6.3.1. Let $\Phi \in \mathrm{C}^2([m,M],\mathbb{R})$ and $\mathcal{J}_{\alpha 2}$ be the functional defined in (6.3.8). Then there exists some $\xi \in [m,M]$ such that*

$$\mathcal{J}_{\alpha 2}(g,\Phi) = \Phi''(\xi)\mathcal{J}_{\alpha 2}(g,\Phi_0) \tag{6.3.9}$$

holds, where $\Phi_0(x) = \frac{x^2}{2}$.

Proof. The idea of the proof is very similar to the proof of the Theorem 6.2.7.

Following the assumptions of our theorem, we have that Φ'' is continuous and that

$$\frac{d-\overline{g}_\alpha}{d-c}G(c,y) + \frac{\overline{g}_\alpha - c}{d-c}G(d,y) - \frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}$$

doesn't change it's positivity on $[m,M]$. For our function Φ the equality (6.3.3) holds, and now applying the integral mean value theorem we get that there exists some $\xi \in [m,M]$ such that

$$\begin{aligned} &\frac{d-\overline{g}_\alpha}{d-c}\Phi(c) + \frac{\overline{g}_\alpha - c}{d-c}\Phi(d) - \frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t} \\ &= \Phi''(\xi)\int_m^M \left[\frac{d-\overline{g}_\alpha}{d-c}G(c,y) + \frac{\overline{g}_\alpha - c}{d-c}G(d,y) - \frac{\int_a^b G(g(t),y)w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right]\mathrm{d}y. \end{aligned} \tag{6.3.10}$$

Calculating the integral on the right side in (6.3.10) we get

$$\frac{d-\overline{g}_\alpha}{d-c}\Phi(c)+\frac{\overline{g}_\alpha-c}{d-c}\Phi(d)-\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}$$
$$=\frac{1}{2}\Phi''(\xi)\left[\frac{d-\overline{g}_\alpha}{d-c}\cdot c^2+\frac{\overline{g}_\alpha-c}{d-c}\cdot d^2-\frac{\int_a^b (g(t))^2 w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right] \quad (6.3.11)$$

and we get the statement (6.3.9) of our theorem. □

Remark 6.3.8. Note that (6.3.11) can also be expressed as

$$\frac{d-\overline{g}_\alpha}{d-c}\Phi(c)+\frac{\overline{g}_\alpha-c}{d-c}\Phi(d)-\frac{\int_a^b \Phi(g(t))w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}$$
$$=\frac{1}{2}\Phi''(\xi)\left[\overline{g}_\alpha(c+d)-cd-\frac{\int_a^b (g(t))^2 w(t)\diamond_\alpha t}{\int_a^b w(t)\diamond_\alpha t}\right].$$

Theorem 6.3.9. *Let* c, d, g *and* w *satisfy the assumptions of Theorem* 6.3.1. *Let* $\Phi, \Psi \in \mathrm{C}^2([m, M], \mathbb{R})$ *and* $\mathcal{J}_{\alpha 2}$ *be the functional defined in* (6.3.8). *Then there exists some* $\xi \in [m, M]$ *such that*

$$\frac{\mathcal{J}_{\alpha 2}(g,\Phi)}{\mathcal{J}_{\alpha 2}(g,\Psi)}=\frac{\Phi''(\xi)}{\Psi''(\xi)}, \quad (6.3.12)$$

provided that the denominator in the left-hand side of (6.3.12) *is nonzero.*

Proof. The proof is very similar to the proof of the Theorem 6.2.9. □

6.4 Exponential convexity and logarithmic convexity

We use an idea from [67] to give an elegant method of producing an n–exponentially convex and exponentially convex functions, applying the functionals $\mathcal{J}_{\alpha 1}$ and $\mathcal{J}_{\alpha 2}$ on a given family of functions with the same property.

Theorem 6.4.1. *Let* $\mathcal{J}_{\alpha i}$, $i = 1, 2$, *be linear functionals defined in* (6.2.7) *and* (6.3.8) *respectively. Let* $\Omega = \{\Phi_\rho :\ \rho \in J\}$, *where* J *is an interval in* $\mathbb{R}$, *be a family of functions* $\Phi_\rho \in \mathrm{C}([m, M], \mathbb{R})$ *such that the function* $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ *is* n*–exponentially convex in the Jensen sense on* J *for every choice of three mutually different points* $x_0, x_1, x_2 \in [m, M]$. *Then* $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ *is an* n*–exponentially convex function in the Jensen sense on* J. *If the function* $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ *is also continuous on* J, *then it is* n*–exponentially convex on* J.

Proof. Define the function $\nu : I \to \mathbb{R}$ by

$$\nu(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \Phi_{r_{jk}}(x)$$

where $\xi_j \in \mathbb{R}$, $r_j, r_k \in J$, $1 \leq j,k \leq n$, $r_{jk} = \frac{r_j + r_k}{2}$ and $\Phi_{r_{jk}} \in \Omega$. Using the assumption that for every choice of three mutually different points $x_0, x_1, x_2 \in [m, M]$ $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is n–exponentially convex in the Jensen sense on J, we obtain that

$$[x_0, x_1, x_2; \nu] = \sum_{j,k=1}^{n} \xi_j \xi_k [x_0, x_1, x_2; \Phi_{r_{jk}}] \geq 0.$$

Therefore ν is convex (and continuous) function on I. Hence $\mathcal{J}_{\alpha i}(g, \nu) \geq 0,\ i = 1, 2,$ which implies that

$$\sum_{j,k=1}^{n} \xi_j \xi_k \mathcal{J}_{\alpha i}(g, \Phi_{r_{jk}}) \geq 0.$$

We conclude that the function $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is n–exponentially convex on J in the Jensen sense.

If $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is continuous on J, then $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is n–exponentially convex by definition. □

The following corollary is immediate consequence of Theorem 6.4.1.

Corollary 6.4.2. *Let $\mathcal{J}_{\alpha i}$, $i = 1, 2$, be linear functionals defined in* (6.2.7) *and* (6.3.8) *respectively. Let $\Omega = \{\Phi_\rho :\ \rho \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions $\Phi_\rho \in \mathrm{C}([m, M], \mathbb{R})$ such that the function $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is exponentially convex in the Jensen sense on J for every choice of three mutually different points $x_0, x_1, x_2 \in [m, M]$. Then $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is an exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is also continuous on J, then it is exponentially convex on J.*

Corollary 6.4.3. *Let $\mathcal{J}_{\alpha i}$, $i = 1, 2$, be linear functionals defined in* (6.2.7) *and* (6.3.8) *respectively. Let $\Omega = \{\Phi_\rho :\ \rho \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions $\Phi_\rho \in \mathrm{C}([m, M], \mathbb{R})$ such that the function $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is 2–exponentially convex in the Jensen sense on J for every choice of three mutually different points $x_0, x_1, x_2 \in [m, M]$. Then the following statements hold:*

(i) *$\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is a 2–exponentially convex function in the Jensen sense on J.*

(ii) *If $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is continuous on J, then it is also 2–exponentially convex on J. If $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is additionallly strictly positive, then it is also log-convex on J.*

(iii) *If* $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ *is strictly positive and differentiable function on* J*, then for every* $p, q, u, v \in J$ *such that* $p \leq u$*,* $q \leq v$*, we have*

$$\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega) \leq \mathcal{M}_{u,v}(g, \mathcal{J}_{\alpha i}, \Omega), \quad i = 1, 2, \tag{6.4.1}$$

where

$$\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega) = \begin{cases} \left(\dfrac{\mathcal{J}_{\alpha i}(g, \Phi_p)}{\mathcal{J}_{\alpha i}(g, \Phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp \left(\dfrac{\frac{d}{dp} \mathcal{J}_{\alpha i}(g, \Phi_p)}{\mathcal{J}_{\alpha i}(g, \Phi_p)} \right), & p = q, \end{cases} \tag{6.4.2}$$

for $\Phi_p, \Phi_q \in \Omega$*.*

Proof. (i) and (ii) are immediate consequences of Theorem 6.4.1.

To prove (iii), let $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ be strictly positive and differentiable, and therefore continuous too. By (ii), the function $\rho \mapsto \mathcal{J}_{\alpha i}(g, \Phi_\rho)$ is log-convex on J and by Remark 2.1.2(e)*, we obtain*

$$\frac{\log \mathcal{J}_{\alpha i}(g, \Phi_p) - \log \mathcal{J}_{\alpha i}(g, \Phi_q)}{p - q} \leq \frac{\log \mathcal{J}_{\alpha i}(g, \Phi_u) - \log \mathcal{J}_{\alpha i}(g, \Phi_v)}{u - v} \tag{6.4.3}$$

for $p \leq u$*,* $q \leq v$*,* $p \neq q$*,* $u \neq v$*, concluding*

$$\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega) \leq \mathcal{M}_{u,v}(g, \mathcal{J}_{\alpha i}, \Omega).$$

Cases $p = q$ *and* $u = v$ *follow from* (6.4.3) *as limit cases.* □

Remark 6.4.4. Note that the results from Theorem 6.4.1, Corollary 6.4.2 and Corollary 6.4.3 still hold when two of the points $x_0, x_1, x_2 \in [m, M]$ coincide, for a family of differentiable functions Φ_ρ such that the function $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is n–exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 2.1.7 and suitable characterization of convexity.

6.5 Examples

In this section we will vary on choice of a family $\Omega = \{\Phi_\rho : \rho \in J\}$ in order to construct different examples of exponentially convex functions and construct some means.

Example 6.5.1. Consider a family of functions

$$\Omega_1 = \{\kappa_\rho : \ \mathbb{R} \to [0, \infty); \ \rho \in \mathbb{R}\}$$

defined by

$$\kappa_\rho(x) = \begin{cases} \dfrac{1}{\rho^2} e^{\rho x}, & \rho \neq 0; \\ \\ \dfrac{1}{2} x^2, & \rho = 0. \end{cases}$$

It is $\frac{d^2}{dx^2}\kappa_\rho(x) = e^{\rho x} > 0$ which shows that κ_ρ is convex on $\mathbb{R}$ for every $\rho \in \mathbb{R}$. From Remark 2.2.6 it follows that $\rho \mapsto \frac{d^2}{dx^2}\kappa_\rho(x)$ is exponentially convex. Therefore $\rho \mapsto [x_0, x_1, x_2; \kappa_\rho]$ is exponentially convex (see [67]) (and so exponentially convex in the Jensen sense). Now using Corollary 6.4.2 we conclude that $\rho \mapsto \mathcal{J}_{\alpha i}(g, \kappa_\rho), \ i = 1, 2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex.

For this family of functions, $\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega)$ from (6.4.2) becomes

$$\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_1) = \begin{cases} \left(\dfrac{\mathcal{J}_{\alpha i}(g, \kappa_p)}{\mathcal{J}_{\alpha i}(g, \kappa_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \\ \exp\left(\dfrac{\mathcal{J}_{\alpha i}(g, id \cdot \kappa_p)}{\mathcal{J}_{\alpha i}(g, \kappa_p)} - \dfrac{2}{p} \right), & p = q \neq 0; \\ \\ \exp\left(\dfrac{\mathcal{J}_{\alpha i}(g, id \cdot \kappa_0)}{3\mathcal{J}_{\alpha i}(g, \kappa_0)} \right), & p = q = 0, \end{cases}$$

and using (6.4.1) we have that it is monotonous in parameters p and q.

If $\mathcal{J}_{\alpha i}, \ i = 1, 2$, are positive, using Theorem 6.2.9 and Theorem 6.3.9 applied for $\Phi = \kappa_p \in \Omega_1$ and $\Psi = \kappa_q \in \Omega_1$, it follows that

$$\aleph_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_1) = \log \mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_1) \qquad i = 1, 2$$

satisfy $\aleph_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_1) \in [m, M]$. If we set $g([a, b]_{\mathbb{T}}) = [m, M]$ then we have that $\aleph_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_1)$ are means (of the function g). Note that by (6.4.1) they are monotonous means.

Example 6.5.2. Consider a family of functions

$$\Omega_2 = \{\beta_\rho : \ (0, \infty) \to \mathbb{R}; \ \rho \in \mathbb{R}\}$$

defined by

$$\beta_\rho(x) = \begin{cases} \dfrac{x^\rho}{\rho(\rho-1)}, & \rho \neq 0,1; \\ -\log x, & \rho = 0; \\ x \log x, & \rho = 1. \end{cases}$$

It is $\frac{d^2}{dx^2}\beta_\rho(x) = x^{\rho-2} = e^{(\rho-2)\log x} > 0$, which shows that β_ρ is convex function for $x > 0$. Also, from Remark 2.2.6 it follows that $\rho \mapsto \frac{d^2}{dx^2}\beta_\rho(x)$ is exponentially convex. Therefore $\rho \mapsto [x_0, x_1, x_2; \beta_\rho]$ is exponentially convex (and so exponentially convex in the Jensen sense). Here we assume that $[m, M] \subset (0, \infty)$, so our family Ω_2 of β_ρ fulfills the conditions of Corollary 6.4.2. In this case $\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega)$ from (6.4.2) becomes

$$\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_2) = \begin{cases} \left(\dfrac{\mathcal{J}_{\alpha i}(g, \beta_p)}{\mathcal{J}_{\alpha i}(g, \beta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\dfrac{1-2p}{p(p-1)} - \dfrac{\mathcal{J}_{\alpha i}(g, \beta_0\beta_p)}{\mathcal{J}_{\alpha i}(g, \beta_p)}\right), & p = q \neq 0,1; \\ \exp\left(1 - \dfrac{\mathcal{J}_{\alpha i}(g, \beta_0^2)}{2\mathcal{J}_{\alpha i}(g, \beta_0)}\right), & p = q = 0; \\ \exp\left(-1 - \dfrac{\mathcal{J}_{\alpha i}(g, \beta_0\beta_1)}{2\mathcal{J}_{\alpha i}(g, \beta_1)}\right), & p = q = 1. \end{cases}$$

If $\mathcal{J}_{\alpha i}$, $i = 1, 2$, are positive, by applying Theorem 6.2.9 and Theorem 6.3.9 for $\Phi = \beta_p \in \Omega_2$ and $\Psi = \beta_q \in \Omega_2$, it follows that for $i = 1, 2$ there exist $\xi_i \in [m, M]$ such that

$$\xi_i^{p-q} = \frac{\mathcal{J}_{\alpha i}(g, \beta_p)}{\mathcal{J}_{\alpha i}(g, \beta_q)}.$$

Since the function $\xi_i \mapsto \xi_i^{p-q}$ is invertible for $p \neq q$, we have

$$m \leq \left(\frac{\mathcal{J}_{\alpha i}(g, \beta_p)}{\mathcal{J}_{\alpha i}(g, \beta_q)}\right)^{\frac{1}{p-q}} \leq M. \tag{6.5.1}$$

Also, $\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_2)$ is continuous, symmetric and monotonous (by (6.4.1)). If we set $g([a, b]_{\mathbb{T}}) = [m, M]$, then we have that

$$m = \min_{t \in [a,b]_{\mathbb{T}}} \{g(t)\} \leq \left(\frac{\mathcal{J}_{\alpha i}(g, \beta_p)}{\mathcal{J}_{\alpha i}(g, \beta_q)}\right)^{\frac{1}{p-q}} \leq \max_{t \in [a,b]_{\mathbb{T}}} \{g(t)\} = M, \quad \text{for } i = 1, 2, \tag{6.5.2}$$

which shows that $\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_2)$ are means (of the function g).

Now we impose one additional parameter r in case $g([a,b]_{\mathbb{T}}) = [m, M]$. For $r \neq 0$ by substituting $g \mapsto g^r$, $p \mapsto p/r$ and $q \mapsto q/r$ in (6.5.2), we get

$$m = \min_{t \in [a,b]_{\mathbb{T}}} \{g^r(t)\} \leq \left(\frac{\mathcal{J}_{\alpha i}(g^r, \beta_p)}{\mathcal{J}_{\alpha i}(g^r, \beta_q)} \right)^{\frac{r}{p-q}} \leq \max_{t \in [a,b]_{\mathbb{T}}} \{g^r(t)\} = M, \quad \text{for } i = 1, 2. \tag{6.5.3}$$

We define new generalized means as follows:

$$\mathcal{M}_{p,q;r}(g, \mathcal{J}_{\alpha i}, \Omega_2) = \begin{cases} \left(\mathcal{M}_{\frac{p}{r},\frac{q}{r}}(g^r, \mathcal{J}_{\alpha i}, \Omega_2) \right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\mathcal{M}_{\frac{p}{r},\frac{q}{r}}(\log g, \mathcal{J}_{\alpha i}, \Omega_1) \right), & r = 0. \end{cases}$$

These new generalized means are also monotonic.

Example 6.5.3. Consider a family of functions

$$\Omega_3 = \{\gamma_\rho : (0, \infty) \to (0, \infty) : \rho \in (0, \infty)\}$$

defined by

$$\gamma_\rho(x) = \begin{cases} \dfrac{\rho^{-x}}{(\log \rho)^2}, & \rho \neq 1; \\ \dfrac{x^2}{2}, & \rho = 1. \end{cases}$$

It is $\frac{d^2}{dx^2}\gamma_\rho(x) = \rho^{-x} > 0$, which shows that γ_ρ is convex function for $\rho > 0$. Also, from Remark 2.2.6 it follows that $\rho \mapsto \frac{d^2}{dx^2}\gamma_\rho(x)$ is exponentially convex. Therefore $\rho \mapsto [x_0, x_1, x_2; \gamma_\rho]$ is exponentially convex (and so exponentially convex in the Jensen sense). Here we assume that $[m, M] \subset (0, \infty)$, so our family Ω_3 of γ_ρ fulfills the conditions of Corollary 6.4.2. In this case $\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega)$ from (6.4.2) becomes

$$\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_3) = \begin{cases} \left(\dfrac{\mathcal{J}_{\alpha i}(g, \gamma_p)}{\mathcal{J}_{\alpha i}(g, \gamma_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathcal{J}_{\alpha i}(g, id \cdot \gamma_p)}{p\mathcal{J}_{\alpha i}(g, \gamma_p)} - \dfrac{2}{p \log p} \right), & p = q \neq 1; \\ \exp\left(\dfrac{-\mathcal{J}_{\alpha i}(g, id \cdot \gamma_1)}{3\mathcal{J}_{\alpha i}(g, \gamma_1)} \right), & p = q = 1, \end{cases}$$

and by (6.4.1) it is monotonous function in parameters p and q. Using Theorem 6.2.9 and Theorem 6.3.9, it follows that for $i = 1, 2$

$$\aleph_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_3) = -L(p, q) \log \mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_3)$$

satisfy $\aleph_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_3) \in [m, M]$. Here $L(p, q)$ is the logarithmic mean defined by: $L(p, q) = \dfrac{p - q}{\log p - \log q}$, $p \neq q$, $L(p, p) = p$.

Example 6.5.4. Consider a family of functions

$$\Omega_4 = \{\delta_\rho : (0, \infty) \to (0, \infty) : \rho \in (0, \infty)\}$$

defined by

$$\delta_\rho(x) = \frac{e^{-x\sqrt{\rho}}}{\rho}.$$

It is $\frac{d^2}{dx^2}\delta_\rho(x) = e^{-x\sqrt{\rho}} > 0$, which shows that δ_ρ is convex function for $\rho > 0$. Also, from Remark 2.2.6 it follows that $\rho \mapsto \frac{d^2}{dx^2}\delta_\rho(x)$ is exponentially convex. Therefore $\rho \mapsto [x_0, x_1, x_2; \delta_\rho]$ is exponentially convex (and so exponentially convex in the Jensen sense). Here we assume that $[m, M] \subset (0, \infty)$, so our family Ω_4 of δ_ρ fulfills the conditions of Corollary 6.4.2. In this case $\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega)$ from (6.4.2) becomes

$$\mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_4) = \begin{cases} \left(\dfrac{\mathcal{J}_{\alpha i}(g, \delta_p)}{\mathcal{J}_{\alpha i}(g, \delta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathcal{J}_{\alpha i}(g, id \cdot \delta_p)}{2\sqrt{p}\mathcal{J}_{\alpha i}(g, \delta_p)} - \dfrac{1}{p}\right), & p = q, \end{cases}$$

and it is monotonous function in parameters p and q by (6.4.1). Using Theorem 6.2.9 and Theorem 6.3.9, it follows that for $i = 1, 2$

$$\aleph_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_4) = -(\sqrt{s} + \sqrt{q}) \log \mathcal{M}_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_4)$$

satisfies $\aleph_{p,q}(g, \mathcal{J}_{\alpha i}, \Omega_4) \in [m, M]$.

Chapter 7

Improvements of the Hermite–Hadamard Inequality

In this chapter we give several refinements of the converses of the Jensen's inequality as well as of the Hermite–Hadamard inequality on time scales. We give mean value theorems and investigate logarithmic and exponential convexity of the linear functionals related to the obtained refinements. We also give several examples which illustrate possible applications for our results. (See [36]).

7.1 Converses of Jensen's inequality

To prove our main results we need the following lemma, which is a simple consequence of [82, Theorem 1].

Lemma 7.1.1. *Let Φ be a convex function on I, $x, y \in I$ and $p, q \in [0,1]$ such that $p+q=1$. Then*

$$\begin{aligned} &\min\{p,q\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right] \\ &\leq p\Phi(x)+q\Phi(y)-\Phi(px+qy) \\ &\leq \max\{p,q\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right]. \end{aligned} \tag{7.1.1}$$

Throughout this chapter, we use the same notations as in Chapter 3.

Theorem 7.1.2. *Assume $f : \mathcal{E} \to [m, M]$ is Δ-integrable, where $[m, M] \subseteq I$, and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. If Φ is a convex function on I, then*

$$\overline{L}_\Delta(\Phi(f),h) \leq \frac{M-\overline{L}_\Delta(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_\Delta(f,h)-m}{M-m}\Phi(M)-\overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi, \tag{7.1.2}$$

where

$$\tilde{f} = \frac{1}{2} - \frac{|f-(m+M)/2|}{M-m}, \quad \delta_\Phi = \Phi(m)+\Phi(M)-2\Phi\left(\frac{m+M}{2}\right). \tag{7.1.3}$$

Moreover, if Φ is concave then inequalities in (7.1.2) *hold in reverse order.*

Proof. Let the functions $p, q : [m, M] \to \mathbb{R}$ be defined by

$$p(x) = \frac{M-x}{M-m}, \quad q(x) = \frac{x-m}{M-m}. \tag{7.1.4}$$

For any $x \in [m, M]$ we can write

$$\Phi(x) = \Phi\left(\frac{M-x}{M-m}m + \frac{x-m}{M-m}M\right) = \Phi(p(x)m + q(x)M).$$

Suppose Φ is convex. By Lemma 7.1.1, we have

$$\Phi(x) \le p(x)\Phi(m) + q(x)\Phi(M) - \min\{p(x), q(x)\}\left(\Phi(m)+\Phi(M)-2\Phi\left(\frac{m+M}{2}\right)\right).$$

Since

$$\min\{x, y\} = \frac{1}{2}(x+y-|x-y|),$$

by replacing x with $f(s)$, for $s \in \mathcal{E}$, we obtain

$$\Phi(f(s)) \le p(f(s))\Phi(m) + q(f(s))\Phi(M) - \tilde{f}(s)\delta_\Phi \tag{7.1.5}$$

where the function $\tilde{f}$ is defined on $\mathcal{E}$ by

$$\tilde{f}(s) = \frac{1}{2} - \frac{|f(s)-(m+M)/2|}{M-m}.$$

Since h is nonnegative Δ-integrable and $L_\Delta(h) > 0$, multiplying (7.1.5) by h, applying integral and then dividing by $L_\Delta(h)$, we obtain

$$\overline{L}_\Delta(\Phi(f), h) \le \overline{L}_\Delta(p(f), h)\Phi(m) + \overline{L}_\Delta(q(f), h)\Phi(M) - \overline{L}_\Delta\left(\tilde{f}, h\right)\delta_\Phi,$$

from which (7.1.2) follows. If Φ is concave, the reverse inequality in (7.1.2) holds immediately by using the fact that if Φ is concave then $-\Phi$ is convex. □

Remark 7.1.3. Theorem 7.1.2 gives refinement of Theorem 3.3.2 as under the required assumptions we have

$$\overline{L}_\Delta\left(\tilde{f}, h\right)\delta_\Phi \ge 0.$$

Theorem 7.1.4. *Assume $\Phi \in \mathrm{C}(I, \mathbb{R})$ is convex and $f : \mathcal{E} \to [m, M]$ is Δ-integrable with $[m, M] \subseteq I$. Also, let $h : \mathcal{E} \to \mathbb{R}$ be nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\begin{aligned}
&\overline{L}_\Delta(\Phi(f), h) - \Phi\left(\overline{L}_\Delta(f, h)\right) \\
&\leq \max_{x\in[m,M]} \left\{ \frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M) - \Phi(x) \right\} - \overline{L}_\Delta\left(\tilde{f}, h\right)\delta_\Phi \\
&= \max_{\sigma\in[0,1]} \left\{ \sigma\Phi(m) + (1-\sigma)\Phi(M) - \Phi(\sigma m + (1-\sigma)M) \right\} - \overline{L}_\Delta\left(\tilde{f}, h\right)\delta_\Phi,
\end{aligned} \tag{7.1.6}$$

where $\tilde{f}$ and δ_Φ are defined in (7.1.3).

Proof. This is an immediate consequence of Theorem 7.1.2. The identity follows from the change of variables $\sigma = (M-x)/(M-m)$, so that for $x \in [m, M]$ we have $\sigma \in [0, 1]$ and $x = \sigma m + (1-\sigma)M$. □

Remark 7.1.5. Arguing as in Remark 7.1.3, (7.1.6) is a refinement of (3.5.1).

Theorem 7.1.6. *Assume $\Phi \in \mathrm{C}(I, \mathbb{R})$, $f : \mathcal{E} \to [m, M]$ is Δ-integrable, where $[m, M] \subseteq I$, and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. If Φ is convex, then*

$$\begin{aligned}
&\overline{L}_\Delta(\Phi(f), h) - \Phi\left(\overline{L}_\Delta(f, h)\right) \\
&\qquad \leq \frac{1}{M-m}\left\{ \left| \frac{m+M}{2} - \overline{L}_\Delta(f, h) \right| + \overline{L}_\Delta\left(\left| \frac{m+M}{2} - f \right|, h \right) \right\}\delta_\Phi,
\end{aligned} \tag{7.1.7}$$

where δ_Φ is defined in (7.1.3)*. Moreover, if Φ is concave then* (7.1.7) *holds in reverse order.*

Proof. Let the functions $p, q : [m, M] \to \mathbb{R}$ be defined as in (7.1.4). Then for any $x \in [m, M]$ we can write

$$\Phi(x) = \Phi(p(x)m + q(x)M).$$

Since $\overline{L}_\Delta(f, h) \in [m, M]$, the above equation implies that

$$\Phi\left(\overline{L}_\Delta(f, h)\right) = \Phi\left(p\left(\overline{L}_\Delta(f, h)\right)m + q\left(\overline{L}_\Delta(f, h)\right)M\right).$$

Suppose Φ is convex. By Lemma 7.1.1, we get

$$\begin{aligned}
\Phi\left(\overline{L}_\Delta(f, h)\right) &\geq p\left(\overline{L}_\Delta(f, h)\right)\Phi(m) + q\left(\overline{L}_\Delta(f, h)\right)\Phi(M) \\
&\quad - \max\left\{ p\left(\overline{L}_\Delta(f, h)\right), q\left(\overline{L}_\Delta(f, h)\right) \right\}\delta_\Phi \\
&= p\left(\overline{L}_\Delta(f, h)\right)\Phi(m) + q\left(\overline{L}_\Delta(f, h)\right)\Phi(M)
\end{aligned} \tag{7.1.8}$$

$$-\left\{\frac{1}{2}+\frac{\left|(m+M)/2-\overline{L}_\Delta(f,h)\right|}{M-m}\right\}\delta_\Phi.$$

Again by Lemma 7.1.1, we get

$$\Phi(f)\le p(f)\Phi(m)+q(f)\Phi(M)-\min\{p(f),q(f)\}\delta_\Phi,$$

which implies that

$$\begin{aligned}\overline{L}_\Delta(\Phi(f),h)&\le \overline{L}_\Delta(p(f),h)\Phi(m)+\overline{L}_\Delta(q(f),h)\Phi(M)\\&\quad-\overline{L}_\Delta(\min\{p(f),q(f)\},h)\delta_\Phi\\&=p\left(\overline{L}_\Delta(f,h)\right)\Phi(m)+q\left(\overline{L}_\Delta(f,h)\right)\Phi(M)\\&\quad-\left\{\frac{1}{2}-\frac{\overline{L}_\Delta(\left|f-(m+M)/2\right|,h)}{M-m}\right\}\delta_\Phi.\end{aligned}\tag{7.1.9}$$

Now, from inequalities (7.1.8) and (7.1.9) we get desired inequality (7.1.7). If Φ is concave, the reverse inequality in (7.1.7) holds immediately by using the fact that if Φ is concave then $-\Phi$ is convex. □

Corollary 7.1.7. *Let all the assumptions of Theorem* 7.1.6 *be satisfied. If* Φ *is convex, then*

$$\overline{L}_\Delta(\Phi(f),h)-\Phi\left(\overline{L}_\Delta(f,h)\right)\le\left\{\frac{1}{2}+\frac{1}{M-m}\left|\frac{m+M}{2}-\overline{L}_\Delta(f,h)\right|\right\}\delta_\Phi.\tag{7.1.10}$$

Moreover, if Φ *is concave then* (7.1.10) *holds in reverse order.*

Proof. Since

$$\frac{1}{M-m}\left|\frac{m+M}{2}-f\right|\le\frac{1}{2},$$

we have

$$\frac{1}{M-m}\overline{L}_\Delta\left(\left|\frac{m+M}{2}-f\right|,h\right)\le\frac{1}{2}.$$

Now inequality (7.1.10) directly follows from Theorem 7.1.6. □

Theorem 7.1.8. *Assume* $\Phi:[m,M]\to\mathbb{R}$ *is a differentiable function such that* Φ' *is strictly increasing on* $[m,M]$*. Suppose* $f:\mathcal{E}\to[m,M]$ *is* Δ*-integrable and* $h:\mathcal{E}\to\mathbb{R}$ *is nonnegative* Δ*-integrable such that* $L_\Delta(h)>0$*. If* $\tilde{f}$ *and* δ_Φ *are defined as in* (7.1.3)*, then*

$$\overline{L}_\Delta(\Phi(f),h)\le\lambda+\Phi\left(\overline{L}_\Delta(f,h)\right)-\overline{L}_\Delta\left(\tilde{f},h\right)\delta_\Phi\tag{7.1.11}$$

holds for some λ *satisfying* $0 < \lambda < (M-m)(\nu - \Phi'(m))$*, where* $\nu = (\Phi(M) - \Phi(m))/(M-m)$*. More precisely* λ *may be determined as follows: Let* $\tilde{x}$ *be the unique solution of the equation* $\Phi'(x) = \nu$*. Then*

$$\lambda = \Phi(m) - \Phi(\tilde{x}) + \nu(\tilde{x} - m)$$

satisfies (7.1.11).

Proof. By Theorem 7.1.4, we have

$$\overline{L}_\Delta(\Phi(f), h) - \Phi\left(\overline{L}_\Delta(f,h)\right) \le \max_{x\in[m,M]} g(x) - \overline{L}_\Delta\left(\tilde{f}, h\right)\delta_\Phi$$

where

$$g(x) = \frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M) - \Phi(x).$$

Then

$$g'(x) = \nu - \Phi'(x),$$

which is strictly decreasing on I with $g'(\tilde{x}) = 0$ for a unique $\tilde{x} \in I$. Consequently $g(x)$ achieves its maximum value at $x = \tilde{x}$. Hence the result follows. □

Remark 7.1.9. Theorem 7.1.8 gives refinement of Theorem 3.5.2.

Corollary 7.1.10. *Suppose* $f : \mathcal{E} \to [m, M]$ *is* Δ*-integrable such that* $[m, M] \subset (0, \infty)$ *and* $h : \mathcal{E} \to \mathbb{R}$ *is nonnegative* Δ*-integrable such that* $L_\Delta(h) > 0$*. Then*

$$\overline{L}_\Delta(f,h) \le \exp\left(\overline{L}_\Delta(\log f, h)\right)\frac{\exp\left(S\left(M/m\right)\right)}{\left[(m+M)^2/4mM\right]^{\overline{L}_\Delta\left(\tilde{f},h\right)}}, \tag{7.1.12}$$

where $S(\cdot)$ *is Specht ratio and* $\tilde{f}$ *is defined in Theorem* 7.1.2.

Proof. This is a special case of Theorem 7.1.8 for $\Phi = -\log$. In this case (7.1.11) becomes

$$-\overline{L}_\Delta(\log f, h) \le \lambda - \log(\overline{L}_\Delta(f,h)) - \overline{L}_\Delta\left(\tilde{f}, h\right)\delta_{-\log},$$

i.e.,

$$\begin{aligned}\overline{L}_\Delta(f,h) &\le \exp\left(\overline{L}_\Delta(\log f, h) + \lambda - \overline{L}_\Delta\left(\tilde{f},h\right)\delta_{-\log}\right)\\ &= \exp\left(\overline{L}_\Delta(\log f, h)\right)\frac{\exp\lambda}{\exp\left(\overline{L}_\Delta\left(\tilde{f},h\right)\delta_{-\log}\right)},\end{aligned}$$

where

$$\delta_{-\log} = -\log m - \log M + 2\log\frac{m+M}{2} = \log\frac{(m+M)^2}{4mM},$$

$$\nu = \frac{\log m - \log M}{M - m}, \quad \tilde{x} = -\frac{1}{\nu} = \frac{M - m}{\log M - \log m}.$$

and

$$\lambda = -\log m + \nu(\tilde{x} - m) + \log \tilde{x} = \log \frac{(M/m)^{m/(M-m)}}{e \log (M/m)^{m/(M-m)}} = S\left(\frac{M}{m}\right),$$

where $S(\cdot)$ is Specht ratio defined by

$$S(a) = \frac{a^{1/(a-1)}}{e \log a^{1/(a-1)}}, \quad a \in (0, \infty) \setminus \{1\}.$$

Considering all this we obtain (7.1.12). □

Corollary 7.1.11. *Suppose $f : \mathcal{E} \to [m, M]$ is Δ-integrable such that $[m, M] \subset (0, \infty)$ and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Then*

$$\begin{aligned} \overline{L}_\Delta(f, h) \leq & \exp\left(\overline{L}_\Delta(\log f, h)\right) + \frac{M - m}{\log(M/m)} S\left(\frac{M}{m}\right) \\ & - \overline{L}_\Delta(\tilde{f}_2, h)(m + M - 2\sqrt{mM}), \end{aligned} \tag{7.1.13}$$

where $S(\cdot)$ is Specht ratio and $\tilde{f}_2$ is defined by

$$\tilde{f}_2 = \frac{1}{2} - \frac{\left|\log f - \log \sqrt{mM}\right|}{\log M - \log m}. \tag{7.1.14}$$

Proof. This is a special case of Theorem 7.1.8 for $\Phi = \exp$ and $f = \log f$. In this case (7.1.11) becomes

$$\overline{L}_\Delta(\exp \log f, h) \leq \lambda + \exp\left(\overline{L}_\Delta(\log f, h)\right) - \overline{L}_\Delta(\tilde{f}_2, h)\delta_{\exp},$$

where

$$\begin{aligned} \delta_{\exp} &= \exp \log m + \exp \log M - 2 \exp \frac{\log m + \log M}{2} = m + M - 2\sqrt{mM}, \\ \nu &= \frac{M - m}{\log M - \log m}, \quad \tilde{x} = \log \nu = \log \frac{M - m}{\log M - \log m} \end{aligned}$$

and

$$\begin{aligned} \lambda &= \exp \log m + \nu(\tilde{x} - \log m) - \exp \tilde{x} \\ &= m + \frac{M - m}{\log M - \log m}\left(\log \frac{M - m}{\log M - \log m} - \log m - 1\right) \\ &= \frac{M - m}{\log(M/m)} S\left(\frac{M}{m}\right). \end{aligned}$$

Considering all this we obtain (7.1.13). □

7.2 Improvements of the Hermite–Hadamard inequality

If Φ is continuous in Theorem 7.1.2, then by combining this theorem with the Theorem 3.2.2, we obtain the refinement of the generalized Hermite–Hadamard inequality (3.3.3). In the following two theorems we give improvements of Theorem 3.3.5.

Theorem 7.2.1. *Assume $f : \mathcal{E} \to [m, M]$ is Δ-integrable, where $[m, M] \subseteq I$, and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Moreover, let p, q be positive numbers such that $p + q > 0$ and*

$$\overline{L}_\Delta(f, h) = \frac{pm + qM}{p + q}$$

holds. If Φ is a convex function on I, then

$$\Phi\left(\frac{pm + qM}{p + q}\right) \leq \overline{L}_\Delta(\Phi(f), h) \leq \frac{p\Phi(m) + q\Phi(M)}{p + q} - \overline{L}_\Delta\left(\tilde{f}, h\right)\delta_\Phi, \tag{7.2.1}$$

where $\tilde{f}$ and δ_Φ are defined as in (7.1.3)*. Moreover, if Φ is concave then inequalities in* (7.2.1) *holds in reverse order.*

Proof. The first inequality in (7.2.1) follows from Theorem 3.2.2 and the second one follows from Theorem 7.1.2. □

Theorem 7.2.2. *Assume $f : \mathcal{E} \to [m, M]$ is Δ-integrable, where $[m, M] \subseteq I$, and $h : \mathcal{E} \to \mathbb{R}$ is nonnegative Δ-integrable such that $L_\Delta(h) > 0$. Moreover, let p, q be positive numbers such that $p + q > 0$ and*

$$\overline{L}_\Delta(f, h) = \frac{pm + qM}{p + q}, \quad 0 < y \leq \frac{M - m}{p + q}\min\{p, q\} \tag{7.2.2}$$

holds. If Φ is a convex function on I, then

$$\begin{aligned} &\Phi\left(\frac{pm + qM}{p + q}\right) \leq \overline{L}_\Delta(\Phi(f), h) \\ &\leq \frac{p\Phi(m) + q\Phi(M)}{p + q} - 2\overline{L}_\Delta\left(\tilde{f}_1, h\right)\left(\frac{p\Phi(m) + q\Phi(M)}{p + q} - \Phi\left(\frac{pm + qM}{p + q}\right)\right), \end{aligned} \tag{7.2.3}$$

where

$$\tilde{f}_1 = \frac{1}{2} - \frac{|f - (pm + qM)/(p + q)|}{2y}. \tag{7.2.4}$$

Moreover, if Φ is concave then (7.2.3) *holds in reverse order.*

Proof. The first inequality in (7.2.3) follows from Theorem 3.2.2. By using (7.2.2), we have

$$m \le \overline{L}_\Delta(f,h) - y < \overline{L}_\Delta(f,h) + y \le M.$$

Suppose $m_1 = \overline{L}_\Delta(f,h) - y$ and $M_1 = \overline{L}_\Delta(f,h) + y$, then

$$\overline{L}_\Delta(f,h) = \frac{\overline{L}_\Delta(f,h) - y + \overline{L}_\Delta(f,h) + y}{2} = \frac{m_1 + M_1}{2}.$$

Now by Theorem 7.2.1 with $p = q = 1$, we obtain

$$\begin{aligned}
\overline{L}_\Delta(\Phi(f),h) &\le \frac{\Phi(\overline{L}_\Delta(f,h) - y) + \Phi(\overline{L}_\Delta(f,h) + y)}{2} \\
&\quad - \overline{L}_\Delta\left(\tilde{f}_1, h\right)\left(\Phi\left(\overline{L}_\Delta(f,h) - y\right) + \Phi\left(\overline{L}_\Delta(f,h) + y\right) - 2\Phi\left(\overline{L}_\Delta(f,h)\right)\right) \\
&= \left(1 - 2\overline{L}_\Delta\left(\tilde{f}_1, h\right)\right)\frac{\Phi\left(\overline{L}_\Delta(f,h) - y\right) + \Phi\left(\overline{L}_\Delta(f,h) + y\right)}{2} \\
&\quad + 2\overline{L}_\Delta\left(\tilde{f}_1, h\right)\Phi\left(\overline{L}_\Delta(f,h)\right).
\end{aligned}$$

Suppose Φ is convex. By Theorem 3.3.2, we get

$$\Phi\left(\overline{L}_\Delta(f,h) - y\right) \le \frac{M - \left(\overline{L}_\Delta(f,h) - y\right)}{M - m}\Phi(m) + \frac{\overline{L}_\Delta(f,h) - y - m}{M - m}\Phi(M),$$

$$\Phi\left(\overline{L}_\Delta(f,h) + y\right) \le \frac{M - \left(\overline{L}_\Delta(f,h) + y\right)}{M - m}\Phi(m) + \frac{\overline{L}_\Delta(f,h) + y - m}{M - m}\Phi(M).$$

Hence

$$\frac{\Phi\left(\overline{L}_\Delta(f,h) - y\right) + \Phi\left(\overline{L}_\Delta(f,h) + y\right)}{2} \le \frac{M - \overline{L}_\Delta(f,h)}{M - m}\Phi(m) + \frac{\overline{L}_\Delta(f,h) - m}{M - m}\Phi(M).$$

If p and q are any nonnegative numbers such that (7.2.2) holds (observe that they are different from those we started with), we obtain

$$\frac{\Phi\left(\overline{L}_\Delta(f,h) - y\right) + \Phi\left(\overline{L}_\Delta(f,h) + y\right)}{2} \le \frac{p\Phi(m) + q\Phi(M)}{p+q}.$$

Considering all this and the fact that $1 - 2\overline{L}_\Delta\left(\tilde{f}_1, h\right) \ge 0$, we deduce

$$\begin{aligned}
\overline{L}_\Delta(\Phi(f),h) &\le \left(1 - 2\overline{L}_\Delta\left(\tilde{f}_1, h\right)\right)\frac{p\Phi(m) + q\Phi(M)}{p+q} + 2\overline{L}_\Delta\left(\tilde{f}_1, h\right)\Phi\left(\overline{L}_\Delta(f,h)\right) \\
&= \frac{p\Phi(m) + q\Phi(M)}{p+q} - 2\overline{L}_\Delta\left(\tilde{f}_1, h\right)\left[\frac{p\Phi(m) + q\Phi(M)}{p+q} - \Phi\left(\frac{pm + qM}{p+q}\right)\right].
\end{aligned}$$

If Φ is concave, the reverse inequality in (7.2.3) holds immediately by using the fact that if Φ is concave then $-\Phi$ is convex. □

From (7.2.3) easily follows a Hammer-Bullen type inequality for multiple Lebesgue Δ-integral.

Corollary 7.2.3. *Let all the assumptions of Theorem* 7.2.2 *be satisfied. If* Φ *is convex, then*

$$\left(1-2\overline{L}_{\Delta}\left(\tilde{f}_1,h\right)\right)\left[\frac{p\Phi(m)+q\Phi(M)}{p+q}-\overline{L}_{\Delta}(\Phi(f),h)\right] \geq 2\overline{L}_{\Delta}\left(\tilde{f}_1,h\right)\left[\overline{L}_{\Delta}(\Phi(f),h)-\Phi\left(\frac{pm+qM}{p+q}\right)\right]. \quad (7.2.5)$$

Moreover, if Φ *is concave then* (7.2.5) *holds in reverse order.*

Proof. It follows directly from Theorem 7.2.2. □

Remark 7.2.4. We can also prove all the results of this section by using the fact that time scales integral is an isotonic linear functional. Using Theorem 3.1.8, Theorem 7.1.2 follows from [72, Theorem 12]; Theorem 7.1.4 follows from [72, Theorem 13]; Theorem 7.1.6 follows from [91, Theorem 8]; Corollary 7.1.7 follows from [91, Theorem 6]; Theorem 7.1.8 follows from [72, Theorem 14]; Corollary 7.1.10 follows from [72, Corollary 2]; Corollary 7.1.11 follows from [72, Corollary 3]; Theorem 7.2.1 follows from [73, Theorem 5]; Theorem 7.2.2 follows from [73, Theorem 6] and Corollary 7.2.3 follows from [73, Corollary 1].

7.3 Mean value theorems

We assume throughout this section and the next section that $f:\mathcal{E}\to[m,M]$ is Δ-integrable and $h:\mathcal{E}\to\mathbb{R}$ is nonnegative Δ-integrable such that $L_{\Delta}(h)>0$. If $\Phi: I\to\mathbb{R}$ is such that $[m,M]\subseteq I$ and $\Phi(f)$ is Δ-integrable, then motivated by Theorems 7.1.2, 7.1.6, and Corollary 7.1.7 we define the linear functionals $\mathcal{J}_{\Delta i},\ i\in\{1,2,3\}$, by

$$\mathcal{J}_{\Delta 1}(\Phi)=\frac{M-\overline{L}_{\Delta}(f,h)}{M-m}\Phi(m)+\frac{\overline{L}_{\Delta}(f,h)-m}{M-m}\Phi(M)-\overline{L}_{\Delta}(\Phi(f),h) - \overline{L}_{\Delta}\left(\tilde{f},h\right)\delta_{\Phi}, \quad (7.3.1)$$

$$\mathcal{J}_{\Delta 2}(\Phi)=\Phi\left(\overline{L}_{\Delta}(f,h)\right)-\overline{L}_{\Delta}(\Phi(f),h) + \frac{1}{M-m}\left\{\left|\frac{m+M}{2}-\overline{L}_{\Delta}(f,h)\right|+\overline{L}_{\Delta}(\left|(m+M)/2-f\right|,h)\right\}\delta_{\Phi}, \quad (7.3.2)$$

$$\mathcal{J}_{\Delta 3}(\Phi) = \Phi\left(\overline{L}_{\Delta}(f,h)\right) - \overline{L}_{\Delta}(\Phi(f),h) + \left\{\frac{1}{2} + \frac{1}{M-m}\left|\frac{m+M}{2} - \overline{L}_{\Delta}(f,h)\right|\right\}\delta_{\Phi}, \quad (7.3.3)$$

where $\tilde{f}$ and δ_{Φ} are defined in (7.1.3). If p, q and $\tilde{f}_1$ are as in Theorems 7.2.1 and 7.2.2, we define linear functionals $\mathcal{J}_{\Delta 4}$ and $\mathcal{J}_{\Delta 5}$ by

$$\mathcal{J}_{\Delta 4}(\Phi) = \frac{p\Phi(m) + q\Phi(M)}{p+q} - \overline{L}_{\Delta}(\Phi(f),h) - \overline{L}_{\Delta}\left(\tilde{f},h\right)\delta_{\Phi}, \quad (7.3.4)$$

$$\mathcal{J}_{\Delta 5}(\Phi) = \frac{p\Phi(m) + q\Phi(M)}{p+q} - \overline{L}_{\Delta}(\Phi(f),h) - 2\overline{L}_{\Delta}\left(\tilde{f}_1,h\right)\left(\frac{p\Phi(m) + q\Phi(M)}{p+q} - \Phi\left(\frac{pm+qM}{p+q}\right)\right). \quad (7.3.5)$$

If Φ is additionally continuous and convex on I, then by using Theorem 7.1.2, Theorem 7.1.6, Corollary 7.1.7, Theorem 7.2.1 and Theorem 7.2.2, respectively, we have

$$\mathcal{J}_{\Delta i}(\Phi) \geq 0, \quad i \in \{1,2,3,4,5\}.$$

Theorem 7.3.1. *Assume $\Phi \in C^2(I,\mathbb{R})$, where $[m,M] \subseteq I$. Then there exists $\xi_i \in [m,M]$, $i \in \{1,2,3,4,5\}$, such that*

$$\mathcal{J}_{\Delta i}(\Phi) = \frac{\Phi''(\xi_i)}{2}\mathcal{J}_{\Delta i}(\Phi_0), \quad i \in \{1,2,3,4,5\} \quad (7.3.6)$$

where $\Phi_0(x) = x^2$.

Proof. We give a proof for the functional $\mathcal{J}_{\Delta 1}$. Since $\Phi \in C^2(I)$ there exists $\Phi_*, \Phi^* \in \mathbb{R}$ such that

$$\Phi_* = \min_{x\in[m,M]} \Phi''(x) \quad \text{and} \quad \Phi^* = \max_{x\in[m,M]} \Phi''(x).$$

Let

$$\phi_1(x) = \frac{\Phi^*}{2}x^2 - \Phi(x) \quad \text{and} \quad \phi_2(x) = \Phi(x) - \frac{\Phi_*}{2}x^2.$$

Then ϕ_1 and ϕ_2 are continuous and convex on $[m,M]$, and we have

$$\mathcal{J}_{\Delta 1}(\phi_1) \geq 0, \quad \mathcal{J}_{\Delta 1}(\phi_2) \geq 0,$$

which implies

$$\frac{\Phi_*}{2}\mathcal{J}_{\Delta 1}(\Phi_0) \leq \mathcal{J}_{\Delta 1}(\Phi) \leq \frac{\Phi^*}{2}\mathcal{J}_{\Delta 1}(\Phi_0).$$

If $\mathcal{J}_{\Delta 1}(\Phi_0) = 0$, there is nothing to prove. Suppose $\mathcal{J}_{\Delta 1}(\Phi_0) > 0$. Then we have

$$\Phi_* \leq \frac{2\mathcal{J}_{\Delta 1}(\Phi)}{\mathcal{J}_{\Delta 1}(\Phi_0)} \leq \Phi^*.$$

Hence, there exists $\xi_1 \in [m, M]$ such that

$$\frac{2\mathcal{J}_{\Delta 1}(\Phi)}{\mathcal{J}_{\Delta 1}(\Phi_0)} = \Phi''(\xi_1),$$

and the result follows. □

Theorem 7.3.2. *Assume $\Phi, \Psi \in C^2(I, \mathbb{R})$, where $[m, M] \subseteq I$. Then there exists $\xi_i \in [m, M]$, $i \in \{1, 2, 3, 4, 5\}$, such that*

$$\frac{\mathcal{J}_{\Delta i}(\Phi)}{\mathcal{J}_{\Delta i}(\Psi)} = \frac{\Phi''(\xi_i)}{\Psi''(\xi_i)}, \quad i \in \{1, 2, 3, 4, 5\}, \tag{7.3.7}$$

provided that the denominators in (7.3.7) *are nonzero.*

Proof. The proof is similar to the proof of the Theorem 6.2.9. □

Remark 7.3.3. If the inverse of the function $\frac{\Phi''}{\Psi''}$ exists, then (7.3.7) gives,

$$\xi_i = \left(\frac{\Phi''}{\Psi''}\right)^{-1} \left(\frac{\mathcal{J}_{\Delta i}(\Phi)}{\mathcal{J}_{\Delta i}(\Psi)}\right), \quad i \in \{1, 2, 3, 4, 5\}.$$

7.4 Exponential convexity and logarithmic convexity

Now we study the log-convexity, n–exponential convexity and exponential-convexity of the functionals $\mathcal{J}_{\Delta i}$, $i \in \{1, 2, 3, 4, 5\}$, similarly as in Section 6.4.

Theorem 7.4.1. *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Assume $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions $\Phi_\rho : I \to \mathbb{R}$ such that $\Phi_\rho(f)$ is Δ-integrable. Suppose the function $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is n–exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0, x_1, x_2 \in I$. Then $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is an n–exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is also continuous on J, then it is n–exponentially convex on J.*

Proof. The proof is similar to the proof of Theorem 6.4.1. □

The following corollary is immediate consequence of Theorem 7.4.1.

Corollary 7.4.2. *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Assume $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions $\Phi_\rho : I \to \mathbb{R}$ such that $\Phi_\rho(f)$ is Δ-integrable. Suppose the function $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0, x_1, x_2 \in I$. Then $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is an exponentially convex function in the Jensen sense on J. If the function $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is also continuous on J, then it is exponentially convex on J.*

Corollary 7.4.3. *Let J be an interval in $\mathbb{R}$ and I be an open interval in $\mathbb{R}$. Assume $\Omega = \{\Phi_\rho : \rho \in J\}$ is a family of functions $\Phi_\rho : I \to \mathbb{R}$ such that $\Phi_\rho(f)$ is Δ-integrable. Suppose the function $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is 2–exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0, x_1, x_2 \in I$. Then the following statements hold:*

(i) *The function $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is 2–exponentially convex in the Jensen sense on J.*

(ii) *If $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is continuous on J, then it is also 2–exponentially convex on J. If $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is additionally strictly positive then it is also log-convex on J.*

(iii) *If $\rho \mapsto \mathcal{J}_{\Delta i}(\Phi_\rho)$, $i \in \{1, 2, 3, 4, 5\}$, is strictly positive differentiable function on J, then for $p \le u$, $q \le v$, $p, q, u, v \in J$, we have*

$$\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega) \le \mathcal{M}_{u,v}(\mathcal{J}_{\Delta i}, \Omega), \quad i \in \{1, 2, 3, 4, 5\} \tag{7.4.1}$$

where

$$\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega) = \begin{cases} \left(\dfrac{\mathcal{J}_{\Delta i}(\Phi_p)}{\mathcal{J}_{\Delta i}(\Phi_q)} \right)^{\frac{1}{p-q}}, & p \ne q; \\ \\ \exp\left(\dfrac{\frac{d}{dp}\mathcal{J}_{\Delta i}(\Phi_p)}{\mathcal{J}_{\Delta i}(\Phi_p)} \right), & p = q. \end{cases} \tag{7.4.2}$$

Proof. The proof is similar to the proof of Corollary 6.4.3. □

Remark 7.4.4. Note that the results from Theorem 7.4.1, Corollary 7.4.2 and Corollary 7.4.3 still hold when two of the points $x_0, x_1, x_2 \in I$ coincide, for a family of differentiable functions Φ_ρ such that the function $\rho \mapsto [x_0, x_1, x_2; \Phi_\rho]$ is n–exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 2.1.7 and suitable characterization of convexity.

Example 7.4.5. Consider the family of functions

$$\Omega_1 = \{\kappa_\rho : \ \mathbb{R} \to [0, \infty); \ \rho \in \mathbb{R}\}$$

defined in Example 6.5.1. Then by using Corollary 7.4.2 we conclude that $\rho \mapsto \mathcal{J}_{\Delta i}(\kappa_\rho), \ i \in \{1, 2, 3, 4, 5\}$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex.

For this family of functions, $\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega), \ i \in \{1, 2, 3, 4, 5\}$, from (7.4.2) becomes

$$\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_1) = \begin{cases} \left(\dfrac{\mathcal{J}_{\Delta i}(\kappa_p)}{\mathcal{J}_{\Delta i}(\kappa_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\dfrac{\mathcal{J}_{\Delta i}(id \cdot \kappa_p)}{\mathcal{J}_{\Delta i}(\kappa_p)} - \dfrac{2}{p} \right), & p = q \neq 0; \\ \exp\left(\dfrac{\mathcal{J}_{\Delta i}(id \cdot \kappa_0)}{3\mathcal{J}_{\Delta i}(\kappa_0)} \right), & p = q = 0, \end{cases}$$

and by (7.4.1) it is monotonous function in parameters p and q. Using Theorem 7.3.2 it follows that for $i \in \{1, 2, 3, 4, 5\}$,

$$\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_1) = \log \mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_1)$$

satisfy $\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_1) \in [m, M]$ which shows that $\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_1)$ are means (of the function f). Note that by (7.4.1) they are monotonous means.

Example 7.4.6. Consider the family of functions

$$\Omega_2 = \{\beta_\rho : \ (0, \infty) \to \mathbb{R}; \ \rho \in \mathbb{R}\}$$

defined in Example 6.5.2. Arguing as in Example 7.4.5, we have $\rho \mapsto \mathcal{J}_{\Delta i}(\beta_\rho), \ i \in \{1, 2, 3, 4, 5\}$, are exponentially convex. In this case $\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega), \ i \in \{1, 2, 3, 4, 5\}$, from (7.4.2) becomes

$$\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_2) = \begin{cases} \left(\dfrac{\mathcal{J}_{\Delta i}(\beta_p)}{\mathcal{J}_{\Delta i}(\beta_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\dfrac{1-2p}{p(p-1)} - \dfrac{\mathcal{J}_{\Delta i}(\beta_p \beta_0)}{\mathcal{J}_{\Delta i}(\beta_p)} \right), & p = q \neq 0, 1; \\ \exp\left(1 - \dfrac{\mathcal{J}_{\Delta i}(\beta_0^2)}{2\mathcal{J}_{\Delta i}(\beta_0)} \right), & p = q = 0; \\ \exp\left(-1 - \dfrac{\mathcal{J}_{\Delta i}(\beta_0 \beta_1)}{2\mathcal{J}_{\Delta i}(\beta_1)} \right), & p = q = 1. \end{cases}$$

As $\mathcal{J}_{\Delta i},\ i \in \{1,2,3,4,5\}$, is positive, by Theorem 7.3.2 for $\Phi = \beta_p \in \Omega_2$ and $\Psi = \beta_q \in \Omega_2$, there exists $\xi_i \in [m, M],\ i \in \{1,2,3,4,5\}$, such that

$$(\xi_i)^{p-q} = \frac{\mathcal{J}_{\Delta i}(\beta_p)}{\mathcal{J}_{\Delta i}(\beta_q)}, \quad i \in \{1,2,3,4,5\}.$$

Since the function $\xi_i \mapsto (\xi_i)^{p-q},\ i \in \{1,2,3,4,5\}$, is invertible for $p \neq q$, we have

$$m \leq \left(\frac{\mathcal{J}_{\Delta i}(\beta_p)}{\mathcal{J}_{\Delta i}(\beta_q)}\right)^{\frac{1}{p-q}} \leq M, \quad i \in \{1,2,3,4,5\}.$$

Also $\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_2), i \in \{1,2,3,4,5\}$, is continuous, symmetric and monotonous (by (7.4.1)), shows that $\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_2),\ i \in \{1,2,3,4,5\}$ is a mean (of the function f).

Example 7.4.7. Consider the family of functions

$$\Omega_3 = \{\gamma_\rho :\ (0,\infty) \to (0,\infty) :\ \rho \in (0,\infty)\}$$

defined in Example 6.5.3. For this family of functions $\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega),\ i \in \{1,2,3,4,5\}$, from (7.4.2) becomes

$$\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_3) = \begin{cases} \left(\dfrac{\mathcal{J}_{\Delta i}(\gamma_p)}{\mathcal{J}_{\Delta i}(\gamma_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathcal{J}_{\Delta i}(id \cdot \gamma_p)}{p\mathcal{J}_{\Delta i}(\gamma_p)} - \dfrac{2}{p \ln p}\right), & p = q \neq 0, 1; \\ \exp\left(\dfrac{-2\mathcal{J}_{\Delta i}(id \cdot \gamma_1)}{3\mathcal{J}_{\Delta i}(\gamma_1)}\right), & p = q = 1, \end{cases}$$

and by (7.4.1) it is monotonous function in parameters s and q. Using Theorem 7.3.2, it follows that for $i \in \{1,2,3,4,5\}$,

$$\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_3) = -L(p,q) \log \mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_3)$$

satisfies $\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_3) \in [m, M]$, which shows that $\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_3)$ is a mean (of the function f). Here $L(p,q)$ is the logarithmic mean defined by $L(p,q) = \dfrac{p-q}{\log p - \log q}$, $p \neq q$, $L(p,p) = p$.

Example 7.4.8. Consider the family of functions

$$\Omega_4 = \{\delta_\rho :\ (0,\infty) \to (0,\infty) :\ \rho \in (0,\infty)\}$$

defined in Example 6.5.4. For this family of functions, $\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega),\ i \in \{1,2,3,4,5\}$, from (7.4.2) becomes

$$\mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_4) = \begin{cases} \left(\dfrac{\mathcal{J}_{\Delta i}(\delta_p)}{\mathcal{J}_{\Delta i}(\delta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \\ \exp\left(-\dfrac{\mathcal{J}_{\Delta i}(id \cdot \delta_p)}{2\sqrt{p}\mathcal{J}_{\Delta i}(\delta_p)} - \dfrac{1}{p}\right), & p = q, \end{cases}$$

and it is monotonous function in parameters p and q by (7.4.1). Using Theorem 7.3.2, it follows that for $i \in \{1,2,3,4,5\}$,

$$\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_4) = -(\sqrt{s} + \sqrt{q}) \log \mathcal{M}_{p,q}(\mathcal{J}_{\Delta i}, \Omega_4)$$

satisfies $\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_4) \in [m, M]$, which shows that $\aleph_{p,q}(\mathcal{J}_{\Delta i}, \Omega_4)$ is a mean (of the function f).

Chapter 8

Cauchy Type Means and Exponential and Logarithmic Convexity for Superquadratic Functions

In this chapter we define positive functionals by using the Jensen's inequality, converse of Jensen's inequality, and Jensen–Mercer's inequality on time scales for superquadratic functions. We give mean-value theorems and introduce related Cauchy type means by using the functionals mentioned above and show the monotonicity of these means. We also show that these functionals are exponentially convex and give some applications of them by using the log-convexity and exponential convexity. (See [34]).

8.1 Mean value theorems

Under the assumptions of Theorems 3.6.2, 3.6.13 and 3.6.16, we define functionals $\mathcal{J}_\Psi, \widetilde{\mathcal{J}}_\Psi$, and $\widehat{\mathcal{J}}_\Psi$ by

$$\begin{aligned}\mathcal{J}_\Psi = &\int_a^b \left[\Psi\left(f(u)\right) - \Psi\left(\left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right)\right]\Delta u \\ &- (b-a)\Psi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right),\end{aligned} \tag{8.1.1}$$

$$\begin{aligned}\widetilde{\mathcal{J}}_\Psi = &(b-a)(\Psi(m)+\Psi(M)) - \int_a^b \Psi(f(t))\Delta t - K \\ &- (b-a)\Psi\left(m+M-\frac{1}{b-a}\int_a^b f(t)\Delta t\right),\end{aligned} \tag{8.1.2}$$

$$\widehat{\mathcal{J}}_\Psi = \frac{M(b-a) - \int_a^b f(t)\Delta t}{M-m}\Psi(m) + \frac{\int_a^b f(t)\Delta t - m(b-a)}{M-m}\Psi(M) - \int_a^b \Psi\left(f(t)\right)\Delta t - R. \tag{8.1.3}$$

From the inequalities (3.6.1), (3.6.9), and (3.6.11), it is clear that, subject to the relevant assumptions, $\mathcal{J}_\Psi, \widetilde{\mathcal{J}}_\Psi$, and $\widehat{\mathcal{J}}_\Psi$ are nonnegative.

In the sequel, we consider the function $\overline{\Psi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$ defined as in (2.3.2).

Theorem 8.1.1. *Let $a, b \in \mathbb{T}$. Suppose $f \in \mathrm{C}_{\mathrm{rd}}([a,b)_\mathbb{T}, [0,\infty))$ and $\Psi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ is such that $\Psi(0) = 0$ and $\overline{\Psi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$. Then*

$$\mathcal{J}_\Psi = \frac{\varrho\Psi''(\varrho) - \Psi'(\varrho)}{\varrho^2}\mathcal{J}_{\Psi_3} \tag{8.1.4}$$

holds for some $\varrho > 0$, provided that $\mathcal{J}_{\Psi_3} \neq 0$, where Ψ_3 is defined in (2.3.4).

Proof. Define

$$\psi_* := \inf_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* := \sup_{x\in(0,\infty)} \overline{\Psi}'(x).$$

Case 1: Suppose

$$\psi_* = \min_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* = \max_{x\in(0,\infty)} \overline{\Psi}'(x).$$

Then

$$\psi_* \leq \frac{x\Psi''(x) - \Psi'(x)}{x^2} \leq \psi^* \quad \text{for all} \quad x > 0. \tag{8.1.5}$$

Hence by Lemma 2.3.5, ϑ_1 and ϑ_2 defined in (2.3.3) are superquadratic. By Theorem 3.6.2, we have $\mathcal{J}_{\vartheta_1}, \mathcal{J}_{\vartheta_2} \geq 0$. Thus, since $\mathcal{J}_{\vartheta_1} = \psi^*\mathcal{J}_{\Psi_3} - \mathcal{J}_\Psi$ and $\mathcal{J}_{\vartheta_2} = \mathcal{J}_\Psi - \psi_*\mathcal{J}_{\Psi_3}$, we obtain

$$\psi_*\mathcal{J}_{\Psi_3} \leq \mathcal{J}_\Psi \leq \psi^*\mathcal{J}_{\Psi_3}. \tag{8.1.6}$$

Now, (8.1.5) and (8.1.6) imply that there exists some $\varrho > 0$ such that (8.1.4) holds.

Case 2: Suppose

$$\psi_* = \min_{x\in(0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* \neq \max_{x\in(0,\infty)} \overline{\Psi}'(x).$$

In this case, ϑ_1 is strictly superquadratic. Therefore $\mathcal{J}_{\vartheta_1} > 0$ and $\mathcal{J}_{\vartheta_2} \geq 0$. Hence

$$\psi_* \leq \frac{x\Psi''(x) - \Psi'(x)}{x^2} < \psi^* \tag{8.1.7}$$

and thus

$$\psi_*\mathcal{J}_{\Psi_3} \leq \mathcal{J}_\Psi < \psi^*\mathcal{J}_{\Psi_3}. \tag{8.1.8}$$

Now, (8.1.7) and (8.1.8) imply that (8.1.4) holds for some $\varrho > 0$.

Case 3: Suppose

$$\psi_* \neq \min_{x \in (0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* = \max_{x \in (0,\infty)} \overline{\Psi}'(x).$$

In this case, ϑ_2 is strictly superquadratic. The rest of the proof is analogous to the proof in Case 2.

Case 4: Suppose

$$\psi_* \neq \min_{x \in (0,\infty)} \overline{\Psi}'(x) \quad \text{and} \quad \psi^* \neq \max_{x \in (0,\infty)} \overline{\Psi}'(x).$$

In this case, ϑ_1 and ϑ_2 both are strictly superquadratic. The rest of the proof is analogous to the proof in Case 2.

In the case where $\psi^* = \infty$ (i.e., $\overline{\Psi}'$ is not bounded above) and ψ_* exists, using just ϑ_2, we obtain

$$\psi_* \leq \frac{x\Psi''(x) - \Psi'(x)}{x^2}$$

in the case of minimum, and strong inequality in the case where ψ_* is infimum. The rest of the proof is as above. The remaining cases can be treated analogously. □

Theorem 8.1.2. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$ such that $\mathcal{J}_{\Psi_3} \neq 0$. Suppose $\Psi, \Phi \in \mathrm{C}^1([0,\infty), \mathbb{R})$ are such that $\Psi(0) = \Phi(0) = 0$ and $\overline{\Psi}, \overline{\Phi} \in \mathrm{C}^1((0,\infty), \mathbb{R})$. Then there exists some $\varrho > 0$ such that*

$$\frac{\mathcal{J}_\Psi}{\mathcal{J}_\Phi} = \frac{\varrho\Psi''(\varrho) - \Psi'(\varrho)}{\varrho\Phi''(\varrho) - \Phi'(\varrho)} \tag{8.1.9}$$

holds, provided that the denominators in (8.1.9) *are nonzero.*

Proof. Define $\chi \in \mathrm{C}^1([0,\infty), \mathbb{R})$ by

$$\chi(x) = \mathcal{J}_\Phi \Psi(x) - \mathcal{J}_\Psi \Phi(x) \quad \text{for} \quad x \geq 0.$$

Then $\overline{\chi} \in \mathrm{C}^1((0,\infty), \mathbb{R})$, $\chi(0) = 0$, and $\mathcal{J}_\chi = 0$. Therefore, by using χ instead of Ψ in Theorem 8.1.1, we obtain that there exists $\varrho > 0$ such that

$$0 = \varrho\chi''(\varrho) - \chi'(\varrho) = \mathcal{J}_\Phi(\varrho\Psi''(\varrho) - \Psi'(\varrho)) - \mathcal{J}_\Psi(\varrho\Phi''(\varrho) - \Phi'(\varrho)),$$

from which (8.1.9) follows. □

Remark 8.1.3. In Theorem 8.1.2, let

$$\mathcal{G}(\varrho) = \frac{\varrho\Psi''(\varrho) - \Psi'(\varrho)}{\varrho\Phi''(\varrho) - \Phi'(\varrho)}$$

and suppose $\mathcal{G}$ is invertible. Then we obtain another mean defined by

$$\varrho = \mathcal{G}^{-1}\left(\frac{\mathcal{J}_\Psi}{\mathcal{J}_\Phi}\right).$$

Theorem 8.1.4. *Let $a, b \in \mathbb{T}$. Suppose $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$, and $\Psi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ is such that $\Psi(0) = 0$ and $\overline{\Psi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$. Then*

$$\widetilde{\mathcal{J}}_{\Psi} = \frac{\varrho\Psi''(\varrho) - \Psi'(\varrho)}{\varrho^2}\widetilde{\mathcal{J}}_{\Psi_3} \tag{8.1.10}$$

holds for some $\varrho > 0$, provided that $\widetilde{\mathcal{J}}_{\Psi_3} \neq 0$.

Proof. The proof is analogous to the proof of Theorem 8.1.1, where, instead of using Theorem 3.6.2, we apply Theorem 3.6.13 to ϑ_1 and ϑ_2. □

Theorem 8.1.5. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$, such that $\widetilde{\mathcal{J}}_{\Psi_3} \neq 0$. Suppose $\Psi, \Phi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ are such that $\Psi(0) = \Phi(0) = 0$ and $\overline{\Psi}, \overline{\Phi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$. Then there exists some $\varrho > 0$ such that*

$$\frac{\widetilde{\mathcal{J}}_{\Psi}}{\widetilde{\mathcal{J}}_{\Phi}} = \frac{\varrho\Psi''(\varrho) - \Psi'(\varrho)}{\varrho\Phi''(\varrho) - \Phi'(\varrho)} \tag{8.1.11}$$

holds, provided that the denominators in (8.1.11) *are nonzero.*

Proof. The proof is analogous to the proof of Theorem 8.1.2, where, instead of using Theorem 8.1.1, we apply Theorem 8.1.4 to χ. □

Remark 8.1.6. In Theorem 8.1.5, let

$$\widetilde{\mathcal{G}}(\varrho) = \frac{\varrho\Psi''(\varrho) - \Psi'(\varrho)}{\varrho\Phi''(\varrho) - \Phi'(\varrho)}$$

and suppose $\widetilde{\mathcal{G}}$ is invertible. Then we obtain another mean defined by

$$\varrho = \widetilde{\mathcal{G}}^{-1}\left(\frac{\widetilde{\mathcal{J}}_{\Psi}}{\widetilde{\mathcal{J}}_{\Phi}}\right).$$

Theorem 8.1.7. *Let $a, b \in \mathbb{T}$. Suppose $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$, and $\Psi \in \mathrm{C}^1([0,\infty),\mathbb{R})$ is such that $\Psi(0) = 0$ and $\overline{\Psi} \in \mathrm{C}^1((0,\infty),\mathbb{R})$. Then*

$$\widehat{\mathcal{J}}_{\Psi} = \frac{\varrho\Psi''(\varrho) - \Psi'(\varrho)}{\varrho^2}\widehat{\mathcal{J}}_{\Psi_3} \tag{8.1.12}$$

holds for some $\varrho > 0$, provided that $\widehat{\mathcal{J}}_{\Psi_3} \neq 0$.

Proof. The proof is analogous to the proof of Theorem 8.1.1, where, instead of using Theorem 3.6.2, we apply Theorem 3.6.16 to ϑ_1 and ϑ_2. □

Theorem 8.1.8. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$, such that $\widehat{\mathcal{J}}_{\Psi_3} \neq 0$. Suppose $\Psi, \Phi \in \mathrm{C}^1([0,\infty), \mathbb{R})$ are such that $\Psi(0) = \Phi(0) = 0$ and $\overline{\Psi}, \overline{\Phi} \in \mathrm{C}^1((0,\infty), \mathbb{R})$. Then there exists some $\varrho > 0$ such that*

$$\frac{\widehat{\mathcal{J}}_{\Psi}}{\widehat{\mathcal{J}}_{\Phi}} = \frac{\varrho \Psi''(\varrho) - \Psi'(\varrho)}{\varrho \Phi''(\varrho) - \Phi'(\varrho)} \tag{8.1.13}$$

holds, provided that the denominators in (8.1.13) *are nonzero.*

Proof. The proof is analogous to the proof of Theorem 8.1.2, where, instead of using Theorem 8.1.1, we apply Theorem 8.1.7 to χ. □

Remark 8.1.9. In Theorem 8.1.8, let

$$\widehat{\mathcal{G}}(\varrho) = \frac{\varrho \Psi''(\varrho) - \Psi'(\varrho)}{\varrho \Phi''(\varrho) - \Phi'(\varrho)}$$

and suppose $\widehat{\mathcal{G}}$ is invertible. Then we obtain another mean defined by

$$\varrho = \widehat{\mathcal{G}}^{-1}\left(\frac{\widehat{\mathcal{J}}_{\Psi}}{\widehat{\mathcal{J}}_{\Phi}}\right).$$

8.2 Generalized means

Firs we recall the definition of generalized means for Cauchy Δ-integrals (see Definition 4.2.1).

Definition 8.2.1. Let $a, b \in \mathbb{T}$. Let $\alpha \in \mathrm{C}(I, \mathbb{R})$ be strictly monotone, where $I \subset \mathbb{R}$ is an interval. If $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, I)$, then the *generalized mean* of f is defined by

$$\mathfrak{M}_{\alpha}(f) = \alpha^{-1}\left(\frac{\int_a^b (\alpha \circ f)(t)\Delta t}{b-a}\right), \tag{8.2.1}$$

provided that (8.2.1) is well defined.

Theorem 8.2.2. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$. Suppose that $\alpha, \beta, \gamma \in \mathrm{C}^2([0,\infty), \mathbb{R})$ are strictly monotone such that*

$$\overline{\alpha \circ \gamma^{-1}}, \overline{\beta \circ \gamma^{-1}} \in \mathrm{C}^1((0,\infty), \mathbb{R}) \quad \text{and} \quad (\alpha \circ \gamma^{-1})(0) = (\beta \circ \gamma^{-1})(0) = 0.$$

If

$$\int_a^b \left((\gamma \circ f)^3(u) - \left| (\gamma \circ f)(u) - \frac{\int_a^b (\gamma \circ f)(t)\Delta t}{b-a} \right|^3 \right) \Delta u - \frac{\left(\int_a^b (\gamma \circ f)(t)\Delta t\right)^3}{(b-a)^2} \neq 0,$$

then

$$\frac{\alpha(\mathfrak{M}_\alpha(f)) - \alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \alpha(\mathfrak{M}_\gamma(f))}{\beta(\mathfrak{M}_\beta(f)) - \beta(\mathfrak{M}_\beta(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f)|)))) - \beta(\mathfrak{M}_\gamma(f))}$$
$$= \frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta) - \alpha'(\zeta)\gamma''(\zeta)) - \alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta) - \beta'(\zeta)\gamma''(\zeta)) - \beta'(\zeta)(\gamma'(\zeta))^2} \tag{8.2.2}$$

holds for some $\zeta \in f([a,b)_\mathbb{T})$*, provided that the denominators in* (8.2.2) *are nonzero.*

Proof. Replace the functions f, Ψ and Φ in Theorem 8.1.2 by $\gamma\circ f, \alpha\circ\gamma^{-1}$ and $\beta\circ\gamma^{-1}$, respectively, so there exists some $\varrho > 0$ such that

$$\frac{\alpha(\mathfrak{M}_\alpha(f)) - \alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \alpha(\mathfrak{M}_\gamma(f))}{\beta(\mathfrak{M}_\beta(f)) - \beta(\mathfrak{M}_\beta(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \beta(\mathfrak{M}_\gamma(f))}$$
$$= \frac{\varrho(\alpha''(\gamma^{-1}(\varrho))\gamma'(\gamma^{-1}(\varrho)) - \alpha'(\gamma^{-1}(\varrho))\gamma''(\gamma^{-1}(\varrho))) - \alpha'(\gamma^{-1}(\varrho))(\gamma'(\gamma^{-1}(\varrho)))^2}{\varrho(\beta''(\gamma^{-1}(\varrho))\gamma'(\gamma^{-1}(\varrho)) - \beta'(\gamma^{-1}(\varrho))\gamma''(\gamma^{-1}(\varrho))) - \beta'(\gamma^{-1}(\varrho))(\gamma'(\gamma^{-1}(\varrho)))^2}.$$

By putting $\gamma^{-1}(\varrho) = \zeta$, there exists some $\zeta \in f([a,b)_\mathbb{T})$ such that (8.2.2) holds. □

Remark 8.2.3. In Theorem 8.2.2, let

$$\mathcal{F}(\zeta) = \frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta) - \alpha'(\zeta)\gamma''(\zeta)) - \alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta) - \beta'(\zeta)\gamma''(\zeta)) - \beta'(\zeta)(\gamma'(\zeta))^2}$$

and suppose $\mathcal{F}$ is invertible. Then, since ζ is in the image of f, we obtain a new mean defined by

$$\mathcal{F}^{-1}\left(\frac{\alpha(\mathfrak{M}_\alpha(f)) - \alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \alpha(\mathfrak{M}_\gamma(f))}{\beta(\mathfrak{M}_\beta(f)) - \beta(\mathfrak{M}_\beta(\gamma^{-1}(|(\gamma\circ f) - \gamma(\mathfrak{M}_\gamma(f))|))) - \beta(\mathfrak{M}_\gamma(f))}\right).$$

Now we recall the definition of generalized power means for Cauchy Δ-integrals (see Definition 4.3.1).

Definition 8.2.4. Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_\mathbb{T}, I)$, where $I \subset \mathbb{R}$ is an interval. If $r \in \mathbb{R}$, then the generalized power mean of f is defined by

$$\mathfrak{M}_r(f) = \begin{cases} \left(\dfrac{\int_a^b f^r(t)\Delta t}{b-a}\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\dfrac{\int_a^b \log f(t)\Delta t}{b-a}\right), & r = 0, \end{cases} \tag{8.2.3}$$

provided that (8.2.3) is well defined.

Corollary 8.2.5. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, I)$ be positive. Suppose $r, l, s > 0$ are such that $r \neq l$, $r \neq 2s$, $l \neq 2s$, and*

$$\int_a^b \left(f^{3s}(u) - \left| f^s(u) - \frac{\int_a^b f^s(t)\Delta t}{b-a} \right|^3 \right) \Delta u - \frac{\left(\int_a^b f^s(t)\Delta t\right)^3}{(b-a)^2} \neq 0.$$

Then

$$\frac{\mathfrak{M}_r^r(f) - \mathfrak{M}_r^r(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^r(f)}{\mathfrak{M}_l^l(f) - \mathfrak{M}_l^l(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^l(f)} = \frac{r(r-2s)}{l(l-2s)} \zeta^{r-l} \tag{8.2.4}$$

holds for some $\zeta \in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.2.4) are nonzero.

Proof. Equation (8.2.4) directly follows from Theorem 8.2.2 by taking $\alpha(x) = x^r$, $\beta(x) = x^l$ and $\gamma(x) = x^s$ in Theorem 8.2.2. □

Remark 8.2.6. From Corollary 8.2.5, since $\zeta \in f([a,b)_{\mathbb{T}})$, we obtain a new mean defined by

$$\mathfrak{M}_{r,l}^{[s]}(f) = \left(\frac{l(l-2s)}{r(r-2s)} \frac{\mathfrak{M}_r^r(f) - \mathfrak{M}_r^r(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^r(f)}{\mathfrak{M}_l^l(f) - \mathfrak{M}_l^l(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^l(f)} \right)^{\frac{1}{r-l}},$$

where $r, l, s > 0$, $r \neq 2s$, $l \neq 2s$. We can extend these means to the limiting cases. To do so, let $r, l, s > 0$. We define

$$\begin{aligned}
\mathfrak{M}_{l,l}^{[s]}(f) &= \exp\left(\frac{P}{Q} - \frac{2(l-s)}{l(l-2s)} \right), \quad l \neq 2s, \\
\mathfrak{M}_{l,2s}^{[s]}(f) = \mathfrak{M}_{2s,l}^{[s]}(f) &= \exp\left(\frac{2sQ}{l(l-2s)P_1} \right)^{\frac{1}{l-2s}}, \quad l \neq 2s, \\
\mathfrak{M}_{2s,2s}^{[s]}(f) &= \exp\left(\frac{Q_1}{2P_1} - \frac{1}{2s} \right),
\end{aligned}$$

where P, Q, P_1 and Q_1 are

$$\begin{aligned}
P =& \frac{1}{b-a} \int_a^b f^l(t) \log f(t) \Delta t - \mathfrak{M}_s^l(f) \log \mathfrak{M}_s(f) \\
&- \frac{1}{s(b-a)} \int_a^b |f^s(t) - \mathfrak{M}_s^s(f)|^{\frac{l}{s}} \log |f^s(t) - \mathfrak{M}_s^s(f)| \Delta t, \\
Q =& \mathfrak{M}_l^l(f) - \mathfrak{M}_l^l(|f^s - \mathfrak{M}_s^s(f)|^{\frac{1}{s}}) - \mathfrak{M}_s^l(f), \\
P_1 =& \frac{1}{b-a} \int_a^b f^{2s}(t) \log f(t) \Delta t - \mathfrak{M}_s^{2s}(f) \log \mathfrak{M}_s(f) \\
&- \frac{1}{s(b-a)} \int_a^b |f^s(t) - \mathfrak{M}_s^s(f)|^2 \log |f^s(t) - \mathfrak{M}_s^s(f)| \Delta t,
\end{aligned}$$

$$
\begin{aligned}
Q_1 =& \frac{1}{b-a}\int_a^b f^{2s}(t)(\log f(t))^2\Delta t - \mathfrak{M}_s^{2s}(f)(\log \mathfrak{M}_s(f))^2 \\
&- \frac{1}{s^2(b-a)}\int_a^b |f^s(t)-\mathfrak{M}_s^s(f)|^2\left(\log|f^s(t)-\mathfrak{M}_s^s(f)|\right)^2\Delta t.
\end{aligned}
$$

Theorem 8.2.7. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$. Suppose $\alpha, \beta, \gamma \in \mathrm{C}^2([0,\infty),\mathbb{R})$ are strictly monotone such that*

$$
\overline{\alpha\circ\gamma^{-1}}, \overline{\beta\circ\gamma^{-1}} \in \mathrm{C}^1((0,\infty),\mathbb{R}) \quad \textit{and} \quad (\alpha\circ\gamma^{-1})(0) = (\beta\circ\gamma^{-1})(0) = 0.
$$

If

$$
\begin{aligned}
&(b-a)((\gamma(m))^3+(\gamma(M))^3) - \int_a^b (\gamma\circ f)^3(t)\Delta t \\
&-(b-a)\left(\gamma(m)+\gamma(M)-\frac{1}{b-a}\int_a^b(\gamma\circ f)(t)\Delta t\right)^3 \\
&-\frac{2}{\gamma(M)-\gamma(m)}\int_a^b \left[((\gamma\circ f)(t)-\gamma(m))(\gamma(M)-(\gamma\circ f)(t))^3\right. \\
&\left.+(\gamma(M)-(\gamma\circ f)(t))((\gamma\circ f)(t)-\gamma(m))^3\right]\Delta t \\
&-\int_a^b\left|(\gamma\circ f)(u)-\frac{1}{b-a}\int_a^b(\gamma\circ f)(t)\Delta t\right|^3\Delta u \neq 0,
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{W_\alpha - X_\alpha - \frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\alpha\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\alpha\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t - Z_\alpha}{W_\beta - X_\beta - \frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\beta\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\beta\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t - Z_\beta} \\
&\qquad\qquad = \frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta)-\alpha'(\zeta)\gamma''(\zeta))-\alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta)-\beta'(\zeta)\gamma''(\zeta))-\beta'(\zeta)(\gamma'(\zeta))^2}
\end{aligned} \tag{8.2.5}
$$

holds for some $\zeta \in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.2.5) *are nonzero, where*

$$
\begin{aligned}
W_\alpha &= \alpha(m)+\alpha(M)-\alpha(\mathfrak{M}_\alpha(f)), \\
X_\alpha &= (\alpha\circ\gamma^{-1})(\gamma(m)+\gamma(M)-\gamma(\mathfrak{M}_\gamma(f))), \\
Z_\alpha &= \alpha(\mathfrak{M}_\alpha(\gamma^{-1}(|(\gamma\circ f)-\gamma(\mathfrak{M}_\gamma(f))|))), \quad Y = \frac{2}{\gamma(M)-\gamma(m)}, \\
\mathfrak{g} &= (\gamma\circ f)-\gamma(m), \quad \mathfrak{h} = \gamma(M)-(\gamma\circ f).
\end{aligned}
$$

Proof. Replace the functions f, Ψ and Φ in Theorem 8.1.5 by $\gamma\circ f, \alpha\circ\gamma^{-1}$ and $\beta\circ\gamma^{-1}$, respectively. The rest of the proof is analogous to the proof of Theorem 8.2.2. □

Remark 8.2.8. In Theorem 8.2.7, let

$$\widetilde{\mathcal{F}}(\zeta)=\frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta)-\alpha'(\zeta)\gamma''(\zeta))-\alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta)-\beta'(\zeta)\gamma''(\zeta))-\beta'(\zeta)(\gamma'(\zeta))^2}$$

and suppose $\widetilde{\mathcal{F}}$ is invertible. Then, since ζ is in the image of f, we obtain a new mean defined by

$$\widetilde{\mathcal{F}}^{-1}\left(\frac{W_\alpha-X_\alpha-\frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\alpha\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\alpha\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-Z_\alpha}{W_\beta-X_\beta-\frac{Y}{b-a}\int_a^b[\mathfrak{g}(t)(\beta\circ\gamma^{-1})(\mathfrak{h}(t))+\mathfrak{h}(t)(\beta\circ\gamma^{-1})(\mathfrak{g}(t))]\Delta t-Z_\beta}\right).$$

Corollary 8.2.9. *Let $a,b\in\mathbb{T}$ and $f\in \mathrm{C_{rd}}([a,b)_{\mathbb{T}},[m,M])$, where $0\le m<M<\infty$. Suppose $r,l,s>0$ are such that $r\neq l$, $r\neq 2s$, $l\neq 2s$, and*

$$\begin{aligned}&(b-a)(m^{3s}+M^{3s})-\int_a^b f^{3s}(t)\Delta t-(b-a)\left(m^s+M^s-\frac{1}{b-a}\int_a^b f^s(t)\Delta t\right)^3\\&-\frac{2}{M^s-m^s}\int_a^b\big[(f^s(t)-m^s)(M^s-f^s(t))^3\\&+(M^s-f^s(t))(f^s(t)-m^s)^3\big]\Delta t-\int_a^b\left|f^s(u)-\frac{1}{b-a}\int_a^b f^s(t)\Delta t\right|^3\Delta u\neq 0.\end{aligned}$$

Then

$$\frac{W_r-X_r-Y_s(\mathfrak{M}_r^r(\mathfrak{g}_s^{\frac1r}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_r^r(\mathfrak{h}_s^{\frac1r}\mathfrak{g}_s^{\frac1s}))-Z_r}{W_l-X_l-Y_s(\mathfrak{M}_l^l(\mathfrak{g}_s^{\frac1l}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_l^l(\mathfrak{h}_s^{\frac1l}\mathfrak{g}_s^{\frac1s}))-Z_l}=\frac{r(r-2s)}{l(l-2s)}\zeta^{r-l}\tag{8.2.6}$$

holds for some $\zeta\in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.2.6) *are nonzero, where*

$$\begin{gathered}W_r=m^r+M^r-\mathfrak{M}_r^r(f),\quad X_r=(m^s+M^s-\mathfrak{M}_s^s(f))^{\frac rs},\\ Z_r=\mathfrak{M}_r^r(|f^s-\mathfrak{M}_s^s(f)|^{\frac1s}),\quad Y_s=\frac{2}{M^s-m^s},\\ \mathfrak{g}_s=f^s-m^s,\quad \mathfrak{h}_s=M^s-f^s.\end{gathered}$$

Remark 8.2.10. From Corollary 8.2.9, since $\zeta\in f([a,b)_{\mathbb{T}})$, we obtain a new mean defined by

$$\widetilde{\mathfrak{M}}_{r,l}^{[s]}(f)=\left(\frac{l(l-2s)}{r(r-2s)}\frac{W_r-X_r-Y_s(\mathfrak{M}_r^r(\mathfrak{g}_s^{\frac1r}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_r^r(\mathfrak{h}_s^{\frac1r}\mathfrak{g}_s^{\frac1s}))-Z_r}{W_l-X_l-Y_s(\mathfrak{M}_l^l(\mathfrak{g}_s^{\frac1l}\mathfrak{h}_s^{\frac1s})+\mathfrak{M}_l^l(\mathfrak{h}_s^{\frac1l}\mathfrak{g}_s^{\frac1s}))-Z_l}\right)^{\frac{1}{r-l}},$$

where $r, l, s > 0$, $r \neq 2s$, $l \neq 2s$. We can extend these means to the limiting cases. To do so, let $r, l, s > 0$. We define

$$\widetilde{\mathfrak{M}}^{[s]}_{l,l}(f) = \exp\left(\frac{\widetilde{P}}{\widetilde{Q}} - \frac{2(l-s)}{l(l-2s)}\right), \quad l \neq 2s,$$

$$\widetilde{\mathfrak{M}}^{[s]}_{l,2s}(f) = \widetilde{\mathfrak{M}}^{[s]}_{2s,l}(f) = \exp\left(\frac{2s\widetilde{Q}}{l(l-2s)\widetilde{P}_1}\right)^{\frac{1}{l-2s}}, \quad l \neq 2s,$$

$$\widetilde{\mathfrak{M}}^{[s]}_{2s,2s}(f) = \exp\left(\frac{\widetilde{Q}_1}{2\widetilde{P}_1} - \frac{1}{2s}\right),$$

where $\widetilde{P}, \widetilde{Q}, \widetilde{P}_1$ and $\widetilde{Q}_1$ are defined by

$$\begin{aligned}
\widetilde{P} =& m^l \log m + M^l \log M - \frac{1}{b-a}\int_a^b f^l(t)\log f(t)\Delta t \\
&- \frac{1}{s}X_l \log(m^s + M^s - \mathfrak{M}^s_s(f)) \\
&- \frac{Y_s}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^{\frac{l}{s}}(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^{\frac{l}{s}}(t)\log(\mathfrak{g}_s(t))]\Delta t \\
&- \frac{1}{s(b-a)}\int_a^b |f^s(t) - \mathfrak{M}^s_s(f)|^{\frac{l}{s}} \log|f^s(t) - \mathfrak{M}^s_s(f)|\Delta t, \\
\widetilde{Q} =& W_l - X_l - Y_s(\mathfrak{M}^l_l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) + \mathfrak{M}^l_l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}})) - Z_l, \\
\widetilde{P}_1 =& m^{2s}\log m + M^{2s}\log M - \frac{1}{b-a}\int_a^b f^{2s}(t)\log f(t)\Delta t \\
&- \frac{1}{s}X_{2s}\log(m^s + M^s - \mathfrak{M}^s_s(f)) \\
&- \frac{Y_s}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)\log(\mathfrak{g}_s(t))]\Delta t \\
&- \frac{1}{s(b-a)}\int_a^b |f^s(t) - \mathfrak{M}^s_s(f)|^2 \log|f^s(t) - \mathfrak{M}^s_s(f)|\Delta t, \\
\widetilde{Q}_1 =& m^{2s}(\log m)^2 + M^{2s}(\log M)^2 - \frac{1}{b-a}\int_a^b f^{2s}(t)(\log f(t))^2\Delta t \\
&- \frac{1}{s^2}X_{2s}(\log(m^s + M^s - \mathfrak{M}^s_s(f)))^2 \\
&- \frac{Y_s}{s^2(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)(\log(\mathfrak{h}_s(t)))^2 + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)(\log(\mathfrak{g}_s(t)))^2]\Delta t \\
&- \frac{1}{s^2(b-a)}\int_a^b |f^s(t) - \mathfrak{M}^s_s(f)|^2(\log|f^s(t) - \mathfrak{M}^s_s(f)|)^2\Delta t.
\end{aligned}$$

Theorem 8.2.11. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$. Suppose $\alpha, \beta, \gamma \in \mathrm{C}^2([0,\infty),\mathbb{R})$ are strictly monotone such that*

$$\overline{\alpha \circ \gamma^{-1}}, \overline{\beta \circ \gamma^{-1}} \in \mathrm{C}^1((0,\infty),\mathbb{R}) \quad \textit{and} \quad (\alpha \circ \gamma^{-1})(0) = (\beta \circ \gamma^{-1})(0) = 0.$$

If

$$\begin{aligned}
&\int_a^b (\gamma \circ f)(t)\Delta t \left((\gamma(M))^2 + (\gamma(m))^2 + \gamma(M)\gamma(m)\right)\\
&- (b-a)\gamma(M)\gamma(m)(\gamma(M)+\gamma(m)) - \int_a^b (\gamma \circ f)^3(t)\Delta t\\
&- \frac{1}{\gamma(M)-\gamma(m)} \int_a^b \left[((\gamma \circ f)(t) - \gamma(m))(\gamma(M) - (\gamma \circ f)(t))^3\right.\\
&\left. +(\gamma(M) - (\gamma \circ f)(t))((\gamma \circ f)(t) - \gamma(m))^3\right] \Delta t \neq 0,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{(b-a)E_\alpha - \int_a^b [\mathfrak{g}(t)(\alpha \circ \gamma^{-1})(\mathfrak{h}(t)) + \mathfrak{h}(t)(\alpha \circ \gamma^{-1})(\mathfrak{g}(t))]\Delta t - (b-a)F_\alpha}{(b-a)E_\beta - \int_a^b [\mathfrak{g}(t)(\beta \circ \gamma^{-1})(\mathfrak{h}(t)) + \mathfrak{h}(t)(\beta \circ \gamma^{-1})(\mathfrak{g}(t))]\Delta t - (b-a)F_\beta}\\
&\qquad = \frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta) - \alpha'(\zeta)\gamma''(\zeta)) - \alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta) - \beta'(\zeta)\gamma''(\zeta)) - \beta'(\zeta)(\gamma'(\zeta))^2} \qquad (8.2.7)
\end{aligned}$$

holds for some $\zeta \in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.2.7) are nonzero, where $\mathfrak{g}$ and $\mathfrak{h}$ are defined as in Theorem 8.2.7 and

$$\begin{aligned}
E_\alpha &= (\gamma(M) - \gamma(\mathfrak{M}_\gamma(f)))\alpha(m) + (\gamma(\mathfrak{M}_\gamma(f)) - \gamma(m))\alpha(M),\\
F_\alpha &= (\gamma(M) - \gamma(m))\alpha(\mathfrak{M}_\alpha(f)).
\end{aligned}$$

Proof. Replace the functions f, Ψ and Φ in Theorem 8.1.5 by $\gamma \circ f, \alpha \circ \gamma^{-1}$ and $\beta \circ \gamma^{-1}$, respectively. The rest of the proof is analogous to the proof of Theorem 8.2.2. □

Remark 8.2.12. In Theorem 8.2.11, let

$$\widehat{\mathcal{F}}(\zeta) = \frac{\gamma(\zeta)(\alpha''(\zeta)\gamma'(\zeta) - \alpha'(\zeta)\gamma''(\zeta)) - \alpha'(\zeta)(\gamma'(\zeta))^2}{\gamma(\zeta)(\beta''(\zeta)\gamma'(\zeta) - \beta'(\zeta)\gamma''(\zeta)) - \beta'(\zeta)(\gamma'(\zeta))^2}$$

and suppose $\widehat{\mathcal{F}}$ is invertible. Then, since ζ is in the image of f, we obtain a new mean defined by

$$\widehat{\mathcal{F}}^{-1}\left(\frac{(b-a)E_\alpha - \int_a^b [\mathfrak{g}(t)(\alpha \circ \gamma^{-1})(\mathfrak{h}(t)) + \mathfrak{h}(t)(\alpha \circ \gamma^{-1})(\mathfrak{g}(t))]\Delta t - (b-a)F_\alpha}{(b-a)E_\beta - \int_a^b [\mathfrak{g}(t)(\beta \circ \gamma^{-1})(\mathfrak{h}(t)) + \mathfrak{h}(t)(\beta \circ \gamma^{-1})(\mathfrak{g}(t))]\Delta t - (b-a)F_\beta}\right).$$

Corollary 8.2.13. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$. Suppose $r, l, s > 0$ are such that $r \neq l$, $r \neq 2s$, $l \neq 2s$, and*

$$\int_a^b f^s(t)\Delta t \left(M^{2s} + m^{2s} + (Mm)^s\right) - (b-a)(Mm)^s(M^s + m^s) - \int_a^b f^{3s}(t)\Delta t$$
$$- \frac{\int_a^b \left[(f^s(t) - m^s)\left(M^s - f^s(t)\right)^3 + (M^s - f^s(t))(f^s(t) - m^s)^3\right]\Delta t}{M^s - m^s} \neq 0.$$

Then

$$\frac{E_r - \mathfrak{M}_r^r(\mathfrak{g}_s^{\frac{1}{r}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_r^r(\mathfrak{h}_s^{\frac{1}{r}}\mathfrak{g}_s^{\frac{1}{s}}) - F_r}{E_l - \mathfrak{M}_l^l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_l^l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}}) - F_l} = \frac{r(r-2s)}{l(l-2s)}\zeta^{r-l} \tag{8.2.8}$$

holds for some $\zeta \in f([a,b)_{\mathbb{T}})$, provided that the denominators in (8.2.8) are nonzero, where $\mathfrak{g}_s$ and $\mathfrak{h}_s$ are defined as in Corollary 8.2.9 and

$$E_r = (M^s - \mathfrak{M}_s^s(f))m^r + (\mathfrak{M}_s^s(f) - m^s)M^r, \quad F_r = (M^s - m^s)\mathfrak{M}_r^r(f).$$

Remark 8.2.14. From Corollary 8.2.13, since $\zeta \in f([a,b)_{\mathbb{T}})$, we obtain a new mean defined by

$$\widehat{\mathfrak{M}}_{r,l}^{[s]}(f) = \left(\frac{l(l-2s)}{r(r-2s)}\frac{E_r - \mathfrak{M}_r^r(\mathfrak{g}_s^{\frac{1}{r}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_r^r(\mathfrak{h}_s^{\frac{1}{r}}\mathfrak{g}_s^{\frac{1}{s}}) - F_r}{E_l - \mathfrak{M}_l^l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_l^l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}}) - F_l}\right)^{\frac{1}{r-l}},$$

where $r, l, s > 0$, $r \neq 2s$, $l \neq 2s$. We can extend these means to the limiting cases. To do so, let $r, l, s > 0$. We define

$$\widehat{\mathfrak{M}}_{l,l}^{[s]}(f) = \exp\left(\frac{\widehat{P}}{\widehat{Q}} - \frac{2(l-s)}{l(l-2s)}\right), \quad l \neq 2s,$$
$$\widehat{\mathfrak{M}}_{l,2s}^{[s]}(f) = \widehat{\mathfrak{M}}_{2s,l}^{[s]}(f) = \exp\left(\frac{2s\widehat{Q}}{l(l-2s)\widehat{P}_1}\right)^{\frac{1}{l-2s}}, \quad l \neq 2s,$$
$$\widehat{\mathfrak{M}}_{2s,2s}^{[s]}(f) = \exp\left(\frac{\widehat{Q}_1}{2\widehat{P}_1} - \frac{1}{2s}\right),$$

where $\widehat{P}, \widehat{Q}, \widehat{P}_1$ and $\widehat{Q}_1$ are defined by

$$\begin{aligned}\widehat{P} =& (M^s - \mathfrak{M}_s^s(f))m^l \log m + (\mathfrak{M}_s^s(f) - m^s)M^l \log M \\ &- \frac{1}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^{\frac{l}{s}}(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^{\frac{l}{s}}(t)\log(\mathfrak{g}_s(t))]\Delta t \\ &- \frac{M^s - m^s}{b-a}\int_a^b f^l(t)\log f(t)\Delta t,\end{aligned}$$

$$\begin{aligned}
\widehat{Q} =& E_l - \mathfrak{M}_l^l(\mathfrak{g}_s^{\frac{1}{l}}\mathfrak{h}_s^{\frac{1}{s}}) - \mathfrak{M}_l^l(\mathfrak{h}_s^{\frac{1}{l}}\mathfrak{g}_s^{\frac{1}{s}}) - F_l,\\
\widehat{P}_1 =& (M^s - \mathfrak{M}_s^s(f))m^{2s}\log m + (\mathfrak{M}_s^s(f) - m^s)M^{2s}\log M\\
&- \frac{1}{s(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)\log(\mathfrak{h}_s(t)) + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)\log(\mathfrak{g}_s(t))]\Delta t\\
&- \frac{M^s - m^s}{b-a}\int_a^b f^{2s}(t)\log f(t)\Delta t,\\
\widehat{Q}_1 =& (M^s - \mathfrak{M}_s^s(f))m^{2s}(\log m)^2 + (\mathfrak{M}_s^s(f) - m^s)M^{2s}(\log M)^2\\
&- \frac{1}{s^2(b-a)}\int_a^b [\mathfrak{g}_s(t)\mathfrak{h}_s^2(t)(\log(\mathfrak{h}_s(t)))^2 + \mathfrak{h}_s(t)\mathfrak{g}_s^2(t)(\log(\mathfrak{g}_s(t)))^2]\Delta t\\
&- \frac{M^s - m^s}{b-a}\int_a^b f^{2s}(t)(\log f(t))^2\Delta t.
\end{aligned}$$

8.3 Exponential convexity and logarithmic convexity

Applying the functional $\mathcal{J}_\Psi$ to the function Ψ_s defined in Lemma 2.3.6, we obtain

$$\begin{aligned}
\mathcal{J}_{\Psi_s} =& \frac{1}{s(s-2)}\left\{\int_a^b \left[f^s(u) - \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s\right]\Delta u\right.\\
&\left. -(b-a)\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s\right\}, \quad s \neq 2
\end{aligned} \tag{8.3.1}$$

and

$$\begin{aligned}
\mathcal{J}_{\Psi_2} =& \frac{1}{2}\left\{\int_a^b \left[f^2(u)\log f(u)\right.\right.\\
&\left. -\left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^2 \log\left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\right]\Delta u\\
&\left. -\frac{1}{b-a}\left(\int_a^b f(t)\Delta t\right)^2 \log\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)\right\}.
\end{aligned} \tag{8.3.2}$$

Theorem 8.3.1. *Let $\mathcal{J}_{\Psi_s}$ be defined as in* (8.3.1)–(8.3.2). *Then*

(i) *for all $n \in \mathbb{N}$ and for all $p_i > 0$, $p_{ij} = \frac{p_i + p_j}{2}$, $1 \le i, j \le n$, the matrix $\left[\mathcal{J}_{\Psi_{p_{ij}}}\right]_{i,j=1}^n$ is positive semidefinite;*

(ii) *the function* $s \mapsto \mathcal{J}_{\Psi_s}$ *is exponentially convex;*

(iii) *if* $\mathcal{J}_{\Psi_s} > 0$, *then the function* $s \mapsto \mathcal{J}_{\Psi_s}$ *is* log-*convex, i.e., for* $0 < r < s < w$, *we have*

$$(\mathcal{J}_{\Psi_s})^{w-r} \leq (\mathcal{J}_{\Psi_r})^{w-s} (\mathcal{J}_{\Psi_w})^{s-r}.$$

Proof. To show (i), let

$$\Lambda(x) = \sum_{i,j=1}^{n} v_i v_j \Psi_{p_{ij}}(x).$$

Then

$$\overline{\Lambda}'(x) = \sum_{i,j=1}^{n} v_i v_j x^{\frac{p_{ij}}{2}-3} = \left(\sum_{i=1}^{n} v_i x^{\frac{p_i-3}{2}} \right)^2 \geq 0$$

and $\Lambda(0) = 0$. Thus Λ is superquadratic. Now using Λ instead of Ψ in (8.1.2), we obtain

$$\mathcal{J}_{\Lambda} = \sum_{i,j=1}^{n} v_i v_j \mathcal{J}_{\Psi_{p_{ij}}} \geq 0. \tag{8.3.3}$$

Hence the matrix $\left[\mathcal{J}_{\Psi_{p_{ij}}}\right]_{i,j=1}^{n}$ is positive semidefinite.

Now we show (ii). Because $\lim_{s \to 2} \mathcal{J}_{\Psi_s} = \mathcal{J}_{\Psi_2}$, the function $s \mapsto \mathcal{J}_{\Psi_s}$ is continuous on $\mathbb{R}_+$. Hence by (8.3.3) and Proposition 2.2.5, the function $s \mapsto \mathcal{J}_{\Psi_s}$ is exponentially convex.

Finally, we show (iii). Because the function $s \mapsto \mathcal{J}_{\Psi_s}$ is exponentially convex, if $\mathcal{J}_{\Psi_s} > 0$, then by Remark 2.2.7, the function $s \mapsto \mathcal{J}_{\Psi_s}$ is log-convex. □

Corollary 8.3.2. *Let* $a, b \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, I)$ *be positive and define*

$$\mathcal{D}_s = \begin{cases} \int_a^b \left[f^s(u) - \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right|^s \right] \Delta u \\ -(b-a) \left(\frac{\int_a^b f(t)\Delta t}{b-a} \right)^s, & s \neq 2 \\ \\ \int_a^b \left[f^2(u) \log f(u) - \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right|^2 \right. \\ \left. \log \left| f(u) - \frac{\int_a^b f(t)\Delta t}{b-a} \right| \right] \Delta u \\ -\frac{1}{b-a} \left(\int_a^b f(t)\Delta t \right)^2 \log \left(\frac{\int_a^b f(t)\Delta t}{b-a} \right), & s = 2. \end{cases}$$

Then

(i) *for* $s > 4$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \geq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s \Delta u$$
$$+ \frac{s(s-2)}{3(b-a)}\left(\frac{3\mathcal{D}_4}{8\mathcal{D}_3}\right)^{s-3}\mathcal{D}_3;$$

(ii) *for* $1 < s < 2$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \leq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s \Delta u$$
$$+ \frac{s(s-2)}{b-a}\left(\frac{\mathcal{D}_2}{2\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\Delta u}\right)^{s-1}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|\Delta u;$$

(iii) *for* $2 < s < 3$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \leq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s \Delta u$$
$$+ \frac{s(s-2)}{2(b-a)}\left(\frac{2\mathcal{D}_3}{3\mathcal{D}_2}\right)^{s-2}\mathcal{D}_2;$$

(iv) *for* $3 < s < 4$,

$$\frac{\int_a^b f^s(t)\Delta t}{b-a} \leq \left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)^s + \frac{1}{b-a}\int_a^b \left|f(u) - \frac{\int_a^b f(t)\Delta t}{b-a}\right|^s \Delta u$$
$$+ \frac{s(s-2)}{3(b-a)}\left(\frac{3\mathcal{D}_4}{8\mathcal{D}_3}\right)^{s-3}\mathcal{D}_3.$$

Proof. The results follow from Theorem 8.3.1 (iii). □

Example 8.3.3. Let us consider the discrete form of $\mathcal{D}_s$. For this, let $[a,b) = \{1,2\}$, $f(1) = x$, $f(2) = y$ such that $y \geq x \geq 0$. Then $\mathcal{D}_s$ becomes

$$\mathcal{D}_s = d_s = x^s + y^s - 2\left(\frac{x+y}{2}\right)^s - 2\left(\frac{y-x}{2}\right)^s.$$

For $s > 4$, we obtain the inequality

$$\begin{aligned} d_s &\geq \frac{s(s-2)}{3}\left(\frac{3d_4}{8d_3}\right)^{s-3} d_3 \\ &= \frac{s(s-2)}{3}\left(\frac{3^2(y+x)^2}{4^2(y+2x)}\right)^{s-3}\frac{(y-x)^2(y+2x)}{2}. \end{aligned}$$

If $3 < s < 4$, we have

$$d_s \leq \frac{s(s-2)}{3}\left(\frac{3^2(y+x)^2}{4^2(y+2x)}\right)^{s-3}\frac{(y-x)^2(y+2x)}{2}.$$

Therefore for $s = 1$, the inequality becomes

$$-(y-x) \leq -\frac{1}{3\cdot 2}\left(\frac{4^2}{3^2}\right)^2\frac{(y+2x)^3(y-x)^2}{(y+x)^4}.$$

Theorem 8.3.4. *Suppose $p, q \in \mathbb{R}$ are such that $1 < p < 2$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $a, b \in \mathbb{T}$ and $f, g \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$ be such that $\int_a^b g^q(t)\Delta t > 0$. Then*

$$\begin{aligned} &\frac{1}{p(p-2)}\left(\left(\left(\int_a^b f^p(t)\Delta t - \int_a^b g(u)h^p(u)\Delta u\right)^{\frac{1}{p}}\left(\int_a^b g^q(t)\Delta t\right)^{\frac{1}{q}}\right)^p \right.\\ &\qquad \left. - \left(\int_a^b f(t)g(t)\Delta t\right)^p\right) \\ &\leq \frac{1}{2^{p-1}}\left(\int_a^b g(u)h(u)\Delta u\right)^{2-p} \\ &\left(\int_a^b g^q(t)\Delta t\left(\int_a^b f^2(t)g^{2-q}(t)\log(f(t)g^{1-q}(t))\Delta t\right.\right. \\ &- \left.\int_a^b g^{2-q}(u)h^2(u)\log(g^{1-q}(u)h(u))\Delta u\right) \\ &- \left.\left(\int_a^b f(t)g(t)\Delta t\right)^2\log\left(\frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right)\right)^{p-1} \end{aligned} \tag{8.3.4}$$

holds, where

$$h(u) = \left|f(u) - g^{q-1}(u)\frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g^q(t)\Delta t}\right|.$$

Proof. In Theorem 8.3.1 (iii), let $r = 1$, $s = p$, $w = 2$, so that $1 < p < 2$. Then we have

$$\left(\mathcal{J}_{\Psi_p}\right)^1 \leq \left(\mathcal{J}_{\Psi_1}\right)^{2-p} \left(\mathcal{J}_{\Psi_2}\right)^{p-1}.$$

By replacing $\dfrac{\int_a^b f(t)\Delta t}{b-a}$ with $\dfrac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}$, where $k \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [0,\infty))$ is such that $\int_a^b k(t)\Delta t > 0$, we get

$$\begin{aligned}
&\frac{1}{p(p-2)}\left(\int_a^b k(t)f^p(t)\Delta t - \int_a^b k(u)\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|^p \Delta u\right.\\
&\qquad\left. - \int_a^b k(t)\Delta t \left(\frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right)^p\right)\\
&\leq \frac{1}{2^{p-1}}\left\{\left(\int_a^b k(t)\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|\Delta u\right)^{2-p}\left(\int_a^b k(t)f^2(t)\log f(t)\Delta t\right.\right.\\
&\qquad - \int_a^b k(u)\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|^2 \log\left|f(u) - \frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right|\Delta u\\
&\qquad\left.\left. - \frac{1}{\int_a^b k(t)\Delta t}\left(\int_a^b k(t)f(t)\Delta t\right)^2 \log\left(\frac{\int_a^b k(t)f(t)\Delta t}{\int_a^b k(t)\Delta t}\right)\right)^{p-1}\right\}.
\end{aligned}$$

Now replacing k by g^q and f by fg^{1-q}, after some calculation, we get the required result. □

Remark 8.3.5. Theorem 8.3.4 refines the time scales Hölder inequality for superquadratic functions (Theorem 3.6.6).

Theorem 8.3.6. *Let $\mathcal{J}_{\Psi_s}$ and $\mathcal{J}_{\Psi_2}$ be positive. Then for $r, l, v, w > 0$ such that $r \leq v$, $l \leq w$, we have*

$$\mathfrak{M}^{[s]}_{r,l}(f) \leq \mathfrak{M}^{[s]}_{v,w}(f). \tag{8.3.5}$$

Proof. Since $\mathcal{J}_{\Psi_s}$ is positive, by Theorem 8.3.1, $\mathcal{J}_{\Psi_s}$ is log-convex. Now by using Remark 2.1.14(b), *for $r, l, v, w > 0$ such that $r \leq v$, $l \leq w$, $r \neq l$, $v \neq w$, we have*

$$\left(\frac{\mathcal{J}_{\Psi_r}}{\mathcal{J}_{\Psi_l}}\right)^{\frac{1}{r-l}} \leq \left(\frac{\mathcal{J}_{\Psi_v}}{\mathcal{J}_{\Psi_w}}\right)^{\frac{1}{v-w}}.$$

By substituting $\frac{r}{s}$ for r, $\frac{l}{s}$ for l, $\frac{u}{s}$ for u, $\frac{v}{s}$ for v, f^s for f and from the continuity of $\mathcal{J}_{\Psi_s}$, we obtain our required result. □

Theorem 8.3.7. *Theorem* 8.3.1 *is still valid if we replace* Ψ_s *by* φ_s *as defined in Lemma* 2.3.7.

Proof. As in the proof of Theorem 8.3.1, consider

$$\Omega(x) = \sum_{i,j=1}^{n} v_i v_j \varphi_{p_{ij}}(x).$$

Then

$$\overline{\Omega}'(x) = \left(\sum_{i=1}^{n} v_i e^{\frac{p_i}{2}x} \right)^2 \geq 0$$

and $\Omega(0) = 0$. Thus Ω is superquadratic. Now using Ω instead of Ψ in (8.1.2), we obtain our required result. □

Corollary 8.3.8. *Let* $a, b \in \mathbb{T}$ *and* $f \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, I)$ *be positive. Let* $r, s \in \mathbb{R}$, $r \neq s$. *Then we have*

$$\mathfrak{M}_{r,s}(f) = \left(\frac{s^3 \left(r \int_a^b f^r(t) \log f(t) \Delta t - A_r - r \int_a^b \mathcal{B}(t) e^{r\mathcal{B}(t)} \Delta t + \int_a^b e^{r\mathcal{B}(t)} \Delta t - 1 \right)}{r^3 \left(s \int_a^b f^s(t) \log f(t) \Delta t - A_s - s \int_a^b \mathcal{B}(t) e^{s\mathcal{B}(t)} \Delta t + \int_a^b e^{s\mathcal{B}(t)} \Delta t - 1 \right)} \right)^{\frac{1}{r-s}},$$

provided that the occurring denominators are nonzero, where

$$A_r = (b-a)(\mathfrak{M}_r^r(f) + \mathfrak{M}_0^r(f) \log(\mathfrak{M}_0^r(f)) - \mathfrak{M}_0^r(f)), \quad \mathcal{B}(t) = \left| \log \left(\frac{f(t)}{\mathfrak{M}_0(f)} \right) \right|.$$

Proof. The proof follows from Theorem 8.1.2 by replacing Ψ, Φ and f with φ_r, φ_s and $\log f$, respectively. □

Remark 8.3.9. For the limiting cases of Cauchy type means defined in Corollary 8.3.8, we have

$$\mathfrak{M}_{s,s}(f) = \exp\left(\frac{B}{C} - \frac{3}{s} \right), \quad s \neq 0 \quad \text{and} \quad \mathfrak{M}_{0,0}(f) = \exp\left(\frac{3B_1}{8C_1} \right),$$

where

$$\begin{aligned} B =& s \left(\int_a^b f^s(t)(\log f(t))^2 \Delta t - (b-a)\mathfrak{M}_0^s(f) \left(\log(\mathfrak{M}_0(f)) \right)^2 \right. \\ & \left. - \int_a^b \mathcal{B}^2(t) e^{s\mathcal{B}(t)} \Delta t \right), \\ C =& s \int_a^b f^s(t) \log f(t) \Delta t - (b-a) A_s - s \int_a^b \mathcal{B}(t) e^{s\mathcal{B}(t)} \Delta t + \int_a^b e^{s\mathcal{B}(t)} \Delta t - 1, \end{aligned}$$

$$B_1 = \int_a^b (\log f(t))^4 \Delta t - (b-a)(\log(\mathfrak{M}_0(f)))^4 - \int_a^b \mathcal{B}^4(t)\Delta t,$$
$$C_1 = \int_a^b (\log f(t))^3 \Delta t - (b-a)(\log(\mathfrak{M}_0(f)))^3 - \int_a^b \mathcal{B}^3(t)\Delta t.$$

Theorem 8.3.10. *Let $\mathcal{J}_{\Psi_s}$ be positive. Then for $r, l, v, w > 0$ such that $r \le v$, $l \le w$, we have*

$$\mathfrak{M}_{r,l}(f) \le \mathfrak{M}_{v,w}(f). \tag{8.3.6}$$

Proof. See the proof of Theorem 8.3.6. □

We can obtain corresponding results for $\widetilde{\mathcal{J}}_{\Psi_s}$ and $\widehat{\mathcal{J}}_{\Psi_s}$ analogously as in the case of $\mathcal{J}_{\Psi_s}$.

Theorem 8.3.11. (i) *For all $n \in \mathbb{N}$ and for all $p_i > 0$, $p_{ij} = \dfrac{p_i + p_j}{2}$, $1 \le i, j \le n$, the matrix $\left[\widetilde{\mathcal{J}}_{\Psi_{p_{ij}}}\right]_{i,j=1}^n$ is positive semidefinite;*

(ii) *the function $s \mapsto \widetilde{\mathcal{J}}_{\Psi_s}$ is exponentially convex;*

(iii) *if $\widetilde{\mathcal{J}}_{\Psi_s} > 0$, then the function $s \mapsto \widetilde{\mathcal{J}}_{\Psi_s}$ is* log-convex, *i.e., for $0 < r < s < w$, we have*

$$\widetilde{\mathcal{J}}_{\Psi_s}^{w-r} \le \widetilde{\mathcal{J}}_{\Psi_r}^{w-s} \widetilde{\mathcal{J}}_{\Psi_w}^{s-r}.$$

Corollary 8.3.12. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m, M])$, where $0 \le m < M < \infty$. Suppose*

$$\widetilde{\mathcal{D}}_s = \begin{cases} (b-a)(m^s + M^s) - \int_a^b f^s(t)\Delta t \\ -(b-a)\left(m + M - \frac{1}{b-a}\int_a^b f(t)\Delta t\right)^s - K_s, & s \ne 2 \\ \\ (b-a)(m^2 \log m + M^2 \log M) - \int_a^b f^2(t) \log f(t)\Delta t \\ -(b-a)\left(m + M - \frac{1}{b-a}\int_a^b f(t)\Delta t\right)^2 \\ \log\left(m + M - \frac{1}{b-a}\int_a^b f(t)\Delta t\right) - K_2, & s = 2, \end{cases}$$

where

$$K_s = \frac{2}{M-m}\int_a^b \left[(f(t)-m)(M-f(t))^s + (M-f(t))(f(t)-m)^s\right]\Delta t$$
$$+ \int_a^b \left| f(u) - \frac{1}{b-a}\int_a^b f(t)\Delta t \right|^s \Delta u$$

and

$$K_2 = \frac{2}{M-m}\int_a^b \big[(f(t)-m)(M-f(t))^2\log(M-f(t)) \\ +(M-f(t))(f(t)-m)^2\log(f(t)-m)\big]\Delta t \\ +\int_a^b \left|f(u)-\frac{1}{b-a}\int_a^b f(t)\Delta t\right|^2 \log\left|f(u)-\frac{1}{b-a}\int_a^b f(t)\Delta t\right|\Delta u.$$

Then

(i) *for* $s>4$,

$$\frac{\widetilde{\mathcal{D}}_s}{s(s-2)} \geq \frac{\widetilde{\mathcal{D}}_3}{3}\left(\frac{3\widetilde{\mathcal{D}}_4}{8\widetilde{\mathcal{D}}_3}\right)^{s-3};$$

(ii) *for* $1<s<2$,

$$\frac{\widetilde{\mathcal{D}}_s}{s(s-2)} \leq -\widetilde{\mathcal{D}}_1\left(-\frac{\widetilde{\mathcal{D}}_2}{2\widetilde{\mathcal{D}}_1}\right)^{s-1};$$

(iii) *for* $2<s<3$,

$$\frac{\widetilde{\mathcal{D}}_s}{s(s-2)} \leq \frac{\widetilde{\mathcal{D}}_2}{2}\left(\frac{2\widetilde{\mathcal{D}}_3}{3\widetilde{\mathcal{D}}_2}\right)^{s-2};$$

(iv) *for* $3<s<4$,

$$\frac{\widetilde{\mathcal{D}}_s}{s(s-2)} \leq \frac{\widetilde{\mathcal{D}}_3}{3}\left(\frac{3\widetilde{\mathcal{D}}_4}{8\widetilde{\mathcal{D}}_3}\right)^{s-3}.$$

Theorem 8.3.13. *Suppose* $p,q\in\mathbb{R}$ *such that* $1<p<2$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Let* $a,b\in\mathbb{T}$ *and* $f,g\in \mathrm{C}_{\mathrm{rd}}([a,b)_{\mathbb{T}},[m,M])$, *where* $0\leq m<M<\infty$, *be such that* $\int_a^b g^q(t)\Delta t>0$. *Then*

$$\frac{1}{p(p-2)}\Bigg(\left(\int_a^b g^q(t)\Delta t\right)^p (m^p+M^p) \\ -\left(\int_a^b g^q(t)\Delta t\right)^{p-1}\int_a^b f^p(t)\Delta t - U_1^p - \frac{2}{M-m}\left(\int_a^b g^q(t)\Delta t\right)^{p-1} U_2 \\ -\int_a^b g(u)h^p(u)\Delta u\left(\int_a^b g^q(t)\Delta t\right)^{p-1}\Bigg) \\ \leq \frac{1}{2^{p-1}}V_1^{2-p}\Bigg((m^2\log m+M^2\log M)\left(\int_a^b g^q(t)\Delta t\right)^2 \tag{8.3.7}$$

$$
\begin{aligned}
&- \int_a^b g^q(t)\Delta t \int_a^b f^2(t) g^{2-q}(t) \log(f(t)g^{1-q}(t))\Delta t \\
&- \left((m+M) \int_a^b g^q(t)\Delta t - \int_a^b f(t)g(t)\Delta t \right)^2 \\
&\log\left(m + M + \frac{\int_a^b f(t)g(t)\Delta t}{\int_a^b g^q(t)\Delta t} \right) - \frac{2}{M-m} \int_a^b g^q(t)\Delta t V_2 \\
&- \int_a^b g^q(t)\Delta t \int_a^b g^{2-q}(u) h^2(u) \log(g^{1-q}(u)h(u))\Delta u \Big)^{p-1},
\end{aligned}
$$

holds, where

$$
\begin{aligned}
U_1 =& (m+M) \int_a^b g^q(t)\Delta t - \int_a^b f(t)g(t)\Delta t, \\
U_2 =& \int_a^b g^q(t) \left(f(t)g^{1-q}(t) - m \right) \left(M - f(t)g^{1-q}(t) \right)^p \Delta t \\
&+ \int_a^b g^q(t) \left(M - f(t)g^{1-q}(t) \right) \left(f(t)g^{1-q}(t) - m \right)^p \Delta t, \\
V_1 =& \int_a^b g(u)h(u)\Delta u + \frac{4}{M-m} \int_a^b g^q(t) \left(M - f(t)g^{1-q}(t) \right) \left(f(t)g^{1-q}(t) - m \right) \Delta t, \\
V_2 =& \int_a^b g^q(t) \Big[\left(f(t)g^{1-q}(t) - m \right) \left(M - f(t)g^{1-q}(t) \right)^2 \log\left(M - f(t)g^{1-q}(t) \right) \\
&+ \left(M - f(t)g^{1-q}(t) \right) \left(f(t)g^{1-q}(t) - m \right)^2 \log\left(f(t)g^{1-q}(t) - m \right) \Big] \Delta t.
\end{aligned}
$$

Theorem 8.3.14. *Let $\widetilde{\mathcal{J}}_{\Psi_s}$ be positive. Then for $r, l, v, w > 0$ such that $r \leq v$, $l \leq w$, we have*

$$
\widetilde{\mathfrak{M}}_{r,l}^{[s]}(f) \leq \widetilde{\mathfrak{M}}_{v,w}^{[s]}(f). \tag{8.3.8}
$$

Theorem 8.3.15. (i) *For all $n \in \mathbb{N}$ and for all $p_i > 0$, $p_{ij} = \dfrac{p_i + p_j}{2}$, $1 \leq i, j \leq n$, the matrix $\left[\widehat{\mathcal{J}}_{\Psi_{p_{ij}}} \right]_{i,j=1}^n$ is positive semidefinite;*

(ii) *the function $s \mapsto \widehat{\mathcal{J}}_{\Psi_s}$ is exponentially convex;*

(iii) *if $\widehat{\mathcal{J}}_{\Psi_s} > 0$, then the function $s \mapsto \widehat{\mathcal{J}}_{\Psi_s}$ is* log-convex, *i.e., for $0 < r < s < w$, we have*

$$
\widehat{\mathcal{J}}_{\Psi_s}^{w-r} \leq \widehat{\mathcal{J}}_{\Psi_r}^{w-s} \widehat{\mathcal{J}}_{\Psi_w}^{s-r}.
$$

Corollary 8.3.16. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}([a,b)_{\mathbb{T}}, [m,M])$, where $0 \le m < M < \infty$. Suppose*

$$\widehat{\mathcal{D}}_s = \begin{cases} \frac{M(b-a)-\int_a^b f(t)\Delta t}{M-m} m^s + \frac{\int_a^b f(t)\Delta t - m(b-a)}{M-m} M^s \\ -R_s - \int_a^b f^s(t)\Delta t, & s \neq 2 \\ \\ \frac{M(b-a)-\int_a^b f(t)\Delta t}{M-m} m^2 \log m + \frac{\int_a^b f(t)\Delta t - m(b-a)}{M-m} M^2 \log M \\ -R_2 - \int_a^b f^2(t) \log f(t) \Delta t, & s = 2, \end{cases}$$

where

$$R_s = \frac{1}{M-m} \int_a^b \left[(f(t)-m)(M-f(t))^s + (M-f(t))(f(t)-m)^s \right] \Delta t$$

and

$$R_2 = \frac{1}{M-m} \int_a^b \Big[(f(t)-m)(M-f(t))^2 \log(M-f(t)) \\ +(M-f(t))(f(t)-m)^2 \log(f(t)-m) \Big] \Delta t.$$

Then

(i) *for $s > 4$,*
$$\frac{\widehat{\mathcal{D}}_s}{s(s-2)} \ge \frac{\widehat{\mathcal{D}}_3}{3} \left(\frac{3\widehat{\mathcal{D}}_4}{8\widehat{\mathcal{D}}_3} \right)^{s-3};$$

(ii) *for $1 < s < 2$,*
$$\frac{\widehat{\mathcal{D}}_s}{s(s-2)} \le -\widehat{\mathcal{D}}_1 \left(-\frac{\widehat{\mathcal{D}}_2}{2\widehat{\mathcal{D}}_1} \right)^{s-1};$$

(iii) *for $2 < s < 3$,*
$$\frac{\widehat{\mathcal{D}}_s}{s(s-2)} \le \frac{\widehat{\mathcal{D}}_2}{2} \left(\frac{2\widehat{\mathcal{D}}_3}{3\widehat{\mathcal{D}}_2} \right)^{s-2};$$

(iv) *for $3 < s < 4$,*
$$\frac{\widehat{\mathcal{D}}_s}{s(s-2)} \le \frac{\widehat{\mathcal{D}}_3}{3} \left(\frac{3\widehat{\mathcal{D}}_4}{8\widehat{\mathcal{D}}_3} \right)^{s-3}.$$

Theorem 8.3.17. *Suppose $p, q \in \mathbb{R}$ such that $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $a, b \in \mathbb{T}$ and $f, g \in \mathrm{C_{rd}}([a, b)_{\mathbb{T}}, [m, M])$, where $0 \leq m < M < \infty$, be such that $\int_a^b g^q(t)\Delta t > 0$. Then*

$$\begin{aligned}
&\frac{1}{p(p-2)}\left(W_1 m^p + W_2 M^p\right. \\
&\left. - \int_a^b g^q(t)\left[\mathfrak{a}(t)\mathfrak{b}^p(t) + \mathfrak{b}(t)\mathfrak{a}^p(t)\right]\Delta t - (M-m)\int_a^b f^p(t)\Delta t\right) \\
&\leq \frac{1}{2^{2p-3}}\left(\int_a^b g^q(t)\mathfrak{a}(t)\mathfrak{b}(t)\Delta t\right)^{2-p}\left(W_1 m^2\log m + W_2 M^2 \log M\right. \\
&- \int_a^b g^q(t)\left[\mathfrak{a}(t)\mathfrak{b}^2(t)\log\mathfrak{b}(t) + \mathfrak{b}(t)\mathfrak{a}^2(t)\log\mathfrak{a}(t)\right]\Delta t \\
&\left. -(M-m)\int_a^b f^2(t)g^{2-q}(t)\Delta t\right)^{p-1},
\end{aligned} \tag{8.3.9}$$

holds, where

$$\mathfrak{a} = fg^{1-q} - m, \quad \mathfrak{b} = M - fg^{1-q},$$

$$W_1 = M\int_a^b g^q(t)\Delta t - \int_a^b f(t)g(t)\Delta t, \quad W_2 = \int_a^b f(t)g(t)\Delta t - m\int_a^b g^q(t)\Delta t.$$

Theorem 8.3.18. *Let $\widehat{\mathcal{J}}_{\Psi_s}$ be positive. Then for $r, l, v, w > 0$ such that $r \leq v$, $l \leq w$, we have*

$$\widehat{\mathfrak{M}}^{[s]}_{r,l}(f) \leq \widehat{\mathfrak{M}}^{[s]}_{v,w}(f). \tag{8.3.10}$$

Remark 8.3.19. Similarly as in Chapter 3, we can apply the theory of isotonic linear functionals. The related results for isotonic linear functionals are given in [3, 4].

Chapter 9

Minkowski and Beckenbach–Dresher Inequalities and Functionals

In this chapter we obtain integral forms of the Minkowski inequality and Beckenbach–Dresher inequality on time scales. Also, we investigate a converse of Minkowski's inequality and several functionals arising from the Minkowski inequality and the Beckenbach–Dresher inequality. (See [35]).

9.1 Minkowski inequalities

Theorem 3.4.6 also holds if we have a finite number of functions. The next theorem gives an inequality of Minkowski type for infinitely many functions. We assume throughout that all occurring integrals are finite.

Theorem 9.1.1. *Let* $(X, \mathcal{K}, \mu_\Delta)$ *and* $(Y, \mathcal{L}, \nu_\Delta)$ *be time scale measure spaces and let* u, v*, and* f *be nonnegative functions on* X, Y*, and* $X \times Y$*, respectively. If* $p \geq 1$*, then*

$$\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} \leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \quad (9.1.1)$$

holds provided all integrals in (9.1.1) *exists. If* $0 < p < 1$ *and*

$$\int_X \left(\int_Y fv\mathrm{d}\nu_\Delta\right)^p u\mathrm{d}\mu_\Delta > 0, \quad \int_Y fv\mathrm{d}\nu_\Delta > 0 \quad (9.1.2)$$

holds, then (9.1.1) *is reversed. If* $p < 0$ *and* (9.1.2) *and*

$$\int_X f^p u \mathrm{d}\mu_\Delta > 0, \tag{9.1.3}$$

hold, then (9.1.1) *is reversed as well.*

Proof. Let $p \geq 1$. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Now, by using Fubini's theorem (Theorem 2.4.11) and Hölder's inequality (Theorem 3.4.2) on time scales, we have

$$\begin{aligned}
&\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) = \int_X H(x)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y) \\
&\leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \\
&= \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}}
\end{aligned}$$

and hence

$$\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y).$$

For $p < 0$ and $0 < p < 1$, the corresponding results can be obtained similarly. □

Remark 9.1.2. Theorem 9.1.1 is a generalization of Theorem 3.4.5 (Minkowski inequality on time scales). Moreover, if $X, Y \subseteq \mathbb{R}^n$ then (9.1.1) becomes

$$\begin{aligned}
&\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu(y)\right)^p u(x)\mathrm{d}\mu(x)\right]^{\frac{1}{p}} \\
&\qquad \leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu(y).
\end{aligned} \tag{9.1.4}$$

In the following theorem we give a converse of Theorem 9.1.1 (integral Minkowski inequality).

Theorem 9.1.3. *Let $(X, \mathcal{K}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be time scale measure spaces and let u, v, and f be nonnegative functions on X, Y, and $X \times Y$, respectively. Suppose*

$$0 < m \leq \frac{f(x,y)}{\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)} \leq M \quad \textit{for all} \quad x \in X,\ y \in Y.$$

If $p \geq 1$, then

$$\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}$$
$$\geq K(p,m,M) \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \quad (9.1.5)$$

provided all integrals in (9.1.5) *exist, where $K(p,m,M)$ is defined by* (3.4.10)*. If $0 < p < 1$ and* (9.1.2) *holds, then* (9.1.5) *is reversed. If $p < 0$ and* (9.1.2) *and* (9.1.3) *hold, then* (9.1.5) *is reversed as well.*

Proof. Let $p \geq 1$. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Then by using Fubini's theorem (Theorem 2.4.11) and the converse Hölder inequality (Theorem 3.4.19) on time scales, we get

$$\begin{aligned}
\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) &= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y) \\
&\geq K(p,m,M) \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{1/p} \\
&\quad \times \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y).
\end{aligned}$$

Dividing both sides by $\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}}$, we obtain (9.1.5). For $0 < p < 1$ and $p < 0$, the corresponding results can be obtained similarly. □

Let the functions f, u, v be defined as in Theorem 9.1.1. Now we define the rth power mean $\overline{\mathcal{M}}_\Delta^{[r]}(f,u)$ of the function f with weight function u and measure μ_Δ by

$$\overline{\mathcal{M}}_\Delta^{[r]}(f,u) = \begin{cases} \left(\frac{\int_X f^r(x,y)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X u(x)\mathrm{d}\mu_\Delta(x)}\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\frac{\int_X \log f(x,y)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X u(x)\mathrm{d}\mu_\Delta(x)}\right), & r = 0, \end{cases} \quad (9.1.6)$$

where $\int_X u\mathrm{d}\mu_\Delta > 0$.

Corollary 9.1.4. *Let $0 < s \leq r$. Then*

$$\overline{\mathcal{M}}_{\Delta}^{[r]}(\overline{\mathcal{M}}_{\Delta}^{[s]}(f, v), u) \geq K\left(\frac{r}{s}, m, M\right) \overline{\mathcal{M}}_{\Delta}^{[s]}(\overline{\mathcal{M}}_{\Delta}^{[r]}(f, u), v).$$

Proof. By putting $p = r/s$ and replacing f by f^s in (9.1.5), raising to the power of $\frac{1}{s}$ and dividing by

$$\left(\int_X u(x) \mathrm{d}\mu_{\Delta}(x)\right)^{\frac{1}{r}} \left(\int_Y v(y) \mathrm{d}\nu_{\Delta}(y)\right)^{\frac{1}{s}},$$

we get the above result. □

9.2 Minkowski functionals

In this section, we will consider some functionals which arise from the Minkowski inequality. Similar results (but not for time scales measure spaces) can be found in [66].

Let f and v be fixed functions satisfying the assumptions of Theorem 9.1.1. Let us consider the functional M_1 defined by

$$\mathsf{M}_1(u) = \left[\int_Y \left(\int_X f^p(x, y) u(x) \mathrm{d}\mu_{\Delta}(x)\right)^{\frac{1}{p}} v(y) \mathrm{d}\nu_{\Delta}(y)\right]^p - \int_X \left(\int_Y f(x, y) v(y) \mathrm{d}\nu_{\Delta}(y)\right)^p u(x) \mathrm{d}\mu_{\Delta}(x),$$

where u is a nonnegative function on X such that all occurring integrals exist. Also, if we fix the functions f and u, then we can consider the functional

$$\mathsf{M}_2(v) = \int_Y \left(\int_X f^p(x, y) u(x) \mathrm{d}\mu_{\Delta}(x)\right)^{\frac{1}{p}} v(y) \mathrm{d}\nu_{\Delta}(y) - \left[\int_X \left(\int_Y f(x, y) v(y) \mathrm{d}\nu_{\Delta}(y)\right)^p u(x) \mathrm{d}\mu_{\Delta}(x)\right]^{\frac{1}{p}},$$

where v is a nonnegative function on Y such that all occurring integrals exist.

Remark 9.2.1. (i) It is obvious that M_1 and M_2 are positive homogeneous, i.e., $\mathsf{M}_1(au) = a\mathsf{M}_1(u)$, and $\mathsf{M}_2(av) = a\mathsf{M}_2(v)$, for any $a > 0$.

(ii) If $p \geq 1$ or $p < 0$, then $\mathsf{M}_1(u) \geq 0$, and if $0 < p < 1$, then $\mathsf{M}_1(u) \leq 0$.

(iii) If $p \geq 1$, then $\mathsf{M}_2(v) \geq 0$, and if $p < 1$ and $p \neq 0$, then $\mathsf{M}_2(v) \leq 0$.

Theorem 9.2.2. (i) *If $p \ge 1$ or $p < 0$, then M_1 is superadditive. If $0 < p < 1$, then M_1 is subadditive.*

(ii) *If $p \ge 1$, then M_2 is superadditive. If $p < 1$ and $p \ne 0$, then M_2 is subadditive.*

(iii) *Suppose u_1 and u_2 are nonnegative functions such that $u_2 \ge u_1$. If $p \ge 1$ or $p < 0$, then*

$$0 \le \mathsf{M}_1(u_1) \le \mathsf{M}_1(u_2), \tag{9.2.1}$$

and if $0 < p < 1$, then (9.2.1) is reversed.

(iv) *Suppose v_1 and v_2 are nonnegative functions such that $v_2 \ge v_1$. If $p \ge 1$, then*

$$0 \le \mathsf{M}_2(v_1) \le \mathsf{M}_2(v_2), \tag{9.2.2}$$

and if $p < 1$ and $p \ne 0$, then (9.2.2) is reversed.

Proof. First we show (i). We have

$$\begin{aligned}
&\mathsf{M}_1(u_1+u_2) - \mathsf{M}_1(u_1) - \mathsf{M}_1(u_2)\\
&= \left[\int_Y \left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad - \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p (u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\\
&\quad - \left[\int_Y \left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad + \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_1(x)\mathrm{d}\mu_\Delta(x)\\
&\quad - \left[\int_Y \left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad + \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_2(x)\mathrm{d}\mu_\Delta(x)\\
&= \left[\int_Y \left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad - \left[\int_Y \left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad - \left[\int_Y \left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p.
\end{aligned}$$

Using the Minkowski inequality (3.6.10) for integrals (Theorem 3.4.6) with p replaced by $1/p$, we have

$$\mathsf{M}_1(u_1+u_2)-\mathsf{M}_1(u_1)-\mathsf{M}_1(u_2)\begin{cases}\geq 0, & p\geq 1 \text{ or } p<0,\\ \leq 0, & 0<p\leq 1.\end{cases} \tag{9.2.3}$$

So, M_1 is superadditive for $p\geq 1$ or $p<0$, and it is subadditive for $0<p\leq 1$. The proof of (ii) is similar: After a simple calculation, we have

$$\begin{aligned}&\mathsf{M}_2(v_1+v_2)-\mathsf{M}_2(v_1)-\mathsf{M}_2(v_2)\\ &= \left[\int_X\left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\ &\quad+\left[\int_X\left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\ &\quad-\left[\int_X\left(\int_Y f(x,y)(v_1+v_2)(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}.\end{aligned}$$

Using the Minkowski inequality (9.1.1) for integrals (Theorem 9.1.1), we have that this is nonnegative for $p\geq 1$ and nonpositive for $p<1$ and $p\neq 0$. Now we show (iii). If $p\geq 1$ or $p<0$, then using superadditivity and positivity of M_1, $u_2\geq u_1$ implies

$$\mathsf{M}_1(u_2)=\mathsf{M}_1(u_1+(u_2-u_1))\geq \mathsf{M}_1(u_1)+\mathsf{M}_1(u_2-u_1)\geq \mathsf{M}_1(u_1),$$

and the proof of (9.2.1) is established. If $0<p<1$, then using subadditivity and negativity of M_1, $u_2\geq u_1$ implies

$$\mathsf{M}_1(u_2)\leq \mathsf{M}_1(u_1)+\mathsf{M}_1(u_2-u_1)\leq \mathsf{M}_1(u_1).$$

The proof of (iv) is similar. □

Remark 9.2.3. From Theorem 9.2.2, we obtain a refinement of the discrete Minkowski inequality given in [66]. Namely, put $X,Y\subseteq\mathbb{N}$ and let u be Δ-measurable on X and v_1 and v_2 be Δ-measurable on Y such that $u(i)=u_i\geq 0$, $i\in X$, $v_1(j)=n_j\geq 0$, $v_2(j)=p_j\geq 0$, $j\in Y$. Then, for fixed f and u, the function M_2 has the form

$$\mathsf{M}_2(v_1)=\sum_{j\in Y}n_j\left(\sum_{i\in X}u_i a_{ij}^p\right)^{1/p}-\left(\sum_{i\in X}u_i\left(\sum_{j\in Y}n_j a_{ij}\right)^p\right)^{1/p},$$

where $f(i,j)=a_{ij}\geq 0$. If $p\geq 1$, then the mapping M_2 is superadditive, and $p_j\geq n_j$ for all $j\in Y$ implies

$$0\leq \sum_{j\in Y}n_j\left(\sum_{i\in X}u_i a_{ij}^p\right)^{1/p}-\left(\sum_{i\in X}u_i\left(\sum_{j\in Y}n_j a_{ij}\right)^p\right)^{1/p}$$

$$\leq \sum_{j\in Y} p_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} p_j a_{ij}\right)^p\right)^{1/p}$$

provided all occurring sums are finite.

Corollary 9.2.4. (i) *Suppose u_1 and u_2 are nonnegative functions such that $Cu_2 \geq u_1 \geq cu_2$, where $c, C \geq 0$. If $p \geq 1$ or $p < 0$, then*

$$c\mathsf{M}_1(u_2) \leq \mathsf{M}_1(u_1) \leq C\mathsf{M}_1(u_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) *Suppose v_1 and v_2 are nonnegative functions such that $Cv_2 \geq v_1 \geq cv_2$, where $c, C \geq 0$. If $p \geq 1$, then*

$$c\mathsf{M}_2(v_2) \leq \mathsf{M}_2(v_1) \leq C\mathsf{M}_2(v_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

Corollary 9.2.5. *If v_1 and v_2 are nonnegative functions such that $v_2 \geq v_1$, then*

$$\begin{aligned}\overline{\mathcal{M}}_{\Delta}^{[0]}\left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y), u\right) - \int_Y \overline{\mathcal{M}}_{\Delta}^{[0]}(f,u)v_1(y)\mathrm{d}\nu_\Delta(y) \\ \leq \overline{\mathcal{M}}_{\Delta}^{[0]}\left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y), u\right) - \int_Y \overline{\mathcal{M}}_{\Delta}^{[0]}(f,u)v_2(y)\mathrm{d}\nu_\Delta(y), \quad (9.2.4)\end{aligned}$$

where $\overline{\mathcal{M}}_{\Delta}^{[0]}(f,u)$ is defined in (9.1.6).

Remark 9.2.6. If the measures are discrete, then from Corollary 9.2.5, we get the following result: Let $u_j, v_i, w_i, a_{ij} > 0$ for all $i \in \{1,\dots,n\}$ and all $j \in \{1,\dots,k\}$. Put $U = \sum_{j=1}^{k} u_j$. If $v_i \leq w_i$ for all $i \in \{1,\dots,n\}$, then

$$\prod_{j=1}^{k}\left(\sum_{i=1}^{n} v_i a_{ij}\right)^{\frac{u_j}{U}} - \sum_{i=1}^{n} v_i \left(\prod_{j=1}^{k} a_{ij}^{\frac{u_j}{U}}\right) \leq \prod_{j=1}^{k}\left(\sum_{i=1}^{n} w_i a_{ij}\right)^{\frac{u_j}{U}} - \sum_{i=1}^{n} w_i \left(\prod_{j=1}^{k} a_{ij}^{\frac{u_j}{U}}\right).$$

This inequality is a refinement of the discrete Hölder inequality

$$\prod_{j=1}^{k}\left(\sum_{i=1}^{n} w_i a_{ij}\right)^{\frac{u_j}{U}} \geq \sum_{i=1}^{n} w_i \left(\prod_{j=1}^{k} a_{ij}^{\frac{u_j}{U}}\right).$$

The next result gives another property of M_1, but a similar result can also be stated for M_2.

Theorem 9.2.7. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave function. Suppose u_1 and u_2 are nonnegative functions such that*

$$\varphi \circ u_1, \quad \varphi \circ u_2, \quad \varphi \circ (\alpha u_1 + (1-\alpha)u_2)$$

are Δ-integrable for $\alpha \in [0,1]$. If $p \geq 1$, then

$$\mathsf{M}_1(\varphi \circ (\alpha u_1 + (1-\alpha)u_2)) \geq \alpha \mathsf{M}_1(\varphi \circ u_1) + (1-\alpha)\mathsf{M}_1(\varphi \circ u_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

Proof. We show this only for $p \geq 1$ as the other case follows similarly. Since φ is concave, we have

$$\varphi(\alpha u_1 + (1-\alpha)u_2)) \geq \alpha\varphi(u_1) + (1-\alpha)\varphi(u_2).$$

Now, from (9.2.1) and (9.2.3), we have

$$\begin{aligned} \mathsf{M}_1(\varphi \circ (\alpha u_1 + (1-\alpha)u_2)) &\geq \mathsf{M}_1(\alpha(\varphi \circ u_1) + (1-\alpha)(\varphi \circ u_2)) \\ &\geq \mathsf{M}_1(\alpha(\varphi \circ u_1)) + \mathsf{M}_1((1-\alpha)(\varphi \circ u_2)) \\ &\geq \alpha\mathsf{M}_1(\varphi \circ u_1) + (1-\alpha)\mathsf{M}_1(\varphi \circ u_2), \end{aligned}$$

and the proof is established. ☐

Let f, u and v be fixed functions satisfying the assumptions of Theorem 9.1.1. Let us define functionals M_3 and M_4 by

$$\begin{aligned} \mathsf{M}_3(A) = \left[\int_Y \left(\int_A f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p \\ - \int_A \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x) \end{aligned}$$

and

$$\begin{aligned} \mathsf{M}_4(B) = \int_B \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \\ - \left[\int_X \left(\int_B f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}, \end{aligned}$$

where $A \subseteq X$ and $B \subseteq Y$.

The following theorem establishes superadditivity and monotonicity of the mappings M_3 and M_4.

Theorem 9.2.8. (i) *Suppose $A_1, A_2 \subseteq X$ and $A_1 \cap A_2 = \emptyset$. If $p \geq 1$ or $p < 0$, then*

$$\mathsf{M}_3(A_1 \cup A_2) \geq \mathsf{M}_3(A_1) + \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) *Suppose $A_1, A_2 \subseteq X$ and $A_1 \subseteq A_2$. If $p \geq 1$ or $p < 0$, then*

$$\mathsf{M}_3(A_1) \leq \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(iii) *Suppose $B_1, B_2 \subseteq Y$ and $B_1 \cap B_2 = \emptyset$. If $p \geq 1$, then*

$$\mathsf{M}_4(B_1 \cup B_2) \geq \mathsf{M}_4(B_1) + \mathsf{M}_4(B_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

(iv) *Suppose $B_1, B_2 \subseteq Y$ and $B_1 \subseteq B_2$. If $p \geq 1$, then*

$$\mathsf{M}_4(B_1) \leq \mathsf{M}_4(B_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

The proof of Theorem 9.2.8 is omitted as it is similar to the proof of Theorem 9.2.2.

Remark 9.2.9. For $p \geq 1$, if S_m is a subset of Y with m elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then we have

$$\mathsf{M}_4(S_m) \geq \mathsf{M}_4(S_{m-1}) \geq \ldots \geq \mathsf{M}_4(S_2) \geq 0$$

and $\mathsf{M}_4(S_m) \geq \max\{\mathsf{M}_4(S_2) : S_2$ is any subset of S_m with 2 elements$\}$.

9.3 Beckenbach–Dresher inequalities

Theorem 9.3.1. *Let $(X, \mathcal{K}, \mu_\Delta), (X, \mathcal{K}, \lambda_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be time scale measure spaces. Suppose u and w are nonnegative functions on X, v is a nonnegative function on Y, f is a nonnegative function on $X \times Y$ with respect to the measure $(\mu_\Delta \times \nu_\Delta)$, and g is a nonnegative function on $X \times Y$ with respect to the measure $(\lambda_\Delta \times \nu_\Delta)$. If*

$$s \geq 1, \quad q \leq 1 \leq p, \quad \text{and} \quad q \neq 0 \tag{9.3.1}$$

or

$$s < 0, \quad p \leq 1 \leq q, \quad \text{and} \quad p \neq 0, \tag{9.3.2}$$

then

$$\frac{\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \leq \int_Y \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} v(y)\mathrm{d}\nu_\Delta(y) \quad (9.3.3)$$

provided all occurring integrals in (9.3.3) *exist. If*

$$0 < s \leq 1, \quad p \leq 1, \quad q \leq 1, \quad \textit{and} \quad p, q \neq 0, \tag{9.3.4}$$

then (9.3.3) *is reversed.*

Proof. Assume (9.3.1) or (9.3.2). By using the integral Minkowski inequality (9.1.1) and Hölder's inequality (3.6.6), we have

$$\begin{aligned}
&\frac{\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\leq \frac{\left[\int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^s}{\left[\int_Y \left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1}{q}} v(y)\mathrm{d}\nu_\Delta(y)\right]^{s-1}} \\
&= \left[\int_Y \left(\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}\right)^{\frac{1}{s}} v(y)\mathrm{d}\nu_\Delta(y)\right]^s \\
&\quad \times \left[\int_Y \left(\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}}\right)^{\frac{1}{1-s}} v(y)\mathrm{d}\nu_\Delta(y)\right]^{1-s} \\
&\leq \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}} \left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}} v(y)\mathrm{d}\nu_\Delta(y).
\end{aligned}$$

If (9.3.4) holds, then the reversed inequality in (9.3.3) can be proved in a similar way. □

Remark 9.3.2. Theorem 9.3.1 is a generalization of Theorem 3.4.9 (Becken–Dresher inequality on time scales).

9.4 Beckenbach–Dresher functionals

Let f, g, u, w be fixed functions satisfying the assumptions of Theorem 9.3.1. We define the Beckenbach–Dresher functional $\mathsf{BD}(v)$ by

$$\mathsf{BD}(v) = \int_Y \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} v(y)\mathrm{d}\nu_\Delta(y) - \frac{\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},$$

where we suppose that all occurring integrals exist.

Theorem 9.4.1. *If* (9.3.1) *or* (9.3.2) *holds, then*

$$\mathsf{BD}(v_1 + v_2) \geq \mathsf{BD}(v_1) + \mathsf{BD}(v_2). \tag{9.4.1}$$

If $v_2 \geq v_1$, then

$$\mathsf{BD}(v_1) \leq \mathsf{BD}(v_2). \tag{9.4.2}$$

If $C, c \geq 0$ and $Cv_2 \geq v_1 \geq cv_2$, then

$$C\mathsf{BD}(v_2) \geq \mathsf{BD}(v_1) \geq c\mathsf{BD}(v_1). \tag{9.4.3}$$

If (9.3.4) *holds, then* (9.4.1), (9.4.2) *and* (9.4.3) *are reversed.*

Proof. Assume (9.3.1) or (9.3.2). Then we have

$$\begin{aligned}
&\mathsf{BD}(v_1 + v_2) - \mathsf{BD}(v_1) - \mathsf{BD}(v_2) \\
&= \frac{\left[\int_X \left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\quad + \frac{\left[\int_X \left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\quad - \frac{\left[\int_X \left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y) + \int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_1(y)\mathrm{d}\nu_\Delta(y) + \int_Y g(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\geq 0,
\end{aligned}$$

where in the last inequality we used (9.3.3) from Theorem 9.3.1. Using Theorem 9.3.1 again, $v_2 \geq v_1$ implies

$$\mathsf{BD}(v_2) = \mathsf{BD}(v_1 + (v_2 - v_1)) \geq \mathsf{BD}(v_1) + \mathsf{BD}(v_2 - v_1) \geq \mathsf{BD}(v_1).$$

The proof of (9.4.3) is similar. If (9.3.4) holds, then the reversed inequalities of (9.4.1), (9.4.2) and (9.4.3) can be proved in a similar way. □

Let f, g, u, v, w be fixed functions. We define a functional BD_1 by

$$\mathsf{BD}_1(A) = \int_A \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} v(y)\mathrm{d}\nu_\Delta(y) \\ - \frac{\left[\int_X \left(\int_A f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_A g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},$$

where $A \subseteq Y$.

For BD_1, the following result holds.

Theorem 9.4.2. (i) *Suppose $A_1, A_2 \subseteq Y$ and $A_1 \cap A_2 = \emptyset$. If* (9.3.1) *or* (9.3.2) *holds, then*

$$\mathsf{BD}_1(A_1 \cup A_2) \geq \mathsf{BD}_1(A_1) + \mathsf{BD}_1(A_2),$$

and if (9.3.4) *holds, then the above inequality is reversed.*

(ii) *Suppose $A_1, A_2 \subseteq Y$ and $A_1 \subseteq A_2$. If* (9.3.1) *or* (9.3.2) *holds, then*

$$\mathsf{BD}_1(A_1) \leq \mathsf{BD}_1(A_2),$$

and if (9.3.4) *holds, then the above inequality is reversed.*

The proof of Theorem 9.4.2 is omitted as it is similar to the proof of Theorem 9.4.1.

Remark 9.4.3. If $S_k \subseteq Y$ has k elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then (9.3.1) or (9.3.2) implies

$$\mathsf{BD}_1(S_m) \geq \mathsf{BD}_1(S_{m-1}) \geq \cdots \geq \mathsf{BD}_1(S_2) \geq 0$$

and $\mathsf{BD}_1(S_m) \geq \max\{\mathsf{BD}_1(S_2) : S_2 \text{ is any subset of } S_m \text{ with 2 elements}\}$, while (9.3.4) implies the reversed inequalities with max replaced by min.

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