Travelling Wave Profiles of (2+1)-dimensional Perturbed and Higher Order cubic-quintic Nonlinear Schrödinger Equations

by

Rafiq Ahmad

A thesis
submitted in partial fulfillment of the
requirements for the degree of
Master of Science
in
Mathematics

School of Natural Sciences,
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August, 2023

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Dedicated

to

My Beloved Parents

and

Honorable Teachers

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Abstract

In this thesis, a detailed study of traveling wave solutions of some higher order nonlinear Schrödinger equations (NLSEs) are discussed. Among these NLSEs, the (2+1)-perturbed nonlinear Schrödinger equation (P-NLSE) in nonlinear fiber optics and higher order cubic-quintic nonlinear Schrödinger equation (CQ-NLSE) are examined. In nonlinear wave motion, a main and recent progress is the discovery of different methods for the solutions of such kind of nonlinear equations. This work motivates the fruitful implementation of three analytical methods, such as tanh-coth, Kudryashov's and sine-cosine methods. These are used to investigate the solitary wave solutions of higher order NLSEs that arise in mathematical physics in a useful and advanced way. We have retrieved trigonometric, hyperbolic, rational and singular solutions. The constraint conditions fall out as an additional product that agree with the existence of the solutions.

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Chapter 1

Introduction

People are compelled to understand what is happening around them because of the earth's revolution around the sun. The more we ponder, the more we learn that each natural process has a model attached to it like a sticker describing its natural characteristics, attributes and connections with other process. Although, it is evident that a basic equation aids comprehension, but nature is much more complex and still infants in our ability to link nature with our system of interpretation and representation. Therefore, some large mathematical equations are utilized to express complex natural processes and the solution of such models is not always guaranteed. If an equation describes a natural law, the solution is not necessarily as straightforward as it appears. Differential equations are the most realistic and widely used representations of various natural processes and their application is required practically in every discipline of science, regardless of subject. Many fundamental rules found in chemistry, physics, biology and other scientific fields are explained by differential models. A quantum mechanical system's wave function is managed by a linear PDE called the Schrödinger equation. Its discovery was a significant milestone in the history of quantum mechanics and is a significant result in the field. The equation is named after an Austrian scientist Erwin Schrödinger, who developed it in 1925 and published it in 1926. This work served as the foundation for his Nobel Prize in 1933 in Physics. A nonlinear form of the Schrödinger equation is the NLSE. It is a classical field equation that primarily describes how light moves through planar waveguides and nonlinear optical fibres.

NLSEs are extremely essential in a variety of domains because nonlinearities in such equations are vital for a realistic representation since they explain the evolution of complicated physical occurrences. NLSEs are investigated from several angles, although researchers are concerned with whether or not their solution exists. Exact solutions will always play a role in understanding various physical processes in numerous scientific domains. NLSEs, used to address real world issues cannot be solved directly. In other words, there are no closed form solutions. Instead, using various numerical methods, solutions can be approximated. However, accurate results are still needed for the verification of the numerical results. In certain natural science fields, it is critical to find exact solutions and to comprehend characteristics of physical laws adequately. Exact solutions of NLSEs have historically played and continue to play a critical part in the scientific understanding of many physical features. There are two categories of integrable and non-integrable NLSEs that are found in theoretical and applied physics and other scientific fields. Many investigators have investigated integrable equations completely because of their logical appearance and remarkable physical properties. The three different types of Schrödinger equations include relativistic equation, time dependent wave equation and time independent wave equation. Every type own its unique physical significance and applications.

- To illustrate how a system changes over time, the time dependent Schrödinger equation is used. This is the most popular form of the equation and it is used in a wide range of physics domains.
- The time independent Schrödinger equation is used to characterise the system's stationary states. In quantum mechanics and atomic physics, this form of the equation is frequently employed.
- The relativistic Schrödinger equation is used to describe systems operating at near-light speeds. In high-energy physics, this version of the equation is employed.

A basic equation that governs any physical phenomenon is easier to comprehend and examine its solutions too. It is critical to keep in mind that the exact solutions can serve as a prototypical model for testing the validity and precision of various numerical and analytic approaches, even if they lack a clear physical description. The creation of exact solutions to nonlinear PDEs is a fundamental element of nonlinear sciences and this discipline has advance significance in recent decades on account of advancement in computer technology. This conflict is crucial to comprehend and interpret the physical appearances. Integrable NLSEs possess special kind of solutions known as solitons. Nonlinear wave processes in plasmas, fluid dynamics and optical fibres are illustrated by kink shaped tanh and bell shaped sech type solutions. Many of these solutions have been developed and symbolic computations are carried out using MAPLE, MATHEMATICA, or other similar software.

1.1 Preliminaries

A travelling wave arises when the media travels in the direction of the wave's propagation. A travelling wave is also related to maintaining a constant speed during its propagation. Such waves have been recorded in a variety of fields, including combustion, which can be generated from a chemical process. In mathematical biology, the apparent impulses in nerve fibres are represented as travelling waves. Traveling waves are used to describe shock characteristics. However, before moving forward, some key terms must be cleared.

1.1.1 Travelling wave solutions

A permanent form solution of wave equation travelling at a constant speed is known as a travelling wave solution. Nonlinear PDEs are often transformed into comparable ODEs in order to discover travelling wave solutions. The solution of travelling wave is given as,

$$P(x,t) = f(x - ct),$$

where c is the wave speed, x and t are space and time variables respectively. There are many different forms of travelling waves and only a few of them will be discussed.

(a) (b)

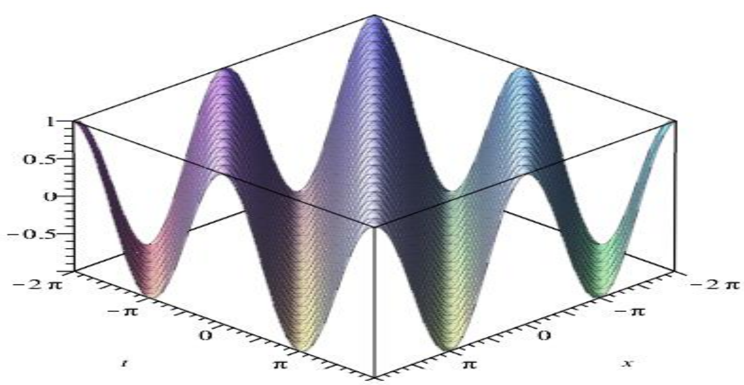
Figure 1.1: Travelling waves in water

1.1.2 Solitary waves

Solitary waves are travelling waves with fixed velocity and shapes that asymptotically approach zero at long distances. The transition of solitary wave is confined in η from one asymptotic state, $\eta = -\infty$ to the other, $\eta = \infty$.

1.1.3 Periodic solutions

Periodic solutions, such as $\cos(x - wt)$, are periodic travelling wave solutions. Periodic solutions can be found using the conventional wave equation $u_{tt} = u_{xx}$.



(a)

Figure 1.2: Graph of a periodic solution $p(x,t) = \cos(x - wt), -2\pi \leq x, t \leq 2\pi$

1.1.4 Kink waves

Kink waves change from one asymptotic state to the next. The kink solution approaches a constant at infinity. The dissipative Burger's equation

$$u_t + 6uu_x = vu_{xx},$$

where v is the coefficient of viscosity.

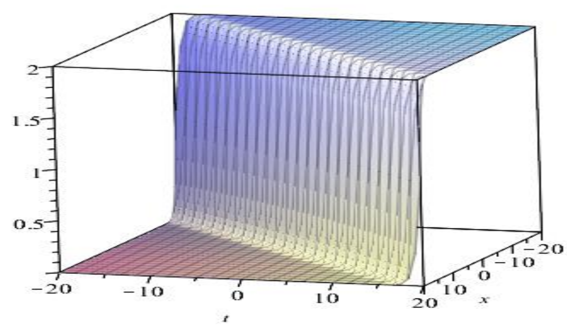


Figure 1.3: Graph of a kink solution $u(x,t) = 1 - \tanh(x - wt), -20 \leq x, t \leq 20$

1.1.5 Peakons

Peakons are single wave solutions with peaks. Except for a peak at the crest's corner, the travelling wave solutions are plane. After interacting peakons are the only solitons that retain their speed and shape. The integrable Camassa-Holm equation gives the following peakon solution,

$$p(x,t) = we^{-\left|(x-wt)\right|}, -2 \leq x, t \leq 2,$$

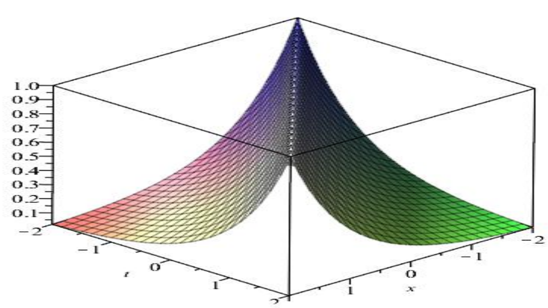


Figure 1.4: Graph of a peakon solution $p(x,t) = e^{-\left|(x-t)\right|}, -2 \leq x, t \leq 2$, with $w = 1$

1.1.6 Compactons

Compactons are solitary waves that reappear with the same consistent shape after colliding with other compactons, which is a unique soliton characteristic.

$$p(x,t) = \cos^{\frac{1}{2}}(x - wt)$$

Compactons have following properties:

- The wavelength of compactons is finite.
- Compactons are compact support solitons.
- Compactons do not have exponential tails.

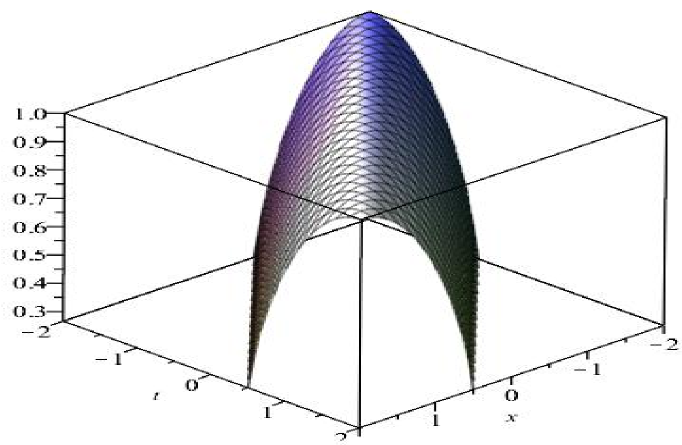


Figure 1.5: Graph of a compacton $p(x,t) = \cos^{\frac{1}{2}}(x - t), -2 \leq x, t \leq 2$, with $w = 1$

1.2 Solitons

A Soliton is travelling wave solution of NLEEs having following properties:

- A wave of permanent form should be visible in the solution.
- While keeping its identity, the solitons interacts with other solitons.

- The solution is confined, which implies that either it dissipates exponentially to zero, as in the KdV equation solitons, or reaches to an infinity, as in the solitary waves of Sine-Gordon's equation.

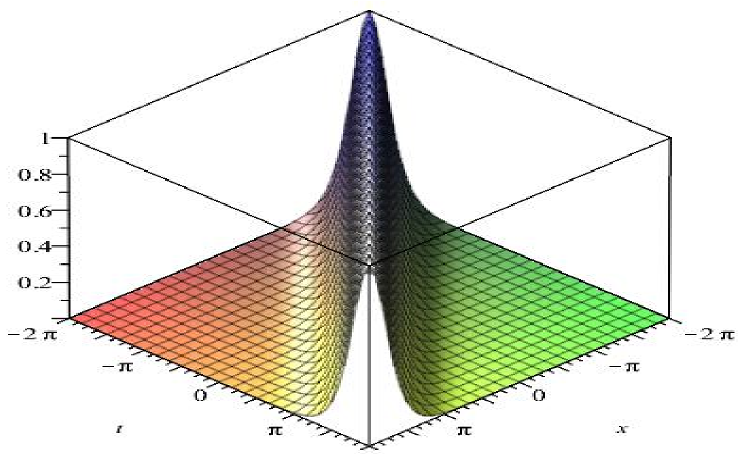


Figure 1.6: Graphical representation of a soliton solution $p(x,t) = \operatorname{sech}^2(x-t), -2\pi \leq x, t \leq 2\pi$

Figure 1.7: Solitary waves in Ocean

1.2.1 Discovery of solitons

In 1834, a Scottish engineer John Scott Russell (1808-1882), made the very first observation of solitary waves in Edinburgh. He noted a large bulge of water waves with same shape moving across the canal. Russell says, "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called wave of translation" [1]. Scott Russell's findings contradicted Isaac Newton's and Daniel Bernoulli's hydrodynamic theories. George Bid-dell Airy and George Gabriel Stokes declined to accept the experimental findings of Scott Russell due to disagreements with previously accepted water wave theories. Lord Rayleigh and Joseph Boussineq [2] did not provide answers to their disagreements until the 1870s. In 1895, the KdV equation was published by Gustav de-Vries and Diederik Korteweg, which included solutions for solitary waves [3]. Using a finite difference approach, In 1965, Bell Labs' Norman Zabusky and Princeton University's Martin Kruskal discovered soliton in the Korteweg de-Vries equation [4]. By definition, Collisions with other solitons have no effect on their form or speed [5]. So, solitary waves on an ocean's surface are near-solitons, but just not exactly, even though when two (overtaking or colliding) solitary waves combine, they become near-solitons, their amplitude changes and an oscillatory remnant is left behind [6].

1.3 Application of solitons

Solitons are used in many fields of pure and applied mathematics, differential equations, differential geometry, Lie algebras, Lie groups and algebraic geometry and other fields.

1.3.1 In Fiber Optics

In fiber optics applications, solitons have been extensively tested. Because of Soliton's stability, without the use of transmitters, long-distance communication is possible and it can also function with higher transmission capacity [7, 8, 9].

1.3.2 In Biology

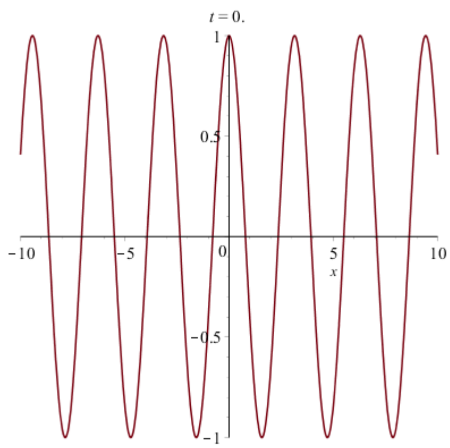
The theory of soliton has been used to explain low-frequency collective motion in the propagation of energy and signal in bio-membranes, as well as in DNA and proteins, the nervous system and other phenomena. A concept in neuroscience posits that signals in the form of density waveform are transmitted to neurons as a kind of solitons. Solitons can be defined as a symmetric energy transfer in bio-molecular systems that emerges as a wave-like transmission of electronic disturbances and associated conformational [10, 11, 12].

1.3.3 In Nuclear Physics

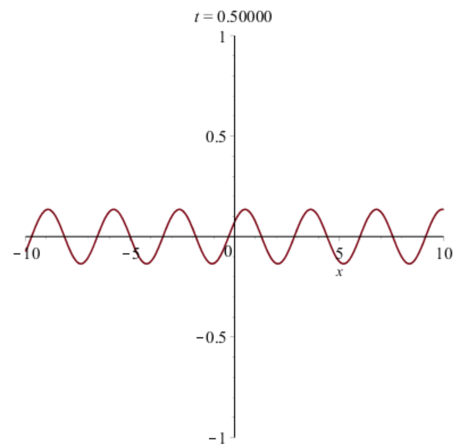
Under certain temperature and energy conditions, the entire nuclear wave function is assumed to be a soliton. Such circumstances are considered to prevail in the nuclei of several nuclear objects, where nuclei do not rebound and proceed each other unaffected, with their soliton waves remaining unchanged as a result of the collision.

1.4 Dispersion and dissipation

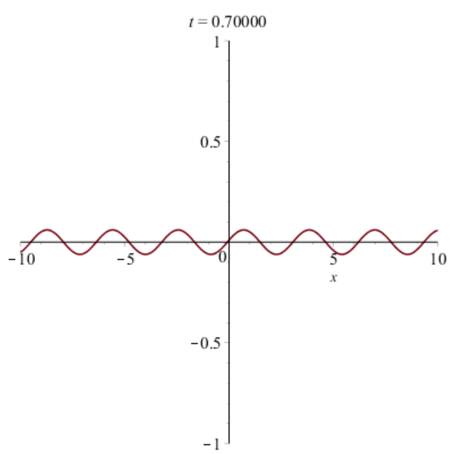
Dispersion is the connection between a wave's phase velocity and its frequency in optics. The phase velocities of waves of different frequencies will differ. Wave dispersion describes the change in phase velocity of water waves as the number of waves change. The mediums with these characteristics are dispersive. A rainbow is a simple illustration of dispersion, which occurs when light is spatially split into wavelength components (different colors). A dissipative wave is one that gradually loses amplitude due to energy loss. The solution of the dissipative equation shows that waves propagate with unity speed. In dispersive media, velocity varies with wave number and in dissipative media, waves propagate at a unity speed. Some 2D graphs are shown to see the dynamic behavior of dissipation terms in PDEs.



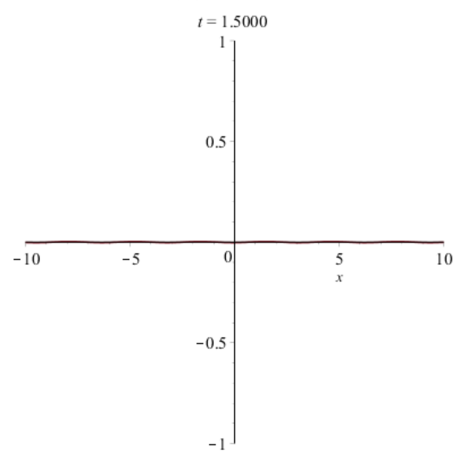
(a) $t = 0$



(b) $t = 0.5$



(c) $t = 0.7$



(d) $t = 1.5$

Figure 1.8: 2D Graphs of solution of dissipative equation

1.5 Literature Review

The study of nonlinear PDEs help researchers to understand complex physical phenomena of nature. NLSEs essentially examine the behavior of nonlinear matter waves in nonlinear transmission systems or Bose-Einstein condensates. Physicists and mathematicians with an interest in this area have been drawn to NLSEs. Developing solitary wave solutions to NLSEs helps in comprehending underlying physics. A natural system can be properly explained by generating the travelling wave solutions of NLSEs. Hence, analytical methodologies are required to generate travelling wave solutions of governing models. Several strategies have been developed during the last few decades. In 1991, Satsuma and Sasa studied a recent type of soliton solution for a higher order NLSE [13]. In 1996, Schurmann constructed the traveling wave solutions of the cubic quintic NLSE [14]. In 2007, Mingliang, Wang, Jinliang Zhang and Xiangzheng Li applied the Sub-ODE method and found solitary wave solutions for higher order NLSE [24]. In 2010, A. Borhanifar and Reza Abzari solved the coupled Schrödinger equations using the differential transformation method [15]. In 2011, Gui-Qiong discussed new types of exact solutions for the fourth order dispersive and cubic-quintic NLSE [16]. In 2011, Anjan and Masood used Lie symmetry analysis to find the stationary solution of the NLSE with log law nonlinearity [17]. In 2011, Dai, C. Q., Wang, X. G. and Wang, Y. Y. found the ultrashort self-similar solutions of the cubic quintic NLSE with distributed coefficients in the in-homogeneous fiber [18]. In 2012, Taghizadeh and Mirzazadeh examined the P-NLSE with Kerr law non linearity using the simplest equation method [19]. In 2015, Li, Tao Xu, Min and Lei Wang discussed the soliton solutions and the dynamical behaviors of a generalized higher order NLSE in optical fibers [20].

Soliton in optical communication systems has become a major topic to discuss throughout time. In 2004, A. M. Wazwaz found the traveling wave solutions of nonlinear equations using the tanh method [21]. In 2006, Chen and Zhang elucidated the behavior of pulses in fiber optics with non linearity and variable dispersion [22]. In 2007, Gao and Tian used symbolic computation and Backlund transformation to investigate the Schrödinger euation in fiber optics [23]. In 2009, Lee used the rational function method to provide analytical solutions to the Jimbo-Miwa equation [25]. In 2010, D. Milovic and A. Biswas discovered dark and brilliant solitons of the generalized nonlinear Schrödinger equation [26]. In 2011, A. Biswas obtained dark solitons with parabolic and dual power law nonlinearities for a generalized nonlinear Schrödinger equation for the second time [27]. In 2017, Arshad, M., Lu, D. and Seadawy, A. R. found the exact dark and bright solitary wave solutions of the higher order cubic–quintic NLSE and discussed its stability [28]. In 2018, N. Raza and A. Javid investigated singular and optical dark solitons for the Biswas-Milovic equation in spatiotemporal dispersion optics [29]. In 2019, N. Raza and A. Javid used the extended trial equation and direct algebraic methods to create the optical dark and dark singular solitons in the (1+2)-dimensional chiral NLSE [30]. S. Yakada, B. Depelair, et al. applied the new extended direct algebraic method to investigate exact solutions for generalized NLSE modeling few cycle pulse propagation in meta materials [31]. In 2019, Chunxiao and Boling discussed the existence of global solutions for the fourth-order NLSEs [32]. In 2020, Saima Arshed and Aqsa Arif found soliton solutions of nonlinear kudryashov's equation and higher-order NLSE [33]. In 2021, Rezazadeh, Hadi, et al. applied the new Kudryashov method to find the optical soliton solutions of the generalized nonautonomous NLSEs [34]. In 2022, Wazwaz found the dark and bright optical solitons of the (2+1)-dimensional P-NLSE in nonlinear optical fibers [35]. In 2022, Esen, H., Secer, A., Ozisik, M. and Bayram, M. discussed the influence of the model's parameters and investigated the analytical soliton solutions of the higher order cubic-quintic NLSE [36]. The thesis is organized as follows:

- In chapter 2, the description of tanh-coth method, Kudryashov's method and sine-cosine method is illustrated.
- In chapter 3, the application of tanh-coth method, Kudryashov's method and sine-cosine method to investigate the solutions of the (2+1)-dimensional P-NLSE in nonlinear fiber optics and higher order CQ-NLSE is given.
- In chapter 4, a summary and conclusion is given.

Chapter 2

Methodologies

In this chapter, we delve into the exploration and analysis of three distinct yet powerful mathematical methods, the tanh-coth, Kudryashov's and the sine-cosine. These methodologies hold supreme importance in the field of applied mathematics and have garnered considerable attention due to their effectiveness in solving a wide range of complex mathematical problems. Throughout this study, we aim to provide a comprehensive overview of each method, elucidating their underlying principles, advantages and limitations. By delving into the intricate details of these methods, we aspire to equip readers with a profound understanding of their applicability and potential in tackling various mathematical challenges. The insights gained from this research are poised to enrich the existing body of knowledge in the domain of mathematical analysis and provide valuable tools for researchers and practitioners alike.

2.1 Description of the tanh-coth method

This segment contains a short explanation of the tanh-coth method.
Let having a nonlinear PDE:

$$L(P, P_t, P_x, P_y, P_{tt}, P_{xx}, P_{yy}, P_{xxx}, ...) = 0, \tag{2.1}$$

using the transformation

$$P(x, y, t) = p(\xi)e^{\iota\eta}, \;\; \xi = x + y - vt \text{ and } \eta = x + y + \omega t,$$

where ω, v are arbitrary constants.
Eq. (2.1) decreases to an ODE

$$Q(p(\xi), p'(\xi), p''(\xi), p'''(\xi), ...) = 0, \tag{2.2}$$

Eq. (2.2) is then integrated as much as all terms have derivatives and integration constants are zeros. Malfliet [1] established the standard tanh approach, in which the tanh is employed as a new variable, because all derivatives of a tanh are represented by the tanh itself. For example, if we set $R = \tanh(\xi)$, then we have

$$\begin{aligned} R &= \tanh(\xi), \\ R' &= 1 - R^2, \\ R'' &= -2R + 2R^3, \\ R''' &= -2 + 8R^2 - 6R^4, \\ R^{(4)} &= 16R - 40R^3 + 24R^5. \end{aligned} \tag{2.3}$$

Moreover, introducing a new independent variable

$$S(\xi) = \tanh(\mu\xi), \;\; \xi = x + y - vt, \tag{2.4}$$

where μ is a wave number, this leads to the change of derivatives

$$\begin{aligned} \frac{d}{d\xi} &= \mu(1 - S^2)\frac{d}{dS}, \\ \frac{d^2}{d\xi^2} &= -2\mu^2 S(1 - S^2)\frac{d}{dS} + \mu^2(1 - S^2)^2\frac{d^2}{dS^2}, \\ \frac{d^3}{d\xi^3} &= 2\mu^3(1 - S^2)(3S^2 - 1)\frac{d}{dS} - 6\mu^3 S(1 - S^2)^2\frac{d^2}{dS^2}\mu^3(1 - S^2)^3\frac{d^3}{dS^3}, \end{aligned} \tag{2.5}$$

$$\frac{d^4}{d\xi^4} = -8\mu^4 S(1-S^2)(3S^2-2)\frac{d}{dS} + 4\mu^4(1-S^2)^2(9S^2-2)\frac{d^2}{dS^2}$$

$$-12\mu^4 S(1-S^2)^3\frac{d^3}{dS^3} + \mu^4(1-S^2)^4\frac{d^4}{dS^4},$$

the tanh-coth method [2] follows the use of finite expansion

$$p(\xi) = R(S) = \sum_{k=0}^{m} a_k S^k + \sum_{k=1}^{m} b_k S^{-k}. \tag{2.6}$$

In tanh-coth method, m is taken as an integer that is positive, that can be found by balancing principle. The standard tanh-coth method can be obtained by substituting $b_k = 0, 1 \leq k \leq m$, in expansion (2.6), the value of m is generally acquired by equating the highest oder linear terms and the highest order nonlinear terms in the final equation. If parameter m is not an integer, then this problem is removed by a transformation technique. In order to calculate the values $a_k (k = 0, ..., m)$, $b_k (k = 1, ..., m)$ and v we get an algebraic system of equations in powers of Y by putting (2.6) into the final ODE. Finally, we get a closed form of an exact solution $p(x, y, t)$. The solutions we find may be solitons, compacton, cuspon, peakon, kink, periodic and travelling wave solutions as well.

2.2 Description of the Kudryashov's method

This segment contains short explanation of the Kudryashov's method.
Consider a nonlinear PDE:

$$L(P, P_t, P_x, P_y, P_{tt}, P_{xx}, P_{yy}, P_{xxx}, ...) = 0, \tag{2.7}$$

here, $P(x, y, t)$ and the partial derivatives of $P(x, y, t)$ also the nonlinear terms are involved in polynomial L.
Applying the following transformation

$$p(x, y, t) = p(\xi)e^{\iota\eta}, \;\; \xi = x + y - vt \;\text{ and }\; \eta = x + y + \omega t, \tag{2.8}$$

where the parameters ω and v are arbitrary. Putting Eq.(2.8), in Eq.(2.7) results to a nonlinear ODE.

$$Q(p(\xi), p'(\xi), p''(\xi), p'''(\xi), ...) = 0, \tag{2.9}$$

the following are the key steps of the Kudryashov's method:

Step 1: Suppose that the solution of Eq. (2.9) can be represented by the following form:

$$p(\xi) = \sum_{i=0}^{n} a_i Z^i(\xi), \tag{2.10}$$

where the constants $a_i (i = 0, 1, ..., n)$ will be obtained algebraically, such that $a_n \neq 0$. While the following Ricatti differential equation is satisfied by the function $Z = Z(\xi)$

$$\frac{dZ}{d\xi} = Z^2(\xi) - Z(\xi), \tag{2.11}$$

it is clear that the solution of Eq. (2.11) is

$$Z(\xi) = \frac{1}{1 + Ae^{\xi}}, \tag{2.12}$$

where A is an integration constant.

Step 2: The positive integer n in Eq. (2.10) can be determined by applying the balancing principle on the highest power nonlinear terms and the highest order derivatives from Eq. (2.9).

Step 3: We put $p(\xi)$ given in Eq. (2.10) and the required derivatives $p'(\xi), p''(\xi), p'''(\xi)$, ... to substitute them in Eq. (2.9) and finally we will obtain the polynomial equation:

$$P[Z(\xi)] = 0. \tag{2.13}$$

Step 4: We select all the terms that have the same algebraic powers of Q from the polynomial equation Eq. (2.13), set them equal to zero and obtain a system of algebraic equations with the set of unknowns $\{a_i(i = 0, 1, ..., n), v, \omega\}$. We can use some calculation software, such as Maple, to solve the system with the natural restrictions of the model and also considering that it is required that $a_n \neq 0$.

Step 5: Finally, substituting the obtained values and Eq. (2.11) in Eq. (2.10), we produce all the feasible solutions of the Eq. (2.9) and therefore those of Eq. (2.7).

2.3 Description of the sine-cosine method

This section contains short explanation of the sine-cosine method.
Having a nonlinear PDE:

$$L(P, P_t, P_x, P_y, P_{tt}, P_{xx}, P_{yy}, P_{xxx}, ...) = 0, \tag{2.14}$$

here, $P(x, y, t)$ and the partial derivatives of $P(x, y, t)$ also the nonlinear terms are involved in polynomial L.
The key steps of the sine-cosine method are as follows:
Step 1: We apply the following transformation:

$$P(x, y, t) = p(\xi)e^{\iota\eta}, \;\; \xi = x + y - vt \text{ and } \eta = x + y + \omega t, \tag{2.15}$$

here, the parameters ω and v are arbitrary. Substituting Eq. (2.15) into Eq. (2.14), we get a nonlinear ODE.

$$Q(p(\xi), p'(\xi), p''(\xi), p'''(\xi), ...) = 0, \tag{2.16}$$

Step 2: In this step, we integrate the ODE (2.16) as many time as possible while assuming that the integration constant is zero.
Step 3: Following the conclusions made in [37], the solution may be set in the form

$$p(x, t) = \lambda \sin^{\beta}[\mu\xi], \quad |\xi| \leq \frac{\pi}{\mu}, \tag{2.17}$$

or in the form

$$p(x, t) = \lambda \cos^{\beta}[\mu\xi], \quad |\xi| \leq \frac{\pi}{2\mu}. \tag{2.18}$$

Where μ, λ and β are variables that will be calculated.
Step 4: We put Eq. (2.17) or Eq. (2.18) into the equation obtained in Eq. (2.16), equate the terms of sine functions when Eq. (2.17) is used, or equate the terms of cosine functions when Eq. (2.18) is used and using Maple, solve the obtained system of algebraic equations to find all feasible solutions of the variables μ, λ and β.

Chapter 3

Travelling Wave Solutions of Higher Order NLSEs

In this chapter, the tanh-coth, Kudryashov's and sine-cosine methods are used to solve higher-order NLSEs that arise in mathematical physics in a useful and advanced way. Higher order (2+1)-dimensional P-NLSE in nonlinear optical fibers and higher order CQ-NLSEs are solved by the tanh-coth, Kudryashov's and sine-cosine methods. A generalized (2+1)-dimensional P-NLSE contains Kerr law nonlinearity, in the existence of nonlinear effects and random dispersion. Second Higher-order NLSE with cubic-quintic nonlinearity (CQNL) terms that define the propagation of ultrashort pulses, third-order dispersion(3OD) and fourth-order dispersion(4OD). These higher-order NLSEs have many applications in the dynamics of traffic flow, fluid mechanics, propagation of light in planar waveguides, nonlinear optical fibers and the study of waves propagating in plasma. Consequently, new exact hyperbolic, periodic, rational and singular solutions of NLSEs are attained.

3.1 Application of tanh-coth method

In this section, a new analytical approach as, tanh-coth method is used for finding travelling wave solutions and solitons for (2+1)-dimensional P-NLSE in nonlinear optical fibres and higher order CQ-NLSE, which have significant implications in a telecommunication system and ultra-fast signal routing.

3.1.1 Exact solutions of (2+1)-dimensional P-NLSE in nonlinear optical fibres

Here, we implement tanh-coth method to find travelling wave solutions of the (2+1)-dimensional P-NLSE in nonlinear optical fibres, which is given as [35]

$$\iota U_t + r_1 U_{xx} + r_2 U_{yy} + U|U|^2 - \iota\left(aU_x - b\left(|U|^2U\right)_x - cU\left(|U|^2\right)_x\right) + U_{xxxx} - U_{yyyy} = 0, \tag{3.1}$$

where $\iota=\sqrt{-1}$ and a, b, c, r_1 and r_2 are arbitrary constants.
The generalized (2+1)-dimensional P-NLSE contains the terms describing self steepening in x and y direction, the terms with self frequency and the 4OD terms.
Applying the following transformation,

$$U(x, y, t) = u(\xi)e^{\iota\eta}, \quad \xi = x + y - vt \quad \text{and} \quad \eta = x + y + \omega t. \tag{3.2}$$

We generate nonlinear ODEs by combining Eq. (3.2) and Eq. (3.1) as follows, then separating the real and imaginary components respectively:

$$(r_1 + r_2)u''(\xi) + (a - \omega - (r_1 + r_2))u(\xi) + (1 - b)u^3(\xi) = 0, \tag{3.3}$$

$$(2(r_1 + r_2) - a - v)u'(\xi) + (2c + 3b)u'(\xi)u^2(\xi) = 0, \tag{3.4}$$

here, the variables a, b, c, r_1 and r_2 are real.
Applying the balancing principle on $u''(\xi)$ and $u^3(\xi)$ from Eq. (3.3), the balancing constant is found to be $m = 1$. Furthermore, the tanh-coth method allows for the employment of the finite expansion,

$$u(\xi) = A_0 + A_1 Y(\xi) + \frac{B_1}{Y(\xi)}. \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.3) and Eq. (3.4) and we obtain the following system by accumulating the coefficients of the similar exponents of $Y(\xi)$ and setting them to zero,

$$-bA_1^3 + 2\mu^2 A_1 r_1 + 2\mu^2 A_1 r_2 + {A_1}^3 = 0,$$

$$-3bA_0{A_1}^2+3A_0{A_1}^2=0,$$

$$-3b{A_0}^2A_1-3b{A_1}^2B_1-A_1r_1-A_1r_2-2\mu^2A_1r_1-2\mu^2A_1r_2+aA_1-\omega A_1$$

$$+3{A_0}^2A_1+3{A_1}^2B_1=0,$$

$$-b{A_0}^3-A_0r_1-A_0r_2+aA_0-\omega A_0+{A_0}^3+6A_0A_1B_1-6bA_0A_1B_1=0,$$

$$-3b{A_0}^2B_1-3b{B_1}^2A_1-B_1r_1-B_1r_2-2\mu^2B_1r_1-2\mu^2B_1r_2+aB_1-\omega B_1$$

$$+3{A_0}^2B_1+3A_1{B_1}^2=0,$$

$$-3b\mu A_0{B_1}^2+3A_0{B_1}^2=0,$$

$$-b{B_1}^3+2\mu^2B_1r_1+2\mu^2B_1r_2+{B_1}^3=0,$$

$$-\mu(3b{A_1}^3+2c{A_1}^3)=0,$$

$$-\mu(6bA_0{A_1}^2+4cA_0{A_1}^2)=0,$$

$$-\mu(3b{A_0}^2A_1-3b{A_1}^3+2c{A_0}^2A_1+3b{A_1}^2B_1-2c{A_1}^3+2c{A_1}^2B_1+2A_1r_1$$

$$+2A_1r_2-aA_1-vA_1)=0,$$

$$-\mu(-6bA_0{A_1}^2-4cA_0{A_1}^2)=0,$$

$$-\mu(-3b{A_0}^2A_1-3b{A_0}^2B_1-3b{A_1}^2B_1-3bA_1{B_1}^2-2c{A_0}^2A_1-2c{A_0}^2B_1-2c{A_1}^2B_1$$

$$-2cA_1{B_1}^2-2A_1r_1-2A_1r_2-2B_1r_1-2B_1r_2+aA_1+vA_1+aB_1+vB_1)=0,$$

$$-\mu(-6bA_0{B_1}^2-4cA_0{B_1}^2)=0,$$

$$-\mu(3b{A_0}^2B_1+3bA_1{B_1}^2-3b{B_1}^3+2c{A_0}^2B_1+2cA_1{B_1}^2-2c{B_1}^3+2B_1r_1+2B_1r_2$$

$$-aB_1-vB_1)=0,\ -\mu(6bA_0{B_1}^2+4cA_0{B_1}^2)=0,$$

$$-\mu(3b{B_1}^3+2c{B_1}^3)=0,$$

where r_1, r_2, μ, ω, v are arbitrary constants.

The outcomes of solving the above system using maple are the following sets of solutions and $A_0=0$:

(i) The first set:

$$B_1=0,\ \ b=-\frac{2}{3}c,\ \ v=2(r_1+r_2)-a,$$

$$A_1 = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{3+2c}}, \ \ \mu = \sqrt{\frac{(r_1 + r_2 - \omega - v)}{2(r_1 + r_2)}}. \tag{3.6}$$

(ii) The second set:

$$A_1 = 0, \ \ b = -\frac{2}{3}c, \ \ v = 2(r_1 + r_2) - a,$$

$$B_1 = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{3+2c}}, \ \ \mu = \sqrt{\frac{(r_1 + r_2 - \omega - v)}{2(r_1 + r_2)}}. \tag{3.7}$$

(iii) The third set:

$$b = -\frac{2}{3}c, \ \ v = 2(r_1 + r_2) - a,$$

$$A_1 = B_1 = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{4(3+2c)}}, \ \ \mu = \sqrt{\frac{(r_1 + r_2 - \omega - v)}{8(r_1 + r_2)}}. \tag{3.8}$$

(iv) The fourth set:

$$b = -\frac{2}{3}c, \ \ v = 2(r_1 + r_2) - a,$$

$$A_1 = B_1 = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{2(3+2c)}}, \ \ \mu = \sqrt{\frac{(r_1 + r_2 - \omega - v)}{4(r_1 + r_2)}}. \tag{3.9}$$

Using above solution sets in (3.5) we get the following general sets of solutions:

$$u_I(\xi) = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{3+2c}} Y(\xi), \quad b = -\frac{2}{3}c, \tag{3.10}$$

$$u_{II}(\xi) = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{3+2c}} Y^{-1}(\xi), \quad b = -\frac{2}{3}c, \tag{3.11}$$

$$u_{III}(\xi) = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{4(3+2c)}} (Y(\xi) + Y^{-1}(\xi)), \quad b = -\frac{2}{3}c, \tag{3.12}$$

$$u_{IV}(\xi) = \sqrt{\frac{3(\omega + v - r_1 - r_2)}{2(3+2c)}} (Y(\xi) + Y^{-1}(\xi)), \quad b = -\frac{2}{3}c. \tag{3.13}$$

If we combine Eqs. (2.4), (3.2), (3.5), (3.1.1) and solution set $u_I(\xi)$, then we get the solution as follows:

$$u_1(x,y,t)=\sqrt{\frac{3(\omega-a+r_1+r_2)}{3+2c}}\left(\tanh\left[\mu(x+y-(2(r_1+r_2)-a)t)\right]\right)e^{(x+y+\omega t)\iota}, \tag{3.14}$$

where

$$\mu=\sqrt{\frac{(a-\omega-r_1-r_2)}{2(r_1+r_2)}}.$$

Considering Eqs. (2.4), (3.2), (3.5), (3.1.1) and (27), then we produce the following solution:

$$u_2(x,y,t)=\sqrt{\frac{3(\omega-a+r_1+r_2)}{3+2c}}\left(\coth\left[\mu(x+y-(2(r_1+r_2)-a)t)\right]\right)e^{(x+y+\omega t)\iota}, \tag{3.15}$$

where

$$\mu=\sqrt{\frac{(a-\omega-r_1-r_2)}{2(r_1+r_2)}}.$$

Assuming Eqs. (2.4), (3.2), (3.5), (3.1.1) and solution set $u_{III}(\xi)$, then we derive the following solution as follows:

$$\begin{aligned}u_3(x,y,t)=\sqrt{\frac{3(\omega-a+r_1+r_2)}{4(3+2c)}}&\Bigg(\tanh\left[\mu(x+y-vt)\right]\\&+\coth\left[\mu(x+y-vt)\right]\Bigg)e^{(x+y+\omega t)\iota},\end{aligned} \tag{3.16}$$

where

$$\mu=\sqrt{\frac{(a-\omega-(r_1+r_2))}{8(r_1+r_2)}},\quad v=(2(r_1+r_2)-a).$$

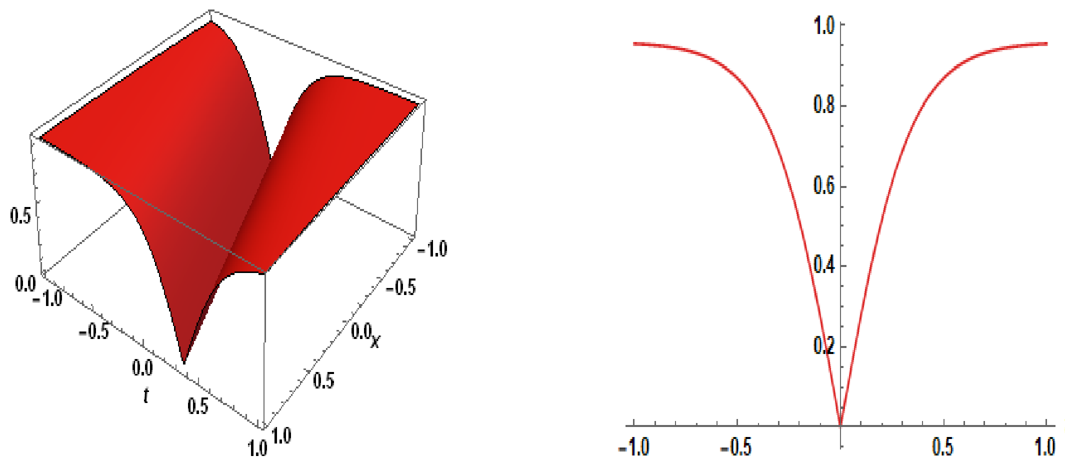
Assuming Eqs. (2.4), (3.2), (3.5), (3.1.1) and solution set $u_{IV}(\xi)$, then we derive the following solution as follows:

$$\begin{aligned}u_4(x,y,t)=\sqrt{\frac{3(\omega-a+r_1+r_2)}{2(3+2c)}}&\Bigg(\tanh\left[\mu(x+y-vt)\right]\\&+\coth\left[\mu(x+y-vt)\right]\Bigg)e^{(x+y+\omega t)\iota},\end{aligned} \tag{3.17}$$

where

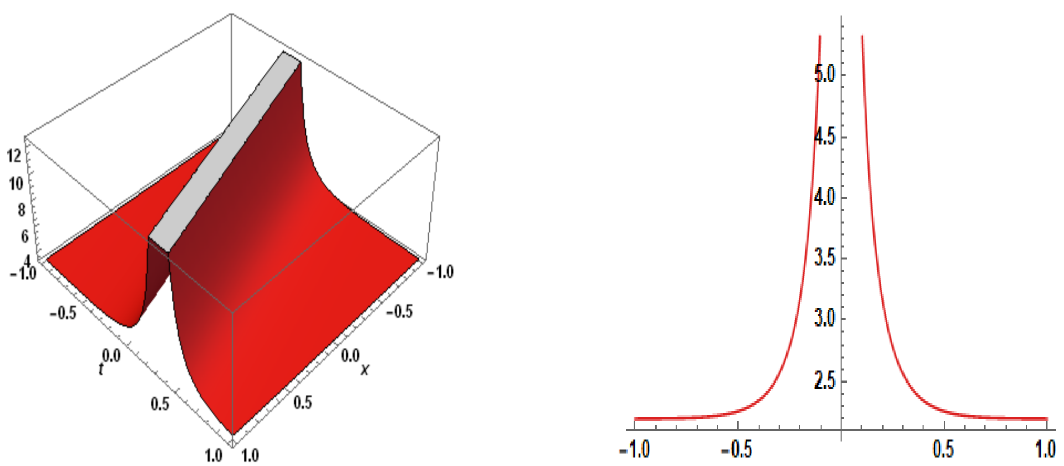
$$\mu = \sqrt{\frac{(a - \omega - (r_1 + r_2))}{4(r_1 + r_2)}}, \ \ v = (2(r_1 + r_2) - a).$$

Graphical illustration of a solution



(a) (b)

Figure 3.1: Graphical representation of a soliton solution $u_1(x, y, t) = -2\sqrt{\frac{3}{13}}e^{\iota(x-5t)}\tanh[3t - x], y = 0, -1 \leq x, t \leq 1$, (a) 3D Plot (b) 2D Plot



(a) (b)

Figure 3.2: Graphical representation of a soliton solution $u_2(x, y, t) = -2\sqrt{\frac{3}{13}}e^{\iota(x-5t)}\coth[3t - x], y = 0, -1 \leq x, t \leq 1$, (a) 3D Plot (b) 2D Plot

3.1.2 Exact Solutions of higher order CQ-NLSE

Here, we implement tanh-coth method to find travelling wave solutions of the higher order CQ-NLSE, which is given as [36]:

$$\iota U_t-\frac{\alpha_1}{2}U_{xx}+\beta_1 U|U|^2-\iota\frac{\alpha_2}{6}U_{xxx}-\frac{\alpha_3}{24}U_{xxxx}+\beta_2\left(|U|^4U\right)-\iota\gamma_1\left(|U|^2U\right)_x-\iota\gamma_2 U\left(|U|^2\right)_x=0, \tag{3.18}$$

where $\iota = \sqrt{-1}$ and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1$ and γ_2 are real parameters.
Applying the following transformation,

$$U(x,t)=u(\xi)e^{\iota\eta}, \quad \xi = px - vt \quad \text{and} \quad \eta = x+\omega t. \tag{3.19}$$

We generate the nonlinear ODEs by combining Eq. (3.19) and Eq. (3.18) as follows, then separating the real and imaginary components respectively:

$$\begin{aligned}\alpha_3 p^4 u^{(4)}(\xi) + 6(2\alpha_1 - 2\alpha_2 - \alpha_3)p^2 u''(\xi) + (24\omega + 4\alpha_2 - 12\alpha_1 + \alpha_3)u(\xi)&\\ -24(\gamma_1+\beta_1)u^3(\xi) - 24\beta_2 u^5(\xi) = 0,&\end{aligned} \tag{3.20}$$

$$\begin{aligned}(\alpha_2+\alpha_3)p^3 u'''(\xi) + (6v + 6\alpha_1 p - 3\alpha_2 p - \alpha_3 p)u'(\xi)&\\ +p(18\gamma_1 + 12\gamma_2)u^2(\xi)u'(\xi) = 0,&\end{aligned} \tag{3.21}$$

here, the parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1$ and γ_2 are arbitrary.
Applying the balancing principle om $u^{(4)}(\xi)$ and $u^5(\xi)$ from Eq. (3.20), the balancing constant is found to be $m = 1$. Furthermore, the tanh-coth method allows for the employment of the finite expansion,

$$u(\xi) = A_0 + A_1 Y(\xi) + \frac{B_1}{Y(\xi)}. \tag{3.22}$$

Putting Eq. (3.22) into Eq. (3.20) and Eq. (3.21) and we obtain the following system by accumulating the coefficients of the similar exponents of $Y(\xi)$ and setting them to zero,

$$24\mu^4 p^4 A_1\alpha_3 - 24A_1^5\beta_2 = 0,$$

$$-120\beta_2 A_0 A_1^4 = 0,$$

$$-40\mu^4p^4A_1\alpha_3+24\mu^2p^2A_1\alpha_1-24\mu^2p^2A_1\alpha_2-12\mu^2p^2A_1\alpha_3-240A_0^2A_1^3\beta_2-120A_1^4B_1\beta_2$$

$$-24A_1^3\beta_1-24A_1^3\gamma_1=0,$$

$$-240A_0^3A_1^2\beta_2-480A_0A_1^3B_1\beta_2-72A_0A_1^2\beta_1-72A_0A_1^2\gamma_1=0,$$

$$16\mu^4p^4A_1\alpha_3-24\mu^2p_A^21\alpha_1+24\mu^2p^2A_1\alpha_2+12\mu^2p^2A_1\alpha_3-120A_0^4A_1\beta_2-720A_0^2A_1^2B_1\beta_2$$

$$-240A_1^3B_1^2\beta_2-72A_0^2A_1\beta_1-72A_0^2A_1\gamma_1-72A_1^2B_1\beta_1-72A_1^2B_1\gamma_1+24\omega_1$$

$$-12A_1\alpha_1+4A_1\alpha_2+A_1\alpha_3=0,$$

$$-24\beta_2A_0^5-480\beta_2A_0^3A_1B_1-720\beta_2A_0A_1^2B_1^2-24\beta_1A_0^3-24\gamma_1A_0^3$$

$$-144\beta_1A_0A_1B_1-144\gamma_1A_0A_1B_1+24\omega A_0-12\alpha_1A_0+4\alpha_2A_0+\alpha_3A_0=0,$$

$$16\mu^4p^4B_1\alpha_3-24\mu^2p^2B_1\alpha_1+24\mu^2p^2B_1\alpha_2+12\mu^2p^2B_1\alpha_3-120A_0^4B_1\beta_2$$

$$-720A_0^2A_1B_1^2\beta_2-240A_1^2B_1^3\beta_2-72A_0^2B_1\beta_1-72A_0^2B_1\gamma_1-72A_1B_1^2\beta_1$$

$$-72A_1B_1^2\gamma_1+24\omega B_1-12B_1\alpha_1+4B_1\alpha_2+B_1\alpha_3=0,$$

$$-240A_0^3B_1^2\beta_2-480A_0A_1B_1^3\beta_2-72A_0B_1^2\beta_1-72A_0B_1^2\gamma_1=0,$$

$$-40\mu^4p^4B_1\alpha_3+24\mu^2p_1^2\alpha_1-24\mu^2p^2B_1\alpha_2-12\mu^2p^2B_1\alpha_3-240A_0^2B_1^3\beta_2$$

$$-120A_1B_1^4\beta_2-24B_1^3\beta_1-24B_1^3\gamma_1=0,$$

$$-120\beta_2A_0B_1^4=0,$$

$$24\mu^4p^4B_1\alpha_3-24B_1^5\beta_2=0,$$

$$6\mu^2p^3A_1\alpha_2+6\mu^2p^3A_1\alpha_3+18pA_1{}^3\gamma 1+12pA_1{}^3\gamma_2=0,$$

$$36p_0A_1^2\gamma_1+24pA_0A_1^2\gamma_2=0,$$

$$-8\mu^2p^3A_1\alpha_2-8\mu^2p^3A_1\alpha_3+18pA_0{}^2A_1\gamma_1+12pA_0{}^2A_1\gamma_2-18pA_1{}^3\gamma_1-12pA_1{}^3\gamma_2$$

$$+18pA_1{}^2B_1\gamma_1+12pA_1{}^2B_1\gamma_2+6pA_1\alpha_1-3pA_1\alpha_2-pA_1\alpha_3+6vA_1=0,$$

$$-36pA_0A_1^2\gamma_1-24pA_0A_1^2\gamma_2=0,$$

$$2\mu^2p^3A_1\alpha_2+2\mu^2p^3A_1\alpha_3+2\mu^2p^3B_1\alpha_2+2\mu^2p^3B_1\alpha_3-18pA_0{}^2A_1\gamma_1-12pA_0{}^2A_1\gamma_2$$

$$-18pA_0{}^2B_1\gamma_1-12pA_0{}^2B_1\gamma_2-18pA_1{}^2B_1\gamma_1-12pA_1{}^2B_1\gamma_2-18pA_1B_1^2\gamma_1-12pA_1B_1^2\gamma_2$$

$$-6pA_1\alpha_1+3pA_1\alpha_2+pA_1\alpha_3-6pB_1\alpha_1+3pB_1\alpha_2+pB_1\alpha_3-6vA_1-6vB_1=0,$$

$$-36pA_0B_1^2\gamma_1-24pA_0B_1^2\gamma_2=0,$$

$$-8\mu^2p^3B_1\alpha_2-8\mu^2p^3B_1\alpha_3+18p{A_0}^2B_1\gamma_1+12p{A_0}^2B_1\gamma_2+18pA_1B_1^2\gamma_1+12pA_1B_1^2\gamma_2$$

$$-18p{B_1}^3\gamma_1-12p{B_1}^3\gamma_2+6pB_1\alpha_1-3pB_1\alpha_2-pB_1\alpha_3+6vB_1=0,$$

$$36pA_0B_1^2\gamma_1+24pA_0B_1^2\gamma_2=0,$$

$$6\mu^2p^3B_1\alpha_2+6\mu^2p^3B_1\alpha_3+18p{B_1}^3\gamma_1+12p{B_1}^3\gamma_2=0.$$

Here, $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\gamma_1$ and γ_2 are arbitrary constants.

The outcomes of solving the above system using maple are the following sets of solutions and $A_0=0$:

(i) The first set:

$$A_1=0,\;\; p=p,\;\; \alpha_2=\alpha_2,\;\; \mu=-\frac{1}{\sqrt{2}p},$$

$$\omega=-\frac{1}{8}\frac{p\alpha_3+8v}{p},\;\; \alpha_1=\frac{1}{3}\frac{2p\alpha_2+p\alpha_3-3v}{p},\;\; B_1=\sqrt{-\frac{2p\alpha_3+3v}{6p\beta_1-4p\gamma_2}},$$

$$\beta_2=\frac{p^2\alpha_3(3\beta_1-2\gamma_2)^2}{4p^2{\alpha_3}^2+12pv\alpha_3+9v^2},\;\; \gamma_1=\frac{1}{3}\frac{3p\alpha_2\beta_1-2p\alpha_2\gamma_2+3p\alpha_3\beta_1-6p\alpha_3\gamma_2-6v\gamma_2}{2p\alpha_3+v}. \tag{3.23}$$

(ii) The second set:

$$B_1=0,\;\; p=p,\;\; \alpha_2=\alpha_2,\;\; \mu=-\frac{1}{\sqrt{2}p},$$

$$\omega=-\frac{1}{8}\frac{p\alpha_3+8v}{p},\;\; \alpha_1=\frac{1}{3}\frac{2p\alpha_2+p\alpha_3-3v}{p},\;\; A_1=\sqrt{-\frac{2p\alpha_3+3v}{6p\beta_1-4p\gamma_2}},$$

$$\beta_2=\frac{p^2\alpha_3(3\beta_1-2\gamma_2)^2}{4p^2{\alpha_3}^2+12pv\alpha_3+9v^2},\;\; \gamma_1=\frac{1}{3}\frac{3p\alpha_2\beta_1-2p\alpha_2\gamma_2+3p\alpha_3\beta_1-6p\alpha_3\gamma_2-6v\gamma_2}{2p\alpha_3+v}. \tag{3.24}$$

(iii) The third set:

$$p=p,\;\; \alpha_2=\alpha_2,\;\; \mu=\frac{\iota}{2p},$$

$$\omega=-\frac{1}{8}\frac{p\alpha_3+8v}{p},\;\; \alpha_1=\frac{1}{3}\frac{2p\alpha_2+p\alpha_3-3v}{p},\;\; A_1=\sqrt{\frac{2p\alpha_3+3v}{6p\beta_1-4p\gamma_2}},\;\; B_1=-A_1,$$

$$\beta_2=\frac{p^2\alpha_3(3\beta_1-2\gamma_2)^2}{4p^2{\alpha_3}^2+12pv\alpha_3+9v^2},\ \ \gamma_1=\frac{1}{3}\frac{3p\alpha_2\beta_1-2p\alpha_2\gamma_2+3p\alpha_3\beta_1-6p\alpha_3\gamma_2-6v\gamma_2}{2p\alpha_3+v}. \tag{3.25}$$

(iv) The fourth set:

$$p=p,\ \ \alpha_2=\alpha_2,\ \ \mu=\frac{1}{2\sqrt{2}p},$$

$$\omega=-\frac{1}{8}\frac{p\alpha_3+8v}{p},\ \ \alpha_1=\frac{1}{3}\frac{2p\alpha_2+p\alpha_3-3v}{p},\ \ A_1=B_1=\sqrt{-\frac{2p\alpha_3+3v}{24p\beta_1-16p\gamma_2}},$$

$$\beta_2=\frac{p^2\alpha_3(3\beta_1-2\gamma_2)^2}{4p^2{\alpha_3}^2+12pv\alpha_3+9v^2},\ \ \gamma_1=\frac{1}{3}\frac{3p\alpha_2\beta_1-2p\alpha_2\gamma_2+3p\alpha_3\beta_1-6p\alpha_3\gamma_2-6v\gamma_2}{2p\alpha_3+v}. \tag{3.26}$$

(v) The fifth set:

$$p=p,\alpha_2=-\alpha_3,\mu=\mu,$$

$$\omega=-\frac{1}{24}\frac{16\mu^4p^5\alpha_3+8\mu^2p^3\alpha_3-48\mu^2p^2v+p\alpha_3+12v}{p},$$

$$\alpha_1=-\frac{1}{3}\frac{p\alpha_3+3v}{p},\ \ \gamma_1=-\frac{2}{3}\gamma_2, A_1=\sqrt{\frac{20\mu^2p^4\alpha_3+p^2\alpha_3-6pv}{6\beta_1-4\gamma_2}}\mu,$$

$$B_1=-A_1,\ \ \beta_2=\frac{4p^2{\alpha_3}^2(3\beta_1-2\gamma_2)^2}{400\mu^4p^6{\alpha_3}^2+40\mu^2p^4{\alpha_3}^2-240\mu^2p^3v\alpha_3+12pv\alpha_3+p^2{\alpha_3}^2+36v^2}. \tag{3.27}$$

(vi) The sixth set:

$$p=p,\ \ \alpha_2=-\alpha_3,\ \ \mu=\frac{1}{p}\sqrt{\frac{p\alpha_3-6v}{10p\alpha_3}},$$

$$\omega=-\frac{1}{600}\frac{11p^2{\alpha_3}^2-732pv\alpha_3+1296v^2}{p^2\alpha_3},\ \ \alpha_1=-\frac{1}{3}\frac{p\alpha_3+3v}{p},$$

$$\gamma_1=-\frac{2}{3}\gamma_2,\ \ \beta_2=-\frac{4}{9}\frac{(9{\beta_1}^2-12\beta_1\gamma_2+4{\gamma_2}^2)p^2\alpha_3}{p^2{\alpha_3}^2-12pv\alpha_3+36v^2},$$

$$A_1=\frac{p\alpha_3-6v}{p}\sqrt[4]{-\frac{9}{3600{\alpha_3}^2{\beta_1}^2-4800{\alpha_3}^2\beta_1\gamma_2+1600{\alpha_3}^2{\gamma_2}^2}},$$

$$B_1=-\frac{20\alpha_3(3\beta_1-2\gamma_2)(p\alpha_3-6v)}{3p}\sqrt[4]{\left(-\frac{9}{3600{\alpha_3}^2{\beta_1}^2-4800{\alpha_3}^2\beta_1\gamma_2+1600{\alpha_3}^2{\gamma_2}^2}\right)^3}. \tag{3.28}$$

(vii) The seventh set:

$$p=p,\alpha_2=-\alpha_3,\mu=\mu,$$

$$\omega = -\frac{1}{24}\frac{256\mu^4p^5\alpha_3 - 16\mu^2p^3\alpha_3 + 96\mu^2p^2v + p\alpha_3 + 12v}{p},$$

$$\alpha_1 = -\frac{1}{3}\frac{p\alpha_3 + 3v}{p}, \quad \gamma_1 = -\frac{2}{3}\gamma_2, \quad A_1 = B_1 = \sqrt{\frac{p^2\alpha_3 - 40\mu^2p^4\alpha_3 - 6pv}{6\beta_1 - 4\gamma_2}}\mu,$$

$$\beta_2 = \frac{4p^2{\alpha_3}^2(3\beta_1 - 2\gamma_2)^2}{1600\mu^4p^6{\alpha_3}^2 - 80\mu^2p^4{\alpha_3}^2 + 480\mu^2p^3v\alpha_3 - 12pv\alpha_3 + p^2{\alpha_3}^2 + 36v^2}. \tag{3.29}$$

Using above solution sets in (3.22) we get the following general sets of solutions

$$u_I(\xi) = \sqrt{-\frac{2p\alpha_3 + 3v}{6p\beta_1 - 4p\gamma_2}}Y^{-1}(\xi), \tag{3.30}$$

$$u_{II}(\xi) = \sqrt{-\frac{2p\alpha_3 + 3v}{6p\beta_1 - 4p\gamma_2}}Y(\xi), \tag{3.31}$$

$$u_{III}(\xi) = \sqrt{\frac{2p\alpha_3 + 3v}{6p\beta_1 - 4p\gamma_2}}\left(Y(\xi) - Y^{-1}(\xi)\right), \tag{3.32}$$

$$u_{IV}(\xi) = \sqrt{-\frac{2p\alpha_3 + 3v}{6p\beta_1 - 4p\gamma_2}}\left(Y(\xi) + Y^{-1}(\xi)\right), \tag{3.33}$$

$$u_V(\xi) = \sqrt{\frac{20\mu^2p^4\alpha_3 + p^2\alpha_3 - 6pv}{6\beta_1 - 4\gamma_2}}\mu\left(Y(\xi) - Y^{-1}(\xi)\right), \tag{3.34}$$

$$u_{VI}(\xi) = \frac{p\alpha_3 - 6v}{p}\left(\sqrt[4]{-\frac{9}{3600{\alpha_3}^2{\beta_1}^2 - 4800{\alpha_3}^2\beta_1\gamma_2 + 1600{\alpha_3}^2{\gamma_2}^2}}Y(\xi)\right.$$

$$\left.-\frac{20\alpha_3(3\beta_1 - 2\gamma_2)}{3}\sqrt[4]{(-\frac{9}{3600{\alpha_3}^2{\beta_1}^2 - 4800{\alpha_3}^2\beta_1\gamma_2 + 1600{\alpha_3}^2{\gamma_2}^2})^3}Y^{-1}(\xi)\right), \tag{3.35}$$

$$u_{VII}(\xi) = \sqrt{\frac{p^2\alpha_3 - 40\mu^2p^4\alpha_3 - 6pv}{6\beta_1 - 4\gamma_2}}\mu\left(Y(\xi) + Y^{-1}(\xi)\right). \tag{3.36}$$

If we combine Eqs. (2.4), (3.19), (3.22), (3.23) and solution set $u_I(\xi)$, then we produce the following solution:

$$u_1(x,t) = \sqrt{-\frac{2p\alpha_3 + 3v}{6p\beta_1 - 4p\gamma_2}}\coth\left[\mu\,(px - vt)\right]e^{\left(x + (-\frac{1}{8}\frac{p\alpha_3 + 8v}{p})t\right)\iota},$$

where

$$\mu=-\frac{1}{\sqrt{2}p}.$$

Considering Eqs. (2.4), (3.19), (3.22), (3.23) and solution set $u_{II}(\xi)$, then we produce the following solution:

$$u_2(x,t)=\sqrt{-\frac{2p\alpha_3+3v}{6p\beta_1-4p\gamma_2}}\tanh\Big[\mu(px-vt)\Big]e^{\left(x+(-\frac{1}{8}\frac{p\alpha_3+8v}{p})t\right)\iota},$$

where

$$\mu=-\frac{1}{\sqrt{2}p}.$$

Assuming Eqs. (2.4), (3.19), (3.22), (3.23) and solution set $u_{III}(\xi)$, then we acquire the following solution:

$$u_3(x,t)=\sqrt{\frac{2p\alpha_3+3v}{6p\beta_1-4p\gamma_2}}\left(\tanh\Big[\mu(px-vt)\Big]-\coth\Big[\mu(px-vt)\Big]\right)e^{\left(x-(\frac{v}{p})t\right)\iota},$$

where

$$\mu=\frac{\iota}{2p}.$$

Assuming Eqs. (2.4), (3.19), (3.22), (3.23) and solution set $u_{IV}(\xi)$, then we acquire the following solution:

$$u_4(x,t)=\sqrt{-\frac{2p\alpha_3+3v}{24p\beta_1-16p\gamma_2}}\Big(\tanh\Big[\mu(px-vt)\Big]+\coth\Big[\mu(px-vt)\Big]\Big)e^{\left(x+(-\frac{1}{8}\frac{p\alpha_3+8v}{p})t\right)\iota},$$

where

$$\mu=\frac{\iota}{2p}.$$

Assuming Eqs. (2.4), (3.19), (3.22), (3.23) and solution set $u_V(\xi)$, then we acquire the following solution:

$$u_5(x,t)=\sqrt{\frac{20\mu^2p^4\alpha_3+p^2\alpha_3-6pv}{6\beta_1-4\gamma_2}}\mu\Big[\tanh\Big[\mu(px-vt)\Big]-\coth\Big[\mu(px-vt)\Big]\Big]e^{(x+\omega t)\iota},$$

where

$$\omega=-\frac{1}{24}\frac{16\mu^4p^5\alpha_3+8\mu^2p^3\alpha_3-48\mu^2p^2v+p\alpha_3+12v}{p}.$$

Assuming Eqs. (2.4), (3.19), (3.22), (3.23) and solution set $u_{VI}(\xi)$, then we acquire the following solution:

$$u_6(x,t) = \frac{p\alpha_3 - 6v}{p}\Bigg(\sqrt[4]{-\frac{9}{3600{\alpha_3}^2{\beta_1}^2 - 4800{\alpha_3}^2\beta_1\gamma_2 + 1600{\alpha_3}^2{\gamma_2}^2}}\tanh\Big[\mu(px - vt)\Big]$$

$$-\frac{20\alpha_3(3\beta_1 - 2\gamma_2)}{3}\sqrt[4]{\left(-\frac{9}{3600{\alpha_3}^2{\beta_1}^2 - 4800{\alpha_3}^2\beta_1\gamma_2 + 1600{\alpha_3}^2{\gamma_2}^2}\right)^3}\coth\Big[\mu(px - vt)\Big]\Bigg)e^{(x+\omega t)\iota},$$

where

$$\omega = -\frac{1}{600}\frac{11p^2{\alpha_3}^2 - 732pv\alpha_3 + 1296v^2}{p^2\alpha_3} \quad and \quad \mu = \frac{1}{p}\sqrt{\frac{p\alpha_3 - 6v}{10p\alpha_3}}.$$

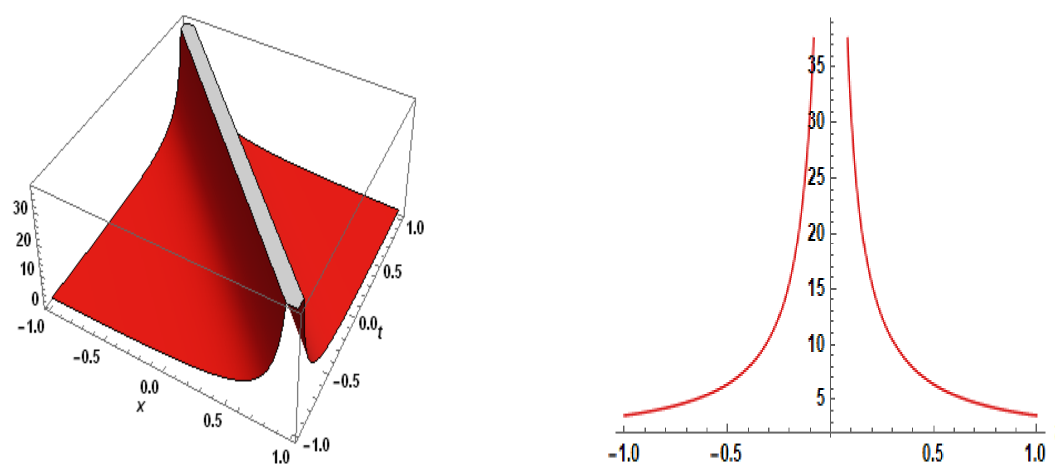
Assuming Eqs. (2.4), (3.19), (3.22), (3.23) and solution set $u_{VII}(\xi)$, then the following solution is derived:

$$u_7(x,t) = \sqrt{\frac{p^2\alpha_3 - 40\mu^2p^4\alpha_3 - 6pv}{6\beta_1 - 4\gamma_2}}\mu\Bigg(\tanh\Big[\mu(px - vt)\Big] + \coth\Big[\mu(px - vt)\Big]\Bigg)e^{(x+\omega t)\iota},$$

where

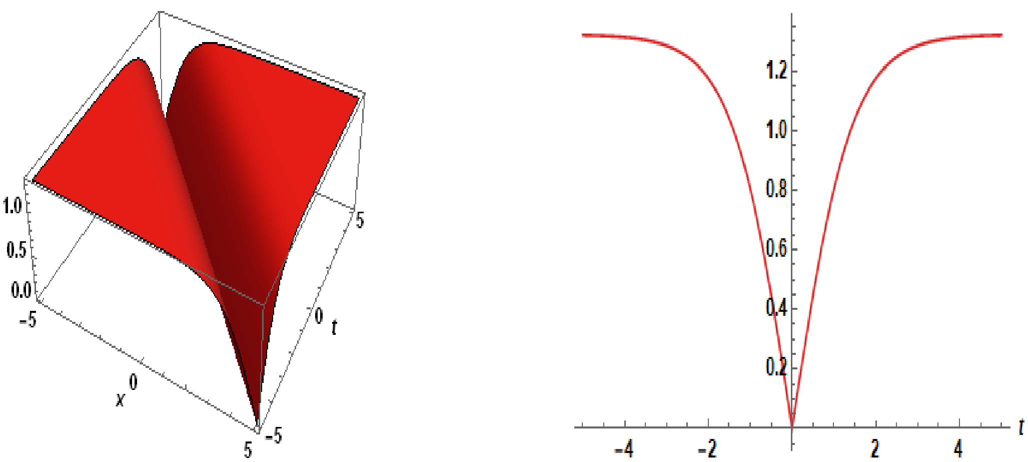
$$\omega = -\frac{1}{24}\frac{256\mu^4p^5\alpha_3 - 16\mu^2p^3\alpha_3 + 96\mu^2p^2v + p\alpha_3 + 12v}{p}.$$

Graphical illustration of a solution



(a) (b)

Figure 3.3: Graphical representation of a soliton solution $u_1(x,t) = \frac{\sqrt{19}}{2}e^{\iota(x+\frac{5t}{4})}(\coth[\frac{-t-x}{\sqrt{2}}]), -1 \leq x, t \leq 1$, (a) 3D Plot (b) 2D Plot



(a) (b)

Figure 3.4: Graphical representation of a soliton solution $u_2(x,t) = \frac{\sqrt{19}}{2}e^{\iota(x+\frac{5t}{4})}(\tanh[\frac{-t-x}{\sqrt{2}}]), -5 \le x, t \le 5$, (a) 3D Plot (b) 2D Plot

3.2 Application of Kudryashov's method

In this section, a new analytical approach as, Kudryashov's method is used for finding travelling wave solutions and solitons for (2+1)-dimensional P-NLSE in nonlinear optical fibres and higher order CQ-NLSE, which have significant implications in telecommunication system and ultra-fast signal routing.

3.2.1 Exact solutions of (2+1)-dimensional P-NLSE in nonlinear optical fibres

Here, we implement Kudryashov's method to find travelling wave solutions of the (2+1)-dimensional P-NLSE in nonlinear optical fibres, which is given as [35]:

$$\iota U_t + r_1 U_{xx} + r_2 U_{yy} + U|U|^2 - \iota\left(aU_x - b\left(|U|^2 U\right)_x - cU\left(|U|^2\right)_x\right) + U_{xxxx} - U_{yyyy} = 0, \tag{3.37}$$

where ι=$\sqrt{-1}$ and a, b, c, r_1 and r_2 are real arbitrary constants.

The generalized (2+1)-dimensional P-NLSE contains the terms describing self steepening in x and y direction, the terms with self frequency and the 4OD terms.

Using the following transformation,

$$U(x, y, t) = u(\xi)e^{\iota\eta}, \quad \xi = x + y - vt \quad \text{and} \quad \eta = x + y + \omega t. \tag{3.38}$$

We generate nonlinear ODEs by combining Eq. (3.38) and Eq. (3.37) as follows, then separating the real and imaginary components respectively:

$$(r_1 + r_2)u''(\xi) + (a - \omega - (r_1 + r_2))u(\xi) + (1 - b)u^3(\xi) = 0, \tag{3.39}$$

$$(2(r_1 + r_2) - a - v)u'(\xi) + (2c + 3b)u^2(\xi)u'(\xi) = 0, \tag{3.40}$$

here, the parameters a, b, c, r_1 and r_2 are arbitrary.
Using the homogeneous balance principle on $u''(\xi)$ and $u^3(\xi)$ from Eq. (3.39), the balancing constant is found to be $m = 1$. Furthermore, the Kudryashov's method allows for the employment of the finite expansion,

$$u(\xi) = A_0 + A_1 Z(\xi). \tag{3.41}$$

Putting Eq. (3.41) into Eq. (3.39) and Eq. (3.40) and we obtain the following system by accumulating the coefficients of the similar exponents of $Z(\xi)$ and setting them to zero,

$$2(r_1 + r_2)A_1 + (1 - b){A_1}^3 = 0,$$

$$-3(r_1 + r_2)A_1 + 3(1 - b)A_0{A_1}^2 = 0,$$

$$(r_1 + r_2)A_1 + (-r_1 - r_2 + a - \omega)A_1 + 3(1 - b){A_0}^2 A_1 = 0,$$

$$(-r_1 - r_2 + a - \omega)A_0 + 3(1 - b){A_0}^3 = 0,$$

$$3b{A_1}^3 + 2c{A_1}^3 = 0,$$

$$6bA_0{A_1}^2 - 3b{A_1}^3 + 4cA_0{A_1}^2 - 2c{A_1}^3 = 0,$$

$$3b{A_0}^2A_1 - 6bA_0{A_1}^2 + 2c{A_0}^2A_1 - 4cA_0{A_1}^2 + 2A_1r_1 + 2A_1r_2 - aA_1 - vA_1 = 0,$$

$$-3b{A_0}^2A_1 - 2c{A_0}^2A_1 - 2A_1r_1 - 2A_1r_2 + aA_1 + vA_1 = 0,$$

where r_1, r_2, μ, ω, v are arbitrary constants.
The outcome of solving the above system using maple is the following set of solution:

$$b = -\frac{2}{3}c, \;\; v = 2(r_1 + r_2) - a, \;\; \omega = \frac{r_1 + r_2 + 2a}{2}, \;\; A_0 = \sqrt{-\frac{3(+r_1 + r_2)}{2(3 + 2c)}}, \;\; A_1 = 2A_0. \tag{3.42}$$

Using above solution set in (3.41) we get the following general set of solution:

$$u(\xi) = \sqrt{-\frac{3(+r_1 + r_2)}{2(3 + 2c)}}(1 + Z(\xi)), \quad b = -\frac{2}{3}c, \tag{3.43}$$

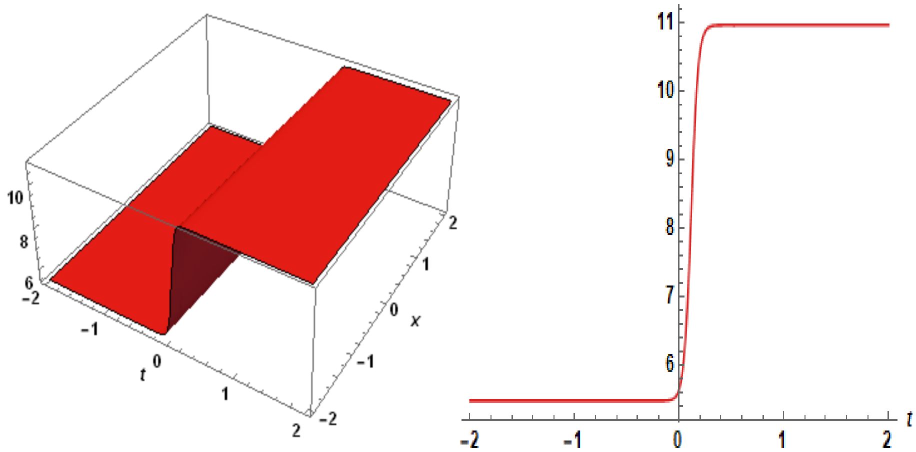
If we combine Eqs. (2.12), (3.38), (3.41), (3.42) and solution set $u(\xi)$, then we acquire the following solution:

$$u_1(x, y, t) = \sqrt{-\frac{3(+r_1 + r_2)}{2(3 + 2c)}}\left(1 + \frac{1}{1 + Ae^{(x+y-(2(r_1+r_2)-a)t)}}\right)e^{(x+y+\omega t)\iota}, \tag{3.44}$$

where

$$\omega = \frac{r_1 + r_2 + 2a}{2}.$$

Graphical illustration of a solution



(a) (b)

Figure 3.5: Graphical representation of a soliton solution $u_1(x, y, t) = \sqrt{\frac{3}{5}} e^{\iota(x-3t)}(1 + \frac{1}{1+3e^{(x-8t)}}), y = 0, -2 \leq x, t \leq 2$, (a) 3D Plot (b) 2D Plot

3.2.2 Exact Solutions of higher order CQ-NLSE

Here, we implement Kudryashov's method to find travelling wave solutions of the higher order CQ-NLSE, which is given as [36]:

$$\iota U_t - \frac{\alpha_1}{2} U_{xx} + \beta_1 U|U|^2 - \iota \frac{\alpha_2}{6} U_{xxx} - \frac{\alpha_3}{24} U_{xxxx} + \beta_2 \left(|U|^4 U\right) - \iota \gamma_1 \left(|U|^2 U\right)_x - \iota \gamma_2 U \left(|U|^2\right)_x = 0, \tag{3.45}$$

where $\iota = \sqrt{-1}$ and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1$ and γ_2 are real parameters.
Applying the following transformation,

$$U(x, t) = u(\xi) e^{\iota \eta}, \quad \xi = px - vt \quad \text{and} \quad \eta = x + \omega t. \tag{3.46}$$

We generate nonlinear ODEs by combining Eq. (3.46) and Eq. (3.45) as follows, then separating the real and imaginary components respectively:

$$\alpha_3 p^4 u^{(4)}(\xi) + 6(2\alpha_1 - 2\alpha_2 - \alpha_3) p^2 u''(\xi) + (24\omega + 4\alpha_2 - 12\alpha_1 + \alpha_3) u(\xi)$$

$$-24(\gamma_1 + \beta_1) u^3(\xi) - 24\beta_2 u^5(\xi) = 0, \tag{3.47}$$

$$(\alpha_2+\alpha_3)\,p^3u'''(\xi)+(6v+6\alpha_1p-3\alpha_2p-\alpha_3p)\,u'(\xi)+p\,(18\gamma_1+12\gamma_2)\,u^2(\xi)u'(\xi)=0, \tag{3.48}$$

here $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1$ and γ_2 are arbitrary constants.

Applying the balancing principle on $u^{(4)}(\xi)$ and $u^5(\xi)$ from Eq. (3.47), the balancing constant is found to be $m=1$. Furthermore, the Kudryashov method allows for the employment of the finite expansion,

$$u(\xi)=A_0+A_1Z(\xi). \tag{3.49}$$

Putting Eq. (3.49) into Eq. (3.47) and Eq. (3.48) and we obtain the following system by accumulating the coefficients of the similar exponents of $Z(\xi)$ and setting them to zero,

$$24p^4A_1\alpha_3-24A_1^5\beta_2=0,$$

$$-60p^4A_1\alpha_3-120A_0A_1^4\beta_2=0,$$

$$50p^4A_1\alpha_3-240A_0^2A_1^3\beta_2+24p^2A_1\alpha_1-24p^2A_1\alpha_2-12p^2A_1\alpha_3-24A_1^3\beta_1-24A_1^3\gamma_1=0,$$

$$-15p^4A_1\alpha_3-240A_0^3A_1^2\beta_2-36p^2A_1\alpha_1+36p^2A_1\alpha_2+18p^2A_1\alpha_3-72A_0A_1^2\beta_1$$

$$+72A_0A_1^2\gamma_1=0,$$

$$p^4A_1\alpha_3-120A_0^4A_1\beta_2+12p^2A_1\alpha_1-12p^2A_1\alpha_2-6p^2A_1\alpha_3-72A_0^2A_1\beta_1-72A_0^2A_1\gamma_1$$

$$+24\omega A_1-12A_1\alpha_1+4A_1\alpha_2+A_1\alpha_3=0,$$

$$-24\beta_2{A_0}^5-24\beta_1{A_0}^3-24\gamma_1{A_0}^3+24\omega A_0-12\alpha_1A_0+4\alpha_2A_0+\alpha_3A_0=0,$$

$$6p^3A_1\alpha_2+6p^3A_1\alpha_3+18pA_1{}^3\gamma_1+12pA_1{}^3\gamma_2=0,$$

$$-12p^3A_1\alpha_2-12p^3A_1\alpha_3+36pA_0{A_1}^2\gamma_1+24pA_0{A_1}^2\gamma_2-18p{A_1}^3\gamma_1-12p{A_1}^3\gamma_2=0,$$

$$7p^3A_1\alpha_2+7p^3A_1\alpha_3+18p{A_0}^2A_1\gamma_1+12p{A_0}^2A_1\gamma_2-36pA_0{A_1}^2\gamma_1$$

$$-24pA_0{A_1}^2\gamma_2+6pA_1\alpha_1-3pA_1\alpha_2-pA_1\alpha_3+6vA_1=0,$$

$$-p^3A_1\alpha_2-p^3A_1\alpha_3-18p{A_0}^2A_1\gamma_1-12p{A_0}^2A_1\gamma_2-6pA_1\alpha_1+3pA_1\alpha_2v$$

$$+pA_1\alpha_3-6vA_1=0,$$

$\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\gamma_1$ and γ_2 are arbitrary constants.

The outcome of solving the above system using maple is the following set of solution:

$$p=\sqrt{2},\;\;\alpha_1=-\frac{1}{6}\frac{9\sqrt{2}v\beta_1-18\gamma_1\sqrt{2}v-18\sqrt{2}v\gamma_2+6\alpha_3\beta_1-24\gamma_1\alpha_3-20\alpha_3\gamma_2}{3\beta_1-2\gamma_2},$$

$$\alpha_2=\frac{3}{2}\frac{3\gamma_1\sqrt{2}v+2\sqrt{2}v\gamma_2-2\alpha_3\beta_1+4\gamma_1\alpha_3+4\alpha_3\gamma_2}{3\beta_1-2\gamma_2},\;\;\beta_2=\frac{2\alpha_3(9{\beta_1}^2-12\beta_1\gamma_2+4{\gamma_2}^2)}{12\sqrt{2}\alpha_3v+9v^2+8{\alpha_3}^2},$$

$$A_1=\sqrt{-\frac{3\sqrt{2}v+4\alpha_3}{3\beta_1-2\gamma_2}},\;\;A_0=-\frac{1}{2}A_1,\;\;\omega=-\frac{1}{8}\frac{36v^3\sqrt{2}+44v{\alpha_3}^2\sqrt{2}+105v^2\alpha_3+8{\alpha_3}^3}{12\sqrt{2}\alpha_3v+9v^2+8\alpha_3^2}. \tag{3.50}$$

Using above solution sets in (3.49) we get the following general sets of solutions:

$$u(\xi)=\sqrt{\frac{3\sqrt{2}v+4\alpha_3}{2\gamma_2-3\beta_1}}\left(-\frac{1}{2}+Z(\xi)\right). \tag{3.51}$$

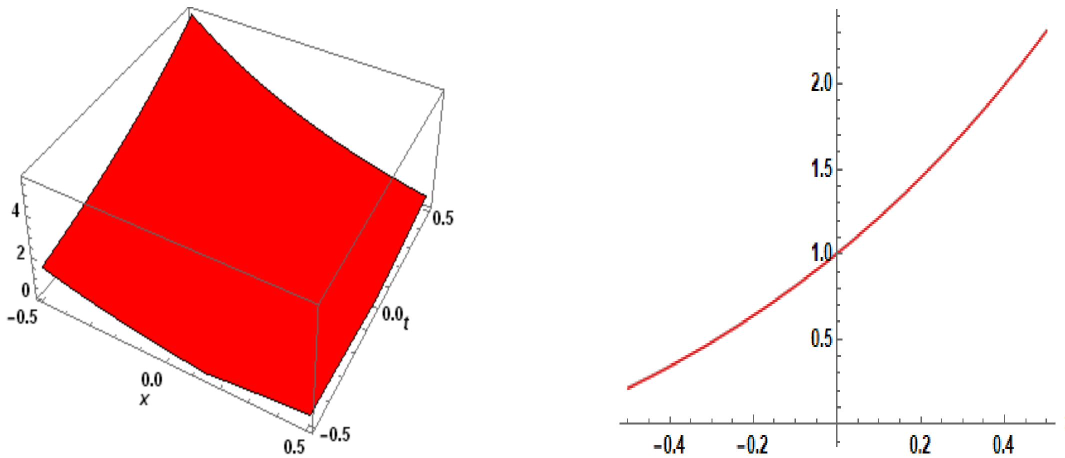
If we combine Eqs. (2.12), (3.46), (3.49), (3.50) and solution set $u(\xi)$, then we acquire the following solutions:

$$u_1(x,t)=\sqrt{\frac{3\sqrt{2}v+4\alpha_3}{2\gamma_2-3\beta_1}}\left(-\frac{1}{2}+\frac{1}{Ae^{(\sqrt{2}x-vt)}}\right)e^{(x+\omega t)\iota}, \tag{3.52}$$

where

$$\omega=-\frac{1}{8}\frac{36v^3\sqrt{2}+44v{\alpha_3}^2\sqrt{2}+105v^2\alpha_3+8{\alpha_3}^3}{12\sqrt{2}\alpha_3v+9v^2+8\alpha_3^2}.$$

Graphical illustration of a solution



(a) (b)

Figure 3.6: Graphical representation of a travelling wave solution $u_1(x,t) = \sqrt{\frac{(16+3\sqrt{2})}{5}} e^{\iota(-\frac{(932+740\sqrt{2})t}{8(137+48\sqrt{2})}+x)}(-\frac{1}{2}+e^{t-\sqrt{2}x})$, $-2 \leq x, t \leq 2$, (a) 3D Plot (b) 2D Plot

3.3 Application of sine-cosine method

In this section, a new analytical approach as, sine-cosine method is used for finding travelling wave solutions, solitons and optical solitons for (2+1)-dimensional P-NLSE in nonlinear optical fibres and Higher order CQ-NLSE, which have significant implications in telecommunication system and ultra-fast signal routing.

3.3.1 Exact solutions of (2+1)-dimensional P-NLSE in nonlinear optical fibres

Here, we implement sine-cosine method to find travelling wave solutions of the (2+1)-dimensional P-NLSE in nonlinear optical fibres, which is given as [35]:

$$\iota U_t + r_1 U_{xx} + r_2 U_{yy} + U|U|^2 - \iota\left(aU_x - b\left(|U|^2 U\right)_x - cU\left(|U|^2\right)_x\right) + U_{xxxx} - U_{yyyy} = 0, \tag{3.53}$$

where $\iota=\sqrt{-1}$ and a, b, c, r_1 and r_2 are arbitrary constants.

The generalized (2+1)-dimensional P-NLSE contains the terms describing self steep-

ening in x and y direction, the terms with self frequency and the 4OD terms.
Applying the following transformation,

$$U(x,y,t) = u(\xi)e^{\iota\eta}, \quad \xi = x + y - vt \quad \text{and} \quad \eta = x + y + \omega t. \tag{3.54}$$

We generate the following nonlinear ODEs by combining Eq. (3.54) and Eq. (3.53), then separating the real and imaginary components respectively:

$$(r_1 + r_2)u''(\xi) + (a - \omega - (r_1 + r_2))u(\xi) + (1 - b)u^3(\xi) = 0, \tag{3.55}$$

$$(2(r_1 + r_2) - a - v)u'(\xi) + (2c + 3b)u^2(\xi)u'(\xi) = 0, \tag{3.56}$$

here, the parameters a, b, c, r_1 and r_2 are arbitrary.
The sine-cosine method allows one to employ the presumed result:

$$u(\xi) = \lambda \cos^\beta[\mu\xi]. \tag{3.57}$$

Substituting Eq. (3.57) into Eq. (3.56) and we obtain the following system by accumulating the coefficients of the similar exponents of $cos^\beta[\mu\xi]$ and setting them to zero,

$$1 - b = 0,$$

$$-\beta^2\mu^2 r_1 - \mu^2 r_2 - r_1 - r_2 + a - \omega = 0,$$

$$\beta^2\mu^2 r_1 + \beta^2\mu^2 r_2 - \beta\mu^2 r_1 - \beta\mu^2 r_2 = 0,$$

$$-3b - 2c = 0,$$

$$-2r_1 - 2r_2 + a + v = 0,$$

where r_1, r_2, ω, v, a are arbitrary constants.
The outcomes of solving the above system using maple are the following sets of solutions:
(i) The first set:

$$\lambda = \lambda, \;\; \mu = \mu, \;\; \beta = 0, \;\; b = 1,$$

$$c = -\frac{3}{2}, \;\; v = 2(r_1 + r_2) - a, \;\; \omega = a - r_1 - r_2. \tag{3.58}$$

(ii) The second set:

$$\lambda = \lambda, \;\; \mu = 0, \;\; \beta = \beta, \;\; b = 1,$$

$$c = -\frac{3}{2}, \;\; v = 2(r_1 + r_2) - a, \;\; \omega = a - r_1 - r_2. \tag{3.59}$$

(iii) The third set:

$$\lambda = \lambda, \;\; \omega = \omega, \;\; \beta = 1, \;\; b = 1,$$

$$c = -\frac{3}{2}, \;\; v = 2(r_1 + r_2) - a, \;\; \mu = \sqrt{\frac{(a-\omega) - (r_1 + r_2)}{(r_1 + r_2)}}. \tag{3.60}$$

Using above solution sets in (3.57) we get the following general sets of solutions:

$$u_I(\xi) = \lambda, \qquad c = -\frac{3}{2}, \tag{3.61}$$

$$u_{II}(\xi) = \lambda \cos\left[\sqrt{\frac{(a-\omega) - (r_1 + r_2)}{(r_1 + r_2)}}\xi\right], \qquad c = -\frac{3}{2}. \tag{3.62}$$

If we combine Eqs. (3.54), (3.57), (3.58) and solution set $u_I(\xi)$, then we acquire the following solution:

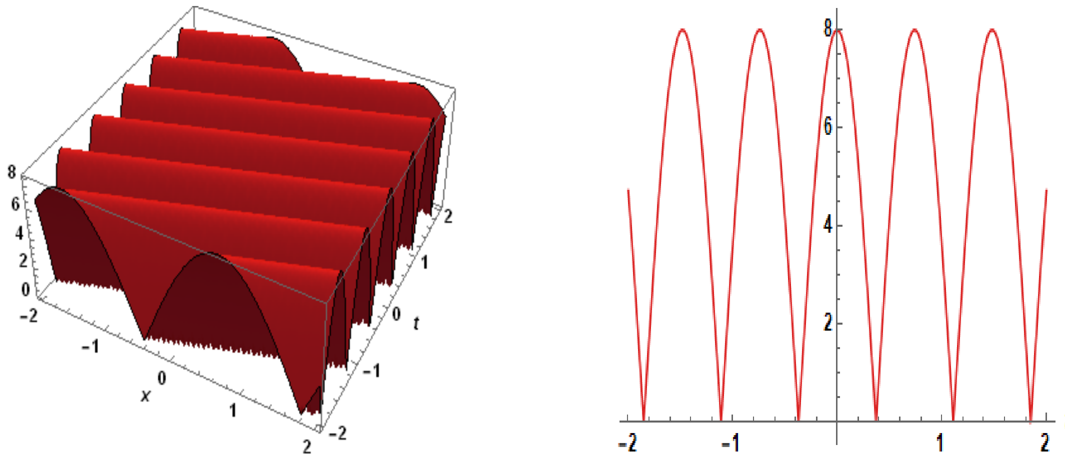
$$u_1(x, y, t) = \lambda e^{(x+y+\omega t)\iota}, \tag{3.63}$$

If we combine Eqs. (3.54), (3.57), (3.60) and solution set $u_{II}(\xi)$, then we acquire the following solution:

$$u_2(x, y, t) = \lambda \cos\left[\sqrt{\frac{(a-\omega) - (r_1 + r_2)}{(r_1 + r_2)}}\Big(x + y - (2(r_1 + r_2) - a)t\Big)\right] e^{(x+y+\omega t)\iota}, \tag{3.64}$$

where λ and ω are arbitrary.

Graphical illustration of a solution



(a) (b)

Figure 3.7: Graphical representation of a periodic solution $u_2(x, y, t) = 8e^{\iota(x-5t)}\cos[\sqrt{2}(x-3t)], y = 0, -2 \leq x, t \leq 2$, (a) 3D Plot (b) 2D Plot

3.3.2 Exact Solutions of Higher order CQ-NLSE

Here, we implement sine-cosine method to find travelling wave solutions of the higher order CQ-NLSE, which is given as [36]:

$$\iota U_t - \frac{\alpha_1}{2}U_{xx} + \beta_1 U|U|^2 - \iota\frac{\alpha_2}{6}U_{xxx} - \frac{\alpha_3}{24}U_{xxxx} + \beta_2\left(|U|^4U\right) - \iota\gamma_1\left(|U|^2U\right)_x - \iota\gamma_2 U\left(|U|^2\right)_x = 0, \tag{3.65}$$

where $\iota = \sqrt{-1}$ and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1$ and γ_2 are real parameters.
Applying the following transformation,

$$U(x,t) = u(\xi)e^{\iota\eta}, \quad \xi = px - vt \quad \text{and} \quad \eta = x + \omega t. \tag{3.66}$$

We generate the following ODEs by combining Eq. (3.66) and Eq. (3.65), then separating the real and imaginary components respectively:

$$\alpha_3 p^4 u^{(4)}(\xi) + 6(2\alpha_1 - 2\alpha_2 - \alpha_3)p^2 u''(\xi) + (24\omega + 4\alpha_2 - 12\alpha_1 + \alpha_3)u(\xi)$$

$$-24(\gamma_1 + \beta_1)u^3(\xi) - 24\beta_2 u^5(\xi) = 0, \tag{3.67}$$

$$(\alpha_2 + \alpha_3)p^3 u'''(\xi) + (6v + 6\alpha_1 p - 3\alpha_2 p - \alpha_3 p)u'(\xi) + p(18\gamma_1 + 12\gamma_2)u^2(\xi)u'(\xi) = 0, \tag{3.68}$$

Here $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1$ and γ_2 are arbitrary constants.

The sine-cosine approach allows for the employment of the finite expansion,

$$u(\xi) = \lambda \cos^{\beta}[\mu\xi]. \tag{3.69}$$

Substituting Eq. (3.69) into Eq. (3.67) and Eq. (3.68) and we obtain the following system by accumulating the coefficients of the similar exponents of $cos^{\beta}[\mu\xi]$ and setting them to zero,

$$-24\beta_2\lambda^4 = 0,$$

$$-24\lambda^2\gamma_1 - 24\lambda^2\beta_1 = 0,$$

$$24\omega - 12\alpha_1 + 4\alpha_2 + \alpha_3\beta^4\mu^4p^4 + \alpha_3 - 12\beta^2\mu^2p^2\alpha_1 + 12\beta^2\mu^2p^2\alpha_2 + 6\beta^2\mu^2p^2\alpha_3 = 0,$$

$$6\alpha_3\beta^3\mu^4p^4 - 8\alpha_3\beta^2\mu^4p^4 + 4\alpha_3\beta\mu^4p^4 - 2\alpha_3\beta^4\mu^4p^4 + 12\beta^2\mu^2p^2\alpha_1 - 12\beta^2\mu^2p^2\alpha_2$$

$$-6\beta^2mu^2p^2\alpha_3 - 12\beta\mu^2p^2\alpha_1 + 12\beta\mu^2p^2\alpha_2 + 6\beta\mu^2p^2\alpha_3 = 0,$$

$$\alpha_3\beta^4\mu^4p^4 - 6\alpha_3\beta\mu^4p^4 + 11\alpha_3\beta^2\mu^4p^4 - 6\alpha_3\beta^3\mu^4p^4 = 0,$$

$$-18p\lambda^2\gamma_1 - 12p\lambda^2\gamma_2 = 0,$$

$$-\beta^2\mu^2p^3\alpha_2 - \beta^2\mu^2p^3\alpha_3 + 3\beta\mu^2p^3\alpha_2 + 3\beta\mu^2p^3\alpha_3 - 2\mu^2p^3\alpha_2 - 2\mu^2p^3\alpha_3 = 0,$$

$$\beta^2\mu^2p^3\alpha_2 + \beta^2\mu^2p^3\alpha_3 - 6p\alpha_1 + 3p\alpha_2 + p\alpha_3 - 6v = 0.$$

Here, $\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, \beta_1$ and β_2 are arbitrary constants.

The outcomes of solving the above system using maple are the following sets of solutions:

(i) The first set:

$$\mu = \mu, \;\; p = p, \;\; \beta = 0, \;\; \lambda = \lambda, \;\; \beta_2 = 0,$$

$$\omega=-\frac{1}{24}\frac{p\alpha_3+12v}{p},\ \ \alpha_1=-\frac{1}{3}\frac{p\alpha_3+3v}{p},\ \ \alpha_2=-\alpha_3,\ \ \beta_1=\frac{2}{3}\gamma_2,\ \ \gamma_1=-\frac{2}{3}\gamma_2 \tag{3.70}$$

(ii) The second set:

$$\mu=0,\ \ p=p,\ \ \beta=\beta,\ \ \lambda=\lambda,\ \ \beta_2=0,$$

$$\omega=\frac{1}{24}\frac{2p\alpha_2+p\alpha_3-12v}{p},\ \ \alpha_1=\frac{1}{6}\frac{3p\alpha_2+p\alpha_3-6v}{p},\ \ \alpha_2=\alpha_2,\ \ \beta_1=\frac{2}{3}\gamma_2,\ \ \gamma_1=-\frac{2}{3}\gamma_2 \tag{3.71}$$

(iii) The third set:

$$\mu=\frac{1}{p}\sqrt{\frac{1}{2}\frac{p\alpha_3-6v}{p\alpha_3}},\ \ p=p,\ \ \beta=2,\ \ \lambda=\lambda,\ \ \beta_2=0,$$

$$\omega=-\frac{1}{24}\frac{p\alpha_3-12v}{p},\ \ \alpha_1=-\frac{1}{3}\frac{p\alpha_3+3v}{p},\ \ \alpha_2=-\alpha_3,\ \ \beta_1=\frac{2}{3}\gamma_2,\ \ \gamma_1=-\frac{2}{3}\gamma_2 \tag{3.72}$$

(iv) The fourth set:

$$\mu=\frac{1}{p}\sqrt{\frac{1}{5}\frac{p\alpha_3-6v}{p\alpha_3}},\ \ p=p,\ \ \beta=3,\ \ \lambda=\lambda,\ \ \beta_2=0,\ \ \gamma_1=-\frac{2}{3}\gamma_2$$

$$\omega=-\frac{1}{150}\frac{4p^2{\alpha_3}^2+102pv\alpha_3-81v^2}{p},\ \ \alpha_1=-\frac{1}{3}\frac{p\alpha_3+3v}{p},\ \ \alpha_2=-\alpha_3,\ \ \beta_1=\frac{2}{3}\gamma_2 \tag{3.73}$$

(v) The fifth set:

$$\mu=\mu,\ \ p=p,\ \ \beta=2,\ \ \lambda=\lambda,\ \ \beta_2=0,\ \ \gamma_1=-\frac{2}{3}\gamma_2$$

$$\omega=\frac{1}{24}\frac{16\mu^4p^5\alpha_3-8\mu^2p^3\alpha_3+p\alpha_3+48v}{p(4\mu^2p^2-3)},\ \ \alpha_1=\frac{1}{6}\frac{8\mu^4p^5\alpha_3-6\mu^2p^3\alpha_3+3p\alpha_3+36v}{p(4\mu^2p^2-3)},$$

$$\alpha_2=-\frac{2(\mu^2p^3\alpha_3-p\alpha_3-3v)}{p(4\mu^2p^2-3)},\ \ \beta_1=\frac{2}{3}\gamma_2 \tag{3.74}$$

(vi) The sixth set:

$$\mu=\mu,\ \ p=p,\ \ \beta=1,\ \ \lambda=\lambda,\ \ \beta_2=0,\ \ \gamma_1=-\frac{2}{3}\gamma_2$$

$$\omega=\frac{1}{24}\frac{2\mu^4p^5\alpha_2+\mu^4p^5\alpha_3-4\mu^2p^3\alpha_2-2\mu^2p^3\alpha_3-12\mu^2p^2v+2p\alpha_2+p\alpha_3-12v}{p},$$

$$\alpha_1=\frac{1}{6}\frac{8\mu^4p^5\alpha_3-6\mu^2p^3\alpha_3+3p\alpha_3+36v}{p(4\mu^2p^2-3)},\ \ \alpha_2=-\frac{2(\mu^2p^3\alpha_3-p\alpha_3-3v)}{p(4\mu^2p^2-3)},\ \ \beta_1=\frac{2}{3}\gamma_2 \tag{3.75}$$

(vii) The seventh set:

$$\mu = \frac{1}{24}\frac{\sqrt{3}\alpha_3}{v}, \;\; p = -\frac{12v}{\alpha_3}, \;\; \beta = 2, \;\; \lambda = \lambda, \;\; \beta_2 = 0,$$

$$\omega = \frac{\alpha_2 + \alpha_3}{3}, \;\; \alpha_1 = \frac{4\alpha_2 + 3\alpha_3}{4}, \;\; \alpha_2 = \alpha_2, \;\; \beta_1 = \frac{2}{3}\gamma_2, \;\; \gamma_1 = -\frac{2}{3}\gamma_2 \tag{3.76}$$

Using above solution sets in (3.69), we get the following general sets of solutions respectively:

$$u_I(x,t) = \lambda, \tag{3.77}$$

$$u_{II}(x,t) = \lambda\cos^2\left[\frac{1}{p}\sqrt{\frac{1}{2}\frac{p\alpha_3 - 6v}{p\alpha_3}}\xi\right], \tag{3.78}$$

$$u_{III}(x,t) = \lambda\cos^3\left[\frac{1}{p}\sqrt{\frac{1}{5}\frac{p\alpha_3 - 6v}{p\alpha_3}}\xi\right], \tag{3.79}$$

$$u_{IV}(x,t) = \lambda\cos^2\left[\mu\xi\right], \tag{3.80}$$

$$u_V(x,t) = \lambda\cos\left[\mu\xi\right], \tag{3.81}$$

$$u_{VI}(x,t) = \lambda\cos^2\left[\frac{1}{24}\frac{\sqrt{3}\alpha_3}{v}\xi\right]. \tag{3.82}$$

If we combine Eqs. (3.66), (3.69), (3.70) and (3.77), then we acquire the following solution:

$$u_1(x,t) = \lambda e^{(x+\omega t)\iota}, \tag{3.83}$$

where

$$\omega = -\frac{1}{24}\frac{p\alpha_3 + 12v}{p}.$$

If we combine Eqs. (3.66), (3.69), (3.72) and (3.78), then we acquire the following solution:

$$u_2(x,t) = \lambda\cos^2\left[\frac{1}{p}\sqrt{\frac{1}{2}\frac{p\alpha_3 - 6v}{p\alpha_3}}(px - vt)\right]e^{(x+\omega t)\iota}, \tag{3.84}$$

where

$$\omega = -\frac{1}{24}\frac{p\alpha_3 - 12v}{p}.$$

If we combine Eqs. (3.66), (3.69), (3.73) and (3.79), then we acquire the following solution:

$$u_3(x,t) = \lambda \cos^3 \left[\frac{1}{p}\sqrt{\frac{1}{2}\frac{p\alpha_3 - 6v}{p\alpha_3}}(px - vt)\right] e^{(x+\omega t)\iota}, \tag{3.85}$$

where

$$\omega = -\frac{1}{150}\frac{4p^2{\alpha_3}^2 + 102pv\alpha_3 - 81v^2}{p}.$$

If we combine Eqs. (3.66), (3.69), (3.74) and (3.80), then we acquire the following solution:

$$u_4(x,t) = \lambda \cos^2[\mu(px - vt)] e^{(x+\omega t)\iota}, \tag{3.86}$$

where

$$\omega = \frac{1}{24}\frac{16\mu^4p^5\alpha_3 - 8\mu^2p^3\alpha_3 + p\alpha_3 + 48v}{p(4\mu^2p^2 - 3)}.$$

If we combine Eqs. (3.66), (3.69), (3.75) and (3.81), then we acquire the following solution:

$$u_5(x,t) = \lambda \cos[\mu(px - vt)] e^{(x+\omega t)\iota}, \tag{3.87}$$

where

$$\omega = \frac{1}{24}\frac{2\mu^4p^5\alpha_2 + \mu^4p^5\alpha_3 - 4\mu^2p^3\alpha_2 - 2\mu^2p^3\alpha_3 - 12\mu^2p^2v + 2p\alpha_2 + p\alpha_3 - 12v}{p}.$$

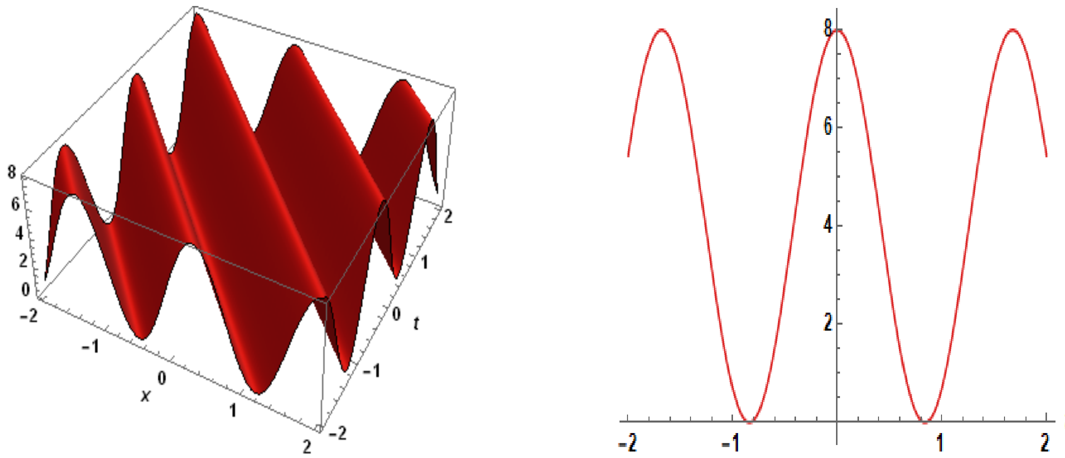
If we combine Eqs. (3.66), (3.69), (3.76) and (3.83), then we acquire the following solution:

$$u_6(x,t) = \lambda \cos^2 \left[\frac{1}{24}\frac{\sqrt{3}\alpha_3}{v}(px - vt)\right] e^{(x+\omega t)\iota}, \tag{3.88}$$

where

$$\omega = \frac{\alpha_2 + \alpha_3}{3}.$$

Graphical illustration of solution



(a) (b)

Figure 3.8: Graphical representation of a periodic solution $u_2(x,t) = 8e^{\iota(-\frac{13t}{24}+x)}\cos^2[\sqrt{\frac{7}{2}}(x+t)], -2 \leq x, t \leq 2$, (a) 3D Plot (b) 2D Plot

Chapter 4

Summary and Conclusion

In this work, we have explored three methods: tanh-coth, Kudryashov's and sine-cosine. The description of these three methods is given in chapter 2. These strategies have been shown to be capable of producing more fruitful and new accurate solitary wave solutions for a variety of NLSEs. A travelling wave transformation is used for transforming nonlinear PDEs into ODEs. In chapter 3, by applying tanh-coth, Kudryashov's and sine-cosine methods to the (2+1)-dimensional P-NLSE in nonlinear fiber optics and higher order CQ-NLSE, we have found travelling wave solutions and exact solutions in the form of trigonometric, hyperbolic, rational and singular functions. Our proposed solutions showed that these three methods are effective and reliable for finding travelling wave solutions.

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