On Jensen's Inequality and Related Results

By

Rishi Naeem
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Thesis Supervisor: Dr. Matloob Anwar

Department of Mathematics
School of Natural Sciences

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Student Name: **Ms. Rishi Naeem** Signature: ____________

Examination Committee:

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	Name	Dr. Sabir Hussain	
	Designation	Associate Professor	
	Official Address	University of Engineering & Technology, Lahore.	Signature

b)	**External Examiner:**		
	Name	Dr. Khuram Ali Khan	
	Designation	Assistant Professor	
	Official Address	Department of Mathematics University of Sargodha, Sargodha.	Signature

c)	**Internal Examiner:**		
	Name	Dr. Rashid Farooq	
	Designation	Associate Professor	
	Official Address	NUST-SNS	Signature

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Dedicated to

My beloved late father: Muhammad Naeem Akhtar *(May Allah shine the brightest light in his grave and grant him the highest rank of Jannah. Those who read this please remember him in your prayers)*

My strong and loving mother: Shamshad Kousar
My loving mother-in-law: Shamim Akhtar and caring father-in-law: Zahoor Ullah.
(May Allah keep them in good health)

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Rishi Naeem

Manuscript delineation

The concept of convex function was first introduced by Jensen in [25]. He said that the chord's midst of the curve, whether continuous or not, lies above or on the curve. This notion has been extended to its familiar form for the continuous case when any point of the chord lie above or on the curve. In present days, the study of convexity excogitate into more valuable theory of functions which can be very helpful in deriving numerous important inequalities of analysis, see [18, 39, 43, 52]. The most significant convex function inequality which is widely known as king is the Jensen's inequality. Young's, Levinson's, the harmonic mean-geometric mean-arithmetic mean inequality, and many other eminent inequalities are particular cases of this known result. Important books on inequalities are: "Inequalities" by Hardy et al. [18], Beckenbach et al. [3] and "Analytical inequalities" by Mitrinović [36]. From 1988-1993, four comprehensive books were published comprising of the application and development of inequalities [6, 37–39]. Fink in [17] gave a brief but comprehensive note on historical background of inequalities, where the progress and growth of mathematical inequalities is given in form of sequences.

The theory of exponentially convex functions was introduced by S. N. Bernstein in [5]; the representation of these functions over the given interval (a, b) as a genre of convex functions was given by D. V. Widder [63]. These functions can be constructed by well known method given in [24]. J. Jakšetić and J. Pečarić in [24] also analysed that linear functionals defined on a family of m-convex functions lead to means of Stolarsky type. This guide to a generalized method of producing exponentially convex functions. J. Pečarić and J. Perić [53] gave the concept of m-exponentially convex functions. Some useful results of m-exponential convexity and logarithmic convexity are given in [2, 7, 8, 19, 23, 27, 32, 33, 50, 51, 55].

In Chapter 1, we give some definitions and notions of convexity, then generalize it to the higher order. It also recalls some basic concepts of m-exponential convexity and logarithmic convexity. In addition, some Jessen and Jensen type functionals and inequalities are also given.

In Chapter 2, the positive functional has been used to investigate the log-convexity and the m-exponential convexity. This positive functional is defined in the form of the difference of the two sides of the inequality given in [49]. We discuss Cauchy and Lagrange type mean value theorems which lead us to Stolarsky type means.

In Chapter 3, we give a refinement of the Jessen inequality. We opt an elegant method of constructing m-exponential convex functions by applying the positive functionals associated with the weighted integral Jensen's inequality and weighted Jessen inequality. Cauchy and Lagrange mean value theorems are also given which enable us to construct means with Stolarsky property.

In Chapter 4, some interesting results have been shown. These results are associated with the refined Jessen's inequality for $m(M)$-ψ-convex functions.

Chapter 5 is the extension of Jessen functional. We also explore logarithmic and exponential convexity. We also give mean value theorems of Cauchy and Lagrange type. Several families of functions are also presented related to our main results.

In Chapter 6, the m-exponential convexity and the log-convexity have been investigated. For this investigation, positive functionals are applied which associate with the refinement of Hermite Hadamard inequality (cited from [45]). With the results that are obtained, some families of functions related to them are presented. Lagrange and Cauchy type mean value theorems have also been given to construct means with Stolarsky property.

In Chapter 7, we report an idea of logarithmic and m-exponentially concave functions and apply this concept on the linear functional related with the Jensen's inequality for generalized Choquet integral. The consequence of these results provide us an interesting application in the probability. We discuss Cauchy and Lagrange type mean value theorems which lead us to Stolarsky type means. We also deduce an interesting result of information theory.

Abstract

A real valued function Ψ on S (S is a set that is convex) is called convex if epigraph of Ψ is a convex set. Alternatively, for any two points $x, y \in S$ the line segment $\mu x + (1-\mu)y$ ($\mu \in [0,1]$) joining these two points on the graph of the function Ψ lies above or on the graph. The well known king of inequalities, that is, the Jensen's inequality is the generalization of the above result. Integral, functional, probabilistic and many other indispensable forms of this fundamental result can be found in literature.

In the dissertation, we introduce some advancements in Jensen's type inequalities. The m-exponential convexity and the log-convexity have been investigated. Positive functionals are used to investigate them. The positive functionals are defined in the form of the difference of two sides of the refined and some known inequalities. We also give an idea of logarithmic and m-exponentially concave functions and apply this concept on the linear functional associated with the Jensen's inequality for generalized Choquet integral. The consequences of obtained results provide us interesting applications in the probability. We also deduce an interesting result of information theory. We discuss Cauchy and Lagrange type mean value theorems which lead us to Stolarsky type means. We also draw some interesting results associated with the refined Jessen's inequality and Lupaş-Beesack-Pečarić (LBP) type inequality for $m(M)$-ψ-convex functions.

Contents

Chapter 1

Introduction and preliminaries

This chapter presents some definitions and notions of convexity, then generalizes it to the higher order. It also recalls some basic concepts of m-exponential convexity (concavity) and logarithmic convexity (concavity). In addition, some Jessen and Jensen type functionals and inequalities are also given.

1.1 Functions of convex type

Convexity arises naturally in the study of functions and hence plays a very important role in many fields of applied and pure mathematics. Different concepts from topology, algebra, geometry and analysis are unified by it. It is considered as an important tool in numerous branches of engineering, in management science and operation research, in mathematical economics, in optimization theory and in statistics. Convexity is the basic concept behind the theory of graphical analysis. Second derivative test in calculus is very useful in recognizing convexity. We recall the definitions and notions about convexity from [52].

Definition 1.1.1. A set $\mathcal{S}$ is convex if for any two points $s_1, s_2 \in \mathcal{S}$ and $\mu \in [0, 1]$

the line segment $\mu s_1 + (1-\mu)s_2 \in \mathcal{S}$.

The following structural objects are the examples of convex and nonconvex sets. The first solid body is a convex set. The middle set is not convex because the line segment between two given points is not contained in the set. The right most set is a hollow structural shape which dose not contain the line segment between any two points of the boundary. So it is not a convex set.

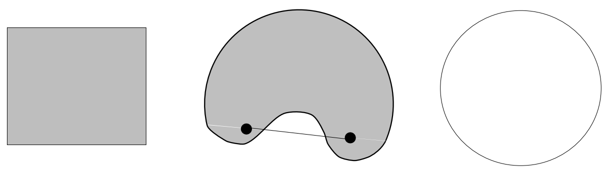

Figure 1.1: Convex and nonconvex sets

Definition 1.1.2. Let $\Psi : [\eta, \zeta](\subseteq \mathbb{R}) \to \mathbb{R}$. The function Ψ is called $\mathcal{J}$-convex or convex in the Jensen sense on $[\eta, \zeta]$ if the inequality

$$\Psi\left(\frac{s_1+s_2}{2}\right) \leq \frac{\Psi(s_1)+\psi(s_2)}{2} \tag{1.1.1}$$

is true for all $s_1, s_2 \in [\eta, \zeta]$.

Definition 1.1.3. Suppose S be a convex set, a function $\Psi : S(\subseteq \mathbb{R}) \to \mathbb{R}$ is convex if

$$\Psi(\mu s_1 + (1-\mu)s_2) \leq \mu\Psi(s_1) + (1-\mu)\Psi(s_2) \tag{1.1.2}$$

for all $s_1, s_2 \in S$ and $\mu \in [0,1]$. If (1.1.2) is strict for $\mu \in (0,1)$ and $s_1 \neq s_2$, then Ψ is strictly convex.

Remark 1.1.1. (i) If we reverse the inequality (1.1.2), then the function Ψ is concave. Moreover, if it is strict for $\mu \in (0,1)$ and for all distinct s_1, s_2, then the function Ψ is strictly concave.

(ii) If $s_1, s_2, s_3 \in S$ such that $s_1 < s_2 < s_3$, then (1.1.2) is equivalent with

$$\begin{vmatrix} s_1 & \Psi(s_1) & 1 \\ s_2 & \Psi(s_2) & 1 \\ s_3 & \Psi(s_3) & 1 \end{vmatrix} \geq 0. \tag{1.1.3}$$

Theorem 1.1.1. *If the second order derivative of $\Psi : [\eta, \zeta] \to \mathbb{R}$ exists on (η, ζ), then Ψ is convex if and only if $\Psi''(s) \geq 0$. Strict convexity holds if $\Psi''(s) > 0$.*

The convex and $\mathcal{J}$-convex functions are interrelated in the context of continuity.

Remark 1.1.2. A $\mathcal{J}$-convex function is convex if it is continuous as well.

Definition 1.1.4. Suppose $S \subseteq \mathbb{R}$ and $\Psi, \Phi : S \to \mathbb{R}$, where Ψ is convex function on S and m is an element of $\mathbb{R}$. If $\Phi - m\Psi$ is a convex mapping on S then Φ is called m-Ψ-lower convex function.

Consider the following classes of functions [11] (see also [44]):

$$\mathfrak{L}(S, m, \Psi) := \{\Phi : S \to \mathbb{R} \, : \, \Phi - m\Psi \text{ is convex on } S\}. \tag{1.1.4}$$

Analogously, for a real number M, M-Ψ-upper convex functions' class is defined as

$$\mathfrak{U}(S, M, \Psi) := \{\Phi : S \to \mathbb{R} \, : \, M\Psi - \Phi \text{ is convex on } S\}. \tag{1.1.5}$$

(m, M)-Ψ-convex functions form another class $\mathfrak{B}(S, m, M, \Psi)$ which is the intersection of the above two classes.

Remark 1.1.3. The concept of h-convex dominated mappings was introduced in [12]: Let $h, k : S(\subseteq \mathbb{R}) \to \mathbb{R}$, where h is a convex function on S. The mapping k is said to be h-convex dominated on set S if the inequality below is true

$$\mu k(s_1) + (1-\mu)k(s_2) - k(\mu s_1 + (1-\mu)s_2) \leq \mu h(s_1) + (1-\mu)h(s_2) - h(\mu s_1 + (1-\mu)s_2)$$

for all s_1, s_2 in S and $\mu \in [0, 1]$.

Note that for $M = 1$, $m = -1$ and $\Psi = h$, (m, M)-Ψ-convex functions are h-convex dominated functions.

1.2 Higher order convex functions

This section generalizes the notion of convexity to the higher order (see for reference [52]).

Definition 1.2.1. For a real valued function $\Psi : [\eta, \zeta] \to \mathbb{R}$, the divided difference of m-th order for the function Ψ at distinct points $s_0, \dots, s_m \in [\eta, \zeta]$ is recursively defined as:

$$[s_i; \Psi] = \Psi(s_i), \quad 0 \le i \le m$$

and

$$[s_0, \dots, s_m; \Psi] = \frac{[s_1, \dots, s_m; \Psi] - [s_0, \dots, s_{m-1}; \Psi]}{s_m - s_0}. \tag{1.2.1}$$

For instance, the divided difference of order $m = 2$ for distinct points $s_0, s_1, s_2 \in [\eta, \zeta]$ is

$$[s_i; \Psi] = \Psi(s_i), \quad (0 \le i \le 2), \tag{1.2.2}$$

$$[s_0, s_1; \Psi] = \frac{\Psi(s_1) - \Psi(s_0)}{s_1 - s_0}, \tag{1.2.3}$$

$$[s_0, s_1, s_2; \Psi] = \frac{[s_1, s_2; \Psi] - [s_0, s_1; \Psi]}{s_2 - s_0}. \tag{1.2.4}$$

Remark 1.2.1. The order of elements $s_0, \dots, s_m$ has no effect on the value $[s_0, \dots, s_m; \Psi]$, that is, its value is independent of the order of $s_i{}'s$ $(0 \le i \le m)$. We may obtained an extension of the above definition by choosing coinciding points. That is, under the assumption that $\Psi'|_{s_0}$ exists if we take the limiting case $s_1 \to s_0$ of (1.2.3) yields:

$$\lim_{s_1 \to s_0} [s_0, s_1; \Psi] = \Psi'(s_0).$$

Now, taking the limiting case $s_1 \to s_0$ of (1.2.4), we get

$$\lim_{s_1 \to s_0} [s_0, s_1, s_2; \Psi] = \frac{\Psi(s_2) - \Psi(s_0) - \Psi'(s_0)(s_2 - s_0)}{(s_2 - s_0)^2}, s_2 \ne s_0$$

assuming that $\Psi'|_{s_0}$ exists. Further, taking the limits $s_i \to s_0$ for $i = 1, 2$ in (1.2.4), yields

$$\lim_{s_2 \to s_0} \lim_{s_1 \to s_0} [s_0, s_1, s_2; \Psi] = \frac{\Psi''(s_0)}{2}$$

assuming that $\Psi''|_{s_0}$ exists.

Generally, the extended form for $m + 1$ points is

$$\lim_{s_m \to s_0} \dots \lim_{s_2 \to s_0} \lim_{s_1 \to s_0} [s_0, \dots, s_m; \Psi] = \frac{\Psi^{(m)}(s_0)}{m!}$$

assuming that $\Psi^{(m)}|_{s_0}$ exists.

It can be easily seen that $[s_0, \dots, s_m; \Psi]$ defined in (1.2.1) is equivalent with

$$[s_0, \dots, s_m; \Psi] = \sum_{i=0}^{m} \frac{\Psi(s_i)}{\prod\limits_{j=0, i \neq j}^{m} (s_i - s_j)}.$$

Definition 1.2.2. For $m \geq 0$, a function $\Psi : [\eta, \zeta] \to \mathbb{R}$ is referred as m-convex on $[\eta, \zeta]$ if and only if for all choices of $(m+1)$ distinct points $s_0, \dots, s_m \in [\eta, \zeta]$, we get

$$[s_0, \dots, s_m; \Psi] \geq 0.$$

The following theorem exhibits a relationship between convexity and differentiability.

Theorem 1.2.1. *If m-th order derivative of a function Ψ on S exists, then Ψ is called m-convex if and only if $\Psi^{(m)}(s) \geq 0$ for $s \in S$.*

1.3 Logarithmic and m-exponential convexity

The advance study of convexity holds gigantic theory of functions, for instance, logarithmic convex functions that follow the laws of comparison of means and are an important instrument in mathematical statistics and special functions (see for example [52]).

Definition 1.3.1. Let S be an interval in $\mathbb{R}$, a positive function $\Psi : S \to \mathbb{R}$ is said to be log-convex if

$$\Psi(\mu s_1 + (1-\mu)s_2) \leq (\Psi(s_1))^{\mu} (\Psi(s_2))^{1-\mu} \tag{1.3.1}$$

for $\mu \in [0,1]$ and for all $s_1, s_2 \in S$.

Alternatively, Ψ is log-convex if $\log \Psi$ is convex.

Definition 1.3.2. A function $\Psi : S \subseteq \mathbb{R} \to \mathbb{R}^+$ is called log-$\mathcal{J}$-convex or log-convex in the Jensen sense if

$$\Psi^2 \left(\frac{s_1 + s_2}{2} \right) \leq \Psi(s_1)\Psi(s_2)$$

is true for each $s_1, s_2 \in S$.

Remark 1.3.1. A log-$\mathcal{J}$-convex function is log-convex if it is continuous also.

S. N. Bernstein gave the the concept of exponential convexity in 1929 [5] and later on these functions were introduced by D. V. Widder [63] as a sub class of convex functions on a specified interval. The concept of m-exponential convexity has been given in [53] which was presented by J. Pečarić and J. Perić.

The forthcoming discussion focuses on some notions and definitions about m-exponentially convex functions (for example, read [53]).

Definition 1.3.3. A real valued function $\Psi : S \to \mathbb{R}$ on an open interval $S \subset \mathbb{R}$ is called m-exponentially $\mathcal{J}$-convex or m-exponentially convex in the Jensen sense if

$$\sum_{j,k=1}^{m} b_j b_k \Psi \left(\frac{s_j + s_k}{2} \right) \geq 0$$

is true for all $b_j \in \mathbb{R}$ and $s_j \in S$, $j = 1, ..., m$.

Remark 1.3.2. It has been observed that 1-exponentially $\mathcal{J}$-convex functions are nonnegative functions. Furthermore, m-exponentially $\mathcal{J}$-convex functions are p-exponentially $\mathcal{J}$-convex for every $p \in \mathbb{N}$ with $p \leq m$.

The notion of positive semi-definite matrices and some basic rule of linear algebra are used to obtain the proposition below.

Proposition 1.3.1. *If a function $\Psi : S \to \mathbb{R}$ is an m-exponentially $\mathcal{J}$-convex, then for every $p \in \mathbb{N}$ with $p \le m$ and $s_1, \ldots, s_m \in S$ the matrix $\left[\Psi\left(\frac{s_i+s_j}{2}\right)\right]_{i,j=1}^{p}$ is a positive semi-definite matrix.*

In particular, $\det\left[\Psi\left(\frac{s_i+s_j}{2}\right)\right]_{i,j=1}^{p} \ge 0$ *for every $p \in \mathbb{N}$ with $p \le m$ and $s_1, \ldots, s_m \in S$.*

Definition 1.3.4. A real valued function $\Psi : S \to \mathbb{R}$ is exponentially $\mathcal{J}$-convex, if it is m-exponentially $\mathcal{J}$-convex for all $m \in \mathbb{N}$.

The (m-exponential) exponentially convex function is defined below.

Definition 1.3.5. A real valued function $\Psi : S \to \mathbb{R}$ is (m-exponential) exponentially convex if it is (m-exponentially $\mathcal{J}$-convex) exponentially $\mathcal{J}$-convex and continuous.

We can draw the following points from the above definition:

Remark 1.3.3.

(i) The above mentioned definition makes it clear that set of all m-exponentially convex functions on S form convex cone.

(ii) Less obvious is that if we take any two m-exponentially convex functions on S then their product is also an m-exponentially convex function on S (one may read [24]).

(iii) m-exponentially convex functions are invariant on taking admissible shifts and translations inside argument of the function. In other words, if $s \mapsto \Psi(s)$ is m-exponentially convex, then $s \mapsto \Psi(s-c)$ and $s \mapsto \Psi(s/\lambda)$ are also m-exponentially convex functions.

Below are some fundamental examples of exponentially convex functions, (one may read [24] for details).

Example 1.3.1.

(i) $\Psi(s) = k$ is exponentially convex on $\mathbb{R}$, for any $k \geq 0$.

(ii) $\Psi(s) = e^{\beta s}$ is exponentially convex on $\mathbb{R}$, for any $\beta \in \mathbb{R}$.

(iii) $\Psi(s) = s^{-\beta}$ is exponentially convex on $(0, \infty)$, for any $\beta > 0$.

Remark 1.3.4. We may note that a positive real valued function $\Psi : S \to \mathbb{R}$ is log-$\mathcal{J}$-convex if and only if it is 2-exponentially $\mathcal{J}$-convex, that is:

$$b_1^2\Psi(s_1) + 2b_1b_2\Psi\left(\frac{s_1+s_2}{2}\right) + b_2^2\Psi(s_2) \geq 0$$

for all $b_1, b_2 \in \mathbb{R}$ and $s_1, s_2 \in S$.

If Ψ is 2-exponentially convex, then Ψ is log-convex. Converse is true if provided that Ψ is continuous also. m-exponentially convex functions are not, in general, exponentially convex. (For details, go through the reference [24]).

Now we write a valuable lemma.

Lemma 1.3.2. *If $\Psi : S \to \mathbb{R}$ is log-convex then for $r, s, t \in S$ with $r < s < t$*

$$(\Psi(s))^{t-r} \leq (\Psi(r))^{t-s}(\Psi(t))^{s-r}. \tag{1.3.2}$$

Proof. See [52], pp. 4. □

Now we present different families of functions to investigate exponential convexity. The following lemma is useful to construct new exponentially convex functions. Since the below mentioned result is the simple consequence of some basic examples and remarks given in [24], so we omit the proof.

Lemma 1.3.3. *(i) Define a real valued function f_l on $S = \mathbb{R}$ by*

$$f_l(s) = \frac{1}{l^2}\exp(ls), \quad (l > 0).$$

It implies the exponential convexity on $(0, \infty)$ of $l \mapsto \frac{d^2}{ds^2} f_l(s)$ for each $s \in S$.

(ii) Let g_l be a real valued function defined on $S = \mathbb{R}^+$ by

$$g_l(s) = \frac{s^l}{l(l-1)}, \quad (l > 1).$$

It implies the exponential convexity on $(1, \infty)$ of the function $l \mapsto \frac{d^2}{ds^2} g_l(s)$ for each $s \in S$.

(iii) Consider a positive real valued function defined on $S = \mathbb{R}^+$ by

$$h_l(s) = \frac{l^{-s}}{(\log l)^2}, \quad (l > 1).$$

It implies the exponential convexity on $(1, \infty)$ of the function $l \mapsto \frac{d^2}{ds^2} h_l(s)$ for each $s \in S$.

(iv) Define a positive real valued function k_l on $S = \mathbb{R}^+$ by

$$k_l(s) = \frac{1}{l}\exp(-s\sqrt{l}), \quad (l > 0).$$

It implies the exponential convexity on $(0, \infty)$ of the function $l \mapsto \frac{d^2}{ds^2} k_l(s)$ for each $s \in S$.

1.4 Jensen type inequalities and their functional approach

The well known king of inequalities, that is, the Jensen inequality is the generalization of (1.1.2) (see [15], [39]). In the theory of inequalities, it is very important due to its various applications in mathematics and statistics. Hölder's, Cauchy's, and arithmetic-geometric-harmonic mean inequalities are some of its special cases.

Theorem 1.4.1. *If* $S \subseteq \mathbb{R}$ *and* $\Psi : S \to \mathbb{R}$ *is convex then for all* $x_i \in S, (1 \leq i \leq n)$ *and* $p_i (1 \leq i \leq n)$ *non negative real such that* $p_1 + \cdots + p_n = 1$ *we have*

$$\Psi\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i \Psi(x_i). \tag{1.4.1}$$

Jensen functional is the difference of the above inequality, written as

$$\Lambda(\Psi) = \sum_{i=1}^{n} p_i \Psi(x_i) - \Psi\left(\sum_{i=1}^{n} p_i x_i\right).$$

The reverse form of Jensen's inequality from [39] stated as:

Theorem 1.4.2. *If* $S \subseteq \mathbb{R}$ *and* $\Psi : S \to \mathbb{R}$ *is convex then for all* $x_i \in S, (1 \leq i \leq n)$ *and if* p *be a real* n*-tuple with* $p_1 > 0, p_i \leq 0$ $(i = 2, \ldots, n)$ $P_n = p_1 + \cdots + p_n > 0$ *and* $\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \in S$*, then*

$$\Psi\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \geq \frac{1}{P_n} \sum_{i=1}^{n} p_i \Psi(x_i). \tag{1.4.2}$$

The theorem below is the classical integral form of Jensen's inequality stated as (see for example [52] or [58]):

Theorem 1.4.3. *If* λ *is a positive measure on a* σ*-algebra* $\mathcal{A}$ *in a set* X*, so that* $\lambda(X) = 1$*. If* $h \in L^1(\lambda)$*,* $h(x) \in S \subseteq \mathbb{R}$ *for all* $x \in X$ *and* $\Psi : S \to \mathbb{R}$ *is convex , then*

$$\Psi\left(\int_X h d\lambda\right) \leq \int_X (\Psi \circ h) d\lambda. \tag{1.4.3}$$

Let $(X, \mathcal{A}, \lambda_1)$ and $(Y, \mathcal{B}, \lambda_2)$ are two probability measure spaces. A (separately) weight function ς is defined as a product-measurable mapping $\varsigma : X \times Y \to [0, \infty)$, (see for example [58]), such that

$$\begin{aligned} &\int_X \varsigma(x, y) d\lambda_1(x) = 1 \text{ (for every } y \in Y), \\ &\int_Y \varsigma(x, y) d\lambda_2(y) = 1 \text{ (for every } x \in X). \end{aligned} \tag{1.4.4}$$

J. Rooin in [57] uses weight function ς and two probability measure spaces to introduce weighted integral Jensen's inequality.

Theorem 1.4.4. [57] Assume that $(X, \mathcal{A}, \lambda_1)$ and $(Y, \mathcal{B}, \lambda_2)$ are two probability measure spaces and ς is a weight function (defined in (1.4.4)). If $S \subseteq \mathbb{R}$, $h \in L^1(\lambda_1)$, $h(x) \in S$ for all $x \in X$ and $\Psi : S \to \mathbb{R}$ is a real convex function, then

$$\int_Y \Psi \left(\int_X h(x) \varsigma(x, y) d\lambda_1(x) \right) d\lambda_2(y)$$

has meaning and

$$\Psi \left(\int_X h d\lambda_1 \right) \leq \int_Y \Psi \left(\int_X h(x) \varsigma(x, y) d\lambda_1(x) \right) d\lambda_2(y) \leq \int_X (\Psi \circ h) d\lambda_1. \tag{1.4.5}$$

Theorem given below is written in [49] which was stated by J. Pečarić and R. Janić.

Theorem 1.4.5. *Assume $\Psi : [0, \infty) \to \mathbb{R}$ ba a nondecreasing convex function, let $(X, \|\cdot\|)$ be a normed space. Then for every $x_i \in V, p_i \geq 0 \, (1 \leq i \leq n)$ such that $P_n = \sum\limits_{i=1}^{n} p_i > 0$, we have*

$$\Psi \left(\frac{1}{P_n} \left\| \sum_{i=1}^{n} p_i x_i \right\| \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \Psi(\|x_i\|). \tag{1.4.6}$$

(1.4.6) gives the following functional acting on nondecreasing convex function Ψ:

$$\Omega(\Psi) = \frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi(\|x_i\|) - \Psi\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right). \tag{1.4.7}$$

The reverse Jensen's inequality and the integral form stated in (1.4.2) and (1.4.3), respectively can also be written in the above given form.

The existing literature considers Hadamard inequality as important and useful inequality. It is as under:

Theorem 1.4.6. *Assume $\Psi : S = [\alpha, \beta] \to \mathbb{R}$ be a convex function, then*

$$\Psi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \Psi(r)dr \leq \frac{\Psi(\alpha)+\Psi(\beta)}{2}. \tag{1.4.8}$$

By considering $r = \frac{\alpha+\beta}{2}$ with its two convex combinations, the refinement of Hadamard inequality is obtained in [45] is as follows:

Theorem 1.4.7. *Consider a closed real interval $S = [\alpha, \beta] \subseteq \mathbb{R}$ with $\alpha < \beta$ and $\gamma, \delta \in S$. Let $\Psi : S \to \mathbb{R}$ is a convex function. Suppose*

$$a = \frac{\gamma-\alpha}{\beta-\alpha}, \quad b = \frac{\beta-\gamma}{\beta-\alpha}, \quad c = \frac{\delta-\alpha}{\beta-\alpha}, \quad d = \frac{\beta-\delta}{\beta-\alpha}. \tag{1.4.9}$$

Then

$$\begin{aligned}\Psi\left(\frac{\alpha+\beta}{2}\right) &\leq a\Psi\left(\frac{\alpha+\gamma}{2}\right) + b\Psi\left(\frac{\gamma+\beta}{2}\right)\\ &\leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Psi(r)dr\\ &\leq \frac{c\Psi(\alpha)+d\Psi(\beta)+\Psi(\delta)}{2} \leq \frac{\Psi(\alpha)+\Psi(\beta)}{2}.\end{aligned} \tag{1.4.10}$$

In the field of potential theory and statistical mechanics, Gustave Choquet in 1953 introduced the notion of Choquet integral with respect to capacity. It has wide applications in machine learning particularly in recognition of patterns, economics, imaging science, and in information fusion. In [61], the Jensen's inequality is established for the Choquet integral that paly a crucial part in the risk aversion theory.

Theorem 1.4.8. [61] Suppose $g : S \to \mathbb{R}$ is an increasing and concave function and $g(0) \geq 0$. Then the Jensen inequality holds for all $X \in L^1_{\mu\nu}$

$$C_{\mu\nu}(g(X)) \leq g(C_{\mu\nu}(X)), \tag{1.4.11}$$

where

$$C_{\mu\nu}(X) = \int_0^\infty \mu(X > t)dt - \int_{-\infty}^0 \nu(X < t)dt$$

if and only if $\mu(A) \leq \bar{\nu}(A)$ for all A.

For other notable literature about Jensen's inequality and the results related to it, see [15, 28–31, 34, 35, 55].

1.5 Jessen type functionals

Let $E(\neq \emptyset)$ and L be a linear class of real valued functions $h : E \to \mathbb{R}$ possessing the properties:

L_1: $h, k \in L \Rightarrow (\gamma h + \delta k) \in L$ for all $\gamma, \delta \in \mathbb{R}$,

L_2: $1 \in L$, that is, if $h(l) = 1$ for some $l \in E$, then $h \in L$.

Take a positive linear functional $B : L \to \mathbb{R}$ possesses the properties:

A_1: $B(\gamma h + \delta k) = \gamma B(h) + \delta B(k)$ for $h, k \in L, \gamma, \delta \in \mathbb{R}$,

A_2: $h \in L, h(l) \geq 0$ on $E \Rightarrow B(h) \geq 0$ (B is positive).

B is said to be normalized if

A_3: $B(1) = 1$.

By a weight function, we mean a mapping $\varsigma : E \times E \to \mathbb{R}^+$ such that

$$\begin{aligned} B\left(\varsigma(x,y)\right) &= 1 \, (\text{for each } y \text{ in } E), \\ Z\left(\varsigma(x,y)\right) &= 1 \, (\text{for each } x \text{ in } E), \end{aligned} \tag{1.5.1}$$

where B and Z satisfy the properties A_1, A_2 and A_3.

For some notable results about isotonic linear functionals and convex functions, these references [4, 13, 46–48] can be seen. Next theorem is the generalized form of Jensen's inequality given by B. Jessen in [26] (see also [52]) for positive linear functionals.

Theorem 1.5.1. *Let $E \neq \emptyset$, $S \subseteq \mathbb{R}$ and L possess properties L_1, L_2 on E. Consider a real valued continuous convex function Ψ on the interval S. If B with the property $B(1) = 1$ be a positive linear functional on L. Then for all $h \in L$ such that $\Psi(h) \in L$ we have $B(h) \in S$ and*

$$\Psi(B(h)) \leq B(\Psi(h)).$$

Here $\Upsilon(\Psi) = B(\Psi(h)) - \Psi(B(h))$ is associated positive linear functionals.

The converse of Jessen's inequality [4] (see also [52]) is stated as:

Theorem 1.5.2. *Assume a convex function Ψ on $S = [\eta, \zeta](-\infty < \eta < \zeta < \infty)$ and let L satisfy properties L_1, L_2 on a nonempty set E. If B with the property $B(1) = 1$ be a positive linear functional on L, then for all $h \in L$ with $\Psi(h) \in L$ (so that $\eta \leq h(l) \leq \zeta$ for all $l \in E$), we have*

$$B(\Psi(h)) \leq \frac{\zeta - B(h)}{\zeta - \eta} \cdot \Psi(\eta) + \frac{B(h) - \eta}{\zeta - \eta} \cdot \Psi(\zeta). \tag{1.5.2}$$

Its functional form is given as:

$$\daleth(\Psi) = \frac{\zeta - B(h)}{\zeta - \eta} \cdot \Psi(\eta) + \frac{B(h) - \eta}{\zeta - \eta} \cdot \Psi(\zeta) - B(\Psi(h)).$$

Various remarkable results related to Jessen are given in [1, 4, 9–11, 14, 54, 62].

1.6 m-exponential and logarithmic concavity

This part consists of some useful notions and definitions of m-exponentially concave functions in the same manner as given for exponentially convex function in [53].

Definition 1.6.1. Suppose S be an open interval of $\mathbb{R}$ and $g : S \to \mathbb{R}$ be a real valued function, then

- g is called m-exponentially $\mathcal{J}$-concave (where $\mathcal{J}$ represents Jensen sense) if

  $$\sum_{j,k=1}^{m} a_j a_k g\left(\frac{y_j + y_k}{2}\right) \leq 0$$

  is true for all $a_i \in \mathbb{R}$ and all $y_i \in S$, $1 \leq i \leq m$.

- If g is continuous and m-exponentially $\mathcal{J}$-concave on S, then it is said to be m-exponentially concave on S.

Definition 1.6.2. Let $g : S \to \mathbb{R}$ be a real valued function, then

- g is called exponentially $\mathcal{J}$-concave, if for all natural numbers m it is m-exponentially $\mathcal{J}$-concave.

- If g is exponentially $\mathcal{J}$-concave and continuous then it is said to be exponentially concave.

Remark 1.6.1. An interesting fact about a positive real valued function $g : S \to \mathbb{R}^+$ is; it is log-$\mathcal{J}$-concave if and only if it is 2-exponentially $\mathcal{J}$-concave, that is, for all $a_1, a_2 \in \mathbb{R}$ and $x, y \in S$ the following holds

$$a_1^2 g(x) + 2a_1 a_2 g\left(\frac{x+y}{2}\right) + a_2^2 g(y) \leq 0.$$

If g is 2-exponentially concave, then g is log-concave. Converse is true if provided that g is continuous also.

A fundamental inequality of log-concave functions stated as follows:

Lemma 1.6.1. *Suppose that $g : S \to \mathbb{R}$ is log-concave then for $l, m, n \in S$ with $l < m < n$*

$$(g(m))^{n-l} \geq (g(l))^{n-m}(g(n))^{m-l}. \tag{1.6.1}$$

Proof. We can easily prove this lemma by [52], pp. 4. □

1.7 Means and its Stolarsky types

In this section, we define mean and discuss its different kinds. Stolarsky type means are also given.

Definition 1.7.1. A function $\mathcal{M} : S \times S \times \ldots \times S \to \mathbb{R}$ ($S \subseteq \mathbb{R}$) is called a mean if for all m-tuples $(s_1, \ldots, s_m)$ of elements of S the following holds

$$\inf\{s_1, \ldots, s_m\} \leq \mathcal{M}(s_1, \ldots, s_m) \leq \sup\{s_1, \ldots, s_m\}.$$

For example, if we consider the Lagrange mean value theorem stated as: Assume a real valued continuous function Ψ which is defined on a closed interval S and differentiable on the interior points of S and let $s_1, s_2 \in S$ with $s_1 < s_2$, there exists at least one ϖ belongs to open interval (s_1, s_2) such that

$$\Psi'(\varpi) = \frac{\Psi(s_2) - \Psi(s_1)}{s_2 - s_1}.$$

From the above equation, the number

$$\varpi = (\Psi')^{-1}\left(\frac{\Psi(s_2)-\Psi(s_1)}{s_2-s_1}\right)$$

is called a mean provided that Ψ' is invertible.

In a similar manner we can define Cauchy's mean from Cauchy's type mean value theorem which states that: Assume Ψ and Φ are real valued continuous functions on $[s_1, s_2]$ and differentiable on (s_1, s_2) with $\Phi(s_2) - \Phi(s_1) \neq 0$, there exists at least one $\varpi \in (s_1, s_2)$ such that

$$\frac{\Psi'(\varpi)}{\Phi'(\varpi)} = \frac{\Psi(s_2)-\Psi(s_1)}{\Phi(s_2)-\Psi(s_1)}.$$

From the above equation, the number

$$\varpi = \left(\frac{\Psi'}{\Phi'}\right)^{-1}\left(\frac{\Psi(s_2)-\Psi(s_1)}{\Phi(s_2)-\Psi(s_1)}\right)$$

is called a mean provided that Ψ'/Φ' is invertible.

There exist some other important means in the literature. One of the well known mean is the arithmetic mean (A.M) defined as $A(\gamma, \delta) = \frac{\gamma+\delta}{2}$. For an integrable function Ψ with $\gamma \leq s \leq \delta$ it is written as

$$A(\Psi; \gamma, \delta) = \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} \Psi(s)ds.$$

For positive γ, δ there are geometric mean (G.M) written as $G(\gamma, \delta) = \sqrt{\gamma\delta}$, identric mean (I.M) defined as

$$I(\gamma, \delta) = \begin{cases} \gamma, & \gamma = \delta; \\ \\ \frac{1}{e}\left(\frac{\delta^{\delta}}{\gamma^{\gamma}}\right)^{\frac{1}{\delta-\gamma}}, & \text{otherwise}, \end{cases} \tag{1.7.1}$$

and m-logarithmic mean

$$L_m(\gamma,\delta) = \begin{cases} \gamma, & \gamma=\delta; \\ \left[\frac{\delta^{m+1}-\gamma^{m+1}}{(m+1)(\delta-\gamma)}\right]^{\frac{1}{m}}, & \text{otherwise.} \end{cases} \tag{1.7.2}$$

Logarithmic mean (L.M) is the limit case of m-logarithmic mean, that is, $\lim_{m\to -1} L_m(\gamma,\delta) = L(\gamma,\delta)$ stated as

$$L(\gamma,\delta) = \begin{cases} \gamma, & \gamma=\delta; \\ \frac{\delta-\gamma}{\ln\delta-\ln\gamma}, & \text{otherwise.} \end{cases} \tag{1.7.3}$$

Now we use (1.4.8) to obtain elementary inequalities based on the above mentioned means.

If we substitute $\Psi(x)=\exp(x)$ in (1.4.8), we may obtain an elementary inequality based on geometric, logarithmic and arithmetic mean. That is,

$$\sqrt{\gamma\delta} \le \frac{\delta-\gamma}{\log\delta-\log\gamma} \le \frac{\gamma+\delta}{2}.$$

Now if we substitute $\Psi(x)=-\log x$ in (1.4.8), we deduce geometric, identric and arithmetic mean inequality:

$$\sqrt{\gamma\delta} \le \frac{1}{e}\left(\frac{\delta^\delta}{\gamma^\gamma}\right)^{\frac{1}{\delta-\gamma}} \le \frac{\gamma+\delta}{2}.$$

The following chain of inequalities is given in [20]:

$$\text{G.M} \le \text{L.M} \le \text{I.M} \le \text{A.M.}$$

Now we give Stolarsky type means which were introduced by Stolarsky in [60] (see also [22, 55]). Let $s, t \in \mathbb{R}$ and γ, δ be positive numbers further $\gamma < \delta$ then

$$\begin{aligned}
\mathcal{M}(\gamma, \delta; s, t) &= \left(\frac{t(\delta^s - \gamma^s)}{s(\delta^t - \gamma^t)}\right)^{\frac{1}{s-t}}, \\
\mathcal{M}(\gamma, \delta; s, 0) &= \left(\frac{\delta^s - \gamma^s}{s(\ln \delta - \ln \gamma)}\right)^{\frac{1}{s}}, \\
\mathcal{M}(\gamma, \delta; s, s) &= \exp\left(-\frac{1}{s}\right)\left(\frac{\gamma^{\gamma^s}}{\delta^{\delta^s}}\right)^{\frac{1}{\gamma^s - \delta^s}}, \\
\mathcal{M}(\gamma, \delta; 0, 0) &= \sqrt{\gamma\delta}
\end{aligned}$$

are means. He further proved that this function is increasing in s and t.

Chapter 2

Exponential convexity for Jensen's type functionals for norms

In this chapter, we discuss m-exponential convexity of the functional

$$f \to \Omega(f) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\|x_i\|) - f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right). \tag{2.0.1}$$

This positive functional is defined in the form of the difference of the two sides of the inequality (1.4.6). We deduce results about log-convexity and exponential convexity. We also discuss Cauchy and Lagrange type mean value theorems which lead us to Stolarsky type means.

2.1 m-Exponential convexity

The new exponentially convex functions can be formed by the following lemma.

Lemma 2.1.1.

(i) *For* $p > 0$ *let* $\varphi_p : [0, \infty) \to \mathbb{R}$ *be defined with*

$$\varphi_p(l) = \frac{e^{pl^2}}{p^2}.$$

Then $p \mapsto \varphi_p(l),\ p \mapsto \frac{d}{dl}\varphi_p(l)$ *and* $p \mapsto \frac{d^2}{dl^2}\varphi_p(l)$ *are exponentially convex on* $(0, \infty)$, *for each* $l \in [0, \infty)$.

(ii) *For* $p > 1$ *let* $\phi_p : [0, \infty) \to \mathbb{R}$ *be defined with*

$$\phi_p(l) = \frac{l^p}{p(p-1)}.$$

Then $p \mapsto \phi_p(l),\ p \mapsto \frac{d}{dl}\phi_p(l)$ *and* $p \mapsto \frac{d^2}{dl^2}\phi_p(l)$ *are exponentially convex on* $(1, \infty)$, *for each* $l \in [0, \infty)$.

Proof. (i) is true from second and third part of Example 1.3.1 and Remark 1.3.3. We get (ii) by similar reasons that used to prove (i) and by observing $l^p = e^{p \ln l}$. □

Now we write a valuable lemma.

Lemma 2.1.2. *Assume* $f : [0, \infty) \to \mathbb{R}$ *is a convex function with* $f'(0) = 0$, *then* f *is an increasing convex function.*

Proof. Assume the function f is convex, then f' is nondecreasing. Since $f'(0) = 0$, we get $f'(x) \geq 0$. This yields f is an increasing convex function. □

We use an effective technique from [24] to construct exponentially and m-exponentially convex functions.

Consider the functional (2.0.1). From Theorem 1.4.5 it follows $\Omega(f) \geq 0$..

Theorem 2.1.3. *Assume* $f \mapsto \Omega(f)$ *is a linear functional defined with* (7.1.2) *and define* $\Phi_1 : (0, \infty) \to \mathbb{R}$ *and* $\Phi_2 : (1, \infty) \to \mathbb{R}$ *with*

$$\Phi_1(p) = \Omega(\varphi_p), \quad \Phi_2(p) = \Omega(\phi_p)$$

where φ_p *and* ϕ_p *are defined in Lemma 2.1.1. Then we get the following statements.*

(i) *The functions* Φ_1 *and* Φ_2 *are continuous on* $(0,\infty)$ *and* $(1,\infty)$*, respectively.*

(ii) *Let* $m \in \mathbb{N}$ *and* $p_1,\ldots,p_m \in (0,\infty)$, $q_1,\ldots,q_m \in (1,\infty)$ *are arbitrary, then the matrices*

$$\left[\Phi_1\left(\frac{p_j+p_k}{2}\right)\right]_{j,k=1}^{m}, \quad \left[\Phi_2\left(\frac{q_j+q_k}{2}\right)\right]_{j,k=1}^{m}$$

are positive semidefinite.

(iii) *The functions* Φ_1 *and* Φ_2 *are exponentially convex on* $(0,\infty)$ *and* $(1,\infty)$*, respectively.*

(iv) *If* $p,q,r \in (0,\infty)$ *are such that* $p<q<r$*, then*

$$\left(\frac{\sum\limits_{i=1}^{n} p_i \exp(q\|x_i\|^2) - P_n \exp\left(\frac{q\|\sum\limits_{i=1}^{n} p_i x_i\|^2}{P_n^2}\right)}{q^2 P_n}\right)^{r-p} \le$$

$$\left(\frac{\sum\limits_{i=1}^{n} p_i \exp(p\|x_i\|^2) - P_n \exp\left(\frac{p\|\sum\limits_{i=1}^{n} p_i x_i\|^2}{P_n^2}\right)}{p^2 P_n}\right)^{r-q} \left(\frac{\sum\limits_{i=1}^{n} p_i \exp(r\|x_i\|^2) - P_n \exp\left(\frac{r\|\sum\limits_{i=1}^{n} p_i x_i\|^2}{P_n^2}\right)}{r^2 P_n}\right)^{q-p};$$

if $s,t,u \in (1,\infty)$ *are such that* $s<t<u$*, then*

$$\left(\frac{\sum\limits_{i=1}^{n} p_i\|x_i\|^t}{t(t-1)P_n} - \frac{(\|\sum_{i=1}^{n} p_i x_i\|)^t}{t(t-1)P_n^t}\right)^{u-s} \le$$

$$\left(\frac{\sum\limits_{i=1}^{n} p_i\|x_i\|^s}{s(s-1)P_n} - \frac{(\|\sum_{i=1}^{n} p_i x_i\|)^s}{s(s-1)P_n^s}\right)^{u-t} \left(\frac{\sum\limits_{i=1}^{n} p_i\|x_i\|^u}{u(u-1)P_n} - \frac{(\|\sum_{i=1}^{n} p_i x_i\|)^u}{u(u-1)P_n^u}\right)^{t-s}$$

Proof. (i) The continuity of the functions $p \mapsto \Phi_i(p),\ i = 1, 2$ is obvious.

(ii) Let $m \in \mathbb{N},\ \xi_j,\ p_j \in \mathbb{R}\ (j = 1, \ldots, m)$ be arbitrary and define auxiliary function $\Psi_1 : [0, \infty) \to \mathbb{R}$ by

$$\Psi_1(l) = \sum_{j,k=1}^{m} \xi_j \xi_k \varphi_{\frac{p_j + p_k}{2}}(l).$$

Now $\Psi_1'(0) = 0$, since $\frac{d}{dl}\varphi_t(0) = 0$, and

$$\Psi_1''(l) = \sum_{j,k=1}^{m} \xi_j \xi_k \frac{d^2}{dl^2} \varphi_{\frac{p_j + p_k}{2}}(l) \geq 0,$$

for $l \geq 0$, by Lemma 2.1.1 which means, by Lemma 2.1.2, that Ψ_1 is increasing convex function. Now, Theorem 1.4.5 implies $\Omega(\Psi_1) \geq 0$. It means

$$\left[\Phi_1\left(\frac{p_j + p_k}{2}\right)\right]_{j,k=1}^{m}$$

is positive semi-definite matrix.

In the similar way, we may define auxiliary function Ψ_2. We conclude

$$\left[\Phi_2\left(\frac{q_j + q_k}{2}\right)\right]_{j,k=1}^{m}$$

is positive semi-definite matrix.

We easily get (iii) and (iv) from (i), (ii) and Lemma 1.3.2. $\square$

From above theorem, we have obtained a valuable corollary. This provides us an exciting link between moments of discrete random variables. It is stated as:

Corollary 2.1.4. *Assume* $(V, \|\cdot\|)$ *be a normed space and let* X *be a discrete random variable defined with* $P(X = x_i) = p_i,\ x_i \in V, p_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} p_i = 1$. *Then, for* $1 < j < k < m$,

$$\{\mathbb{E}[\|X\|^k] - (\|\mathbb{E}[X]\|)^k\}^{m-j} \leq C(j,k,m)\{\mathbb{E}[\|X\|^j] - (\|\mathbb{E}[X]\|)^j\}^{m-k}\{\mathbb{E}[\|X\|^m] - (\|\mathbb{E}[X]\|)^m\}^{k-j},$$

where

$$C(j,k,m) = \frac{\begin{pmatrix} k \\ 2 \end{pmatrix}^{m-j}}{\begin{pmatrix} j \\ 2 \end{pmatrix}^{m-k} \begin{pmatrix} m \\ 2 \end{pmatrix}^{k-j}}. \tag{2.1.1}$$

Theorem 7.1.3 also sets the following model:

Theorem 2.1.5. *Suppose that $\mathcal{C} \subset \mathbb{R}$ be an open interval, and $\Gamma = \{\eta_t | t \in \mathcal{C}\}$ be a family of continuous function defined on $S \subseteq [0, \infty)$, such that the function $\frac{d}{dl}\eta_t(0) = 0,\ t \in C$, and $t \mapsto \frac{d^2}{dl^2}\eta_t(l)$ is m-exponentially convex on $\mathcal{C}$ for any $l \in S$. Consider the functional $f \mapsto \Omega(f)$ as given in (7.1.2). Then $t \mapsto \Omega(\eta_t)$ is an m-exponentially convex function on $\mathcal{C}$.*

Remark 2.1.1. In the above theorem, we can add the other features of Theorem 7.1.3.

2.2 Mean value theorems

The lemma given below will be very helpful.

Lemma 2.2.1. *Let $f \in C^2([0,a])$, with $f'(0) = 0$. Denote $\mathfrak{d} = \inf_{l \in [0,a]} f''(l)$, $\mathfrak{D} = \sup_{l \in [0,a]} f''(l)$. Then the functions $f_1, f_2 : I \to \mathbb{R}^+$ defined by*

$$\begin{aligned} f_1(l) &= \frac{\mathfrak{D}}{2} l^2 - f(l) \\ f_2(l) &= f(l) - \frac{\mathfrak{d}}{2} l^2 \end{aligned} \tag{2.2.1}$$

are convex and nondecreasing.

Proof. The functions f_1, f_2 satisfy the conditions of Lemma 2.1.2 and the result follows. ∎

Theorem 2.2.2. *Let $x_i \in X, p_i \geq 0\,(i = 1, 2, \ldots n)$ such that $P_n = \sum_{i=1}^{n} p_i > 0$. Let $f \in C^2([0,a])$ with $f'(0) = 0$, where $\max_i \|x_i\| < a$. Then there exists $\varpi \in [0,a]$ such that*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(\|x_i\|) - f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right) = \varrho f''(\varpi), \tag{2.2.2}$$

where

$$\varrho = \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i \|x_i\|^2 - \left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right)^2\right].$$

Proof. Denote $\mathfrak{D} = \max_{t\in[0,a]} f''(t)$ and $\mathfrak{d} = \min_{t\in[0,a]} f''(t)$. Then the functions $f_1, f_2 : [0,a] \to \mathbb{R}$ as in Lemma 6.2.1, are convex and nondecreasing. This means that $\Omega(f_1),\ \Omega(f_2) \geq 0$, that is,

$$\varrho\mathfrak{d} \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\|x_i\|) - f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right) \leq \varrho\mathfrak{D}, \tag{2.2.3}$$

Now by (2.2.2) the Bolzano intermediate theorem ensures that $\varpi \in [0,a]$. □

Corollary 2.2.3. *Let $x_i \in X, p_i \geq 0\,(i = 1, 2, \ldots n)$ such that $P_n = \sum_{i=1}^{n} p_i > 0$. Let $f, g \in C^2([0,a])$, $f'(0) = g'(0) = 0$, where $\max_i \|x_i\| < a$. Then there exists $\varpi \in [0,a]$ such that the following holds*

$$g''(\varpi)\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(\|x_i\|) - f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right)\right] = f''(\varpi)\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i g(\|x_i\|) - g\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right)\right]. \tag{2.2.4}$$

Proof. Consider the auxiliary function $k \in C^2([0,a])$ defined with $k(l) = c_1 f(l) - c_2 g(l)$, where c_1 and c_2 are defined by

$$c_1 = \frac{1}{P_n}\sum_{i=1}^{n} p_i g(\|x_i\|) - g\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right), \tag{2.2.5}$$

and

$$c_2 = \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\|x_i\|) - f\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right). \tag{2.2.6}$$

It is obvious $\Omega(k) = 0$. Further, since $k'(0) = 0$, from Theorem 2.2.2 it follows that there exists some $\varpi \in [0, a]$ such that the following holds

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i k(\|x_i\|) - k\left(\frac{1}{P_n}\left\|\sum_{i=1}^{n} p_i x_i\right\|\right) = \varrho k''(\varpi). \tag{2.2.7}$$

The left-hand side of this equation equals to zero, whereas $\varrho \neq 0$. It yields $k''(\varpi) = 0$. $\square$

Remark 2.2.1. From (2.2.4) we can define various kinds of means provided that f''/g'' has an inverse. Mathematically, we can write it as:

$$\varpi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\Omega(f)}{\Omega(g)}\right). \tag{2.2.8}$$

Particularly, if we substitute $f(l) = \phi_p(l),\ g(l) = \phi_q(l)$ in (2.2.4) (the functions ϕ_p are defined in Lemma 2.1.1), then we get

$$\mu(p,q;\Omega) = \begin{cases} \frac{1}{P_n}\left(\frac{q(q-1)}{p(p-1)}\frac{P_n^{p-1}\sum\limits_{i=1}^{n} p_i\|x_i\|^p - \left(\left\|\sum_{i=1}^{n} p_i x_i\right\|\right)^p}{P_n^{q-1}\sum\limits_{i=1}^{n} p_i\|x_i\|^q - \left(\left\|\sum_{i=1}^{n} p_i x_i\right\|\right)^q}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{1-2p}{p(p-1)} + \frac{P_n^{p-1}\sum\limits_{i=1}^{n} p_i\|x_i\|^p \ln\|x_i\| - \|\sum\limits_{i=1}^{n} p_i x_i\|^p \ln(\sum\limits_{i=1}^{n}\|p_i x_i\|/P_n)}{P_n^{p-1}\sum\limits_{i=1}^{n} p_i\|x_i\|^p - \|\sum\limits_{i=1}^{n} p_i x_i\|^p}\right), & p = q \neq 1. \end{cases}$$

Chapter 3

Weighted Jessen and Jensen inequality and exponential convexity

In this chapter, m-exponential convexity of the positive functionals associated with the weigted integral Jensen's inequality (1.4.5) is examined. The respective positive functional are:

$$\Omega(\Psi) = \int_Y \Psi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \Psi\left(\int_X hd\lambda_1\right), \tag{3.0.1}$$

$$\digamma(\Psi) = \int_X (\Psi \circ h)d\lambda_1 - \int_Y \Psi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y). \tag{3.0.2}$$

To construct means with Stolarsky property, Cauchy and Lagrange type mean value theorems are also given. The next section will introduce a refinement in Jessen's inequality via weight function and m-exponential convexity of such functions is discussed that is associated with these linear functionals. Numerous families of functions have been given related to obtained results and mean value theorems are also given.

3.1 m-exponential convexity of weighted integral Jensen's inequality and Stolarsky type means

Here we establish (m-exponential) exponentially convex functions by applying an elegant method from [24].

In the following theorem and its corollaries we use notations as: $\mathcal{C}$ is a real open interval, S is any real interval, $\Gamma = \{g_l \mid l \in \mathcal{C}\}$ is a family of functions defined on S. $\Omega(\Psi)$ is as given in (3.0.1). Then we have the following useful results which produce new m-exponentially convex functions.

Theorem 3.1.1. *Assume the function $l \mapsto [s_1, s_2, s_3; g_l]$ is m-exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Omega(g_l)$ is an m-exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. In addition, the continuity of this function implies the m-exponential convexity on $\mathcal{C}$.*

Proof. Assume $l_i, l_j \in \mathcal{C}, l_{ij} = \frac{l_i+l_j}{2}$ and $a_i, a_j \in \mathbb{R}$ for $i, j \in \{1, \ldots, m\}$. Consider the function Δ on S defined as

$$\Delta(s) = \sum_{i,j=1}^{m} a_i a_j g_{l_{ij}}(s).$$

The hypothesis of m-exponential $\mathcal{J}$-convexity of $l \mapsto [s_1, s_2, s_3; g_l]$ yields

$$[s_1, s_2, s_3; \Delta] = \sum_{i,j=1}^{m} a_i a_j [s_1, s_2, s_3; g_{l_{ij}}] \geq 0,$$

implying the convexity of Δ on S. Hence $\Omega(\Delta) \geq 0$. Further the linearity of Ω, gives

$$\sum_{i,j=1}^{m} a_i a_j \Omega(g_{l_{ij}}) \geq 0.$$

We conclude that the function $l \mapsto \Omega(g_l)$ is an m-exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. □

We have obtained the corollaries below from above theorem.

Corollary 3.1.2. *Suppose that the function $l \mapsto [s_1, s_2, s_3; g_l]$ is exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Omega(g_l)$ is an exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. In addition, the continuity of this function implies the exponential convexity on $\mathcal{C}$.*

Corollary 3.1.3. *Suppose that the function $l \mapsto [s_1, s_2, s_3; g_l]$ is 2-exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then the following statements hold*

i) *The continuity of the function $l \mapsto \Omega(g_l)$ implies the 2-exponential convexity of $l \mapsto \Omega(g_l)$ on $\mathcal{C}$, and hence the function is log-convex. That is, for $l, r, s \in \mathcal{C}$ such that $r < s < l$ the following holds*

$$\Omega^{l-r}(g_s) \leq \Omega^{l-s}(g_r)\Omega^{s-r}(g_l).$$

ii) *Assume that the function $l \mapsto \Omega(g_l)$ on $\mathcal{C}$ is strictly positive and differentiable. Then for $l \leq u$ and $r \leq v$, $(l, r, u, v \in \mathcal{C})$ yields*

$$\tau(l, r, \Omega) \leq \tau(u, v; \Omega),$$

where

$$\tau(l, r; \Omega) = \begin{cases} \left(\frac{\Omega(g_l)}{\Omega(g_r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \\ e^{\frac{\frac{d}{dl}(\Omega(g_l))}{\Omega(g_l)}}, & \textit{otherwise}. \end{cases} \tag{3.1.1}$$

Proof. **i)** This is a simple consequence of Theorem 3.1.1 and Remark 1.3.4.
ii) The log-convexity of $l \mapsto \Omega(g_l)$ on $\mathcal{C}$ follows from (i). It implies the convexity of $l \mapsto \log \Omega(g_l)$ on $\mathcal{C}$. By applying [53, Proposition 3.2] with $l \leq u, r \leq v$, we get

$$\frac{\log \Omega(g_l) - \log \Omega(g_r)}{l - r} \leq \frac{\log \Omega(g_u) - \log \Omega(g_v)}{u - v}, \tag{3.1.2}$$

thus we have

$$\tau(l, r, \Omega) \leq \tau(u, v; \Omega).$$

The cases $l = r$ and $u = v$ follow from (6.1.2) as limiting cases. $\square$

Remark 3.1.1. The following positive functionals are useful in defining the basic inequality of log-convex functions.

$$\Omega(f_l) = \frac{1}{l^2}\left(\int_Y \exp\left(l\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \exp\left(l\int_X hd\lambda_1\right)\right).$$

$$\Omega(g_l) = \frac{1}{l(l-1)}\left(\int_Y \left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)^l d\lambda_2(y) - \left(\int_X hd\lambda_1\right)^l\right).$$

$$\Omega(h_l) = \frac{1}{(\log l)^2}\left(\int_Y l^{-\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)}d\lambda_2(y) - l^{-\left(\int_X hd\lambda_1\right)}\right).$$

$$\Omega(k_l) = \frac{1}{l}\left(\int_Y \exp\left(-\sqrt{l}\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \exp\left(-\sqrt{l}\int_X hd\lambda_1\right)\right).$$

Theorem 3.1.4. *Suppose* $\Omega(\Psi)$ *be the linear functional defined by* (3.0.1) *and consider the function* $\phi_i : (0,\infty) \to \mathbb{R}$ *for* $i = 1, 4$ *and* $\phi_i : (1,\infty) \to \mathbb{R}$ *for* $i = 2, 3$ *defined as*

$\phi_1(l) = \Omega(f_l),\ \phi_2(l) = \Omega(g_l),\ \phi_3(l) = \Omega(h_l),\ \phi_4(l) = \Omega(k_l)$, *where* f_l, g_l, h_l *and* k_l *are defined in Lemma 1.3.3. Then*

(i) The functions ϕ_i *are continuous on* $(0,\infty)$ *for* $i = 1, 4$ *and continuous on* $(1,\infty)$ *for* $i = 2, 3$.

(ii) If $m \in \mathbb{N}$, $l_1, \ldots, l_m \in (0,\infty)$ *for* $i = 1, 4$ *and* $l_1, \ldots, l_m \in (1,\infty)$ *for* $i = 2, 3$. *Then*

$$\left[\phi_i\left(\frac{l_j + l_k}{2}\right)\right]_{j,k=1}^m$$

are positive semidefinite matrices.

(iii) ϕ_i *are exponentially convex on* $(0, \infty)$ *when* $i = 1, 4$ *and on* $(1, \infty)$ *for* $i = 2, 3$.

(iv) Let $l, r, s \in (0, \infty)$ *for* $i = 1, 4$ *and* $r, s, l \in (1, \infty)$ *for* $i = 2, 3$ *with* $l < r < s$, *then*

$$(\phi_i(r))^{s-l} \leq (\phi_i(l))^{s-r} (\phi_i(s))^{r-l},$$

where $\phi_i(l)$ *for* $i = 1, 2, 3, 4$ *are defined in Remark 3.1.1.*

(v) If the functions ϕ_i *are differentiable and strictly positive on* $(0, \infty)$ *for* $i = 1, 4$ *and on* $(1, \infty)$ *for* $i = 2, 3$. *Then for every* $l, r, u, v \in (0, \infty)$ *for* $i = 1, 4$ *and* $l, r, u, v \in (1, \infty)$ *for* $i = 2, 3$ *such that* $l \leq u$ *and* $r \leq v$, *we have*

$$\tau(l, r, \phi_i) \leq \tau(u, v; \phi_i),$$

where

$$\tau(l, r; \phi_i) = \begin{cases} \left(\frac{\phi_i(l))}{\phi_i(r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \\ \exp\left(\frac{\frac{d}{dl}(\phi_i(l))}{\phi_i(l)}\right), & \text{otherwise.} \end{cases} \tag{3.1.3}$$

Proof. (i) The continuity of these functions is obvious.

(ii) Let $d_j, l_j \in \mathbb{R}$, where $j = 1, \ldots, m$ (m is any natural number). Define the auxiliary function Δ_1 on $S = \mathbb{R}$ by

$$\Delta_1(s) = \sum_{j,k=1}^{m} d_j d_k f_{\frac{l_j + l_k}{2}}(s).$$

Since

$$\Delta_1''(s) = \sum_{j,k=1}^{m} d_j d_k \frac{d^2}{ds^2} f_{\frac{l_j + l_k}{2}}(s) \geq 0$$

for $s \in S$ by Lemma 1.3.3. This implies Δ_1 is convex. Now Theorem 1.4.4 implies that $\Omega(\Delta_1) \geq 0$. This means that

$$\left[\phi_1\left(\frac{l_j + l_k}{2}\right)\right]_{j,k=1}^{m}$$

is a positive semidefinite matrix.

To prove the remaining positive semidefinite matrices, we can define the auxiliary functions Δ_i for $i = 2, 3, 4$ in the similar manner.

(iii) and (iv) are simple consequence of (i), (ii) and Lemma 1.3.2. We can easily prove (v) by using basic inequality of log-convex functions given in part (iv). $\square$

In the following lemma we assume $S = [\mu, \nu] \subseteq \mathbb{R}$ and in the next theorems we further assume S is a compact real interval.

Lemma 3.1.5. [52] *Suppose $\Psi \in C^2(S)$ and $\Psi : S \to \mathbb{R}$, Ψ'' is bounded. Assume $\mathfrak{d} = \inf_{l\in S} \Psi''(l)$, $\mathfrak{D} = \sup_{l\in S} \Psi''(l)$. Then the functions $\Psi_1, \Psi_2 : S \to \mathbb{R}$ defined by*

$$\begin{aligned} \Psi_1(l) &= \frac{\mathfrak{D}}{2} l^2 - \Psi(l) \\ \Psi_2(l) &= \Psi(l) - \frac{\mathfrak{d}}{2} l^2 \end{aligned} \tag{3.1.4}$$

are convex.

Theorem 3.1.6. *Assume $\Psi : S \to \mathbb{R}, \Psi \in C^2(S)$. If $(X, \mathcal{A}, \lambda_1)$ and $(Y, \mathcal{B}, \lambda_2)$ are two probability measure spaces and ς is a weight function (defined in (1.4.4)) such that $h \in L^1(\lambda_1), h(x) \in S$ for all $x \in X$. Then there exists $\varpi \in S$ such that*

$$\int_Y \Psi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right) d\lambda_2(y) - \Psi\left(\int_X h d\lambda_1\right) = \zeta \Psi''(\varpi), \tag{3.1.5}$$

where

$$\zeta = \frac{1}{2}\left[\int_Y \left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)^2 d\lambda_2(y) - \left(\int_X h d\lambda_1\right)^2\right].$$

Proof. Suppose $\mathfrak{D} = \max_{l\in S} \Psi''(l)$ and $\mathfrak{d} = \min_{l\in S} \Psi''(l)$. Then the convexity of the functions $\Psi_1, \Psi_2 : S \to \mathbb{R}$ follows from Lemma 3.1.5. Since they are also continuous. Apply left-hand side of the inequality (1.4.5), we get

$$\int_Y \Psi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right) d\lambda_2(y) - \Psi\left(\int_X h d\lambda_1\right) \leq \zeta \mathfrak{D},$$

and

$$\int_Y \Psi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \Psi\left(\int_X hd\lambda_1\right) \geq \zeta\mathfrak{d}.$$

Now combining these two inequalities and since Ψ'' is continuous, there exists $\varpi \in S\,(\mathfrak{d} \leq \Psi''(\varpi) \leq \mathfrak{D})$ such that (3.1.5) holds. □

Theorem 3.1.7. *Assume $\Phi, \Psi : S \to \mathbb{R}$ and $\Phi, \Psi \in C^2(S)$. If $(X, \mathcal{A}, \lambda_1)$ and $(Y, \mathcal{B}, \lambda_2)$ are two probability measure spaces and ς is a weight function (defined in* (1.4.4)*) such that $h \in L^1(\lambda_1), h(x) \in S$ for all $x \in X$. Then $\varpi \in S$ exists that ensures*

$$\begin{aligned}&\Psi''(\varpi)\left[\int_Y \Phi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \Phi\left(\int_X hd\lambda_1\right)\right]\\ &= \Phi''(\varpi)\left[\int_Y \Psi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \Psi\left(\int_X hd\lambda_1\right)\right].\end{aligned} \tag{3.1.6}$$

Proof. Suppose a function $k \in C^2(S)$ defined as $k = c_1\Phi - c_2\Psi$, where c_1 and c_2 are defined by

$$c_1 = \int_Y \Psi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \Psi\left(\int_X hd\lambda_1\right), \tag{3.1.7}$$

and

$$c_2 = \int_Y \Phi\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - \Phi\left(\int_X hd\lambda_1\right). \tag{3.1.8}$$

As $k \in C^2(S)$, applying Theorem 3.1.6 on k ensures that there exists some $\varpi \in S$ such that the following holds

$$\int_Y k\left(\int_X h(x)\varsigma(x,y)d\lambda_1(x)\right)d\lambda_2(y) - k\left(\int_X hd\lambda_1\right) = \zeta k''(\varpi). \tag{3.1.9}$$

The left-hand side of this equation equals to zero, the term ζ on the right-hand side is non zero, so we have that $k''(\varpi) = 0$. Thus the assertion of our theorem follows directly. □

Remark 3.1.2. If Φ''/Ψ'' is invertible, then means of various kinds can be defined by (3.1.6). That is,

$$\varpi = \left(\frac{\Phi''}{\Psi''}\right)^{-1}\left(\frac{\Omega(\Phi)}{\Omega(\Psi)}\right). \tag{3.1.10}$$

If we apply Cauchy type mean value Theorem 3.1.7 on functions $\Phi = f_l$ and $\Psi = f_r$ (defined in Lemma 1.3.3), it yields

$$M(l, r; \Omega) = \log \tau(l, r; \Omega)$$

satisfy

$$\mu \leq M(l, r; \Omega) \leq \nu,$$

where

$$\tau(l, r; \Omega) = \begin{cases} \left(\frac{\Omega(f_l)}{\Omega(f_r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \exp\left(\frac{\Omega(id.f_l)}{\Omega(f_l)}\right) \cdot \exp\left(-\frac{2}{l}\right), & l = r \neq 0. \end{cases}$$

If we set $\mu = \min_{l\in[\mu,\nu]}\{f(l)\}$ and $\nu = \max_{l\in[\mu,\nu]}\{f(l)\}$, then

$$\min_{l\in[\mu,\nu]}\{f(l)\} \leq M(l, r; \Omega) \leq \max_{l\in[\mu,\nu]}\{f(l)\},$$

which shows that $M(l, r; \Omega)$ are means of $f(l), l \in [\mu, \nu]$. If $l \leq u, r \leq v$ where $l, r, u, v \in \mathbb{R}$, then it yields by Theorem 3.1.4 that these means are monotonic.

Now, if we apply Cauchy type mean value Theorem 3.1.7 on functions $\Phi = g_l$ and $\Psi = g_r$ (defined in Lemma 1.3.3), this yields that there exists $\varpi \in [\mu, \nu]$ such that

$$\varpi^{l-r} = \frac{\Omega(g_l)}{\Omega(g_r)}.$$

Since the function $\varpi \mapsto \varpi^{l-r}$ is invertible for $l \neq r$, we get

$$\mu \leq \left(\frac{\Omega(g_l)}{\Omega(g_r)}\right)^{\frac{1}{l-r}} \leq \nu.$$

If we set $\mu = \min_{l\in[\mu,\nu]}\{f(l)\}$ and $\nu = \max_{l\in[\mu,\nu]}\{f(l)\}$, then in this case $\tau(l,r;\Omega)$ are means of $f(l), l \in [\mu,\nu]$, where

$$\tau(l,r;\Omega) = \begin{cases} \left(\frac{\Omega(g_l)}{\Omega(g_r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \exp\left(\frac{1-2l}{l(l-1)}\right) \cdot \exp\left(-\frac{\Omega(g_0 g_l)}{\Omega(g_l)}\right), & l = r \neq 1. \end{cases}$$

Note that in this case, the monotonicity property also holds for $\tau(l,r;\Omega)$.

Remark 3.1.3. We can construct similar results for the positive functional $\digamma(\Psi)$ defined in (3.0.2). Moreover, by introducing suitable weight functions, J. Rooin gave the refinement of the discrete Jensen's inequality and few of its applications [see [56]]. We can prove similar results for the positive functionals associated with this refined form of discrete Jensen's inequality.

3.2 Weighted reverse Jensen's inequality and its applications

Throughout section, we assume a convex set $S \subset V$, where V is a real vector space, $x_1, x_2, \ldots, x_n \in S$. Let $\mu_1, \lambda_1 > 0, \mu_i, \lambda_i \leq 0\,(i = 2, \ldots, n)$ and $U_m = \sum_{i=1}^{m} \mu_i, \Lambda_n = \sum_{j=1}^{n} \lambda_j > 0$. Also, we suppose that $\varsigma : \{(i,j) : 1 \leq i \leq m,\, 1 \leq j \leq n\} \to [0,\infty)$ is a weight function such that

$$\frac{1}{U_m}\sum_{i=1}^{m} \varsigma(i,j)\mu_i = 1 \quad (j = 1, \ldots, n),$$

and

$$\frac{1}{\Lambda_n}\sum_{j=1}^{n} \varsigma(i,j)\lambda_j = 1 \quad (i = 1, \ldots, m).$$

Now we refine converse Jensen's inequality via weight functions.

Lemma 3.2.1. *If ς is a weight function, then*

$$\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right) \geq \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma(i,j)\mu_i x_i\right) \geq \frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(x_i). \quad (3.2.1)$$

Proof. By using reversed Jensen's inequality

$$\begin{aligned}\frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma(i,j)\mu_i x_i\right) &\geq \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\frac{1}{U_m}\sum_{i=1}^{m}\varsigma(i,j)\mu_i\Psi(x_i)\\ &= \frac{1}{U_m}\sum_{i=1}^{m}\left(\frac{1}{\Lambda_n}\sum_{j=1}^{n}\varsigma(i,j)\lambda_j\right)\mu_i\Psi(x_i) = \frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(x_i),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma(i,j)\mu_i x_i\right) &\leq \Psi\left(\frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\frac{1}{U_m}\sum_{i=1}^{m}\varsigma(i,j)\mu_i x_i\right)\\ &= \Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\left(\frac{1}{\Lambda_n}\sum_{j=1}^{n}\varsigma(i,j)\lambda_j\right)\mu_i x_i\right)\\ &= \Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right).\end{aligned}$$

□

Now, we refine the reversed Jensen's inequality by using weight functions as follows:

Theorem 3.2.2. *Suppose that ς_1 and ς_2 are two weight functions, then*

(i)

$$\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right) \geq \psi_{\varsigma_1,\varsigma_2}(l) \geq \frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(x_i), \quad (3.2.2)$$

where

$$\psi_{\varsigma_1,\varsigma_2}(l) = \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j)+l\varsigma_2(i,j)]\mu_i x_i\right)\ (0\le l\le 1). \tag{3.2.3}$$

(ii) For each $1\le j\le n$*, the function* $l\to\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j)+l\varsigma_2(i,j)]\mu_i x_i\right)$ $(0\le l\le 1)$ *and thus,* $\psi_{\varsigma_1,\varsigma_2}$ *is convex.*

(iii)

$$\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right)\ge\int_0^1\psi_{\varsigma_1,\varsigma_2}(l)dl\ge\frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(x_i). \tag{3.2.4}$$

Particularly, assume $S\subseteq\mathbb{R}$*. Then*

$$\begin{aligned}\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right)&\ge\frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j A\left(\Psi;\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i x_i,\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i x_i\right)\\&\ge\frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(x_i),\end{aligned} \tag{3.2.5}$$

where $A(\Psi;\gamma,\delta)$ *is the arithmetic mean for an integrable function* Ψ *on the interval* $[\gamma,\delta]$*.*

Proof. (i) The mapping $(i,j)\to(1-l)\varsigma_1(i,j)+l\varsigma_2(i,j)$, where $1\le i\le m, 1\le j\le n$ is a weight function for each $0\le l\le 1$. So (3.2.2) follows from (3.2.1).

(ii) Let a and b are two nonnegative real numbers with $a+b=1$ and $0\le l_1,l_2\le 1$. Then for each $1\le j\le n$ yields the convexity of $\psi_{\varsigma_1,\varsigma_2}$.

(iii) The function $\psi_{\varsigma_1,\varsigma_2}$ is Riemann integrable on $[0,1]$ because it is bounded and convex on this interval. Hence by integrating (3.2.2) yields (3.2.4).

Particularly, assume $S \subseteq \mathbb{R}$. Then by applying change of variables $u = \frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j) + l\varsigma_2(i,j)]\mu_i x_i$, we get the desired result.

□

Now we give some applications of our results.

Theorem 3.2.3. *For positive numbers $x_1, \ldots, x_n$, we have*

$$\left(\prod_{i=1}^{m} x_i^{\mu_i}\right)^{\frac{1}{U_m}} \geq \left(\prod_{j=1}^{n}\left[I\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i x_i, \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i x_i\right)\right]^{\lambda_i}\right)^{\frac{1}{\Lambda_n}} \geq \frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i, \tag{3.2.6}$$

where I is the identric mean.

Proof. Consider a real valued function Ψ on $(0,\infty)$ defined as $\Psi(t) = -\ln t$. Clearly Ψ is convex and for positive real numbers γ, δ, $A(\Psi;\gamma,\delta) = -\ln I(\gamma,\delta)$. Then

$$\frac{1}{\Lambda_n}\sum_{j=1}^{n} A\left(\Psi;\gamma,\delta\right) = -\ln\left(\prod_{j=1}^{n}\left[I\left(\gamma,\delta\right)\right]^{\lambda_j}\right)^{\frac{1}{\Lambda_n}},$$

where $\gamma = \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i x_i$, $\delta = \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i x_i$. It can be seen that

$$\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right) = -\ln\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right), \quad \frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(x_i) = -\ln\left(\prod_{i=1}^{m} x_i^{\mu_i}\right)^{\frac{1}{U_m}}.$$

Substituting the above values in (3.2.5) provides the required result. □

Theorem 3.2.4. *For $x_i \in (0, \frac{1}{2}]$ $(1 \leq i \leq m)$ and $A_m = \sum_{i=1}^{m}\mu_i x_i$, $G_m = \prod_{i=1}^{m} x_i^{\mu_i}$ and $A'_m = \sum_{i=1}^{m}\mu_i(1-x_i)$, $G'_m = \prod_{i=1}^{m}(1-x_i)^{\mu_i}$ are the arithmetic and geometric* *means of x_i and $1-x_i$, respectively. Then*

$$\frac{A'_m}{A_m} \geq \left(\prod_{j=1}^{n} \left[\frac{I\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i(1-x_i), \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i(1-x_i)\right)}{I\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i x_i, \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i x_i\right)} \right] \right)^{\frac{1}{\Lambda_n}} \geq \left(\frac{G'_m}{G_m}\right)^{\frac{1}{U_m}}, \tag{3.2.7}$$

where the identric mean is defined in (1.7.1).

Proof. Consider a convex mapping $\Psi(t) = \ln \frac{1-t}{t}$ on $(0, \frac{1}{2}]$, and $A(\Psi; \gamma, \delta) = \ln \frac{I(1-\gamma, 1-\delta)}{I(\gamma,\delta)}$ $\gamma, \delta \in (0,1)$. We then have

$$\frac{1}{\Lambda_n}\sum_{j=1}^{n} A\left(\Psi;\gamma,\delta\right) = \ln \left(\prod_{j=1}^{n} \left[I\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i[1-x_i], \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i[1-x_i] \right) \right]^{\lambda_i} \right)^{\frac{1}{\Lambda_n}},$$

where $\gamma = \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i x_i$, $\delta = \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i x_i$. We substitute the above equation in (3.2.5), and by considering

$$\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_i x_i\right) = \ln\frac{A'_m}{A_m}, \ \frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(x_i) = \ln\left(\frac{G'_m}{G_m}\right)^{\frac{1}{U_m}},$$

the desired result is obtained. □

Theorem 3.2.5. *Assume (X, A, λ) is a measure space, $p \geq 1$, and $h_1, \ldots, h_m \in L^p(\lambda)$, then*

$$\begin{aligned} \left\| \frac{1}{U_m}\sum_{i=1}^{m}\mu_i h_i \right\|_p^p &\geq \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j \left\| L_p^p\left(\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_i|h_i|, \frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_i|h_i| \right) \right\|_1 \\ &\geq \frac{1}{U_m}\sum_{i=1}^{m}\mu_i \left\|h_i\right\|_p^p, \end{aligned} \tag{3.2.8}$$

where L_p is the p-logarithmic mean.

Proof. Consider the convex function $\Psi : L^p \to \mathbb{R}$ defined as: $\Psi(h) = \|h\|_p^p$. We observe that the mapping $X \times [0,1] \to \mathbb{R}$ with $(x,l) \to \frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j) + l\varsigma_2(i,j)]\mu_i h_i(x)$ is product measurable. Since $\left|\frac{1}{U_m}\sum_{i=1}^{m}\mu_i h_i\right| \leq \frac{1}{U_m}\sum_{i=1}^{m}\mu_i |h_i|$ and for $1 \leq i \leq m$, the L^p-norm of h_i and $|h_i|$ are equal. It is sufficient to consider $h_i \geq 0$ $(1 \leq i \leq m)$. By applying the change of variable $u = \frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j) + l\varsigma_2(i,j)]\mu_i h_i$ and using Fubini's theorem yield

$$\begin{aligned}\int_0^1 \psi_{\varsigma_1,\varsigma_2}(l)dl &= \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j \int_0^1\int_X \left(\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j) + l\varsigma_2(i,j)]\mu_i h_i\right)^p d\mu dl \\ &= \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j \int_X L_p^p\left(\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j) + l\varsigma_2(i,j)]\mu_i h_i\right) d\mu \\ &= \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j \left\|L_p^p\left(\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j) + l\varsigma_2(i,j)]\mu_i h_i\right)\right\|_1,\end{aligned}$$

which by substituting in (3.2.5), we get the required result. □

Remark 3.2.1. [56] Assume (X, A, λ) is a finite measure space and let Z be the vector space of all measurable functions on X with point-wise operations. Let $S \subseteq Z$ contains all non negative measurable functions on X, then S is convex. Since the function $l \to \frac{l}{l+1}$ $(l \geq 0)$ is concave, the mapping $\Psi : S \to \mathbb{R}$ with

$$\Psi(h) = \int_X \frac{h}{1+h} d\lambda \ \ (h \in S) \tag{3.2.9}$$

is concave.

Theorem 3.2.6. *Under the notation of Remark 3.2.1, let $h_1, \ldots, h_m \in S$, and Ψ is* *defined in* (3.2.9)*, then*

$$\frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi(h_i) \geq \lambda(X) - \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\left\|L^{-1}\left(1+\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_ih_i, 1+\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_ih_i\right)\right\|_1$$
$$\geq \Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_ih_i\right), \tag{3.2.10}$$

where L is the logarithmic mean.

Proof. It is obvious $X\times[0,1]\to\mathbb{R}$ with $(x,l)\to\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j)+l\varsigma_2(i,j)]\mu_ix_i$ is product measurable. Since Ψ is concave, so $-\Psi$ is convex. Thus from (3.2.5), yields

$$\Psi\left(\frac{1}{U_m}\sum_{i=1}^{m}\mu_ih_i\right) \geq \int_0^1\psi_{\varsigma_1,\varsigma_2}(l)dl \geq \frac{1}{U_m}\sum_{i=1}^{m}\mu_i\Psi\left(h_i\right),$$

where

$$\int_0^1\psi_{\varsigma_1,\varsigma_2}(l)dl = \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\int_X\int_0^1\frac{\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j)+l\varsigma_2(i,j)]\mu_ih_i}{1+\frac{1}{U_m}\sum_{i=1}^{m}[(1-l)\varsigma_1(i,j)+l\varsigma_2(i,j)]\mu_ih_i}dld\lambda$$
$$= \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\int_X\frac{1}{\delta-\gamma}\int_\gamma^\delta\left(1-\frac{1}{1+l}\right)dld\lambda(x)$$
$$= \lambda(X) - \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\int_X\frac{1}{\delta-\gamma}\ln\frac{1+\delta}{1+\gamma}d\lambda$$
$$= \lambda(X) - \frac{1}{\Lambda_n}\sum_{j=1}^{n}\lambda_j\left\|L^{-1}\left(1+\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_ih_i, 1+\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_ih_i\right)\right\|_1,$$

where $\gamma=\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_1(i,j)\mu_ih_i(x)$ and $\delta=\frac{1}{U_m}\sum_{i=1}^{m}\varsigma_2(i,j)\mu_ih_i(x)$. □

3.3 m-exponential convexity of weighted Jessen's functionals and mean value theorems

The following theorem is the refinement of Jessen's inequality.

Theorem 3.3.1. *Assume L satisfies properties L_1 and L_2 on a nonempty set E, and suppose that Ψ is a continuous convex function on an interval $S \subseteq \mathbb{R}$. If B and Z are positive linear functionals with $B(1) = Z(1) = 1$ and ς is a weight function (defined in* (1.5.1)*) then for all $h, h \cdot \varsigma \in L$ such that $\Psi(B(h \cdot \varsigma)), \Psi(h) \in L$ we have $B(h \cdot \varsigma), B(h) \in S$ and*

$$\Psi(B(h)) \leq Z(\Psi(B(h \cdot \varsigma))) \leq B(\Psi(h)). \tag{3.3.1}$$

Proof.

$$Z(\Psi(B(h \cdot \varsigma))) \geq \Psi(Z(B(h \cdot \varsigma))) \geq \Psi(Z(B(h))) = \Psi(B(h)).$$

Suppose $S = [c, d]$. Since $c \leq h(l) \leq d$ for every $l \in E$, it yields $c \leq B(h) \leq d$ and $c \leq B(h \cdot \varsigma) \leq d$. For arbitrary but a positive fixed ε there exist $\mu, \nu \in \mathbb{R}$ such that for $\varrho = \mu\varrho_0 + \nu\varrho_1$ ($\varrho_j(l) = l^j$ for $j = 0, 1$) we then have

(a) $\varrho \leq \Psi$, it yields $\varrho(B(h)) \leq \Psi(B(h \cdot \varsigma))$.

(b) $\varrho(B(h)) \geq \Psi(B(h \cdot \varsigma)) - \epsilon$ which implies $Z(\varrho(B(h))) \geq Z(\Psi(B(h \cdot \varsigma))) - \varepsilon$.

(If $c < B(h) < d$ and $c < B(h \cdot \varsigma) < d$ or if $h, h \cdot \varsigma$ have finite derivatives in $[c, d]$, we can replace (b) by $\varrho(B(h)) = \Psi(B(h \cdot \varsigma))$). Now (a) implies $\varrho \circ h \leq \Psi \circ h$; hence

$$\begin{aligned} B(\Psi \circ h) \geq B(\varrho \circ h) &= \mu + \nu B(h) \\ &= Z(\varrho(B(h))) \geq Z(\Psi(B(h \cdot \varsigma))) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, hence we complete the proof. □

If $\varsigma = 1$, then we deduce the following result.

Corollary 3.3.2. *Assume L satisfies properties L_1 and L_2 on a nonempty set E, and assume that Ψ is a continuous convex function on an interval $S \subseteq \mathbb{R}$. If B and Z are positive linear functionals with $B(1) = Z(1) = 1$, then for all $h \in L$ such that $\Psi(B(h)), \Psi(h) \in L$ we have $B(h) \in S$ and*

$$\Psi(B(h)) \leq Z(\Psi(B(h))) \leq B(\Psi(h)). \tag{3.3.2}$$

Remark 3.3.1. We consider the following functionals under the assumption of Theorem 3.3.1

$$\Lambda(\Psi) = Z(\Psi(B(h \cdot \varsigma))) - \Psi(B(h)). \tag{3.3.3}$$

$$\daleth(\Psi) = B(\Psi(h)) - Z(\Psi(B(h \cdot \varsigma))). \tag{3.3.4}$$

Then $\Lambda(\Psi)$ and $\Lambda'(\Psi)$ are positive.

In the following theorem and its corollaries we use notations as: $\mathcal{C}$ is a real open interval, S is any interval, $\Theta = \{g_l \mid l \in \mathcal{C}\}$ is a family of continuous functions defined on S. $\Lambda(\Psi)$ is as given in Remark 3.3.1. Then the following results produce new m-exponentially convex functions. Since the proofs are analogous to those in the continuous case given in the previous section, so we omit the proofs.

Theorem 3.3.3. *Suppose the function $l \mapsto [s_1, s_2, s_3; g_l]$ is m-exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Lambda(g_l)$ is an m-exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. In addition, the continuity of this function implies m-exponential convexity on $\mathcal{C}$.*

Corollaries stated below have been obtained from the above theorem.

Corollary 3.3.4. *Suppose the function $l \mapsto [s_1, s_2, s_3; g_l]$ is exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Lambda(g_l)$ is an exponentially*

$\mathcal{J}$*-convex function on* $\mathcal{C}$*. In addition, the continuity of this function implies the exponential convexity on* $\mathcal{C}$*.*

Corollary 3.3.5. *Suppose the function* $l \mapsto [s_1, s_2, s_3; g_l]$ *is 2-exponentially* $\mathcal{J}$*-convex on* $\mathcal{C}$*, where* s_1, s_2, s_3 *are distinct points of* S*. Then* $l \mapsto \Lambda(g_l)$ *is 2-exponentially* $\mathcal{J}$*-convex function on* $\mathcal{C}$*. In addition, the continuity of this function implies the exponential convexity on* $\mathcal{C}$ *and thus the function is log-convex. That is, for* $l, r, s \in \mathcal{C}$ *such that* $r < s < l$*. the following holds*

$$\Lambda^{l-r}(g_s) \leq \Lambda^{l-s}(g_r)\Lambda^{s-r}(g_l).$$

Now we present different families of functions to investigate exponential convexity.

Remark 3.3.2. The following positive functionals are useful in defining the basic inequality of log-convex functions.

$$\Lambda(f_l) = \frac{1}{l^2}\left(Z\left(\exp\left(lB(h\cdot\varsigma)\right)\right) - \exp(lB(h))\right).$$
$$\Lambda(g_l) = \frac{1}{l(l-1)}\left(Z\left((B(h\cdot\varsigma))^l\right) - (B(h))^l\right).$$
$$\Lambda(h_l) = \frac{1}{(\log l)^2}\left(Z\left(l^{-B(h\cdot\varsigma)}\right) - l^{-B(h)}\right).$$
$$\Lambda(k_l) = \frac{1}{l}\left(Z\left(\exp\left(-B(h\cdot\varsigma)\sqrt{l}\right)\right) - \exp\left(-B(h)\sqrt{l}\right)\right).$$

Theorem 3.3.6. *Let* $\Lambda(\Psi)$ *be the linear functional defined by* (3.3.3) *and consider the function* $\phi_i : (0,\infty) \to \mathbb{R}$ *for* $i = 1, 4$ *and* $\phi_i : (1,\infty) \to \mathbb{R}$ *for* $i = 2, 3$ *defined by*

$\phi_1(l) = \Lambda(f_l),\ \phi_2(l) = \Lambda(g_l),\ \phi_3(l) = \Lambda(h_l),\ \phi_4(l) = \Lambda(k_l)$*, where* f_l, g_l, h_l *and* k_l *are defined in Lemma 1.3.3. Then*

(i) The functions ϕ_i *are continuous on* $(0,\infty)$ *for* $i = 1, 4$ *and continuous on* $(1,\infty)$ *for* $i = 2, 3$*.*

(ii) *If* $m \in \mathbb{N}$, $l_1, \ldots, l_m \in (0,\infty)$ *for* $i = 1,4$ *and* $l_1, \ldots, l_m \in (1,\infty)$ *for* $i = 2,3$, *then the matrices*

$$\left[\phi_i\left(\frac{l_j + l_k}{2}\right)\right]_{j,k=1}^{m}$$

are positive semidefinite.

(iii) *The functions* ϕ_i *are exponentially convex on* $(0,\infty)$ *for* $i = 1,4$ *and exponentially convex on* $(1,\infty)$ *for* $i = 2,3$.

(iv) *Let* $l, r, s \in (0,\infty)$ *for* $i = 1,4$ *and* $r, s, l \in (1,\infty)$ *for* $i = 2,3$ *with* $l < r < s$, *then*

$$(\phi_i(r))^{s-l} \leq (\phi_i(l))^{s-r}\,(\phi_i(s))^{r-l},$$

where $\phi_i(l)$ *for* $i = 1,2,3,4$ *are defined in Remark 3.3.2.*

The Cauchy and Lagrange type mean value theorems are stated below.

Theorem 3.3.7. *Assume* L *satisfies properties* L_1 *and* L_2 *on a nonempty set* E, *and let* $\Psi : S = [a,b] \to \mathbb{R}, \Psi \in C^2(S)$, *where* $S \subseteq \mathbb{R}$ *is a compact real interval. If* B *and* Z *are positive linear functionals with* $B(1) = Z(1) = 1$, *and* ς *is a weight function (defined in* (1.5.1)*) then for all* $h, h \cdot \varsigma \in L$ *such that* $\Psi(B(h \cdot \varsigma)), (B(h \cdot \varsigma))^2 \in L$ *there exists some* $\varpi \in S$ *such that the following holds*

$$Z(\Psi(B(h \cdot \varsigma))) - \Psi(B(h)) = \frac{\Psi''(\varpi)}{2}\left[Z((B(h \cdot \varsigma))^2) - (B(h))^2\right]. \tag{3.3.5}$$

Theorem 3.3.8. *Assume* L *satisfies properties* L_1 *and* L_2 *on a nonempty set* E, *and let* $\Phi, \Psi : S = [a,b] \to \mathbb{R}, \Phi, \Psi \in C^2(S)$, *where* $S \subseteq \mathbb{R}$ *is a compact real interval. If* B *and* Z *are positive linear functionals with* $B(1) = 1$ *and* $Z(1) = 1$, *and* ς *is a weight function (defined in* (1.5.1)*) then for all* $h, h \cdot \varsigma \in L$ *such that*

$\Phi(B(h\cdot\varsigma)), \Psi(B(h\cdot\varsigma)), (B(h\cdot\varsigma))^2 \in L$ *and* $Z((B(h\cdot\varsigma))^2) - (B(h))^2 \neq 0$ *there exists some* $\varpi \in S$ *such that the following holds*

$$\Psi''(\varpi)\left[Z(\Phi(B(h\cdot\varsigma))) - \Phi(B(h))\right] = \Phi''(\varpi)\left[Z(\Psi(B(h\cdot\varsigma))) - \Psi(B(h))\right]. \quad (3.3.6)$$

Remark 3.3.3. We can construct similar results for the positive functional $\daleth(\Psi)$ defined in (3.3.4).

Chapter 4

Weighted Jessen's inequality for $m(M)$-ψ-convex functions and related results

In analysis, positive linear functionals are very useful as they are natural objects. By introducing weight function, an important functional form of Jensen's inequality, that is, the Jessen inequality is given in the previous chapter.

This chapter discusses few interesting results related with weighted Jessen inequality (3.3.1) for $m(M)$-ψ-convex functions. Throughout chapter, we assume ς is a weight function which satisfies (1.5.1) and use the following notations:

$$\begin{aligned}
F_1 &= Z(\psi(B(h\cdot\varsigma))) - \psi(B(h)),\\
F_2 &= Z(\Phi(B(h\cdot\varsigma))) - \Phi(B(h)),\\
F_3 &= B(\psi(h)) - Z(\psi(B(h\cdot\varsigma))),\\
F_4 &= B(\Phi(h)) - Z(\Phi(B(h\cdot\varsigma))).
\end{aligned}$$

4.1 Refined Jessen's inequality

Theorem 4.1.1. *Suppose that $\psi : S \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $h : E \to S$ such that $\psi(B(h \cdot \varsigma)), \psi(h), h \cdot \varsigma, h \in L$, ς is weight function and $B, Z : L \to \mathbb{R}$ are isotonic linear and normalised functionals.*

(*i*) *Assume $\Phi \in \mathfrak{L}(S, m, \psi)$ and $\Phi(B(h \cdot \varsigma)), \Phi(h) \in L$, then*

$$mF_1 \leq F_2, \tag{4.1.1}$$

$$mF_3 \leq F_4. \tag{4.1.2}$$

(*ii*) *Assume $\Phi \in \mathfrak{U}(S, M, \psi)$ and $\Phi(B(h \cdot \varsigma)), \Phi(h) \in L$, then*

$$F_2 \leq MF_1, \tag{4.1.3}$$

$$F_4 \leq MF_3. \tag{4.1.4}$$

(*iii*) *Assume $\Phi \in \mathfrak{B}(S, m, M, \psi)$ and $\Phi(B(h \cdot \varsigma)), \Phi(h) \in L$, then above inequalities* (4.1.1)-(4.1.4) *hold.*

Proof. (i) Since $\Phi \in \mathfrak{L}(S, m, \psi)$ and $\Phi(B(h \cdot \varsigma)) \in L$, this yield $\Phi - m\psi$ is convex and $(\Phi - m\psi) \circ B(h \cdot \varsigma) \in L$. For the convex function $\Phi - m\psi$, the refined Jessen's inequality (3.3.1) is applied to obtain the required results.
(ii) Similarly, we observe $\Phi(B(h \cdot \varsigma)) \in L$ and $\Phi \in \mathfrak{U}(S, M, \psi)$. It yields $M\psi - \Phi$ is convex and $(M\psi - \Phi) \circ B(h \cdot \varsigma) \in L$.
(iii) It can be easily observed that this is true from (i) and (ii). □

We have obtained the below corollary from above theorem.

Corollary 4.1.2. *Assume the second order derivative of convex function $\psi : S \subseteq \mathbb{R} \to \mathbb{R}$ exists on $\overset{\circ}{S}$ (interior of S) and $h : E \to S$ such that $\psi(B(h \cdot \varsigma)), \psi(h), h \cdot \varsigma, h \in L$, ς is weight function and $B, Z : L \to \mathbb{R}$ are isotonic linear and normalised functionals.*

(i) *If second order derivative of* $\Phi : S \to \mathbb{R}$ *exists and for given real number* m, $\Phi''(l) \geq m\psi''(l), l \in \overset{\circ}{S}$. *Then* (4.1.1) *and* (4.1.2) *are satisfied under the assumption that* $\Phi(B(h \cdot \varsigma)), \Phi(h) \in L$.

(ii) *If second order derivative of* $\Phi : S \to \mathbb{R}$ *exists and for given real number* m, $\Phi''(l) \leq M\psi''(l), l \in \overset{\circ}{S}$. *Then* (4.1.3) *and* (4.1.4) *are satisfied under the assumption that* $\Phi(B(h \cdot \varsigma)), \Phi(h) \in L$.

(iii) *If second order derivative of* $\Phi : S \to \mathbb{R}$ *exists and for given real number* m, $m\psi''(l) \leq \Phi''(l) \leq M\psi''(l), l \in \overset{\circ}{S}$. *Then* (4.1.1)-(4.1.4) *are satisfied under the assumption that* $\Phi(B(h \cdot \varsigma)), \Phi(h) \in L$.

The following propositions are some particular cases of the above corollary.

Proposition 4.1.3. *Let the second order derivative of* $\Phi : S \to \mathbb{R}$ *exists on* $\overset{\circ}{S}$.

(i) *For* $\inf_{l \in \overset{\circ}{S}} \Phi''(l) = s > -\infty$, *we have*

$$\begin{aligned} \frac{s}{2}\left[Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right] &\leq F_2, \\ \frac{s}{2}\left[B(h^2) - Z([B(h \cdot \varsigma)]^2)\right] &\leq F_4, \end{aligned} \tag{4.1.5}$$

under the assumption that $\Phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^2, h^2, \Phi \circ h, h \cdot \varsigma, h \in L$.

(ii) *For* $\sup_{l \in \overset{\circ}{S}} \Phi''(l) = S < \infty$, *we have*

$$\begin{aligned} F_2 &\leq \frac{S}{2}\left[Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right], \\ F_4 &\leq \frac{S}{2}\left[B(h^2) - Z([B(h \cdot \varsigma)]^2)\right], \end{aligned} \tag{4.1.6}$$

under the assumption that $\Phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^2, h^2, \Phi \circ h, h \cdot \varsigma, h \in L$.

(iii) *For* $-\infty < s \leq \Phi''(l) \leq S < \infty, l \in \overset{\circ}{S}$, *both* (4.1.5) *and* (4.1.6) *hold, under the assumption that* $\Phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^2, h^2, \Phi \circ h, h \cdot \varsigma, h \in L$.

The proof is followed by applying the Corollary 4.1.2 for $\psi(l) = \frac{1}{2}l^2$ and $m = s, M = S$.

Proposition 4.1.4. *Suppose that the the second order derivative of* $\Phi : S \subseteq \mathbb{R}^+ \to \mathbb{R}$ *exists on* $\overset{\circ}{S}$. *Let* $h_q : S \to \mathbb{R}$ *defined by* $h_q(l) = \Phi''(l)l^{2-q}$, *where* $q \in (-\infty, 0) \cup (1, \infty)$.

(i) For $\inf_{l \in \overset{\circ}{S}} h_q(l) = k > -\infty$, *we get*

$$\begin{aligned} \frac{k}{q(q-1)} \left[Z\left([B(h \cdot \varsigma)]^q \right) - [B(h)]^q \right] &\leq F_2, \\ \frac{k}{q(q-1)} \left[B(h^q) - Z([B(h \cdot \varsigma)]^q) \right] &\leq F_4, \end{aligned} \tag{4.1.7}$$

assuming that $\phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^q, f^q, \Phi \circ h, h \cdot \varsigma, h \in L$.

(ii) For $\sup_{l \in \overset{\circ}{S}} h_q(l) = K < \infty$, *we have*

$$\begin{aligned} F_2 &\leq \frac{K}{q(q-1)} \left[Z\left([B(h \cdot \varsigma)]^q \right) - [B(h)]^q \right], \\ F_4 &\leq \frac{K}{q(q-1)} \left[B(h^q) - Z([B(h \cdot \varsigma)]^q) \right], \end{aligned} \tag{4.1.8}$$

assuming that $\Phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^q, f^q, \Phi \circ h, h \cdot \varsigma, h \in L$.

(iii) For $-\infty < k \leq h_q(l) \leq K < \infty, l \in \overset{\circ}{S}$, *both* (4.1.7) *and* (4.1.8) *hold, under the assumption that* $\Phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^q, h^q, \Phi \circ h, h \cdot \varsigma, h \in L$.

Proof. (i) Define the auxiliary function $g_q(l) = \Phi(l) - \frac{k}{q(q-1)} l^q$. Then

$$\begin{aligned} g_q''(l) &= \Phi''(l) - kl^{q-2} = l^{q-2}(l^{2-q}\Phi''(l) - k) \\ &= l^{q-2}(h_q(l) - k) \geq 0. \end{aligned}$$

which implies g_q is convex, that is, $\Phi \in \mathfrak{L}(S, \frac{k}{q(q-1)}, (\cdot)^q)$. We deduce (4.1.7) by applying Corollary 4.1.2. Analogously, we can prove (ii), and (iii) follows by (i) and (ii). □

We state another result.

Proposition 4.1.5. *Assume that the second order derivative of mapping* $\Phi : S \subseteq \mathbb{R}^+ \to \mathbb{R}$ *exists on* $\overset{\circ}{S}$. *Define* $z(l) = l^2\Phi''(l), l \in S$.

(i) For $\inf_{l\in\overset{\circ}{S}} z(l) = \gamma > -\infty$, *we then have*

$$\begin{aligned} \gamma\left[\ln[B(h)] - Z\left(\ln\left[B(h\cdot\varsigma)\right]\right)\right] &\leq F_2, \\ \gamma\left[Z\left(\ln\left[B(h\cdot\varsigma)\right]\right) - B(\ln h)\right] &\leq F_4, \end{aligned} \tag{4.1.9}$$

assuming that $\Phi(B(h\cdot\varsigma)), \ln\left[B(h\cdot\varsigma)\right], \ln h, \Phi\circ h, h\cdot\varsigma, h \in L$ *and* $B(h\cdot\varsigma), B(h) > 0$.

(ii) For sup$_{l\in\overset{\circ}{S}} z(l) = \Gamma < \infty$, *we get*

$$\begin{aligned} F_2 &\leq \Gamma\left[\ln[B(h)] - Z\left(\ln\left[B(h\cdot\varsigma)\right]\right)\right], \\ F_4 &\leq \Gamma\left[Z\left(\ln\left[B(h\cdot\varsigma)\right]\right) - B(\ln h)\right], \end{aligned} \tag{4.1.10}$$

assuming that $\Phi(B(h\cdot\varsigma)), \ln\left[B(h\cdot\varsigma)\right], \ln h, \Phi\circ h, h\cdot\varsigma, h \in L$ *and* $B(h\cdot\varsigma), B(h) > 0$.

(iii) For $-\infty < \gamma \leq z(l) \leq \Gamma < \infty$ *with* $l \in \overset{\circ}{S}$, *both* (4.1.11) *and* (4.1.12) *hold, under the assumption that* $\Phi(B(h\cdot\varsigma)), \ln\left[B(h\cdot\varsigma)\right], \ln h, \Phi\circ h, h\cdot\varsigma, h \in L$ *and* $B(h\cdot\varsigma), B(h) > 0$.

Define the auxiliary function $g(l) = \Phi(l) + \gamma \ln l$. Analogously as in the proof of Proposition 4.1.4, we get the required results.

Proposition 4.1.6. *Assume that the second order derivative of* $\Phi : S \subseteq \mathbb{R}^+ \to \mathbb{R}$ *exists on* $\overset{\circ}{S}$. *Define* $\sigma(l) = l\Phi''(l), l \in S$.

(i) For $\inf_{l\in \overset{\circ}{S}} \sigma(l) = \xi > -\infty$*, we have*

$$\begin{aligned} \xi \left[B(h\cdot\varsigma)Z\left(\ln\left[B(h\cdot\varsigma)\right]\right) - B(h)\ln B(h)\right] &\leq F_2, \\ \xi \left[B(h\ln h) - B(h\cdot\varsigma)Z\left(\ln\left[B(h\cdot\varsigma)\right]\right)\right] &\leq F_4, \end{aligned} \tag{4.1.11}$$

assuming that $\Phi(B(h\cdot\varsigma)), \ln\left[B(h\cdot\varsigma)\right], h\ln h, \Phi\circ h, h\cdot\varsigma, h \in L$ *and* $B(h\cdot\varsigma), B(h) > 0$.

(ii) For $\sup_{l\in \overset{\circ}{S}} \sigma(l) = \Xi < \infty$*, we get*

$$\begin{aligned} F_2 &\leq \Xi\left[B(h\cdot\varsigma)Z\left(\ln\left[B(h\cdot\varsigma)\right]\right) - B(h)\ln B(h)\right], \\ F_4 &\leq \Xi\left[B(h\ln h) - B(h\cdot\varsigma)Z\left(\ln\left[B(h\cdot\varsigma)\right]\right)\right], \end{aligned} \tag{4.1.12}$$

assuming that $\Phi(B(h\cdot\varsigma)), \ln\left[B(h\cdot\varsigma)\right], h\ln h, \Phi\circ h, h\cdot\varsigma, h \in L$ *and* $B(h\cdot\varsigma), B(h) > 0$.

(iii) For $-\infty < \xi \leq \sigma(l) \leq \Xi < \infty$ *with* $l \in \overset{\circ}{S}$*, both* (4.1.11) *and* (4.1.12) *hold, under the assumption that* $\Phi(B(h\cdot\varsigma)), \ln\left[B(h\cdot\varsigma)\right], h\ln h, \Phi\circ h, h\cdot\varsigma, h \in L$ *and* $B(h\cdot\varsigma), B(h) > 0$.

Define the auxiliary mapping $g(l) = \Phi(l) - \xi l\ln l$. Analogously as in the proof of Proposition 4.1.4, we get the desired results.

4.2 Some applications

From Proposition 4.1.3, we know that

$$\frac{s}{2}\left[Z\left(\left[B(h\cdot\varsigma)\right]^2\right) - \left[B(h)\right]^2\right] \leq F_2 \leq \frac{S}{2}\left[Z\left(\left[B(h\cdot\varsigma)\right]^2\right) - \left[B(h)\right]^2\right]. \tag{4.2.1}$$

and

$$\frac{s}{2}\left[B(h^2) - Z(\left[B(h\cdot\varsigma)\right]^2)\right] \leq F_4 \leq \frac{S}{2}\left[B(h^2) - Z(\left[B(h\cdot\varsigma)\right]^2)\right], \tag{4.2.2}$$

by assuming that the second order derivative of $\Phi : S \subseteq \mathbb{R} \to \mathbb{R}$ exists on $\overset{\circ}{S}, -\infty < s \leq \Phi''(l) \leq S < \infty, l \in \overset{\circ}{S}, h : E \to S, \Phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^2, h^2, \Phi \circ h, h \cdot \varsigma, h \in L$ and ς is a weight function $B, Z : L \to \mathbb{R}$ are isotonic linear and normalised functionals.

Now, we use (4.2.1) and (4.2.2) and various functions to give some useful applications.

Proposition 4.2.1. *Suppose that* $m, M \in \mathbb{R}$, $0 < \alpha \leq h \leq \beta < \infty$ *and* $0 < \alpha \leq h \cdot \varsigma \leq \beta < \infty$. *Then from* (4.2.1) *and* (4.2.2) *with* $\Phi : [\alpha, \beta] \to \mathbb{R}, \Phi(l) = -\ln l$, *we then have*

$$\begin{aligned} \frac{1}{2\beta^2}\left[Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right] &\leq \ln[B(h)] - Z\left(\ln[B(h \cdot \varsigma)]\right) \\ &\leq \frac{1}{2\alpha^2}\left[Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right], \end{aligned} \tag{4.2.3}$$

and

$$\begin{aligned} \frac{1}{2\beta^2}\left[B(h^2) - Z([B(h \cdot \varsigma)]^2)\right] &\leq Z\left(\ln[B(h \cdot \varsigma)]\right) - B[\ln(h)] \\ &\leq \frac{1}{2\alpha^2}\left[B(h^2) - Z([B(h \cdot \varsigma)]^2)\right], \end{aligned} \tag{4.2.4}$$

assuming that $\ln[B(h \cdot \varsigma)], [B(h \cdot \varsigma)]^2, h^2, \ln h, h \cdot \varsigma, h \in L$ *and* $B(h), B(h \cdot \varsigma) > 0$.

We observe inequality (4.2.3) is equivalent to

$$\begin{aligned} \exp\left[\frac{1}{2\beta^2}\left(Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right)\right] &\leq \frac{B(h)}{\exp\left[Z\left(\ln[B(h \cdot \varsigma)]\right)\right]} \\ &\leq \exp\left[\frac{1}{2\alpha^2}\left(Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right)\right], \end{aligned} \tag{4.2.5}$$

Proposition 4.2.2. *For* $q \in (-\infty, 0) \cup (1, \infty)$, *if we apply* (4.2.1) *and* (4.2.2) *with* $\Phi : [\alpha, \beta] \to \mathbb{R}, \Phi(l) = l^q$. *Then for* $q > 2$ *we have*

$$\begin{aligned} \frac{q(q-1)}{2}\alpha^{q-2}\left[Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right] &\leq Z\left([B(h \cdot \varsigma)]^q\right) - [(B(h))]^q \\ &\leq \frac{q(q-1)}{2}\beta^{q-2}\left[Z\left([B(h \cdot \varsigma)]^2\right) - [B(h)]^2\right], \end{aligned} \tag{4.2.6}$$

and

$$\frac{q(q-1)}{2}\alpha^{q-2}\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right]\leq B(h^q)-Z\left([(B(h\cdot\varsigma)]^q\right)\\\leq\frac{q(q-1)}{2}\beta^{q-2}\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right]. \tag{4.2.7}$$

Now, for $q\in(-\infty,0)\cup(1,2)$ *we have*

$$\frac{q(q-1)}{2}\beta^{q-2}\left[Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right]\leq Z\left([B(h\cdot\varsigma)]^q\right)-[(B(h))]^q\\\leq\frac{q(q-1)}{2}\alpha^{q-2}\left[Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right], \tag{4.2.8}$$

and

$$\frac{q(q-1)}{2}\beta^{q-2}\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right]\leq B(h^q)-Z\left([(B(h\cdot\varsigma)]^q\right)\\\leq\frac{q(q-1)}{2}\alpha^{q-2}\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right], \tag{4.2.9}$$

under the assumption that $[B(h\cdot\varsigma)]^q,[B(h\cdot\varsigma)]^2,h^2,h^q,h\cdot\varsigma,h\in L$.

Proposition 4.2.3. *For* $\Phi:[\alpha,\beta]\to\mathbb{R},\Phi(l)=l\ln l$, (4.2.1) *and* (4.2.2) *imply the following results:*

$$\frac{1}{2\beta}\left[Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right]\leq B(h\cdot\varsigma)Z\left(\ln[B(h\cdot\varsigma)]\right)-B(h)\ln[B(h)]\\\leq\frac{1}{2\alpha}\left[Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right], \tag{4.2.10}$$

and

$$\frac{1}{2\beta}\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right]\leq B(h\ln h)-B(h\cdot\varsigma)Z\left(\ln[B(h\cdot\varsigma)]\right)\\\leq\frac{1}{2\alpha}\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right], \tag{4.2.11}$$

assuming that $\ln[B(h\cdot\varsigma)],[B(h\cdot\varsigma)]^2,h^2,h\ln h,h\cdot\varsigma,h\in L$ *and* $B(h),B(h\cdot\varsigma)>0$.

It can be noted that (4.2.10) is equivalent to

$$\begin{aligned}\exp\left[\frac{1}{2\beta}\left(Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right)\right]&\le\frac{\exp[B(h\cdot\varsigma)Z\left(\ln[B(h\cdot\varsigma)]\right)]}{[B(h)]^{B(h)}}\\&\le\exp\left[\frac{1}{2\alpha}\left(Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right)\right].\end{aligned}\tag{4.2.12}$$

Proposition 4.2.4. *Assume* $-\infty<\alpha\le h\le\beta<\infty$ *and* $-\infty<\alpha\le h\cdot\varsigma\le\beta<\infty$. *The inequalities* (4.2.1) *and* (4.2.2) *for* $\Phi(l)=e^l, l\in\mathbb{R}$ *imply the following*

$$\begin{aligned}\frac{1}{2}\exp(\alpha)\left[Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right]&\le Z\left(\exp(B(h\cdot\varsigma))\right)-\exp(B(h))\\&\le\frac{1}{2}\exp(\beta)\left[Z\left([B(h\cdot\varsigma)]^2\right)-[B(h)]^2\right],\end{aligned}\tag{4.2.13}$$

and

$$\begin{aligned}\frac{1}{2}\exp(\alpha)\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right]&\le B(\exp(h))-Z\left(\exp(B(h\cdot\varsigma))\right)\\&\le\frac{1}{2}\exp(\beta)\left[B(h^2)-Z([B(h\cdot\varsigma)]^2)\right],\end{aligned}\tag{4.2.14}$$

assuming that $\exp(B(h\cdot\varsigma)), [B(h\cdot\varsigma)]^2, h^2, \exp(h), h\cdot\varsigma, h\in L$.

From Proposition 4.1.4, we have

$$\frac{k}{q(q-1)}\left[Z\left([B(h\cdot\varsigma)]^q\right)-[B(h)]^q\right]\le F_2\le\frac{K}{q(q-1)}\left[Z\left([B(h\cdot\varsigma)]^q\right)-[B(h)]^q\right],\tag{4.2.15}$$

and

$$\frac{k}{q(q-1)}\left[B(h^q)-Z([B(h\cdot\varsigma)]^q)\right]\le F_4\le\frac{K}{q(q-1)}\left[B(h^q)-Z([B(h\cdot\varsigma)]^q)\right]\tag{4.2.16}$$

by assuming that the second order derivative of $\Phi:S\subseteq\mathbb{R}^+\to\mathbb{R}$ exists on $\overset{\circ}{S}, k\le\Phi''(l)l^{2-q}\le K, l\in\overset{\circ}{S}, h:E\to S, \Phi(B(h\cdot\varsigma)), [B(h\cdot\varsigma)]^q, h^q, \Phi\circ h, h\cdot\varsigma, h\in L$ and ς is a weight function $B,Z:L\to\mathbb{R}$ are isotonic linear and normalised functionals.

Proposition 4.2.5. *If* $0 < \alpha \leq h \leq \beta < \infty$ *and* $0 < \alpha \leq h \cdot \varsigma \leq \beta < \infty$ *and we assume that* $\Phi(l) = -\ln l$*. Then for* $q \in (-\infty, 0)$ *we get*

$$\begin{aligned} \frac{\alpha^{-q}}{q(q-1)} \left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right] &\leq \ln[B(h)] - Z\left(\ln[B(h\cdot\varsigma)]\right) \\ &\leq \frac{\beta^{-q}}{q(q-1)} \left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right], \end{aligned} \tag{4.2.17}$$

and

$$\begin{aligned} \frac{\alpha^{-q}}{q(q-1)} \left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right] &\leq Z\left(\ln[B(h\cdot\varsigma)]\right) - B(\ln(h)) \\ &\leq \frac{\beta^{-q}}{q(q-1)} \left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right]. \end{aligned} \tag{4.2.18}$$

Now for $q \in (1, \infty)$*, we obtain*

$$\begin{aligned} \frac{\beta^{-q}}{q(q-1)} \left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right] &\leq \ln[B(h)] - Z\left(\ln[B(h\cdot\varsigma)]\right) \\ &\leq \frac{\alpha^{-q}}{q(q-1)} \left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right], \end{aligned} \tag{4.2.19}$$

and

$$\begin{aligned} \frac{\beta^{-q}}{q(q-1)} \left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right] &\leq Z\left(\ln[B(h\cdot\varsigma)]\right) - B(\ln(h)) \\ &\leq \frac{\alpha^{-q}}{q(q-1)} \left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right] \end{aligned} \tag{4.2.20}$$

by assuming that $\ln[B(h\cdot\varsigma)], [B(h\cdot\varsigma)]^q, h^q, \ln h, h\cdot\varsigma, h \in L$ *and* $B(h), B(h\cdot\varsigma) > 0$.

Proposition 4.2.6. *If* $0 < \alpha \leq h \leq \beta < \infty$ *and* $0 < \alpha \leq h \cdot \varsigma \leq \beta < \infty$ *and we suppose* $\Phi(l) = l \ln l$*. Then for* $q \in (-\infty, 0)$ *we get*

$$\begin{aligned} \frac{\alpha^{1-q}}{q(q-1)} \left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right] &\leq B(h\cdot\varsigma) Z\left(\ln[B(h\cdot\varsigma)]\right) - B(h)\ln[B(h)] \\ &\leq \frac{\beta^{1-q}}{q(q-1)} \left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right], \end{aligned} \tag{4.2.21}$$

and

$$\frac{\alpha^{1-q}}{q(q-1)}\left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right] \le B(h\ln h) - B(h\cdot\varsigma)Z\left(\ln[B(h\cdot\varsigma)]\right) \le \frac{\beta^{1-q}}{q(q-1)}\left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right]. \tag{4.2.22}$$

Now for $q \in (1,\infty)$, we obtain

$$\frac{\beta^{1-q}}{q(q-1)}\left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right] \le B(h\cdot\varsigma)Z\left(\ln[B(h\cdot\varsigma)]\right) - B(h)\ln[B(h)] \le \frac{\alpha^{1-q}}{q(q-1)}\left[Z\left([B(h\cdot\varsigma)]^q\right) - [B(h)]^q\right], \tag{4.2.23}$$

and

$$\frac{\beta^{1-q}}{q(q-1)}\left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right] \le B(h\ln h) - B(h\cdot\varsigma)Z\left(\ln[B(h\cdot\varsigma)]\right) \le \frac{\alpha^{1-q}}{q(q-1)}\left[B(h^q) - Z([B(h\cdot\varsigma)]^q)\right]. \tag{4.2.24}$$

under the assumption that $\ln[B(h\cdot\varsigma)], [B(h\cdot\varsigma)]^q, h^q, h\ln h, h\cdot\varsigma, h \in L$ *and* $B(h), B(h\cdot\varsigma) > 0$.

Finally, we derive the following results by using Proposition 4.1.5.

$$\gamma\left[\ln[B(h)] - Z\left(\ln\left[B(h\cdot\varsigma)\right]\right)\right] \le F_2 \le \Gamma\left[\ln[B(h)] - Z\left(\ln\left[B(h\cdot\varsigma)\right]\right)\right], \tag{4.2.25}$$

and

$$\gamma\left[Z\left(\ln\left[B(h\cdot\varsigma)\right]\right) - B(\ln h)\right] \le F_4 \le \Gamma\left[Z\left(\ln\left[B(h\cdot\varsigma)\right]\right) - B(\ln h)\right] \tag{4.2.26}$$

by assuming that the second order derivative $\Phi : S \subseteq \mathbb{R}^+ \to \mathbb{R}$ exists on $\overset{\circ}{S}$, $-\infty < \gamma \le l^2\Phi''(l) \le \Gamma < \infty, \Phi(B(h\cdot\varsigma)), \ln\left[B(h\cdot\varsigma)\right], \ln h, \Phi\circ h, h\cdot\varsigma, h \in L$ and $B(h\cdot\varsigma), B(h) > 0$.

Proposition 4.2.7. *Let $0 < \alpha \leq h \leq \beta < \infty$ and $0 < \alpha \leq h \cdot \varsigma \leq \beta < \infty$ and consider $\Phi(l) = l \ln l$, the inequalities (4.2.25) and (4.2.26) yield*

$$\begin{aligned} \alpha \left[\ln[B(h)] - Z\left(\ln\left[B(h \cdot \varsigma)\right]\right)\right] &\leq B(h \cdot \varsigma) Z\left(\ln[B(h \cdot \varsigma)]\right) - B(h) \ln[B(h)] \\ &\leq \beta \left[\ln[B(h)] - Z\left(\ln\left[B(h \cdot \varsigma)\right]\right)\right], \end{aligned} \tag{4.2.27}$$

and

$$\begin{aligned} \alpha \left[Z\left(\ln\left[B(h \cdot \varsigma)\right]\right) - B(\ln h)\right] &\leq B(h \ln(h)) - B(h \cdot \varsigma) Z\left(\ln[B(h \cdot \varsigma)]\right) \\ &\leq \beta \left[Z\left(\ln\left[B(h \cdot \varsigma)\right]\right) - B(\ln h)\right] \end{aligned} \tag{4.2.28}$$

under the assumption that $\ln\left[B(h \cdot \varsigma)\right], \ln h, h \ln h, h \cdot \varsigma, h \in L$ and $B(h \cdot \varsigma), B(h) > 0$.

It can be observed that the inequality (4.2.27) is equivalent to

$$\begin{aligned} \left[\frac{[B(h)]}{\exp\left[Z\left(\ln\left[B(h \cdot \varsigma)\right]\right)\right]}\right]^{\alpha} &\leq \frac{\exp\left[Z\left(\ln[B(h \cdot \varsigma)]\right)^{B(h \cdot \varsigma)}\right]}{[B(h)]^{B(h)}} \\ &\leq \left[\frac{B(h)}{\exp\left[Z\left(\ln\left[B(h \cdot \varsigma)\right]\right)\right]}\right]^{\beta}. \end{aligned} \tag{4.2.29}$$

Chapter 5

Generalized form of Jessen type functionals and exponential convexity

In this paper, we introduce an extension of Jessen functional and investigate logarithmic and exponential convexity. We also present mean value theorems of Cauchy and Lagrange type. Several families of functions are also presented related to our main results.

Throughout chapter, we assume ς is a weight function which satisfies (1.5.1).

5.1 Extension of the Jessen's functional

Now we prove the counterpart of the inequality $\Psi(B(h)) \leq Z(\Psi(B(h \cdot \varsigma)))$ (see Theorem 3.3.1) for compact interval $S = [\eta, \zeta]$.

Theorem 5.1.1. *Assume* Ψ *be a convex function on* $S = [\eta, \zeta]\,(-\infty < \eta < \zeta < \infty)$*. Let* L *satisfies properties* L_1, L_2 *on a nonempty set* E*,* ς *is weight function and*

B, Z are isotonic linear normalized functional on L, then for all $h \cdot \varsigma \in L$ such that $\Psi(B(h \cdot \varsigma)) \in L$ (so that $\eta \leq B(h \cdot \varsigma) \leq \zeta$), we have

$$Z(\Psi(B(h \cdot \varsigma))) \leq \frac{\zeta - B(h \cdot \varsigma)}{\zeta - \eta}\Psi(\eta) + \frac{B(h \cdot \varsigma) - \eta}{\zeta - \eta}\Psi(\zeta). \tag{5.1.1}$$

Proof. The definition of convex function implies

$$\Psi(b) \leq \frac{c-b}{c-a}\Psi(a) + \frac{b-a}{c-a}\Psi(c) \quad (a \leq b \leq c, a < c).$$

Now set $a = \eta, b = B(h \cdot \varsigma), c = \zeta$ give

$$\Psi(B(h \cdot \varsigma)) \leq \frac{\zeta - B(h \cdot \varsigma)}{\zeta - \eta}\Psi(\eta) + \frac{B(h \cdot \varsigma) - \eta}{\zeta - \eta}\Psi(\zeta).$$

Since Z is isotonic linear and normalized functional, (5.1.1) holds. □

The next theorems are our main findings.

Theorem 5.1.2. *Assume Ψ be a convex function on $S = [\eta, \zeta]\,(-\infty < \eta < \zeta < \infty)$. Suppose L satisfies properties L_1, L_2 on a nonempty set E, ς is weight function and B, Z are isotonic linear normalized functional on L, then for all $h, h \cdot \varsigma \in L$ such that $\Psi(B(h \cdot \varsigma)), \Psi(\eta + \zeta - B(h \cdot \varsigma)) \in L$ (so that $\eta \leq B(h \cdot \varsigma) \leq \zeta$), we have*

$$\Psi(\eta + \zeta - B(h)) \leq \Psi(\eta) + \Psi(\zeta) - Z(\Psi(B(h \cdot \varsigma))). \tag{5.1.2}$$

Proof. The function $\Phi : [\eta, \zeta] \to \mathbb{R}$ defined as $\Phi(t) = \Psi(\eta + \zeta - t), t \in [\eta, \zeta]$ and the function Ψ are convex and continuous. Then by the left hand side of the inequality of (3.3.1), we have

$$\psi(B(h)) \leq Z(\psi(B(h \cdot \varsigma))).$$

That is,

$$\Psi(\eta + \zeta + B(h)) \leq Z(\Psi(\eta + \zeta - B(h \cdot \varsigma))).$$

Applying Theorem 5.1.1, we obtain

$$
\begin{aligned}
Z(\Psi(\eta+\zeta-B(h\cdot\varsigma))) &\le \frac{\zeta-B(h\cdot\varsigma)}{\zeta-\eta}\Phi(\eta)+\frac{B(h\cdot\varsigma)-\eta}{\zeta-\eta}\Phi(\zeta)\\
&\le \frac{\zeta-B(h\cdot\varsigma)}{\zeta-\eta}\Psi(\zeta)+\frac{B(h\cdot\varsigma)-\eta}{\zeta-\eta}\Psi(\eta)\\
&= \Psi(\eta)+\Psi(\zeta)-\left[\frac{\zeta-B(h\cdot\varsigma)}{\zeta-\eta}\Psi(\eta)+\frac{B(h\cdot\varsigma)-\eta}{\zeta-\eta}\Psi(\zeta)\right]\\
&\le \Psi(\eta)+\Psi(\zeta)-Z(\Psi(B(h\cdot\varsigma))).
\end{aligned}
$$

The last statement is obtained by observing that if Ψ is concave then $-\Psi$ is convex and B, Z are linear on L. □

Theorem 5.1.3. *Assume Ψ be a convex function on $S=[\eta,\zeta]$ $(-\infty<\eta<\zeta<\infty)$. Suppose that L satisfies properties L_1, L_2 on a nonempty set E, ς is weight function and B, Z are isotonic linear normalised functional on L, then for all $h\cdot\varsigma\in L$ such that $\Psi(h), \Psi(\eta+\zeta-B(h\cdot\varsigma))\in L$ (so that $\eta\le h(l)\le\zeta$ for all $l\in E$), we have*

$$
Z(\Psi(\eta+\zeta-B(h\cdot\varsigma)) \le \Psi(\eta)+\Psi(\zeta)-B(\Psi(h)). \tag{5.1.3}
$$

Proof. Similar to the proof of Theorem 5.1.2 we can prove it by using right hand side of the inequality (3.3.1) and using Theorem 1.5.2 instead of Theorem 5.1.1. □

5.2 Exponential convexity

This section contains the investigation of the exponential and logarithmic convexity of the functionals that are associated with the extension of Jessen functional given in the previous section.

Remark 5.2.1. We consider the following functionals under the assumption of Theorems 5.1.2 and 5.1.3, respectively.

$$\Omega(\Psi) = \Psi(\eta) + \Psi(\zeta) - Z(\Psi(B(h \cdot \varsigma))) - \Psi(\eta + \zeta - B(h)), \tag{5.2.1}$$

$$\beth(\Psi) = \Psi(\eta) + \Psi(\zeta) - B(\Psi(h)) - Z(\Psi(\eta + \zeta - B(h \cdot \varsigma)). \tag{5.2.2}$$

Then $\Omega(\Psi)$ and $\Omega'(\Psi)$ are positive.

In the following theorem and its corollaries we use notations as: $\mathcal{C}$ is a real open interval, S is any real interval, $\Theta = \{g_l \mid l \in \mathcal{C}\}$ is a family of continuous functions defined on S. $\Omega(\Psi)$ is as given in Remark 5.2.1. Then the new m-exponentially convex functions are produced by the useful results given below.

Theorem 5.2.1. *Assume the function $l \mapsto [s_1, s_2, s_3; g_l]$ is m-exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Omega(g_l)$ is an m-exponentially $\mathcal{J}$-convex on $\mathcal{C}$. In addition, the continuity of this function implies the m-exponential convexity on $\mathcal{C}$.*

Proof. Suppose $l_i, l_j \in \mathcal{C}, l_{ij} = \frac{l_i + l_j}{2}$ and $a_i, a_j \in \mathbb{R}$ for $i, j \in \{1, 2, ..., m\}$ $(m \in \mathbb{N})$. Define the function Δ on S by

$$\Delta(s) = \sum_{i,j=1}^{m} a_i a_j g_{l_{ij}}(s).$$

Being the linear combination of continuous functions, Δ is continuous. The hypothesis of m-exponential $\mathcal{J}$-convexity of the function $l \mapsto [s_1, s_2, s_3; g_l]$ gives

$$[s_1, s_2, s_3; \Delta] = \sum_{i,j=1}^{m} a_i a_j [s_1, s_2, s_3; g_{l_{ij}}] \geq 0,$$

which implies the convexity of Δ on S. In addition, the linearity of Ω gives

$$\sum_{i,j=1}^{m} a_i a_j \Omega(g_{l_{ij}}) \geq 0.$$

We conclude that the function $l \mapsto \Omega(g_l)$ is an m-exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. $\square$

We have obtained the corollaries below from the above theorem.

Corollary 5.2.2. *Assume the function $l \mapsto [s_1, s_2, s_3; g_l]$ is exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Omega(g_l)$ is an exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. In addition, the continuity of this function implies the exponential convexity on $\mathcal{C}$.*

Corollary 5.2.3. *Assume the function $l \mapsto [s_1, s_2, s_3; g_l]$ is 2-exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Omega(g_l)$ is 2-exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. In addition, the continuity of this function implies the 2-exponential convexity on C, and thus the function is log-convex. That is, for $l, r, s \in C$ such that $r < s < l$ the following holds*

$$\Omega^{l-r}(g_s) \leq \Omega^{l-s}(g_r)\Omega^{s-r}(g_l).$$

Proof. This is directly obtained from Theorem 5.2.1 and Remark 1.3.4. $\square$

To define the basic inequality of log-convex functions we present positive functionals.

Remark 5.2.2. The following positive functionals are useful in defining the basic inequality of log-convex functions.

$$\Omega(f_l) = \frac{1}{l^2}\left(\exp(l\eta) + \exp(l\zeta) - Z\left(\exp\left(lB(h \cdot \varsigma)\right)\right) - \exp(l\eta + l\zeta - lB(h))\right).$$

$$\Omega(g_l) = \frac{1}{l(l-1)}\left(\eta^l + \zeta^l - Z\left((B(h \cdot \varsigma))^l\right) - (\eta + \zeta - B(h))^l\right).$$

$$\Omega(h_l) = \frac{1}{(\log l)^2}\left(l^{-\eta} + l^{-\zeta} - Z\left(l^{-B(h \cdot \varsigma)}\right) - l^{B(h)-\eta-\zeta}\right).$$

$$\Omega(k_l) = \frac{1}{l}\left(\exp(-\eta\sqrt{l}) + \exp(-\zeta\sqrt{l}) - Z\left(\exp\left(-B(h \cdot \varsigma)\sqrt{l}\right)\right) - \exp\left((B(h) - \eta - \zeta)\sqrt{l}\right)\right).$$

Theorem 5.2.4. *Let $\Omega(\Psi)$ be the linear functional defined by* (5.2.1) *and consider the functions $\phi_i : (0, \infty) \to \mathbb{R}$ for $i = 1, 4$ and $\phi_i : (1, \infty) \to \mathbb{R}$ for $i = 2, 3$ defined as*

$\phi_1(l) = \Omega(f_l), \ \phi_2(l) = \Omega(g_l), \ \phi_3(l) = \Omega(h_l), \ \phi_4(l) = \Omega(k_l)$. *Then*

(i) The functions ϕ_i are continuous on $(0, \infty)$ for $i = 1, 4$ and continuous on $(1, \infty)$ for $i = 2, 3$.

(ii) Suppose $m \in \mathbb{N}$, $l_1, \ldots, l_m \in (0, \infty)$ for $i = 1, 4$ and $l_1, \ldots, l_m \in (1, \infty)$ for $i = 2, 3$. Then

$$\left[\phi_i\left(\frac{l_j + l_k}{2}\right)\right]_{j,k=1}^{m}$$

are positive semidefinite matrices.

(iii) ϕ_i's are exponentially convex on $(0, \infty)$ when $i = 1, 4$ and on $(1, \infty)$ for $i = 2, 3$.

(iv) Let $l, r, s \in (0, \infty)$ for $i = 1, 4$ and $r, s, l \in (1, \infty)$ for $i = 2, 3$ with $l < r < s$, then

$$(\phi_i(r))^{s-l} \le (\phi_i(l))^{s-r} (\phi_i(s))^{r-l},$$

where $\phi_i(l)$ for $i = 1, 2, 3, 4$ are defined in Remark 5.2.2.

Proof. (i) The continuity of these functions is obvious.

(ii) Let $d_j, l_j \in \mathbb{R}$, where $j = 1, \ldots, m$ (m is any natural number). Define the auxiliary function Δ_1 on $S = \mathbb{R}$ by

$$\Delta_1(s) = \sum_{j,k=1}^{m} d_j d_k f_{\frac{l_j + l_k}{2}}(s).$$

Since

$$\Delta_1''(s) = \sum_{j,k=1}^{n} d_j d_k \frac{d^2}{ds^2} f_{\frac{l_j + l_k}{2}}(s) \ge 0$$

for $s \in S$ by Lemma 1.3.3. This implies Δ_1 is convex. Now Theorem 5.1.2 implies that $\Omega(\Delta_1) \geq 0$. This means that

$$\left[\phi_1\left(\frac{l_j + l_k}{2}\right)\right]_{j,k=1}^{m}$$

is a positive semidefinite matrix.

To prove the remaining positive semidefinite matrices, we can define the auxiliary functions Δ_i for $i = 2, 3, 4$ in the similar manner.

We easily get (iii) and (iv) from (i), (ii) and Lemma 1.3.2. □

Remark 5.2.3. We can construct similar results for the positive functional $\beth(\Psi)$ defined in (5.2.2).

5.3 Mean value theorems

The lemma given below will be very helpful to state the mean value theorems of Cauchy and Lagrange type.

Lemma 5.3.1. [52] *Let $\Psi : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$, be such that $\Psi \in C^2(S)$, Ψ'' is bounded and $\mathfrak{d} = \inf_{l \in S} \Psi''(l)$, $\mathfrak{D} = \sup_{l \in S} \Psi''(l)$. Then the functions $\Psi_1, \Psi_2 : S \to \mathbb{R}$ defined by*

$$\begin{aligned} \Psi_1(l) &= \frac{\mathfrak{D}}{2} l^2 - \Psi(l), \\ \Psi_2(l) &= \Psi(l) - \frac{\mathfrak{d}}{2} l^2 \end{aligned} \tag{5.3.1}$$

are convex.

Theorem 5.3.2. *Let L satisfies properties L_1 and L_2 on a nonempty set E, and let $\Psi : S \to \mathbb{R}, \Psi \in C^2(S)$, where $S = [\eta, \zeta] \subseteq \mathbb{R} \, (-\infty < \eta < \zeta < \infty)$. If B, Z are isotonic linear normalised functionals and ς is a weight function (defined in (1.5.1)),* *then for all $h, h \cdot \varsigma \in L$ such that $\Psi(B(h\cdot\varsigma)), (B(h\cdot\varsigma))^2 \in L$ there exists some $\varpi \in S$ at which*

$$\Psi(\eta) + \Psi(\zeta) - Z(\Psi(B(h \cdot \varsigma))) - \Psi(\eta + \zeta - B(h)) = \alpha \Psi''(\varpi), \tag{5.3.2}$$

where

$$\alpha = \frac{1}{2}\left[\eta^2 + \zeta^2 - (\eta + \zeta - B(h))^2 - Z([B(h \cdot \varsigma)]^2)\right].$$

Proof. Assume $\mathfrak{D} = \max_{l \in S} \Psi''(l)$ and $\mathfrak{d} = \min_{l \in S} \Psi''(l)$. Then Lemma 6.2.1 gives the convexity of the functions $\Psi_1, \Psi_2 : S \to \mathbb{R}$. Since they are also continuous. Apply Theorem 5.1.2, we get

$$\Psi(\eta) + \Psi(\zeta) - Z(\Psi(B(h \cdot \varsigma))) - \Psi(\eta + \zeta - B(h)) \leq \alpha \mathfrak{D}, \tag{5.3.3}$$

and

$$\Psi(\eta) + \Psi(\zeta) - Z(\Psi(B(h \cdot \varsigma))) - \Psi(\eta + \zeta - B(h)) \geq \alpha \mathfrak{d}. \tag{5.3.4}$$

Now combining these two inequalities and since Ψ'' is continuous, there exists $\varpi \in S$ $(\mathfrak{d} \leq \Psi''(\varpi) \leq \mathfrak{D})$ such that (5.3.2) holds. □

Theorem 5.3.3. *Assume L satisfies properties L_1 and L_2 on a nonempty set E, and suppose $\Phi, \Psi : S \to \mathbb{R}, \Phi, \Psi \in C^2(S)$, where $S = [\eta, \zeta] \subseteq \mathbb{R}\,(-\infty < \eta < \zeta < \infty)$. If B, Z are isotonic linear normalised functionals and ς is a weight function (defined in* (1.5.1)*), then for all $h, h \cdot \varsigma \in L$ such that $\Phi(B(h\cdot\varsigma)), \Psi(B(h\cdot\varsigma)), (B(h\cdot\varsigma))^2 \in L$ and $\eta^2 + \zeta^2 - (\eta + \zeta - B(h))^2 - Z([B(h\cdot\varsigma)]^2) \neq 0$ there exists some $\varpi \in S$ such that the following holds*

$$\begin{aligned} &\Psi''(\varpi)\left[\Phi(\eta) + \Phi(\zeta) - Z(\Phi(B(h \cdot \varsigma))) - \Phi(\eta + \zeta - B(h))\right] \\ &= \Phi''(\varpi)\left[\Psi(\eta) + \Psi(\zeta) - Z(\Psi(B(h \cdot \varsigma))) - \Psi(\eta + \zeta - B(h))\right]. \end{aligned} \tag{5.3.5}$$

Proof. Suppose the function $k \in C^2(S)$, $k = c_1\Phi - c_2\Psi$, where c_1 and c_2 are defined by

$$c_1 = \Psi(\eta) + \Psi(\zeta) - Z(\Psi(B(h \cdot \varsigma))) - \Psi(\eta + \zeta - B(h)), \tag{5.3.6}$$

and

$$c_2 = \Phi(\eta) + \Phi(\zeta) - Z(\Phi(B(h \cdot \varsigma))) - \Phi(\eta + \zeta - B(h)). \tag{5.3.7}$$

As $k \in C^2(S)$, applying Theorem 5.3.2 on the function k ensures the existence of some $\varpi \in S$ such that the following holds

$$k(\eta) + k(\zeta) - Z(k(B(h \cdot \varsigma))) - k(\eta + \zeta - B(h)) = \alpha k''(\varpi). \tag{5.3.8}$$

The left-hand side of the above equation is equal to zero. Since α is non zero, so we have that $k''(\varpi) = 0$. Thus the assertion of our theorem follows directly. □

Similarly we can define mean value theorems for Theorem 5.1.3. Here we omit the proofs.

Theorem 5.3.4. *Assume L satisfies properties L_1 and L_2 on a nonempty set E, and assume $\Psi : S \to \mathbb{R}, \Psi \in C^2(S)$, where $S = [\eta, \zeta] \subseteq \mathbb{R}\,(-\infty < \eta < \zeta < \infty)$. If B, Z are isotonic linear normalised functionals and ς is a weight function (defined in (1.5.1)), then for all $h \cdot \varsigma \in L$ such that $\Psi(\eta+\zeta-B(h\cdot\varsigma)), \Psi(h), (\eta+\zeta-B(h\cdot\varsigma))^2, h^2 \in L$ there exists some $\varpi \in S$ such that the following holds*

$$\Psi(\eta) + \Psi(\zeta) - B(\Psi(h)) - Z(\Psi(\eta + \zeta - B(h \cdot \varsigma))) = \beta \Psi''(\varpi), \tag{5.3.9}$$

where

$$\beta = \frac{1}{2}\left[\eta^2 + \zeta^2 - B(h^2) - Z([\eta + \zeta - B(h \cdot \varsigma)]^2)\right].$$

Theorem 5.3.5. *Assume L satisfies properties L_1 and L_2 on a nonempty set E, and suppose $\Phi, \Psi : S \to \mathbb{R}, \Phi, \Psi \in C^2(S)$, where $S = [\eta, \zeta] \subseteq \mathbb{R}\,(-\infty < \eta < \zeta < \infty)$. If B, Z are isotonic linear normalised functionals and ς is a weight function (defined in (1.5.1)), then for all $h \cdot \varsigma \in L$ such that $\Phi(\eta + \zeta - B(h \cdot \varsigma)), \Psi(\eta + \zeta - B(h \cdot \varsigma)), \Phi(h), \Psi(h), (\eta+\zeta-B(h\cdot\varsigma))^2, h^2 \in L$ and $\eta^2+\zeta^2-B(h^2)-Z([\eta+\zeta-B(h\cdot\varsigma)]^2) \neq 0$*

there exists some $\varpi \in S$ *such that the following holds*

$$\begin{aligned} &\Psi''(\varpi)\left[\Phi(\eta)+\Phi(\zeta)-B(\Phi(h))-Z(\Phi(\eta+\zeta-B(h\cdot\varsigma)))\right] \\ &=\Phi''(\varpi)\left[\Psi(\eta)+\Psi(\zeta)-B(\Psi(h))-Z(\Psi(\eta+\zeta-B(h\cdot\varsigma)))\right]. \end{aligned} \tag{5.3.10}$$

5.4 LBP type inequality for $m(M)$-ψ-convex functions

The result below is related to counterpart of the inequality $\psi(B(h)) \leq Z(\psi(B(h\cdot\varsigma)))$ (see (5.1.1)) over compact interval $S = [\eta, \zeta]$ for $m(M)$-ψ-convex functions. In this section, we use:

$$\begin{aligned} T_1 &= \frac{\zeta - B(h\cdot\varsigma)}{\zeta-\eta}\psi(\eta) + \frac{B(h\cdot\varsigma)-\eta}{\zeta-\eta}\psi(\zeta) - Z(\psi(B(h\cdot\varsigma))), \\ T_2 &= \frac{\zeta - B(h\cdot\varsigma)}{\zeta-\eta}\Phi(\eta) + \frac{B(h\cdot\varsigma)-\eta}{\zeta-\eta}\Phi(\zeta) - Z(\Phi(B(h\cdot\varsigma))). \end{aligned}$$

Theorem 5.4.1. *Suppose* $\psi : S = [\eta, \zeta](\subseteq \mathbb{R}) \to \mathbb{R}$ *is a convex function and* $h : E \to S$ *such that* $\psi(B(h\cdot\varsigma)), h\cdot\varsigma \in L$, ς *is weight function and* $B, Z : L \to \mathbb{R}$ *are isotonic linear normalised functionals.*

(*i*) *Assume* $\Phi \in \mathfrak{L}(S, m, \psi)$ *and* $\Phi(B(h\cdot\varsigma)) \in L$, *then*

$$mT_1 \leq T_2. \tag{5.4.1}$$

(*ii*) *Assume* $\Phi \in \mathfrak{U}(S, M, \psi)$ *and* $\Phi(B(h\cdot\varsigma)) \in L$, *then*

$$T_2 \leq MT_1. \tag{5.4.2}$$

(*iii*) *Assume* $\Phi \in \mathfrak{B}(S, m, M, \psi)$ *and* $\Phi(B(h\cdot\varsigma)) \in L$, *then both* (5.4.1) *and* (5.4.2) *hold.*

Proof. (i) Since $\Phi \in \mathfrak{L}(S, m, \psi)$ and $\Phi(B(h \cdot \varsigma)) \in L$, this yield $\Phi - m\psi$ is convex and $(\Phi - m\psi) \circ B(h \cdot \varsigma) \in L$. For the convex function $\Phi - m\psi$, Theorem 5.1.1 is applied to obtain the required results.

(ii) Similarly, we observe $\Phi(B(h \cdot \varsigma)) \in L$ and $\Phi \in \mathfrak{U}(S, M, \psi)$. It yields $M\psi - \Phi$ is convex and $(M\psi - \Phi) \circ B(h \cdot \varsigma) \in L$.

(iii) It can be easily observed that this is true from (i) and (ii). □

The above theorem is used to obtain the below useful corollary.

Corollary 5.4.2. *Assume the second order derivative of convex function $\psi : S \subseteq \mathbb{R} \to \mathbb{R}$ exists on $\overset{\circ}{S}$ and $h : E \to S$ such that $\psi(B(h \cdot \varsigma)), h \cdot \varsigma \in L$, ς is weight function and $B, Z : L \to \mathbb{R}$ are isotonic linear and normalised functionals.*

(i) *If second order derivative of $\Phi : S \to \mathbb{R}$ exists and for given real number m, $\Phi''(l) \geq m\psi''(l), l \in \overset{\circ}{S}$. Then* (5.4.1) *holds under the assumption that $\Phi(B(h \cdot \varsigma)) \in L$.*

(ii) *If second order derivative of $\Phi : S \to \mathbb{R}$ exists and for given real number m, $\Phi''(l) \leq M\psi''(l), l \in \overset{\circ}{S}$. Then* (5.4.2) *is true if $\Phi(B(h \cdot \varsigma)) \in L$.*

(iii) *If second order derivative of $\Phi : S \to \mathbb{R}$ exists and for given real number m, $m\psi''(l) \leq \Phi''(l) \leq M\psi''(l), l \in \overset{\circ}{S}$. Then both* (5.4.1) *and* (5.4.2) *hold, under the assumption that $\Phi(B(h \cdot \varsigma)) \in L$.*

The following propositions are some particular cases of the above corollary.

Proposition 5.4.3. *Let the second order derivative of $\Phi : S \subseteq \mathbb{R} \to \mathbb{R}$ exists on $\overset{\circ}{S}$.*

(i) *For $\inf_{l \in \overset{\circ}{S}} \Phi''(l) = s > -\infty$, we get*

$$\frac{s}{2}\left[(\eta + \zeta)B(h \cdot \varsigma) - \eta\zeta - Z([B(h \cdot \varsigma)]^2)\right] \leq T_2. \tag{5.4.3}$$

under the assumption that $\Phi(B(h \cdot \varsigma)), [B(h \cdot \varsigma)]^2, h \cdot \varsigma \in L$.

(*ii*) *For* $\sup_{l\in\overset{\circ}{S}} \Phi''(l) = S < \infty$, *we obtain*

$$T_2 \leq \frac{S}{2}\left[(\eta+\zeta)B(h\cdot\varsigma) - \eta\zeta - Z([B(h\cdot\varsigma)]^2)\right]. \tag{5.4.4}$$

assuming that $\Phi(B(h\cdot\varsigma)), [B(h\cdot\varsigma)]^2, h\cdot\varsigma \in L$.

(*iii*) *For* $-\infty < s \leq \Phi''(l) \leq S < \infty, l \in \overset{\circ}{S}$, *both* (5.4.3) *and* (5.4.4) *hold, under the assumption that* $\Phi(B(h\cdot\varsigma)), [B(h\cdot\varsigma)]^2, h\cdot\varsigma \in L$.

Proof. Define the auxiliary function $g(l) := \Phi(l) - \frac{1}{2}sl^2$. Then $g''(l) = \Phi''(l) - s \geq 0$ yields g is convex, or, equivalently, $\Phi \in \mathfrak{L}(S, \frac{1}{2}s, (\cdot)^2)$. The below inequality is obtained by using Corollary 5.4.2.

$$m\left[\frac{\zeta - B(h\cdot\varsigma)}{\zeta-\eta}\eta^2 + \frac{B(h\cdot\varsigma)-\eta}{\zeta-\eta}\zeta^2 - Z([B(h\cdot\varsigma)]^2)\right] \leq T_2,$$

which is clearly equivalent to (5.4.3). Analogously, we can prove (ii), and (iii) follows by (i) and (ii). □

Another useful finding has been given as;

Proposition 5.4.4. *Assume that the second order derivative of* $\Phi : S \subseteq \mathbb{R}^+ \to \mathbb{R}$ *exists on* $\overset{\circ}{S}$. *Let* $q \in (-\infty, 0) \cup (1, \infty)$ *and define* $h_q : S \to \mathbb{R}$, $h_q(l) = \phi''(l)l^{2-q}$.

(*i*) *For* $\inf_{l\in\overset{\circ}{S}} h_q(l) = k > -\infty$, *we have*

$$\frac{k}{q(q-1)}\left[qL_{q-1}^{q-1}(\eta,\zeta)B(h\cdot\varsigma) - \eta\zeta(q-1)L_{q-2}^{q-2}(\eta,\zeta) - Z([B(h\cdot\varsigma)]^q)\right] \leq T_2. \tag{5.4.5}$$

assuming that $\Phi(B(h\cdot\varsigma)), [B(h\cdot\varsigma)]^q, h\cdot\varsigma \in L$, *and* L_q *is the* q*-logarithmic mean.*

(*ii*) *For* $\sup_{l\in\overset{\circ}{S}} h_q(l) = K < \infty$, *we have*

$$T_2 \geq \frac{K}{q(q-1)}\left[qL_{q-1}^{q-1}(\eta,\zeta)B(h\cdot\varsigma) - \eta\zeta(q-1)L_{q-2}^{q-2}(\eta,\zeta) - Z([B(h\cdot\varsigma)]^q)\right]. \tag{5.4.6}$$

under the assumption that $\Phi(B(h\cdot\varsigma)), [B(h\cdot\varsigma)]^q, h\cdot\varsigma \in L$.

(*iii*) *For* $-\infty < k \leq \Phi''(l) \leq K < \infty, l \in \overset{\circ}{S}$, *both* (5.4.5) *and* (5.4.6) *hold, assuming that* $\Phi(B(h\cdot\varsigma)), [B(h\cdot\varsigma)]^q, h\cdot\varsigma \in L$.

Proof. (i) Define the auxiliary mapping $g_q(l) = \Phi(l) - \frac{k}{q(q-1)}l^q$. Then

$$\begin{aligned} g_q''(l) &= \Phi''(l) - kl^{q-2} = l^{q-2}(l^{2-q}\Phi''(l) - k) \\ &= l^{q-2}(h_q(l) - k) \geq 0. \end{aligned}$$

which implies g_q is convex or, equivalently, $\Phi \in \mathfrak{L}(S, \frac{k}{q(q-1)}, (\cdot)^q)$. We Apply Corollary 5.4.2 to deduce (5.4.5). Analogously, we can prove (ii), and (iii) follows by (i) and (ii). □

Below proposition is also true.

Proposition 5.4.5. *Suppose that the second order derivative of* $\Phi : S \subseteq \mathbb{R}^+ \to \mathbb{R}$ *exists on* $\overset{\circ}{S}$. *Define* $z(l) = l^2\Phi''(l), l \in S$.

(*i*) *For* $\inf_{l\in\overset{\circ}{S}} z(l) = \gamma > -\infty$, *we have*

$$\gamma\left[Z(\ln[B(h\cdot\varsigma)]) + \ln\left[I\left(\frac{1}{\eta}, \frac{1}{\zeta}\right)\right] + 1 - \frac{B(h\cdot\varsigma)}{L(\eta,\zeta)}\right] \leq T_2. \tag{5.4.7}$$

under the assumption that $\Phi(B(h\cdot\varsigma)), \ln[B(h\cdot\varsigma)], h\cdot\varsigma \in L$. *L and I denote the logarithmic and identric mean, respectively.*

(*ii*) *For sup*$_{l\in\overset{\circ}{S}} z(l) = \Gamma < \infty$, *we get*

$$T_2 \leq \Gamma\left[Z(\ln[B(h\cdot\varsigma)]) + \ln\left[I\left(\frac{1}{\eta}, \frac{1}{\zeta}\right)\right] + 1 - \frac{B(h\cdot\varsigma)}{L(\eta,\zeta)}\right]. \tag{5.4.8}$$

assuming that $\Phi(B(h\cdot\varsigma)), \ln[B(h\cdot\varsigma)], h\cdot\varsigma \in L$.

(iii) *For* $-\infty < \gamma \leq z(l) \leq \Gamma < \infty$ *for* $l \in \overset{\circ}{S}$, *both* (5.4.7) *and* (5.4.8) *hold, under the assumption that* $\Phi(B(h \cdot \varsigma)), \ln[B(h \cdot \varsigma)], h \cdot \varsigma \in L$.

Proof. Define the auxiliary mapping $g(l) = \Phi(l) + \gamma \ln l$. Analogously as in the proof of Proposition 5.4.4, we get the required results. □

Finally, the following proposition holds.

Proposition 5.4.6. *Suppose that the second order derivative of* $\Phi : S \subseteq \mathbb{R}^+ \to \mathbb{R}$ *exists on* $\overset{\circ}{S}$. *Define* $\sigma(l) = l\Phi''(l), l \in S$.

(i) *For* $\inf_{l \in \overset{\circ}{S}} \sigma(l) = \xi > -\infty$, *we have*

$$\xi \left[B(h \cdot \varsigma) \ln I(\eta, \zeta) - \frac{G^2(\eta, \zeta)}{L(\eta, \zeta)} + B(h \cdot \varsigma) - B(h \cdot \varsigma) Z(\ln[B(h \cdot \varsigma)]) \right] \leq T_2 \tag{5.4.9}$$

assuming that $\Phi(B(h \cdot \varsigma)), \ln[B(h \cdot \varsigma)], h \cdot \varsigma \in L$. G *and* L *denote geometric and logarithmic mean, respectively.*

(ii) *For* $\sup_{l \in \overset{\circ}{S}} \sigma(l) = \Xi < \infty$, *then*

$$T_2 \leq \Xi \left[B(h \cdot \varsigma) \ln I(\eta, \zeta) - \frac{G^2(\eta, \zeta)}{L(\eta, \zeta)} + B(h \cdot \varsigma) - B(h \cdot \varsigma) Z(\ln[B(h \cdot \varsigma)]) \right] \tag{5.4.10}$$

under the assumption that $\Phi(B(h \cdot \varsigma)), \ln[B(h \cdot \varsigma)], h \cdot \varsigma \in L$.

(iii) *For* $-\infty < \xi \leq \sigma(l) \leq \Xi < \infty$ *for* $l \in \overset{\circ}{S}$, *both* (5.4.9) *and* (5.4.10) *hold, under the assumption that* $\Phi(B(h \cdot \varsigma)), \ln[B(h \cdot \varsigma)], h \cdot \varsigma \in L$.

Proof. Define the auxiliary mapping $g(l) = \Phi(l) - \xi l \ln l$. Analogously as in the proof of Proposition 5.4.4, we get the required results. □

Chapter 6

m-Exponential convexity of refinements of Hermite-Hadamard's inequality

In this chapter, m-exponential convexity of the functions related with the refinement of Hermite Hadamard inequality (1.4.10) is examined. In addition, the results about exponential and log-convexity are deduced. To construct means with Stolarsky property, Cauchy and Lagrange type mean value theorems are also given.

Below are the functional forms of inequalities (1.4.9):

Remark 6.0.1.

$$\Gamma_1(\psi) = a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\gamma+\beta}{2}\right) - \psi\left(\frac{\alpha+\beta}{2}\right). \tag{6.0.1}$$

$$\Gamma_2(\psi) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(r)dr - a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\gamma+\beta}{2}\right). \tag{6.0.2}$$

$$\Gamma_3(\psi) = \frac{1}{2}\left[c\psi(\alpha) + d\psi(\beta) + \psi(\delta)\right] - \frac{1}{\beta-\alpha}\int_\alpha^\beta \psi(r)dr. \tag{6.0.3}$$

$$\Gamma_4(\psi) = (1-c)\psi(\alpha) + (1-d)\psi(\beta) - \psi(\delta). \tag{6.0.4}$$

Then for $1 \leq j \leq 4$, $\Gamma_j(\psi)$ are positive.

6.1 Exponential convexity

Now we apply positive functionals (given in the previous section) on a given family of functions to investigate m-exponential and exponential convexity.

In the following theorem and its corollaries we use notations as: $\mathcal{C}$ is a real open interval, S is any real interval, $\Xi = \{g_l \mid l \in \mathcal{C}\}$ is a family of functions defined on S. $\Gamma_1(\Psi)$ is as given in Remark 6.0.1. Then we have the following useful results which produce new m-exponentially convex functions.

Theorem 6.1.1. *Suppose that the function $l \mapsto [s_1, s_2, s_3; g_l]$ is m-exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Gamma_1(g_l)$ is an m-exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. In addition, the continuity of this function implies the m-exponential convexity on $\mathcal{C}$.*

Proof. Suppose l_j, l_k be the elements of $\mathcal{C}$, $l_{jk} = \frac{l_j + l_k}{2}$ and c_j, c_k are real numbers for $j, k = 1, \ldots, m$. Consider the function Δ on S as

$$\Delta(s) = \sum_{j,k=1}^{m} c_j c_k g_{l_{jk}}(s).$$

The hypothesis of function $l \mapsto [s_1, s_2, s_3; g_l]$ is m-exponentially $\mathcal{J}$-convex thus yields

$$[s_1, s_2, s_3; \Delta] = \sum_{j,k=1}^{m} c_j c_k [s_1, s_2, s_3; g_{l_{jk}}] \geq 0.$$

This implies the convexity of Δ on S. Thus, we get $\Gamma_1(\Delta)$ is non negative. By the linear property of Γ_1 we have

$$\sum_{j,k=1}^{m} c_j c_k \Gamma_1\left(g_{l_{jk}}\right) \geq 0,$$

concluding the m-exponentially $\mathcal{J}$-convexity of function $l \mapsto \Gamma_1(g_l)$ on $\mathcal{C}$. $\square$

The above model results the following outcomes.

Corollary 6.1.2. *Suppose that the function $l \mapsto [s_1, s_2, s_3; g_l]$ is exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then $l \mapsto \Gamma_1(g_l)$ is an exponentially $\mathcal{J}$-convex function on $\mathcal{C}$. In addition, the continuity of this function implies the exponential convexity on $\mathcal{C}$.*

Corollary 6.1.3. *Suppose that the function $l \mapsto [s_1, s_2, s_3; g_l]$ is 2-exponentially $\mathcal{J}$-convex on $\mathcal{C}$, where s_1, s_2, s_3 are distinct points of S. Then below statements are true:*

i) *The continuity of the function $l \mapsto \Gamma_1(g_l)$ implies the 2-exponential convexity of $l \mapsto \Gamma_1(g_l)$ on $\mathcal{C}$, which concludes the log-convexity stated as:*

$$\Gamma_1^{l-r}(g_s) \leq \Gamma_1^{l-s}(g_r)\Gamma_1^{s-r}(g_l)$$

for $l, r, s \in \mathcal{C}$ with $r < s < l$.

ii) *Assume that the function $l \mapsto \Gamma_1(g_l)$ on $\mathcal{C}$ is strictly positive and its first order derivative also exists, then for $l \leq u$ and $r \leq v$, $(l, r, u, v \in \mathcal{C})$ yields*

$$\varsigma(l, r; \Gamma_1) \leq \varsigma(u, v; \Gamma_1),$$

where

$$\varsigma(l, r; \Gamma_1) = \begin{cases} \left(\frac{\Gamma_1(g_l)}{\Gamma_1(g_r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \\ \exp\left(\frac{\frac{d}{dl}(\Gamma_1(g_l))}{\Gamma_1(g_l)}\right), & \textit{otherwise}. \end{cases} \tag{6.1.1}$$

Proof. **i)** This is directly obtained from Remark 1.3.4 and Theorem 6.1.1.

ii) (i) follows the log-convexity of function $l \mapsto \Gamma_1(g_l)$ on $\mathcal{C}$, which yields $l \mapsto \log \Gamma_1(g_l)$ is a convex function on $\mathcal{C}$. Now for $l \leq u$ and $r \leq v$, applying [53, Proposition 3.2] we obtained

$$\frac{\log\Gamma_1(g_l)-\log\Gamma_1(g_r)}{l-r}\leq\frac{\log\Gamma_1(g_u)-\log\Gamma_1(g_v)}{u-v}. \tag{6.1.2}$$

It yields

$$\varsigma(l,r;\Gamma_1)\leq\varsigma(u,v;\Gamma_1).$$

By applying limit on (8) follows the remaining cases. □

Remark 6.1.1. In defining fundamental inequality of logarithmic convexity these described positive functionals are very useful.

$$\Gamma_1(f_l)=\frac{1}{l^2}\left[a\exp\left(\frac{l(\alpha+\gamma)}{2}\right)+b\exp\left(\frac{l(\beta+\gamma)}{2}\right)-\exp\left(\frac{l(\alpha+\beta)}{2}\right)\right].$$

$$\Gamma_1(g_l)=\frac{l}{(l-1)2^l}\left[a(\alpha+\gamma)^l+b(\gamma+\beta)^l-(\alpha+\beta)^l\right].$$

$$\Gamma_1(h_l)=\frac{1}{(\log l)^2}\left[al^{-\frac{1}{2}(\alpha+\gamma)}+bl^{-\frac{1}{2}(\beta+\gamma)}-l^{-\frac{1}{2}(\alpha+\beta)}\right].$$

$$\Gamma_1(k_l)=\frac{1}{l}\left[a\exp\left(-\frac{1}{2}(\alpha+\gamma)\sqrt{l}\right)+b\exp\left(-\frac{1}{2}(\beta+\gamma)\sqrt{l}\right)-\exp\left(-\frac{1}{2}(\alpha+\beta)\sqrt{l}\right)\right].$$

Theorem 6.1.4. *Consider the linear functional $\Gamma_1(\psi)$ stated in (6.0.1). Now for i=1,4 and i=2,3 let's define $\theta_i:(0,\infty)\to\mathbb{R}$ and $\theta_i:(1,\infty)\to\mathbb{R}$, respectively, as*

$$\theta_1(l)=\Gamma_1(f_l),\quad \theta_2(l)=\Gamma_1(g_l),\quad \theta_3(l)=\Gamma_1(h_l),\quad \theta_4(l)=\Gamma_1(k_l).$$

We then have:

(i) For $i=1,4$ and $i=2,3$, θ_i functions preserve continuity on $(0,\infty)$ and $(1,\infty)$, respectively.

(ii) Let $m\in\mathbb{N}$, $l_i\in(0,\infty)$ and $l_i\in(1,\infty)$ $(1\leq i\leq m)$ for $i=1,4$ and $i=2,3$, respectively. This implies that the below matrices are positive semidefinite.

$$\left[\theta_i\left(\frac{l_j+l_k}{2}\right)\right]_{j,k=1}^{m}.$$

(iii) The exponential convexity holds for θ_i functions on $(0,\infty)$ and $(1,\infty)$ for $i = 1,4$ and $i = 2,3$, respectively.

(iv) Suppose $l,r,s \in (0,\infty)$ and $l,r,s \in (1,\infty)$ for $i = 1,4$ and $i = 2,3$, respectively. It yields

$$(\theta_i(r))^{s-l} \leq (\theta_i(l))^{s-r}(\theta_i(s))^{r-l}, \qquad (l < r < s).$$

(v) Assume that θ_i functions are strictly positive and their first order derivative also exist on $(0,\infty)$ and $(1,\infty)$ for $i = 1,4$ and $i = 2,3$, respectively. Then for $l \leq u$ and $r \leq v$, where $l,r,u,v \in (0,\infty)$ and $l,r,u,v \in (1,\infty)$ for $i = 1,4$ and $i = 2,3$, respectively yield

$$\varsigma(l,r;\theta_i) \leq \varsigma(u,v;\theta_i),$$

with

$$\varsigma(l,r;\theta_i) = \begin{cases} \left(\frac{\theta_i(l))}{\theta_i(r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \\ \exp\left(\frac{\frac{d}{dl}(\theta_i(l))}{\theta_i(l)}\right), & \text{otherwise.} \end{cases} \tag{6.1.3}$$

Proof. (i) The functions $l \to \theta_i(l)$ $(1 \leq i \leq 4)$ are obviously continuous.

(ii) For natural number m and c_j, c_k are real numbers for $j,k = 1,\ldots,m$, consider the function Δ_1 on $S = \mathbb{R}$ defined as

$$\Delta_1(s) = \sum_{j,k=1}^{m} c_j c_k f_{\frac{l_j+l_k}{2}}(s).$$

Now for $s \in S$ Lemma 1.3.3 yields

$$\Delta_1''(s) = \sum_{j,k=1}^{m} c_j c_k \frac{d^2}{ds^2} f_{\frac{l_j+l_k}{2}}(s) \geq 0.$$

This yields the convexity of Δ_1. Theorem 1.4.7 results that $\Gamma_1(\Delta_1)$ is non negative. It implies that the following matrix is a positive semidefinite matrix:

$$\left[\theta_1\left(\frac{l_j+l_k}{2}\right)\right]_{j,k=1}^{m}.$$

Analogously, the auxiliary functions Δ_i ($i \in \{2,3,4\}$) may be defined are helpful in proving rest of the positive semidefinite matrices.

(i), (ii) and Lemma 1.3.2 simply yield (iii) and (iv). Part (iv) is simply used to prove (v). □

6.2 Mean value theorems

Below lemma is important in proving our results.

Lemma 6.2.1. [52] *Consider $S = [\alpha, \beta](\subseteq \mathbb{R})$, $\psi \in C^2(S)$. Suppose $\psi : S \to \mathbb{R}$, ψ'' is bounded and let $\mathfrak{d} = \inf_{s\in S} \psi''(s), \mathfrak{D} = \sup_{s\in S} \psi''(s)$. It implies the convexity of the real functions ψ_1, ψ_2 defined over the set S as*

$$\begin{aligned}\psi_1(s) &= \frac{\mathfrak{D}}{2}s^2 - \psi(s)\\ \psi_2(s) &= \psi(s) - \frac{\mathfrak{d}}{2}s^2.\end{aligned} \tag{6.2.1}$$

Theorem 6.2.2. *Suppose $S = [\alpha, \beta](\subseteq \mathbb{R})$ is compact and assume a real function ψ on S, where $\psi \in C^2(S)$. Consider the points $\gamma, \delta \in S$, and a, b, c, d are defined in* (1.4.9). *It implies the existence of point $\varpi \in S$ such that*

$$a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\beta+\gamma}{2}\right) - \psi\left(\frac{\alpha+\beta}{2}\right) = \varepsilon\psi''(\varpi), \tag{6.2.2}$$

where

$$\varepsilon = \frac{1}{2}\left[a\left(\frac{\alpha+\gamma}{2}\right)^2 + b\left(\frac{\beta+\gamma}{2}\right)^2 - \left(\frac{\alpha+\beta}{2}\right)^2\right].$$

Proof. Suppose $\mathfrak{d} = \min_{s\in S} \psi''(s), \mathfrak{D} = \max_{s\in S} \psi''(s)$. Lemma 6.2.1 follows the convexity of functions $\psi_1, \psi_2 : S \to \mathbb{R}$; the continuity property also holds for ψ_1 and ψ_2. Now using the leftmost inequality of (1.4.10) yields

$$a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\beta+\gamma}{2}\right) - \psi\left(\frac{\alpha+\beta}{2}\right) \leq \varepsilon\mathfrak{D},$$

and

$$a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\beta+\gamma}{2}\right) - \psi\left(\frac{\alpha+\beta}{2}\right) \geq \varepsilon\mathfrak{d}.$$

Joining the above two inequalities and using the fact that second order derivative of ψ is continuous, results the existence of a point ϖ in S with $\mathfrak{d} \leq \psi''(\varpi) \leq \mathfrak{D}$. This proves the required result. □

Theorem 6.2.3. *$S = [\alpha, \beta](\subseteq \mathbb{R})$ is compact and assume two real functions ψ, χ on S, where $\psi, \chi \in C^2(S)$. Consider the points $\gamma, \delta \in S$, and a, b, c, d are defined in* (1.4.9). *It implies the existence of point $\varpi \in S$ such that*

$$\begin{aligned} &\psi''(\varpi)\left[a\chi\left(\frac{\alpha+\gamma}{2}\right) + b\chi\left(\frac{\beta+\gamma}{2}\right) - \chi\left(\frac{\alpha+\beta}{2}\right)\right] \\ &= \chi''(\varpi)\left[a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\beta+\gamma}{2}\right) - \psi\left(\frac{\alpha+\beta}{2}\right)\right]. \end{aligned} \tag{6.2.3}$$

Proof. Define a function $\vartheta \in C^2(S)$ by $\vartheta = e_1\chi - e_2\psi$, where

$$e_1 = a\psi\left(\frac{\alpha+\gamma}{2}\right) + b\psi\left(\frac{\beta+\gamma}{2}\right) - \psi\left(\frac{\alpha+\beta}{2}\right) \tag{6.2.4}$$

and

$$e_2 = a\chi\left(\frac{\alpha+\gamma}{2}\right) + b\chi\left(\frac{\beta+\gamma}{2}\right) - \chi\left(\frac{\alpha+\beta}{2}\right). \tag{6.2.5}$$

As the function $\vartheta \in C^2(S)$ and implementing this function to Theorem 6.2.2 implies the existence of a point $\varpi \in S$ such as

$$a\vartheta\left(\frac{\alpha+\gamma}{2}\right) + b\vartheta\left(\frac{\beta+\gamma}{2}\right) - \vartheta\left(\frac{\alpha+\beta}{2}\right) = \varepsilon\vartheta''(\varpi). \tag{6.2.6}$$

The expression on right side of this equation is non zero, whereas the one on the left side is zero. Thus it follows, $\vartheta''(\varpi) = 0$ concluding the required result. □

Remark 6.2.1. We may describe different types of means by applying (6.2.3) under the assumption that χ''/ψ'' is invertible.

Such as,

$$\varpi = \left(\frac{\chi''}{\psi''}\right)^{-1} \left(\frac{\Gamma_1(\chi)}{\Gamma_1(\psi)}\right). \tag{6.2.7}$$

Applying mean value Theorem 6.2.3 (Cauchy kind) on $\chi = f_l, \psi = f_r$ (given by Lemma 1.3.3). This implies

$$Q(l, r; \Gamma_1) = \log \varsigma(l, r; \Gamma_1)$$

provide

$$\alpha \leq Q(l, r; \Gamma_1) \leq \beta.$$

Thus $Q(l, r; \Gamma_1)$ is a mean. Now suppose l, r, u and v are real numbers such as $l \leq u, r \leq v$ then Theorem 6.1.4 results that this mean is monotonic.

$$\varsigma(l, r; \Gamma_1) = \begin{cases} \left(\frac{\Gamma_1(f_l)}{\Gamma_1(f_r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \exp\left(\frac{\Gamma_1(id.f_l)}{\Gamma_1(f_l)}\right) \cdot \exp\left(-\frac{2}{l}\right), & l = r \neq 0. \end{cases}$$

Furthermore, applying mean value Theorem 6.2.3 (Cauchy kind) on $\chi = g_l, \psi = g_r$ (given by Lemma 1.3.3). This implies the existence of an element $\varpi \in S$ so that

$$\varpi^{l-r} = \frac{\Gamma_1(g_l)}{\Gamma_1(g_r)}.$$

For distinct points l, r, we obtain

$$\alpha \leq \left(\frac{\Gamma_1(g_l)}{\Gamma_1(g_r)}\right)^{\frac{1}{l-r}} \leq \beta$$

provided that $\varpi \mapsto \varpi^{l-r}$ is invertible. This provides $\varsigma(l, r; \Gamma_1)$ is a mean which is monotonic as well, where

$$\varsigma(l, r; \Gamma_1) = \begin{cases} \left(\frac{\Gamma_1(g_l)}{\Gamma_1(g_r)}\right)^{\frac{1}{l-r}}, & l \neq r; \\ \exp\left(-\frac{\Gamma_1(g_0 \cdot g_l)}{\Gamma_1(g_l)}\right) \cdot \exp\left(\frac{1-2l}{l(l-1)}\right), & l = r \neq 1. \end{cases}$$

Remark 6.2.2. Analogous result can also be constructed for $\Gamma_j(\psi)$, $j = 2, 3, 4$ stated in Remark 6.0.1.

Chapter 7

Jensen's inequality for generalized Choquet integral and exponential concavity

This chapter applies the notion of m-exponentially concave functions on the functional Λ which is associated with the Jensen's inequality for generalized Choquet integral (1.4.11). Its outcomes lead to useful results about log-concavity and exponential concavity. The consequence of the above result provide us an interesting application in the probability. The Stolarsky type means and Cauchy and Lagrange type mean value theorems are discussed in section 3. In the last section, few applications about information and probability theory are given.

7.1 m-Exponential concavity

The following lemma is very useful.

Lemma 7.1.1. *(i) Let $\phi_p : [0, \infty) \to \mathbb{R}$, with $p \in [0, 1]$ defined as*

$$\phi_p(x) = x^p.$$

Then $p \mapsto \phi_p(x)$, and $p \mapsto \frac{d^2}{dx^2}\phi_p(x)$ are increasing and concave for each $x \in [0, \infty)$.

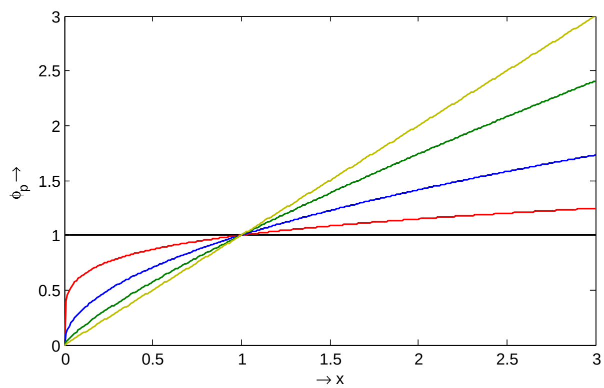

Figure 7.1: The graph of $\phi_p(x) = x^p$ for $p = 0, 0.2, 0.5, 0.8, 1$

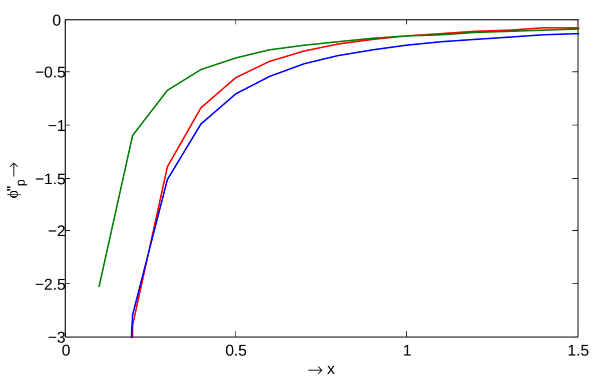

Figure 7.2: The graph of $\phi_p''(x) = p(p-1)x^{p-2}$ for $p = 0, 0.2, 0.5, 0.8, 1$

(ii) Let $\psi : [1, \infty) \to \mathbb{R}$ *is defined as*

$$\psi(x) = e^{-\frac{1}{x}}.$$

Then $x \mapsto \psi(x)$ *and* $x \mapsto \frac{d^2}{dx^2}\psi(x)$ *are increasing and concave for each* $x \in [1, \infty)$,

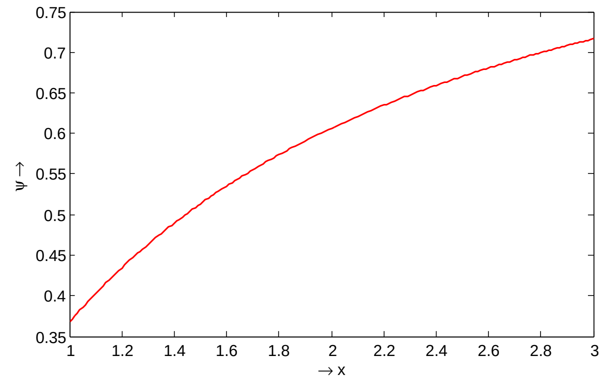

Figure 7.3: The graph of $\psi(x) = e^{-\frac{1}{x}}$

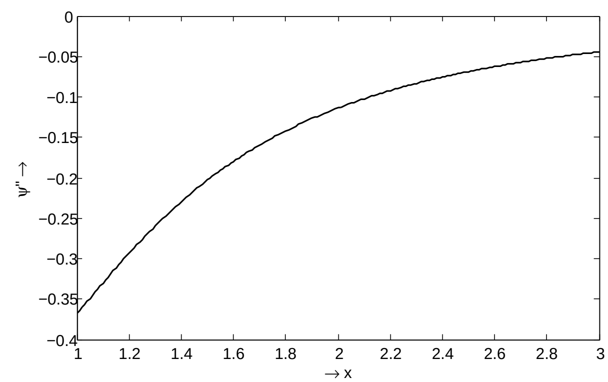

Figure 7.4: The graph of $\psi''(x) = e^{-\frac{1}{x}}(1 - 2x)/x^4$

It can be observed from Figures 1, 2, 3 and 4 that the above functions are increasing and concave.

The following lemma is very important in proving our results.

Lemma 7.1.2. *Assume $g : [0, a] \to \mathbb{R}$ be a concave function with*

$$\sup_{l \in [0,a]} g(l) = g(a), \tag{7.1.1}$$

then g is an increasing concave function.

Proof. Assume on contrary g is not an increasing function, then there exists $x, y \in [0, a]$ such that $0 \leq x < y < a$ with $g(x) \geq g(y)$. It yields $g(y) > g(a)$, but $\sup_{l \in [0,a]} g(l) = g(a)$. We get a contradiction. □

Remark 7.1.1. In Lemma 7.1.2, if $g : [0, \infty) \to \mathbb{R}$ then (7.1.1) is equivalent with

$$\lim_{a \to \infty} \sup_{l \in [0,a]} g(l) \to \infty.$$

The below given functional is acting on an increasing concave functions:

$$g \mapsto \Lambda(g) = C_{\mu\nu}(g(X)) - g(C_{\mu\nu}(X)). \tag{7.1.2}$$

Theorem 1.4.8 implies $\Lambda(g) \leq 0$.

Theorem 7.1.3. *Suppose that $g \mapsto \Lambda(g)$ is a linear functional given in* (7.1.2) *and let $\Delta_1 : [0, \infty) \to \mathbb{R}$ and $\Delta_2 : [1, \infty) \to \mathbb{R}$ defined as*

$$\Delta_1(p) = \Lambda(\phi_p), \quad \Delta_2(x) = \Lambda(\psi(x))$$

where the functions ϕ_p and ψ are given in Lemma 7.1.1. Then following assertions hold.

(i) Functions Δ_i are continuous on $[0, \infty)$ for $i = 1$, and $[1, \infty)$ for $i = 2$.

(ii) For natural numbers m and $1 \leq i \leq m$, $u_i \in [0,\infty)$, $v_i \in [1,\infty)$, the following matrices

$$\left[\Delta_1\left(\frac{u_j+u_k}{2}\right)\right]_{j,k=1}^{m}, \ \left[\Delta_2\left(\frac{v_j+v_k}{2}\right)\right]_{j,k=1}^{m}$$

are negative semidefinite.

(iii) The functions Δ_i are exponentially concave on $[0,\infty)$ for $i=1$ and, $[1,\infty)$ for $i=2$.

(iv) Assume $l,m,n \in [0,1]$ with $l<m<n$, then

$$(C_{\mu\nu}(X^m)-(C_{\mu\nu}(X))^m)^{n-l} \geq \left(C_{\mu\nu}(X^l)-(C_{\mu\nu}(X))^l\right)^{n-m} \times (C_{\mu\nu}(X^n)-(C_{\mu\nu}(X))^n)^{m-l};$$

Assume $r,s,t \in [1,\infty)$ with $r<s<t$, then

$$\left(C_{\mu\nu}\left(\exp\left(\frac{-1}{X}\right)\right)-\exp\left(\frac{-1}{C_{\mu\nu}(X)}\right)\right)^{t-r}$$

$$\geq \left(C_{\mu\nu}\left(\exp\left(\frac{-1}{X}\right)\right)-\exp\left(\frac{-1}{C_{\mu\nu}(X)}\right)\right)^{t-s} \times \left(C_{\mu\nu}\left(\exp\left(\frac{-1}{X}\right)\right)-\exp\left(\frac{-1}{C_{\mu\nu}(X)}\right)\right)^{s-r}.$$

Proof. (i) The functions Δ_i, $i=1,2$ are continuous (obvious).
(ii) For each natural numbers m and $\eta_j, u_j \in \mathbb{R}$, $(1 \leq j \leq m)$ be arbitrary and define auxiliary function $\Theta_1 : [0,\infty) \to \mathbb{R}$ by

$$\Theta_1(x) = \sum_{j,k=1}^{m} \eta_j \eta_k \phi_{\frac{u_j+u_k}{2}}(x).$$

Now $\lim_{a\to\infty} \sup_{x\in[0,a]} \phi_p(x) \to \infty$, and Lemma 7.1.1 implies

$$\Theta_1''(x) = \sum_{j,k=1}^{m} \eta_j \eta_k \frac{d^2}{dx^2} \phi_{\frac{u_j+u_k}{2}}(x) \leq 0,$$

provided that η_j and η_k have same signs and $x \geq 0$. Further Lemma 7.1.2 indicates that Θ_1 is an increasing concave function. Theorem 1.4.8 suggests $\Lambda(\Theta_1) \leq 0$. This yields

$$\left[\Delta_1\left(\frac{u_j+u_k}{2}\right)\right]_{j,k=1}^{m}$$

is negative semidefinite matrix.

By defining an auxiliary function Θ_2 in similar manner, we can conclude that

$$\left[\Delta_2\left(\frac{v_j+v_k}{2}\right)\right]_{j,k=1}^{m}$$

is negative semidefinite matrix.

(i), (ii) and Lemma 1.6.1 simply yield (iii) and (iv). □

The following model about m-exponential concavity based on Theorem 7.1.3 stated as follows:

Theorem 7.1.4. *Assume $\mathcal{C}$ be an open interval of $\mathbb{R}$, and define a family of continuous functions $\Upsilon = \{\rho_t | t \in C\}$ on $S \subseteq [0, \infty)$, such that $\lim_{a\to\infty}\sup_{x\in[0,a]}\rho_t(x) \to \infty$ and $t \mapsto \frac{d^2}{dx^2}\rho_t(x)$ is m-exponentially concave on $\mathcal{C}$ for every $x \in S$. Assume the functional $g \mapsto \Lambda(g)$ as stated in (7.1.2). Then $t \mapsto \Lambda(\rho_t)$ is an m-exponentially concave function on $\mathcal{C}$.*

Remark 7.1.2. In the above theorem, we can easily add other key components of Theorem 7.1.3.

7.2 Mean value theorems

The mean value theorems will be constructed with the help of below lemma.

Lemma 7.2.1. *Assume* $g \in C^2([0,a])$, *such that* $\sup_{l\in[0,a]} g(l) = g(a)$. *Suppose* $\mathfrak{d} = \inf_{l\in[0,a]} g''(l)$, $\mathfrak{D} = \sup_{l\in[0,a]} g''(l)$. *Then the functions* $\varphi_i : S \to \mathbb{R}$ *for* $i = 1, 2$ *described as*

$$\begin{aligned} \varphi_1(l) &= g(l) - \frac{\mathfrak{D}}{2} l^2 \\ \varphi_2(l) &= \frac{\mathfrak{d}}{2} l^2 - g(l) \end{aligned} \tag{7.2.1}$$

are concave and increasing functions.

Proof. Since the functions φ_1, φ_2 meet the hypothesis of Lemma 7.1.2, so we obtain the desired result. □

Theorem 7.2.2. *For all* $X \in L^1_{\mu\nu}$ *and* $g \in C^2([0,a])$, $\sup_{x\in[0,a]} g(x) = g(a)$. *Then there exists* $0 \le \varpi \le a$ *such that*

$$C_{\mu\nu}\left(g(X)\right) - g\left(C_{\mu\nu}(X)\right) = \sigma g''(\varpi), \tag{7.2.2}$$

where

$$\sigma = \frac{1}{2}\left[C_{\mu\nu}\left(X^2\right) - \left(C_{\mu\nu}\left(X\right)\right)^2\right].$$

Proof. Suppose $\mathfrak{D} = \max_{x\in[0,a]} g''(x)$ and $\mathfrak{d} = \min_{x\in[0,a]} g''(x)$. Then the functions $\varphi_1, \varphi_2 : [0,a] \to \mathbb{R}$ given in Lemma 7.2.1, are concave and increasing. It yields $\Lambda(\varphi_1)$, $\Lambda(\varphi_2) \le 0$, that is,

$$\sigma\mathfrak{d} \le C_{\mu\nu}\left(g(X)\right) - g\left(C_{\mu\nu}(X)\right) \le \sigma\mathfrak{D}. \tag{7.2.3}$$

Now by Bolzano theorem of intermediate value, there exists $\varpi \in [0,a]$ from (7.2.2). □

Corollary 7.2.3. *For all* $X \in L^1_{\mu\nu}$ *Let* $f, g \in C^2([0,a])$ *are concave functions with* $\sup_{x\in[0,a]} f(x) = f(a), \sup_{x\in[0,a]} g(x) = g(a)$ *and* $f(0) = g(0) = 0$. *Then* $0 \le \varpi \le a$ *exists which ensures*

$$g''(\varpi)\left[C_{\mu\nu}\left(f(X)\right) - f\left(C_{\mu\nu}(X)\right)\right] = f''(\varpi)\left[C_{\mu\nu}\left(g(X)\right) - g\left(C_{\mu\nu}(X)\right)\right] \tag{7.2.4}$$

under the assumption that the denominators are not equal to zero.

Proof. Assume an auxiliary function $\chi \in C^2([0,a])$ defined with $\chi(x) = a_1 f(x) - a_2 g(x)$, where a_1 and a_2 are defined by

$$a_1 = C_{\mu\nu}\left(g(X)\right) - g\left(C_{\mu\nu}(X)\right), \tag{7.2.5}$$

and

$$a_2 = C_{\mu\nu}\left(f(X)\right) - f\left(C_{\mu\nu}(X)\right). \tag{7.2.6}$$

Since $\Lambda(\chi) \leq 0$ and $\sup_{x\in[0,a]} \chi(x) = \chi(a)$. Now by using Theorem 7.2.2, there is some ϖ $(0 \leq \varpi \leq a)$ exists which ensures

$$C_{\mu\nu}\left(\chi(X)\right) - \chi\left(C_{\mu\nu}(X)\right) = \sigma\chi''(\varpi). \tag{7.2.7}$$

At right-hand side $\sigma \neq 0$ whereas at left-hand side, the whole expression is zero. So we conclude $\chi''(\varpi) = 0$. $\square$

Remark 7.2.1. With the existence of inverse of g''/f'', the means of different kinds can be defined by (7.2.4), that is,

$$\varpi = \left(\frac{g''}{f''}\right)^{-1}\left(\frac{\Lambda(g)}{\Lambda(f)}\right). \tag{7.2.8}$$

Let's use particularly $g(x) = \phi_n(x),\ f(x) = \phi_p(x)$ in (7.2.4) (ϕ_p functions are defined in Lemma 7.1.1), the following expressions are obtained:

$$\gamma(n,p;\Lambda) = \begin{cases} \left(\frac{(C_{\mu\nu}(X^n) - (C_{\mu\nu}(X))^n)^n}{C_{\mu\nu}(X^p) - (C_{\mu\nu}(X))^p}\right)^{\frac{1}{n-p}}, & n \neq p; \\ \exp\left(\frac{(C_{\mu\nu}(X))^n \ln(C_{\mu\nu}(X)) - X^n \ln X (C_{\mu\nu}(X^n))'}{C_{\mu\nu}(X^n) - (C_{\mu\nu}(X))^n}\right), & n = p = 1. \end{cases}$$

7.3 Applications

Theorem 7.1.3 results a significant application in the theory of probability and a captivating link can also be seen among the moments of random variables.

Corollary 7.3.1. *Let $(S, \mathcal{F})$ be a measurable space and assume X be an integrable real-valued random variable and $P = \mu = \nu$ with a finite expectation $\mathbb{E}_P(X)$, where P is the probability measure, then for $0 < m < n < p < 1$*

$$\begin{aligned}&\{\mathbb{E}_P\left(X^n\right) - (\mathbb{E}_P(X))^n\}^{p-m}\\&\geq \{\mathbb{E}_P\left(X^m\right) - (\mathbb{E}_P(X))^m\}^{p-n}\{\mathbb{E}_P\left(X^p\right) - (\mathbb{E}_P(X))^p\}^{n-m}.\end{aligned} \tag{7.3.1}$$

The Jensen type inequality stated below is a very strong and useful tool in statistics.

Proposition 7.3.2. *For $m > 1/2$, we get*

$$\mathbb{E}_P X^m \leq (\mathbb{E}_P X)^m + \left[\left(\mathbb{E}_P \sqrt[6]{X}\right)^2 + \left(\mathbb{E}_P \sqrt[6]{X}\right)\sqrt[6]{\mathbb{E}_P X} + \sqrt[3]{\mathbb{E}_P X}\right]^{3m-1/2} \times \left[\mathbb{E}_P\left(\sqrt[6]{X}\right) - \sqrt[6]{\mathbb{E}_P X}\right].$$

Proof. Choosing $j = 1/6, k = 1/2$ in (7.3.1) yield the required result. □

$D_\alpha(P||Q)$ (Rényi divergence) and $D_{KL}(P||Q)$ (Kullback-Leibler divergence) are two most important quantities in the theory of information and probability. These quantities are defined as follows:

$$D_\alpha(P||Q) = \frac{1}{\alpha - 1}\ln\int p^\alpha q^{1-\alpha} d\mu \ \ (\alpha > 1)$$

$$D_{KL}(P||Q) = \int p \ln\frac{p}{q} d\mu,$$

where p and q are probability mass functions. (For details and applications see [16] and [59].)

The following proposition is a relation between Kullback-Leibler and Rényi divergence.

Proposition 7.3.3. *$D_\alpha(P||Q) \geq D_{KL}(P||Q)$, where $\alpha > 1$*

Proof. Since $\log g$ is concave, from Theorem 2.3 of [52], we get

$$\log\left(\int_X pg d\mu\right) \geq \int_X p \log g d\mu.$$

Substitute $g(t) = \left(\frac{p}{q}\right)^{\alpha-1}(t)$, we get the required result. □

Conclusion

This dissertation is based on following four aspects; some advancement in Jensen's type inequalities, the exponentially convex functions and their construction in a similar way given by J. Jakšetić and J. Pečarić, exponential concavity, mean value theorems which led us to Stolarsky type means.

Firstly, the positive functional (defined in the form of the difference of the two sides of the known Jensen's inequality for norms) has been used to investigate the m-exponential convexity and the log-convexity. We gave an application in the probability which was the consequence of our theorem. This gave an interesting connection between moments of discrete random variables. We also gave an important advancement in the Jessen and reverse Jensen inequality by introducing weight function. We opted an elegant method of constructing m-exponential convex functions by applying the positive functionals associated with the weighted integral Jensen's inequality and weighted Jessen inequality. Some interesting results that are associated with this refined Jessen's inequality for $m(M)$-ψ-convex have been shown. We have introduced the counterpart of the refined Jessen inequality and investigate logarithmic and exponential convexity. We also gave LBP inequality for $m(M)$-ψ-convex functions. We have deduced a useful corollary and presented some of its particular cases by defining some auxiliary functions. These expressions contained some well known means which depict their usefulness. Moreover, we investigated the m-exponential convexity and the log-convexity by using positive functionals which were associated with the refinement of Hermite Hadamard inequality. We also gave an idea of logarithmic and m-exponentially concave functions. We have applied this concept on the linear functional associated with the Jensen's inequality for generalized Choquet integral. The consequence of obtained results provided us significant application in the theory of probability and a captivating link can also be seen among the moments of random variables. $D_\alpha(P||Q)$ (Rényi divergence) and $D_{KL}(P||Q)$ (Kullback-Leibler divergence) are two most important quantities in the theory of information and probability. We also gave a useful result that showed a relation between Kullback-Leibler and Rényi divergence. Furthermore, cauchy and Lagrange mean value theorems are also given which enable us to construct means with Stolarsky property. Several families of functions have also been presented related to our main results.

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