Applications of Geometrothermodynamics in Non–Standard Theories of Gravity

by

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Dedicated

To My Family

and

Prof. Hernando Quevedo

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Abstract

Black holes are one of the most interesting objects of study in physics. Classically, they trap everything including light. They are characterized by an event horizon which encloses a curvature singularity. A study of black hole physics shows that a black hole behaves as a thermodynamic system, with the area of the event horizon playing the role of the entropy and a geometric quantity called surface gravity as the temperature of the black hole.

It is possible to introduce differential geometric concepts in ordinary thermodynamics. The most known structures were postulated by Weinhold and Ruppeiner who introduced Riemannian metrics in the space of equilibrium states of a thermodynamic system. These geometric structures can obviously be applied in black hole thermodynamics. Unfortunately, the results are not in agreement with ordinary thermodynamics which is manifestly Legendre invariant. To overcome this inconsistency, the theory of geometrothermodynamics was proposed. It incorporates arbitrary Legendre transformations into the geometric structure of the equilibrium space in an invariant manner.

In this thesis I have studied the thermodynamics of various black holes in different gravity theories by means of thermodynamic Riemannian curvatures. The thermodynamics of black holes is reformulated within the context of the formalism of geometrothermodynamics. This reformulation is shown to be invariant with respect to Legendre transformations and to allow several equivalent representations. Legendre invariance allows to explain a series of contradictory results known in the literature from the use of Weinhold's and Ruppeiner's thermodynamic metrics for black holes.

I present a brief review of classical and black hole thermodynamics and the basic mathematical elements of geometrothermodynamics in the first chapter. Then I present a systematic and consistent construction of geometrothermodynamics by using Riemannian contact geometry for the phase manifold and harmonic maps for the equilibrium manifold in section 1.11.

In chapter two, using the formalism of geometrothermodynamics, I investigate the geometric properties of the equilibrium manifold for diverse thermodynamic systems. Starting from Legendre invariant metrics of the phase manifold, I derive thermodynamic metrics for the equilibrium manifold whose curvature becomes singular at those points where phase transitions of first and second order occur.

I present the thermodynamics and the thermodynamic geometries of charged rotating BTZ black holes in the third chapter. The thermodynamics of these black holes is investigated within the context of the Weinhold and Ruppeiner geometries and the formalism of geometrothermodynamics. Considering the behavior of the heat capacity and the Hawking temperature, I show that Weinhold and Ruppeiner geometries cannot describe completely the thermodynamics of these black holes and of their limiting case of vanishing electric charge. In contrast, the Legendre invariance imposed on the metric in geometrothermodynamics allows one to describe these black holes and their limiting cases in a consistent and invariant manner.

In the fourth chapter, the thermodynamic properties of five-dimensional static and spherically symmetric black holes in Einstein-Gauss-Bonnet theory are investigated. To formulate the thermodynamics of these black holes I use the Bekenstein-Hawking entropy relation and, alternatively, a modified entropy formula which follows from the first law of thermodynamics of black holes. The results of both approaches are not equivalent.

Chapter five is devoted to the study of thermodynamic geometries of the most general static, spherically symmetric, topological black holes of the Hořava–Lifshitz gravity. In particular, I show that a Legendre invariant metric derived in the context of geometrothermodynamics for the equilibrium manifold reproduces correctly the phase transition structure of these black holes. Moreover, the limiting cases in which the mass, the entropy or the Hawking temperature vanish are also accompanied by curvature singularities which indicate the limit of applicability of the thermodynamics and the geometrothermodynamics of black holes. The Einstein limit and the case of a black hole with flat horizon are also investigated.

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- H Quevedo, A Sánchez, S Taj and A Vázquez, *"Phase transitions in Geometrothermodynamics"*, *Gen. Rel. Grav.* **43** (2011) 1153.
- H Quevedo, A Sánchez, S Taj and A Vázquez, *"Curvature as a Measure of Thermodynamic Interaction"*, *J. Korean Phys. Soc.* **57** (2010) 646.
- M Akbar, H Quevedo, K Saifullah, A Sánchez and S. Taj, *"Thermodynamic Geometry of Charged Rotating BTZ Black Holes"*, *Phys. Rev.* **D 83** (2011) 084031.
- S Taj and H Quevedo, *"Geometrothermodynamics of higher dimensional black holes in Einstein-Gauss-Bonnet theory"*, *Gen. Rel. Grav.* (submitted); arXiv:1104.3195[hep–th].
- H Quevedo, A Sánchez, S Taj and A Vázquez, *"Geometrothermodynamics in Hořava-Lifshitz gravity"*, *J. Phys. A* (To appear); arXiv:1101.4494[hep–th].

Chapter 1

Introduction

One of the most interesting aspects of modern theoretical physics is its direct relation with different areas of mathematics. General Relativity (GR), for instance, can be considered as an application of differential geometry. Once a metric tensor is given which is compatible with a torsion-free connection and satisfies the Einstein field equations, the corresponding curvature turns out to be a measure of gravitational interaction. This fascinating result combines apparently different concepts of geometry and physics, allowing us to study gravity by measuring the curvature of spacetime. In fact, this result can conceptually be generalized to include all the known field interactions in nature. The electromagnetic, weak and strong interactions can classically be described by using the Minkowski metric and a gauge connection. In all the cases, the resulting gauge curvature can be considered as a measure of the corresponding field interaction [54].

In particular, the four known interactions of nature can be described in terms of geometrical concepts. Indeed, Einstein proposed the astonishing principle *"field strength = curvature"* to understand the physics of the gravitational field (see, for instance, [54, 93]). In this case, *"curvature"* means the curvature of a Riemannian manifold. In GR, the connection involved is unique as a consequence of the assumption that the torsion tensor vanishes. The idea of this construction can be represented schematically as *metric → Christoffel symbols → Riemann curvature = gravitational field strength*.

The second element of GR is Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

which established for the first time the amazing principle *"geometry = energy"* in some sense, where G is the gravitational constant and Λ is the cosmological constant. The conceptual fundamentals of this principle were very controversial; however, experimental evidence has shown its correctness and all modern generalizations of Einstein's theory follow the same principle. On the other hand, since the field strength can be considered as a measure of the gravitational interaction, we conclude that the entire idea of GR can be summarized in the principle *"gravitational interaction = curvature"* which holds for all known forces of nature [118].

Although thermodynamics is based entirely upon empirical results which are satisfied under certain conditions in almost any macroscopic system, the geometric approach to thermodynamics has proved to be useful and illuminating. In very broad terms, one can say that in a thermodynamic system all the known forces act among the particles that constitute the system. Due to the large number of particles involved in the system, only a statistical approach is possible, from which average values for the physical quantities of interest are derived. The question arises whether it is also possible to find a geometric construction for which the principle *"curvature = thermodynamic interaction"* holds. The formalism of geometrothermodynamics (GTD) [112] satisfies this condition. (I shall discuss it in detail in section 1.11.) First, I must mention that the interpretation of thermodynamic interaction is based upon the statistical approach to thermodynamics in which all the properties of the system can be derived from the explicit form of the corresponding Hamiltonian [61] and the interaction between the particles of the system is described by the potential part of the Hamiltonian. Consequently, if the potential vanishes, the system has zero thermodynamic interaction. In this thesis, I present the metric structures which satisfy this condition for systems with no thermodynamic interaction as well as for systems characterized by interaction with phase transitions in non-standard theories of gravity.

It is not an exaggeration to say that one of the most exciting predictions of Einstein's theory of gravitation is that there may exist black holes: A black hole is, by definition, a region in spacetime in which the gravitational field is so strong that it precludes even light from escaping to infinity. A black hole is formed when a massive body contracts to a size less than the radius at which the escape velocity is the speed of light. When the massive body gets charged the geometry becomes more complicated. The inclusion of rotation also increases the complexity.

The term *"black hole"* was introduced by Wheeler in 1964 for the first time and it was publicized in 1967 [144], although the theoretical study of these objects has quite a long history. It has frequently been supposed by theoretical physicists that Newton's theory of gravitation predicts the black holes [93]. The idea of a body so massive that even light could not escape was first put forward by John Michell in 1783 [90] and Pierre–Simon de Laplace in 1796 [81] within the framework of the Newtonian theory. Such *"dark stars"*, as these were named by Michell, were largely ignored in the nineteenth century, since it was not understood how a wave such as light could be influenced by gravity. In 1915, Albert Einstein published his theory of GR, a new theory of gravitation that made fundamental predictions on the effect of gravity on light. Only a few months after the publication of GR, Karl Schwarzschild solved the Einstein field equations (1.1) by assuming a static and spherically symmetric geometry, obtaining what is now called the Schwarzschild solution or metric [131]:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1.2}$$

where $d\Omega^2$ is the line element of the two-sphere and $r_s = 2GM/c^2$ is called the Schwarzschild radius.

The inclusion of an electric charge Q also yields an exact static spherical solution which is known as the Reissner-Nordström black hole (1918) when $Q^2 \leq GM^2$ [123]. Finally, an angular momentum J can be added to the system yielding a stationary, axisymmetric solution which, when $Q = 0$, is known as the Kerr solution (1963) [77] and is called the Kerr-Newman solution (1965) when $Q \neq 0$ [101]. These solutions describe a black hole when the condition $\frac{J^2 c^2}{G^2 M^2} + \frac{Q^2}{G} \leq M^2$ is satisfied. The equality sign corresponds to the extreme black hole solutions which have zero temperature. When the above constraint is violated we have a naked singularity, a visible singularity not surrounded by a horizon.

The uniqueness theorems state that the only static or stationary solutions of the Einstein–Maxwell equations that are asymptotically flat and have regular horizons are the above solutions characterized only by the parameters M, Q and J. All the other parameters that specified the initial state before the formation of the black hole are radiated away during the creation process. This simple description of a black hole is summarized by the well-known metaphoric statement of Wheeler: *A black hole has no hair.*

The theory of black holes has been strongly connected to the theory of gravitational collapse after the work of Oppenheimer and Snyder [103] and Penrose's theorem [109]: a realistic, slightly non–spherical complete collapse leads unavoidably to the formation of a black hole and a singularity [62]. In the context of gravitational collapse towards a black hole, two important conjectures have been formulated, the cosmic censorship and the hoop conjectures. The cosmic censorship conjecture [108] forbids the existence of naked singularities, while the hoop conjecture [137] states that black holes form when and only when a mass M gets compacted into a region whose circumference in every direction is less than its Schwarzschild circumference $4\pi GM/c^2$. During the last few decades, a great effort has been initiated in order to detect the energetic astrophysical processes predicted to be powered by black holes.

One of the most remarkable achievements in black hole physics was the realization that black holes have temperature T and entropy S. The discovery by Bekenstein [16] that the behavior of the horizon area of a black hole resembles the behavior of the entropy of a classical thermodynamic system initiated an intensive and still ongoing investigation of what is now called *black hole thermodynamics* [14,44,69]. Several attempts have been made in order to describe the thermodynamic behavior of black holes. Indeed Bardeen, Carter and Hawking [14], supported by previous work of Christodoulou [40] and Penrose [110], have shown that black holes obey the so–called four laws of black hole mechanics. I shall discuss these in detail in the section 1.7.

The geometry of thermodynamics has been the subject of moderate research since the original works by Gibbs [58] and Caratheodory [33]. During the last few decades several attempts have been made to introduce differential geometric concepts in ordinary thermodynamics. Results have been achieved in many different approaches.

Riemannian geometry was first introduced in statistical physics and thermodynamics in 1945 by Rao [122]. On the other hand, Riemannian geometry in the space of equilibrium states was introduced in 1976 by Weinhold [141] and by Ruppeiner in 1979 [125,126]. The Ruppeiner metric is similar to the Rao metric used in mathematical statistics and is conformally equivalent to Weinhold's metric. The study of the relation between the phase space and the metric structures of the space of equilibrium states led to the result that Weinhold's and Ruppeiner's thermodynamic metrics are not invariant [98, 127], i.e. the geometric properties of the space of equilibrium states are different when different thermodynamic potentials are used. This result clearly contradicts ordinary equilibrium thermodynamics which is manifestly invariant. Moreover, the question whether the curvature of the space of equilibrium states can be considered as a measure for thermodynamic interaction remained unanswered.

In order to incorporate certain requirements in Riemannian structures at the level of the phase space and the equilibrium space, the formalism of GTD was proposed [112] and applied to different thermodynamic systems [2, 113–119]. The main motivation for introducing the formalism of GTD was to formulate a geometric approach which takes into account the fact that in ordinary thermodynamics the description of a system does not depend on the choice of thermodynamic potential. The main goal of this thesis is to present metric structures which satisfy these conditions for systems with no thermodynamic interaction as well as for systems characterized by interaction with phase transitions in non standard theories of gravity.

The task of quantizing GR is one of the outstanding problems of modern theoretical physics. Attempts to reconcile quantum theory and GR date back to the 1930s (for a historical review see [124,134]). The obstacles to quantize gravity are in part technical. GR is a complicated nonlinear and non-renormalizable theory. Moreover, viewed as an ordinary field theory, it has a coupling constant $G^{1/2}$. The problem of finding a consistent quantum theory of gravity goes deeper. GR is a geometric theory of spacetime and quantizing gravity means quantizing spacetime itself. Faced with such problems, it is natural to look for simpler models that share the important conceptual features of GR while avoiding some of the computational difficulties.

Einstein gravity in a $(2+1)$-dimensional spacetime [35, 36] has some unusual features that clearly differentiate it from the $(3+1)$-dimensional Einstein gravity [15, 48, 59, 60]. Any vacuum solution with $\Lambda = 0$ is *flat* and any vacuum solution with non-vanishing cosmological constant has constant curvature. This follows from the fact that the Weyl tensor in 2+1 dimensions is identically zero. The $(2+1)$-dimensional spacetime has no local degrees of freedom and thus its dynamics is substantially different from the one of the (3+1)-dimensional case. In particular, there is no Newtonian limit, that is there is no gravitational force between masses. Note, however, that the absence of gravitational dynamics in GR does not imply that $(2+1)$-dimensional spacetimes are trivial and uninteresting.

In what concerns black hole solutions in $(2+1)$-dimensional Einstein gravity, quite surprisingly (since the $(2+1)$-dimensional spacetime is quite poor at the dynamical level), Bañados, Teitelboim and Zanelli [12] found a black hole solution (the BTZ black hole), with mass M and angular momentum J, that is asymptotically Anti de Sitter (AdS). The existence of this black hole gets even more remarkable when one realizes that the BTZ metric has constant curvature and thus there can be no curvature singularity at the origin. The extension to include a radial electric field in the BTZ black hole, has been done by Clément [41] and Martínez, Teitelboim and Zanelli [86]. In chapter 3 of this thesis, I will discuss the thermodynamics and the thermodynamic geometries of BTZ and charged rotating BTZ black holes in $(2+1)$-dimensional gravity [2].

Shortly after Einstein's announcement of his theory of GR, physicists initiated efforts to develop a unified field theory. In the 1920s Kaluza and Klein [76, 80] for the first time unified gravity and electromagnetism by suggesting the existence of an extra dimension, besides the usual 3+1 dimensions. Indeed, they realized that 5-dimensional vacuum GR contained $(3+1)$-dimensional GR in the presence of an electromagnetic field together with Maxwell's laws of electromagnetism and an equation for a scalar field. The dominant view is that the extra dimension is not observed on experimentally accessible energy scales because it is too small. The Kaluza-Klein mechanism has also unified matter and geometry since the photon that is present in the (3+1)-dimensional spacetime is a manifestation of empty 5-dimensional spacetime. The key point to the Kaluza-Klein unification is the realization that to a $(3+1)$-dimensional gauge symmetry (e.g., the $U(1)$ gauge invariance of Maxwell theory) corresponds a geometric symmetry (an invariance with respect to coordinate transformations) in the extra dimension. Thus, the main achievement of Kaluza-Klein proposal demonstrated that different uncorrelated phenomena that occur in $(3+1)$-dimensional spacetime can be manifestations of the same 5-dimensional theory. From then onwards the theories of strong and weak nuclear interactions have been developed and several attempts to unify the $(3+1)$-dimensional theories in a single $D > 5$ higher-dimensional theory have been tried with some successes. (For a nice review see [104].)

Not surprisingly, the mere extension of GR by considering extra spacelike dimensions can immediately lead to very non-trivial alterations in the theory [79, 111]. According to the basic principles of GR, higher dimensional gravity is described by theories containing higher powers of the curvature [84]. In five dimensions, the most general theory leading to second order field equations for the metric is the so–called Einstein-Gauss-Bonnet (EGB) theory, which contains quadratic powers of the curvature [82]. The inclusion of additional structure in the gravitational action increases even further the diversity of the models available and gives rise to a rich phenomenology. The black holes in EGB gravity have interesting properties. Although some part of EGB gravity resembles GR, the other part does not. Some properties of these black holes are reviewed in chapter 4 of this thesis.

The Kaluza-Klein idea of extra spacetime dimensions continues to pervade current attempts to unify the fundamental forces, but in ways somewhat different from that originally visualized. Recently Petr Hořava [65] suggested a new candidate for quantum field theory of gravity. This theory is a non-relativistic power-counting renormalizable theory in four dimensions, which admits the Lifshitz scale invariance in time and space that reduces to Einstein's GR at large scales [66]. The Hořava theory has received a great deal of attention and since its formulation various properties and characteristics have been extensively analyzed. I shall review the Hořava-Lifshitz gravity and investigate the thermodynamic geometries of the most general static, spherically symmetric, topological black holes of Hořava's gravity in chapter 5 of this thesis. In particular, I will show that a Legendre invariant metric derived in the context of GTD for the equilibrium manifold reproduces correctly the phase transition structure of these black holes [119].

Henceforth in this thesis I use the units in which $c = \hbar = 8G = k_B = 1$.

1.1 Classical Thermodynamics

Since this thesis is written for a readership of mathematicians who may not have studied thermodynamics, I provide a brief review of the basic concepts of this subject.

Thermodynamics is the science of energy conversion involving heat and other forms of energy, most notably mechanical work. It studies and interrelates the macroscopic variables, such as temperature, volume and pressure, which describe physical thermodynamic systems. Historically, thermodynamics developed out of the need to understand the relation between heat and chemical transformations and the conversion of heat into mechanical energy. Later Maxwell [87], Boltzmann [21] and others placed the theory on more secure footing, providing much greater clarity, by regarding the macroscopic variables as driving force for the microscopic (molecular) mechanical variables.

Perhaps the most striking feature of matter is the incredible simplicity with which it can be characterized. Thermodynamics is a phenomenological theory of matter. As such, it draws its results directly from experiments. Classical thermodynamics only deals with equilibrium states of a system, each of which corresponds to a set of indistinguishable microstates. The following is a list of some concepts, which will be used [67, 107].

- A *thermodynamic system* is a certain quantity of matter or the space which is under thermodynamic study or analysis. The system is covered by the boundary, which may be notional or real and the area beyond the boundary is called *the surroundings.* The boundary of the system can be fixed or it can be movable. The exchange of mass or energy or both can occur between the system and surroundings.

- There are three mains types of thermodynamic systems. A system in which the transfer of mass as well as energy can take place across its boundary is called an *open system.* A system in which the transfer of energy takes place across its boundary with the surroundings, but no transfer of mass takes place is known as a *closed system.* A system in which neither the transfer of mass nor that of energy takes place across boundary with the surroundings is known as a *isolated system.*

- *Thermodynamic variables* are measurable thermodynamic quantities associated with the thermodynamic system, such as the pressure P, the volume V, the temperature T, the magnetic field H, etc. These macroscopic variables always occur in conjugate pairs of extensive and intensive variables. A thermodynamic quantity is said to be *extensive* if it is directly proportional to the system size or the amount of material in the system under consideration and is said to be *intensive* if it does not depend on the system size or the amount of material in the system

- Related to the intensive variable pressure P the extensive variable is volume V. The pair define the mechanical energy; work PdV (in Joules) is done by the system on the surroundings if its volume is increased by dV in equilibrium at pressure P. The pair (P, V) is the most familiar pair of the *mechanical state variables.*

- A *thermodynamic state* is specified by the set of values of all the thermodynamic variables necessary for the description of the system.

- Quantities, such as temperature T and pressure P, which return to the same value whenever the system returns to the same equilibrium state are called *state variables.*

- *Thermodynamic equilibrium* prevails when the thermodynamic state of the system does not change with time. In thermodynamics a state automatically means a state in equilibrium unless otherwise specified.

- The amount of heat required to raise the temperature T by 1 degree when the volume occupied by the system is kept constant, is called *heat capacity* at constant volume and is denoted by C_V. Heat capacity is an extensive variable.

- The *equation of state* (EOS) is a functional relationship among the thermodynamic variables for a system in equilibrium. If P, V and T are the only thermodynamic variables of the system under consideration, the EOS takes the form

$$f(P, V, T) = 0, \tag{1.3}$$

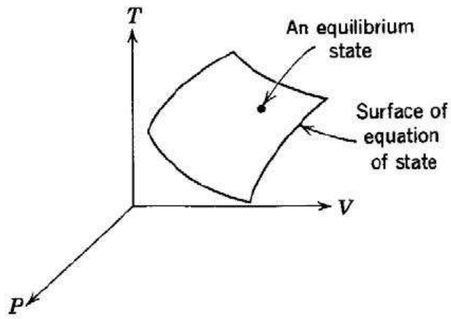

Figure 1.1: Geometrical representation of the equation of state.

which reduces the number of independent variables of the system from three to two. The function f is assumed to be given as a part of the specifications of the system. It is customary to represent the state of such a system by a point in a three dimensional $PVT-$space. The EOS then defines a surface in this space, as shown in Fig.(1.1). Any point lying on the surface represents a state in equilibrium.

- A *thermodynamic transformation* is a change of state. If the initial state is an equilibrium state, the transformation can be brought about only by the changes in the external conditions of the system. The transformation is quasi-static if the external condition changes so slowly that at any moment the system is approximately in equilibrium. It is reversible if the transformation retraces its history in time when the external condition does so.

- The $PV-$*diagram* of a system is the projection of the surface of the equation of state on to the $PV-$plane. Every point on the $PV-$diagram therefore represents an equilibrium state.

- A heat *reservoir*, or simply reservoir, is a system so large that the gain or loss of any finite amount of heat does not change its temperature significantly.

- A system is *thermally isolated* if no heat exchange can take place between it and the surroundings. Thermal isolation may be achieved by surrounding a system with an *adiabatic wall*. Any transformation the system can undergo in thermal isolation is said to take place adiabatically.

- The concept of *work* is taken over from mechanics. For a system with parameters P, V and T, the work done $đW$ by a system in an infinitesimal transformation in which the volume increased by dV is given by

$$đW = PdV. \tag{1.4}$$

- The ideal gas is an important idealized thermodynamic system. Experimentally all gases behave in a universal way when they are sufficiently dilute. The ideal gas is an idealization of this limiting behavior. The parameters for an ideal gas are pressure P, volume V, temperature T and number of molecules N. The *EOS* of an ideal gas is given by

$$PV = NRT, \tag{1.5}$$

where R is the gas constant and its value is 8.315 joule/deg.

1.2 The Laws of Thermodynamics

The laws of thermodynamics form an axiomatic basis of thermodynamics. They are deceptively the simplest statements, each implying many properties of thermal behavior which are not at all self evident and which were accepted after a lot of careful experimentation. These laws are a complete set of logically sufficient axioms, from which the rest of thermodynamics can be derived. For completeness I also include the zeroth law of thermodynamics. The laws of thermodynamics define the rules of temperature equivalence (zeroth law, 1786), energy conservation (first law, 1837), entropy tendencies (second law, 1865) and conditions for an absence of temperature (third law, 1906) [11]. The combined law of thermodynamics, sometimes called the Gibbs fundamental equation, is the combination of the four laws in one expression (for a historical review of the laws of classical thermodynamics, see for instance [133]).

The Zeroth Law of Thermodynamics: The zeroth law of thermodynamics is a generalization of the principle of thermal equilibrium between thermodynamic systems, in contact. A system is said to be in thermal equilibrium when it experiences no net change in thermal energy. Suppose A, B and C are three distinct thermodynamic systems, then it states that:

"If systems A and B are in thermal equilibrium with system C, then system A is in thermal equilibrium with system B".

This is also known as the principle of temperature measurement. Practically the Zeroth law means that all the three thermodynamic systems are at the same temperature and it forms the basis for comparison of temperatures.

The zeroth law as stated above was formulated by Ralph Fowler in 1931 [78]. The theoretical framework for the zeroth law was first enunciated by Scottish physicist Joseph Black in his 1786 Lectures on Chemistry [19]. Most references, however, state that Irish physicist James Maxwell's *"law of equal temperatures"*, published almost 100 years after Black in 1871, was the first formulation of what is now called the zeroth law [87]. The term *"zeroth law"* was coined jointly by Fowler and Edward Guggenheim in 1939 [56].

The First Law of Thermodynamics: The first law of thermodynamics can be stated in several ways:

"Energy can be neither created nor destroyed. It can only change form."

"In any process in an isolated system, the total energy remains the same".

"For a thermodynamic cycle the net heat supplied to the system equals the net work done by the system".

It is more than just a definition. In an arbitrary thermodynamic transformation let ΔQ denote the net amount of heat absorbed by the system and ΔW the net amount of work done by the system. The first law of thermodynamics states that the quantity ΔU, defined by

$$\Delta U = \Delta Q - \Delta W, \tag{1.6}$$

is the same for all transformations leading from a given initial state to a given final state. This immediately defines a state function U, called the internal energy. It is defined only up to an arbitrary additive constant. It is not difficult to see that U is an extensive variable; if a system in equilibrium is divided into two equal parts, each part has half the internal energy U of the original system.

Julius Robert von Mayer and James Prescott Joule are the two well known names associated with the conventional first law of thermodynamics [133]. The statement of the principle of *"conservation of energy"* appeared due to Mayor whereas the credit for accurate experimental determination of the principle of *"mechanical equivalent of heat"* goes to Joule. Both the principles are different from each other. The experimental foundation of the first law is Joule's demonstration of the equivalence between heat and mechanical energy – the feasibility of converting mechanical work completely into heat. The inclusion of heat as a form of energy leads naturally to the inclusion of heat in the statement of the conservation of energy. The first law is precisely such a statement [133]. However, the first explicit statement of the first law of thermodynamics was given by Rudolf Clausius in 1837:

"There is a state function U, called 'energy', whose differential equals the work exchanged with the surroundings during an adiabatic process."

In an infinitesimal transformation, any change in U, dU, is the difference between heat added $đQ$ and the work done by the system, $đW$,

$$dU = đQ - đW = đQ - PdV. \tag{1.7}$$

where dU is a perfect differential $đQ$ and $đW$ are imperfect ones (that is they are non-integrable). Note the convention used here: $đQ$ is the heat *added* to the system, $đW$ is the work *done* by the system. This set of equations is called *the first law of thermodynamics.*

It states that mechanical work and heat are two forms of energy and must be lumped together when the change in the internal energy of the system is computed.

Consider a thermodynamic system with variables P, T and V. Any pair of these three variables can be chosen as the independent variables that completely specify the state of the thermodynamic system. The other variable is then determined by the *EOS*. Consider $U = U(P, V)$, then

$$dU = \left(\frac{\partial U}{\partial P}\right)_V dP + \left(\frac{\partial U}{\partial V}\right)_P dV. \tag{1.8}$$

The requirement that dU is exact leads to the result

$$\frac{\partial}{\partial V}\left[\left(\frac{\partial U}{\partial P}\right)_V\right]_P = \frac{\partial}{\partial P}\left[\left(\frac{\partial U}{\partial V}\right)_P\right]_V. \tag{1.9}$$

The Second Law of thermodynamics: The laws of thermodynamics have a negative quality that distinguishes them from other laws of physics, which makes direct, positive, experimental proof quite difficult. The first law may be stated by saying that energy cannot be destroyed. This sort of statement is much harder to demonstrate than the positive statement of any other law of physics.

The second law also has this negative quality and the lack of direct verification is even more difficult than with the first law. One way of phrasing the second law is that:

"The spontaneous tendency of a system to go toward thermodynamic equilibrium cannot be reversed without at the same time changing some organized energy, work into some disorganized energy, heat."

No single experiment can assure the validity of this statement. The theory of thermodynamics has been and still is, successful in interpreting and predicting all thermal phenomenon so far.

German scientist Rudolf Clausius is credited with the first formulation of the second law, now known as the *Clausius's principle*:

"There exist no thermodynamic transformation whose sole effect is to extract a quantity of heat from a colder reservoir and to deliver it to a hotter reservoir".

It is logically equivalent to another, apparently different statement, called *Kelvin's principle*:

"There exist no thermodynamic transformation whose sole effect is to extract a quantity of heat from a given heat reservoir and convert it entirely into work".

Carnot Engine: An engine that does all the things required by definition in a reversible way is called a *Carnot engine*. A *Carnot cycle* operates between two temperatures, a hotter one T_h, that of the heat source and a colder one, T_c, that of the heat sink. Any sort of material can be used, not just one having U as a function of T only. The cycle consists of four quasistatic processes:

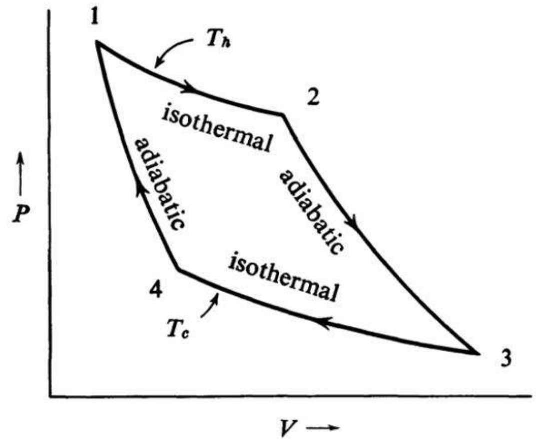

Figure 1.2: Example of Carnot cycle, plotted in PV–plane.

- an isothermal expansion from 1 to 2, at temperature T_h, withdrawing heat ΔQ_{12} from the source and doing work ΔW_{12} (not necessarily equal to ΔQ_{12});
- an adiabatic expansion from 2 to 3, doing further work ΔW_{23} but with no change in heat and ending up at T_c;
- an isothermal compression at T_c from 3 to 4 requiring work $\Delta W_{34} = -\Delta W_{43}$ to be done on the system and contributing heat $\Delta Q_{34} = -\Delta W_{43}$ to the heat sink at temperature T_c ending at the state 4;
- finally, an adiabatic compression from 4 to 1, requiring work $\Delta W_{14} = -\Delta W_{41}$ ($\Delta Q_{41} = 0$) to be done on the system to bring it back to state 1, ready for another cycle, see Fig.(1.2).

This cycle, of course, does not convert all the heat drawn from the reservoir at T_h into work, some of it is dumped as unused heat into sink at T_c. The net work done by the engine per cycle is the area inside the closed region 1234 in Fig.(1.2) and which, according to the first law is equal to the $\Delta Q_{12} + \Delta W_{34} = \Delta Q_{12} - \Delta W_{43}$. Since $dU = 0$ for any cyclic transformation. The efficiency η with which heat is withdrawn and converted to work is equal to the ratio between the work produced and the heat withdrawn

$$\eta = 1 - \frac{\Delta W_{43}}{\Delta Q_{12}}. \tag{1.10}$$

There are a large number of Carnot cycles, all operating between T_h and T_C to generate work involving different substances with different *EOS*. Since all the processes are quasistatic, the cycle is reversible; it can be run backward. One way of stating second law is to say that all the Carnot cycles operating between temperatures T_h and T_c have *the same efficiency*. Another way is to say that

"*No engine operating between two given temperatures is more efficient than a Carnot engine.*"

By now I have mentioned four different phrasings of the second law. All these statements are equivalent, see for instance [67, 94].

Entropy: a Thermal-State Variable Since any quasistatic, reversible cycle can be considered as a sum of Carnot cycles, for any such cycle the integral of the quantity $đQ/T$ around the whole cycle is zero. For any thermodynamic state function $Z(x, y)$, the integral is zero, as long as all of the parts of the path are reversible processes. Therefore, at equilibrium, the quantity $dS = đQ/T$ is a perfect differential; where $đQ$ is the heat given to the system in an elementary reversible process and T is the temperature of the thermodynamic system during the process. The integral of this perfect differential, $S(x, y)$, is a state variable and is called the *entropy* of the system. It is an extensive variable, proportional to n. Therefore, for the intensive variable T the related extensive variable is S. Thus there is an integrating function for $đQ$, if heat $đQ$ is absorbed in a reversible process; it is reciprocal of the thermodynamic temperature T.

Choose any arbitrary fixed state O of the thermodynamic system as reference state. The entropy $S(A)$ for any state A is defined as

$$S(A) = \int_O^A \frac{đQ}{T}. \tag{1.11}$$

This perfect differential measures the change dS in state variable S, the entropy. The difference in the entropy of two states is defined by

$$S(A) - S(B) = \int_B^A \frac{đQ}{T}, \tag{1.12}$$

where the path of integration is any reversible path between A and B. However, there is no such integrating factor for $đQ$ for an irreversible process. Thus, for a reversible process, the integral $\int \frac{đQ}{T}$ is independent of the path and it depends only on the initial and final states. Another way of stating the second law is that for all closed cycles the integral around the cycle

$$\oint \frac{đQ}{T} \leq 0, \tag{1.13}$$

where the equality holds for reversible cycles.

The entropy S is the extensive variable that pairs with T as V with P. The heat absorbed by the system in a reversible process is $đQ = TdS$. This product has the dimensions of energy and equals the amount of heat given to the system, at temperature T in the thermodynamic scale, during a reversible quasistatic process which produces a change dS in the entropy of the system [67, 94].

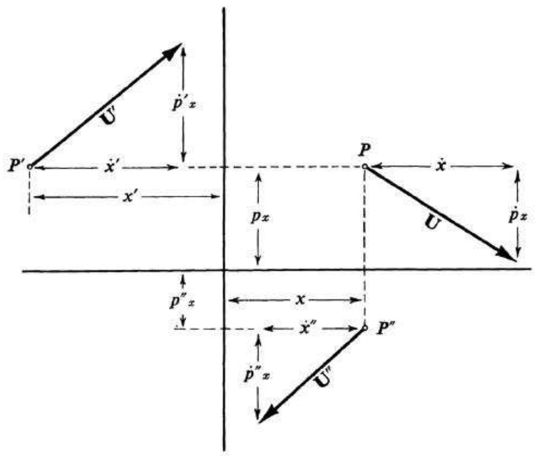

Figure 1.3: Representation of a one-dimensional molecule's position (x, p_x) and its velocity $\mathbf{U} = (\dot{x}, \dot{p}_x)$.

The Third Law of Thermodynamics: The third law of thermodynamics is the least known of all the laws. It refers to the absolute zero temperature on the Kelvin scale. The second law enables us to define the entropy of a substance up to an arbitrary additive constant but does not uniquely determine the difference in entropy of two states A and B, if A refers to one substance and B to another. The third law was independently uncovered by Walther Nernst in 1906 and formulated by Max Planck in 1911 for such a determination. The third law has been formulated in a variety of ways. Two (essentially equivalent) formulations, due to Nernst, state that:

"Isothermal reversible processes become isentropic in the limit of zero temperature,"

and

"It is impossible to reduce the temperature of any system to the absolute zero in a finite number of operations."

A stronger version, proposed by Planck, states that:

"The entropy of a system at absolute zero is a constant, which may be taken to be zero."

The generality of this statement lies in the facts that: (a) it refers to any system; and (b) that $S \to 0$ as $T \to 0$, regardless of the values of any other parameter of which S may be a function. It is obvious that the third law determines the entropy of any thermodynamic system uniquely.

1.3 The Phase Space and Legendre Transformations

In mathematics and physics, the *phase space*, introduced by Willard Gibbs in 1901, is the space of all possible states of a physical system; by *"state"* we do not simply mean the

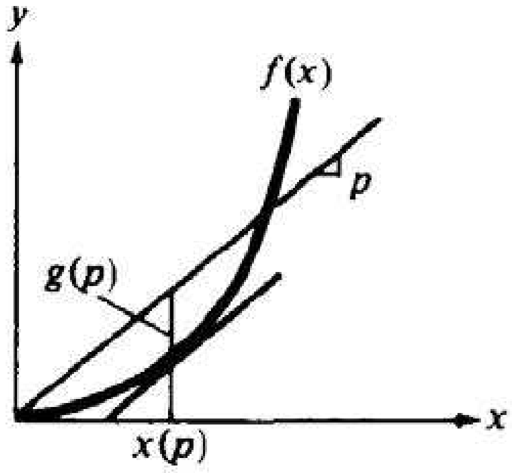

Figure 1.4: Legendre Transformation.

positions x of all the objects in the system (which would occupy *physical space* or *configuration space*), but also their *velocities* or *momenta* p (which would occupy *momentum space*). One needs both the position and momentum of system in order to determine the future behavior of that system. In particular, the specification of a molecule's momentum and position is sufficient to determine completely its future motion [107]. Thus the status of an individual molecule at any instant can be defined in terms of six coordinates, x, y, z, p_x, p_y, p_z (and, perhaps, of a seventh, time). This six-dimensional space is called the *phase space*. Each molecule is pictured as following a trajectory in phase space, between collisions, as it moves under the influence of the general field of force represented by the potential energy [94].

Consider a particle of mass m, constrained to move along a line, so its phase space is two dimensional, the coordinates being position x and momentum p_x , as shown in Fig.(1.3). If the particle is at point P, it will not stay there; its velocity $\mathbf{U}$ in phase space will have two components, $\dot{x}$ and $\dot{p}_x$. There is a direct relationship between the x component of molecules's *velocity* and *position* in phase space,

$$x = \frac{\dot{p}_x}{m}. \tag{1.14}$$

The Legendre transformations: The Legendre transform or Legendre transformation (LT), named after Adrien-Marie Legendre, is a very useful mathematical tool: it transforms functions on a vector space to functions on the dual space [9].

Let $y = f(x)$ be a convex function, $f''(x) > 0$. The LT of the function f is a new function g of a new variable p, which is constructed in the following way (see Fig.(1.4)). We draw the graph of f in the xy-plane. Let p be a given number. Consider the straight line $y = px$. We take the point $x = x(p)$ at which we are measuring the maximal distance between the line $y = px$ and $f(x)$: for each p the function $px - f(x) = F(p, x)$ has a maximum with respect to x at the point $x(p)$. Now we define

$$\mathcal{L}(f)(p) = g(p) = F(p, x(p)). \tag{1.15}$$

The point $x(p)$ is defined by the extremal condition

$$\frac{\partial F}{\partial x} = 0, \tag{1.16}$$

that is $f'(x) = p$. The convexity condition ensures that there is only one of these critical points $x(p)$ and that it is a maximum.
If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $\nabla^2 f > 0$. The LT is given by

$$\mathcal{L}(f)(p) = max_x(px - f(x)). \tag{1.17}$$

Example 1: If $f(x) = x^2$, then $g(p) = \mathcal{L}(f)(p) = p^2/4$.
Example 2: If $f(v) = \frac{1}{2}mv^2$, then $g(p) = \mathcal{L}(f)(p) = p^2/(2m)$.

The LT is its own inverse and it uses maximization as the transformation procedure. The transform is especially well behaved if $f(x)$ is a convex function. The LT is an application of the duality relationship between points and lines. The LT is used extensively in mechanics (taking us from Lagrangians to Hamiltonians and back) and thermodynamics (relating energy to other thermodynamic potentials) [9].

1.4 The Chemical and Thermodynamic Potentials

The chemical potential μ is the state variable conjugate to n, the number of moles of matter in the system under consideration. Consider a thermodynamic system containing n constituent species. Its total internal energy U is postulated to be a function of the entropy S, the volume V and the number of particles of each species $n_1, ..., n_k$. The chemical potential of the i-th species, μ_i is defined as the partial derivative

$$\mu_i = \left(\frac{\partial U}{\partial n_i}\right)_{S,V,n_{j\neq i}}, \tag{1.18}$$

where the subscripts simply emphasize that the entropy, volume and the other particle numbers are to be kept constant.

The internal energy U has many properties of a potential function. For a reversible process

$$dU = TdS - PdV + \sum_i \mu_i dn_i. \tag{1.19}$$

Thus, the internal energy U is the potential appropriate when all the extensive variables

Name	Symbol	Formula	Variables
Internal energy	U	$\int (TdS - PdV + \sum_i \mu_i dn_i)$	$S, V, \{n_i\}$
Helmholtz free energy	F	$U - TS$	$T, V, \{n_i\}$
Enthalpy	H	$U + PV$	$T, V, \{n_i\}$
Gibbs free energy	G	$U + PV - TS$	$T, P, \{n_i\}$
Grand potential	Ω	$U - TS - \sum_i \mu_i dn_i$	$T, V, \{\mu_i\}$

Table 1.1: **Thermodynamic potentials.** Where T = temperature, S = entropy, P = pressure, V = volume, n_i is the number of particles of type i in the system and μ_i is the chemical potential for an i-type particle. For the sake of completeness, the set of all n_i are also included as natural variables, although they are sometimes ignored.

are the independent variables. If U is expressed as a function of the extensive variables S, V, n, then its partials are given by

$$\left(\frac{\partial U}{\partial S}\right)_{V,n} = T, \quad \left(\frac{\partial U}{\partial V}\right)_{S,n} = -P, \quad \left(\frac{\partial U}{\partial n}\right)_{S,V} = \mu, \tag{1.20}$$

are intensive variables, which thus play a role analogous to the components of forces. Since value of a second partial is independent of the order of application of the derivatives. A most useful set of relationships between the partials of the intensive variables is

$$\left(\frac{\partial T}{\partial V}\right)_{S,n} = \left[\frac{\partial}{\partial V}\left(\frac{\partial U}{\partial S}\right)_{V,n}\right]_{S,n}, \tag{1.21}$$

$$\left[\frac{\partial}{\partial S}\left(\frac{\partial U}{\partial V}\right)_{S,n}\right]_{V,n} = -\left(\frac{\partial P}{\partial S}\right)_{V,n}, \tag{1.22}$$

$$\left(\frac{\partial T}{\partial n}\right)_{S,V} = \left(\frac{\partial \mu}{\partial S}\right)_{S,n}, \quad -\left(\frac{\partial P}{\partial n}\right)_{S,V} = \left(\frac{\partial \mu}{\partial V}\right)_{S,n}. \tag{1.23}$$

These derivative interrelationships are called *Maxwell's relations.* Since each of the partials is a state variable, they can be integrated along any convenient reversible path to obtain differences in the value of various intensive state variables, between different equilibrium states.

Suppose that X_i and Y_i to be conjugate pairs of mechanical variables and the Y_i to be the natural variables of the internal energy. Since all of the natural variables of the internal energy U are extensive quantities, i.e.,

$$\lambda U(X_i) = U(\lambda X_i), \tag{1.24}$$

so that

$$\frac{\partial}{\partial \lambda} U(\lambda X_i) = U(X_i), \tag{1.25}$$

then

$$U(X_i) = \sum_j \left(\frac{\partial U}{\partial X_j} \right) X_j = \sum_i Y_j X_j. \tag{1.26}$$

From the equation of state, we then have:

$$U = TS - PV + \sum_i \mu_i n_i, \tag{1.27}$$

where the intensive variables T, P, etc., are all expressed as functions of the extensive variable set. This equation is called *Euler's equation.* When all the intensive variables are explicitly given in terms of the extensive variables, for a particular system its thermal properties can all be calculated from the expression for U, by appropriate transformation and differentiation [94].

The fact that we have both a differential and an integrated equation for U implies that we have another differential relationship between the intensive and extensive variables. Taking the differential of Euler's equation we have

$$dU = \sum Y_j dX_j + \sum X_j dY_j. \tag{1.28}$$

Since change in the internal energy U in a reversible process is given by

$$dU = \sum Y_j dX_j, \tag{1.29}$$

subtracting Eqs.(1.28) and (1.29) we get

$$\sum_j X_j dY_j = 0, \tag{1.30}$$

or

$$\sum_i n_i d\mu_i = V dP - S dT. \tag{1.31}$$

In particular, for 1 mole of a single-component material ($n = 1$),

$$d\mu = -s dT + v dP, \tag{1.32}$$

where s and v are the values of entropy and volume per mole of material (for example $v = M/\rho$, where M is the molecular weight and ρ is its density). This differential equation for chemical potential μ of a substance is called the *Gibbs–Duhem equation.* It states that the sum of the imperfect differentials on the right-hand side turns out to be a perfect differential of the state variable we call thermodynamic potential. In general μ increases with the increase of pressure and decreases as the temperature increases.

The Thermodynamic Potentials: The term *thermodynamic potential* derives from

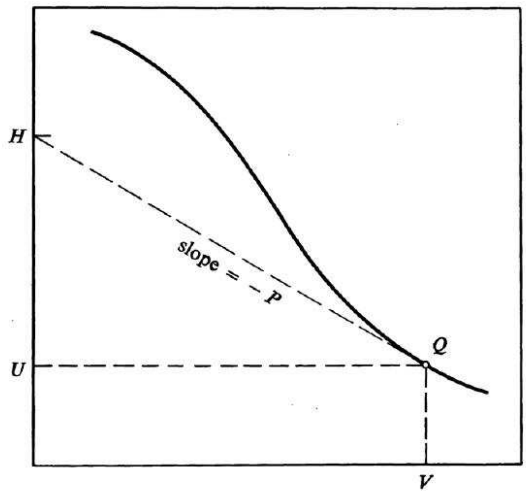

Figure 1.5: Legendre Transformation from U as a function V to H as a function of P.

an analogy with mechanical potential energy. They are extensive state variables of dimensions of energy. Their purpose is to allow for simple treatment of equilibrium for systems interacting with the environment. The concept of thermodynamic potentials was introduced by Pierre Duhem in 1886. The main thermodynamic potential which has a physical interpretation is the internal energy, U. Other thermodynamic potentials are LT of the internal energy $U(S, V, n)$ with respect to natural variables S, V or n [94]. Five common thermodynamic potentials are listed in Table 1.1.

For example, the enthalpy H is LT of $U = U(S, V, n_i)$ with respect to $-PV$. Geometrically, the LT from U to H involves the pair of variables (P, V). For a specific value of V, the function $U(V)$, has a slope

$$\frac{dU}{dV} = P(V), \tag{1.33}$$

which defines the tangent, HQ (see Fig.(1.5)). The tangent HQ has an intercept on the U axis of $H = U + PV$. Since

$$dU = -PdV, \quad dH = dU + PdV + VdP, \tag{1.34}$$

we see that

$$\frac{dH(P)}{dP} = V(P). \tag{1.35}$$

Thus enthalpy H is the potential that has P as a basic variable of V.

A mnemonic device that can be used to work out relationships between one thermodynamic function and another is given in Fig.(1.6). The four variables T, S, Y, X are at the corners of square and the four related potentials are at the sides. The variables adjacent to a potential are the natural ones for the potential and the arrows relate other variables to the partials of the potential.

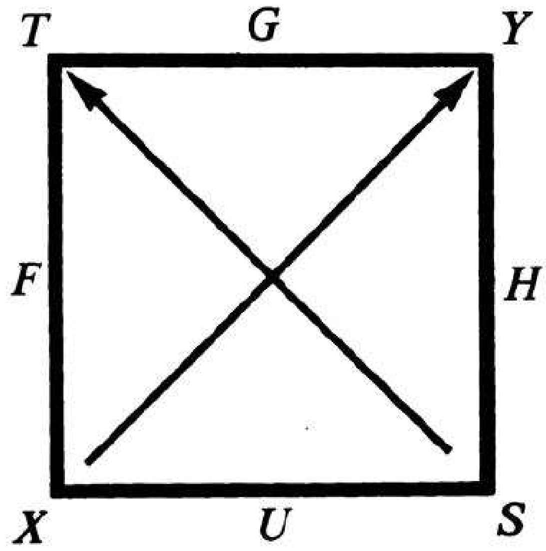

Figure 1.6: Diagram illustrating the thermodynamic potentials and their partials with respect to their natural variables and the Maxwell relations connecting the partials of these variables, for a two-variable system.

1.5 Thermodynamic Description of Phase Transitions

Every substance we know about can exist in several different forms, called *phases*. *Phase diagrams* show the preferred physical states of matter at different temperatures and pressure. Within each phase, the material is uniform with respect to its chemical composition and physical state. Water can be a vapor, liquid or solid. Change of phases comes abruptly, as we know ordinary water is liquid at room temperature and atmospheric pressure, but if cooled below 273.15K it solidifies; and if heated above 373.15K it vaporizes. At each of these temperatures the material undergoes a precipitous change of properties–a *"phase transition"*. The measurement of the external conditions at which the transformation occurs is termed as the *phase transition point.* At high pressures solid water undergoes several additional phase transitions from one solid form to another. These distinguishable solid phases, designated as "ice I", "ice II", "ice III", ..., differ in crystal structure and in essentially all thermodynamic properties. The *"phase diagram"* of water is shown in Fig.(1.7).

In solid state the entropy S and thermodynamic potential increases as temperature T increases. Its heat capacity goes to zero at zero temperature, however C_V rises at high temperatures. However, if we add more and more heat quasistatically to the crystalline solid holding the pressure constant at some moderate value, its temperature rises until finally it melts, turning into a liquid. The temperature T_m at which melting occurs depends on the pressure and the amount of heat required to melt 1 mole of the crystal is called

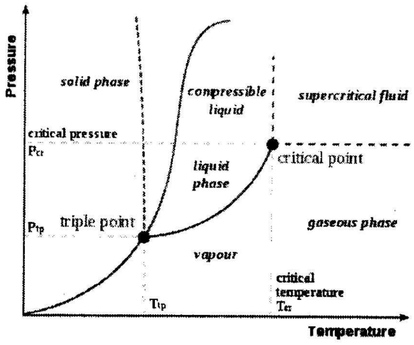

Figure 1.7: Phase diagram for water.

the *latent heat of melting* L_m, which is a function of pressure [94]. If heat is added to the liquid, its temperature will increase until another phase change occurs–the liquid *evaporates*. Again the temperature remains constant at the temperature of vaporization T_V until all the liquid is converted into vapor.

Since the temperature of melting T_m is nearly independent of pressure, whereas the temperature of vaporization T_V is strongly dependent on P. Therefore as P decreases, the two curves one for T_V, the other for T_m, converge. As shown in Fig.(1.8), where the curve AB is the melting-point curve and AC is the vaporization-point curve. The two meet at the *triple point* A, which is the only point where solid, liquid and vapor can coexist in equilibrium. Below this pressure the liquid is not a stable phase and along the curve OA the solid transform directly into the vapor (sublimation). Along this curve the latent heat of sublimation

$$L_s = L_m + L_v, \tag{1.36}$$

at the triple point. The dashed lines are the lines of intersection of the PVT surfaces by planes parallel to the PT plane for different values of V.

As pressure increases, keeping $T = T(P)$, so that the difference between volume and entropy of gas and liquid diminish until at C, the *critical point*, there ceases to be any distinction between liquid and gas and hence the curve AC terminates.

If a phase change involves a major rearrangement of structure of the substance, resulting in change of volume, viscosity and so on. Since such changes involve energy input or output to produce a finite amount of heat, the latent heat L, is required to raise the temperature of the substance from just below the transition temperature to just above it. Since

$$C_P = T\frac{\partial S}{\partial T}\bigg|_P, \tag{1.37}$$

the heat capacity of the substance becomes infinite at transition [34]. Such transitions are called *phase changes of the first kind.*

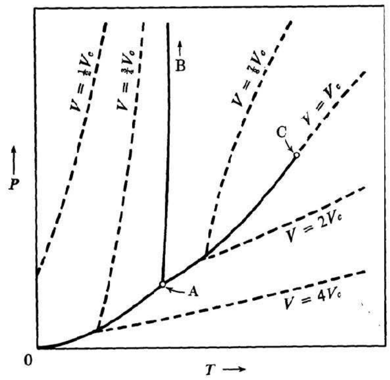

Figure 1.8: Phase diagram for a material that expands upon melting. Solid lines are the curves for phase change, dashed lines those for constant volume.

If the changes of phase involve a different kind of ordering in a crystal lattice or the appearance of superfluid, some of which may be a simple change of slope of S against T at the transition point. In this case the heat capacity changes discontinuously but does not become infinite at transition temperature. Such changes are called *phase changes of the second kind.*

There is still another kind of phase transition that does not fit into the ongoing classification. An example of this is the transition between the two kinds of liquid helium, which is called a *lambda transition.*

1.6 Statistical Mechanics: The Role of Entropy

Statistical Mechanics uses laws of probability for dealing with large populations, to the study of the thermodynamic behavior of systems composed of a large number of particles and provides a framework for relating the microscopic properties of individual atoms and molecules to the macroscopic bulk properties of materials. Physical quantities which describe a macroscopic body in equilibrium are, almost always, close to their mean values. However, there are certain small deviations from the mean values, which is the natural behavior of the system under consideration. We call these deviations *thermodynamic fluctuations.* The problem of concern is how to find the probability of these deviations.

In statistical mechanics, *entropy* is, in some sense, a measure of the number of ways in which a system may be arranged, often taken to be a measure of *"disorder"*. Specifically, the entropy of the system is given by the natural logarithm of the number of microstates Ω_m (that are consistent with the macrostate), i.e.,

$$S = \ln \Omega_m. \tag{1.38}$$

The microstate Ω_m will be a function of the macrostate, namely $\Omega_m = \Omega_m(U, V, N)$ where

U, V and N are the extensive parameters of the system. The Eq.(1.38) is very important as it provides the basic connection between macroscopic (thermodynamic) entropy and statistical microscopic physics (number of states). The thermodynamic definition of entropy provides the experimental definition, while the statistical one extends the concept, providing an explanation and a deeper understanding of its nature. Clearly, $S = 0$ when $\Omega_m = 1$ meaning that there is no entropy present when there is only one exact microstate (hence no disorder).

Using the first law we can also express the entropy as a thermodynamic potential

$$dS = \frac{1}{T}dU + \frac{P}{T}dV - \frac{\mu}{T}dn, \tag{1.39}$$

which is just the differential form of the entropy. If the entropy $S = S(U, V, n)$ is known, then complete knowledge of all the thermodynamic parameters can be obtained. Furthermore, for isolated systems (where $dQ_{reversible} = 0$) in equilibrium

$$dS = 0 \Leftrightarrow S = S_{maximum} \tag{1.40}$$

and for irreversible processes

$$dS > 0. \tag{1.41}$$

So the state of equilibrium is defined as the state of maximum entropy.

In 1907 Einstein inverted Eq.(1.38) and obtained

$$\Omega_m = e^S, \tag{1.42}$$

which is associated to the probability distribution as

$$P \propto e^S. \tag{1.43}$$

By Taylor expanding the entropy about the fluctuation quantity x up to the second order, we obtain

$$S(x) = S(0) - \frac{1}{2}\beta x^2, \tag{1.44}$$

where

$$\beta = -\left.\frac{\partial^2 S}{\partial x^2}\right|_0. \tag{1.45}$$

Substituting Eq.(1.44) into Eq.(1.42) we obtain

$$P(x) = Ae^{-\frac{1}{2}\beta x^2}, \tag{1.46}$$

with A being a normalization constant and the normalization condition is given by

$$\int P(x)dx = 1. \tag{1.47}$$

Thermodynamic system	**Black hole**
Temperature, T	Surface gravity, κ
Energy, U	Black hole's mass M
Entropy, S	Area of event horizon, A

Table 1.2: Analogy between thermodynamic parameters and black hole's parameters.

Since integration is over all space, hence by using Gaussian integration formula

$$A = \sqrt{\frac{\beta}{2\pi}}. \tag{1.48}$$

Thus the probability distribution of the various values of the fluctuations is:

$$P(x) = \sqrt{\frac{\beta}{2\pi}} e^{-\frac{1}{2}\beta x^2}. \tag{1.49}$$

This probability distribution is categorized as a *Gaussian distribution.* It reaches a maximum when $x = 0$ and decreases rapidly and symmetrically as $|x|$ increases. The mean squared fluctuation is defined as

$$< x^2 >= \int x^2 P(x) dx = \frac{1}{\beta}. \tag{1.50}$$

The Gaussian distribution for more than one variable reads

$$P = \frac{\sqrt{\beta}}{(2\pi)^{n/2}} e^{-\frac{1}{2}\beta_{ij} x_i x_j}, \tag{1.51}$$

where

$$\beta_{ij} = -\frac{\partial^2 S}{\partial x_i \partial x_j}, \quad \beta = |\beta_{ij}|. \tag{1.52}$$

Note that $\beta_{ij} = \beta_{ji}$.

1.7 Black Hole Thermodynamics

Black hole thermodynamics has continued to fascinate researchers since Hawking's discovery of the thermal radiation from black holes [68, 69] because it provides a real connection between gravity and quantum mechanics. The relationship between geometrical properties of the event horizon and thermodynamic quantities clearly indicates a relationship between properties of the spacetime geometry and some kind of quantum physics. The thermodynamic behavior should have a statistical interpretation in quantum gravity and hence it provides clues to the form of the quantization of gravity. (For more discussions on black hole physics see, for instance, [55].)

1.8 Four Laws of Black Hole Thermodynamics

The laws of black hole thermodynamics are purely classical and resemble the usual four laws of thermodynamics presented in section 1.1, if one admits that surface area A of the horizon is proportional to the entropy S of the black hole, as suggested by Bekenstein [16], and that the surface gravity κ is proportional to the black hole temperature T [69]. At first, this resemblance was only an analogy since if the black hole had a temperature it would have to radiate, in a clear contradiction with the known classical fact that nothing could escape from the black hole horizon. However, in a revolutionary work, Hawking [69], using a semiclassical treatment in which the gravitational field of the black hole is treated classically but the matter is treated quantum mechanically, has shown that the black holes do indeed radiate a spectrum characteristic of a blackbody.

In this section we discuss thermodynamic properties of black hole thermodynamics established by Bardeen, Carter and Hawking [14]. Some results from the classical black hole mechanics will be displayed in a suggestive form such that the analogy between these laws and the laws of classical thermodynamics is transparent.

By its very definition, a classical black hole cannot emit anything, so it seems useless to attempt to associate a nonzero temperature with it. On the other hand, there must be some relationship between dM, the change in the mass of a black hole and dA, the change in its horizon area. When $dA = 0$ the change of energy of the black hole is

$$dM = \Omega dJ + \phi dQ, \tag{1.53}$$

where J and Q are the angular momentum and charge of the hole and Ω and ϕ are the angular velocity and electric potential of the horizon. It is like the first law of thermodynamics but with the heat flow term $dQ = TdS$ missing.

In 1971 Stephen Hawking [69] showed that the area, A of the event horizon of a black hole can never decrease (though it can remain constant) in any process:

$$dA \geq 0. \tag{1.54}$$

The area of the event horizon increases when (1) mass increases and (2) spin decreases. It was later noted by Bekenstein [16] that this result is analogous to the statement of the second law of classical thermodynamics, namely that the total entropy, S of a closed system never decreases in any process:

$$dS \geq 0. \tag{1.55}$$

With these arguments it is legitimate to establish the laws of black hole mechanics in parallel to the laws of classical thermodynamics (section 1.1) by using parameters of the black hole (see Table 1.2 and 1.3) as follows:

Zeroth Law: The zeroth law states that:

Law	Thermodynamic system	Black hole
Zeroth Law	T constant on a body in thermal equilibrium	κ constant over a black hole's event horizon
First Law	$dU = TdS - PdV + \mu dn$	$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \phi dQ$
Second Law	$dS \geq 0$	$dA \geq 0$
Third Law	$T = 0$ cannot be reached	$\kappa = 0$ cannot be reached

Table 1.3: Analogy between the laws of thermodynamic and the laws of black hole mechanics.

"The surface gravity κ of a stationary black hole is constant everywhere on the surface of the event horizon".

Although κ is defined locally on the horizon, it turns out that it is always constant over the horizon of a stationary black hole. This constancy is reminiscent of the Zeroth Law of thermodynamics which states that the temperature is uniform everywhere over a system in thermal equilibrium. The surface gravity is related to the *Hawking temperature* T_H by

$$T_H = \frac{\kappa}{2\pi}. \tag{1.56}$$

First Law: The first law (an energy conservation law) states that when one throws an infinitesimal amount of matter into a stationary black hole described by M, J and Q, it will evolve into a new stationary black hole in such a way that the change in the hairs of the system satisfies

$$dM = \left(\frac{\kappa}{8\pi}\right) dA + \text{“}\textit{work terms}\text{”}, \tag{1.57}$$

or

$$dM = T_H dS_{BH} + \text{“}\textit{work terms}\text{”}. \tag{1.58}$$

It is readily seen that the above equations are analogous to the first law of thermodynamics, i.e.

$$dE = TdS + \text{“}\textit{work terms}\text{”}. \tag{1.59}$$

The entropy of the black hole is thus represented by a quarter of the area of the event horizon, that is

$$S_{BH} = \frac{A}{4}. \tag{1.60}$$

This quantity, S_{BH}, is known as the *Bekenstein-Hawking entropy.* The factor $\frac{1}{4}$ was found by Hawking [70] based on the application of the quantum field theory to the black holes which shows that they will absorb and emit particles as if they were thermal black bodies with the Hawking temperature, T_H. The *"work terms"* are given differently depending on the type of the black holes. For the Kerr-Newman black hole family, the first law would be

$$dM = \left(\frac{\kappa}{8\pi}\right) dA + \Omega dJ + \phi dQ, \tag{1.61}$$

where Ω is the angular velocity of the hole and ϕ is the electric potential which are defined by

$$\Omega = \frac{\partial M}{\partial J}, \qquad \phi = \frac{\partial M}{\partial Q}. \tag{1.62}$$

Second Law: The second law states that:

"In any classical process, the area of a black hole, A and hence its entropy S_{BH}, do not decrease;"

$$dS_{BH} \geq 0. \tag{1.63}$$

This statement is analogous to the second law of classical thermodynamics which is consequently obtainable from the Hawking's area theorem. In classical thermodynamics and black hole physics, the second law indicates the irreversibility inherent in the system under consideration. In thermodynamics, the law of nondecreasing entropy signifies that the part of the internal energy U that cannot be transformed into work grows with time. Quite similarly, the law of nondecreasing area of a black hole signifies that the fraction of a black hole's internal energy that cannot be extracted grows with time.

The second law of black hole mechanics can be violated if the quantum effects are taken into account, namely that the area of the event horizon can be reduced via Hawking radiation. It is essential that the black hole radiation is thermal in nature, therefore generating a rise in entropy in the surrounding region. The *generalized entropy*, S' was introduced by Bekenstein [17] to account for this sort of entropy. It is defined as the sum of the black hole's entropy, S_{BH} and the entropy of the surrounding matter, S_m

$$S' = S_{BH} + S_m. \tag{1.64}$$

This statement is known as the *Generalized Second Law* (GSL):

$$dS' \geq 0. \tag{1.65}$$

The ordinary second law seems to fail when the matter is dropped into a black hole because according to classical GR, the matter will disappear into a spacetime singularity, in this manner the total entropy of the universe decreases as there is no compensation for the lost entropy. The GSL keeps the law of entropy valid as the total entropy of the universe still increases when the matter falls into the black hole.

Third Law: Finally, the third law states that:

"It is impossible by any procedure, no matter how idealized, to reduce the black hole temperature to zero by a finite sequence of reversible processes."

However, the extremal black holes, for example the Kerr black holes do have $\kappa = 0$ thus zero temperature (absolute zero) but non-zero entropy. To actually reduce the surface gravity to zero is merely an idealized case because it is forbidden by the Cosmic censorship conjecture. The surface gravity of the Kerr-Newman black hole is given by

$$\kappa = \frac{4\pi}{A}\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}, \tag{1.66}$$

and thus $\kappa = 0$ implies $\frac{J^2}{M^2} + Q^2 = M^2$, which is the condition for an extreme black hole. If the state $\kappa < 0$ could be reached we would have a naked singularity. Hence, the cosmic censorship conjecture stated by Penrose plays the role of the third law.

Israel emphasized that it is difficult to define the meaning of *"finite sequence of reversible processes"* considering only quasi-static processes [71]. He proposed and proved the following version of the third law:

A non-extremal black hole cannot become extremal at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in the neighborhood of the outer apparent horizon.

1.9 Thermodynamics as Geometry

One of the first applications of differential geometry [49] in thermodynamics is due to Gibbs [58] and Charatheodory [33]. Results have been achieved in two different approaches.

The first approach, developed by Hermann [63] and Mrugala [95, 96], uses the natural contact structure of the so-called phase space $\mathcal{T}$. Extensive and intensive thermodynamic variables are taken together with the thermodynamic potential to constitute well-defined coordinates on $\mathcal{T}$. The space of thermodynamic equilibrium states $\mathcal{E}$, a subspace of $\mathcal{T}$, is defined by means of a smooth embedding map $\phi : \mathcal{E} \rightarrow \mathcal{T}$. This implies that each system possesses its own space $\mathcal{E}$. On the other hand, it is always possible to introduce the fundamental Gibbs 1-form on $\mathcal{T}$ which, when projected on $\mathcal{E}$ with the pullback of ϕ, generates the first law of thermodynamics and the conditions for thermodynamic equilibrium. Furthermore, on $\mathcal{T}$ it is also possible to consider Riemannian structures [64, 136]. The second one consists of introducing metric structures on the space of thermodynamic equilibrium states $\mathcal{E}$.

During the last few decades several attempts have been made in order to introduce differential geometric concepts in ordinary thermodynamics. In the next section I will discuss basic ideas and concepts of thermodynamic geometry from the viewpoint of Riemannian geometry and different approaches to study the geometry of black holes.

1.10 Thermodynamics: A Riemannian Geometric Model

It has been known, particularly since the early works of Gibbs, that the analysis of thermodynamic systems can be facilitated with the help of graphical and geometrical methods. It was Weinhold [141] who first realized that the *"geometry"* of thermodynamic phase spaces lacks an intrinsic *metric* structure. He introduced in the equilibrium space a Riemannian metric defined in terms of the second derivatives of the internal energy U, which is mass M in the case of black holes, with respect to entropy S and other extensive variables N^a of the thermodynamic system under consideration. The *Weinhold metric* is given by

$$g_{ij}^{W} = \partial_i \partial_j U(S, N^a), \tag{1.67}$$

However, the geometry based on this metric seems to be meaningless in the context of pure equilibrium thermodynamics. Later on Ruppeiner [125] argued that if the theory of fluctuations is included in the axioms of equilibrium thermodynamics, there exists a corresponding Riemannian metric which enables us to represent thermodynamic systems by Riemannain manifolds and associate the curvature of these manifolds with the interactions. He introduced a Riemannian metric structure in thermodynamic fluctuation theory and related it to the second derivatives of the entropy. This geometric structure was used to find the significance of the distance between equilibrium states and to study the thermodynamics of equilibrium systems. It was observed by Ruppeiner [126] that in thermodynamic fluctuation theory the Riemannian curvature of the Ruppeiner metric measures the complexity of the underlying statistical mechanical model. This concept is associated to probabilities, i.e., the less probable a fluctuation between states, the further apart they are. This can be recognized if one considers g_{ij} in the distance formula (line element) between the two equilibrium states

$$ds^2 = g_{ij} dx_i dx_j, \tag{1.68}$$

where the matrix of coefficients g_{ij} is the symmetric metric tensor and we have $\beta ij = g_{ij}$. The *Ruppeiner metric* is defined as

$$g_{ij}^{R} = -\partial_i \partial_j S(M, N^a), \tag{1.69}$$

where S is the entropy, M denotes the energy and N^a are other extensive variables of the system such as electric charge Q and angular momentum J. The Ruppeiner geometry is conformally related to the Weinhold geometry [97, 127] as

$$ds_R^2 = \frac{1}{T} ds_W^2, \tag{1.70}$$

where T is the temperature of the system under consideration. Eq.(1.70) often provides a more convenient way to compute the Ruppeiner metric [2].

One of the aims of the application of geometry in thermodynamics is to describe phase transitions in terms of curvature singularities and to interpret curvature as a measure of thermodynamic interaction. Since the proposal of Weinhold, many investigations have been carried out to understand the thermodynamic geometry of various thermodynamic systems. The Weinhold and the Ruppeiner geometries have been analyzed in a number of black hole families to study phase space, critical behavior and stability properties [6–8, 26, 53, 72, 73, 89, 91, 130, 132]. In some particular cases, it was found that the Weinhold and the Ruppeiner geometries carry information about the phase transitions structure. In fact, this is true in the case of the ideal gas, whose curvature vanishes and the van der Waals gas for which the thermodynamic curvature becomes singular at those points where phase transitions occur.

Unfortunately, the results obtained are contradictory in the case of black holes. For instance, for the Kerr black hole the Weinhold metric predicts no phase transitions at all [6] whereas the Ruppeiner metric, with a very specific thermodynamic potential, predicts phase transitions that are compatible with the results of standard black hole thermodynamics [132]. Nevertheless, a change of the thermodynamic potential affects the Ruppeiner geometry in such a way that the resulting curvature singularity does not correspond to a phase transition. Another example is provided by the BTZ black hole for which the curvature of the equilibrium space turns out to be flat [26, 89, 130]. This flatness is usually interpreted as a consequence of the lack of thermodynamic interaction. However, if one applies an invariant approach the resulting manifold is curved [113].

The problem of using the Weinhold or the Ruppeiner metrics in equilibrium space is that the results can depend on the choice of thermodynamic potential, i.e., the results are not invariant with respect to LT [98, 128]. These results indicate that, in the case of black holes, geometry and thermodynamics are compatible only for a specific thermodynamic potential. However, it is well known that ordinary thermodynamics does not depend on the thermodynamic potential, i.e., it is invariant with respect to LT. The formalism of GTD incorporates Legendre invariance into the geometric structures of the phase space and equilibrium space so that the results do not depend on the choice of thermodynamic potential. The phase transition structure contained in the heat capacity of black holes [44] becomes completely integrated in the scalar curvature of the Legendre invariant metric so that a curvature singularity corresponds to a phase transition.

1.11 Geometrothermodynamics: A Consistent Approach to the Geometry of Thermodynamics

In this section I present the main geometric structures which are necessary for the formulation of GTD [112].

The formalism of GTD was developed in order to unify in a consistent manner the geometric properties of the phase space and the space of equilibrium states [112]. Legendre invariance plays an important role in this formalism. As an approach GTD allows us to handle thermodynamic systems in terms of geometric structures which are invariant with respect to LT, i.e., independent of the thermodynamic potential [112]. GTD explains why Weinhold's and Ruppeiner's metrics are not suitable to describe thermodynamics in an independent manner [115]. It can be used to derive Legendre invariant metrics which, in particular, describe black hole thermodynamics in a unified and consistent manner [2, 5, 113–121, 135, 139, 140].

The idea is easily applicable. We first consider an arbitrary metric on the phase space whose pullback induces a metric on the space of equilibrium states. Then we derive the conditions for the invariance of these metrics under an arbitrary LT, where the contact structure is considered to be compatible with the Riemannian structure. GTD unifies the thermodynamics and geometry either in the space of equilibrium states or at the level of the phase space [112].

In analytic geometry, a thermodynamic system is described by an equation of state $f(E^a, I^a) = 0$ which determines a surface in the space with coordinates $\{E^a, I^a\}$. One of the most important contributions of analytic geometry to the understanding of thermodynamics is the identification of points of phase transitions with extremal points of the surface $f(E^a, I^a) = 0$. More detailed descriptions of these contributions can be found in standard textbooks on thermodynamics (see, for instance, [34]). This approach, however, implies that the equation $f(E^a, I^a) = 0$ must be determined experimentally. Here I present an alternative approach in which a thermodynamic system is described by an extremal hypersurface, satisfying a system of differential equations. This alternative approach provides a solid mathematical structure to thermodynamics and opens the possibility of finding extremal surfaces and investigating their thermodynamic properties by analyzing their geometric structure.

1.11.1 Harmonic Maps

The subject of harmonic maps is vast and has many applications. Harmonic maps were defined and named by F. B. Fuler, who also gave important examples and asked significant questions [57]. I first consider relevant aspects of harmonic functions on Euclidean space; then I will give a general introduction to harmonic maps on Riemannian manifolds [50].

Harmonic functions on Euclidean spaces: Harmonic functions f on an open domain W of $\mathbb{R}^m$ are solutions of the Laplace equation

$$\Delta f = 0, \tag{1.71}$$

where

$$\Delta := \frac{\partial^2}{(\partial x^1)^2} + + \frac{\partial^2}{(\partial x^m)^2}, \quad (x^1, ..., x^m) \in W. \tag{1.72}$$

The operator Δ is called the *Laplace operator* or *Laplacian* after P. S. Laplace.

The Dirichlet principle: *Dirichlet's principle*, also known as *Thomson's principle*, states that the function f minimizes the functional

$$E_W(f) := \frac{1}{2}\int_W \sum_{\alpha=1}^{m} \left(\frac{\partial f}{\partial x^\alpha}(x)\right)^2 dx^m = \frac{1}{2}\int_W \sum_{\alpha=1}^{m} |df_x|^2 dx^m, \tag{1.73}$$

where $d^m x := dx^1 ... dx^m$. The harmonic functions are *critical points* (also called *extremals*) of the *Dirichlet energy functional*(1.73).

Harmonic Maps on Riemannian Manifolds: Harmonic maps are solutions to a natural geometrical variational problem. This notion was motivated from some fundamental concepts of differential geometry, such as geodesics, minimal surfaces and harmonic functions. They are nonlinear extensions of harmonic functions. Just like harmonic functions, harmonic maps are critical points of the *Dirichlet energy functional*, of maps between two Riemannian manifolds [138].

Consider two (pseudo)-Riemannian manifolds (M, γ) and (M', γ') of dimension m and m', respectively. Let the base manifold M be coordinatized by x^α $(\alpha, \beta, \gamma, ... = 1, 2, ..., m)$ and M' by x'^μ $(\mu, \nu, \lambda, ... = 1, 2, ..., m')$, so that the metrics on M and M' can be, in general, smooth functions of the corresponding coordinates, i.e., $\gamma = \gamma(x)$ and $\gamma' = \gamma'(x')$. A harmonic map is a smooth map $\varphi : M \to M'$, or in coordinates $\varphi : x \longmapsto x'$ so that x' becomes a function of x and the x's satisfy the field equations following from the action [92]

$$S = \frac{1}{2}\int d^m x \sqrt{|\det(\gamma)|}\ \gamma^{\alpha\beta}(x)\ \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\gamma'_{\mu\nu}(x')\ , \tag{1.74}$$

which sometimes is called the *"Dirichlet energy functional"* of the harmonic map φ. Using the calculus of variations for the action (1.74), the field equations are

$$\frac{1}{\sqrt{|\det(\gamma)|}}\frac{\partial}{\partial x^\beta}\left(\sqrt{|\det(\gamma)|}\gamma^{\alpha\beta}\frac{\partial x'^\mu}{\partial x^\alpha}\right) + \Gamma^\mu{}_{\nu\lambda}\ \gamma^{\alpha\beta}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\lambda}{\partial x^\beta} = 0\ , \tag{1.75}$$

where $\Gamma^\mu{}_{\nu\lambda}$ are the Christoffel symbols associated to the metric $\gamma'_{\mu\nu}$ of the target manifold N. If $\gamma'_{\mu\nu}$ is a flat metric, one can choose Cartesian-like coordinates such that $\gamma'_{\mu\nu} = \chi_{\mu\nu} = \mathrm{diag}(\pm 1, ..., \pm 1)$, the field equations become linear and the harmonic map is linear [50,51]. Harmonic maps appear in many different contexts; e.g.,

(a) Constant maps are harmonic.

(b) If dim $M = 1$, then the harmonic maps are the geodesics of M'.

(c) If $M' = \mathbb{R}$, they are the harmonic functions on M.

(d) If $M' = S'$(= the unit circle), then the harmonic maps are canonically identified with the harmonic 1-forms on M with integral periods.

(e) If $dimM = 2$, they include (parametric representations of) the minimal surfaces of M'; the energy is the Dirichlet-Douglas integral.
(f) If M is a Riemannian submanifold of M' of minimal volume, then the inclusion map $i : M \to M'$ is harmonic.

1.11.2 Curvature as a Measure of the Thermodynamic Interaction

Now I show that harmonic maps are the correct mathematical tool to investigate the properties of the phase manifold and its submanifolds which contain the information about the physical states of thermodynamic systems. I present a systematic and consistent construction of the formalism of GTD by using Riemannian contact geometry for the definition of the thermodynamical phase manifold and harmonic maps for the definition of the equilibrium manifold [118]. In fact, I will use the method of harmonic maps, which has been extensively used in field theories and appears naturally in the context of GTD. This approach allows us to interpret any thermodynamic system as a hypersurface in the equilibrium space completely determined by the field theoretical approach of harmonic maps.

1.11.3 The Thermodynamic Phase Manifold

The main element of GTD is the thermodynamic phase manifold which is a Riemannian contact manifold whose contact structure and metric are invariant with respect to LT. The first step to introduce the language of differential geometry in thermodynamics is the definition of the $(2n+1)$-dimensional differential manifold $\mathcal{T}$ and its tangent manifold $T\mathcal{T}$. Let $\mathcal{V} \subset T\mathcal{T}$ be a field of hyperplanes. It can be shown that $\mathcal{V} = \ker \Theta$, where Θ is a non-vanishing differential 1-form [39]. If the Frobenius integrability condition $\Theta \wedge d\Theta = 0$ is satisfied, the hyperplane field $\mathcal{V}$ is completely integrable. On the other hand, if $\Theta \wedge d\Theta \neq 0$, then $\mathcal{V}$ is non-integrable. In the limiting case $\Theta \wedge (d\Theta)^n \neq 0$, the hyperplane field $\mathcal{V}$ becomes maximally non-integrable and it is said to define a contact structure on $\mathcal{T}$. The pair $(\mathcal{T}, \mathcal{V})$ is usually known as a contact manifold [23, 49] and sometimes it is also denoted as $(\mathcal{T}, \Theta)$ to emphasize the role of the contact form Θ.

Let G be a non-degenerate metric on $\mathcal{T}$. The set $(\mathcal{T}, \Theta, G)$ defines a Riemannian contact manifold. Notice that the contact manifold $(\mathcal{T}, \Theta)$ is uniquely defined in the following sense. The condition $\Theta \wedge (d\Theta)^n \neq 0$ is independent of Θ; in fact, it is a property of $\mathcal{V} = \ker \Theta$. If another 1-form Θ' generates the same $\mathcal{V}$, it must be of the form $\Theta' = f\Theta$, where $f : \mathcal{T} \to \mathbb{R}$ is a smooth non-vanishing function. The Riemannian metric G, however, is completely arbitrary. I will use this freedom to select only those metrics which are invariant under the LT.

Suppose the thermodynamic phase space $\mathcal{T}$ can be coordinatized by the set of independent coordinates $Z^A = \{\Phi, E^a, I^a\}$, $a = 1, ..., n$, $A = 0, 1, ..., 2n$. Here Φ represents the thermodynamic potential and E^a and I^a are the extensive and intensive thermodynamic variables, respectively. The positive integer n indicates the number of thermodynamic degrees of freedom of the system.

Notice that in the manifold $\mathcal{T}$ all the coordinates Φ, E^a and I^a must be completely independent and thus thermodynamic systems cannot be described in $\mathcal{T}$. An LT is defined as [9]

$$\{Z^A\} \longrightarrow \{\widetilde{Z}^A\} = \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\} \tag{1.76}$$

$$\Phi = \tilde{\Phi} - \delta_{kl}\tilde{E}^k\tilde{I}^l \ , \quad E^i = -\tilde{I}^i, \quad E^j = \tilde{E}^j, \quad I^i = \tilde{E}^i, \quad I^j = \tilde{I}^j \ , \tag{1.77}$$

where $i \cup j$ is any disjoint decomposition of the set of indices $\{1, ..., n\}$ and $k, l = 1, ..., i$. In particular, for $i = \{1, ..., n\}$ and $i = \emptyset$ we obtain the total LT and the identity, respectively. LT are a special case of contact transformations which leave invariant the contact structure of $\mathcal{T}$. Legendre invariance guarantees that the geometric properties of G do not depend on the thermodynamic potential used in its construction. In these particular coordinates, let the fundamental Gibbs 1-form be defined on $\mathcal{T}$ as

$$\Theta = d\Phi - \delta_{ab}I^a dE^b \ , \quad \delta_{ab} = \text{diag}(1, 1, ..., 1) \ , \tag{1.78}$$

an expression which is manifestly invariant with respect to LT (1.77). Consequently, the contact manifold $(\mathcal{T}, \Theta)$ is Legendre invariant and so will be the Riemannian contact manifold $(\mathcal{T}, \Theta, G)$, if we demand Legendre invariance of the metric G.

Any Riemannian contact manifold $(\mathcal{T}, \Theta, G)$ whose components are Legendre invariant is called a thermodynamic *phase manifold* and constitutes the starting point for the description of thermodynamic systems in terms of geometric concepts. I would like to emphasize the fact that Legendre invariance is an important condition that guarantees that the description does not depend on the choice of the thermodynamic potential, a property that is essential in ordinary thermodynamics.

1.11.4 Geodesics of the Phase Manifold

Consider a base manifold with $\dim(M) = 1$ and identify the target manifold with the thermodynamic phase manifold $(\mathcal{T}, \Theta, G)$. Then, the field Eqs.(1.75) reduce to the geodesic equations

$$\frac{d^2 Z^A}{d\lambda^2} + \Gamma^A_{\ BC}\frac{dZ^B}{d\lambda}\frac{dZ^C}{d\lambda} = 0 \ , \tag{1.79}$$

where λ is an affine parameter and $\Gamma^A_{\ BC}$ are the Christoffel symbols of the phase manifold metric G. Since any Legendre invariant G has a non-zero curvature, these geodesic equations are highly non-linear and difficult to solve in general. Preliminary results indicate that geodesics of the phase manifold represent families of thermodynamic systems that can be investigated in the context of GTD.

1.11.5 The Equilibrium Manifold

Consider the harmonic map $\varphi : \mathcal{E} \rightarrow \mathcal{T}$, where $\mathcal{E}$ is a subspace of the phase manifold $(\mathcal{T}, \Theta, G)$ and $\dim(\mathcal{E}) = n$. For the sake of concreteness, let us assume that the extensive variables $\{E^a\}$ are the coordinates of $\mathcal{E}$. Then, in terms of coordinates the harmonic embedding map reads $\varphi : \{E^a\} \longmapsto \{Z^A(E^a)\} = \{\Phi(E^a), E^a, I^a(E^a)\}$. Moreover, the pullback φ^* of the harmonic map induces canonically a metric g on $\mathcal{E}$ by means of

$$g = \varphi^*(G) \ , \quad \text{i.e.,} \quad g_{ab} = \frac{\partial Z^A}{\partial E^a}\frac{\partial Z^B}{\partial E^b} G_{AB} = Z^A_{,a} Z^B_{,b} G_{AB} \ . \tag{1.80}$$

If we assume that the metric γ of the base manifold coincides with the induced metric g, the action (1.74) reduces to

$$S = \frac{n}{2}\int d^n E \sqrt{|\det(g)|} \ , \tag{1.81}$$

and the field equations become

$$\frac{1}{\sqrt{|\det(g)|}} \left(\sqrt{|\det(g)|}\ g^{ab} Z^A_{,a} \right)_{,b} + \Gamma^A_{BC} Z^B_{,b} Z^C_{,c} g^{bc} = 0 \ . \tag{1.82}$$

The action (1.81) corresponds to the volume element of the submanifold $\mathcal{E} \subset \mathcal{T}$ and, consequently, the field Eqs.(1.82) represent the condition for $\mathcal{E}$ to be an extremal hypersurface in the phase manifold [140]. If the harmonic map satisfies the condition

$$\varphi^*(\Theta) = \varphi^*(d\Phi - \delta_{ab} I^a dE^b) = 0, \tag{1.83}$$

the pair $(\mathcal{E}, g)$ is called the *space of thermodynamic equilibrium states* (equilibrium manifold). The last condition is equivalent to

$$d\Phi = I_a dE^a \ , \quad \frac{\partial \Phi}{\partial E^a} = I_a \ . \tag{1.84}$$

The first of these equations corresponds to the first law of thermodynamics, whereas the second one is usually known as the condition for thermodynamic equilibrium [34]. Thus the harmonic map $\varphi : \mathcal{E} \rightarrow \mathcal{T}$ defines the equilibrium manifold $(\mathcal{E}, g)$ as an extremal submanifold of the phase manifold $(\mathcal{T}, \Theta, G)$ in which the first law of thermodynamics and the equilibrium conditions for a given system with fundamental equation $\Phi = \Phi(E^a)$ hold.

In the GTD formalism, Eq.(1.84) also means that the intensive thermodynamic variables are dual to the extensive ones. Notice that the mapping φ as defined above implies that the equation $\Phi = \Phi(E^a)$ must be explicitly given. In standard thermodynamics this is known as *the fundamental equation* from which all the equations of state can be derived [34, 67]. In this representation, the second law of thermodynamics is equivalent to the convexity condition on the thermodynamic potential

$$\partial^2 \Phi / \partial E^a \partial E^b \geq 0, \tag{1.85}$$

if Φ coincides with the internal energy of the system [63]. The smooth map φ induces in a canonical manner a metric g (thermodynamic metric) in the equilibrium manifold $\mathcal{E}$ by means of $g = \varphi^*(G)$. Consequently, $\mathcal{E}$ is a Riemannian manifold with a non–degenerate metric g.

The thermodynamic potential must satisfy the homogeneity condition

$$\Phi(\lambda E^a) = \lambda^\beta \Phi(\lambda E^a), \tag{1.86}$$

for constant parameters λ and β. Differentiating this homogeneity condition with respect to λ at $\lambda = 1$ and using (1.84), we get

$$\beta\Phi = \delta_{ab} I^a E^b, \tag{1.87}$$

which is known as *Euler's identity.* Calculating the exterior derivative of (1.87) and using (1.84), we obtain *the generalized Gibbs-Duhem* relation

$$(1-\beta)\delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0. \tag{1.88}$$

The classical expressions for Euler's identity (1.27) and Gibbs-Duhem (1.31) relation are obtained from the above equations by putting $\beta = 1$.

1.11.6 Geodesics of the Equilibrium Manifold

Consider a base manifold with $\dim(M) = 1$ and identify the target manifold M' with the equilibrium manifold $(\mathcal{E}, g)$ defined above. Then, the field equations reduce to the geodesic equations in the equilibrium manifold

$$\frac{d^2 E^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dE^b}{d\tau} \frac{dE^c}{d\tau} = 0 \ , \tag{1.89}$$

where $\Gamma^a{}_{bc}$ are the Christoffel symbols of the thermodynamic metric g and τ is an arbitrary affine parameter along the geodesic. The solutions to the geodesic equations depend on the explicit form of the thermodynamic metric g which, in turn, depends on the fundamental equation $\Phi = \Phi(E^a)$. Therefore, a particular thermodynamic system leads to a specific set of geodesic equations whose solutions depend on the properties of the system. Not all the solutions need to be physically realistic since, in principle, there could be geodesics that connect equilibrium states that are not compatible with the laws of thermodynamics. Those geodesics which connect physically meaningful states will represent quasi-static thermodynamic processes. Therefore, a quasi-static process can be seen as a dense succession of equilibrium states. This is in agreement with the standard interpretation of quasi-static processes in ordinary thermodynamics [34]. The affine parameter τ can be used to label each of the equilibrium states which are part of a geodesic. Because of its intrinsic freedom, the affine parameter can be chosen in such a way that it increases as the entropy of a quasi-static process increases. This opens the possibility of interpreting the affine parameter as a *"time"* parameter with a specific direction which coincides with the direction of entropy increase. This has been shown explicitly in the case of the ideal gas [121].

From the above discussion I conclude that that harmonic maps play an important role in the formalism of GTD. They can be used to derive geodesic equations in different spaces and to introduce in a consistent and invariant way the concept of equilibrium manifold. It turns out that for a given fundamental equation of the form $\Phi = \Phi(E^a)$, GTD provides an invariant approach to construct the corresponding equilibrium manifold whose points represent equilibrium states. The harmonic map which determines the equilibrium manifold generates also a system of differential equations that determine extremal hypersurfaces in the phase manifold. This construction allows us to investigate the properties of the curvature of the equilibrium manifold and to propose it as an invariant measure of thermodynamic interaction. The thermodynamic curvature not only measures the interaction of the thermodynamic system, but also becomes singular at those points where phase transitions occur [118]. Thus, GTD represents an invariant geometric formalism of standard thermodynamics that resembles the famous principle *"curvature = interaction"* valid for all known forces of nature.

In GTD, to describe a thermodynamic system it is necessary to specify a metric G for the phase manifold $\mathcal{T}$ and a fundamental equation $\Phi = \Phi(E^a)$. These ingredients allows us to construct explicitly the pair $(\mathcal{E}, g)$ that constitutes the equilibrium manifold whose geometric properties should be related to the thermodynamic properties of the specific system. In chapter 2 I will present different metrics G which generate equilibrium manifolds for different thermodynamic systems. It is worth mentioning that GTD allows us to implement easily different thermodynamic representations.

Finally, I mention that the geometry of the metric $g = \phi^*(G)$ is invariant with respect to arbitrary diffeomorphisms performed on $\mathcal{E}$. This can be shown by considering explicitly the components of g in terms of the components of G and applying arbitrary LT on G. This important property allows us to consider variational principles in GTD that impose additional conditions on the metric structures [139]. It is the only consistent and invariant formalism that correctly describes geometry of thermodynamics. Indeed, it correctly reproduces the thermodynamic behavior of black holes in different gravity theories, which is the main subject of this thesis.

Chapter 2

Phase Transitions in Geometrothermodynamics

In this chapter I will present metrics G which generate equilibrium manifolds for different thermodynamic systems. In order to describe a thermodynamic system in GTD it is necessary to specify the metric G for the phase manifold $\mathcal{T}$ and a fundamental equation $\Phi = \Phi(E^a)$. These ingredients allow us to construct explicitly the pair $(\mathcal{E}, g)$ that constitutes the equilibrium manifold whose geometric properties should be related to the thermodynamic properties of the specific system.

2.1 Metrics used in Geometrothermodynamics

As mentioned before, the only freedom in the construction of the phase manifold is in the choice of the metric G. Legendre invariance implies a series of algebraic conditions for the metric components G_{AB} [112] and it can be shown that these conditions are not trivially satisfied. For instance, a straightforward computation shows that the flat metric $G = \delta_{AB} dZ^A dZ^B$ is not invariant with respect to LT (1.77). It then follows that the phase space is necessarily curved. A detailed analysis of the Legendre invariance conditions show that the metric

$$G = (d\Phi - I_a dE^a)^2 + \lambda\,(E_a I_a)^{2k+1} dE^a dI^a \ , \quad E_a = \delta_{ab} E^b \ , \quad I_a = \delta_{ab} I^b \ , \tag{2.1}$$

where λ is an arbitrary real constant and k is an integer, is invariant with respect to partial and total LT. To our knowledge this is the most general metric satisfying the conditions of Legendre invariance. The corresponding scalar curvature

$$R = \frac{2}{\lambda^2} \left\{ \left[\sum_{a=1}^{n} (E_a I_a)^{-2k-1} \right]^2 - 3 \sum_{a \neq b}^{n} (E_a I_a E_b I_b)^{-2k-1} \right\} \ , \tag{2.2}$$

shows that the manifold is curved in general.

Furthermore, the phase manifold metric

$$G = (d\Phi - I_a dE^a)^2 + \lambda (E_a I^a)^{2k+1} (\chi_{bc} dE^b dI^c) \ , \tag{2.3}$$

where χ_{ab} is a constant diagonal tensor, satisfies the conditions that follow from a total LT (1.77). The corresponding curvature is rather cumbersome and cannot be written in a compact form; however, an inspection of its explicit form shows that it is always different from zero.

The metrics (2.1) and (2.3) are the most general Legendre invariant metrics found so far and contain other known metrics as particular cases [112]. The LT impose very strong conditions on the components G_{AB}; indeed, Eq.(1.77) shows that such a transformation can change an extensive variable to the negative of the corresponding intensive variable. This implies that only very specific combinations of extensive and intensive variables can be invariant under LT.

From the description given above it follows that if we limit ourselves to the case of total LT, then there exists a class of metrics,

$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\xi_{ab} E^a I^b)(\chi_{cd} dE^c dI^d), \quad \eta_{ab} = diag(-1, 1, ..., 1) \tag{2.4}$$

parameterized by the diagonal constant tensors ξ_{ab} and η_{ab}, which is invariant for several choices of these free tensors. In fact, since ξ_{ab} and η_{ab} must be constant and diagonal it seems reasonable to express them in terms of the usual Euclidean and pseudo–Euclidean metrics $\delta_{ab} = diag(1, ..., 1)$ and $\eta_{ab} = diag(-1, 1, ..., 1)$ respectively. Then, for instance, the choice

$$\xi_{ab} = \delta_{ab}, \quad \chi_{ab} = \eta_{ab}, \tag{2.5}$$

corresponds to a Legendre invariant metric which has been used to describe the geometric properties of systems with first order phase transitions [112, 116]. Moreover, the choice

$$\xi_{ab} = \delta_{ab}, \quad \chi_{ab} = \delta_{ab}, \tag{2.6}$$

turned out to describe correctly second order phase transitions especially in black hole thermodynamics [116]. The additional choice

$$\xi_{ab} = \frac{1}{2}(\delta_{ab} - \eta_{ab}), \quad \eta_{ab} = \delta_{ab}, \tag{2.7}$$

can be used to handle in a geometric manner second order phase transitions and also the thermodynamic limit $T \to 0$ [2, 135]. Obviously, for a given thermodynamic system it is very important to choose the appropriate metric in order to describe correctly the thermodynamic properties in terms of the geometric properties in GTD.

The pullback ϕ of the Legendre invariant metric (2.1) generates the following thermodynamic metric

$$g = \lambda (E_a \Phi^a)^{(2k+1)} \delta_{ab} \Phi_{bc} dE^a dE^c, \tag{2.8}$$

where

$$\Phi_a = \frac{\partial\Phi}{\partial E_a}, \quad \Phi_{bc} = \frac{\partial^2\Phi}{\partial E^b \partial E^c}, \tag{2.9}$$

which can be shown to be invariant with respect to arbitrary (partial and total) LT. On the other hand, the metric (2.3) of the phase manifold generates the thermodynamic metric

$$g = \lambda(\xi^b_a E^a \Phi_b)(\chi^b_a \Phi_{bc} dE^a dE^c), \tag{2.10}$$

where

$$\chi^b_a = \xi_{ac}\delta^{bc}, \quad \chi^b_a = \chi_{ac}\delta^{bc}, \tag{2.11}$$

which is invariant with respect to total LT. In the case of (2.7), the pullback φ^* induces on $\mathcal{E}$, by means of $g = \varphi^*(G)$, the thermodynamic metric

$$g = \frac{1}{2}\left[E^a\left(\frac{\partial\Phi}{\partial E^a} - \eta_{ab}\delta^{bc}\frac{\partial\Phi}{\partial E^c}\right)\right]\left(\eta_{ab}\delta^{bc}\frac{\partial^2\Phi}{\partial E^c\partial E^d}dE^a dE^d\right). \tag{2.12}$$

Notice that the explicit components of the thermodynamic metric g can be calculated in a straightforward manner once the fundamental equation $\Phi = \Phi(E^a)$ is explicitly given.

However, in the formalism of GTD the metric

$$g_0 = \frac{\partial^2\Phi}{\partial E^a \partial E^b} dE^a dE^b, \tag{2.13}$$

is generated as

$$g_0 = \varphi(G_0) = \varphi^*(\delta_{ab} dE^a dI^b), \tag{2.14}$$

where the metric G_0 is not Legendre invariant. This implies that the results obtained by using the metric g_0 can depend on the choice of thermodynamic potential and, consequently, can lead to contradictory results. In particular, for $\Phi = U$ (internal energy), g_0 is equivalent to the Weinhold metric and for $\Phi = S$, g_0 is the Ruppeiner metric, which are not Legendre invariant.

An important issue to be addressed when imposing invariance conditions on metrics is the one related to the existence of solutions. In our case, the point is whether there exist metrics which satisfy the condition of invariance under LT. In fact, it is easy to construct an invariant $(2n + 1)$-dimensional metric by defining the non-degenerate metric as the *"square"* of the Gibbs 1-form (1.78) and the invariance of this metric follows from the invariance of the Gibbs 1-form.

2.2 Geometrothermodynamic Representations

In the context of GTD, it is also possible to consider any representation of the fundamental equation $\Phi = \Phi(E^a)$. The mapping ϕ can be defined in each case, independently of the chosen thermodynamic potential. On the other hand, since only Legendre invariant structures on $\mathcal{T}$ and $\mathcal{E}$ are considered, the characteristics of the underlying geometry for a given thermodynamic system will be independent of the thermodynamic potential. This is in agreement with standard thermodynamics. However, LT allow us to introduce a set of seven additional thermodynamic potentials which depend on different combinations of extensive and intensive variables. The complete set of thermodynamic potentials can be written as

$$\begin{aligned} M =& M(S,J,Q), \\ & M_1 = M_1(T,J,Q) = M - TS, \\ & M_2 = M_2(S,\Omega,Q) = M - \Omega J, \\ & M_3 = M_3(S,J,\phi) = M - \phi Q, \\ & M_4 = M_4(T,\Omega,Q) = M - TS - \Omega J, \\ & M_5 = M_5(T,J,\phi) = M - TS - \phi Q, \\ & M_6 = M_6(S,\Omega,\phi) = M - \Omega J - \phi Q, \\ & M_7 = M_7(T,\Omega,\phi) = M - TS - \Omega J - \phi Q. \end{aligned} \tag{2.15}$$

Thus, for any representation of the fundamental equation, one has the freedom of choosing anyone of the potentials $M, M_1, ..., M_7$, without affecting the thermodynamic properties of thermodynamic systems.

The mathematical tools presented in the last sections allow us to define geometric structures in an invariant way. In particular, the curvature of the thermodynamic metric g should represent the thermodynamic interaction independently of the thermodynamic potential. In fact, this is not a trivial condition from a geometric point of view. For instance, a geometric analysis of black hole thermodynamics by using metrics introduced *ad hoc* in the equilibrium manifold leads to contradictory results [6, 26, 89, 130, 132].

2.3 Systems with Second Order Phase Transitions

Using the choice (2.5), the non-degenerate metric

$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\eta_{cd} dE^c dI^d), \ \eta_{ab} = \text{diag}(-1, 1, ..., 1) \tag{2.16}$$

is invariant with respect to total LT (1.77) and consequently, can be used to describe the geometric properties of the phase manifold $\mathcal{T}$. The smooth map $\varphi : \{E^a\} \mapsto \{\Phi(E^a), E^a, I^a(E^a)\}$, satisfying $\varphi^*(d\Phi - \delta_{ab} I^a dE^b) = 0$, induces the thermodynamic metric

$$g = \varphi^*(G) = \left(E^c \frac{\partial \Phi}{\partial E^c} \right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d \right) \tag{2.17}$$

that can be explicitly calculated once the fundamental equation $\Phi = \Phi(E^a)$ is specified. Notice that by virtue of the equilibrium conditions (1.84) and Euler's identity (1.87), the conformal factor of the thermodynamic metric (2.17) turns out to be proportional to the thermodynamic potential Φ. This means that in general the conformal factor can assumed different from zero.

For concreteness, assume that $E^1 = S$ is the entropy of a system with two thermodynamic degrees of freedom ($n = 2$). Then, the dual intensive variable $I^1 = \partial\Phi/\partial S = T$ represents the temperature and the thermodynamic metric (2.17) reduces to

$$g = (ST + E^2 I^2)\left[-\Phi_{SS} dS^2 + \Phi_{22}(dE^2)^2\right] \,, \tag{2.18}$$

where a subscript represents derivation with respect to the corresponding coordinate.

To describe the phase transition structure of the system I use the heat capacity

$$C = T\left(\frac{\partial T}{\partial S}\right) = \frac{\Phi_S}{\Phi_{SS}}, \tag{2.19}$$

so that phase transitions of second order occur at those points where C diverges. For the calculation of concrete examples it is necessary to specify the fundamental equation $\Phi = \Phi(S, E^2)$. In particular, I am interested in analyzing the thermodynamic properties of black hole configurations [44] for which usually $\Phi = M$ is the mass (energy) of the black hole and E^2 is an additional extensive variable like electric charge Q, angular momentum J, etc. In this case, the fundamental equation is essentially equivalent to the entropy–area relation $S = S(M, E^2) = kA$, where A is the area of the horizon and k is a constant that depends on the dimension of the spacetime. However, it turns out that it is not always possible to express the fundamental equation in the form $M = M(S, E^2)$ and the entropy–area relation $S = S(M, E^2)$ must be used. In ordinary thermodynamics this corresponds to a change from the energy representation to the entropy representation. GTD allows us to perform changes of representations in a simple manner. In fact, to obtain the entropy representation of the metric (2.18)the Gibbs 1-form (1.78) of the phase space can be rewritten as

$$\Theta_S = dS - \frac{1}{T}dM + \frac{I^2}{T}dE^2, \tag{2.20}$$

Therefore, the first law of thermodynamics becomes

$$dM = TdS + I^2 dE^2, \tag{2.21}$$

as

$$dS = (1/T)dM - (I^2/T)dE^2, \tag{2.22}$$

and identify M and E^2 as the new extensive variables, whereas $1/T$ and $-I^2/T$ represent the corresponding new dual intensive variables. Replacing these new variables in the general expression (2.17), the metric is given by

$$g = \left(MS_M + E^2 S_2\right)\left[-S_{MM} dM^2 + S_{22}(dE^2)^2\right] \,, \tag{2.23}$$

Black hole	Fundamental equation
RN	$S=\pi(M+\sqrt{M^2-Q^2})^2$
RNAdS	$M=\frac{(D-2)\omega_{D-2}}{16\pi}\left(\frac{4S}{\omega_{D-2}}\right)^{\frac{D-1}{D-2}}\left[\frac{1}{l^2}+\left(\frac{\omega_{D-2}}{4S}\right)^{\frac{2}{D-2}}+\frac{2\pi^2Q^2}{(D-2)(D-3)S^2}\right]$
Kerr	$S=2\pi(M^2+\sqrt{M^4-J^2})$
KAdS	$M=\frac{D-2}{2\pi}\left(\frac{\omega_{D-2}}{2^D}\right)^{\frac{1}{D-2}}S^{\frac{D-3}{D-2}}\left(1+\frac{4\pi^2J^2}{S^2}\right)^{\frac{1}{D-2}}$
BTZ	$M=\frac{S^2}{16\pi^2l^2}+\frac{4\pi^2J^2}{S^2}$
BTZCS	$M=\frac{1}{8\pi^2k^2}\left[S^2+8\pi^2kJ+\frac{S}{l}\sqrt{(l^2-k^2)(S^2+16\pi^2kJ)}\right]$
BTZTF	$S=2\sqrt{2}\pi l\left[M+\left(M^2-\frac{J^2}{l^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}-\frac{3}{2}\ln 2\sqrt{2}\pi l\left[M+\left(M^2-\frac{J^2}{l^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$

Table 2.1: **Black holes with two degrees of freedom.** Here I use the following notations: RN = Reissner-Nordström, RNAdS = Reissner-Nordström-Anti-de-Sitter, KAdS = Kerr-Anti-de-Sitter, BTZ = Bañados-Teitelboim-Zanelli, BTZCS = Bañados-Teitelboim-Zanelli-Chern-Simons, BTZTF = Bañados-Teitelboim-Zanelli black hole with thermal fluctuations, M = mass, S=entropy, Q = charge, J = angular momentum, $\Lambda=-1/l^2$= cosmological constant and $\omega_{D-2}=2\pi^{\frac{D-1}{2}}/\Gamma[\frac{D-1}{2}]$ is the volume of the unit $(D-2)$–sphere, k = Chern-Simons coupling constant.

which allows us to investigate the properties of the same thermodynamic system in the entropy representation.

The above thermodynamic metrics (2.18) and (2.23) have been used to investigate the black hole configurations listed in Table 2.1. In all the cases GTD is mathematically consistent and reproduces the thermodynamic behavior of the black holes. In fact, the scalar curvature of the equilibrium manifold $(\mathcal{E},g)$ is in all cases different from zero, indicating the presence of non-trivial thermodynamic interaction. Furthermore, in Table 2.2 I present the heat capacity and the scalar curvature for each of the black holes contained in Table 2.1. It follows that at those points where the heat capacity diverges the scalar curvature of the equilibrium manifold becomes singular. Consequently, a second order phase transition is characterized by a curvature singularity. This shows that indeed the curvature of the equilibrium manifold can be considered as a measure of thermodynamic interaction.

BH	Heat Capacity	Scalar curvature
RN	$-\frac{2\pi^2 r_+^2(r_+-r_-)}{(r_+-3r_-)}$	$\frac{(r_+^2-3r_-r_++6r_-^2)(r_++3r_-)(r_+-r_-)^2}{\pi^2 r_+^3(r_+^2+3r_-^2)^2(r_+-3r_-)^2}$
RNAdS	$\frac{(D-2)S\left[\frac{D-1}{l^2}+(D-3)\left(\frac{\omega_{D-2}}{4S}\right)^{\frac{2}{D-2}}-\frac{2\pi^2Q^2}{(D-2)S^2}\right]}{\frac{D-1}{l^2}-(D-3)\left(\frac{\omega_{D-2}}{4S}\right)^{\frac{2}{D-2}}+\frac{2(2D-5)\pi^2Q^2}{(D-2)S^2}}$	$\frac{\mathcal{N}^{RNAdS}}{\left[\frac{D-1}{l^2}-(D-3)\left(\frac{\omega_{D-2}}{4S}\right)^{\frac{2}{D-2}}+\frac{2(2D-5)\pi^2Q^2}{(D-2)S^2}\right]^2}$
Kerr	$\frac{2\pi^2 r_+(r_++r_-)^2(r_+-r_-)}{r_+^2-6r_+r_--3r_-^2}$	$\frac{(3r_+^3+3r_+^2r_-+17r_+r_-^2+9r_-^3)(r_+-r_-)^3}{2\pi^2 r_+^2(r_++r_-)^4(r_+^2-6r_+r_--3r_-^2)^2}$
KAdS	$-\frac{(D-2)S[3S^2+20\pi^2J^2-D(S^2+4\pi^2J^2)](S^2+4\pi^2J^2)}{3S^4+24\pi^2J^2S^2+240\pi^4J^4-D(S^4+48\pi^4J^4)}$	$\frac{\mathcal{N}^{KAdS}}{[3S^4+24\pi^2J^2S^2+240\pi^4J^4-D(S^4+48\pi^4J^4)]^2}$
BTZ	$\frac{4\pi r_+(r_+^2-r_-^2)}{r_+^2+3r_-^2}$	$-\frac{3}{2}\frac{l^4}{(r_+^2+3r_-^2)^2}$
BTZCS	$\frac{4\pi(l^2-k^2)r_+^2(r_+^2-r_-^2)}{l[lr_+(r_+^2+3r_-^2)-kr_-(r_-^2+3r_+^2)]}$	$\frac{\mathcal{N}^{BTZCS}}{l^2[lr_+(r_+^2+3r_-^2)-kr_-(r_-^2+3r_+^2)]^2}$
BTZTF	$\frac{4\pi r_+(r_+^2-r_-^2)}{r_+^2+3r_-^2}$,	$\frac{(r_+^2-r_-^2)^2(5r_+^4-6r_+^2r_-^2+9r_-^4)}{4\pi^2 r_+^4(r_+^2+3r_-^2)^3}$

Table 2.2: **GTD of black holes with two degrees of freedom.** Here I use the following notations: r_+ = radius of the exterior horizon, r_- = radius of the interior horizon. The function $\mathcal{N}$ represents in each case the numerator of the scalar curvature which is a well-behaved function at the points where the denominator vanishes. For more details see [116].

The fact that the above mentioned results are invariant with respect to LT explains some contradictory results [8,26,91] that follow when the equilibrium manifold is equipped with metrics strongly associated to a particular thermodynamic potential.

The $(3+1)$-dimensional Kerr-Newman black hole represents a thermodynamic system with three degrees of freedom. Indeed, from the entropy-area relation one can derive the fundamental equation [44]

$$S = \pi\left(2M^2 - Q^2 + 2\sqrt{M^4 - M^2Q^2 - J^2}\right) , \qquad (2.24)$$

which depends on the extensive variables M, J and Q. The second order phase transitions are determined by the corresponding heat capacity:

$$C = -\frac{4TM^3S^3}{2M^6 - 3M^4Q^2 - 6M^2J^2 + Q^2J^2 + 2(M^4 - M^2Q^2 - J^2)^{3/2}} . \qquad (2.25)$$

According to the general expression for the metric g, as given in Eq.(2.17), the equilibrium manifold is $(2+1)$-dimensional and the corresponding Legendre invariant metric reduces to

$$g_{ab}^{KN} = (MS_M + QS_Q + JS_J) \begin{pmatrix} S_{MM} & 0 & 0 \\ 0 & -S_{QQ} & -S_{QJ} \\ 0 & -S_{QJ} & -S_{JJ} \end{pmatrix} . \tag{2.26}$$

Inserting, here, the expression for the entropy (2.24), we obtain a rather cumbersome metric which cannot be written in a compact form. Nevertheless, the scalar curvature can be shown to have the form $R^{KN} = \mathcal{N}^{KN}/\mathcal{D}^{KN}$, where

$$\mathcal{D}^{KN} = \left[2M^6 - 3M^4Q^2 - 6M^2J^2 + Q^2J^2 + 2(M^4 - M^2Q^2 - J^2)^{3/2}\right]^2 , \tag{2.27}$$

and $\mathcal{N}^{KN}$ is a function which is always positive in the black-hole region $M^4 \geq M^2Q^2 + J^2$. Since the denominators of the heat capacity and the scalar curvature coincide, I conclude that there exist curvature singularities at those points where second order phase transitions occur. This result reinforces the interpretation of the curvature of the manifold $\mathcal{E}$ as a measure of thermodynamic interaction.

An additional test for the thermodynamic metrics (2.18) and (2.23) consists in calculating the curvature of systems with no thermodynamic interaction or no phase transitions. Table 2.3 contains the fundamental equations for some ordinary thermodynamic systems which I investigated from the point of view of GTD.

Table 2.4 contains the results of our analysis. In the case of the ideal gas and its paramagnetic generalization, which have no thermodynamic interaction, the curvature vanishes and the equilibrium manifold becomes flat. In fact, one can show the general result that any generalization of the ideal gas whose fundamental equation can be separate in its variables as

$$S = S_1(U) + S_2(V) + S_3(E^3) + ... \tag{2.28}$$

always generates a flat thermodynamic metric [140].

The 1-dimensional Ising model [83] generates a metric whose curvature is non-zero, indicating the presence of thermodynamic interaction and regular everywhere, indicating the lack of second order phase transitions. Consequently, GTD reproduces at the level of the curvature the properties associated with the thermodynamic interaction between the particles of the above mentioned systems.

2.4 Systems with First Order Phase Transitions

In order to study the system with first order phase transitions, consider the choice (2.6) thus, the non–degenerate and Legendre invariant metric is given by

$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\delta_{cd} dE^c dI^d). \tag{2.29}$$

This metric is also invariant with respect to total LT (1.77) and can be used to describe the geometric properties of $\mathcal{T}$. If one assume that the equilibrium manifold $\mathcal{E} \subset \mathcal{T}$ has as

System	Fundamental equation
IG	$S = N\left(\ln\frac{V}{N} + \frac{3}{2}\ln\frac{U}{N}\right)$
PaIG	$S = N\left[\ln\frac{V}{N} + \frac{3}{2}\ln\frac{U}{N} - \frac{3}{2}\ln\left(1 - \frac{2\mathcal{M}^2}{N^2}\right) - \frac{3}{\mu^2}\frac{\mathcal{M}^2}{N^2}\right]$
1IM	$F = -J - T\ln\left(\cosh\frac{H}{T} + \sqrt{\sinh^2\frac{H}{T} + e^{-4\frac{J}{T}}}\right)$

Table 2.3: **Ordinary thermodynamic systems.** I use the following notations: IG = ideal gas, PaIG = paramagnetic ideal gas, 1IM = 1–dimensional Ising model, U = energy , V = volume, N = total number of molecules, $\mathcal{M}$ = magnetization, H = magnetic field, T = Temperature, J = spin interaction parameter.

coordinates the extensive variables $\{E^a\}$, the smooth embedding map $\mathcal{E} \to \mathcal{T}$, satisfying $\varphi^*(d\Phi - \delta_{ab}I^a dE^b) = 0$, can be used to generate in a canonical way the thermodynamic metric

$$g = \varphi^*(G) = (E^c\Phi_c)\left(\frac{\partial^2\Phi}{\partial E^a \partial E^b} dE^a dE^b\right), \tag{2.30}$$

for the equilibrium manifold $\mathcal{E}$. In the case of a thermodynamic systems with only two degrees of freedom ($n = 2$), let us introduce the extensive variables $\Phi = U$ = internal energy, $E^1 = S$ = entropy and $E^2 = V$ = volume, together with the intensive variables T = temperature and P = pressure. The first law of thermodynamics (1.84) reads $dU = TdS - PdV$ and the explicit form of the thermodynamic metric is

$$g = (ST - PV)\left(U_{SS}dS^2 + 2U_{SV}dSdV + U_{VV}dV^2\right), \tag{2.31}$$

or, equivalently, in the entropy representation

$$g = \frac{1}{T}(U + VP)\left(S_{UU}dU^2 + 2S_{UV}dUdV + S_{VV}dV^2\right) . \tag{2.32}$$

Probably, the best–known thermodynamic system with a very rich structure of first order phase transitions is the van der Waals fluid [34]. For the sake of generality, I will use here the van der Waals fundamental equation together with the theorem of corresponding states in order to recast the fundamental equation in an invariant form applicable to all fluids

$$\overline{S} = \ln\left(3\overline{V} - 1\right) + \frac{3}{2}\ln\left(\overline{U} + \frac{3}{\overline{V}}\right) , \tag{2.33}$$

where $\overline{U} = U/U_c$ and $\overline{V} = V/V_c$, with $U_c = 4a/9b$, $V_c = 3b$ and a and b are the van der Waals constants. The heat capacity following from this fundamental equation turns out to be constant and, consequently, no second order phase transitions can occur. However,

System	Heat Capacity	Curvature scalar
IG	$\frac{3}{2}N$	$R = 0$
PaIG	$\frac{3}{2}N\left(1 - \frac{2\mathcal{M}^2}{N^2}\right)$	$R = 0$
1IM	$-\frac{1}{T^2}\left\{\frac{M(T,H)}{I^3\left[I+\cosh\frac{H}{T}\right]} - \frac{N(T,H)^2}{I^2\left[I+\cosh\frac{H}{T}\right]^2}\right\}$	$R \neq 0$

Table 2.4: **GTD of some ordinary systems.** Here $I = I(T, H, J)$ is function with non-zero values. The Ricci scalar for the Ising model cannot be put in a compact form, but a numerical analysis shows that it is everywhere regular.

the critical points determined by the roots of the equation

$$\overline{PV}^3 - 3\overline{V} + 2 = 0\,, \tag{2.34}$$

correspond to first order phase transitions [34], where $\overline{P} = P/P_c$, with $P_c = a/27b^2$, is the reduced pressure.

From the fundamental Eq.(2.33) it is then straightforward to compute the thermodynamic metric in the entropy representation:

$$\begin{aligned} g^{vdW} &= -\frac{9\left(5\overline{UV}^2 - \overline{UV} - 3\overline{V} + 3\right)}{4(3\overline{V}-1)(\overline{UV}+3)^3\overline{V}^2}\Bigg\{\overline{V}^4 d\overline{U}^2 - 6\overline{V}^2 d\overline{U}d\overline{V} + \\ &+ 9\left[\frac{2(\overline{UV}+3)(\overline{UV}^4 - 6\overline{V}^2 + 6\overline{V} - 1)}{3(3\overline{V}-1)^2} - 1\right]d\overline{V}^2\Bigg\}\,. \end{aligned} \tag{2.35}$$

The 2–dimensional equilibrium manifold turns out to be curved in general, indicating that the particles of the fluid interact thermodynamically. Furthermore, the scalar curvature of the above metric can be written in the form

$$R = \frac{\mathcal{N}^{vdW}}{\left(\overline{PV}^3 - 3\overline{V} + 2\right)^2}, \tag{2.36}$$

where $\mathcal{N}^{vdW}$ is a function of $\overline{U}$ and $\overline{V}$ that is well–behaved at the points where the denominator vanishes. The scalar curvature diverges at the critical points determined by Eq.(2.34). Consequently, a first order phase transition can be interpreted geometrically as a curvature singularity. This is in accordance with our intuitive interpretation of thermodynamic curvature.

It is worth noticing that the metric (2.30) can also be used to describe the properties of ordinary systems. Indeed, a straightforward computation of the corresponding thermodynamic metrics, using the fundamental equations of the ordinary systems listed in Table 2.3, leads to results equivalent to those reviewed in Table 2.4.

2.5 Conclusion

GTD is a differential geometric formalism whose objective is to describe in an invariant manner the properties of thermodynamic systems in terms of geometric concepts. In this chapter, I presented different thermodynamic metrics to describe the properties of thermodynamic systems. I used two different thermodynamic metrics which can be canonically derived from the non-degenerate Legendre invariant metric

$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\chi_{cd} dE^c dI^d), \tag{2.37}$$

of the phase manifold $\mathcal{T}$, where χ_{ab} is an arbitrary constant diagonal tensor. If $\chi_{ab} = \eta_{ab}$, the resulting thermodynamic metric can be used to describe the properties of systems characterized by second order phase transitions. This was shown in particular for several black hole configurations in diverse theories and dimensions. In the Euclidean case, $\chi_{ab} = \delta_{ab}$, the corresponding thermodynamic metric was shown to correctly reproduce the structure of phase transitions of first order in the specific case of a fundamental equation which describes realistic gases and fluids.

The formalism of GTD indicates that phase transitions occur at those points where the thermodynamic curvature is singular. The singularities represent critical points where the geometric description of GTD does not hold anymore and must give place to a more general approach. In ordinary thermodynamics the situation is similar; near the points of phase transitions equilibrium thermodynamics is not valid and non-equilibrium models must be implemented. The results show that the metric structure of the phase manifold $\mathcal{T}$ determines the type of systems that can be described by a specific thermodynamic metric: An Euclidean structure describes systems with first order phase transitions, whereas a pseudo-Euclidean structure describes systems with second order phase transitions. At the moment an explanation for this result is unknown.

Chapter 3

Thermodynamic Geometry of Charged Rotating BTZ Black Holes

The spacetime of a black hole in 2+1 dimensions with negative cosmological constant Λ provides an example of a lower-dimensional toy model which shares many of the important conceptual issues of GR in 3+1 dimensions, but avoids some of the difficulties found in mathematical computations. This spacetime is known as the BTZ black hole [12] and it warrants attention in its own right (for a review, see [36]). A key feature of this model lies in the simplicity of its construction. It is a spacetime with constant negative curvature and is obtained as a discrete quotient of $(2+1)$-dimensional AdS space [13]. The BTZ spacetime is free of phase transitions. Even so, all characteristic features of black holes such as the event horizon and Hawking radiation are present so that this model is a genuine black hole. Furthermore, despite its simplicity, the BTZ black holes plays an outstanding role in many of the recent developments in string theory, specially in the context of the AdS/CFT conjecture [142].

One of the most interesting aspects of black holes is related to their thermodynamic properties. In the case of the BTZ black holes, the extensive thermodynamic variables are the mass M, angular momentum J and entropy S which is proportional to the horizon area. The intensive variables are the angular velocity Ω and the Hawking temperature T at the horizon.

In this chapter I present the thermodynamics and the thermodynamic geometries of charged rotating BTZ (CR-BTZ) black holes in (2+1)-dimensional gravity. The thermodynamics of these systems is investigated within the context of the Ruppeiner and Weinhold thermodynamic geometries and the formalism of GTD. Considering the behavior of the heat capacity and the Hawking temperature, these geometries cannot describe completely the thermodynamics of these black holes and of their limiting case of vanishing electric charge. In contrast, the Legendre invariance imposed on the metric in GTD allows one to describe the CR-BTZ black holes and their limiting cases in a consistent and invariant manner [2].

3.1 General Relativity in 2+1 Dimensions

Let us briefly describe how GR is modified in 2+1 dimensions [36]. The Einstein-Hilbert actions becomes

$$S = \frac{1}{2\pi}\int d^3x\sqrt{-g}(R-2\Lambda) + S_m, \tag{3.1}$$

where S_m is the matter part of the action. As in 3+1 dimensions, the resulting Euler-Lagrange equations are the standard Einstein field equations (1.1). These field equations are written in covariant form and are invariant under the action of the group of diffeomorphisms of the spacetime. The choice of the energy-momentum tensor $T_{\mu\nu}$ completely determines the Ricci tensor $R_{\mu\nu}$, but in general it does not determine the Riemann tensor $R_{\mu\nu\gamma\delta}$. This is the fundamental difference between $(3+1)$-dimensional and $(2+1)$-dimensional gravities. The Weyl tensor (or conformal tensor) is defined to be the tensor $C_{\mu\nu\gamma\delta}$. In n-dimensions, with $n \geq 3$, the Weyl tensor can be written as follows

$$C_{\mu\nu\gamma\delta} = R_{\mu\nu\gamma\delta} + \frac{2}{n-2}(g_{\mu[\gamma}R_{\delta]\nu} - g_{\nu[\gamma}R_{\delta]\mu}) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\gamma}g_{\delta]\nu}. \tag{3.2}$$

In 2+1 dimensions:

$$C_{\mu\nu\gamma\delta} = R_{\mu\nu\gamma\delta} + g_{\mu\gamma}R_{\nu\delta} + g_{\nu\gamma}R_{\mu\delta} - g_{\nu\gamma}R_{\mu\delta} - g_{\mu\delta}R_{\nu\gamma} + \frac{1}{2}(g_{\mu\gamma}g_{\nu\delta} - g_{\mu\delta}g_{\nu\delta})R. \tag{3.3}$$

The Weyl tensor in GR provides curvature to the spacetime when the Ricci tensor is zero. In GR the source of the Ricci tensor is the energy-momentum of the local matter distribution. If the matter distribution is zero then the Ricci tensor will be zero. However the spacetime is not necessarily flat in this case since the Weyl tensor contributes curvature to the Riemann curvature tensor and so the gravitational field is not zero in spacetime void situations. This term allows gravity to propagate in regions where there is no matter/energy source.

In 2+1 dimensions the space geometry is determined by the Riemann curvature tensor $R_{\mu\nu\gamma\delta}$. In 2+1 dimensions the latter has six independent components, exactly as many as the associated Ricci tensor. This means that the space time geometry can be expressed in terms of the Ricci tensor

$$R_{\mu\nu\gamma\delta} = g_{\mu\gamma}R_{\nu\delta} + g_{\nu\gamma}R_{\mu\delta} - g_{\nu\gamma}R_{\mu\delta} - g_{\mu\delta}R_{\nu\gamma} - \frac{1}{2}(g_{\mu\gamma}g_{\nu\delta} - g_{\mu\delta}g_{\nu\delta})R. \tag{3.4}$$

As a result the Weyl tensor (3.3), the traceless part of $R_{\mu\nu\gamma\delta}$, vanishes identically and the gravitational field has no dynamical degrees of freedom. One concludes that $R_{\mu\nu\gamma\delta}$ is completely determined by $T_{\mu\nu}$ and by the cosmological constant Λ. In particular, regions of spacetime with $T_{\mu\nu} = 0$ are regions of constant curvature, with $R_{\mu\nu\gamma\delta} = \Lambda(g_{\mu\gamma}g_{\nu\delta} - g_{\mu\delta}g_{\nu\delta})$ and $R = 6\Lambda$. Thus every solution of the vacuum Einstein field equations (1.1) with $\Lambda = 0$ is *flat* and every solution with a non-vanishing cosmological constant has constant curvature. Physically a $(2+1)$-dimensional spacetime has no local degrees of freedom: curvature is concentrated at the location of matter and there are no gravitational waves. GR in a (2+1)-dimensional spacetime has no Newtonian limit in the sense that there is no gravitational force between static point sources. (For more discussion on $(2+1)$-dimensional gravity see [36].)

In order to quantize $(2+1)$-dimensional GR, one needs to first understand the classical solutions of the Einstein field equations (1.1). Indeed, many of the best-understood approaches to quantization start with particular representations of the space of solutions. There are two fundamental approaches to classical GR in 2+1 dimensions. The first of these, based on the Arnowitt-Deser-Misner (ADM) decomposition of the metric, is familiar from $(3+1)$-dimensional gravity [10]. The main new feature in this case is that for certain topologies, one can find the general solution of the constraints. The second approach is also similar to a $(3+1)$-dimensional formalism. It starts with the first-order field equations which becomes substantially simpler in 2+1 dimensions. In both cases, the goal is to set up the field equations in a manner that permits a complete characterization of the classical solutions.

The unified treatment of space and time is a cornerstone of GR. Sometimes it is useful to introduce an explicit–although largely arbitrary – division of spacetime into spatial and temporal directions. Such a division is described by the ADM formalism [10]. The ADM decomposition of spacetime into space and time furnishes a natural setting for the initial value problem and it underlies Wheeler's *"geometrodynamical"* picture of classical GR as the dynamics of evolving spatial geometries. It provides a canonical description of the gravitational phase space and it leads to a Hamiltonian version of classical GR.

3.2 The BTZ $(2+1)$-Dimensional Black Holes

Consider a spacetime manifold with the topology $[0,1] \times \Sigma$ which represents a segment of a Universe between an initial surface $\{0\} \times \Sigma$ and final surface $\{1\} \times \Sigma$, where both the initial and final surfaces are assumed to be spacelike and Σ is an open or closed two-surface [35]. The ADM approach to $(2+1)$-dimensional GR starts with a slicing of the spacetime manifold M into constant time surfaces Σ_t endowed with a coordinate system $\{x^i\}$ and an induced metric $g_{ij}(t, x^i)$. In order to obtain the $(2+1)$-dimensional geometry, one must describe the way nearby time slices Σ_t and Σ_{t+dt} fit together. To do so, start at a point on Σ_t with coordinates x^i and displace it infinitesimally in the direction normal to

Σ_t [36]. The resulting change in proper time can be written as

$$d\tau = Ndt, \tag{3.5}$$

where $N(t, x^i)$ is the *lapse function*. Such a displacement will shift the time coordinate and alter the spatial coordinates as well. To allow for such a possibility, we write

$$x^i(t + dt) = x^i(t) - N^i dt, \tag{3.6}$$

where $N^i(t, x^i)$ is the *shift vector*. By the Lorentzian version of the Pythagoras theorem, the interval between points (t, x^i) and $(t + dt, x^i + dx^i)$ is then

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^i dt). \tag{3.7}$$

The Eq.(3.7) is the ADM form of the metric. It is customary in the ADM formalism to establish a new set of conventions that emphasize the role of the surface Σ. Henceforth in this section, spatial indices $i, j, ...$ will be lowered and raised with the spatial metric g_{ij} and its inverse g^{ij} and not with the full spacetime metric. Note that the components of g^{ij} are not simply the spatial components of the full three-metric $g^{\mu\nu}$; the inverse of the metric tensor (3.7)

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^j}{N^2} & \left(g^{ij} - \frac{N^i N^j}{N^2}\right) \end{pmatrix}. \tag{3.8}$$

This convention can cause confusion at first, but it simplifies later notation.

The geometry of the slice Σ_t, $(t \in [0, 1])$ comprises of two elements: the intrinsic geometry of the slice as a two-manifold and the extrinsic geometry, which describes the embedding of Σ_t in the spacetime M [35]. Just as the intrinsic geometry is determined by the behavior of vectors tangent to Σ_t under parallel transport, the extrinsic geometry is determined by the behavior of vectors normal to Σ_t. In particular, the extrinsic curvature K_{ij} of a surface Σ is defined by

$$K_{\mu\nu} = -\nabla_\mu n_\nu + n_\mu n^\rho \nabla_\rho n_\nu, \tag{3.9}$$

where ∇ is the full three-dimensional covariant derivative and n^μ is the unit normal to Σ. In the ADM decomposition (3.7), the normal to Σ_t has components $n_\mu = (N, 0, 0)$. Therefore the expression for the intrinsic curvature becomes

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} -^{(2)} \partial_i N_j -^{(2)} \partial_j N_i). \tag{3.10}$$

A general stationary, axially symmetric $(2 + 1)$-dimensional metric can be written in the form

$$ds^2 = -N(r)^2dt^2 + f(r)^2dr^2 + r^2(d\phi - N^\phi(r)dt)^2. \tag{3.11}$$

The spatial metric g_{ij} takes the form

$$g_{ij} = \begin{pmatrix} f^2 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{3.12}$$

The only non-zero components of the Ricci tensor are

$$^{(2)}R_{rr} = \frac{f'}{fr}, \tag{3.13}$$

$$^{(2)}R_{\phi\phi} = \frac{rf'}{f^3}, \tag{3.14}$$

and hence

$$\sqrt{^{(2)}g}\,^{(2)}R = 2\frac{f'}{f^2}. \tag{3.15}$$

Since the metric (3.11) is static, the extrinsic curvature becomes

$$K_{ij} = -\frac{1}{2N}(^{(2)}\nabla_j N_i +^{(2)} \nabla_i N_j), \tag{3.16}$$

and the only non-vanishing component is

$$K_{r\phi} = -\frac{r^2}{2N}(N^\phi)'. \tag{3.17}$$

The corresponding canonical momentum is

$$\pi^r_\phi = -\frac{r^3}{2Nf}(N^\phi)'. \tag{3.18}$$

The momentum constraint can be evaluated as

$$^{(2)}\nabla_j \pi^{ij} = 0 = g^{il}\partial_K \pi^k_l - \frac{1}{2}g^{il}(\partial_l g_{jk})\pi^{jk}. \tag{3.19}$$

Since g_{jk} has only diagonal elements and π^{jk} is entirely off-diagonal, the last term of the last equation vanishes. Hence

$$\pi^r_\phi = constant = A(say). \tag{3.20}$$

The Hamiltonian constraint equation becomes

$$\frac{2A^2 f}{r^3} - 2\frac{f'}{f} = 0. \tag{3.21}$$

Whose solution is

$$\frac{1}{f^2} = B^2 + \frac{A^2}{r^2}, \tag{3.22}$$

where B^2 is a constant of integration which is required to be positive to ensure that f^2 remains positive for large values of r. To proceed further one of the dynamical equations of motion coming from varying g_{ij} in the action is required. The Hamiltonian constraint is

$$\mathfrak{H} = 2fr(\pi^{\phi r})^2 - 2\frac{f'}{f^2}, \tag{3.23}$$

the momentum constraint is independent of f and all time derivatives vanish, so the action is

$$I_{\rm eff} \sim -\int dt \int dr \Big\{ 2Nrf(\pi^{\phi r})^2 - 2N\frac{f'}{f^2} \Big\} + \text{terms independent of } f. \tag{3.24}$$

The field equation obtained by varying f is thus

$$\frac{N'}{f^2} + Nr(\pi^{r\phi})^2 = 0. \tag{3.25}$$

Combining the last two Eqs.(3.23) and (3.25) with the constraint $H = 0$, we obtain

$$\frac{N'}{N} = -\frac{f'}{f}, \tag{3.26}$$

or

$$N = f^{-1}, \tag{3.27}$$

up to a constant factor that can be absorbed by a suitable rescaling of the time coordinate t. In order to determine N^ϕ we use (3.18):

$$(N^\phi)' = \frac{2Nf}{r^3}\pi^r_\phi = -2\frac{A}{r^3}, \tag{3.28}$$

so

$$N^\phi = C + \frac{A}{r^2}. \tag{3.29}$$

These solutions are considered for $C = 0$, since otherwise the metric has nonphysical asymptotic behavior.

Substituting Eqs.(3.22), (3.27) and (3.29) in (3.11), we finally obtain

$$\begin{aligned} ds^2 &= -\Big(B^2 + \frac{A^2}{r^2}\Big)dt^2 + \Big(B^2 + \frac{A^2}{r^2}\Big)^{-1}dr^2 + r^2\Big(d\phi + \frac{A}{r^2}dt\Big)^2, & (3.30)\\ &= -\Big(Bdt - \frac{A}{B}d\phi\Big)^2 + \Big(B^2 + \frac{A^2}{r^2}\Big)^{-1}dr^2 + \Big(r^2 + \frac{A^2}{B^2}\Big)d\phi^2. & (3.31)\end{aligned}$$

This metric can be put in a slightly more standard form by defining

$$\tilde{r} = \frac{1}{B^2}(A^2 + B^2 r)^{1/2}, \tag{3.32}$$

and hence as a result of above rescaling, we get

$$ds^2 = -\Big(dt - \frac{A}{B}d\phi\Big)^2 + d\tilde{r}^2 + B^2\tilde{r}^2 d\phi^2. \tag{3.33}$$

To find a physical interpretation for the constants of A and B, it is useful to examine the ADM equations of motion in the presence of sources, treating the conical singularity at $r = 0$ as a point particle. In the presence of matter, the field equations obtained from the variation of N and N' become

$$\begin{aligned} H &= -\sqrt{^{(2)}g}T^0_0, & (3.34)\\ H_i &= -\sqrt{^{(2)}g}T^0_i. & (3.35)\end{aligned}$$

and the mass of an isolated source is then

$$m = \int d^2x \sqrt{{}^{(2)}g} T^0_0 = -\int d^2x H. \tag{3.36}$$

The only term in the Hamiltonian constraint that has a chance of behaving peculiarly at $r = 0$ is the spatial curvature ${}^{(2)}R$. Under these conditions the curvature can be written as

$$\int_\Sigma d^2x \sqrt{{}^{(2)}g} \, {}^{(2)}R = \int_{\partial\Sigma} d\phi v^\perp = 2\pi v^\perp, \tag{3.37}$$

where it is evident from (3.18) that

$$v^\perp = -\frac{2}{f} + \text{const} \sim -2B + \text{const}, \tag{3.38}$$

as $r \to \infty$. One can fix the constant by noting that when $B = 1$ and $A = 0$, the metric g_{ij} is that of flat Euclidean two-space for which the integral (3.37) vanish. Hence

$$v^\perp = 2 - 2B = \frac{\beta}{\pi}, \tag{3.39}$$

and the total curvature integral is 2β.

Considering the axially symmetric spacetime with negative cosmological constant $\Lambda = -1/l^2$, for which the spacetime is asymptotically AdS [35]. The Hamiltonian constraint (3.21) is now

$$\frac{2A^2 f}{r^3} - 2\frac{f'}{f^2} - \frac{2r}{l^2} f = 0, \tag{3.40}$$

which has its solution

$$\frac{1}{f^2} = B^2 + \frac{A^2}{r^2} + \frac{r^2}{l^2}. \tag{3.41}$$

The equation of motion for N now becomes

$$\frac{N'}{f^2} + Nr(\pi^{\phi r})^2 - \frac{Nr}{l^2} = 0. \tag{3.42}$$

The solution however is still $N = f^{-1}$.

Renaming some of the constants, we obtain a metric

$$ds^2 = -N^2 dt^2 + r^2(d\phi^2 + N^\phi dt)^2 + N^{-2} dr^2, \tag{3.43}$$

with

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi = -\frac{J}{2r^2}. \tag{3.44}$$

The two constants of integration M and J appearing are the conserved charges associated with asymptotic invariance under time displacements (mass) and rotational invariance (angular momentum), respectively. These charges are given by flux integrals through a large circle at spacelike infinity [12]. This spacetime is the $(2+1)$-dimensional black hole of BTZ. It has an event horizon at $r = r_+$ and an inner horizon $r = r_-$, where

$$r_\pm^2 = \frac{l^2}{2}\Big[M \pm \Big(M^2 - \frac{J^2}{l^2}\Big)\Big], \tag{3.45}$$

are the zeros of the lapse function N.

3.3 The Charged Rotating BTZ Black Holes

The rotating BTZ solutions with the incorporation of the charge Q, i.e., the charged rotating BTZ (CR-BTZ) black hole solutions [13, 86] in 2+1 spacetime dimensions are derived from the action [1, 86]

$$I = \frac{1}{2\pi} \int dx^3 \sqrt{-g} \left(R + 2\Lambda - \frac{\pi}{2} F_{\mu\nu} F^{\mu\nu} \right) \ . \tag{3.46}$$

The corresponding line element for the CR-BTZ solution is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\phi - \frac{J}{2r^2} dt \right)^2 \ , \tag{3.47}$$

with lapse function:

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r \ . \tag{3.48}$$

Here M and J are the mass and angular momentum respectively and Q is the charge carried by the black hole. The horizons of the CR-BTZ metric correspond to the roots of the lapse function $f(r)$. Depending on these roots there are three cases for the CR-BTZ configuration [4, 24, 25]:

1. Usual CR-BTZ black hole when two distinct real roots exist;
2. Extreme CR-BTZ black hole in case of two repeated real roots;
3. Naked CR-BTZ singularity when no real roots exist.

I shall investigate the first case in this thesis. In terms of the exterior horizon radius r_+, the black hole mass and the angular momentum are given respectively by

$$M = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2} - \frac{\pi}{2} Q^2 \ln(r_+), \tag{3.49}$$

and

$$J = 2r_+ \sqrt{M - \frac{r_+^2}{l^2} - \frac{\pi}{2} Q \ln(r_+)}. \tag{3.50}$$

The corresponding angular velocity on the horizon takes the form

$$\Omega = 2r_+^2 \frac{\partial M}{\partial J}\bigg|_{r=r_+} = \frac{J}{2r_+^2} = \frac{1}{r_+} \sqrt{M - \frac{r_+^2}{l^2} - \frac{\pi}{2} Q \ln(r_+)}. \tag{3.51}$$

The Hawking temperature T at the black hole horizon is

$$T = \frac{1}{4\pi} \frac{df}{dr} = \frac{1}{4\pi} \left(\frac{2r_+}{l^2} - \frac{J^2}{2r_+^3} - \frac{\pi}{2} \frac{Q^2}{r_+} \right), \tag{3.52}$$

and the electric potential is given by

$$\phi = \frac{\partial M}{\partial Q}\bigg|_{r=r_+} = -\pi Q \ln r_+. \tag{3.53}$$

Furthermore, using the fundamental postulate of black hole thermodynamics, the entropy of the CR-BTZ black hole is defined as

$$S = 4\pi r_+ . \tag{3.54}$$

In terms of this entropy, the corresponding thermodynamic fundamental equation and the temperature for the CR-BTZ black hole are given respectively by

$$M = \left(\frac{S}{4\pi l}\right)^2 + \left(\frac{2\pi J}{S}\right)^2 - \frac{\pi Q^2}{2}\ln\frac{S}{4\pi}, \tag{3.55}$$

and

$$T = \left(\frac{\partial M}{\partial S}\right)_{J,Q} = \frac{S}{8\pi^2 l^2} - \frac{8\pi^2 J^2}{S^3} - \frac{\pi Q^2}{2S}. \tag{3.56}$$

The thermodynamic quantities T, S, J, Q and M obey the first law of thermodynamics [3]

$$dM = TdS + \Omega dJ + \phi dQ. \tag{3.57}$$

An important quantity for the analysis of the thermodynamic properties is the heat capacity of the CR-BTZ black hole [4], $C_{J,Q} = (\partial M/\partial T)_{J,Q}$, which is given by

$$C_{J,Q} = S\frac{S^4 - 4\pi^3 l^2 Q^2 S^2 - 64\pi^4 l^2 J^2}{S^4 + 4\pi^3 l^2 Q^2 S^2 + 192\pi^4 l^2 J^2}, \tag{3.58}$$

or using the horizon radius r_+ as coordinate, by

$$C_{J,Q} = 4\pi r_+ \frac{4r_+^4 - \pi l^2 Q^2 r_+^2 - l^2 J^2}{4r_+^4 + \pi l^2 Q^2 r_+^2 + 3l^2 J^2}. \tag{3.59}$$

Furthermore, the capacitances

$$\widetilde{C}_{S,Q} = \left(\frac{\partial J}{\partial \Omega}\right)_{S,Q}, \tag{3.60}$$

and

$$\widetilde{C}_{S,J} = \left(\frac{\partial Q}{\partial \phi}\right)_{S,J}, \tag{3.61}$$

can also contain important information about the thermodynamic behavior of black holes. From the fundamental Eq.(3.55), we obtain

$$\widetilde{C}_{S,Q} = \frac{S^2}{4\pi}, \quad \widetilde{C}_{S,J} = -\frac{1}{4\pi \ln(S/4\pi)} = -\frac{1}{4\pi\ln(r_+)}. \tag{3.62}$$

From expressions (3.52) and (3.54), I conclude that the condition

$$S^4 - 4\pi^3 l^2 Q^2 S^2 - 64\pi^4 l^2 J^2 > 0 \tag{3.63}$$

must be satisfied in order for the temperature to be positive definite, a requirement which follows from the standard laws of black hole thermodynamics. From the above condition and the expression (3.58), it follows that the heat capacity is always positive definite.

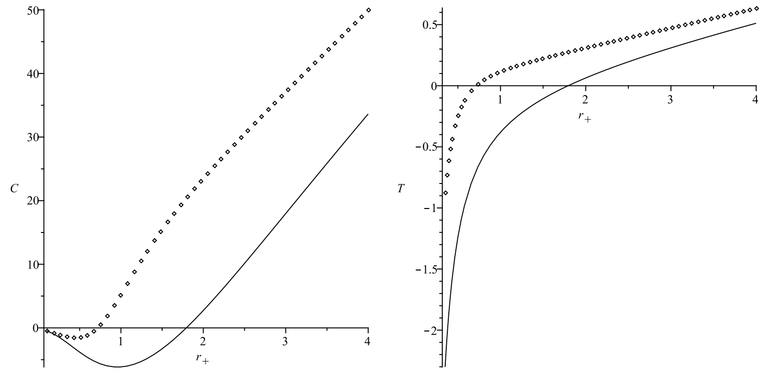

Figure 3.1: Behavior of the heat capacity and temperature as functions of the event horizon radius r_+ of a CR-BTZ black hole with $Q = 2$, $J = 1$ and $l = 1$. Temperature and heat capacity vanish at $r_+ \approx 1.79$. The unstable region ($C_{J,Q} < 0$) corresponds to an unphysical region with negative temperature. To illustrate the contribution of the charge I include the plots (dotted curves) for the case of the non-static BTZ black hole with $Q = 0$ [26,113].

This is an important observation which implies that a CR-BTZ black hole with a positive definite temperature must be a thermodynamically stable configuration. In fact, a change of sign of the heat capacity is usually associated with a drastic change of the stability properties of a thermodynamic system; a negative heat capacity represents a region of instability whereas the stable domain is characterized by a positive heat capacity.

It is worth mentioning that the heat capacity is a regular function for all real positive values of the exterior horizon radius. In fact, the denominator of the expression (3.59) is always positive and, consequently, $C_{J,Q}$ is a regular function, except in the pathological case where $S = J = Q = 0$. On the other hand, in black hole thermodynamics, divergences of the heat capacity are associated with second-order phase transitions. This implies that a CR-BTZ black hole cannot undertake a phase transition associated with a divergence of the heat capacity. The above observations demonstrate that the CR-BTZ black hole is a completely stable thermodynamic system with no phase transition structure.

However, the capacitance $\widetilde{C}_{S,J}$ becomes singular for a black hole with horizon radius $r_+ = 1$, at which the electric potential ϕ vanishes. This indicates the presence of a second-order phase transition. In fact, since the heat capacity can be written as

$$C_{J,Q} = \left(\frac{T}{\partial^2 M / \partial S^2} \right)_{J,Q} , \tag{3.64}$$

a second-order phase transition takes place if the condition

$$\frac{\partial^2 M}{\partial S^2} = 0, \tag{3.65}$$

is satisfied. Therefore, for the capacitance

$$\widetilde{C}_{S,J} = \left(\frac{T}{\partial^2 M / \partial Q^2} \right)_{S,J} , \tag{3.66}$$

by analogy the singularity situated at

$$\frac{\partial^2 M}{\partial^2 Q} = 0, \tag{3.67}$$

can be considered as a second-order phase transition.

The above observations demonstrate that the CR-BTZ black hole is a completely stable thermodynamic system with no phase transition structure following from the heat capacity $C_{J,Q}$. The only second-order phase transition can occur for the horizon radius $r_+ = 1$ at which the capacitance $\widetilde{C}_{S,J}$ diverges. In this chapter, I will use this fact in order to test different geometric descriptions of the thermodynamics of the CR-BTZ black hole. For selected values of J, Q and l, Fig.(3.1) shows the behavior of the heat capacity and temperature for a charged black hole (solid line) and for a neutral black hole (dotted line). The comparison of both curves shows that the charge essentially increases the value of the horizon radius at which the heat capacity and temperature vanish. As the value of the horizon radius increases the contribution of the charge decreases. Finally, for very large values of the horizon radius the heat capacities and the temperatures coincide, indicating that the contribution of the charge is negligible.

One would expect that the limiting case $T \to 0$ or, equivalently, $C \to 0$ corresponds to an extreme black hole with only one horizon of radius, say, r_*. To analyze this question it is necessary to find the domain of parameters for which the equation $f(r) = 0$ allows only one positive real root and to calculate the value of T for this domain. However, the equation $f(r) = 0$ cannot be solved analytically because of the presence of the logarithmic term $\ln r$. An alternative procedure consists in solving the equation $T = 0$ for r^2 to obtain

$$r^2 := r_*^2 = \frac{\pi l^2 Q^2}{8}(1+\eta) \ , \quad \eta = \sqrt{1 + \frac{16 J^2}{\pi^2 l^2 Q^4}} \ , \tag{3.68}$$

and introducing this solution into the equation $f(r) = 0$ to obtain the value of the mass at this radius, i. e.,

$$M = \frac{\pi Q^2}{4}\left[\eta - \ln(1+\eta) - \ln\frac{\pi l^2 Q^2}{8}\right] \ , \tag{3.69}$$

where J^2 is replaced by using the definition of the auxiliary parameter η. Now the question is whether the last expression represents a physical mass, i.e. whether it is positive. A numerical analysis shows that for any value $\eta > 1$, a condition that follows from the definition of η, there always exists a combination of values for Q and l such that M is positive. Fig.(3.2) shows an example of the behavior of the mass for a fixed value of the parameter l. Therefore, the limit $T \to 0$ indeed corresponds to an extreme black hole.

Notice that the above numerical analysis is necessary only if the solution must correspond to a black hole with positive mass. If this condition is not imposed, a simpler analysis is possible. The only solution of the equation $f'(r) = 0$ is $r = r_*$ and $f(r)$ is a smooth function for $r \in (0, \infty)$ with $\lim_{r\to 0,+\infty} f(r) = +\infty$, the equation $f(r) = 0$ has

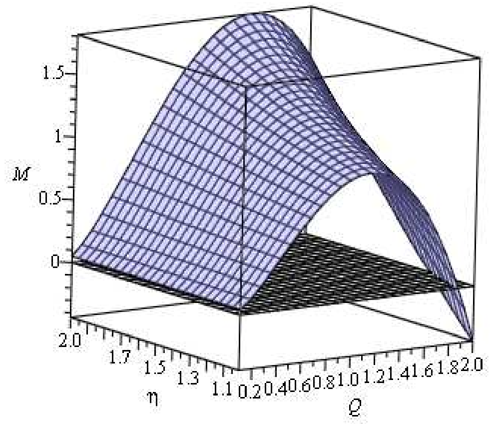

Figure 3.2: The mass M of an extreme CR-BTZ black hole with horizon radius $r_* = \pi l^2 Q^2(1+\eta)/8$ as a function of the charge Q and the angular parameter $\eta > 1$. Here, I select $l = 1$ as a representative value. The plane $M = 0$ is plotted to visualize the region where $M > 0$.

only one solution for an extreme black hole $r = r_{ebh}$ if and only if $f'(r_{ebh}) = 0$. On the other hand, since

$$T = 0 \iff f'(r_+) = 0 \tag{3.70}$$

this immediately proves that any black hole with zero temperature is extremal.

3.4 Weinhold and Ruppeiner Geometries

Now I construct the thermodynamic geometry of the CR-BTZ black hole by using the Weinhold metric (1.67). In this case the extensive variables are $N^r = \{J, Q\}$ so that the general Weinhold metric becomes

$$\begin{aligned} ds_W^2 = &\left(\frac{\partial^2 M}{\partial S^2}\right) dS^2 + \left(\frac{\partial^2 M}{\partial J^2}\right) dJ^2 + \left(\frac{\partial^2 M}{\partial Q^2}\right) dQ^2 + 2\left(\frac{\partial^2 M}{\partial S \partial J}\right) dSdJ \\ &+ 2\left(\frac{\partial^2 M}{\partial J \partial Q}\right) dJdQ + 2\left(\frac{\partial^2 M}{\partial Q \partial S}\right) dQdS, \end{aligned} \tag{3.71}$$

and in the special case of the CR-BTZ black hole

$$\begin{aligned} ds_W^2 = &\left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 J^2}{S^4} + \frac{\pi Q^2}{2S^2}\right) dS^2 + \frac{8\pi^2}{S^2} dJ^2 - \pi \ln\left(\frac{S}{4\pi}\right) dQ^2 \\ &- \frac{32\pi^2 J}{S^3} dSdJ - 2\frac{\pi Q}{S} dSdQ \ . \end{aligned} \tag{3.72}$$

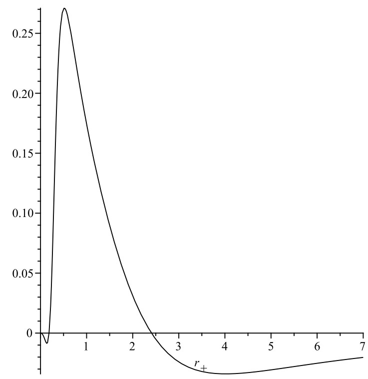

Figure 3.3: Thermodynamic curvature for the Weinhold (R_W) geometry as a function of the event horizon radius, r_+, of the CR-BTZ black hole. Here, the free parameters are chosen as $Q = 2$, $J = 1$ and $l = 1$. The curvature is completely regular in the entire domain of r_+.

The corresponding scalar curvature is given by

$$R_W = \frac{l^2 r_+^2 \Big[-4r_+^4 (1 + 2\ln r_+ + 4\ln {r_+}^2) + \pi l^2 Q^2 (9 + 2\ln r_+) {r_+}^2 + J^2 l^2 (1 + 2\ln r_+) \Big]}{\Big[-4\, {r_+}^4 \ln r_+ - \pi\, Q^2 l^2 (\ln r_+ + 2) {r_+}^2 + J^2 l^2 \ln r_+ \Big]^2} . \tag{3.73}$$

The general behavior of the scalar curvature for the Weinhold geometry is illustrated in Fig.(3.3). The thermodynamic curvature is regular for all positive values of the horizon radius. At the value $r_+ \approx 1.79$ (with $l = 1$, $J = 1$, $Q = 2$), at which the temperature vanishes, the scalar curvature is $R_W \approx 0.0527$. Moreover, it is positive and regular in the interval $0.185 < r_+ < 1.79$, a region where the temperature is negative and in the interval $1.79 \leq r_+ < 2.4$, a region where the temperature is positive. This means that the Weinhold thermodynamic curvature cannot differentiate between a CR-BTZ black hole with positive temperature and a similar configuration with negative temperature.

I now investigate the limiting case of a vanishing charge. The additional extensive variable in this case is $N^r = \{J\}$ so that the Weinhold metric reduces to

$$ds_W^2 = \left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 J^2}{S^4} \right) dS^2 - \frac{32\pi^2 J}{S^3} dS dJ + \frac{8\pi^2}{S^2} dJ^2 \ , \tag{3.74}$$

and the corresponding scalar curvature becomes

$$R_W = 16\, \frac{\pi^2 l^2 S^6}{(S^4 - 64\, \pi^4 J^2 l^2)^2} \ . \tag{3.75}$$

A true curvature singularity exists at the value $S^4 = 64\, \pi^4 J^2 l^2$ that, according to Eq.(3.63) with $Q = 0$, corresponds to the the limit of vanishing temperature or, equivalently, to the extreme black hole limit. This result shows that the Weinhold thermodynamic curvature in this case correctly describes the transition from a region with positive and well-defined temperature to a region with an unphysical negative temperature. This is in contrast to what I obtained in the case of a charged black hole in which the Weinhold thermodynamic curvature is not able to recognize the transition to an extreme black hole with zero temperature.

Now I consider the Ruppeiner geometry. A direct computation of the Ruppeiner metric (1.69) cannot be performed because it is not possible to rewrite explicitly the fundamental Eq.(3.55) in the the form $S = S(M, J, Q)$. Nevertheless, indirect methods of computation are available. Indeed, if one assume the invariance of the line element under a change of thermodynamic potential, the relationship (1.70) can be used to derive the Ruppeiner metric from the Weinhold metric. Then,

$$ds_R^2 = \frac{dS^2}{S} + \frac{\pi}{T}\left[\frac{8\pi}{S^2}\left(\frac{2J}{S}dS - dJ\right)^2 + \left(\frac{Q}{S}dS - dQ\right)^2 - \left(1 + \ln\frac{S}{4\pi}\right)dQ^2\right] . \tag{3.76}$$

A second indirect and equivalent method to compute the components of the Ruppeiner metric consists in applying the chain rule for partial derivatives. For instance, from the definition of the temperature given in Eq.(3.56) one obtains that

$$g^R_{MM} = -\left(\frac{\partial^2 S}{\partial M^2}\right)_{J,Q} = -\left(\frac{\partial(1/T)}{\partial M}\right)_{J,Q} = \frac{1}{T^3}\left(\frac{\partial T}{\partial S}\right)_{J,Q} . \tag{3.77}$$

This means that

$$g^R_{MM} = \frac{1}{T^3} g^W_{SS}, \tag{3.78}$$

or, using the appropriate coordinates in the corresponding line elements,

$$g^R_{MM} = \frac{1}{T} g^W_{MM}, \tag{3.79}$$

as stated in Eq.(1.70). The remaining metric components can be calculated in a similar manner. For instance,

$$g^R_{JJ} = -\left(\frac{\partial^2 S}{\partial J^2}\right)_{M,Q} = -\frac{\partial}{\partial J}\left(\frac{1}{\partial J/\partial S}\right)_{M,Q} = -\left(\frac{\partial J}{\partial S}\right)^{-3}\left(\frac{\partial(\partial J/\partial S)}{\partial S}\right)_{M,Q} , \tag{3.80}$$

and

$$g^R_{QQ} = -\left(\frac{\partial^2 S}{\partial Q^2}\right)_{M,J} = -\left(\frac{\partial Q}{\partial S}\right)^{-3}\left(\frac{\partial(\partial Q/\partial S)}{\partial S}\right)_{M,J} , \tag{3.81}$$

etc. None of these computations requires the explicit use of the equation $S = S(M, J, Q)$.

The thermodynamic curvature scalar R_R of the metric (3.76) turns out to be nonzero, i.e., the space of its thermodynamic equilibrium states is non-flat. The explicit form of R_R cannot be written in a compact form. Therefore, I perform a numerical analysis of its behavior and the result is illustrated in Fig.(3.4).

The singularity located at $r_+ \approx 1.79$ represents the limit for which the heat capacity vanishes and the temperature becomes negative. This shows that the Ruppeiner thermodynamic curvature describes correctly the behavior of the CR-BTZ black hole.

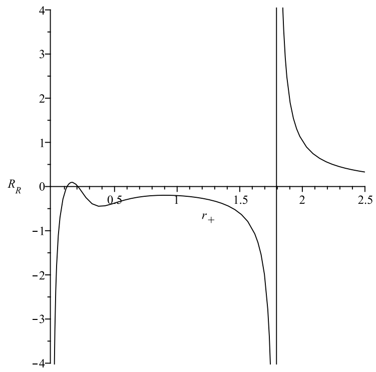

Figure 3.4: Thermodynamic curvature, R_R, of the Ruppeiner geometry as a function of the event horizon, r_+, for a CR-BTZ black hole with $J = l = 1$, $Q = 2$. The only singularity is located at $r_+ \approx 1.79$.

In the limiting case of a vanishing charge, it is possible to rewrite the fundamental Eq.(3.49) as $S = S(M, J)$ in the following manner

$$S = 4\pi r_+ = \pi\sqrt{8Ml^2\left(1+\sqrt{1-\frac{J^2}{M^2l^2}}\right)}, \tag{3.82}$$

so that the Ruppeiner metric can be computed by using the definition (1.69). Then

$$ds_R^2 = -\frac{\pi l^2}{(r_+^2 - r_-^2)^3}\left[r_+(r_+^2 + 3r_-^2)(l^2dM^2 + dJ^2) - 2lr_-(3r_+^2 + r_-^2)dMdJ\right] , \tag{3.83}$$

where

$$r_\pm^2 = \frac{l^2M}{2}\left(1 \pm \sqrt{1-\frac{J^2}{l^2M^2}}\right) . \tag{3.84}$$

A straightforward calculation shows that the curvature of this metric vanishes identically, indicating the absence of thermodynamic interaction, i.e., the thermodynamic variables M and J do not generate thermodynamic interaction. This is a peculiar result because, as we have seen above, the Ruppeiner geometry correctly describes the thermodynamic behavior of the CR-BTZ black hole. This implies that only the charge Q acts as a source of thermodynamic interaction in the Ruppeiner geometry. It seems that there is no specific reason for the existence of this difference between thermodynamic variables of this particular black hole configuration.

3.5 Geometrothermodynamics of the CR–BTZ Black Holes

For the geometric description of the thermodynamics of the CR-BTZ black hole in GTD, I first introduce the 7-dimensional phase space $\mathcal{T}$ with coordinates M, S, J, Q, T, Ω and ϕ, a contact 1-form

$$\Theta = dM - TdS - \Omega dJ - \phi dQ, \tag{3.85}$$

which satisfies the condition $\Theta \wedge (d\Theta)^3 \neq 0$ and a Legendre invariant metric

$$G = (dM - TdS - \Omega dJ - \phi dQ)^2 + TS(-dTdS + d\Omega dJ + d\phi dQ). \tag{3.86}$$

This particular metric is a special case of a metric used in [113] to describe the region of positive temperature of the BTZ black hole.

Let $\mathcal{E}$ be a $(2+1)$-dimensional subspace of $\mathcal{T}$ with coordinates $E^a = (S, Q, J)$, $a = 1, 2, 3$, defined by means of a smooth mapping $\varphi : \mathcal{E} \to \mathcal{T}$. The subspace $\mathcal{E}$ is called the space of equilibrium states if $\varphi^*(\Theta) = 0$, where φ^* is the pullback of φ. Furthermore, a metric structure g is naturally induced on $\mathcal{E}$ by applying the pullback on the metric G of $\mathcal{T}$, i.e., $g = \varphi^*(G)$. It is clear that the condition $\varphi^*(\Theta) = 0$ leads immediately to the first law of thermodynamics of black holes as given in Eq.(3.57). It also implies the existence of the fundamental equation $M = M(S, Q, J)$ and the conditions of thermodynamic equilibrium Eqs. (3.51)-(3.53). Moreover, the induced metric

$$g = \varphi^*(G) = S\frac{\partial M}{\partial S}\left(-\frac{\partial^2 M}{\partial S^2}dS^2 + \frac{\partial^2 M}{\partial J^2}dJ^2 + \frac{\partial^2 M}{\partial Q^2}dQ^2\right), \tag{3.87}$$

determines all the geometric properties of the equilibrium space $\mathcal{E}$. In the above expression I used the Euler identity to simplify the form of the conformal factor. In order to obtain the explicit form of the metric it is only necessary to specify the thermodynamic potential M as a function of S, J and Q as given in Eq.(3.55). Another advantage of the use of GTD is that it allows us to easily implement different thermodynamic representations of the fundamental equation, given as $M = M(S, Q, J), S = S(M, Q, J), Q = Q(S, M, J)$ or $J = J(S, M, Q)$ and redefine the coordinates in $\mathcal{T}$ and the smooth mapping ϕ in such a way that the condition $\varphi^*(\Theta) = 0$ generates on $\mathcal{E}$ the corresponding fundamental equation in the $S-$, $Q-$, or the $J-$representation, respectively. The results obtained with different representations of the same fundamental equation are completely equivalent.

For the CR-BTZ black hole, using the fundamental equation $M = M(S, J, Q)$ given in Eq.(3.55), the thermodynamic metric can be written as

$$g = \frac{S^4 - 64\,\pi^4 J^2 l^2 - 4\,\pi^3 l^2 Q^2 S^2}{8\pi^2 l^2 S^2}\left[-\left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 l^2}{S^4} + \frac{\pi Q^2}{2S^2}\right)dS^2 + \frac{8\pi^2}{S^2}dJ^2 - \pi \ln\frac{S}{4\pi}dQ^2\right]. \tag{3.88}$$

The corresponding thermodynamic curvature turns out to be nonzero and is given by

$$R_{GTD} = \frac{2l^4 r_+^4}{D_1 D_2}\left[\frac{1}{\ln^2 r_+} + \frac{1}{D_1 D_2^2}\left(\frac{N_1}{\ln r_+} + N_0\right)\right] \tag{3.89}$$

where

$$D_1 = 4r_+^4 + \pi l^2 Q^2 r_+^2 + 3l^2 J^2\ , \quad D_2 = 4r_+^4 - l^2 J^2 - \pi l^2 Q^2 r_+^2\ ,$$

$$\begin{aligned} N_0 = 4[&6\,J^6 l^6 + 23\,\pi\,J^4 l^6 Q^2 r_+^2 + l^4 J^2\left(15\,\pi^2 l^2 Q^4 + 8\,J^2\right) r_+^4 \\ &+ 4\pi l^4 Q^2\left(14\,J^2 + \pi^2 Q^4 l^2\right) r_+^6 + 4l^2\left(\pi^2 Q^4 l^2 - 40\,J^2\right) r_+^8 \\ &- 16\,\pi\,Q^2 l^2 r_+^{10} + 128\,r_+^{12}]\ , \end{aligned}$$

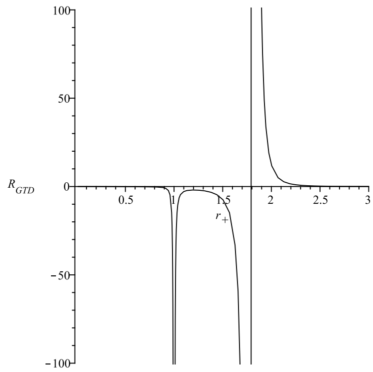

Figure 3.5: Thermodynamic curvature of the CR-BTZ black hole (R_{GTD}) as a function of event horizon r_+. A typical behavior is depicted for the specific values $Q = 2$, $J = l$ and $l = 1$.

$$N_1 = 2[-15\, J^6 l^6 - 6\,\pi\, J^4 l^6 Q^2 r_+^2 - 3l^4 J^2 \left(\pi^2 l^2 Q^4 - 20 J^2\right) r_+^4 + 128\pi J^2 l^4 Q^2 r_+^6 - 4l^2 \left(4J^2 - 5\pi^2 l^2 Q^4\right) r_+^8 + 96\,\pi\, Q^2 l^2 r_+^{10} + 64\, r_+^{12}]\ .$$

One can see from the expression for the scalar curvature that the curvature singularities are situated at those values of the parameters where $D_1 = 0$, $D_2 = 0$ or $\ln r_+ = 0$. For real values of the parameters the condition $D_1 = 4r_+^4 + \pi l^2 Q^2 r_+^2 + 3l^2 J^2 = 0$ cannot be satisfied. In fact, this term appears in the denominator of the heat capacity (3.59) and determines the absence of phase transitions of the CR-BTZ black hole. The singularities determined by the roots of the equation $D_2 = 4r_+^4 - l^2 J^2 - \pi l^2 Q^2 r_+^2 = 0$ coincide with the points where $T = 0$ or, equivalently, where the heat capacity (3.59) vanishes. This implies that the no physical region of negative temperatures is isolated from the allowed region with positive temperatures by a true curvature singularity. The third singularity located at $\ln r_+ = 0$ can be interpreted as a critical point that is not determined by the heat capacity (3.59). In fact, at $r_+ = 1$ the second derivative of the mass $\partial^2 M/\partial Q^2 = 0$, indicating either the transition into a region of instability or a second order phase transition. The singular behavior of the GTD scalar curvature is illustrated in Fig.(3.5).

Let us now consider the limiting case of vanishing charge. The geometrothermodynamic metric reduces to

$$g = \frac{S^4 - 64\,\pi^4 J^2 l^2}{8\pi^2 l^2 S^2}\left[-\left(\frac{1}{8\pi^2 l^2} + \frac{24\pi^2 l^2}{S^4}\right) dS^2 + \frac{8\pi^2}{S^2} dJ^2\right], \tag{3.90}$$

and the corresponding scalar curvature can be written as

$$R_{GTD} = \frac{256\, l^4 \pi^4 S^8}{\left(S^4 + 192\,\pi^4 J^2 l^2\right)^2 \left(S^4 - 64\,\pi^4 J^2 l^2\right)}. \tag{3.91}$$

The behavior of this scalar and the temperature is depicted in Fig.(3.6). It follows that in general a curvature singularity appears when the condition $S^4 - 64\,\pi^4 J^2 l^2 = 0$ is satisfied which, according to Eq.(3.56) with $Q = 0$, corresponds to a zero temperature. I conclude

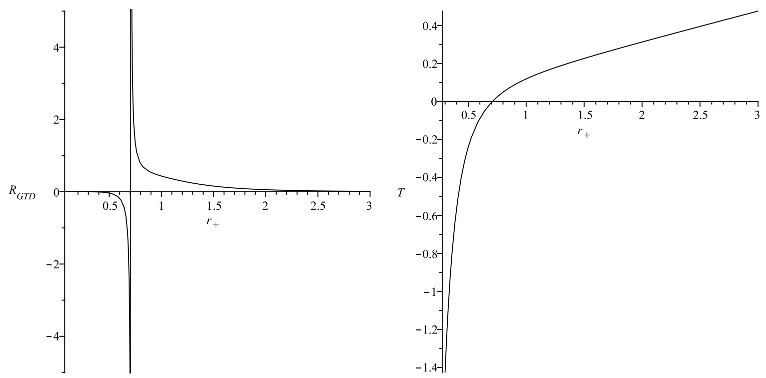

Figure 3.6: Behavior of the scalar curvature in GTD and temperature as functions of the event horizon radius $r_+ = S/4\pi$ of a neutral rotating BTZ black hole with $J = 1$ and $l = 1$. The curvature singularity coincides with the point of zero temperature.

that the invariant metric proposed in GTD correctly describes the limiting case of a neutral BTZ black hole.

3.6 Conclusion

In this chapter, I analyzed the thermodynamics and the thermodynamic geometry of the charged rotating BTZ black holes. By considering the behavior of the heat capacity and the Hawking temperature, I found that this black hole configuration is free of phase transitions and stable. In fact, the instability region is characterized by a non-physical negative temperature. Moreover, a numerical analysis is performed which shows that in the limiting case of zero temperature the black hole becomes extreme.

I analyzed the thermodynamic geometry based on the Weinhold metric and found that the corresponding thermodynamic curvature is free of singularities in the entire equilibrium manifold. This result is not in accordance with the analysis of the behavior of the heat capacity and the Hawking temperature that indicates the presence of an unphysical region with negative temperature for $r_+ \leq 1.79$ (the additional parameters are chosen as $l = 1,\ J = 1$ and $Q = 2$). The Weinhold geometry does not describe correctly the thermodynamic geometry in this specific case. However, in the limiting case of a vanishing electric charge there exists a true curvature singularity that is located at the point where the temperature vanishes. It is not clear why the presence of an electric charge cannot be handled correctly in the context of the Weinhold thermodynamic metric.

Although it is not possible to calculate explicitly the Ruppeiner metric, it can be derived from the Weinhold metric by using a conformal transformation with the inverse of the temperature as the conformal factor. A numerical analysis of the Ruppeiner thermodynamic curvature shows that it is smooth and well-behaved in the region $r_+ > 1.79$, with a true curvature singularity situated at $r_+ \approx 1.79$. This is exactly the value of the horizon radius at which the Hawking temperature vanishes. This result indicates that the Ruppeiner geometry correctly describes the thermodynamics of the CR-BTZ black hole. However, in the limiting case of vanishing electric charge the Ruppeiner metric turns out to be flat. Since a vanishing thermodynamic curvature is usually interpreted as indicating the absence of thermodynamic interaction, it is not clear why Ruppeiner geometry correctly describes the thermodynamics of the CR-BTZ black hole but fails in the limiting neutral case.

Finally, the properties of a Legendre invariant metric proposed in the context of GTD are analyzed. In this case, the curvature can be calculated explicitly and it turns out that it possesses a true curvature singularity at those points where the Hawking temperature vanishes, and at the horizon radius $r_+ = 1$ where a secondorder phase transition is located. In the entire region where the CR-BTZ black hole corresponds to a stable thermodynamic system with no phase transition structure, the thermodynamic curvature of GTD is described by a smooth function of all the thermodynamic variables. In the limiting case of vanishing electric charge, the metric proposed in GTD is also able to correctly describe the thermodynamic properties of the black hole configuration in the sense that it is finite and smooth in the region where the black hole is stable, but possesses a true curvature singularity at the point where the temperature vanishes. Since the Weinhold and Ruppeiner metrics are not invariant with respect to LT, I conclude that the Legendre invariance imposed in the context of GTD is an important property to describe geometrically the thermodynamics of black holes without intrinsic contradictions.

Chapter 4

Geometrothermodynamics of Five–Dimensional Black Holes in Einstein-Gauss-Bonnet Theory

High-dimensional GR is perhaps one of the oldest and most elegant ways for unifying all interactions in physics. Higher-derivative curvature terms naturally occur, such as in the quantum field theory in curved space [18] and in the effective low-energy action of string theories. In GR because of the nonlinearity of the Einstein field equations (1.1), it is quite difficult to find exact analytical solutions with these higher derivative terms. In most cases, one has to adopt some approximation methods or find solutions numerically.

In five dimensions, the most general theory leading to second order field equations for the metric is the EGB theory, which contains quadratic powers of the curvature. The most general action of the EGB theory is obtained by adding the Gauss-Bonnet (GB) invariant and the matter Lagrangian L_{matter} to the Einstein-Hilbert action

$$I = \kappa \int d^5x\sqrt{-g}\left(R + \alpha L_{GB} + L_{matter}\right), \tag{4.1}$$

where

$$L_{GB} = (R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}), \tag{4.2}$$

is the GB Lagrangian, κ is related to the Newton constant and α is the GB coupling constant having dimensions of $(length)^2$. Here R, $R_{\mu\nu}$ and $R_{\alpha\beta\gamma\delta}$ are respectively, the Ricci scalar, the Ricci tensor and the Riemann tensor. The gravity theory with a GB term was originally proposed by Lanczos [82], independently rediscovered by Lovelock [84], (See [146] and [147] for more discussions on EGB gravity).

GB extensions of GR have been motivated from a string theoretical point of view as a version of higher-dimensional gravity, since this sort of modification also appears in low energy effective actions of string theory. The pioneering work in this regard belongs to Boulware and Deser [22]. They obtained the most general static black hole solutions in EGB theory. The GB term yields non trivial dynamics in dimensions greater than or equal to five. It has some remarkable features. For instance, in higher dimensions, it is the most general quadratic correction which preserves the property that the equations of motion involve only second order derivatives of the metric [84]. Despite being quadratic in the Riemann tensor (and Ricci tensor), terms containing more than two partial derivatives of the metric cancel out, making the Euler-Lagrange equations second order quasilinear partial differential equations in the metric. Consequently, there are no additional dynamical degrees of freedom, as in $f(R)$ gravity. However, in $3+1$ dimensions, the GB term is topological in nature and it does not enter the dynamics [129]. This is the most general combination of tensors that satisfies the following conditions [84]:

- it is symmetric;
- it depends only on the metric and its first two derivatives;
- it has vanishing divergence;
- it is linear in the second derivatives of the metric.

If we go to higher dimensions ($D \geq 5$), it turns out that these conditions are satisfied by a linear combination of the metric, the Einstein tensor and the Lovelock tensor [82, 84]. The Lovelock tensor arises from the variation of the GB term in the above action (4.1). The EGB gravity is the natural generalisation of GR to five or six dimensions. The GB term is important from both physical and geometrical points of view; it naturally arises as the next leading order of the α-expansion of the heterotic superstring theory (α^{-1} is the string tension) [146] and plays a fundamental role in Chern–Simons gravitational theories [38].

This chapter deals with the thermodynamic properties of five-dimensional static and spherically symmetric black holes in Einstein-Maxwell-Gauss-Bonnet (EMGB) theory with and without incorporating the effects of cosmological constant and in Einstein-Yang-Mills-Gauss-Bonnet (EYMGB) theory. It exhibits the various stable and unstable phases of the black holes in these modified gravity theories. To formulate the thermodynamics of these black holes the Bekenstein-Hawking entropy relation and, alternatively, a modified entropy formula which follows from the first law of thermodynamics of black holes have been used.

4.1 Spherically Symmetric Black Holes in EMGB Gravity

In the case of the EGB gravity minimally coupled to the electromagnetic field, the matter component of the action (4.1) is given by

$$L_{matter} = F_{\alpha\beta}F^{\alpha\beta} , \tag{4.3}$$

where

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \tag{4.4}$$

is the electromagnetic tensor field, A_μ is the vector potential. The gravitational and electromagnetic field equations obtained by varying the action (4.1) with respect to $g_{\mu\nu}$ and $F_{\mu\nu}$ are

$$G_{\mu\nu} - \alpha H_{\mu\nu} = T_{\mu\nu}, \quad \nabla_{\mu\nu} F^{\mu}_{\nu} = 0, \tag{4.5}$$

where $T_{\mu\nu}$ is the electromagnetic stress energy momentum tensor given by

$$T_{\mu\nu} = 2F^{\lambda}_{\mu} F_{\lambda\sigma} - \frac{1}{2} F_{\lambda\nu} F^{\lambda\sigma} g_{\mu\nu}, \tag{4.6}$$

and

$$H_{\mu\nu} = 2\left(RR_{\mu\nu} - 2R_{\mu\lambda}R^{\lambda}_{\nu} - 2R^{\sigma\delta}R_{\mu\nu\sigma\delta} + R^{\alpha\beta\sigma}_{\mu}R_{\nu\alpha\beta\sigma}\right) - \frac{1}{2} g_{\mu\nu} L_{GB}. \tag{4.7}$$

(Note that Eq.(4.5) does not contain any derivative of the curvature term, so derivatives of the metric tensor higher than two do not appear. Thus, the GB gravity is a special case of higher derivative gravity.)

Spherically symmetric black holes of the EMGB theory have been investigated very intensively as possible scenarios for the realization of the low energy limit of certain string theories. A particular solution which contains as a special case a black hole spacetime obtained by using the following five-dimensinal static spherically symmetric line element [37] (see also [31, 99])

$$ds^2 = -G(r)dt^2 + \frac{dr^2}{G(r)} + r^2 d\Omega_3^2 \ , \tag{4.8}$$

where $d\Omega_3^2$ is the metric of a three-dimensional hypersurface with constant curvature $6k$ which has the explicit form

$$d\Omega_3^2 = \begin{cases} d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2 d\theta_3^2) \ , & (k=1) \\ d\theta_1^2 + \sinh^2\theta_1(d\theta_2^2 + \sin^2\theta_2 d\theta_3^2) \ , & (k=-1) \\ \alpha^{-1}dx^2 + d\phi_1^2 + d\phi_2^2 \ , & (k=0) \ . \end{cases} \tag{4.9}$$

Here, the coordinate x has the dimension of length while the angular coordinates are dimensionless as usual and

$$(\theta_1, \theta_2) \in [0, \pi], \quad (\theta_3, \phi_1, \phi_2) \in [0, 2\pi]. \tag{4.10}$$

If we assume that there exists a charge Q at $r = 0$, such that it is a point charge for $k = 1$ and is the charge density of a line charge for $k = 0$. Then the vector potential A_μ can be

$$A_\mu = \phi(r)\delta^0_\mu, \tag{4.11}$$

where

$$\phi(r) = -\frac{Q}{2r^2}. \tag{4.12}$$

The metric function $G(r)$ can be obtained by solving the field Eqs.(4.5) as

$$G(r) = k + \frac{r^2}{4\alpha}\left[1 \pm \sqrt{1 + \frac{8\alpha(M + 2\alpha|k|)}{r^4} - \frac{8\alpha Q^2}{3r^6}}\right], \tag{4.13}$$

where the geometric mass $M + 2\alpha|k|$ is different from that of Einstein gravity for $k = \pm 1$. Thus, the mass of a five-dimensional spacetime in GB gravity for $k = \pm 1$, differs from that of GR by a term which is proportional to $6\alpha|k|$. Note that the GB term decreases the mass of the spacetime for $\alpha < 0$ and increases the mass for $\alpha > 0$. It is worthwhile to mention that this occurs only for the five-dimensional spacetime. For higher-dimensional solutions in EMGB gravity see, for instance [47]. This solution is well defined if the expression within the square root is positive definite. For the solution (4.13) of the EMGB theory to describe a black hole it is necessary that the condition $G(r) = 0$ be satisfied. For the special case $k = +1$, the roots of this equation are

$$r_\pm = \frac{1}{2}\left[\sqrt{M + \frac{2Q}{\sqrt{3}}} \pm \sqrt{M - \frac{2Q}{\sqrt{3}}}\right] , \tag{4.14}$$

independent of the value of the coupling constant α. It turns out that in some cases these radii determine naked singularities [47]. However, the specific case with $\alpha < 0$ and $k = +1$ corresponds to a solution which is asymptotically de Sitter and represents a black hole with an event horizon situated at

$$r_+ = \frac{1}{2}\left[\sqrt{M + \frac{2Q}{\sqrt{3}}} + \sqrt{M - \frac{2Q}{\sqrt{3}}}\right] , \tag{4.15}$$

provided $(\frac{Q}{M})^2 \leq \frac{3}{4}$. It is remarkable to note that there exist no asymptotically de Sitter solutions for $k = 0$ and $k = -1$.

It is interesting to mention that this specific black hole solution is asymptotically de Sitter although the cosmological constant does not appear explicitly in the action (4.1). This is a particular characteristic of the EGB theory in five dimensions [47]. Moreover, the fact that the radius of the event horizon does not depend on the value of the coupling constant α leads to interesting thermodynamic consequences. In fact, it will be shown later that the thermodynamics of the black holes by using the Bekenstein-Hawking entropy relationship differs completely from the one obtained by using a modified entropy relation in which the coupling constant appears explicitly.

4.1.1 Geometrothermodynamics with the Bekenstein–Hawking Entropy Relation

In order to study thermodynamic properties of the black hole solution described above first I will use the classical (Bekenstein-Hawking) definition of black hole entropy and in

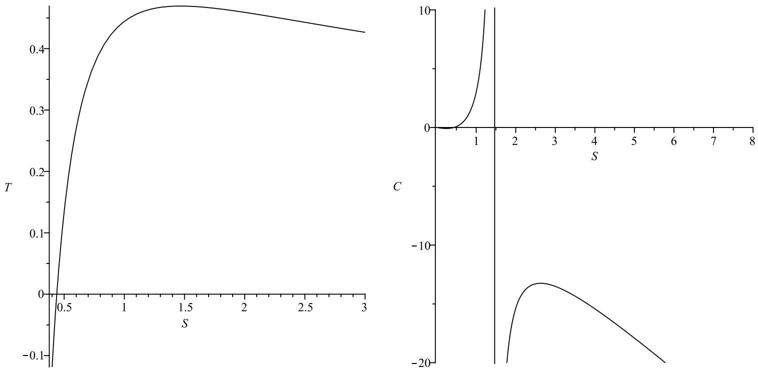

Figure 4.1: Temperature and heat capacity of a charged black hole in EGB theory as a function of the Bekenstein-Hawking entropy S for $Q = 1$.

the next subsection 4.1.2 I will define the modified form of entropy for EGB theory by adding an extra term $(6\alpha r)$. The surface area of the event horizon is given by

$$A = r_+^3 \int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi} \int_{\psi=0}^{2\pi} \sin^2\theta \sin\phi d\theta d\phi d\psi = 2\pi^2 r_+^3 \ . \tag{4.16}$$

Choosing the constants appropriately, the Bekenstein-Hawking entropy [52] of the black hole is given by

$$S = r_+^3, \tag{4.17}$$

representing the fundamental equation that contains all the thermodynamic information. In the mass representation, $M = M(S, Q)$, for the black hole solution presented above this fundamental equation can be rewritten as

$$M = S^{\frac{2}{3}} + \frac{1}{3}\frac{Q^2}{S^{\frac{2}{3}}} \ . \tag{4.18}$$

Using the energy conservation law of the black hole

$$dM = TdS + \phi dQ, \tag{4.19}$$

one obtains the temperature and electric potential of the black hole on the event horizon as

$$T = \frac{2}{9}\frac{3S^{\frac{4}{3}} - Q^2}{S^{\frac{5}{3}}}, \tag{4.20}$$

and

$$\phi = \frac{2}{3}\frac{Q}{S^{\frac{2}{3}}}. \tag{4.21}$$

In the positive domain $(S^{4/3} > Q^2/3)$, the temperature increases rapidly as a function of the entropy S until it reaches its maximum value at $S^{4/3} = 5Q^2/3$. Then, as the entropy increases, the temperature becomes a monotonically decreasing function. This behavior is shown in Fig.(4.1).

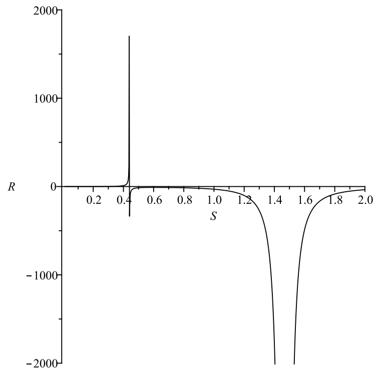

Figure 4.2: Thermodynamic curvature of a charged black hole in EGB theory as a function of the Bekenstein-Hawking entropy S for $Q = 1$. The curvature singularities are located at the points $T = 0$ and $C \to \infty$.

According to Davies [44–46], the phase transition structure of the black hole can be derived from the heat capacity, $C = T\partial S/\partial T$, which in this case is given by

$$C_Q = -3S\left(\frac{3S^{\frac{4}{3}} - Q^2}{3S^{\frac{4}{3}} - 5Q^2}\right). \tag{4.22}$$

In the physical region with $3S^{4/3} - Q^2 > 0$, i. e., the region with positive temperature, the heat capacity is positive in the interval $Q^2 < 3S^{4/3} < 5Q^2$, indicating that the black hole is stable in this region. At the maximum value of the temperature which occurs at $3S^{4/3} - 5Q^2 = 0$, the heat capacity diverges and changes spontaneously its sign from positive to negative. This indicates the presence of a second order phase transition which is accompanied by a transition into a region of instability, see Fig.(4.1). These are the main features of the thermodynamic behavior of the charged spherically symmetric black hole (4.13).

For the geometrothermodynamic approach to black hole thermodynamics all what is needed is the fundamental equation, $M = M(S, Q)$ as given in Eq.(4.18). Then, from the general metric (2.12) with $\Phi = M$ and $E^a = (S, Q)$, the thermodynamic metric of the equilibrium manifold is given by

$$g = \frac{4}{27}\frac{3S^{4/3} - Q^2}{S^{4/3}}\left(\frac{3S^{4/3} - 5Q^2}{9S^2}dS^2 + dQ^2\right). \tag{4.23}$$

The corresponding scalar curvature is given by

$$R = -\frac{243\ S^{8/3}}{(3S^{4/3} - Q^2)(3S^{4/3} - 5Q^2)^2}. \tag{4.24}$$

A first singularity is situated at the roots of the equation $3S^{4/3} - Q^2 = 0$, i.e., at the points where the temperature vanishes. The second singularity corresponds to the roots of $3S^{4/3} - 5Q^2 = 0$. According to the expression for the heat capacity (4.22), these are exactly the points where phase transitions take place and the temperature reaches its maximum value. It follows that the geometrothermodynamic curvature of the metric (4.23) reproduces correctly the thermodynamic behavior near the points of zero temperature as well as near the points of phase transitions as shown in Fig.(4.2).

The thermodynamic geometry of this black hole was also studied in [37] using the Ruppeiner geometry. It turns out that the Ruppeiner metric is flat in this case and, consequently, cannot reproduce the behavior at the places where phase transitions occur or where the temperature becomes negative.

4.1.2 Geometrothermodynamics with a Modified Entropy Relation

Usually the entropy of black holes satisfies the so-called area formula, i.e, the black hole entropy equals one-quarter of the horizon area. In gravity theories in higher dimensions and with higher powered curvature terms, however, the entropy of black holes does not necessarily satisfy the area formula and other possibilities can be considered to define entropy. For instance, in [28] a simple method was suggested to get the black hole entropy, by assuming that black holes, considered as genuine thermodynamic systems, must obey the first law of thermodynamics. That is a black hole solution, parameterized by the mass M or, alternatively, by the outer horizon radius r_+ and the temperature T, satisfies the first law of thermodynamics

$$dM = TdS + \mu_i dQ^i, \tag{4.25}$$

where Q^i are the additional charges of the black hole and μ_i are the corresponding chemical potentials. If the mass and the temperature can be calculated by using standard methods, the integration of the first law yields the modified entropy formula

$$S = \int_0^{r_+} T^{-1} \left(\frac{\partial M}{\partial r_+}\right)_{Q_i} dr_+ + S_0 \ , \tag{4.26}$$

where the additive integration constant S_0 can be fixed by imposing the condition that the entropy goes to zero in the case of an extreme black hole or when the area of the horizon vanishes. Notice that in the integration (4.26) the charges Q_i must be considered as constants. In [42], the modified entropy was computed for an $n-$dimensional generalization of the black hole solution (4.13) with the result

$$S = \frac{\Omega_K r_+^{n-2}}{4G}\left[1 + \frac{2\tilde{\alpha}k(n-2)}{(n-4)r_+^2}\right] \ , \tag{4.27}$$

where Ω_k denotes the spatial volume element and $\tilde{\alpha} = (n-2)(n-3)\alpha$. In the case $n = 5$, the modified entropy reduces to

$$S = r_H^3 + 6k\tilde{\alpha}r_+ \ , \tag{4.28}$$

where suitable units were chosen and I set $S_0 = 0$ for simplicity. Notice that the contribution of the correction term vanishes for $k = 0$ so that the GB term has no effect on the

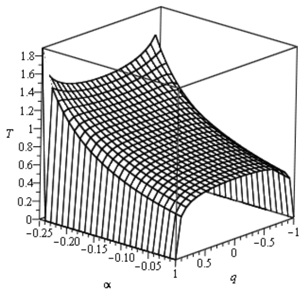

Figure 4.3: Modified temperature as function of the specific charge q and the coupling parameter $\tilde{\alpha}$ for a fixed value of the mass $M = 1$.

expression for the entropy which reduces in this case to the standard area formula. Moreover, the modified entropy formula does not contain the additional charges Q_i explicitly, but only implicitly through the horizon radius r_+. So one can assume the validity of the modified entropy (4.28) regardless of the nature of the additional charges.

For the black hole solution (4.13) the modified entropy formula (with $k = 1$) gives

$$S = \frac{1}{8}\left[\sqrt{M + \frac{2Q}{\sqrt{3}}} + \sqrt{M - \frac{2Q}{\sqrt{3}}}\right]^3 + 3\tilde{\alpha}\left[\sqrt{M + \frac{2Q}{\sqrt{3}}} + \sqrt{M - \frac{2Q}{\sqrt{3}}}\right] . \tag{4.29}$$

In this case the fundamental equation is of the form $S = S(M, Q)$ and cannot be rewritten explicitly as $M = M(S, Q)$. This means that for the further analysis I must use the entropy representation. This is not a problem for the formalism of GTD which can be applied to any arbitrary representation. In fact, for the entropy representation we only need to consider the fundamental 1-form as

$$\Theta_S = dS - \frac{1}{T}dM + \frac{\phi}{T}dQ, \tag{4.30}$$

so that the thermodynamic potential is $\Phi = S$. The coordinates of the equilibrium manifold are $E^a = (M, Q)$ and the equilibrium conditions become

$$\frac{1}{T} = \frac{\partial S}{\partial M} , \quad \frac{\phi}{T} = -\frac{\partial S}{\partial Q} . \tag{4.31}$$

From the above expressions one obtains the temperature and electric potential of the black hole on the event horizon as

$$T = \frac{8}{3}\frac{\sqrt{M(1-q^2)}}{(4\tilde{\alpha} + M + M\sqrt{1-q^2})(\sqrt{1+q} + \sqrt{1-q})} , \tag{4.32}$$

$$\phi = \frac{2}{\sqrt{3}}\frac{\sqrt{1+q} - \sqrt{1-q}}{\sqrt{1+q} + \sqrt{1-q}} , \tag{4.33}$$

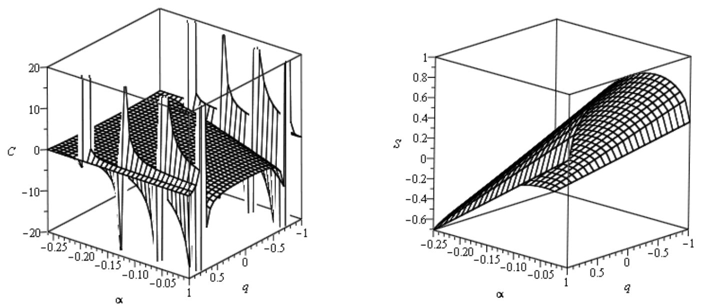

Figure 4.4: Modified heat capacity and entropy as functions of the specific charge q and the coupling parameter $\tilde{\alpha}$ for a fixed value of the mass $M = 1$.

where $q = \frac{2Q}{\sqrt{3}M}$ represents a rescaled specific charge that satisfies the condition $q^2 \leq 1$. Furthermore, to find out the phase transitions structure we must find the points where the heat capacity ($C_Q = T\partial S/\partial T|_Q = -(\partial S/\partial M)^2/(\partial^2 S/\partial M^2)|_Q$)

$$C_Q = -\frac{3}{4}\frac{\sqrt{M}\sqrt{1-q^2}(\sqrt{1+q}+\sqrt{1-q})(4\tilde{\alpha}+M+M\sqrt{1-q^2})^2}{M(1-3q^2)+(4\tilde{\alpha}+M)\sqrt{1-q^2}-8\tilde{\alpha}}, \tag{4.34}$$

diverges. Since the coupling constant $\tilde{\alpha}$ is negative, the temperature function turns out to be positive definite only for certain ranges of $\tilde{\alpha}$ and q. In Fig.(4.3), I choose a particular range of values of $\tilde{\alpha} \in [-1/4, 0]$ in which the modified temperature is positive. The behavior of the modified heat capacity and entropy in the same range of values is depicted in Fig.(4.4). Notice that the entropy is not positive definite in this interval; however, it is possible to choose the arbitrary constant S_0 in Eq.(4.26) so that the modified entropy is always positive and the expressions for the modified temperature and heat capacity remain unchanged.

An interesting result of using the modified entropy is that the phase transition structure now depends on the value of the coupling constant $\tilde{\alpha}$. For instance, if I choose it as $\tilde{\alpha} = -1/4$, the heat capacity is as illustrated in Fig.(4.5) (left plot). In this case, the heat capacity is represented by a negative smooth function with no singularities in the interval $-1 \leq q \leq 1$, indicating that the black hole is a completely unstable thermodynamic system with no phase transition structure. This behavior changes drastically, if I choose a different value of the coupling constant. In fact, Fig.(4.5) (right plot) illustrates the singular behavior of the heat capacity in the case $\tilde{\alpha} = -1/10$. The black hole undergoes a second order phase transition at $q \approx \pm 0.82$. In the interval $-0.82 \leq q \leq 0.82$, the configuration is unstable because the heat capacity is negative. Outside this interval, however, the black hole is stable. I conclude that the coupling constant $\tilde{\alpha}$ can induce a second order phase transition in an unstable black hole in such a way that the resulting configuration is a stable black hole for certain values of the specific charge q.

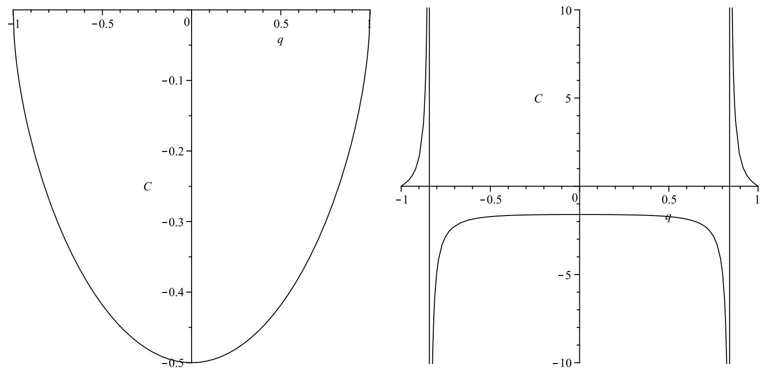

Figure 4.5: The heat capacity as a function of the specific charge q for $\tilde{\alpha} = -1/4$ (left plot) and $\tilde{\alpha} = -1/10$ (right plot). In both cases the mass is $M = 1$.

Now I investigate this black hole configuration in the context of GTD. As mentioned above, the coordinates of the equilibrium manifold are $E^a = (M, Q)$ and the thermodynamic potential $\Phi = S$ is given by the fundamental Eq.(4.29). Then, the thermodynamic metric for the equilibrium manifold $\mathcal{E}$ is given by

$$g = -\frac{3r_+(r_+^2 + 2\tilde{\alpha})}{8\sqrt{1-q^2}\, M^2}\left\{ \frac{3r_+[2r_+^2 - 3q^2 M + 4\tilde{\alpha}(\sqrt{1-q^2} - 2)]}{2(1-q^2)^{3/2}}\, dM^2 \right. \\ \left. + \sqrt{M}\left[\frac{4\tilde{\alpha} + (2+q)M}{(1+q)^{3/2}} + \frac{4\tilde{\alpha} + (2-q)M}{(1-q)^{3/2}}\right] dQ^2 \right\} . \tag{4.35}$$

The behavior of the corresponding scalar curvature is shown in Fig.(4.6) for two different values of the coupling constant. The plot on the left shows the case $\tilde{\alpha} = -1/4$ and corresponds to the case of an unstable configuration as shown in Fig.(4.5) (left plot). It is clear that the curvature is represented by a smooth function that is free of singularities in the entire domain of the specific charge, except at $q = \pm 1$ where the temperature vanishes (see Fig.(4.3)). The plot on the right illustrates the behavior for $\tilde{\alpha} = -1/10$ and shows two curvature singularities at $q \approx \pm 0.82$ which are the points where the phase transition occurs (see right plot in Fig.(4.5)). In this case, it is also possible to show that an additional curvature singularity (not plotted) exists in the limiting case $q \to \pm 1$, indicating the blow up of the approach as $T \to 0$.

4.2 Spherically Symmetric Black Holes in EMGB Gravity with Cosmological Constant

In the case of the EMGB theory with cosmological constant, the matter component of the action (4.1) is given by

$$L_{matter} = F_{\alpha\beta}F^{\alpha\beta} - 2\Lambda\ , \quad F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}\ , \tag{4.36}$$

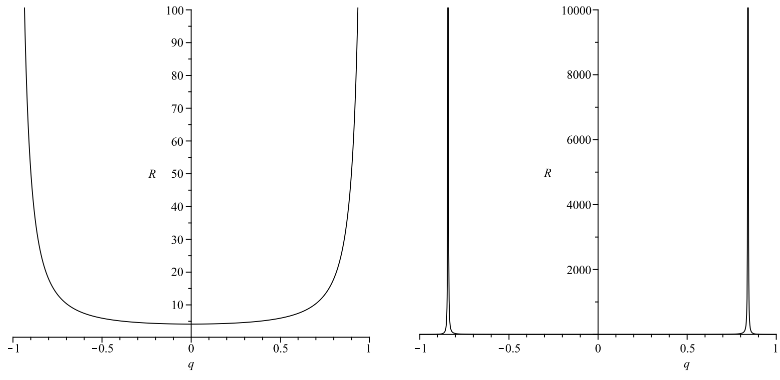

Figure 4.6: Thermodynamic curvature as a function of the rescaled specific charge q. For $\tilde{\alpha} = -1/4$ (left plot) the curvature is completely regular, except at $q = \pm 1$. Curvature singularities appear for $q \approx 0.82$ and $\tilde{\alpha} = -1/10$ (right plot).

where $F_{\alpha\beta}$ represents the electromagnetic Faraday tensor. By variational principle (i.e., $\delta S = 0$) we obtain the Einstein-Maxwell equations with GB term

$$G_{\mu\nu} + \lambda g_{\mu\nu} = T^{(EM)}_{\mu\nu} + T^{(GB)}_{\mu\nu}, \tag{4.37}$$

where $T^{(EM)}_{\mu\nu}$ is the usual electromagnetic energy-momentum tenor given by

$$T^{(EM)}_{\mu\nu} = F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{4.38}$$

and

$$T^{(GB)}_{\mu\nu} = \alpha\left(4R^{\alpha\beta}R_{\mu\alpha\nu\beta} - 2R_{\mu\alpha\beta\gamma}R^{\alpha\beta\gamma}_{\nu} + 4R_{\mu\alpha}R^{\alpha}_{\nu} - 2RR_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R_{GB}\right), \tag{4.39}$$

is the effective tensor associated with the quadratic GB-term.

A five-dimensional spherically symmetric solution in EMGB grvity with Λ was obtained by Wiltshire [143], using the metric ansatz (4.8) with $k = +1$ and the metric function is

$$G(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha}\sqrt{1 + \frac{8\alpha M}{r^4} - \frac{8\alpha Q^2}{3r^6} + \frac{4\alpha\Lambda}{3}}\ . \tag{4.40}$$

If α is very small then the metric coefficient $G(r)$ can be approximately written as

$$G(r) \backsimeq 1 - \frac{2m}{\pi r^2} + \frac{Q^2}{3r^4}\frac{r^2}{l^2} + O(\alpha), \tag{4.41}$$

where $l = \sqrt{\frac{6}{\Lambda}}$ is the curvature scalar. So, in the limit $\alpha \to 0$ the above solution reduces to the usual five-dimensional Einstein-Maxwell solution (i.e., RN solution) with a cosmological constant.

The two parameters $M(>0)$ and Q are identified as the mass and electric charge of the system. The limit of vanishing cosmological constant generates a solution contained in Eq.(4.13) with the minus sign in front of the square root and a redefined mass parameter.

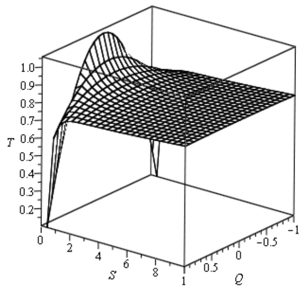

Figure 4.7: Behavior of the temperature in the range $Q \in [-1, 1]$ and $S \in [0, 10]$ for fixed value of the cosmological constant $\Lambda = -1$.

In this limit, however, the resulting solution does not describe a black hole, but a naked singularity.

For the solution (4.40) to describe a black hole spacetime, it is necessary that the expression inside the square root must be positive and the function $G(r)$ vanishes on the horizon radius, i. e.,

$$1 + \frac{8\alpha M}{r_+^4} - \frac{8\alpha Q^2}{3r_+^6} + \frac{4\alpha\Lambda}{3} > 0 \ , \quad 1 + \frac{r_+^2}{4\alpha} - \frac{r_+^2}{4\alpha}\sqrt{1 + \frac{8\alpha M}{r_+^4} - \frac{8\alpha Q^2}{3r_+^6} + \frac{4\alpha\Lambda}{3}} = 0 \ . \quad (4.42)$$

Moreover, to guarantee that the mass of the black hole is always positive (see below) we demand that the coupling constant α and the cosmological constant Λ must be positive and positive definite, respectively. In this section I will limit the analysis to this range of parameters, so that the black holes determined by the function (4.40) turns out to be asymptotically AdS.

4.2.1 Geometrothermodynamics with the Bekenstein–Hawking Entropy Relation

The thermodynamics of EMGB black hole with cosmological constant is essentially an extension of the RNdS black hole due to GB theory in the canonical ensembles. The condition $G(r_+) = 0$ implies that

$$\frac{\Lambda}{3} r_+^6 - 2r_+^4 + 2\left(M - 2\alpha\right) r_+^2 - \frac{2}{3}Q^2 = 0 \ . \quad (4.43)$$

Moreover, as I mentioned in section 4.1.1, with the appropriate choice of units the Bekenstein-Hawking entropy of the black hole is given by $S = r_+^3$. Then, the corresponding thermodynamic fundamental equation in the mass representation becomes

$$M = 2\alpha + S^{2/3} + \frac{Q^2}{3S^{2/3}} - \frac{\Lambda}{6} S^{4/3} \ . \quad (4.44)$$

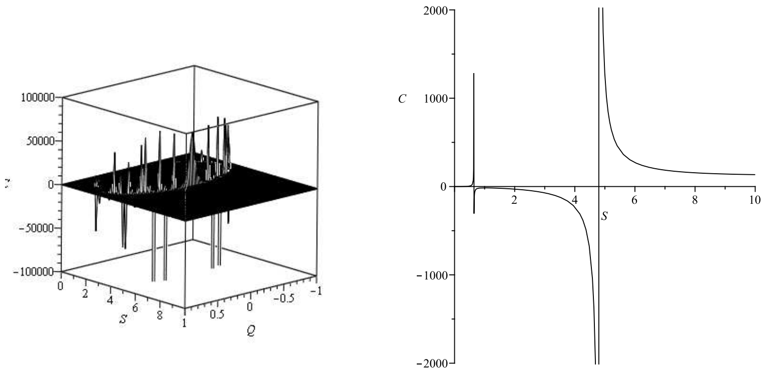

Figure 4.8: Behavior of the heat capacity in the range $Q \in [-1, 1]$ and $S \in [0, 10]$ for fixed value of the cosmological constant $\Lambda = -1$. The right plot shows the details of the phase transition structure for the particular charge $Q = 1/2$.

Notice that to guarantee the positiveness of the mass, we must assume that $\alpha > 0$ and $\Lambda < 0$.

Using the energy conservation law of the black hole

$$dM = TdS + \phi dQ, \tag{4.45}$$

one obtains the temperature and electric potential of the black hole on the event horizon as

$$T = \frac{2}{9} \frac{3\,S^{4/3} - \Lambda\,S^2 - Q^2}{S^{5/3}} \,, \tag{4.46}$$

and

$$\phi = \frac{2Q}{3S^{\frac{2}{3}}}. \tag{4.47}$$

Now, for a given charge, the heat capacity has the expression

$$C_Q = 3S \left(\frac{3S^{\frac{4}{3}} - \Lambda S^2 - Q^2}{5Q^2 - 3S^{\frac{4}{3}} - \Lambda S^2} \right) \,. \tag{4.48}$$

The expression for the temperature (4.46) shows that it is positive only in the range $3\,S^{4/3} - \Lambda\,S^2 > Q^2$. Consequently, the heat capacity can take either positive or negative values, indicating the possibility of stable and unstable states for the black hole. In fact, the expression for the heat capacity exhibits a very rich structure in the range where the temperature is positive. In Fig.(4.7), a particular range was chosen to show the behavior of the temperature. Similar behavior for heat capacity is shown in Fig.(4.8), the plot on the right shows for the particular value $Q = 1/2$ two different phase transitions at $S \approx 0.6$ and $S \approx 4.9$. The first one corresponds to a transition from a stable state ($C > 0$) to an unstable state ($C < 0$). The second one represents a second order phase transition in which the black hole becomes a stable system again.

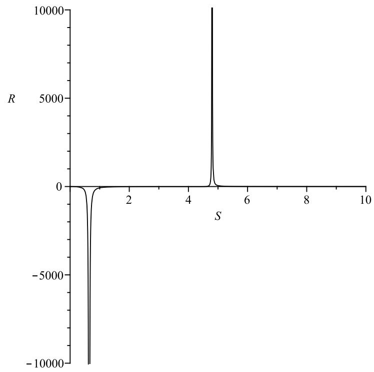

Figure 4.9: Curvature singularities in the equilibrium manifold of the black hole (4.40) with $\Lambda = -1$ and $Q = 1/2$. The singularities are located at $S \approx 0.6$ and $S \approx 4.9$.

The fundamental equation is in this case given as $M = M(S, Q)$ in Eq.(4.44). Then, the associated coordinates to the equilibrium manifold $\mathcal{E}$ are $E^a = (S, Q)$ and $\Phi = M$ is the thermodynamic potential. The thermodynamic metric can then be written as

$$g = \frac{4}{27S^{4/3}}(3S^{4/3} - \Lambda S^2 - Q^2)\left[\frac{1}{9S^2}(3S^{4/3} + \Lambda S^2 - 5Q^2)dS^2 + dQ^2\right] . \tag{4.49}$$

A straightforward computation results in the following scalar curvature:

$$R = \frac{27}{2}\frac{S^{7/3}N(S,Q,\Lambda)}{\left(3\,S^{4/3} - Q^2 - \Lambda\,S^2\right)^3\left(3\,S^{4/3} - 5\,Q^2 + \Lambda\,S^2\right)^2}, \tag{4.50}$$

with

$$\begin{aligned} N(S,Q,\Lambda) = \quad & 42\,Q^2S^{7/3}\Lambda - 34\,SQ^4\Lambda - 5\,S^3Q^2\Lambda^2 - 18\,Q^4\sqrt[3]{S} - 7\,S^5\Lambda^3 \\ + \quad & 36\,S^{11/3}\Lambda + 15\,S^{13/3}\Lambda^2 - 162\,S^3 + 108\,Q^2S^{5/3}\ . \end{aligned} \tag{4.51}$$

From the expression for the scalar curvature it is obvious that the singularities are located at the points satisfying $3\,S^{4/3} - Q^2 - \Lambda\,S^2 = 0$, which coincide with the points where $T \to 0$ and at the points satisfying $3\,S^{4/3} - 5\,Q^2 + \Lambda\,S^2 = 0$, which are the points where $C_Q \to \infty$. For the particular case $\Lambda = -1$ and $Q = 1/2$ the singularities are shown in Fig.(4.9); their locations clearly coincide with the points where second order phase transitions occur, see right plot in Fig.(4.8).

4.2.2 Geometrothermodynamics with a Modified Entropy Relation

The modified entropy relation (4.28), with $k = +1$, cannot be solved in this case to obtain an explicit fundamental equation $M = M(Q, S)$. We must therefore consider the implicit fundamental equation determined by the relationships

$$S = r_+^3 + 6\tilde{\alpha}r_+\ , \quad M = \frac{\tilde{\alpha}}{3} + \frac{Q^2}{3r_+^2} + r_+^2 - \frac{\Lambda}{6}r_+^4\ . \tag{4.52}$$

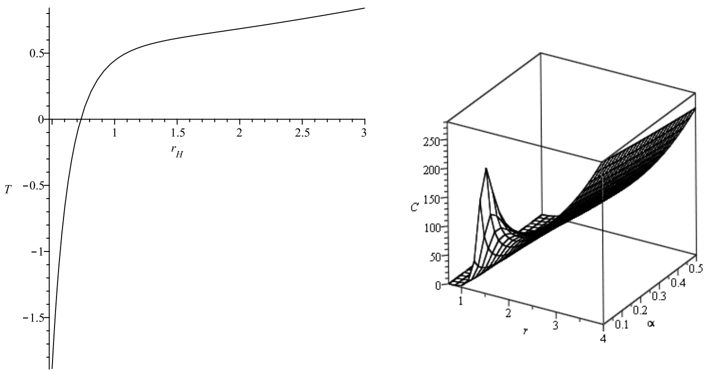

Figure 4.10: Temperature and heat capacity for a black hole of the EMGB theory with cosmological constant $\Lambda = -1$, charge $Q = 1$ and coupling constant $\tilde{\alpha} = 1/4$ (for the temperature) and $\tilde{\alpha} \in (0, 0.5)$ (for the heat capacity). The temperature is positive for $r_H > r_H^{min} \approx 0.73$.

Then, the main thermodynamic variables can then be expressed as

$$T = \frac{2(3r_+^4 - Q^2 - \Lambda r_+^6)}{9r_+^3(r_+^2 + 2\tilde{\alpha})}, \tag{4.53}$$

$$\phi = \frac{2Q}{3r_+^2}, \tag{4.54}$$

$$C_Q = \frac{3r_+(r_+^2 + 2\tilde{\alpha})^2(3r_+^4 - Q^2 - \Lambda r_+^6)}{6\tilde{\alpha}Q^2 + 5Q^2r_+^2 + 6\tilde{\alpha}r_+^4 - 3r_+^6(1 + 2\tilde{\alpha}\Lambda) - \Lambda r_+^8}\ . \tag{4.55}$$

For a physically reasonable configuration we demand the positiveness of the temperature; this implies that $3r_+^4 - \Lambda r_+^6 > Q^2$. For a given value of Λ and Q, this condition determines a minimum horizon radius r_+^{min} for which the temperature is positive. Moreover, from the expression for the heat capacity (4.55) and from the condition of positive temperature, it follows that if the condition

$$\tilde{\alpha}|\Lambda| \geq \frac{1}{2} \tag{4.56}$$

is satisfied, the heat capacity is positive and, consequently, all possible black hole configurations are stable. This is an interesting condition that relates two fundamental constants, namely, the tension of the string, proportional to $\tilde{\alpha}^{-1}$ and the cosmological constant Λ.

For the range $\tilde{\alpha}|\Lambda| < 1/2$ where unstable states in principle can exist, let us consider the parameters $\Lambda = -1$ and $Q = 1$. This choice together with the positiveness condition of the temperature fix the value of $r_+^{min} \approx 0.73$ (see left plot in Fig.(4.10)). Notice that the value of r_H^{min} does not depend on the value of the coupling constant $\tilde{\alpha}$. In Fig.(4.11), the behavior of the heat capacity is explored for the entire range $\tilde{\alpha} \in (0, 1/2)$, according to the condition $\tilde{\alpha}|\Lambda| < 1/2$. One can see that the heat capacity is represented by a smooth positive function in the entire domain. All the black hole configurations are found stable in this case.

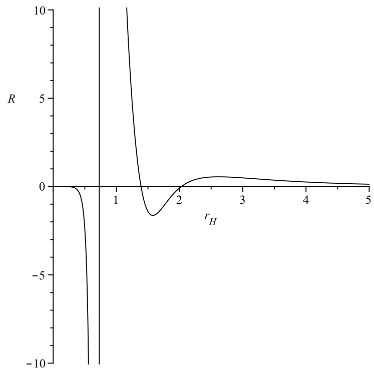

Figure 4.11: Thermodynamic curvature for a black hole of the EMGB theory with cosmological constant $\Lambda = -1$, charge $Q = 1$ and coupling constant $\tilde{\alpha} = 1/4$.

I now investigate the geometric properties of the equilibrium manifold. According to the implicit fundamental equation, the thermodynamic metric (2.12) can be written as

$$g = -f_1(r_H, Q, \tilde{\alpha}, \Lambda)\Big\{ \left(5Q^2 r_+^2 + 6\tilde{\alpha} r_+^4 - 3r_+^6(1+2\tilde{\alpha}\Lambda) - \Lambda r_+^8\right) dS^2 + 9r_H \left(r_H^2 + 2\tilde{\alpha}\right)^2 dQ^2 \Big\}, \tag{4.57}$$

where

$$f_1(r_H, Q, \tilde{\alpha}, \Lambda) = \frac{4}{243} \frac{(r_H^2 + 6\tilde{\alpha})(3r_+^4 - Q^2 - \Lambda r_+^6)}{r_H^6 (r_H^2 + 2\tilde{\alpha})^4}. \tag{4.58}$$

The expression for the scalar is quite cumbersome but it can schematically be represented as

$$R = \frac{N(r_+, \Lambda, Q, \tilde{\alpha})}{(3r_+^4 - Q^2 - \Lambda r_+^6)^3 \left(6\tilde{\alpha}Q^2 + 5Q^2 r_+^2 + 6\tilde{\alpha} r_+^4 - 3r_+^6(1+2\tilde{\alpha}\Lambda) - \Lambda r_+^8\right)^2 (r_H^2 + 6\tilde{\alpha})^3}\,, \tag{4.59}$$

where $N(r_+, Q, \Lambda, \tilde{\alpha})$ is a finite function in the entire domain of definition. From the expression for the scalar curvature (4.59), the temperature (4.53) and heat capacity (4.55), it follows that singularities can take place only at those points where $T \to 0$ or $C \to \infty$. In Fig.(4.11), the behavior of the scalar curvature is shown for a particular choice of the parameters. One can see that a singularity occurs at the point where the temperature vanishes. The singularity situated at $(r_H^2 + 6\tilde{\alpha}) = 0$ corresponds to the limit $S \to 0$ which indicates the breakdown of the thermodynamic picture of the black hole and, hence of GTD. No other singularities exist because the heat capacity is finite in this domain.

4.3 Spherically Symmetric Black Holes in EYMGB Gravity

In an attempt to associate a geometric structure to the electromagnetic field, Yang and Mills [145] used in 1953 the concept of a principal fiber bundle with the Minkowski spacetime as the base manifold and the symmetry group $U(1)$ as the standard fiber to demonstrate that the Faraday tensor can be interpreted as the curvature of this particular fiber bundle. It is well known that the weak interaction and the strong interaction can be represented as the curvature of a principal fiber bundle with a Minkowski base manifold and the standard fiber $SU(2)$ and $SU(3)$, respectively. This has opened the possibility of fixing the background metric in accordance with the desired properties of the base manifold and selecting different connections as local cross-sections of the principal fiber bundle. This interesting geometrical approach constitutes the base for constructing the modern gauge theories that are used to describe the physics of the electromagnetic, weak and strong interactions. In this section, I will show that geometrothermodynamic metric (2.12) can be fixed as the background metric in accordance with the desired properties of the base manifold and it is possible to interpret the thermodynamic interaction as the curvature of a Legendre invariant Riemannian manifold. I first describe the black hole solution [88] and its properties in EYMGB gravity and then study the geometry of the black hole thermodynamics in the subsequent sections.

The five-dimensional spherically symmetric solution obtained recently by Mazharimousavi and Halisoy [88] has the metric ansatz

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega_3^2 \ . \tag{4.60}$$

In this case the metric on the unit three sphere $d\Omega_3^2$ is given by

$$d\Omega_3^2 = \frac{1}{4}(d\theta^2 + d\phi^2 + d\psi^2 - 2\cos\theta d\phi d\psi), \tag{4.61}$$

with

$$\theta \in [0, \pi], \quad (\phi, \psi) \in [0, 2\pi]. \tag{4.62}$$

For the Yang-Mills field the energy momentum tensor is given by

$$T_{\mu\nu} = 2F_{\mu}^{i\alpha}F_{\nu\alpha}^{i} - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}^{i}F^{i\alpha\beta}, \tag{4.63}$$

where $F_{\alpha\beta}^{i}$ are the Yang-Mills field 2-forms such that

$$F_{\alpha\beta}^{i}F^{i\alpha\beta} = 6Q^2/r^4, \tag{4.64}$$

represents the matter Lagrangian in the general action (4.1). The modified Einstein equations in EYMGB theory are just the same as (1.1) and the solution is given by

$$B(r) = 1 + \frac{r^2}{4\alpha} - \sqrt{1 + \frac{M}{2\alpha} + \frac{r^4}{16\alpha^2} + \frac{Q^2}{\alpha}\ln r} \ , \tag{4.65}$$

where M is an integration constant to be identified as the mass and Q is the only non-zero gauge charge.

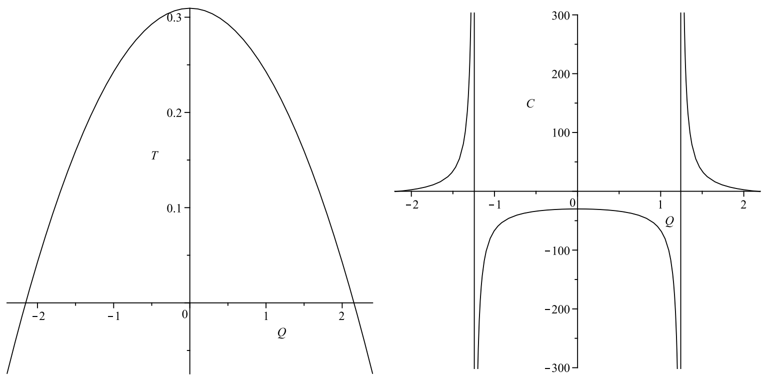

Figure 4.12: Temperature and the heat capacity in terms of the Yang-Mills charge Q. To illustrate the thermodynamic behavior, the Bekenstein-Hawking entropy was chosen as $S = 10$.

This black hole solution of the EYMGB theory is well defined for all r if the GB coupling parameter α is positive definite. For $\alpha < 0$, the spacetime has a curvature singularity at the hypersurface $r = r_s$, where r_s is the largest root of $B(r) = 0$. The event horizon radius r_+ satisfies the equation $B(r_+) = 0$ and is given by

$$M = r_+^2 - 2Q^2 \ln r_+ \ . \tag{4.66}$$

4.3.1 Geometrothermodynamics with the Bekenstein–Hawking Entropy Relation

In suitable units, the entropy S of the black hole is given by $S = r_+^3$, where $A = 2\pi^2 r_+^3$ is the surface area of the event horizon. According to Eq.(4.66), the thermodynamic fundamental equation in the M-representation becomes

$$M = S^{\frac{2}{3}} - \frac{2}{3} Q^2 \ln S \ . \tag{4.67}$$

Thus, the expressions for the thermodynamic quantities, namely, the temperature and the electric potential are given by

$$T = \frac{2}{3} \frac{S^{2/3} - Q^2}{S} \ , \quad \phi = -\frac{4}{3} Q \ln S \ . \tag{4.68}$$

It follows that for the temperature of the black hole to be positive the charge must satisfy the condition $Q < S^{1/3}$. Moreover, for a fixed value of the entropy, the maximum temperature is reached at the value $Q = 0$, indicating that the Yang-Mills charge reduces the temperature of the black hole. This behavior is illustrated in Fig.(4.12).

Now, for a fixed charge, the heat capacity is given by the expression

$$C_Q = -3S \frac{S^{2/3} - Q^2}{S^{2/3} - 3Q^2} \ . \tag{4.69}$$

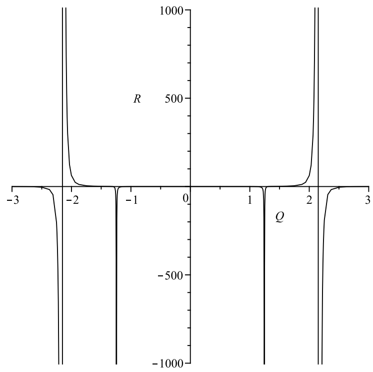

Figure 4.13: The thermodynamic curvature with $S = 10$ as a function of the Yang-Mills charge Q. Singularities are due to the existence of a second order phase transition at $Q \approx \pm 1.24$ or to the vanishing of the temperature at $Q \approx \pm 2.15$.

In the region $Q < S^{1/3}$, where the temperature is positive, the heat capacity diverges at those points where $Q = S^{1/3}/\sqrt{3}$, indicating the existence of a second order phase transition. In the interval $S^{1/3}/\sqrt{3} < |Q| < S^{1/3}$, the heat capacity is positive (and $T > 0$), i.e., the black hole configuration is stable in this interval. Furthermore, the heat capacity is negative within the interval $0 < |Q| < S^{1/3}/\sqrt{3}$ which corresponds to an unstable thermodynamic configuration. Since the heat capacity at $Q = 0$ is negative, I conclude that the addition of a Yang-Mills charge Q to an unstable neutral black hole not only reduces its temperature, but also changes its heat capacity until it becomes positive and the system becomes stable, if the charge is sufficiently large. The transition from an unstable state to a stable state is accompanied by a second order phase transition. This thermodynamic behavior is illustrated in Fig.(4.12).

To investigate the geometry of the corresponding equilibrium manifold, I use the general metric (2.12) with the fundamental equation $M = M(S, Q)$ as given in Eq.(4.67). Then,

$$g = \frac{4}{27}\left(S^{2/3} - Q^2\right)\left(\frac{S^{2/3} - 3Q^2}{S^2} dS^2 - 6 \ln S \, dQ^2\right) . \qquad (4.70)$$

The corresponding thermodynamic curvature scalar can be represented as

$$R = \frac{N(S, Q)}{(S^{2/3} - Q^2)^3 (S^{2/3} - 3Q^2)^2 \ln^2 S}, \qquad (4.71)$$

where $N(S, Q)$ is a well-behaved function of its arguments. We see that there are several places where true curvature singularities can exist. First, if $Q = S^{1/3}$ the curvature scalar diverges and, as described above, the temperature vanishes. Then, at $Q = S^{1/3}/\sqrt{3}$ there exists a singularity whose location coincides with the values at which the heat capacity diverges and second order phase transitions occur. Finally, if $\ln S \to 0$ the curvature scalar diverges. I interpret this additional singularity as related to a second order phase

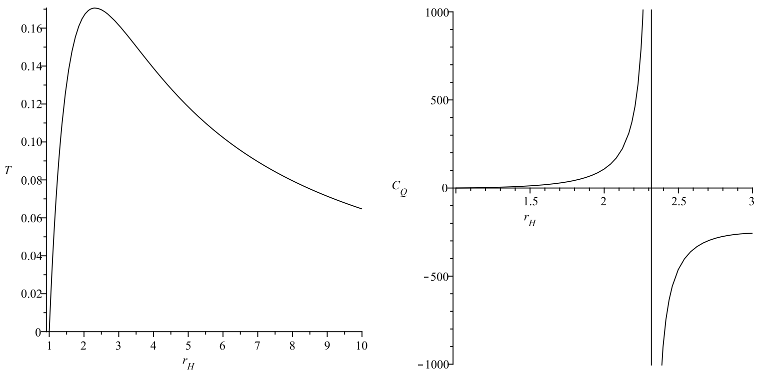

Figure 4.14: Temperature and heat capacity in terms of the horizon radius r_+ of a black in the EYMGB theory. Here the modified entropy relation is used with $Q = 1$ and $\tilde{\alpha} = 1$

transition which is not contained in C_Q. In fact, in analogy to the heat capacity C_Q defined in Eq.(4.69), I can introduce the capacitance

$$C_S \equiv \left(\frac{\partial Q}{\partial \Phi}\right)_S = \left(\frac{\partial \Phi}{\partial Q}\right)_S^{-1} = \left(\frac{\partial^2 M}{\partial Q^2}\right)_S^{-1} . \tag{4.72}$$

Then, from the fundamental Eq.(4.67)

$$C_S = -\frac{3}{4 \ln S}, \tag{4.73}$$

so that in the limit $S \to 1$, a second order phase transition occurs. This proves the physical origin of the additional singularity of the thermodynamic curvature. The behavior of this thermodynamic curvature is depicted in Fig.(4.13).

4.3.2 Geometrothermodynamics with a Modified Entropy Relation

In this case, the thermodynamic fundamental equation cannot be written explicitly. Therefore, I use the implicit equation $S = S(M, Q)$ determined by the expressions

$$S = r_+^3 + 6\tilde{\alpha} r_H \ , \quad \text{and} \quad M = r_+^2 - 2Q^2 \ln r_+ \ . \tag{4.74}$$

Then, the thermodynamic variables are given by

$$T = \frac{2}{3} \frac{r_+^2 - Q^2}{r_+(r_+^2 + 2\tilde{\alpha})} \ , \tag{4.75}$$

$$\phi = 4Q \ln r_+ \ , \tag{4.76}$$

$$C_Q = \frac{3r_+(r_+^2 - Q^2)(r_+^2 + 2\tilde{\alpha})^2}{-r_+^4 + (2\tilde{\alpha} + 3Q^2) r_+^2 + 2Q^2\tilde{\alpha}} \ . \tag{4.77}$$

Notice that in this case the condition for a positive definite temperature reads $r_H^2 > Q^2$. Moreover, the explicit presence of the coupling constant $\tilde{\alpha}$ in the heat capacity leads to the possibility of modifying the phase transition structure of the black hole by changing

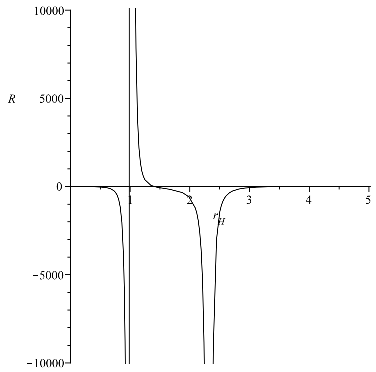

Figure 4.15: Scalar curvature in terms of the horizon radius r_+ of a black in the EYMGB theory with $Q = 1$ and $\tilde{\alpha} = 1$.

the value of the GB coupling constant. Indeed, the expression for the heat capacity (4.77) diverges for

$$r_H^2 = \frac{3}{2}\,Q^2 + \tilde{\alpha} \pm \frac{1}{2}\,\sqrt{9\,Q^4 + 20\,\tilde{\alpha}\,Q^2 + 4\,\tilde{\alpha}^2}\ , \tag{4.78}$$

indicating that for a given value of the Yang-Mills charge it is possible to find a range of values of $\tilde{\alpha}$ for which second order phase transitions take place. This behavior is illustrated in Fig.(4.14).

For the thermodynamic system determined by the fundamental Eq.(4.74), the Legendre invariant metric (2.12) is given by

$$g = -\frac{4}{27}\frac{(r_+^2 + 6\tilde{\alpha})(r_+^2 - Q^2)}{r_+^2(r_+^2 + 2\tilde{\alpha})^4}\Big\{ \left(-r_+^4 + (2\tilde{\alpha} + 3Q^2)r_+^2 + 2Q^2\tilde{\alpha}\right) dS^2 \\ + 18 r_+^2 (r_+^2 + 2\tilde{\alpha})^3 \ln r_+ dQ^2 \Big\}. \tag{4.79}$$

The expression for the scalar curvature can be schematically written as

$$R = \frac{N(r_+, Q, \tilde{\alpha})}{\left[-r_+^4 + (2\tilde{\alpha} + 3Q^2)r_+^2 + 2Q^2\tilde{\alpha}\right]^2 (r_+^2 - Q^2)^3 (r_+^2 + 6\tilde{\alpha})^3 (\ln r_+)^2}\ , \tag{4.80}$$

where $N(r_+, Q, \tilde{\alpha})$ is a function that is finite at those points where the denominator vanishes. We see that curvature singularities occur at $r_+^2 = Q^2$, which is the point where the temperature vanishes and at the roots (4.78) which determine the points where second order phase transitions occur C_Q. The singularity situated at $\ln r_+ \to 0$ corresponds to a second order phase transition determined by the capacitance $C_S \equiv (\partial Q/\partial \Phi)_S = -1/(4 \ln r_+)$, according to Eq.(4.74). Finally, the singularity situated at $(r_+^2 + 6\tilde{\alpha}) = 0$ corresponds to the limit $S \to 0$ which indicates the breakdown of the thermodynamic picture of the black hole and, consequently, of GTD. A particular example of location of these curvature singularities is depicted in Fig.(4.15).

4.4 Conclusion

In this chapter, I analyzed the thermodynamics of static spherically symmetric black holes in the five dimensional EGB theory and its generalizations including an electromagnetic Maxwell field, a cosmological constant and a Yang-Mills field. To investigate the thermodynamics of these black holes I used two different approaches. The first one is based upon the Bekenstein-Hawking entropy relation, according to which the entropy of a black hole is proportional to the area of its event horizon. The second approach uses as a starting point a modified entropy relation which follows from the assumption that black holes satisfy the first law of thermodynamics in higher dimensions. The two approaches are not equivalent since the corresponding thermodynamic variables exhibit completely different behaviors. In particular, I noticed that the thermodynamics of black holes based upon the modified entropy formula depends on the value of the coupling constant of the GB term that appears in the action of the theory. Phase transitions appear which depend on the explicit value of the coupling constant and change the stability properties of the black holes. This kind of phase transitions is absent when the Bekenstein-Hawing entropy relation is used.

For all the black holes analyzed in this chapter, I used the formalism of GTD to find the geometric properties of the corresponding manifolds of equilibrium states. Once the thermodynamic fundamental equation of the black is given, a standard procedure of GTD allows us to compute the explicit form of the thermodynamic metric that describes the geometric properties of the equilibrium manifold. It turns out that the thermodynamic metrics depend on the entropy relation used to construct the thermodynamics of the black holes under consideration. The thermodynamic metrics obtained from the Bekenstein-Hawking relation are different from those obtained by using the modified entropy formula. Nevertheless, in all the analyzed cases I found that the curvature singularities of the equilibrium manifold are located at those points where the heat capacity diverges and the black hole undergoes a second order phase transition. Moreover, curvature singularities also appear at those points where the temperature vanishes, indicating the limit of applicability of black hole thermodynamics and of GTD.

I conclude that the formalism of GTD can be used in the EGB theory for five dimensional black holes to describe the thermodynamic properties in terms of geometric concepts, regardless of the entropy relation used to formulate the thermodynamics. In particular, all the black holes analyzed in the EGB theory can be characterized by non-flat equilibrium manifolds. This means that all those black holes possess an intrinsic non trivial thermodynamic interaction. Moreover, the thermodynamic interaction can be represented by means of the curvature of the equilibrium manifold, the points where the heat capacity diverges and, consequently, second order phase transitions occur, are represented in GTD by curvature singularities, indicating the limit of applicability of the thermodynamics and

GTD to black holes.

Chapter 5

Geometrothermodynamics in Hořava–Lifshitz Gravity

One of the biggest difficulties in attempts toward the theory of quantum gravity is the fact that GR is non-renormalizable. This would imply loss of theoretical control and predictability at high energies. In 2009 Petr Hořava formulated a new theory of gravity to evade this difficulty by invoking a Lifshitz-type anisotropic scaling at high energy [65, 66]. (I shall discuss it in detail in section 5.1.) This theory is called *Hořava–Lifshitz* (HL) *gravity*.

HL gravity , being a less restricted theory than GR, requires the temporal and spatial coordinates to be treated on different grounds. The theory itself is invariant under the so-called *foliation-preserving diffeomorphism*, which is a combination of global time reparametrizations and spatial diffeomorphisms. It is a field theoretical model which can be interpreted as a complete theory of gravity in the ultraviolet (UV) limit. The model is renormalizable and non-relativistic in the UV regime [102]. Moreover, it reduces to Einstein's gravity theory with a cosmological constant in the infrared (IR) limit.

5.1 Hořava–Lifshitz Gravity: Review and Basic Equations

Now I explain basics of HL gravity, such as power-counting argument, Lorentz symmetry, basic quantities, action and equations of motion.

Power–counting: Let us begin with heuristically explaining the usual power-counting argument in field theory. As the simplest example, consider a scalar field with the canonical kinetic term:

$$\frac{1}{2}\int dtd^3x\dot{\psi}^2, \tag{5.1}$$

where a dot represents differentiation with respect to the time coordinate. The scaling dimension of the scalar field ψ is determined by demanding that the kinetic term be invariant under the scaling

$$t \to lt, \quad x \to lx, \quad \psi \to l^{-s}\psi \tag{5.2}$$

where l is an arbitrary number and s is the scaling dimension to be determined. The invariance of the kinetic term under the scaling gives $s = 1$. In other words, the scalar field scales like energy. Thus n-th order interaction term behaves as

$$\int dtd^3x\dot{\psi}^2 \propto E^{(1+3-ns)}, \tag{5.3}$$

where E is the energy scale of the system under consideration. Here, the minus sign in the exponent comes from -1 in $E \to l^{-1}E$. Now, it is expected that we have a good theoretical control of UV regime, i.e., high E, behaviors if the exponent is non-positive. Since $s = 1$, this condition leads to $n \leq 4$. This is the power-counting renormalizability condition [65].

Unfortunately, Einstein gravity is not power-counting renormalizable. This is because the curvature is a highly nonlinear function of the metric. The nonrenormalizability is one of the difficulties in attempts to quantize GR.

Lorentz symmetry: As already stated, HL gravity is power-counting renormalizable. To evade such argument the basic idea is very simple, abandoning Lorentz symmetry and invoking a different kind of scaling in the UV regime [66]. The scaling invoked here, often called *anisotropic scaling* or *Lifshitz scaling*, is

$$x^i \to lx^i, \ t \to l^z t, \tag{5.4}$$

where z is a number called *dynamical critical exponent* and $i = 1, 2, 3$.

Let us now see how the power-counting argument changes if the scaling is anisotropic as in (5.4). Invariance of the canonical kinetic term (5.1) under this scaling leads to

$$s = \frac{3-z}{2}. \tag{5.5}$$

This of course recovers the previous result $s = 1$ for $z = 1$. The interesting fact here is that $s = 0$ if $z = 3$. This implies that, if $z = 3$, the amplitude of quantum fluctuations of ψ does not change as the energy scale of the system changes. The n-th order interaction term behaves as

$$\int dtd^3x\dot{\psi}^2 \propto E^{\frac{(z+3-ns)}{z}}. \tag{5.6}$$

For $z = 3$ (and thus $s = 0$), the exponent is negative for any n and, therefore, any nonlinear interactions are power-counting renormalizable. For $z > 3$, the theory is power-counting super–renormalizable [85]. From the above consideration, it is expected that gravity may become renormalizable if the anisotropic scaling with $z \geq 3$ is realized in the UV regime.

Basic Equations: Since the Lorentz invariance is not respected, the time coordinate

t and the spatial coordinates $x^i(i = 1, 2, 3)$ are treated separately. The fundamental symmetry of the theory is the invariance under space-independent time reparametrization and time-dependent spatial diffeomorphism:

$$t \to t'(t), \quad x \to \vec{x}(t, \vec{x}). \tag{5.7}$$

The time-dependent spatial diffeomorphism allows an arbitrary change of spatial coordinates on each constant time surface. However, the time reparametrization here is not allowed to depend on spatial coordinates. As a result, unlike GR, in HL gravity the foliation of spacetime by constant time hypersurfaces is not just a choice of coordinates but is a physical entity. Indeed, the foliation is preserved by the symmetry transformation (5.7). For this reason, the map (5.7) is called *foliation preserving diffeomorphism.* In addition to the foliation preserving diffeomorphism invariance, we assume that the theory is invariant under the *spatial parity* $\vec{x} \to -\vec{x}$ and the *time reflection* $t \to -t$.

The breaking of the four-dimensional diffeomorphism invariance, allows for a different treatment of the kinetic and potential terms for the metric. Thus, although the kinetic term is quadratic in time derivatives of the metric, the potential has higher-order space derivatives. In particular, the UV behavior of the potential is determined by the square of the Cotton tensor of the three-dimensional geometry. At large distances, higher derivative terms do not contribute and the theory reduces to standard GR. Indeed at $\lambda = 1$, which is an IR fixed point, standard GR is recovered [20].

Since HL gravity breaks general four-dimensional covariance and splits it into three-dimensional covariance plus reparametrization invariance of time. It is therefore convenient to formulate it in the $(3+1)$–ADM formalism, where an arbitrary metric can be written in the form

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{5.8}$$

where N^2 is the lapse function and N^i represents the shift. The shift N^i and the metric g_{ij} depend on both the time coordinate t and the spatial coordinates $\vec{x}$, the lapse function N is assumed to be a function of the time only. This condition on the lapse function is called the *projectability condition.* The projectability condition stems from the foliation preserving diffeomorphism. The lapse function represents a gauge freedom associated with the space-independent time reparametrization $t \to t'(t)$ and, thus, it is fairly natural to restrict it to be space independent.

Action: Then, the HL action is written as [85],

$$\mathcal{I}_{HL} = \int \mathcal{L}_{HL}\; dt\; d^3x\,, \tag{5.9}$$

where

$$\mathcal{L}_{HL} = \sqrt{g}N \left[\frac{2}{\kappa}\Big(K_{ij}K^{ij} - \lambda K^2\Big) + \frac{\kappa^2\mu^2(\Lambda R - 3\Lambda^2)}{8(1-3\lambda)} + \frac{\kappa^2\mu^2(1-4\Lambda)}{32(1-3\lambda)}R^2 - \right.$$
$$\left. - \frac{\kappa^2}{2\omega^4}\left(C_{ij} - \frac{\mu\omega^2}{2}R_{ij}\right)\left(C^{ij} - \frac{\mu\omega^2}{2}R^{ij}\right)\right] . \tag{5.10}$$

Here R_{ij} and R are the three-dimensional Ricci tensor and curvature scalar, respectively. Moreover, for a spacelike hypersurface with a fixed time the extrinsic curvature K_{ij} and the Cotton tensor C_{ij} are given by the expressions

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \ , \quad C^{ij} = \epsilon^{ikl}\nabla_k \left(R^j{}_l - \frac{1}{4}\ R\ \delta^j{}_l \right) . \tag{5.11}$$

Finally, κ^2, λ, μ, ω and Λ are constant parameters. In (5.10), the first two terms are the kinetic terms, while the others give the potential of the theory in the so-called "*detailed-balanced*" form.

In general, the IR vacuum in Hořava's theory is AdS spacetime. It was found that the Schwarzschild–AdS black hole solution is not recovered in the IR limit, although Einstein's theory with cosmological model was obtained at the level of the action [105]. This difficulty was solved by introducing an additional parameter which modifies the IR behavior [27,85, 100]. The vacuum of this theory turns out to be the AdS spacetime. In order to obtain a Minkowski vacuum in the IR sector, one can consider an additional term ($\mu^4 R$) in the original action. This generalization is known as the deformed HL model.

This chapter explores the thermodynamic geometries of the most general static, spherically symmetric, topological black holes of the HL gravity. A Legendre invariant metric derived in the context of GTD for the equilibrium manifold reproduces correctly the phase transition structure of these black holes. Moreover, the limiting cases in which the mass, the entropy or the Hawking temperature vanish are also analyzed which indicate the limit of applicability of the thermodynamics and the GTD of black holes. I then investigate the Einstein limit and the case of a black hole with flat horizon in this gravity theory.

5.2 Topological Black Hole Solutions in HL Gravity

Consider now the spherically symmetric line element

$$ds^2 = -\tilde{N}^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2 \,, \tag{5.12}$$

where $d\Omega_k^2$ is the line element of the 2–dimensional Einstein space with constant curvature $2k$. Substituting the metric (5.12) into the action (5.9), we obtain [29],

$$\mathcal{I}_{HL} = \frac{\kappa^2\mu^2\Omega_k}{8(1-3\lambda)} \int \tilde{N}\left[\frac{(\lambda-1)}{2}F'^2 - \frac{2\lambda}{r}FF' + \frac{(2\lambda-1)}{r^2}F^2\right]\ dt\ dr \,, \tag{5.13}$$

where a prime denotes the derivative with respect to r and F is defined as,

$$F(r) = k - \Lambda r^2 - f(r)\,. \tag{5.14}$$

The variation of (5.13) leads to the following set of equations

$$\left(\frac{2\lambda}{r}F - (\lambda-1)F'\right)\tilde{N}' + (\lambda-1)\left(\frac{2}{r^2}F - F''\right)\tilde{N} \;=\; 0\,, \tag{5.15}$$

$$(\lambda-1)r^2F'^2 - 4\lambda rFF' + 2(2\lambda-1)F^2 \;=\; 0\,, \tag{5.16}$$

whose solution is

$$F(r) = \alpha r^s\,, \quad \tilde{N} = \gamma r^{1-2s}\,, \tag{5.17}$$

where α and γ are integration constants and s is given by

$$s = \frac{2\lambda - \sqrt{2(3\lambda-1)}}{\lambda-1}\,. \tag{5.18}$$

This solution was obtained recently by Cai, Cao and Ohta (CCO) [30]. In general, the value of s with a positive sign in front of the square root is also a solution of the above equations. However, in this case the asymptotic properties of the solution are not compatible with the properties of a black hole spacetime. In the allowed interval $\lambda > 1/3$, i. e. for $s \in (-1, 2)$, the above solution is asymptotically AdS and describes the gravitational field of a static black hole.

In order to obtain the thermodynamic variables of the CCO black hole, it is necessary to use the canonical Hamilton formulation for the corresponding thermodynamic ensemble [29]. According to this Hamiltonian approach, the mass of the black hole is

$$M = \frac{c^3\gamma\Omega_k l^{2-2s}}{16\pi G}\left(\frac{1+s}{2-s}\right)\left(\frac{k + \frac{r_+^2}{l^2}}{(\frac{r_+}{l})^s}\right)^2\,, \tag{5.19}$$

where $l^2 = -1/\Lambda$ represents the radius of curvature. Moreover, the Hawking temperature is given by

$$T = \frac{\gamma}{4\pi r_+^{2s}}\left[(2-s)\frac{r_+^2}{l^2} - ks\right]\,. \tag{5.20}$$

Finally, integrating the first law of thermodynamics, $dM = TdS + \mu_i dQ^i$, for constant values of the additional thermodynamic variables Q^i, the entropy associated with the black hole is obtained as

$$S = \frac{c^3\Omega_k l^2}{4G}\left(\frac{1+s}{2-s}\right)\left(\frac{r_+^2}{l^2} + k\ln\frac{r_+^2}{l^2}\right)\,. \tag{5.21}$$

The entropy S is defined up to an additive constant that can be chosen arbitrarily in order to avoid zero or negative values. Here, r_+ represents the radius of the exterior horizon which is a function of M and l determined by the algebraic equation

$$r_+^s - \frac{\mathbb{A}}{M^{\frac{1}{2}}l^2}r_+^2 - \frac{\mathbb{A}k}{M^{\frac{1}{2}}} = 0\,, \quad \mathbb{A} = \frac{\kappa\mu\gamma^{\frac{1}{2}}\Omega_k^{\frac{1}{2}}}{2^{\frac{7}{4}}[3\lambda-1]^{\frac{1}{4}}}\,. \tag{5.22}$$

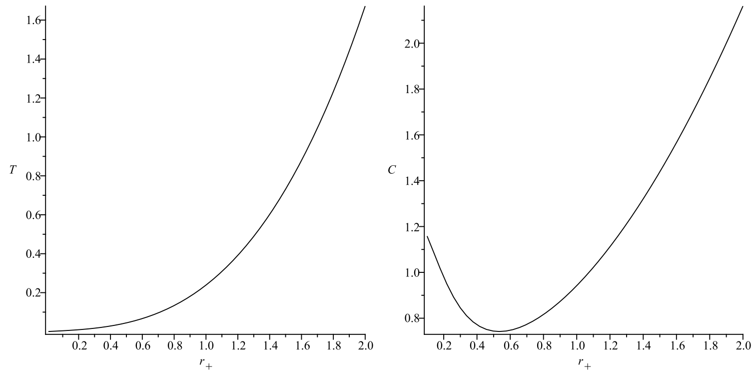

Figure 5.1: Temperature and heat capacity of the topological black hole. I choose $k = 1$, $l = 1$ and $s = -1/2$ so that $T > 0$ for $r_+ > 0$. Moreover, $\gamma = 1$ and $\Omega_k c^3/(4G) = 1$ for simplicity. The heat capacity is a smooth function in the entire region $r_+ > 0$ with a minimum located at $r_+ \approx 0.53$.

Using the expressions (5.19) and (5.20), we obtain the heat capacity

$$C = \left(\frac{\partial M}{\partial r_+}\right)\left(\frac{\partial T}{\partial r_+}\right)^{-1} = \frac{c^3\Omega_k l^2}{4G}\left(\frac{1+s}{2-s}\right)\frac{\left(k+\frac{r_+^2}{l^2}\right)\left((2-s)\frac{r_+^2}{l^2}-ks\right)}{(s-1)(s-2)\frac{r_+^2}{l^2}+ks^2}\ . \tag{5.23}$$

According to Davies [44–46], second order phase transitions take place at those points where the heat capacity diverges, i. e., for

$$\frac{r_+^2}{l^2} = \frac{ks^2}{(s-1)(2-s)}\ . \tag{5.24}$$

Then, I conclude that phase transitions can occur only for $k = 1$ and $s \in (1, 2)$ and for $k = -1$ and $s \in (-1, 1)$. For all the remaining values of k and s the corresponding black hole cannot undergo a phase transition. Notice that the phase transition condition (5.24) must be considered together with the inequality

$$\frac{r_+^2}{l^2} > \frac{ks}{2-s} \tag{5.25}$$

that follows from the condition $T > 0$ from Eq.(5.20). The behavior of heat capacity and temperature of the topological black hole is depicted in Fig.(5.1).

5.3 Weinhold and Ruppeiner Geometries

According to Eqs.(5.19)–(5.21), the mass of the CCO black hole is a function of the entropy S and the curvature radius l. Although the entropy is clearly a thermodynamic variable, the thermodynamic nature of the radius of curvature is not so obvious. Nevertheless, a detailed analysis [32] of the thermodynamic properties of AdS black holes reveal that indeed it is possible to consider the cosmological constant as a well-defined thermodynamic variable. Here, I follow this result and assume that the radius of curvature is a thermodynamic variable.

Let us consider first the Weinhold metric Eq.(1.67). Since in the case of black holes the internal energy is represented by the mass M, the Weinhold metric becomes

$$g^W = M_{SS}dS^2 + 2M_{Sl}dSdl + M_{ll}dl^2 \ , \tag{5.26}$$

where $M_S = \partial M/\partial S$, etc. Since the expression for M as given in Eq.(5.19), does not contain S explicitly, it is necessary to use r_+ as a coordinate and its relation to S and l by means of Eqs.(5.20) and (5.21). Then, we obtain

$$g^W = M_{SS}S_{r_+}^2 dr_+^2 + 2\left(M_{SS}S_l + M_{Sl}\right)S_{r_+}dr_+dl + \left(M_{SS}S_l^2 + 2M_{Sl}S_l + M_{ll}\right)dl^2 \ . \tag{5.27}$$

Introducing the thermodynamic Eqs.(5.19)–(5.21), the explicit metric components are given by

$$g^W_{r_+r_+} = \frac{c^3\Omega_k\gamma}{4\pi G}\left(\frac{1+s}{2-s}\right)l^{4-2s}\left(\frac{r_+}{l}\right)^{-2-2s}\left(k+\frac{r_+^2}{l^2}\right)\left(ks^2+(s^2-3s+2)\frac{r_+^2}{l^2}\right) \ , \tag{5.28}$$

$$\begin{aligned} g_{lr_+} = \ & \frac{c^3\Omega_k\gamma}{4\pi G r_+}\left(\frac{1+s}{2-s}\right)l^{-2s}\left(\frac{r_+}{l}\right)^{-1-2s}\Bigg\{k\left(ks^2+(s^2-3s+2)\frac{r_+^2}{l^2}\right)\ln\frac{r_+^2}{l^2} \\ & +(s-2)\frac{r_+^2}{l^2}\left(\frac{r_+^2}{l^2}-k(s-2)\right)-k^2s^2\Bigg\} \ , \end{aligned} \tag{5.29}$$

$$\begin{aligned} g^W_{ll} = \ & \frac{c^3\Omega_k\gamma}{8\pi G(k+r_+^2/l^2)r_+^{2s}}\left(\frac{1+s}{2-s}\right)\Bigg\{2k^2\left(ks^2+(s^2-3s+2)\frac{r_+^2}{l^2}\right)\ln^2\frac{r_+^2}{l^2} \\ & +4k\left[(s-2)\frac{r_+^2}{l^2}\left[\frac{r_+^2}{l^2}-k(s-2)\right]-k^2s^2\right]\ln\frac{r_+^2}{l^2}+3\frac{r_+^6}{l^6}+k(11-4s)\frac{r_+^4}{l^4} \\ & +k^2(2s^2-10s+13)\frac{r_+^2}{l^2}+k^3(1+2s^2)\Bigg\} \ . \end{aligned} \tag{5.30}$$

Moreover, the curvature scalar can be expressed as

$$R^W = \frac{N^W}{D^W} \ , \quad D^W = (s^2-s-2)r_+^6+k(s^2+8s-8)l^2r_+^4+k^2(s^2-3s+2)l^4r_+^2+k^3s^2l^6 \ , \tag{5.31}$$

where N^W is a function of r_+ and l which is finite at those points where the denominator vanishes. The singularities are determined by the roots of the equation $D^W = 0$. It is easy to see that the solutions of this equation do not coincide with the points where the heat capacity diverges. The concrete example of a stable black hole in which a curvature singularity exists at a point where the heat capacity is regular as shown in Fig.(5.2). I conclude that the Weinhold curvature fails to reproduce the phase transition structure of the CCO topological black hole.

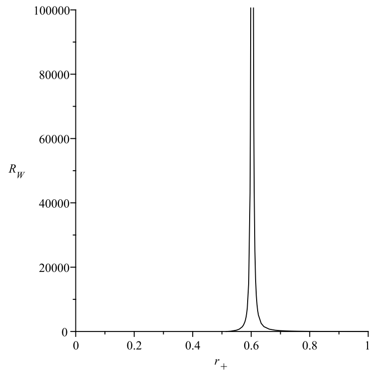

Figure 5.2: The curvature of the topological black hole. Here, I chose $k = 1$, $l = 1$ and $s = -1/2$ as representative values. Moreover, $\gamma = 1$ and $\Omega_k c^3/(4G) = 1$ for simplicity. The curvature shows a singularity at $r_+ \approx 0.60$.

Now I consider the Ruppeiner metric

$$g^R = S_{MM} dM^2 + 2S_{Ml} dM dl + S_{ll} dl^2 \ , \tag{5.32}$$

which in terms of the coordinates r_+ and l becomes

$$g^R = S_{MM} M_{r_+}^2 dr_+^2 + 2\left(S_{Ml} + S_{MM} M_l\right) M_{r_+} dr_+ dl + \left(S_{MM} M_l^2 + 2S_{Ml} M_l + S_{ll}\right) dl^2 \ . \tag{5.33}$$

Using the expressions for the thermodynamic variables (5.19) and (5.21), we obtain

$$g^R_{r_+r_+} = \frac{c^3\Omega_k}{G}\left(\frac{1+s}{2-s}\right)\frac{l^2}{r_+^2}\left(k+\frac{r_+^2}{l^2}\right)\frac{\left[s^2\left(k+\frac{r_+^2}{l^2}\right)+(2-3s)\frac{r_+^2}{l^2}\right]}{\left[s\left(k+\frac{r_+^2}{l^2}\right)-2\frac{r_+^2}{l^2}\right]} \ , \tag{5.34}$$

$$g^R_{lr_+} = -\frac{c^3\Omega_k}{G}\left(\frac{1+s}{2-s}\right)\frac{l}{r_+}\left(k+\frac{r_+^2}{l^2}\right)\frac{\left[s^2k\left(k-\frac{r_+^2}{l^2}\right)-k(s-2)\frac{r_+^2}{l^2}-(2s^2-7s+6)\frac{r_+^4}{l^4}\right]}{\left[s\left(k+\frac{r_+^2}{l^2}\right)-2\frac{r_+^2}{l^2}\right]^2} \ , \tag{5.35}$$

$$\begin{aligned} g^R_{ll} &= \frac{c^3\Omega_k}{2G}\left(\frac{1+s}{2-s}\right)\frac{1}{\left[s\left(k+\frac{r_+^2}{l^2}\right)-2\frac{r_+^2}{l^2}\right]^3}\Bigg\{2ks\frac{r_+^2}{l^2}\left[(11s-19)\frac{r_+^4}{l^4}+7k(2s-1)\frac{r_+^2}{l^2}\right. \\ &+ \left. k^2(1+7s)\right] + k\ln\left(\frac{r_+^2}{l^2}\right)\left[(s^3-6s^2+12s-8)\frac{r_+^6}{l^6}+ks(3s^2-12s+12)\frac{r_+^4}{l^4}\right. \\ &+ \left. 3k^2s^2(s-2)\frac{r_+^2}{l^2}+k^3s^3\right] + k\left[(20-3s^3)\frac{r_+^6}{l^6}-k(20+9s^3)\frac{r_+^4}{l^4}+k^2(4-9s^3)\frac{r_+^2}{l^2}\right. \\ &+ \left. k^3s^2(2-3s)\right] + (6s^2-22s+20)\frac{r_+^8}{l^8}\Bigg\} \ . \end{aligned} \tag{5.36}$$

From these expressions for the metric functions it is then straightforward to find the scalar curvature

$$R^R = \frac{N^R}{D^R} \ , \quad D^R = (s+1)l^{10}\left[A\ln\left(\frac{r_+^2}{l^2}\right)-B\right]^2\left(k+\frac{r_+^2}{l^2}\right)^2\left[s\left(k+\frac{r_+^2}{l^2}\right)-2\frac{r_+^2}{l^2}\right]^3 , \tag{5.37}$$

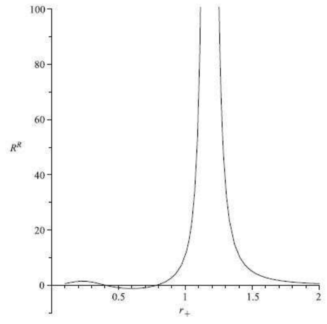

Figure 5.3: Thermodynamic curvature of the Ruppeiner metric. This case corresponds to the choice: $k = 1$, $l = 1$, $s = -1/2$, $\gamma = 1$ and $\Omega_k c^3/(4G) = 1$. A singularity exists at $r_+ \approx 1.18$ which does not correspond to a phase transition.

with

$$A = kl^6\left[\frac{r_+^4}{l^4}(s^3 - 5s^2 + 8s - 4) + \frac{r_+^2}{l^2}(2s^2 - 5s + 2)ks + k^2s^3\right] , \tag{5.38}$$

$$B = kl^6\left[\frac{r_+^4}{l^4}(3s^3 - 13s^2 + 16s - 4) + \frac{r_+^2}{l^2}(6s^2 - 15s + 6)ks + 3k^2s^3\right] + r_+^6(2s^2 - 8s + 8) . \tag{5.39}$$

The curvature singularities are determined by the roots of the equation $D^R = 0$ which do not coincide with the points where the heat capacity (5.23) shows second order phase transitions. To illustrate the behavior of the curvature I analyze the particular case with $k = 1$, $l = 1$ and $s = -1/2$ for which the temperature is always positive and the heat capacity is a smooth positive function which corresponds to a stable black hole configuration. The numerical analysis of this case is depicted in Fig.(5.3).

5.4 Geometrothermodynamics of the CCO Black Holes

The formalism of GTD is invariant and, consequently, one can choose any arbitrary thermodynamic potential Φ in any arbitrary representation to describe the thermodynamics of a black hole. Take, for instance, $\Phi = M$ for the CCO topological black holes presented in section 5.2. The coordinates of the 5-dimensional phase manifold can be chosen as $Z^A = (M, S, l, T, L)$, where T is the temperature dual to S and L is the dual of the curvature radius l. The fundamental 1-form is then $\Theta = dM - TdS - ldL$ and the Legendre invariant metric (2.4) is written as

$$G = \Theta^2 + ST\left(-dSdT + dldL\right) . \tag{5.40}$$

The smooth map $\varphi : \mathcal{E} \to \mathcal{T}$ or in coordinates $\varphi : (S, l) \mapsto [M(S, l), S, l, T(S, l), L(S, l)]$ determines the equilibrium manifold $\mathcal{E}$ with metric

$$g^{GTD} = \varphi^*(G) = SM_S\left(-M_{SS}dS^2 + M_{ll}dl^2\right) , \tag{5.41}$$

on which the first law of thermodynamics $dM = TdS + ldL$ and the equilibrium conditions

$$T = \frac{\partial M}{\partial S} \equiv M_S \ , \quad L = \frac{\partial M}{\partial l} \equiv M_l \tag{5.42}$$

hold. As mentioned above, the fundamental equation $M = M(S,l)$ cannot be written explicitly and so I use r_+ instead of S as a coordinate. Then,

$$g^{GTD} = SM_S\left[-M_{SS}S_{r_+}^2 dr_+^2 - 2M_{SS}S_{r_+}S_l dldr_+ + (M_{ll} - M_{SS}S_l^2)dl^2\right] \ . \tag{5.43}$$

Using the expressions for the mass and the entropy, we obtain

$$\begin{aligned} g^{GTD} &= \frac{c^3\Omega_k^2\gamma^2 r_+^{-4s}}{16\pi^2 G}\left(\frac{s+1}{2-s}\right)\left[(s-2)\frac{r_+^2}{l^2} + ks\right] S(r_+,l)\Bigg\{\left[(s-1)(s-2)\frac{r_+^2}{l^2} + ks^2\right] \\ &\times \left[\left(1 + \frac{kl^2}{r_+^2}\right)dr_+^2 + 2\frac{kl}{r_+}\left(\ln\frac{r_+^2}{l^2} - 1\right)dldr_+\right] + C(r_+,l)dl^2\Bigg\} \ , \end{aligned} \tag{5.44}$$

where

$$C(r_+,l) = \frac{2k^2\left(\ln\frac{r_+^2}{l^2} - 1\right)^2\left[(s-1)(s-2)\frac{r_+^2}{l^2} + ks^2\right] - \left(k^2 + 3\frac{r_+^4}{l^4}\right)\left(k + \frac{r_+^2}{l^2}\right)}{2\left(k + \frac{r_+^2}{l^2}\right)}. \tag{5.45}$$

The curvature scalar corresponding to the metric (5.43) is found to be

$$R^{GTD} = \frac{N^{GTD}}{D^{GTD}}, \tag{5.46}$$

where

$$\begin{aligned} D^{GTD} &= \left(k^2 + 3\frac{r_+^4}{l^4}\right)^2\left(k + \frac{r_+^2}{l^2}\right)^{13}\left(\frac{r_+^2}{l^2} + k\ln\frac{r_+^2}{l^2}\right)^3 \\ &\times \left[(2-s)\frac{r_+^2}{l^2} - ks\right]^4\left[(s-1)(s-2)\frac{r_+^2}{l^2} + ks^2\right]^2 . \end{aligned} \tag{5.47}$$

Here N^{GTD} is a function of r_+ and l that is finite at those points where the denominator vanishes. There are several curvature singularities in this case. The first one occurs if $k + r_+^2/l^2 = 0$ and corresponds to the limit $M \to 0$, as follows from Eq.(5.19). A second singularity is located at the roots of the equation $r_+^2 + k\ln r_+^2 = 0$ and can be interpreted from Eq.(5.21) as the limit $S \to 0$. Moreover, according to Eq.(5.20), the singularity situated at $r_+^2/l^2 = ks/(2-s)$ corresponds to the limit $T \to 0$.

Finally, if $(s-1)(s-2)r_+^2/l^2 + ks^2 = 0$ a singularity occurs that, according to Eq.(5.23), coincides with the limit $C \to \infty$, i.e., with the points where second order phase transitions take place. Clearly, the singularities at which the mass, the entropy or the temperature vanish must be considered as unphysical and indicate the limit of applicability of the thermodynamics of black holes.

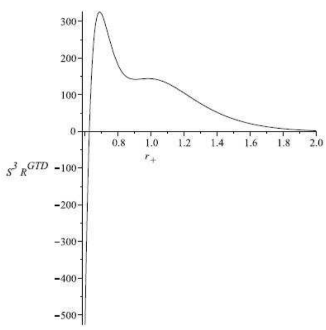

Figure 5.4: Thermodynamic curvature of the GTD metric. This case corresponds to the choice: $k = 1$, $l = 1$, $s = -1/2$, $\gamma = 1$ and $\Omega_k c^3/(4G) = 1$. The expression $S^3 R^{GTD}$ is plotted to avoid the unphysical singularity at $S = 0$.

The thermodynamic curvature in GTD for the case $k = 1$, $l = 1$ and $s = -1/2$ shows a singularity at the value $r_+^2 + \ln r_+^2 = 0$, i. e., for $r_+ \approx 0.75$, which corresponds to the limit $S \to 0$. The behavior of the curvature $S^3 R^{GTD}$ to avoid the unphysical singularity as $S \to 0$ is illustrated in Fig.(5.4). We see that in the analyzed interval no curvature singularities appear. This is in accordance with the behavior of the heat capacity which in the same interval is free of phase transitions as illustrated in Fig.(5.2). A curvature singularity can be observed for $r_+ \to 0$ which indicates the break down of the black hole configuration and, consequently, of its thermodynamics.

I conclude that the curvature obtained within the formalism of GTD correctly describes the thermodynamic behavior of topological black hoes in HL gravity.

5.4.1 The Einstein Limit of the CCO Black Holes

Einstein's theory of gravity with cosmological constant is obtained from the HL gravity in the limit $\lambda \to 1$. The CCO topological black holes reduce in this case to a single black hole configuration with $s = 1/2$, whereas the corresponding thermodynamic variables are written as

$$M = \frac{c^3 \Omega_k l}{16\pi G} \frac{l}{r_+} \left(k + \frac{r_+^2}{l^2} \right)^2 , \tag{5.48}$$

$$S = \frac{c^3 \Omega_k l^2}{4G} \left(\frac{r_+^2}{l^2} + k \ln \frac{r_+^2}{l^2} \right) , \tag{5.49}$$

$$T = \frac{\gamma}{8\pi r_+} \left(3\frac{r_+^2}{l^2} - k \right) , \tag{5.50}$$

and

$$C = \left(\frac{\partial M}{\partial r_+} \right) \left(\frac{\partial T}{\partial r_+} \right)^{-1} = \frac{c^3 \Omega_k l^2}{2G} \frac{\left(k + \frac{r_+^2}{l^2} \right) \left(3\frac{r_+^2}{l^2} - k \right)}{\left(k + 3\frac{r_+^2}{l^2} \right)} . \tag{5.51}$$

It follows that the second order phase transitions take place at the points where the condition $3r_+^2/l^2 + k = 0$ is satisfied.

Inserting the expressions (5.48) and (5.49) into the metric (5.43), we obtain

$$g^{GTD} = \frac{1}{32}\left(\frac{c^3\Omega_k\gamma}{4G}\right)^2 \left(\frac{r_+^2}{l^2} + k\ln\frac{r_+^2}{l^2}\right)\left(k - 3\frac{r_+^2}{l^2}\right)\frac{l^2}{r_+^2}\left(A_1 dr_+^2 + 2A_2 dl dr_+ + A_3 dl^2\right) , \tag{5.52}$$

with

$$A_1 = \frac{l^2}{r_+^2}\left(k + \frac{r_+^2}{l^2}\right)\left(k + 3\frac{r_+^2}{l^2}\right) , \tag{5.53}$$

$$A_2 = k\frac{l}{r_+}\left(\ln\frac{r_+^2}{l^2} - 1\right)\left(k + 3\frac{r_+^2}{l^2}\right) , \tag{5.54}$$

$$A_3 = \frac{1}{k + \frac{r_+^2}{l^2}}\left[k^2\left(k + 3\frac{r_+^2}{l^2}\right)\left(\ln\frac{r_+^2}{l^2} - 2\right)\ln\frac{r_+^2}{l^2} - 6\frac{r_+^4}{l^4}\left(k + \frac{r_+^2}{l^2}\right) - k^2\left(k - \frac{r_+^2}{l^2}\right)\right] . \tag{5.55}$$

The corresponding scalar curvature can be expressed as

$$R^{GTD} = \frac{N^{GTD}}{D^{GTD}} , \tag{5.56}$$

$$D^{GTD} = \left(k + \frac{r^2}{l^2}\right)^4\left(\frac{r^2}{l^2} + k\ln\frac{r^2}{l^2}\right)^3\left(3\frac{r^2}{l^2} - k\right)^3\left(3\frac{r^4}{l^4} + k^2\right)^2\left(3\frac{r^2}{l^2} + k\right)^2 , \tag{5.57}$$

where the numerator function N^{GTD} is finite at all the points where the denominator vanishes. From this expression we can see that the roots of the equation $3r_+^2/l^2 + k = 0$ determine curvature singularities which coincide with the points where second order phase transitions occur ($C \to \infty$). Additional singularities occur if $r_+^2/l^2 + k = 0$, $r_+^2/l^2 + k\ln(r_+^2/l^2) = 0$, or $3r_+^2/l^2 - k = 0$ which correspond to the limits $M \to 0$, $S \to 0$ or $T \to 0$, respectively.

5.4.2 The Limiting Black Hole with Flat Horizon

Thermodynamics of a CCO black hole with a flat horizon ($k = 0$) is described by the following variables [29]

$$M = \frac{c^3\gamma\Omega_k l^{2-2s}}{16\pi G}\left(\frac{1+s}{2-s}\right)\left(\frac{r_+}{l}\right)^{2(2-s)} , \tag{5.58}$$

$$S = \frac{c^3\Omega_k l^2}{4G}\left(\frac{1+s}{2-s}\right)\left(\frac{r_+}{l}\right)^2 , \tag{5.59}$$

$$T = \frac{\gamma l^{-2s}}{4\pi}(2-s)\left(\frac{r_+}{l}\right)^{2-2s} , \tag{5.60}$$

and

$$C = \left(\frac{\partial M}{\partial r_+}\right)\left(\frac{\partial T}{\partial r_+}\right)^{-1} = \frac{c^3\Omega_k}{4G}\left(\frac{1+s}{2-s}\right)\frac{r_+^2}{s-1} = \frac{S}{s-1} . \tag{5.61}$$

From the expression for the heat capacity we see that this black hole is free of phase transitions.

Using the relations (5.58) and (5.59), the thermodynamic metric (5.43) is written as

$$g^{GTD} = \frac{(s-2)r_+^6}{8}\left(\frac{c^3\gamma\Omega_k}{4\pi G}\right)^2\left(\frac{1+s}{s-2}\right)^2\left(\frac{r_+}{l}\right)^{-4s}$$
$$\times\left[2(s-1)(s-2)l^{-4(s+1)}dr_+^2 - 3l^{-2(3+2s)}r_+^2 dl^2\right] , \qquad (5.62)$$

for which the curvature $R = 0$, indicating that no phase transition structure exists. This is in accordance with the result obtained above from the study of the heat capacity.

According to GTD, a flat equilibrium manifold is a consequence of the lack of thermodynamic interaction. This can be understood in the following way. For this special case, Eq.(5.59) indicates that the horizon radius is

$$r_+ = \left[\frac{4G}{\Omega_k c^3}\left(\frac{2-s}{1+s}\right)S\right]^{\frac{1}{2}} , \qquad (5.63)$$

so that Eq.(5.58) generates the explicit fundamental equation

$$M = \frac{\gamma}{4\pi}\left(\frac{4G}{\Omega_k c^3}\right)^{1-s}\left(\frac{2-s}{1+s}\right)^{1-s}\frac{S^{2-s}}{l^2} , \qquad (5.64)$$

which, in turn, can be rewritten as

$$(2-s)\ln S = \ln M + \ln l^2 + \ln S_0 , \quad S_0 = \frac{4\pi}{\gamma}\left(\frac{c^3\Omega_0}{4G}\frac{1+s}{2-s}\right)^{1-s} . \qquad (5.65)$$

So, the entropy function can be separated in the extensive variables M and l. On the other hand, all thermodynamic potentials that possess the property of being separable have been shown [139] to correspond to systems with no thermodynamic interaction and zero thermodynamic curvature. This is an indication that the statistical internal structure of a CCO black hole with flat horizon is equivalent to that of an ideal gas which is the main example of a system with no intrinsic thermodynamic interaction. I notice that in this limiting case the Weinhold and Ruppeiner geometries are flat too, indicating that there exists a statistical analogy between a black hole with flat horizon and an ideal gas.

5.5 Conclusion

In this chapter, I applied the formalism of GTD to describe the thermodynamics of the CCO topological black holes in the Hořava-Lifshitz model of quantum gravity. In the thermodynamic phase manifold a particular Riemannian metric is introduced whose main property is its invariance with respect to LT. The Legendre invariant metric induces in a canonical manner a thermodynamic metric in the equilibrium manifold which is defined as a submanifold of the thermodynamic phase manifold.

I used the expressions of the main thermodynamic variables of the CCO black holes in order to compute the explicit form of the thermodynamic metric of the equilibrium manifold. The corresponding thermodynamic curvature turned out to be nonzero in general, indicating the presence of thermodynamic interaction. Moreover, it was shown that the phase transitions which are characterized by divergencies of the heat capacity are described in GTD by curvature singularities in the equilibrium manifold. I also studied the thermodynamics of the CCO black holes by using Weinhold and Ruppeiner geometries and found that they fail to describe the corresponding phase transition structure.

It is found that the geometrothermodynamic equilibrium manifold of the CCO black holes present additional curvature singularities which correspond to the vanishing of the mass, entropy and Hawking temperature. In general, the vanishing of these thermodynamic variables is an indication of the limit of applicability of black hole thermodynamics. So I conclude that the formalism of GTD breaks down, with curvature singularities, exactly at those points where black hole thermodynamics fails.

In the context of the GTD of the CCO black holes, the limit of Einstein gravity and of a black holes with flat horizons is also analyzed. In both cases the obtained results are consistent with the thermodynamics of the respective black hole configurations. It turned out that the equilibrium manifold of black holes with flat horizons is flat. The flatness of the equilibrium manifold is interpreted as a consequence of the lack of intrinsic thermodynamic interaction. This property resembles the statistical behavior of an ideal gas.

Chapter 6

Summary and Future Directions

Geometrical methods always played an important role in thermodynamics. They not only facilitate the analysis of thermodynamic systems of equilibrium states, but also give a better understanding and deeper insight into the mathematical structure of the theory. The empirical laws of phenomenological thermodynamics can be expressed in a mathematically rigorous and concise way if one uses the language of differential and contact geometry. This approach to problems of equilibrium thermodynamics was originated by Hermann [63] and further developed by Mrugala [95].

Another approach to the geometry of thermodynamics is based on the concept of the distance between thermodynamic states. On a purely phenomenological level, it was initiated by Weinhold [141] who introduced a sort of Riemannian metric into the space of thermodynamic parameters by means of a scalar product of some reference vectors, tangent to the manifold of thermodynamic states. Many authors later discussed the physical consequences which resulted from the Weinhold's approach.

Ruppeiner [125] included the theory of fluctuations in the axioms of thermodynamics and showed that this leads to a Riemannian metric on a manifold of equilibrium states. Elements of this metric were represented by the second moments of fluctuations of some parameters. He also proposed to connect the Riemann curvature of the thermodynamic manifold with interparticle effective strength of interaction in the system. However, it turned out that the two metrics introduced by Weinhold and Ruppeiner are conformally equivalent [97, 127].

These two approaches have been the subject of analysis since the 1970s. The multiple applications of Ruppeiner's metric have been reviewed in [72, 73, 132]. Weinhold [141] and Ruppeiner [126] argued that the thermodynamic curvature of metrics (1.67) and (1.69) can be used as a measure of thermodynamic interaction. This argument was supported by the result that the thermodynamic curvature is zero for an ideal gas and nonvanishing for a van der Waals gas with singularities at those points where phase transitions occur. However, the correctness of these results was the subject of many discussions because they depend on the thermodynamic potential. Similar problems were found in the case of black hole thermodynamics where, for instance, the thermodynamic curvature of the RN black hole is zero in the entropy representation and nonzero in the energy representation. This unpleasant property means that a given thermodynamic system has different properties, when different thermodynamic potentials are used.

Recently, the formalism of GTD was proposed as an approach which allows us to handle thermodynamic systems in terms of geometric structures which are invariant with respect to LT, i.e., independent of the thermodynamic potential [112]. It explains why Weinhold's and Ruppeiner's metrics are not suitable to describe thermodynamics in an independent manner [115]. The formalism of GTD can be used to derive Legendre invariant metrics which, in particular, describe black hole thermodynamics in a unified and consistent manner [2, 116, 119, 135].

In this thesis, the thermodynamics of black holes is reformulated within the context of the formalism of GTD. Using this formalism, different thermodynamic metrics are derived for the space of equilibrium states of black holes which reproduces the thermodynamic behavior of black holes in standard (chapter 2) and non-standard (chapters 3-5) theories of gravity. The results shows that the curvature of derived thermodynamic metric can be used as a measure of thermodynamic interaction for black holes.

In Einstein-Maxwell theory, the space of equilibrium states,equipped with thermodynamic metric (2.12) derived in the context of GTD, becomes singular at those points where phase transitions occur and it is flat in the limit of extreme black holes that is when the two horizons coincide. This indicates that the derived thermodynamic metric is well-defined in the region $M^4 - M^2Q^2 - J^2$, except at the phase transition points where it becomes singular. Outside this region, the thermodynamic metric is not well-defined when the fundamental equation becomes complex and cannot be used to generate the geometric Riemannian structure of the space of equilibrium states. This is an indication that the thermodynamic description of black holes cannot be extended into the region of naked singularities. This is also an indication that classical thermodynamics cannot be used for black holes of the size of the Planck length, which is the extremal limit of applicability one would expect for classical thermodynamics.

In this thesis, I assumed Davies [44–46] formulation of phase transitions for black holes. However, the interpretation of divergences in specific heats as phase transitions is not definitely settled and is still a subject of debate [43, 74, 75, 106]. In fact, what is really needed is a microscopic description which would couple to the macroscopic thermodynamics of black holes. However, such a microscopic description must be related to a theory of quantum gravity which is still far from being formulated in a consistent and unified manner. In the meantime, one can only use the intuitive interpretation of phase transitions as it is known in classical thermodynamics. The thermodynamic metrics proposed in this thesis are intuitively simple and satisfy the mathematical compatibility conditions of GTD. It would be interesting to investigate the stability of the metrics derived in this thesis, especially in the different scenarios available in black hole thermodynamics. Legendre invariance is an important element of GTD. It limits the number of metrics that can be used to describe classical thermodynamics in terms of geometric concepts. It is also essential in order to obtain results that are independent of the choice of extensive variables and thermodynamic potential. A different point of view, in which for a given thermodynamic system there exists a preferred thermodynamic potential [115], is necessary in order to explain the vanishing of Ruppeiner's thermodynamic curvature in cases where thermodynamic interaction is present. The classical thermodynamics, which is Legendre invariant, must not be changed when one tries to represent it in terms of geometric structures.

It is also hoped that GTD can be used further to investigate the black holes. In particular, it is now possible to ask the question about the statistical model which in the corresponding limit leads to the black hole thermodynamics we are now correctly handling in GTD. In fact, it is known that from a given partition function of a statistical system it is possible to construct a thermodynamic metric (the Fisher-Rao metric) which, however, is not Legendre invariant. It seems reasonable to ask the question whether it is possible to derive the partition function from a given invariant thermodynamic metric. Preliminary results indicate that the answer to this question is affirmative. In this way, starting from the invariant metric which describes black holes thermodynamics, one would obtain a partition function that corresponds to a specific statistical model of black holes. This would represent an alternative way of exploring the *"interior"* of black holes without using any of the known approaches to quantum gravity. Another possible application of GTD is in the context of non-equilibrium thermodynamics. This formalism allows us to handle correctly, from the mathematical point of view, the dependence of the thermodynamic variables in terms of spacetime coordinates. In fact, there are several possibilities to introduce such a dependence. The question is whether these mathematical models could be used to propose a consistent model for non-equilibrium thermodynamics.

It is important to note that the GTD analysis may be used to find classical statistical models for black holes also in the higher dimensional theories, to analyze their physical and geometric properties, to explore the possibility of generalizing them to include quantum statistical models, using the standard approach of quantum statistical physics, and to investigate possible quantum effects.

The formalism of GTD may be used to find the statistical ensembles of black holes in higher dimensional theories, for instance EGB theory, by using, for instance, the Euclidean action method [32], and to analyze the physical differences between the canonical, grand-canonical and micro-canonical ensembles. It would be interesting to find a generalization of the Fisher-Rao statistical metric which coincides in the *"thermodynamic limit"* with the invariant thermodynamic metrics obtained in [135], to investigate the geodesics and the curvature of this generalized metric, considering its statistical origin and the different ensembles. In this context, it will be necessary to define the thermodynamic limit in accordance to GTD.

Furthermore, using the resulting statistical ensembles of black holes and the standard method explained in [67], one can propose statistical quantum ensembles for black holes in higher dimensional gravity theories. Using the formalism of GTD and the definition of *"thermodynamic limit"*, one could derive the statistical quantum metrics for black holes. Finally, one can compare the different versions of thermodynamic, statistical and quantum statistical metrics in order to find possible *"quantum effects"* at the level of the geodesics and curvature of the equilibrium space.

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