Some Aspects of Spacetime Coordinates

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A Thesis
Submitted in Partial Fulfillment of the requirements for the Degree of DOCTOR OF PHILOSOPHY in MATHEMATICS

Centre for Advanced Mathematics and Physics
National University of Sciences and Technology, Islamabad, Pakistan.

2012

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Dedicated

to

My parents and family

Acknowledgements

My entire gratitude goes to Allah Almighty who enabled me to finish my PhD thesis.

In the first place I would like to record my gratitude to my PhD supervisor, Professor Azad Akhter Siddiqui for his supervision, advice, and guidance from the very early stage of this research. His truly research intuition has made him a constant oasis of ideas and passions in research, which exceptionally inspired my growth as a researcher. I am indebted to him more than he knows. One simply could not wish for a better or friendlier supervisor.

I am thankful to many people who taught me at different stages of my life: my high school teachers, my university teachers, and my graduate teachers in Attock. I am also grateful to researchers and scholars abroad who cooperated openhandedly in my research.

My special thanks go to the Rector NUST and both former and present Principals CAMP, Professor Asghar Qadir and Professor Azad Akhter Siddiqui who provided an impressive research environment in the centre. I owe a great gratitude to HEC who funded my studies. I am grateful to the HEC digital library for helping me in many different ways.

I wish to thank my best friends Dr M. Jamil and Tanzeel-ur-Rehman for helping me get through the difficult times, and for all the emotional support and comraderie, they provided.

My parents deserve special mention for their inseparable support and prayers. My father, Riaz Hussain Shah and mother, Taqeer Fatima, in the first place are the persons who put the fundament of my learning character. Shahzad, Wajahat and Tauseef Zahra! Many thanks for being supportive and caring siblings.

I don't find words to express appreciation to my wife Toheed Zahra whose dedication, love and persistent confidence in me, took the load off my shoulder. My little kids Hammad, Ammad and Tatheer Zahra also inspired me with their innocent understanding of my work.

Finally, I would like to thank everybody who was important to the successful realization of this thesis.

S. M. JAWWAD RIAZ

Abstract

In this thesis, some aspects of spacetime coordinates are presented. After discussing some non-singular coordinates for the Schwartzschild, the Reissner-Nordström and the Kerr black hole spacetimes, non-singular Kruskal-like coordinates for different cases of general circularly symmetric black holes in $(2+1)$ dimensions are constructed. The approach is further extended to construct non-singular coordinates for the rotating BTZ black hole. As Kruskal-like coordinates do not remove the coordinate singularity for the extreme BTZ spacetime geometry, the possibility of obtaining Carter-like coordinates is discussed. It is found that these coordinates also do not remove the coordinate singularity for this geometry.

The Double-null form has great importance in general relativity (GR), especially in solar-terrestrial relationships, investigation of black hole spacetimes, formulating the Newman-Penrose formalism and Numerical Relativity etc. In Chapter 3, three dimensional spacetimes are classified according to the possibility of converting them to double-null form. It is found that a class of $(2+1)-$dimensional spacetimes in which coefficient g_{02} or g_{12} or both are non-zero, cannot be transformed to the double-null form.

In black hole thermodynamics, it has been shown earlier for different spacetimes that the Einstein field equations at the horizon can be expressed as the first law of black hole thermodynamics. In Chapter 4, a simpler approach, using the concept of foliation is developed to obtain such results. Using this simpler approach, thermodynamic identities are established for the Schwarzschild, the Reissner-Nordström, the Kerr, and the Kerr-Newmann black holes. An important aspect of this approach is that one has to essentially deal with an $(n-1)-$dimensional induced metric for an $n-$dimensional spacetime, which significantly simplifies the calculations to obtain such results.

List of publications from this thesis

[1] Syed Muhammad Jawwad Riaz and Azad A. Siddiqui
"Non-singular coordinates for circularly symmetric black holes in (2+1) dimensions" Gen. Rel. Grav. (2011) 43 : 1167-1178 ; DOI 10.1007/s10714-010-1058-5

[2] Azad A. Siddiqui1, Syed Muhammad Jawwad Riaz and M. Akbar
"Foliation and the First Law of Black hole Thermodynamics" Chin. Phys. Lett. Vol. 28, No. 5 (2011) 050401

[3] Syed Muhammad Jawwad Riaz and Azad A. Siddiqui
"Foliation and the First Law of Black Hole Thermodynamics for Kerr and Kerr-Newman Spacetimes" (Submitted to Chin. Phys. Lett.)

[4] Syed Muhammad Jawwad Riaz and Azad A. Siddiqui
"Double Null form for (2+1) dimensional Space times" (Submitted to Gen. Rel. Grav.)

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Chapter 1

Introduction

Among experimental sciences, Physics is the science of measurement. Position has basic importance among all measurements. A coordinate system (CS) is needed to measure accurate position of an object in space (or spacetime). When we choose a CS, actually we are assigning a label to each point in space. As the laws of physics are invariant, no matter what CS is chosen, therefore, to compare measurements between two different coordinate systems, we need a set of equations which relate the two different labels assigned to the same physical point. These kinds of relations are called "coordinate transformations" which relate the two coordinate systems. The simplest known CS is the cartesian CS, with three mutually-perpendicular axes, usually labeled by x, y and z. However, it is not always the best choice. It is usually most convenient to choose a CS that possess the same symmetries as the situation being modeled. Thus for physical problems possessing spherical symmetry, spherical polar coordinates are our best choice. In spherical coordinates, a point is still labeled by three numbers, i-e. the distance from the origin and two angular coordinates. Coordinate transformations relate the labels of a point in one CS to the labels of the same point in the other CS.

In Physics, the theory of relativity is considered to be one of the greatest achievements. The "Special Theory", which exclude gravity, was put forward by Einstein in 1905 to explain some troubling facts that had arisen in the study of electromagnetism. In particular, his postulate that the speed of light in vacuum is constant for all observers forced scientists to throw away many commonsense assumptions like the absolute nature of the passage of time. About a decade later, Einstein published the "General Theory of Relativity" (GR), which describes all motions as relative. Time is no longer absolute and uniform. It is taken as an added dimension along with curved space and depends on the velocity. GR catalyzed and added an essential depth of knowledge to the science of Physics. With relativity, Cosmology and Astrophysics predict extraordinary astronomical phenomena such as black holes, neutron stars, and gravitational waves. Different coordinate systems are used in black hole physics to remove the coordinate singularities to explore black hole spacetimes. In theoretical and experimental work on solar-terrestrial relationships, a variety of coordinate systems are used to display satellite trajectories, boundary locations, and vector field measurements. The need for more than one coordinate system arises from the fact that often various physical processes are understood better, experimental data more ordered or calculations are easily performed in one CS than in the other. Therefore, in those situations it is necessary to transform from one to another of these CS.

The plan of the thesis is as follows:
In the remaining sections of Chapter 1, different non-singular coordinates for the Schwarzschild, the Reissner-Nordström (RN) and the Kerr black hole spacetimes are given along with their Carter-Penrose (CP) diagrams. In Chapter 2, non-singular coordinates are obtained for different cases of general circularly symmetric black holes in $(2+1)$ dimensions are presented. The approach is further extended to construct non-singular coordinates for the rotating Banados-Teitelboim-Zanelli (BTZ) black hole along with the CP diagrams. In Chapter 3, after giving the importance of the double-null form, their existence for $(2+1)$-dimensional spacetimes is discussed. In Chapter 4, after brief description of black hole thermodynamics and foliation, a simple procedure is developed using the concept of foliation to express Einstein's field equation as the first law of black hole thermodynamics for the Schwartzschild, the RN, the Kerr, and the Kerr-Newman black hole spacetimes. The thesis concludes in Chapter 5 with a brief summery and discussion. In particular, some unsolved problems and further lines of investigation are mentioned.

1.1 Black Holes

Thermonuclear reactions in a star create pressure which supports the star against gravity. The pressure reduces due to exhaustion of nuclear fuel, as a result the balance between pressure and gravity can no longer be maintained and the star begins to contract. If the mass of the star is sufficient, so that the inward force of gravity overcomes all outward acting forces, then the collapse continues. The volume of the star continues to decrease and, therefore, the density continues to increase. Hence the escape velocity, within a trapped surface, exceeds the velocity of light. Then a star becomes a black hole.

Now the question is, what volume of a given mass is needed to trap even light and what would be the geometry of the space surrounding such a mass. Shortly after description of gravity as a geometric property of space and time (GR), in 1916, a German Physicist, K. Schwarzschild [1] calculated what the space surrounding a point mass would look like, and calculated the value of the radial parameter, r, of the trapped surface from the point, now called the Schwarzschild radius. In 1939, J. R. Oppenheimer and H. Synder [2] showed that a cold and sufficiently massive star must collapse indefinitely. In 1967, J. A. Wheeler named such an object a "black hole" [3] and Ruffini [4] identified the first observed black hole in Cygnus XI.

The basic idea of Einstein's theory of gravitation consists in geometrizing the gravitational force. Einstein postulated that a particle would travel on the straightest available path (called the geodesic) in spacetime [5]. This law should replace Newton's laws of motion. The straightness of the path depends on the spacetime curvature, therefore, gravitation can be expressed in terms of spacetime curvature. In Newtonian terms, gravitation is due to the presence of matter. According to special relativity there is no essential difference between matter and energy. Thus, in relativistic terms, the geometry of spacetime is related to the distribution of matter through the Einstein's field equations

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor which is symmetric, i.e. $G_{\mu\nu} = G_{\nu\mu}$, and vanishes when the spacetime is flat and $T_{\mu\nu}$ is the energy-momentum tensor which can be thought of as the source for the gravitational field. It is a divergence free tensor, i-e.$\nabla T_{\mu\nu} = 0$. The coupling constant $\kappa = 8\pi G/c^4$ where G is gravitational constant and c is the speed of light. The complete form of the Einstein field equations contain an extra term, called the cosmological constant, denoted by Λ. It was found recently to be extremely small and is responsible for the present accelerated expansion of the Universe. Thus the Einstein field equations with cosmological constant, Λ, can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.2}$$

The right and left hand sides of equation (1.2) provide information about the physics and the geometry of the spacetime respectively. Although these equations look simple in appearance, they are complicated and it is difficult to find an exact solution of these equations. These equations are second order, non-linear partial differential equations in the metric tensor.

In classical GR, the *no-hair* conjecture states that "all black hole solutions of the Einstein-Maxwell equations of gravitation and electromagnetism can be completely characterized by three parameters". These are:

(1) Mass, m

(2) Charge, Q

(3) Spin, usually the angular momentum, a, per unit mass.

Using this characterization, three general classes of black holes are:

(a) Static black holes with no charge and no rotation, described by the Schwarzschild solution

(b) Black holes having electrical charge, described by the RN solution

(c) Stationary rotating black holes, described by the Kerr solution.

1.2 The Schwarzschild Black Hole

When one is faced with a difficult set of mathematical equations, the first step is to look for special cases that are simple to solve. Such an approach often yields insight into the most interesting and physically relevant situations. As for any other theory of mathematical physics, it is also true for General Relativity. Therefore, for the first application of the theory of general relativity, consider a solution to the field equations that is time independent and spherically symmetric. Such a situation describes the field of gravity found outside the sun, for example. Since our interest is in the field outside the matter distribution, one can simplify things even further by restricting to the matter-free regions of space in the vicinity of some mass. In relativity, it means that one can find a solution to the problem using the vacuum field equations and ignoring the stress-energy tensor. That type of solution, found in 1916 by Schwarzschild, is named after him. The vacuum solution of the field equations (1.1), obtained by K. Schwarzschild [1], for spherically symmetric gravitational fields due to a point mass, m, is given (in gravitational units $G = c = 1$) by the metric,

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2, \tag{1.3}$$

where $f(r) = (1 - \frac{2m}{r})$, m is the mass of the Schwarzschild black hole and $d\Omega^2 = (d\theta^2 + sin^2\theta d\phi^2)$ is the line element of the $2-$sphere.

The Schwarzschild spacetime is asymptotically flat, as the metric tends to the Minkowski metric in polar coordinates as $r \to \infty$

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2. \tag{1.4}$$

A spacetime is *stationary* if and only if there exists a timelike coordinate t such that the metric is independent of t. In other words, Schwarzschild spacetime possesses time translation symmetry, since the metric remains unchanged by time translation $t \to t + t_0$, where t_0 is an arbitrary constant. Clearly the Schwarzschild metric (1.3) is independent of the timelike coordinate t and is, therefore, stationary.

A spacetime is *static* if and only if the coordinates are chosen so that in addition to time translation symmetry, the spatial coordinates do not change when move along tangent vector g_k. It requires that the tangent vector, g_k, be orthogonal to all the spatial tangent vectors $g_k \cdot g_\mu = g_{k\mu} = 0$, for $\mu \neq k$. Evidently the Schwarzschild metric (1.3) is also static. It is *spherically symmetric*. This is evident from the fact that the angular part $r^2 d\Omega^2$ of the metric is the metric of a $2-$sphere of radius r. The radius, r, in Schwarzschild coordinates is defined such that the area of the $2-$sphere measured by an observer at rest in Schwarzschild coordinates is $4\pi r^2$.

1.2.1 Singularities of the Schwarzschild Black Hole

The Schwarzschild metric (1.3) shows unusual behavior at $r = 2m$. For $r > 2m$, $g_{tt} > 0$ and $g_{rr} < 0$. However, the signs of these components of the metric (1.3) reverse for $r < 2m$. Therefore, a world line along the t-axis has $ds^2 < 0$ and describes a spacelike curve. However, a world line along the r-axis has $ds^2 > 0$ and describes a timelike curve. The space and time character of the coordinates have reversed. This shows that a massive particle inside the Schwarzschild radius $r = 2m$ can not remain stationary at $r =$ constant.

At $r = 2m$, g_{tt} is zero. The fact that g_{tt} vanishes reveals that the surface $r = 2m$ is an infinite redshift surface. Obviously something unusual is going on. However, nothing unusual appears to $g_{\theta\theta}$ and $g_{\phi\phi}$, while g_{rr} behaves very badly. In fact, $g_{rr} \to \infty$ as $r \to 2m$. When a mathematical expression tends to infinity at some point, that point is called a *singularity*. In GR, the presence of a singularity must be explored very carefully. The first question to ask is whether a singularity is physically real or it is due to a bad choice of coordinates we have made. For the Schwarzschild geometry, the singularity at $r = 2m$ is due to a bad choice of coordinates and is a *coordinate singularity*. By using a different set of coordinates, it is possible to write the metric (1.3) in such a way that the singularity at $r = 2m$ is removed. However, the singularity at $r = 0$ is due to infinite curvature and cannot be removed by a change in coordinates and is an *essential singularity*. The Riemann tensor is helpful for determining whether a singularity is essential or coordinate. If the curvature invariants become infinite the singularity is essential. Constructing scalars from the Riemann tensor one could check if they become infinite somewhere. Infinitely many scalars could be constructed from the Riemann tensor. However, symmetry considerations can be used to show that there are only finitely many independent scalars. All others can be expressed in terms of these. Thus one only needs to construct the simplest scalars. These are

$$\mathcal{R}_1 = g^{ab} R_{ab} = R \,, \tag{1.5}$$

$$\mathcal{R}_2 = R^{ab}_{cd} R^{cd}_{ab} \,, \tag{1.6}$$

$$\mathcal{R}_3 = R^{ab}_{cd} R^{cd}_{ef} R^{ef}_{ab} \,. \tag{1.7}$$

If the above defined independent curvature invariants, $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are all finite the singularity (if any) is coordinate, otherwise the singularity is essential. For the Schwarzschild metric (1.3), the invariants are

$$\mathcal{R}_1 = 0 \,, \tag{1.8}$$

$$\mathcal{R}_2 = \frac{48m^2}{r^6} \,, \tag{1.9}$$

and

$$\mathcal{R}_3 = \frac{64m^3}{r^6} \,. \tag{1.10}$$

The invariants $\mathcal{R}_2$ and $\mathcal{R}_3$ have infinite value at $r = 0$, but remain finite at $r = 2m$. Thus the singularity at $r = 2m$ is a coordinate singularity and can be removed by changing to an appropriate coordinate system.

Before proceeding to the construction of a non-singular coordinate system for the Schwarzschild spacetime, it is useful to first see the procedure used in its developing form, without the complications introduced by the spacetime curvature. This procedure is to use *null coordinates*

$$v = \frac{1}{\sqrt{2}}(t + r),\; u = \frac{1}{\sqrt{2}}(t - r), \tag{1.11}$$

so that

$$t = \frac{1}{\sqrt{2}}(v + u),\; r = \frac{1}{\sqrt{2}}(v - u), \tag{1.12}$$

and the flat spacetime represented by the Minkowski metric as

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2, \tag{1.13}$$

takes the form

$$ds^2 = 2dvdu - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2. \tag{1.14}$$

The metric tensor components in these coordinates are

$$g_{01} = g_{10} = 1,\; g_{22} = -r^2,\; g_{33} = -r^2 sin^2\theta,\; g_{\mu\nu} = 0\,, \tag{1.15}$$

and its determinant is $r^4 sin^2\theta$. The coordinates (v, u) are called null because $g_{00} = g_{11} = 0$. The use of null coordinates is not crucial for constructing non-singular coordinates but has been discussed because they are commonly used for spacetime diagrams.

1.2.2 The Eddington-Finkelstein Coordinates

The Schwarzschild black hole is studied further by examining the behavior of light cones near surface $r = 2m$. For the radial light rays ($d\theta = d\phi = 0$), the Schwarzschild metric (1.3) takes the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)}. \tag{1.16}$$

The paths of light rays are studied by setting $ds^2 = 0$, which leads to a relationship, expressing the slope of a light cone

$$\frac{dt}{dr} = \pm \left(1 - \frac{2m}{r}\right)^{-1}. \tag{1.17}$$

Notice that Eq.(1.17) as $r \to \infty$

$$\frac{dt}{dr} = \pm 1. \tag{1.18}$$

Therefore in this limit one can recover the motion of light rays in flat space (integration gives $t = \pm r$ modulo a constant, just one could expect for light cones in Minkowski space). Now examine the behavior as one approaches smaller r, specifically approaching $r = 2m$. It will be helpful to examine the positive sign, which corresponds to outward going radial null rays. Then one can write Eq.(1.17) as

$$\frac{dt}{dr} = \frac{r}{r - 2m}. \tag{1.19}$$

Observe that as $r \to 2m$, $dt/dr \to \infty$. This indicates that the light cones are becoming narrow as one approaches $r = 2m$ (at $r = 2m$, the lines become vertical). This effect is shown in Fig.(1.1). The solution of Eq.(1.19) is

$$t = r + 2m \ln |\frac{r - 2m}{2m}|. \tag{1.20}$$

This shows that the Schwarzschild time approaches $\pm\infty$ logarithmically, as null rays approach $r = 2m$. In Schwarzschild coordinates, radially falling light rays never appear to cross the horizon $r = 2m$.

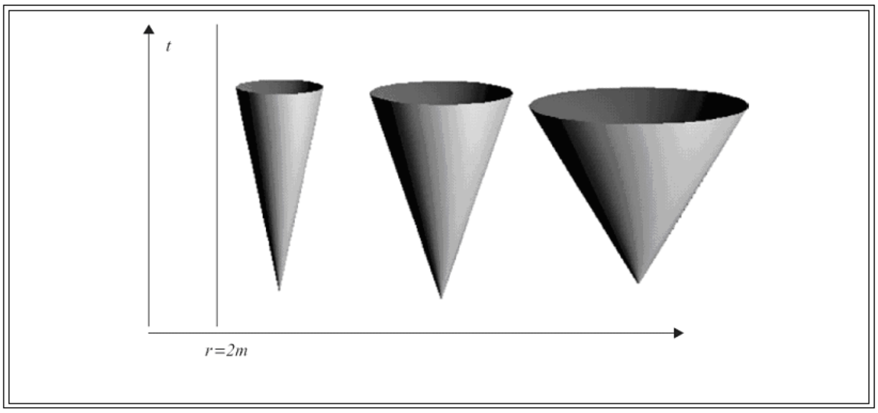

Figure 1.1: In Schwarzschild coordinates, light cones close up approaching $r = 2m$.

The first attempt to get rid of the coordinate singularity of the Schwarzschild metric was made by Eddington [6] and re-discovered by Finkelstein [7]. They carried out a transformation of the time coordinate (1.20), which seems to show that in falling light rays pass through the Schwarzschild horizon. For this purpose, they introduced a new radial coordinate, r^*, as

$$r^* = r + 2m1n\left(\frac{r}{2m} - 1\right), \tag{1.21}$$

along with two retarded and advanced null coordinates

$$u = t - r^* \quad and \quad v = t + r^*. \tag{1.22}$$

From Eq.(1.21), one can write

$$
\begin{aligned}
dr^* &= dr + \frac{dr}{(r/2m-1)}\,, \\
&= \frac{(r/2m-1)}{(r/2m-1)}dr + \frac{dr}{(r/2m-1)} = \left(\frac{r}{2m}\right)\frac{dr}{(r/2m-1)}\,, \\
&= \frac{dr}{(1-2m/r)}\,.
\end{aligned} \tag{1.23}
$$

Use the advanced coordinate, v, of Eq.(1.22) to write

$$
dt = dv - dr^* = dv - \frac{dr}{(1-2m/r)}\,. \tag{1.24}
$$

Taking squares, Eq.(1.24) takes the form

$$
dt^2 = dv^2 - 2\frac{dvdr}{(1-2m/r)} + \frac{dr^2}{(1-2m/r)^2}\,. \tag{1.25}
$$

Using Eq.(1.25), the Schwarzschild metric (1.3) takes the advanced Eddington-Finkelstein form of the metric

$$
ds^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2. \tag{1.26}
$$

The metric (1.26) is no longer singular at $r = 2m$, while the curvature singularity at $r = 0$ is clearly seen.

Once again, consider the radial paths of light rays by setting $d\theta = d\phi = 0$ and $ds^2 = 0$. This time one finds that

$$
\left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr = 0. \tag{1.27}
$$

On dividing both sides by dv^2, we obtain

$$
\left(1 - \frac{2m}{r}\right) - 2\frac{dr}{dv} = 0. \tag{1.28}
$$

If one sets $r = 2m$, $dr/dv = 0$ and one can integrate it to find $r(v) =$ constant, which describes light rays that stay right where they are, neither outgoing nor incoming. Eq.(1.28) can be written as

$$\frac{dv}{dr} = \frac{2}{(1 - 2m/r)}. \tag{1.29}$$

Integrating both sides to have

$$v(r) = 2(r + 2m1n|r - 2m|) + const. \tag{1.30}$$

Eq.(1.30) gives us the paths that radial light rays follow using (v, r) coordinates. If $r > 2m$, then as r increases, v increases. This describes the behavior one could expect for outgoing radial light rays. If $r < 2m$, as r decreases, v increases. So the light rays are ingoing. In these coordinates, light cones no longer become increasingly narrow and they make it past the line $r = 2m$. However, the fact that the time and radial coordinates reverse their characters inside $r = 2m$ means that light cones tilt over in this region (see Fig.(1.2)).

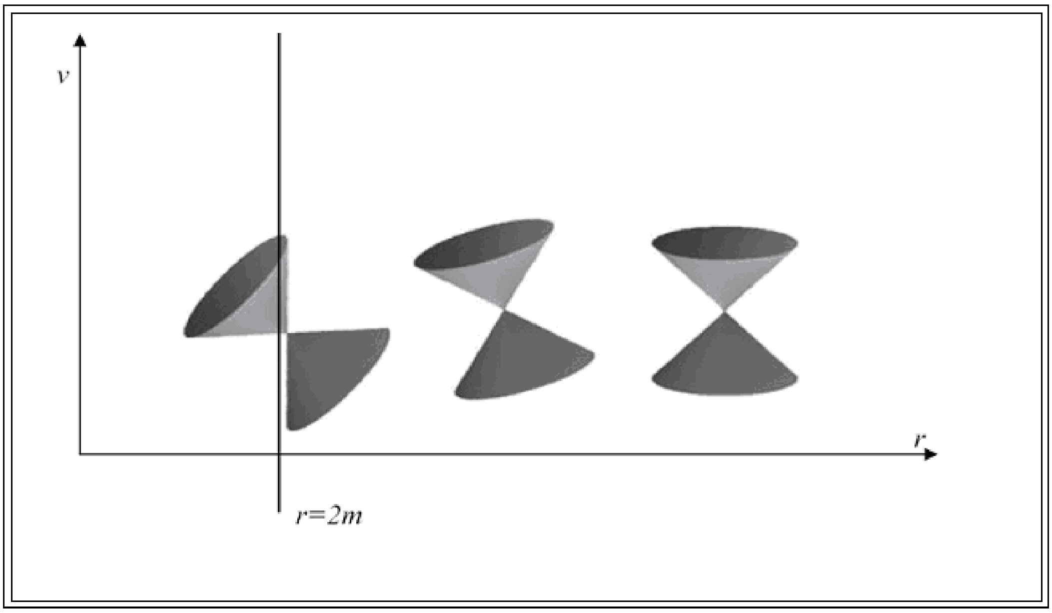

Figure 1.2: In Eddington-Finkelstein coordinates, light cones tip over as r gets smaller approaching $r = 2m$.

Using the retarded time of Eq.(1.22), the Schwarzschild metric (1.3) takes the retarded

Eddington-Finkelstein form of the metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dvdr - r^2 d\Omega^2. \qquad (1.31)$$

In short, the following points are concluded:

(a) There is no singularity in Eqs.(1.26) and (1.31) at $r = 2m$, showing that $r = 2m$ is only a coordinate singularity.

(b) Advanced Eddington-Finkelstein coordinates (v, r) give a pathological description of outgoing radial null cones. Similarly retarded Eddington-Finkelstein coordinates (u, r) give a pathological description of ingoing radial null cones.

(c) Moving towards the direction of smaller values of r, light cones begin to tip over and squeeze up.

(d) For $r < 2m$, future-directed lightlike and timelike curves are directed towards $r = 0$.

(e) In retarded coordinate, u, the Schwarzschild metric (1.3) takes the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2 d\Omega^2. \qquad (1.32)$$

Using v and u simultaneously, the metric (1.3) takes the double-null form

$$ds^2 = (1 - \frac{2m}{r})dudv - r^2 d\Omega^2. \qquad (1.33)$$

Thus Eddington-Finkelstein coordinates are well suited for describing the geometry over the region $2m < r < \infty$ and $-\infty < t < \infty$. However, another coordinate system is needed to describe the interior region of the surface $r = 2m$.

1.2.3 The Kruskal Coordinates

To describe the interior region of the surface $r = 2m$, of the Schwarzschild black hole, the Kruskal coordinates [8] can be used. In these coordinates Kruskal exponentiated the advanced and retarded coordinates (v, u) given by Eq.(1.22) and A Qadir [5] introduced two constants α, β such that the entire manifold of the maximally extended Schwarzschild solution is covered by a single coordinate patch

$$V = \alpha e^{v/\beta}, U = -\alpha e^{u/\beta}, \tag{1.34}$$

so that we have

$$VU = -\alpha^2 e^{\sqrt{2}r^*/\beta}. \tag{1.35}$$

Using Eq.(1.21) in Eq.(1.35), V and U are related to r by

$$VU = -\alpha^2 \left|\frac{r}{2m} - 1\right|^{2\sqrt{2}m/\beta} e^{\sqrt{2}r/\beta}. \tag{1.36}$$

Choose $\beta = 2\sqrt{2}m$ to get simplest form of the Eq.(1.36). For this choice of β, the metric (1.3) in Kruskal coordinates (U, V) takes the form

$$ds^2 = \frac{16m^2}{\alpha^2 r} e^{-r/2m} dU dV - r^2 d\Omega^2, \tag{1.37}$$

which is perfectly well defined and no longer singular at the horizon $r = 2m$ as determinant $= -(256m^4 sin\theta/\alpha^2)^2$ is non-zero.

In the Kruskal coordinates, the Schwarzschild spacetime is shown in Fig.(1.3). The Kruskal diagram shows the four regions I, II, III and IV bounded by event horizons. Regions I and III represent exterior region of the Schwarzschild black hole in which $r > 2m$. While regions II and IV are two identical, but time-reversed regions in which the physical singularity $r = 0$ lies and represent interior regions of the Schwarzschild black hole in which $0 < r < 2m$. Event horizons are solid straight lines at 45 degrees passing through the center of the diagram. t =constant are dotted lines pass through the center of the diagram at various angles. On the right side (of region I) lines with inclinations 45° and -45° represent $t = +\infty$ and $t = -\infty$ respectively. Also, r =constant curves are hyperbolae to which the lines with inclinations 45° and -45° are asymptotic. From the diagram, in regions I and III

the vertical hyperbolae are timelike world lines and the horizontal straight line is the spacelike hypersurface $t = 0$. In regions II and IV the vertical straight line is timelike and the horizontal hyperbolae are spacelike hypersurfaces. The horizontal hyperbolae end at $r = 0$, which is an essential singularity as is clear from Eq.(1.37), thus the diagram ends at this hyperbola.

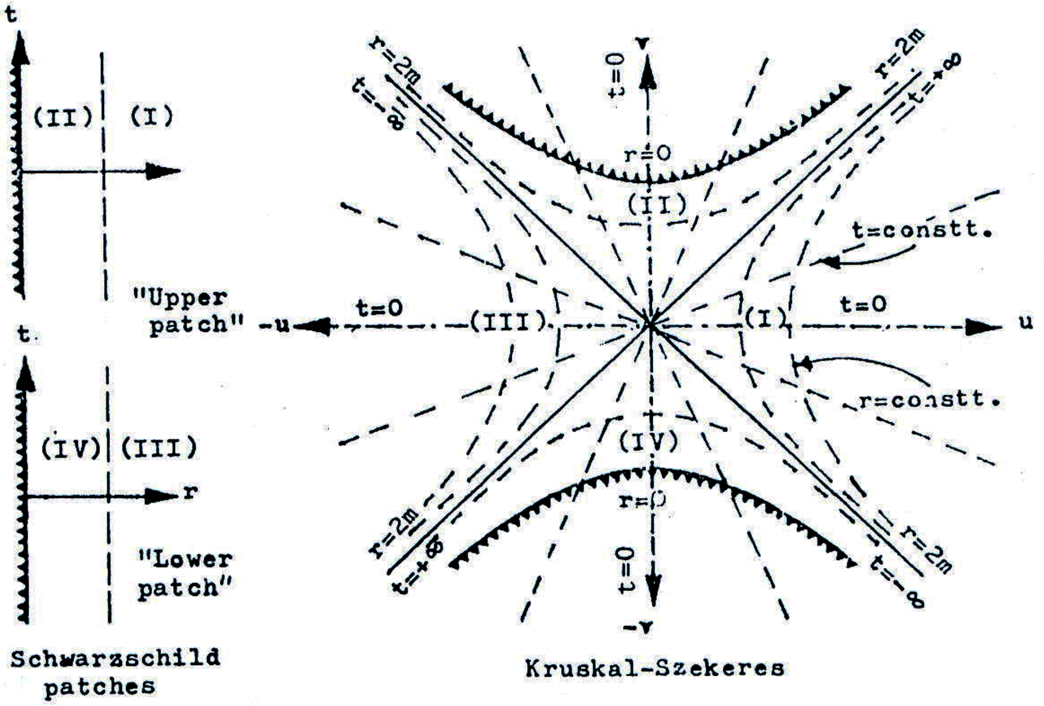

Figure 1.3: The transformation of the Schwarzschild vaccum geometry between Schwarzschild and Kruskal-Szekeres coordinates [9].

1.2.4 The Kruskal-Szekeres Coordinates

Another useful coordinate system called"Kruskal-Szekeres coordinates" can be obtained from the Kruskal coordinates given in previous subsection, which has a timelike coordinate T and a spacelike coordinate R, defined by

$$T = \frac{1}{\sqrt{2}}(V - U),\ R = \frac{1}{\sqrt{2}}(V + U). \tag{1.38}$$

In (T, R) coordinates, with $\alpha = 1$, the metric (1.37) takes the form

$$ds^2 = \frac{4m}{r}e^{-r/2m}(dT^2 - dR^2) - r^2 d\Omega^2. \tag{1.39}$$

Using the definitions of V and U, Eqs.(1.38) give

$$T = \sqrt{2}\left|\frac{r}{2m} - 1\right|^{1/2} e^{r/4m} sinh(t/4m), R = \sqrt{2}\left|\frac{r}{2m} - 1\right|^{1/2} e^{r/4m} cosh(t/4m), \tag{1.40}$$

where T varies from $-\infty$ to $+\infty$ and R from 0 to ∞. The inverse transformations are

$$R^2 - T^2 = 2\left(\frac{r}{2m} - 1\right) e^{r/2m},\ T/R = tanh(t/4m). \tag{1.41}$$

In (T, R) coordinates, outgoing and ingoing null geodesics are given by $T =$ constant and $R =$ constant respectively. Furthermore, the future and past horizons are given by the equation $TR = 0$. The curvature singularity at $r = 0$ is given by the equation $TR = 2$. Hence these coordinate covers the entire Schwarzschild spacetime and are well-behaved everywhere outside the physical singularity.

1.2.5 Compactified Kruskal-Szekeres Coordinates

Coordinates that have a finite range of values are called compactified coordinates and are useful in tracing spacetime diagrams. In order to see the entire spacetime diagram at a glance, it is required to bring regions at infinity (in space and time) to a finite position. Such representations of spacetimes are called *block diagrams* and are generally known as CP diagrams. For the Schwarzschild geometry, compactification of the Kruskal-Szekeres coordinates is achieved by defining [5]

$$\Psi = tan^{-1}(T + R) + tan^{-1}(T - R),\, \xi = tan^{-1}(T + R) + tan^{-1}(T - R). \tag{1.42}$$

The compactified Schwarzschild spacetime in (Ψ, ξ) coordinates is given in Fig(1.4) and the maximal extension in Fig.(1.5). The lower and upper boundaries in Fig(1.5) represent the past and future essential singularities respectively. The lower and upper right edges are past null infinity, $\mathfrak{I}^-$, and future null infinity, $\mathfrak{I}^+$ respectively. The past null infinity is reached by past directed null lines while the future null infinity is reached by future directed null lines. The bottom and upper right vertex are the past timelike infinity, I^-, and the future timelike infinity, I^+, respectively. These are reached by timelike lines going to the past or future, which do not come from or go into $r = 0$. Finally the right vertex is spacelike infinity, I^0. t = constant curves in region I go from I^- to I^+ and r = constant curves in this region go from the center of the diagram to I^0. A similar description is possible for other regions.

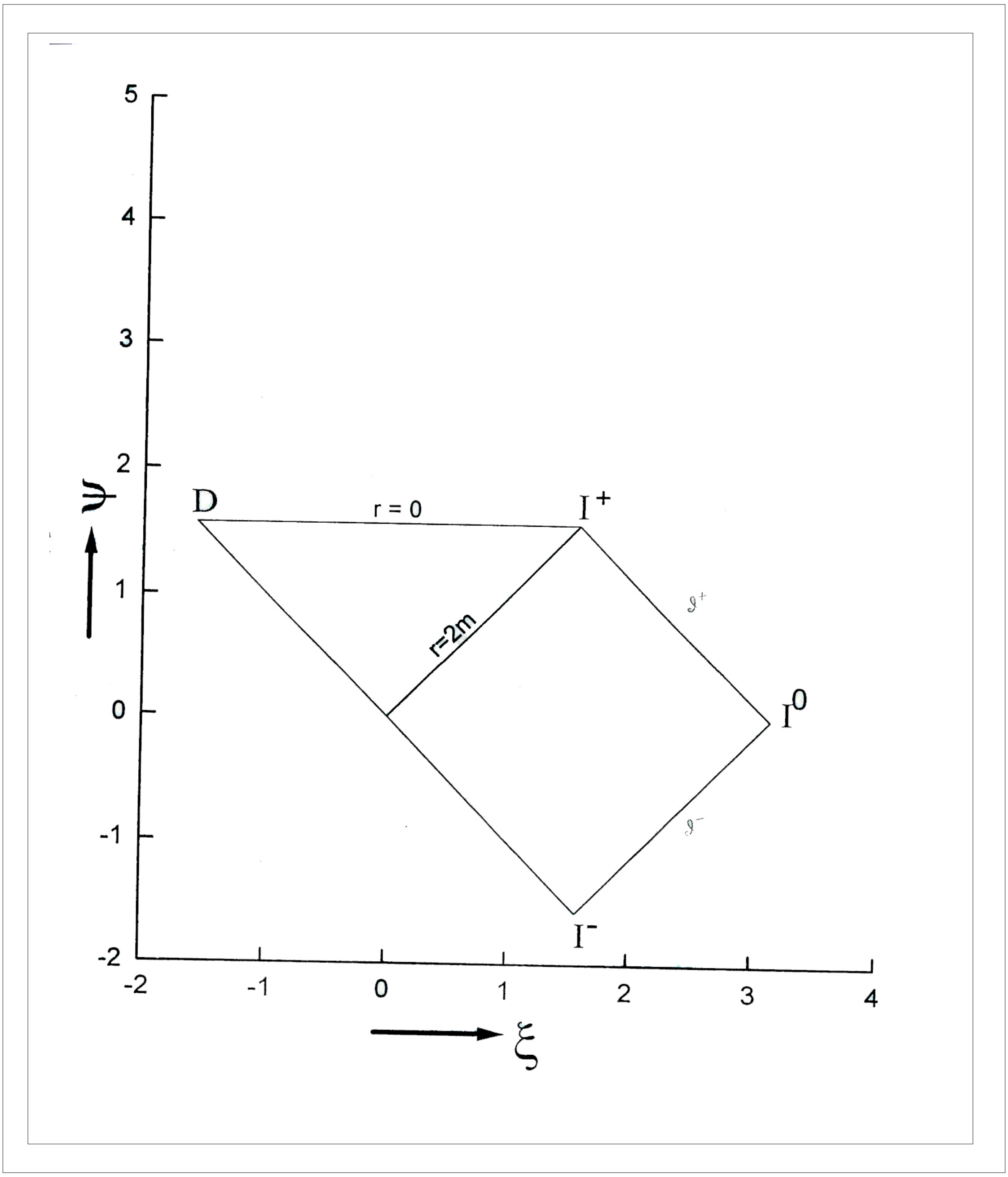

Figure 1.4: Schwarzschild spacetimes in $(\Psi, \xi, \theta, \phi)$ coordinates [9].

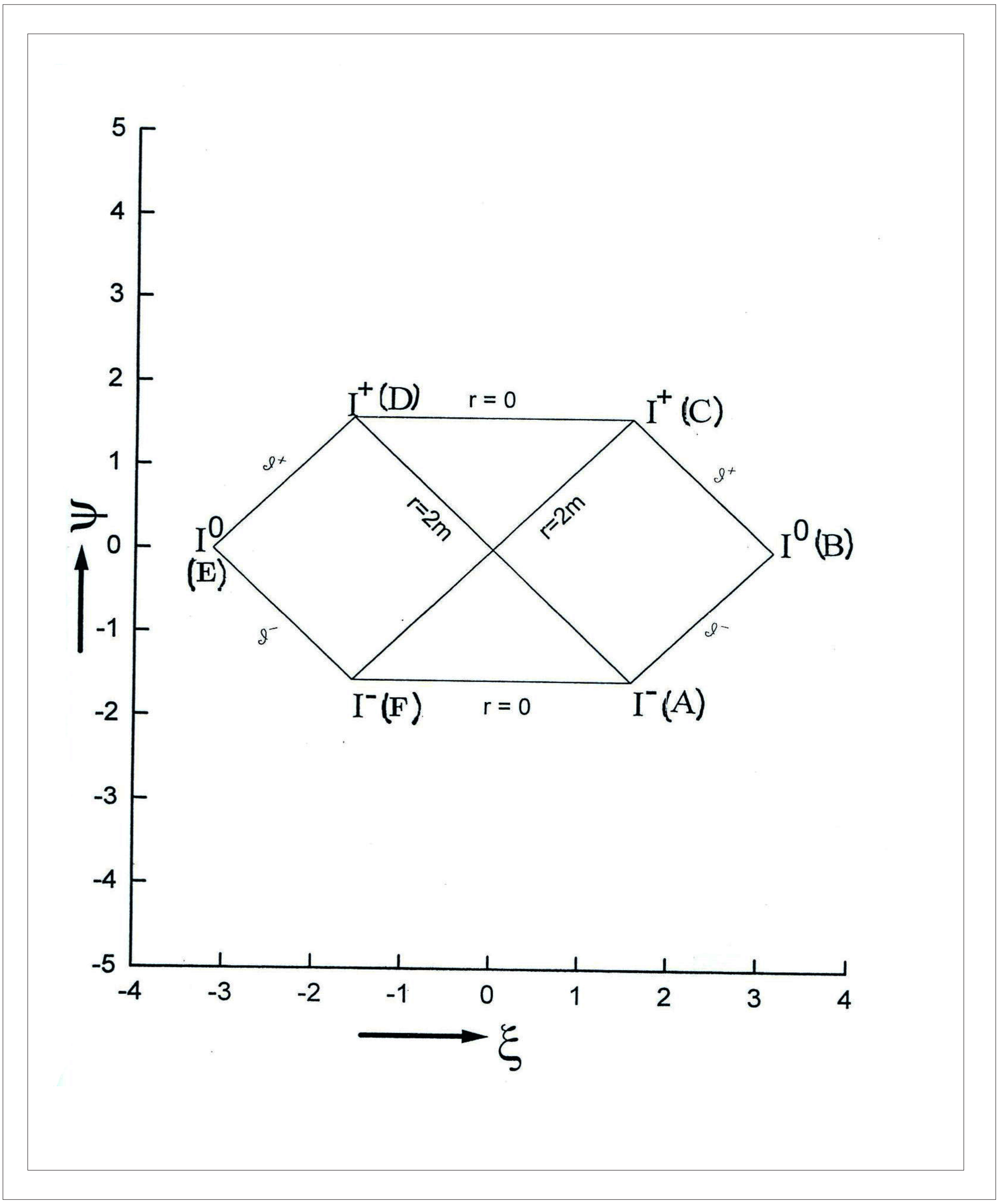

Figure 1.5: The Maximal extension of the Schwarzschild spacetime in $(\Psi, \xi, \theta, \phi)$ coordinates [9].

1.3 The Reissner-Nordström Black Hole

Reissner [10] and Nordström [11] obtained the solution of Eqs.(1.1), with a non-vanishing energy-momentum tensor arising from the electromagnetic field, which describes the field outside a spherically symmetric massive charged point, called the RN black hole and is given by

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 - \frac{1}{\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)} dr^2 - r^2 d\Omega^2, \tag{1.43}$$

where m and Q are mass and charge (in gravitational units) of the black hole respectively. The metric (1.43) looks like the Schwarzschild metric (1.3) with the replacement

$$m \to m(r) = m - \frac{Q^2}{2r}. \tag{1.44}$$

For the RN black hole, apart from the essential singularity at $r = 0$, there are two horizons

$$r_\pm = m \pm \sqrt{m^2 - Q^2}, \tag{1.45}$$

called the *outer* and *inner horizons*, respectively.

The RN time coordinate, t, is timelike outside the outer horizon, $r > r_+$, spacelike between the horizons $r_- < r < r_+$ and again timelike inside the inner horizon $r < r_-$. Conversely, the radial coordinate r is spacelike outside the outer horizon, $r > r_+$, timelike between the horizons $r_- < r < r_+$ and again spacelike inside the inner horizon $r < r_-$. This physical behavior is similar to that of the Schwarzschild geometry.

For the RN metric, the invariants given by Eqs. (1.6) and (1.7) are

$$\mathcal{R}_2 = \frac{48m^2}{r^6}\left(1 - \frac{2Q^2}{mr} + \frac{7Q^4}{6m^2r^2}\right), \tag{1.46}$$

$$\mathcal{R}_3 = \frac{64m^3}{r^9}\left(1 - \frac{3Q^2}{mr} + \frac{7Q^4}{2m^2r^2} - \frac{7Q^6}{4m^3r^3}\right). \tag{1.47}$$

Notice that both invarients $\mathcal{R}_2$ and $\mathcal{R}_3$ remain finite at $r = r_\pm$ and become infinite at $r = 0$. Therefore, $r = r_\pm$ are coordinate singularities while $r = 0$ is an essential singularity.

1.3.1 The Kruskal like Coordinates

To construct non-singular coordinates, like the Schwarzschild black hole case, it is possible to go through the steps: RN coordinates → Eddington-Finkelstein like coordinates → Kruskal-Szekeres like coordinates → Compactified Kruskal-Szekeres like Coordinates. With the previous experience of using Eddington-Finklestein coordinates in the Schwarzschild geometry, we can bypass these coordinates and proceed directly to the Kruskal like coordinates.

In terms of r_- and r_+, $f(r) = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)$ can be written as

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2}. \tag{1.48}$$

In order to remove the coordinate singularities, define r^* as

$$r^* = \int \frac{dr}{f(r)} = r + \frac{r_+^2}{(r_+ - r_-)} \ln \left|\frac{r - r_+}{k_1}\right| - \frac{r_-^2}{(r_+ - r_-)} \ln \left|\frac{r - r_-}{k_2}\right|,$$

where k_1 and k_2 are arbitrary constant of integration, which are obviously most conveniently chosen to be r_+ and r_- respectively. Using r^* to define null coordinates u and v, the definition of the Kruskal coordinates given by Eqs.(1.34) here yields

$$V = \alpha e^{t/\beta} e^{-r/\beta} \left|\frac{r}{r_+} - 1\right|^{\frac{r_+^2}{\beta(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{-\frac{r_-^2}{\beta(r_+ - r_-)}},$$

$$U = -\alpha e^{-t/\beta} e^{-r/\beta} \left|\frac{r}{r_+} - 1\right|^{\frac{r_+^2}{\beta(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{-\frac{r_-^2}{\beta(r_+ - r_-)}}.$$

V and U are related to r by

$$VU = -\alpha^2 e^{-2r/\beta} \left| \frac{r}{r_+} - 1 \right|^{\frac{2r_+^2}{\beta(r_+ - r_-)}} \left| \frac{r}{r_-} - 1 \right|^{-\frac{2r_-^2}{\beta(r_+ - r_-)}}.$$

Using these coordinates the metric (1.43) takes the form

$$ds^2 = \frac{2\beta^2}{\alpha^2}(1 - \frac{r_+}{r})(1 - \frac{r_-}{r})e^{\frac{2r}{\beta}} \left| \frac{r}{r_+} - 1 \right|^{-\frac{2r_+^2}{\beta(r_+ - r_-)}} \left| \frac{r}{r_-} - 1 \right|^{-\frac{2r_-^2}{\beta(r_+ - r_-)}} dUdV - r^2 d\Omega^2. \quad (1.49)$$

Observe that Kruskal like coordinates (V, U) do not remove both the singularities simultaneously. For this geometry, two separate coordinate patches are needed to cover the entire region of the CP diagram. One for the region $0 < r < r_+$ and the other for $r_- < r < \infty$. Thus, for coordinates, (V_+, U_+), regular at $r = r_+$ choose

$$\beta = \frac{2r_+^2}{(r_+ - r_-)},$$

and

$$\alpha = \frac{2r_+ r_-}{(r_+ - r_-)}.$$

Write $(1 - \frac{r_+}{r})$ as $(\frac{r_+}{r})(\frac{r}{r_+} - 1)$, the metric (1.49) takes the form

$$ds^2 = \frac{2r_+^2}{r^2} \left(\frac{r}{r_-} - 1 \right)^{\frac{(r_+^2 + r_-^2)}{r_+^2}} e^{-\frac{r(r_+ - r_-)}{r_+^2}} dU_+ dV_+ - r^2 d\Omega^2. \quad (1.50)$$

Clearly the metric (1.50) is non-singular at $r = r_+$. However, these coordinates do not remove the singularity at $r = r_-$.

Construct coordinates (V_-, U_-), to remove the singularity at $r = r_-$ by choosing $\beta = \frac{-2r_-^2}{(r_+ - r_-)}$. Again writing $(1 - \frac{r_-}{r})$ as $(\frac{r_-}{r})(\frac{r}{r_-} - 1)$, the metric (1.49) takes the form

$$ds^2 = \frac{2r_-^2}{r^2} \left(\frac{r}{r_+} - 1 \right)^{\frac{(r_+^2 + r_-^2)}{r_-^2}} e^{-\frac{r(r_+ - r_-)}{r_-^2}} dU_- dV_- - r^2 d\Omega^2. \quad (1.51)$$

Clearly the metric (1.51) is non-singular at $r = r_-$. Here at least two coordinate patches are required to cover the RN geometry from infinity to the singularity at $r = 0$. Whereas one coordinate patch is enough for the entire CP diagram for the Schwarzschild geometry.

1.3.2 Compactified Kruskal-Szekeres like Coordinates

The Kruskal picture is not easy to draw because of the two coordinate systems and the fact that one region stretches off to infinity, while the other hits a singularity at a finite position. To trace the CP diagram for the RN geometry, define coordinates analogues to the compactified Kruskal-Szekeres coordinates [5], as

$$\psi_+ = tan^{-1}(v_+ + u_+) + tan^{-1}(v_+ - u_+), \quad \xi_+ = tan^{-1}(v_+ + u_+) - tan^{-1}(v_+ - u_+), \quad (r_- < r < \infty) \tag{1.52}$$

$$\psi_- = tan^{-1}(v_- + u_-) + tan^{-1}(v_- - u_-), \quad \xi_- = tan^{-1}(v_- + u_-) - tan^{-1}(v_- - u_-), \quad (0 < r < r_+) \tag{1.53}$$

where the generalized Kruskal-Szekeres coordinates are,

$$v_+ = \alpha_+ exp\left(\frac{r}{\beta_+}\right) \left|\frac{r}{r_+} - 1\right|^{\frac{r_+^2}{\beta_+(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{-\frac{r_-^2}{\beta_+(r_+ - r_-)}} sinh\left(\frac{t}{\beta_+}\right), \tag{1.54}$$

$$u_+ = \alpha_+ exp\left(\frac{r}{\beta_+}\right) \left|\frac{r}{r_+} - 1\right|^{\frac{r_+^2}{\beta_+(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{-\frac{r_-^2}{\beta_+(r_+ - r_-)}} cosh\left(\frac{t}{\beta_+}\right), \tag{1.55}$$

$$v_- = \alpha_+ exp\left(\frac{r}{\beta_-}\right) \left|\frac{r}{r_+} - 1\right|^{\frac{r_+^2}{\beta_-(r_+ - r_-)}} \left|\frac{r}{r_-} - 1\right|^{-\frac{r_-^2}{\beta_-(r_+ - r_-)}} sinh\left(\frac{t}{\beta_-}\right), \tag{1.56}$$

$$u_- = \alpha_+ exp\left(\frac{r}{\beta_-}\right)\left|\frac{r}{r_+}-1\right|^{\frac{r_+^2}{\beta_-(r_+-r_-)}}\left|\frac{r}{r_-}-1\right|^{-\frac{r_-^2}{\beta_-(r_+-r_-)}} cosh\left(\frac{t}{\beta_-}\right), \tag{1.57}$$

where

$$\beta_+ = \frac{2r_+^2}{(r_+-r_-)}, \qquad \beta_- = \frac{-2r_-^2}{(r_+-r_-)}, \tag{1.58}$$

and $\alpha_\pm$ are constants. The values of α_+ and α_- are chosen so that the hypersurfaces in (v_+, u_+) and (v_-, u_-) coordinates match properly at a point, r, between r_- and r_+. The coordinates (v_+, u_+) are non-singular at $r = r_+$ and are used for $r_- < r < \infty$. Similarly, (v_-, u_-) are non-singular at $r = r_-$ and are used for $0 < r < r_+$. $r = r_\pm$ correspond to $v_+ = u_+ = 0$ and $v_- = u_- = 0$ respectively. The implicit relation between these coordinates and r is given by

$$u_+^2 - v_+^2 = \alpha_+^2 exp\left(\frac{2r}{\beta_+}\right)\left|\frac{r}{r_+}-1\right|^{\frac{2r_+^2}{\beta_+(r_+-r_-)}}\left|\frac{r}{r_-}-1\right|^{-\frac{2r_-^2}{\beta_+(r_+-r_-)}} = -tan\left(\frac{\psi_+ + \xi_+}{2}\right) tan\left(\frac{\psi_+ - \xi_+}{2}\right), \tag{1.59}$$

and

$$u_-^2 - v_-^2 = \alpha_-^2 exp\left(\frac{2r}{\beta_-}\right)\left|\frac{r}{r_+}-1\right|^{\frac{2r_+^2}{\beta_-(r_+-r_-)}}\left|\frac{r}{r_-}-1\right|^{-\frac{2r_-^2}{\beta_-(r_+-r_-)}} = -tan\left(\frac{\psi_- + \xi_-}{2}\right) tan\left(\frac{\psi_- - \xi_-}{2}\right). \tag{1.60}$$

The resulting diagram, Fig.(1.6), in compactified Kruskal-Szekeres like coordinates, is the CP diagram of the RN geometry. Fig.(1.7) shows the maximal extension. Notice that the essential singularity in this case is timelike in contrast to the Schwarzschild essential singularity which is spacelike.

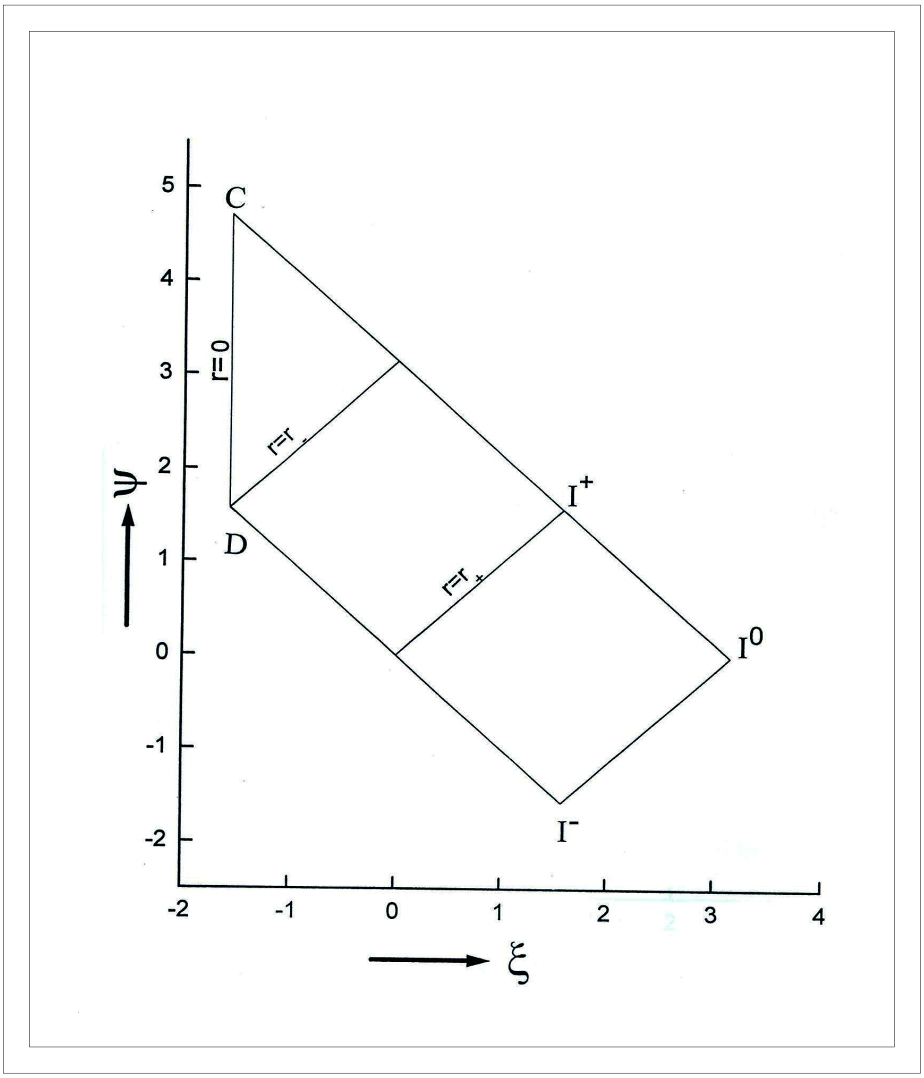

Figure 1.6: Reissner-Nordströn spacetime in $(\Psi, \xi, \theta, \phi)$ coordinates [9].

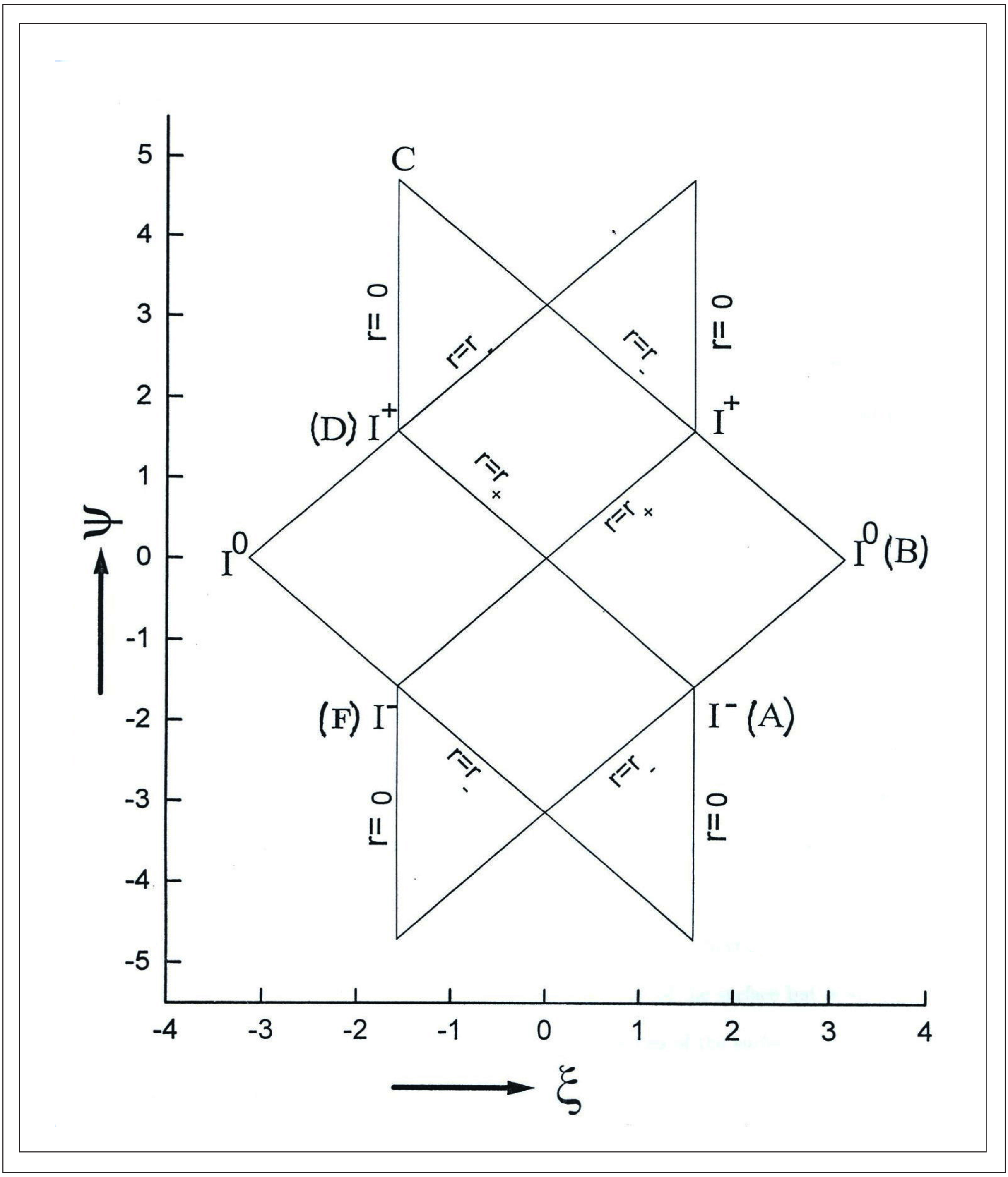

Figure 1.7: Maximal extension of the Reissner-Nordströn spacetime in $(\Psi, \xi, \theta, \phi)$ coordinates [9].

1.3.3 The Extreme Reissner-Nordström Black Hole

From Eq.(1.45), it is clear that $r_+ + r_- = 2m$. As $Q \to 0$, $r_+ \to 2m$ and $r_- \to 0$. In this limit, the RN black hole tends to become the Schwarzschild black hole, the outer horizon becomes the Schwarzschild event horizon while the inner horizon collapses on to the essential singularity. The previous analysis breaks down at $r_+ = r_- = m$ when $Q^2 = m^2$. This is called the extreme Reissner-Nordström (eRN) case.

The metric for the the eRN spacetime in (t, r, θ, φ) coordinates is

$$ds^2 = \left(1 - \frac{m}{r}\right)^2 dt^2 - \frac{dr^2}{(1 - \frac{m}{r})^2} - r^2 d\Omega^2. \tag{1.61}$$

Metric (1.61) is singular at $r = 0$ and m, the former being an essential and the latter a coordinate singularity.

1.3.4 Non-Existence of Kruskal like Coordinates for the eRN Black Hole

To avoid the coordinate singularity for a black hole in *spherical* coordinates for a spacetime, one defines r^* as

$$r^* = \int \sqrt{\frac{g_{11}}{g_{00}}} dr, \tag{1.62}$$

which is the generalization of (1.21), and the advanced and retarded coordinates [12] (v, u) as $v = t + r^*$ and $u = t - r^*$. Then the Kruskal like coordinates (V, U) are given by $V = \alpha e^{\beta v}$ and $U = -\alpha e^{-\beta u}$, where α and β are constants.

For the eRN geometry r^* is

$$r^* = r - \frac{rm}{r - m} + 2m1n \left| \frac{r - m}{m} \right|, \tag{1.63}$$

which has two different types of singularities, namely a logarithmic and a pole divergence, unlike the usual RN geometry where both divergence are logarithmic. There is an even more important difference, namely that for the usual RN geometry these singularities are at two different values of r, whereas for the eRN case both are at $r = m$. For the usual RN spacetime one can choose suitable value of β to remove either singularity from the metric, but for the eRN spacetime both singularities can not be removed simultaneously by the usual procedure as the eRN metric in (V, U) coordinates is

$$ds^2 = \frac{1}{4m^2} e^{-\frac{r}{2m}} e^{\frac{r}{r-m}} dV dU - r^2 d\Omega^2, \tag{1.64}$$

which is singular at $r = m$. There are no Kruskal like coordinates known for the extreme case.

De Felice and Clarke [13] noted the above problem and introduced the following coordinates

$$\tilde{t} = tan^{-1}V + tan^{-1}U, \quad \tilde{r} = tan^{-1}V - tan^{-1}U. \tag{1.65}$$

In these coordinates the eRN metric takes the form

$$ds^2 = \frac{(r-m)^2}{r^2} 4m^2 sin^{-1}(\tilde{t} + \tilde{r}) sin^{-1}(\tilde{t} + \tilde{r}) \left(d\tilde{t^2} - d\tilde{r^2} \right) - r^2 d\Omega^2. \tag{1.66}$$

The term $(r - m)^2$ in the metric tends to zero as the horizon is approached. The metric is not regular there and it may not even be defined for particular values of $\tilde{t}$ and $\tilde{r}$. For example as $V \to 1$ and $U \to 0$, $\tilde{t}$ and $\tilde{r}$ going to $\frac{\pi}{4}$ and, therefore, $sin^{-1}(\tilde{t} + \tilde{r})$ in the metric (1.66) becomes undefined.

1.3.5 The Carter Coordinates for the eRN Black Hole

To avoid the coordinate singularity at $r = m$, Carter introduce the coordinates [14]

$$\psi = tan^{-1}v + cot^{-1}w, \quad \xi = tan^{-1}v - cot^{-1}w, \tag{1.67}$$

where $v = t + r^*$ and $w = -t + r^*$, with r^* given as

$$r^* = \int \frac{dr}{(1 - \frac{m}{r})^2}. \tag{1.68}$$

In the $(\psi, \xi, \theta, \varphi)$ coordinates Eq.(1.61) take the form

$$ds^2 = \frac{(r-m)^2}{4r^2} sec^2\left(\frac{\psi+\xi}{2}\right) cosec^2\left(\frac{\psi-\xi}{2}\right)(d\psi^2 - \xi^2) - r^2 d\Omega^2, \tag{1.69}$$

or

$$ds^2 = \frac{(r-m)^2}{4r^2}(1+v^2)(1+w^2)(d\psi^2 - \xi^2) - r^2 d\Omega^2. \tag{1.70}$$

This metric is not manifestly regular at $r = m$, as one can not put $r = m$ in Eq.(1.69) directly. However, as $r \to m, (r - m) \to 0$, either $v \to 0$ and $w \to -\infty$, or $v \to \infty$ and $w \to 0$, one can set $w \to -\infty$, or $v \to \infty$ in such a way that $(r - m)^2(1 + w^2)$ or $(r - m)^2(1 + v^2)$ remains finite. The product is finite but can only be evaluated using L'Hospital's rule, which make these coordinates inconvenient for numerical calculations [5].

1.3.6 The RN Black Hole with Charge Exceeding Mass

The RN spacetime with $Q > m$, has no horizons. The change in geometry from extremal to one with no horizon is discontinuous. In other words there is no way to pack a black hole with more charge then its mass. Indeed if we try to force additional charge into an extremal black hole, then the work so done increases its mass so that the charge Q does not exceed its mass m.

Real fundamental particles nevertheless have charge far exceeding their mass. e.g, the charge to mass ratio of a proton is

$$\frac{e}{m_p} \approx 10^{18}$$

where e is the charge in Coulomb and m_p is the mass of the photon in Planck unit. However, the Schwarzschild radius of such a fundamental particle is far smaller then its Compton wavelenght. Therefore, quantum mechanics, nor GR, governs the structure of these fundamental particle.

1.4 The Kerr Black Hole

As an important generalization of the Schwarzschild metric, Kerr [11] presented the first axisymmetric asymptotically flat, non-static but stationary metric, describing a spinning black hole, now called a Kerr black hole. This metric is interpreted as the field generated by a material source at rest, having angular momentum. The Kerr metric, in the so-called Boyer-Lindquist coordinates, has the form

$$ds^2 = \frac{\Delta^2}{\rho^2}\left[dt - asin^2\theta d\phi\right]^2 + \frac{sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - adt\right]^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2, \quad (1.71)$$

where

$$\Delta^2 = r^2 - 2mr + a^2,$$

is the horizon function and

$$\rho^2 = r^2 + a^2cos^2\theta.$$

In metric (1.71), a is the angular momentum per unit mass of the source of the Kerr field. For $a = 0$ (absence of rotation) the Kerr metric (1.71) reduces to the Schwarzschild metric (1.3). The generalization of the Kerr metric for a spinning black hole with charge is called the charged Kerr or Kerr-Newmann black hole [12].

For the rotating gravitational source, represented by the Kerr metric (1.71), it is needed to investigate the essential and coordinate singularities. Since the metric, given by Eq. (1.71), is off-diagonal, simply putting $g_{00} = 0$ will not determine the singularities of this metric. The horizon of the Kerr black hole rotates, observed by a distant observer, therefore it is incorrect to try to solve for the location of the horizon by assuming that it is at rest. The world line of photon sitting on the horizon and battling against inflow of space, remains at fixed radius r and polar angle θ, but it moves in time t and azimuthal angle ϕ. The photon $4-$velocity is $v^\mu = \{v^0, 0, 0, v^3\}$, and the condition that it is on null geodesics is

$$0 = v_\mu v^\mu = g_{\mu\upsilon} v^\mu v^\upsilon = g_{00}(v^0)^2 + 2g_{03}v^0 v^3 + g_{33}(v^3)^2.$$

It has solutions provided the determinant of the 2×2 matrix of the metric coefficients in t and ϕ is less then or equal to zero. The determinant is

$$g = g_{00}g_{33} - g_{03}^2 = -\sin^2\theta\Delta^2. \tag{1.72}$$

Thus if $\Delta^2 \geq 0$, there exist null geodesics such that a photon can be instantaneously at rest in r and θ, whereas if $\Delta^2 < 0$, then no null geodesics exist. The boundary

$$\Delta = 0$$

defines *outer* and *inner* horizons at

$$r = r_\pm = m\left[1 \pm \sqrt{1 - \frac{a^2}{m^2}}\right]. \tag{1.73}$$

Here to determine the nature of the singularity is not so easy because calculating the independent, and not identically zero, components of the Riemann tensor is very tedious. After some very messy calculations, it is found that the singularities given by Eq. (1.73) are coordinate singularities, while there is an essential singularity at $r = a$, $\theta = \frac{\pi}{2}$, called a ring singularity.

It is possible to construct Kruskal-Szekres like coordinates here, like the RN case, by getting rid of the zeroes of g_{11} [5]. For this purpose one would only need to find r^* by integrating $r^2 dr/\Delta^2$. Here writing $\Delta = (r - r_+)(r - r_-)$ with $r_\pm$ given by Eq.(1.73), and using exactly the same formulae as before.

Chapter 2

Non-singular Coordinates for $(2+1)$-Dimensional Black Hole Spacetimes

2.1 Introduction

Lower dimensional gravity is used as an arena for investigating various problems that arise in four dimensions but are not solvable there. Those that have been investigated include quantum gravity in three dimensions [15] and black hole evaporation in two dimensions [16]. Also black hole solutions of the Einstein field equations in $(2+1)$ dimension share many important features with $(3+1)-$dimensional black holes. Both have an event horizon. They occur as an endpoint of gravitational collapse. Both show mass inflation and have non-vanishing Hawking temperature and interesting thermodynamic features [17, 18]. Black holes in $(2+1)$ dimensions provide a simple toy model for a number of studies including super-string and super-gravity theories. Another important aspect of $(2+1)$ dimensional spacetime is that it significantly simplifies the calculations in Numerical Relativity. The reason for the simplicity in $(2+1)$ dimensions lies in the fact that, in $(3+1)$ dimensional spacetime, the curvature tensor decomposes into a curvature scalar, R, a Ricci tensor, $R_{\mu\nu}$, and a remaining trace-free Weyl tensor, $C^{\sigma}{}_{\mu\nu\rho}$, whereas in $(2+1)$ dimensions, the Weyl tensor vanishes identically and the full curvature tensor is determined by the Ricci tensor and its trace as

$$R_{\mu\nu\rho\delta} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \quad (2.1)$$

Thus the fundamental difference between (2+1) and (3+1) dimensional spacetimes originates in the fact that the curvature tensor in (2+1) dimensions depends linearly on the Ricci tensor. Therefore, the structure of $(2+1)$ dimensional gravity is simple enough to allow a number of exact computations which are impractical in $(3+1)$ dimensions [19]. Work on $(2+1)$-dimensional gravity dates back to 1963 and occasional articles appeared over the next twenty years [20]. Credit for the recent growth of interest goes to Deser, Jackiw, and T Hooft [21], who examined the classical and quantum dynamics of the point source.
In the following sections non-singular coordinates for some circularly symmetric black holes in $(2+1)$ dimensions, and rotating BTZ black hole are presented.

2.2 Non-singular Coordinates for Some Circularly Symmetric $(2+1)-$Dimensional Black Holes

The circularly symmetric black hole solutions in $(2+1)-$dimensional gravity with a cosmological constant for a null fluid collapse, whose horizon structure depends on the value of k, where k is a proportionality constant between the fluid pressure, P, and energy density, ρ, is given by Viqar Husain [22] as

$$ds^2 = -F(r,v)dv^2 + 2dvdr + r^2d\theta^2, \tag{2.2}$$

where $0 < r < \infty$ is the radial coordinate, $-\infty < v < \infty$ is an advanced time coordinate, $0 \leq \theta \leq 2\pi$ is the angular coordinate and

$$F(r,v) = (1 - 2g(v) - 2h(v)r^{1-k} - \Lambda r^2). \tag{2.3}$$

Here $g(v)$, $h(v)$ are arbitrary functions and $\Lambda = -1/l^2$ (l is the radius of curvature). The metric (2.2) represents static BTZ black hole for $k = 1$ and represents black hole with multiple horizon for $k < 1$ [22].
In the following sub-sections non-singular Kruskal-like coordinates are presented for $k = 1$ and $k = 0$ cases.

2.2.1 Case 1: when $k = 1$, $g(v) = A > 0$ and $h(v) = 0$

This case represents the static BTZ black hole, the metric (2.2) takes the form

$$ds^2 = -F(r)dv^2 + 2dvdr + r^2 d\theta^2, \tag{2.4}$$

where $F(r) = (1 - 2A - \Lambda r^2)$.
In double null coordinates (v, u), metric (2.4) takes the form

$$ds^2 = -F(r)dudv + r^2 d\theta^2, \tag{2.5}$$

where $F(r) = (1 - 2A - \Lambda r^2)$. Its zeros are at $r = \pm l\sqrt{2A - 1} = \pm c$. Hence the metric (2.5) is singular at $r = \pm c$, so we can write $F(r) = \frac{(r-c)(r+c)}{l^2}$. In order to remove the coordinate singularity, we define r^* as

$$r^* = \int \frac{1}{F(r)} dr = \frac{l^2}{2c} \ln \left| \frac{r - c}{r + c} \right|.$$

Coordinates analogous to the Kruskal-like coordinates are then obtained as $V = \alpha e^{\beta v}$ and $U = -\alpha e^{-\beta u}$. Using these coordinates the metric (2.5) takes the form

$$ds^2 = \frac{(r - c)(r + c)}{l^2 \alpha^2 \beta^2} \left| \frac{r + c}{r - c} \right|^{\frac{l^2 \beta}{c}} dV dU + r^2 d\theta^2. \tag{2.6}$$

Set $\beta = c/l^2$, to get

$$ds^2 = \frac{l^2 (r + c)^2}{\alpha^2 c^2} dV dU + r^2 d\theta^2, \tag{2.7}$$

which is non-singular at $r = c$.
Coordinates U and V are related with r by

$$VU = -\alpha^2 \left| \frac{r-c}{r+c} \right|.$$

As $r \to c$, $V, U \to 0$.

Now we introduce Kruskal-Szekres-like coordinates

$$\xi = V + U, \;\; \eta = V - U.$$

In (ξ, η) coordinates the metric (2.7) takes the form

$$ds^2 = \frac{l^2(r+c)^2}{4\alpha^2 c^2}(d\xi^2 - d\eta^2) + r^2 d\theta^2, \tag{2.8}$$

where ξ and η are related with r by

$$\xi^2 - \eta^2 = -\alpha^2 \left(\frac{r-c}{r+c} \right).$$

Thus (ξ, η) coordinates cover the whole manifold and are non-singular at $r = c$.

2.2.2 Case 2: when $k = 0$, $g(v) = A > 0$ and $h(v) = B > 0$

In this case the $F(r, v)$ given by Eq.(2.3) takes the form

$$F(r) = (1 - 2A - 2Br - \Lambda r^2). \tag{2.9}$$

The zeros of $F(r)$ are given by

$$r_\pm = l^2(B \pm \sqrt{l^2 B^2 + 2A - 1}).$$

Now $F(r)$ can be written as, $F(r) = \frac{(r-r_+)(r-r_-)}{l^2}$. In order to remove the coordinate singularities at $r = r_\pm$, define r^* as

$$r^* = \int \frac{1}{F(r)} dr = \frac{l^2}{(r_+ - r_-)} \ln \left| \frac{r - r_+}{r - r_-} \right|.$$

Coordinates analogous to the Kruskal-like coordinates are then obtained as $V = \alpha e^{\beta v}$ and $U = -\alpha e^{-\beta u}$. Using these coordinates the metric (2.5) takes the form

$$ds^2 = \frac{4l^2(r - r_-)}{\alpha^2(r_+ - r^-)^2} dV dU + r^2 d\theta^2, \tag{2.10}$$

which is non-singular at $r = r_+$ but it is still singular at $r = r_-$. V and U are related to r by

$$VU = -\alpha^2 \left| \frac{r - r_+}{r - r_-} \right|,$$

$V, U \to 0$ as $r \to r_+$.

In Kruskal-Szekres-like coordinates metric (2.10) takes the form

$$ds^2 = \frac{l^2(r - r_-)}{\alpha^2(r_+ - r^-)^2} (d\xi^2 - d\eta^2) + r^2 d\theta^2, \tag{2.11}$$

where ξ and η are related with r by

$$\xi^2 - \eta^2 = -\alpha^2 \left(\frac{r - r_+}{r - r_-} \right).$$

The coordinate system (V, U) covers only the region $r_- < r < \infty$ of the whole manifold. Similarly by defining coordinates analogous to the Kruskal-like coordinates, $V = -\alpha e^{-\beta v}$ and $U = \alpha e^{\beta u}$, the singularity at $r = r_-$ in the region $0 < r < r_+$ of the spacetime can be removed.

2.3 Rotating BTZ (Banados-Teitelboim-Zanelli) Black Hole

M. Banados, C. Teitelboim, and J. Zanelli discovered, in 1992, the black hole solution of Einstein's equations with a negative cosmological constant in $(2+1)$ dimensions, called BTZ

(Banados-Teitelboim-Zanelli) black hole [23]. The BTZ solution is guaranteed by the $(2+1)$ dimensional version of Birkhoff's theorem [24]. This discovery was rather surprising as there was no speculation at that time that there would exist a black hole solution in $(2+1)$ dimensions. It attracted much interest in recent years because of its classical and quantum properties [25]. The line element of the rotating BTZ black hole can be written as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\phi - \frac{J}{2r^2}dt)^2 \,, \tag{2.12}$$

where the function $f(r)$ is given by

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \,, \tag{2.13}$$

with $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \phi \leq 2\pi$. The two constants M and J are dimensionless mass and angular momentum respectively. Note that $l = (-\Lambda)^{\frac{-1}{2}}$ is the radius of curvature that provides the length scale necessary to have horizons in a theory in which the mass is dimensionless. Horizons of the rotating BTZ metric are zeros of the function $f(r)$. Depending on these zeros, there are three different cases of the rotating BTZ metric:
(1) Two distinct real roots, usual rotating BTZ black hole
(2) Repeated real roots, extreme BTZ (EBTZ) black hole
(3) No real root, naked rotating BTZ singularity.

2.3.1 Non-singular Coordinates for the Usual Rotating BTZ Black Hole

Considering the first case when $M^2 > J^2/l^2$. For this case $f(r)$ has two distinct real roots. These roots are given by

$$r_{\pm}^2 = \frac{Ml^2}{2}\left[1 \pm \sqrt{1 - \frac{J^2}{M^2l^2}}\right] . \tag{2.14}$$

The Ricci scalar, R, for the rotating BTZ metric is given by

$$R = 6/l^2 \,, \tag{2.15}$$

which is independent of r. This suggests that $r = r_{\pm}$ are coordinate singularities. Checking the second curvature invariant confirms this expectation. In terms of r_- and r_+ , $f(r)$ can be written as

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}. \tag{2.16}$$

In order to remove the coordinate singularities, we define r^* as

$$r^* = \int \frac{1}{f(r)} dr = \frac{l^2 r_+}{2(r_+^2 - r_-^2)} \ln \left|\frac{r - r_+}{r + r_+}\right| - \frac{l^2 r_-}{2(r_+^2 - r_-^2)} \ln \left|\frac{r - r_-}{r + r_-}\right|. \tag{2.17}$$

Like the RN case, two separate coordinate patches are needed to cover the entire region of the CP diagram of the rotating BTZ black hole: one for the region $0 < r < r_+$ and the other for $r_- < r < \infty$. Coordinates analogous to the Kruskal-like coordinates, for $0 < r < r_+$, are given by $V_- = \alpha e^{\beta v}$ and $U_- = -\alpha e^{-\beta u}$, where the advanced and retarded coordinates [12] are $v = t + r^*$ and $u = t - r^*$, α and β are positive constants. Using Kruskal-like coordinates the metric (2.12) takes the form

$$\begin{aligned} ds^2 \;\; = \;\; & \frac{J^2}{16\beta^2 r^2} \left\{ \frac{1}{V_-^2} dV_-^2 + \frac{1}{U_-^2} dU_-^2 \right\} + \left\{ \frac{J^2}{8\alpha^2 \beta^2 r^2} - \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\alpha^2 \beta^2 l^2 r^2} \right\} \\ & \times \left| \frac{r - r_-}{r + r_-} \right|^{\frac{\beta l^2 r_-}{2(r_+^2 - r_-^2)}} \left| \frac{r - r_+}{r + r_+} \right|^{\frac{-\beta l^2 r_+}{2(r_+^2 - r_-^2)}} dV_- dU_- - \frac{J}{2\beta} \left\{ \frac{1}{V_-} dV_- - \frac{1}{U_-} dU_- \right\} d\phi + r^2 d\phi^2, \end{aligned} \tag{2.18}$$

where

$$V_- = \alpha e^{\beta t} \left| \frac{r - r_+}{r + r_+} \right|^{\frac{-\beta l^2 r_+}{2(r_+^2 - r_-^2)}} \left| \frac{r - r_-}{r + r_-} \right|^{\frac{\beta l^2 r_-}{2(r_+^2 - r_-^2)}},$$

$$U_- = -\alpha e^{\beta t} \left| \frac{r - r_+}{r + r_+} \right|^{\frac{-\beta l^2 r_+}{2(r_+^2 - r_-^2)}} \left| \frac{r - r_-}{r + r_-} \right|^{\frac{\beta l^2 r_-}{2(r_+^2 - r_-^2)}}.$$

V_- and U_- are related to r by

$$V_-U_- = -\alpha^2 \left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{(r_+^2-r_-^2)}}.$$

It follows that V_-, $U_- \to 0$ as $r \to r_-$ but the metric (2.18) remains singular at $r = r_+$. The coordinate system (V_-, U_-) covers only the region $0 < r < r_+$ of the whole manifold. This region is shown in Fig.(2.1).

Coordinates, analogous to the Kruskal-like coordinates, for $r_- < r < \infty$ are given by $V_+ = -\alpha e^{-\beta v}$ and $U_+ = \alpha e^{\beta u}$. Using these coordinates the metric (2.12) takes the form

$$\begin{aligned} ds^2 &= \frac{J^2}{16\beta^2 r^2}\left\{\frac{1}{V_+^2}dV_+^2 + \frac{1}{U_+^2}dU_+^2\right\} + \left\{\frac{J^2}{8\alpha^2\beta^2 r^2} - \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2}\right\} \\ &\times \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} dV_+ dU_+ + r^2 d\phi^2 - \frac{J}{2\beta}\left\{\frac{1}{V_+}dV_+ - \frac{1}{U_+}dU_+\right\} d\phi, \end{aligned} \tag{2.19}$$

where

$$V_+ = -\alpha e^{-\beta t} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}},$$

$$U_+ = \alpha e^{\beta t} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}}.$$

V_+ and U_+ are related to r by

$$V_+U_+ = -\alpha^2 \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{(r_+^2-r_-^2)}}.$$

It follows that V_+, $U_+ \to 0$ as $r \to r_+$ but the metric (2.19) remains singular at $r = r_-$. The coordinate system (V_+, U_+) covers only the region $r_- < r < \infty$ of the whole manifold. This region is shown in Fig.(2.2).

Kruskal-Szekres-like coordinates for the region $0 < r < r_+$, are defined as

$$\xi_- = V_- + U_-,\ \eta_- = V_- - U_-.$$

Using these coordinates, the metric (2.18) takes the form

$$\begin{aligned} ds^2 &= -[-\frac{J^2(\xi_-^2+\eta_-^2)}{8\beta^2 r^2(\xi_-^2-\eta_-^2)^2} + \frac{1}{4}(-\frac{J^2}{8\alpha^2\beta^2 r^2} + \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2}) \\ & \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{(r_+^2-r_-^2)}} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{(r_+^2-r_-^2)}}]d\xi_-^2 + [\frac{J^2(\xi_-^2+\eta_-^2)}{8\beta^2 r^2(\xi_-^2-\eta_-^2)^2} \\ & +\frac{1}{4}(-\frac{J^2}{8\alpha^2\beta^2 r^2} + \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2}) \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{(r_+^2-r_-^2)}} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{(r_+^2-r_-^2)}}]d\eta_-^2 \\ & +\frac{J^2\xi_-\eta_-}{2\beta^2 r^2(\xi_-^2-\eta_-^2)}d\xi_- d\eta_- + r^2 d\phi^2 - \frac{J}{\beta}[\frac{\eta_-}{\xi_-^2-\eta_-^2}d\xi_- - \frac{\xi_-}{\xi_-^2-\eta_-^2}d\eta_-]d\phi, \quad (2.20) \end{aligned}$$

here

$$\xi_- = 2\alpha\left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{2(r_+^2-r_-^2)}} \sinh(\beta t)\,,$$

$$\eta_- = 2\alpha\left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{2(r_+^2-r_-^2)}} \cosh(\beta t)\,,$$

and the inverse transformations are

$$t = \frac{1}{\beta}\tanh^{-1}(\frac{\xi_-}{\eta_-})\,,$$

$$\xi_-^2 - \eta_-^2 = -4\alpha^2\left[\frac{r-r_+}{r+r_+}\right]^{\frac{-\beta l^2 r_+}{(r_+^2-r_-^2)}} \left[\frac{r-r_-}{r+r_-}\right]^{\frac{\beta l^2 r_-}{(r_+^2-r_-^2)}}.$$

For the region $r_- < r < \infty$, define

$$\xi_+ = V_+ + U_+,\ \eta_+ = V_+ - U_+$$

Using these coordinates the metric (2.19) takes the form

$$\begin{aligned} ds^2 &= -[-\frac{J^2(\xi_+^2+\eta_+^2)}{8\beta^2 r^2(\xi_+^2-\eta_+^2)^2} + \frac{1}{4}(-\frac{J^2}{8\alpha^2\beta^2 r^2} + \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2}) \\ & \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{(r_+^2-r_-^2)}} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{(r_+^2-r_-^2)}}]d\xi_+^2 + [\frac{J^2(\xi_+^2+\eta_+^2)}{8\beta^2 r^2(\xi_+^2-\eta_+^2)^2} \\ & + \frac{1}{4}(-\frac{J^2}{8\alpha^2\beta^2 r^2} + \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2})|\frac{r-r_-}{r+r_-}|^{\frac{-\beta l^2 r_-}{(r_+^2-r_-^2)}}|\frac{r-r_+}{r+r_+}|^{\frac{\beta l^2 r_+}{(r_+^2-r_-^2)}}]d\eta_+^2 \\ & + \frac{J^2\xi_+\eta_+}{2\beta^2 r^2(\xi_+^2-\eta_+^2)}d\xi_+ d\eta_+ + r^2 d\phi^2 - \frac{J}{\beta}[\frac{\eta_+}{\xi_+^2-\eta_+^2}d\xi_+ - \frac{\xi_+}{\xi_+^2-\eta_+^2}d\eta_+]d\phi, \quad (2.21) \end{aligned}$$

where

$$\xi_+ = 2\alpha \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}} \sinh(\beta t)\,,$$

$$\eta_+ = 2\alpha \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}} \cosh(\beta t)\,,$$

and the inverse transformations are

$$t = \frac{1}{\beta}\tanh^{-1}(\frac{\xi_+}{\eta_+})\,,$$

$$\xi_+^2 - \eta_+^2 = -4\alpha^2 \left[\frac{r-r_+}{r+r_+}\right]^{\frac{\beta l^2 r_+}{(r_+^2-r_-^2)}} \left[\frac{r-r_-}{r+r_-}\right]^{\frac{-\beta l^2 r_-}{(r_+^2-r_-^2)}}.$$

The values of α and β are chosen so that the coordinates (ξ_-, η_-) and (ξ_+, η_+) are matched at a point r between r_- and r_+. The coordinates (ξ_-, η_-) are non-singular at $r = r_-$. These

are used for $0 < r < r_+$. Similarly (ξ_+, η_+) are non-singular at $r = r_+$. These are used for $r_- < r < \infty$. $r = r_\pm$ corresponds to $\xi_- = \eta_- = 0$ and $\xi_+ = \eta_+ = 0$ respectively.
Notice that the particular case $J = 0$, which corresponds to static BTZ black hole, we can recover the results presented in subsection (2.2.1).

2.3.2 Non-Existence of Carter-like Coordinate for the EBTZ Black Hole

For the extreme case the metric (2.12) takes the form

$$ds^2 = \left(\frac{r_e^4}{r^2 l^2} - g(r)\right) dt^2 + \frac{1}{g(r)} dr^2 - \frac{2r_e^2}{l} dt d\phi + r^2 d\phi^2, \tag{2.22}$$

where $g(r) = \frac{(r^2 - r_e^2)^2}{r^2 l^2}$ and $r_e^2 = \frac{Ml^2}{2}$. This metric is singular at $r = r_e$. Like the eRN case [86], Kruskal-like coordinates do not remove the singularity in EBTZ case also. Hence, to avoid the coordinate singularity, we introduce Carter-like coordinates [14]

$$\psi = tan^{-1}(\frac{v}{l}) + cot^{-1}(\frac{w}{l}),$$

$$\xi = tan^{-1}(\frac{v}{l}) - cot^{-1}(\frac{w}{l}),$$

here $v = t + r^*$ and $w = -t + r^*$, with r^* given by

$$r^* = \int \frac{r^2 l^2}{(r^2 - r_e^2)^2} dr = \frac{l^2}{4r_e} ln \left| \frac{r - r_e}{r + r_e} \right| - \frac{l^2}{2} \frac{r}{(r^2 - r_e^2)}.$$

In (ψ, ξ, ϕ) coordinates the metric (2.22) takes the form

$$ds^2 = \frac{1}{4r^2} \left\{ (f_1 + f_2) d\psi^2 + (f_1 - f_2) d\xi^2 + \frac{r_e^4}{2} f_3 f_4 d\psi d\xi \right\} - \frac{r_e^2}{2} (f_3 d\psi + f_4 d\xi) d\phi + r^2 d\phi^2, \tag{2.23}$$

where

$$f_1 = \frac{r_e^4}{4} \{\sec^4(\frac{\psi + \xi}{2}) + \csc^4(\frac{\psi - \xi}{2})\},$$

$$f_2 = \{\frac{r_e^4}{2} - (r^2 - r_e^2)^2\} sec^2(\frac{\psi+\xi}{2}) \csc^2(\frac{\psi-\xi}{2}) \, ,$$

$$f_3 = \sec^2(\frac{\psi+\xi}{2}) + \csc^2(\frac{\psi-\xi}{2}) \, ,$$

and

$$f_4 = sec^2(\frac{\psi+\xi}{2}) - \csc^2(\frac{\psi-\xi}{2}) \, .$$

Note that ψ and ξ are related to the radial parameter by

$$tan(\frac{\psi+\xi}{2}) + cot(\frac{\psi-\xi}{2}) - l \left[\frac{1}{2r_e} ln|\frac{r-r_e}{r+r_e}| - \frac{r}{(r^2-r_e^2)} \right] = 0. \qquad (2.24)$$

The determinant of the metric (2.23) is

$$|g_{ab}| = -\frac{(r-r_e)^4 (r+r_e)^4}{16r^2} sec^4(\frac{\psi+\xi}{2}) \csc^4(\frac{\psi-\xi}{2}) \, , \qquad (2.25)$$

where

$$\csc^4(\frac{\psi-\xi}{2}) = (1 + \frac{w^2}{l^2})^2 = (1 + \frac{1}{l^2}(-t + r^*)^2)^2. \qquad (2.26)$$

or

$$
\begin{aligned}
\csc^4(\frac{\psi-\xi}{2}) &= (\frac{l^2}{4r_e}[(r-r_e)\ln(r-r_e)-(r-r_e)\ln(r+r_e)-r_e-\frac{r_e(r-r_e)}{(r+r_e)}])^4 \\
&\quad -4(r-r_e)(\frac{l^2}{4r_e}[(r-r_e)\ln(r-r_e)-(r-r_e)\ln(r+r_e)-r_e \\
&\quad -\frac{r_e(r-r_e)}{(r+r_e)}])^3 t+2(r-r_e)^2(\frac{l^2}{4r_e}[(r-r_e)\ln(r-r_e)-(r-r_e)\ln(r+r_e) \\
&\quad -r_e-\frac{r_e(r-r_e)}{(r+r_e)}])^2(l^2+3t^2)-4(r-r_e)^4(\frac{l^2}{4r_e}[(r-r_e)\ln(r-r_e) \\
&\quad -(r-r_e)\ln(r+r_e)-r_e-\frac{r_e(r-r_e)}{(r+r_e)}])(l^2+t^2)t+(r-r_e)^4(l^2+t^2)^2.
\end{aligned}
\tag{2.27}
$$

Notice that as $r \to r_e$ and $\psi \to \xi$, the value of the determinant is finite $-\frac{r_e^2 l^4}{256}\sec^4(\xi)$, but the metric (2.23) is still not regular at $r = r_e$. Therefore, Carter-like coordinates do not remove the singularity in extreme BTZ case.

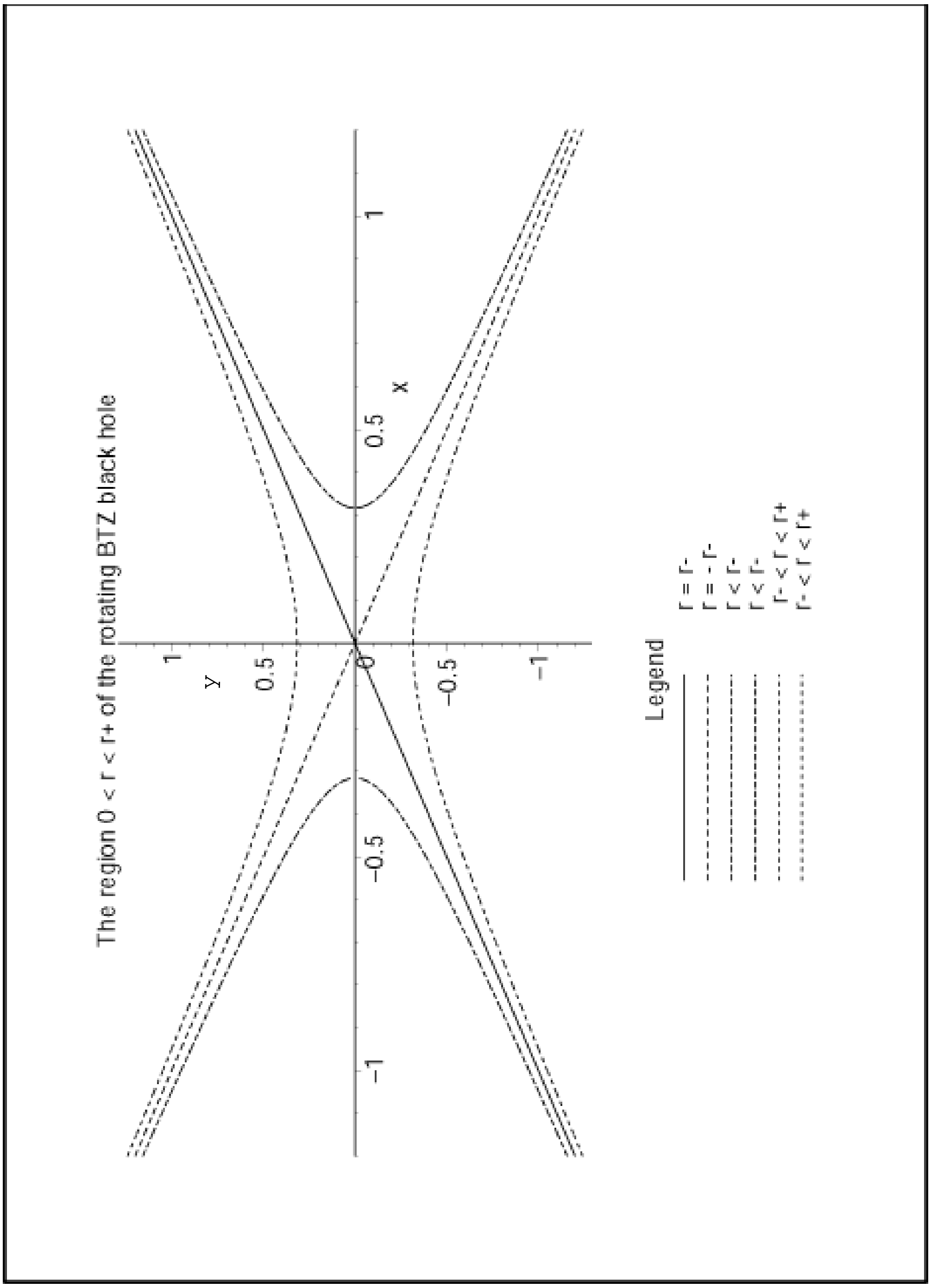

Figure 2.1: The region $0 < r < r_+$ of the rotating BTZ black hole. η_- is along x-axis and ξ_- is along y-axis.

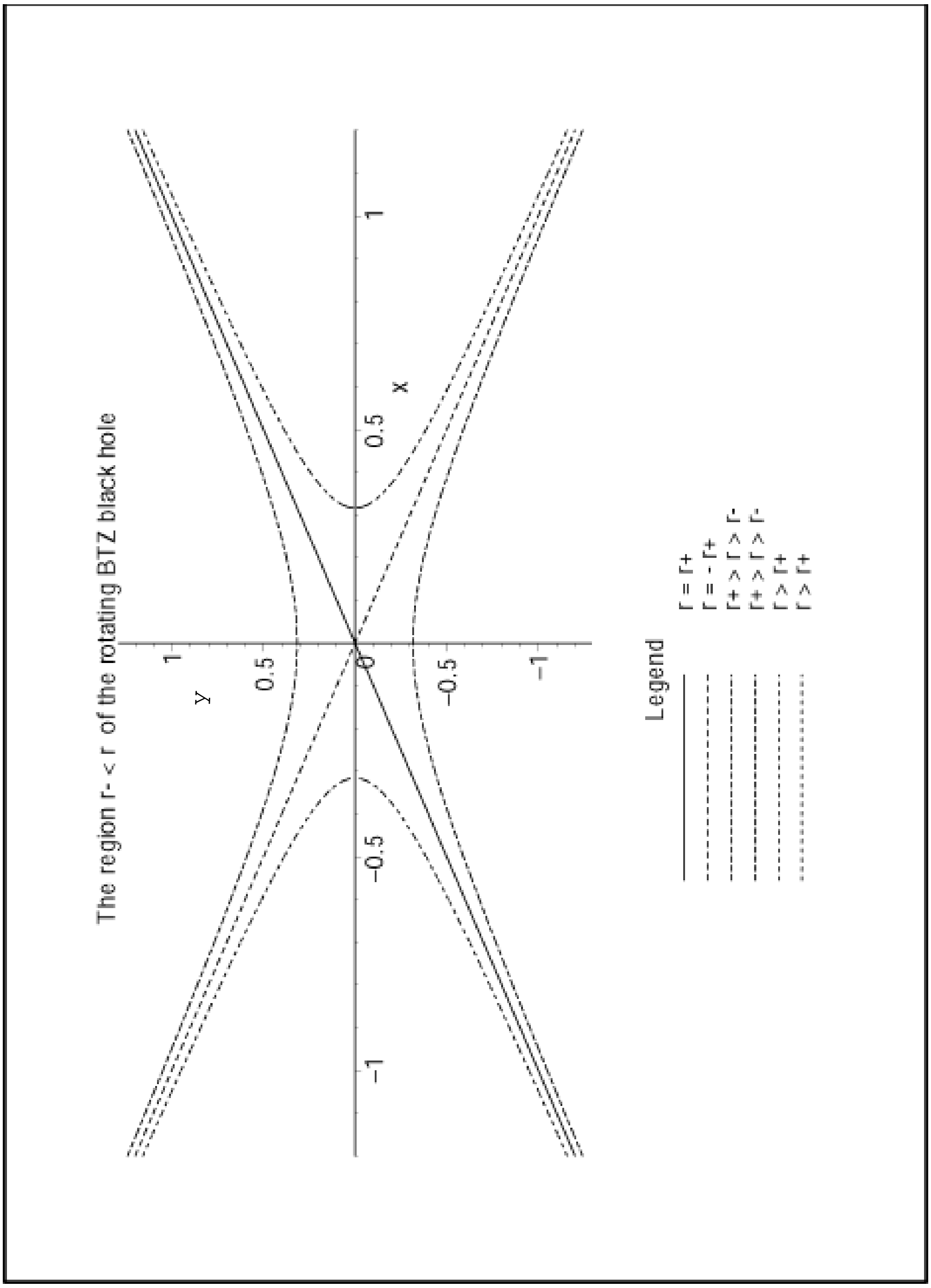

Figure 2.2: The region $r > r_-$ of the rotating BTZ black hole. η_+ is along x-axis and ξ_+ is along y-axis.

Chapter 3

Double-Null Form for $(2+1)$-Dimensional Spacetimes

Different forms of spacetime metrices are used in the study of solar-terrestrial relationships, investigation of black hole spacetimes, the Newman-Penrose (NP) formalism and Numerical Relativity etc. The need for more than one forms arises from the fact that often various physical processes are more understood, experimental data more ordered, or calculations more easily performed in one form than in another. The double-null form for $(2+1)$ dimensional spacetime is

$$ds^2 = 2\widetilde{g}_{01}dudv + \widetilde{g}_{22}d\phi^2, \tag{3.1}$$

where $\widetilde{g}_{01}$ and $\widetilde{g}_{22}$ depend on u, v and ϕ.

The double-null form is widely used by Howard in his investigation of black hole spacetimes [26] and also in earlier work by Roman and Bergman [27]. It is also helpful in simplification of different calculations in NP formalism which was developed by Ezra T. Newman and Roger Penrose [28, 29, 30, 31]. In case of vacuum space-times, the NP formalism simplifies dramatically in double-null form, which allows many theorems to be proven easily.

Double-null form also plays a key role in Numerical Relativity which is applied to many areas, such as cosmological models, perturbed black holes and neutron stars [32, 33, 34]. In numerical analysis, careful attention is paid to the stability and convergence of the numerical solution which is usually possible using double-null form of the spacetime. Apart from NP formalism and Numerical Relativity, the double-null form also helps to formulate light-cone gauge of string theory which is a great attempt to reunite quantum mechanics and general relativity [35, 36, 37, 38, 39]. In the following sections we discuss existence of double-null form for (2+1) dimensional spacetimes.

3.1 Existence of Double-Null Form for (2+1)-Dimensional Spacetimes

Consider the general spacetime metric in three dimensions

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\phi^2 + 2g_{01}dtdr + 2g_{02}dtd\phi + 2g_{12}drd\phi\,, \tag{3.2}$$

where $g_{00}, g_{11}, g_{22}, g_{01}, g_{02}$ and g_{12} depend on t, r and ϕ.
In order to transform metric (3.2) into the double-null form (3.1), consider $t = t(u, v)$, $r = r(u, v)$ and use the coordinate transformations

$$\widetilde{g}_{ab} = \frac{\partial x^l}{\partial \widetilde{x}^a}.\frac{\partial x^m}{\partial \widetilde{x}^b}.g_{lm}\,, \tag{3.3}$$

where x^l, x^m and $\widetilde{x}^a$, $\widetilde{x}^b$ refer to the (t, r, ϕ) and (u, v, ϕ) coordinates respectively, with $l, m, a, b = 0, 1, 2$, to obtain the following system of partial differential equations

$$\widetilde{g}_{00} = \left(\frac{\partial t}{\partial u}\right)^2 g_{00} + \left(\frac{\partial r}{\partial u}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{01}, \tag{3.4}$$

$$\widetilde{g}_{11} = \left(\frac{\partial t}{\partial v}\right)^2 g_{00} + \left(\frac{\partial r}{\partial v}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{01}, \tag{3.5}$$

$$\widetilde{g}_{22} = g_{22}, \tag{3.6}$$

$$\widetilde{g}_{01} = \left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{00} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{11} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right) g_{01}, \tag{3.7}$$

$$\widetilde{g}_{02} = \left(\frac{\partial t}{\partial u}\right) g_{02} + \left(\frac{\partial r}{\partial u}\right) g_{12}, \tag{3.8}$$

$$\widetilde{g}_{12} = \left(\frac{\partial t}{\partial v}\right) g_{02} + \left(\frac{\partial r}{\partial v}\right) g_{12}. \tag{3.9}$$

Requiring $\widetilde{g}_{00} = \widetilde{g}_{11} = \widetilde{g}_{02} = \widetilde{g}_{12} = 0$ in the above system of Eqs.(3.4) to (3.9), we find

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{00} + \left(\frac{\partial r}{\partial u}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{01} = 0, \tag{3.10}$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{00} + \left(\frac{\partial r}{\partial v}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{01} = 0, \tag{3.11}$$

$$g_{22} = \widetilde{g}_{22}, \tag{3.12}$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{00} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{11} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right) g_{01} = \widetilde{g}_{01}, \tag{3.13}$$

$$\left(\frac{\partial t}{\partial u}\right) g_{02} + \left(\frac{\partial r}{\partial u}\right) g_{12} = 0, \tag{3.14}$$

$$\left(\frac{\partial t}{\partial v}\right) g_{02} + \left(\frac{\partial r}{\partial v}\right) g_{12} = 0. \tag{3.15}$$

Requiring the Jacobian,

$$J = \left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) - \left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial u}\right), \tag{3.16}$$

of the transformations from (t, r, ϕ) to (u, v, ϕ) to be non-zero, Eqs.(3.14) and (3.15) have only trivial solution

$$g_{02} = g_{12} = 0.$$

This shows that if the metric (3.2) contains g_{02} or g_{12} then it cannot be transformed into the double-null form (3.1). Therefore, the most general form of a $(2+1)$ dimensional spacetime metric that possibly can be transformed into the double null form is

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\phi^2 + 2g_{01}dtdr. \tag{3.17}$$

Now in order to obtain specific transformations to transform metric (3.17) in double-null form we consider the following cases:
(1) $g_{00}, g_{11}, g_{22} \neq 0$ and $g_{01} = 0$
(2) $g_{00}, g_{22}, g_{01} \neq 0$ and $g_{11} = 0$
(3) $g_{11}, g_{22}, g_{01} \neq 0$ and $g_{00} = 0$
(4) $g_{00}, g_{11}, g_{22}, g_{01} \neq 0$.

3.1.1 Case 1: $g_{00}, g_{11}, g_{22} \neq 0$ and $g_{01} = 0$

The system of Eqs.(3.10) to (3.13) becomes

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{00} + \left(\frac{\partial r}{\partial u}\right)^2 g_{11} = 0, \tag{3.18}$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{00} + \left(\frac{\partial r}{\partial v}\right)^2 g_{11} = 0, \tag{3.19}$$

$$g_{22} = \widetilde{g}_{22}, \tag{3.20}$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{00} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{11} = \widetilde{g}_{01}. \tag{3.21}$$

Requiring the Jacobian (3.16) to be non-zero, Eqs.(3.18) and (3.19) imply

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) + \left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial u}\right) = 0, \tag{3.22}$$

which gives the following transformations

$$t = \Phi(u \pm v), \;\; r = \Psi(u \mp v), \tag{3.23}$$

where Φ and Ψ are arbitrary functions.

3.1.2 Case 2: $g_{00}, g_{22}, g_{01} \neq 0$ and $g_{11} = 0$

In this case the system of Eqs.(3.10) to (3.13) becomes

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{00} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{01} = 0, \tag{3.24}$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{00} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{01} = 0, \tag{3.25}$$

$$g_{22} = \widetilde{g}_{22}, \tag{3.26}$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right)g_{00}+\left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v}+\frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right)g_{01}=\widetilde{g}_{01}. \tag{3.27}$$

Eqs.(3.24) and (3.25) can be written as

$$\frac{\partial t}{\partial u}\left(\frac{\partial t}{\partial u}g_{00}+2\frac{\partial r}{\partial u}g_{01}\right)=0 \tag{3.28}$$

and

$$\frac{\partial t}{\partial v}\left(\frac{\partial t}{\partial v}g_{00}+2\frac{\partial r}{\partial v}g_{01}\right)=0. \tag{3.29}$$

Eqs.(3.28) and (3.29) are satisfied for the following cases:
(a) $\frac{\partial t}{\partial u}=\frac{\partial t}{\partial v}=0$
(b) $\frac{\partial t}{\partial u}g_{00}+2\frac{\partial r}{\partial u}g_{01}=\frac{\partial t}{\partial v}g_{00}+2\frac{\partial r}{\partial v}g_{01}=0\Rightarrow g_{01}=g_{00}=0$
(c) $\frac{\partial t}{\partial u}=\frac{\partial t}{\partial v}g_{00}+2\frac{\partial r}{\partial v}g_{01}=0$
(d) $\frac{\partial t}{\partial v}=\frac{\partial t}{\partial u}g_{00}+2\frac{\partial r}{\partial u}g_{01}=0.$
In cases (a) and (b), the double-null form is not possible as they lead to contradictions $\widetilde{g}_{01}=0$ and $g_{00}=0=g_{01}$ respectively. However, existence of double-null form is possible for cases (c) and (d).

3.1.3 Case 3: $g_{11},g_{22},g_{01}\neq 0$ and $g_{00}=0$

In this case the system of Eqs.(3.10) to (3.13) takes the form

$$\left(\frac{\partial r}{\partial u}\right)^2 g_{11}+2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right)g_{01}=0, \tag{3.30}$$

$$\left(\frac{\partial r}{\partial v}\right)^2 g_{11}+2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right)g_{01}=0, \tag{3.31}$$

$$g_{22} = \widetilde{g}_{22}, \tag{3.32}$$

$$\left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{11} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right) g_{01} = \widetilde{g}_{01}. \tag{3.33}$$

Eqs.(3.30) and (3.33) can be written as

$$\frac{\partial r}{\partial u}\left(\frac{\partial r}{\partial u} g_{11} + 2\frac{\partial t}{\partial u} g_{01}\right) = 0 \tag{3.34}$$

and

$$\frac{\partial r}{\partial v}\left(\frac{\partial r}{\partial v} g_{11} + 2\frac{\partial t}{\partial v} g_{01}\right) = 0. \tag{3.35}$$

Eqs.(3.34) and (3.35) are satisfied for the following cases:
(a) $\frac{\partial r}{\partial u} = \frac{\partial r}{\partial v} = 0$
(b) $\frac{\partial r}{\partial u} g_{11} + 2\frac{\partial t}{\partial u} g_{01} = \frac{\partial r}{\partial v} g_{11} + 2\frac{\partial t}{\partial v} g_{01} = 0 \Rightarrow g_{01} = g_{11} = 0$
(c) $\frac{\partial r}{\partial u} = \frac{\partial r}{\partial v} g_{11} + 2\frac{\partial t}{\partial v} g_{01} = 0$
(d) $\frac{\partial r}{\partial v} = \frac{\partial r}{\partial u} g_{11} + 2\frac{\partial t}{\partial u} g_{01} = 0.$
Cases (a) and (b), again lead to contradictions $\widetilde{g}_{01} = 0$ and $g_{11} = 0 = g_{01}$ respectively, however, existence of double-null form is possible for cases (c) and (d).

3.1.4 Case 4: $g_{00}, g_{11}, g_{22}, g_{01} \neq 0$

In this case, the following system of partial differential equations

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{00} + \left(\frac{\partial r}{\partial u}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{01} = 0, \tag{3.36}$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{00} + \left(\frac{\partial r}{\partial v}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{01} = 0, \tag{3.37}$$

$$g_{22} = \widetilde{g}_{22}, \tag{3.38}$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{00} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{11} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right) g_{01} = \widetilde{g}_{01}, \tag{3.39}$$

is to be solved, along with requiring the Jacobian (3.16) to be non-zero, to obtain the transformations which will give the required double-null form.

Chapter 4

Foliation and the First Law of Black Hole Thermodynamics

4.1 Thermal Physics

One of the most electrifying and rapidly developing branches of the study of black holes is "black hole thermodynamics". The law of conservation of energy is the basis of the subject. Historically, thermodynamics is developed to understand the energy conversion involving heat and other forms of energy. Later Maxwell, Boltzmann and others gave this theory a sound base to give a better comprehension regarding the macroscopic variables as driving force for the microscopic mechanical variables. Classical thermodynamics deals only with equilibrium states of a system. The laws of thermodynamics form an axiomatic basis of thermodynamics and were accepted after a lot of careful experimentation. These laws are a complete set of logically sufficient axioms, from which the rest of thermodynamics can be extracted. The laws of thermodynamics explain temperature equivalence (zeroth law), energy conservation (first law), entropy tendencies (second law) and condition for an absence of temperature (third law). The four laws of thermodynamics are [40]-[46]:

1) **Zeroth Law of Thermodynamics**

"Objects in thermal equilibrium with a third object are in thermal equilibrium with each other". This is also known as the principle of temperature measurement. It enables us to define the parameter, temperature as heat flows from a higher temperature to lower temperature and so there is no heat flow when two systems are at the same temperature. It was formulated

by Ralph Fowler in 1931, as stated above. Scottish physicist Joseph Black in 1738 gave the theoretical framework for the zeroth law. Almost after 100 years of Joseph Black, Scottish physicist James Clerk Maxwell published *"laws of equal temperature"* in 1871, which was considered the first formulation by most references. The term *"zeroth law"* was coined by Fowler and Edward in 1939.

In classical thermodynamics absolute zero is the temperature at which an ideal gas would have zero volume. From the kinetic theory of gases and Boltzmanns statistical mechanics, the absolute temperature, T, is related to the energy density, E, of random motion of the particles in the system by the formula [47]

$$E = \frac{3}{2}nkT, \tag{4.1}$$

where n is the number of moles per unit volume of particles in the system and k is the Boltzmanns constant, which has a value $1.3810 - 16ergs/^{\circ}K$.

2) **First Law of Thermodynamics**

"In an isolated system the net deficit of energy on inter-converting other forms of energy is equivalent to the gain of heat energy".

For example, if at a given pressure, there is a decrease of volume of a system, this will equal the increase in heat of that system (all other physical quantities being held fixed).

In an arbitrary thermodynamic transformation let ΔQ denote the net amount of heat absorbed by the system and ΔW the net amount of work done by the system. The first law states that the internal energy ΔU, defined by

$$\Delta U = \Delta Q - \Delta W,$$

is the same for all transformations leading from a given initial state to a given final state.

It can be stated as *"Energy can neither be created nor destroyed. It can only change form."* However, in 1837, Rudolf Clausius, gave the first explicit statement of the first law of thermodynamics as

"there is a state function U, called energy, whose differential equals the work exchanged with

the surroundings during an adiabatic process."

In an infinitesimal transformation, any change in U, dU, is the difference between heat added dQ and the work done by the system, dW,

$$dU = dQ - dW = dQ - PdV.$$

where dU is a perfect differential dQ and dW are imperfect ones (that is they are non integrable). dQ is the heat added to the system and dW is the work done by the system. It states that mechanical work and heat are two forms of energy and must be lumped together when the change in the internal energy of the system is computed.

3) **Second Law of Thermodynamics**

"The over-all change in heat energy of an isolated system is non-negative in any interconversion of energy".

If we use a temperature difference to work a heat engine, the amount of work done by it will not be greater than the energy used, but it can be less. Reversible processes are those in which it is equal and the others are called irreversible. In statistical terms the extent of disorder of the system, measured by a parameter called entropy (and generally denoted by S), must be non-decreasing in any process.

It can be stated as the spontaneous tendency of a system to go towards thermodynamic equilibrium cannot be reversed without at the same time changing some organized energy, work into some disorganized energy, heat. The laws of thermodynamics have a negative quality that differentiates them from other laws of physics. The first law may be stated by saying that *energy cannot be destroyed*. German scientists Rudolf Clausius is the first who gives the first formulation of the second law, now known as the Clausius's principle:

"There exists no thermodynamic transformation whose sole effect is to extract a quantity of heat from a colder reservoir and to deliver it to a hotter reservoir".

It is logically equivalent to another statement called Kelvin's principle:

"There exists no thermodynamic transformation whose sole effect is to extract a quantity of heat from a given heat reservoir and convert it entirely into work".

Carnot Engine:

It is a hypothetical engine that operates on the reversible *Carnot cycle*. A carnot cycle operates between two temperatures, a higher one T_h, that of the heat source and a lower one, T_c, that of the heat sink. The cycle consist of two adiabatic and two isothermal processes. This cycle does not convert all the heat drawn from the reservoir at T_h into work, some of it is dumped as unused heat into sink at T_c. The net work done by the engine per cycle is the area enclosed by the two adiabatic and two isothermal curves. The Carnot efficiency, η, is given by:

$$\eta = \frac{T_h - T_c}{T_h} \times 100.$$

Second law can also be stated as:

"No engine operating between two given temperatures is more efficient than a Carnot engine".

The above four different phrasings of second law are equivalent [48].

4) **Third Law of Thermodynamics**

It is the least known of all the laws. This law enables us to define the entropy of a substance up to an arbitrary additive constant but does not uniquely determine the difference in entropy of two states. It was independently uncovered by Walther Nernst in 1906 and Max Planck formulated it in 1911. It can be stated in a variety of ways. Two formulations due to Nernst are:

"Isothermal reversible processes become isentropic in the limit of zero temperature".

and

"It is impossible to reduce the temperature of any system to the absolute zero in a finite number of operations". This law can be rephrased as

"The absolute zero of temperature can never be achieved by a finite number of isothermal (constant temperature) and adiabatic (constant heat) cooling steps".

In other words, it takes an infinite number of successive isothermal and adiabatic steps to reach zero temperature.

Thermodynamic systems which are not isolated, a law, known as Stefans law, deals with them. The darker a body the more efficient it is at absorbing or emitting radiation. The idealized limit of a perfect radiator is called a "black body". Stefans law states that *"the total energy density, ρ, radiated by a black body is proportional to the fourth power of the temperature"*,

$$\rho = \sigma T^4, \tag{4.2}$$

where $\sigma \approx 7.6x10^{15} erg/cm^3/^\circ K^4$ is the Stefan-Boltzmann constant. This was stated as a purely empirical result. An attempt at spectral analysis of this energy radiation led to confusing results [47]. For long wave-lengths Wein found that $E \propto \lambda^{-2}$, where λ is the wave-length of the black body radiation being observed. However, for high frequencies Rayleigh and Jeans found that $\rho \propto V^3 e^{-\nu\tau}$, where ν is the frequency of the black body radiation being observed and ν is proportional to its absolute temperature. In 1900 Max Planck deduced a formula for the energy density of radiation if it is emitted and absorbed by matter in discrete quanta of $h\nu$, where h is called Planck's constant. This energy density distribution, which Planck verified experimentally, known as the Planck spectrum, is

$$\rho(\nu, T) = \frac{8\pi c^3 h\nu^3}{e^{\frac{h\nu}{kT}} - 1}. \tag{4.3}$$

On integration over all ν this gives Stefans law. In the long wavelength limit it gives Weins law and in the high frequency limit the Rayleigh-Jeans law. It clearly provides a more fundamental understanding of radiation from black bodies and is generally regarded as the birth of the quantum theory.

4.1.1 Black Hole Thermodynamics

The connection between black holes and thermodynamics started from an argument presented by Penrose about reversing the entropy increase by using black holes [49]. To extract rest energy from non-usable objects, Penrose started with the consideration that objects could be lowered from a safe distance into the black hole. As it would be lowered it would wind up a spring. At the surface of the hole it would be released and the freed spring could then do some useful work. He soon realized, the natural next step was to fill up a box with non-usable thermal radiation and lower the box into the black hole, as before, then open it near the surface to let the thermal energy out. Due to the equivalence of mass and energy this would allow the relaxed spring to do useful work. He would, in fact, not only have "got something from nothing" but would have got rid of the thermal pollution far from the hole.

Penrose and Floyd (1971) [50], went on to construct a mechanism to extract energy from rotating black holes. The essence of the Floyd-Penrose process is to break a compound object inside the ergosphere of a Kerr black hole in such a way that one part falls into the hole and the other goes away to infinity. Since the locally-rest-particles appear to travel faster than light as viewed from infinity. Applying energy conservation, as seen from infinity the outgoing particle appears to have more energy than the original object did, even counting the rest energy in. As such the rotating black hole has lost energy in all. This can be arranged if the incoming object has angular momentum in the same direction to the rotation of the black hole.

At the time when Penrose presented this argument it seemed unbelievable that energy could ever be extracted from a black hole. However, it soon became clear that the Kerr black hole stored some of its energy in its ergosphere which is outside the black hole. Thus no energy really came out of the black hole itself, it merely appeared to lose mass on account of some of the supposed mass being stored as energy outside the black hole. Ruffini and his student, Christodoulou [51], demonstrated that there is an irreducible mass that a black hole has and extra energy of electromagnetic nature can be stored in it.

4.1.2 Laws of Black Hole Thermodynamics

In comparing the laws of black hole thermodynamics in GR with the ordinary laws of thermodynamics, it should be noted that the black hole uniqueness theorems [52] establish that stationary black holes are characterized by a small number of parameters analogous to the state parameters of ordinary thermodynamics. In the corresponding laws:
a) The role of energy, E, is played by the mass, M, of the black hole.
b) The role of temperature, T, is played by a constant times the surface gravity, κ, of the black hole.
c) The role of Entropy, S, is played by a constant times the area, A, of the black hole.

The fact that E and M represent the same physical quantities provides a strong mathematical analogy between the laws of black hole thermodynamics and ordinary laws of thermodynamics that might be of physical significance. As argued in [53], this can not be the case in classical GR. The physical temperature of the black hole is zero, so there can be no physical relationship between T and κ. Therefore, it would be inconsistent to assume a physical relationship between S and A. However, the situation changes when Quantum effects are considered [54].

In 1971 Stephen Hawking [55], stated that the area, A, of the event horizon of a black hole can never decrease (but can remain constant) in any process, mathematically:

$$\Delta A \geq 0.$$

It was later noted by Bekenstein [56] that this result is analogous to the statement of the ordinary second law of thermodynamics, namely, that the total entropy, S, of a closed system never decrease in any process,

$$\Delta S \geq 0. \tag{4.4}$$

With these arguments it is legitimate to establish the laws of *black hole thermodynamics* in parallel to the laws of ordinary thermodynamics by using parameters of the black hole.

Zeroth Law:

In order to discuss zeroth law, it is fruitful to define Hawking radiation. A thermal radiation is predicted to be emitted by black holes known as "Hawking radiation", due to quantum mechanical effects. It is named after the physicist Stephen Hawking, who provided a theoretical argument for its existence in 1974.

"The event horizon is described by a quantity κ, the surface gravity, which remains constant over the event horizon". It is called *zeroth law.*

The relationship between the surface gravity and the physical temperature of the black hole

(Hawking temperature) is expressed by

$$T_H = \frac{\kappa}{2\pi}.$$

For the Schwarzschild black hole, where $\kappa = \frac{1}{4M}$, the Hawking temperature becomes

$$T_H = \frac{1}{8\pi M}.$$

In case of the rotating Kerr black hole, the Hawking temperature is

$$T_H = (\frac{\kappa}{8\pi}) = 2\left(1 + \frac{M}{\sqrt{M^2 - a^2}}\right)^{-1}\left(\frac{1}{8\pi M}\right),$$

where $a = \frac{J}{M}$. For the charged non-rotating RN black hole, one has

$$T_H = \frac{\kappa}{8\pi} = \left(1 - \frac{Q^4}{r_+^4}\right)\frac{1}{8\pi M}.$$

First Law:

The first law of black hole thermodynamics, like the first law of ordinary thermodynamics, is a statement representing energy conservation. Energy conservation holds only if there is time translational invariance. The difference from usual thermodynamics arises because we can conceive of an "isolated system" as one with adequate thermal insulation. There can never be gravitational insulation and hence we can not really think of the equivalent of an isolated system in the context of black holes. Nevertheless, we can treat a black hole as an approximately isolated system. In fact, for actual use of thermodynamics we accept the fact that there are no real isolated systems but only approximations to them. The first law of black hole thermodynamics then states that:

" the net energy including rest energy going into a black hole equals the sum of increase in the rest energy of the black hole spacetime and the energy radiated away from the in falling object".

When a black hole goes from one stationary state to another, this law deals with the mass (energy) change, dM. In simple words this law is an identity relating the change in mass, M, angular momentum, J, and area , A, of the horizon of a black hole spacetime when it is perturbed. The variation of these quantities always satisfy

$$dM = (\frac{\kappa}{8\pi})dA + work\ terms, \tag{4.5}$$

or

$$dM = TdS_{BH} + work\ terms, \tag{4.6}$$

where S_{BH} is entropy of the black hole.

Second Law:

In any classical process, the area of the event horizon does not decrease, mathematically

$$dA \geq 0.$$

Also the black hole entropy, S_{BH}, never decreases . If the quantum effect is taken into account, namely that the area of the event horizon can be reduced via Hawking radiation, the second law of back hole thermodynamics can be violated . It is important to note that naturally the black hole radiation is thermal, therefore, generates a rise in the entropy in the surrounding. The *generalized entropy*, $S^{'}$, introduced by Bekenstein [56] is defined as

$$S^{'} = S_{BH} + S_m,$$

where S_{BH} is the black hole entropy and S_m is the entropy of the surrounding matter. The statement known as *generalized Second Law* (GSL) is

$$\Delta S^{'} \geq 0.$$

The ordinary second law of black hole thermodynamics seems to fail when a matter is dropped into a black hole because according to classical GR, the matter will disappear into a spacetime singularity, in this way the net entropy of the universe decreases as there is no compensation for the lost entropy. The GSL keeps the law of entropy valid as the total entropy of the universe still increases when that matter falls into the black hole.

Third Law:

Within a finite time the limit $\kappa = 0$ cannot be reached, i-e, it is not possible how many processes we do, we will never reach the limit $\kappa = 0$. However, the extremal black holes, the Kerr black hole in which $\frac{a}{M} = 1$, do have $\kappa = 0$ thus zero temperature (absolute zero) but non-zero entropy. To actually reduce the surface gravity to zero is merely an idealized case because it is forbidden by the cosmic censorship conjecture.

It is important to note that the close mathematical analogy of the zeroth, first and second laws of thermodynamics to corresponding laws of classical black hole thermodynamics is broken by the Plank-Nernst form of the third law of thermodynamics [57], which states that $S \to 0$ as $T \to 0$. The analogy of this law fails in black hole thermodynamics, since there exist extremal black holes (black hole with $\kappa = 0$) with finite area. However there is a good reason to believe that the "Plank-Nernst theorem" should not be viewed as a fundamental law of thermodynamics, but rather as a property of the density of the states near the ground state in the thermodynamic limit, which happens to be valid for commonly studied materials. Indeed, examples can be given of ordinary systems that violate the Plank-Nernst form of the third law in a manner very similar to the violations of the analog of this law that occur for black holes [58]. As far as, we are concerned with physical theories regarding the laws of black hole thermodynamics, thereby a number of *open questions* can be raised [59]-[61], such as:

1) Is the black hole entropy real or subjective?

2) Where does it appear on or near the horizon or deep in the hole?

3) What dynamical mechanism makes the black hole entropy a universal function, independent of the hole past history or detailed internal condition?

When quantum effects are considered, one can ask about:

4) Can the black hole entropy be derived from quantum mechanical considerations?

5) What happens to the black hole entropy after the black hole has evaporated, due to the effect of Hawking radiation? Will all the information disappear after the evaporation?

These frequently asked questions are embarrassing, because we do not know with our present knowledge how to answer them precisely. Nevertheless, it is hoped that success in modern theory of gravity would be the key to answer–if not all–some of these open questions.

4.1.3 Foliation

The word *foliation* is from a Latin original *folia*, which means leaf. The concept of foliation or slicing develops in seventeenth century from the theory of differential equations where the trajectories of solution space can be considered as the leaves of the foliation. Poincare, in late nineteenth century, was able to develop methods for the study of global, qualitative properties of dynamical systems in situations where the explicit solution methods had failed. He discovered that the study of the geometry of the space of the trajectories of a dynamical system reveals complex phenomena. He gave strong impetus to the topological methods and emphasized the qualitative nature of the phenomena, which led to the subject of foliation.

The foliation of an n-dimensional manifold, M, is a decomposition of M into submanifolds, all being of the same dimension, p. The submanifolds are the leaves of the foliation. The co-dimension, q, of a foliation is defined as $q = n - p$. A foliation of co-dimension one is called a foliation by *hypersurfaces*. The pioneers of foliation theory were Reeb [62] and Ehresmann [63], the former, in particular, coined the term foliation.

The simplest understood cases of foliation are when $p = q = 1$, e.g. the two dimensional xy-plane can be foliated by the straight lines, $y = mx + c$, with c taken as the parameter and any fixed m. One can also foliate the xy-plane by circles, $x^2 + y^2 = a^2 (0 < a < \infty)$, but in this case the origin is left out unless the degenerate circle, $a = 0$, is included.

A foliation of a manifold is said to be *complete* if it covers the entire manifold by a sequence of non-intersecting sub-manifolds. For example, a disc of radius a can be completely foliated by circles but it can not be completely foliated by squares as there would be some portion of the disc left uncovered.

Hypersurfaces (timelike, null or spacelike) could be used to foliate a spacetime. For this approach, we have to specify some geometrical property that this family of hypersurfaces must satisfy. A local or global time parameter, varying from one hypersurface to another, is then provided. However, there is no guarantee that a complete foliation can be so achieved. An example of the first approach is the requirement that the hypersurfaces look *flat* to an observer, locally i-e. have zero intrinsic curvature. An example of the global approach is the requirement that this family of hypersurfaces have zero or constant mean extrinsic curvature.

4.1.4 Foliation of a Black Hole Spacetime

Traditionally black holes are understood as embedded in asymptotically flat spacetime. It is taken for granted that the spacetime is not compact. A classical black hole is a region from inside of which not even light can escape to infinity. Therefore, an *infinity* to escape to is required. There is a problem, if the spacetime did not tend to Minkowski space far away from the source, or more dramatically, if the universe was closed so that there was nothing sufficiently far away from the source. In a closed universe, as there is no infinity to escape to, the distingtion between *inside* and *outside* a black hole is ambiguous, or alternatively, the distinction between the black hole singularity and the final cosmological singularity is not clear.

Penrose, pointed out that it should be possible to regard the black hole singularity as a part of the final singularity [64, 65]. He suggested that there should exist a foliation of the spacetime by a sequence of spacelike hypersurfaces which would approach the singularity smoothly without cutting it anywhere. Thus the entire spacetime would be foliated. The limit of some parameter going to some specific value should yield the entire singularity.

Initially hypersurfaces of zero mean extrinsic curvature are used to foliate a spacetime called foliation by maximal slicing [66]. Foliating even the most simple of spacetimes, the Schwarzschild spacetime, the hypersurfaces did not pass through all the spacetime points [67]. This meant that either Penrose's conjecture was incorrect or the maximal slicing procedure was inappropriate for the purpose. Later York slicing procedure [67, 68] was thought to be more appropriate for the purpose. In this slicing hypersurfaces are defined to have some given constants mean extrinsic curvature. After that there has been a lot of work on foliating different spacetimes [69]-[88].

4.2 Field Equations and the First Law of Black Hole Thermodynamics

Black hole horizon acts as a boundary of the spacetime and blocks any physical information to flow out to the rest of world. This led Bekenstein [89] to claim that the black holes must hold non-zero entropy since they withhold information from the outside observer. This interpretation became unambiguous when Hawking [90] showed that a black hole can emit thermal radiation with a temperature proportional to its surface gravity at the black hole horizon and with an entropy proportional to its horizon area [91]. The Hawking temperature $T = \kappa/2\pi$ and the black hole entropy $S = A/4G$ are connected through the identity $dE = TdS$, usually called first law of black hole thermodynamics [89, 90, 91]. In more general, the first law of black hole thermodynamics is related with the energy change by $dE = TdS + workterms$. For Kerr-Newman black hole family, the first law of black hole thermodynamics is given by

$$dE = TdS + \Omega dJ + \Phi dQ, \tag{4.7}$$

where $\Omega = \frac{\partial M}{\partial J}$ is the angular velocity and $\Phi = \frac{\partial M}{\partial Q}$ is the electric potential. The above equation indicates that the thermodynamic interpretation of the Einstein field equations is possible near horizon because black hole solutions are obtained from Einstein's field equations and the geometric quantities of the spacetime metric are related to the thermodynamic quantities. Jacobson [92] was able to find the Einstein field equations by employing the first law of thermodynamics $\delta Q = TdS$ along with proportionality of entropy to the horizon area of the black hole. In case of Einstein gravity [93], as well as for a more general Lanczos-Lovelock theories of gravity, Paranjape Sarkar and Padmanabhan [94] found that it is possible to interpret the field equations for a special class of spherically symmetric spacetimes as a thermodynamic identity $TdS = dE + PdV$ near black hole horizon. For a more general situation in Einstein gravity, Kothawala, Sarkar and Padmanabhan [95] found that the field equations near any spherically symmetric horizon can be expressed as a thermal identity $TdS = dE + PdV$ and extended their approach for stationary axis-symmetry horizons and time dependent evolving horizon and found, in both cases, that the structure of the Einstein equations near horizon can also be expressed as a thermodynamic identity under the virtual displacement of the horizon. However, it was soon realized that the notions of temperature and entropy can also be associated with several other types of horizons which advocate a generic thermodynamic behavior of horizons [96, 97, 98, 99]. In case of apparent horizon of FRW universe, Cai and Kim [100] are able to derive the Friedmann equations of (n+1)-dimensional Friedman-Robertson-Walker (FRW) universe with any spatial curvature by applying the first law of thermodynamics ($TdS = -dE$) to the apparent horizon. Also by using the entropy expression of a static spherically symmetric black hole in the Gauss-Bonnet gravity and in more general Lovelock gravity, they reproduce the corresponding Friedmann equations. The possible extensions to the scalar-tensor gravity and $f(R)$ gravity theory have been studied in reference [101]. However, in more general context, Akbar etal [102] have shown that thermodynamic interpretation of Friedmann equations, describing the dynamics of the universe, is possible at apparent horizon of FRW universe and showed that it is possible to interpret the differential form of Friedmann equations near apparent horizon as a universal form $dE = TdS + WdV$ in the Einstein theory of general relativity as well as for a wider class of Gauss-Bonnet and Lovelock theories of gravity. These thermodynamic interpretation of gravitational dynamics at horizons needs further investigation for understanding it at a deeper level [93, 105]. In recent years, (2+1)-dimensional BTZ (Banados-Teitelboim-Zanelli) black holes have drawn a lot of attention as simplified models for exploring conceptual issues relating to the black hole thermodynamics (see, e.g.,[19, 106]). The thermodynamic interpretation of field equations for static as well as non-static BTZ black hole near horizon is presented in [107] and for charged rotating BTZ black hole in [18]. Further interesting work in this direction, can be seen in [103, 104].

In the next subsection, using the concept of foliation, we present an elegant and simpler way of obtaining relationship between the Einstein field equations at the horizon and the first law of black hole thermodynamics. Instead of obtaining field equations of black hole spacetime and analyzing thermal interpretation at the horizon, here the main idea is to consider a foliation so that the horizon corresponds to a particular hypersurface. Then we work out the field equations for the induced metric of the hypersurfaces and obtain the earlier thermal analysis for the hypersurface corresponding to the horizon. The main advantage of our procedure is that instead of dealing with the field equations of an $n-$dimensional spacetime, we deal with the field equations of $(n-1)-$dimensional induced metric. In the coming subsections, we present our procedure for the Schwarzschild, RN, Kerr and Kerr-Newman spacetimes.

4.2.1 The Schwarzschild Black Hole

The thermal quantities associated with the event horizon of the Schwarzschild black hole Eq.(1.3) are the temperature $T = \frac{f\prime(r)}{4\pi} \mid_{horizon} = \frac{1}{8\pi m}$ and the entropy $S = \frac{A}{4} = 4\pi m^2$ (where A is the horizon area).

Consider $r =$ constant (say k) hypersurfaces, so that the black hole horizon corresponds to a particular hypersurface. The induced metric of the hypersurfaces is then given by

$$ds_h^2 = f(k)dt^2 - k^2(d\theta^2 + sin^2\theta d\phi^2). \tag{4.8}$$

The $(0,0)$-component of the Einstein tensor given by Eq.(1.1) with $\kappa = -8\pi$ for the induced metric (4.8) is

$$G_{00} = \frac{f(k)}{k^2} = \frac{1 - \frac{2m}{k}}{k^2}. \tag{4.9}$$

Treating k as a parameter and considering an arbitrary small displacement dk, the $(0,0)$-component of the Einstein field equations (1.1) for the induced metric (4.8) with $T_{00} = 0$ gives

$$\frac{dk}{2} - \frac{m}{2\pi k^2} d(\pi k^2) = 0. \tag{4.10}$$

For the hypersurface given by $k = 2m$ the above Eq.(4.10) yields the first law

$$dm = TdS. \tag{4.11}$$

It is important to note that $k = 2m$, which corresponds to the black hole event horizon, is the only hypersurface satisfying the first law.

4.2.2 The RN Black Hole

The thermal quantities associated with the RN black hole (1.43) at the outer horizon ($r = r_+ = m + \sqrt{m^2 - Q^2}$) are the temperature $T = \frac{f\prime(r)}{4\pi} \mid_{r=r_+} = \frac{(mr_+ - Q^2)}{2\pi r_+^3}$, entropy $S = \frac{A}{4} = \pi r_+^2$ (where A is the horizon area) and electric potential $\Phi = \frac{\partial m}{\partial Q} \mid_{r=r_+} = \frac{Q}{r_+}$.

We again consider $r =$ constant (say k) hypersurfaces, so that the black hole horizons correspond to particular hypersurfaces. The induced metric of the hypersurfaces takes the same form as given by Eq.(4.8), with $f(k)$ now given by

$$f(k) = 1 - \frac{2m}{k} + \frac{Q^2}{k^2}. \tag{4.12}$$

The $(0, 0)$-component of the Einstein tensor (1.1) for the induced metric (4.8) with $f(k)$ given by (4.12) is

$$G_{00} = \frac{f(k)}{k^2} = \frac{1 - \frac{2m}{k} + \frac{Q^2}{k^2}}{k^2}. \tag{4.13}$$

Again treating k as a parameter and considering an arbitrary small displacement dk, the $(0, 0)$-component of the Einstein field Eqs.(1.1) for the induced metric with $T_{00} = -P$ gives

$$\frac{dk}{2} - \frac{mdk}{k} + \frac{Q^2 dk}{2k^2} = 4\pi P k^2 dk, \tag{4.14}$$

where P corresponds to the pressure of the source. Also at the horizon $m = \frac{r_+}{2} + \frac{Q^2}{2r_+^2}$, which gives

$$dm = \frac{dr_+}{2} - \frac{Q^2}{2r_+^2}dr_+ + \frac{Q}{r_+}dQ. \qquad (4.15)$$

Now using Eq.(4.15) in Eq.(4.14) and simplifying at the horizon, $k = r_+$, we obtain

$$dm - \frac{(mr_+ - Q^2)}{2\pi r_+^3}d(\pi r_+^2) - \frac{Q}{r_+}dQ = Pd(\frac{4}{3}\pi r_+^3), \qquad (4.16)$$

which can be expressed in terms of the thermal quantities, defined after Eq.(4.12), as the first law

$$dm = TdS + \Phi dQ + PdV, \qquad (4.17)$$

where $V = \frac{4}{3}\pi r_+^3$ is the volume enclosed by the horizon.

4.2.3 The Kerr Black Hole

The Vaccum solution of the field equations (1.1), for axis-symmetric gravitational field due to a mass, M, called the Kerr black hole [108] is given by (1.71). Consider r =constant (say k), hypersurfaces, so that the black hole horizons correspond to particular hypersurfaces. The induced metric of the hypersurfaces is given by

$$ds_h^2 = -\frac{\Delta^2}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}(adt - (k^2 + a^2)d\phi)^2, \qquad (4.18)$$

where $\Delta^2 = (k^2 + a^2) - 2mk$, $\rho^2 = k^2 + a^2\cos^2\theta$.

The Kerr black hole is associated with temperature, T, entropy, S, and angular velocity, Ω, as

$$T = \frac{K}{2\pi} = \frac{1}{2\pi}\lim_{r\longrightarrow k}\sqrt{\frac{-g^{11}}{g^{00}}}(\frac{(g^{00})'}{g^{00}}) = \frac{m(k^2 - a^2)}{2\pi(k^2 + a^2)^2}, \qquad (4.19)$$

$$S = \pi(k^2 + a^2), \tag{4.20}$$

$$\Omega = \lim_{r \longrightarrow k} \frac{-g^{13}}{g^{33}} = \frac{a}{k^2 + a^2}. \tag{4.21}$$

At the horizon $m = \frac{k^2+a^2}{2k}$, which gives

$$dm = \frac{k^2 - a^2}{2k^2} dk. \tag{4.22}$$

From $J = am$ and Eq.(4.22), we have

$$dJ = \frac{a(k^2 - a^2)}{2k^2} dk. \tag{4.23}$$

Also

$$TdS = -\frac{(a^2 - k^2)}{2(k^2 + a^2)} dk, \tag{4.24}$$

and

$$\Omega dJ = \frac{a^2(k^2 - a^2)}{2k^2(k^2 + a^2)} dk. \tag{4.25}$$

The $(0, 0)$-component of the Einstein tensor for the induced metric (4.18) is

$$\frac{2mk - k^2}{2k^2} + \frac{4mka^2}{2k^2(k^2 + a^2)} - \frac{12m^2k^2a^2}{2k^2(k^2 + a^2)^2} = 0. \tag{4.26}$$

Subtracting and adding $\frac{mk(k^2-a^2)}{(k^2+a^2)^2}$, we have

$$\frac{2mk-k^2}{2k^2}+\frac{4mka^2}{2k^2(k^2+a^2)}-\frac{12m^2k^2a^2}{2k^2(k^2+a^2)^2}-\frac{mk(k^2-a^2)}{(k^2+a^2)^2}+\frac{mk(k^2-a^2)}{(k^2+a^2)^2}=0. \qquad (4.27)$$

Now treating k as a parameter and multiplying both sides of Eq.(4.27) by an arbitrary small displacement dk, and using Eq.(4.24), we get

$$\frac{2mk-k^2}{2k^2}dk+\frac{4mka^2}{2k^2(k^2+a^2)}dk-\frac{12m^2k^2a^2}{2k^2(k^2+a^2)^2}dk-TdS+\frac{mk(k^2-a^2)}{(k^2+a^2)^2}dk=0.$$

For the hypersurfaces given by $k=\frac{k^2+a^2}{2m}$, we have

$$\frac{-a^2(k^2-a^2)}{2k^2(k^2+a^2)}dk+\frac{k^2}{2k^2}dk-\frac{a^2}{2k^2}dk-TdS=0.$$

Combining second and third term to get

$$\frac{-a^2(k^2-a^2)}{2k^2(k^2+a^2)}dk+\frac{k^2-a^2}{2k^2}dk-TdS=0. \qquad (4.28)$$

Using Eqs.(4.22) and (4.25) in Eq.(4.28), we have

$$dm=TdS+\Omega dJ.$$

It is important to note that $k^2+a^2-2mk=0$, which corresponds to the black hole event horizons, is the only hypersurfaces satisfying the first law.

4.2.4 The Kerr-Newmann Black Hole

The solution of the Einstein-Maxwell equations for axis symmetric gravitational field due to a charge rotating mass, m, with cosmological constant equals zero, called the Kerr-Newman black hole is given (in gravitational units G=c=1) by the metric (1.71), with

$$\Delta^2=(r^2+a^2)-2mr+Q^2$$

and

$$\rho^2=r^2+a^2\cos^2\theta.$$

Consider $r =$ constant (say k), hypersurfaces, so that the black hole horizons correspond to a particular hypersurfaces. The induced metric of the hypersurfaces is given by

$$ds_h^2 = -\frac{\Delta^2}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}(adt - (k^2 + a^2)d\phi)^2, \tag{4.29}$$

where $\Delta^2 = (k^2 + a^2) - 2Mk + Q^2$, $\rho^2 = k^2 + a^2\cos^2\theta$.

The Kerr-Newman spacetime is associated with temperature, T, entropy, S, and angular velocity, Ω, as

$$T = \frac{K}{2\pi} = \frac{1}{2\pi}\lim_{r\longrightarrow k}\sqrt{\frac{-g^{11}}{g^{00}}}(\frac{(g^{00})'}{g^{00}}) = \frac{m(k^2 - a^2) - Q^2k}{2\pi(k^2 + a^2)^2}, \tag{4.30}$$

$$S = \pi(k^2 + a^2), \tag{4.31}$$

$$\Omega = \lim_{r\longrightarrow k}\frac{-g^{13}}{g^{33}} = \frac{a}{k^2 + a^2}. \tag{4.32}$$

At the event horizon, the mass, m, is given by

$$m = \frac{(k^2 + a^2) + Q^2}{2k},$$

which gives

$$dm = \frac{k^2 - a^2 - Q^2}{2k^2}dk + \frac{Q}{k}dQ. \tag{4.33}$$

From $J = am$ and using Eq.(4.33), we have

$$dJ = adm = \frac{a(k^2 - a^2 - Q^2)}{2k^2}dk + \frac{aQ}{k}dQ. \tag{4.34}$$

From Eqs.(4.30) and (4.31), we get

$$TdS = \frac{mk(k^2 - a^2) - Q^2k^2}{(k^2 + a^2)^2}dk. \tag{4.35}$$

Using Eq.(4.32) and Eq.(4.34), we have

$$\Omega dJ = \frac{a}{k^2 + a^2}dJ = \frac{a^2(k^2 - a^2 - Q^2)}{2k^2(k^2 + a^2)}dk + \frac{a^2Qk}{k^2(k^2 + a^2)}dQ, \tag{4.36}$$

Now the electrostatic potential and the differential of horizon's volume are given by [12]

$$\phi = \frac{kQ}{k^2 + a^2}, \tag{4.37}$$

and

$$dV = 4\pi(k^2 + a^2)dk. \tag{4.38}$$

The $(0, 0)$-component of the Einstein tensor for the induced metric (4.29) is

$$\frac{2mk - k^2 - Q^2}{(k^2 + a^2)^2} + \frac{2a^2(2mk - Q^2)}{(k^2 + a^2)^3} - \frac{3a^2[4m^2k^2 + Q^2(Q^2 - 4mk)]}{(k^2 + a^2)^4} = -8\pi P, \tag{4.39}$$

where P is $(0, 0)$-component of the Stress-energy tensor.

Now treating k as a parameter and considering an arbitrary small displacement dk and multiplying both sides of Eq.(4.39) by $\frac{-(k^2+a^2)}{2}dk$, we have

$$-\frac{2mk - k^2 - Q^2}{2(k^2 + a^2)}dk - \frac{a^2(2mk - Q^2)}{(k^2 + a^2)^2}dk + \frac{3a^2[4m^2k^2 + Q^2(Q^2 - 4mk)]}{2(k^2 + a^2)^3}dk = P(4\pi(k^2 + a^2)dk). \tag{4.40}$$

For the hypersurfaces given by $2mk = k^2 + a^2 + Q^2$, we have

$$-\frac{a^2}{2(k^2+a^2)}dk - \frac{a^2}{(k^2+a^2)}dk + \frac{3a^2[(k^2+a^2+Q^2)^2 + Q^2(Q^2-4mk)]}{2(k^2+a^2)^3}dk = PdV. \quad (4.41)$$

Simplifying, we have

$$-\frac{a^2}{2(k^2+a^2)}dk + \frac{a^2}{2(k^2+a^2)}dk + \frac{a^2Q^2}{2(k^2+a^2)^2}dk + \frac{5a^2Q^2}{2(k^2+a^2)^2}dk - \frac{6a^2Q^2}{2(k^2+a^2)^2}dk = PdV. \quad (4.42)$$

Adding and subtracting $\frac{k^2}{2(k^2+a^2)}dk$ and $\frac{Q^2}{2(k^2+a^2)}dk$ in Eq.(4.42) and simplifying, we have

$$-\frac{a^2}{2(k^2+a^2)}dk + \frac{a^2}{2(k^2+a^2)}dk - \frac{k^2}{2(k^2+a^2)}dk + \frac{Q^2}{2(k^2+a^2)}dk + \frac{k^2}{2(k^2+a^2)}dk - \frac{Q^2}{2(k^2+a^2)}dk = PdV. \quad (4.43)$$

From 2nd, 3rd and 4th terms we get $-TdS$, so we have

$$-TdS - \frac{a^2}{2(k^2+a^2)}dk + \frac{k^2}{2(k^2+a^2)}dk - \frac{Q^2}{2(k^2+a^2)}dk = PdV.$$

Adding and subtracting $\frac{a^4}{2k^2(k^2+a^2)}dk$, $\frac{a^2Q^2}{2k^2(k^2+a^2)}dk$ and $\frac{a^2Qk}{k^2(k^2+a^2)}dQ$, we get

$$-TdS - \frac{a^2}{2(k^2+a^2)}dk + \frac{a^4}{2k^2(k^2+a^2)}dk + \frac{a^2Q^2}{2k^2(k^2+a^2)}dk - \frac{a^2Qk}{k^2(k^2+a^2)}dQ$$
$$-\frac{a^4}{2k^2(k^2+a^2)}dk - \frac{a^2Q^2}{2k^2(k^2+a^2)}dk + \frac{a^2Qk}{k^2(k^2+a^2)}dQ + \frac{k^2}{2(k^2+a^2)}dk - \frac{Q^2}{2(k^2+a^2)}dk = PdV. \quad (4.44)$$

Simplifying 2nd, 3rd, 4th and 5th terms, we have

$$-TdS - \Omega dJ - \frac{a^4}{2k^2(k^2+a^2)}dk - \frac{a^2Q^2}{2k^2(k^2+a^2)}dk + \frac{a^2Qk}{k^2(k^2+a^2)}dQ + \frac{k^2}{2(k^2+a^2)}dk - \frac{Q^2}{2(k^2+a^2)}dk = PdV.$$

Adding and subtracting $\frac{k^2-a^2-Q^2}{2k^2}dk + \frac{Q}{k}dQ$ and using Eq.(4.33) we get

$$-TdS - \Omega dJ + dm - \frac{k^2 - a^2 - Q^2}{2k^2}dk - \frac{Q}{k}dQ + \frac{a^2Qk}{k^2(k^2+a^2)}dQ - \frac{a^4}{2k^2(k^2+a^2)}dk$$
$$-\frac{a^2Q^2}{2k^2(k^2+a^2)}dk + \frac{k^2}{2(k^2+a^2)}dk - \frac{Q^2}{2(k^2+a^2)}dk = PdV. \quad (4.45)$$

Simplifying 5th and 6th term and using Eq.(4.37) we finally have

$$dm = TdS + \Omega dJ + \phi dQ + PdV.$$

It is important to note that $k^2 + a^2 + Q^2 - 2mk = 0$, which corresponds to the black hole event horizons, is the only hypersurfaces satisfying the first law.

Chapter 5

Conclusion

In this thesis, we have presented:

In Chapter 1, after discussing singularities of the Schwarzschild black hole spacetime, suitable coordinates are used to remove the coordinate singularities of the Schwarzschild, the RN, and the Kerr black hole spacetimes. Also non-existence of the Kruskal-like coordinates for the eRN black hole is discussed. It is further noticed that the Carter-like coordinates for the eRN back hole are inconvenient to be used for numerical calculations.

In Chapter 2, after presenting non-singular Kruskal-like coordinates for different cases of general circularly symmetric black holes in $(2+1)$ dimensions, non-singular Kruskal-Szekeres-like coordinates for the usual rotating BTZ black hole are constructed. It is observed that they cannot remove both the singularities simultaneously. For this, like the RN black hole, we need two separate coordinate patches to cover a full block of the CP diagram. In case of the rotating EBTZ black hole, Carter-like coordinates do not remove the coordinate singularities and are not regular at the horizon.

In Chapter 3, using coordinate transformation, existence of double-null form for different classes of general spacetimes metrics in $(2+1)$ dimensions have been obtained. It has been observed that the existence depends on two coefficients g_{02} and g_{12} of the metric. A class of three dimensional spacetimes in which coefficient g_{02} or g_{12} or both are non-zero cannot be transformed to the double-null form. Further investigation on the remaining classes of three dimensional spacetimes yields:

1) Classes $g_{00}, g_{11}, g_{22} \neq 0$, $g_{01} = 0$ and $g_{00}, g_{11}, g_{22}, g_{01} \neq 0$ can be transformed into the double-null form and the required transformations have been obtained.

2) The class $g_{00}, g_{22}, g_{01} \neq 0$, $g_{11} = 0$, satisfying the conditions

a) $\frac{\partial t}{\partial u} = \frac{\partial t}{\partial v} g_{00} + 2\frac{\partial r}{\partial v} g_{01} = 0$

or

b) $\frac{\partial t}{\partial v} = \frac{\partial t}{\partial u} g_{00} + 2\frac{\partial r}{\partial u} g_{01} = 0$

can also be transformed into the double-null form, but specific transformations have not been obtained in this case.

However, the spacetimes satisfying conditions

c) $\frac{\partial t}{\partial u} = \frac{\partial t}{\partial v} = 0$

or

d) $\frac{\partial t}{\partial u} g_{00} + 2\frac{\partial r}{\partial u} g_{01} = \frac{\partial t}{\partial v} g_{00} + 2\frac{\partial r}{\partial v} g_{01} = 0$

cannot be transformed into the double-null form.

3) The class $g_{11}, g_{22}, g_{01} \neq 0$, $g_{00} = 0$, satisfying the conditions

a) $\frac{\partial r}{\partial u} = \frac{\partial r}{\partial v} g_{11} + 2\frac{\partial t}{\partial v} g_{01} = 0$

or

b) $\frac{\partial r}{\partial v} = \frac{\partial r}{\partial u} g_{11} + 2\frac{\partial t}{\partial u} g_{01} = 0$.

can also be transformed into the double-null form, but specific transformations have not been obtained in this case.

However, the spacetimes satisfying conditions

c) $\frac{\partial r}{\partial u} = \frac{\partial r}{\partial v} = 0$

or

d) $\frac{\partial r}{\partial u} g_{11} + 2\frac{\partial t}{\partial u} g_{01} = \frac{\partial r}{\partial v} g_{11} + 2\frac{\partial t}{\partial v} g_{01} = 0$

cannot be transformed into the double-null form.

4) Class $g_{00}, g_{11}, g_{22}, g_{01} \neq 0$, the system of partial differential equations (3.36) to (3.39) is to be solved to obtain transformations that transform the class into the double-null form.

In Chapter 4, a new approach to express the Einstein field equation as the first law of black hole thermodynamics at the horizon of the Schwarzschild, RN, Kerr and Kerr-Newmann black holes have been presented. In this approach, foliations of these spacetimes have been used such that the horizon corresponds to a particular leaf (hypersurface) of the foliation. Then we work out the field equations of the induced metric of the hypersurfaces and obtain that the field equations for the induced metric, at the horizon, can be expressed as the first law of black hole thermodynamics. The important aspect of this approach is that here we have to essentially deal with $(n-1)-$dimensional induced metric for an $n-$dimensional spacetime, which significantly simplifies the calculations to obtain such results. It will be interesting in future to extend this approach to the other black hole geometries.

In addition to the work mentioned, it will be interesting to investigate the following problems in future:

a) Turning to the EBTZ black hole case, there are no Kruskal-like coordinates known for this spacetime. The Carter-like coordinates for this geometry are not regular at the horizon, therefore, a non-singular coordinate system for the EBTZ black hole is needed.

b) *Double-Null Form for* $(3+1)-$*dimensional Spacetimes*
Consider the general spacetime metric in four dimensions

$$\begin{aligned} ds^2 &= g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 + 2g_{01}dtdr + 2g_{02}dtd\theta \\ &\quad +2g_{03}dtd\phi + 2g_{12}drd\theta + 2g_{13}drd\phi + 2g_{23}d\theta d\phi, \end{aligned} \tag{5.1}$$

where $g_{00}, g_{11}, g_{22}, g_{33}, g_{01}, g_{02}, g_{03}, g_{12}, g_{13}$ and g_{23} depend on t, r, θ and ϕ.
In order to transform metric (5.1) into the double-null form (3.1), consider $t = t(u, v)$, $r = r(u, v)$ and use the coordinate transformations (3.33), where now x^l, x^m and $\widetilde{x}^a$, $\widetilde{x}^b$ refer to the (t, r, θ, ϕ) and (u, v, θ, ϕ) coordinates respectively, with $l, m, a, b = 0, 1, 2, 3$, and also requiring $\widetilde{g}_{00} = \widetilde{g}_{11} = \widetilde{g}_{22} = \widetilde{g}_{02} = \widetilde{g}_{03} = \widetilde{g}_{12} = \widetilde{g}_{13} = \widetilde{g}_{23} = 0$, to obtain the following system

of partial differential equations

$$\left(\frac{\partial t}{\partial u}\right)^2 g_{00} + \left(\frac{\partial r}{\partial u}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial u}\right) g_{01} = 0, \tag{5.2}$$

$$\left(\frac{\partial t}{\partial v}\right)^2 g_{00} + \left(\frac{\partial r}{\partial v}\right)^2 g_{11} + 2\left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial v}\right) g_{01} = 0, \tag{5.3}$$

$$g_{22} = 0, \tag{5.4}$$

$$g_{33} = \widetilde{g}_{33}, \tag{5.5}$$

$$\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial v}\right) g_{00} + \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) g_{11} + \left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right) g_{01} = \widetilde{g}_{01}, \tag{5.6}$$

$$\left(\frac{\partial t}{\partial u}\right) g_{02} + \left(\frac{\partial r}{\partial u}\right) g_{12} = 0, \tag{5.7}$$

$$\left(\frac{\partial t}{\partial v}\right) g_{02} + \left(\frac{\partial r}{\partial v}\right) g_{12} = 0, \tag{5.8}$$

$$\left(\frac{\partial t}{\partial u}\right) g_{03} + \left(\frac{\partial r}{\partial u}\right) g_{13} = 0, \tag{5.9}$$

$$\left(\frac{\partial t}{\partial v}\right) g_{03} + \left(\frac{\partial r}{\partial v}\right) g_{13} = 0, \tag{5.10}$$

$$g_{23} = 0. \tag{5.11}$$

Requiring the Jacobian,

$$J = \left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) - \left(\frac{\partial t}{\partial v}\right)\left(\frac{\partial r}{\partial u}\right), \tag{5.12}$$

of the transformations (t, r, θ, ϕ) to $(u, v, \theta, \phi)$to be non-zero, Eqs. (5.7) to (5.10), only have trivial solution

$$g_{02} = g_{12} = g_{13} = g_{03} = 0.$$

This proves that if the metric (5.1) contains g_{02}, g_{12}, g_{03}, g_{13}, g_{22} or g_{23} then it cannot be transformed into the double-null form (3.1).

Hence the class of $(3+1)-$dimensional spacetimes in which any one of the coefficients g_{02}, g_{12}, g_{03}, g_{13}, g_{23} or g_{22} is non-zero, cannot be transformed to the double-null form. It will be interesting to adopt the procedure used in Chapter 3, to explore the double-null form for other remaining classes of $(3+1)-$dimensional spacetimes.

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Gen Relativ Gravit (2011) 43:1167–1178
DOI 10.1007/s10714-010-1058-5

RESEARCH ARTICLE

Non-singular coordinates for circularly symmetric black holes in 2 + 1 dimensions

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Received: 21 January 2010 / Accepted: 12 July 2010 / Published online: 3 August 2010

Abstract In this paper we present non-singular coordinates for the rotating BTZ (Banados–Teitelboim–Zanelli) black hole. The approach is further extended to construct non-singular coordinates for different cases of general circularly symmetric black holes in 2 + 1 dimensions.

Keywords BTZ black hole · Kruskal coordinates · General relativity · Carter coordinates

1 Introduction

The BTZ solution is the most general black hole solution in three dimensions, as is guaranteed by the (2 + 1) dimensional version of Birkhoff's theorem [1]. It attracted much interest in recent years because of its classical and quantum properties (for review see [2]).

The Schwarzschild space time has a coordinate singularity, in (t, r) coordinates, at $r = 2m$. Kruskal (or Kruskal–Szekeres) coordinates are defined for this space–time to remove the coordinate singularity in such a way that we can directly insert $r = 2m$ in the metric coefficients and obtain a finite answer ($\frac{\pm 16m^2}{e}$) (as the coordinates

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are manifestly regular at the horizon). The RN space–time for ($Q < m$) has two coordinate singularities (r_- and r_+) in (t, r) coordinates. One can avoid each of these singularities by appropriately defining an analogue of the Kruskal coordinates [3], but cannot remove both simultaneously. As such, for this analogue we need two separate coordinate patches to cover a full block of the Carter–Penrose (CP) diagram.

The plane of the paper is as follows. In the Sect. 2 we discuss coordinate singularities of the rotating BTZ black hole. In Sect. 3 we present non-singular Kruskal–Szekeres like coordinates for the usual rotating BTZ black hole. In Sect. 4 we present non-singular Kruskal–Szekeres like coordinates for different cases of general circularly symmetric black holes in 2 + 1 dimensions. The conclusion and discussion is given in the last section.

2 Rotating BTZ black hole

The rotating BTZ black hole [4–8] is a solution of the standard Einstein–Maxwell equations in (2 + 1) dimensions, with a negative cosmological constant. The line element of the BTZ black hole can be written as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2, \tag{1}$$

where the function $f(r)$ is given by

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \tag{2}$$

with $-\infty < t < \infty, 0 < r < \infty$ and $0 \leq \phi \leq 2\pi$.

Horizons of the rotating BTZ metric are roots of the function $f(r)$. Depending on these roots, there are three different cases of the rotating BTZ metric:

(1) Two distinct real roots, usual rotating BTZ black hole;
(2) Repeated real roots, extreme BTZ black hole;
(3) No real root, naked rotating BTZ singularity.

Considering the first case when $M^2 > J^2/l^2$. For this case $f(r)$ has two distinct real roots. These roots are given by

$$r_\pm^2 = \frac{Ml^2}{2}\left[1 \pm \sqrt{1 - \frac{J^2}{M^2l^2}}\right]. \tag{3}$$

The Ricci scalar R for the rotating BTZ metric is given by

$$R = 6/l^2,$$

which is independent of r. This suggests that $r = r_\pm$ are coordinate singularities. Checking the second curvature invariant confirms this expectation. In terms of r_- and

r_+, $f(r)$ can be written as

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}. \tag{4}$$

3 Non-singular coordinates for the usual rotating BTZ black hole

In order to remove the coordinate singularities, we define r^* as

$$r^* = \int \frac{1}{f(r)} dr = \frac{l^2 r_+}{2(r_+^2 - r_-^2)} \ln \left| \frac{r - r_+}{r + r_+} \right| - \frac{l^2 r_-}{2(r_+^2 - r_-^2)} \ln \left| \frac{r - r_-}{r + r_-} \right|. \tag{5}$$

We define Kruskal-like coordinates and observe that they cannot remove both the singularities simultaneously. For this we need two separate coordinate patches to cover a full block of the CP diagram: one for the region $0 < r < r_+$ and the other for $r_- < r < \infty$. Coordinates analogous to the Kruskal-like coordinates, for $0 < r < r_+$, are given by $V_- = \alpha e^{\beta v}$ and $U_- = -\alpha e^{-\beta u}$, where the advanced and retarded coordinates [9] are $v = t + r^*$ and $u = t - r^*$, α and β are positive constants. Using these coordinates the metric (1) takes the form

$$\begin{aligned} ds^2 = &\frac{J^2}{16\beta^2 r^2} \left\{ \frac{1}{V_-^2} dV_-^2 + \frac{1}{U_-^2} dU_-^2 \right\} + \left\{ \frac{J^2}{8\alpha^2\beta^2 r^2} - \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\alpha^2 \beta^2 l^2 r^2} \right\} \\ &\times \left| \frac{r - r_-}{r + r_-} \right|^{\frac{\beta l^2 r_-}{2(r_+^2 - r_-^2)}} \left| \frac{r - r_+}{r + r_+} \right|^{\frac{-\beta l^2 r_+}{2(r_+^2 - r_-^2)}} dV_- dU_- \\ &- \frac{J}{2\beta} \left\{ \frac{1}{V_-} dV_- - \frac{1}{U_-} dU_- \right\} d\phi + r^2 d\phi^2, \end{aligned} \tag{6}$$

where

$$\begin{aligned} V_- &= \alpha e^{\beta t} \left| \frac{r - r_+}{r + r_+} \right|^{\frac{-\beta l^2 r_+}{2(r_+^2 - r_-^2)}} \left| \frac{r - r_-}{r + r_-} \right|^{\frac{\beta l^2 r_-}{2(r_+^2 - r_-^2)}}, \\ U_- &= -\alpha e^{\beta t} \left| \frac{r - r_+}{r + r_+} \right|^{\frac{-\beta l^2 r_+}{2(r_+^2 - r_-^2)}} \left| \frac{r - r_-}{r + r_-} \right|^{\frac{\beta l^2 r_-}{2(r_+^2 - r_-^2)}}. \end{aligned} \tag{7}$$

V_- and U_- are related to r by

$$V_- U_- = -\alpha^2 \left| \frac{r - r_+}{r + r_+} \right|^{\frac{-\beta l^2 r_+}{(r_+^2 - r_-^2)}} \left| \frac{r - r_-}{r + r_-} \right|^{\frac{\beta l^2 r_-}{(r_+^2 - r_-^2)}}. \tag{8}$$

As $r \to r_-$, V_-, $U_- \to 0$ but the metric (6) remains singular at $r = r_+$. The coordinate system (V_-, U_-) covers only the region $0 < r < r_+$ of the whole manifold. This region is shown in Fig. 1. Coordinates, analogous to the Kruskal-like coordinates, for

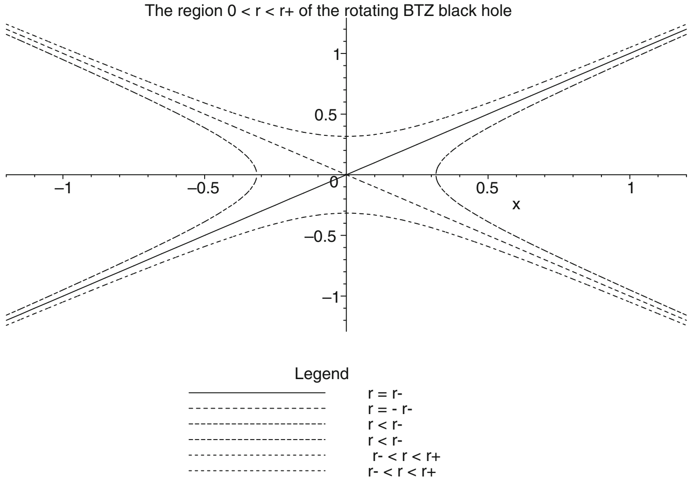

Fig. 1 η_- is along x-axis and ξ_- is along y-axis

$r_- < r < \infty$ are given by $V_+ = -\alpha e^{-\beta v}$ and $U_+ = \alpha e^{\beta u}$. Using these coordinates the metric (1) takes the form

$$ds^2 = \frac{J^2}{16\beta^2 r^2}\left\{\frac{1}{V_+^2}dV_+^2 + \frac{1}{U_+^2}dU_+^2\right\} + \left\{\frac{J^2}{8\alpha^2\beta^2 r^2} - \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2}\right\}$$
$$\times \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} dV_+ dU_+ + r^2 d\phi^2$$
$$-\frac{J}{2\beta}\left\{\frac{1}{V_+}dV_+ - \frac{1}{U_+}dU_+\right\} d\phi, \tag{9}$$

where

$$V_+ = -\alpha e^{-\beta t}\left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}},$$
$$U_+ = \alpha e^{\beta t}\left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2-r_-^2)}}. \tag{10}$$

V_+ and U_+ are related to r by

$$V_+ U_+ = -\alpha^2 \left|\frac{r-r_+}{r+r_+}\right|^{\frac{\beta l^2 r_+}{(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{-\beta l^2 r_-}{(r_+^2-r_-^2)}}. \tag{11}$$

It follows that as $r \to r_+$, $V_+, U_+ \to 0$. We see that the metric (9) remains singular at $r = r_-$. The coordinate system (V_+, U_+) covers only the region $r_- < r < \infty$ of the whole manifold. This region is shown in Fig. 2.

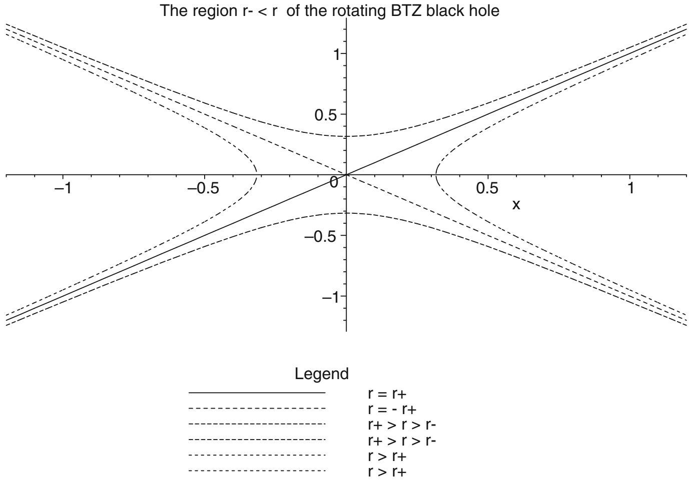

Fig. 2 η_+ is along x-axis and ξ_+ is along y-axis

We introduce space like and time like coordinates for the region $0 < r < r_+$, as $\xi_- = V_- + U_-$, $\eta_- = V_- - U_-$.

Using these coordinates the metric (6) takes the form

$$ds^2 = -\left[-\frac{J^2(\xi_-^2+\eta_-^2)}{8\beta^2 r^2(\xi_-^2-\eta_-^2)^2} + \frac{1}{4}\left(-\frac{J^2}{8\alpha^2\beta^2 r^2} + \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2}\right)\right.$$
$$\left.\times \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{(r_+^2-r_-^2)}} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{(r_+^2-r_-^2)}}\right] d\xi_-^2$$
$$+\left[\frac{J^2(\xi_-^2+\eta_-^2)}{8\beta^2 r^2(\xi_-^2-\eta_-^2)^2} + \frac{1}{4}\left(-\frac{J^2}{8\alpha^2\beta^2 r^2} + \frac{(r^2-r_+^2)(r^2-r_-^2)}{\alpha^2\beta^2 l^2 r^2}\right)\right.$$
$$\left.\times \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{(r_+^2-r_-^2)}} \left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{(r_+^2-r_-^2)}}\right] d\eta_-^2$$
$$+\frac{J^2\xi_-\eta_-}{2\beta^2 r^2(\xi_-^2-\eta_-^2)} d\xi_- d\eta_- + r^2 d\phi^2 - \frac{J}{\beta}\left[\frac{\eta_-}{\xi_-^2-\eta_-^2} d\xi_- - \frac{\xi_-}{\xi_-^2-\eta_-^2} d\eta_-\right] d\phi, \tag{12}$$

here

$$\xi_- = 2\alpha \left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{2(r_+^2-r_-^2)}} \sinh(\beta t),$$
$$\eta_- = 2\alpha \left|\frac{r-r_+}{r+r_+}\right|^{\frac{-\beta l^2 r_+}{2(r_+^2-r_-^2)}} \left|\frac{r-r_-}{r+r_-}\right|^{\frac{\beta l^2 r_-}{2(r_+^2-r_-^2)}} \cosh(\beta t), \tag{13}$$

and the inverse transformations are

$$t = \frac{1}{\beta} \tanh^{-1}\left(\frac{\xi_-}{\eta_-}\right), \tag{14}$$

and

$$\xi_-^2 - \eta_-^2 = -4\alpha^2 \left[\frac{r - r_+}{r + r_+}\right]^{\frac{-\beta l^2 r_+}{(r_+^2 - r_-^2)}} \left[\frac{r - r_-}{r + r_-}\right]^{\frac{\beta l^2 r_-}{(r_+^2 - r_-^2)}}. \tag{15}$$

For the region $r_- < r < \infty$, we define space like and time like coordinates as $\xi_+ = V_+ + U_+, \eta_+ = V_+ - U_+$.

Using these coordinates the metric (9) will takes the form

$$\begin{aligned} ds^2 = &-\Bigg[-\frac{J^2(\xi_+^2 + \eta_+^2)}{8\beta^2 r^2(\xi_+^2 - \eta_+^2)^2} + \frac{1}{4}\Bigg(-\frac{J^2}{8\alpha^2\beta^2 r^2} \\ &+ \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\alpha^2\beta^2 l^2 r^2}\Bigg)\left|\frac{r - r_-}{r + r_-}\right|^{\frac{-\beta l^2 r_-}{(r_+^2 - r_-^2)}} \left|\frac{r - r_+}{r + r_+}\right|^{\frac{\beta l^2 r_+}{(r_+^2 - r_-^2)}}\Bigg] d\xi_+^2 \\ &+ \Bigg[\frac{J^2(\xi_+^2 + \eta_+^2)}{8\beta^2 r^2(\xi_+^2 - \eta_+^2)^2} + \frac{1}{4}\Bigg(-\frac{J^2}{8\alpha^2\beta^2 r^2} \\ &+ \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\alpha^2\beta^2 l^2 r^2}\Bigg)\left|\frac{r - r_-}{r + r_-}\right|^{\frac{-\beta l^2 r_-}{(r_+^2 - r_-^2)}} \left|\frac{r - r_+}{r + r_+}\right|^{\frac{\beta l^2 r_+}{(r_+^2 - r_-^2)}}\Bigg] d\eta_+^2 \\ &+ \frac{J^2 \xi_+ \eta_+}{2\beta^2 r^2(\xi_+^2 - \eta_+^2)} d\xi_+ d\eta_+ + r^2 d\phi^2 - \frac{J}{\beta}\left[\frac{\eta_+}{\xi_+^2 - \eta_+^2} d\xi_+ - \frac{\xi_+}{\xi_+^2 - \eta_+^2} d\eta_+\right] d\phi, \end{aligned} \tag{16}$$

here

$$\begin{aligned} \xi_+ &= 2\alpha \left|\frac{r - r_+}{r + r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2 - r_-^2)}} \left|\frac{r - r_-}{r + r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2 - r_-^2)}} \sinh(\beta t), \\ \eta_+ &= 2\alpha \left|\frac{r - r_+}{r + r_+}\right|^{\frac{\beta l^2 r_+}{2(r_+^2 - r_-^2)}} \left|\frac{r - r_-}{r + r_-}\right|^{\frac{-\beta l^2 r_-}{2(r_+^2 - r_-^2)}} \cosh(\beta t), \end{aligned} \tag{17}$$

and the inverse transformations are

$$t = \frac{1}{\beta} \tanh^{-1}\left(\frac{\xi_+}{\eta_+}\right), \tag{18}$$

$$\xi_+^2 - \eta_+^2 = -4\alpha^2 \left[\frac{r - r_+}{r + r_+}\right]^{\frac{\beta l^2 r_+}{(r_+^2 - r_-^2)}} \left[\frac{r - r_-}{r + r_-}\right]^{\frac{-\beta l^2 r_-}{(r_+^2 - r_-^2)}}. \tag{19}$$

The coordinates (ξ_-, η_-) are non-singular at $r = r_-$ and are used for the region $0 < r < r_+$. Similarly (ξ_+, η_+) are non-singular at $r = r_+$ and are used for the region $r_- < r < \infty$. The values of α and β are chosen so that the coordinates (ξ_-, η_-) and (ξ_+, η_+) are matched properly at a point r between r_- and r_+. $r = r_\pm$ correspond to $\xi_+ = \eta_+ = 0$ and $\xi_- = \eta_- = 0$ respectively.

4 Non-singular coordinates for some cases of the circularly symmetric 2 + 1 dimensional black holes

The metric [10]

$$ds^2 = -(1 - 2g(v) - 2h(v)r^{1-k} - \Lambda r^2)dv^2 + 2dvdr + r^2 d\theta^2, \tag{20}$$

where $g(v)$, $h(v)$ are arbitrary functions and $\Lambda = -1/l^2$, is a solution of the 2 + 1 dimensional Einstein equations for the null fluid stress energy tensor with $P = k\rho$.
Case 1: As discussed in [10] for $k = 1$, $g(v) = A > 0$ and $h(v) = 0$, the metric (20) takes the form

$$ds^2 = -F(r)dv^2 + 2dvdr + r^2 d\theta^2, \tag{21}$$

where $F(r) = 1 - 2A - \Lambda r^2$.

In double null coordinates (u, v) the metric (21) takes the form

$$ds^2 = -F(r)dudv + r^2 d\theta^2. \tag{22}$$

The only real positive root of the function $F(r)$ is $c = l\sqrt{2A - 1}(A > 1/2)$, where the metric (22) is singular. In order to remove this coordinate singularity, we define r^* as

$$r^* = \int \frac{1}{F(r)}dr = \frac{l^2}{2c} \ln \left| \frac{r - c}{r + c} \right|. \tag{23}$$

Coordinates analogous to the Kruskal–Szekeres like coordinates, defined by

$$V = \alpha e^{v/\beta}, \quad U = -\alpha e^{-u/\beta}, \tag{24}$$

where $v = t + r^*$, $u = t - r^*$ and α is a positive constant. In these coordinates (with $\beta = l^2/c$) the metric (22) takes the form

$$ds^2 = \frac{l^2(r + c)^2}{\alpha^2 c^2} dV dU + r^2 d\theta^2, \tag{25}$$

here

$$V = \alpha e^{t/\beta} \sqrt{\left| \frac{r - c}{r + c} \right|}, \quad U = -\alpha e^{-t/\beta} \sqrt{\left| \frac{r - c}{r + c} \right|}. \tag{26}$$

and U and V are related with r by

$$VU = -\alpha^2 \left| \frac{r-c}{r+c} \right|. \tag{27}$$

In space like and time like coordinates $\xi = V + U$, $\eta = V - U$, the metric (25) takes the form

$$ds^2 = \frac{l^2(r+c)^2}{4\alpha^2 c^2}(d\xi^2 - d\eta^2) + r^2 d\theta^2, \tag{28}$$

where

$$\xi = \alpha\sqrt{\left|\frac{r-c}{r+c}\right|}\sinh\left(\frac{t}{\beta}\right), \quad \eta = \alpha\sqrt{\left|\frac{r-c}{r+c}\right|}\cosh\left(\frac{t}{\beta}\right), \tag{29}$$

and the inverse transformations are

$$t = \beta \tanh^{-1}\left(\frac{\xi}{\eta}\right), \quad \xi^2 - \eta^2 = -\alpha^2\left(\frac{r-c}{r+c}\right). \tag{30}$$

These coordinates (ξ, η) cover the whole manifold and are non-singular at $r = c$.
Case 2: As discussed in [10] for $k = 0$, $g(v) = A > 0$ and $h(v) = B > 0$, the metric (20) takes the similar form as given by Eq. (21) with $F(r)$ now given by

$$F(r) = 1 - 2A - 2rB + r^2/l^2. \tag{31}$$

$F(r)$ has real roots if $l^2B^2 \geq 1 - 2A$. If $A < 1/2$ we have two positive real roots otherwise only one positive real root exists. The extreme case corresponds to $l^2B^2 = 1 - 2A$. The two positive real roots of $F(r)$ are given by

$$r_{\pm} = l^2 B \pm \sqrt{l^4B^2 - l^2(1-2A)}.$$

In order to remove the coordinate singularities at $r = r_{\pm}$, define r^* as

$$r^* = \int \frac{1}{F(r)} dr = \frac{l^2}{(r_+ - r_-)} \ln\left|\frac{r - r_+}{r - r_-}\right|.$$

Here the Kruskal–Szekeres like coordinates do not remove both the singularities simultaneously and we need two coordinate patches, one for the region $r_- < r < \infty$ and another for the region $0 < r < r_+$. Using Kruskal–Szekeres like coordinates, Eq. (24), the metric (22) takes the form

$$ds^2 = \frac{4l^2(r - r_-)}{\alpha^2(r_+ - r_-)^2} dV dU + r^2 d\theta^2, \tag{32}$$

which is non-singular at $r = r_+$ but it is still singular at $r = r_-$.

Here

$$V = \alpha\, e^{\frac{t(r_+ - r_-)}{2l^2}} \sqrt{\left|\frac{r - r_+}{r - r_-}\right|}, \quad U = -\alpha\, e^{\frac{-t(r_+ - r_-)}{2l^2}} \sqrt{\left|\frac{r - r_+}{r - r_-}\right|}. \tag{33}$$

V and U are related to r by

$$VU = -\alpha^2 \left|\frac{r - r_+}{r - r_-}\right|. \tag{34}$$

In space like and time like coordinates $\xi = V + U$, $\eta = V - U$, metric (32) takes the form

$$ds^2 = \frac{l^2(r - r_-)}{\alpha^2(r_+ - r_-)^2}(d\xi^2 - d\eta^2) + r^2 d\theta^2, \tag{35}$$

here

$$\xi = \alpha\sqrt{\left|\frac{r - r_+}{r - r_-}\right|} \sinh\left(\frac{t(r_+ - r_-)}{2l^2}\right), \quad \eta = \alpha\sqrt{\left|\frac{r - r_+}{r - r_-}\right|} \cosh\left(\frac{t(r_+ - r_-)}{2l^2}\right), \tag{36}$$

and the inverse transformations are

$$t = \frac{2l^2}{(r_+ - r_-)} \tanh^{-1}\left(\frac{\xi}{\eta}\right), \quad \xi^2 - \eta^2 = -\alpha^2\left(\frac{r - r_+}{r - r_-}\right). \tag{37}$$

The coordinate system (ξ, η) covers only the region $r_- < r < \infty$ of the whole manifold. Following the similar approach as adopted in Sect. 3, one can easily construct the non-singular coordinates for the region $0 < r < r_+$ also.

5 Conclusion and discussion

We have presented the Kruskal-like coordinates that remove coordinate singularities of the usual rotating BTZ black hole. In case of the rotating extreme BTZ black hole the metric (1) takes the form

$$ds^2 = \left(\frac{r_e^4}{r^2 l^2} - g(r)\right) dt^2 + \frac{1}{g(r)} dr^2 - \frac{2r_e^2}{l} dt d\phi + r^2 d\phi^2, \tag{38}$$

where $g(r) = \frac{(r^2 - r_e^2)^2}{r^2 l^2}$ and $r_e^2 = \frac{Ml^2}{2}$. This metric is singular at $r = r_e$. We have not been able to find Kruskal (or Kruskal–Szekres) like coordinates for this metric. Hence, to avoid the coordinate singularity, we introduce the Carter-like coordinates [11]

$$\psi = \tan^{-1}\left(\frac{v}{l}\right) + \cot^{-1}\left(\frac{w}{l}\right), \quad \xi = \tan^{-1}\left(\frac{v}{l}\right) - \cot^{-1}\left(\frac{w}{l}\right), \tag{39}$$

here $v = t + r^*$ and $w = -t + r^*$, with r^* given by

$$r^* = \int \frac{r^2 l^2}{(r^2 - r_e^2)^2} dr = \frac{l^2}{4r_e} \ln\left|\frac{r - r_e}{r + r_e}\right| - \frac{l^2}{2} \frac{r}{(r^2 - r_e^2)}. \tag{40}$$

In the (ψ, ξ, ϕ) coordinates the metric (38) takes the form

$$\begin{aligned} ds^2 = \frac{1}{4r^2} &\left\{ (f_1 + f_2) d\psi^2 + (f_1 - f_2) d\xi^2 + \frac{r_e^4}{2} f_3 f_4 d\psi d\xi \right\} \\ &- \frac{r_e^2}{2} (f_3 d\psi + f_4 d\xi) d\phi + r^2 d\phi^2, \end{aligned} \tag{41}$$

where

$$\begin{aligned} f_1 &= \frac{r_e^4}{4} \left\{ \sec^4\left(\frac{\psi + \xi}{2}\right) + \csc^4\left(\frac{\psi - \xi}{2}\right) \right\}, \\ f_2 &= \left\{ \frac{r_e^4}{2} - (r^2 - r_e^2)^2 \right\} \sec^2\left(\frac{\psi + \xi}{2}\right) \csc^2\left(\frac{\psi - \xi}{2}\right), \\ f_3 &= \sec^2\left(\frac{\psi + \xi}{2}\right) + \csc^2\left(\frac{\psi - \xi}{2}\right), \end{aligned}$$

and

$$f_4 = \sec^2\left(\frac{\psi + \xi}{2}\right) - \csc^2\left(\frac{\psi - \xi}{2}\right). \tag{42}$$

Note that ψ and ξ are related to the radial parameter by

$$\tan\left(\frac{\psi + \xi}{2}\right) + \cot\left(\frac{\psi - \xi}{2}\right) - l\left[\frac{1}{2r_e} \ln\left|\frac{r - r_e}{r + r_e}\right| - \frac{r}{(r^2 - r_e^2)}\right] = 0. \tag{43}$$

The determinant of the metric (41) is

$$|g_{ab}| = -\frac{(r - r_e)^4 (r + r_e)^4}{16r^2} \sec^4\left(\frac{\psi + \xi}{2}\right) \csc^4\left(\frac{\psi - \xi}{2}\right) \quad (a, b = 0, 1, 2).$$

Now

$$\csc^4\left(\frac{\psi - \xi}{2}\right) = \left(1 + \frac{w^2}{l^2}\right)^2 = \left(1 + \frac{1}{l^2}(-t + r^*)^2\right)^2, \tag{44}$$

on simplifying, we have

$$\begin{aligned}
&\csc^4\left(\frac{\psi-\xi}{2}\right)\\
&=\left(\frac{l^2}{4r_e}\left[(r-r_e)\ln(r-r_e)-(r-r_e)\ln(r+r_e)-r_e-\frac{r_e(r-r_e)}{(r+r_e)}\right]\right)^4\\
&\quad-4(r-r_e)\left(\frac{l^2}{4r_e}\left[(r-r_e)\ln(r-r_e)-(r-r_e)\ln(r+r_e)-r_e-\frac{r_e(r-r_e)}{(r+r_e)}\right]\right)^3 t\\
&\quad+2(r-r_e)^2\left(\frac{l^2}{4r_e}\left[(r-r_e)\ln(r-r_e)-(r-r_e)\ln(r+r_e)-r_e-\frac{r_e(r-r_e)}{(r+r_e)}\right]\right)^2\\
&\quad\times(l^2+3t^2)\\
&\quad-4(r-r_e)^4\left(\frac{l^2}{4r_e}\left[(r-r_e)\ln(r-r_e)-(r-r_e)\ln(r+r_e)-r_e-\frac{r_e(r-r_e)}{(r+r_e)}\right]\right)\\
&\quad\times(l^2+t^2)t\\
&\quad+(r-r_e)^4(l^2+t^2)^2. \qquad (45)
\end{aligned}$$

As $r \to r_e$ and $\psi \to \xi$, the value of the determinant is finite $-\frac{r_e^2 l^4}{256}\sec^4(\xi)$, but the metric (41) is still singular at $r = r_e$. Carter-like coordinates for this geometry cannot remove the coordinate singularity and are manifestly not regular at the horizon. However these coordinates work in the case of extreme Reissner–Nordstörm black hole, there they give finite determinant and are regular at the horizon. Non-singular Kruskal-like coordinates are also presented for different cases of the general circularly symmetric 2 + 1 dimensional black holes.

Whereas the non-rotating BTZ metric can be easily transformed into the double-null form, and has the same singularities as in the (t, r) coordinates, here we have dealt with the rotating BTZ metric. We tried to solve the Einstein equations for this case but have not been unsuccessful so far. We have also tried to find the coordinate transformation to transform the metric in the desired form. The attempts that we made indicate that perhaps the rotating-BTZ metric may not be transformed to the required form. However, this has not been proved formally. Even if we manage to prove the non-existence, the result may not be pertinent here.

Acknowledgments The authors are very grateful to Prof. A. Qadir for his valuable comments, suggestions and involvement throughout this work. Authors are also thankful to the unknown Referees for their comments that have improved the manuscript significantly.

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CHIN. PHYS. LETT. Vol. 28, No. 5 (2011) 050401

Foliation and the First Law of Black Hole Thermodynamics

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(Received 10 January 2011)

There has been lots of interest in exploring the thermodynamic properties at the horizon of a black hole spacetime. It has been shown earlier that for different spacetimes, the Einstein field equations at the horizon can be expressed as the first law of black hole thermodynamics. Using the idea of foliation, we develop a simpler procedure to obtain such results. We consider r = constant slices, for the Schwarzschild and Reissner–Nordstrom black hole spacetimes. The Einstein field equations for the induced 3-dimensional metrics of the hypersurfaces are expressed in thermodynamic quantities under the virtual displacements of the hypersurfaces. As expected, it is found that the field equations of the induced metric corresponding to the horizon can be written as a first law of black hole thermodynamics. It is to be mentioned here that our procedure is much easier, to obtain such results, as here one has to essentially deal with $(n-1)$-dimensional induced metric for an n-dimensional spacetime.

PACS: 04.70.Dy, 05.70.−a, 98.80.Jk DOI: 10.1088/0256-307X/28/5/050401

Splitting a space into a sequence of subspaces, such that every point in the space lies in one and only one of the subspaces, is called a *foliation*.[1] The foliation of an n-dimensional manifold, M, is a decomposition of M into submanifolds, all being of the same dimension, p. The submanifolds are the leaves of the foliation. The co-dimension q of a foliation is defined as $q = n - p$. A foliation of co-dimension 1 is called a foliation by *hypersurfaces*. Some physically reasonable solutions to the Einstein field equations are singular and represent black hole spacetimes.[2] These spacetimes have special significance because of the horizons in their geometry. To analyze the dynamics of such geometries, one often foliates[3−22] the spacetime by a sequence of null or space-like hypersurfaces. Here we have used r = constant foliations for the Schwarzschild and Reissner Nordstrom black holes to discuss the first law of thermodynamics at their horizons.

The relationship between the Einstein field equations and the first law of black hole thermodynamics was first presented by Jacobson,[23] then Padmanabhan[24] made a general formalism for understanding the thermodynamics of horizons. Since then there has been a lot of work in this direction for different spacetime geometries.[25−35]

In this Letter, using foliation, we present an elegant and simpler way of obtaining relationship between the Einstein field equations at the horizon and the first law of black hole thermodynamics. Instead of obtaining field equations of black hole spacetime and analyzing thermal interpretation at the horizon, here the main idea is to consider a foliation so that the horizon corresponds to a particular hypersurface. Then we work out the field equations for the induced metric of the hypersurfaces and obtain the earlier thermal analysis for the hypersurface corresponding to the horizon. The main advantage of our procedure is that instead of dealing with the field equations of an n-dimensional spacetime, we deal with the field equations of $(n-1)$-dimensional induced metric.

The static, spherically symmetric solution to the Einstein field equations

$$G_{ab} = -8\pi T_{ab}, \quad (a, b = 0, 1, 2, 3) \tag{1}$$

known as the Schwarzschild black hole is given in gravitational units $c = G = 1$ as

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2}$$

where $f(r) = 1 - \frac{2m}{r}$, m is the mass of the Schwarzschild black hole, G_{ab} and T_{ab} are Einstein's and Stress-energy tensors respectively. The thermal quantities associated with the event horizon of the above black hole spacetime are the temperature $T = \frac{f'(r)}{4\pi}\,|_{\rm horizon} = \frac{1}{8\pi m}$ and the entropy $S = \frac{A}{4} = 4\pi m^2$, where A is the horizon area.

We consider a foliation of the Schwarzschild black hole spacetime by r = constant (say k) hypersurfaces, so that the black hole horizon corresponds to a particular hypersurface. The induced metric of the hypersurfaces is given by

$$ds_h^2 = f(k)dt^2 - k^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3}$$

In order to look for the dynamics of the hypersurfaces, we consider $(0,0)$-component of the Einstein

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tensor for the induced metric which is

$$G_{00} = \frac{f(k)}{k^2} = \frac{1-\frac{2m}{k}}{k^2}. \quad (4)$$

Now treating k as a parameter and considering an arbitrary small displacement dk, the $(0,0)$-component of the Einstein field equation (1) for the induced metric with $T_{00}=0$ gives

$$\frac{dk}{2} - \frac{m}{2\pi k^2} d(\pi k^2) = 0. \quad (5)$$

For the hypersurface given by $k=2m$, Eq. (5) yields the first law

$$dm = TdS. \quad (6)$$

It is important to note that $k=2m$, which corresponds to the black hole event horizon, is the only hypersurface satisfying the first law.

Reissner and Nordstrom obtained the solution to Eq. (1), with a non-vanishing energy-momentum tensor arising from the electromagnetic field, which describes the field outside a spherically symmetric massive charged point, called the Reissner–Nordstrom (RN) black hole, is given in gravitational units by Eq. (2), with $f(r)$ now given by

$$f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad (7)$$

where m and Q are mass and charge of the black hole respectively. The thermal quantities associated with the RN black hole at the outer horizon ($r=r_+=m+\sqrt{m^2-Q^2}$) are the temperature $T=\frac{f'(r)}{4\pi}\mid_{r=r_+}=\frac{(mr_+-Q^2)}{2\pi r_+^3}$, entropy $S=\frac{A}{4}=\pi r_+^2$ (where A is the horizon area) and electric potential $\Phi=\frac{\partial m}{\partial Q}\mid_{r=r_+}=\frac{Q}{r_+}$.

We again consider a foliation of the RN spacetime by $r=$ constant (say k) hypersurfaces, so that the black hole horizon corresponds to a particular hypersurface. The induced metric of the hypersurfaces takes the same form as given by Eq. (3), with f now given by Eq. (7).

In order to look for the dynamics of the hypersurfaces, consider $(0,0)$-component of the Einstein tensor for the induced metric

$$G_{00} = \frac{f(k)}{k^2} = \frac{1-\frac{2m}{k}+\frac{Q^2}{k^2}}{k^2}. \quad (8)$$

Again treating k as a parameter and considering an arbitrary small displacement dk, the $(0,0)$-component of the Einstein field equation (1) for the induced metric with $T_{00}=-P$, after multiplying it with dk, gives

$$\frac{dk}{2} - \frac{mdk}{k} + \frac{Q^2dk}{2k^2} = 4\pi Pk^2dk, \quad (9)$$

where P corresponds to the pressure of the source. Also at the horizon $m=\frac{r_+}{2}+\frac{Q^2}{2r_+^2}$, which gives

$$dm = \frac{dr_+}{2} - \frac{Q^2}{2r_+^2}dr_+ + \frac{Q}{r_+}dQ. \quad (10)$$

Now substituting Eq. (10) into Eq. (9) and simplifying at the horizon, $k=r_+$, we obtain

$$dm - \frac{(mr_+-Q^2)}{2\pi r_+^3}d(\pi r_+^2) - \frac{Q}{r_+}dQ = Pd(\frac{4}{3}\pi r_+^3), \quad (11)$$

which can be expressed in terms of the thermal quantities defined after Eq. (7) as the first law

$$dm = TdS + \Phi dQ + PdV, \quad (12)$$

where $V=\frac{4}{3}\pi r_+^3$ is the volume enclosed by the horizon.

In summary, we have presented a new approach to express the Einstein field equation as the first law of black hole thermodynamics at the horizon of the Schwarzschild and the Reissner–Nordstrom black holes. In our approach, we have used the foliations of these spacetimes such that the horizon corresponds to a particular leaf (hypersurface) of our foliation. Then we work out the field equations of the induced metric of the hypersurfaces and show that the field equations for the induced metric, at the horizon, can be expressed as the first law of black hole thermodynamics. The important aspect of our approach is that here we have to essentially deal with $(n-1)$-dimensional induced metric for an n-dimensional spacetime, which significantly simplifies the calculations to obtain such results. Here we have dealt with the Schwarzschild and the Reissner–Nordstrom black hole spacetimes which are static and spherically symmetric. It will be interesting to extend this approach to the Kerr and the Kerr–Newmann black holes, which are not spherically symmetric and also to other black hole geometries.

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